

SOLVABLE QUOTIENTS OF SUBDIRECT PRODUCTS OF PERFECT GROUPS ARE NILPOTENT

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ABSTRACT. We prove the statement in the title and exhibit examples of quotients of arbitrary nilpotency class. This answers a question by D. F. Holt.

1. INTRODUCTION

A subgroup S of a direct product $G_1 \times \cdots \times G_n$ is said to be *subdirect* if its projection to each of the factors G_i is surjective. Subdirect products have recently been a focus of interest from combinatorial and algorithmic point of view; see, for example, [1]. They are also frequently used in computational finite group theory, notably to construct perfect groups; see [6].

In September 2017 D. F. Holt asked on the group pub forum whether a solvable quotient of a subdirect product of perfect groups is necessarily abelian. The question was re-posted on mathoverflow [5] where Holt gave some background: “This problem arose in a study of the complexity of certain algorithms for finite permutation and matrix groups. The group S in the applications is the kernel of the action of a transitive but imprimitive permutation (or matrix) group on a block system. So in that situation the G_i are all isomorphic, and $\text{Aut}(S)$ induces a transitive action on the direct factors G_i ($1 \leq i \leq n$).”

In the present note we start from the observation that every solvable quotient of a subdirect product of perfect groups is nilpotent, proved in Section 2, and then proceed to construct several examples demonstrating that the nilpotency class can be arbitrarily high. Specifically, in Section 5 we exhibit an infinite group G such that for every d there is a subdirect subgroup of G^{2^d} with a nilpotent quotient of degree d . In Section 6 we exhibit a perfect subgroup G of the special linear group $\text{SL}_6(\mathbb{F}_4)$ such that G^4 contains a subdirect subgroup with a nilpotent quotient of class 2. This example is generalized in Section 7 to a perfect subgroup G of the special linear group $\text{SL}_{2n}(\mathbb{F}_4)$ such that $G^{2^{n-1}}$ contains a subdirect subgroup with

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a nilpotent quotient of class $n - 1$. These examples are all based on a particular subdirect product construction, which is described in Section 3, and whose lower central series is computed in Section 4.

Throughout the paper we will use the following notation. Let \mathbb{N} denote the set of positive integers. For $n \in \mathbb{N}$, we write $[n]$ to denote $\{1, \dots, n\}$. For elements x, y of a group G we write $x^y := y^{-1}xy$ and $[x, y] := x^{-1}y^{-1}xy$. The higher commutators in a group G will always be assumed to be left associated, i.e.

$$[x_1, \dots, x_k] := [[x_1, x_2], \dots, x_k], \quad [N_1, \dots, N_k] := [[N_1, N_2], \dots, N_k]$$

for $x_i \in G$ and $N_i \trianglelefteq G$. Furthermore, for $k \geq 1$ we let

$$\gamma_k(G) := \underbrace{[G, \dots, G]}_k, \quad \gamma_1(G) := G,$$

denote the k th term in the lower central series of G and, for $N \trianglelefteq G$ and $k \geq 0$, write

$$[G, {}_k N] := [G, \underbrace{N, \dots, N}_k], \quad [G, {}_0 N] := G.$$

The following two facts about commutators for a group G with a normal subgroup N will frequently be used ($k, l \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}$):

- [C1] $[\gamma_k(N), \gamma_l(N)] \leq \gamma_{k+l}(N)$;
- [C2] $[[G, {}_m N], \gamma_l(N)] \leq [G, {}_{m+l} N]$.

The first can be found in [8, 5.1.11(i)]. The second follows by induction on l . The case $l = 1$ holds by the definition of $[G, {}_k N]$ and the fact that $\gamma_1(N) = N$. For $l > 1$ we have

$$\begin{aligned} [[G, {}_m N], \gamma_l(N)] &\leq \left[[[G, {}_m N], N], \gamma_{l-1}(N) \right] \cdot \left[[[G, {}_m N], \gamma_{l-1}(N)], N \right] \\ &\hspace{15em} \text{by the Three Subgroup Lemma} \\ &= [[G, {}_{m+1} N], \gamma_{l-1}(N)] \cdot \left[[[G, {}_m N], \gamma_{l-1}(N)], N \right] \\ &\leq [G, {}_{m+l} N] \cdot [[G, {}_{m+l-1} N], N] \text{ by induction} \\ &= [G, {}_{m+l} N]. \end{aligned}$$

2. SOLVABLE QUOTIENTS MUST BE NILPOTENT

That solvable quotients of subdirect products of perfect groups are necessarily nilpotent is an easy consequence of the following more general observation.

Theorem 2.1. *Let $N \trianglelefteq S \leq_{\text{sd}} G_1 \times \dots \times G_n$ such that N, S are both subdirect products of groups G_1, \dots, G_n . Then S/N is nilpotency of class at most $n - 1$.*

Proof. For $i = 1, \dots, n$, let $K_i \trianglelefteq S$ be the kernel of the projection of S onto G_i . Notice that $\bigcap_{i=1}^n K_i = 1$. Moreover we claim that

$$K_i N = S \text{ for all } i = 1, \dots, n. \quad (1)$$

For the inclusion \supseteq , let $s \in S$. Since the projection of N onto G_i is onto by assumption, we have $b \in N$ such that $b_i = s_i$. Then sb^{-1} is in K_i which proves (1). It now follows that

$$\gamma_n(S) = [K_1 N, \dots, K_n N] \leq [K_1, \dots, K_n] N.$$

Since $[K_1, \dots, K_n] \leq \bigcap_{i=1}^n K_i = 1$, we obtain $\gamma_n(S) \leq N$. \square

Remark 2.2. The analogue of Theorem 2.1 holds, with exactly the same proof, in any congruence modular variety when the normal subgroup N is replaced by a congruence ν of the subdirect product S such that the join of ν with any projection kernel ρ_i ($i \leq n$) yields the total congruence on S . In fact, then S/ν is supernilpotent of class at most $n - 1$. We refer the reader to [3] for the definition of commutators of congruences and to [7] for supernilpotence and higher commutators in this general setting. It then follows that the analogue of Theorem 2.1 holds for rings, K -algebras, Lie algebras, and loops.

Corollary 2.3. *Let $S \leq_{\text{sd}} G_1 \times \dots \times G_n$ be a subdirect product of perfect groups G_1, \dots, G_n , and let $N \trianglelefteq S$ be a normal subgroup of S . If S/N is solvable, then S/N is nilpotent of class at most $n - 1$.*

Proof. For $i = 1, \dots, n$, let N_i denote the projection of N on G_i . Since each G_i is perfect and G_i/N_i is solvable by assumption, we obtain $G_i = N_i$ and the result follows from Theorem 2.1. \square

3. A SUBDIRECT CONSTRUCTION

By Corollary 2.3, every solvable quotient of the subdirect product of two perfect groups is abelian. In what follows, we are going to show that in fact there exist subdirect powers of a perfect group G with quotients of arbitrarily large nilpotency class. To this end we introduce a construction that is based on higher commutator relations from universal algebra [7].

Let G be a group, and let $d \in \mathbb{N}$. We will be working in the direct product of 2^d copies of G with components indexed by the power set $\mathcal{P} := \mathcal{P}([d])$. The set \mathcal{P} will be linearly ordered by the short-lex ordering; specifically

$$A <_{\text{sl}} B \text{ iff } |A| < |B| \text{ or } (|A| = |B| \text{ and } \min(A \setminus B) < \min(B \setminus A)).$$

Now, for any $A \subseteq [d]$, $u \in G$ and $K \subseteq G$, define $\Delta_A(u) \in G^{\mathcal{P}}$ by

$$\Delta_A(u)(B) := \begin{cases} u & \text{if } A \subseteq B \\ 1 & \text{otherwise.} \end{cases}$$

and

$$\Delta_A(K) := \{\Delta_A(u) : u \in K\}.$$

The following is an immediate consequence of the above definition:

Lemma 3.1. *If $K, L \trianglelefteq G$ and $A, B \subseteq [d]$, then*

$$[\Delta_A(K), \Delta_B(L)] = \Delta_{A \cup B}([K, L]).$$

Now, for a normal subgroup $N \trianglelefteq G$ and for $k \in [d+1]$ we define the following subset of $G^{\mathcal{P}}$:

$$\Gamma_k := \Gamma_k(G, N) := \prod_{|A| < k} \Delta_A([G, {}_{|A|}N]) \cdot \prod_{|A| \geq k} \Delta_A(\gamma_{|A|}(N)). \quad (2)$$

Here the factors in the products are ordered by short-lex on \mathcal{P} .

Lemma 3.2. *Let $A, B \subseteq [d]$. Then*

- (i) $[[G, {}_{|A|}N], [G, {}_{|B|}N]] \leq [G, {}_{|A \cup B|}N]$,
- (ii) $[[G, {}_{|A|}N], \gamma_{|B|}(N)] \leq [G, {}_{|A \cup B|}N]$ for $B \neq \emptyset$ and
- (iii) $[\gamma_{|A|}(N), \gamma_{|B|}(N)] \leq \gamma_{|A \cup B|}(N)$ for $A, B \neq \emptyset$.

Proof. Immediate from [C1] and [C2]. \square

Lemma 3.3. $\Gamma_k(G, N)$ is a subdirect subgroup of $G^{\mathcal{P}}$ for each $k \in [d+1]$ and $N \trianglelefteq G$.

Proof. First notice that all factors in the definition of Γ_k are subgroups. That Γ_k projects onto each coordinate follows from the presence of $\Delta_{\emptyset}([G, {}_0N]) = \Delta_{\emptyset}(G)$, the diagonal of $G^{\mathcal{P}}$.

To prove that Γ_k is a subgroup, consider two generic factors $\Delta_A(K)$ and $\Delta_B(L)$ in its definition for $A <_{sl} B$. That is, $K = [G, {}_{|A|}N]$ if $|A| < k$ and $K = \gamma_{|A|}(N)$ otherwise, and likewise for L . By Lemma 3.1, we obtain

$$\begin{aligned} \Delta_B(L) \cdot \Delta_A(K) &= \Delta_A(K) \cdot \Delta_B(L) \cdot [\Delta_B(L), \Delta_A(K)] \\ &= \Delta_A(K) \cdot \Delta_B(L) \cdot \Delta_{A \cup B}([K, L]). \end{aligned}$$

By Lemma 3.2, we have $[K, L] \leq [G, {}_{|A \cup B|}N]$ if $|A \cup B| < k$; further $[K, L] \leq \gamma_{|A \cup B|}(N)$ if $|A \cup B| \geq k$. Hence $\Delta_{A \cup B}([K, L])$ is contained in another factor of the product defining Γ_k . Since $B \leq_{sl} A \cup B$, the result follows. \square

Remark 3.4. In reference to Holt's motivation, we note that the automorphism group of $S := \Gamma_1$ acts transitively on the index set \mathcal{P} . More precisely, for $i \in [d]$, the permutation on \mathcal{P} that maps a subset A of $[d]$ to its symmetric difference with $\{i\}$ induces an automorphism r_i on S . This can be easily seen by the action of r_i on the generators $\Delta_A(u)$ of S : $r_i(\Delta_A(u)) = \Delta_A(u)$ if $i \notin A$; else $r_i(\Delta_A(u)) = \Delta_{A \setminus \{i\}}(u) \cdot \Delta_A(u)^{-1}$. Then $\langle r_1, \dots, r_d \rangle \leq \text{Aut}(S)$ is isomorphic to \mathbb{Z}_2^d and acts regularly on the set of direct factors of S .

4. THE LOWER CENTRAL SERIES

Continuing with the construction from the previous section, we now let $S := \Gamma_1$ and determine its lower central series:

Lemma 4.1. *If $G' = G$, then $\gamma_k(S) = \Gamma_k$ for all $k \in [d + 1]$.*

Proof. We need to show that

$$[\Gamma_k, \Gamma_1] = \Gamma_{k+1} \quad \text{for } k \in [d]. \quad (3)$$

For the inclusion \leq it is sufficient to prove that

$$[\Delta_A(K), \Delta_B(L)] = \Delta_{A \cup B}([K, L]) \leq \Gamma_{k+1}, \quad (4)$$

where

$$K = \begin{cases} [G, {}_{|A|}N] & \text{if } |A| < k, \\ \gamma_{{}_{|A|}}(N) & \text{otherwise,} \end{cases} \quad L = \begin{cases} G & \text{if } B = \emptyset, \\ \gamma_{{}_{|B|}}(N) & \text{otherwise.} \end{cases}$$

Note that (4) follows from the claim that

$$[K, L] \leq \begin{cases} [G, {}_{|A \cup B|}N] & \text{if } |A \cup B| < k + 1, \\ \gamma_{{}_{|A \cup B|}}(N) & \text{otherwise.} \end{cases} \quad (5)$$

The assertion for $|A \cup B| < k + 1$ follows from Lemma 3.2 except when $|A| = k$ and $B \subseteq A$. In this case we use [C2] to obtain

$$[K, L] = [\gamma_{{}_{|A|}}(N), L] \leq [G, \gamma_{{}_{|A|}}(N)] \leq [G, {}_{|A|}N] = [G, {}_{|A \cup B|}N].$$

Claim (5) for $|A \cup B| \geq k + 1$ is also immediate from Lemma 3.2 and the observation that $[G, {}_{|A \cup B|}N] \leq \gamma_{{}_{|A \cup B|}}(N)$ for $A \cup B \neq \emptyset$. Hence (5), (4), and the inclusion \subseteq of (3) are proved.

For the inclusion \supseteq of (3) we show that

$$\Delta_A(K) \subseteq [\Gamma_k, \Gamma_1], \quad (6)$$

where

$$K = \begin{cases} [G, {}_{|A|}N] & \text{if } |A| < k + 1 \\ \gamma_{{}_{|A|}}(N) & \text{if } |A| \geq k + 1. \end{cases}$$

Now, if $A = \emptyset$, the assumption that $G = G'$ yields

$$\Delta_\emptyset(K) = \Delta_\emptyset(G) = \Delta_\emptyset([G, G']) = [\Delta_\emptyset(G), \Delta_\emptyset(G)] \subseteq [\Gamma_k, \Gamma_1].$$

Assume now that $A \neq \emptyset$, and write it as $A = B \cup C$, where $|B| = |A| - 1$ and $|C| = 1$. If $|A| < k + 1$, then $|B| < k$ and by Lemma 3.1

$$\Delta_A(K) = \Delta_{B \cup C}([G, {}_{|B|+|C|}N]) = [\Delta_B([G, {}_{|B|}N]), \Delta_C(N)] \subseteq [\Gamma_k, \Gamma_1].$$

Likewise, if $|A| \geq k + 1$, then $|B| \geq k$ and

$$\Delta_A(K) = \Delta_{B \cup C}(\gamma_{{}_{|B|+|C|}}(N)) = [\Delta_B(\gamma_{{}_{|B|}}(N)), \Delta_C(N)] \subseteq [\Gamma_k, \Gamma_1].$$

This completes the proof of (6), the inclusion \supseteq of (3) and the lemma. \square

We can now prove our main result:

Theorem 4.2. *Let G be a perfect group, $d \in \mathbb{N}$, and N a normal subgroup such that $\gamma_d(N) > [G, {}_dN]$. Then $S = \Gamma_1(G, N)$ is a subdirect subgroup of G^{2^d} which has a nilpotent quotient of class d .*

Proof. That S is subdirect was proved in Lemma 3.3. By Lemma 4.1 and (2), we have

$$\begin{aligned}\gamma_d(S) &= \Gamma_d = \prod_{|A| < d} \Delta_A([G, {}_{|A|}N]) \cdot \Delta_{[d]}(\gamma_d(N)), \\ \gamma_{d+1}(S) &= \Gamma_{d+1} = \prod_{A \in \mathcal{P}} \Delta_A([G, {}_{|A|}N]).\end{aligned}$$

Hence the kernels of the projections on $\mathcal{P} \setminus \{[d]\}$ of $\gamma_d(S)$ and $\gamma_{d+1}(S)$, respectively, are $\Delta_{[d]}(\gamma_d(N))$ and $\Delta_{[d]}([G, {}_dN])$, respectively. Since the assumption $\gamma_d(N) > [G, {}_dN]$ yields

$$\Delta_{[d]}(\gamma_d(N)) > \Delta_{[d]}([G, {}_dN]),$$

we obtain $\gamma_d(S) > \gamma_{d+1}(S)$. Thus $S/\gamma_{d+1}(S)$ is indeed d -nilpotent but not $(d-1)$ -nilpotent. \square

Remark 4.3. The 2018 version of the GAP library ‘Perfect Groups’ by Felsch, Holt and Plesken [4, 2] provides a list of all perfect groups whose sizes are less than 10^6 excluding 11 sizes. Unfortunately none of the groups G in this library have a normal subgroup N with $\gamma_d(N) > [G, {}_dN]$ for $d > 1$. Hence Theorem 4.2 does not yield non-abelian nilpotent quotients of $\Gamma_1(G, N)$ for them.

5. AN INFINITE EXAMPLE

In this section we demonstrate the existence of a perfect group G satisfying the assumptions of Theorem 4.2. More precisely, we will construct an infinite group G with a normal subgroup N such that

$$\gamma_d(N) > [G, {}_dN] \text{ for all } d \in \mathbb{N}. \quad (7)$$

Let X be a countably infinite alphabet with elements

$$X := \{x_{i,w} : i \in \mathbb{N}, w \in \{0,1\}^*\} \cup \{y_i : i \in \mathbb{N}\}.$$

Here $\{0,1\}^*$ denotes the set of all words over $\{0,1\}$ including the empty word ϵ . Let F be the free group on X . The derived subgroup F' is free by Nielsen’s Theorem, and it is routine to see that it has a free basis Z which contains all the commutators $[u, v]$ with $u, v \in X$, $u \neq v$. For $i \in \mathbb{N}$, $w \in \{0,1\}^*$ set

$$h(x_{i,w}) := [x_{i,w0}, x_{i,w1}] \in Z,$$

and extend h to a bijection $h : X \rightarrow Z$, and then to an isomorphism $h : F \rightarrow F'$.

Now let $F_i := F$ for $i \in \mathbb{N}$, and let G be the limit of the directed system

$$F_1 \xrightarrow{h} F_2 \xrightarrow{h} F_3 \xrightarrow{h} \dots$$

Thus, there exist embeddings $h_i : F_i \rightarrow G$ ($i \in \mathbb{N}$) such that $h_i = h_{i+1}h$, $h_1(F_1) \leq h_2(F_2) \leq \dots$ and $G = \bigcup_{i \in \mathbb{N}} h_i(F_i)$.

The group G is perfect, since $h_i(F_i) = h_{i+1}(F_{i+1})'$. Also, from $h_1(F_1) = h_i(F_i)^{(i-1)}$, it follows that $N := h_1(F_1) \trianglelefteq G$. We claim that (7) holds for this G and N .

Suppose to the contrary that $\gamma_d(N) = [G, {}_dN]$ for some d . Consider the element

$$u := [x_{1,\epsilon}, x_{2,\epsilon}, \dots, x_{d,\epsilon}] \in \gamma_d(F_1).$$

Then $h_1(u) \in \gamma_d(N)$ and hence $h_1(u) \in [G, {}_dN]$. It follows that

$$h_1(u) = v_1 v_2 \dots v_k,$$

where

$$v_i = c_i^{-1} [g_i, n_{i1}, \dots, n_{id}]^{e_i} c_i$$

for $e_i \in \{-1, 1\}$, $c_i, g_i \in G$, $n_{ij} \in N$ for all $i \leq k, j \leq d$. From $G = \bigcup_{t \in \mathbb{N}} h_t(F_t)$ it follows that there exists $t \in \mathbb{N}$ such that $c_i, g_i \in h_t(F_t)$ for all $i = 1, \dots, k$. Therefore we have

$$h_1(u) \in [h_t(F_t), {}_d h_1(F_1)].$$

Since $h_1 = h_t h^{t-1}$ and $h_t : F_t \rightarrow G$ is an injection, this implies

$$h^{t-1}(u) \in [F_t, {}_d h^{t-1}(F_1)].$$

Recall $F_1 = F_t = F$ and $h^{t-1}(F) = F^{(t-1)}$ to obtain

$$h^{t-1}(u) \in [F, {}_d F^{(t-1)}]. \quad (8)$$

Note that

$$h^{t-1}(x_{i,\epsilon}) = h^{t-2}([x_{i,0}, x_{i,1}]) = h^{t-3}([[x_{i,00}, x_{i,01}], [x_{i,10}, x_{i,11}]]) = \dots$$

generates $F^{(t-1)}$ as a fully invariant subgroup of F . Hence

$$h^{t-1}(u) = [h^{t-1}(x_{1,\epsilon}), \dots, h^{t-1}(x_{d,\epsilon})]$$

generates $\gamma_d(F^{(t-1)})$ as a fully invariant subgroup. Since $[F, {}_d F^{(t-1)}]$ is also fully invariant, (8) implies the inclusion \leq in

$$\gamma_d(F^{(t-1)}) = [F, {}_d F^{(t-1)}].$$

The converse inclusion \geq is trivial. Since F is free of countable rank, this implies that

$$\gamma_d(H^{(t-1)}) = [H, {}_d H^{(t-1)}]$$

for *all* countable (not necessarily perfect) groups H . But this is false, one counter-example being the group of upper unitriangular $n \times n$ matrices over any field for sufficiently large n . This contradiction establishes Claim (7).

6. A FINITE EXAMPLE

In this section we exhibit a finite group G with a normal subgroup N such that $[N, N] > [G, N, N]$.

Let $\mathbb{F} := \mathbb{F}_4$ be the field of order 4, and let $\alpha \in \mathbb{F}$ be a primitive element; recall that $\alpha^2 = \alpha + 1$. Recall that the special linear group $S := \mathrm{SL}_2(\mathbb{F})$ is perfect [8, 3.2.8]; in fact, S is isomorphic to the alternating group A_5 . Let us denote by I the identity of S . Also let

$$T := \{A \in \mathbb{F}^{2 \times 2} : \mathrm{tr}(A) = 0\},$$

the set of all matrices of trace 0.

In what follows we will work in the special linear group $\mathrm{SL}_6(\mathbb{F})$. Its elements will be considered and written as 3×3 block matrices over $\mathbb{F}^{2 \times 2}$. First define the following three subgroups of $\mathrm{SL}_6(\mathbb{F})$, where the omitted entries are understood to be 0-blocks:

$$H := \left\{ \begin{pmatrix} A & & \\ & A & \\ & & A \end{pmatrix} : A \in S \right\},$$

$$M := \left\{ \begin{pmatrix} I & B & D \\ & I & C \\ & & I \end{pmatrix} : B, C, D \in \mathbb{F}^{2 \times 2}, \mathrm{tr}(B) = \mathrm{tr}(C) = 0 \right\},$$

$$N := \left\{ \begin{pmatrix} I & bI & D \\ & I & cI \\ & & I \end{pmatrix} : b, c \in \mathbb{F}, D \in \mathbb{F}^{2 \times 2} \right\},$$

and then set

$$G := HM.$$

Lemma 6.1. *The following hold for G , N and M as defined above:*

- (i) $M \trianglelefteq G \leq \mathrm{SL}_6(\mathbb{F})$.
- (ii) G is perfect.
- (iii) $N \trianglelefteq G$.
- (iv) $[N, N] > [G, N, N]$.

We note that G has size $60 \cdot 4^{10} = 62\,914\,560$ which exceeds the maximal size of perfect groups listed in the ‘Perfect Groups’ GAP library [2].

In the proof of the above results, we will make use of the following two technical observations. Throughout what follows we will use the standard exponentiation notation for matrix conjugation: $B^A := A^{-1}BA$.

Lemma 6.2. *The set T , considered as an \mathbb{F} -module, is spanned by the set*

$$U := \{-B + B^A : B \in T, A \in S\}.$$

Proof. For every $x \in \mathbb{F}$ we have

$$\begin{pmatrix} x & x^2 \\ 0 & x \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in U,$$

and hence

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \alpha & \alpha^2 \\ 0 & \alpha \end{pmatrix} \in \mathbb{F}U.$$

By symmetry $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathbb{F}U$, and also $I = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathbb{F}U$. Since T is spanned by I , $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, the assertion follows. \square

Lemma 6.3. *The \mathbb{F} -module spanned by the set*

$$U := \{BC : B, C \in T\}$$

is the entire $\mathbb{F}^{2 \times 2}$.

Proof. The subset

$$\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \cdot \left\{ I, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

of U contains the natural basis of $\mathbb{F}^{2 \times 2}$. \square

Proof of Lemma 6.1. (i) For

$$X := \begin{pmatrix} A & & \\ & A & \\ & & A \end{pmatrix} \in H \text{ and } Y := \begin{pmatrix} I & B & D \\ & I & C \\ & & I \end{pmatrix} \in M \quad (9)$$

we have

$$Y^X = \begin{pmatrix} I & B^A & D^A \\ & I & C^A \\ & & I \end{pmatrix}.$$

From $\text{tr}(B^A) = \text{tr}(B) = 0$ and $\text{tr}(C^A) = \text{tr}(C) = 0$ it follows that $Y^X \in M$. Thus, H normalizes M , and both assertions follow.

(ii) Since $H \cong S$ is perfect, we have

$$[G, G] = [HM, HM] = [H, H][H, M][M, M] = H[H, M]M'.$$

So it suffices to show that

$$[H, M] = M. \quad (10)$$

Now, for $X \in H$, $Y \in M$ as in (9), we have

$$Y^{-1} = \begin{pmatrix} I & -B & BC - D \\ & I & -C \\ & & I \end{pmatrix}$$

and hence

$$[Y, X] = Y^{-1}Y^X = \begin{pmatrix} I & -B + B^A & D^A - BC^A + BC - D \\ & I & -C + C^A \\ & & I \end{pmatrix}. \quad (11)$$

Using Lemma 6.2 and (11) twice, first with $C = D = 0$, and then with $B = C = 0$, we see that

$$\begin{pmatrix} I & P & 0 \\ & I & 0 \\ & & I \end{pmatrix}, \begin{pmatrix} I & 0 & 0 \\ & I & Q \\ & & I \end{pmatrix} \in [H, M] \text{ for all } P, Q \in T.$$

Multiplying these two matrices in both orders yields

$$\begin{pmatrix} I & P & 0 \\ & I & Q \\ & & I \end{pmatrix}, \begin{pmatrix} I & P & PQ \\ & I & Q \\ & & I \end{pmatrix} \in [H, M] \text{ for all } P, Q \in T. \quad (12)$$

Hence also

$$\begin{pmatrix} I & 0 & PQ \\ & I & 0 \\ & & I \end{pmatrix} = \begin{pmatrix} I & P & 0 \\ & I & Q \\ & & I \end{pmatrix}^{-1} \begin{pmatrix} I & P & PQ \\ & I & Q \\ & & I \end{pmatrix} \in [H, M]. \quad (13)$$

Using Lemma 6.3, it follows that

$$\begin{pmatrix} I & 0 & R \\ & I & 0 \\ & & I \end{pmatrix} \in [H, M] \text{ for all } R \in \mathbb{F}^{2 \times 2}. \quad (14)$$

Multiplying the first matrix from (12) together with (14) yields

$$\begin{pmatrix} I & P & R \\ & I & Q \\ & & I \end{pmatrix} = \begin{pmatrix} I & P & 0 \\ & I & Q \\ & & I \end{pmatrix} \begin{pmatrix} I & 0 & R \\ & I & 0 \\ & & I \end{pmatrix} \in [H, M]$$

for all $P, Q \in T$, $R \in \mathbb{F}^{2 \times 2}$, proving Claim (10) and part (ii) of the lemma.

(iii) Again, let $X \in H$ and $Y \in M$ be as in (9), and also let

$$Z := \begin{pmatrix} I & bI & E \\ & I & cI \\ & & I \end{pmatrix} \in N.$$

Then

$$Z^X = \begin{pmatrix} I & bI & E^A \\ & I & cI \\ & & I \end{pmatrix} \text{ and } Z^Y = \begin{pmatrix} I & bI & bC - cB + E \\ & I & cI \\ & & I \end{pmatrix}$$

are both in N . Hence $N \trianglelefteq G$.

(iv) To compute $[G, N]$, note that, for $X \in H, Y \in M, Z \in N$ as above, we have:

$$\begin{aligned} [Z, X] &= Z^{-1}Z^X = \begin{pmatrix} I & 0 & -E + E^A \\ & I & 0 \\ & & I \end{pmatrix}, \\ [Z, Y] &= Z^{-1}Z^Y = \begin{pmatrix} I & 0 & bC - cB \\ & I & 0 \\ & & I \end{pmatrix}. \end{aligned} \quad (15)$$

Recalling Lemma 6.2 it follows that

$$[G, N] = \left\{ \begin{pmatrix} I & 0 & P \\ & I & 0 \\ & & I \end{pmatrix} : P \in T \right\}. \quad (16)$$

From here it is immediate that $[G, N, N]$ is trivial. On the other hand, for $Y \in N$ with $B = b'I, C = c'I$ in (15) we obtain

$$[Z, Y] = \begin{pmatrix} I & 0 & (bc' - b'c)I \\ & I & 0 \\ & & I \end{pmatrix}.$$

Hence

$$[N, N] = \left\{ \begin{pmatrix} I & 0 & dI \\ & I & 0 \\ & & I \end{pmatrix} : d \in \mathbb{F} \right\}$$

is strictly bigger than $[G, N, N]$. \square

7. A FAMILY OF FINITE EXAMPLES

The construction from the previous section can be readily adapted to yield, for every $d \in \mathbb{N}$, a finite perfect group G and a normal subgroup N such that $\gamma_d(N) > [G, {}_dN]$. We just give an outline.

Let $n \in \mathbb{N}$ be arbitrary. The field \mathbb{F} , group S and set T will be as in the previous section. This time we will work in the group $\mathrm{SL}_{2n}(\mathbb{F})$, which we will treat as consisting of $n \times n$ block-matrices with 2×2 entries.

For a matrix $A \in S$ we will write $\Delta(A)$ to denote the block diagonal matrix having n copies of A along the main diagonal. Also, for arbitrary $A \in \mathbb{F}^{2 \times 2}$ and $i, j \in \{1, \dots, n\}, i \neq j$, we will use $E_{ij}(A)$ to denote the matrix which has the identity blocks on the main diagonal, has (i, j) block equal to A , and has zero blocks elsewhere.

The basic ingredients for our construction are the following subgroups of $\mathrm{SL}_{2n}(\mathbb{F})$. Let

$$H := \{\Delta(A) : A \in S\} \cong S.$$

The second subgroup M consists of all block upper unitriangular matrices with arbitrary matrices of trace 0 on the super-diagonal, i.e.,

$$M := \{(A_{ij}) : A_{ii} = I, A_{i,i+1} \in T, A_{ij} \in \mathbb{F}^{2 \times 2} (j - i > 1), A_{ij} = 0 (i < j)\}.$$

The third subgroup N consists of those elements of M for which all the blocks on the super-diagonal belong to the center of S , i.e,

$$N := \{(A_{ij}) \in M : A_{i,i+1} = a_i I (a_i \in \mathbb{F})\}.$$

Finally, we let

$$G := HM.$$

Lemma 7.1. *The following hold for G , N and M as defined above:*

- (i) $M \trianglelefteq G \leq \mathrm{SL}_{2n}(\mathbb{F})$.
- (ii) G is perfect.
- (iii) $N \trianglelefteq G$.
- (iv) $\gamma_d(N) > [G, {}_d N]$ for $d < n$.

Proof. Items (i) and (iii) follow exactly as for Lemma 6.1. For proving (ii), we again just need

$$[H, M] = M.$$

For this, let $A \in S$, $B \in T$ and $E_{i,i+1}(B) \in M$ for $1 \leq i \leq n-1$, and note that

$$[E_{i,i+1}(B), \Delta(A)] = E_{i,i+1}(-B + B^A).$$

By Lemma 6.2 it follows that

$$\{E_{i,i+1}(P) : P \in T, 1 \leq i \leq n-1\} \subseteq [H, M]. \quad (17)$$

As in (13) we obtain

$$[E_{i,i+1}(P), E_{i+1,i+2}(Q)] = E_{i,i+2}(PQ).$$

By Lemma 6.3 it follows that

$$\{E_{i,i+2}(R) : R \in \mathbb{F}^{2 \times 2}, 1 \leq i \leq n-2\} \subseteq [H, M].$$

A straightforward inductive argument now shows that

$$\{E_{ij}(R) : R \in \mathbb{F}^{2 \times 2}, j - i > 1\} \subseteq [H, M]. \quad (18)$$

From (17) and (18) it easily follows that $[H, M] = M$, as claimed.

For (iv), we obtain as in (16)

$$[G, N] = \{(A_{ij}) \in N : A_{i,i+1} = 0, A_{i,i+2} \in T\}.$$

Assuming $n > d$, from this it is easy to see that

$$[G, {}_d N] = \{(A_{ij}) : A_{ij} = 0 (0 < j - i \leq d), A_{i,i+d+1} \in T\}.$$

Finally,

$$\gamma_d(N) = \{(A_{ij}) \in N : A_{ij} = 0 (0 < j - i \leq d-1), A_{i,i+d} = a_i I (a_i \in \mathbb{F})\},$$

and therefore $\gamma_d(N) > [G, {}_dN]$, as required. \square

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