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STABILITY AND CYCLES IN A COBWEB MODEL WITH HETEROGENEOUS EXPECTATIONS

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We investigate the dynamics of a cobweb model with heterogeneous beliefs, generalizing the example of Brock and Hommes (1997). We examine situations where the agents form expectations by using either rational expectations, or a type of adaptive expectations with limited memory defined from the last two prices. We specify conditions that generate cycles. These conditions depend on a set of factors that includes the intensity of switching between beliefs and the adaption parameter. We show that both Flip bifurcation and Neimark–Sacker bifurcation can occur as primary bifurcation when the steady state is unstable.

Keywords: Bounded rationality, Cobweb model, Flip bifurcation, Neimark–Sacker bifurcation

1. INTRODUCTION

In relation to economic modeling, there has been a lengthy and continuing debate about formation of expectations. Although the rational expectations hypothesis plays a major role in dynamic macroeconomic research, papers that model expectations relaxing that assumption are increasing, but few of these investigate the dynamics in any detail. The cobweb model of Brock and Hommes (1997) first gave a satisfying exposition on both accounts, that is, a rigorous

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foundation for heterogeneous beliefs and a systematic dynamical study. The expectation formation arises from rational choice between various costly forecasts. The concept of adaptively rational equilibrium dynamics (ARED), in which market equilibrium dynamics is coupled to the choice of prediction of learning strategies, is introduced. Brock and Hommes then showed that this type of expectation formation can generate inherent instability for the ARED, leading to possible complex motions. The present paper further develops this approach by considering a different set of forecasts and aims at characterizing such instability.

Over the past decade, a growing number of papers have dealt with the role of heterogeneous expectations in generating instability (Chiarella and He, 1998, 2001; Franke and Neseman, 1999; Goeree and Hommes, 2000; Hommes, 1991). While economic implications of these studies are obvious for some specific markets,[1] most papers, including ours, are based on the simple cobweb model, as it is one of the most tractable models involving market dynamics.

The framework and the economic import of these papers, including ours, are close to those of Brock and Hommes (1997).[2]

Let us first consider the framework. Expectation formation is modeled as a rational economic decision. Indeed, producers choose between two methods of predicting prices depending on their performance, namely a costly sophisticated predictor and a costless unsophisticated predictor.[3] The predictor's performance is defined as the net realized profits in the most recent period less the cost associated with using the predictor. Depending on this performance, each producer may at every period switch from one predictor to another. For producers as a whole, this switching process, which is perfectly endogenous, may occur at various levels of intensity.

Let us now turn to the economic meaning of this class of models (Branch, 2002; Brock and Hommes, 1997; Lasselle et al., 2003). Under the previous assumptions on the expectation formation and the ARED concept, the instability of the steady state is generated by a simple but powerful mechanism which can be intuitively described as follows.

On the one hand, when the price is close to its steady-state value, very few agents use the most sophisticated predictor, since its cost exceeds the benefits of its forecast. Therefore, the distance between the current price and its steady-state value grows large over time.

On the other hand, while its cost is significant, the sophisticated predictor provides a better net return when the current price is far from its steady-state value. Thus, the distance between the two prices gets smaller over time.

Let us illustrate this mechanism[4] in the model of Brock and Hommes (1997). Suppose that at time t the current price is close to but greater than its steady-state value and the vast majority of agents use the naïve expectations predictor. As a result, the supply at $t+1$ is mainly evaluated from p_t, but the demand is computed from the current price at $t+1$. As the dynamics in the cobweb model is inherently oscillatory, the current price at $t+1$ will be less than the steady-state value. The

same reasoning is true for the following period. The current price at $t + 2$ will be greater than its steady-state value.

Consequently, price oscillations are endogenously generated in the steady-state neighborhood.

The immediate steps in research can then be either to look for stability conditions for convergence of the price dynamics and their consequences in the model, as did Branch (2002), or to characterize the steady-state instability, as pioneered by Brock and Hommes (1997). It is indeed well known that any complete dynamical analysis should begin with that characterization, as it can lead to complicated dynamical phenomena studied from bifurcation. When a bifurcation occurs, the qualitative properties of the dynamical system in the vicinity of steady state have been modified following a small change in value of one of the parameters of the model. At the critical value of the parameter, there exists one steady state. However, if the parameter increases beyond that critical value, even if the perturbation is small, then there exist cycles.

Brock and Hommes (1997) showed that this mechanism could lead to highly complex dynamics. They focused on a bifurcation route to chaos. On this route, the primary bifurcation can only be a Flip bifurcation; that is, the equilibrium time paths exhibit attracting cycles of period 2.

The main contribution of our paper is to show that this mechanism can lead not only to the possibility of stable cycles of period two, but also to attracting limit cycles through primary bifurcations. Indeed, we show that when the steady state is unstable, supercritical Flip bifurcation as well as supercritical Neimark–Sacker bifurcation can occur for a set of parameters. The existence of these two types of attracting cycles is directly linked to our definition of the expectation functions.

While Brock and Hommes (1997) assume that costly rational expectations are competing with costless naïve expectations, we replace the latter by costless adaptive expectations. More precisely, we assume that adaptive expectations are a weighted average of the last two prices. This relationship, which is crucial for our results, is a reasonable alternative possibility to the relationships assumed by Brock and Hommes (1997) and by Branch (2002), for instance. It was already present in Hommes's cobweb model (1998) with homogenous and adaptive expectations.[5] Allowing for adaptive expectations, we consider its implications using the evolutionary framework of Brock and Hommes (1997).[6]

The costless adaptive predictor used by us is more sophisticated than the naïve one but is still relatively unsophisticated. It may be a reasonable forecasting strategy for boundedly rational agents in some situations,[7] such as those in which the marginal expected gains from more refined prediction methods gains exceed the extra cost of these methods. According to proponents[8] of Bounded Rationality Theory, such as Simon (1957) or Baumol and Quandt (1964), it may be justified as follows. First, as suggested by Simon (1957), individuals have a limited capacity to store and process information. They can lose or forget information quickly.

We can then imagine that beyond two periods they do not keep the information about prices. Second, agents could also believe that the prices observed more than two periods ago would have no impact (or so little impact) on future prices that it is not necessary to take account of that information. Third, one could conjecture that the extra cost in keeping and taking that information into account would exceed the extra benefit to be obtained. Therefore, it would be "economically rational" not to take these earlier prices into account in the prediction function.

Given the existing literature derived from Brock and Hommes (1997), our model allows us to derive two new results.

First, the model of Brock and Hommes (1997) becomes a special case of our model. Indeed, the naïve expectations they consider correspond to our adaptive expectations when all weight is put on the most recent price. As we consider an expectation function with two lags, the dimension of the dynamical system of our model increases from 2 to 3. Due to this change, we are able to demonstrate the existence of a new type of primary bifurcation, namely a primary Neimark–Sacker bifurcation.[9]

Second, our conclusion is more cautious than that of Branch (2002). Branch (2002) considers a more generalized setting than Brock and Hommes (1997) and us. Indeed, he examines in detail the stability properties of the cobweb model when agents can choose between three predictors: the rational expectations predictor, the naïve predictor, and adaptive beliefs. On pp. 77–78, he studies a model close to ours where agents choose between a costly predictor and a costless adaptive predictor defined as a weighted average of the most recent price and the most recent forecast. This scheme requires as much memory as our scheme based on a weighted average of the two most recent prices. One of his main conclusions (Theorem 8, p. 77) states that the stability conditions of the steady state are broader when adaptive expectations put "enough" weight on the past. As our model is simpler than his, our conclusion is more specific. First, the stability zone is wider when the agents base their adaptive expectations on both past prices with more weight on the most recent price. In other words, the "size" of the stability region is nonmonotonic in the adaption parameter. Second, the instability of the steady state may lead to stable cycles. On the one hand, these cycles may appear when the agents put "enough" weight on the current price (cycles occurring through a Flip bifurcation). On the other hand, stable cycles can also occur when the agents put "reduced" weight on the most recent price (cycles occurring through the Neimark–Sacker bifurcation). We conclude that the adaptive predictor is stabilizing relative to naïve expectations and there exists a critical parameter value related to the switching process that can induce a bifurcation regardless of the weight on past information in the adaptive predictor.

The paper is organized as follows. The cobweb model and its dynamics under rational versus adaptive expectations are presented in Section 2. The stability conditions of the steady state and of periodic equilibria are stated in Section 3. Section 4 concludes.

2. THE COBWEB MODEL WITH RATIONAL VERSUS ADAPTIVE EXPECTATIONS

We present an extension of the model of Brock and Hommes (1997) that focuses on the case of rational versus naïve expectations. The only two changes to their framework are the following. On the one hand, we consider the introduction of an adaptive expectation function with two lags rather than naïve expectations. On the other hand, the analysis is based on the relative number of agents using rational expectations compared to the number of agents using adaptive expectations, denoted by n_1. Although the second change is just a matter of presentation, the first change, through small, leads to significant differences in results. To make the results comparable with these of Brock and Hommes (1997), we follow their setup closely.

Supply decisions are made by choosing the output that maximizes expected profits subject to the one-period production lag. That is,

$$\max_q \left[p^e_{t+1}\, q - c(q) \right],$$

where $c(q)$ is the cost function, which is increasing in q.

Price expectations, p^e_{t+1}, are formed by choosing a predictor from a set of expectation functions. Given this heterogeneity in expectation formation, market supply is a weighted sum of the supply decisions of the heterogeneous agents. The weights are simply the proportion of agents using a specific predictor. That is, in our model each agent chooses between two predictors, $H_j \in \{H_1,\ H_2\}$, where each predictor depends upon a vector of past prices $\mathbf{P}_t = (p_t,\ p_{t-1}, \ldots, p_0)$. The fractions of agents using one of the two predictors, $n_{j,t}(p_t, \mathbf{H}(\mathbf{P}_{t-1}))$ depend on the current price and on the vectors of previous predictors:

$$\mathbf{H}(\mathbf{P}_{t-1}) = (H_1(\mathbf{P}_{t-1}),\, H_2(\mathbf{P}_{t-1})).$$

Therefore, market equilibrium is given by the equation

$$D(p_{t+1}) = \sum_{j=1}^{2} n_{j,t}(p_t, \mathbf{H}(\mathbf{P}_{t-1}))\, S(H_j(\mathbf{P}_t)),$$

where $D(.)$ is the demand function and $S(.)$ is the supply function.

To keep the model analytically tractable, we assume linear demand and supply. Therefore let $D(p_t) = F - Bp_t$ be the demand and $S(H_j(\mathbf{P}_t)) = bH_j(\mathbf{P}_t)$ be the supply, with $F, B, b \in R_+$.

Without loss of generalization of the stability properties, we set F equal to zero. Market equilibrium is determined by the condition

$$D(p_{t+1}) = n_{1,t}\, S(H_1(\mathbf{P}_t)) + n_{2,t}\, S(H_2(\mathbf{P}_t)), \tag{1}$$

where the two predictor functions are defined as

$$H_1(\mathbf{P}_t) = p_{t+1} \quad \text{with cost } C \geq 0, \tag{2}$$

$$H_2(\mathbf{P}_t) = \tau p_t + (1-\tau)p_{t-1} \quad \text{with } 0 < \tau < 1 \text{ and no cost.} \tag{3}$$

Each period, after observing the new price and assessing the accuracy of their forecasts, producers update their prediction of the next period's price. The evolution of the proportion of agents using a particular predictor is given by

$$n_{j,t+1} = \mathrm{Exp}(\beta U_{j,t+1}) \Big/ \sum_{j=1}^{2} \mathrm{Exp}(\beta U_{j,t+1}). \tag{4}$$

$U_{j,t+1}$ is a measure of the welfare associated with a certain predictor.

The variable β parameterizes preferences over profits. The larger β, the more likely a producer is to switch to an expectation with slightly higher returns. Brock and Hommes call this the "intensity of choice" parameter. Assume that the measure of the welfare is equal to realized net profits in the last period; then we obtain

$$U_{j,\,t+1} = \pi_j[p_{t+1},\ \mathbf{H}(\mathbf{P}_t)],$$

where

$$\pi_j[p_{t+1},\ \mathbf{H}(\mathbf{P}_t)] = p_{t+1}S[H_j(\mathbf{P}_t)] - c\{S[H_j(\mathbf{P}_t)]\} - C_j.$$

C_j is the fixed cost associated with H_j. The cost of production is a simple quadratic cost function $c(q) = q^2/(2b)$. The profit functions for producers using each predictor are respectively

$$\pi_1(p_{t+1},\ p_{t+1}) = \frac{b}{2}p_{t+1}^2 - C, \tag{5}$$

$$\pi_2(p_{t+1},\ p_t,\ p_{t-1}) = \frac{b}{2}[\tau p_t + (1-\tau)p_{t-1}][2p_{t+1} - (\tau p_t + (1-\tau)p_{t-1})]. \tag{6}$$

Plugging these into (4) leads to the law of motion for the two predictors,

$$n_{1,\,t+1} = \mathrm{Exp}\left[\beta\left(\frac{b}{2}p_{t+1}^2 - C\right)\right] \Big/ Z_{t+1}, \tag{7}$$

$$n_{2,t+1} = \mathrm{Exp}\left\{\beta\frac{b}{2}\,[\tau p_t + (1-\tau)p_{t-1}]\,[2p_{t+1} - (\tau p_t + (1-\tau)p_{t-1})]\right\} \Big/ Z_{t+1}, \tag{8}$$

where $Z_{t+1} = \sum_{j=1}^{2} \mathrm{Exp}(\beta\pi_{j,t+1})$ and $n_{1,\,t+1} + n_{2,\,t+1} = 1$.

The cobweb model with rational and adaptive expectations is a system (S) of nonlinear difference equations that governs the law of motion of price,

$$p_{t+1} = \phi(p_t,\ p_{t-1},\ n_{1,t}), \tag{9}$$

and the law of motion of the proportion of agents using the rational expectation predictor,

$$n_{1,t+1} = \varphi(p_t,\ p_{t-1},\ n_{1,t}), \tag{10}$$

where

$$\phi(p_t,\ p_{t-1},\ n_{1,t}) = A(n_{1,t})\,[\tau p_t + (1-\tau)\,p_{t-1}]\,,$$
$$A(n_{1,t}) = b(n_{1,t} - 1)/(B + b\,n_{1,t}),$$

and

$$\varphi(p_t,\ p_{t-1},\ n_{1,t}) = \frac{1}{1 + \mathrm{Exp}\left(-\frac{\beta}{2}\{b[(\tau p_t + (1-\tau)p_{t-1})^2(A(n_{1,t}) - 1)^2] - 2C\}\right)}.$$

Since (9) and (10) are, respectively, a second-order difference equation and a first-order difference equation, the system (S) can be rewritten as a system of three first-order difference equations (S′):

$$h_{t+1} = p_t, \tag{11}$$

$$p_{t+1} = \phi(h_t,\ p_t,\ n_{1,t}), \tag{12}$$

$$n_{1,t+1} = \varphi(h_t,\ p_t,\ n_{1,t}). \tag{13}$$

The stability or the instability of the steady state issued from the system (S′) formed by equations (11), (12), and (13) can be directly investigated by looking at the Jacobian matrix of (S′) taken at the steady state. These stability properties will be studied in the following section.

3. STABILITY AND CYCLES

A simple computation shows that the system (S′) has a unique steady state $E = (0,\ 0,\ \bar{n}_1(\beta) = 1/[1 + \mathrm{Exp}(\beta\, C)])$. To ease the presentation, let us assume that $C = 0$ or $C = 1$. When $C = 0$, the agents have free access to the sophisticated predictor.

Remark. $\partial A(\bar{n}_1(\beta))/\partial\,\beta < 0$. The proof is left to the reader.

PROPOSITION 1. *Assume that the slopes of the supply and the demand satisfy* $b/B > 1$. *When the information costs are nil, the steady state is* $E = (0,\ 0,\ \bar{n}_1(\beta) = 1/2)$ *and is always locally asymptotically stable.*

The proof is left to the reader.

PROPOSITION 2 (Local Stability of the Steady State). *Let* $b/B>1$ *and* $C=1$. *There exists a unique* $E = (0, 0, \bar{n}_1(\beta))$, *where* $\bar{n}_1(\beta) = 1/(1 + \operatorname{Exp}\beta)$ *with the following properties*:

(i) $\exists\, \beta_1$ *such that*:
  (a) $\forall\, 0 \le \beta < \beta_1$ *and* $\forall\, 2/3 < \tau < 1$, *E is locally asymptotically stable.*
  (b) $\forall\, \beta > \beta_1$ *and for all* $2/3 < \tau < 1$, *E is locally unstable.*

(ii) $\exists\, \beta_2$ *such that*:
  (a) $\forall\, 0 \le \beta < \beta_2$ *and* $\forall\, \tau \in (0, 1/2 \cup 1/2, 2/3)$, *E is locally asymptotically stable.*
  (b) $\forall\, \beta > \beta_2$ *and* $\forall\, \tau \in (0,\ 1/2) \cup (1/2, 2/3)$, *E is locally unstable.*

Proof. See Appendix. ■

PROPOSITION 3 (Primary Bifurcations of the Steady State). *Let* $b/B > 1$ *and* $C = 1$.

(i) *Fix* $2/3 < \tau < 1$. *When* $\beta = \beta_1$, *the system undergoes a supercritical Flip bifurcation.*
(ii) *Fix* $\tau \in (0,\ 1/2) \cup (1/2, 2/3)$. *When* $\beta = \beta_2$, *the system undergoes a Neimark–Sacker bifurcation. Moreover, the Neimark–Sacker bifurcation is supercritical on some* $\tau \in (0, 1/2) \cup (1/2, 2/3)$.
(iii) *When* $\tau = 1/2$, *the system is in strong resonance* 1:3.
(iv) *When* $\tau = 2/3$, *the system undergoes a codim-2 bifurcation.*

Proof. See Appendix. ■

The dynamical analysis depends on a set of parameters composed of the adaption parameter, τ, the intensity of choice, β, and the slopes of the demand and the supply, B and b. For specific combinations of these parameters, the steady state can lose its stability, giving birth to periodic equilibria; that is to say, the system undergoes a primary bifurcation and stabilizing fluctuations in prices can appear. We are able to prove analytically when these bifucations arise.

Our propositions enlighten the complex relationship between the adaptation parameter and the intensity of choice in the cobweb model with rational vs. adaptive expectations. On the one hand, as we shall see in some of our forthcoming illustrations, there is a nonmonotonic relation between the "size" of the stability region of the steady state and the adaptation parameter; that is, a higher weight on the most recent price will not necessarily lead to a larger stability region. Indeed, beyond some critical values of the intensity of choice, as more weight is placed on the most recent information, the former must decrease or else the steady state will become locally unstable. In other words, the speed of the movement from one

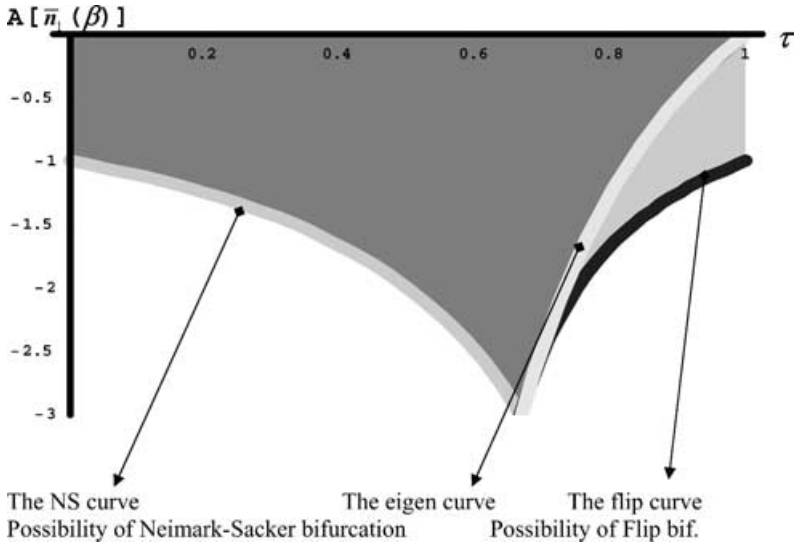

FIGURE 1. Local stability of the steady state.

predictor to the other predictor is balanced with the adaptation parameter. On the other hand, for specific values of the intensity of choice, regardless of the weight on past information in the adaptive predictor, stable cycles in prices can appear.

Consequently, the substitution of naïve expectations by adaptive expectations in the cobweb model with heterogeneous expectations can not only create a more stable environment but also foster the possibility of stabilizing cycles.

The following figures illustrate our propositions and facilitate the understanding of our findings.

Figures 1 and 2 show how the stability of the steady state depends on the parameters values. Up to three curves are drawn: the eigen curve, the flip curve, and the NS curve. On each curve, β is at its critical value, to which is associated a specific value of τ. The eigen curve consists of parameter values for which the eigenvalues of the Jacobian matrix evaluated at the unique steady state change from real to complex. The flip curve consists of parameter values for which one of the eigenvalues is equal to –1. It represents the possibility of Flip bifurcation as a primary bifurcation. The NS curve consists of parameter values for which complex eigenvalues have moduli equal to 1. It represents the possibility of Neimark–Sacker bifurcation as a primary bifurcation. The flip curve and the NS curve intersect when $\tau = 2/3$. Finally, the unique steady state is locally asymptotically stable in the shaded region, where all the moduli of the eigenvalues are less than 1.

In Figure 1 the three curves are plotted in the $(\tau, A[\bar{n}_1(\beta)])$ plane. We choose $A[\bar{n}_1(\beta)]$ for the vertical axis for two reasons. On the one hand, this coefficient allows us to distinguish the two areas where the nonzero eigenvalues are either

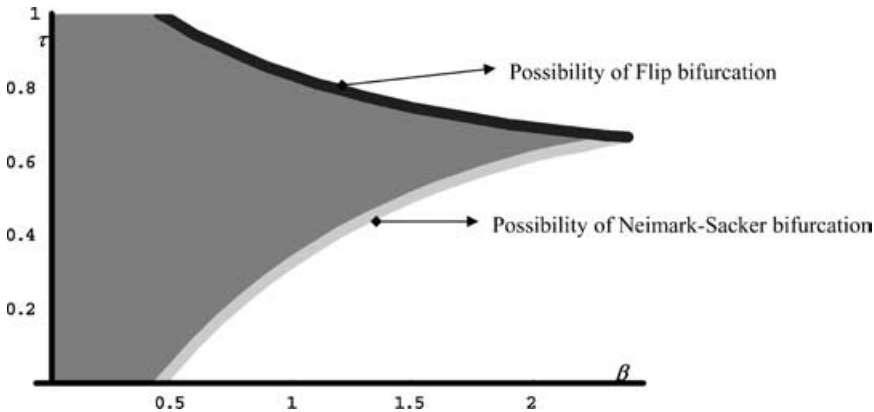

FIGURE 2. Local stability of the steady state when $B = 0.3$ and $b = 1.35$.

real or complex. On the other hand, it is the coefficient in the law of motion of the prices.

We can point out two facts. First, whatever the value of the adaption parameter, the steady state can be asymptotically stable, but the size of the stability of the region is nonmonotonic in τ. Second, the system can undergo a bifurcation, but the possibility of primary bifurcation rests on specific values between τ and β. Indeed the two parameters are jointly dependent; that is, to the critical value of the intensity of choice corresponds a specific value of the adaptive parameter. Let us develop these facts from Figure 2.

Figure 2 illustrates the nonmonotonic relationship between β and τ. It plots the flip and NS curves in the (β, τ)-plane for specific values of the parameters of the demand and the supply, $B = 0.3$ and $b = 1.35$. It shows overall that the adaptive expectations (when $\tau \in (0, 1)$) are less destabilizing for the market than the naïve expectations ($\tau = 1$).

Whatever the value of the adaptation parameter, the unique steady state can be locally asymptotically stable for small values of the intensity of choice. But as the intensity of choice is increasing, there is a need for a more balanced weighting between the two prices to ensure this stability. Note that this "more" balanced weighting is not a completely balanced weighting. Indeed, the most recent information must count for around 2/3 in the adaptive expectations function. So our evolutionary framework adds a new feature. There exists a favorable trade-off between information and the speed of movement between the predictors. More balanced information captured on each side of the steady state can increase the speed of movement between predictors without destabilizing the market.

To illustrate this main point of our paper, let us apply the mechanism described in the Introduction. To begin with, let us remind ourselves of three facts. First, the instability in the cobweb model is characterized by oscillations around its unique

steady state. Second, our cheap predictor rests on two periods, so it captures the most recent information on each side of this unique steady state. Third, adaptive expectations dampen the oscillations.

Now suppose that at time t the current price is close to but greater than its steady-state value and a vast majority of agents use the adaptive expectations predictor. The supply in $t+1$ is mainly evaluated from p_t and p_{t-1}, but the demand is computed from the current price in $t+1$. As the dynamics in the cobweb model is inherently oscillatory, the current price in $t+1$ will be less than the steady-state value. But it will be higher than if naïve expectations were the costless predictor. The same reasoning is true for the following period. The current price in $t+2$ will be greater than its steady-state value, but less than the value that can be found if naïve expectations were the costless predictor. Consequently, price oscillations are more dampened in our model in the steady-state neighborhood. The second parameter of our model, the intensity of choice, which inherently fosters divergent dynamics in the model, can thus increase without damaging the stability. We may say that the process of switching predictors in this model enhances stability of the model.

As we shall see in the following figures, as the set of parameters varies, the local stability of the steady state can be transformed and for fixed sets of parameters it can lead to stabilizing cycles.

Figure 3 assembles several graphs and illustrates Proposition 3(ii). Notably, we can see a limit cycle for specific values of the parameters in the $(p(t-1), p(t))$-plane (recall that $p(t-1) = h(t)$). The initial conditions are $h_0 = 0.2$, $p_0 = 1$, and $n_{1,0} = 0.5$; the parameters are as follows: $\tau = 0.628$, $\beta - 2.11272$, $C = 1$, $B = 0.3$, $b = 1.35$.

One could then wonder what happens to the dynamics of the current price p_t or of the current proportion of agents using the rational expectations predictor $n_{1,t}$ when the intensity of choice β increases (for a given τ).

Let us first consider a value of τ greater than $2/3$. Figures 4a and 4b show the bifurcation diagrams of p (4a) and n_1 (4b) with respect to β for a fixed value of τ ($\tau = 0.8$). As β increases between 1.5 and 10, the system can undergo a variety of period-doubling. The primary bifurcation occurs around $\beta = 1.18$. The unique steady state in prices loses its stability and becomes a cycle of period 2; the uniqueness of n_1 can disappear. As β takes higher values, there is a possibility of periodic attractors.

Let us now consider a lower value of τ ($\tau = 0.6$) and assess the dynamical behavior of our variables as β takes larger and larger values.[10]

Figures 4c and 4d show the bifurcation diagrams of p (4c) and n_1 (4d) with respect to β. As β increases between 1.5 and 10, the system can undergo a variety of bifurcation. The primary bifurcation occurs around $\beta = 1.94$. The unique steady state in prices loses its stability and becomes a limit cycle. The uniqueness of n_1 disappears.

For large values of β, there is a possibility of periodic attractors, as illustrated in the graphs assembled in Figure 5. From these graphs, we can note that the switch

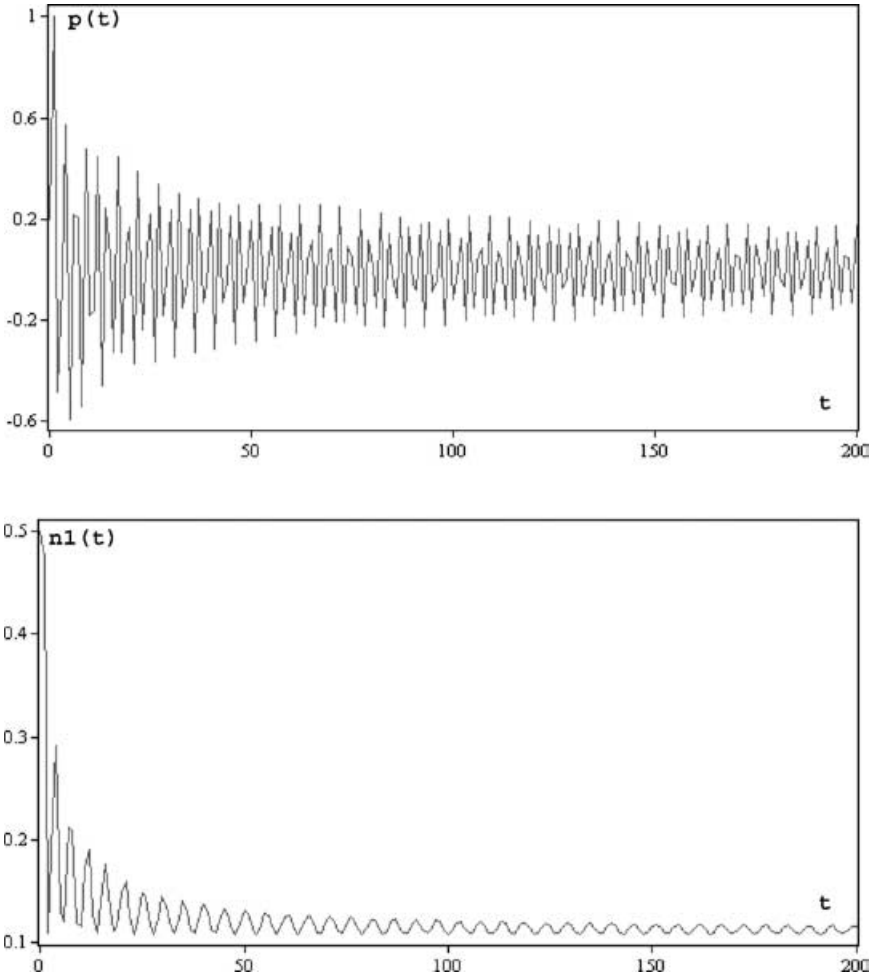

FIGURE 3. Limit cycle in prices and $\bar{n}_1 = 0.107867$, $\tau = 0.628$, $\beta = 2.11272$, $C = 1$, $B = 0.3$, $b = 1.35$.

from the sophisticated predictor to the cheap predictor becomes more and more irregular as the intensity of choice increases.

The phenomena first shown by Brock and Hommes (1997) exists in our model and confirms the possibility of *a rational route to randomness*.

First, as illustrated in the time-series graphs in Figure 5, there exist two different patterns. The first pattern is featured in the vicinity of the steady state. Most agents then use the cheap predictor. As a result, the price dynamics diverges from its steady-state value. They will keep forming their expectations of the future price from the adaptive predictor until it becomes profitable to buy the rational expectations forecast. The second pattern then occurs. Most agents use the rational

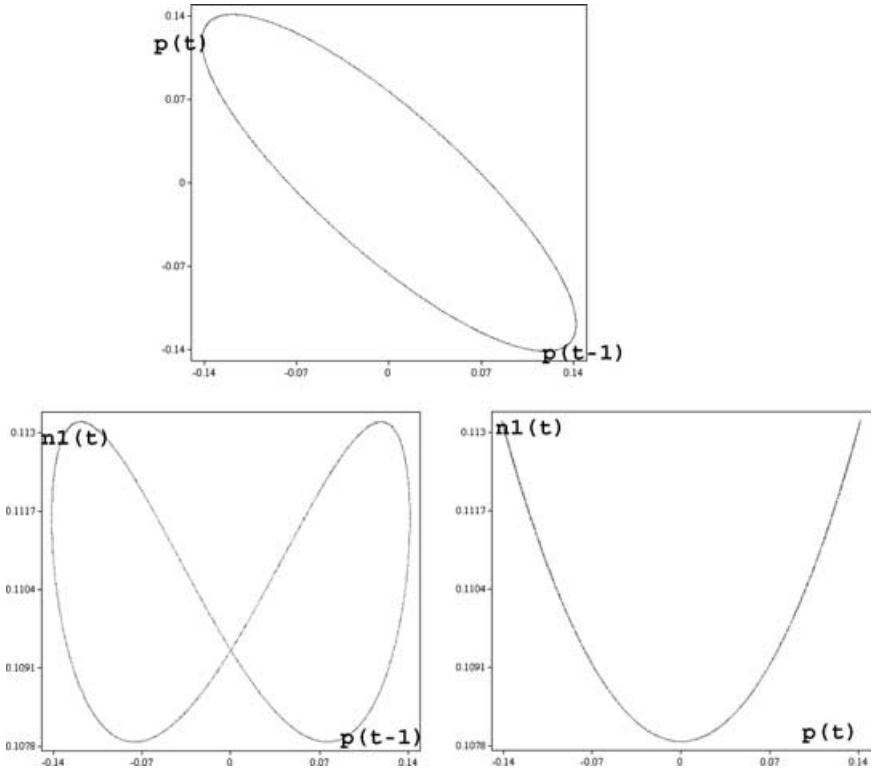

FIGURE 3. Continued.

expectations predictor, causing a speedy convergence towards the steady state. Note that the change between the two patterns is irregular and each pattern is more or less lengthy.

Second, the experiments show that the Lyapunov characteristic exponent is positive for large values of the intensity of choice when the adaptation parameter takes some high or low values, implying the possibility of chaotic behavior in our model.

4. CONCLUDING COMMENTS

Our paper shows how relevant the adaptation parameter can be to the dynamical study of the steady state in the cobweb model with heterogeneous beliefs with evolutionary updating. Associated with a set of parameters (which notably includes the slopes of supply and demand, the intensity of choice between predictors, the cost, and the features of each predictor), we establish the conditions for local stability and instability of the steady state. This allows us to demonstrate the possible emergence of stable cycles. In other words, expectations

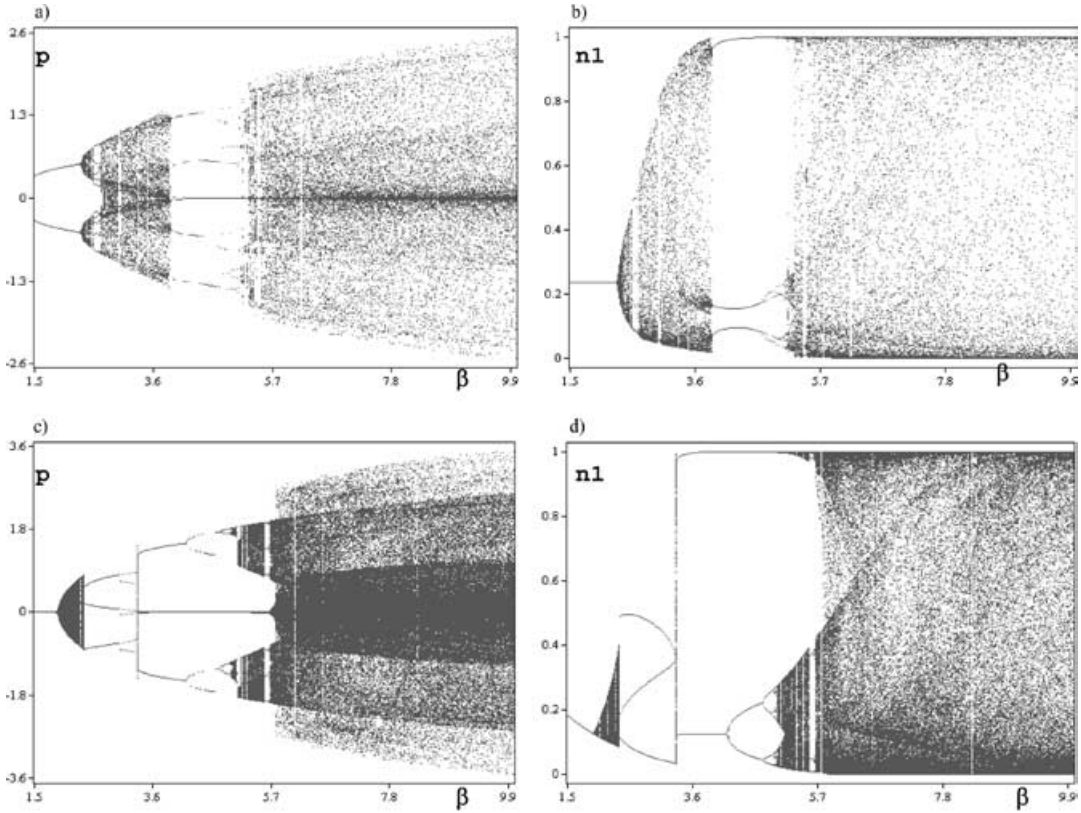

FIGURE 4. Bifurcation diagrams: (a, b) $\tau = 0.8$, $C = 1$, $B = 0.3$, $b = 1.35$; (c, d) $\tau = 0.6$, $C = 1$, $B = 0.3$, $b = 1.35$.

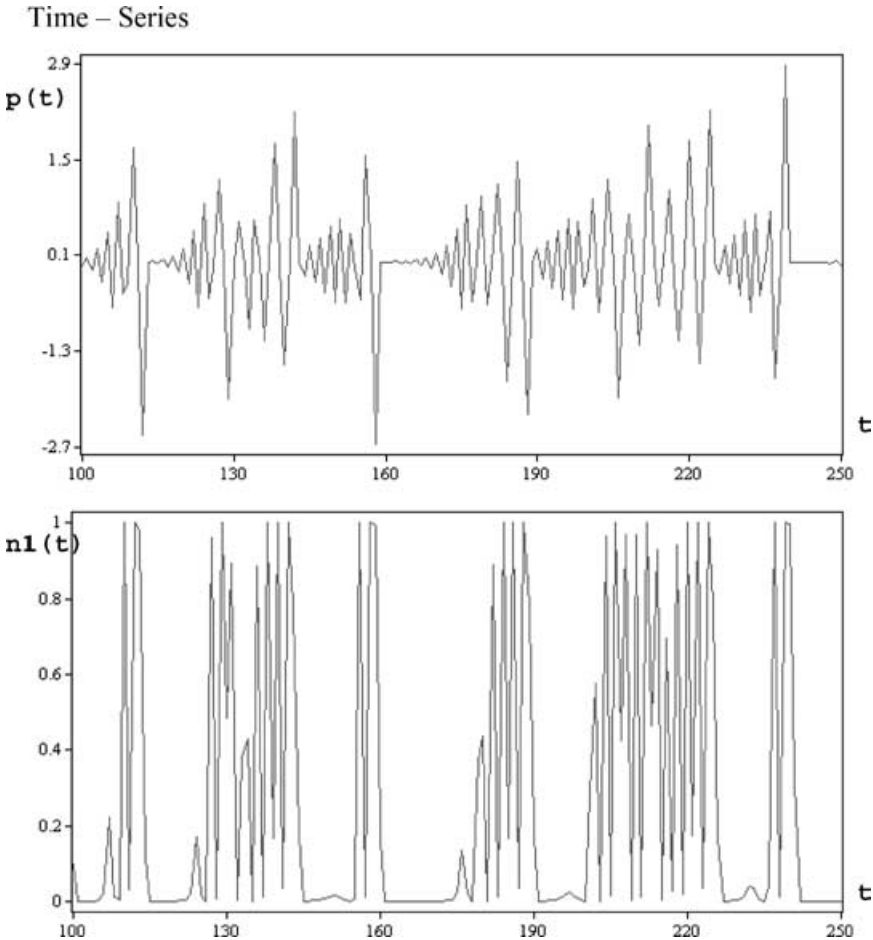

FIGURE 5. $\tau = 0.6$, $C = 1$, $B = 0.3$, $b = 1.35$, $\beta - 7$.

may, by themselves and when their formation is modeled as an economic decision, be sufficient to generate endogenous fluctuations in this evolutionary framework.

Future research could investigate in a more systematic way how the features of the predictors could generate stable periodic equilibria consistent with heterogeneous expectations. One could also investigate the effects of another type of measure of the welfare associated with a certain predictor, $U_{j,t+1}$, in our model. Indeed, one could assume that this measure was a weighted average of the two most recent net profits and see if our results changed. Numerical simulations could show if this new feature can also lead to a rational route to randomness.

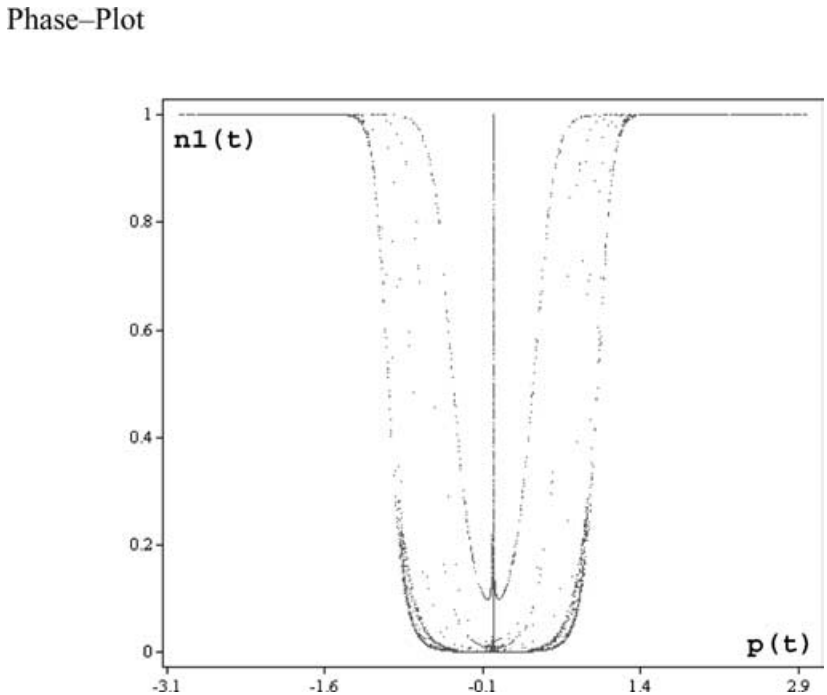

FIGURE 5. Continued.

NOTES

1. See for instance Frankel and Froot (1990) for concerns related to the foreign exchange market.
2. See also Brock and Hommes (1995).
3. A basic but necessary assumption used in the literature on this topic is the local instability of the steady state when all agents use the cheap predictor.
4. Brock used the Samuelson's boat parable to illustrate this mechanism (refer to Brock's interview by Woodford (2000)).
5. There is no endogenous switching process; the supply curve is nonlinear.
6. A similar formulation is also used in the cobweb model of Chiarella and He (1998).
7. The reference to bounded rationality is quite common in the literature on heterogeneous expectations. See for instance Tisdell (1996) or Hommes (2000).
8. The type of rational economic decision-making underlying our model is more akin to that of Baumol and Quandt than to that of Simon. The former treats the problem as an optimizing one. The latter considers it as a "satisficing" one. However, our model includes elements of both ideas.
9. See Proposition 3. For a mathematical exposition of bifurcations, we refer to Kuznetsov (2000).
10. Similar behavior can be observed for lower values of τ (e.g., 0.17). In these cases, the agents put a heavy (and perhaps unrealistic) weight on the less recent information in the unsophisticated predictor. The experiments show that the switch between the two predictors becomes more and more irregular for some small values of β.

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APPENDIX

Proof of Proposition 2. We just need to study the stability properties of the steady state $E = (0, 0, \bar{n}_1(\beta) = 1/(1 + \operatorname{Exp}\beta))$. The steady state is asymptotically stable when all the absolute values of the real eigenvalues or all the moduli of the complex eigenvalues of the Jacobian matrix at E are less than 1 (Azariadis, 1993).

The Jacobian matrix at E is

$$J = \begin{pmatrix} 0 & 1 & 0 \\ A(\bar{n}_1(\beta))(1-\tau) & A(\bar{n}_1(\beta))(\tau) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In what follows, we will denote $\bar{n}_1(\beta)$ by $\bar{n}_1$, keeping in mind that the relative weight of agents using rational expectations depends on the intensity of choice β.

(i) If $A(\bar{n}_1)\tau^2 + 4(1-\tau) < 0 \Leftrightarrow A(\bar{n}_1) < -4(1-\tau)/\tau^2$, then there are three eigenvalues, 0 and

$$\lambda_{1,2} = \frac{A(\bar{n}_1)\tau \mp \sqrt{A(\bar{n}_1)[A(\bar{n}_1)\tau^2 + 4(1-\tau)]}}{2}.$$

Study of λ_1:

$$\lambda_1 < -1 \Leftrightarrow \frac{A(\bar{n}_1)\tau - \sqrt{A(\bar{n}_1)[A(\bar{n}_1)\tau^2 + 4(1-\tau)]}}{2} < -1$$

$$\Leftrightarrow A(\bar{n}_1)\tau + 2 < \sqrt{A(\bar{n}_1)[A(\bar{n}_1)\tau^2 + 4(1-\tau)]}.$$

If $A(\bar{n}_1)\tau + 2 < 0 \Leftrightarrow A(\bar{n}_1) < -2/\tau$, this inequality is always true and then $\lambda_1 < -1$ whatever τ.

Let us now assume that $A(\bar{n}_1) \geq -2/\tau$ and let us find the conditions for which $-1 < \lambda_1 < 0$. We have

$$-A(\bar{n}_1)\tau - 2 < -\sqrt{A(\bar{n}_1)[A(\bar{n}_1)\tau^2 + 4(1-\tau)]}$$

$$\Leftrightarrow A(\bar{n}_1) > -1/(2\tau - 1) \quad \text{if } \tau > 1/2$$

(note that $-1/(2\tau - 1) > -2/\tau$ when $\tau > 2/3$)

$$\Leftrightarrow (-3\tau + 2)/[\tau(2\tau - 1)] < 0 \quad \text{if } \tau > 2/3.$$

So we have shown that when $-2/\tau < -1/(2\tau - 1) < A(\bar{n}_1) < -4(1-\tau)/\tau^2$ and $\tau > 2/3$, then $-1 < \lambda_1 < 0$.

Study of λ_2: It is easy to check that $-1 < \lambda_2 < 0$.

(ii) If $A(\bar{n}_1)\tau^2 + 4(1-\tau) > 0$, then there are three eigenvalues: 0 and

$$\lambda_{1,2} = \frac{A(\bar{n}_1)\tau \pm i\sqrt{-A(\bar{n}_1)[A(\bar{n}_1)\tau^2 + 4(1-\tau)]}}{2}.$$

Study of the modulus:

$$|\lambda_{1,2}| = \sqrt{\left(\frac{A(\bar{n}_1)\tau}{2}\right)^2 + \frac{1}{4}\{-A(\bar{n}_1)[A(\bar{n}_1)\tau^2 + 4(1-\tau)]\}}$$

$$= \sqrt{-(1-\tau)A(\bar{n}_1)}\ |\lambda_{1,2}| < 1 \Leftrightarrow A(\bar{n}_1) > -1/(1-\tau)$$

(note that $-1/(1-\tau) > -4(1-\tau)/\tau^2$ when $0 < \tau < 2/3$).

■

Proof of Proposition 3 (We Follow Kuznetsov (2000)). Our system (S′) is three-dimensional and needs to be rewritten so that the steady state is at the origin:

$$h_{t+1} = p_t,$$

$$p_{t+1} = \phi(h_t,\ p_t,\ n_{1,t}),$$

$$n_{1,t+1} = \varphi(h_t,\ p_t,\ n_{1,t}).$$

Let us denote $m_t = n_{1,t} - \bar{n}_1$. Then the system (S′) becomes the following system (S1):

$$h_{t+1} = p_t, \tag{A.1}$$

$$p_{t+1} = \phi(h_t,\ p_t,\ m_t + \bar{n}_1), \tag{A.2}$$

$$m_{t+1} = \varphi(h_t,\ p_t,\ m_t + \bar{n}_1) - \bar{n}_f = \psi(h_t,\ p_t,\ m_t)\,. \tag{A.3}$$

The steady state is then (0, 0, 0).

Let us denote (S1) as a discrete–time dynamical system:

$$x \to f(x) \tag{A.4}$$

We can write this system as

$$\tilde{x} = Jx + F(x),\ x \in R^3, \tag{A.5}$$

where J is the Jacobian matrix of (A.4) at the steady state and $F(x) = O(\|x\|^2)$ is a smooth function. Let us represent its Taylor expansion in the form

$$F(x) = \frac{1}{2}B(x, x) + \frac{1}{6}C(x, x, x) + O(\|x\|^4),$$

where $B(x, y)$ and $C(x, y, z)$ are multilinear functions.

Let us first consider the Flip case (Proposition 3i). In that case, $A(\bar{n}_1) = -1/(2\tau - 1)$ and $\tau \in (2/3, 1)$.

The Jacobian matrix J of (A.4) at the steady state is

$$J = \begin{pmatrix} 0 & 1 & 0 \\ (\tau - 1)/(2\tau - 1) & -\tau/(2\tau - 1) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

There are three eigenvalues: $0, -1$, and $(\tau - 1)/(2\tau - 1)$. The corresponding critical eigenspace is one-dimensional and spanned by an eigenvector $q \in R^3$ such that $Jq = -q$, where $q^T = (1/\sqrt{2}, -1/\sqrt{2}, 0)$. Let $s \in R^3$ be the adjoint eigenvector; that is, $J^T s = -s$, where J^T is the transposed matrix of J. Normalize s with respect to q such that $\langle s, q\rangle = 1$, where $s^T = \sqrt{2}/(2 - 3\tau)(1 - \tau, 2\tau - 1, 0)$.

The bilinear function $B\,(x,\ y)$, defined for two vectors $x = (x_1, x_2, x_3)^T$ and $y = (y_1, y_2, y_3)^T \in R^3$, can be partitioned into three elements,

$$B\,(x,\ y) = \begin{pmatrix} 0 \\ x_3 B_p^{1,3}\, y_1 + x_3 B_p^{2,3}\, y_2 + x_1 B_p^{1,3}\, y_3 + x_2 B_p^{2,3}\, y_3 \\ x_1 B_m^{1,1}\, y_1 + x_2 B_m^{1,2}\, y_1 + x_1 B_m^{1,2}\, y_2 + x_2 B_m^{2,2}\, y_2 \end{pmatrix},$$

where $B_p^{1,3} = (1 - \tau)A'(\bar{n}_1)$, $B_p^{2,3} = \tau A'(\bar{n}_1)$, $B_m^{1,1} = (\tau - 1)^2\,\sigma$, $B_m^{1,2} = \tau\,(1 - \tau)\,\sigma$, and $B_m^{2,2} = (\tau)^2\,\sigma$, with $\sigma = \beta b\ \mathrm{Exp}[\beta](A(\bar{n}_1 - 1))^2/(1 + \mathrm{Exp}[\beta])^2$ and

$$A'(\bar{n}_1) = \frac{2\tau b}{(2\tau - 1)}\left(\frac{1}{B + b\,\bar{n}_1}\right).$$

We leave to the reader to show that none of the elements of $C(x, y, z)$ is relevant for us.

Following Kuznetsov, the map (A.5) can be transformed to the normal form,

$$\tilde{\varepsilon} = -\varepsilon + \chi(0)\varepsilon^3 + O(\varepsilon^4),$$

where

$$\chi(0) = \frac{1}{6}\langle s, C(q,q,q)\rangle - \frac{1}{2}\langle s, B(q, (J - \mathrm{Id})^{-1}B(q,q))\rangle = -\frac{1}{4}\frac{(1-2\tau)^4}{(2-3\tau)}\sigma A'(\bar{n}_f).$$

We denote by Id the Identity matrix.

Thus, the critical normal form coefficient $\chi(0)$, which determines the nondegeneracy of the Flip bifurcation and allows us to predict the direction of bifurcation of the two-period cycle, is always positive when $\tau > 2/3$. Therefore, the Flip bifurcation is nondegenerate and always supercritical.

Let us now consider the Neimark–Sacker case (Proposition 3(ii)). In that case, $A(\bar{n}_1) = -1/(1-\tau)$ and $0 < \tau < 1/2$ and $1/2 < \tau < 2/3$.

The Jacobian matrix $J$of (A.4) at the steady state is

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -\tau/(1-\tau) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

There are three eigenvalues: 0 and

$$\lambda_{1,2} = -\frac{\tau}{2(1-\tau)} \pm i\frac{\sqrt{(2-\tau)(2-3\tau)}}{2(1-\tau)} = \mathrm{Re}(\lambda) \pm i\,\mathrm{Im}(\lambda).$$

J has a simple pair of eigenvalues on the unit circle $\lambda_{1,2} = e^{\pm i\theta_0}$ with $\pi/2 < \theta_0 < \pi$ and $\theta_0 \neq 2\pi/3$. Let $q \in C^3$ be a complex eigenvector corresponding to λ_1,

$$Jq = e^{i\theta_0}q, \qquad J\bar{q} = e^{-i\theta_0}\bar{q},$$

$q^T = (1, \mathrm{Re}(\lambda) + i\,\mathrm{Im}(\lambda), 0)$ and $\bar{q}^T = (1, \mathrm{Re}(\lambda) - i\,\mathrm{Im}(\lambda), 0)$. Introduce also the adjoint eigenvector $s \in C^3$ having the properties

$$J^T s = e^{-i\theta_0}s \qquad \text{and} \qquad J^T\bar{s} = e^{i\theta_0}\bar{s}$$

and satisfying the normalization

$$\langle s, q\rangle = 1,$$

where $\langle s, q\rangle = \sum_{i=1}^{3}\bar{s}_i q_i$ is the standard product in C^3,

$$\bar{s}^T = \frac{1}{2\,\mathrm{Im}(\lambda)i}(-\mathrm{Re}(\lambda) + i\,\mathrm{Im}(\lambda), 1, 0).$$

Following Kuznetsov, we know that in the absence of strong resonances, that is,

$$e^{ik\theta_0} \neq 1, \quad \text{for } k = 1, 2, 3, 4,$$

the map (A.5) can be transformed into

$$\tilde{z} = e^{i\theta_0}z(1 + \kappa(0)|z|^2) + O(u^4),$$

with $\alpha(0) = \mathrm{Re}\,\kappa(0)$, which determines the direction of the bifurcation of a closed invariant curve. This real number can be computed by the following invariant formula:

$$\alpha(0) = \frac{1}{2}\mathrm{Re}\{e^{-i\theta_0}[\langle s, C(q, q, \bar{q})\rangle + 2\langle s, B(q, (\mathrm{Id} - J)^{-1}B(q, \bar{q}))\rangle$$
$$+ \langle s, B(\bar{q}, (e^{2i\theta_0}\mathrm{Id} - J)^{-1}B(q, q))\rangle]\}.$$

Therefore,

$$\alpha(0) = \frac{1}{2}\mathrm{Re}\left\{A'(\bar{n}_1)\sigma\left[(\mathrm{Re}(\lambda)\tau)((\mathrm{Im}(\lambda)\tau)^2 - 3L^2) + L(L^2 - 3(\mathrm{Im}(\lambda)\tau)^2)\right.\right.$$
$$\left.\left. + \frac{(L^2 + (\mathrm{Im}(\lambda)\tau)^2)^2}{2}\right]\right\},$$

where $L = \tau - 1 - \tau^2/[2(\tau - 1)]$.

The coefficient $\alpha(0)$ is always negative when $\tau \in (0,\ 0.203817) \cup (0.59299, 2/3$. Therefore, the Neimark–Sacker bifurcation is nondegenerate and supercritical on these intervals.

When $\tau = 0.5$, Lasselle et al. (2003) establish that $A(\bar{n}_f) = -2$ and $\theta_0 = 2\pi/3$. The stationary equilibrium then undergoes a strong resonance 1:3 (see Kuznetsov (2000), p. 397).

When $\tau = 2/3$, the two curves of the Neimark–Sacker bifurcation and of the Flip bifurcation intersect. The steady state has a double –1 eigenvalue, a codim-2 bifurcation occurs (see Frouzakis et al. (1991), p. 85). ■