

GENERATING THE FULL TRANSFORMATION SEMIGROUP USING ORDER PRESERVING MAPPINGS

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(Received 8 October, 2002; accepted 2 February, 2003)

Abstract. For a linearly ordered set X we consider the relative rank of the semigroup of all order preserving mappings \mathcal{O}_X on X modulo the full transformation semigroup \mathcal{T}_X . In other words, we ask what is the smallest cardinality of a set A of mappings such that $\langle \mathcal{O}_X \cup A \rangle = \mathcal{T}_X$. When X is countably infinite or well-ordered (of arbitrary cardinality) we show that this number is one, while when $X = \mathbb{R}$ (the set of real numbers) it is uncountable.

2000 Mathematics Subject Classification. 20M20, 06A05.

1. Introduction. For a semigroup S , the ‘classical’ idea of rank is concerned with finding minimum size generating sets for S ; see [6] or [10]. When working with a finitely generated semigroup S determining the rank of S , denoted $\text{rank}(S)$, is a natural consideration. However, for an uncountable semigroup S the rank of S is $|S|$, and so the classical notion of rank provides us with no information. We introduce a different rank property which allows us to ‘measure’, from a certain perspective, a given semigroup with respect to some distinguished subsemigroups. For a semigroup S , if $A \subseteq S$ then we call the minimum cardinality of a set B such that

$$\langle A \cup B \rangle = S,$$

the *relative rank of S modulo A* . Alternatively, we may refer to this cardinality as the *relative rank of A in S* ; which we denote by $\text{rank}(S : A)$. This subject has been studied in the context of groups in [2] and [14]. In these papers, so called large subgroups of the symmetric group \mathcal{S}_X , over an infinite set X , were considered. The relative rank of the full transformation semigroup \mathcal{T}_X , over an infinite set X , modulo various standard subsemigroups was first considered in [7]. It was shown in [11] that the relative rank of \mathcal{T}_X modulo \mathcal{S}_X is two. In the same paper it was shown that the relative rank of the set of all idempotent maps on X in \mathcal{T}_X is, also, two. Sierpiński [15] showed that any countable set of maps from X to X is contained in a 2-generated subsemigroup of \mathcal{T}_X . An alternative proof of this was given by Banach [1]; see also [8]. An immediate corollary of this result is that the relative rank of a subset of \mathcal{T}_X is either uncountable or at most two. The corresponding result, that any countable set of permutations is contained in a 2-generated subgroup of \mathcal{S}_X , was given some years later in [3]. The

analogues of these results in the semigroup of all binary relations, the semigroup of all partial maps, and the symmetric inverse semigroup were proven in [8].

In this paper we consider the relative rank of the full transformation semigroup \mathcal{T}_X , where (X, \leq) is an infinite linearly ordered set, modulo the subsemigroup \mathcal{O}_X of all order preserving maps on X . Recall, that a map $\alpha \in \mathcal{T}_X$ is *order preserving* if

$$x \leq y \text{ implies } x\alpha \leq y\alpha,$$

for all $x, y \in X$.

For finite X (of size n , say) the semigroup \mathcal{O}_X has been studied extensively. Its order is $\binom{2n-1}{n-1}$, its rank, in the classical sense, is n , it is idempotent generated and its idempotent rank is $2n - 2$ (see [4] and [9]). Furthermore, it is easy to see that

$$\text{rank}(\mathcal{T}_X : \mathcal{O}_X) = 2.$$

Indeed, it is well-known that it is possible to generate \mathcal{T}_X using elements of \mathcal{S}_X and an arbitrary map α with the property that $|\text{im}(\alpha)| = n - 1$. If we choose $\alpha \in \mathcal{O}_X$ then the result follows from the observation that $\mathcal{O}_X \cap \mathcal{S}_X = \{1_X\}$ and the fact that $\text{rank}(\mathcal{S}_X) = 2$.

In [8] the case where $X = \mathbb{N}$ (with the usual ordering) was considered, and it was shown that

$$\text{rank}(\mathcal{T}_{\mathbb{N}} : \mathcal{O}_{\mathbb{N}}) = 1.$$

In this paper we build on the above example and show that

$$\text{rank}(\mathcal{T}_X : \mathcal{O}_X) = 1,$$

when X is an arbitrary countable linearly ordered set or an arbitrary well-ordered set (of any cardinality) but that $\text{rank}(\mathcal{T}_X : \mathcal{O}_X)$ can be uncountable for some (uncountable, non-well-ordered) linearly ordered sets X .

2. Countable linearly ordered sets. In this section we prove the following result.

THEOREM 2.1. *Let X be a countable linearly ordered set. The relative rank of \mathcal{T}_X modulo \mathcal{O}_X is one.*

For the remainder of this section X will be a fixed countably infinite linearly ordered set. For $x, y \in X$ with $x < y$ we define

$$\begin{aligned} [x, y] &= \{z \in X : x \leq z \leq y\}, & (x, y) &= \{z \in X : x < z < y\}, \\ (x, y] &= \{z \in X : x < z \leq y\}, & [x, y) &= \{z \in X : x \leq z < y\}. \end{aligned}$$

We call these sets *intervals*, delimited by x and y . An element $d \in X$ is called *discrete* if there exist $x, y \in X$ with $d \in (x, y)$ such that $(x, d) = (d, y) = \emptyset$. We call an element $r \in X$ *right isolated* if for every $x \in X$ such that $x < r$ we have $(x, r) \neq \emptyset$ but there exists $y \in X$ with $y > r$ such that $(r, y) = \emptyset$. Note that, in fact, if r is right isolated, then $|(x, r)| = |X|$ for all $x < r$. Perhaps a better name for an element with the above properties is a right isolated left limit point, but for the sake of convenience we shall use the shorter name. *Left isolated* elements are defined analogously. Note that if X has a smallest or largest element then this element is not right or left isolated or discrete with

the current definitions. To remedy this, if $x_0 \in X$ is the smallest element of X and there exists $y \in X$ such that $y \neq x_0$ and $(x_0, y) = \emptyset$ then we shall call x_0 discrete; otherwise we call x_0 left isolated. Analogously, if X has a largest element then it is either discrete or right isolated. Finally, an element $t \in X$ (which is neither the largest nor the smallest element of X) is called a *limit point* if $(x, t) \neq \emptyset$ and $(t, y) \neq \emptyset$ for every $x < t$ and for every $y > t$.

We now start a sequence of lemmas, leading to the proof of Theorem 2.1. Throughout we take $\mathbb{N} = \{1, 2, \dots\}$.

LEMMA 2.2. *Let X be a countable linearly ordered set consisting entirely of limit points, and let λ be any function from X to \mathbb{N} . Then there exists an order preserving injection α from X to X such that $|X \setminus \text{im}(\alpha)| = |X| (= |\text{im}(\alpha)|)$ satisfying $x\alpha\lambda \geq x\lambda$, for all $x \in X$.*

Proof. We start by finding an interval $I \subseteq X$ such that for every $x, y, z \in I$, with $x < y$, there exists $t \in (x, y)$ such that $t\lambda \geq z\lambda$. Let J be an arbitrary interval. One of the following alternatives holds:

- (i) for every subinterval of J delimited by $x, y \in J$, with $x < y$, and for every $n \in \mathbb{N}$ there exists $z \in (x, y)$ such that $z\lambda > n$; or
- (ii) there exists a subinterval $K \subseteq J$ and there exists $n \in \mathbb{N}$ such that $x\lambda \leq n$, for each $x \in K$.

If condition (i) holds then the interval J has the required property and we let $I = J$. Assume that condition (ii) holds. If for all $x, y \in K$, with $x < y$, there exists $z \in (x, y)$ such that $z\lambda = n$, then K satisfies the necessary condition and we let $I = K$. Otherwise, there exists a subinterval $K_1 \subseteq K$ such that $x\lambda \leq n - 1$, for all $x \in K_1$. We consider K_1 in the same way as we have just considered K , so that if for all $x, y \in K_1$, with $x < y$, there exists $z \in (x, y)$ such that $z\lambda = n - 1$ then we let $I = K_1$. Otherwise, there exists a subinterval $K_2 \subseteq K_1$ such that $x\lambda \leq n - 2$, for all $x \in K_2$. We repeat this process to give the sequence of non-empty subintervals:

$$K = K_0 \supseteq K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots,$$

where $x\lambda \leq n - i$ for every $x \in K_i$. Note that this process always terminates with $i = n - 1$ at latest.

Let $(a, b) \subseteq I$ be an arbitrary interval. Note that since a and b are both limit points we have $|(a, b)| = |X|$. We define $\alpha \in \mathcal{O}_X$ inductively as follows. First we define α on the points a and b , so that $b\alpha = b$ and $a\alpha = c$, where $c \in (a, b)$ with $c\lambda \geq a\lambda$. Such a point c exists from the defining property of I .

Next, we enumerate the elements of $[a, b]$:

$$b = e_0, a = e_1, e_2, e_3, \dots$$

For $k \in \mathbb{N}$, our inductive hypothesis is that the elements $e_0\alpha, e_1\alpha, e_2\alpha, \dots, e_k\alpha \in [c, b]$ are defined so that α is injective, order preserving and satisfies $e_i\alpha\lambda \geq e_i\lambda$ for every $i \in \{0, 1, \dots, k\}$. To define $e_{k+1}\alpha$, we find the largest $e_i \in [a, b]$ with $e_i < e_{k+1}$, where $i \leq k$, and the smallest $e_j \in [a, b]$ with $e_{k+1} < e_j$, where $j \leq k$. Since $i, j \leq k$ the elements $e_i\alpha$ and $e_j\alpha$ are already defined and $e_i\alpha < e_j\alpha$. From the definition of I there exists $y \in (e_i\alpha, e_j\alpha)$ such that $y\lambda \geq e_{k+1}\lambda$ and so we define $e_{k+1}\alpha = y$. Finally, for $x < a$ and for $x > b$ we let $x\alpha = x$. Note that since a is a limit point and $[a, c) \cap \text{im}(\alpha) = \emptyset$ we have $|X \setminus \text{im}(\alpha)| = |X|$. □

We use the above lemma to prove a more general result for linearly ordered countably infinite sets which contain elements that are not limit points.

LEMMA 2.3. *Let X be a countable linearly ordered set. Then there exists an order preserving injection α from X to X such that $|X \setminus \text{im}(\alpha)| = |X| (= |\text{im}(\alpha)|)$.*

Proof. There are two cases to consider.

Case 1. *There is an infinite sequence of consecutive discrete points in X .* It is clear that any such sequence is either strictly increasing or strictly decreasing. Without loss of generality, we assume that $\{y_1 < y_2 < \dots\} \subseteq X$ is an increasing sequence of consecutive discrete points. Define α from X to X by

$$x\alpha = \begin{cases} y_{2i} & x = y_i \ (i \in \mathbb{N}) \\ x & x \notin \{y_1, y_2, \dots\}. \end{cases}$$

The map α is an order preserving injection and $X \setminus \text{im}(\alpha) = \{y_1, y_3, \dots\}$.

Case 2. *No infinite sequence of consecutive discrete points exists.* Without loss of generality assume that X has no smallest or largest point. (Indeed, if X had a smallest point, say, then it would start as $x_1 < x_2 < \dots < x_k < l$, where each x_i is a discrete point and l is a left isolated point. But then we can consider $X \setminus \{x_1, \dots, x_k, l\}$.)

Consider an arbitrary right isolated point r . Then there exists x_1 such that $r < x_1$ and $(r, x_1) = \emptyset$. Clearly, x_1 is either a discrete point or a left isolated point. If x_1 is discrete then there exists $x_2 > x_1$ such that $(x_1, x_2) = \emptyset$. By assumption we can continue this for only finitely many steps to obtain a finite sequence $r, x_1, x_2, \dots, x_k, l$ where each x_i is a discrete point and l is left isolated. Thus for every right isolated $r \in X$ there exists a corresponding left isolated $l \in X$ such that the interval (r, l) is finite. Let ρ be the equivalence relation with equivalence classes $\{t\}$, where t is a limit point, and $\{r, x_1, x_2, \dots, x_k, l\}$ where x_i is a discrete point for each $i \in \{1, 2, \dots, k\}$, l is left isolated and r is right isolated. Let $\overline{X} = X/\rho$, the quotient of X by ρ . The order on X induces a (linear) order on \overline{X} :

$$x/\rho \leq y/\rho \text{ if and only if } x/\rho = y/\rho \text{ or } x' < y' \text{ for all } x' \in x/\rho, y' \in y/\rho.$$

We claim that every point x/ρ of \overline{X} is a limit point. We have two cases to consider, when $x/\rho = \{x\}$ and when $x/\rho = \{r, x_1, x_2, \dots, x_k, l\}$, for some $k \geq 0$.

In the first case, we have that x is a limit point in X . Let $y \in X$ with $y/\rho < x/\rho$. We show that $(y/\rho, x/\rho) \neq \emptyset$. From the definition of the order on \overline{X} we have $y < x$ and so $|(y, x)| = |X|$. It follows that there exists $z \in (y, x)$ such that $z \neq x$ and $z \notin y/\rho$, since y/ρ is finite. Since x/ρ is a singleton this implies that $z/\rho \in (y/\rho, x/\rho)$ and so $(y/\rho, x/\rho) \neq \emptyset$, as required. An analogous argument shows that for any $z \in X$ with $x/\rho < z/\rho$ we have $(x/\rho, z/\rho) \neq \emptyset$. It follows that x/ρ is a limit point.

In the second case, note that for any $y, z \in X$ such that $y/\rho < x/\rho < z/\rho$ we have $y < r < l < z$. Since r is a right isolated point it follows that (y, r) is infinite. Again, since y/ρ and r/ρ are finite there exists $t \in (y, r)$ such that $t \notin y/\rho$ and $t \notin r/\rho$. This implies that $t/\rho \in (y/\rho, r/\rho) = (y/\rho, x/\rho)$ and so $(y/\rho, x/\rho) \neq \emptyset$, as required. An analogous argument shows that $(x/\rho, z/\rho) \neq \emptyset$. It follows that x/ρ is a limit point.

We now label each element of \overline{X} according to the size of its class, so that we may apply Lemma 2.2. More precisely, we define $\lambda : \overline{X} \rightarrow \mathbb{N}$ by

$$(x/\rho)\lambda = |x/\rho|.$$

By Lemma 2.2 there exists an order preserving bijection $\bar{\alpha}$ from \bar{X} to \bar{X} satisfying

$$(x/\rho)\bar{\alpha}\lambda \geq (x/\rho)\lambda,$$

such that $\bar{X} \setminus \text{im}(\bar{\alpha})$ is infinite. We shall now ‘lift’ the function $\bar{\alpha}$ to a function $\alpha : X \rightarrow X$ as follows:

$$x\alpha = \begin{cases} y & \text{if } x \text{ is a limit point and } (x/\rho)\bar{\alpha} = y/\rho = \{y\} \text{ where } y \text{ is a limit point,} \\ r & \text{if } x \text{ is a limit point and } (x/\rho)\bar{\alpha} = \{r, x_1, x_2, \dots, x_k, l\}, \\ x'_i & \text{if } x = x_i \text{ in } \{r = x_0, x_1, \dots, x_k, x_{k+1} = l\} \in \bar{X} \text{ and} \\ & (x/\rho)\bar{\alpha} = \{r' = x'_0, x'_1, \dots, x'_s, x'_{s+1} = l'\} \text{ for } s \geq k. \end{cases}$$

In the final case, since $\bar{\alpha}$ satisfies $(x/\rho)\bar{\alpha}\lambda = |(x/\rho)\bar{\alpha}| \geq |x/\rho| = (x/\rho)\lambda$, it is clear that, under $\bar{\alpha}$, the image of $x/\rho = \{r = x_0, x_1, \dots, x_k, x_{k+1} = l\}$ must be a set with at least $k + 2$ elements. Note that, $x\alpha \in (x/\rho)\bar{\alpha}$ for every $x \in X$.

For arbitrary $x, y \in X$, with $x < y$, we show that $x\alpha < y\alpha$ and hence α is order preserving and injective. There are two cases to consider. Firstly, if $x/\rho \neq y/\rho$ then $(x/\rho)\bar{\alpha} < (y/\rho)\bar{\alpha}$, since $\bar{\alpha}$ is order preserving and injective. It follows from the definition of the order on \bar{X} that $z < t$, for all $z \in (x/\rho)\bar{\alpha}$ and for all $t \in (y/\rho)\bar{\alpha}$, and hence $x\alpha < y\alpha$. Secondly, if

$$x/\rho = \{r = x_0, x_1, \dots, x_k, x_{k+1} = l\} = y/\rho$$

then $x = x_i$ and $y = x_j$ for $i < j$. By definition we have

$$(x/\rho)\bar{\alpha} = \{r' = x'_0, x'_1, \dots, x'_s, x'_{s+1} = l'\} = (y/\rho)\bar{\alpha},$$

where $s \geq k$ and $x\alpha = x'_i < x'_j = y\alpha$, as required. Note that $\text{im}(\alpha) \subseteq \bigcup \text{im}(\bar{\alpha})$, and since $\bar{X} \setminus \text{im}(\bar{\alpha})$ is infinite, it follows that $X \setminus \text{im}(\alpha)$ is infinite. □

The next two lemmas allow us to use methods similar to those in the proof of [8, Example 1.6] to encode an arbitrary map into an order preserving map.

LEMMA 2.4. *Let Y be a countably infinite linearly ordered set. Then there exists $Z \subseteq Y$ such that either $Z \cong \mathbb{Z}^+$ or $Z \cong \mathbb{Z}^-$*

Proof. There are two cases to consider.

Case 1. *All the points in Y are discrete points.* We construct Z as follows. Let $z_1 \in Y$ be arbitrary. Then at least one of the sets $\{y \in Y : y > z_1\}$ or $\{y \in Y : y < z_1\}$ is infinite. Without loss of generality we assume that

$$|\{y \in Y : y > z_1\}| = |Y|.$$

We may choose z_2, z_3, \dots so that $(z_1, z_2) = \emptyset$, $(z_2, z_3) = \emptyset$, etc. Then $Z = \{z_1, z_2, z_3, \dots\} \cong \mathbb{Z}^+$.

Case 2. *There exists a left isolated point, a right isolated point or a limit point in Y .* Without loss of generality, we assume that there is a right isolated point $r \in Y$. Let $z_1 < r$ be arbitrary, since r is right isolated we have $(z_1, r) \neq \emptyset$. Hence we may choose $z_2 \in (z_1, r)$. Continue to choose $z_3 \in (z_2, r)$, $z_4 \in (z_3, r)$, etc. Then $Z = \{z_1, z_2, z_3, \dots\} \cong \mathbb{Z}^+$. □

LEMMA 2.5. *Let X be a countable linearly ordered infinite set. Let $Z \subseteq X$ be such that $Z \cong \mathbb{Z}^+$ or $Z \cong \mathbb{Z}^-$, and let α be an order preserving map from Z to Z . Then there exists $\beta \in \mathcal{O}_X$ such that $\beta|_Z = \alpha$.*

Proof. We assume, without loss of generality, that $Z = \{z_1 < z_2 < \dots\} \cong \mathbb{Z}^+$ and let α be an order preserving map from Z to Z . We define $\beta \in \mathcal{T}_X$ by

$$x\beta = \begin{cases} x & \text{if } x < z_1, \\ z_i\alpha & \text{if } x \in [z_i, z_{i+1}), \\ x & \text{if } x > z_i, \text{ for every } i \in \mathbb{N}. \end{cases}$$

It is easy to verify that β is order preserving and $\beta|_Z = \alpha$. □

We are now in a position to prove the main result of this section.

Proof of Theorem 2.1. Let $Y \subseteq X$ such that $|Y| = |X \setminus Y| = |X|$ and let β be an order preserving bijection from X to $X \setminus Y$; these exist by Lemma 2.3. Assume, without loss of generality, that there exists $Z \subseteq Y$ such that $Z \cong \mathbb{Z}^+$ and let $Z = \{z_1 < z_2 < \dots\}$, as described in Lemma 2.4. Since $X \setminus Y$ and Z have the same cardinality there exists a bijection ϵ from $X \setminus Y$ to Z . Note that $\beta\epsilon$ is a bijection from X to Z . Let δ be any mapping from Z to X such that

$$z_{p_k}^j \delta = z_k \epsilon^{-1} \beta^{-1},$$

where p_k is the k th prime and $j \in \mathbb{N}$ arbitrary. Let $\Delta \in \mathcal{T}_X$ be any mapping such that

$$x\Delta = \begin{cases} x\epsilon & \text{if } x \in X \setminus Y \\ x\delta & \text{if } x \in Z \\ \text{arbitrary} & \text{if } x \in Y \setminus Z \end{cases}$$

Let $\alpha \in \mathcal{T}_X$ be arbitrary. We show that α can be generated using Δ and elements of \mathcal{O}_X . We define γ from Z to Z inductively. First we define γ on z_1 , so that if $z_1 \epsilon^{-1} \beta^{-1} \alpha \beta \epsilon = z_k$ then $z_1 \gamma = z_{p_k}$. For $t > 1$ we assume that γ is defined and order preserving on z_1, \dots, z_{t-1} . To define $z_t \gamma$ we first let

$$M = \max\{i : z_i \in \{z_1, \dots, z_{t-1}\} \gamma\}.$$

Since $\beta\epsilon$ is a bijection from X to Z there exists $x \in X$ such that

$$z_t \epsilon^{-1} \beta^{-1} = x$$

and there exists $z_s \in Z$ such that $(x\alpha)\beta\epsilon = z_s$. We choose $j \in \mathbb{N}$ such that $p_s^j > M$ and we define $z_t \gamma = z_{p_s^j}$. Our choice of j ensures that γ is order preserving. Let $\eta \in \mathcal{T}_X$ be an extension of γ to an element of \mathcal{O}_X , as described in Lemma 2.5.

We claim that

$$\alpha = \beta\Delta\eta\Delta.$$

Let $x \in X$ be arbitrary. Since $\beta\Delta$ is a bijection (from X to Z) there exists a unique element $z_i \in Z$ such that $x\beta\Delta = x\beta\epsilon = z_i$. Analogously, there exists a unique element $z_s \in Z$ such that $x\alpha\beta\Delta = x\alpha\beta\epsilon = z_s$. Hence

$$x\beta\Delta\eta\Delta = x\beta\epsilon\eta\Delta = z_i\eta\Delta = z_i\gamma\Delta = z_{p_s^j}\Delta = z_{p_s^j}\delta = z_s\epsilon^{-1}\beta^{-1} = x\alpha.$$

We have shown that $\alpha \in \langle \mathcal{O}_X, \Delta \rangle$ and so $\text{rank}(\mathcal{T}_X : \mathcal{O}_X) = 1$. □

3. Well-ordered sets. In this section we extend the result of the previous section to well-ordered sets of arbitrary cardinality. Recall that an ordered set (X, \leq) is *well-ordered* if every subset of X contains a least element. We start by introducing some standard results concerning well-ordered sets which we shall use later. For more details see [5], [12] or [13].

For an arbitrary $x \in X$ we call the set $s(x) = \{y \in X : y < x\}$ the *initial segment* of x . It is well-known that for any two well-ordered sets X and Y either X is isomorphic to Y , X is isomorphic to an initial segment of Y or Y is isomorphic to an initial segment of X ; see for example [12, Theorem 1]. This induces a natural (well) ordering on the class of all well-ordered sets, so that

$$X \leq Y \text{ if and only if } X \cong Y \text{ or } X \text{ is isomorphic to an initial segment of } Y. \quad (1)$$

A natural reformulation of this result relates a well-ordered set to its subsets.

PROPOSITION 3.6. *Each subset of a well-ordered set X is either isomorphic to X or to an initial segment of X .*

Another useful and natural consequence of the order on the class of all well-ordered sets is:

PROPOSITION 3.7. *No well-ordered set is isomorphic to an initial segment of itself.*

For a proof see [12, Lemma 2.2].

In light of the previous section a natural question to ask is whether, or not, there exists an uncountable set X for which the relative rank of \mathcal{T}_X modulo \mathcal{O}_X is countable? The next result answers this question in the affirmative.

LEMMA 3.8. *Let X be an arbitrary infinite set and let Ω denote the least well-ordered set of cardinality $|X|$. Then the relative rank of \mathcal{T}_Ω modulo \mathcal{O}_Ω is one.*

Proof. Let Ω_x ($x \in \Omega$) denote subsets of Ω such that $|\Omega_x| = |\Omega|$ and $\Omega_x \cap \Omega_y = \emptyset$ whenever $x \neq y$. Since each Ω_x has cardinality $|X|$ and Ω is the smallest well-ordered set of cardinality $|X|$, it follows from Proposition 3.6 and Proposition 3.7 that $\Omega_x \cong \Omega$, for all $x \in \Omega$. Fix a mapping $\mu \in \mathcal{T}_\Omega$ such that

$$\Omega_y \mu = y \quad (y \in \Omega).$$

For an arbitrary $\alpha \in \mathcal{T}_\Omega$, we define a map $\beta \in \mathcal{O}_\Omega$ by transfinite induction as follows. If $x\alpha = y$ then we define $x\beta = z$, where $z \in \Omega_y$ and $z > t\beta$ for every $t < x$. Such an element z exists since $\Omega_y (\cong \Omega)$ is not isomorphic to an initial segment of Ω , by Proposition 3.7. For an arbitrary $x \in \Omega$, if $x\alpha = y$ then we have

$$x\beta\mu = z\mu = y = x\alpha,$$

and so $\alpha \in \langle \mathcal{O}_\Omega, \mu \rangle$. It follows that $\mathcal{T}_\Omega = \langle \mathcal{O}_\Omega, \mu \rangle$ and in particular $\text{rank}(\mathcal{T}_\Omega : \mathcal{O}_\Omega) = 1$, as required. \square

We use this lemma to prove the main result of this section:

THEOREM 3.9. *Let X be an arbitrary well-ordered set. The relative rank of \mathcal{T}_X modulo \mathcal{O}_X is one.*

Proof. Let T be the smallest well-ordered set of cardinality $|X|$. By Proposition 3.6 either there exists an initial segment of X which is isomorphic to T or $X \cong T$. In

the latter case the result follows by Lemma 3.8. In the former case, let Y denote the initial segment of X isomorphic to T and let $Z = X \setminus Y$. By the same argument as in the proof of Lemma 3.8, we may find pairwise disjoint sets $Y_1, Y_2, Y_3 \subseteq Y$ such that $Y = Y_1 \cup Y_2 \cup Y_3$ and $Y_i \cong Y$, for each i . By Lemma 3.8 there exists a map $\epsilon \in \mathcal{T}_{Y_2}$ such that $\langle \mathcal{O}_{Y_2}, \epsilon \rangle = \mathcal{T}_{Y_2}$. We define a map $\sigma : \mathcal{T}_{Y_2} \rightarrow \mathcal{T}_X$ such that for $x \in X$ and $\rho \in \mathcal{T}_{Y_2}$

$$(x)(\rho\sigma) = \begin{cases} x & x \in Z, \\ y\rho & x \in Y, \text{ where } y = \min\{z \in Y_2 : z \geq x\}. \end{cases}$$

In the second case, note that such an element y always exists since Y_2 is well-ordered and is not a subset of an initial segment of Y . It is easy to see that $\mathcal{O}_{Y_2}\sigma \subseteq \mathcal{O}_X$. Since $Y \cong Y_1$ and $y \leq z$, for every $y \in Y$ and for every $z \in Z$, we may define an injection $\beta \in \mathcal{O}_X$ from X to $Y_1 \cup Z$ such that $Y\beta = Y_1$ and $Z\beta = Z$. Since $Y_1 \cup Z$ has the same cardinality as Y_2 we may find a bijection δ from $Y_1 \cup Z$ to Y_2 . We let $\bar{\mu}$ be any order preserving bijection from Y_2 to Y_3 and define $\mu \in \mathcal{T}_X$ by

$$x\mu = \begin{cases} x & x > y, \text{ for all } y \in Y_2; \\ y\bar{\mu} & y = \min\{z \in Y_2 : z \geq x\}. \end{cases}$$

In fact, μ is order preserving. In order to see this, it is enough to note that $\{x : x > y \text{ for all } y \in Y_2\} = Z$ since Y_2 , being isomorphic to Y , is not contained in an initial segment of Y . From the definition it is obvious that μ restricted to either Z or $Y = X \setminus Z$ is order preserving. In addition, $Y\mu = Y_2\mu = Y_3 \subseteq Y$, $Z\mu = Z$ and $y < z$ for all $y \in Y, z \in Z$.

Next, we define a map $\gamma : Y_3 \rightarrow X$ by

$$y\gamma = y\bar{\mu}^{-1}\delta^{-1}\beta^{-1}.$$

Note that the domains of the maps δ, ϵ and γ are disjoint and that their union is X itself. Hence we may define a map $\eta \in \mathcal{T}_X$ by

$$x\eta = \begin{cases} x\delta & x \in Y_1 \cup Z \\ x\epsilon & x \in Y_2 \\ x\gamma & x \in Y_3. \end{cases}$$

We claim that η together with \mathcal{O}_X generates \mathcal{T}_X . Let $\alpha \in \mathcal{T}_X$ be arbitrary. First note that for any $\theta \in \mathcal{T}_{Y_2}$ there exists $\rho \in \langle \mathcal{O}_{Y_2}\sigma, \eta \rangle \subseteq \langle \mathcal{O}_X, \eta \rangle (\subseteq \mathcal{T}_X)$ such that $\rho \upharpoonright_{Y_2} = \theta$. This follows from $(\zeta\sigma) \upharpoonright_{Y_2} = \zeta$ for all $\zeta \in \mathcal{T}_{Y_2}$, $\eta \upharpoonright_{Y_2} = \epsilon$ and $\langle \mathcal{O}_{Y_2}, \epsilon \rangle = \mathcal{T}_{Y_2}$.

In particular, since $\beta\delta$ is a bijection from X to Y_2 , we may find a map $\rho \in \langle \mathcal{O}_X, \eta \rangle$ such that $\rho \upharpoonright_{Y_2} = \delta^{-1}\beta^{-1}\alpha\beta\delta$. Then, for an arbitrary $x \in X$, with $x\beta\delta = y \in Y_2$ we have

$$\begin{aligned} x\beta\eta\rho\mu\eta &= (x\beta\delta)\rho\mu\eta = y\rho\mu\eta = y\delta^{-1}\beta^{-1}\alpha\beta\delta\mu\eta = (x\alpha)\beta\delta\mu\eta \\ &= (x\alpha\beta\delta\mu)\gamma = x\alpha\beta\delta\mu\bar{\mu}^{-1}\delta^{-1}\beta^{-1} = x\alpha, \end{aligned}$$

and so $\alpha \in \langle \mathcal{O}_X, \eta \rangle$ and $\mathcal{T}_X = \langle \mathcal{O}_X, \eta \rangle$. □

4. Concluding remarks. We can extend the main result of the last section to a wider family of linearly ordered sets by means of the following lemma.

LEMMA 4.10. *Let X be an infinite linearly ordered set such that there exists a subset $Y \subseteq X$ with $|Y| = |X|$ and where any order preserving map from Y to Y can be extended to an order preserving map from X to X . Then $\text{rank}(\mathcal{T}_Y : \mathcal{O}_Y) \leq 2$ implies $\text{rank}(\mathcal{T}_X : \mathcal{O}_X) \leq 2$.*

Proof. By assumption, there exist $\epsilon', \delta' \in \mathcal{T}_Y$ such that $\langle \mathcal{O}_Y, \epsilon', \delta' \rangle = \mathcal{T}_Y$, and for every $\alpha \in \mathcal{O}_Y$ there exists $\eta \in \mathcal{O}_X$ such that $\eta|_Y = \alpha$. Let $\epsilon, \delta \in \mathcal{T}_X$ be any mappings such that $\epsilon|_Y = \epsilon'$ and $\delta|_Y = \delta'$. Let $\beta : X \rightarrow Y$ be any bijection and let $\gamma \in \mathcal{T}_X$ be arbitrary. We show that it is possible to generate γ using elements of \mathcal{O}_X and four other mappings. Since $\langle \mathcal{O}_Y, \epsilon', \delta' \rangle = \mathcal{T}_Y$ we see that for any map $\alpha \in \mathcal{T}_Y$ there exists $\mu \in \langle \mathcal{O}_X, \epsilon, \delta \rangle$ such that $\mu|_Y = \alpha$. In particular, there exists $\mu \in \langle \mathcal{O}_X, \epsilon, \delta \rangle$ such that

$$\mu|_Y = \beta^{-1}\gamma\beta.$$

Let ν be any extension of β^{-1} to an element of \mathcal{T}_X . For an arbitrary $x \in X$ if $x\beta = y$ then

$$x\beta\mu\nu = y\mu\nu = y\beta^{-1}\gamma\beta\nu = y\beta^{-1}\gamma = x\gamma.$$

We have shown that $\gamma \in \langle \mathcal{O}_X, \delta, \epsilon, \beta, \nu \rangle$ and so $\mathcal{T}_X = \langle \mathcal{O}_X, \delta, \epsilon, \beta, \nu \rangle$. It follows from [8, Corollary 1.2] that $\text{rank}(\mathcal{T}_X : \mathcal{O}_X) \leq 2$. □

COROLLARY 4.11. *If X is an arbitrary linearly ordered set, such that there exists a well-ordered subset $Y \subseteq X$, with $|Y| = |X|$, then $\text{rank}(\mathcal{T}_X : \mathcal{O}_X) \leq 2$.*

Proof. Let α' be an order preserving map from Y to Y . Then $\alpha \in \mathcal{T}_X$ defined by

$$x\alpha = \begin{cases} x & x > y \text{ for all } y \in Y \\ y\alpha' & y = \min\{z \in Y : z \geq x\}, \end{cases}$$

is an order preserving map from X to X . The result follows by Theorem 3.9 and Lemma 4.10. □

Having found the relative rank of $\mathcal{T}_{\mathbb{N}}$ modulo $\mathcal{O}_{\mathbb{N}}$ and the relative rank of $\mathcal{T}_{\mathbb{Q}}$ modulo $\mathcal{O}_{\mathbb{Q}}$ a natural question to ask is: what is the relative rank of $\mathcal{T}_{\mathbb{R}}$ modulo $\mathcal{O}_{\mathbb{R}}$?

EXAMPLE 4.12. The relative rank of $\mathcal{T}_{\mathbb{R}}$ modulo $\mathcal{O}_{\mathbb{R}}$ is uncountable. We show this by proving that the cardinality of the semigroup of all order preserving mappings on the reals \mathbb{R} is $2^{\aleph_0} < 2^{2^{\aleph_0}} = |\mathcal{T}_{\mathbb{R}}|$. For an arbitrary $\alpha \in \mathcal{O}_{\mathbb{R}}$ we show that α is discontinuous at only countably many points in \mathbb{R} . Let $D = \{x : \alpha \text{ is discontinuous at } x\}$. For $x \in D$, let $a_x = \sup_{t < x} \{t\alpha\}$ and let $b_x = \inf_{t > x} \{t\alpha\}$. Next, define $\beta : D \rightarrow G$, where G is the family of all open subsets of \mathbb{R} , by:

$$x\beta = (a_x, b_x).$$

Observe that the family $\{x\beta : x \in D\}$ consists of non-empty pairwise disjoint open sets, hence

$$|\{x\beta : x \in D\}| \leq \aleph_0.$$

The map β is injective and so

$$|D| = |\{x\beta : x \in D\}| \leq \aleph_0.$$

Next, we show that α is almost determined by its rational points. For $x \in \mathbb{R} \setminus \mathbb{Q}$ define $s_x = \sup_{q \in \mathbb{Q}} \{ q\alpha : q < x \}$ and $t_x = \inf_{q \in \mathbb{Q}} \{ q\alpha : q > x \}$ and observe that

$$x\alpha \in [s_x, t_x],$$

since α is an order preserving map. Since α is discontinuous at only countably many points there are only countably many intervals $[s_x, t_x]$ which are not singletons. It follows that there are only 2^{\aleph_0} maps in $\mathcal{O}_{\mathbb{R}}$.

The cardinality of the set \mathcal{P} of all order preserving mappings $\mathbb{Q} \rightarrow \mathbb{R}$ is 2^{\aleph_0} . Since every element α of $\mathcal{O}_{\mathbb{R}}$ is almost determined by $\alpha|_{\mathbb{Q}} \in \mathcal{P}$ it follows that $|\mathcal{O}_{\mathbb{R}}| = 2^{\aleph_0}$ too.

We conclude the paper with the following two questions:

OPEN PROBLEM 4.13. *Is it true that if $\text{rank}(\mathcal{T}_X : \mathcal{O}_X) \leq 2$ for a linearly ordered set X then there exists $Y \subseteq X$ such that $|X| = |Y|$ and Y , or Y^R (the set Y with the order reversed), is well-ordered?*

OPEN PROBLEM 4.14. *Does there exist an infinite linearly ordered set X such that $\text{rank}(\mathcal{T}_X : \mathcal{O}_X) = 2$?*

ACKNOWLEDGEMENTS. The authors are grateful to an anonymous referee for his/her comments. The second and third authors would also like to thank Dr Lars Olsen for helpful discussions. The third author acknowledges partial support from INTAS 99/1224.

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