On the Rate of Convergence of $\left(\frac{\|f\|_p}{\|f\|_\infty}\right)^p$ as $p \to \infty$

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Abstract. Let $(X, \mathcal{E}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be a measurable function such that $\|f\|_p < \infty$ for all $p \geq 1$ and $\|f\|_\infty > 0$. In this paper, we describe the rate of convergence of $\left(\frac{\|f\|_p}{\|f\|_\infty}\right)^p$ as $p \to \infty$.

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1. Statement of Results

Let $(X, \mathcal{E}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be a measurable function such that $\|f\|_p < \infty$ for all $p \geq 1$ and $\|f\|_\infty > 0$. There are many results describing the limiting behaviour of $\|f\|_p$ as $p \to \infty$. For example, it is well-known that

$$\frac{\|f\|_p}{\|f\|_\infty} \to 1 \text{ as } p \to \infty;$$

see [2, p. 201] for a proof of this and some related results. However, other limiting behaviours may also be of interest. For example, in the study of the regularity of solutions to the Navier–Stokes equation, it is sometimes of interest to know the limiting behaviour of the p'th power of $\frac{\|f\|_p}{\|f\|_\infty}$, i.e. it is of interest to know how $\left(\frac{\|f\|_p}{\|f\|_\infty}\right)^p$ behaves for large values of p; see, for example, [5, Equation (38)] for a more detailed discussion of this. We first note that it is not difficult to show that $\left(\frac{\|f\|_p}{\|f\|_\infty}\right)^p$ converges as $p \to \infty$. More precisely, if we let

$$E_f = \left\{ |f| = \|f\|_\infty \right\}, \tag{1.1}$$

i.e. E_f is the extremum set of f, then

$$\left(\frac{\|f\|_p}{\|f\|_\infty}\right)^p \to \mu(E_f) \text{ as } p \to \infty. \tag{1.2}$$

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Indeed, to see this, note that if we write $F_f = \{|f| < \|f\|_\infty\}$, then

$$\|f\|_p^p = \int_{E_f} |f|^p \, d\mu + \int_{F_f} |f|^p \, d\mu = \|f\|_\infty^p \, \mu(E_f) + \int_{F_f} |f|^p \, d\mu \,,$$

and so

$$\left(\frac{\|f\|_p}{\|f\|_\infty} \right)^p = \mu(E_f) + \int_{F_f} \left(\frac{|f|}{\|f\|_\infty} \right)^p \, d\mu. \tag{1.3}$$

However, clearly $(\frac{|f|}{\|f\|_\infty})^p 1_{F_f} \to 0$ pointwise as $p \to \infty$. Also, $(\frac{|f|}{\|f\|_\infty})^p 1_{F_f} \leq \frac{|f|}{\|f\|_\infty}$ for all $p \geq 1$ and $\int \frac{|f|}{\|f\|_\infty} \, d\mu = \frac{\|f\|_1}{\|f\|_\infty} < \infty$. It follows immediately from this and the Dominated Convergence theorem that

$$\int_{F_f} \left(\frac{|f|}{\|f\|_\infty} \right)^p \, d\mu \to 0 \quad \text{as } p \to \infty. \tag{1.4}$$

Finally, (1.2) follows immediately from (1.3) and (1.4).

We will now show that it is, in fact, possible to compute the rate of convergence in (1.2). This is the main result in this note and the statement of Theorem 1.1 below.

Theorem 1.1. *Let $(X, \mathcal{E}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be a measurable function such that $\|f\|_p < \infty$ for all $p \geq 1$ and $\|f\|_\infty > 0$. Write*

$$
\begin{aligned}
\underline{a}_f &= \liminf_{r \searrow 0} \frac{\log \mu\left(\{1 - r < \frac{|f|}{\|f\|_\infty} < 1\}\right)}{\log r} \,, \\
\overline{a}_f &= \limsup_{r \searrow 0} \frac{\log \mu\left(\{1 - r < \frac{|f|}{\|f\|_\infty} < 1\}\right)}{\log r} .
\end{aligned}
\tag{1.5}
$$

If we put

$$\Delta_f(p) = \left| \left(\frac{\|f\|_p}{\|f\|_\infty} \right)^p - \mu(E_f) \right|, \tag{1.6}$$

where E_f is defined in (1.1), then

$$\underline{a}_f \leq \liminf_{p \to \infty} \frac{\log \Delta_f(p)}{-\log p} \leq \limsup_{p \to \infty} \frac{\log \Delta_f(p)}{-\log p} \leq \overline{a}_f.$$

The proof of Theorem 1.1 is given in Sect. 2.

Remark. Theorem 1.1 shows that for each $\varepsilon > 0$, there is a number $p_\varepsilon \geq 1$, such that

$$p^{-\overline{a}_f - \varepsilon} \leq \Delta_f(p) \leq p^{-\underline{a}_f + \varepsilon},$$

for all $p \geq p_\varepsilon$. In particular, if $\underline{a}_f = \overline{a}_f = a_f$, then

$$p^{-a_f - \varepsilon} \leq \Delta_f(p) \leq p^{-a_f + \varepsilon}.$$

Loosely speaking, this says that $\Delta_f(p)$ behaves roughly like p^{-a_f} for large values of p, i.e. the rate at which $(\frac{\|f\|_p}{\|f\|_\infty})^p$ converges to $\mu(E_f)$ is roughly equal to p^{-a_f}.

Remark. Let $(X, \mathcal{E}, \mu)$ be a measure space. In Theorem 1.1 we assume that the function $f : X \to \mathbb{R}$ satisfies the following two conditions: (1) $\|f\|_p < \infty$ for all $p \geq 1$ and (2) $\|f\|_\infty > 0$. We will now briefly discuss what happens if these conditions are not satisfied.

Regarding condition (2). Of course, if condition (2) is not satisfied, i.e., if $\|f\|_\infty = 0$, then f is the null function, whence $\|f\|_p = \|f\|_\infty = 0$ for all $p \geq 1$, and it follows from this that the ratio $\frac{\|f\|_p}{\|f\|_\infty}$ equals $\frac{0}{0}$ for all $p \geq 1$. In particular, we conclude that the ratio $\frac{\|f\|_p}{\|f\|_\infty}$ is undefined for all $p \geq 1$, and the problem of computing the rate of convergence of $\left(\frac{\|f\|_p}{\|f\|_\infty}\right)^p$ as $p \to \infty$ is meaningless.

Regarding condition (1). It is not difficult to see that the conclusion in Theorem 1.1 remains valid even if condition (1) is replaced by the following slightly weaker condition: there is a real number $p_0 \geq 1$ such that $\|f\|_p < \infty$ for all $p \geq p_0$. If this condition is not satisfied, i.e., if there is no number $p_0 \geq 1$ such that $\|f\|_p < \infty$ for all $p \geq p_0$, then it is not difficult to see that $\|f\|_p = \infty$ for all sufficiently large real numbers $p \geq 1$. However, simple examples show that $\|f\|_\infty$ can be either finite or infinity (for example, let $X = \mathbb{R}$ and let μ be Lebesgue measure; if the function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 1$ for all $x \in \mathbb{R}$, then $\|f\|_p = \infty$ for all real numbers $p \geq 1$, but $\|f\|_\infty = 1 < \infty$, and if the function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x$ for all $x \in \mathbb{R}$, then $\|f\|_p = \infty$ for all real numbers $p \geq 1$, but $\|f\|_\infty = \infty$). If $\|f\|_\infty < \infty$, then it follows that $\frac{\|f\|_p}{\|f\|_\infty} = \infty$ for all sufficient large real numbers p, whence $\left(\frac{\|f\|_p}{\|f\|_\infty}\right)^p = \infty$ for all sufficient large real numbers p, and the problem of computing the rate of convergence of $\left(\frac{\|f\|_p}{\|f\|_\infty}\right)^p$ as $p \to \infty$ is, therefore, trivial. On the other hand, if $\|f\|_\infty = \infty$, then it follows that the ratio $\frac{\|f\|_p}{\|f\|_\infty}$ equals $\frac{\infty}{\infty}$ for all sufficiently large real numbers p. In particular, we conclude that if $\|f\|_\infty = \infty$, then the ratio $\frac{\|f\|_p}{\|f\|_\infty}$ is undefined for all sufficiently large real numbers p, and the problem of computing the rate of convergence of $\left(\frac{\|f\|_p}{\|f\|_\infty}\right)^p$ as $p \to \infty$ is meaningless.

In several important and natural cases the numbers $\underline{a}_f$ and $\overline{a}_f$ can be computed explicitly. This is the content of the next corollary.

Corollary 1.2. *Let X be an open subset of $\mathbb{R}$ and let μ denote the Lebesgue measure on X. Let $f : X \to \mathbb{R}$ be a measurable function such that $\|f\|_p < \infty$ for all $p \geq 1$ and $\|f\|_\infty > 0$, and let E_f and $\Delta_f(p)$ be defined in (1.1) and (1.6), respectively.*

If $E_f = \{x_1, \ldots, x_n\}$ is finite and that there are polynomials P_i and Q_i for $i = 1, \ldots, n$ and a positive number $\delta > 0$ such that

$$P_i(x_i - x) \leq f(x_i) - f(x) \leq Q_i(x_i - x),$$

for all i and all $x \in B(x_i, \delta) \cap X$ (i.e. near x_i, the graph of f is "sandwiched" between the graphs of P_i and Q_i), then

$$\frac{1}{\max_i \deg Q_i} \leq \liminf_{p \to \infty} \frac{\log \Delta_f(p)}{-\log p} \leq \limsup_{p \to \infty} \frac{\log \Delta_f(p)}{-\log p} \leq \frac{1}{\max_i \deg P_i}.$$

In particular, if $E_f = \{x_0\}$ consists of just one element and there are polynomials P and Q with $\deg P = \deg Q = N$ and a positive number $\delta > 0$ such that

$$P(x_0 - x) \leq f(x_0) - f(x) \leq Q(x_0 - x),$$

for all $x \in B(x_0, \delta) \cap X$, then

$$\lim_{p \to \infty} \frac{\log \Delta_f(p)}{-\log p} = \frac{1}{N}.$$

Proof. In this case, it is not difficult to see that $\underline{a}_f = \frac{1}{\max_i \deg Q_i}$ and $\overline{a}_f = \frac{1}{\max_i \deg P_i}$ where the numbers $\underline{a}_f$ and $\overline{a}_f$ are defined in (1.5), and the result therefore follows immediately from Theorem 1.1. $\square$

Remark. The statement in Theorem 1.1 is related to local dimensions of measures. If λ a Borel probability measure on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, then the lower and upper local dimensions of λ at x are defined by

$$\underline{\dim}_{\mathsf{loc}}(\lambda; x) = \liminf_{r \searrow 0} \frac{\log \lambda(B(x, r))}{\log r},$$

and

$$\overline{\dim}_{\mathsf{loc}}(\lambda; x) = \limsup_{r \searrow 0} \frac{\log \lambda(B(x, r))}{\log r},$$

respectively. If the lower and upper local dimension of λ at x coincide, then we write $\dim_{\mathsf{loc}}(\lambda; x)$ for the common value, i.e. we write

$$\dim_{\mathsf{loc}}(\lambda; x) = \lim_{r \searrow 0} \frac{\log \lambda(B(x, r))}{\log r},$$

provided the limit exists. The detailed study of the local dimensions of measures is known as multifractal analysis and has received enormous interest during the past 20 years; the reader is refereed to the texts by Falconer [1] or Pesin [6] for a more thorough discussion of this. It is now generally believed by experts that local dimensions provide important information about the geometric properties of measures.

We will now describe the relation between the statement in Theorem 1.1 and local dimensions of measures. Let $(X, \mathcal{E}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be a measurable function such that $\|f\|_p < \infty$ for all $p \geq 1$ and $\|f\|_\infty > 0$. Define the function $\Phi_f : X \to \mathbb{R}$ by

$$\Phi_f = \frac{|f|}{\|f\|_\infty} 1_{\{|f| < \|f\|_\infty\}},$$

and let μ_f denote the distribution of Φ_f, i.e. μ_f is the Borel probability measure on $\mathbb{R}$ defined by

$$\mu_f(B) = \mu(\Phi_f^{-1}(B)),$$

for Borel subsets B of $\mathbb{R}$. It is clear that if $r > 0$, then

$$\mu\left(\left\{1 - r < \frac{|f|}{\|f\|_\infty} < 1\right\}\right) = \mu(\Phi_f^{-1}B(1,r)) = \mu_f(B(1,r)),$$

and the statement in Theorem 1.1, therefore, says that

$$\underline{\dim}_{\mathsf{loc}}(\mu_f; 1) \leq \liminf_{p\to\infty} \frac{\log \Delta_f(p)}{-\log p} \leq \limsup_{p\to\infty} \frac{\log \Delta_f(p)}{-\log p} \leq \overline{\dim}_{\mathsf{loc}}(\mu_f; 1).$$

In particular, if the local dimension $\dim_{\mathsf{loc}}(\mu_f; 1)$ of μ_f at 1 exists, then

$$\lim_{p\to\infty} \frac{\log \Delta_f(p)}{-\log p} = \dim_{\mathsf{loc}}(\mu_f; 1).$$

2. Proof of Theorem 1.1

We first prove two auxiliary results that will be used in the proof of Theorem 1.1, namely Lemmas 2.2 and 2.4. Lemma 2.2 provides an alternative expressing for the p'th moment of a measure. This expression will allow us to bound $\Delta_f(p)$ by an integral of the form $\int_{1-\delta}^1 pu^p(1-u)^a \, du$ for suitable choices of δ and a, and Lemma 2.4 establishes the asymptotic behaviour of the integral $\int_{1-\delta}^1 pu^p(1-u)^a \, du$ as $p \to \infty$. Before stating and proving the first main auxiliary result, namely Lemma 2.2, we recall the following well-known result from analysis:

Lemma 2.1. *Let X be a separable metric space and let m be a Borel measure on X. If $f : X \to [0, \infty)$ is a positive Borel function, then*

$$\int f \, dm = \int_0^\infty m(\{f \geq t\}) \, dt.$$

Proof. This result is proven in [3, Theorem 1.15]. $\qquad\square$

Lemma 2.2. *Let μ be a Borel probability measure on $[0, 1]$. Fix $0 < \delta < 1$. Then there is a function $h : [1, \infty) \to \mathbb{R}$ such that*

$$\int x^p \, d\mu(x) = \int_{1-\delta}^1 pu^{p-1} \mu([u, 1]) \, du + h(p)$$

and $|h(p)| \leq (1-\delta)^p$ for all $p \geq 1$.

Proof. It now follows from Lemma 2.1 that

$$\int x^p \, d\mu(x) = \int_0^\infty \mu\big(\{x \in [0,1] \,|\, x^p \geq t\}\big) \, dt = \int_0^\infty \mu\big(\{x \in [0,1] \,|\, x \geq t^{\frac{1}{p}}\}\big) \, dt. \tag{2.1}$$

Introducing the substitution $u = t^{\frac{1}{p}}$ into the integral in (2.1), it now follows that

$$\int x^p \, d\mu(x) = \int_0^\infty pu^{p-1} \mu\big(\{x \in [0,1] \,|\, x \geq u\}\big) \, du,$$

and the assumption $\operatorname{supp} \mu \subseteq [0,1]$, therefore, implies that

$$\int x^p \, \mathrm{d}\mu(x) = \int_0^1 pu^{p-1} \, \mu([u,1]) \, \mathrm{d}u \,, \tag{2.2}$$

It follows immediately from (2.2) that

$$\int x^p \, \mathrm{d}\mu(x) = \int_{1-\delta}^1 pu^{p-1} \, \mu([u,1]) \, \mathrm{d}u + h(p) \,,$$

where $h(p) = \int_0^{1-\delta} pu^{p-1} \, \mu([u,1]) \, \mathrm{d}u$. In particular, we conclude that $|h(p)| \leq \int_0^{1-\delta} pu^{p-1} \, \mathrm{d}u = (1-\delta)^p$ for all $p \geq 1$. $\qquad \square$

Next, we state and prove the second main auxiliary result, namely Lemma 2.4. In order to prove Lemma 2.4 we first prove Lemma 2.3 below.

Lemma 2.3. *Fix $0 < \delta < 1$ and $a > 0$. Then there are functions $f, g : [1, \infty) \to \mathbb{R}$ and a real number c such that*

$$\int_{1-\delta}^1 pu^{p-1}(1-u)^a \, du = c \, f(p) \, p^{-a} + g(p)$$

and $f(p) \to 1$ as $p \to \infty$ and $|g(p)| \leq (1-\delta)^p$ for all $p \geq 1$.

Proof. Define the function $f : [1, \infty) \to \mathbb{R}$ and the real number c by $f(p) = p^a \frac{\Gamma(p+1)}{\Gamma(p+a+1)}$ and $c = \Gamma(a+1)$, and note that it follows from [4, p. 119] that $f(p) \to 1$ as $p \to \infty$.

Also, define the function $g : [1, \infty) \to \mathbb{R}$ by $g(p) = -\int_0^{1-\delta} pu^{p-1}(1-u)^a \, du$, and note that $|g(p)| \leq \int_0^{1-\delta} pu^{p-1}(1-u)^a \, du \leq \int_0^{1-\delta} pu^{p-1} \, du = (1-\delta)^p$ for all $p \geq 1$.

Finally, observe that it follows from [4, p. 36, (1.10)] that $\int_0^1 u^{p-1}(1-u)^a \, du = \frac{\Gamma(p)\Gamma(a+1)}{\Gamma(p+a+1)}$, whence

$$\int_{1-\delta}^1 pu^{p-1}(1-u)^a \, du = \int_0^1 pu^{p-1}(1-u)^a \, du - \int_0^{1-\delta} pu^{p-1}(1-u)^a \, du$$

$$= p\frac{\Gamma(p)\Gamma(a+1)}{\Gamma(p+a+1)} + g(p)$$

$$= \frac{\Gamma(p+1)\Gamma(a+1)}{\Gamma(p+a+1)} + g(p)$$

$$= c \, f(p) \, p^{-a} + g(p),$$

for all $p \geq 1$. $\qquad \square$

Lemma 2.4. *Fix $0 < \delta < 1$, $a > 0$. Let $h : [1, \infty) \to \mathbb{R}$ be a function and assume that $|h(p)| \leq (1-\delta)^p$ for all $p \geq 1$. Then*

$$\lim_{p \to \infty} \frac{\log \left(\int_{1-\delta}^1 pu^{p-1}(1-u)^a \, du + h(p) \right)}{-\log p} = a.$$

Proof. It follows from Lemma 2.3 there are functions $f, g : [1, \infty) \to \mathbb{R}$ and a real number c such that

$$\int_{1-\delta}^1 p u^{p-1}(1-u)^a \, du = c \, f(p) \, p^{-a} \, + \, g(p),$$

and $f(p) \to 1$ as $p \to \infty$ and $|g(p)| \le (1-\delta)^p$ for all $p \ge 1$. In particular, this shows that

$$\int_{1-\delta}^1 p u^{p-1}(1-u)^a \, du \, + \, g(p) = c \, f(p) \, p^{-a} \, + \, g(p) \, + \, h(p) = p^{-a} \, \varphi(p),$$

where the function $\varphi : [1, \infty) \to \mathbb{R}$ is defined by $\varphi(q) = c \, f(p) + p^a g(p) + p^a h(p)$, and so

$$\frac{\log \left(\int_{1-\delta}^1 p u^{p-1}(1-u)^a \, du \, + \, h(p) \right)}{-\log p} = a - \frac{\log \varphi(p)}{\log p}. \tag{2.3}$$

However, we clearly have $|p^a g(p)| \le p^a (1-\delta)^p \to 0$ as $p \to \infty$ and $|p^a h(p)| \le p^a (1-\delta)^p \to 0$ as $p \to \infty$, and so $\varphi(p) = c \, f(p) + p^a g(p) + p^a h(p) \to c$ as $p \to \infty$. The desired result follows from this and (2.3). $\square$

We now turn towards the proof of Theorem 1.1.

Proof of Theorem 1.1. Note that it follows from (1.3) that

$$\Delta_f(p) = \left| \left(\frac{\|f\|_p}{\|f\|_\infty} \right)^p - \mu(E_f) \right| = \int_{\{|f| < \|f\|_\infty\}} \left(\frac{|f|}{\|f\|_\infty} \right)^p \, d\mu. \tag{2.4}$$

Next, as in the remark following the statement of Corollary 1.2, define $\Phi_f : X \to \mathbb{R}$ by $\Phi_f = \frac{|f|}{\|f\|_\infty} 1_{\{|f| < \|f\|_\infty\}}$, and let μ_f denote the distribution of Φ_f, i.e., μ_f is the Borel probability measure on $\mathbb{R}$ defined by $\mu_f(B) = \mu(\Phi_f^{-1}(B))$ for Borel subsets B of $\mathbb{R}$. It now follows from (2.4) and the definition of μ_f that μ_f is a Borel probability measure on $[0, 1]$ with

$$\Delta_f(p) = \int_{\{|f| < \|f\|_\infty\}} \left(\frac{|f|}{\|f\|_\infty} \right)^p \, d\mu = \int \Phi_f^p \, d\mu = \int x^p \, d\mu_f(x), \tag{2.5}$$

and

$$\mu \left(\left\{ 1 - r < \frac{|f|}{\|f\|_\infty} < 1 \right\} \right) = \mu(\Phi_f^{-1} B(1, r)) = \mu_f(B(1, r)), \tag{2.6}$$

for $r > 0$. Also, recall (see the Remark following the statement of Corollary 1.2) that the lower and upper local dimension of μ_f at 1 are defined by

$$\underline{\dim}_{\mathsf{loc}}(\mu_f; 1) = \liminf_{r \searrow} \frac{\log \mu_f(B(1, r))}{\log r},$$

and

$$\overline{\dim}_{\mathsf{loc}}(\mu_f; 1) = \limsup_{r \searrow} \frac{\log \mu_f(B(1, r))}{\log r},$$

respectively. It follows from (2.5) and (2.6) that the statement in Theorem 1.1 can be reformulated as

$$\underline{\dim}_{\mathsf{loc}}(\mu_f; 1) \leq \liminf_{p \to \infty} \frac{\log \int x^p \, \mathrm{d}\mu_f(x)}{-\log p}$$

$$\leq \limsup_{p \to \infty} \frac{\log \int x^p \, \mathrm{d}\mu_f(x)}{-\log p} \leq \overline{\dim}_{\mathsf{loc}}(\mu_f; 1). \qquad (2.7)$$

We will now prove (2.7). Fix $\varepsilon > 0$ with $0 < \varepsilon < \underline{\dim}_{\mathsf{loc}}(\mu_f; 1)$, and note that we can choose $\delta_\varepsilon > 0$ such that

$$\underline{\dim}_{\mathsf{loc}}(\mu_f; 1) - \varepsilon \leq \frac{\log \mu_f(B(1, r))}{\log r} \leq \overline{\dim}_{\mathsf{loc}}(\mu_f; 1) + \varepsilon, \qquad (2.8)$$

for all $0 < r < \delta_\varepsilon$. It now follows from Lemma 2.2 that there is a function $h_\varepsilon : [1, \infty) \to \mathbb{R}$ such that

$$\int x^p \, \mathrm{d}\mu_f(x) = I_\varepsilon(p) + h_\varepsilon(p), \qquad (2.9)$$

where

$$I_\varepsilon(p) = \int_{1-\delta_\varepsilon}^1 p u^{p-1} \, \mu_f([u, 1]) \, \mathrm{d}u,$$

and $|h_\varepsilon(p)| \leq (1 - \delta_\varepsilon)^p$ for all $p \geq 1$.

We will now estimate $I_\varepsilon(p)$. For brevity we write $\underline{\alpha}_\varepsilon = \underline{\dim}_{\mathsf{loc}}(\mu; \mathbf{1}) - \varepsilon$ and $\overline{\alpha}_\varepsilon = \overline{\dim}_{\mathsf{loc}}(\mu; \mathbf{1}) + \varepsilon$. Observe that if $u \in (1 - \delta_\varepsilon, 1)$, then $1 - u \leq \delta_\varepsilon$, whence (using (2.8)) $\underline{\alpha}_\varepsilon = \underline{\dim}_{\mathsf{loc}}(\mu_f; 1) - \varepsilon \leq \frac{\log \mu_f(B(\mathbf{1}, 1-u))}{\log(1-u)} \leq \overline{\dim}_{\mathsf{loc}}(\mu_f; 1) - \varepsilon = \overline{\alpha}_\varepsilon$ and so $\mu_f([u, 1]) = \mu_f(B(1, 1 - u)) \leq (1 - u)^{\underline{\alpha}_\varepsilon}$ and $\mu_f([u, 1]) = \mu_f(B(1, 1 - u)) \geq (1 - u)^{\overline{\alpha}_\varepsilon}$. This clearly implies that

$$I_\varepsilon(p) = \int_{1-\delta_\varepsilon}^1 p u^{p-1} \, \mu_f([u, 1]) \, \mathrm{d}u \leq \int_{1-\delta_\varepsilon}^1 p u^{p-1} \, (1 - u)^{\underline{\alpha}_\varepsilon} \, \mathrm{d}u ,$$

$$(2.10)$$

$$I_\varepsilon(p) = \int_{1-\delta_\varepsilon}^1 p u^{p-1} \, \mu_f([u, 1]) \, \mathrm{d}u \geq \int_{1-\delta_\varepsilon}^1 p u^{p-1} \, (1 - u)^{\overline{\alpha}_\varepsilon} \, \mathrm{d}u .$$

Combining (2.9) and (2.10) yields

$$\int x^p \, \mathrm{d}\mu_f(x) \leq \int_{1-\delta_\varepsilon}^1 p u^{p-1} \, (1 - u)^{\underline{\alpha}_\varepsilon} \, \mathrm{d}u \, + \, h_\varepsilon(p) ,$$

$$(2.11)$$

$$\int x^p \, \mathrm{d}\mu_f(x) \geq \int_{1-\delta_\varepsilon}^1 p u^{p-1} \, (1 - u)^{\overline{\alpha}_\varepsilon} \, \mathrm{d}u \, + \, h_\varepsilon(p) ,$$

where $|h_\varepsilon(p)| \leq (1 - \delta_\varepsilon)^p$ for all $p \geq 1$, and $\underline{\alpha}_\varepsilon, \overline{\alpha}_\varepsilon > 0$. It therefore follows from Lemma 2.4 and 2.11 that

$$\liminf_{p \to \infty} \frac{\log \int x^p \, \mathrm{d}\mu_f(x)}{-\log p} \geq \lim_{p \to \infty} \frac{\log \left(\int_{1-\delta_\varepsilon}^1 pu^{p-1} (1-u)^{\underline{\alpha}_\varepsilon} \, \mathrm{d}u \, + \, h_\varepsilon(p) \right)}{-\log p}$$
$$= \underline{\alpha}_\varepsilon = \underline{\dim}_{\mathsf{loc}}(\mu_f; 1) - \varepsilon,$$

and

$$\limsup_{p \to \infty} \frac{\log \int x^p \, \mathrm{d}\mu_f(x)}{-\log p} \leq \lim_{p \to \infty} \frac{\log \left(\int_{1-\delta_\varepsilon}^1 pu^{p-1} (1-u)^{\overline{\alpha}_\varepsilon} \, \mathrm{d}u \, + \, h_\varepsilon(p) \right)}{-\log p}$$
$$= \overline{\alpha}_\varepsilon = \overline{\dim}_{\mathsf{loc}}(\mu_f; 1) + \varepsilon$$

for all $\varepsilon > 0$. Letting $\varepsilon \searrow 0$ now gives the desired result. $\qquad\square$

References

[1] Falconer, K.J.: Fractal Geometry-Mathematical Foundations and Applications. Wiley, Hoboken (1990)

[2] Hewitt, E., Stromberg, K.: Real and Abstract Analysis. A Modern Treatment of the Theory of Functions of a Real Variable. Springer, New York (1965)

[3] Mattila, P.: Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability, Cambridge Studies in Advanced Mathematics, vol. 44. Cambridge University Press, Cambridge (1995)

[4] Olver, F.W.J.: Asymptotics and Special Functions, Computer Science and Applied Mathematics. Academic Press, New York (1974)

[5] Tran, C., Yu, X.: A geometrical approach to the problem of Navier–Stokes regularity. (2018) **(Preprint)**

[6] Pesin, Y.: Dimension Theory in Dynamical Systems. Contemporary Views and Applications. The University of Chicago Press, Chicago (1997)

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