# EMBEDDING RIGHT-ANGLED ARTIN GROUPS INTO BRIN-THOMPSON GROUPS

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ABSTRACT. We prove that every finitely-generated right-angled Artin group can be embedded into some Brin-Thompson group $nV$. It follows that many other groups can be embedded into some $nV$ (e.g., any finite extension of any of Haglund and Wise's special groups), and that various decision problems involving subgroups of $nV$ are unsolvable.

## 1. INTRODUCTION

The **Brin-Thompson groups** $nV$ are an infinite family of groups that act by homeomorphisms on Cantor spaces. They were first defined by Matt Brin in [7], and can be viewed as higher-dimensional generalizations of the group $V$ $(= 1V)$ defined by Richard J. Thompson (see [10] for an introduction to Thompson's groups). The groups $nV$ are finitely presented [19] and simple [9], and for $1 \leq j < k$ it is known that $jV$ embeds into $kV$ [7], but also that $jV$ and $kV$ are not isomorphic [3].

Recall that a **simple graph** is a finite graph with no loops (a loop is an edge from a vertex to itself) and no mulitple edges (any two distinct vertices admit at most one edge between them). Given a simple graph $\Gamma$, the associated **right-angled Artin group** $A_\Gamma$ has one generator $g_v$ for each vertex $v$ of $\Gamma$, with one relation of the form $g_v g_w = g_w g_v$ for each pair of vertices $v, w$ that are connected by an edge. This class of groups has been studied extensively (see [11] for an introduction).

Our main result is the following theorem:

**Theorem 1.1.** *For any simple graph $\Gamma$, there exists an $n \geq 1$ so that the right-angled Artin group $A_\Gamma$ embeds isomorphically into $nV$.*

Specifically, we prove that $A_\Gamma$ embeds into $nV$ for $n = |V| + |E^c|$, where $V$ is the set of vertices of $\Gamma$ and $E^c$ is the set of complementary edges, i.e. the set of all pairs of vertices that are *not* connected by an edge.

Recall the definition of Bleak and Salazar-Díaz in [5] that a group $G$ is **demonstrable for a group** $K$ of homeomorphisms of a space $X$ if there is an embedding $\hat{G}$ of $G$ into $K$ so that there is an open set $U$ in $X$ so that for all $\hat{g} \neq 1_K \in \hat{G}$ we have $U \cdot \hat{g} \cap U = \emptyset$ (we call the image group $\hat{G}$ a demonstrative subgroup of $K$ with demonstration set $U$). Bleak and Salazar-Díaz also say that such a group $K$ of homeomorphisms of a space $X$ **acts with local realisation** if given any open set $U \subset X$, there is a subgroup $K_U$ of $K$ supported only on $U$ so that $K_U \cong K$. We now state a theorem of [5].

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**Theorem:** [Bleak–Salazar-Díaz, Proposition 3.6] *Let $K$ be a group of homeomorphisms of a space $X$ so that $K$ acts with local realisation, and let $G$ be demonstrable for $K$. Then $K \wr G$ embeds in $K$.*

The embedding $A_\Gamma \to nV$ that we construct is demonstrative for $nV$. Moreover, $nV$ acts on the Cantor cube with local realization, so we obtain the following.

**Theorem 1.2.** *For any simple graph $\Gamma$, there exists an $n \geq 1$ so that the standard restricted wreath product $nV \wr A_\Gamma \cong \left(\bigoplus_{A_\Gamma} nV\right) \rtimes A_\Gamma$ embeds isomorphically into $nV$.*

The Krasner-Kaloujnine theorem [21] states that any extension of a group $G$ by a group $H$ is contained in the (unrestricted) wreath product $H \wr G$. Since, for all $n \geq 1$, all finite groups embed in $nV$ in a demonstrative way, Theorem 1.1 and Krasner-Kaloujnine theorem show the following.

**Corollary 1.3.** *For any finite graph $\Gamma$ there is a natural number $n$ so that every finite extension of $A_\Gamma$ embeds into $nV$.*

Recall that we say a group $G$ **virtually embeds** in a group $H$ if $G$ admits a finite index subgroup $F$ so that $F$ embeds in $H$. Note also that any group that virtually embeds into $nV$ embeds isomorphically into $nV$. This follows from the fact that finite groups embed demonstratively into $V$ (and hence into $nV$ [5, Lemma 3.3] by taking products across the cantor spaces in the extraneous dimensions), combined with the argument above. In particular, any group that virtually embeds into a right-angled Artin group embeds isomorphically into some $nV$.

**Corollary 1.4.** *If $G$ is a surface group or a graph braid group, then there exists an $n \geq 1$ so that $G$ embeds into $nV$.*

*Proof.* Droms, Servatius, and Servatious prove that the fundamental group of a surface of genus five embeds into $A_\Gamma$ for $\Gamma$ a 5-cycle [24], and it follows that all hyperbolic surface groups embed into $10V$. Neunhöffer, the second and third author observe that all non-hyperbolic surface groups embed into $V$ [4].

As for graph braid groups, Crisp and Wiest prove that all graph braid groups embed in some right-angled Artin group [15]. □

Note that the braid groups $B_n$ do not belong to the family of graph braid groups. We do not know whether braid groups can be embedded into $nV$.

Haglund and Wise have shown that the fundamental group of any "special" cube complex embeds in a right-angled Artin groups [17]. These are known as **special groups**, and any group that has a special subgroup of finite index is **virtually special**.

**Corollary 1.5.** *For any virtually special group $G$, there exists an $n \geq 1$ so that $G$ embeds isomorphically into $nV$. This includes:*

(1) *All finitely generated Coxeter groups* [18].
(2) *Many word hyperbolic groups, including all one-relator groups with torsion* [25].
(3) *All limit groups* [25].
(4) *Many 3-manifold groups, including the fundamental groups of all compact 3-manifolds that admit a Riemannian metric of nonpositive curvature* [23], *as well as all finite-volume hyperbolic 3-manifolds* [1].

Bridson has recently proven several undecidability results for right-angled Artin groups [6]. These have the following consequences for the Brin-Thompson groups.

**Corollary 1.6.** *There exists an $n \geq 1$ with the following properties. First, the isomorphism problem for finitely presented subgroups of $nV$ is unsolvable. Second, there exists a subgroup $H \leq nV$ that has unsolvable subgroup membership problem and unsolvable conjugacy problem.*

As far as we know, none of the decision problems mentioned in this corollary have been settled for Thompson's group $V$. Thus, it is conceivable that the statement of this corollary holds for $n = 1$.

In general, the bound $n = |V| + |E^c|$ for $A_\Gamma$ embedding into $nV$ is far from sharp. For example, our method proves that a free group of rank $k$ embeds into $nV$ for $n = k(k+1)/2$, but in fact all such groups embed into $V$. However, Bleak and Salazar-Díaz show that $\mathbb{Z}^2 * \mathbb{Z}$ does not embed into $V$, and hence the only right-angled Artin groups that embed into $V$ are direct products of free groups [5]. This leads to the following conjecture.

**Conjecture 1.7.** *A right-angled Artin group $A_\Gamma$ embeds into $nV$ if and only if $A_\Gamma$ does not contain $\mathbb{Z}^{n+1} * \mathbb{Z}$.*

**Remark 1.8.** We have several remarks.

(1) Firstly, we note that Corwin and Haymaker in [12] use the main result of Bleak–Salazar-Díaz to show that the only obstruction to a right-angled Artin group embedding into $V$ is the existence of a subgroup isomorphic to $\mathbb{Z}^2 * \mathbb{Z}$, thus verifying the $n = 1$ case of the conjecture above.

(2) If the conjecture is true, this would imply that the right-angled Artin group for a 5-cycle embeds into $2V$, and hence all surface groups would embed into $2V$ as well.

(3) Hsu and Wise proved that right-angled Artin groups can be embedded into $\mathrm{SL}_n(\mathbb{Z})$ (see [20]) and Grigorchuk, Sushanski and Romankov showed that $\mathrm{SL}_n(\mathbb{Z})$ can be realized using synchronous automata. We observe that one can use Theorem 1.1 coupled with the embedding Theorem 5.2 in [2] to recover a weaker version of this result, showing that right-angled Artin groups can be realized using asynchronous automata.

## 2. Right-Angled Artin Groups

Given a finite graph $\Gamma$ with vertex set $V_\Gamma = \{v_1, \ldots, v_n\}$ and edge set $E$, the corresponding **right-angled Artin group** $A_\Gamma$ is defined by the presentation

$$A_\Gamma = \big\langle g_1, \ldots, g_n \;\big|\; g_i g_j = g_j g_i \text{ for all } \{v_i, v_j\} \in E \big\rangle.$$

For example, if $\Gamma$ has no edges, then $A_\Gamma$ is a free group on $n$ generators. Similarly, if $\Gamma$ is a complete graph, then $A_\Gamma$ is a free abelian group of rank $n$. See [11] for a general introduction to these groups.

We need a version of the ping-pong lemma for actions of right-angled Artin groups. The following is a slightly modified version of the ping-pong lemma for right-angled Artin groups stated in [13] (also see [22]).

**Theorem 2.1** (Ping-Pong Lemma for Right-Angled Artin Groups)**.** *Let $A_\Gamma$ be a right-angled Artin group with generators $g_1, \ldots, g_n$ acting on a set $X$. Suppose that there exist subsets $\{S_i^+\}_{i=1}^n$ and $\{S_i^-\}_{i=1}^n$ of $X$, with $S_i = S_i^+ \cup S_i^-$, satisfying the following conditions:*

(1) *$g_i(S_i^+) \subseteq S_i^+$ and $g_i^{-1}(S_i^-) \subseteq S_i^-$ for all $i$.*
(2) *If $g_i$ and $g_j$ commute (with $i \neq j$), then $g_i(S_j) = S_j$.*
(3) *If $g_i$ and $g_j$ do not commute, then $g_i(S_j) \subseteq S_i^+$ and $g_i^{-1}(S_j) \subseteq S_i^-$.*
(4) *There exists a point $x \in X - \bigcup_{i=1}^n S_i$ such that $g_i(x) \in S_i^+$ and $g_i^{-1}(x) \in S_i^-$ for all $i$.*

*Then the action of $A_\Gamma$ on $X$ is faithful.* □

Indeed, if $U$ is any subset of $X - \bigcup_{i=1}^n S_i$ such that $g_i(U) \subseteq S_i^+$ and $g_i^{-1}(x) \in S_i^-$, then all of the sets $\{g(U) \mid g \in A_\Gamma\}$ are disjoint. In the case where $X$ is a topological space and $U$ is an open set, this means that the action of $A_\Gamma$ on $X$ is demonstrative in the sense of [5].

## 3. The Groups $nV$ and $XV$

Given a finite alphabet $\Sigma$, let $\Sigma^\omega$ denote the space of all strings of symbols from $\Sigma$ under the product topology, and let $\Sigma^*$ denote the set of all finite strings of symbols from $\Sigma$, including the empty string.

A **Cantor cube** is any finite product $X = \Sigma_1^\omega \times \cdots \times \Sigma_n^\omega$, where $\Sigma_1, \ldots, \Sigma_n$ are finite alphabets with at least two symbols each. Given any tuple $(\alpha_1, \ldots, \alpha_n) \in \Sigma_1^* \times \cdots \times \Sigma_n^*$, the corresponding **subcube** $X(\alpha_1, \ldots, \alpha_n)$ of $X$ is the set of all points $(x_1, \ldots, x_n) \in X$ such that $x_i$ begin with $\alpha_i$ for each $i$. Note that $X$ is homeomorphic to $X(\alpha_1, \ldots, \alpha_n)$ via the map

$$(x_1, \ldots, x_n) \;\mapsto\; (\alpha_1 \cdot x_1, \ldots, \alpha_n \cdot x_n)$$

where $\cdot$ denotes concatenation. More generally, any two subcubes $X(\alpha_1, \ldots, \alpha_n)$ and $X(\beta_1, \ldots, \beta_n)$ of $X$ have a **canonical homeomorphism** between them given by prefix replacement, i.e.

$$(\alpha_1 \cdot x_1, \ldots, \alpha_n \cdot x_n) \;\mapsto\; (\beta_1 \cdot x_1, \ldots, \beta_n \cdot x_n).$$

A **rearrangement** of a Cantor cube $X$ is any homeomorphism of $X$ obtained through the following procedure:

(1) Choose a partition $D_1, \ldots, D_k$ of the domain $X$ into finitely many subcubes.
(2) Choose another partition $R_1, \ldots, R_k$ of the range $X$ into the same number of subcubes.
(3) Define a homeomorphism $h\colon X \to X$ piecewise by mapping each $D_i$ to $R_i$ via a canonical homeomorphism.

The rearrangements of $X$ form a group under composition, which we refer to as $XV$. In the case where $X = (\{0,1\}^\omega)^n$, the group $XV$ is known as the **Brin-Thompson group $nV$**.

**Proposition 3.1.** *If $X = \Sigma_1^\omega \times \cdots \times \Sigma_n^\omega$ is any Cantor cube, then the rearrangement group $XV$ of $X$ embeds into the Brin-Thompson group $nV$.*

*Proof.* For each $i$, let $\lambda_{i,1}, \lambda_{i,2}, \ldots, \lambda_{i,m_i}$ denote the symbols of $\Sigma_i$. For each $i$ we choose a complete binary prefix code $\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,m_i}$ with $m_i$ different codewords. That is, we choose a finite rooted binary tree with $m_i$ leaves, and we let $\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,m_i}$ be the binary addresses of these leaves. Then the map

$$\lambda_{i,j_1} \cdot \lambda_{i,j_2} \cdot \lambda_{i,j_3} \cdots \;\mapsto\; \alpha_{i,j_1} \cdot \alpha_{i,j_2} \cdot \alpha_{i,j_3} \cdots$$

gives a homeomorphism from $\Sigma_i^\omega$ to $\{0,1\}^\omega$. Taking the Cartesian product gives a homeomorphism $X \to (\{0,1\}^\omega)^n$ which maps each subcube of $X$ to a subcube of $(\{0,1\}^\omega)^n$, and it is easy to check that conjugating $XV$ by this homeomorphism yields a subgroup of $nV$. □

## 4. Embedding Right-Angled Artin Groups

The goal of this section is to prove Theorem 1.1. That is, we wish to embed any right-angled Artin group into some $nV$.

Let $A_\Gamma$ be a right-angled Artin group with generators $g_1, \ldots, g_n$. For convenience, we assume that none of the generators $g_i$ lie in the center of $A_\Gamma$. For in this case $A_\Gamma \cong A'_{\Gamma'} \times \mathbb{Z}$ for some right-angled Artin group $A'_{\Gamma'}$ with fewer generators, and since $sV \times \mathbb{Z}$ embeds in $sV$, any embedding $A'_{\Gamma'} \to kV$ yields an embedding $A_\Gamma \to kV$.

Let $P$ be the set of all pairs $\{i,j\}$ for which $g_i g_j \neq g_j g_i$, and note that each $i \in \{1, \ldots, n\}$ lies in at least one element of $P$. Let $X$ be the following Cantor cube:

$$X \;=\; \prod_{i=1}^{n} \{0,1\}^\omega \;\times \prod_{\{i,j\}\in P} \{i,j,\emptyset\}^\omega.$$

Our goal is to prove the following theorem.

**Theorem 4.1.** *The group $A_\Gamma$ embeds into $XV$, and hence embeds into $kV$ for $k = n + |P|$.*

We begin by establishing some notation:

(1) For each point $x \in X$, we will denote its components by $\{x_i\}_{i\in\{1,\ldots,n\}}$ and $\{x_{ij}\}_{\{i,j\}\in P}$.
(2) Given any $i \in \{1,\ldots,n\}$ and $\alpha \in \{0,1\}^*$, let $C_i(\alpha)$ be the subcube consisting of all $x \in X$ for which $x_i$ begins with $\alpha$. Let $L_{i,\alpha}\colon X \to C_i(\alpha)$ be the canonical homeomorphism, i.e. the map that prepends $\alpha$ to $x_i$.
(3) For each $i \in \{1,\ldots,n\}$, let $P_i$ be the set of all $j$ for which $\{i,j\} \in P$, and let $S_i$ be the subcube consisting of all $x \in X$ such that $x_{ij}$ begins with $i$ for all $j \in P_i$. Let $F_i\colon X \to S_i$ be the canonical homeomorphism, i.e. the map that prepends $i$ to $x_{ij}$ for each $j \in P_i$.
(4) Let $S_{ii} = F_i(S_i) = F_i^2(X)$, i.e. the subcube consisting of all $x \in X$ such that $x_{ij}$ begins with $ii$ for each $j \in P_i$.

Now, for each $i \in \{1,\ldots,n\}$, define a homeomorphism $h_i\colon X \to X$ as follows:

(1) $h_i$ maps $X - S_i$ to $(S_i - S_{ii}) \cap C_i(10)$ via $L_{i,10} \circ F_i$.
(2) $h_i$ is the identity on $S_{ii}$.
(3) $h_i$ maps $(S_i - S_{ii}) \cap C_i(1)$ to $(S_i - S_{ii}) \cap C_i(11)$ via $L_{i,1}$.
(4) $h_i$ maps $(S_i - S_{ii}) \cap C_i(01)$ to $X - S_i$ via $F_i^{-1} \circ L_{i,01}^{-1}$.
(5) $h_i$ maps $(S_i - S_{ii}) \cap C_i(00)$ to $(S_i - S_{ii}) \cap C_i(0)$ via $L_{i,0}^{-1}$.

Note that the five domain pieces form a partition of $X$, and each is the union of finitely many subcubes. Similarly, the five range pieces form a partition of $X$, and each is the union of finitely many subcubes. Since each of the maps is a restriction of a canonical homeomorphism, it follows that $h_i$ is an element of $XV$.

**Proposition 4.2.** *For each $i, j \in \{1, \ldots, n\}$, if $g_i$ and $g_j$ commute, then so do $h_i$ and $h_j$.*

*Proof.* Observe that $h_i(x)$ is completely determined by $x_i$ and $\{x_{ij}\}_{j \in P_i}$, and only changes these coordinates of $x$. If $g_i$ and $g_j$ commute, then the relevant sets of coordinates for $h_i$ and $h_j$ do not overlap, and hence $h_i$ and $h_j$ commute. □

Thus we can define a homomorphism $\Phi\colon A_\Gamma \to XV$ by $\Phi(g_i) = h_i$ for each $i$.

**Proposition 4.3.** *The homomorphism $\Phi$ is injective.*

*Proof.* For each $i$, let $S_i^+ = S_i \cap C_i(1)$, and let $S_i^- = S_i \cap C_i(0)$. These two sets form a partition of $S_i$, with

$$h_i(S_i^+) = S_i \cap C_i(11) \subseteq S_i^+ \qquad \text{and} \qquad h_i^{-1}(S_i^-) = S_i \cap C_i(00) \subseteq S_i^-.$$

Now suppose we are given two generators $g_i$ and $g_j$. If $g_i$ and $g_j$ commute, then clearly $h_i(S_j) = S_j$. If $g_i$ and $g_j$ do not commute, then $S_j \subseteq X - S_i$, and therefore $h_i(S_j) \subseteq S_i^+$ and $h_i^{-1}(S_j) \subseteq S_i^-$.

Finally, let $x$ be an point in $X$ such that $x_{ij}$ starts with $\emptyset$ for all $\{i, j\} \in P$. Then $x \in X - S_i$ for all $i$, so $h_i(x) \in S_i^+$ and $h_i^{-1}(x) \in S_i^-$. The homomorphism $\Phi$ is thus injective by Theorem 2.1. □

This proves our main theorem. Note further that, if $U$ is the open subset of $X$ consisting of all points $x \in X$ such that $x_{ij}$ starts with $\emptyset$ for all $\{i, j\} \in P$, then $h_i(U) \subseteq S_i^+$ and $h_i^{-1}(U) \subseteq S_i^-$, and therefore all of the sets $\{g(U) \mid g \in A_\Gamma\}$ are disjoint. Thus the action of $A_\Gamma$ on $X$ is demonstrative in the sense of [5], and it follows that the conjugate action of $A_\Gamma$ on $(\{0,1\}^\omega)^k$ as a subgroup of $kV$ is demonstrative as well.

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