Research Article

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Topological 2-generation of automorphism groups of countable ultrahomogeneous graphs

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Abstract: A countable graph is *ultrahomogeneous* if every isomorphism between finite induced subgraphs can be extended to an automorphism. Woodrow and Lachlan showed that there are essentially four types of such countably infinite graphs: the random graph, infinite disjoint unions of complete graphs K_n with $n \in \mathbb{N}$ vertices, the K_n-free graphs, finite unions of the infinite complete graph K_ω, and duals of such graphs. The groups $\mathrm{Aut}(\Gamma)$ of automorphisms of such graphs Γ have a natural topology, which is compatible with multiplication and inversion, i.e. the groups $\mathrm{Aut}(\Gamma)$ are topological groups. We consider the problem of finding minimally generated dense subgroups of the groups $\mathrm{Aut}(\Gamma)$ where Γ is ultrahomogeneous. We show that if Γ is ultrahomogeneous, then $\mathrm{Aut}(\Gamma)$ has 2-generated dense subgroups, and that under certain conditions given $f \in \mathrm{Aut}(\Gamma)$ there exists $g \in \mathrm{Aut}(\Gamma)$ such that the subgroup generated by f and g is dense. We also show that, roughly speaking, g can be chosen with a high degree of freedom. For example, if Γ is either an infinite disjoint union of K_n or a finite union of K_ω, then g can be chosen to have any given finite set of orbit representatives.

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1 Introduction

Finding minimal generating sets for groups and semigroups is a classical problem in the literature. If G is a group, X is a subset of G, and $\langle X\rangle$ denotes the subgroup generated by X, then

$$|X| \leq |\langle X\rangle| \leq \sum_{n\in\mathbb{N}} |X|^n.$$

Hence if X is finite, then $\langle X\rangle$ is either finite or countable, and if X is infinite then $\langle X\rangle$ has the same cardinality as X. Therefore, if G is uncountable and X is a generating set for G, it follows that $|G| = |X|$. In this case, no new information is captured by knowing the cardinality of a minimal generating set. In this paper, we are concerned with generating certain subgroups of a particular class of uncountable groups. More precisely, we are interested in generating dense subgroups of certain uncountable topological groups, i.e. groups with a topology where multiplication and inversion are continuous. For example, if the set of integers $\mathbb{Z}$ has the discrete topology and $\mathrm{Sym}(\mathbb{Z})$ denotes the symmetric group on $\mathbb{Z}$, then $\mathrm{Sym}(\mathbb{Z})$ is a topological group with the

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subspace topology inherited from the product topology on $\mathbb{Z}^{\mathbb{Z}}$. It is relatively straightforward to show that the subgroup of Sym($\mathbb{Z}$) generated by the permutation f defined by $(i)f = i + 1$ for all $i \in \mathbb{Z}$, and any transposition $(j\ j + 1)$ is dense in Sym($\mathbb{Z}$). In other words, Sym($\mathbb{Z}$) has a 2-generated dense subgroup, and it can be shown that it has no cyclic dense subgroups.

The problem of whether a given uncountable topological group has a finitely generated dense subgroup has been extensively studied in the literature. An early result is that of Prasad [11] and Grząślewicz [4] who independently showed that, in some sense, almost every pair of elements of the group of all invertible measure preserving transformations of the unit interval generate a dense subgroup. The following groups have been shown to have dense 2-generated subgroups: the automorphisms of the random graph (defined below) [9], isometries of the Urysohn space [12], the homeomorphisms and measure-preserving homeomorphisms of the Cantor space, and the automorphism group of the infinitely splitting rooted tree [7].

In this paper, the problems we are concerned with were motivated by the following notion and results from finite group theory: A group G is said to be $\frac{3}{2}$*-generated* if for any non-identity element $x \in G$ there is an element $y \in G$ such that $G = \langle x, y\rangle$. It is a classical theorem of Piccard [10] that the symmetric group on any finite set of size greater than 4 is $\frac{3}{2}$-generated. Stein [13] and Woldar [14] showed that the finite simple groups are $\frac{3}{2}$-generated.

Darji and Mitchell [1] showed that for any non-identity element $f \in$ Sym($\mathbb{Z}$) there is an element $g \in$ Sym($\mathbb{Z}$) such that $\langle f, g\rangle$ is a dense subgroup of Sym($\mathbb{Z}$). Analogous results for the group of automorphisms of the random graph and the order-automorphisms of the rationals $\mathbb{Q}$ were established in [2]. This paper could be viewed as a continuation of [1] and [2]. We will consider the automorphism groups of the ultrahomogeneous graphs from the perspective of generating dense subgroups.

2 Statement of the main results

The purpose of this section is to state the main theorem of this paper.

A *graph* Γ is a set of vertices and undirected edges between those vertices. Two vertices of a graph are *adjacent* if there is an edge between them. The *complete graph* K_n is the graph with $n \in \mathbb{N}$ vertices and an edge between every pair of distinct vertices. The complete graph with a countable infinite set of vertices is denoted K_ω. If Γ and Δ are graphs with disjoint sets of vertices (and hence edges), then the *disjoint union* of Γ and Δ is the graph whose vertices and edges are the unions of the vertices and edges, respectively, of Γ and Δ, and no additional edges. The *dual* Δ of a graph Γ has the same vertices as Γ and has an edge between every pair of two distinct vertices which are not adjacent in Γ. If U is a set of vertices of a graph Γ, then the *subgraph induced* by U is the graph with vertices U and edges between $u \in U$ and $v \in U$ if and only if u and v are adjacent in Γ.

If Γ is a graph, then we say that Γ satisfies the *Alice's restaurant property* if for every pair of disjoint finite subsets U and V of vertices of Γ there exists a vertex $w \in \Gamma \setminus (U \cup V)$ such that w is adjacent to every vertex in U and to no vertex in V. Classical results (for example [3]) show that there exists a countable infinite graph with the Alice's restaurant property and that any two countably infinite graphs with the property are isomorphic. As such we refer to any such graph as the *random graph*, denoted R.

A graph is K_n*-free* if none of the subgraphs induced by sets consisting of $n \in \mathbb{N}$ vertices is a complete graph. Obviously, for this definition to be meaningful n must be at least 2. If $n \in \mathbb{N}$ is fixed and Γ is a K_n-free graph, then we say that Γ has the *Alice's restaurant property for* K_n*-free graphs* if for every pair of disjoint induced subgraphs U and V of Γ where U is K_{n-1}-free, there exists a vertex $w \in \Gamma \setminus (U \cup V)$ such that w is adjacent to every vertex in U and to no vertex in V. Again, countably infinite graphs satisfying the Alice's restaurant property for K_n-free graphs, $n > 1$, exist, any two such graphs are isomorphic, and we refer to any such graph as the *universal* K_n*-free graph*, denoted $H(n)$.

Although it is not relevant for this paper, the universal K_n-free graphs, $n > 1$, and the random graph are the *Fraïssé limits* of the classes of finite K_n-free graphs and finite graphs, respectively; see [5] for more details about Fraïssé limits.

If Γ and Δ are graphs, then a function $f : \Gamma \to \Delta$ is an *isomorphism* if f is a bijection which maps adjacent vertices in Γ to adjacent vertices in Δ. An isomorphism from a graph Γ to itself is an *automorphism*, and the group of all automorphisms of Γ is denoted by $\mathrm{Aut}(\Gamma)$. A countable graph is *ultrahomogeneous* if every isomorphism between finite induced subgraphs can be extended to an automorphism. Woodrow and Lachlan showed that there are essentially four types of such countably infinite graphs, described in the following theorem.

Theorem 2.1 (cf. [8]). *The countable ultrahomogeneous graphs up to isomorphism are the following:*
(i) *the random graph R,*
(ii) *the K_n-free universal graph $H(n)$ for every $n \in \mathbb{N}$, $n \geq 3$,*
(iii) *the graph ωK_n consisting of the disjoint union of countably many copies of K_n for every $n \in \mathbb{N}$,*
(iv) *the graph nK_ω consisting of the disjoint union of $n \in \mathbb{N}$ copies of K_ω for $n \geq 2$,*
and the duals of these graphs.

In this paper, we are primarily concerned with the groups of automorphisms of the graphs in Theorem 2.1. Since the automorphism group of a graph and its dual are equal, it will suffice to consider the graphs in Theorem 2.1 (i)–(iv), and not their duals.

Suppose that Γ is a graph. If ϕ is an isomorphism between finite induced subgraphs of Γ, then we denote the domain of ϕ by $\mathrm{dom}(\phi)$, and the range by $\mathrm{ran}(\phi)$. The groups $\mathrm{Aut}(\Gamma)$ of automorphisms of such graphs Γ have a natural topology with basis consisting of the sets

$$[\phi] := \{f \in \mathrm{Aut}(\Gamma) : (x)f = (x)\phi \text{ for all } x \in \mathrm{dom}(\phi)\},$$

where ϕ is an isomorphism of finite induced subgraphs of Γ. If X is any subset of $\mathrm{Aut}(\Gamma)$, then we denote by $X^{<\omega}$ the set of isomorphisms between finite induced subgraphs of Γ with an extension in X. The set $\{[\phi] : \phi \in \mathrm{Aut}(\Gamma)^{<\omega}\}$ is the basis for the topology on $\mathrm{Aut}(\Gamma)$ given above. It can be shown that multiplication, thought of as a function from $\mathrm{Aut}(\Gamma) \times \mathrm{Aut}(\Gamma)$, with the product topology, to $\mathrm{Aut}(\Gamma)$, is continuous with respect to this topology, and that the inversion function ${}^{-1} : \mathrm{Aut}(\Gamma) \to \mathrm{Aut}(\Gamma)$ is also continuous. As such, $\mathrm{Aut}(\Gamma)$ is a topological group. The topology on $\mathrm{Aut}(\Gamma)$ is completely-metrizable, i.e. there exists a complete metric inducing the topology on $\mathrm{Aut}(\Gamma)$. A subset of a topological space is *dense* if it has non-empty intersection with every open set. The basis defined above is countable, and so $\mathrm{Aut}(\Gamma)$ is separable, and hence a Polish group. A topological space is a *Baire space* if every countable intersection of open dense sets is dense. If X is a Baire space and $Y \subseteq X$, then Y is a *comeagre* subset of X if Y contains an intersection of open dense sets. Since $\mathrm{Aut}(\Gamma)$ is a Polish space, it is a Baire space; see [6, Theorem 8.4]. It is well known (see, for example, [6, Theorem 3.11]) that G_δ subspaces of Polish spaces are Polish, it is also easy to show that in a metric space every closed set is a G_δ set. Hence every G_δ subspace and every closed subspace of $\mathrm{Aut}(\Gamma)$ is Polish, and thus Baire.

In this paper, we consider the problem of finding minimally generated dense subgroups of the groups $\mathrm{Aut}(\Gamma)$ where Γ is an ultrahomogeneous graph. In particular, we show that, under certain assumptions, if $f \in \mathrm{Aut}(\Gamma)$, then there exists a Baire subspace of $\mathrm{Aut}(\Gamma)$ containing a comeagre set C with the property that every $g \in C$ generates a dense subgroup together with f. If $f \in \mathrm{Aut}(\Gamma)$ is arbitrary, then the subspaces we will consider are

$$\begin{aligned} D_f &= \{g \in \mathrm{Aut}(\Gamma) : \langle f, g\rangle \text{ is dense in } \mathrm{Aut}(\Gamma)\},\\ \mathcal{I}(\Gamma) &= \{g \in \mathrm{Aut}(\Gamma) : g \text{ has no finite orbits}\},\\ \mathcal{I}_\Sigma(\Gamma) &= \{g \in \mathcal{I}(\Gamma) : \Sigma \subset \Gamma \text{ is a set of orbit representatives for } g\}, \end{aligned} \tag{2.1}$$

where the *set of orbit representatives* of an automorphism g consists of exactly one vertex in every orbit of g.

Suppose that Γ is a graph consisting of the disjoint union of countably many copies of K_n or finitely many copies of K_ω. We denote by $L_1, L_2, \ldots$ the connected components of Γ. Every $f \in \mathrm{Aut}(\Gamma)$ induces a permutation $\overline{f}$ of the indices of the connected components of Γ, $\mathbb{N}$ or $\{1, 2, \ldots, n\}$, which is defined by

$$(i)\overline{f} = j \quad \text{if} \quad (L_i)f = L_j.$$

If $f \in \mathrm{Aut}(nK_\omega)$ is a non-identity element and $\Sigma \subseteq nK_\omega$, then we define

$$\begin{aligned} \mathcal{A}_f &= \{g \in \mathrm{Aut}(nK_\omega) : \langle \overline{f}, \overline{g} \rangle = S_n\}, \\ \mathcal{A}_{f,\Sigma} &= \{g \in \mathcal{A}_f : \Sigma \subseteq nK_\omega \text{ is a set of orbits representatives for } g\}. \end{aligned} \tag{2.2}$$

If $n \neq 4$ then, by a classical theorem [10], we have $\mathcal{A}_f \neq \varnothing$ for all non-identity f.

In the next section, we will show that $\mathcal{I}(\Gamma)$ and $\mathcal{I}_\Sigma(\Gamma)$ are Baire spaces with the subspace topology in $\mathrm{Aut}(\Gamma)$, and that $\mathcal{A}_{f,\Sigma}$ and $\mathcal{A}_f$ are Baire subspaces of $\mathrm{Aut}(nK_\omega)$.

The main results of this paper are contained in the following theorem.

Theorem 2.2. *We have the following results:*
(i) $D_f \cap \mathcal{I}(H(n))$ *is comeagre in* $\mathcal{I}(H(n))$ *for all* $f \in \mathrm{Aut}(H(n))$ *with infinite support.*
(ii) $D_f \cap \mathcal{I}_\Sigma(\omega K_n)$ *is comeagre in* $\mathcal{I}_\Sigma(\omega K_n)$ *for all* $f \in \mathrm{Aut}(\omega K_n)$ *such that the support of* $\overline{f}$ *is infinite and* Σ *is a finite subset of* ωK_n.
(iii) *Suppose that* $f \in \mathrm{Aut}(nK_\omega)$ *is such that for every finite subset* Γ *of* nK_ω *which is setwise stabilised by* f *there are components* L *and* L' *of* nK_ω *such that* $|L \cap \Gamma| \neq |L' \cap \Gamma|$. *Then* $D_f \cap \mathcal{A}_{f,\Sigma}$ *is comeagre in* $\mathcal{A}_{f,\Sigma}$ *for every finite subset* Σ *of* nK_ω.

The analogue of Theorem 2.2 (i) for the random graph was proven in [2, Theorem 1.6] and for the symmetric group in [1, Theorem 3.3].

The paper is organised as follows: In Section 3 we define some further notation and give some results that are common to the proofs of the three parts of Theorem 2.2. We prove the three parts of Theorem 2.2 in the final three sections of the paper.

3 Preliminaries

We denote the cardinality of the natural numbers by ω and suppose that $\mathbb{N} = \{0, 1, \dots\}$.

A *graph* Γ is a pair $(V(\Gamma), E(\Gamma))$ of sets: $V(\Gamma)$ of *vertices* and $E(\Gamma) \subseteq \{\{x, y\} : x, y \in V(\Gamma) \text{ and } x \neq y\}$ of edges. Where appropriate we identify Γ and $V(\Gamma)$ so that we may write $x \in \Gamma$ to mean x is a vertex of Γ. If $\{x, y\}$ is an edge of a graph Γ, then we say that x and y are *adjacent* in Γ. If x is a vertex of Γ, then the subgraph induced by the set of all vertices adjacent to x is denoted $N(x)$.

Suppose that $f : X \to Y$ for some sets X and Y. Then we refer to X and Y as the *domain* and *range* of f, denoted by $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$. If $Z \subseteq \mathrm{dom}(f)$, then we define $(Z)f = \{(z)f : z \in Z\}$. If $f : X \to Y$ and $Z \subseteq X$, then the *restriction* of f to Z is the function $f|_Z : Z \to Y$ such that $(z)f|_Z = (z)f$ for all $z \in Z$. We say that f is an *extension* of any of its restrictions. We refer to any isomorphism between finite induced subgraphs of a graph Γ as a *partial isomorphism of* Γ.

If f and g are arbitrary bijections, then we define their composition

$$f \circ g : \mathrm{dom}(f) \cap (\mathrm{dom}(g) \cap \mathrm{ran}(f))f^{-1} \to \mathrm{ran}(g) \cap (\mathrm{dom}(g) \cap \mathrm{ran}(f))g$$

to be $(x)f \circ g = ((x)f)g$ whenever $(x)f \in \mathrm{dom}(g)$. We denote the composite $f \circ f^{-1}$ by f^0, being the identity on $\mathrm{dom}(f)$.

If f is a bijection and $x \in \mathrm{dom}(f) \cup \mathrm{ran}(f)$, we define the *component* of x under f to be the set

$$\{(x)f^k : k \in \mathbb{Z} \text{ and } x \in \mathrm{dom}(f^k)\}.$$

A component of a bijection f is *complete* if $(x)f^k$ is defined for every $k \in \mathbb{Z}$. A component that is not complete is *incomplete*. If $f : X \to X$ is a permutation, then every component of f is complete and in this context complete components are called *orbits*.

Next, we will show that $\mathcal{I}(\Gamma)$ and $\mathcal{I}_\Sigma(\Gamma)$ (as defined in (2.1)) are Baire spaces with the subspace topology in $\mathrm{Aut}(\Gamma)$ for any countably infinite graph Γ, and that $\mathcal{A}_{f,\Sigma}$ and $\mathcal{A}_f$ (defined in (2.2)) are Baire subspaces of $\mathrm{Aut}(nK_\omega)$.

If Γ is any countably infinite graph and $f \in \mathrm{Aut}(\Gamma) \setminus \mathcal{I}(\Gamma)$, then f has a finite orbit O and hence $[f|_O]$ is a subset of $\mathrm{Aut}(\Gamma) \setminus \mathcal{I}(\Gamma)$. In other words, $\mathcal{I}(\Gamma)$ is closed, and hence Baire.

Lemma 3.1. *The subset $\mathcal{A}_f$ of $\mathrm{Aut}(nK_\omega)$ is a Baire space.*

Proof. Let $g \in \mathrm{Aut}(nK_\omega) \setminus \mathcal{A}_f$. Then $\langle \overline{f}, \overline{g} \rangle \neq S_n$. Let $\Gamma \subseteq nK_\omega$ be a finite set containing at least one vertex in every connected component of nK_ω. Then for all $h \in [g|_\Gamma]$ we have that $\overline{h} = \overline{g}$ and thus $h \notin \mathcal{A}_f$. Therefore, the open set $[g|_\Gamma]$ is a subset of $\mathrm{Aut}(nK_\omega) \setminus \mathcal{A}_f$ and thus $\mathcal{A}_f$ is closed, and hence Baire. □

It follows immediately from the next lemma and the preceding discussion that $\mathcal{I}_\Sigma(\Gamma)$ and $\mathcal{A}_{f,\Sigma}$ are Baire.

Lemma 3.2. *Let Ω be countable, let T be a subspace of $\mathrm{Sym}(\Omega)$, and let $\Sigma \subseteq \Omega$ be finite. If T is Baire, then*

$$T_\Sigma = \{f \in T : \Sigma \text{ is a set of orbit representatives of } f\}$$

is also Baire.

Proof. Let K be the set of those $g \in T$ such that distinct elements of Σ belong to different orbits of g. We will show that K is a closed subset of T. If $T = K$, then K is closed in T. Otherwise, let $g \in T \setminus K$. Then there exist $x, y \in \Sigma$ and $m \in \mathbb{N}$ such that $(x)g^m = y$. If $\Gamma = \{(x)g^i : 0 \le i \le m\}$, then $[g|_\Gamma] \cap T$ is a subset of $T \setminus K$. Hence $T \setminus K$ is open, and so K is closed. Since closed subspaces of Baire spaces are Baire, it follows that K is Baire.

If $x \in \Omega$ is arbitrary, then we denote by A_x the set of all those $g \in K$ such that the orbit of x under g has non-trivial intersection with Σ. Then $T_\Sigma = \bigcap_{x\in\Omega} A_x \subseteq K$. Suppose that $g \in A_x$. Then there are $n \in \mathbb{Z}$ and $y \in \Sigma$ such that $(y)g^n = x$. If $\Gamma' = \{(y)g^i : 0 \le i \le n \text{ or } n \le i \le 0\}$, then $[g|_{\Gamma'}] \cap K$ is a subset of A_x and so A_x is open in K for all x. Therefore T_Σ, being a G_δ subset of K, is Baire. □

We end this section by stating two lemmas that we will use repeatedly later in the paper. We omit the easy proof of the first lemma.

Lemma 3.3. *Let Γ be any graph. Then for every $f \in \mathrm{Aut}(\Gamma)$ and any $p \in \mathrm{Aut}(\Gamma)^{<\omega}$, we have that*

$$\{g \in \mathrm{Aut}(\Gamma) : \langle f, g \rangle \cap [p] \neq \varnothing\}$$

is an open set in $\mathrm{Aut}(\Gamma)$.

Lemma 3.4. *Let Γ be any graph, let $f \in \mathrm{Aut}(\Gamma)$, and let $S \subseteq \mathrm{Aut}(\Gamma)$ be such that every $q \in S^{<\omega}$ has an extension in S with only finitely many orbits. If $D_f \cap S_\Sigma$ is dense in S_Σ for every finite $\Sigma \subseteq \Gamma$, then $D_f \cap S$ is comeagre in S.*

Proof. Since $\{[q] : q \in \mathrm{Aut}(\Gamma)^{<\omega}\}$ is a basis for the topology on $\mathrm{Aut}(\Gamma)$, it follows that

$$D_f \cap S = \{g \in S : \langle f, g \rangle \text{ is dense in } \mathrm{Aut}(\Gamma)\} = \bigcap_{p\in\mathrm{Aut}(\Gamma)^{<\omega}} \{g \in S : \langle f, g \rangle \cap [p] \neq \varnothing\}.$$

Since $\{g \in S : \langle f, g \rangle \cap [p] \neq \varnothing\}$ is open in S by Lemma 3.3, it suffices to show that $\{g \in S : \langle f, g \rangle \cap [p] \neq \varnothing\}$ is dense in S for all $p \in \mathrm{Aut}(\Gamma)^{<\omega}$.

Let $q \in S^{<\omega}$. By the hypothesis, there is $g \in S$ which extends q and has a finite number of orbits. Let Σ be a set of orbit representatives of g. Then $q \in S_\Sigma^{<\omega}$. Since $D_f \cap S_\Sigma$ is dense in S_Σ there is $h \in [q]$ such that $h \in D_f \cap S_\Sigma$. In other words, $\langle f, h \rangle$ is dense in $\mathrm{Aut}(\Gamma)$ and so $\{g \in S : \langle f, g \rangle \cap [p] \neq \varnothing\}$ is dense in S. □

4 K_n-free graphs

In this section, we will consider the ultrahomogeneous K_n-free graphs, denoted by $H(n)$ for $n \ge 3$. The case $n = 2$ gives a graph with no edges and its automorphism group is just the symmetric group on countably many points, which was already considered in [1].

If for $x \in H(n)$ the subgraph $N(x)$ has a subgraph Γ isomorphic to K_{n-1}, then $\Gamma \cup \{x\}$ is isomorphic to K_n, which is impossible. Hence $N(x)$ is K_{n-1}-free for every vertex $x \in H(n)$. We will repeatedly make use of this fact without reference.

Let U and V be finite disjoint subsets of vertices of $H(n)$ such that U is K_{n-1}-free. Then, by the Alice's restaurant property for $H(n)$, there is a vertex $w \notin U \cup V$ such that there are no edges between w and V, and there is an edge between u and w for all $u \in U$. In other words, $N(w) \cap (U \cup V) = U$.

The purpose of this section is to prove Theorem 2.2 (i), which we restate for the sake of convenience.

Theorem 4.1. *Let $f \in \operatorname{Aut}(H(n))$ have infinite support. Then $D_f \cap \mathcal{I}(H(n))$ is comeagre in $\mathcal{I}(H(n))$.*

We will proceed by first proving a number of technical results. First, we will show that the set $D_f \cap \mathcal{I}$ can be written as a countable intersection of sets of a certain type. The rest of the argument will then be dedicated to showing that these sets are open and dense.

Lemma 4.2. *Let $\mathcal{P} \subseteq \operatorname{Aut}(H(n))^{<\omega}$ be such that $p \in \mathcal{P}$ if and only if $\operatorname{dom}(p) \cap \operatorname{ran}(p) = \varnothing$ and there are no edges between $\operatorname{dom}(p)$ and $\operatorname{ran}(p)$. Then*

$$D_f \cap \mathcal{I}(H(n)) = \bigcap_{p \in \mathcal{P}} \{g \in \mathcal{I}(H(n)) : \langle f, g\rangle \cap [p] \neq \varnothing\}.$$

Proof. Recall that

$$\begin{aligned} D_f \cap \mathcal{I}(H(n)) &= \{g \in \mathcal{I}(H(n)) : \langle f, g\rangle \text{ is dense in } \operatorname{Aut}(H(n))\} \\ &= \bigcap_{q \in \operatorname{Aut}(H(n))^{<\omega}} \{g \in \mathcal{I}(H(n)) : \langle f, g\rangle \cap [q] \neq \varnothing\}. \end{aligned}$$

($\subseteq$) This follows immediately since $\mathcal{P} \subseteq \operatorname{Aut}(H(n))^{<\omega}$.

($\supseteq$) Let

$$g \in \bigcap_{p \in \mathcal{P}} \{g \in \mathcal{I}(H(n)) : \langle f, g\rangle \cap [p] \neq \varnothing\}$$

and suppose that $q \in \operatorname{Aut}(H(n))^{<\omega}$. By repeated application of the Alice's restaurant property we can find a subgraph Γ of $H(n)$ such that Γ is isomorphic to $\operatorname{dom}(q)$, $\Gamma \cap (\operatorname{dom}(q) \cup \operatorname{ran}(q)) = \varnothing$, and such that there are no edges between Γ and $\operatorname{dom}(q) \cup \operatorname{ran}(q)$. Let p be the isomorphism between $\operatorname{dom}(q)$ and Γ. Since $H(n)$ is ultrahomogeneous, we have that $p \in \operatorname{Aut}(H(n))^{<\omega}$. Then $\operatorname{dom}(p) = \operatorname{dom}(q)$, $\operatorname{ran}(p) = \operatorname{dom}(p^{-1}q) = \Gamma$ and $\operatorname{ran}(p^{-1}q) = \operatorname{ran}(q)$. Hence $p, p^{-1}q \in \mathcal{P}$. By the choice of g there are $h_1, h_2 \in \langle f, g\rangle$ such that $h_1 \in [p]$ and $h_2 \in [p^{-1}q]$. Therefore $h_1h_2 \in [q]$ and $h_1h_2 \in \langle f, g\rangle$, thus $\langle f, g\rangle \cap [q] \neq \varnothing$. Since q was arbitrary,

$$g \in \bigcap_{q \in \operatorname{Aut}(H(n))^{<\omega}} \{g \in \mathcal{I}(H(n)) : \langle f, g\rangle \cap [q] \neq \varnothing\}. \qquad \square$$

The following lemma provides a condition under which it is possible to extend a given partial isomorphism of $H(n)$ to another partial isomorphism of $H(n)$. Although we will only apply the following lemma to the graphs $H(n)$, we state it for arbitrary ultrahomogeneous graphs, since the proof is not harder in the general case.

Lemma 4.3. *Let Γ be an ultrahomogeneous graph, let $q \in \operatorname{Aut}(\Gamma)^{<\omega}$, and let $x, y \in \Gamma$. Suppose that $x \notin \operatorname{dom}(q)$ and $N(y) \cap \operatorname{ran}(q) = (N(x))q$. Then $q \cup \{(x, y)\} \in \operatorname{Aut}(\Gamma)^{<\omega}$.*

Proof. Since Γ is ultrahomogeneous, it is sufficient to show that $q \cup \{(x, y)\}$ is an isomorphism between two subgraphs of Γ. By the hypothesis, q is an isomorphism and so it suffices to show that there is an edge between vertices x and $z \in \operatorname{dom}(q)$ if and only if there is an edge between vertices y and $(z)q$. Let $z \in \operatorname{dom}(q)$. Then there is an edge between z and x if and only if $z \in N(x)$ which is equivalent to $(z)q \in N(y) \cap \operatorname{ran}(q)$, i.e. there is an edge between $(z)q$ and y. □

The following easy corollary shows that any incomplete component of an isomorphism of $H(n)$ can be extended.

Corollary 4.4. *Let $q \in \operatorname{Aut}(H(n))^{<\omega}$, let $x \in H(n) \setminus \operatorname{dom}(q)$, and let $\Sigma \subseteq H(n)$ be finite. Then there is an element $y \in H(n) \setminus (\{x\} \cup \Sigma)$ such that $q \cup \{(x, y)\} \in \operatorname{Aut}(H(n))^{<\omega}$.*

Proof. Let $U = (N(x))q$ and let $V = (\operatorname{ran}(q) \cup \{x\} \cup \Sigma) \setminus U$. Since $N(x)$ is K_{n-1}-free and q is a partial isomorphism, U is also K_{n-1}-free. Hence by the Alice's restaurant property, there is $y \in H(n) \setminus (\operatorname{ran}(q) \cup \{x\} \cup \Sigma)$ such that $N(y) \cap \operatorname{ran}(q) = (N(x))q$. Therefore, we are done by Lemma 4.3. □

Let q be a partial isomorphism of $H(n)$ such that q has no complete components, set Σ to be $\mathrm{dom}(q) \cup \mathrm{ran}(q)$, and let $x \in H(n) \setminus \mathrm{dom}(q)$. Then by Corollary 4.4 there is a partial isomorphism h of $H(n)$ extending q such that $x \in \mathrm{dom}(h)$ and h has no complete components. Repeatedly applying Corollary 4.4 in a back and forth argument, we may deduce that there is an $r \in \mathfrak{I}(H(n))$ extending q, which gives us the following lemma.

Corollary 4.5. *Let $q \in \mathrm{Aut}(H(n))^{<\omega}$. Then $q \in \mathfrak{I}(H(n))^{<\omega}$ if and only if q has no complete components.*

The following two technical lemmas form the essential part of the proof of Theorem 4.1.

Lemma 4.6. *Let $q \in \mathfrak{I}(H(n))^{<\omega}$ be such that $\mathrm{ran}(q) \cup \mathrm{dom}(q) = \Delta \cup \Gamma$, where $\Delta \cap \Gamma = \varnothing$ and Γ is the union of incomplete components of q of fixed length m. Let $x, y \notin \mathrm{dom}(q) \cup \mathrm{ran}(q)$ be such that*

$$N(x) \cap \Delta \subseteq \mathrm{dom}(q^{2m}) \quad \text{and} \quad (N(x) \cap \Delta)q^{2m} = N(y) \cap \Delta$$

and let $\Sigma_1, \Sigma_2 \subseteq H(n) \setminus \Gamma$ be finite subsets such that $\Sigma_1 \cap \mathrm{ran}(q) = \varnothing$ and $\Sigma_2 \cap \mathrm{dom}(q) = \varnothing$. Then there are $x_1, \dots, x_{2m-1} \in H(n) \setminus \Sigma_1 \cup \Sigma_2$ such that there are no edges between x_i and $\Sigma_1 \cup \Sigma_2$ for all $i \in \{1, \dots, 2m-1\}$, and

$$q \cup \{(x_i, x_{i+1}) : 0 \le i \le 2m-1\} \in \mathfrak{I}(H(n))^{<\omega},$$

where $x_0 = x$ and $x_{2m} = y$.

Proof. Define $q_0 = q$, $x_0 = x$ and

$$\Gamma_i = \mathrm{dom}(q_i) \cup \mathrm{ran}(q_i) \cup \Sigma_1 \cup \Sigma_2 \cup \{x, y\}$$

for all i. Suppose that for $i \in \{0, \dots, m-1\}$ there is an extension $q_i \in \mathfrak{I}(H(n))^{<\omega}$ of q_0 such that

$$q_i = q_0 \cup \{(x_j, x_{j+1}) : 0 \le j \le i - 1r\}$$

with $x_0 \notin \mathrm{ran}(q_i)$, $x_i \notin \mathrm{dom}(q_i)$, $y \notin \mathrm{ran}(q_i) \cup \mathrm{dom}(q_i)$, and

$$x_j \notin \Sigma_1 \cup \Sigma_2 \cup \Delta, \tag{4.1}$$
$$N(x_j) \cap (\Sigma_1 \cup \Sigma_2 \cup \{x_0, \dots, x_{j-1}, y\}) = \varnothing, \tag{4.2}$$
$$N(x_i) \cap \Gamma_i = (N(x_0) \cap \Gamma_0)q_i^i \tag{4.3}$$

for all $j \in \{1, \dots, i\}$.

If $i = 0$, then we have that $x_0, y \notin \mathrm{dom}(q_0) \cup \mathrm{ran}(q_0)$ and (4.1), (4.2), and (4.3) are trivially satisfied.

Suppose that $i > 0$. Let $U = (N(x_i))q_i \subseteq \Gamma_i$ and $V = \Gamma_i \setminus U$. If $N(x_i)$ contains a subgraph isomorphic to K_{n-1}, then the subgraph together with x_i forms K_n, which is impossible. Hence $N(x_i)$ is K_{n-1}-free and since q_i is an isomorphism, U is also K_{n-1}-free. Therefore the sets U and V satisfy the hypothesis of the Alice's restaurant property and thus there is a vertex $x_{i+1} \in H(n) \setminus \Gamma_i$ such that there is an edge between x_{i+1} and every vertex in U and there are no edges between x_{i+1} and V, i.e. $N(x_{i+1}) \cap \Gamma_i = U$. Also it follows from $\mathrm{ran}(q_i) \subseteq \Gamma_i$ that

$$N(x_{i+1}) \cap \mathrm{ran}(q_i) = (N(x_{i+1}) \cap \Gamma_i) \cap \mathrm{ran}(q_i) = U \cap \mathrm{ran}(q_i) = (N(x_i))q_i.$$

Then

$$q_{i+1} = q_i \cup \{(x_i, x_{i+1})\} = q_0 \cup \{(x_j, x_{j+1}) : 0 \le j \le i\} \in \mathrm{Aut}(H(n))^{<\omega}$$

by Lemma 4.3, and so $q_{i+1} \in \mathfrak{I}(H(n))^{<\omega}$ by Corollary 4.5. Since $x_{i+1} \notin \Gamma_i$, we have that $x_{i+1} \notin \{x_0, x_i, y\}$, implying that $x_0 \notin \mathrm{ran}(q_{i+1})$, $x_{i+1} \notin \mathrm{dom}(q_{i+1})$, and $y \notin \mathrm{ran}(q_{i+1}) \cup \mathrm{dom}(q_{i+1})$. It also follows from $\mathrm{dom}(q_i) \subseteq \Gamma_i$ and (4.3) that

$$N(x_{i+1}) \cap \Gamma_i = U = (N(x_i))q_i = (N(x_i) \cap \Gamma_i)q_i = (N(x_0) \cap \Gamma_0)q_i^{i+1}. \tag{4.4}$$

Since $\Sigma_1 \cup \Sigma_2 \cup \Delta \subseteq \Gamma_i$ and x_{i+1} is chosen outside the set Γ_i, it follows that $x_{i+1} \notin \Sigma_1 \cup \Sigma_2 \cup \Delta$. Then $x_j \notin \Sigma_1 \cup \Sigma_2 \cup \Delta$ for all $j \in \{1, \dots, i+1\}$ by (4.1).

We will now show that (4.2) holds for $j = i+1$. First of all note that $x_0, y \notin \mathrm{ran}(q_i)$, and since $U \subseteq \mathrm{ran}(q_i)$ we have that $x_0, y \notin U$. From (4.2) we may deduce that $x_j \notin N(x_i)$ and thus $x_{j+1} \notin (N(x_i))q_i = U$ for all

$j \in \{0, \ldots, i-1\}$, i.e. $\{x_0, \ldots, x_i, y\} \cap U = \varnothing$. It follows from the hypothesis that $\Sigma_1 \cap \operatorname{ran}(q_0) = \varnothing$, and so (4.1) implies that $\Sigma_1 \cap \operatorname{ran}(q_i) = \varnothing$. Since $U \subseteq \operatorname{ran}(q_i)$, we have that

$$(\Sigma_1 \cup \{x_0, \ldots, x_i, y\}) \cap U = \varnothing.$$

It only remains to show that $\Sigma_2 \cap U = \varnothing$. Suppose $z \in \Sigma_2 \cap U$. Then $z \in (N(x_0) \cap \Gamma_0)q_i^{i+1}$ by (4.4). Then $z \in \operatorname{ran}(q_i)$ and by the above we have $z \neq x_j$ for all $j \in \{0, \ldots, i\}$, thus $z \in \operatorname{ran}(q_0) \subseteq \Gamma \cup \Delta$. However by the hypothesis of the lemma, we have $\Sigma_2 \subseteq H(n) \setminus \Gamma$, implying that $z \in \Delta$. Since $x_j \notin \Delta$ for all $j \in \{1, \ldots, i\}$ by (4.1) and $x_0 \notin \Delta$ by the hypothesis of the lemma, it follows that the incomplete component of q_0 containing z was not extended in q_i. Moreover, Δ is a union of incomplete components of q_0 whence

$$(z)q_i^{-(i+1)} \in N(x_0) \cap \Delta.$$

Also from $\Sigma_2 \cap \operatorname{dom}(q_0) = \varnothing$ and (4.1) we may deduce that $\Sigma_2 \cap \operatorname{dom}(q_i) = \varnothing$ and so $z \notin \operatorname{dom}(q_i)$. It also follows from the hypothesis of the lemma that $(z)q_i^{-(i+1)} \in \operatorname{dom}(q_i^{2m})$. Then $z \in \operatorname{dom}(q_i^{2m-(i+1)})$, which is impossible since $i + 1 < 2m$. Hence,

$$U \cap (\Sigma_1 \cup \Sigma_2 \cup \{x_0, \ldots, x_i, y\}) = \varnothing.$$

Since

$$(\Sigma_1 \cup \Sigma_2 \cup \{x_0, \ldots, x_i, y\}) \subseteq \Gamma_{i+1},$$

we have that

$$\begin{aligned} N(x_{i+1}) \cap (\Sigma_1 \cup \Sigma_2 \cup \{x_0, \ldots, x_i, y\}) &= (N(x_{i+1}) \cap \Gamma_{i+1}) \cap (\Sigma_1 \cup \Sigma_2 \cup \{x_0, \ldots, x_i, y\}) \\ &= U \cap (\Sigma_1 \cup \Sigma_2 \cup \{x_0, \ldots, x_i, y\}) = \varnothing. \end{aligned}$$

Hence (4.1) and (4.2) are satisfied, and so it only remains to verify (4.3). It is routine to verify that $\operatorname{dom}(q_{i+1}^{i+1}) \setminus \operatorname{dom}(q_i^{i+1}) = \{x_0\}$. It follows from $x_0 \notin N(x_0)$ and (4.4) that

$$N(x_{i+1}) \cap \Gamma_i = (N(x_0) \cap \Gamma_0)q_{i+1}^{i+1}.$$

Since $\Gamma_{i+1} = \Gamma_i \cup \{x_{i+1}\}$ and $x_{i+1} \notin N(x_{i+1})$, we have

$$N(x_{i+1}) \cap \Gamma_i = N(x_{i+1}) \cap \Gamma_{i+1} = (N(x_0) \cap \Gamma_0)q_{i+1}^{i+1}.$$

Therefore, q_{i+1} satisfies the inductive hypothesis. Thus by induction on i, there is $q_m \in \mathfrak{I}(H(n))^{<\omega}$ such that

$$q_m = q_0 \cup \{(x_j, x_{j+1}) : 0 \le j \le m-1\}, \quad x_0 \notin \operatorname{ran}(q_m), \quad x_m \notin \operatorname{dom}(q_m), \quad y \notin \operatorname{ran}(q_m) \cup \operatorname{dom}(q_m),$$

and q_m satisfies (4.1), (4.2) and (4.3).

Note that if $z \in \Sigma_1 \cup \Sigma_2 \setminus \{x, y\}$, then $z \notin \Gamma$ and by (4.1) either $z \notin \operatorname{dom}(q_m) \cup \operatorname{ran}(q_m)$ or $z \in \Delta$. Hence,

$$\begin{aligned} N(x_m) \cap \Gamma_m = (N(x_0) \cap \Gamma_0)q_m^m &= \Big((N(x_0) \cap (\Gamma \cup \{x, y\} \cup \Sigma_1 \cup \Sigma_2 \setminus \Delta)) \cup (N(x_0) \cap \Delta)\Big)q_m^m \\ &= (N(x_0) \cap \Delta)q_m^m, \end{aligned} \tag{4.5}$$

since $x \notin N(x_0)$, $y \notin \operatorname{dom}(q_m)$ and all incomplete components of q on Γ are of length m.

The next step is to inductively construct an extension $h = q_{2m} \in \mathfrak{I}(H(n))^{<\omega}$ of q_m. Suppose that for $i \in \{m, \ldots, 2m-2\}$ there is an extension $q_i \in \mathfrak{I}(H(n))^{<\omega}$ of the form $q_i = q_m \cup \{(x_j, x_{j+1}) : m \le j \le i-1\}$ such that $x_0 \notin \operatorname{ran}(q_i)$, $x_i \notin \operatorname{dom}(q_i)$, $y \notin \operatorname{dom}(q_i) \cup \operatorname{ran}(q_i)$, and

$$x_j \notin \Sigma_1 \cup \Sigma_2 \cup \Delta, \tag{4.1}$$

$$N(x_j) \cap (\Sigma_1 \cup \Sigma_2 \cup \{x_0, \ldots, x_{j-1}, y\}) = \varnothing, \tag{4.2}$$

$$N(x_i) \cap \Gamma_i = (N(y) \cap \operatorname{dom}(q_i^{i-2m}))q_i^{i-2m} \tag{4.6}$$

for all $j \in \{1, \ldots, i\}$.

We will now show that q_m satisfies the inductive hypothesis. Note that (4.1) and (4.2) are the same as before, so we only need to verify (4.6). Since no incomplete components of q_0 which intersect Δ were extended in q_m, condition (4.1) implies that $(\Delta)q_0^k = (\Delta)q_m^k$ for any $k \in \mathbb{Z}$. It follows from the hypothesis of the lemma that

$$(N(x_0) \cap \Delta)q_m^m \subseteq \mathrm{dom}(q_m^m) \quad \text{and} \quad (N(x_0) \cap \Delta)q_m^m = (N(y) \cap \Delta)q_m^{-m}.$$

Hence by (4.5), we have

$$N(x_m) \cap \Gamma_m = (N(x_0) \cap \Delta)q_m^m = (N(y) \cap \Delta)q_m^{-m}.$$

Suppose that $z \in N(y) \cap \mathrm{dom}(q_m^{-m})$. Then

$$z \in \mathrm{dom}(q_m) \cup \mathrm{ran}(q_m) = \Gamma \cup \Delta \cup \{x_0, \dots, x_m\}.$$

Note that all incomplete components of q_m intersecting Γ not trivially, are of length m. Hence $z \in \Delta \cup \{x_m\}$ and by (4.2) we have that $x_m \notin N(y)$, thus $z \in \Delta$. Therefore,

$$N(y) \cap \mathrm{dom}(q_m^{-m}) \subseteq N(y) \cap \Delta,$$

and so

$$N(x_m) \cap \Gamma_m = (N(y) \cap \Delta)q_m^{-m} = (N(y) \cap \mathrm{dom}(q_m^{-m}))q_m^{-m}.$$

Hence q_m satisfies (4.6) and the inductive hypothesis is satisfied for $i = m$.

Let $U = (N(x_i))q_i$ and $V = \Gamma_i \setminus U$. The sets U and V satisfy the hypothesis of the Alice's restaurant property and thus we can find $x_{i+1} \in H(n) \setminus \Gamma_i$ with $N(x_{i+1}) \cap \Gamma_i = U = (N(x_i))q_i$. Then $N(x_{i+1}) \cap \mathrm{ran}(q_i) = (N(x_i))q_i$, and so

$$q_{i+1} = q_i \cup \{(x_i, x_{i+1})\} \in \mathcal{I}(H(n))^{<\omega}$$

by Lemma 4.3 and Corollary 4.5. Since $x_{i+1} \notin \Gamma_i$, we have that $x_{i+1} \notin \{x_0, x_i, y\}$ implying that $x_0 \notin \mathrm{ran}(q_{i+1})$, $x_{i+1} \notin \mathrm{dom}(q_{i+1})$, and $y \notin \mathrm{dom}(q_{i+1}) \cup \mathrm{ran}(q_{i+1})$.

Since $\mathrm{dom}(q_i) \subseteq \Gamma_i$, it follows from (4.6) that

$$N(x_{i+1}) \cap \Gamma_i = U = (N(x_i))q_i = (N(x_i) \cap \Gamma_i)q_i = (N(y) \cap \mathrm{dom}(q_i^{i-2m}))q_i^{i+1-2m}. \tag{4.7}$$

Since $\Sigma_1 \cup \Sigma_2 \cup \Delta \subseteq \Gamma_i$ and x_{i+1} is chosen outside the set Γ_i, it follows that $x_{i+1} \notin \Sigma_1 \cup \Sigma_2 \cup \Delta$. Then $x_j \notin \Sigma_1 \cup \Sigma_2 \cup \Delta$ for all $j \in \{1, \dots, i+1\}$.

We will now show that (4.2) holds for $j = i+1$. First of all note that $x_0, y \notin \mathrm{ran}(q_i)$ and since $U \subseteq \mathrm{ran}(q_i)$, we have that $x_0, y \notin U$. From (4.2) we may deduce that $x_j \notin N(x_i)$, and thus $x_{j+1} \notin (N(x_i))q_i = U$ for all $j \in \{0, \dots, i-1\}$, i.e. $\{x_0, \dots, x_i, y\} \cap U = \varnothing$. It follows from the hypothesis that $\Sigma_1 \cap \mathrm{ran}(q_0) = \varnothing$, and so (4.1) implies $\Sigma_1 \cap \mathrm{ran}(q_i) = \varnothing$. Since $(\Sigma_1 \cup \{x_0, y\}) \cap \mathrm{ran}(q_i) = \varnothing$ and $U \subseteq \mathrm{ran}(q_i)$, it follows that

$$(\Sigma_1 \cup \{x_0, \dots, x_i, y\}) \cap U = \varnothing.$$

It remains to show that $\Sigma_2 \cap U = \varnothing$. Suppose $z \in \Sigma_2 \cap U$. Then

$$z \in (N(y) \cap \mathrm{dom}(q_i^{i-2m}))q_i^{i+1-2m}$$

by (4.7). Note that $z \in U \subseteq \mathrm{ran}(q_i)$. Also we showed in the previous paragraph that $z \neq x_j$ for all $j \in \{0, \dots, i\}$. Hence, $z \in \mathrm{ran}(q_0) \subseteq \Gamma \cup \Delta$. However, by the hypothesis of the lemma, we have $\Sigma_2 \subseteq H(n) \setminus \Gamma$, implying that $z \in \Delta$. Since $x_j \notin \Delta$ for all $j \in \{1, \dots, i\}$ by (4.1) and $x_0 \notin \Delta$ by the hypothesis of the lemma, it follows that the incomplete component of q_0 containing z was not extended in q_i. Moreover, Δ is a union of incomplete components of q_0, and $z \in \mathrm{dom}(q_i^{2m-(i+1)})$, so

$$(z)q_i^{2m-(i+1)} \in N(y) \cap \Delta.$$

By the hypothesis of the lemma, we have $(z)q_i^{2m-(i+1)} \in \mathrm{ran}(q_i^{2m})$. Then there is $u \in \mathrm{dom}(q_i^{2m})$ such that $(z)q_i^{2m-(i+1)} = (u)q_i^{2m}$, and so

$$z = (u)q_i^{i+1} \in \mathrm{dom}(q_i^{2m-(i+1)}).$$

Hence $z \in \operatorname{dom}(q_i)$, since $2m > i + 1$. However, $z \in \Sigma_2$ and so $z \notin \operatorname{dom}(q_0)$, implying that $z \in \{x_0, \dots, x_i\}$, which contradicts (4.1). Hence,

$$U \cap (\Sigma_1 \cup \Sigma_2 \cup \{x_0, \dots, x_i, y\}) = \varnothing,$$

and since

$$(\Sigma_1 \cup \Sigma_2 \cup \{x_0, \dots, x_i, y\}) \subseteq \Gamma_{i+1},$$

we have for all $j \in \{1, \dots, i+1\}$ that

$$N(x_j) \cap (\Sigma_1 \cup \Sigma_2 \cup \{x_0, \dots, x_{j-1}, y\}) = \varnothing.$$

It is routine to verify that

$$\operatorname{dom}(q_{i+1}^{i+1-2m}) \setminus \operatorname{dom}(q_i^{i+1-2m}) = \{x_{i+1}\}.$$

Since $x_{i+1} \notin N(y)$, it follows from (4.7) that

$$N(x_{i+1}) \cap \Gamma_i = (N(y) \cap \operatorname{dom}(q_i^{i-2m}))q_{i+1}^{i+1-2m}.$$

It is routine to check that

$$\operatorname{dom}(q_{i+1}^{i+1-2m}) \setminus \operatorname{dom}(q_i^{i-2m}) \subseteq \{x_1, \dots, x_{i+1}\}.$$

Then by (4.2), we have that $x_j \notin N(y)$ for all $j \in \{x_0, \dots, i+1\}$. Hence,

$$N(x_{i+1}) \cap \Gamma_i = (N(y) \cap \operatorname{dom}(q_{i+1}^{i+1-2m}))q_{i+1}^{i+1-2m}.$$

From the definition of Γ_{i+1} we obtain that $\Gamma_{i+1} = \Gamma_i \cup \{x_{i+1}\}$. However, $x_{i+1} \notin N(x_{i+1})$, thus

$$N(x_{i+1}) \cap \Gamma_{i+1} = (N(y) \cap \operatorname{dom}(q_{i+1}^{i+1-2m}))q_{i+1}^{i+1-2m}.$$

Therefore, q_{i+1} satisfies the inductive hypothesis and hence we obtain

$$q_{2m-1} = q_0 \cup \{(x_j, x_{j+1}) : 0 \le j \le 2m-2\} \in \operatorname{Aut}(H(n))^{<\omega}$$

such that $y \notin \operatorname{dom}(q_{2m-1}) \cup \operatorname{ran}(q_{2m-1})$, $x_j \notin \Sigma_1 \cup \Sigma_2$, there are no edges between x_j and $\Sigma_1 \cup \Sigma_2$ for all $j \in \{1, \dots, 2m-1\}$, and

$$N(x_{2m-1}) \cap \Gamma_{2m-1} = (N(y))q_{2m-1}^{-1}.$$

Therefore,

$$h = q_{2m} = q_{2m-1} \cup \{(x_{2m-1}, y)\} \in \mathfrak{I}(H(n))^{<\omega}$$

by Lemma 4.3 and Corollary 4.5 as required. □

Lemma 4.7. *Let $q \in \mathfrak{I}(H(n))^{<\omega}$ and let $p \in \mathcal{P}$ be such that the sets $\operatorname{dom}(q) \cup \operatorname{ran}(q)$ and $\operatorname{dom}(p) \cup \operatorname{ran}(p)$ are disjoint. Then there are an extension $h \in \mathfrak{I}(H(n))^{<\omega}$ of q and $m \in \mathbb{N}$ such that h^{2m} extends p.*

Proof. If necessary by extending q, using Corollary 4.4, we may assume that all of the components of q have length m for some $m \in \mathbb{N}$.

Let $\operatorname{dom}(p) = \{x_1, \dots, x_d\}$ for some $d \in \mathbb{N}$, let $q_0 = q$ and $\Gamma = \operatorname{dom}(q_0) \cup \operatorname{ran}(q_0)$. We will now inductively define $q_i \in \mathfrak{I}(H(n))^{<\omega}$, and once they are defined let $\Delta_i = \operatorname{dom}(q_i) \cup \operatorname{ran}(q_i) \setminus \Gamma$ for $i \in \{0, \dots, d\}$. Suppose that for some $k \in \{0, \dots, d-1\}$ we have defined $q_k \in \mathfrak{I}(H(n))^{<\omega}$, an extension of q_0 such that both Γ and Δ_k are unions of incomplete components of q_k, incomplete components of q_k contained in Δ_k are of length $2m+1$, and the following are true:

$$x_j, (x_j)p \notin \operatorname{dom}(q_k) \cup \operatorname{ran}(q_k), \tag{4.8}$$

$$(x_i)q_k^{2m} = (x_i)p,$$

$$N(x_j) \cap \Delta_k \subseteq \operatorname{dom}(q_k^{2m}), \tag{4.9}$$

$$(N(x_j) \cap \Delta_k)q_k^{2m} = N((x_j)p) \cap \Delta_k \tag{4.10}$$

for all $i \in \{1, \dots, k\}$ and $j \in \{k+1, \dots, d\}$.

Let $\Sigma_1 = \mathrm{dom}(p)$ and $\Sigma_2 = \mathrm{ran}(p)$. We will show that the hypothesis of Lemma 4.6 is satisfied by q_k, x_{k+1}, $(x_{k+1})p$, Σ_1, and Σ_2. First of all, note that $x_{k+1}, (x_{k+1})p \notin \mathrm{dom}(q_k) \cup \mathrm{ran}(q_k)$ by condition (4.8). Also by the hypothesis of the lemma, we have $\Sigma_1, \Sigma_2 \subseteq H(n) \setminus \Gamma$. Note that

$$N(x_{k+1}) \cap \Delta \subseteq \mathrm{dom}(q_k^{2m}) \quad \text{and} \quad (N(x_{k+1}) \cap \Delta)q_k^{2m} = N((x_{k+1})p) \cap \Delta$$

immediately follow from conditions (4.9) and (4.10). Hence to apply Lemma 4.6 we only need to verify that $\Sigma_1 \cap \mathrm{ran}(q_k) = \Sigma_2 \cap \mathrm{dom}(q_k) = \varnothing$. We will do so in the next two paragraphs.

First, we show that $x_i \notin \mathrm{ran}(q_k)$ for all $i \in \{1, \dots, d\}$. Suppose that $x_i \in \mathrm{dom}(q_k) \cup \mathrm{ran}(q_k)$; by the inductive hypothesis we can deduce that $i \leq k$. Since $\mathrm{dom}(p) \cap \Gamma = \varnothing$ by the hypothesis of the lemma, it then follows that $x_i \in \Delta_k$. Therefore, x_i is on an incomplete component of length $2m + 1$ and $x_i \in \mathrm{dom}(q_k^{2m})$ by the inductive hypothesis, implying that $x_i \in \mathrm{dom}(q_k) \setminus \mathrm{ran}(q_k)$. Hence, $\Sigma_1 \cap \mathrm{ran}(q_k) = \varnothing$.

The argument that $\Sigma_2 \cap \mathrm{dom}(q_k) = \varnothing$ is similar to above. Let $(x_i)p \in \Sigma_2$. Suppose that

$$(x_i)p \in \mathrm{dom}(q_k) \cup \mathrm{ran}(q_k).$$

Then we can deduce that $i \leq k$. Since $\mathrm{ran}(p) \cap \Gamma = \varnothing$ by the hypothesis of the lemma, it then follows that $(x_i)p \in \Delta_k$. Therefore, $(x_i)p$ is on an incomplete component of length $2m + 1$ and $(x_i)p \in \mathrm{ran}(q_k^{2m})$ by the inductive hypothesis, implying that $(x_i)p \in \mathrm{ran}(q_k) \setminus \mathrm{dom}(q_k)$.

Hence by Lemma 4.6 there is an extension $q_{k+1} \in \mathcal{I}(H(n))^{<\omega}$ of q_k such that

$$q_{k+1} = q_k \cup \{(y_i, y_{i+1}) : 0 \leq i \leq 2m - 1\}, \quad y_0 = x_{k+1}, \quad y_{2m} = (x_{k+1})p,$$

there are no edges between y_i and $\Sigma_1 \cup \Sigma_2$, and $y_i \notin \Sigma_1 \cup \Sigma_2$ for $i \in \{1, \dots, 2m - 1\}$. Then by the choice of Σ_1, Σ_2, and the definition of q_{k+1}, we have

$$x_j, (x_j)p \notin \mathrm{dom}(q_{k+1}) \cup \mathrm{ran}(q_{k+1}), \quad (x_i)q_{k+1}^{2m} = (x_i)p$$

for all $i \in \{1, \dots, k + 1\}$ and $j \in \{k + 2, \dots, d\}$. It also follows from the definition of q_{k+1} that

$$\Delta_{k+1} = \Delta_k \cup \{y_i : 0 \leq i \leq 2m\}$$

and thus Δ_{k+1} is a union of incomplete components of q_{k+1} each of length $2m + 1$.

Let $j \in \{k + 2, \dots, d\}$ and let $z \in N(x_j) \cap \Delta_{k+1}$. If $z \in \Delta_k$, then by the inductive hypothesis, we have

$$z \in \mathrm{dom}(q_k^{2m}) \subseteq \mathrm{dom}(q_{k+1}^{2m}) \quad \text{and} \quad (z)q_{k+1}^{2m} = (z)q_k^{2m} \in N((x_j)p) \cap \Delta_k \subseteq N((x_j)p) \cap \Delta_{k+1}.$$

Otherwise, $z \in \Delta_{k+1} \setminus \Delta_k$. Hence $z = y_t$ for some $t \in \{0, \dots, 2m\}$. However, y_t is such that there are no edges between y_t and $\mathrm{dom}(p)$ for $t \in \{1, \dots, 2m - 1\}$. Then z is either y_0 or y_{2m}. Since $p \in \mathcal{P}$, there are no edges between $x_j \in \mathrm{dom}(p)$ and $y_{2m} = (x_{k+1})p \in \mathrm{ran}(p)$. Hence $z = y_0$ and thus $z \in \mathrm{dom}(q_{k+1}^{2m})$. Since $z \in N(x_j)$ there is an edge between x_j and $z = y_0 = x_{k+1}$. Then it follows from the fact that p is an isomorphism that there is an edge between $(x_j)p$ and $(x_{k+1})p$. Hence,

$$(z)q_{k+1}^{2m} = y_{2m} = (x_{k+1})p \in N((x_j)p) \cap \Delta_{k+1}.$$

Since z was arbitrary, we have

$$N(x_j) \cap \Delta_{k+1} \subseteq \mathrm{dom}(q_{k+1}^{2m}) \quad \text{and} \quad (N(x_j) \cap \Delta_{k+1})q_{k+1}^{2m} \subseteq N((x_j)p) \cap \Delta_{k+1}.$$

Let $z \in N((x_j)p) \cap \Delta_{k+1}$. If $z \in \Delta_k$, then it follows from the inductive hypothesis that

$$z \in N((x_j)p) \cap \Delta_k = (N(x_j) \cap \Delta_k)q_k^{2m} \subseteq (N(x_j) \cap \Delta_{k+1})q_{k+1}^{2m}.$$

Otherwise, $z = y_j$ for some $j \in \{0, \dots, 2m\}$. Similarly to above, we have $z = y_{2m} = (x_{k+1})p$ and since p is an isomorphism, we obtain $(z)q_{k+1}^{-2m} = y_0 = x_{k+1} \in N(x_j)$. Hence $z \in (N(x_j) \cap \Delta_{k+1})q_{k+1}^{2m}$ as $x_{k+1} \in \Delta_{k+1}$, and so

$$(N(x_j) \cap \Delta_{k+1})q_{k+1}^{2m} = N((x_j)p) \cap \Delta_{k+1}$$

for all $j \in \{k + 2, \dots, d\}$.

Thus, q_{k+1} satisfies the inductive hypothesis and by induction there is an extension $h = q_d \in \mathcal{I}(H(n))^{<\omega}$ of q such that h^{2m} is an extension of p. □

Finally, we prove the main result of this section.

Theorem 4.1. *Let $f \in \operatorname{Aut}(H(n))$ have infinite support. Then $D_f \cap \mathcal{I}(H(n))$ is comeagre in $\mathcal{I}(H(n))$.*

Proof. By Lemma 4.2, we have

$$D_f \cap \mathcal{I}(H(n)) = \bigcap_{p \in \mathcal{P}} \{g \in \mathcal{I}(H(n)) : \langle f, g\rangle \cap [p] \neq \varnothing\},$$

and $\{g \in \operatorname{Aut}(H(n)) : \langle f, g\rangle \cap [p] \neq \varnothing\}$ is open by Lemma 3.3, thus it is enough to show that

$$\{g \in \mathcal{I}(H(n)) : \langle f, g\rangle \cap [p] \neq \varnothing\}$$

is dense in $\mathcal{I}(H(n))$ for all $p \in \mathcal{P}$.

Fix $p \in \mathcal{P}$ and let $q \in \mathcal{I}(H(n))^{<\omega}$. If necessary by extending q, using Corollary 4.4, we may assume that all of the components of q have length m for some $m \in \mathbb{N}$, and that $\operatorname{ran}(p) \cup \operatorname{dom}(p) \subseteq \operatorname{dom}(q)$. Suppose that $\operatorname{ran}(q) \setminus \operatorname{dom}(q) = \{x_{1,0}, x_{2,0}, \dots, x_{d,0}\}$. Let $q_{1,0} = q$ and once $q_{i,j}$ is defined let $\Gamma_{i,j} = \operatorname{dom}(q_{i,j}) \cup \operatorname{ran}(q_{i,j})$ for all i, j. We will perform an induction on the elements of the set

$$\{(i, j) : i \in \{1, \dots, d\} \text{ and } j \in \{0, \dots, m\}\},$$

ordered lexicographically, to construct $q_{d,m} \in \mathcal{I}(H(n))^{<\omega}$ of the form

$$q_{d,m} = q_{1,0} \cup \{(x_{i,j}, x_{i,j+1}) : 1 \leq i \leq d \text{ and } 0 \leq j \leq m-1\}$$

such that $x_{i,j} \in \operatorname{supp}(f)$ and $(x_{i,j})f \notin \operatorname{ran}(q_{d,m}) \cup \operatorname{dom}(q_{d,m})$ for all i and all $j \geq 1$. In order to shorten the rest of the proof, once we have defined $q_{i,m}$ for some $i < d$, we will set $q_{i+1,0} = q_{i,m}$, and similarly we denote $\Gamma_{i,-1} = \varnothing$ for all i.

Suppose that for $k \in \{1, 2, \dots, d\}$ and $t \in \{0, 1, \dots, m-1\}$ we have

$$q_{k,t} = q_{1,0} \cup \{(x_{i,j}, x_{i,j+1}) : 1 \leq i \leq k \text{ and } 0 \leq j \leq t-1\} \in \mathcal{I}(H(n))^{<\omega}$$

such that $x_{k,t} \in \operatorname{supp}(f)$ and

$$x_{k,t} \notin \Gamma_{k,t-1} \cup (\Gamma_{k,t-1})f \cup (\Gamma_{k,t-1})f^{-1}.$$

Choose $x \in \operatorname{supp}(f)$ such that $x \notin \Gamma_{k,t}$ which is possible since $\operatorname{supp}(f)$ is infinite. Then by the Alice's restaurant property there is a vertex

$$y \notin \Gamma_{k,t} \cup (\Gamma_{k,t})f^{-1} \cup \{x, (x)f\}$$

such that there is an edge between x and y, and there are no edges between y and $\Gamma_{k,t} \cup (\Gamma_{k,t})f^{-1} \cup \{(x)f\}$. Let

$$U = (N(x_{k,t}))q_{k,t} \cup \{y\} \quad \text{and} \quad V = (\Gamma_{k,t} \cup (\Gamma_{k,t})f \cup (\Gamma_{k,t})f^{-1} \cup \{(y)f\}) \setminus U.$$

Since $(N(x_{k,t}))q_{k,t}$ is K_{n-1}-free and there are no edges between y and $(N(x_{k,t}))q_{k,t}$, the set U is also K_{n-1}-free. Hence by Alice's restaurant property there is a vertex

$$x_{k,t+1} \notin \Gamma_{k,t} \cup (\Gamma_{k,t})f \cup (\Gamma_{k,t})f^{-1} \cup \{y, (y)f\}$$

such that $N(x_{k,t+1}) \cap (U \cup V) = U$. It follows from $\operatorname{ran}(q_{k,t}) \subseteq \Gamma_{k,t}$ and $y \notin \Gamma_{k,t}$ that

$$N(x_{k,t+1}) \cap \operatorname{ran}(q_{k,t}) = U \cap \operatorname{ran}(q_{k,t}) = ((N(x_{k,t}))q_{k,t} \cup \{y\}) \cap \operatorname{ran}(q_{k,t}) = (N(x_{k,t}))q_{k,t},$$

and so

$$q_{k,t+1} = q_{k,t} \cup \{(x_{k,t}, x_{k,t+1})\} \in \mathcal{I}(H(n))^{<\omega}$$

by Lemma 4.3 and Corollary 4.5.

It follows from f being an automorphism and the existence of an edge between x and y, that there is an edge between $(x)f$ and $(y)f$. However, there is no edge between y and $(x)f$, thus it follows that $y \in \operatorname{supp}(f)$. The vertex y was chosen so that $y \notin \Gamma_{k,t} \cup (\Gamma_{k,t})f^{-1}$, and so $y, (y)f \notin \Gamma_{k,t}$. Since $(N(x_{k,t}))q_{k,t} \subseteq \Gamma_{k,t}$ and $y \neq (y)f$, it follows that $(y)f \notin U$. By the choice of $x_{k,t+1}$ there is an edge between $x_{k,t+1}$ and y and there are no edges between $x_{k,t+1}$ and $(y)f$, thus $x_{k,t+1} \in \operatorname{supp}(f)$. Hence $q_{k,t+1}$ satisfies the inductive hypothesis.

This way we can obtain $q_{d,m} \in \mathcal{I}(H(n))^{<\omega}$ such that for all i and all $j \geq 1$, we have

$$x_{i,j} \notin \Gamma_{i,j-1} \cup (\Gamma_{i,j-1})f \cup (\Gamma_{i,j-1})f^{-1}.$$

Hence, $(x_{i,j})f \notin \Gamma_{i,j-1}$. Also if $(x_{i,j})f = x_{i',j'}$, where $(i,j) < (i',j')$ lexicographically, then $x_{i',j'} \in (\Gamma_{i,j})f$, which is impossible. Therefore, $(x_{i,j})f \notin \operatorname{ran}(q_{d,m}) \cup \operatorname{dom}(q_{d,m})$ and thus

$$((\operatorname{dom}(q))q_{k,m}^m f) \cap (\operatorname{dom}(q_{k,m}) \cup \operatorname{ran}(q_{k,m})) = \varnothing.$$

Since $p \in \mathcal{P}$ and $\mathcal{P}$ is closed under conjugation, we have

$$u = (q_{k,m}^m f)^{-1} p q_{k,m}^m f \in \mathcal{P}.$$

Recall that $\operatorname{ran}(p) \cup \operatorname{dom}(p) \subseteq \operatorname{dom}(q)$, thus the partial isomorphisms $q_{k,m}$ and u satisfy the hypothesis of Lemma 4.7. Hence there are an extension $h \in \mathcal{I}(H(n))^{<\omega}$ of $q_{k,m}$ and $l \in \mathbb{Z}$ such that h^{2l} extends u. Therefore, $h^m f h^{2l} (h^m f)^{-1}$ extends p and thus

$$\{g \in \mathcal{I}(H(n)) : \langle f, g \rangle \cap [p] \neq \varnothing\} \cap [q] \neq \varnothing.$$

Since $q \in \mathcal{I}(H(n))^{<\omega}$ was arbitrary, we get that

$$\{g \in \mathcal{I}(H(n)) : \langle f, g \rangle \cap [p] \neq \varnothing\}$$

is dense in $\mathcal{I}(H(n))$. □

5 Infinitely many finite complete graphs: ωK_n

In this section, we consider the ultrahomogeneous graphs ωK_n for $n \in \mathbb{N}$, $n > 0$. Throughout the section, we assume that $n \in \mathbb{N}$, $n > 0$, is fixed and that the connected components of ωK_n are $\{L_i : i \in \mathbb{Z}\}$. First, we prove a couple of technical results.

We begin by characterising the elements of $\mathcal{I}_\Sigma(\omega K_n)$ in a lemma analogous to Corollary 4.5.

Lemma 5.1. *Let $q \in \operatorname{Aut}(\omega K_n)^{<\omega}$ be such that $\operatorname{dom}(q)$ is a union of connected components and there is $\Sigma \subseteq \operatorname{dom}(q)$ which intersects every component of q in exactly one vertex. Then $q \in \mathcal{I}_\Sigma(\omega K_n)^{<\omega}$ if and only if $\overline{q}$ has no complete components.*

Proposition 5.2. *Let Σ be a finite subset of ωK_n. Then $\mathcal{I}_\Sigma(\omega K_n)$ is non-empty if and only if $|\Sigma|$ is a multiple of n and if $r = |\Sigma|/n$, there is partition $\{P_1, \dots, P_r\}$ of $\mathbb{Z}$ such that P_i is infinite and*

$$\sum_{j \in P_i} |L_j \cap \Sigma| = n$$

for all $i \in \{1, \dots, r\}$.

Proof. ($\Rightarrow$) Let $f \in \mathcal{I}_\Sigma(\omega K_n)$. If $x \in L_i$ and $(x)f \in L_j$, then, since f is an automorphism, $(L_i)f = L_j$. Moreover, if $(L_i)f^m = L_i$ for some $m \in \mathbb{Z}$, then $(L_i)f^{rm} = L_i$ for all $r \in \mathbb{Z}$, and since L_i is finite, f would have a finite cycle. Hence $(L_i)f^m \neq L_i$ for all $m \in \mathbb{Z}$, and so every vertex in L_i is on a separate orbit of f.

Let $k_1, \dots, k_r \in \mathbb{Z}$ be orbit representatives of $\overline{f}$. Since for every orbit of $\overline{f}$ there are n orbits in f, it follows that $rn = |\Sigma|$. It follows from Lemma 5.1 that $\overline{f}$ has no complete component. So $(L_{k_i})f^m = (L_{k_{i'}})f^{m'}$ if and only if $i = i'$ and $m = m'$. Hence,

$$n = \Big|\Sigma \cap \Big(\bigcup_{m \in \mathbb{Z}} (L_{k_i})f^m\Big)\Big| = \Big|\bigcup_{m \in \mathbb{Z}} (\Sigma \cap (L_{k_i})f^m)\Big| = \sum_{m \in \mathbb{Z}} |\Sigma \cap (L_{k_i})f^m|$$

for all $i \in \{1, \dots, r\}$. Let $P_i = \{(k_i)\overline{f}^m : m \in \mathbb{Z}\}$ where $i \in \{1, \dots, r\}$. Then $\{P_1, \dots, P_r\}$ is the required partition.

(⇐) For $i \in \{1, \dots, r\}$ let $P_i = \{k_{i,j} : j \in \mathbb{Z}\}$. Define $f \in \mathcal{I}_\Sigma(\omega K_n)$ to be such that

$$(k_{i,j})\overline{f} = k_{i,j+1}$$

for all $i \in \{1, \dots, r\}$ and $j \in \mathbb{Z}$ by inductively defining f on $\bigcup_{j\in\mathbb{Z}} L_{k_{i,j}}$ for each i independently.

Let $i \in \{1, \dots, r\}$ be arbitrary. Then $|L_{k_{i,0}} \cap \Sigma| + |L_{k_{i,1}} \cap \Sigma| \le n$. Since $L_{k_{i,0}}$ and $L_{k_{i,1}}$ are both of size n, there exists a bijection $q_1 : L_{k_{i,0}} \to L_{k_{i,1}}$ such that for every $x \in L_{k_{i,0}}$ at most one of the points x and $(x)q_1$ is in Σ. Suppose that for some $m \in \mathbb{N}$ we have a bijection

$$q_{2m+1} : \bigcup_{j=-m}^{m} L_{k_{i,j}} \to \bigcup_{k=-m+1}^{m+1} L_{k_{i,j}}$$

such that every incomplete component of q_{2m+1} intersects Σ in at most one point.

Let $t = \sum_{j=-m}^{m+1} |L_{k_{i,j}} \cap \Sigma|$. Then there are $n - t$ incomplete components of q_{2m+1} which have empty intersection with Σ. Since

$$\sum_{j=-m-1}^{m+1} |L_{i(j,r)} \cap \Sigma| \le n,$$

it follows that $|L_{k_{i,-m-1}} \cap \Sigma| \le n - t$. Hence there exists a bijection $\phi : L_{k_{i,-m-1}} \to L_{k_{i,-m}}$ such that for every $x \in L_{k_{i,-m-1}} \cap \Sigma$ the value $(x)\phi$ belongs to an incomplete component of q_{2m+1} which contains no points from Σ. If we set $q_{2m+2} = q_{2m+1} \cup \phi$, then every incomplete component of q_{2m+2} intersects Σ in at most one point. Similarly we can extend q_{2m+2} to q_{2m+3} by adding a bijection from $L_{k_{i,m+1}}$ to $L_{k_{i,m+2}}$.

Hence by induction,

$$f_i = \bigcup_{m\in\mathbb{Z}} q_{2m+1}$$

is an automorphism of $\bigcup_{j\in\mathbb{Z}} L_{k_{i,j}}$ and every orbit of f_i intersects Σ exactly once. The required f is then just the function $\bigcup_{i=1}^{r} f_i$. □

Lemma 5.3. *Let $\Sigma \subseteq \omega K_n$ be finite, and let $\mathcal{F}$ consist of those $g \in \operatorname{Aut}(\omega K_n)^{<\omega}$ where the sets $\operatorname{dom}(g)$ and $\operatorname{ran}(g)$ are disjoint, both are unions of connected components of ωK_n, and $\overline{g}$ does not have any complete components. Then*

$$D_f \cap \mathcal{I}_\Sigma(\omega K_n) = \bigcap_{p\in\mathcal{F}} \{g \in \mathcal{I}_\Sigma : \langle f, g\rangle \cap [p] \ne \varnothing\}.$$

Proof. Recall that

$$D_f \cap \mathcal{I}_\Sigma(\omega K_n) = \{g \in \mathcal{I}_\Sigma(\omega K_n) : \langle f, g\rangle \text{ is dense in } \operatorname{Aut}(\omega K_n)\} = \bigcap_{q\in\operatorname{Aut}(\omega K_n)^{<\omega}} \{g \in \mathcal{I}_\Sigma(\omega K_n) : \langle f, g\rangle \cap [q] \ne \varnothing\}.$$

(⊆) This follows immediately, since $\mathcal{F} \subseteq \operatorname{Aut}(\omega K_n)^{<\omega}$.

(⊇) Let

$$g \in \bigcap_{p\in\mathcal{F}} \{g \in \mathcal{I}_\Sigma(\omega K_n) : \langle f, g\rangle \cap [p] \ne \varnothing\},$$

let $q \in \operatorname{Aut}(\omega K_n)^{<\omega}$, and let

$$\Gamma = \bigcup \{L_i : \operatorname{dom}(q) \cap L_i \ne \varnothing\}.$$

If $h \in \operatorname{Aut}(\omega K_n)$ is an extension of q, $x \in L_i$, and $(x)h \in L_j$, then $(L_i)h = L_j$. Hence $(\Gamma)h$ is a union of connected components of ωK_n. Let $r = h|_\Gamma$. Then $[r] \subseteq [q]$.

Let Γ be a subgraph of ωK_n such that Γ is isomorphic to $\operatorname{dom}(r)$, $\Gamma \cap (\operatorname{dom}(r) \cup \operatorname{ran}(r)) = \varnothing$, and such that there are no edges between Γ and $\operatorname{dom}(r) \cup \operatorname{ran}(r)$. Let p be any isomorphism between $\operatorname{dom}(r)$ and Γ. Note that since $\operatorname{dom}(r)$ is a union of connected components of ωK_n, so is Γ. Since ωK_n is ultrahomogeneous, we have that $p \in \operatorname{Aut}(\omega K_n)^{<\omega}$. Then we obtain $\operatorname{dom}(p) = \operatorname{dom}(r)$, $\operatorname{ran}(p) = \operatorname{dom}(p^{-1}r) = \Gamma$ and $\operatorname{ran}(p^{-1}r) = \operatorname{ran}(r)$. Hence, $p, p^{-1}r \in \mathcal{F}$. By the choice of g there are $h_1, h_2 \in \langle f, g\rangle$ such that $h_1 \in [p]$ and $h_2 \in [p^{-1}r]$. Therefore, $h_1h_2 \in [r] \subseteq [q]$ and $h_1h_2 \in \langle f, g\rangle$, thus $\langle f, g\rangle \cap [q] \ne \varnothing$. Since q was arbitrary,

$$g \in \bigcap_{q\in\operatorname{Aut}(\omega K_n)^{<\omega}} \{g \in \mathcal{I}_\Sigma(\omega K_n) : \langle f, g\rangle \cap [q] \ne \varnothing\}.$$ □

We will now prove Theorem 2.2 (ii), which we restate for the sake of convenience.

Theorem 5.4. *Let $f \in \mathrm{Aut}(\omega K_n)$ be such that $\mathrm{supp}(\bar{f})$ is infinite and let Σ be a finite subset of ωK_n. Then $D_f \cap \mathcal{I}_\Sigma(\omega K_n)$ is comeagre in $\mathcal{I}_\Sigma(\omega K_n)$.*

Proof. If $\mathcal{I}_\Sigma(\omega K_n)$ is empty, then the result holds trivially. So for the remainder of the proof, we will suppose that $\mathcal{I}_\Sigma(\omega K_n)$ is non-empty.

By Lemma 5.3, we have

$$D_f \cap \mathcal{I}_\Sigma(\omega K_n) = \bigcap_{p \in \mathcal{F}} \{g \in \mathcal{I}_\Sigma(\omega K_n) : \langle f, g \rangle \cap [p] \neq \varnothing\},$$

and by Lemma 3.3 the set $\{g \in \mathcal{I}_\Sigma(\omega K_n) : \langle f, g \rangle \cap [p] \neq \varnothing\}$ is open, so it suffices to show that the aforementioned set is dense in $\mathcal{I}_\Sigma(\omega K_n)$.

Let $p \in \mathcal{F}$ and let $q \in \mathcal{I}_\Sigma(\omega K_n)^{<\omega}$. We will show that there exists an extension $h \in \mathcal{I}_\Sigma(\omega K_n)^{<\omega}$ of q such that every extension $g \in \mathcal{I}_\Sigma(\omega K_n)$ of h satisfies $\langle f, g \rangle \cap [p] \neq \varnothing$. If necessary, by extending q, we can assume without loss of generality that $\mathrm{dom}(q)$ is a union of connected components of ωK_n, and that q has $|\Sigma|$ incomplete components each of some fixed length m, and that $\Sigma \cup \mathrm{dom}(p) \cup \mathrm{ran}(p) \subseteq \mathrm{dom}(q)$. Then $\mathrm{ran}(q) \setminus \mathrm{dom}(q)$ is a union of connected components $L_{1,0}, \ldots, L_{N,0}$ for some $N \in \mathbb{N}$.

Let $q_{1,0} = q$ and once $q_{i,j}$ is defined let $\Gamma_{i,j} = \mathrm{dom}(q_{i,j}) \cup \mathrm{ran}(q_{i,j})$. Suppose there is $i \in \{0, \ldots, m-1\}$ such that $q_{1,i} \in \mathcal{I}_\Sigma(\omega K_n)^{<\omega}$ is defined in a way such that $\mathrm{dom}(q_{1,i})$ is a union of connected components and $(x)q_{1,i}^j \in L_{1,j}$ for all $x \in L_{1,0}$ and $j \in \{1, \ldots, i\}$. Since $\bar{f}$ has infinite support, there exists a connected component $L_{1,i+1}$ of ωK_n such that $(L_{1,i+1})f \neq L_{1,i+1}$ and

$$L_{1,i+1} \cap (\Gamma_{1,i} \cup (\Gamma_{1,i})f \cup (\Gamma_{1,i})f^{-1}) = \varnothing.$$

Let $\phi : L_{1,i} :\to L_{1,i+1}$ be a bijection and let $q_{1,i+1} = q_{1,i} \cup \phi$. Then $q_{1,i+1} \in \mathcal{I}_\Sigma(\omega K_n)^{<\omega}$ by Lemma 5.1. Also by definition of $q_{1,i+1}$ the set $\mathrm{dom}(q_{1,i+1}) = \mathrm{dom}(q_{1,i}) \cup L_{1,i}$ is a union of connected components and $(x)q_{1,i+1}^{i+1} \in L_{1,i+1}$ for all $x \in L_{1,0}$. Hence by induction there is $q_{1,m} \in \mathcal{I}_\Sigma(\omega K_n)^{<\omega}$ such that $\mathrm{dom}(q_{1,m})$ is a union of connected components of ωK_n and $(x)q_{1,m}^j \in L_{1,j}$ for all $x \in L_{1,0}$ and $j \in \{1, \ldots, m\}$.

Let $q_{2,0} = q_{1,m}$ and suppose for some $i \in \{2, \ldots, N\}$ that there is an extension $q_{i,0} \in \mathcal{I}_\Sigma(\omega K_n)^{<\omega}$ of q such that $\mathrm{dom}(q_{i,0})$ is a union of connected components of ωK_n and $(x)q_{i,0}^k \in L_{j,k}$ for all $x \in L_{j,0}$, all $j \in \{1, \ldots, i-1\}$, and all $k \in \{1, \ldots, m\}$. The same argument as before can be used to define an extension $q_{i,m} \in \mathcal{I}_\Sigma(\omega K_n)^{<\omega}$ of q such that $\mathrm{dom}(q_{i,m})$ is a union of connected components of ωK_n and $(x)q_{i,m}^k \in L_{j,k}$ for all $x \in L_{j,0}$, all $j \in \{1, \ldots, i\}$, and all $k \in \{1, \ldots, m\}$. Hence by induction, $\mathrm{dom}(q_{N,m})$ is a union of connected components of ωK_n and $(x)q_{N,m}^k \in L_{j,k}$ for all $x \in L_{j,0}$, all $j \in \{1, \ldots, N\}$, and all $k \in \{1, \ldots, m\}$.

We will show that $q_{N,m}$ is the desired extension of q. Let $r = q_{N,m}$. If $x \in L_{i,0}$ for some $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, m\}$, then

$$(x)r^j \in L_{i,j} \subseteq \Gamma_{i,j} \tag{5.1}$$

and so by the choice of $L_{i,j}$ we have $(x)r^j \notin \Gamma_{i,j-1} \cup (\Gamma_{i,j-1})f \cup (\Gamma_{i,j-1})f^{-1}$ for all $j \in \{1, \ldots, m\}$. In particular,

$$(x)r^j f \notin \Gamma_{i,j-1} \quad \text{and} \quad (x)r^j f^{-1} \notin \Gamma_{i,j-1} \tag{5.2}$$

for all $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, m\}$.

Let $x \in L_{i,0}$ and $y \in L_{j,0}$ for any $i, j \in \{1, \ldots, N\}$. We will show that $((x)r^k)f \neq (y)r^l$ for all $k \in \{1, \ldots, m\}$ and $l \in \{-m+1, \ldots, m\}$. If $i = j$ and $k = l$, then, since $(x)r^k, (y)r^l \in L_{i,k}$ by (5.1) and $(L_{i,k})f \neq L_{i,k}$ by the choice of $L_{i,j}$, it follows that $((x)r^k)f \neq (y)r^l$. Hence we may assume that $(i, k) \neq (j, l)$. There are three cases to consider.

If $l \leq 0$, then

$$(y)r^l \in \mathrm{dom}(q_{1,0}) \cup \mathrm{ran}(q_{1,0}) = \Gamma_{1,0} \subseteq \Gamma_{i,k}$$

and $(x)r^k f \notin \Gamma_{i,k}$ by (5.2), and so $(x)r^k f \neq (y)r^l$.

Suppose that $i > j$ and $l > 0$, or $i = j$ and $k > l > 0$. Then $(y)r^l \in \Gamma_{j,l}$ by (5.1). By the assumption of this case, we obtain $\Gamma_{j,l} \subseteq \Gamma_{i,k-1}$ and $((x)r^k)f \notin \Gamma_{i,k-1}$ by (5.2). Thus $((x)r^k)f \neq (y)r^l$ in this case.

Suppose that $i < j$ and $l > 0$, or $i = j$ and $k < l$. Then $\Gamma_{i,k} \subseteq \Gamma_{j,l-1}$. Since $((y)r^l)f^{-1} \notin \Gamma_{j,l-1}$ by (5.2), it follows that $((y)r^l)f^{-1} \notin \Gamma_{i,k}$, and so $((x)r^k)f \neq (y)r^l$. Therefore, in all three cases $((x)r^k)f \notin \mathrm{ran}(r) \cup \mathrm{dom}(r)$.

Recall that $\operatorname{dom}(p) \cup \operatorname{ran}(p) \subseteq \operatorname{dom}(q)$ and that every point in $\operatorname{dom}(q)$ can be expressed as $(x)r^j$ for some $x \in \bigcup_{i=1}^{N} L_{i,0}$ and $j \in \{-m+1, \dots, -1\}$. Define $u = (r^m f)^{-1} p (r^m f)$. Since $\overline{p}$ has no complete components, the same is true for $\overline{u}$. Also

$$\operatorname{dom}(u) \cup \operatorname{ran}(u) \subseteq \{((x)r^j)f : 1 \leq j \leq m \text{ and } x \in L_{i,0} \text{ for some } i\}$$

and hence $(\operatorname{dom}(u) \cup \operatorname{ran}(u)) \cap (\operatorname{dom}(r) \cup \operatorname{ran}(r)) = \varnothing$.

Suppose $\operatorname{dom}(u) \setminus \operatorname{ran}(u) = \bigcup_{k=1}^{M} L_{i_k}$ and let n_k be the largest integer such that $(L_{i_k})u^{n_k}$ is defined for some $k \in \{1, \dots, M\}$. Define v to be an extension of u by bijections $(L_{i_k})u^{n_k} \to L_{i_{k+1}}$ for all $k \in \{1, \dots, M-1\}$. Then the domain of v is a union of connected components of the graph and v has no complete components, since neither p nor u do. Finally, choose any bijection $\psi : L_{N,m} \to L_{i_1}$ and define $h = r \cup \psi \cup v$. Then the number of components in h is $|\Sigma|$ and so $h \in \mathcal{I}_\Sigma(\omega K_n)^{<\omega}$ by Lemma 5.1. Let $g \in \mathcal{I}_\Sigma(\omega K_n)$ be an extension of h. By definition of u we have that $(h^m f)h(h^m f)^{-1}$ extends p, thus $\langle f, g\rangle \cup [p] \neq \varnothing$ and $g \in [q]$. Therefore, the set $\{g \in \mathcal{I}_\Sigma(\omega K_n) : \langle f, g\rangle \cap [p] \neq \varnothing\}$ is dense in $\mathcal{I}_\Sigma(\omega K_n)$. □

The following is an immediate corollary of Lemma 3.4.

Corollary 5.5. *Let $f \in \operatorname{Aut}(\omega K_n)$ be such that $\operatorname{supp}(\overline{f})$ is infinite. Then $D_f \cap \mathcal{I}(\omega K_n)$ is comeagre in $\mathcal{I}(\omega K_n)$.*

6 Finitely many infinite complete graphs: nK_ω

In this section, we will consider the ultrahomogeneous graph nK_ω for a fixed $n \in \mathbb{N}$ such that $n \geq 2$. Throughout this section, let $L_1, L_2, \dots, L_n$ be the connected components of nK_ω. Recall that, if $f \in \operatorname{Aut}(nK_\omega)$ and $\Sigma \subseteq nK_\omega$ is finite, then

$$\mathcal{A}_f = \{g \in \operatorname{Aut}(nK_\omega) : \langle \overline{f}, \overline{g}\rangle = S_n\}$$

and

$$\mathcal{A}_{f,\Sigma} = \{g \in \mathcal{A}_f : \Sigma \text{ is a set of orbit representatives of } g\}.$$

To specify when $\mathcal{A}_f$ is non-empty, we require the following classical theorem.

Proposition 6.1 (cf. [10]). *Let $a \in S_n$ be a non-identity element and let $n \in \mathbb{N}$ be such that $n \neq 4$, or $n = 4$ and $a \notin \{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. Then there exists $b \in S_n$ such that $\langle a, b\rangle = S_n$.*

We see by Proposition 6.1, that $\mathcal{A}_f \neq \varnothing$ if and only if $n \neq 4$, or $n = 4$ and $\overline{f} \notin \{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$.

Next, we show that $\mathcal{A}_f$ and $\mathcal{A}_{f,\Sigma}$ are Baire spaces and thus we can consider their comeagre subsets.

Lemma 6.2. *Let $\Sigma \subseteq nK_\omega$ be finite. Then $\mathcal{A}_f$ is closed and $\mathcal{A}_{f,\Sigma}$ is a Baire space.*

Proof. Let $g \in \operatorname{Aut}(nK_\omega) \setminus \mathcal{A}_f$. Then $\langle \overline{f}, \overline{g}\rangle \neq S_n$. Let $\Gamma \subseteq nK_\omega$ be a finite set such that $L_i \cap \Gamma \neq \varnothing$ for all $i \in \{1, \dots, n\}$. Then for all $h \in [g|_\Gamma]$ we have that $\overline{h} = \overline{g}$ and thus $h \notin \mathcal{A}_f$. Therefore, the open set $[g|_\Gamma]$ is a subset of $\operatorname{Aut}(nK_\omega) \setminus \mathcal{A}_f$ and thus $\mathcal{A}_f$ is closed, and hence Baire. Then, by Lemma 3.2, $\mathcal{A}_{f,\Sigma}$ is a Baire space. □

The following lemma combined with Lemma 6.2 demonstrates that D_f is not dense, and thus not comeagre, in any set which is not contained in $\mathcal{A}_f$.

Lemma 6.3. *If $g \in \operatorname{Aut}(nK_\omega)$ is such that $\langle f, g\rangle$ is dense in $\operatorname{Aut}(nK_\omega)$, then $\langle \overline{f}, \overline{g}\rangle = S_n$. In other words, $D_f \subseteq \mathcal{A}_f$.*

Proof. Let $g \in D_f$. Then $\langle f, g\rangle$ is dense in $\operatorname{Aut}(nK_\omega)$. Let $\sigma \in S_n$ be arbitrary. Then it is straightforward to verify that there is $q \in \operatorname{Aut}(nK_\omega)^{<\omega}$ such that $\overline{q} = \sigma$. Since $\langle f, g\rangle$ is dense, it follows that there is a product $h \in \langle f, g\rangle$ which extends q. Therefore $\sigma = \overline{h} \in \langle \overline{f}, \overline{g}\rangle$ which implies that $g \in \mathcal{A}_f$. □

Let $f \in \operatorname{Aut}(nK_\omega)$. Then f is called *non-stabilizing* if for all $\Gamma \subsetneq nK_\omega$, all $x \in \Gamma$, and all $q \in \mathcal{A}_f^{<\omega}$ there is $g \in [q] \cap \mathcal{A}_f$ such that $(x)h \notin \Gamma$ for some $h \in \langle f, g\rangle$. We say that $f \in \operatorname{Aut}(nK_\omega)$ is *stabilizing* if it is not non-stabilizing.

Proposition 6.4. *Let $f \in \mathrm{Aut}(nK_\omega)$ be such that $\mathcal{A}_f \neq \varnothing$. Then f is stabilizing if and only if there is a finite subset Γ of nK_ω such that f stabilises Λ setwise and*

$$|L_i \cap \Lambda| = |L_j \cap \Lambda|$$

for all $i, j \in \{1, 2, \ldots, n\}$.

Proof. ($\Rightarrow$) Let f be a stabilizing automorphism of nK_ω. By the definition of being non-stabilizing, there are $\Delta \subsetneq nK_\omega$, $x \in \Delta$ and $q \in \mathcal{A}_f^{<\omega}$ such that for all $g \in [q] \cap \mathcal{A}_f$ and all $h \in \langle f, g\rangle$ we have that $(x)h \in \Delta$. If necessary, by taking an extension of q, we may assume without loss of generality that $\overline{q} \in S_n$. Fix any $g \in [q] \cap \mathcal{A}_f$ and let $\Gamma = \{(x)h : h \in \langle f, g\rangle\} \subseteq \Delta$. Then the subgroup $\langle f, g\rangle$ stabilises Γ. Hence f also setwise stabilises Γ. Let $i, j \in \{1, \ldots, n\}$ be arbitrary. Since $g \in \mathcal{A}_f$ we may choose $h \in \langle f, g\rangle$ such that $(i)\overline{h} = j$. By the definition, Γ is setwise stabilised by h and thus

$$(L_i \cap \Gamma)h \subseteq L_j \cap \Gamma \quad \text{and} \quad (L_j \cap \Gamma)h^{-1} \subseteq L_i \cap \Gamma,$$

as both h and h^{-1} are bijections. It follows that $|L_i \cap \Gamma| = |L_j \cap \Gamma|$. Since $\langle f, g\rangle$ also setwise stabilises $nK_\omega \setminus \Gamma$, the same argument shows that $|L_i \cap (nK_\omega \setminus \Gamma)| = |L_j \cap (nK_\omega \setminus \Gamma)|$.

Finally, suppose that both Γ and $nK_\omega \setminus \Gamma$ are infinite. Then for every $i \in \{1, \ldots, n\}$ the sets

$$(\Gamma \cap L_i) \setminus (\mathrm{dom}(q) \cup \mathrm{ran}(q)) \quad \text{and} \quad ((nK_\omega \setminus \Gamma) \cap L_i) \setminus (\mathrm{dom}(q) \cup \mathrm{ran}(q))$$

are non-empty. Hence for every $i \in \{1, \ldots, n\}$ there are $x \in L_i \cap \Gamma$ and an extension $g \in \mathrm{Aut}(nK_\omega)$ of q such that $(x)g \in nK_\omega \setminus \Gamma$, contradicting the choice of Γ. Therefore either Γ or $nK_\omega \setminus \Gamma$ is finite and since both sets are stabilised setwise by f, one of them is the required set Λ.

($\Leftarrow$) Let $m = |L_i \cap \Lambda|$ for any, and all, $i \in \{1, 2, \ldots, n\}$ and let $L_i \cap \Lambda = \{y(i, j) : 1 \leq j \leq m\}$. Since $\mathcal{A}_f$ is non-empty, there is $\sigma \in S_n$ such that $\langle \overline{f}, \sigma\rangle = S_n$. Define a finite isomorphism $q : \Lambda \to \Lambda$ such that

$$(y(i, j))q = y((i)\sigma, j)$$

for all $j \in \{1, \ldots, m\}$. Then $\overline{q} = \sigma$ and so $q \in \mathcal{A}_f^{<\omega}$. Moreover, Λ is a union of cycles of q and hence $\langle f, g\rangle$ stabilises Λ for any $g \in [q]$. Therefore, f is stabilizing. □

The following theorem is a restatement of Theorem 2.2 (iii) and it is the main result in this section.

Theorem 6.5. *Let $f \in \mathrm{Aut}(nK_\omega)$. Then f is non-stabilizing if and only if D_f is comeagre in $\mathcal{A}_f$. Furthermore, if f is non-stabilizing and Σ is any finite subset of nK_ω, then $D_f \cap \mathcal{A}_{f,\Sigma}$ is comeagre in $\mathcal{A}_{f,\Sigma}$.*

If f is stabilizing and $D_f \cap \mathcal{A}_{f,\Sigma}$ is comeagre in $\mathcal{A}_{f,\Sigma}$ for all Σ, then by Lemma 3.4, we have that $D_f \cap \mathcal{A}_f$ is comeagre in $\mathcal{A}_f$ and so, by Theorem 6.5, we obtain that f is non-stabilizing, which is a contradiction. Hence if f is stabilizing, then there exists Σ such that $D_{f,\Sigma} \cap \mathcal{A}_{f,\Sigma}$ is not comeagre in $\mathcal{A}_{f,\Sigma}$. It is therefore natural to ask: for which stabilizing f and finite sets Σ is $D_f \cap \mathcal{A}_{f,\Sigma}$ comeagre in $\mathcal{A}_{f,\Sigma}$?

We will prove Theorem 6.5 in a series of lemmas. We begin by showing several ways to extend partial isomorphisms in $\mathcal{A}_{f,\Sigma}^{<\omega}$, which we will have to do ad infinitum in the proof of Theorem 6.5.

The first lemma follows immediately from the definitions and the proof is omitted.

Lemma 6.6. *Let $q \in \mathrm{Aut}(nK_\omega)^{<\omega}$ be such that $\overline{q} \in S_n$ and let $h = q \cup \{(x, y)\}$. Then $h \in \mathrm{Aut}(nK_\omega)^{<\omega}$ if and only if there is $a \in \{1, \ldots, n\}$ such that $x \in L_a \setminus \mathrm{dom}(q)$ and $y \in L_{(a)\overline{q}} \setminus \mathrm{ran}(q)$.*

Roughly speaking, in the next lemma, we show how to extend a partial isomorphism with a set of orbit representative to an automorphism with the same set of orbit representatives.

Lemma 6.7. *Let $q \in \mathrm{Aut}(nK_\omega)^{<\omega}$ be such that $\overline{q} \in S_n$ and let Σ be a finite subset of $\mathrm{dom}(q)$ such that $|\Sigma \cap C| \leq 1$ for every component C of q, with equality holding if C is complete. Suppose that for each $i \in \{1, \ldots, n\}$ there is $j \in \{1, \ldots, n\}$ such that $(j)\overline{q}^m = i$ for some $m \in \mathbb{Z}$ and $L_j \cap \Sigma$ contains a point in an incomplete component of q. Then there is an extension $g \in \mathrm{Aut}(nK_\omega)$ of q such that Σ is a set of orbit representatives of g, every incomplete component of q is contained in an infinite orbit of g, and $(x)g \notin \mathrm{dom}(q)$ for all $x \in \mathrm{ran}(q) \setminus \mathrm{dom}(q)$.*

Proof. For each $x \in \operatorname{ran}(q) \setminus \operatorname{dom}(q)$ there is $a \in \{1, \dots, n\}$ such that $x \in L_a$ and there is

$$y \in L_{(a)\overline{q}} \setminus (\operatorname{dom}(q) \cup \operatorname{ran}(q)).$$

Then by Lemma 6.6, the mapping $q' = q \cup \{(x, y)\}$ is in $\operatorname{Aut}(nK_\omega)^{<\omega}$ and $(x)q' = y \notin \operatorname{dom}(q)$. Repeating this for each vertex in $\operatorname{ran}(q) \setminus \operatorname{dom}(q)$ we obtain an extension $q'' \in \operatorname{Aut}(nK_\omega)^{<\omega}$ of q such that $(x)q'' \notin \operatorname{dom}(q)$ for all $x \in \operatorname{ran}(q) \setminus \operatorname{dom}(q)$. Hence $(x)g = (x)q'' \notin \operatorname{dom}(q)$ for every extension $g \in \operatorname{Aut}(nK_\omega)$ of q'' and every $x \in \operatorname{ran}(q) \setminus \operatorname{dom}(q)$.

Suppose that O is an incomplete component of q'' such that $O \cap \Sigma = \varnothing$. Let $y \in O \cap \operatorname{dom}(q'') \setminus \operatorname{ran}(q'')$. Then there is $a \in \{1, \dots, n\}$ such that $y \in L_a$. It follows from the hypothesis that there are $b \in \{1, \dots, n\}$, $y_0 \in L_b \cap \operatorname{ran}(q'') \setminus \operatorname{dom}(q'')$ such that the component of q'' containing y_0 intersects Σ non-trivially, and $m \in \mathbb{N}$ such that $(b)\overline{q}^m = a$. Successively for each $i \in \{1, \dots, m-1\}$ choose

$$y_i \in L_{(b)\overline{q}^i} \setminus (\operatorname{dom}(q'') \cup \operatorname{ran}(q'') \cup \{y_1, \dots, y_{i-1}\}),$$

and let $y_m = y$. Then by repeated application of Lemma 6.6 we have that

$$q'' \cup \{(y_{i-1}, y_i) : 1 \le i \le m\} \in \operatorname{Aut}(nK_\omega)^{<\omega}.$$

If we repeat this for every incomplete component of q'' which has empty intersection with Σ, we obtain an extension $q_0 \in \operatorname{Aut}(nK_\omega)^{<\omega}$ of q'' such that every component of q'' intersect Σ in exactly one point.

Let $nK_\omega = \{x_i : i \in \mathbb{N}\}$ and suppose that for some $j \in \mathbb{N}$ we have a $q_j \in \operatorname{Aut}(nK_\omega)^{<\omega}$ such that incomplete components of q are contained in incomplete components of q_j, Σ consists of exactly one point from every component of q_j, and

$$\{x_1, \dots, x_j\} \subseteq \operatorname{dom}(q_j) \cap \operatorname{ran}(q_j).$$

Suppose $x_{j+1} \notin \operatorname{dom}(q_j) \cap \operatorname{ran}(q_j)$. There are three cases to consider.

Suppose that $x_{j+1} \in \operatorname{ran}(q_j) \setminus \operatorname{dom}(q_j)$. Then by Lemma 6.6 there is a one-point extension

$$q_{j+1} = q_j \cup \{(x_{j+1}, y)\} \in \operatorname{Aut}(nK_\omega)^{<\omega}$$

for some $y \notin \operatorname{dom}(q_j) \cup \operatorname{ran}(q_j)$. Suppose that $x_{j+1} \in \operatorname{dom}(q_j) \setminus \operatorname{ran}(q_j)$. Then by Lemma 6.6 there is a one-point extension

$$q_{j+1}^{-1} = q_j^{-1} \cup \{(x_{j+1}, y)\} \in \operatorname{Aut}(nK_\omega)^{<\omega}$$

for some $y \notin \operatorname{dom}(q_j) \cup \operatorname{ran}(q_j)$.

Finally, suppose that $x_{j+1} \in L_a \setminus (\operatorname{dom}(q_j) \cup \operatorname{ran}(q_j))$ for some a. It follows from the hypothesis that there are $b \in \{1, \dots, n\}$, $y_0 \in L_b \cap \operatorname{ran}(q_j) \setminus \operatorname{dom}(q_j)$ such that the component of q_j containing y_0 intersects Σ non-trivially and $m \in \mathbb{N}$ such that $(b)\overline{q}^m = a$. Successively for each $i \in \{1, \dots, m-1\}$ choose

$$y_i \in L_{(b)\overline{q}^i} \setminus (\operatorname{dom}(q_j) \cup \operatorname{ran}(q_j) \cup \{y_1, \dots, y_{i-1}\}).$$

Also let $y_m = x_{j+1}$. Then by repeated application of Lemma 6.6 we have that

$$q_j \cup \{(y_{i-1}, y_i) : 1 \le i \le m\} \in \operatorname{Aut}(nK_\omega)^{<\omega}.$$

Now, we get the first case and we can define q_{j+1} as before.

In all three cases, we have defined an extension q_{j+1} satisfying the inductive hypothesis. Let

$$g = \bigcup_{j \in \mathbb{N}} q_j.$$

Then $g \in \operatorname{Aut}(nK_\omega)$ and since the orbits of g are in one to one correspondence with incomplete components of q_0, it follows that Σ is a set of orbit representatives. □

Corollary 6.8. *Let $q \in \mathcal{A}_{f,\Sigma}^{<\omega}$ be such that $\Sigma \subseteq \operatorname{dom}(q)$ and $\overline{q} \in S_n$. Then there is an extension $g \in \mathcal{A}_{f,\Sigma}$ of q such that every incomplete component of q is contained in an infinite orbit of g and $(x)g \notin \operatorname{dom}(q)$ for all $x \in \operatorname{ran}(q) \setminus \operatorname{dom}(q)$.*

Proof. Since $q \in \mathcal{A}_{f,\Sigma}^{<\omega}$, the set Σ intersects every incomplete component of q in at most one point and every complete component in exactly one point.

If $i \in \{1, \dots, n\}$ is arbitrary, then, since every extension h of q in $\mathcal{A}_{f,\Sigma}$ has $|\Sigma|$ orbits, it follows that there is at least one infinite orbit of h with points in L_i. Since Σ is a set of orbit representatives, there exists $x \in \Sigma \cap L_j$ for some $j \in \{1, \dots, n\}$ such that $(j)\overline{q}^m = i$ for some $m \in \mathbb{Z}$. In particular, x is on an incomplete component of q and so q satisfies the hypothesis of Lemma 6.7 from which the corollary follows. □

In the next lemma, as a further consequence of Lemma 6.7, we show that the direct implication of the first part of Theorem 6.5 is a consequence of the second part.

Lemma 6.9. *Let $f \in \operatorname{Aut}(nK_\omega)$ be such that $D_f \cap \mathcal{A}_{f,\Sigma}$ is comeagre in $\mathcal{A}_{f,\Sigma}$ for all finite sets $\Sigma \subseteq nK_\omega$. Then D_f is comeagre in A_f.*

Proof. Suppose $q \in \mathcal{A}_f^{<\omega}$. If necessary by extending q, we can assume that $\overline{q} \in S_n$. Then all extensions $h \in \operatorname{Aut}(nK_\omega)^{<\omega}$ of q are also in $\mathcal{A}_f^{<\omega}$. For all $i \in \{1, \dots, n\}$, let $x_i \in L_i \setminus (\operatorname{dom}(q) \cup \operatorname{ran}(q))$. Then, applying Lemma 6.6 repeatedly we can construct an extension $h \in \mathcal{A}_f^{<\omega}$ of q such that each vertex x_i is on an incomplete component of h. Fix any $\Sigma \subseteq nK_\omega$ such that Σ intersects every component of h exactly once. Since $\overline{h} \in S_n$ for each $i \in \{1, \dots, n\}$, there is an incomplete component containing x_i and by the choice of Σ there is $j \in \{1, \dots, n\}$ such that $\Sigma \cap L_j$ is non-empty and $(j)\overline{h}^m = i$ for some $m \in \mathbb{Z}$. Then by Lemma 6.7 there is an extension g of q with finitely many orbits. Therefore, we are done by Lemma 3.4. □

Lemma 6.10. *Let $q \in \mathcal{A}_{f,\Sigma}^{<\omega}$ be such that $\Sigma \subseteq \operatorname{dom}(q)$ and $\overline{q} \in S_n$. Suppose $h = q \cup \{(x, y)\} \in \operatorname{Aut}(nK_\omega)^{<\omega}$ for some $x \notin \operatorname{dom}(q)$ and $y \notin \operatorname{dom}(q) \cup \operatorname{ran}(q)$ such that $x \neq y$. Then $h \in \mathcal{A}_{f,\Sigma}^{<\omega}$.*

Proof. Since $q \in \operatorname{Aut}(nK_\omega)^{<\omega}$, there is $r \in \operatorname{Aut}(nK_\omega)^{<\omega}$ extending q, such that $x \in \operatorname{ran}(r) \setminus \operatorname{dom}(r)$. By Corollary 6.8 there is $g \in \mathcal{A}_{f,\Sigma}$ such that every incomplete component of r is contained in an infinite orbit of g and $(x)g \notin \operatorname{dom}(r)$, and so $(x)g \notin \operatorname{dom}(q)$. If g extends h, then we are done; so we assume that $(x)g \neq y$. Note that if $(x)g = x$, then $\{x\}$ is an orbit of g and therefore $x \in \Sigma$. However, $\Sigma \subseteq \operatorname{dom}(q)$, which contradicts the assumption that $x \notin \operatorname{dom}(q)$. Hence, $(x)g \neq x$.

Since $x \notin \operatorname{dom}(q)$ and g is an extension of q, it follows that $(x)g \notin \operatorname{ran}(q)$. Then $(x)g, y \notin \operatorname{dom}(q) \cup \operatorname{ran}(q)$ and since $h \in \operatorname{Aut}(nK_\omega)^{<\omega}$ and $(x)g \in \operatorname{Aut}(nK_\omega)$, it follows that $(x)g$ and y are in the same connected component of nK_ω. Then the transposition $((x)g\ y)$ swapping $(x)g$ and y is in $\operatorname{Aut}(nK_\omega)$ and so

$$g' = ((x)g\ y)g((x)g\ y).$$

It follows from $(x)g \neq x$, $(x)g \neq y$, and $(x)g, y \notin \operatorname{dom}(q) \cup \operatorname{ran}(q)$ that g' is an extension of h. Therefore, $h \in \mathcal{A}_{f,\Sigma}^{<\omega}$. □

Lemma 6.11. *Let $q \in \mathcal{A}_{f,\Sigma}^{<\omega}$ be such that $\Sigma \subseteq \operatorname{dom}(q)$ and let A, B be distinct incomplete components of q such that at most one of A and B intersects Σ non-trivially. Suppose that*

$$\overline{q|_{\operatorname{dom}(q)\setminus A}} = \overline{q|_{\operatorname{dom}(q)\setminus B}} \in S_n$$

and let $h = q \cup \{(x, y)\} \in \operatorname{Aut}(nK_\omega)^{<\omega}$ for some $x \in A$ and $y \in B$. Then $h \in \mathcal{A}_{f,\Sigma}^{<\omega}$.

Proof. Since $h \in \operatorname{Aut}(nK_\omega)^{<\omega}$, it follows that $x \notin \operatorname{dom}(q)$ and $y \notin \operatorname{ran}(q)$.

Assume without loss of generality that $B \cap \Sigma = \varnothing$ and $B = \{y_1, \dots, y_m\}$ for some $m \in \mathbb{N}$ such that $y_1 = y$ and $(y_i)q = y_{i+1}$ for all $i \in \{1, \dots, m-1\}$. The proof of the case when $B \cap \Sigma \neq \varnothing$ can be obtained by applying the argument below to q^{-1} and h^{-1}.

For $k \in \{1, \dots, m\}$ we will show that there is $h_k \in \mathcal{A}_{f,\Sigma}^{<\omega}$ extending h_{k-1} such that $\Sigma \subseteq \operatorname{dom}(h_k)$, $\overline{h_k} \in S_n$, and

$$(x)h_k^i = y_i \text{ for } 1 \leq i \leq k, \quad y_k \notin \operatorname{dom}(h_k), \quad \text{and} \quad y_i \notin \operatorname{dom}(h_k) \cup \operatorname{ran}(h_k) \text{ for } k < i.$$

If $k = 1$, then we define $h_1 = h|_{\operatorname{dom}(h)\setminus B}$. By Lemma 6.10, it follows that

$$h_1 = q|_{\operatorname{dom}(q)\setminus B} \cup \{(x, y)\} \in \mathcal{A}_{f,\Sigma}^{<\omega},$$

and so h_1 satisfies the required conditions.

Suppose $k > 1$. Then by Lemma 6.10 we have that $h_{k+1} = h_k \cup \{(y_k, y_{k+1})\} \in \mathcal{A}^{<\omega}_{f,\Sigma}$. Since

$$\operatorname{dom}(h_{k+1}) = \operatorname{dom}(h_k) \cup \{y_k\} \quad \text{and} \quad \operatorname{ran}(h_{k+1}) = \operatorname{ran}(h_k) \cup \{y_{k+1}\},$$

it follows that h_{k+1} satisfies the required conditions.

Therefore after repeating this process m times, we obtain $h_m \in \mathcal{A}^{<\omega}_{f,\Sigma}$ which extends h_1. It follows from the definition of h_m that $h_m = h$. □

Now we can characterize when the set $\mathcal{A}_{f,\Sigma}$ is non-empty.

Lemma 6.12. *Let $f \in \operatorname{Aut}(nK_\omega)$ and let Σ be a finite subset of nK_ω. Then $\mathcal{A}_{f,\Sigma}$ is non-empty if and only if there exists $\sigma \in S_n$ such that $\langle \overline{f}, \sigma\rangle = S_n$ and for all $i \in \{1, \dots, n\}$, we have*

$$\Big(\bigcup_{j\in\mathbb{Z}} L_{(i)\sigma^j}\Big) \cap \Sigma \neq \varnothing.$$

Proof. (⇒) Suppose that $g \in \mathcal{A}_{f,\Sigma}$. Since $g \in \mathcal{A}_{f,\Sigma} \subseteq \mathcal{A}_f$, it follows from the definition of $\mathcal{A}_f$ that $\langle \overline{f}, \overline{g}\rangle = S_n$. Let $i \in \{1, \dots, n\}$. Then there are $x \in \Sigma$ and $m \in \mathbb{N}$ such that $(x)g^m \in L_i$, since Σ is a set of orbit representatives. Hence,

$$x \in L_{(i)\overline{g}^{-m}} \subseteq \bigcup_{j\in\mathbb{Z}} L_{(i)\overline{g}^j}.$$

(⇐) It is routine to show that there is $q \in \operatorname{Aut}(nK_\omega)^{<\omega}$ such that $\Sigma \subseteq \operatorname{dom}(q)$, $\overline{q} = \sigma$ and q has precisely $|\Sigma|$ many components, all of which are incomplete and Σ intersects them in precisely one point. Since all components of q are incomplete, this satisfies the hypothesis of Lemma 6.7 and hence there is an extension $g \in \mathcal{A}_{f,\Sigma}$ of q. □

By Proposition 6.1, in the case that $n \geq 3$, there exists $\sigma \in S_n$ such that $\langle \overline{f}, \sigma\rangle = S_n$ if and only if $n \neq 4$ and $\overline{f} \neq \operatorname{id}$, or $n = 4$ and $\overline{f} \notin \{\operatorname{id}, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$.

In the next lemma, we give a decomposition of $D_f \cap \mathcal{A}_{f,\Sigma}$ as an intersection of sets that we will later prove to be open and dense under the hypothesis of Theorem 6.5.

Lemma 6.13. *Let $\mathcal{P} \subseteq \operatorname{Aut}(nK_\omega)^{<\omega}$ be such that $p \in \mathcal{P}$ if and only if $\operatorname{dom}(p)$ and $\operatorname{ran}(p)$ are disjoint and $\overline{p} = \operatorname{id}$. Then*

$$D_f \cap \mathcal{A}_{f,\Sigma} = \bigcap_{p\in\mathcal{P}} \{g \in \mathcal{A}_{f,\Sigma} : \langle f, g\rangle \cap [p] \neq \varnothing\}.$$

Proof. Recall that

$$D_f \cap \mathcal{A}_{f,\Sigma} = \{g \in \mathcal{A}_{f,\Sigma} : \langle f, g\rangle \text{ is dense in } \operatorname{Aut}(nK_\omega)\} = \bigcap_{q\in\operatorname{Aut}(nK_\omega)^{<\omega}} \{g \in \mathcal{A}_{f,\Sigma} : \langle f, g\rangle \cap [q] \neq \varnothing\}.$$

(⊆) This follows immediately since $\mathcal{P} \subseteq \operatorname{Aut}(nK_\omega)^{<\omega}$.

(⊇) Let

$$g \in \bigcap_{p\in\mathcal{P}} \{g \in \mathcal{A}_{f,\Sigma} : \langle f, g\rangle \cap [p] \neq \varnothing\},$$

and let $q \in \operatorname{Aut}(nK_\omega)^{<\omega}$ be arbitrary. Since $g \in \mathcal{A}_{f,\Sigma}$, there is $h \in \langle f, g\rangle$ such that $\overline{h} = \overline{q}^{-1}$.

Let $p \in \operatorname{Aut}(nK_\omega)^{<\omega}$ be such that $\overline{p} = \operatorname{id}$, $\operatorname{dom}(p) = \operatorname{dom}(hq)$ and $\operatorname{ran}(p) \cap (\operatorname{dom}(hq) \cup \operatorname{ran}(hq)) = \varnothing$. Then $\operatorname{dom}(p^{-1}hq) = \operatorname{ran}(p)$ and $\operatorname{ran}(p^{-1}hq) = \operatorname{ran}(hq)$, so $p, p^{-1}hq \in \mathcal{P}$. Hence there are $h_1, h_2 \in \langle f, g\rangle$ such that $h_1 \in [p]$ and $h_2 \in [p^{-1}hq]$. Therefore $h^{-1}h_1h_2 \in [q]$, so

$$g \in \bigcap_{q\in\operatorname{Aut}(nK_\omega)^{<\omega}} \{g \in \mathcal{A}_{f,\Sigma} : \langle f, g\rangle \cap [q] \neq \varnothing\},$$

as required. □

Let w be a freely reduced word over the alphabet $\{\alpha, \beta\}$, this means $w = \alpha^{n_1}\beta^{n_2}\cdots\beta^{n_{2N}}$ for some $N \in \mathbb{N}$ and $n_1, \dots, n_{2N} \in \mathbb{Z}$ with $n_i \neq 0$ for all $i \in \{2, \dots, 2N-1\}$. Also let $f \in \operatorname{Aut}(nK_\omega)$ be fixed and suppose that $p \in \operatorname{Aut}(nK_\omega)^{<\omega}$. Then define

$$w(p) = p^{n_1}f^{n_2}p^{n_3}\cdots p^{n_{2N-1}}f^{n_{2N}},$$

where the product on the right-hand side is the usual product of partial permutations. Now notice that $\mathrm{Aut}(nK_\omega) \cup \mathrm{Aut}(nK_\omega)^{<\omega}$ forms a subsemigroup of the semigroup of all isomorphisms between finite induced subgraphs of nK_ω. Hence, if we denote by $F_{\alpha,\beta}$ the free group on the alphabet $\{\alpha, \beta\}$, then $w(p)$ is simply the image of w under the semigroup homomorphism

$$\phi : F_{\alpha,\beta} \to \mathrm{Aut}(nK_\omega) \cup \mathrm{Aut}(nK_\omega)^{<\omega}$$

such that $(\alpha)\phi = p$ and $(\beta)\phi = f$.

Lemma 6.14. *Let $n \in \mathbb{N}$ be such that $n > 1$ and let $f \in \mathrm{Aut}(nK_\omega)$ be non-stabilizing. If $n = 2$ and $\overline{f} = \mathrm{id}$, then further suppose that $\mathrm{fix}(f)$ is finite. Let $\Gamma, \Delta \subseteq nK_\omega$ be finite and disjoint and let $q \in \mathcal{A}^{<\omega}_{f,\Sigma}$ be such that $\overline{q} \in S_n$ and $\mathrm{ran}(q) \cap \Delta = \varnothing$. Then there are an extension $h \in \mathcal{A}^{<\omega}_{f,\Sigma}$ of q and $w \in F_{\alpha,\beta}$ such that*

$$\overline{w(h)} = \mathrm{id}, \quad \mathrm{ran}(h) \cap \Delta = \varnothing, \quad \Gamma \subseteq \mathrm{dom}(w(h)), \quad \textit{and} \quad (\Gamma)w(h) \cap \mathrm{dom}(h) = \varnothing.$$

Moreover, $(\Gamma)w(h)h^m \cap \mathrm{dom}(q) = \varnothing$ for all $m \in \mathbb{Z}$, i.e. no vertex in $(\Gamma)w(h)$ is on an incomplete component of h, which extends an incomplete component of q.

The proof of Lemma 6.14 is rather involved, so before giving its proof we will demonstrate how the lemma can be used to prove Theorem 6.5.

First we prove an easy special case of Theorem 6.5.

Lemma 6.15. *Let $f \in \mathrm{Aut}(2K_\omega)$ be non-stabilising such that $\overline{f} = \mathrm{id}$ and $\mathrm{fix}(f)$ is infinite, and let $\Sigma \subseteq 2K_\omega$ be finite. Then $D_f \cap \mathcal{A}_{f,\Sigma}$ is comeagre in $\mathcal{A}_{f,\Sigma}$.*

Proof. By Lemmas 3.3 and 6.13 we only need to show that $\{g \in \mathcal{A}_{f,\Sigma} : \langle f, g\rangle \cap [p] \neq \varnothing\}$ is dense in $\mathcal{A}_{f,\Sigma}$ for all $p \in \mathcal{P}$. Let $q \in \mathcal{A}^{<\omega}_{f,\Sigma}$ and suppose, without loss of generality, that $\mathrm{dom}(p) \cup \mathrm{ran}(p) \cup \Sigma \subseteq \mathrm{dom}(q)$ and $\overline{q} \in S_2$. Since $\overline{f} = \mathrm{id}$, it follows that $\overline{q} = (1\ 2)$.

Let L_1 and L_2 be the connected components of $2K_\omega$. If necessary by relabelling the connected components, we may assume that $L_2 \cap \mathrm{fix}(f)$ is infinite. It follows from Proposition 6.4 that if f has a finite cycle contained in L_1, then f is stabilising. Hence all of the cycles of f contained in L_1 are infinite.

Let $m_1 \in \mathbb{Z}$ be such that $(L_1 \cap \mathrm{dom}(p))f^{m_1}$ is disjoint from $\mathrm{dom}(q) \cup \mathrm{ran}(q)$. By Lemmas 6.6 and 6.10 there is an extension $q_1 \in \mathcal{A}^{<\omega}_{f,\Sigma}$ of q such that

$$(\mathrm{dom}(p))f^{m_1} \subseteq \mathrm{dom}(q_1) \quad \text{and} \quad (L_1 \cap \mathrm{dom}(p))f^{m_1}q_1 \subseteq \mathrm{fix}(f) \setminus \mathrm{dom}(q_1),$$

which is possible since $L_2 \cap \mathrm{fix}(f)$ is infinite and $(L_1)f^{m_1}q_1 \subseteq L_2$. The extension q_1 can be chosen so that components of q_1 containing any vertices from $(L_1 \cap \mathrm{dom}(p))f^{m_1}$ do not extend any of the components of q. Since $(L_2 \cap \mathrm{dom}(p))f^{m_1}q_1 \subseteq L_1$, there is $m_2 \in \mathbb{Z}$ such that $(L_2 \cap \mathrm{dom}(p))f^{m_1}q_1f^{m_2}$ is disjoint from $\mathrm{dom}(q_1) \cup \mathrm{ran}(q_1)$. Hence, $(\mathrm{dom}(p))f^{m_1}q_1f^{m_2} \cap \mathrm{dom}(q_1) = \varnothing$.

Let $m_3 \in \mathbb{Z}$ be such that $(L_1 \cap \mathrm{ran}(p))f^{m_3}$ is disjoint from $\mathrm{dom}(q_1) \cup \mathrm{ran}(q_1)$. By Lemmas 6.6 and 6.10 there is an extension $q_2 \in \mathcal{A}^{<\omega}_{f,\Sigma}$ of q_1 such that

$$(\mathrm{ran}(p))f^{m_3} \subseteq \mathrm{ran}(q_2), \quad (L_1 \cap \mathrm{ran}(p))f^{m_3}q_2^{-1} \subseteq \mathrm{fix}(f) \setminus \mathrm{ran}(q_2),$$

and $(\mathrm{dom}(p))f^{m_1}q_2f^{m_2}$ is disjoint from $\mathrm{dom}(q_2)$. The extension q_2 can be chosen so that components of q_2 containing any vertices from $(L_1 \cap \mathrm{ran}(p))f^{m_3}$ do not extend any of the components of q_1, and also that every vertex of $(L_1 \cap \mathrm{ran}(p))f^{m_3}$ is on a different incomplete component of q_2. Then there is $m_4 \in \mathbb{Z}$ such that $(L_2 \cap \mathrm{ran}(p))f^{m_3}q_2^{-1}f^{m_4}$ is disjoint from $\mathrm{dom}(q_2) \cup \mathrm{ran}(q_2) \cup (\mathrm{dom}(p))f^{m_1}q_2f^{m_2}$. Hence,

$$(\mathrm{dom}(p))f^{m_1}q_2f^{m_2} \cap \mathrm{dom}(q_2) = \varnothing \quad \text{and} \quad (\mathrm{ran}(p))f^{m_3}q_2^{-1}f^{m_4} \cap \mathrm{ran}(q_2) = \varnothing.$$

Let $\mathrm{dom}(p) = \{x_1, \dots, x_k\}$. Then for all $i \in \{1, \dots, k\}$ there are

$$y_i \in 2K_\omega \setminus \big(\mathrm{dom}(q_2) \cup \mathrm{ran}(q_2) \cup (\mathrm{dom}(p))f^{m_1}q_2f^{m_2} \cup (\mathrm{ran}(p))f^{m_3}q_2^{-1}f^{m_4}\big)$$

such that

$$h' = q_2 \cup \{((x_i)f^{m_1}q_2f^{m_2}, y_i) : 1 \le i \le k\} \in \mathcal{A}^{<\omega}_{f,\Sigma}$$

by Lemmas 6.6 and 6.10. Let A be the incomplete component of h' containing $(x_1)f^{m_1}q_2f_{m_2}$ and let B be the incomplete component of h' containing $(x_1)pf^{m_3}q_2^{-1}f^{m_4}$. Then $y_1 \in A$, and so $|A| \geq 2$. If $|B| = 1$, then

$$h' \cup \{(y_1, (x_1)pf^{m_3}q_2^{-1}f^{m_4})\} \in \mathcal{A}_{f,\Sigma}^{<\omega}$$

by Lemmas 6.6 and 6.10, as $(x_1)f^{m_1}q_2f^{m_2}$ and $(x_1)pf^{m_3}q_2^{-1}f^{m_4}$ are in the same connected component of $2K_\omega$. If $(x_1)p \in L_2$, then by the choice of m_4, we have

$$(x_1)pf^{m_3}q_2^{-1}f^{m_4} \notin \operatorname{dom}(h') \cup \operatorname{ran}(h'),$$

and so $|B| = 1$; we have already considered this case. Suppose that $|B| \geq 2$. Then $(x_1)p \in L_1$ and by the choice of q_2, the incomplete component of h' containing $(x_1)pf^{m_3}q_2^{-1}f^{m_4}$, in other words B, does not extend an incomplete component of q_1. Since A is an incomplete component of q_1 with y_1 adjoined, it follows that B intersects Σ trivially, and A and B are distinct. Hence,

$$\overline{h'|_{\operatorname{dom}(h')\setminus A}} = \overline{h'|_{\operatorname{dom}(h')\setminus B}} = (1\ 2),$$

and thus $h' \cup \{(y_1, (x_1)pf^{m_3}q_2^{-1}f^{m_4})\} \in \mathcal{A}_{f,\Sigma}^{<\omega}$ by Lemma 6.11. Repeating this argument for $i \in \{2, \dots, k\}$, it can be shown that

$$h = q_2 \cup \{((x_i)f^{m_1}q_2f^{m_2}, y_i), (y_i, (x_i)pf^{m_3}q_2^{-1}f^{m_4}) : 1 \leq i \leq k\} \in \mathcal{A}_{f,\Sigma}^{<\omega}.$$

Hence $f^{m_1}gf^{m_2}g^2f^{-m_4}gf^{-m_3} \in [p]$ for every $g \in [h] \cap \mathcal{A}_{f,\Sigma}$. Therefore, $\{g \in \mathcal{A}_{f,\Sigma} : \langle f, g\rangle \cap [p] \neq \varnothing\}$ intersects $[q]$ non-trivially, and since q was arbitrary, is dense in $\mathcal{A}_{f,\Sigma}$. □

Next, we give the proof of Theorem 6.5 modulo the proof of Lemma 6.14, which is given in the next section.

Proof of Theorem 6.5. If $\mathcal{A}_f = \varnothing$, then f is non-stabilizing and D_f is comeager in $\mathcal{A}_f$. Hence we may assume that $\mathcal{A}_f \neq \varnothing$.

Suppose that f is stabilizing. By the definition, there are $\Gamma \subsetneq nK_\omega$, $x \in \Gamma$ and $q \in \mathcal{A}_f^{<\omega}$ such that for all $g \in [q] \cap \mathcal{A}_f$ and all $h \in \langle f, g\rangle$ we have that $(x)h \in \Gamma$. Let $y \notin \Gamma$. Then $p = \{(x, y)\} \in \operatorname{Aut}(nK_\omega)^{<\omega}$. Then $\langle f, g\rangle \cap [p] = \varnothing$ and thus $g \notin D_f$, implying that D_f is not dense in $\mathcal{A}_f$. Hence $D_f \cap \mathcal{A}_f$ is not comeagre in $\mathcal{A}_f$.

If f is non-stabilising and Σ is a finite subset of nK_ω, then it suffices, by Lemma 6.9, to show that $D_f \cap \mathcal{A}_{f,\Sigma}$ is comeagre in $\mathcal{A}_{f,\Sigma}$. If $\mathcal{A}_{f,\Sigma} = \varnothing$, the result is trivial. Hence we may assume that $\mathcal{A}_{f,\Sigma} \neq \varnothing$. If $n = 2$, $\overline{f} = \operatorname{id}$, and $\operatorname{fix}(f)$ is infinite we are done by Lemma 6.15. Hence we may additionally assume that $n \geq 2$, and that if $n = 2$ and $\overline{f} = \operatorname{id}$, then $\operatorname{fix}(f)$ is finite.

By Lemmas 3.3 and 6.13 we only need to show that $\{g \in \mathcal{A}_{f,\Sigma} : \langle f, g\rangle \cap [p] \neq \varnothing\}$ is dense in $\mathcal{A}_{f,\Sigma}$ for all $p \in \mathcal{P}$. Let $q \in \mathcal{A}_{f,\Sigma}^{<\omega}$ and suppose, without loss of generality, that $\operatorname{dom}(p) \cup \operatorname{ran}(p) \cup \Sigma \subseteq \operatorname{dom}(q)$ and $\overline{q} \in S_n$.

Apply Lemma 6.14 with $\Delta = \varnothing$ and $\Gamma = \operatorname{dom}(p)$. Then there is an extension $q_1' \in \mathcal{A}_{f,\Sigma}^{<\omega}$ of q and $\omega_1 \in F_{\alpha,\beta}$ such that

$$\overline{\omega_1(q_1')} = \operatorname{id}, \quad \operatorname{dom}(p) \subseteq \operatorname{dom}(\omega_1(q_1')), \quad \text{and} \quad (\operatorname{dom}(p))\omega_1(q_1') \cap \operatorname{dom}(q_1') = \varnothing.$$

Suppose $(\operatorname{dom}(p))\omega_1(q_1') \setminus \operatorname{ran}(q_1')$ is non-empty. Let $y \in (\operatorname{dom}(p))\omega_1(q_1') \setminus \operatorname{ran}(q_1')$ and let $a \in \{1, \dots, n\}$ be such that $y \in L_a$. Then there is

$$x \in L_{(a)\overline{q_1'}^{-1}} \setminus (\operatorname{dom}(q_1') \cup \operatorname{ran}(q_1') \cup (\operatorname{dom}(p))\omega_1(q_1')).$$

It follows from Lemma 6.6 that $q_1'' = q_1' \cup \{(x, y)\} \in \operatorname{Aut}(nK_\omega)^{<\omega}$ and thus in $\mathcal{A}_{f,\Sigma}^{<\omega}$ by Lemma 6.10. Then

$$\overline{\omega_1(q_1'')} = \operatorname{id}, \quad \operatorname{dom}(p) \subseteq \operatorname{dom}(\omega_1(q_1'')), \quad \text{and} \quad (\operatorname{dom}(p))\omega_1(q_1'') \cap \operatorname{dom}(q_1'') = \varnothing.$$

Moreover,

$$\big|(\operatorname{dom}(p))\omega_1(q_1') \setminus \operatorname{ran}(q_1')\big| > \big|(\operatorname{dom}(p))\omega_1(q_1'') \setminus \operatorname{ran}(q_1'')\big|,$$

and if we do this extension for every vertex in $(\operatorname{dom}(p))\omega_1(q_1') \setminus \operatorname{ran}(q_1')$, we can define an extension $q_1 \in \mathcal{A}_{f,\Sigma}^{<\omega}$ of q_1' such that

$$\overline{\omega_1(q_1)} = \operatorname{id}, \quad \operatorname{dom}(p) \subseteq \operatorname{dom}(\omega_1(q_1)), \quad \text{and} \quad (\operatorname{dom}(p))\omega_1(q_1) \subseteq \operatorname{ran}(q_1) \setminus \operatorname{dom}(q_1). \tag{6.1}$$

Hence every vertex in $(\operatorname{dom}(p))\omega_1(q_1)$ is on an incomplete component of q_1.

If $\Delta = (\mathrm{dom}(p))\omega_1(q_1)$ and $\Gamma = \mathrm{ran}(p)$, then $\mathrm{ran}(q_1^{-1}) = \mathrm{dom}(q_1)$ and Δ are disjoint. Hence by Lemma 6.14, there is an extension $q_2^{-1} \in \mathcal{A}_{f,\Sigma}^{<\omega}$ of q_1^{-1} and $\omega_2' \in F_{\alpha,\beta}$ such that

$$\overline{\omega_2'(q_2^{-1})} = \mathrm{id}, \quad \mathrm{ran}(q_2^{-1}) \cap (\mathrm{dom}(p))\omega_1(q_1) = \varnothing,$$
$$\mathrm{ran}(p) \subseteq \mathrm{dom}(\omega_2'(q_2^{-1})), \quad (\mathrm{ran}(p))\omega_2'(q_2^{-1}) \cap \mathrm{dom}(q_2^{-1}) = \varnothing,$$

and no vertex in $(\mathrm{ran}(p))\omega_2'(q_2^{-1})$ is on an incomplete component of q_2^{-1} extending an incomplete component of q_1^{-1}.

Since $\mathrm{dom}(p) \subseteq \mathrm{dom}(\omega_1(q_1))$ by (6.1) and q_2 is an extension of q_1, it follows that

$$(\mathrm{dom}(p))\omega_1(q_1) = (\mathrm{dom}(p))\omega_1(q_2).$$

Let $\omega_2 \in F_{\alpha,\beta}$ be such that $\omega_2(q_2) = \omega_2'(q_2^{-1})$, i.e. replace every occurrence of α in ω_2' by α^{-1} and vice versa. Then

$$\overline{\omega_2(q_2)} = \mathrm{id}, \quad \mathrm{dom}(q_2) \cap (\mathrm{dom}(p))\omega_1(q_2) = \varnothing,$$
$$\mathrm{ran}(p) \subseteq \mathrm{dom}(\omega_2(q_2)), \quad (\mathrm{ran}(p))\omega_2(q_2) \cap \mathrm{ran}(q_2) = \varnothing,$$

and no vertex in $(\mathrm{ran}(p))\omega_2(q_2)$ is on an incomplete components of q_2 extending an incomplete component of q_1.

For $j \in \{1, \dots, l\}$ let $\{i(j,k) : k \in \{1, \dots, m_j\}\}$ be the orbits of $\overline{q_2}$ and suppose that $(i(j,k))\overline{q_2} = i(j, k+1)$ for all $j \in \{1, \dots, l\}$ and all $k \in \{1, \dots, m_j - 1\}$. For each $i \in \{1, \dots, n\}$ choose

$$x_i \in L_i \setminus (\mathrm{dom}(q_2) \cap \mathrm{ran}(q_2) \cup (\mathrm{dom}(p))\omega_1(q_2) \cup (\mathrm{ran}(p))\omega_2(q_2)),$$

and also for all $j \in \{1, \dots, l\}$ choose

$$x_{i(j,m_j+1)} \in L_{i(j,1)} \setminus (\{x_{i(j,1)}\} \cup \mathrm{dom}(q_2) \cap \mathrm{ran}(q_2) \cup (\mathrm{dom}(p))\omega_1(q_2) \cup (\mathrm{ran}(p))\omega_2(q_2)).$$

Then

$$h_0 = q_2 \cup \{(x_{i(j,k)}, x_{i(j,k+1)}) : j \in \{1, \dots, l\} \text{ and } k \in \{1, \dots, m_j\}\} \in \mathrm{Aut}(nK_\omega)^{<\omega}$$

by Lemma 6.6 and also $h_0 \in \mathcal{A}_{f,\Sigma}^{<\omega}$ by Lemma 6.10. Let P be an arbitrary incomplete component of h_0. Since $x_{i(j,k)} \notin \mathrm{dom}(q_2) \cup \mathrm{ran}(q_2)$ for all j and all k, it follows that P is either a subset of

$$K = \{x_{i(j,k)} : j \in \{1, \dots, l\} \text{ and } k \in \{1, \dots, m_j + 1\}\}$$

or disjoint from K. If $P \subseteq K$, then $q_2 \subseteq h_0|_{\mathrm{dom}(h_0)\setminus P}$, and so $\overline{h_0|_{\mathrm{dom}(h_0)\setminus P}} = \overline{q_2} \in S_n$. Otherwise $P \cap K = \varnothing$, and so $\{x_i : i \in \{1, \dots, n\}\} \subseteq \mathrm{dom}(h_0) \setminus P$. Hence,

$$\{(x_{i(j,k)}, x_{i(j,k+1)}) : j \in \{1, \dots, l\} \text{ and } k \in \{1, \dots, m_j\}\} \subseteq h_0|_{\mathrm{dom}(h_0)\setminus P},$$

which implies that $\overline{h_0|_{\mathrm{dom}(h_0)\setminus P}} = \overline{q_2} \in S_n$. It follows from the choice of vertices x_i and $x_{i(j,m_j+1)}$, that

$$\overline{\omega_2(h_0)} = \mathrm{id}, \quad \mathrm{dom}(h_0) \cap (\mathrm{dom}(p))\omega_1(h_0) = \varnothing,$$
$$\mathrm{ran}(p) \subseteq \mathrm{dom}(\omega_2(h_0)), \quad (\mathrm{ran}(p))\omega_2(h_0) \cap \mathrm{ran}(h_0) = \varnothing.$$

Let k be the order of $\overline{q} \in S_n$. We will now inductively construct an extension $h \in \mathcal{A}_{f,\Sigma}^{<\omega}$ of h_0 (and hence of q) such that $(x)\omega_1(h)h^k\omega_2(h)^{-1} = (x)p$ for all $x \in \mathrm{dom}(p)$. Let $\mathrm{dom}(p) = \{x_1, \dots, x_d\}$, and suppose that for $j \in \{0, \dots, k-2\}$ we have an extension $h_j \in \mathcal{A}_{f,\Sigma}^{<\omega}$ of h_0 such that

$$\mathrm{dom}(h_j) \cap (\mathrm{dom}(p))\omega_1(h_j)h_j^j = \varnothing, \quad (\mathrm{ran}(p))\omega_2(h_j) \cap \mathrm{ran}(h_j) = \varnothing,$$

and $\mathrm{dom}(p)$ and $\mathrm{ran}(p)$ are contained in $\mathrm{dom}(\omega_1(h_j)h_j^j)$ and $\mathrm{dom}(\omega_2(h_j))$ respectively.

Note that if $j = 0$, the inductive hypothesis is satisfied since h_0^0 is an identity on $\mathrm{dom}(h_0)$, the $\mathrm{dom}(h_0)$ is disjoint from $(\mathrm{dom}(p))\omega_1(h_0)$, the set $\mathrm{ran}(h_0)$ is disjoint from $(\mathrm{ran}(p))\omega_2(h_j) \cap \mathrm{ran}(h_j)$, and $\mathrm{dom}(p)$ and $\mathrm{ran}(p)$ are contained in $\mathrm{dom}(\omega_1(h_0))$ and $\mathrm{dom}(\omega_2(h_0))$ respectively.

Suppose $j > 0$. For all $i \in \{1, \dots, d\}$ let $y_i = (x_i)\omega_1(h_j)h_j^j$ and suppose that $a_i \in \{1, \dots, n\}$ such that $y_i \in L_{a_i}$. Then for each successive $i \in \{1, \dots, d\}$ choose

$$z_i \in L_{(a_i)\overline{h_j}} \setminus \big(\mathrm{dom}(h_j) \cup \mathrm{ran}(h_j) \cup \{y_1, \dots, y_d\} \cup \{z_1, \dots, z_{i-1}\} \cup (\mathrm{ran}(p))\omega_2(h_j)\big).$$

We define $h_{j+1} = h_j \cup \{(y_i, z_i) : 1 \le i \le d\}$. Since $z_i \in L_{(a_i)\overline{h_j}}$, by Lemma 6.6 we have $h_{j+1} \in \mathrm{Aut}(nK_\omega)^{<\omega}$ and hence $h_{j+1} \in \mathcal{A}^{<\omega}_{f,\Sigma}$ by Lemma 6.10. Note that the choice of z_i implies that none of the incomplete components of h_j are amalgamated in h_{j+1}.

It is easy to see that $\mathrm{dom}(h_{j+1}) = \mathrm{dom}(h_j) \cup \{y_1, \dots, y_d\}$ and $\mathrm{ran}(h_{j+1}) = \mathrm{ran}(h_j) \cup \{z_1, \dots, z_d\}$. Since $(x_i)\omega_1(h_{j+1})h_{j+1}^j = (x_i)\omega_1(h_j)h_j^j$ for all $i \in \{1, \dots, d\}$, we have

$$(x_i)\omega_1(h_{j+1})h_{j+1}^{j+1} = (x_i)\omega_1(h_{j+1})h_{j+1}^{j}h_{j+1} = (x_i)\omega_1(h_j)h_j^j h_{j+1} = (y_i)h_{j+1} = z_i \notin \mathrm{dom}(h_{j+1}).$$

Hence,

$$\mathrm{dom}(h_{j+1}) \cap (\mathrm{dom}(p))\omega_1(h_{j+1})h_{j+1}^{j+1} = \varnothing \quad \text{and} \quad \mathrm{dom}(p) \subseteq \omega_1(h_{j+1})h_{j+1}^{j+1}.$$

It follows from $\mathrm{ran}(p) \subseteq \omega_2(h_0)$, that $(\mathrm{ran}(p))\omega_2(h_{j+1}) = (\mathrm{ran}(p))\omega_2(h_j)$, and so

$$(\mathrm{ran}(p))\omega_2(h_{j+1}) \cap \mathrm{ran}(h_j) = \varnothing.$$

Since $z_i \notin (\mathrm{ran}(p))\omega_2(h_j)$ for all $i \in \{1, \dots, d\}$, it also follows that $(\mathrm{ran}(p))\omega_2(h_{j+1}) \cap \mathrm{ran}(h_{j+1}) = \varnothing$. Finally, $\mathrm{dom}(p)$ and $\mathrm{ran}(p)$ are contained in $\mathrm{dom}(\omega_1(h_{j+1})h_{j+1}^{j+1})$ and $\mathrm{dom}(\omega_2(h_{j+1}))$ respectively, and so h_{j+1} satisfies the inductive hypothesis.

By induction on j, we obtain an extension $h_{k-1} \in \mathcal{A}^{<\omega}_{f,\Sigma}$ of h_0 (and thus q) such that

$$\mathrm{dom}(h_{k-1}) \cap (\mathrm{dom}(p))\omega_1(h_{k-1})h_{k-1}^{k-1} = \varnothing, \quad (\mathrm{ran}(p))\omega_2(h_{k-1}) \cap \mathrm{ran}(h_{k-1}) = \varnothing, \tag{6.2}$$

and $\mathrm{dom}(p)$ and $\mathrm{ran}(p)$ are contained in $\mathrm{dom}(\omega_1(h_{k-1})h_{k-1}^{k-1})$ and $\mathrm{dom}(\omega_2(h_{k-1}))$ respectively.

Define h to be

$$h_{k-1} \cup \big\{\big((x_i)\omega_1(h_{k-1})h_{k-1}^{k-1}, ((x_i)p)\omega_2(h_{k-1})\big) : 1 \le i \le d\big\}.$$

Recall that k is the order of $\overline{q}$. Since h_{k-1} is an extension of q and $\overline{q} \in S_n$, it follows that $\overline{h_{k-1}} = \overline{q}$, thus $\overline{h_{k-1}^k} = \mathrm{id}$. Also $\omega_1(h_{k-1})$ and $\omega_2(h_{k-1})$ are extensions of $\omega_1(q_1)$ and $\omega_2(q_2)$ respectively, hence

$$\overline{\omega_1(h_{k-1})} = \overline{\omega_1(q_1)} = \mathrm{id} = \overline{\omega_2(q_2)} = \overline{\omega_2(h_{k-1})}.$$

Then x_i, $(x_i)\omega_1(h_{k-1})h_{k-1}^k$, and $((x_i)p)\omega_2(h_{k-1})$ are in the same connected component of nK_ω for all i. Thus it follows from Lemma 6.6 and (6.2), that $h \in \mathrm{Aut}(nK_\omega)^{<\omega}$.

We will now show that h can be obtained from h_{k-1} by repeated applications of Lemma 6.11, and so $h \in \mathcal{A}^{<\omega}_{f,\Sigma}$. First of all, note that $\Sigma \subseteq \mathrm{dom}(q) \subseteq \mathrm{dom}(h_{k-1})$ and that no incomplete components of h_0, and thus of q_2, were amalgamated in h_{k-1}. According to Lemma 6.14, q_2 was chosen so that $((x_i)p)\omega_2(q_2)$ is not on an incomplete component of q_2 extending an incomplete component of q_1 for all $i \in \{1, \dots, d\}$. Hence the vertex $((x_i)p)\omega_2(h_{k-1})$ is not on an incomplete component of h_{k-1} extending an incomplete component of q_1 for all $i \in \{1, \dots, d\}$. Also since $\Sigma \subseteq \mathrm{dom}(q) \subseteq \mathrm{dom}(q_1)$, it follows that the intersection of any incomplete component of h_{k-1} containing a vertex in $(\mathrm{ran}(p))\omega_2(h_{k-1})$ and Σ is empty.

By (6.1) every vertex in $(\mathrm{dom}(p))\omega_1(q_1)$ is on an incomplete component of q_1 and since $\omega_1(h_{k-1})h_{k-1}^{k-1}$ is defined on $\mathrm{dom}(p)$ it follows that every vertex in

$$(\mathrm{dom}(p))\omega_1(h_{k-1})h_{k-1}^{k-1}$$

is on an incomplete component of h_{k-1} extending an incomplete component of q_1. Hence incomplete components of h_{k-1} containing vertices $(\mathrm{ran}(p))\omega_2(h_{k-1})$ are distinct from the incomplete components of h_{k-1} containing the vertices $(\mathrm{dom}(p))\omega_1(h_{k-1})h_{k-1}^{k-1}$. Also recall that for every incomplete component P of h_0 we have that

$$\overline{h_0|_{\mathrm{dom}(h_0)\setminus P}} \in S_n.$$

Since h_{k-1} is an extension of h_0 and no incomplete components of h_0 were amalgamated, for any incomplete component Q of h_{k-1}, we have

$$\overline{h_{k-1}|_{\mathrm{dom}(h_{k-1})\setminus Q}} \in S_n.$$

Thus we can apply Lemma 6.11 to show that $h \in \mathcal{A}_{f,\Sigma}^{<\omega}$.

Finally, h was defined so that

$$\omega_1(h)h^k\omega_2(h)^{-1} \in [p]$$

and thus any extension $g \in [h] \cap \mathcal{A}_{f,\Sigma}$ also satisfies $g \in \{r \in \mathcal{A}_{f,\Sigma} : \langle f, r\rangle \cap [p] \neq \varnothing\}$. Therefore,

$$\{g \in \mathcal{A}_{f,\Sigma} : \langle f, g\rangle \cap [p] \neq \varnothing\}$$

is dense in $\mathcal{A}_{f,\Sigma}$ as required. □

Proof of Lemma 6.14

The purpose of this section is to prove Lemma 6.14. First we prove a technical result relating to the behaviour of a non-stabilizing isomorphism f of nK_ω. Recall that $f \in \mathrm{Aut}(nK_\omega)$ is called *non-stabilizing* if for all $\Gamma \subsetneq nK_\omega$, all $x \in \Gamma$ and all $q \in \mathcal{A}_f^{<\omega}$ there is $g \in [q] \cap \mathcal{A}_f$ such that $(x)h \notin \Gamma$ for some $h \in \langle f, g\rangle$.

Let $f \in \mathrm{Aut}(nK_\omega)$ be non-stabilizing and let $x \in nK_\omega$. Then for every $q \in \mathrm{Aut}(nK_\omega)^{<\omega}$ there is $g \in [q] \cap \mathcal{A}_f$ such that $(x)h \notin \mathrm{dom}(q)$ for some $h \in \langle f, g\rangle$. It follows that there are $N \in \mathbb{N}$ and $m_1, m_2, \dots, m_{2N} \in \mathbb{Z}$ such that $(x)\prod_{i=1}^{N} g^{m_{2i-1}}f^{m_{2i}} \notin \mathrm{dom}(q)$. If we assume that the length of the product $\sum_{i=1}^{2N}|m_i|$ is minimal, then the image of x under any proper prefix of the product $\prod_{i=1}^{N} g^{m_{2i-1}}f^{m_{2i}}$ belongs to $\mathrm{dom}(q)$. Therefore,

$$(x)\prod_{i=1}^{N} q^{m_{2i-1}}f^{m_{2i}} = (x)\prod_{i=1}^{N} g^{m_{2i-1}}f^{m_{2i}} \in nK_\omega \setminus \mathrm{dom}(q).$$

In the next lemma we show that the powers m_{2i-1} of q in the above equation can be chosen to be positive.

Lemma 6.16. *Let f be non-stabilizing and let $x \in nK_\omega$. Then for every $q \in \mathrm{Aut}(nK_\omega)^{<\omega}$ there are $N \in \mathbb{N}$ and $m_1, m_2, \dots, m_{2N} \in \mathbb{Z}$ such that $m_1, m_3, \dots, m_{2N-1} > 0$ and*

$$(x)\prod_{i=1}^{N} q^{m_{2i-1}}f^{m_{2i}} \in nK_\omega \setminus \mathrm{dom}(q).$$

Proof. By the discussion above there are $K \in \mathbb{N}$ and $k_1, k_2, \dots, k_{2K} \in \mathbb{Z}$ such that

$$(x)\prod_{i=1}^{K} q^{k_{2i-1}}f^{k_{2i}} \in nK_\omega \setminus \mathrm{dom}(q).$$

Suppose that $M \in \{0, 1, \dots, K\}$ is the least value such that $(x)\prod_{i=1}^{M} q^{k_{2i-1}}f^{k_{2i}}$ is on an incomplete component of q, where $M = 0$ in the case that x is on an incomplete component. Then $y_t = (x)\prod_{i=1}^{t} q^{k_{2i-1}}f^{k_{2i}}$ is on a complete component of q for all $t \in \{0, \dots, M-1\}$. It follows that there exist $m_{2t+1} > 0$ such that $(y_t)q^{m_{2t+1}} = (y_t)q^{k_{2t+1}}$ for all $t \in \{0, \dots, M-1\}$. Additionally, define $m_{2i} = k_{2i}$ for all $i \in \{1, \dots, M\}$.

By the choice of M, we know that

$$y = (x)\prod_{i=1}^{M} q^{m_{2i-1}}f^{m_{2i}} = (x)\prod_{i=1}^{M} q^{k_{2i-1}}f^{k_{2i}}$$

is in an incomplete component of q. Hence there is z in the incomplete component of y under q such that $z \in \mathrm{ran}(q) \setminus \mathrm{dom}(q)$ and there is $m_{2M+1} \geq 0$ such that $(y)q^{m_{2M+1}} = z \notin \mathrm{dom}(q)$. Therefore,

$$(x)\Big(\prod_{i=1}^{M} q^{m_{2i-1}}f^{m_{2i}}\Big)q^{m_{2M+1}} \in nK_\omega \setminus \mathrm{dom}(q),$$

as required. □

For the proofs of the next three lemmas we require the following notation: First of all, recall that for a fixed $f \in \operatorname{Aut}(nK_\omega)$, if $p \in \operatorname{Aut}(nK_\omega)^{<\omega}$ and $w = \alpha^{n_1}\beta^{n_2}\cdots\beta^{n_{2N}} \in F_{\alpha,\beta}$ for some $N \in \mathbb{N}$ and $n_1, \dots, n_{2N} \in \mathbb{Z}$, then

$$w(p) = p^{n_1}f^{n_2}p^{n_3}\cdots p^{n_{2N-1}}f^{n_{2N}},$$

where the product on the right-hand side is the usual product of partial permutations. Let $\Gamma, \Theta, \Phi, \Delta \subseteq nK_\omega$ be finite subsets, let $p \in \mathcal{A}_{f,\Sigma}^{<\omega}$ and let $w \in F_{\alpha,\beta}$. Suppose $x \in \Gamma$ and define $w_{p,x}$ to be the largest prefix of w such that $x \in \operatorname{dom}(w_{p,x}(p))$ and let $w_{p,x}$ be the empty word if there are no such prefix. To make the notation less cluttered, whenever possible, we will identify the word $w_{p,x}$ with its realisation in $\operatorname{Aut}(nK_\omega)^{<\omega}$, in other words with the partial isomorphism $w_{p,x}(p)$. To avoid confusion, if $w, w' \in F_{\alpha,\beta}$, we denote that w and w' are equal by $w \equiv w'$. Note that if $w_{p,x}$ is a proper prefix of w (i.e. $|w_{p,x}| < |w|$), since f is an isomorphism, we have that $(x)w_{p,x} \notin \operatorname{dom}(p)$ and $w_{p,x}\alpha$ is a prefix of w.

Suppose that $\Theta \subseteq \Gamma$. Then we say that p satisfies $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$ if the following conditions are satisfied:
(1) $\overline{w(p)} = \mathrm{id}$,
(2) $\operatorname{ran}(p) \cap \Delta = \varnothing$,
(3) $\operatorname{dom}(w(p)) \cap \Gamma = \Theta$,
(4) the image of Θ under $w(p)$ is disjoint from $\operatorname{dom}(p)$,
(5) $(x)w_{p,x} \neq (y)w_{p,y}$ for all $x, y \in \Gamma$ such that $x \neq y$,
(6) $(x)w_{p,x}p^m \in nK_\omega \setminus \Phi$ for all $x \in \Gamma$ and $m \in \mathbb{Z}$ such that $x \in \operatorname{dom}(w_{p,x}p^m)$.
Finally, define $\mathbf{b}(w)$ to be the total number of occurrences of β and β^{-1} in the freely reduced word w.

Using the definition of $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$ we can now restate Lemma 6.14. In the case that $\Gamma = \Theta$, it follows that $w_{p,x} = w$ for all $x \in \Gamma$. Hence in this case, (5) is a consequence of $w(p)$ being a finite isomorphism.

Lemma 6.17. *Let $n \in \mathbb{N}$ be such that $n > 1$ and let $f \in \operatorname{Aut}(nK_\omega)$ be non-stabilising. If $n = 2$ and $\overline{f} = \mathrm{id}$, then further suppose that $\operatorname{fix}(f)$ is finite. Let $\Gamma, \Delta \subseteq nK_\omega$ be finite and disjoint, and let $q \in \mathcal{A}_{f,\Sigma}^{<\omega}$ be such that $\overline{q} \in S_n$ and $\operatorname{ran}(q) \cap \Delta = \varnothing$. Then there is an extension $h \in \mathcal{A}_{f,\Sigma}^{<\omega}$ of q and $w \in F_{\alpha,\beta}$ satisfying $\mathcal{S}(\Gamma, \Gamma, \operatorname{dom}(q), \Delta, w)$.*

The proof of Lemma 6.17 will be split into three parts. We say that a word $w \in F_{\alpha,\beta}$ *starts with* a letter $\gamma \in \{\alpha, \beta\}$ if there is $w' \in F_{\alpha,\beta}$ such that $w = \gamma w'$.

Lemma 6.18. *Let $n \in \mathbb{N}$ be such that $n > 1$ and let $f \in \operatorname{Aut}(nK_\omega)$ be non-stabilising. If $n = 2$ and $\overline{f} = \mathrm{id}$, then further suppose that $\operatorname{fix}(f)$ is finite. Let $\Gamma, \Delta \subseteq nK_\omega$ be finite, and let $q \in \mathcal{A}_{f,\Sigma}^{<\omega}$ be such that $\operatorname{ran}(q) \cap \Delta = \varnothing$. Then there is an extension $h \in \mathcal{A}_{f,\Sigma}^{<\omega}$ of q and $w \in F_{\alpha,\beta}$ not containing α^{-1} and starting with α such that h satisfies $\mathcal{S}(\Gamma, \varnothing, \operatorname{dom}(q), \Delta, w)$.*

Proof. If necessary by extending q, using Lemma 6.6 and Lemma 6.10, we may assume that $\overline{q} \in S_n$ and $\Sigma, \Gamma \subseteq \operatorname{dom}(q)$. In the case that $n = 2$ and $\overline{f} = \mathrm{id}$, we also assume that $\operatorname{fix}(f) \subseteq \operatorname{dom}(q)$.

Let $d = |\Gamma|$. We will now inductively define a sequence $q_0, \dots, q_d \in \mathcal{A}_{f,\Sigma}^{<\omega}$ of extensions of q, and a sequence $\lambda^{(0)}, \dots, \lambda^{(d)}$ of words in $F_{\alpha,\beta}$ so that $h = q_d$ and $w = \lambda^{(d)}$ are as required. Let $q_0 = q$, let $\Gamma_0 = \varnothing$, and let $\lambda^{(0)} = \alpha$. Suppose that for some $j \in \{0, \dots, d-1\}$ we have $\Gamma_j \subseteq \Gamma$, a word $\lambda^{(j)}$ in $F_{\alpha,\beta}$ starting with α and not containing α^{-1}, and $q_j \in \mathcal{A}_{f,\Sigma}^{<\omega}$ such that $|\Gamma_j| = j$ and the following hold:
(I) $\operatorname{ran}(q_j) \cap \Delta = \varnothing$,
(II) $(u){\lambda^{(j)}}_{q_j,u} \neq (v){\lambda^{(j)}}_{q_j,v}$ for all $u, v \in \Gamma_j$ with $u \neq v$,
(III) $(u){\lambda^{(j)}}_{q_j,u}q_j^m \notin \operatorname{dom}(q)$ for all $m \in \mathbb{Z}$ such that $u \in \operatorname{dom}({\lambda^{(j)}}_{q_j,u}q_j^m)$ and all $u \in \Gamma_j$,
(IV) ${\lambda^{(j)}}_{q_j,u} \not\equiv \lambda^{(j)}$ for all $u \in \Gamma_j$.

Let $x \in \Gamma \setminus \Gamma_j$ be arbitrary and let $\Gamma_{j+1} = \Gamma_j \cup \{x\}$. The first step in the proof is to find $v \in F_{\alpha,\beta}$ such that $x \notin \operatorname{dom}(\lambda^{(j)}v\alpha(q_j))$ and to find $m \in \mathbb{N}$ such that $m > |\lambda^{(j)}v|$ such that we can define

$$\lambda^{(j+1)} \equiv \lambda^{(j)}v\alpha^m\beta\alpha. \tag{6.3}$$

In order to define v consider two cases. If $x \in \operatorname{dom}(\lambda^{(j)}(q_j))$, then by Lemma 6.16 there is $v \in F_{\alpha,\beta}$ such that α^{-1} is not contained in v and the image of x under $\lambda^{(j)}v(q_j)$ is in $nK_\omega \setminus \operatorname{dom}(q_j)$. Otherwise, $x \notin \operatorname{dom}(\lambda^{(j)}(q_j))$, in which case let v be the empty word. Hence in both cases,

$$x \notin \operatorname{dom}(\lambda^{(j)}v\alpha(q_j)). \tag{6.4}$$

To define m we will again consider two separate cases. If $n = 2$ and $\overline{f} = \mathrm{id}$, let $m > |\lambda^{(j)}\nu|$ be arbitrary. Otherwise, either $n = 2$ and $\overline{f} = (1\ 2)$ or $n \geq 3$. Let $L_1, \dots, L_n$ be the connected components of nK_ω and let $a \in \{1, \dots, n\}$ so that $x \in L_a$. Consider any extension $g \in \mathrm{Aut}(nK_\omega)$ of q_j, and let b be the image of a under the permutation $\overline{(\lambda^{(j)}\nu)(g)}$. Since $\overline{q_j} \in S_n$, it follows that b is independent of the extension g. We will show that in this case we can choose $m > |\lambda^{(j)}\nu|$ to be such that

$$(b)\overline{q_j}^m \in \mathrm{supp}(\overline{f}). \tag{6.5}$$

If $n = 2$ and $\overline{f} = (1\ 2)$, then any $m > |\lambda^{(j)}\nu|$ satisfies (6.5). Let $n \geq 3$ be arbitrary, and let O be the orbit of $\overline{q_j}$ containing b. Suppose that $\overline{f}$ fixes O pointwise. If $|O| \leq 2$, then since $n \geq 3$, there is $c \in \{1, \dots, n\} \setminus O$, and so $(b\ c) \notin \langle \overline{f}, \overline{q_j} \rangle$. If $|O| \geq 3$, then the symmetric group on $|O|$ is not cyclic, and so there is a $\sigma \in S_n$ such that $\mathrm{supp}(\sigma) \subseteq O$ and $\sigma \notin \langle \overline{q_j}|_O \rangle$. Then $\sigma \notin \langle \overline{f}, \overline{q_j} \rangle$. However, both cases are impossible since $\langle \overline{f}, \overline{q_j} \rangle = S_n$. Hence $\overline{f}$ does not fix O pointwise. Hence we may choose $m > |\lambda^{(j)}\nu|$ to satisfy (6.5). Let $\lambda^{(j+1)}$ be as in (6.3). For brevity, denote the prefix $\lambda^{(j)}\nu\alpha^m\beta$ of $\lambda^{(j+1)}$ by ρ.

Next we show how to construct $q_{j+1} \in \mathcal{A}^{<\omega}_{f,\Sigma}$ from q_j. In order to do so, we need to consider a possible complication, namely the existence of $y \in \Gamma_j$ such that $(y)\lambda^{(j+1)}{}_{q_j,y} = (x)\lambda^{(j+1)}{}_{q_j,x}$. The case where such y does not exist is slightly easier and can be proved in a very similar fashion, simply ignoring any mention of y in the following argument (to be more precise (i), (ii), (iv), and (v) are exactly the same, (vii) and (viii) are unnecessary, and in (iii) and (vi) the vertex u can be any vertex in the set Γ_j). Hence we will omit this case. Suppose there is $y \in \Gamma_j$ such that $(y)\lambda^{(j+1)}{}_{q_j,y} = (x)\lambda^{(j+1)}{}_{q_j,x}$. It follows from (II) that such y is unique. Since $\lambda^{(j+1)}{}_{q_j,x}$ is a partial isomorphism and $x \neq y$, it follows that $\lambda^{(j+1)}{}_{q_j,x} \neq \lambda^{(j+1)}{}_{q_j,y}$ and so $\lambda^{(j+1)}{}_{q_j,x} \not\equiv \lambda^{(j+1)}{}_{q_j,y}$. Condition (IV) implies that $\lambda^{(j)}{}_{q_j,y}$ is a proper prefix of $\lambda^{(j)}$ and so $y \notin \mathrm{dom}(\rho(q_j))$. Also from (6.4), we have that

$$|\lambda^{(j+1)}{}_{q_j,x}| \leq |\lambda^{(j)}\nu| < |\rho|. \tag{6.6}$$

Hence $|\lambda^{(j+1)}{}_{q_j,x}|, |\lambda^{(j+1)}{}_{q_j,y}| < |\rho|$. There are two cases to consider: either

$$|\lambda^{(j+1)}{}_{q_j,x}| > |\lambda^{(j+1)}{}_{q_j,y}| \quad \text{or} \quad |\lambda^{(j+1)}{}_{q_j,x}| < |\lambda^{(j+1)}{}_{q_j,y}|.$$

Consider $|\lambda^{(j+1)}{}_{q_j,x}| > |\lambda^{(j+1)}{}_{q_j,y}|$. We proceed by inductively constructing a sequence $r_0, \dots, r_{|\rho|}$ of extensions of q_j so that $r_0 = q_j$ and $r_{|\rho|}$ is the required q_{j+1}. Let $r_0 = q_j$. For $k \in \{0, \dots, |\rho|\}$ let the inductive hypothesis be as follows: there is an extension $r_k \in \mathcal{A}^{<\omega}_{f,\Sigma}$ of r_{k-1} (or q_j if $k = 0$) such that the following hold:

(i) $k \leq |\lambda^{(j+1)}{}_{r_k,x}| \leq |\rho|$.

(ii) $\mathrm{ran}(r_k) \cap \Delta = \varnothing$.

(iii) $\lambda^{(j+1)}{}_{r_k,u} \equiv \lambda^{(j)}{}_{q_j,u}$ for $u \in \Gamma_j \setminus \{y\}$.

(iv) $(u)\lambda^{(j+1)}{}_{r_k,u} \neq (v)\lambda^{(j+1)}{}_{r_k,v}$ for $u, v \in \Gamma_j$ with $u \neq v$.

(v) $(u)\lambda^{(j+1)}{}_{r_k,u} r_k^m \in nK_\omega \setminus \mathrm{dom}(q)$ for all $m \in \mathbb{Z}$ such that $u \in \mathrm{dom}(\lambda^{(j+1)}{}_{r_k,u} r_k^m)$ and all $u \in \Gamma_j$.

(vi) $(x)\lambda^{(j+1)}{}_{r_k,x} \notin \mathrm{dom}(r_k) \cup \{(u)\lambda^{(j)}{}_{q_j,u} : u \in \Gamma_j \setminus \{y\}\}$. Moreover, if $k > 0$ and we can write $\lambda^{(j+1)}{}_{r_k,x} \equiv \tau\beta^i$ for some $i \in \mathbb{Z} \setminus \{0\}$, and $\tau \in F_{\alpha,\beta}$ such that τ ends with a letter α and the image of x under $\tau(r_k)$ is in $\mathrm{supp}(f^i)$, then $(x)\lambda^{(j+1)}{}_{r_k,x} \notin \mathrm{dom}(r_k) \cup \mathrm{ran}(r_k)$.

(vii) If $k > 0$ and $(x)\lambda^{(j+1)}{}_{r_{k-1},x} \neq (y)\lambda^{(j+1)}{}_{r_{k-1},y}$ then $(x)\lambda^{(j+1)}{}_{r_k,x} \neq (y)\lambda^{(j+1)}{}_{r_k,y}$.

(viii) $|\lambda^{(j+1)}{}_{r_k,x}| > |\lambda^{(j+1)}{}_{r_k,y}|$. Moreover, if $(x)\lambda^{(j+1)}{}_{r_k,x} = (y)\lambda^{(j+1)}{}_{r_k,y}$, then $|\lambda^{(j+1)}{}_{r_k,y}| \geq |\lambda^{(j+1)}{}_{q_j,y}| + k$.

First we demonstrate that the base case $k = 0$ holds. Condition (i) is satisfied by r_0 by (6.6) and condition (ii) is satisfied because $r_0 = q_j$ satisfies (I). Since q_j satisfies (IV), we have that $\lambda^{(j)}{}_{q_j,u} \not\equiv \lambda^{(j)}$ and thus $u \notin \mathrm{dom}(\lambda^{(j)}(q_j))$ which then implies that $\lambda^{(j+1)}{}_{q_j,u} \equiv \lambda^{(j)}{}_{q_j,u}$ for all $u \in \Gamma_j$. Hence (iii) is satisfied by r_0. Since $\lambda^{(j+1)}{}_{q_j,u} \equiv \lambda^{(j)}{}_{q_j,u}$ for all $u \in \Gamma_j$, conditions (iv) and (v) are the same as conditions (II) and (III) respectively. Recall that $(x)\lambda^{(j+1)}{}_{q_j,x} \in nK_\omega \setminus \mathrm{dom}(q_j)$ by the definition of $\lambda^{(j+1)}{}_{q_j,x}$ and that if

$$(x)\lambda^{(j+1)}{}_{q_j,x} = (u)\lambda^{(j+1)}{}_{q_j,u},$$

where $u \in \Gamma_j$, then $u = y$ by (II). Hence r_0 satisfies the first part of (vi), while r_0 satisfies the second part of (vi), the whole of (vii), and the second part of (viii) trivially, since $k = 0$. Finally, the first part of (viii) is just the assumption of this case. Therefore, r_0 satisfies the inductive hypothesis.

Next we show how to obtain r_{k+1} from r_k. Suppose that for some $k \in \{0, \dots, |\rho| - 1\}$ we have $r_k \in \mathcal{A}^{<\omega}_{f,\Sigma}$ which satisfies (i)–(viii). We consider the case $\lambda^{(j+1)}{}_{r_k,x} \equiv \rho$ and $\lambda^{(j+1)}{}_{r_k,x}$ being a proper prefix of ρ separately.

Case 1. We begin by considering the case where ${\lambda^{(j+1)}}_{r_k,x}$ is a proper prefix of ρ. Let $z = (x){\lambda^{(j+1)}}_{r_k,x}$. Since ${\lambda^{(j+1)}}_{r_k,x}$ is a proper prefix of $\lambda^{(j+1)}$, it follows that $z \notin \operatorname{dom}(r_k)$ and ${\lambda^{(j+1)}}_{r_k,x}\alpha$ is a prefix of $\lambda^{(j+1)}$. Recall that $\mathbf{b}(\lambda^{(j+1)})$ is the total number of occurrences of letters β and β^{-1} in the word $\lambda^{(j+1)} \in F_{\alpha,\beta}$. Let $c \in \{1, \dots, n\}$ be so that $z \in L_c$, and choose

$$z' \in L_{(c)\overline{r_k}} \setminus \bigcup_{i=-\mathbf{b}(\lambda^{(j+1)})}^{\mathbf{b}(\lambda^{(j+1)})} (\Delta \cup \{(u){\lambda^{(j+1)}}_{r_k,u} : u \in \Gamma_j\} \cup \operatorname{dom}(r_k) \cup \operatorname{ran}(r_k) \cup \{z\})f^{-i}.$$

Since $z \notin \operatorname{dom}(r_k)$ and $z' \notin \operatorname{dom}(r_k) \cup \operatorname{ran}(r_k)$, it follows from Lemmas 6.6 and 6.10 that

$$r_{k+1} = r_k \cup \{(z, z')\} \in \mathcal{A}^{<\omega}_{f,\Sigma}.$$

Then there is some $i \in \mathbb{Z}$ such that

$${\lambda^{(j+1)}}_{r_{k+1},x} \equiv {\lambda^{(j+1)}}_{r_k,x}\alpha\beta^i. \tag{6.7}$$

Hence,

$$|{\lambda^{(j+1)}}_{r_{k+1},x}| > |{\lambda^{(j+1)}}_{r_k,x}| \geq k.$$

We will now show that ${\lambda^{(j+1)}}_{r_{k+1},x}$ is a prefix of ρ. Suppose that ${\lambda^{(j+1)}}_{r_{k+1},x}$ is not a prefix of ρ. Since ${\lambda^{(j+1)}}_{r_{k+1},x}$ is a prefix of $\lambda^{(j+1)}$, it follows that ${\lambda^{(j+1)}}_{r_{k+1},x} = \lambda^{(j+1)}$. Hence the fact that $\lambda^{(j+1)} \equiv \rho\alpha$ and (6.7) imply that

$${\lambda^{(j+1)}}_{r_k,x}\alpha\beta^i \equiv {\lambda^{(j+1)}}_{r_{k+1},x} = \lambda^{(j+1)} = \rho\alpha,$$

thus $i = 0$ and ${\lambda^{(j+1)}}_{r_k,x} = \rho$, which contradicts the assumption of this case. Therefore, ${\lambda^{(j+1)}}_{r_{k+1},x}$ is prefix of ρ and so (i) is satisfied by r_{k+1}.

It follows from the definition of r_{k+1} that

$$\operatorname{dom}(r_{k+1}) = \operatorname{dom}(r_k) \cup \{z\} \quad \text{and} \quad \operatorname{ran}(r_{k+1}) = \operatorname{ran}(r_k) \cup \{z'\}. \tag{6.8}$$

Since the vertex z' was chosen outside Δ, we have that (ii) is satisfied by r_{k+1}.

Let $u \in \Gamma_j \setminus \{y\}$. It follows from (vi) for r_k that

$$z = (x){\lambda^{(j+1)}}_{r_k,x} \neq (u){\lambda^{(j)}}_{q_j,u}$$

and since r_k satisfies (iii), it follows that $z \neq (u){\lambda^{(j+1)}}_{r_k,u}$. Also ${\lambda^{(j+1)}}_{r_k,u} = \lambda^{(j)}_{q_j,u}$ is a proper prefix of $\lambda^{(j)}$, and so a proper prefix of $\lambda^{(j+1)}$ by (iii) and (IV). Then $(u){\lambda^{(j+1)}}_{r_k,u} \notin \operatorname{dom}(r_k)$ and ${\lambda^{(j+1)}}_{r_k,u}\alpha$ is a prefix of $\lambda^{(j+1)}$, and thus $(u){\lambda^{(j+1)}}_{r_k,u} \notin \operatorname{dom}(r_{k+1})$ by (6.8). Hence ${\lambda^{(j+1)}}_{r_{k+1},u} \equiv {\lambda^{(j+1)}}_{r_k,u}$, and since r_k satisfies (iii), we have

$${\lambda^{(j+1)}}_{r_{k+1},u} \equiv {\lambda^{(j+1)}}_{r_k,u} \equiv {\lambda^{(j)}}_{q_j,u}. \tag{6.9}$$

Therefore, r_{k+1} satisfies (iii).

In order to prove that r_{k+1} satisfies (iv), we consider two cases. Suppose that

$$z = (x){\lambda^{(j+1)}}_{r_k,x} \neq (y){\lambda^{(j+1)}}_{r_k,y}.$$

It follows by (i) and (viii) that $|{\lambda^{(j+1)}}_{r_k,y}| < |\rho|$. Hence, ${\lambda^{(j+1)}}_{r_k,y}$ is a proper prefix of $\lambda^{(j+1)}$ and so

$$(y){\lambda^{(j+1)}}_{r_k,y} \notin \operatorname{dom}(r_k)$$

and ${\lambda^{(j+1)}}_{r_k,y}\alpha$ is a prefix of $\lambda^{(j+1)}$, and so $(y){\lambda^{(j+1)}}_{r_k,y} \notin \operatorname{dom}(r_{k+1})$ by (6.8). Hence ${\lambda^{(j+1)}}_{r_{k+1},y} \equiv {\lambda^{(j+1)}}_{r_k,y}$, in other words

$$z \neq (y){\lambda^{(j+1)}}_{r_k,y} \implies {\lambda^{(j+1)}}_{r_{k+1},y} \equiv {\lambda^{(j+1)}}_{r_k,y}. \tag{6.10}$$

Combining this with the previous paragraph, we obtain ${\lambda^{(j+1)}}_{r_{k+1},u} \equiv {\lambda^{(j+1)}}_{r_k,u}$ for all $u \in \Gamma_j$. Therefore, r_{k+1} satisfies (iv), since r_k does.

Otherwise, suppose that

$$z = (x){\lambda^{(j+1)}}_{r_k,x} = (y){\lambda^{(j+1)}}_{r_k,y}.$$

Since we have $(z')f^i \notin \mathrm{dom}(r_{k+1})$ for all $i \in \{-\mathbf{b}(\rho), \dots, \mathbf{b}(\rho)\}$, by the choice of z' and (6.8), there exists $i \in \{-\mathbf{b}(\rho), \dots, \mathbf{b}(\rho)\}$ such that $(y)\lambda^{(j+1)}{}_{r_{k+1},y} = (z')f^i$, and so $\lambda^{(j+1)}{}_{r_{k+1},y} \equiv \lambda^{(j+1)}{}_{r_k,y}\alpha\beta^i$, in other words

$$(x)\lambda^{(j+1)}{}_{r_k,x} = (y)\lambda^{(j+1)}{}_{r_k,y} \implies \lambda^{(j+1)}{}_{r_{k+1},y} \equiv \lambda^{(j+1)}{}_{r_k,y}\alpha\beta^i \text{ for some } i \in \{-\mathbf{b}(\rho), \dots, \mathbf{b}(\rho)\}. \tag{6.11}$$

The vertex z' was chosen so that $(z')f^i \neq (u)\lambda^{(j+1)}{}_{r_k,u}$ for all $u \in \Gamma_j \setminus \{y\}$. Since $\lambda^{(j+1)}{}_{r_{k+1},u} \equiv \lambda^{(j+1)}{}_{r_k,u}$ for all $u \in \Gamma_j \setminus \{y\}$ and r_k satisfies (iv), it then follows that r_{k+1} satisfies (iv).

Let $u \in \Gamma_j \setminus \{y\}$ be arbitrary. Then $(u)\lambda^{(j+1)}{}_{r_{k+1},u} = (u)\lambda^{(j+1)}{}_{r_k,u}$ by (6.9). Since $z' \notin \mathrm{dom}(r_k)$, no two components of r_k become subsets of the same component of r_{k+1}. It follows that, for any $m \in \mathbb{Z}$, the vertex $(u)\lambda^{(j+1)}{}_{r_{k+1},u}r_{k+1}^m$ equals either $(u)\lambda^{(j+1)}{}_{r_k,u}r_k^m$ or z', neither of which belongs to $\mathrm{dom}(q)$. Hence (v) holds for all $u \in \Gamma_j \setminus \{y\}$.

By (6.10), if $z \neq (y)\lambda^{(j+1)}{}_{r_k,y}$ then $\lambda^{(j+1)}{}_{r_{k+1},y} \equiv \lambda^{(j+1)}{}_{r_k,y}$, and so using the argument of the previous paragraph, we obtain that

$$(y)\lambda^{(j+1)}{}_{r_{k+1},y}r_{k+1}^m \notin \mathrm{dom}(q)$$

for all $m \in \mathbb{Z}$. Hence to show that r_{k+1} satisfies (v) it remains to consider the case where

$$z = (x)\lambda^{(j+1)}{}_{r_k,x} = (y)\lambda^{(j+1)}{}_{r_k,y}.$$

It follows from (6.11) that $(y)\lambda^{(j+1)}{}_{r_{k+1},y} = (z')f^i$ for some $i \in \{-\mathbf{b}(\rho), \dots, \mathbf{b}(\rho)\}$. If $(z')f^i \notin \mathrm{dom}(r_{k+1}) \cup \mathrm{ran}(r_{k+1})$, then no component of r_{k+1}, and thus q, contains the vertex $(z')f^i = (y)\lambda^{(j+1)}{}_{r_{k+1},y}$, and so r_{k+1} satisfies (v). Suppose that $(z')f^i \in \mathrm{dom}(r_{k+1}) \cup \mathrm{ran}(r_{k+1})$. But z' was chosen so that

$$(z')f^i \notin \mathrm{dom}(r_k) \cup \mathrm{ran}(r_k) \cup \{z\},$$

which implies $(z')f^i = z'$ and so $(y)\lambda^{(j+1)}{}_{r_{k+1},y} = z'$. From its definition, the component of r_{k+1} containing $z' = (y)\lambda^{(j+1)}{}_{r_{k+1},y}$ is the component of r_k containing $z = (y)\lambda^{(j+1)}{}_{r_k,y}$ together with the vertex z'. In other words,

$$(y)\lambda^{(j+1)}{}_{r_{k+1},y}r_{k+1}^m$$

equals $(y)\lambda^{(j+1)}{}_{r_k,y}r_k^m$ or z', if defined. Since $(y)\lambda^{(j+1)}{}_{r_k,y}r_k^m \in nK_\omega \setminus \mathrm{dom}(q)$ for all $m \in \mathbb{Z}$, it follows that $(y)\lambda^{(j+1)}{}_{r_{k+1},y}r_{k+1}^m \notin \mathrm{dom}(q)$ for all $m \in \mathbb{Z}$. Thus r_{k+1} satisfies condition (v).

By (6.7), we have

$$\lambda^{(j+1)}{}_{r_{k+1},x} \equiv \lambda^{(j+1)}{}_{r_k,x}\alpha\beta^i$$

for some $i \in \{-\mathbf{b}(\rho), \dots, \mathbf{b}(\rho)\}$. Hence,

$$(x)\lambda^{(j+1)}{}_{r_{k+1},x} = (z')f^i \notin \mathrm{dom}(r_k) \cup \{z\} \cup \{(u)\lambda^{(j+1)}{}_{r_k,u} : u \in \Gamma_j\}$$

by the choice of z'. By (iii), we have $(u)\lambda^{(j+1)}{}_{r_k,u} = (u)\lambda^{(j+1)}{}_{q_j,u}$ for all $u \in \Gamma_j \setminus \{y\}$ and $\mathrm{dom}(r_{k+1}) = \mathrm{dom}(r_k) \cup \{z\}$, and so the first part of (vi) is satisfied by r_{k+1}. To check the second part of (vi), suppose that $\lambda^{(j+1)}{}_{r_{k+1},x} \equiv \tau\beta^i$ for some $i \in \mathbb{Z} \setminus \{0\}$ and $\tau \in F_{\alpha,\beta}$ such that τ ends with a letter α and the image of x under $\tau(r_{k+1})$ is in $\mathrm{supp}(f^i)$. Then, by (6.7), we have $\tau = \lambda^{(j+1)}{}_{r_k,x}\alpha$ and the last part of the assumption from the previous sentence becomes $z' = (x)\lambda^{(j+1)}{}_{r_k,x}r_{k+1} \in \mathrm{supp}(f^i)$. Then $(x)\lambda^{(j+1)}{}_{r_{k+1},x} = (z')f^i \neq z'$. Since $(z')f^i \notin \mathrm{dom}(r_k) \cup \mathrm{ran}(r_k) \cup \{z\}$ by the choice of z', it follows from (6.8) that

$$(x)\lambda^{(j+1)}{}_{r_{k+1},x} \notin \mathrm{dom}(r_{k+1}) \cup \mathrm{ran}(r_{k+1}).$$

Therefore, r_{k+1} satisfies (vi).

By (6.10) if $z = (x)\lambda^{(j+1)}{}_{r_k,x} \neq (y)\lambda^{(j+1)}{}_{r_k,y}$, then $\lambda^{(j+1)}{}_{r_{k+1},y} \equiv \lambda^{(j+1)}{}_{r_k,y}$, so $(y)\lambda^{(j+1)}{}_{r_k,y} = (y)\lambda^{(j+1)}{}_{r_{k+1},y}$. It follows from (6.7) that there is $i \in \{-\mathbf{b}(\rho), \dots, \mathbf{b}(\rho)\}$ so that $(x)\lambda^{(j+1)}{}_{r_{k+1},x} = (z')f^i$. Hence by the choice of z' we have

$$(x)\lambda^{(j+1)}{}_{r_{k+1},x} = (z')f^i \neq (y)\lambda^{(j+1)}{}_{r_k,y} = (y)\lambda^{(j+1)}{}_{r_{k+1},y},$$

and so (vii) holds for r_{k+1}.

Finally, we will show that r_{k+1} satisfies (viii). Suppose that

$$(y)\lambda^{(j+1)}{}_{r_k,y} \neq (x)\lambda^{(j+1)}{}_{r_k,x}.$$

Then $\lambda^{(j+1)}{}_{r_{k+1},y} \equiv \lambda^{(j+1)}{}_{r_k,y}$ by (6.10). Since $|\lambda^{(j+1)}{}_{r_k,x}| < |\lambda^{(j+1)}{}_{r_{k+1},x}|$ and r_k satisfies (viii), it follows that r_{k+1} satisfies (viii) as well. The other case is when $(y)\lambda^{(j+1)}{}_{r_k,y} = (x)\lambda^{(j+1)}{}_{r_k,x}$. Then

$$\lambda^{(j+1)}{}_{r_{k+1},y} \equiv \lambda^{(j+1)}{}_{r_k,y}\alpha\beta^i$$

for some $i \in \{-\mathbf{b}(\rho), \ldots, \mathbf{b}(\rho)\}$ by (6.11). Since $\lambda^{(j+1)}{}_{r_k,y}$ is a proper prefix of $\lambda^{(j+1)}{}_{r_k,x}$ by (viii) applied to r_k, it follows that $\lambda^{(j+1)}{}_{r_k,y}\alpha$ is a prefix of $\lambda^{(j+1)}{}_{r_k,x}$, and so $\lambda^{(j+1)}{}_{r_k,x} = \lambda^{(j+1)}{}_{r_k,y}\alpha\beta^{i'}$ for some $i' \in \mathbb{Z}$. Suppose that $\lambda^{(j+1)}{}_{r_{k+1},y}$ is not a prefix of $\lambda^{(j+1)}{}_{r_k,x}$, in other words either $i > 0$ and $i' \in \{0, \ldots, i-1\}$; or $i < 0$ and $i' \in \{i+1, \ldots, 0\}$. Then either $\lambda^{(j+1)}{}_{r_k,x}\beta$ or $\lambda^{(j+1)}{}_{r_k,x}\beta^{-1}$ must be a prefix of $\lambda^{(j+1)}$, which contradicts (6.7). Hence $\lambda^{(j+1)}{}_{r_{k+1},y}$ is a prefix of $\lambda^{(j+1)}{}_{r_k,x}$, and thus

$$|\lambda^{(j+1)}{}_{r_{k+1},y}| \leq |\lambda^{(j+1)}{}_{r_k,x}| < |\lambda^{(j+1)}{}_{r_{k+1},x}|.$$

Therefore, r_{k+1} satisfies the first part of (viii).

In order to show the second part of (viii), suppose that $(x)\lambda^{(j+1)}{}_{r_{k+1},x} = (y)\lambda^{(j+1)}{}_{r_{k+1},y}$. Since (vii) holds for r_{k+1} we have that $(x)\lambda^{(j+1)}{}_{r_k,x} = (y)\lambda^{(j+1)}{}_{r_k,y}$ and thus

$$|\lambda^{(j+1)}{}_{r_k,y}| \geq |\lambda^{(j+1)}{}_{q_j,y}| + k$$

by (viii) for r_k. Also (6.11) implies that $|\lambda^{(j+1)}{}_{r_k,y}| < |\lambda^{(j+1)}{}_{r_{k+1},y}|$. Therefore,

$$|\lambda^{(j+1)}{}_{r_{k+1},y}| \geq |\lambda^{(j+1)}{}_{q_j,y}| + k + 1$$

and thus r_{k+1} satisfies (viii) and hence this case is complete.

Case 2. Suppose $\lambda^{(j+1)}{}_{r_k,x} \equiv \rho$. It follows from (6.4) that $|\lambda^{(j+1)}{}_{r_0,x}| < |\rho|$, and so $k > 0$. Let $r_{k+1} = r_k$. Then r_{k+1} trivially satisfies conditions (i)–(vii) and the first part of condition (viii). To show the second part of (viii) we will consider two cases. Suppose that $n = 2$ and $\overline{f} = \mathrm{id}$. Since $\lambda^{(j+1)} \equiv \rho\alpha$ and $\lambda^{(j+1)}{}_{r_k,x} \equiv \rho$ it follows that the image of x under $\rho(r_k)$ is $(x)\lambda^{(j+1)}{}_{r_k,x} \in nK_\omega \setminus \mathrm{dom}(r_k)$. Let $t \in nK_\omega$ be the image of x under $\lambda^{(j)}\nu\alpha^m(r_k)$. Then $\rho \equiv \lambda^{(j)}\nu\alpha^m\beta$ implies that $(t)f$ is the image of x under ρ, and so if $t \in \mathrm{fix}(f)$, we have

$$t = (t)f = (x)w_{r_k,x} \in nK_\omega \setminus \mathrm{dom}(r_k)$$

by the assumption that $w_{r_k,x} = \rho$. However, we have assumed at the beginning of the proof that $\mathrm{fix}(f) \subseteq \mathrm{dom}(q)$, which is a contradiction since $\mathrm{dom}(q) \subseteq \mathrm{dom}(r_k)$. Hence, $t \in \mathrm{supp}(f)$. Otherwise, either $n = 2$ and $\overline{f} = (1\ 2)$, or $n \geq 3$. Recall that $a, b \in \{1, \ldots, n\}$ are such that $x \in L_a$ and b is the image of a under $\overline{\lambda^{(j)}\nu(r_k)}$. Then the image of a under $\overline{\lambda^{(j)}\nu\alpha^m(r_k)}$ is in $\mathrm{supp}(\overline{f})$ by (6.5), and so the image of x under $\lambda^{(j)}\nu\alpha^m(r_k)$ is in $\mathrm{supp}(f)$ in both cases. Hence it follows from the second part of (vi) that

$$(x)\lambda^{(j+1)}{}_{r_k,x} \notin \mathrm{dom}(r_k) \cup \mathrm{ran}(r_k). \tag{6.12}$$

Next, using (6.12), we will show that $(x)\lambda^{(j+1)}{}_{r_k,x} \neq (y)\lambda^{(j+1)}{}_{r_k,y}$, which then implies that r_{k+1} satisfies the second half of (viii), and this case will be complete. Suppose that $(x)\lambda^{(j+1)}{}_{r_k,x} = (y)\lambda^{(j+1)}{}_{r_k,y}$. Since

$$\lambda^{(j+1)}{}_{r_k,x} \equiv \rho \equiv \lambda^{(j)}\nu\alpha^m\beta \quad \text{and} \quad |\lambda^{(j+1)}{}_{r_0,x}| \leq |\lambda^{(j)}\nu|$$

by (6.4), the fact that at any inductive step incomplete components of q_j were extended by at most one point implies that $k \geq m$. Since m was chosen so that $m > |\lambda^{(j)}\nu|$, and r_k satisfies (viii), we have

$$|\rho| = |\lambda^{(j+1)}{}_{r_k,x}| \geq |\lambda^{(j+1)}{}_{r_k,y}| \geq |\lambda^{(j+1)}{}_{q_j,y}| + k > m > |\lambda^{(j)}\nu|.$$

Hence $\lambda^{(j+1)}{}_{r_k,y}$ is a prefix of ρ, and $\lambda^{(j)}\nu$ is a prefix of $\lambda^{(j+1)}{}_{r_k,y}$. The former and the fact that $\lambda^{(j+1)} = \rho\alpha$ imply that $\lambda^{(j+1)}{}_{r_k,y}$ is a proper prefix of $\lambda^{(j+1)}$, and so $\lambda^{(j+1)}{}_{r_k,y}\alpha$ is a prefix of $\lambda^{(j+1)}$ and $y \notin \mathrm{dom}(\lambda^{(j+1)})$. Since

$\lambda^{(j+1)} \equiv \lambda^{(j)}\nu\alpha^m\beta\alpha$, there is $i \in \{1, \dots, m-1\}$ such that ${\lambda^{(j+1)}}_{r_k,y} \equiv \lambda^{(j)}\nu\alpha^i$. Hence, $(y){\lambda^{(j+1)}}_{r_k,y} \in \operatorname{ran}(r_k)$. But this contradicts (6.12) and so we conclude that $(x){\lambda^{(j+1)}}_{r_k,x} \neq (y){\lambda^{(j+1)}}_{r_k,y}$. Therefore, r_{k+1} satisfies the second part of (viii), since $r_{k+1} = r_k$, as required.

Hence by induction there is $q_{j+1} = r_{|\rho|} \in \mathcal{A}^{<\omega}_{f,\Sigma}$ satisfying conditions (i)–(viii). We will now show that q_{j+1} satisfies (I)–(IV).

It follows from (ii) that q_{j+1} satisfies (I). Suppose that $(x){\lambda^{(j+1)}}_{q_{j+1},x} = (y){\lambda^{(j+1)}}_{q_{j+1},y}$. Then by (i) and (viii) we have

$$|\rho| = |{\lambda^{(j+1)}}_{q_{j+1},x}| > |{\lambda^{(j+1)}}_{q_{j+1},y}| \geq |{\lambda^{(j+1)}}_{q_j,y}| + |\rho|,$$

which is a contradiction. Hence it follows from (iii), (iv), and (vi) that q_{j+1} satisfies (II). It follows from (v) that we only need to verify (III) for x. From (i) we have that ${\lambda^{(j+1)}}_{q_{j+1},x} \equiv \rho$, and so

$$(x){\lambda^{(j+1)}}_{q_{j+1},x} \notin \operatorname{dom}(q_{j+1}) \cup \operatorname{ran}(q_{j+1})$$

by (vi) and the choice of ρ, and so (III) holds for q_{j+1}. Finally, condition (IV) follows from (i), (iii), (viii) and the fact that q_j satisfies (IV). Therefore, q_{j+1} satisfies the inductive hypothesis.

Consider the case where $|{\lambda^{(j+1)}}_{q_j,x}| < |{\lambda^{(j+1)}}_{q_j,y}|$. The above argument applies if we switch the roles of x and y, i.e. let $\Gamma_j' = \Gamma_j \cup \{y\} \setminus \{x\}$ and $\lambda^{(j)\prime} \equiv \lambda^{(j+1)}$. Then q_j, $\lambda^{(j)\prime}$ and Γ_j' satisfy conditions (I)–(IV) and we can proceed as before.

Hence by induction there is $h = q_d$ satisfying (I)–(IV). Since $\langle \overline{f}, \overline{h}\rangle = S_n$ there is $w \in F_{\alpha,\beta}$ which does not contain α^{-1}, $\lambda^{(d)}$ is a prefix of w, and $\overline{w(h)} = \mathrm{id}$. Then from (I)–(IV) it follows that conditions (2), (3), (5) and (6) of $\mathcal{S}(\Gamma, \varnothing, \operatorname{dom}(q), \Delta, w)$ are satisfied by h. Since $\Theta = \varnothing$, condition (4) of $\mathcal{S}(\Gamma, \varnothing, \operatorname{dom}(q), \Delta, w)$ follows trivially from (3) of $\mathcal{S}(\Gamma, \varnothing, \operatorname{dom}(q), \Delta, w)$. Hence h satisfies $\mathcal{S}(\Gamma, \varnothing, \operatorname{dom}(q), \Delta, w)$. □

The next lemma is the second step in the proof of Lemma 6.17.

Lemma 6.19. *Let $n \in \mathbb{N}$ be such that $n > 1$, let $f \in \operatorname{Aut}(nK_\omega)$ be non-stabilising, let $q \in \mathcal{A}^{<\omega}_{f,\Sigma}$ be such that $\overline{q} \in S_n$, and let $w \in F_{\alpha,\beta}$ be a word which does not contain α^{-1} and which starts with α. Suppose $\Gamma, \Phi, \subseteq \operatorname{dom}(q)$, $\Theta \subseteq \Gamma$, and $x \in \Gamma \setminus \Theta$. If q satisfies $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$, then there is an extension $h \in \mathcal{A}^{<\omega}_{f,\Sigma}$ of q such that h satisfies $\mathcal{S}(\Gamma, \Theta \cup \{x\}, \Phi, \Delta, w)$.*

Proof. For all $k \in \{0, \dots, |w|\}$, define ρ_k to be a prefix of w of length k. Recall that for all $u \in \Gamma$ we identify the word $w_{q,u}$ with its realisation $w_{q,u}(u)$. In the same way, if q_k is a partial isomorphism, we identify the word ρ_k with the partial isomorphism $\rho_k(q_k)$.

It follows from condition (3) of $\mathcal{S}(\Gamma, \Theta, \Phi, \operatorname{dom}(q), w)$ and the fact that $x \in \Gamma \setminus \Theta$ that $x \notin \operatorname{dom}(w(q))$, and so $w_{q,x}$ is a proper prefix of w. Let M be such that $M - 1 = |w_{q,x}|$, or in other words M is the smallest non-negative integer such that $x \notin \operatorname{dom}(\rho_M(q))$. Then $M \leq |w|$. Since $x \in \Gamma \subseteq \operatorname{dom}(q)$ and w starts with α, it follows that $M > 1$, and so $M \in \{2, \dots, |w|\}$. Since $w_{q,x}$ is a proper prefix of w, it follows that $w_{q,x}\alpha$ is a prefix of w and $(x)w_{q,x} \in nK_\omega \setminus \operatorname{dom}(q)$. Hence $\rho_M = \rho_{M-1}\alpha$ and the image of x under $\rho_{M-1}(q)$ is in $nK_\omega \setminus \operatorname{dom}(q)$.

We will inductively construct a sequence $q_{M-1} = q, q_M, \dots, q_{|w|} \in \mathcal{A}^{<\omega}_{f,\Sigma}$ such that if $j \in \{M, \dots, |w|\}$, then q_j is an extension of q_{j-1} and the following conditions are satisfied:

(i) $\operatorname{ran}(q_j) \cap \Delta = \varnothing$,
(ii) $w_{q_j,u} \equiv w_{q,u}$ and $(u)w_{q_j,u} \in nK_\omega \setminus \operatorname{dom}(q_j)$ for all $u \in \Gamma \setminus \{x\}$,
(iii) $(x)\rho_j f^i \in nK_\omega \setminus \operatorname{dom}(q_j)$ for all $i \in \{-\mathbf{b}(w) + \mathbf{b}(\rho_j), \dots, \mathbf{b}(w) - \mathbf{b}(\rho_j)\}$,
(iv) $(x)w_{q_j,x} \neq (u)w_{q_j,u}$ for all $u \in \Gamma \setminus \{x\}$,
(v) $(u)w_{q_j,u}q_j^m \in nK_\omega \setminus \Phi$ for all $u \in \Gamma$ and for all $m \in \mathbb{Z}$ such that $u \in \operatorname{dom}(w_{q_j,u}q_j^m)$.

Then $h = q_{|w|}$ will be the required extension of q.

Let y be the image of x under $\rho_{M-1} = w_{q,x}$ and suppose $y \in L_a$ for some $a \in \{1, \dots, n\}$. Recall that $\mathbf{b}(w)$ is the number of occurrences of letters β and β^{-1} in the word w. We may choose

$$z \in L_{(a)\overline{q}} \setminus \bigcup_{i=-\mathbf{b}(w)}^{\mathbf{b}(w)} (\operatorname{dom}(q) \cup \operatorname{ran}(q) \cup \{y\} \cup \Delta \cup \{(u)w_{q,u} : u \in \Gamma\})f^{-i}$$

and define $q_M = q \cup \{(y, z)\}$. Then $q_M \in \mathcal{A}^{<\omega}_{f,\Sigma}$ by Lemmas 6.6 and 6.10 since

$$y \notin \operatorname{dom}(q) \quad \text{and} \quad z \notin \operatorname{dom}(q) \cup \operatorname{ran}(q).$$

First, we will show that q_M satisfies conditions (i) to (v). Since $\operatorname{ran}(q_M) = \operatorname{ran}(q) \cup \{z\}$ and z was chosen outside Δ, it follows that q_M satisfies (i). Let $u \in \Gamma \setminus \{x\}$. If $u \notin \Theta$, then, from (3) of $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$, we have $u \notin \operatorname{dom}(w(q))$ and so $w_{q,u}$ is a proper prefix of w. It follows that $(u)w_{q,u} \in nK_\omega \setminus \operatorname{dom}(q)$. On the other hand, if $u \in \Theta$, then $w_{q,u} = w$ and $(u)w_{q,u} \in nK_\omega \setminus \operatorname{dom}(q)$ by (3) and (4) of $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$. Hence in both cases $(u)w_{q,u} \in nK_\omega \setminus \operatorname{dom}(q)$. Since

$$\operatorname{dom}(q_M) \setminus \operatorname{dom}(q) = \{y\} \quad \text{and} \quad (u)w_{q,u} \neq (x)w_{q,x} = y$$

by part (5) of $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$, it follows that $(u)w_{q,u} \in nK_\omega \setminus \operatorname{dom}(q_M)$ and so $w_{q_M,u} \equiv w_{q,u}$, proving (ii). Let $i \in \{-\mathbf{b}(w) + \mathbf{b}(\rho_M), \dots, \mathbf{b}(w) - \mathbf{b}(\rho_M)\}$. Since $\operatorname{dom}(q_M) = \operatorname{dom}(q) \cup \{y\}$, it follows from the choice of z that

$$(x)\rho_M f^i = (y)q_M f^i = (z)f^i \in nK_\omega \setminus \operatorname{dom}(q_M).$$

Hence q_M satisfies condition (iii). Let $u \in \Gamma \setminus \{x\}$. Note that since q_M satisfies (iii) there is $k \in \{-\mathbf{b}(w), \dots, \mathbf{b}(w)\}$ such that $w_{q_M,x} = w_{q,x}\alpha\beta^k$, and so $(x)w_{q_M,x} = (z)f^k$. It follows from the choice of z and the fact that q_M satisfies (ii) that

$$(x)w_{q_M,x} = (z)f^k \neq (u)w_{q,u} = (u)w_{q_M,u}.$$

Hence q_M satisfies (iv).

Finally, to show that q_M satisfies (v) consider two cases, namely $u = x$ and $u \in \Gamma \setminus \{x\}$. Suppose that $u = x$ and $m \in \mathbb{Z}$ are such that $x \in \operatorname{dom}(w_{q_M,x}q_M^m)$. As shown before, we have $(x)w_{q_M,x} = (z)f^k$ for some $k \in \{-\mathbf{b}(w), \dots, \mathbf{b}(w)\}$. From the choice of z it follows that

$$(x)w_{q_M,x} = (z)f^k \notin \operatorname{dom}(q) \cup \operatorname{ran}(q) \cup \{y\}.$$

Suppose $(z)f^k \neq z$. Then $(x)w_{q_M,x} = (z)f^k \notin \operatorname{dom}(q_M) \cup \operatorname{ran}(q_M)$, and so $m = 0$. Since $\Phi \subseteq \operatorname{dom}(q) \subseteq \operatorname{dom}(q_M)$, this implies that

$$(x)w_{q_M,x}q_M^m = (z)f^k \in nK_\omega \setminus \Phi.$$

Suppose that $(z)f^k = z$, in other words $(x)w_{q_M,x} = z$. Since $z \notin \operatorname{dom}(q_M)$, it follows that $x \notin \operatorname{dom}(w_{q_M,x}q_M^m)$ for all $m > 0$. If $m = 0$ then

$$(x)w_{q_M,x}q_M^m = (z)f^k \in nK_\omega \setminus \Phi$$

by the choice of z and since $\Phi \subseteq \operatorname{dom}(q)$. Suppose that $m < 0$. Then $m + 1 \leq 0$ and it follows from the definition of q_M that $\operatorname{dom}(q_M^{m+1})$ is either $\operatorname{dom}(q^{m+1})$ or $\operatorname{dom}(q^{m+1}) \cup \{(z)q_M^{m+1}\}$. Note that $y \in \operatorname{dom}(q_M^{m+1})$ implies $y \in \operatorname{dom}(q^{m+1})$. It follows from (6) of $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$ that

$$(x)w_{q_M,x}q_M^m = (z)q_M^m = (y)q_M^{m+1} = (y)q^{m+1} = (x)w_{q,x}q^{m+1} \in nK_\omega \setminus \Phi.$$

Hence q_M satisfies (v) for $u = x$.

Suppose that $u \in \Gamma \setminus \{x\}$ and $m \in \mathbb{Z}$ is such that $u \in \operatorname{dom}(w_{q_M,u}q_M^m)$. Since q_M satisfies (ii), it follows that $(u)w_{q_M,u} = (u)w_{q,u}$. If $m \leq 0$, or $m > 0$ and there is no $m' \in \{0, \dots, m-1\}$ with $(u)w_{q,u}q^{m'} = y$, then

$$(u)w_{q_M,u}q_M^m = (u)w_{q,u}q^m \in nK_\omega \setminus \Phi$$

by (6) of $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$. Otherwise, $m > 0$ and there is $m' \in \{0, \dots, m-1\}$ such that $(u)w_{q,u}q^{m'} = y$, in which case

$$(u)w_{q_M,u}q_M^{m'+1} = z \notin \operatorname{dom}(q_M).$$

Hence $m = m' + 1$, and since $\Phi \subseteq \operatorname{dom}(q) \subseteq \operatorname{dom}(q_M)$, it follows that $(u)w_{q_M,u}q_M^m \in nK_\omega \setminus \Phi$. Therefore, q_M satisfies (v) and thus the inductive hypothesis holds.

In the case where $M = |w|$, the partial automorphism $q_{|w|}$ already satisfies conditions (i) to (v). Hence suppose that $M < |w|$ and suppose that for some $j \in \{M, \dots, |w| - 1\}$ there is an extension $q_j \in \mathcal{A}_{f,\Sigma}^{<\omega}$ of q_{j-1} satisfying conditions (i) to (v). We have two cases to consider, namely either $\rho_{j+1} = \rho_j\beta^\varepsilon$ or $\rho_{j+1} = \rho_j\alpha^\varepsilon$ for some $\varepsilon \in \{-1, 1\}$.

First consider the case $\rho_{j+1} = \rho_j\beta^{\varepsilon}$, where $\varepsilon \in \{-1, 1\}$. Let $q_{j+1} = q_j$. Then conditions (i), (ii), (iv), and (v) are trivially satisfied by q_{j+1}. In order to show that q_{j+1} satisfies (iii), let $i \in \mathbb{Z}$ be such that

$$i \in \{-\mathbf{b}(w) + \mathbf{b}(\rho_{j+1}), \ldots, \mathbf{b}(w) - \mathbf{b}(\rho_{j+1})\}.$$

Then $|i + \varepsilon| \leq \mathbf{b}(w) - \mathbf{b}(\rho_{j+1}) + 1 = \mathbf{b}(w) - \mathbf{b}(\rho_j)$, and so

$$(x)\rho_{j+1}f^i = (x)\rho_j f^{i+\varepsilon} \in nK_\omega \setminus \operatorname{dom}(q_j) = nK_\omega \setminus \operatorname{dom}(q_{j+1}).$$

Hence q_{j+1} satisfies condition (iii), and so the induction hypothesis.

Otherwise, $\rho_{j+1} = \rho_j\alpha^{\varepsilon}$ for some $\varepsilon \in \{-1, 1\}$, and so $\rho_{j+1} = \rho_j\alpha$ since w does not contain α^{-1}. Let $y = (x)\rho_j$ and let $a \in \{1, \ldots, n\}$ be such that $y \in L_a$. Choose

$$z \in L_{(a)\overline{q_j}} \setminus \bigcup_{i=-\mathbf{b}(w)}^{\mathbf{b}(w)} (\operatorname{dom}(q_j) \cup \operatorname{ran}(q_j) \cup \{y\} \cup \Delta \cup \{(u)w_{q_j,u} : u \in \Gamma\})f^{-i}.$$

Since $y \notin \operatorname{dom}(q_j)$ by (iii) and $z \notin \operatorname{dom}(q_j) \cup \operatorname{ran}(q_j)$, it follows from Lemmas 6.6 and 6.10 that

$$q_{j+1} = q_j \cup \{(y, z)\} \in \mathcal{A}^{<\omega}_{f,\Sigma}.$$

Observe that

$$\operatorname{dom}(q_{j+1}) = \operatorname{dom}(q_j) \cup \{y\} \quad \text{and} \quad \operatorname{ran}(q_{j+1}) = \operatorname{ran}(q_j) \cup \{z\}. \tag{6.13}$$

The vertex z was chosen so that $z \notin \Delta$, and so q_{j+1} satisfies (i).

It follows from (iii) that $x \in \operatorname{dom}(\rho_j)$ and $x \notin \operatorname{dom}(\rho_{j+1}(q_j))$, thus $w_{q_j,x} \equiv \rho_j$. Let $u \in \Gamma \setminus \{x\}$. Since q_j satisfies (iv), we have

$$(u)w_{q_j,u} \neq (x)w_{q_j,x} = (x)\rho_j = y.$$

It then follows from $(u)w_{q_j,u} \in nK_\omega \setminus \operatorname{dom}(q_j)$ and (6.13) that

$$(u)w_{q_j,u} \in nK_\omega \setminus \operatorname{dom}(q_{j+1}),$$

and so $w_{q_{j+1},u} \equiv w_{q_j,u}$. Then $(u)w_{q_{j+1},u} \in nK_\omega \setminus \operatorname{dom}(q_{j+1})$, and since q_j satisfies (ii), it follows that q_{j+1} also satisfies (ii).

Let $i \in \{-\mathbf{b}(w), \ldots, \mathbf{b}(w)\}$. Then by (6.13) and the fact that z was chosen so that $(z)f^i \notin \operatorname{dom}(q_j) \cup \{y\}$, we have

$$(x)\rho_{j+1}f^i = (z)f^i \in nK_\omega \setminus \operatorname{dom}(q_{j+1}).$$

Hence q_{j+1} satisfies (iii).

From the fact that q_{j+1} satisfies (iii), it follows that $w_{q_{j+1},x} \equiv w_{q_j,x}\alpha\beta^k$ for some $k \in \{-\mathbf{b}(w), \ldots, \mathbf{b}(w)\}$, and so

$$(x)w_{q_{j+1},x} = (z)f^k. \tag{6.14}$$

By the choice of z and the fact that q_{j+1} satisfies (ii), we have

$$(x)w_{q_{j+1},x} = (z)f^k \neq (u)w_{q_j,u} = (u)w_{q_{j+1},u}$$

for every $u \in \Gamma \setminus \{x\}$. Hence q_{j+1} satisfies (iv).

Finally, to show that q_{j+1} satisfies (v) consider two cases, namely $u = x$ and $u \in \Gamma \setminus \{x\}$. Suppose that $u = x$ and $m \in \mathbb{Z}$ is such that $x \in \operatorname{dom}(w_{q_{j+1},x}q^m_{j+1})$. From the choice of z and (6.14) it follows that

$$(x)w_{q_{j+1},x} = (z)f^k \notin \operatorname{dom}(q) \cup \operatorname{ran}(q) \cup \{y\}.$$

Suppose $(z)f^k \neq z$. Then

$$(x)w_{q_{j+1},x} = (z)f^k \notin \operatorname{dom}(q_{j+1}) \cup \operatorname{ran}(q_{j+1}),$$

and so $m = 0$, in which case $\Phi \subseteq \operatorname{dom}(q) \subseteq \operatorname{dom}(q_{j+1})$ implies that

$$(x)w_{q_{j+1},x}q^m_{j+1} = (z)f^k \in nK_\omega \setminus \Phi.$$

Suppose that $(z)f^k = z$, in other words $(x)w_{q_{j+1},x} = z$. Since $z \notin \mathrm{dom}(q_{j+1})$, it follows that $x \notin \mathrm{dom}(w_{q_{j+1},x}q_{j+1}^m)$ for all $m > 0$. If $m = 0$, then

$$(x)w_{q_{j+1},x}q_{j+1}^m = (z)f^k \in nK_\omega \setminus \Phi$$

by the choice z and since $\Phi \subseteq \mathrm{dom}(q)$. Suppose that $m < 0$. Then $m + 1 \leq 0$ and it follows from the definition of q_{j+1} that $\mathrm{dom}(q_{j+1}^{m+1})$ is either $\mathrm{dom}(q_j^{m+1})$ or $\mathrm{dom}(q_j^{m+1}) \cup \{(z)q_{j+1}^{m+1}\}$. Note that $y \in \mathrm{dom}(q_{j+1}^{m+1})$ implies $y \in \mathrm{dom}(q_j^{m+1})$. It follows from (6) of $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$ that

$$(x)w_{q_{j+1},x}q_{j+1}^m = (z)q_{j+1}^m = (y)q_{j+1}^{m+1} = (y)q_j^{m+1} = (x)w_{q_j,x}q^{m+1} \in nK_\omega \setminus \Phi.$$

Hence q_{j+1} satisfies (v) for $u = x$.

Suppose that $u \in \Gamma \setminus \{x\}$ and $m \in \mathbb{Z}$ such that $u \in \mathrm{dom}(w_{q_{j+1},u}q_{j+1}^m)$. Since q_j and q_{j+1} satisfy (ii), it follows that $(u)w_{q_{j+1},u} = (u)w_{q,u} = (u)w_{q_j,u}$. If $m \leq 0$, or $m > 0$ and there is no $m' \in \{0, \dots, m-1\}$ with $(u)w_{q_j,u}q^{m'} = y$, then

$$(u)w_{q_{j+1},u}q_{j+1}^m = (u)w_{q_j,u}q_j^m \in nK_\omega \setminus \Phi$$

since q_j satisfies (v). Otherwise, $m > 0$ and there is $m' \in \{0, \dots, m-1\}$ such that $(u)w_{q_j,u}q_j^{m'} = y$, in which case

$$(u)w_{q_{j+1},u}q_{j+1}^{m'+1} = z \notin \mathrm{dom}(q_{j+1}).$$

Hence $m = m' + 1$, and since $\Phi \subseteq \mathrm{dom}(q) \subseteq \mathrm{dom}(q_{j+1})$, it follows that $(u)w_{q_{j+1},u}q_{j+1}^m \in nK_\omega \setminus \Phi$. Therefore, q_{j+1} satisfies (v) and thus the inductive hypothesis.

By induction there is $h = q_{|w|} \in \mathcal{A}_{f,\Sigma}^{<\omega}$ satisfying (i)–(v). We will show that h satisfies $\mathcal{S}(\Gamma, \Theta \cup \{x\}, \Phi, \Delta, w)$ and will refer to parts (1) to (6) of this condition by writing (1) to (6), where appropriate, without reference to $\mathcal{S}(\Gamma, \Theta \cup \{x\}, \Phi, \Delta, w)$ in the rest of the proof.

Since h is an extension of q and $\overline{q} \in S_n$, it follows that $\overline{h} = \overline{q}$. Hence

$$\overline{w(h)} = \mathrm{id},$$

and so h satisfies (1). Since h satisfies (i) and (v), it also satisfies (2) and (6). Since $w = \rho_{|w|}$, condition (iii) implies that $x \in \mathrm{dom}(w(h))$, and so $x \in \mathrm{dom}(w(h)) \cap \Gamma$. If $u \in \Gamma \setminus \{x\}$, then $w_{h,u} \equiv w_{q,u}$ by (ii), and so

$$u \in \mathrm{dom}(w(h)) \cap \Gamma \quad \text{if and only if} \quad u \in \mathrm{dom}(w(q)) \cap \Gamma.$$

Therefore, $\mathrm{dom}(w(h)) \cap \Gamma = \Theta \cup \{x\}$ as q satisfies $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$, in other words h satisfies (3). By (iii) the image of x under $w(h)$ is in $nK_\omega \setminus \mathrm{dom}(h)$, and by (ii) the image of $u \in \Theta$ under $w(h) = w_{q,u}$ is also in $nK_\omega \setminus \mathrm{dom}(h)$. Hence h satisfies condition (4). It then follows from (ii), (iv) and the fact that q satisfies (5) of $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$ that $(u)w_{h,u} = (v)w_{h,v}$ only if $u = v$ for all $u, v \in \Gamma$, and thus h satisfies (5). Hence h satisfies $\mathcal{S}(\Gamma, \Theta \cup \{x\}, \Phi, \Delta, w)$, as required. □

Proof of Lemma 6.17. If necessary by extending q, using Lemmas 6.6 and 6.10, we can assume $\Gamma \subseteq \mathrm{dom}(q)$.

Let $d = |\Gamma|$. By Lemma 6.18, there is a freely reduced word $w \in F_{\alpha,\beta}$ not containing α^{-1} and starting with α, and an extension $q_0 \in \mathcal{A}_{f,\Sigma}^{<\omega}$ of q satisfying $\mathcal{S}(\Gamma, \varnothing, \mathrm{dom}(q), \Delta, w)$. Suppose that for some $j \in \{0, 1, \dots, d-1\}$ we have already extended $q = q_0$ to $q_j \in \mathcal{A}_{f,\Sigma}^{<\omega}$ such that there is $\Gamma_j \subseteq \Gamma$ with $|\Gamma_j| = j$, and such that q_j satisfies $\mathcal{S}(\Gamma, \Gamma_j, \mathrm{dom}(q), \Delta, w)$. Let $x \in \Gamma \setminus \Gamma_j$ and let $\Gamma_{j+1} = \Gamma_j \cup \{x\}$. Then, by condition (3) of $\mathcal{S}(\Gamma, \Gamma_j, \mathrm{dom}(q), \Delta, w)$, we have $x \notin \mathrm{dom}(w(q_j))$. Hence if we let $\Theta = \Gamma_j$ and $\Phi = \mathrm{dom}(q)$ then by Lemma 6.19 there is an extension $q_{j+1} \in \mathcal{A}_{f,\Sigma}^{<\omega}$ of q_j satisfying $\mathcal{S}(\Gamma, \Gamma_{j+1}, \mathrm{dom}(q), \Delta, w)$.

Therefore, by induction on j we obtain $h = q_d \in \mathcal{A}_{f,\Sigma}^{<\omega}$ which satisfies $\mathcal{S}(\Gamma, \Gamma, \mathrm{dom}(q), \Delta, w)$, as required for the conclusion. □

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