AUTOMATIC S-ACTS AND INVERSE SEMIGROUP PRESENTATIONS

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Contents

Declaration

I, Erzsébet Rita Dombi, hereby certify that this thesis, which is approximately 66000 words in length, has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

Signature .... **Name** Erzsébet Rita Dombi **Date** 03/09/04

I was admitted as a research student in September 2001 and as a candidate for the degree of Ph.D.; the higher study for which this is a record was carried out in the University of St Andrews between 2001 and 2004.

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I hereby certify that Erzsébet Rita Dombi has fulfilled the conditions of the Resolutions and Regulations appropriate for the degree of Ph.D. in the University of St Andrews and that the candidate is qualified to submit this thesis in application for that degree.

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Abstract

To provide a general framework for the theory of automatic groups and semigroups, we introduce the notion of an automatic semigroup act. This notion gives rise to a variety of definitions for automaticity depending on the set chosen as a semigroup act. Namely, we obtain the notions of automaticity, Schützenberger automaticity, $\mathcal{R}$- and $\mathcal{L}$-class automaticity, etc. We discuss the basic properties of automatic semigroup acts. We show that if S is a semigroup with local right identities, then automaticty of a semigroup act is independent of the choice of both the generators of S and the generators of the semigroup act. We also discuss the equality problem of automatic semigroup acts. To give a geometric approach, we associate a directed labelled graph to each S-act and introduce the notion of the fellow traveller property in the associated graph. We verify that if S is a regular semigroup with finitely many idempotents, then Schützenberger automaticity is characterized by the fellow traveller property of the Schützenberger graph. We also verify that a Schützenberger automatic regular semigroup with finitely many idempotents is finitely presented. We end Chapter 3 by proving that an inverse free product of Schützenberger automatic inverse semigroups is Schützenberger automatic.

In Chapter 4, we first introduce the notion of finite generation and finite presentability with respect to a semigroup action. With the help of these concepts we give a necessary and sufficient condition for a semidirect product of a semilattice by a group to be finitely generated and finitely presented as an inverse semigroup. We end Chapter 4 by giving a necessary and sufficient condition for the semidirect product of a semilattice by a group to be Schützenberger automatic.

Chapter 5 is devoted to the study of HNN extensions of inverse semigroups from finite generation and finite presentability point of view. Namely, we give necessary and sufficient conditions for finite presentability of Gilbert's and Yamamura's HNN extension of inverse semigroups. The majority of the results contained in Chapter 5 are the result of a joint work with N.D. Gilbert and N. Ruškuc.

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Chapter 1

Introduction

In order to obtain certain properties of (infinite) semigroups or groups quickly and efficiently with the help of computers, encoding them with finitely many pieces of information is important. One way of encoding a semigroup or a group is to find generators and relations between words on the generators that describe the semigroup or group in question. This is the subject of the theory of semigroup and group presentations. Another way of encoding is in terms of generators and finitely many finite state automata. This approach led to the theory of automatic groups and semigroups.

The theory of automatic groups was introduced in the early 1980s by Epstein *et al* [9]. The main aim was to introduce a large class of groups having solvable word problem. Automatic groups have many pleasant properties. They are finitely presented, their automaticity does not depend on the choice of the finite generating set, and most importantly they are characterized by a geometric property, called the fellow traveller property of their Cayley graphs.

That the (language theoretic) notion of an automatic group can be naturally extended to semigroups was observed by Campbell *et al* [5] and the study of automatic semigroups was initiated. Investigation began into whether properties of automatic groups carry over in the case of automatic semigroups [5]. Automatic semigroups have solvable word problem. On the other hand, it turned out that automatic semigroups behave more wildly than automatic groups. Automatic semigroups are not necessarily finitely presented [5, Example 4.4], automaticity of semigroups does depend on the choice of generators [5, Example 4.5]. Moreover a geometric characterization of automatic semigroups with the help of their Cayley graphs does not exists.

The response to the above results was twofold. On one hand alternative notions of automaticity have been considered for semigroups [34], [19]. On the other hand semigroup classes closely related to groups have been considered. For example, regarding automatic completely simple semigroups, the group theoretic results generalize. That is, automatic completely simple semigroups are finitely presented, automaticity does not depend on the choice of generators and fellow traveller property of their Cayley graph characterizes them. In [12] it is proved that automaticity of monoids does not depend on the choice of the generating set.

The main purpose of Chapter 3 is to give a general framework for the theory of automatic semigroups and groups and to extend the known results. In Section 1, we introduce the notion of an automatic semigroup act or automatic S-act for short. Choosing the S-act to be a set closely related to the semigroup S, we arrive at a wide variety of notions of automaticity. If the S-act is the semigroup S, then we obtain the usual notion of automaticity. By choosing the S-act to be $S/\mathcal{R}$ or $S/\mathcal{L}$, we introduce the notion of $\mathcal{R}$-class and $\mathcal{L}$-class automaticity. With the assumption that S is a regular semigroup and the S-act is an $\mathcal{R}$-class of S we introduce the notion of Schützenberger automaticity. Combining Schützenberger automaticity and $\mathcal{R}$-class automaticity we introduce the notion of a strongly Schützenberger automatic semigroup. In Section 3, we illustrate these notions by giving examples.

In Sections 4 and 5, we discuss basic properties of automatic S-acts and compare the notion of automaticity and Schützenberger automaticity of regular semigroups.

In Section 6, we show that if S is a semigroup with local right identities, then automaticity of an S-act is independent of the choice of generators of S and also of the choice of generators of the S-act. A consequence of this result is that automaticity of a regular semigroup is independent of the choice of generators.

In Sections 7-9 we discuss a geometric approach. To each S-act we associate a directed labelled graph and introduce the notion of the fellow traveller property in that graph. We show that if an S-act is automatic then the fellow traveller property holds in the associated graph. In Section 8, we investigate the conditions under which the converse holds. In particular, we verify that if S is a regular semigroup with finitely many idempotent elements, then the fellow traveller property of a Schützenberger graph of an $\mathcal{R}$-class R does characterize Schützenberger automaticity of R. In Section 9, we give an example of a semigroup S with infinitely many idempotents, in which there is an $\mathcal{R}$-class whose Schützenberger graph possesses the fellow traveller property but is not Schützenberger automatic.

As mentioned above, automatic semigroups and groups have solvable word problem. In Section 10, we introduce the equality problem for S-acts, and show that the equality problem is solvable in automatic S-acts.

Free products of automatic groups and semigroups are automatic. In Section 11, we verify that inverse free product of Schützenberger automatic inverse semigroups is Schützenberger automatic. The proof of this result is based on the description of the Schützenberger graphs of the inverse free product of two inverse semigroups given by Jones *et al* [25]. The underlying tool of the results in [25] is Stephen's procedure for constructing Schützenberger graphs with respect to an inverse semigroup presentation [35].

An extensive study of semigroup presentations began in the last decade. One of the main areas of research is the study of presentations of semigroup constructions [33], [1], [7]. Naturally, the main purpose is to give a necessary and sufficient condition for the semigroup construction to be finitely presented. In the last two chapters we investigate two inverse semigroup constructions: one of them is the semidirect product of a semilattice by a group; the other is an HNN extension (both in the sense of Gilbert and in the sense of Yamamura) of an inverse semigroup.

We begin Chapter 4 by introducing the concept of finite generation and finite presentability with respect to a semigroup action. With the help of these concepts we give a necessary and sufficient condition for a semidirect product of a semilattice by a group to be finitely generated and to be finitely presented as an inverse semigroup. We end Chapter 4 by giving a necessary and sufficient condition for a semidirect product of a semilattice by a group to be Schützenberger automatic.

The construction of an HNN extension was first introduced when Higman, Neumann and Neumann studied embeddability questions of groups. The construction was first considered for semigroups by Howie [20]. In recent years, two alternative notions for HNN extensions of inverse semigroups have been given. One of these is due to Yamamura [36], the other one to Gilbert [14]. We consider finite presentability of HNN extensions of groups and inverse semigroups both in the sense of Gilbert and in the sense of Yamamura. Considering Yamamura's HNN extension we see that the group theoretic result generalizes. The main result of Chapter 5 is to give a necessary and sufficient condition for Gilbert's HNN extension of an inverse semigroup to be finitely presented. Sections 3-8 of Chapter 5 contain results of a joint work with N.D.Gilbert and N.Ruškuc, and are accepted for publication.

Chapter 2

Preliminaries

We begin by reviewing three main mathematical subjects under consideration in this thesis. These are: semigroup theory, the theory of finite state automata and regular languages and the theory of automatic groups and semigroups. To give a detailed survey on these subjects is beyond the scope of the thesis. Our aim is to introduce the basic definitions and results that are going to be used in the third chapter. For further details in semigroup theory we refer the reader to [22], in the theory of finite automata and regular languages [28],[21] and in automatic groups and semigroup [9] and [5]. At the beginning of each subsequent chapter we give a further brief introduction containing the necessary definitions and results to make the chapter self-contained.

2.1 Semigroups

We summarize some basic definitions and results from semigroup theory we will refer to in the thesis.

Basic definitions

A non-empty set S together with an associative binary operation, usually called *multiplication* is called a *semigroup*. Let T be a non-empty subset of a semigroup S. We say that T is a *subsemigroup* of S, if T is closed with respect to the multiplication. If S is a semigroup and X is a non-empty subset of S, then the intersection of all subsemigroups of S containing X is the *subsemigroup* $\langle X \rangle$ *generated by* X. If $\langle X \rangle = S$, then we say that S is generated by X.

Let X be a non-empty set. Let X^+ denote the set of all finite words $(x_1 \ldots x_n)$, where $x_i \in X$ for all $1 \leq i \leq n$. The integer $1 \leq n$ is the *length* $|w|$ of the word $(x_1 \ldots x_n)$. The set X^+ can be equipped with the following associative operation, called concatenation:

$$(x_1 \dots x_n)(y_1 \dots y_m) = (x_1 \dots x_n y_1 \dots y_m).$$

The semigroup obtained is called the *free semigroup on* X. We recall [33, proposition 1.1].

Proposition 2.1.1. *Let X be a non-empty set and S be a semigroup. Then every mapping $f : X \to S$ extends uniquely to a homomorphism $\varphi : X^+ \to S$. If S is generated by X, then φ is surjective.*

Let S be a semigroup generated by $X \subseteq S$. Let $\varphi : X^+ \to S$ denote the homomorphism extending the identity map $\iota : X \to S$. For any two words $u, v \in X^+$, we write $u \equiv v$, if u and v are identical as words, and we write $u = v$, if u and v represent the same element of S, that is, if $u\varphi = v\varphi$.

Let S be a semigroup. We say that $1 \in S$ is an *identity element*, if $1s = s1 = s$ holds for all $s \in S$. It is easy to see that a semigroup can have at most one identity element. If a semigroup S contains an identity element, then we call S a *monoid.* If S does not have an identity element, then a monoid can be formed from S in the following way. We choose a symbol $1 \notin S$ and define $1s = s1 = s$ for all $s \in S$ and define $11 = 1$. The monoid obtained is denoted by S^1.

Let S be a semigroup with at least two elements. We say that $0 \in S$ is a *zero element,* if $0s = s0 = 0$ holds for all $s \in S$. A semigroup S can have at most one zero element. If S does not have a zero element, then a semigroup with zero element can be obtained in the following way. We choose a symbol $0 \notin S$ and define $0s = s0 = 0$ for all $s \in S$ and define $00 = 0$. The semigroup obtained is denoted by S^0.

We say that $e \in S$ is an *idempotent,* if $ee = e$. The set of idempotents of S will be denoted by $E(S)$.

Let S be a semigroup and R be a relation on S. We say that R is *left (right) compatible* with the multiplication, if whenever $s, t \in S$ are such that $s \; R \; t$, then for all $a \in S$, $as \; R \; at$, $(sa \; R \; ta)$ holds. We say that R is *compatible* with the multiplication, if whenever $s_1, s_2, t_1, t_2 \in S$ are such that $s_1 \; R \; t_1$ and $s_2 \; R \; t_2$ hold, then $s_1 s_2 \; R \; t_1 t_2$ holds. A left (right) compatible equivalence relation is called a *left (right) congruence*, and a compatible equivalence relation is called a *congruence* on S.

Before we give examples for a left and a right congruence on a semigroup S, we introduce the following notions. Let A be a non-empty subset of a semigroup S. We say that A is a *left (right) ideal* in S, if for all $s \in S$ and $a \in A$, $sa \in A$, $(as \in A)$ holds. If A is a left and a right ideal in S, then we say that A is an *ideal* in S. If A is an ideal of S so that $A \subset S$, then say that S is a *proper ideal* of S. We say that S is *simple*, if it has no proper ideals. If $a \in S$, then the smallest left (right) ideal of S containing a is $S^1a = \{sa \mid s \in S^1\}$ $(aS^1 = \{as \mid s \in S^1\})$, which we call the *principal left (right) ideal generated by a*. The smallest ideal containing a is $S^1aS^1 = \{sat \mid s, t \in S^1\}$, which we call the *principal ideal generated by a*.

Let S be a semigroup. We define the following equivalence relations on S. For all $s, t \in S$, we define $s \mathcal{L} t$, if and only if s and t generate the same principal left ideal, that is, if and only if $S^1s = S^1t$. Similarly, for all $s, t \in S$, we define $s \mathcal{R} t$, if and only if s and t generate the same principal right ideal, that is, if and only if $sS^1 = tS^1$. Clearly, $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence on S. It can be easily seen that $\mathcal{L}$ and $\mathcal{R}$ commutes, and so $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ is the smallest equivalence containing $\mathcal{L}$ and $\mathcal{R}$. We denote by $\mathcal{H}$ the intersection of $\mathcal{L}$ and $\mathcal{R}$. If $\mathcal{H}$ is trivial, then we call S a *combinatorial* or *aperiodic* semigroup. We introduce our final equivalence relation. For all $s, t \in S$, we define $s \mathcal{J} t$, if and only if s and t generate the same principal ideal, that is, if and only if $S^1sS^1 = S^1tS^1$. We note that S is a simple semigroup if and only if $\mathcal{J} = S \times S$. It is immediate that $\mathcal{H} \subseteq \mathcal{R}, \mathcal{L} \subseteq \mathcal{J}$, and so $\mathcal{D} \subseteq \mathcal{J}$ holds as well. We call the relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$ *Green's relations*. We denote the $\mathcal{L}$-class [$\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$, $\mathcal{J}$-class] of an element $s \in S$ by L_s $[R_s, H_s, D_s, J_s]$. Since $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$, the $\mathcal{D}$-class of $s \in S$, can be visualized in a convenient way by the egg-box picture:

	L_s			
R_s	$\cdot s$			

The box represents a $\mathcal{D}$-class, namely D_s, the rows correspond to $\mathcal{R}$-classes, the columns to $\mathcal{L}$-classes, and each cell represents an $\mathcal{H}$-class.

Lemma 2.1.2. *Let S be a semigroup and $e \in E(S)$. Then $er = r$ for all $r \in R_e$ and $le = l$ for all $l \in L_e$.*

Proof. See [22, Proposition 2.3.3].

With the help of left and right translations, a bijection can be given between $\mathcal{L}$-, $\mathcal{R}$- and $\mathcal{H}$-classes of a semigroup contained within a $\mathcal{D}$-class.

Lemma 2.1.3 (Green's lemma). *Let S be a semigroup and $s, t \in S$ so that $s \mathcal{L} t$. Assume that $s = at$ and $t = bs$. Then the mappings*

$$\lambda_a : R_s \to R_t;\ x \mapsto bx \quad \text{and} \quad \lambda_b : R_t \to R_s;\ y \mapsto ay$$

are mutually inverse $\mathcal{L}$-class preserving bijections.

Proof. See [22, Lemma 2.2.2].

A *subgroup* of a semigroup S is a subsemigroup, which is a group. With the help of Green's lemma the following propositions can be proved.

Proposition 2.1.4. *The maximal subgroups of a semigroup S coincide with the $\mathcal{H}$-classes of S which contain an idempotent. Each subgroup of S is contained in an $\mathcal{H}$-class of S.*

Proof. See [22, Corollary 2.2.6].

Proposition 2.1.5. *Let S be a semigroup, and $s, t \in S$ such that $s \mathcal{D} t$. Then $st \in R_s \cap L_t$ if and only if $L_s \cap R_t$ contains an idempotent.*

Proof. See [22, Proposition 2.3.7].

We illustrate Proposition 2.1.5 with the help of the egg-box picture:

	L_t		L_s
R_s	$\cdot st$		$\cdot s$
R_t	$\cdot t$		$\cdot e$

Regular semigroups

Throughout the thesis we shall be interested in regular semigroups. We first generalize the group theoretic notion of an inverse. Let S be a semigroup. We say that $s \in S$ is *regular*, if there exists $x \in S$ so that $s = sxs$. We say that x is an *inverse* of s, if $s = sxs$ and $x = xsx$ hold. It can be easily seen, that if $s \in S$ is regular and $s = sxs$, then $s' = xsx$ is an inverse of s. Thus, we may deduce that $s \in S$ is regular if and only if s has an inverse. An element can have more than one inverse. The set of all inverses of an element will be denoted by $V(s)$. We call S a *regular semigroup,* if every element of S is regular. Note that if $s' \in V(s)$, then $ss', s's \in E(S)$ and $ss' \mathcal{R}\ s\ \mathcal{L}\ s's$. Thus in a regular semigroup, every $\mathcal{L}$- and $\mathcal{R}$-class contains an idempotent. Also we have

Proposition 2.1.6. *If $s \in S$ is regular, then every element of D_s is regular.*

Proof. See [22, Proposition 2.3.1].

Let $s \in S$, where S is a regular semigroup. The locations of the idempotents in D_s determines the location of inverses of s. Namely, we have

Proposition 2.1.7. *Let S be regular semigroup and $s \in S$. Then*

(1) For all $s' \in V(s)$, we have $s \mathcal{D} s'$.

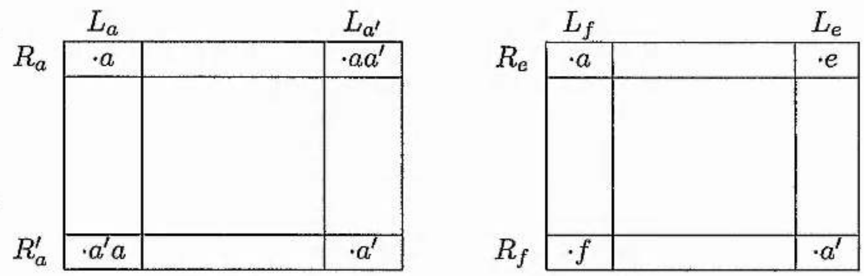
(2) For all $s' \in V(s)$, we have $ss' \in R_s \cap L_{s'}$ and $s's \in L_s \cap R_{s'}$.

(3) If $e, f \in E(S)$ and $e \mathcal{R} s \mathcal{L} f$, then there exists $s' \in V(s)$ such that $ss' = e$ and $s's = f$.

(4) No $\mathcal{H}$-class contains more than one inverse of s.

Proof. See [22, Theorem 2.3.4].

We illustrate the results of the above proposition with the help of the egg-box picture:



The Green's relations in a regular semigroup can be described in the following way:

Proposition 2.1.8. *Let S be a regular semigroup and $s, t \in S$. Then*

(1) $s \mathcal{L} t$, if and only if $s's = t't$ for some $s' \in V(s)$, $t' \in V(t)$.

(2) $s \mathcal{R} t$, if and only if $ss' = tt'$ for some $s' \in V(s)$, $t' \in V(t)$.

(3) $s \mathcal{H} t$, if and only if $s's = t't$ and $ss' = tt'$ for some $s' \in V(s)$, $t' \in V(t)$.

Proof. See [22, Proposition 2.4.1].

Let S be a regular semigroup and $s, t \in S$. To determine an inverse of st in terms of inverses of s and t, we need to introduce the notion of a sandwich set.

Definition 2.1.9. Let S be a regular semigroup. The *sandwich set* of $e, f \in E(S)$ is the non-empty set

$$S(e, f) = \{h \in E(S) \mid h = he = fh,\ ehf = ef\}.$$

Proposition 2.1.10. *Let S be a regular semigroup, $s, t \in S$. Let $s' \in V(s)$, $t' \in V(t)$ and $h \in S(s's, tt')$. Then $t'hs' \in V(st)$.*

Proof. See [22, Theorem 2.5.4].

With the help of Proposition 2.1.5, the following proposition can be verified.

Proposition 2.1.11. *Let S be a regular semigroup, $e \in E(S)$ and consider $G = H_e$. Let $s, t \in R_e$ such that $s \mathcal{H} t$. Then $Gs = Gt$ hold.*

We give a brief summary on certain regular semigroup classes that will appear in the thesis.
We say that S is a *completely regular semigroup*, if every $\mathcal{H}$-class in S is a group. A completely regular simple semigroup is called a *completely simple semigroup*. It can be verified, that if S is a completely simple semigroup, then eSe is a group for all $e \in E(S)$. Completely simple semigroups play an important role in the description of the structure of completely regular semigroups. Namely, if S is a completely regular semigroup, then $S/\mathcal{J}$ is a semilattice, and each $\mathcal{J}$-class is a completely simple subsemigroup of S. In other words, each completely regular semigroup is a semilattice of completely simple semigroups. The Rees-Sushkevich Theorem, which we now introduce, describes the structure of completely simple semigroups, and so a finer picture can be obtained from completely regular semigroups.

Proposition 2.1.12 (Rees-Sushkevich). *Let G be a group, Λ and I non-empty sets. Let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in G. Let $S = I \times G \times \Lambda$, and define a multiplication on S by*

$$(i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda j}h, \mu).$$

Then S is a completely simple semigroup. Conversely, every completely simple semigroup is isomorphic to a semigroup constructed in this way.

Proof. See [22, Theorem 3.3.1].

The semigroup $I \times G \times \Lambda$ with the given multiplication is denoted by $\mathcal{M}[G, I, \Lambda; P]$ and is called a Rees matrix semigroup over G.

In a regular semigroup an element can have more than one inverse. We say that S is an *inverse semigroup*, if every element has exactly one inverse. The inverse of an element s is denoted by s^{-1}. It can be easily seen that if S is an inverse semigroup and $s, t \in S$, then $(st)^{-1} = t^{-1}s^{-1}$, $(s^{-1})^{-1} = s$ hold.

Proposition 2.1.13. *Let S be a semigroup. Then the following are equivalent.*

(1) S is an inverse semigroup.

(2) S is regular and $E(S)$ forms a semilattice.

(3) Every $\mathcal{R}$- and $\mathcal{L}$-class contains exactly one idempotent.

Proof. See [22, Theorem 5.5.1].

There is a natural partial order defined on inverse semigroups, which possesses pleasant properties. Let S be an inverse semigroup and $s, t \in S$. We say that $s \leq t$ if and only if $s = et$ for some $e \in E(S)$. It can be easily seen, that $s \leq t$ if and only if $s^{-1} \leq t^{-1}$. Furthermore, if $s_1 \leq t_1$ and $s_2 \leq t_2$, then $s_1 s_2 \leq t_1 t_2$.

2.2 Regular languages and finite state automata

We will often refer to a finite non-empty set A as an alphabet. We may wish to adjoin an element λ to A^+, called the empty-word, to form the free monoid A^*. The length of the empty-word will be defined to be 0. A subset of $L \subseteq A^*$ will be called a *language over A*. If L and K are languages over A, then $L \cap K,\ L \cup K, L \setminus K, LK = \{uv \mid u \in L,\ v \in K\}$, $L^* = \{\lambda\} \cup L \cup LL \cup LLL \cup \dots$ and $L^+ = L \cup LL \cup LLL \cup \dots$ are also languages over A. We often write $L + K$ instead of $L \cup K$.

A *deterministic finite state automaton* is a quintuple $\mathcal{A} = (\Sigma, A, \mu, p, T)$, where Σ is a finite non-empty set called the set of states, A is an alphabet, $\mu\colon \Sigma \times A \to \Sigma$ is a function, called the *transition function*, $p \in \Sigma$ is called the *initial state*, $T \subseteq \Sigma$ is the set of *terminal* or *final states*. We extend the transition function to a mapping $\mu^*\colon \Sigma \times A^* \to \Sigma$ as follows:

$$(q, \lambda) \mapsto q;\ (q, a) \mapsto \mu(q, a);\ (q, aw) \mapsto \mu^*(\mu(q, a), w),$$

where $q \in \Sigma,\ a \in A,\ w \in A^*$. We say that a word $w = a_1 \dots a_n$ over A is *recognized* by the automaton $\mathcal{A}$ if $\mu^*(p, w) \in T$. The *language $L(\mathcal{A})$ recognized (or accepted) by the automaton $\mathcal{A}$* is the set of all elements $w \in A^*$ that are recognized by $\mathcal{A}$.

A deterministic finite state automaton $\mathcal{A} = (\Sigma, A, \mu, p, T)$ can be visualized as a directed labelled graph, whose vertices are elements of Σ and there is an arrow from state q to state r with label a, if $\mu(q, a) = r$. If $w = a_1 \dots a_n \in L(\mathcal{A})$, then we say that the path

$$p \xrightarrow{a_1} q_1 \xrightarrow{a_2} \dots \xrightarrow{a_{n-1}} q_{n-1} \xrightarrow{a_n} r$$

is successful.

A *non-deterministic finite state automaton* is a quintuple $\mathcal{A} = (\Sigma, A, \mu, p, T)$, where Σ is a finite non-empty set called the set of states, A is a finite non-empty set called the alphabet, $\mu\colon \Sigma \times (A \cup \{\lambda\}) \to P(\Sigma)$, where $P(\Sigma)$ denotes the set of all subsets of Σ , is a function, called the *transition function*, $p \in \Sigma$ is called the *initial state* and $T \subseteq \Sigma$. The elements of T are called *terminal* or *final states*. To define the language accepted by this automaton, we need the following notion. Let w be a word over $A \cup \{\lambda\}$. The *value of* w is the word obtained from w by deleting all occurrences of λ in w. A *path* in $\mathcal{A}$ is a sequence

$$q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \dots \xrightarrow{a_{n-1}} q_{n-1} \xrightarrow{a_n} q_n,$$

where $q_i \in \mu(q_{i-1}, a_{i-1})$, $(1 \leq i \leq n)$ and we say that $a_1 \dots a_n$ is the *label of the path.* We say that w is accepted by $\mathcal{A}$, if there exists a path from the initial state to a final state whose label is w or a word with the same value as w. The *language $L(\mathcal{A})$ recognized (or accepted) by the automaton $\mathcal{A}$* is the set of all elements $w \in A^*$ that are recognized by $\mathcal{A}$. Similarly to deterministic finite state automata, non-deterministic finite state automata can be visualized as directed labelled graphs.

Let A be an alphabet. A *regular expression over A* is a special type of word over $A \cup \{\emptyset, (,), +, \cdot, ^*\}$. Namely, $\emptyset, \lambda, a$, where $a \in A$ are regular expressions. If r, r_1, r_2 are regular expressions, then so are $(r_1 + r_2), (r_1 \cdot r_2)$ and $(r)^*$. Every regular expression is formed in this way. Each regular expression r defines a language $L(r)$. Namely, we define $L(\emptyset) = \emptyset$, $L(\lambda) = \{\lambda\}, L(a) = \{a\}$, where $a \in A$. We define $L(r_1 + r_2) = L(r_1) + L(r_2)$, $L(r_1 \cdot r_2) = L(r_1)L(r_2)$ and $L(r^*) = (L(r))^*$, where r, r_1, r_2 are regular expressions. A language defined by a regular expression is called a *regular language.* The first major result in automata theory is due to Kleene, Rabin and Scott. For the proof, see for example Theorem 4.1.2 and Theorem 5.2.1 in [28].

Theorem 2.2.1 (Kleene, Rabin, Scott). *Let A be an alphabet and L be a language over A. Then the following are equivalent:*

(1) L is recognized by a deterministic finite state automaton.

(2) L is recognized by a non-deterministic finite state automaton.

(3) L is a regular language.

The following lemma provides a necessary condition for a language to be regular.

Lemma 2.2.2 (Pumping lemma). *Let L be an infinite regular language over the alphabet A. Then, there exists a positive integer N such that every word $w \in L$ with length* *greater or equal to N can be factorized as $w = xyz$, where $|xy| \leq N, |y| \geq 1$ and $xy^n z \in L$ for all $n \geq 0$.*

Proof. See [28, Theorem 2.6.1].

Next, we recall Proposition 2.2 in [5].

Proposition 2.2.3. *Let X and Y be finite sets. Then the following hold:*

(1) $\emptyset$, X^+ and X^* are regular languages.

(2) Any finite subset of X^* is a regular language.

(3) If $K \subseteq X^*$ and $L \subseteq Y^*$ are regular languages, then $K \cup L$, $K \cap L$, $K - L, KL, K^*$ and $K^{\text{rev}} = \{x_1 \dots x_n \mid x_n \dots x_1 \in K\}$ are regular languages.

(4) If $K \subseteq X^*$ is a regular language and $\varphi : X^+ \to Y^+$ is a semigroup homomorphism, then $K\varphi$ is a regular language.

(5) If $L \subseteq Y^*$ is a regular language and $\varphi : X^+ \to Y^+$ is a semigroup homomorphism, then $L\varphi^{-1}$ is a regular language.

Before we finish this section, we introduce the following notion of automata theory.

Definition 2.2.4. Let A be an alphabet and L be a language over A. Let $v \in A^*$. The *left quotient of L by v* is the language

$$K = \{w \in A^* \mid vw \in L\}.$$

Similarly, the *right quotient of L by v* is the language

$$N = \{w \in A^* \mid wv \in L\}.$$

The following lemma will prove useful.

Lemma 2.2.5. *Let L be a regular language over the alphabet A and let $v \in A^*$. Then the left (right) quotient of L by v is also a regular language.*

Proof. See [28, Proposition 7.5.5].

2.3 Automatic groups and semigroups

Let X be a finite set, and let \$ be a symbol not contained in X. Let $X(2,\$) = ((X \cup \$) \times (X \cup \$)) \setminus \{(\$,\$)\}$. Define $\delta_X : X^* \times X^* \to X(2,\$)^*$ by

$$(a_1 \dots a_n, b_1 \dots b_m)\delta_X = \begin{cases} (a_1,b_1)\dots(a_n,b_n) & \text{if } n = m, \\ (a_1,b_1)\dots(a_n,b_n)(\$,b_{n+1})\dots(\$,b_m) & \text{if } n < m, \\ (a_1,b_1)\dots(a_m,b_m)(a_{m+1},\$)\dots(a_n,\$) & \text{if } n > m. \end{cases}$$

We recall [5, Proposition 2.2].

Proposition 2.3.1. *Let X be a finite set.*

(1) If $K, L \subseteq X^*$ are regular languages, then $(K \times L)\delta_X$ is a regular language.

(2) If $U \subseteq (X^* \times X^*)\delta_X$ is a regular language, then

$$\{u \in X^* \mid (u,v)\delta_X \in U \text{ for some } v \in X^*\}$$

is also a regular language.

We will also make use of [5, Proposition 2.4] and of [5, Proposition 2.3] :

Proposition 2.3.2. *If L is a regular language over X, then $\{(w,w) \mid w \in L\}\delta_X$ is a regular language over $X(2,\$)$.*

Proposition 2.3.3. *Let L and K be regular languages over $X(2,\$)$. Then the language*

$$W = \{(u,w) \in X^* \times X^* \mid \text{there exists } v \in X^* \text{ such that } (u,v)\delta_X \in L,\ (v,w)\delta_X \in K\}\delta_X$$

is regular.

Let S be a semigroup. Assume that the set $X \subseteq S$ generates S and let $\varphi : X^+ \to S$ denote the homomorphism extending the identity map $\iota : X \to S$. As introduced, for any two words $u, v \in X^+$, we write $u \equiv v$, if u and v are identical as words, and we write $u = v$, if u and v represent the same element of S, that is, if $u\varphi = v\varphi$. We say that the word problem is solvable in S, if an algorithm can be given with the help of which it can be decided whether two words over X represent the same element of S.

Definition 2.3.4. Let S be a semigroup generated by a finite set X. Let L be a regular language over X. We say that (X, L) is an *automatic structure* for S, if the following conditions hold:

(1) $L\varphi = S$;

(2) $L_= = \{(u, v) \in L \times L \mid u = v\}\delta_X$ is a regular language;

(2) $L_x = \{(u, v) \in L \times L \mid ux = v\}\delta_X$ is a regular language for all $x \in X$.

If S has an automatic structure (X, L), then we say that S is an *automatic semigroup*. We say that (X, L) is an *automatic structure with uniqueness* for S, if whenever $u, v \in L$ are such that $u = v$, then $u \equiv v$ holds.

Considering groups as semigroups, we obtain the notion of an automatic group.

Definition 2.3.5. Let G be a group generated a finite set X as a semigroup. Let L be a regular language over X. We say that (X, L) is an *automatic structure* for G, if the following conditions hold:

(1) $L\varphi = G$;

(2) $L_= = \{(u, v) \in L \times L \mid u = v\}\delta_X$ is a regular language;

(2) $L_x = \{(u, v) \in L \times L \mid ux = v\}\delta_X$ is a regular language for all $x \in X$.

If G has an automatic structure (X, L), then we say that G is an *automatic group*. We say that (X, L) is an *automatic structure with uniqueness* for G, if whenever $u, v \in L$ are such that $u = v$, then $u \equiv v$ holds.

We now recall [5, Corollary 5.5].

Proposition 2.3.6. *If S is an automatic semigroup (group), then there exists an automatic structure with uniqueness.*

Automatic groups can be described with the help of a geometric property of their Cayley graphs. First, we summarize some basic notions concerning the Cayley graph of a group.

Let G be a group generated by a set X. We assume that X is closed under taking inverses. The *right Cayley graph* $\Gamma = \Gamma_X(G)$ of G is a directed labelled graph, whose vertices are elements of G, and there is an arrow from g to h with label x precisely when $h = gx$. Clearly, there is an arrow from g to h with label x precisely when there is an arrow from h to g with label x^{-1}. We define a *directed path* between two vertices g and h of Γ to be a sequence of edges:

$$g = g_0 \xrightarrow{x_1} g_1 \xrightarrow{x_2} g_2 \cdots g_{n-1} \xrightarrow{x_n} g_n = h$$

and say that the *length of the path* is n. Clearly Γ is a connected graph, and there exists a directed path between any two vertices of Γ. Let $g, h \in G$. We define the distance $d_\Gamma(g, h)$ between g and h to be the length of the shortest path connecting g and h. Before we define the fellow traveller property, we introduce the following notation. If $u \equiv x_1 \dots x_m$ and $t \geq 1$, then we let

$$u(t) = \begin{cases} x_1 \dots x_t & \text{if } t \leq m, \\ x_1 \dots x_m & \text{if } t \geq m. \end{cases}$$

Definition 2.3.7. Let G be a group generated by a finite set X. Let L be a regular language over X and assume that $L\varphi = G$. We say that the Cayley graph $\Gamma = \Gamma_X(G)$ possesses *the fellow traveller property* with respect to L, if there exists a constant $k \in \mathbb{N}$, such that whenever $u, v \in L$ are such that $d_\Gamma(u, v) \leq 1$, then $d_\Gamma(u(t), v(t)) \leq k$ for all $t \geq 1$.

The following result yields a powerful tool in proving results about automatic groups.

Proposition 2.3.8. *Let G be a group. Then G is automatic if and only if the fellow traveller property holds in the Cayley graph of G with respect to some regular language L.*

Proof. See [9, Theorem 2.3.5].

We recall the following nice properties of automatic groups.

Proposition 2.3.9. *Let G be an automatic group and assume that (X, L) forms an automatic structure for G. Then the following hold:*

(1) If Y also generates G as a semigroup, then there exists a regular language K over Y such that (Y, K) forms an automatic structure for G.

(2) The group G is finitely presented.

(3) The word problem is solvable in quadratic time.

Proof. See [9, Theorem 2.4.1, Theorem 2.3.12, Theorem 2.3.10].

Next, we discuss some of the basic properties of automatic semigroups. Let S be a group generated by a set X. The (right) Cayley graph $\Gamma_X(S) = \Gamma$ of S is a directed labelled graph, whose vertices are elements of S, and there is an arrow from s to t with label x precisely when $s = tx$. Certain subgraphs, called Schützenberger graphs of the Cayley graph of S will play an important role in the thesis. Let R be an $\mathcal{R}$-class of S. The *Schützenberger graph* of R is the Cayley graph of S restricted to R. In other words, the vertices of the Schützenberger graph of R are elements of R, and there is an arrow from s to t, $(s,t \in R)$ with label x, if $s = tx$.

The aim of giving a geometric characterization of automatic semigroups with the help of the Cayley graph raises a few question. First of all the Cayley graph of a semigroup is not necessarily connected. Even, if two vertices s and t are in the same connected component, then there does not necessarily exists a directed path between them. These considerations lead to the decision to ignore the direction of the edges in the Cayley graph of S. A *path* between two vertices s and t of Γ is a sequence of edges:

$$s = s_0 \overset{x_1}{\text{———}} s_1 \overset{x_2}{\text{———}} s_2 \ldots s_{n-1} \overset{x_n}{\text{———}} s_n = t$$

such that either (s_i, x_i, s_{i+1}) or (s_{i+1}, x_i, s_i) is an arrow in the Cayley graph of S, and say that the *length of the path* is n. For any two vertices s and t we define the *distance* $d_\Gamma(s,t)$ between s and t to be the length of the shortest path connecting s and t and say that the *distance is infinite* if s and t belong to different components of Γ.

Definition 2.3.10. Let S be a semigroup generated by a finite set X. Let Γ be the Cayley graph of S with respect to X. Let L be a regular language over X and assume that $L\varphi = S$. We say that Γ possesses the *fellow traveller property* with respect to L, if there exists a constant $k \in \mathbb{N}$, such that whenever $u, v \in L$ are such that $d_\Gamma(u,v) \leq 1$, then $d_\Gamma(u(t), v(t)) \leq k$ for all $t \geq 1$.

Proposition 3.12 of [5] tells us:

Proposition 2.3.11. *Let S be an automatic semigroup. Then the fellow traveller property holds in the Cayley graph of S with respect to some regular language L.*

The converse of Proposition 2.3.11 does not necessarily hold. The fellow traveller property holds in any semigroup with a zero element. To be more precise, if S is a semigroup with a zero element, then the distance between any two vertices in the Cayley graph of S is less then or equal to two. Thus, to demonstrate that the converse of Proposition 2.3.11 does not hold, one needs to take a semigroup with zero that is not automatic. Such a semigroup can be formed by adjoining a zero element to a group that is not automatic.

Automatic semigroups behave more wildly than automatic groups. An example is given in [5, Example 4.4] for a semigroup that is automatic but not finitely presented. Another example [5, Example 4.5] shows that a semigroup can be automatic with respect to one generating set, but not automatic with respect to another one. On the other hand, if we consider monoids, then automaticity is independent of the choice of the generators

[12]. Also, if we consider semigroup classes that are somehow related to groups, such as consider completely simple semigroups, then the group theoretic results generalize [4].

Proposition 2.3.12. *Let S be an automatic semigroup generated by the set X. Then the word problem is solvable in quadratic time.*

Proof. See [5, Corollary 3.7].

Consider the alphabet $X = \{x_1, x_2, \ldots, x_{n-1}, x_n\}$ and choose an ordering $x_1 < x_2 < \ldots < x_{n-1} < x_n$. Let $u, v \in X^+$. We define $u < v$ if and only if u is shorter then v, or they have the same length, and u comes before v in the lexicographical order. This ordering is called the *shortlex order* on X^+.

Proposition 2.3.13. *Let (X, L) be an automatic structure for a semigroup S. Let $<$ denote the shortlex order on X^+. Then the language*

$$K = \{u \mid \text{if } (u,v) \in L_= \text{ then } u < v\}$$

is regular.

Proof. See [5, Proposition 5.4].

The following proposition will also prove useful.

Proposition 2.3.14. *Let (X, L) be an automatic structure for a semigroup S and let $w \in X^+$. Then $L_w = \{(u,v) \mid u, v \in L, uw = v\}\delta_X$ is a regular language.*

Proof. See [5, Proposition 3.2].

The notion of a padded product of regular languages is introduced in [10]:

Definition 2.3.15. Let X be an alphabet and let M, N be regular languages over $(X^* \times X^*)\delta_X$. The *padded product* of M and N is the language

$$M \odot N = \{(u_1u_2, v_1v_2) \mid (u_1, v_1)\delta_X \in M,\ (u_2, v_2)\delta_X \in N\}\delta_X.$$

Theorem 3.3 of [10] gives a sufficient condition on the padded product of regular languages to be regular.

Proposition 2.3.16. *Let X be an alphabet and let M, N be regular languages over $(X^* \times X^*)\delta_X$. If there exists a constant c such that for any two words $u, v \in X^*$ the following property holds:*

$$(u,v)\delta_X \in M \qquad \text{then} \qquad ||u| - |v|| \leq c,$$

then the padded product $M \odot N$ is a regular language.

Chapter 3

Automatic semigroup acts

3.1 Definitions

Finite state automata are easy to handle computationally. Therefore, some recent research has aimed at encoding algebraic structures in terms of finite state automata. However, there are different ways to define what it means for an algebraic structure to be automatic. Epstein *et al.* developed a theory of automatic groups, Campbell *et al.* realized that the notion of an automatic group can be naturally extended to semigroups and investigated to what extent the theory of automatic groups can be generalized to semigroups. It turned out that automatic semigroups do not have as nice properties as automatic groups, which led researchers to modify the definition of an automatic semigroup, see for example [19],[34].

We mention that Khoussainov and Nerode represent another viewpoint of defining an algebraic structure automatic [26]. Automatic groups (semigroups) in the sense of Epstein *et al.* (Campbell *et al.*), viewed as unary structures – where the unary operations are multiplication by the generators on the right – are automatic in the sense of Khoussainov and Nerode.

Our aim in this chapter is to give a general framework for the theory of automatic semigroups introduced by Cambell *et al.* and to extend the known results. To achieve this goal we are going to define automaticity of a very simple structure, called an S-act. One of the reasons we are going to work with S-acts is that semigroups and groups can be naturally considered as S-acts and S-acts proved to be a useful tool in studying semigroups. On the other hand, certain sets – for example an $\mathcal{R}$- or an $\mathcal{L}$-class – can be associated with a semigroup S, which can be viewed as a S-acts. This will allow us to introduce other notions of automaticity, for example Schützenberger automaticity, $\mathcal{R}$-class automaticity.

Throughout this chapter S will denote a semigroup, X a generating set for S. We assume that $X \subseteq S$ and denote by $\varphi : X^+ \to S$ the homomorphism extending the identity map $\iota : X \to S$.

Let $S = \langle X \rangle$ be a semigroup and A be a non-empty set. A *right action (left action) of S on A* is a function $f\colon A \times S \to A$; $(a, s) \mapsto a.s$ such that $(a.s).t = a.(st)$ $((a.s).t = a.(ts))$ holds for all $s, t \in S$, and we say that *A is a right S-act (left S-act)*. We say that A is an *S-act*, if it is either a left or a right S-act. Although it might seem a little unorthodox to write left actions on the right, we will do so. The main reason for this is that we stick to the rule in this chapter that a product $s_1s_2 \dots s_n$, $(s_i \in S,\ 1 \leq i \leq n)$ is read from left to right. If we would write left actions of S on A on the left, then considering $(s_1s_2 \dots s_n).a$ would make us read $s_1s_2 \dots s_n$ from right to left.

The subset $A_0 = \{a_1, a_2, \dots, a_n, \dots\}$ of A is said to *generate the S-act A*, if

$$\bigcup_{a_j \in A_0} a_j.S = A.$$

If A_0 can be chosen to be a finite set, then we say that the S-act A is *finitely generated*. If A_0 can be chosen to be a one element set, then we say that the S-act A is *cyclic*. Clearly, if A_0 generates the S-act A, then it might happen that there exists $s, t \in S$ so that $s \neq t$ and $a_i.s = a_j.t$ for some $a_i, a_j \in A_0$. To generalize the notion and basic properties (for example uniqueness of an automatic structure) of an automatic group or semigroup, we would like to use only as many elements of S as are necessary to generate the S-act A. The following straightforward lemma is a crucial observation that will lead to the defintion of an automatic S-act.

Lemma 3.1.1. *Let A be an S-act. Then A_0 generates A if and only if there exist subsets $T_1, T \dots, T_n, \dots$ of S such that*

$$\bigcup_{a_j \in A_0} a_j.T_j = \bigcup_{a_j \in A_0} \{a_j.t \mid t \in T_j\} = A.$$

If there exist regular languages $L_1, L_2, \dots, L_n, \dots$ over X such that $L_j\varphi = T_j$, then we say that *A is regularly generated by A_0 and $\{L_1, L_2, \dots, L_n, \dots\}$*, and use the notation

$$A = \bigcup_{a_j \in A_0} a_j.L_j\varphi = \{a_j.l\varphi \mid l \in L_j\}.$$

Following semigroup and group theoretical conventions, we will also write $a_i.u = a_j.v$ and $a_i.(u \cdot x) = a_j.v$, $(u, v) \in L_i \times L_j$ instead of $a_i.u\varphi = a_j.v\varphi$ and $a_i.(u \cdot x)\varphi = a_j.v\varphi$, since the context will always make it clear over which alphabet the regular languages are being taken, and what the homomorphism from the set of all words over that alphabet to S is.

Before we give the definition of an automatic S-act A, we give examples for right and left S-acts.

Example 3.1.2. Let S be a semigroup and ξ a right congruence on S. Define $f : S/\xi \times S \to S/\xi$; $(s\xi, t) \mapsto (st)\xi$. Since ξ is a right congruence, f is a well defined function, and clearly defines a right action of S on the factor set S/ξ.

Example 3.1.3. Let S be a semigroup and ξ a left congruence on S. Define $f : S/\xi \times S \to S/\xi$; $(s\xi, t) \mapsto (ts)\xi$. Since ξ is a left congruence, f is a well defined function, and clearly defines a left action of S on the factor set S/ξ.

Example 3.1.4. Any semigroup S can be considered as a right S-act, where the action is right multiplication, that is $f : S \times S \to S$, $(s,t) \mapsto s \cdot t$. For convenience, when considering a semigroup S as a right S-act, we write $s \cdot t$ instead of $s.t$.

Example 3.1.5. Let S be a semigroup. Adjoin a zero element $\mathbf{0}$ to S and extend multiplication of S to $S^0 = S \cup \{\mathbf{0}\}$ in the usual way; $s\mathbf{0} = \mathbf{0}s = \mathbf{0}$ for all $s \in S$ and $\mathbf{0} \cdot \mathbf{0} = \mathbf{0}$. Let T be a subset of S with the condition that if

$$t \cdot (s_1 s_2) \in T, \text{ where } t \in T \text{ and } s_1, s_2 \in S, \text{ then } t \cdot s_1 \in T. \tag{3.1}$$

Let $T^0 = T \cup \{\mathbf{0}\}$. Then one can naturally consider T^0 as a right S^0-act by defining $f: T^0 \times S^0 \to T^0$ as:

$$(t,s) \mapsto \begin{cases} t \cdot s & \text{if } ts \in T, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

To verify that $(t \cdot s_1) \cdot s_2 = t \cdot (s_1 \cdot s_2)$ indeed holds for all $s_1, s_2 \in S$, we consider the following two cases:

(i) If $t \cdot s_1 \notin T$, then $t \cdot s_1 = \mathbf{0}$ and it follows that $(t \cdot s_1) \cdot s_2 = \mathbf{0} \cdot s_2 = \mathbf{0}$ for all $s_2 \in S$. On the other hand, by condition (3.1), $t \cdot (s_1 s_2) \notin T$, hence $t \cdot (s_1 s_2) = \mathbf{0}$, proving that $t \cdot (s_1 s_2) = (t \cdot s_1) \cdot s_2$.

(ii) If $t \cdot s_1 \in T$, then $(t \cdot s_1) \cdot s_2 = t \cdot (s_1 \cdot s_2)$ is immediate.

Our final example is a special case of the last example. This important subcase is the basis of the definition of a Schützenberger automatic regular semigroup.

Example 3.1.6. Let S be a semigroup and adjoin a zero element $\mathbf{0}$ to S as in Example 3.1.5. Let R be a $\mathcal{R}$-class of S. We verify that R fulfills (3.1), hence a right action of S^0 on R^0 can be defined as in Example 3.1.5. Let $r \in R,\ s_1, s_2 \in S$, such that $r \cdot (s_1 s_2) \in R$. Then $r\ \mathcal{R}\ rs_1s_2$ and by the definition of $\mathcal{R}$, there exists $t \in S$ such that $r = rs_1s_2 \cdot t$, that is $r = rs_1 \cdot s_2t$. On the other hand $rs_1 = r \cdot s_1$, verifying that $r\ \mathcal{R}\ rs_1$.

Next we introduce the notion of an automatic S-act A.

Definition 3.1.7. Let S be a semigroup generated by a finite set X. Let A be an S-act and $A_0 = \{a_1, a_2, \ldots, a_n\}$ be a finite subset of A. Let $L_1, L_2, \ldots, L_n$ be regular languages over X. We say that $(X, L_1, \ldots, L_n)$ *forms an automatic structure for A with respect to the generating set* A_0, if the following conditions hold:

(1) A is regularly generated by A_0 and $\{L_1, \ldots, L_n\}$, that is:

$$\bigcup_{j=1}^{n} a_j.L_j\varphi = A;$$

(2) $L_{(a_i,a_j)_=} = \{(u, v) \in L_i \times L_j \mid a_i.u = a_j.v\}\delta_X$ is a regular language for all $(a_i, a_j) \in A_0 \times A_0$;

(3) $L_{(a_i,a_j)_x} = \{(u, v) \in L_i \times L_j \mid a_i.(u \cdot x) = a_j.v\}\delta_X$ is a regular language for all $(a_i, a_j) \in A_0 \times A_0$ and $x \in X$.

If an S-act A has an automatic structure $(X, L_1, \ldots, L_n)$ with respect to the generating set A_0, then we say that A is *automatic with respect to the generating set* A_0. An automatic structure $(X, L_1, \ldots, L_n)$ with respect to the generating set A_0 of an S-act A is said to be *with uniqueness*, if for all $a_i, a_j \in A_0,\ a_i.u = a_j.v,\ ((u, v) \in L_i \times L_j)$ implies $a_i = a_j$ and $u \equiv v$.

The following two lemmas will prove useful. Both of the lemmas and their proofs are based on [5, Proposition 3.1.].

Lemma 3.1.8. *Let $(X, L_1, L_2, \ldots, L_n)$ be an automatic structure for the S-act A with respect to the generating set $A_0 = \{a_1, a_2, \ldots, a_n\}$. Let $b \in A$ and $x \in X$. Then for each i, $(1 \leq i \leq n)$, the language $\{w \in L_i \mid a_i.(w \cdot x) = b\}$ is regular.*

Proof. Since $\bigcup_{j=1}^{n} a_j.L_j\varphi = A$, there exists – and hence we can fix – a language L_k and $u \in L_k$ such that $a_k.u = b$. Choose a language L_i. It might happen that $\{w \in L_i \mid a_i.(w \cdot x) = b\} = \emptyset$, in which case it is a regular language by definition. Assume that $\{w \in L_i \mid a_i.(w \cdot x) = b,\ x \in X\} \neq \emptyset$ and let $v \in X^+$. Then $(v,u)\delta_X \in L_{(a_i,a_k)_x}$ if and only if $v \in L_i$ and $a_i.(v \cdot x) = a_k.u$. By Propositions 2.2.3 and 2.3.1, the language

$$K = \{(w,u) \mid w \in L_i,\ a_i.(w \cdot x) = b\}\delta_X = L_{(a_i,a_k)_x} \cap \{(v,u) \mid v \in X^+\}\delta_X$$

is regular. Making use of Proposition 2.3.1, we obtain that

$$\begin{aligned}\{w \in X^+ \mid (w,v)\delta_X \in K \text{ for some } v \in X^+\} &= \{w \in X^+ \mid (w,u)\delta_X \in K\} \\ &= \{w \in L_i \mid a_i.(w \cdot x) = b\}\end{aligned}$$

is a regular language. □

Similarly we can prove the following:

Lemma 3.1.9. *Let $(X, L_1, L_2, \ldots, L_n)$ be an automatic structure for the S-act A with respect to the generating set $A_0 = \{a_1, a_2, \ldots, a_n\}$ and let $b \in A$. Then for each i, $(1 \leq i \leq n)$, the set $\{w \in L_i \mid a_i.w = b\}$ is regular.*

Now we claim that the definition of an automatic S-act is indeed a generalization of the notion of an automatic semigroup. As in Example 3.1.4, when considering a semigroup S as a right S-act, we assume that the action is right multiplication and we write $s \cdot t$ instead of $s.t$.

Proposition 3.1.10. *A semigroup S generated by a finite set X is automatic if and only if it is automatic as a right S-act with respect to the generating set X.*

Proof. $(\Rightarrow)$ Assume that (X, L) forms an automatic structure for S, where $X = \{x_1, \ldots, x_n\}$. Then $L\varphi = S$ and the languages $L_=$ and L_x $(x \in X)$ are regular. By Lemma 2.2.5, for each j, $(1 \leq j \leq n)$, the language

$$L_j = \{w \in X^+ \mid x_j w \in L\}$$

is regular. We assume that none of the languages L_j $(1 \leq j \leq n)$ are empty. We clearly have that

$$\bigcup_{j=1}^{n} x_j \cdot L_j\varphi = L\varphi = S,$$

and hence S is regularly generated by X and $L_1, \ldots, L_n$. We claim that $(X, L_1, \ldots, L_n)$ is an automatic structure for the right S-act S with respect to the generating set X. By definition,

$$\begin{aligned} K_{(x_i,x_j)_=} &= \{(u,v) \in L_i \times L_j \mid x_i \cdot u = x_j \cdot v\}\delta_X \\ &= \{(u,v)\delta_X \in X(2,\$) \mid (x_i,x_j)(u,v)\delta_X \in L_=\}, \end{aligned}$$

and so it is a regular language by Lemma 2.2.5. Similarly,

$$\begin{aligned} K_{(x_i,x_j)_x} &= \{(u,v) \in L_i \times L_j \mid x_i \cdot u \cdot x = x_j \cdot v\}\delta_X \\ &= \{(u,v)\delta_X \in X(2,\$) \mid (x_i,x_j)(u,v)\delta_X \in L_x\} \end{aligned}$$

is a regular language by Lemma 2.2.5.

($\Leftarrow$) Conversely, assume that $(X, L_1, \ldots, L_n)$ is an automatic structure for the right S-act S with respect to the generating set X. Then $\bigcup_{j=1}^n x_j \cdot L_j\varphi = S$ and so $K = \bigcup_{j=1}^n x_j L_j$ is a regular language for which $K\varphi = S$. We show that (X, K) is an automatic structure for S. By definition,

$$\begin{aligned} K_= &= \{(u,v) \mid u,v \in K,\ u = v\}\delta_X \\ &= \{(u,v) \mid u \equiv x_i k,\ v \equiv x_j l,\ (k,l) \in L_i \times L_j,\ x_i k = x_j l\}\delta_X \\ &= \bigcup_{1 \le i,j \le n} (x_i,x_j) L_{(x_i,x_j)_=}, \end{aligned}$$

which is a finite union of regular languages, and hence is regular. Similarly, for each $x \in X$, the language

$$\begin{aligned} K_x &= \{(u,v) \mid u,v \in K,\ u \cdot x = v\}\delta_X \\ &= \{(u,v) \mid u \equiv x_i k,\ v \equiv x_j l,\ (k,l) \in L_i \times L_j,\ x_i k x = x_j l\}\delta_X \\ &= \bigcup_{1 \le i,j \le n} (x_i,x_j) L_{(x_i,x_j)_x} \end{aligned}$$

is regular, since it is a finite union of regular languages. □

For monoids the following proposition holds.

Proposition 3.1.11. *A monoid M is automatic if and only if it is automatic as a right M-act with respect to the generating set $\{1\}$.*

Proof. That $\{1\}$ is a generating set for the M-act M is immediate, since $1 \cdot M = M$. The lemma now follows from the definitions. □

3.2 Schützenberger automatic regular semigroups

It was proved in [8], that free inverse semigroups are not automatic. On the other hand, free inverse semigroups have solvable word problem [22], [27]. These facts motivate us to introduce a new definition of automaticity for regular semigroups, namely that of a Schützenberger automatic regular semigroup. We formulate the new notion in the framework of automatic S-acts, where the S-acts are $\mathcal{R}$-classes of the semigroup. In the next section, we show that free inverse semigroups are Schützenberger automatic.

Being Schützenberger automatic means that the regular semigroup satisfies a collection of local properties. As indicated above these local properties are defined on the $\mathcal{R}$-classes of the regular semigroup. By introducing $\mathcal{R}$-class automaticity in terms of left actions of S, we connect the $\mathcal{R}$-classes of S. Finally, combining the definition of Schützenberger automaticity and $\mathcal{R}$-class automaticity, we introduce the notion of a strongly Schützenberger automatic regular semigroup.

Throughout this chapter S will denote a semigroup, X a generating set for S. We assume that $X \subseteq S$ and denote by $\varphi : X^+ \to S$ the homomorphism extending the identity map $\iota : X \to S$.

Before we give the definition of a Schützenberger automatic regular semigroup, we turn our attention to the right S^0-act R^0 given in Example 3.1.6.

Proposition 3.2.1. *Let S be a regular semigroup generated by a finite set X, and let R be an $\mathcal{R}$-class of S. Then the following are equivalent:*

(S1) There exists a regular language L over $X^0 = X \cup \{0\}$ such that (X^0, L) forms an automatic structure for the right S^0-act R^0 with respect to some one element generating set $\{s\}$.

(S2) There exists a regular language K over X and an element $s \in R$ such that the following conditions hold:

(i) $s \cdot K\varphi = R$;

(ii) $K_= = \{(u,v) \mid u,v \in K,\ s \cdot u = s \cdot v\}\delta_X$ is a regular language;

(iii) $K_x = \{(u,v) \mid u,v \in K,\ s \cdot (u \cdot x) = s \cdot v\}\delta_X$ is a regular language for all $x \in X$.

(S3) There exists a regular language N over X such that the following conditions hold:

(i) $N\varphi = R$;

(ii) $N_= = \{(u,v) \mid u,v \in N,\ u = v\}\delta_X$ *is a regular language;*

(iii) $N_x = \{(u,v) \mid u,v \in N,\ u \cdot x = v\}\delta_X$ *is a regular language for all* $x \in X$.

Proof. $(S1) \Rightarrow (S2)$ Assume that (X^0, L) is an automatic structure for the right S^0-act R^0 with respect to the generating set $\{s\}$. Then $s \cdot L\varphi = R^0$. By Lemma 3.1.9, $L' = \{l \in L \mid s \cdot l = \mathbf{0}\}$ is a regular language, hence $K = L - L'$ is also a regular language and $s \cdot K\varphi = R$. It is also obvious that

$$\begin{aligned} K_= &= \{(u,v) \mid u,v \in K,\ s \cdot u = s \cdot v\}\delta_X \\ &= L_= - \{(u,v) \mid u,v \in L'\}\delta_X \\ &= L_= - (L' \times L')\delta_X, \end{aligned}$$

and so $K_=$ is a regular language. To show that K_x is a regular language, note that

$$K_x = L_x - (L' \times L')\delta_X - \{(u,v) \in L \times L' \mid s \cdot (u \cdot x) = s \cdot v\}\delta_X.$$

Let $L'' = \{u \in L \mid s \cdot (u \cdot x) = \mathbf{0}\}$. By Lemma 3.1.8, L'' is a regular language. It follows that $\{(u,v) \in L \times L' \mid s \cdot (u \cdot x) = s \cdot v\}\delta_X = (L'' \times L')\delta_X$ is also a regular language, and hence we may deduce that K_x is a regular language.

$(S2) \Rightarrow (S3)$ Assume that there exists a regular language K over X such that conditions $(i) - (iii)$ of $(S2)$ hold. Let w be a word over X representing s. Then we have $R = s \cdot K\varphi = (wK)\varphi$. Define $N = wK$. It is obvious that N is a regular language and that $N\varphi = R$. Moreover

$$\begin{aligned} N_= &= \{(u,v) \mid u,v \in N,\ u = v\}\delta_X \\ &= \{(u,v) \mid u \equiv wk_1,\ v \equiv wk_2,\ wk_1 = wk_2\}\delta_X \\ &= \{(u,v) \mid u \equiv wk_1,\ v \equiv wk_2,\ s \cdot k_1 = s \cdot k_2\}\delta_X \\ &= (w,w)\delta_X K_=, \end{aligned}$$

hence is a regular language. Similarly we obtain that

$$N_x = \{(u,v) \mid u,v \in N,\ u \cdot x = v\}\delta_X = (w,w)\delta_X K_x$$

is a regular language for all $x \in X$.

$(S3) \Rightarrow (S1)$ Assume that there exists a regular language N over X such that conditions $(i) - (iii)$ of $(S3)$ hold. Adjoin a zero element $\mathbf{0}$ to S and extend multiplication in the usual way. Define a right action of S^0 on R^0 as before:

$$(r,s) \mapsto \begin{cases} r \cdot s & \text{if } rs \in R, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Let e be an idempotent element of R. Such an element of R exists, since S is a regular semigroup. Consider the regular language $L = N \cup \{\mathbf{0}\}$. Recall that e is a left identity in its $\mathcal{R}$-class. Then $e \cdot L\varphi = e \cdot (N \cup \{\mathbf{0}\})\varphi = R^0$. Since $N\varphi = R$, we obtain that

$$\begin{aligned} L_= &= \{(u,v) \mid u,v \in L,\ e\cdot u = e\cdot v\}\delta_X \\ &= \{(u,v) \mid u,v \in N,\ u = v\}\delta_X \cup \{(\mathbf{0},\mathbf{0})\} \\ &= N_= \cup \{(\mathbf{0},\mathbf{0})\}. \end{aligned}$$

Similarly, if $x \in X$ then

$$L_x = \{(u,v) \mid u,v \in L,\ e\cdot u\cdot x = e\cdot v\}\delta_X = N_x \cup \{(\mathbf{0},\mathbf{0})\},$$

and we have that

$$L_{\mathbf{0}} = \{(u,v) \mid u,v \in L,\ e\cdot u\cdot \mathbf{0} = e\cdot v\}\delta_X = (\mathbf{0},\mathbf{0})$$

which verifies that $(X^{\mathbf{0}}, L)$ is indeed an automatic structure for the right S^0-act R^0 with respect to the generating set $\{e\}$. $\square$

We note that in the proof of Proposition 3.2.1 the assumption that S is a regular semigroup was only used to deduce that the $\mathcal{R}$-class R contains an idempotent element. Hence, Proposition 3.2.1 can be rephrased for all such $\mathcal{R}$-classes of an arbitrary semigroup that contains an idempotent. On the other hand, if we assume that S is a semigroup in which every $\mathcal{R}$-class contains an idempotent, then we have that S has to be a regular semigroup.

We introduce now the definition of a Schützenberger automatic regular semigroup S.

Definition 3.2.2. Let S be a regular semigroup generated by a finite set X, and L be a regular language over X. Let R be an $\mathcal{R}$-class of S. We say that (X, L) is a *Schützenberger automatic structure for* R, if the following conditions are satisfied:

(i) $L\varphi = R$;

(ii) $L_= = \{(u,v) \mid u,v \in L,\ u = v\}\delta_X$ is a regular language;

(iii) $L_x = \{(u,v) \mid u,v \in L,\ u\cdot x = v\}\delta_X$ is a regular language for all $x \in X$.

If an $\mathcal{R}$-class R has a Schützenberger automatic structure (X, L), then we say that R is Schützenberger automatic. A Schützenberger automatic structure (X, L) of an $\mathcal{R}$-class R is said to be *with uniqueness*, if $u = v$ $(u, v \in L)$ implies $u \equiv v$. Define a regular semigroup S to be *Schützenberger automatic*, if all of its $\mathcal{R}$-classes are Schützenberger automatic.

Although Proposition 3.2.1 provides three different descriptions of a Schützenberger automatic regular semigroup, we will usually work with $(S2)$ and $(S3)$. Clearly conditions of $(S3)$ seem to be the easiest to check. On the other hand, the advantage of $(S2)$ is that if $k \in K$, then for every prefix $\tilde{k}$ of k, $s \cdot \tilde{k} \in R_s$, a property that is not necessarily true when considering description $(S3)$. That is if $n \in N$ and $\tilde{n}$ is a prefix of n, then usually $\tilde{n} \notin R$.

The next proposition tells us, that the notion of a Schützenberger automatic regular semigroup is in fact a generalization of the notion of an automatic group.

Proposition 3.2.3. *Let G be a group. Then the following are equivalent:*

(1) G is automatic.

(2) The right G-act G is automatic.

(3) G is Schützenberger automatic.

Proof. (1) $\Leftrightarrow$ (2) See Proposition 3.1.10.

(1) $\Leftrightarrow$ (3) Follows from the fact that if G is a group then $\mathcal{R} = G \times G$. $\square$

Schützenberger automaticity of a regular semigroup S is a collection of local properties. To link the $\mathcal{R}$-classes of S we define $\mathcal{R}$-class automaticity in terms of left S-actions on the partially ordered set $S/\mathcal{R}$.

First we need to define a left action of a semigroup S on $S/\mathcal{R}$. Let

$$f \colon S/\mathcal{R} \times S \to S/\mathcal{R};\ (R_t, s) \mapsto R_{st}.$$

Since R is a left congruence on S, f clearly defines a left S-act.

For our purposes, we will define $\mathcal{R}$-class automaticity of a semigroup S when S is a monoid.

Definition 3.2.4. Let S be a monoid generated by a finite X. Let L be a regular language over X. We say that (X, L) *is an $\mathcal{R}$-class automatic structure for $S/\mathcal{R}$*, if (X, L) is an automatic structure for the left S-act $S/\mathcal{R}$ with respect to the generating set R_1. That is,

(1) $R_1 \cdot L\varphi = S/\mathcal{R}$;

(2) $L_= = \{(u, v) \mid u, v \in L,\ R_1.u = R_1.v\}\delta_X$ is a regular language;

(3) $L_x = \{(u, v) \mid u, v \in L,\ R_1.(u \cdot x) = R_1.v\}\delta_X$ is a regular language for all $x \in X$.

Combining the notions of Schützenberger automaticity and $\mathcal{R}$-class automaticity we introduce the notion of a strongly Schützenberger automatic regular semigroup.

Definition 3.2.5. We say that a regular semigroup is *strongly Schützenberger automatic*, if it is Schützenberger automatic, and $\mathcal{R}$-class automatic.

3.3 Examples

In this section we give several examples for automatic S-acts. Throughout this section S will denote a semigroup, X a finite generating set for S. For the sake of simplicity we assume that $X \subseteq S$ and denote by $\varphi : X^+ \to S$ the homomorphism extending the identity map $\iota : X \to S$.

Proposition 3.3.1. *Let A be a finite S-act so that the action $f : A \times S \to A$ is surjective. Then A is automatic.*

Proof. Let $A = \{a_1, \ldots, a_n\}$ and for each $a_i \in A$, fix an element $t_i \in S$ and an element $b_i \in A$ such that $a_i = b_i.t_i$. Since f is a surjective map, such elements exist. Let $T = \{t_1, \ldots, t_n\}$. For each $t_i \in T$ fix a word w_i over X for which $w_i\varphi = t_i$ and let $L_i = \{w_i\}$. Clearly A is regularly generated by $L_1, \ldots, L_n$ and $\{b_1, \ldots, b_n\}$. On the other hand the languages $L_{(b_i,b_j)_=}$ and $L_{(b_i,b_j)_x}$ $(x \in X)$ are finite, and hence they are regular, proving that A is indeed automatic. $\square$

Corollary 3.3.2. *Let S be a monoid such that $S/\mathcal{R}$ is finite. Then S is $\mathcal{R}$-class automatic.*

Proof. The left action $f : S/\mathcal{R} \times S \to S/\mathcal{R}$ is clearly surjective. Hence if $S/\mathcal{R}$ is finite, then $S/\mathcal{R}$ is an automatic left S-action by Proposition 3.3.1, proving that S is indeed $\mathcal{R}$-class automatic. $\square$

Corollary 3.3.3. *Let R be a finite $\mathcal{R}$-class of a regular semigroup S. Then R is Schützenberger automatic.*

Proof. If R is a finite $\mathcal{R}$-class of S, then the right S^0-act R^0 is finite, moreover the right action $f : R^0 \times S^0 \to R^0$ is surjective. Hence by Proposition 3.3.1, the right S^0-act R^0 is automatic. Making use of Proposition 3.2.1 we obtain that R is Schützenberger automatic. $\square$

We have the following immediate consequence of Corollary 3.3.3:

Corollary 3.3.4. *Let S be a regular semigroup whose $\mathcal{R}$-classes are finite. Then S is Schützenberger automatic.*

Combining Corollaries 3.3.2 and 3.3.3 we obtain:

Corollary 3.3.5. *Let S be a finite regular semigroup. Then S is strongly Schützenberger automatic.*

Next, we give two examples for infinite automatic S-acts.

Example 3.3.6. Let A be the double infinite chain with an identity adjoined on top:

$$\ldots < e_{-1} < e_0 < e_1 < \ldots < \mathbf{1}.$$

Let $G = \langle g \rangle$ be the infinite cyclic group. Let $n, k \in \mathbb{Z}$ and define a right action of G on A in the following way:

$$f : A \times G \to A; \quad (e_n, g^k) \mapsto e_n.g^k = e_{n+k}; \quad (\mathbf{1}, g^k) \mapsto \mathbf{1}.g^k = \mathbf{1}.$$

We claim that the right G-act A is automatic. (We note that the defined action can be considered as a left action of G on A as well and it can be verified that the left G-act A is automatic.)

Let $X = \{g, g^{-1}\}$, $A_0 = \{e_0, \mathbf{1}\}$, $L_0 = g^+ \cup (g^{-1})^+ \cup \{gg^{-1}\}$ and $L_1 = \{g\}$. By definition $e_n = e_0.g^n$, $n \in \mathbb{Z}$, thus A is regularly generated by A_0 and the regular languages L_0, L_1. Clearly we have that the languages $L_{(e_0,\mathbf{1})_=}, L_{(e_0,\mathbf{1})_x}, L_{(\mathbf{1},e_0)_=}, L_{(\mathbf{1},e_0)_x}$ $(x \in X)$ are empty, and hence they are regular. Moreover we have that

$$\begin{aligned}
L_{(\mathbf{1},\mathbf{1})_=} &= \{(u,v) \in L_1 \times L_1 \mid \mathbf{1}.u = \mathbf{1}.v\}\delta_X = (g,g);\\
L_{(\mathbf{1},\mathbf{1})_x} &= \{(u,v) \in L_1 \times L_1 \mid \mathbf{1}.(u \cdot x) = \mathbf{1}.v\}\delta_X = (g,g);\\
L_{(e_0,e_0)_=} &= \{(u,v) \in L_0 \times L_0 \mid e_0.u = e_0.v\}\delta_X\\
&= (g,g)^+\delta_X \cup (g^{-1},g^{-1})^+\delta_X \cup (gg^{-1},gg^{-1})\delta_X;\\
L_{(e_0,e_0)_g} &= \{(u,v) \in L_0 \times L_0 \mid e_0.(u \cdot g) = e_0 \cdot v\}\delta_X\\
&= (g,g)^+\delta_X(\$,g) \cup (g^{-1},g^{-1})(g^{-1},g^{-1})^+\delta_X(g^{-1},\$)\\
&\quad \cup (g^{-1},g)(\$,g^{-1}) \cup (g,g)(g^{-1},\$)\\
L_{(e_0,e_0)_{g^{-1}}} &= \{(u,v) \in L_0 \times L_0 \mid e_0.(u \cdot g^{-1}) = e_0 \cdot v\}\delta_X\\
&= (g^{-1},g^{-1})^+\delta_X(\$,g^{-1}) \cup (g,g)(g,g)^+\delta_X(g,\$)\\
&\quad \cup (g,g)(\$,g^{-1}) \cup (g,g^{-1})(g^{-1},\$).
\end{aligned}$$

These languages are regular, and hence we obtain that (X, L_0, L_1) forms an automatic structure with respect to the generating set A_0 for A. $\square$

Example 3.3.7. Let B be the infinite antichain

$$\ldots, e_{-1}, e_0, e_1, \ldots.$$

Adjoin an identity $\mathbf{1}$ on top and a zero $\mathbf{0}$ on bottom, that is for each $e_i \in B$, $\mathbf{0} < e_i < \mathbf{1}$ holds. Let A denote the lattice obtained. Let $G = \langle g \rangle$ be the infinite cyclic group. Let $n, k \in \mathbb{Z}$ and define a right action of G on A in the following way:

$$f : A \times G \to A; \quad (e_n, g^k) \mapsto e_{n+k}; \quad (\mathbf{1}, g^k) \mapsto \mathbf{1}; \quad (\mathbf{0}, g^k) \mapsto \mathbf{0}.$$

We claim that the right G-act A is automatic. (We note that the defined action can be considered as a left action of G on A as well and it can be verified that the left G-act A is automatic.)

Let $X = \{g, g^{-1}\}$, $A_0 = \{e_0, \mathbf{1}, \mathbf{0}\}$, $L_0 = g^+ \cup (g^{-1})^+ \cup \{gg^{-1}\}$, $L_1 = L_2 = \{g\}$. By definition, $e_n = e_0.g^n$, $n \in \mathbb{Z}$, thus A is regularly generated by A_0 and the regular languages L_0, L_1, L_2. Let

$$(a, b) \in \{(\mathbf{1}, \mathbf{0}), (\mathbf{0}, \mathbf{1}), (e_0, \mathbf{1}), (\mathbf{1}, e_0), (e_0, \mathbf{0}), (\mathbf{0}, e_0)\}.$$

Clearly, the languages $L_{(a,b)_=}, L_{(a,b)_x}, L_{(a,b)_=}, L_{(a,b)_x}$ $(x \in X)$ are empty, and hence they are regular. Moreover we have that

$$\begin{aligned}
L_{(\mathbf{1},\mathbf{1})_=} &= \{(u, v) \in L_1 \times L_1 \mid \mathbf{1}.u = \mathbf{1}.v\}\delta_X = (g, g); \\
L_{(\mathbf{1},\mathbf{1})_x} &= \{(u, v) \in L_1 \times L_1 \mid \mathbf{1}.(u \cdot x) = \mathbf{1}.v\}\delta_X = (g, g);
\end{aligned}$$

for all $x \in X$. Similarly

$$\begin{aligned}
L_{(\mathbf{0},\mathbf{0})_=} &= \{(u, v) \in L_2 \times L_2 \mid \mathbf{0}.u = \mathbf{0}.v\}\delta_X = (g, g); \\
L_{(\mathbf{0},\mathbf{0})_x} &= \{(u, v) \in L_2 \times L_2 \mid \mathbf{0}.(u \cdot x) = \mathbf{0}.v\}\delta_X = (g, g);
\end{aligned}$$

for all $x \in X$. Also we have that

$$\begin{aligned}
L_{(e_0,e_0)_=} &= \{(u, v) \in L_0 \times L_0 \mid e_0.u = e_0.v\}\delta_X \\
&= (g, g)^+\delta_X \cup (g^{-1}, g^{-1})^+\delta_X \cup (gg^{-1}, gg^{-1})\delta_X; \\
L_{(e_0,e_0)_g} &= \{(u, v) \in L_0 \times L_0 \mid e_0.(u \cdot g) = e_0 \cdot v\}\delta_X \\
&= (g, g)^+\delta_X(\$, g) \cup (g^{-1}, g^{-1})(g^{-1}, g^{-1})^+\delta_X(g^{-1}, \$) \\
&\quad \cup (g^{-1}, g)(\$, g^{-1}) \cup (g, g)(g^{-1}, \$) \\
L_{(e_0,e_0)_{g^{-1}}} &= \{(u, v) \in L_0 \times L_0 \mid e_0.(u \cdot g^{-1}) = e_0 \cdot v\}\delta_X \\
&= (g^{-1}, g^{-1})^+\delta_X(\$, g^{-1}) \cup (g, g)(g, g)^+\delta_X(g, \$) \\
&\quad \cup (g, g)(\$, g^{-1}) \cup (g, g^{-1})(g^{-1}, \$)
\end{aligned}$$

These languages are regular, and hence we obtain that (X, L_0, L_1, L_2) forms an automatic structure with respect to the generating set A_0 for A. $\square$

Example 3.3.8. The bicyclic monoid is strongly Schützenberger automatic.

To verify this, consider the bicyclic monoid given by the monoid presentation $B = \langle p, q \mid pq = 1\rangle$. We let $X = \{p, q\}$ and denote by $\varphi : X^+ \to S$ the homomorphism extending the identity map $\iota : X \to S$. Each element $s \in B$ can be written as $s = q^i p^j$, $(i, j \in \mathbb{N})$, moreover $q^i p^j \mathrel{\mathcal{R}} q^n p^m$ if and only if $i = n$. To show that B is $\mathcal{R}$-class automatic, consider the regular language $K = q^+ \cup qp$. Clearly $R_1.K\varphi = B/\mathcal{R}$. On the other hand

$$\begin{aligned}
K_= &= \{(u,v) \in K \times K \mid R_1.u = R_1.v\}\delta_X = \{(u,u) \mid u \in K\}\delta_X,\\
K_p &= \{(u,v) \in K \times K \mid R_1.u \cdot p = R_1.v\}\delta_X = (q,q)^+\delta_X(q,\$) \cup (qp,qp)\delta_X,\\
K_q &= \{(u,v) \in K \times K \mid R_1.u \cdot q = R_1.v\}\delta_X = (q,q)^+\delta_X(\$,q) \cup (qp,q)\delta_X,
\end{aligned}$$

and so (X, K) indeed forms an automatic structure for the left B-act $B/\mathcal{R}$.

To show that B is Schützenberger automatic, consider an arbitrary $\mathcal{R}$-class R of B. Assume that $q^i p^j \in R$. As we have noted above, any element of R is of the form $q^i p^k$, $k \in \mathbb{N}$. Let $L = q^i p^*$. Clearly $L\varphi = R$ and we have that

$$\begin{aligned}
L_= &= \{(u,v) \in L \times L \mid u = v\}\delta_X = \{(u,u) \mid u \in L\}\delta_X,\\
L_p &= \{(u,v) \in L \times L \mid u \cdot p = v\}\delta_X = (q,q)^i\delta_X(p,p)^*\delta_X(\$,p),\\
L_q &= \{(u,v) \in L \times L \mid u \cdot q = v\}\delta_X = (q,q)^i\delta_X(p,p)^*\delta_X(p,\$),
\end{aligned}$$

proving that (X, L) is a Schützenberger automatic structure for R. Since R was an arbitrary $\mathcal{R}$-class of B, it follows that B is Schützenberger automatic. $\square$

Example 3.3.9. Polycyclic monoids are strongly Schützenberger automatic.

The polycyclic monoid on n generators $(n \geq 2)$ is given by the presentation

$$P_n = \langle p_1, \ldots, p_n, p_1^{-1}, \ldots, p_n^{-1} \mid p_i p_i^{-1} = 1, p_i p_j^{-1} = 0 \text{ for } i \neq j\rangle.$$

Consider the generating set $X = \{0, 1, p_1, \ldots, p_n, p_1^{-1}, \ldots, p_n^{-1}\}$ for P_n. Denote by $\varphi : X^+ \to P_n$ the homomorphism extending the identity map $\iota : X \to P_n$. To make notation convenient, we let $Y = \{p_1, \ldots, p_n\}$ and $Y^{-1} = \{p_1^{-1}, \ldots, p_n^{-1}\}$. Every non-zero element of P_n can be uniquely written as $u^{-1}v$, where $u, v \in Y^+$. Moreover $u^{-1}v \mathrel{\mathcal{R}} x^{-1}y$, $(u, v, x, y \in$

Y^+) if and only if u and x are identitcal words ([27, Chapter 9.3]). To show that P_n is $\mathcal{R}$-class automatic, we consider the regular language $K = \{0,1\} \cup (Y^{-1})^+$. Clearly $R_1.K\varphi = P_n/\mathcal{R}$. On the other hand, we have that

$$K_= = \{(u,v) \in K \times K \mid R_1.u = R_1.v\}\delta_X = \{(u,u) \mid u \in K\}\delta_X = K_1,$$
$$K_0 = \{(u,v) \in K \times K \mid R_1.u \cdot 0 = R_1.v\}\delta_X = (0,0) \cup (1,0) \cup ((Y^{-1})^+ \times \{0\})\delta_X,$$

and hence they are regular languages. To show that K_{p_i} and $K_{p_i^{-1}}$ are regular languages for all $1 \leq i \leq n$, we make the following observations. By Lemma 2.2.5, the language

$$M_i = \{u \in (Y^{-1})^+ \mid u \equiv p_i^{-1}w\}$$

is regular, and so by Proposition 2.2.3, the language

$$N_i = (Y^{-1})^+ \setminus M_i^{\mathrm{rev}}$$

is also regular. By Proposition 2.3.2, the language $A = \{(u,u) | u \in (Y^{-1})^*\}\delta_X$ is regular. Using the above introduced notation, we obtain that

$$K_{p_i} = \{(u,v) \in K \times K \mid R_1.u \cdot p_i = R_1.v\}\delta_X = (N_i \times \{0\})\delta_X \cup A(p_i^{-1}, \$) \cup (0,0) \cup (1,1)$$

and

$$K_{p_i^{-1}} = \{(u,v) \in K \times K \mid R_1.u \cdot p_i^{-1} = R_1.v\}\delta_X = A(\$, p_i^{-1}) \cup (0,0) \cup (1, p_i^{-1})$$

are regular languages. Hence, we may deduce that P_n is indeed $\mathcal{R}$-class automatic.

Next, we show that P_n is Schützenberger automatic. Clearly the $\mathcal{R}$-class of 0 is Schützenberger automatic. Consider an $\mathcal{R}$-class $R \neq R_0$ of P_n and assume that $u^{-1}v \in R$, where $u, v \in Y^+$. As we mentioned before, every element of R is of the form $u^{-1}w$, where $w \in Y^*$. Consider the regular language $L = u^{-1}Y^*$. Clearly $L\varphi = R$, moreover

$$L_= = \{(v,w) \in L \times L \mid v = w\}\delta_X = \{(v,v) \mid v \in L\}\delta_X = L_1,$$
$$L_0 = \{(v,w) \in L \times L \mid v \cdot 0 = w\}\delta_X = \emptyset$$

To prove that L_{p_i} and $L_{p_i^{-1}}$ are regular languages for all $1 \leq i \leq n$, we make the following observations. By Lemma 2.2.5, the language

$$\overline{M}_j = \{v \in Y^+ \mid v \equiv wp_j\}$$

is regular, and so by Proposition 2.2.3, the language

$$N_i = \bigcup_{j=1, i \neq j}^{n} \overline{M}_j$$

is also regular. By Proposition 2.3.2, the language $B = \{(v,v) | v \in Y^*\}\delta_X$ is regular. Using the above introduced notation, we obtain that

$$L_{p_i} = \{(v,w) \in L \times L \mid v \cdot p_i = w\}\delta_X = (u^{-1}, u^{-1})\delta_X B(\$, p_i)$$
$$L_{p_i^{-1}} = \{(v,w) \in L \times L \mid v \cdot p_i^{-1} = w\}\delta_X = (u^{-1}, u^{-1})\delta_X B(p_i, \$) \cup (u^{-1}N_i \times \{0\})\delta_X,$$

proving that (X, L) is a Schützenberger automatic structure for R. It follows that B is Schützenberger automatic. □

Proposition 3.3.10. *Let X be a non-empty finite set. Then the following hold:*

(1) The free band and the free semilattice on X are strongly Schützenberger automatic.

(2) The free completely simple semigroup on X is strongly Schützenberger automatic.

(3) The free inverse semigroup on X is Schützenberger automatic.

Proof. (1) It is known that if a band or a semilattice B is finitely generated then it is finite. The assertion hence follows from Corollary 3.3.5.

(2) We first describe the free completely simple semigroup on X. Fix $z \in X$ and let $Y = \{p_{xy} \mid x, y \in X \setminus \{z\}\}$. Let $Z = X \cup Y$. Consider the free group $FG(Z)$ and denote its identity element by 1. For all $x \in X$ let $p_{zx} = p_{xz} = 1$, and consider the $X \times X$-matrix P with (x, y) entry p_{xy}. Then the free completely simple semigroup on X is isomorphic to the Rees matrix semigroup $CS_X = \mathcal{M}(X, FG(Z), X; P)$. Since X is finite, CS_X has finitely many $\mathcal{R}$-classes, and hence is $\mathcal{R}$-class automatic by Corollary 3.3.4. In Section 9, we will verify that a completely simple semigroup is Schützenberger automatic if and only if all of its maximal subgroups are automatic. Clearly the maximal subgroups of CS_X are isomorphic to the free group $FG(Z)$, which is known to be automatic, and hence the assertion follows.

(3) It is known that the $\mathcal{R}$-classes of the free inverse semigroup $FIS(X)$ are finite, and hence by Corollary 3.3.4, $FIS(X)$ is indeed Schützenberger automatic. □

3.4 Basic properties

One naturally expects automatic S-acts to bear certain properties: for example, having automatic structures with uniqueness, invariance under the change of generators of S and of the S-act A, etc. In this section we discuss the concept of an automatic structure with uniqueness and the invariance of $\mathcal{R}$-classes being Schützenberger automatic within a $\mathcal{D}$-class. Finally, in preparation for the fifth section (discussion of automaticity being invariant under the change of generators), we introduce three technical lemmas. Throughout this section S will denote a semigroup, X a finite generating set for S. We assume that $X \subseteq S$ and denote by $\varphi : X^+ \to S$ the homomorphism extending the identity map $\iota : X \to S$. We begin this section with the following observation.

Lemma 3.4.1. *Let A be an S-act regularly generated by $L_1, \dots, L_n \subseteq X^+$ and A_0. Assume that $L_{(a_i,a_j)_=}$, $a_i, a_j \in A_0$ is a regular language. Then $L_{(a_j,a_i)_=}$ is also a regular language.*

Proof. Let $\mathcal{A} = (\Sigma, X(2,\$), \mu, p, F)$ be a finite state automaton accepting the language $L_{(a_i,a_j)_=}$. Consider the automaton $\mathcal{B} = (\Sigma, X(2,\$), \tilde{\mu}, p, F)$, where $\tilde{\mu}(q,(x,y)) = \mu(q,(y,x))$. Since $(u,v) \in L_{(a_i,a_j)_=}$ if and only if $(v,u) \in L_{(a_j,a_i)_=}$, it is clear, that $\mathcal{B}$ is the automaton accepting $L_{(a_j,a_i)_=}$. $\square$

Automatic structure with uniqueness

We prove in two steps that if A is an automatic S-act, then there exists an automatic structure with uniqueness. In the first step we show that if A is an automatic S-act, then there exists an automatic structure $(X, K_1, \dots, K_m)$ with respect to a generating set $B = \{a_1, \dots, a_m\}$, such that $a_i.K_i\varphi \cap a_j.K_j\varphi = \emptyset$ for all $1 \leq i,j \leq m$, $i \neq j$. That is, $a_i.u = a_j.v$, $(u,v) \in K_i \times K_j$ implies $a_i = a_j$. In the second step – following the semigroup and group theoretical approach to the problem, by ordering the alphabet X and introducing *shortlex order* on X^+ – we construct regular languages $L_1, \dots, L_m$ from $K_1, \dots, K_m$, such that $(X, L_1, \dots, L_m)$ will form an automatic structure with uniqueness for A.

Lemma 3.4.2. *Let $(X, L_1, \dots, L_n)$ be an automatic structure with respect to the generating set $A_0 = \{a_1, \dots, a_n\}$ for the S-act A. Then there exists a subset $B = \{b_1, \dots, b_m\} \subseteq A_0$ and regular languages $K_1, \dots, K_m$ such that the following hold:*

(i) $b_i.K_i\varphi \cap b_j.K_j\varphi = \emptyset$ *for all* $1 \leq i,j \leq m$, $i \neq j$.

(ii) $(X, K_1, \dots, K_m)$ *is an automatic structure for A with respect to the generating set B.*

Proof. For each i $(1 \leq i \leq n-1)$, we let

$$\widetilde{L}_i = \bigcup_{j=i+1}^{n} \{u \in L_i \mid (u,v)\delta_X \in L_{(a_i,a_j)_=} \text{ for some } v \in L_j\}.$$

Since $L_{(a_i,a_j)_=}$ is a regular language for each j $(i+1 \leq j \leq n)$, the languages

$$\{u \in L_i \mid (u,v)\delta_X \in L_{(a_i,a_j)_=} \text{ for some } v \in L_j\}$$

are regular by Proposition 2.3.1. Thus $\widetilde{L}_i$ is a regular language, since it is a finite union of regular languages. Define for each i $(1 \leq i \leq n-1)$

$$K_i = L_i - \widetilde{L}_i \qquad \text{and let} \qquad K_n = L_n.$$

Note that we only kept those elements u of L_i for which there exists no generator a_j, $i < j$ of A and $v \in L_j$ such that $a_i.u = a_j.v$. By Proposition 2.2.3, $K_1, K_2, \ldots, K_n$ are regular languages. It might happen that $K_i = \emptyset$, in which case a_i is a surplus generator, since then, for all $u \in L_i$, there exists $v \in L_j$, $i < j$, such that $a_i.u = a_j.v$. Let $B = \{a_i \in A_0 \mid K_i \neq \emptyset\}$.

That $a_i.K_i\varphi \cap a_j.K_j\varphi = \emptyset$, where $a_i, a_j \in B$, $i \neq j$, $1 \leq i,j \leq n$ follows from the definition of the regular languages $K_1, \ldots, K_m$. To give a more detailed reason: the equation $a_i.u = a_j.v$, where $(u,v) \in K_i \times K_j$ cannot hold, since if $i < j$, then by the definition of K_i, u should have been removed from L_i, and similarly if $j < i$, then v should have been removed from L_j.

To prove that an automatic structure for A is obtained with respect to the generating set B, we first verify that we did not subtract necessary elements from $L_1, \ldots, L_n$, that is we show that

$$\bigcup_{a_j \in B} a_j.K_j\varphi = A.$$

Let $b \in A$. Since $(X, L_1, \ldots, L_n)$ is an automatic structure for A with respect to the generating set A_0, there exists $a_i \in A_0$ and $u \in L_i$ such that $a_i.u\varphi = b$. If $u \in K_i$, then we are finished. If $u \notin K_i$, then there exists $a_j \in A_0$ and $v \in L_j$ such that $a_i.u = a_j.v$ and $i < j$. Obviously we can suppose that j is the greatest such index, hence it follows by the definition of K_j that $v \in K_j$, hence $b = a_j.v\varphi \in a_j.K_j\varphi$.

It is immediate that for all $(a_i, a_j) \in B \times B$,

$$K_{(a_i,a_j)_=} = \{(u,v) \in K_i \times K_j \mid a_i.u = a_j.v\}\delta_X = L_{(a_i,a_j)_=} \cap (K_i \times K_j)\delta_X,$$

hence is a regular language. Similarly we obtain that for all $x \in X$,

$$K_{(a_i,a_j)_x} = \{(u,v) \in K_i \times K_j \mid a_i.(u \cdot x) = a_j.v\}\delta_X = L_{(a_i,a_j)_x} \cap (K_i \times K_j)\delta_X,$$

is a regular language. $\square$

We have the following immediate consequence of Lemma 3.4.2.

Corollary 3.4.3. *If A is an automatic S-act, then there exists an automatic structure $(X, K_1, \ldots, K_m)$ with respect to some generating set B, with the help of which A can be partitioned into m subsets $A_1, \ldots, A_m$; that is to say that*

$$\bigcup_{j=1}^{m} A_j = A \quad \text{and} \quad A_i \cap A_j = \emptyset \ \text{ for all } 1 \leq i,j \leq m,\ i \neq j.$$

Proof. Let A be an automatic S-act. Making use of Lemma 3.4.2, we may assume that $(X, K_1, \ldots, K_m)$ is an automatic structure with respect to a generating set $B = \{b_1, \ldots, b_m\}$, such that $b_i.K_i\varphi \cap b_j.K_j\varphi = \emptyset$ for all $1 \leq i,j \leq m$, $i \neq j$. Define $A_i = b_i.K_i\varphi$. It is immediate that $A_i \cap A_j = \emptyset$ for all $1 \leq i,j \leq m$, $i \neq j$. Furthermore, since $\bigcup_{j=1}^{m} b_j.K_j\varphi = A$, we also have that $\bigcup_{j=1}^{m} A_j = A$. $\square$

Before we turn to the second step, we recall the definition of the *shortlex ordering*. Let X be an alphabet, and choose an ordering $x_1 < x_2 < \ldots < x_n$ on X. The shortlex ordering $<$ on X^+ is defined as follows:

$v < w \Leftrightarrow v$ is shorter then w, or they have the same length
and v comes before w in the lexicographical order.

Proposition 3.4.4. *If A is an automatic S-act, then A has an automatic structure with uniqueness.*

Proof. Let $(X, K_1, \ldots, K_n)$ be an automatic structure with respect to the generating set $A_0 = \{a_1, \ldots, a_n\}$ for the S-act A. By Lemma 3.4.2, we may assume that $a_i.K_i\varphi \cap a_j.K_j\varphi = \emptyset$ for all $1 \leq i,j \leq n$, $i \neq j$. Hence, if $a_i.u = a_j.v$, $u \in K_i, v \in K_j$, then $a_i = a_j$ and $u, v \in K_i$. To construct an automatic structure $(X, L_1, \ldots, L_n)$ with uniqueness, we need to reduce the number of elements of each regular language K_i, $(1 \leq i \leq n)$ in such a way that for each $a \in (a_i.K_i\varphi)$ we have exactly one $w \in L_i \subseteq K_i$ such that $a_i.w\varphi = a$. Define

$$L_i = \{w \in K_i \mid \text{ if } (w,v) \in K_{(a_i,a_i)_=} \text{ then } w < v \text{ in the shortlex order}\}.$$

By Proposition 2.3.13, L_i is a regular language. Furthermore, we clearly have that

$$\bigcup_{j=1}^{n} a_j.L_j\varphi = A.$$

If $a_i.u = a_j.v$, $(u,v) \in L_i \times L_j$, then as before $a_i = a_j$, $u, v \in L_i$, and hence by the definition of L_i, $u \equiv v$. It follows that

$$L_{(a_i,a_j)_=} = \begin{cases} \emptyset & \text{if } i \neq j, \\ \{(u,u) \mid u \in L_i\}\delta_X & \text{if } i = j. \end{cases}$$

Thus $L_{(a_i,a_j)_=}$ is a regular language for all $(a_i, a_j) \in A_0 \times A_0$ by Proposition 2.3.2. Moreover,

$$L_{(a_i,a_j)_x} = K_{(a_i,a_j)_x} \cap (L_i \times L_j)\delta_X,$$

and hence is a regular language. We may deduce that $(X, L_1, \ldots, L_n)$ is indeed an automatic structure with uniqueness with respect to the generating set A_0 for the S-act A. □

An immediate consequence of Proposition 3.4.4:

Corollary 3.4.5. *If S is an $\mathcal{R}$-class automatic semigroup, then there exists an $\mathcal{R}$-class automatic structure for $S/\mathcal{R}$ with uniqueness.*

Also we have:

Corollary 3.4.6. *If R is a Schützenberger automatic $\mathcal{R}$-class of a regular semigroup S, then there exists a Schützenberger automatic structure for R with uniqueness.*

Proof. Let R be a Schützenberger automatic $\mathcal{R}$-class of a regular semigroup S. Then by Proposition 3.2.1, the right S^0-act R^0 is automatic with respect to some generating set $\{s\}$, hence there exists an automatic structure (X, L) with uniqueness for the right S^0-act R^0 with respect to $\{s\}$. Following the construction of the regular languages K and N in Proposition 3.2.1, we have that $K = L - L'$, where $L' = \{l \in L \mid s \cdot l = \mathbf{0}\}$ and $N = w(L - L')$, where w is a word over X representing $s = es$. Since (X, L) is an automatic structure with uniqueness, and $N_= = (w,w)\delta_X K_=$, we obtain that

$$(u,v) \in N_= \Leftrightarrow (u,v) \in (w,w)\delta_X(L_= \cap (K \times K)\delta_X) \Leftrightarrow u \equiv v,$$

hence (X, N) is a Schützenberger automatic structure with uniqueness for R. □

Schützenberger automatic $\mathcal{R}$-classes in a $\mathcal{D}$-class.

We will focus on Schützenberger automatic $\mathcal{R}$-classes of a regular semigroup S. To be more precise, we show that the property of being Schützenberger automatic is invariant within a $\mathcal{D}$-class of a regular semigroup S. Recall that if $s \in S$, then R_s denotes the $\mathcal{R}$-class of s.

Proposition 3.4.7. *Let S be a regular semigroup and R be a Schützenberger automatic $\mathcal{R}$-class of S. Let $s \in R$, $r \in S$ such that $s\ \mathcal{D}\ r$. Then R_r is Schützenberger automatic.*

Proof. Suppose that (X, L) is a Schützenberger automatic structure for R. Then $L\varphi = R$ and the languages $L_=$, L_x $(x \in X)$ are regular. On the other hand, since $s\ \mathcal{D}\ r$, there exists $t \in S^1$ such that $s\ \mathcal{R}\ t\ \mathcal{L}\ r$. In particular, we have that $r = k \cdot t$ for some $k \in S^1$ and it follows by Green's lemma, that we can give an $\mathcal{L}$-class preserving bijection

$$\xi : R_t \to R_r;\ x \mapsto k \cdot x.$$

Let w be a word over X representing k, and let $K = wL$. Clearly K is a regular language. We show that (X, K) is a Schützenberger automatic structure for R_r. Since ξ is a bijection and $L\varphi = R_t$, we obviously have that $K\varphi = (wL)\varphi = k \cdot L\varphi = R_r$. Also,

$$\text{if} \quad wl_1 = wl_2 \quad \text{then} \quad l_1 = l_2$$

holds, since if $wl_1 = k \cdot l_1\varphi = k \cdot l_2\varphi = wl_2$, that is if $(l_1\varphi)\xi = (l_2\varphi)\xi$, then $l_1\varphi = l_2\varphi$, since ξ is injective. Similarly,

$$\text{if} \quad wl_1x = wl_2,\ (x \in X) \quad \text{then} \quad l_1x = l_2.$$

Thus we obtain that

$$\begin{aligned} K_= &= \{(u, v) \mid u, v \in K,\ u = v\}\delta_X \\ &= \{(u, v) \mid u \equiv wl_1,\ v \equiv wl_2,\ (l_1, l_2 \in L),\ wl_1 = wl_2\}\delta_X = (w, w)\delta_X L_= \end{aligned}$$

is a regular language. Similarly, we obtain that

$$\begin{aligned} K_x &= \{(u, v) \mid u, v \in K,\ u \cdot x = v\}\delta_X \\ &= \{(u, v) \mid u \equiv wl_1,\ v \equiv wl_2,\ (l_1, l_2 \in L),\ wl_1x = wl_2\}\delta_X = (w, w)\delta_X L_x \end{aligned}$$

is a regular language for all $x \in X$. □

Technical lemmas

We are going to give two lemmas that will prove useful later on. First, we generalize Proposition 3.2 of [5].

Lemma 3.4.8. *Let $(X, L_1, \ldots, L_n)$ be an automatic structure with respect to the generating set A_0 for the S-act A. Let $w \in X^+$ and $a_i, a_k \in A_0$. Then the language*

$$L_{(a_i, a_k)_w} = \{(u, v) \in L_i \times L_k |\ a_i.uw = a_k.v\}\delta_X$$

is regular.

Proof. Let $w \equiv x_1 \dots x_t$ and fix $a_i, a_k \in A_0$. Let $L = \bigcup_{j=1}^{n} L_j$. Consider the set

$$K_1 = \bigcup_{j=1}^{n} L_{(a_i,a_j)_{x_1}}.$$

Since $(X, L_1, \dots, L_n)$ is an automatic structure with respect to the generating set A_0 for A, K_1 is a regular language. We note that

$$(u, v_1)\delta_X \in K_1 \iff (u, v_1) \in L_i \times L \text{ and } a_i.(u \cdot x_1) = a_j.v_1,$$
$$\text{where } a_j \in A_0,\ v_1 \in L_j.$$

Since K_1 is a regular language, the language

$$N_1 = \{v_1 \in L \mid (u, v_1)\delta_X \in K_1 \text{ for some } u \in L_i\}$$

is regular by Proposition 2.3.1. Moreover the set N_1 determines the following index set

$$I_1 = \{j \mid 1 \leq j \leq n \text{ there exists } v \in L_j \cap N_1\}.$$

We next define the languages K_2, N_2 and the index set I_2. Then we give a recursive definition for K_l, N_l, I_l $(1 \leq l < t)$. Let

$$K_2 = \bigcup_{m \in I_1} (\bigcup_{j=1}^{n} L_{(a_m,a_j)_{x_2}}).$$

Since K_2 is a finite union of regular languages it is a regular language. We note that

$$\begin{aligned}(v_1, v_2)\delta_X \in K_2 &\iff (v_1, v_2) \in N_1 \times L \text{ and } a_m.(v_1 \cdot x_2) = a_j.v_2 \\ &\quad\ \text{where } m \in I_1,\ v_1 \in L_m \cap N_1 \text{ and } a_j \in A_0,\ v_2 \in L_j. \\ &\iff \text{there exists } u \in L_i \text{ such that } a_i.(ux_1x_2) = a_j.v_2, \text{ where} \\ &\quad\ a_j \in A_0,\ v_2 \in L_j.\end{aligned}$$

Since K_2 is a regular language, the set

$$N_2 = \{v_2 \in L \mid (v_1, v_2)\delta_X \in K_2 \text{ for some } v_1 \in N_1\}$$

forms a regular language by Proposition 2.3.1. We consider the following index set determined by N_2:

$$I_2 = \{j \mid 1 \leq j \leq n \text{ there exists } v \in L_j \cap N_2\}.$$

Giving the promised recursive definition for $K_l, N_l, I_l,\ (2 \le j < t)$, we let

$$K_l = \bigcup_{m \in I_{l-1}} (\bigcup_{j=1}^{n} L_{(a_m,a_j)_{x_l}}).$$

Clearly K_l is a regular language. We note that

$$\begin{aligned}(v_{l-1}, v_l)\delta_X \in K_l \iff & (v_{l-1}, v_l) \in N_{l-1} \times L \text{ and } a_m.(v_{l-1} \cdot x_l) = a_j.v_l \\ & \text{where } m \in I_{l-1},\ v_{l-1} \in L_m \cap N_{l-1} \text{ and } a_j \in A_0,\ v_l \in L_j. \\ \iff & \text{there exists } u \in L_i \text{ such that } a_i.(ux_1x_2 \dots x_l) = a_j.v_l, \\ & \text{where } a_j \in A_0,\ v_l \in L_j.\end{aligned}$$

We have that the language

$$N_l = \{v_l \in L \mid (v_{l-1}, v_l) \in K_l \text{ for some } v_{l-1} \in N_{l-1}\}$$

is regular. The index set determined by N_l is

$$I_l = \{j \mid 1 \le j \le n \text{ there exists } v \in L_j \cap N_l\}.$$

Finally we consider the set

$$K_t = \bigcup_{m \in I_{t-1}} L_{(a_m,a_k)_{x_t}}.$$

Clearly K_t is a regular language, since it is a finite union of regular languages. We note that

$$\begin{aligned}(v_{t-1}, v)\delta_X \in K_t \iff & (v_{t-1}, v) \in N_{t-1} \times L_k \text{ and } a_m.(v_{t-1} \cdot x_t) = a_k.v \\ & \text{where } m \in I_{t-1} \text{ and } v_{t-1} \in L_m. \\ \iff & \text{there exists } u \in L_i \text{ such that } a_i.ux_1x_2 \dots x_t = a_k.v \\ & \text{and } v \in L_k.\end{aligned}$$

Now the languages

$$\begin{aligned}K_{x_1x_2} &= \{(u, v_2) \in L_i \times N_2 \mid \text{ there exists } v_1 \in N_1 \text{ such that} \\ &\quad (u, v_1) \in K_{x_1} \text{ and } (v_1, v_2) \in K_{x_2}\}\delta_X \\ K_{x_1x_2x_3} &= \{(u, v_3) \in L_i \times N_3 \mid \text{ there exists } v_2 \in N_2 \text{ such that} \\ &\quad (u, v_2) \in K_{x_1x_2} \text{ and } (v_2, v_3) \in K_{x_3}\}\delta_X \\ &\ \vdots \\ K_{x_1 \dots x_t} &= \{(u, v) \in L_i \times L_k \mid \text{ there exists } v_{t-1} \in N_{t-1} \text{ such that} \\ &\quad (u, v_{t-1}) \in K_{x_1x_2 \dots x_{t-1}} \text{ and } (v_{t-1}, v) \in K_{x_t}\}\delta_X\end{aligned}$$

are regular by Proposition 2.3.3. Clearly, if $(u,v) \in K_{x_1\ldots x_t}$, then $(u,v) \in L_{(a_i,a_k)_w}$. Let $(u,v) \in L_{(a_i,a_k)_w}$. Since $\bigcup_{j=1}^{n} a_j.L_j\varphi = A$, we can construct the following sequence

$$\begin{aligned} a_i.ux_1 &= a_{j_1}.v_1, & v_1 &\in L_{j_1} \\ a_{j_1}.v_1x_2 &= a_{j_2}.v_2, & v_2 &\in L_{j_2} \\ &\vdots & &\vdots \\ a_{j_{t-1}}.v_{t-1}x_t &= a_k.v, & v &\in L_k \end{aligned}$$

where $(u,v_i)\delta_X \in K_{x_1\ldots x_i}$ $(1 \le i \le t-1)$, and so $(u,v) \in K_{x_1\ldots x_t}$. Hence we may deduce that $K_{x_1\ldots x_t} = L_{(a_i,a_k)_w}$, and so $L_{(a_i,a_k)_w}$ is indeed a regular language. $\square$

Finally we introduce the following lemma:

Lemma 3.4.9. *Let $(X, L_1, \ldots, L_n)$ be an automatic structure with uniqueness for the S-act A, where the generating set is $A_0 = \{a_1, \ldots, a_n\}$. Assume that $a_n = a_i.u,\ u \in L_i,\ n \neq i$. Let $a_j \in A_0,\ j \neq n$ and $x \in X$. Then the languages*

$$\begin{aligned} &(a)\ L'_{(a_i,a_j)_=} = \{(v,w) \mid v \in uL_n,\ w \in L_j,\ a_i.v = a_j.w\}\delta_X; \\ &(b)\ L'_{(a_j,a_i)_=} = \{(w,v) \mid w \in L_j,\ v \in uL_n,\ a_j.w = a_i.v\}\delta_X; \\ &(c)\ L'_{(a_i,a_j)_x} = \{(v,w) \mid v \in uL_n,\ w \in L_j,\ a_i.(v\cdot x) = a_j.w\}\delta_X; \\ &(d)\ L'_{(a_j,a_i)_x} = \{(w,v) \mid w \in L_j,\ v \in uL_n,\ a_j.(w\cdot x) = a_i.v\}\delta_X \end{aligned}$$

are regular.

Proof. (a) Let $(X, L_1, \ldots, L_n)$ be an automatic structure with uniqueness for the S-act A, where the generating set is $A_0 = \{a_1, \ldots, a_n\}$. Assume that $a_n = a_i.u,\ u \in L_i,\ n \neq i$. Then,

$$\begin{aligned} L'_{(a_i,a_j)_=} &= \{(v,w) \mid v \in uL_n,\ w \in L_j,\ a_i.v = a_j.w\}\delta_X \\ &= \{(u\tilde{v},w) \mid \tilde{v} \in L_n,\ w \in L_j,\ a_i.u\tilde{v} = a_j.w\}\delta_X \\ &= \{(u\tilde{v},w) \mid \tilde{v} \in L_n,\ w \in L_j,\ a_n.\tilde{v} = a_j.w\}\delta_X. \end{aligned}$$

Hence if $(v,w)\delta_X \in L'_{(a_i,a_j)_=}$, then $v = u\tilde{v}$ for some $\tilde{v} \in L_n$ and $(\tilde{v},w)\delta_X \in L_{(a_n,a_j)_=}$. Since $(X, L_1, \ldots, L_n)$ is an automatic structure with uniqueness it follows that $a_n = a_j$ and $\tilde{v} \equiv w$. Since by assumption $a_j \neq a_n$, we may deduce that $L'_{(a_i,a_j)_=} = \emptyset$, hence it is a regular language.

(b) The proof is similar to that of part (a).

(c) First we note that

$$\begin{aligned} L'_{(a_i,a_j)_x} &= \{(v,w) \mid v \in uL_n,\ w \in L_j,\ a_i.(v \cdot x) = a_j.w\}\delta_X \\ &= \{(u\tilde{v},w) \mid \tilde{v} \in L_n,\ w \in L_j,\ a_i.(u\tilde{v} \cdot x) = a_j.w\}\delta_X \\ &= \{(u\tilde{v},w) \mid \tilde{v} \in L_n,\ w \in L_j,\ a_n.(\tilde{v} \cdot x) = a_j.w\}\delta_X. \end{aligned}$$

Hence

$$(v,w) \in L'_{(a_i,a_j)_x} \iff v \equiv u\tilde{v},\ \tilde{v} \in L_n \text{ and } (\tilde{v},w)\delta_X \in L_{(a_n,a_j)_x}.$$

Let

$$\mathcal{A}_1 = (\Sigma_1, X, \mu_1, p_0, T_1)$$

be a finite state automaton accepting the regular language uL_n. We assume that $|T_1| = 1$. Let

$$\mathcal{A}_2 = (\Sigma_2, X(2,\$), \mu_2, q_0, T_2)$$

be a finite state automaton accepting the regular language $L_{(a_n,a_j)_x}$. With the help of $\mathcal{A}_1$ and $\mathcal{A}_2$ we want to construct a finite state automaton $\mathcal{A}$ accepting $L'_{(a_i,a_j)_x}$. Assume that the length of the word u is m ($a_i.u = a_n$).

We make some remarks. If we input the word $(x_1 \ldots x_{t_1}, y_1 \ldots y_{t_2})\delta_X$ into $\mathcal{A}$, then we require $\mathcal{A}$ to check:

(P1) whether $x_1 \ldots x_{t_1}$ is an element of uL_n or not;

(P2) whether $(x_{m+1} \ldots x_{t_1}, y_1 \ldots y_{t_2})\delta_X$ is an element of $L_{(a_n,a_j)_x}$ or not.

The first requirement can be checked with the help of $\mathcal{A}_1$ and the second one with the help of $\mathcal{A}_2$. Note also that $\mathcal{A}_2$ need to be set to work after reading through the first m (the length of u) letters of the input word. That is we have to construct an automaton in such a way that it

- recognizes that we have read the first m letters;
- stacks in the first instance $y_1 \ldots y_m \in X(2,\$)$, so that after reading through the first m letters $(x_1 \ldots x_m, y_1 \ldots y_m)\delta_X$, we can set to work $\mathcal{A}_2$ by inputting the letter (x_{m+1}, y_1). At this stage, the stack has to change to $y_2 \ldots y_{m+1}$ if $y_{m+1} \neq \$$ and to $y_2 \ldots y_m$ otherwise. Hence the stack always contains words of length less then or equal to m.

- empty its stack by the end of the procedure. In other words we want our automaton to read all of $(x_{m+1}\dots x_{t_1}, y_1\dots y_{t_2})\delta_X$ with the help of $\mathcal{A}_2$. To be more precise, we need to take into consideration that we may have the following situation occur: The word $(x_1\dots x_{t_1}, y_1\dots y_{t_2})\delta_X$ is already read by the automaton $\mathcal{A}$, but there is still a word $y_k\dots y_{t_2}$ of length less then or equal to m stacked, so that the word $(\$, y_k)\dots(\$, y_{t_2})$ is still waiting to be"processed by $\mathcal{A}_2$". To be able to do that, we need to lengthen the word $(x_1\dots x_{t_1}, y_1\dots y_{t_2})\delta_X$ by attaching the letter $(\$,\$)$ at most m times, to make sure that the empty word is stacked at the end of the procedure.

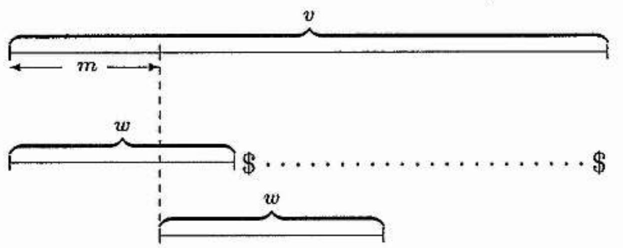
Depending on how many times the letter $(\$,\$)$ needs to be attached to the word

$$(x_1,\dots x_{t_1}, y_1\dots y_{t_2})\delta_X$$

so that property (P2) could be checked, we let

$$K_0 = \{(v,w)\delta_X \in L'_{(a_i,a_j)_x} \mid |v| - m > |w|\}.$$

Illustrating this case:

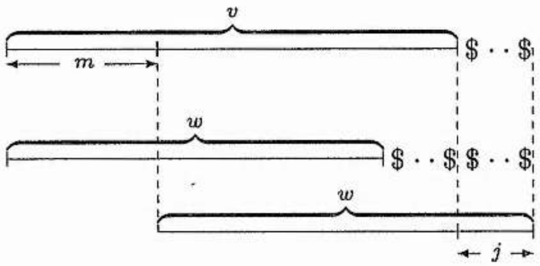


That is K_0 contains words $(v,w)\delta_X \in L'_{(a_i,a_j)X}$ for which properties (P1) and (P2) can be checked without attaching the letter $(\$,\$)$.

For $1 \le j \le m-1$ we let

$$K_j = \{(v,w)\delta_X \in L'_{(a_i,a_j)_x} \mid |v| - m + j = |w|\}.$$

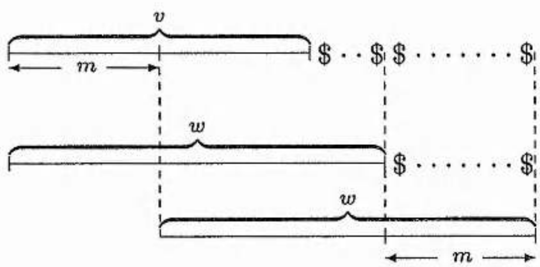
Illustrating this case:



That is K_j contains words $(v, w)\delta_X \in L'_{(a_i,a_j)X}$ for which properties (P1) and (P2) can be checked by attaching j times the letter $(\$, \$)$.

We let

$$K_m = \{(v, w)\delta_X \in L'_{(a_i,a_j)_x} \mid |v| \dot{-} m < |w|\}.$$



That is K_m contains words $(v, w)\delta_X \in L'_{(a_i,a_j)X}$ for which properties (P1) and (P2) can be checked by attaching at least m times the letter $(\$, \$)$.

It is immediate that if $(v, w)\delta_X \in L'_{(a_i,a_j)_x}$, then $(v, w) \in K_j$ for some $1 \leq j \leq m$, since the length of $v = u\tilde{v} \in uL_n$ has to be greater then m. Thus,

$$\bigcup_{j=1}^{m} K_j = L'_{(a_i,a_j)_x}. \tag{3.2}$$

We first verify that the languages $N_j = K_j(\$, \$)^j$, $(0 \leq j \leq m)$ are regular. For N_j we let

$$\mathcal{L}_j = \{0, 1, \ldots, m\};$$
$$\mathcal{I}_j = \{w \in X^\star \mid |w| \leq m\};$$
$$\mathcal{J}_j = \{0, 1, \ldots, j\}.$$

Attach a failure state $\overline{FS}$ to Σ_1 and denote $\Sigma_1' = \Sigma \cup \overline{FS}$. We will explain why this step was needed when we discuss how the transition function ν_j of the below defined automaton works. Consider the automaton $\mathcal{B}_j = (Q_j, X(2,\$) \cup (\$,\$), \nu_j, r_j, F_j)$ where

$$\begin{aligned} Q_j &= (\Sigma_1' \times \Sigma_2 \times \mathcal{L}_j \times \mathcal{I}_j \times \mathcal{J}_j) \cup FS; \\ r_j &= (p_0, q_0, 0, \lambda, 0); \\ F_j &= T_1 \times T_2 \times \{m\} \times \{\lambda\} \times \{j\}, \end{aligned}$$

where λ denotes the empty word. Before we define the transition function ν_j, we introduce the following notations: for $w = y_1 \dots y_t \in \mathcal{I}_j$ we let

$$\overline{w} = \begin{cases} y_2 \dots y_t & \text{if } w \neq \lambda \\ w & \text{if } w = \lambda \end{cases}$$

and define $\overline{w}\ \$ = \overline{w}$.

Let $(p, q, l, w, k) \in Q_j$, where $w = y_1 \dots y_t$ and let $(x, y) \in X(2,\$)$. Recall that $\mathcal{A}_1$ is a finite state automaton with exactly one final state. Let

$$p_1 = \begin{cases} \mu_1(p, x) & \text{if } x \neq \$ \\ p & \text{if } x = \$,\ p = T_1 \\ \overline{FS} & \text{otherwise} \end{cases}$$

and let

$$q_1 = \begin{cases} q & \text{if } l < m \\ \mu_2(q, (x, y_1)) & \text{if } w \neq \lambda,\ l = m \\ \mu_2(q, (x, \$)) & \text{if } w = \lambda,\ x \neq \$,\ l = m \end{cases}$$

The transition function ν_i in given in the following tableau:

$\nu_j((p,q,l,w,k),(x,y))$		
$(p_1, q_1, l+1, wy, k)$	if $y \neq \$,\ l < m,\ k = 0$;	(R1)
$(p_1, q_1, l+1, w, k)$	if $y = \$,\ l < m,\ k = 0$;	(R2)
$(p_1, q_1, l, \overline{w}y, k)$	if $x \neq \$$ or $w \neq \lambda,\ l = m,\ k = 0$	(R3)
$(p_1, q_1, l, \overline{w}, k+1)$	if $x = \$,\ y = \$,\ l = m,\ k < j,\ w \neq \lambda$	(R4)
FS	if $x \neq \$,\ y \neq \$,\ k > 0$	(R5)
FS	if $x = \$,\ y = \$,\ k < j,\ w = \lambda$	(R6)
FS	if $(p, q, l, w, k) = FS$	(R7)

We now explain in more detail how the transition function works:

With rules (R1) and (R2), we

- count how many letters of the input word we have read so far up to m;
- fill up the stack with a word of length less then or equal to m;
- begin to check with the help of $\mathcal{A}_1$, whether property (P1) holds for the input word.

With rule (R3), we
- continue checking whether property (P1) holds for the input word;
- begin checking with the help of $\mathcal{A}_2$, whether property (P2) holds for the input word.

With rule (R4), we
- continue checking whether property (P2) holds for the input word by emptying the stack;
- begin counting the number of $(\$,\$)$ letters of the input word.

With rule (R5), we assure that
- words are not accepted in which a letter $(\$,\$)$ is followed by a letter (x,y), $x \neq \$$ or $y \neq \$$.

With rule (R6), we assure that
- the stack empties exactly when we read through j times the letter $(\$,\$)$ consecutively.

With rules (R1)-(R3), with the definition of p_1 and with the help of the assumption that $\mathcal{A}_1$ is an automaton with one final state we assure that
- if $(v,w)\delta_X$ is accepted by $\mathcal{B}_j$, then $v \in uL_n$, that is v does not contain a letter $\$$ followed by a letter $x \neq \$$.

With rule (R1)-(R4), we assure that
- the input word satisfies property (P2).

We may now deduce that a word w is accepted by $\mathcal{B}_j$ if and only if $w \in N_j$, hence for each $1 \leq j \leq m$ the language N_j is regular. By Proposition 2.2.3, $N_j^{\text{rev}} = (\$,\$)^j K_j^{\text{rev}}$ is also a regular language. Making use of Lemma 2.2.5, we may deduce that K_j^{rev}, and hence K_j is a regular language for each $1 \leq j \leq m$. By (3.2) we have that $L'_{(a_i,a_j)_x}$ is a finite union of regular languages, and hence is a regular language.

(d) The proof is similar to that of part (c). Basically we need to redefine $\mathcal{A}_2$ to be the automaton accepting the regular language $L_{(a_j,a_n)_x}$. □

3.5 Automatic versus Schützenberger automatic.

Considering a regular semigroup S, the property of being Schützenberger automatic is a collection of local properties – namely it is required that all $\mathcal{R}$-classes are Schützenberger automatic – meanwhile being automatic is a global property. In this section we discuss how the two notions compare. Proposition 3.2.3 tells us that a group is automatic if and only if it is Schützenberger automatic. We verify that under certain conditions automatic regular semigroups are Schützenberger automatic. Moreover, we give an example that shows that even strongly Schützenberger automatic regular semigroups might not be automatic. Throughout this section S will denote a semigroup, X a finite generating set for S. We assume that $X \subseteq S$ and denote by $\varphi : X^+ \to S$ the homomorphism extending the identity map $\iota : X \to S$.

Proposition 3.5.1. *Let S be an automatic regular semigroup with automatic structure (X, L). Assume that for every $\mathcal{R}$-class R of S, there exists a regular language $K \subseteq L$ such that $K\varphi = R$. Then S is Schützenberger automatic.*

Proof. Let (X, L) be an automatic structure for S with uniqueness, $s \in S$, and consider the $\mathcal{R}$-class of s, R_s. By assumption, there exists a regular language $K \subseteq L$ such that $K\varphi = R_s$. We see that (X, K) is a Schützenberger automatic structure for R_s, since

$$K_= = \{(u,v) \mid u,v \in K,\ u = v\}\delta_X = L_= \cap (K \times K)\delta_X$$

and

$$K_x = \{(u,v) \mid u,v \in K,\ u \cdot x = v\}\delta_X = L_x \cap (K \times K)\delta_X$$

are regular languages which completes the proof. $\square$

Corollary 3.5.2. *Let S be an automatic regular semigroup with finitely many $\mathcal{R}$-classes. Then S is Schützenberger automatic.*

Proof. Let (X, L) be an automatic structure for S. Let $S/\mathcal{R} = \{R_1, R_2, \ldots, R_n\}$ and choose idempotents $e_1, e_2, \ldots, e_n$ so that $e_i \in R_i$. Let R be an $\mathcal{R}$-class of S with idempotent e. We show that the assumption of Proposition 3.5.1 is satisfied, that is we construct a regular language $K \subseteq L$ such that $K\varphi = R$. Consider the automaton $\mathcal{A} = (\Sigma, X, \mu, p, F)$, where

$$\Sigma = \bigcup_{i=1}^{n} R_{e_i} \cup \{p\}; \quad F = \{R\}; \quad \mu\colon (R_i, x) \to R_{xe_i} \quad \text{and } \mu\colon (p, x) \to R_x.$$

Since $\mathcal{R}$ is a left congruence, μ is well defined. Note that if a word $w \equiv x_1x_2\ldots x_n$ is accepted by this automaton, that is if

$$p \xrightarrow{x_1} R_{x_1} \xrightarrow{x_2} R_{(x_2x_1)} \quad \cdots \quad R_{(x_{n-1}\ldots x_1)} \xrightarrow{x_n} R_{(x_n\ldots x_1)} = R$$

is a successful path, then $w^{\mathrm{rev}} \in R$. Let M be the language accepted by this automaton and let $M' = M^{\mathrm{rev}}$. By Proposition 2.2.3, $K = M' \cap L$ is a regular language, moreover we have that $K\varphi = R$. The assertion follows by Proposition 3.5.1. $\square$

The following counterexample shows that the reverse statement does not hold even if S is strongly Schützenberger automatic.

Example 3.5.3. Let $S = [Y_2; G_1, G_2, \varphi]$, be a Clifford semigroup where the building blocks are Y_2, the two element chain, $G_1 = \langle a, b \mid \ \rangle$ the free group of rank 2, $G_2 = \langle c, d, \mid c^2 = d^2 = 1 \rangle$ the free product of two cyclic groups of order two, and $\varphi \colon G_1 \to G_2$ is the homomorphism defined by $a\varphi = c, b\varphi = d$. Then S is Schützenberger automatic, but it is not automatic.

Indeed, since S has two $\mathcal{R}$-classes, G_1 and G_2 we obtain by Corollary 3.3.2, that S is $\mathcal{R}$-class automatic. Moreover, the groups G_1 and G_2 are known to be automatic groups, hence by Proposition 3.2.3 they are Schützenberger automatic and we may deduce that S is strongly Schützenberger automatic. However, it is shown in [4] that S is not automatic.

3.6 Invariance under the change of generators.

The following problem is essential to consider when talking about automatic S-acts:

Main Problem: Let $(X, L_1, L_2, \ldots, L_n)$ be an automatic structure with respect to the generating set A_0 for the S-act A. Assume that the finite set Y also generates S and that the finite set B also generates the S-act A. Do there exist regular languages $K_1, \ldots, K_m$ over Y such that $(Y, K_1, \ldots, K_m)$ forms an automatic structure with respect to the generating set B for the S-act A?

We can split the above problem into two subproblems:

Changing the generators of S. Let $(X, L_1, L_2, \ldots, L_n)$ be an automatic structure with respect to the generating set A_0 for the S-act A. Assume that the finite set Y also generates S. Do there exist regular languages $K_1, \ldots, K_n$ over Y such that $(Y, K_1, \ldots, K_n)$ forms an automatic structure with respect to the same generating set A_0 for the S-act A?

Changing the generators of A. Let $(X, L_1, L_2, \ldots, L_n)$ be an automatic structure with respect to the generating set A_0 for the S-act A. Assume that the finite set B also generates the S-act A. Do there exist regular languages $K_1, \ldots, K_m$ over X such that $(X, K_1, \ldots, K_m)$ forms an automatic structure with respect to the generating set B for the S-act A?

Giving a positive answer to both of the subproblems certainly gives a positive answer to our main problem. We can first change the generating set of S and then the generating set of the S-act A.

It is known that automaticity of semigroups depends on the choice of the finite generating set. Hence with Proposition 3.1.10 in mind, we expect that in general we cannot change the set of generators either of S or of A without losing the property of being automatic. We will verify in this section that if S is a semigroup with local right identities, then an affirmative answer can be given to both of the subproblems, and hence to our main problem.

As in the previous sections S will denote a semigroup, X a finite generating set for S. We assume that $X \subseteq S$ and denote by $\varphi : X^+ \to S$ the homomorphism extending the identity map $\iota : X \to S$. Also, $S^0 = S \cup \{\mathbf{0}\}$ denotes the semigroup obtained by adjoining a zero element $\mathbf{0}$ to S, and $X^0 = X \cup \{\mathbf{0}\}$.

We say that a semigroup S has *local right identities*, if for all $s \in S$ there exists $e \in E(S)$ such that $se = s$. We note here that regular semigroups have local right identities.

Changing the generators of the semigroup

Proposition 3.6.1. *Let $S = \langle X \rangle$ be a semigroup with local right identities and assume that $(X, L_1, L_2, \ldots, L_n)$ is an automatic structure with respect to the generating set A_0 for the S-act A. Assume that the finite set Y also generates S. Then there exist regular languages $K_1, \ldots, K_n$ over Y such that $(Y, K_1, \ldots, K_n)$ forms an automatic structure with respect to the same generating set A_0 for the S-act A.*

The proof of Proposition 3.6.1 follows from Lemmas 3.6.2, 3.6.4 and 3.6.5.

Lemma 3.6.2. *Let $S = \langle X \rangle$ be a semigroup with local right identities and assume that $(X, L_1, L_2, \ldots, L_n)$ forms an automatic structure with respect to the generating set $A_0 = \{a_1, \ldots, a_n\}$ for the S-act A. Assume that the finite set Y also generates S. Then there exists a finite set $Y' \supseteq Y$ and regular languages $K_1, \ldots, K_n$ over Y' such that $(Y', K_1, \ldots, K_n)$ forms an automatic structure with respect to the same generating set A_0 for the S-act A.*

Proof. Let $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_m\}$. Let $\psi : Y^+ \to S$ be the homomorphism extending the identity map on Y. Fix words $v_1, v_2, \ldots, v_k$ over Y such that $x_i\varphi = v_i\psi$, $(1 \le i \le k)$. Let $Y' = Y \cup \{e_1, \ldots, e_m\}$, where $y_j\psi \cdot e_j = y_j\psi$ for all $1 \le j \le m$.

Clearly, since $Y' \supseteq Y$, we have that Y' also generates S and that there exists a surjective homomorphism $\varsigma : (Y')^+ \to S$ such that $\varsigma|_Y = \psi$. Making use of the fact that e_i is a right identity for $y_i\psi$, we have that for each word v_i there exists $f_i \in \{e_1, \ldots, e_m\}$ such that $(v_i(f_i)^t)\varsigma = v_i\varsigma$, $t \in \mathbb{N}$. Hence, by attaching the appropriate local right identities to the words $v_1, \ldots, v_k$, we obtain words $w_1, \ldots, w_k$ over Y' of equal length such that

$$x_i\varphi = w_i\varsigma = v_i\varsigma = v_i\psi. \tag{3.3}$$

Consider the map

$$\tilde{\xi}\colon X \to (Y')^+, \ x_i \mapsto w_i,$$

and extend it to a semigroup homomorphism $\xi : X^+ \to (Y')^+$. In particular we have that the diagram

$$\begin{tikzcd} X^+ \arrow[rr, "\xi"] \arrow[dr, "\varphi"'] & & (Y')^+ \arrow[dl, "\varsigma"] \\ & S & \end{tikzcd}$$

is commutative, since

$$x\varphi = v_i\psi = v_i\varsigma = w_i\varsigma = (x\xi)\varsigma, \tag{3.4}$$

hence $\varphi = \xi\varsigma$. Let

$$K_i = L_i\xi \quad (1 \le i \le n).$$

By Proposition 2.2.3, K_i is a regular language for all $(1 \le i \le n)$. We verify that

$$(Y', K_1, K_2, \ldots, K_n)$$

is an automatic structure with respect to the generating set A_0 for the S-act A.

Since $(X, L_1, L_2, \ldots, L_n)$ is an automatic structure for the S-act A, we have that $\bigcup_{j=1}^{n} a_j.L_j\varphi = A$. To show that $\bigcup_{j=1}^{n} a_j.K_j\varsigma = A$, let $a \in A$. Then there exists $a_i \in A_0$ and $u \in L_i$, such that $a_i.u\varphi = a$. On the other hand $u\varphi = (u\xi)\varsigma$ by (3.4) and also by definition $u\xi \in K_i$. Thus we obtain that $a_i.(u\xi)\varsigma = a$ and we deduce that

$$\bigcup_{j=1}^{n} a_j.K_j\varsigma = A.$$

Let $(a_i, a_j) \in A_0 \times A_0$. We show that the languages $K_{(a_i,a_j)_=}$ and $K_{(a_i,a_j)_y}$, $y \in Y'$ are regular. Let $\theta : X(2,\$)^+ \to Y'(2,\$)^+$ be the semigroup homomorphism induced by the map

$$(x, y) \mapsto (x, y)(\xi \times \xi)\delta_{Y'}, \quad (x, y) \in X \times X.$$

Since for each $1 \leq i, j \leq k$ the length of $x_i\xi \equiv w_i$ and $x_j\xi \equiv w_j$ is the same, we have that for all $(u', v') \in X^+ \times X^+$

$$(u', v')\delta_X\theta = (u'\xi, v'\xi)\delta_{Y'} = (u', v')(\xi \times \xi)\delta_{Y'}$$

and we may deduce that

$$\begin{aligned} K_{(a_i,a_j)_=} &= \{(u, v) \in K_i \times K_j \mid a_i.u = a_j.v\}\delta_{Y'} \\ &= \{(u'\xi, v'\xi) \mid (u', v') \in L_i \times L_j,\ a_i.u'\xi = a_j.v'\xi\}\delta_{Y'} \\ &= \{(u', v') \in L_i \times L_j \mid a_i.u' = a_j.v'\}(\xi \times \xi)\delta_{Y'} \\ &= \{(u', v') \in L_i \times L_j \mid a_i.u' = a_j.v'\}\delta_X\theta = L_{(a_i,a_j)_=}\theta, \end{aligned}$$

hence is a regular language by Proposition 2.2.3. Let $y \in Y'$ and $w \in X^+$ such that $w\xi \equiv y$. Then,

$$\begin{aligned} K_{(a_i,a_j)_y} &= \{(u, v) \in K_i \times K_j \mid a_i.(u \cdot y) = a_j.v\}\delta_{Y'} \\ &= \{(u'\xi, v'\xi) \mid (u', v') \in L_i \times L_j,\ a_i.(u'\xi \cdot y) = a_j.v'\xi\}\delta_{Y'} \\ &= \{(u'\xi, v'\xi) \mid (u', v') \in L_i \times L_j,\ a_i.(u' \cdot w)\xi = a_j.v'\xi\}\delta_{Y'} \\ &= \{(u', v') \in L_i \times L_j \mid a_i.(u' \cdot w) = a_j.v'\}(\xi \times \xi)\delta_{Y'} \\ &= \{(u', v') \in L_i \times L_j \mid a_i.(u' \cdot w) = a_j.v'\}\delta_X\theta = L_{(a_i,a_j)_w}\theta, \end{aligned}$$

hence is regular by Lemma 3.4.8 and Proposition 2.2.3. □

Definition 3.6.3. Let $(X, L_1, \ldots L_n)$ be an automatic structure for the S-act A. Let $L = \bigcup_{j=1}^n L_j$. We say that a *letter* $x \in X$ *is dispensable in* L, if whenever a word $w \in L$ contains the letter x, the word $\widetilde{w}$ obtained from w by deleting all occurrences of x in w represents the same element of S as w.

We note that in Lemma 3.6.2, all supplementary letters added to Y are dispensable in $K = \bigcup_{j=1}^n K_j$. To be more accurate: let $w \in K$. Then $w = (x_{i_1} \ldots x_{i_t})\xi = w_{i_1} \ldots w_{i_t}$ for some $(x_{i_1} \ldots x_{i_t}) \in L_i$. Recall that w_{i_k} is obtained from v_{i_k} by attaching an appropriate local right identity (supplementary letter) to it, and also by (3.3) $v_{i_j}\varsigma = w_{i_j}\varsigma$. That is deleting a supplementary letter from w means substituting certain w_{i_j}'s by v_{i_j}, which does not have any affect on the element represented by w.

Before we finish proving Proposition 3.6.1 we invoke the following technical lemma which follows from the proofs of [12, Lemma 3.1] and [9, Theorem 2.4.1.].

Lemma 3.6.4. *Let S be a semigroup generated by $X = \{x_1, \ldots, x_m, e\}$. Assume that $Y = \{x_1, \ldots, x_m\}$ also generates S and let $e = x_{i_1} \ldots x_{i_k}$. Define $\psi : X^+ \to Y^+$ as follows: For each word w, let $w\psi$ be the word obtained from w by substituting all kth occurrences of e by the word $x_{i_1} \ldots x_{i_k}$ and delete all other occurrences. Then the following hold:*

(1) If L is a regular language over X, then $L\psi$ is a regular language over Y.

(2) If J is a regular language over $X(2, \$)$, then

$$(J^l)^r = \{(u\psi, v\psi) \mid (u,v)\delta_X \in J\}\delta_Y$$

is a regular language over $Y(2, \$)$.

Lemma 3.6.5. *Let $(X, K_1, \ldots, K_n)$ be an automatic structure for the S-act A with respect to the generating set $A_0 = \{a_1, \ldots, a_n\}$, where $X = \{x_1, \ldots, x_m, e\}$. Let $K = \bigcup_{j=1}^n K_j$. Assume that e is dispensable in K and that $Y = \{x_1, \ldots, x_m\}$ also generates S. Then there exist regular languages $L_1, \ldots, L_n$ over Y such that $(Y, L_1, \ldots, L_n)$ is an automatic structure for the S-act A.*

Proof. Assume that $e = y_1 \ldots y_k$, $y_i \in Y$ $(1 \le i \le k)$. Let $\psi \colon Y^+ \to S$ be the homomorphism extending the identity map on $\iota : Y \to S$. Define the map $\varsigma \colon X^+ \to Y^+$ in the following way. For each word $w \in X^+$, replace all kth occurrences of e by $y_1 \ldots y_k$ and delete all other occurrences of e in w. Let $L_i = K_i\varsigma$, $(1 \le i \le n)$. By Lemma 3.6.4(1), L_i is a regular language for each $1 \le i \le n$. Let a_i, $a_j \in A_0$. To show that $(X, L_1, \ldots, L_n)$ forms an automatic structure with respect to the generating set A_0 for A, we need to verify that the languages $L_{(a_i,a_j)_=}$, $L_{(a_i,a_j)_y}$ $(y \in Y)$ are regular. First we note that since e is dispensable in K we have by definition for all $u' \in K$ that $(u'\varsigma)\psi = u'\varphi$, hence by letting $u \equiv u'\varsigma$ we obtain

$$a_i.u' = a_i.u'\varphi = a_i.(u'\varsigma)\psi = a_i.u\psi = a_i.u.$$

Now we may deduce using Lemma 3.6.4 (2) that

$$\begin{aligned} L_{(a_i,a_j)_=} &= \{(u',v') \in L_i \times L_j \mid a_i.u' = a_j.v'\}\delta_Y \\ &= \{(u',v') \mid (u',v') = (u\varsigma, v\varsigma) \text{ for some } (u,v) \in K_i \times K_j, \\ &\qquad\qquad a_i.u = a_j.v\}\delta_Y \\ &= \{(u\varsigma, v\varsigma) \in L_i \times L_j \mid (u,v)\delta_X \in K_{(a_i,a_j)_=}\}\delta_Y \\ &= (K_{(a_i,a_j)_=}{}^l)^r \end{aligned}$$

is a regular language. Let $y \in Y$ and $w \in X^+$ such that $y\psi = w\varphi$. Then

$$\begin{aligned} L_{(a_i,a_j)_y} &= \{(u',v') \in L_i \times L_j \mid a_i.(u' \cdot y) = a_j.v'\}\delta_Y \\ &= \{(u',v') \mid (u',v') = (u\varsigma, v\varsigma) \text{ for some } (u,v) \in K_i \times K_j, \\ &\qquad a_i.(u \cdot w) = a_j.v\}\delta_Y \\ &= \{(u\varsigma, v\varsigma) \in L_i \times L_j \mid (u,v)\delta_X \in K_{(a_i,a_j)_w}\}\delta_Y \\ &= ({K_{(a_i,a_j)_w}}^l)^r \end{aligned}$$

is a regular language by Lemma 3.4.8 and Lemma 3.6.4(2). □

Proof of Proposition 3.6.1. Assume that $(X, L_1, L_2, \ldots, L_n)$ is an automatic structure with respect to the generating set A_0 for the S-act A. Assume that the finite set Y also generates S. Then, according to the proof of Lemma 3.6.2, there exists a finite set $Y' \supseteq Y$ and regular languages $N_1, \ldots, N_n$ over Y' such that $(Y', N_1, \ldots, N_n)$ forms an automatic structure for A with generating set A_0. Moreover, all letters added to Y are dispensable in $N = \bigcup_{j=1}^n N_j$. Making use of Lemma 3.6.5 we can remove the supplementary letters added to Y one by one and obtain regular languages $K_1, \ldots, K_n$ such that $(Y, K_1, \ldots, K_n)$ forms an automatic structure with respect to the generating set A_0 for the S-act A. □

Corollary 3.6.6. *Let $S = \langle X \rangle$ be a regular semigroup and R be an $\mathcal{R}$-class of S. Assume that (X, L) is a Schützenberger automatic structure for R and that the finite set Y also generates S. Then there exists a regular language K over Y such that (Y, K) is a Schützenberger automatic structure for R.*

Proof. Assume that (X, L) is a Schützenberger automatic structure for the $\mathcal{R}$-class R of the regular semigroup S. Then by Proposition 3.2.1 there exists a regular language L_1 over $X^0 = X \cup \{0\}$ such that (X^0, L_1) forms an automatic structure for the right S^0-act R^0 with respect to a one element generating set $\{s\}$. Since Y generates S, we clearly have that $Y \cup \{0\}$ generates S^0. Applying Proposition 3.6.1, we deduce that there exists a regular language L_2 over Y^0 such that (Y^0, L_2) forms an automatic structure for R^0 with generating set $\{s\}$. Pulling back the argument for the $\mathcal{R}$-class R with the help of Proposition 3.2.1 we conclude that there exists a regular language K over Y such that (Y, K) forms a Schützenberger automatic structure for R. □

Changing the generators of the act.

Proposition 3.6.7. *Let S be a semigroup with local right identities. Let $(X, L_1, \ldots, L_n)$ be an automatic structure with respect to the generating set A_0 for the S-act A. Assume that the finite set B also generates A. Then there exist regular languages $K_1, K_2, \ldots, K_m$ over X, such that $(X, K_1, \ldots, K_m)$ forms an automatic structure with respect to the generating set B for the S-act A.*

The proof of the proposition is based on Lemmas 3.6.8, 3.6.9.

Lemma 3.6.8. *Let S be a semigroup with local right identities and $(X, L_1, \ldots, L_n)$ be an automatic structure with respect to the generating set A_0 for the S-act A. Let $a_{n+1} \in A$ and $B = A_0 \cup \{a_{n+1}\}$. Then there exist regular languages $K_1, \ldots, K_{n+1}$ over X such that $(X, K_1, \ldots, K_{n+1})$ forms an automatic structure with respect to the generating set B for the S-act A.*

Proof. By Proposition 3.4.4 we can assume that $(X, L_1, \ldots, L_n)$ is an automatic structure with uniqueness for the S-act A. Let $a_{n+1} \in A$. Then there exists a unique generator $a_i \in A_0$ and a unique word $u \in L_i$ such that $a_i.u\varphi = a_{n+1}$. Let e be a local right identity of S such that $u\varphi \cdot e = u\varphi$. Let w be a word over X representing e and $L_{n+1} = \{w\}$. Note that $(uw)\varphi = u\varphi$, and so

$$a_{n+1} = a_i.u\varphi = a_i.(u \cdot w)\varphi = (a_i.u\varphi) \cdot w\varphi = a_{n+1}.w\varphi.$$

That is

$$a_{n+1}.w = a_{n+1} \tag{3.5}$$

holds. We clearly have that $\bigcup_{j=1}^{n+1} a_j.L_j\varphi = A$. To show that $(X, L_1, \ldots, L_{n+1})$ indeed forms an automatic structure with respect to the generating set B we need to verify that for all $1 \leq j \leq n$ the languages $L_{(a_j,a_{n+1})_=}$, $L_{(a_j,a_{n+1})_x}$ and $L_{(a_{n+1},a_j)_x}$ are regular.

Since $(X, L_1, \ldots, L_n)$ is an automatic structure with uniqueness and since $a_{n+1}.w = a_{n+1}$ by (3.5), we have that

$$a_j.v = a_{n+1}.w \;\Leftrightarrow\; a_j.v = a_{n+1} \;\Leftrightarrow\; a_j = a_i \;\text{ and } v \equiv u.$$

Thus $L_{(a_j,a_{n+1})_=} = \emptyset$ for all $1 \leq j \leq n$, $i \neq j$ and $L_{(a_i,a_{n+1})_=} = (u, w)\delta_X$, and hence they are regular.

Next we consider the language $L_{(a_j,a_{n+1})_x}$ $(1 \leq j \leq n)$. Let $L'_j = \{v \in L_j \mid a_j.(v \cdot x) = a_{n+1}\}$. By Lemma 3.1.8, L'_j is a regular language, and so

$$\begin{aligned} L_{(a_j,a_{n+1})_x} &= \{(v, w) \in L_j \times L_{n+1} \mid a_j.(v \cdot x) = a_{n+1}.w\}\delta_X \\ &= \{(v, w) \in L_j \times L_{n+1} \mid a_j.(v \cdot x) = a_{n+1}\}\delta_X \\ &= (L'_j \times L_{n+1})\delta_X = (L'_j \times \{w\})\delta_X \end{aligned}$$

is a regular language by Proposition 2.3.1.

Finally consider the language $L_{(a_{n+1},a_j)_x}$. Let $c = a_{n+1}.(w{\cdot}x)$ and $L''_j = \{v \in L_j \mid a_j.v = c\}$. By Lemma 3.1.9, L''_j is a regular language, and hence

$$\begin{aligned} L_{(a_{n+1},a_j)_x} &= \{(w,v) \in L_{n+1} \times L_j \mid a_{n+1}.(w \cdot x) = a_j.v\}\delta_X \\ &= \{(w,v) \in L_{n+1} \times L_j \mid c = a_j.v\}\delta_X \\ &= (L_{n+1} \times L''_j)\delta_X = (\{w\} \times L''_j)\delta_X \end{aligned}$$

is a regular language by Proposition 2.3.1, proving that $(X, L_1, L_2, \ldots, L_{n+1})$ is indeed an automatic structure with respect to the generating set B for the S-act A. □

Lemma 3.6.9. *Let S be a semigroup with local right identities and let $(X, L_1, \ldots, L_n)$ be an automatic structure with respect to the generating set $A_0 = \{a_1 \ldots, a_n\}$ for the S-act A. Assume that $B = \{a_1, \ldots, a_{n-1}\}$ also generates the S-act A. Then there exist regular languages $K_1, \ldots, K_{n-1}$ over X such that $(X, K_1, \ldots, K_{n-1})$ is an automatic structure with respect to the generating set B for the S-act A.*

Proof. Let $(X, L_1, \ldots, L_n)$ be an automatic structure with uniqueness with respect to the generating set $A_0 = \{a_1 \ldots, a_n\}$ for the S-act A. Assume that $B = \{a_1, \ldots, a_{n-1}\}$ also generates the S-act A. Then there exists a unique generator $a_i \in B$ and $u \in L_i$ such that $a_i.u\varphi = a_n$. Define

$$K_j = L_j \quad \text{for all} \quad (1 \le j \le n-1,\ i \ne j) \quad \text{and} \quad K_i = L_i \cup uL_n.$$

By Proposition 2.2.3, K_i is a regular language, and we clearly have that

$$\bigcup_{j=1}^{n-1} a_j.K_j\varphi = A.$$

To prove that $(X, K_1, \ldots, K_{n-1})$ forms an automatic structure with respect to the generating set B, we need to verify that the languages $K_{(a_i,a_j)_=}, K_{(a_i,a_j)_x}$ and $K_{(a_j,a_i)_x}$, $x \in X$, $(1 \le j \le n-1)$, are regular, since for all $j \ne i$, $(1 \le j \le n-1)$ $L_j = K_j$.

Case 1. If $a_j \ne a_i$, then

$$\begin{aligned} K_{(a_i,a_j)_=} &= \{(v,w) \in K_i \times K_j \mid a_i.v = a_j.w\}\delta_X \\ &= L_{(a_i,a_j)_=} \cup \{(v,w) \in uL_n \times L_j \mid a_i.v = a_j.w\}\delta_X, \end{aligned}$$

is a regular language by Lemma 3.4.9 and by the fact that $(X, L_1, \ldots, L_n)$ is an automatic structure for A.

Case 2. If $a_j = a_i$, then

$$\begin{aligned}K_{(a_i,a_i)_=} &= \{(v,w) \in K_i \times K_i \mid a_i.v = a_i.w\}\delta_X \\ &= L_{(a_i,a_j)_=} \cup \{(v,w) \in uL_n \times L_i \mid a_i.v = a_i.w\}\delta_X \\ &\quad \cup \{(w,v) \in L_i \times uL_n \mid a_i.w = a_i.v\}\delta_X \cup (u,u)\delta_X L_{(a_n,a_n)_=} \\ &= L_{(a_i,a_j)_=} \cup (u,u)\delta_X L_{(a_n,a_n)_=},\end{aligned}$$

is a regular language by Lemma 3.4.9 and by the fact that $(X, L_1, \ldots, L_n)$ is an automatic structure for A.

Case 3. If $a_j \neq a_i$, then

$$\begin{aligned}K_{(a_i,a_j)_x} &= \{(v,w) \in K_i \times K_j \mid a_i.(v \cdot x) = a_j.w\}\delta_X \\ &= L_{(a_i,a_j)_x} \cup \{(v,w) \in uL_n \times L_j \mid a_i.(v \cdot x) = a_j.w\}\delta_X\end{aligned}$$

and

$$\begin{aligned}K_{(a_j,a_i)_x} &= \{(w,v) \in K_j \times K_i \mid a_j.(w \cdot x) = a_i.v\}\delta_X \\ &= L_{(a_j,a_i)_x} \cup \{(w,v) \in L_j \times uL_n \mid a_j.(w \cdot x) = a_i.v\}\delta_X\end{aligned}$$

are regular languages by Lemma 3.4.9 and by the fact that $(X, L_1, \ldots, L_n)$ is an automatic structure for A.

Case 4. If $a_j = a_i$, then

$$\begin{aligned}K_{(a_i,a_i)_x} &= \{(v,w) \in K_i \times K_i \mid a_i.(v \cdot x) = a_j.w\}\delta_X \\ &= L_{(a_i,a_i)_x} \cup \{(v,w) \in uL_n \times L_i \mid a_i.(v \cdot x) = a_i.w\}\delta_X \\ &\quad \cup \{(w,v) \in L_i \times uL_n \mid a_i.(w \cdot x) = a_i.v\}\delta_X \cup (u,u)\delta_X L_{(a_n,a_n)_x},\end{aligned}$$

is a regular language by Lemma 3.4.9, Proposition 2.2.3 and the fact that $(X, L_1, \ldots, L_n)$ is an automatic structure for A. □

Proof of Proposition 3.6.7. Assume that $(X, L_1, \ldots, L_n)$ is an automatic structure with respect to the generating set $A_0 = \{a_1 \ldots, a_n\}$ for the S-act A. Assume that $B = \{b_1, \ldots, b_m\}$ also generates A. Then, by Lemma 3.6.8, there exist regular languages $N_1, \ldots, N_{n+m}$ over X, such that $(X, N_1, \ldots, N_{n+m})$ forms an automatic structure with respect to the generating set $C = A_0 \cup B$ for the S-act A. Since B also generates the S-act A, by applying Lemma 3.6.9 we can subtract the generators of A_0 from C one by one, so that we obtain regular languages $K_1, \ldots, K_m$ over X, such that $(X, K_1, \ldots, K_m)$ forms an automatic structure with respect to the generating set B for the S-act A. $\square$

Proposition 3.6.7 has the following consequences, which provide a generalization of [9, Theorem 1.1] and of [4, Proposition 1.4].

Corollary 3.6.10. *Let S be a semigroup with local right identities. Assume that (X, L) is an automatic structure for S and that the finite set Y also generates S. Then there exists a regular language K over Y such that (Y, K) forms an automatic structure for S.*

Proof. Let (X, L) be an automatic structure for the semigroup S, where S has local right identities. Assume that $X = \{x_1, \ldots, x_n\}$. Then by Proposition 3.1.10, S is an automatic right S-act with generating set X. In particular we have that there exist regular languages $L_1, \ldots, L_n$ over X such that $(X, L_1, \ldots, L_n)$ forms an automatic structure with respect to the generating set X for the right S-act S.

Assume that $Y = \{y_1, \ldots, y_m\}$ also generates the semigroup S. Then as shown in Proposition 3.1.10, Y generates S as an S-act as well, hence by Proposition 3.6.7, there exist regular languages $K_1, \ldots, K_m$ such that $(Y, K_1, \ldots, K_m)$ is an automatic structure with respect to the generating set Y for the S-act S. Making use of Proposition 3.1.10 repeatedly, we obtain that there exists a regular language K over Y such that (Y, K) forms an automatic structure for the semigroup S. $\square$

Corollary 3.6.11. *Let S be a regular semigroup. Assume that (X, L) is an automatic structure for S and that the finite set Y also generates S. Then there exists a regular language K over Y such that (Y, K) forms an automatic structure for S.*

Proof. Since every regular semigroup has local right identities, our corollary immediately follows from Corollary 3.6.10. $\square$

We will make use of the following useful fact later on.

Proposition 3.6.12. *Let S be a regular semigroup generated by a finite set X and R be an $\mathcal{R}$-class of S. Let $s, t \in R$. Assume that (X^0, L) forms an automatic structure with respect to the generating set $\{s\}$ for the right S^0-act R^0. Then there exists a regular language K over X^0, such that (X^0, K) forms an automatic structure with respect to the generating set $\{t\}$ for R^0.*

Proof. Since (X^0, L) forms an automatic structure with respect to the generating set $\{s\}$ for R^0, we have that $s \cdot L\varphi = R^0$ and that the languages

$$L_= = \{(u,v) \mid u,v \in L,\ s \cdot u = s \cdot v\}\delta_X$$

$$L_x = \{(u,v) \mid u,v \in L,\ s \cdot ux = s \cdot v\}\delta_X \quad (x \in X^0)$$

are regular. Since $s \mathcal{R} t$, we also have that $s = tt's$, $t' \in V(t)$. Let w be a word over X such that $w\varphi = t's$ and let $K = wL$. Note that $s = t \cdot w\varphi$. Clearly K is a regular language over X^0 and $t \cdot K\varphi = t \cdot (wL)\varphi = s \cdot L\varphi = R^0$. We verify that (X^0, K) is an automatic structure with respect to the generating set $\{t\}$ for R^0. Note that $(k_1, k_2) \in K_=$ if and only if $k_1 \equiv wl_1$ and $k_2 \equiv wl_2$ for some $l_1, l_2 \in L$, and $t \cdot wl_1 = t \cdot wl_2$. Since $t \cdot w = s$, it follows that $t \cdot wl_1 = t \cdot wl_2$ if and only if $s \cdot l_1 = s \cdot l_2$ if and only if $(l_1, l_2) \in L_=$. Hence, we have that $K_= = (w,w)\delta_X L_=$ and so it is regular. Similarly we have that $K_x = (w,w)\delta_X L_x$, $x \in X^0$ proving that (X^0, K) is an automatic structure with respect to the generating set $\{t\}$ for R^0. □

To summarize the main result of this section, we have:

Theorem 3.6.13. *Let $S = \langle X \rangle$ be a semigroup with local right identities and assume that $(X, L_1, L_2, \dots, L_n)$ forms an automatic structure with respect to the generating set A_0 for the S-act A. Assume that the finite set Y also generates S and that the finite set B also generates the S-act A. Then there exist regular languages $K_1, \dots, K_m$ over Y such that $(Y, K_1, \dots, K_m)$ forms an automatic structure with respect to the generating set B for the S-act A.*

3.7 Fellow traveller property I.

In this section we first associate a directed labelled graph $\Gamma_X(A,S)$ to each S-act A and introduce the notion of distance in $\Gamma_X(A,S)$. With these tools, we give the definition of the fellow traveller property and claim that the introduced notion is a generalization of the fellow traveller property given for semigroups and groups. Finally we prove in this section that if A is an automatic S-act, then $\Gamma_X(A,S)$ possesses the fellow traveller property.

As before, S will denote a semigroup, X a finite generating set for S. We assume that $X \subseteq S$ and we denote by $\varphi : X^+ \to S$ the homomorphism extending the identity map $\iota : X \to S$. We denote by S^0 the semigroup obtained by adjoining a zero element $\mathbf{0}$ to S, and we let $X^0 = X \cup \{\mathbf{0}\}$. If S is a group, then we will assume that X is closed under taking inverses.

The associated graph

Intuitively we can think of an S-act A as a directed labelled graph $\Gamma_X(A,S)$, in which the vertices are elements of A, the labels are elements of X and there is an arrow from a to b with label x precisely when $a.x = b$. We write the arrows of $\Gamma_X(A,S)$ as ordered triples (a,x,b) indicating that $a.x = b$. We let $\mathcal{V}(\Gamma_X(A,S))$ denote the set of vertices and $\mathcal{A}(\Gamma_X(A,S))$ denote the set of arrows of $\Gamma_X(A,S)$. Clearly $\Gamma_X(A,S)$ is not necessarily a connected graph.

We define a *path* between two vertices a and b of $\Delta = \Gamma_X(A,S)$ to be a sequence of edges:

$$a = a_0 \overset{x_1}{\text{———}} a_1 \overset{x_2}{\text{———}} a_2 \ldots a_{n-1} \overset{x_n}{\text{———}} a_n = b$$

such that either $(a_i, x_i, a_{i+1}) \in \mathcal{A}(\Delta)$ or $(a_{i+1}, x_i, a_i) \in \mathcal{A}(\Delta)$, $(0 \leq i \leq n-1)$ and say that the *length of the path* is n. For $a, b \in \mathcal{V}(\Delta)$, we define the *distance* $d_\Delta(a,b)$ between a and b to be the length of the shortest path connecting a and b and say that the *distance is infinite* if a and b belong to different components of Δ.

We give a list of S-acts A, when $\Gamma_X(A,S)$ or a graph related to $\Gamma_X(A,S)$ is a well-known graph.

Example 3.7.1. If A is the right S-act S (see Example 3.1.4), then $\Delta = \Gamma_X(A,S)$ is the right Cayley graph $\Gamma = \Gamma_X(S)$ of S. Bearing in mind the definition of distance in the Cayley graph of a semigroup introduced in the Preliminaries, we have for all $a, b \in S$ that $d_\Delta(a,b) = d_\Gamma(a,b)$. If S is a group then Δ is a connected graph and $(g,x,h) \in \mathcal{A}(\Delta)$ if and only if $(h, x^{-1}, g) \in \mathcal{A}(\Delta)$. In other words, any two vertices are connected via a directed path. To make it clear whether we consider S as a semigroup or a right S-act we write Cayley graph $\Gamma_X(S)$ of S and the graph $\Gamma_X(S,S)$ respectively.

Example 3.7.2. If A is the right S^0-act R^0, where R is an $\mathcal{R}$-class of S (see Example 3.1.6), then $\Gamma_X(R^0, S^0)$ restricted to R is the Schützenberger graph $S\Gamma_X(R)$ of R.

Fellow traveller property

Recall the following notation. If $u \equiv x_1 \ldots x_m$ and $t \geq 1$, then

$$u(t) = \begin{cases} x_1 \ldots x_t & \text{if } t \leq m, \\ x_1 \ldots x_m & \text{if } t \geq m. \end{cases}$$

Definition 3.7.3. Let A be an S-act regularly generated by $L_1, \ldots, L_n$ and the set $A_0 = \{a_1, \ldots, a_n\}$. The graph $\Delta = \Gamma_X(A,S)$ is said to have the *fellow traveller property with respect to L_i, L_j and a_i, a_j*, if there exists a constant $k \in \mathbb{N}$ such that whenever $(u,v) \in$

$L_i \times L_j$ with $d_\Delta(a_i.u, a_j.v) \leq 1$, then $d_\Delta(a_i.u(t), a_j.v(t)) \leq k$ for all $t \geq 1$. We say that $\Gamma_X(A, S)$ possesses the *fellow traveller property* with respect to $L_1, \ldots, L_n$ and A_0, if it possesses the fellow traveller property with respect to any two regular languages L_i, L_j and corresponding generators a_i, a_j.

We have seen that if S is a semigroup then $\Gamma_X(S, S)$ is the Cayley graph of S. Now we show that the fellow traveller property for S-acts is a generalization of the fellow traveller property given for semigroups and groups.

Proposition 3.7.4. *Let S be a semigroup generated by a finite set $X = \{x_1, \ldots, x_n\}$. Then the Cayley graph of S possesses the fellow traveller property with respect to some regular language L if and only if $\Gamma_X(S, S)$ possesses the fellow traveller property with respect to some regular languages $L_1, \ldots, L_n$ and X.*

Proof. ($\Rightarrow$) Assume that the Cayley graph $\Gamma = \Gamma_X(S)$ of S has the fellow traveller property with respect to a language L. Then $L\varphi = S$ and there exists a constant $k \in \mathbb{N}$ such whenever $d_\Gamma(u, v) \leq 1$ with $u, v \in L$ then $d_\Gamma(u(t), v(t)) \leq k$ for all $t \geq 1$. As in Proposition 3.1.10, the languages $L_i = \{u \in X^+ \mid x_i u \in L\}$ $(1 \leq i \leq n)$ and X regularly generates the S-act S. Let $\Delta = \Gamma_X(S, S)$. Choose languages L_i, L_j and let $(u, v) \in L_i \times L_j$ such that $d_\Delta(x_i \cdot u, x_j \cdot v) \leq 1$. Bearing in mind that $d_\Delta(a, b) = d_\Gamma(a, b)$ for all $a, b \in S$, (see Example 3.7.1) we have that $d_\Gamma(x_i u, x_j v) \leq 1$, and hence $d_\Gamma((x_i u)(t), (x_j v)(t)) \leq k$ holds for all $t \geq 1$. In particular we have that $d_\Delta(x_i \cdot (u(t)), x_j \cdot (v(t))) \leq k$ for all $t \geq 1$, proving that the fellow traveller property holds in Δ with respect to L_i, L_j and x_i, x_j. Since L_i, L_j were arbitrarily chosen, we may deduce that $\Gamma_X(S, S)$ possesses the fellow traveller property.

($\Leftarrow$) Assume that $\Delta = \Gamma_X(S, S)$ possesses the fellow traveller property with respect to some regular languages $L_1, \ldots, L_n$ and X. Then $\bigcup_{j=1}^n x_j L_j \varphi = S$ and for any two chosen languages L_i, L_j and corresponding generators x_i, x_j, there exists a constant $k \in \mathbb{N}$ such that whenever $d_\Delta(x_i \cdot u, x_j \cdot v) \leq 1$ with $(u, v) \in L_i \times L_j$ then $d_\Delta(x_i \cdot (u(t)), x_j \cdot (v(t))) \leq k$ for all $t \geq 1$. As in Proposition 3.1.10, we let $L = \bigcup_{j=1}^n x_j L_j$. Then $L\varphi = S$. We claim that the Cayley graph $\Gamma = \Gamma_X(S)$ has the fellow traveller property with respect to L. Let $N \in \mathbb{N}$ be a constant such that for any two generators x_k and x_m that are connected via a path in Δ the distance $d_\Delta(x_i, x_j) \leq N$. Let $M = \max(k, N)$. Assume that $d_\Gamma(u, v) \leq 1$, $(u, v \in L)$. Then $u = x_i \cdot \tilde{u}$ and $v = x_j \cdot \tilde{v}$, where $x_i, x_j \in X$ and $(\tilde{u}, \tilde{v}) \in L_i \times L_j$. It follows that $d_\Delta(x_i \cdot \tilde{u}, x_j \cdot \tilde{v}) \leq 1$ holds and we obtain that for all $t \geq 1$, $d_\Delta(x_i \cdot (\tilde{u}(t)), x_j \cdot (\tilde{v}(t))) \leq k \leq M$ holds. Since $d_\Delta(a, b) = d_\Gamma(a, b)$ for all $a, b \in S$, we have that for all $t \geq 1$, $d_\Gamma(x_i \cdot (\tilde{u}(t)), x_j \cdot (\tilde{v}(t))) \leq k \leq M$. To finish the proof we need to verify that $d_\Gamma(x_i, x_j) \leq M$ or equivalently that x_i and x_j are connected in $\Gamma = \Delta$. But the latter fact follows by our assumptions, since $x_i \cdot (\tilde{u} \cdot x) = x_j \cdot \tilde{v}$ holds for some $x \in X \cup \{\lambda\}$, where λ denotes the empty word. That is, we have the following path in Δ

$$x_i \xrightarrow{\quad \tilde{u} \quad} x_i \cdot \tilde{u} \xrightarrow{x} x_i \cdot \tilde{u} \cdot x = x_j \cdot \tilde{v} \xleftarrow{\quad \tilde{v} \quad} x_j$$

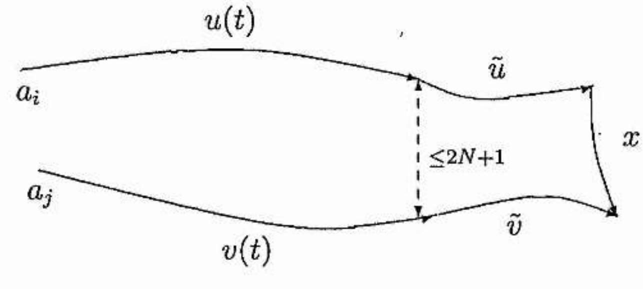
We may deduce that the Cayley graph of S possesses the fellow traveller property with respect to L. $\square$

Next we verify that if A is an automatic S-act, then $\Gamma_X(A, S)$ possesses fellow traveller property. We follow the group and semigroup theoretical proofs.

Proposition 3.7.5. *Let S be a semigroup generated by a finite set X. Let A be an automatic S-act. Then there exist regular languages $L_1, \ldots, L_n$ over X, such that $\Gamma_X(A, S)$ has the fellow traveller property with respect to $L_1, \ldots, L_n$ and A_0.*

Proof. Let $(X, L_1, \ldots, L_n)$ be an automatic structure for A with respect to the generating set $A_0 = \{a_1, \ldots, a_n\}$. For each regular language $L_{(a_i,a_j)_x}$, $x \in X \cup \{=\}$ consider a finite state automaton $\mathcal{A}_{(a_i,a_j)_x}$ accepting it and choose a constant $N \in \mathbb{N}$, such that N is greater then the number of states of any of the automata defined. Let $\Delta = \Gamma_X(A, S)$.

Choose regular languages L_i, L_j and assume that $d_\Delta(a_i.u, a_j.v) \leq 1$ holds, where $(u, v) \in L_i \times L_j$. Without loss of generality we can assume that $a_i.(u \cdot y) = a_j.v$ for some $y \in X \cup \{\lambda\}$, where λ denotes the empty word. Then $(u, v)\delta_X$ is accepted by the automaton $\mathcal{A}_{(a_i,a_j)_x}$, where $x = y$ if $y \in X$ and x is the symbol $=$ if $y = \lambda$. Start reading the word $(u, v)\delta_X$ in $\mathcal{A}_{(a_i,a_j)_x}$ and assume that after reading the first t letters $(u(t), v(t))\delta_X$, we are at state q. Let $(\tilde{u}, \tilde{v})\delta_X$ be the shortest word over $X(2, \$)$ such that reading $(\tilde{u}, \tilde{v})\delta_X$ from state q, we arrive at a final state of $\mathcal{A}_{(a_i,a_j)_x}$. Clearly, the length of $(\tilde{u}, \tilde{v})\delta_X$ is less then N, since the number of states of the considered automaton is less then N. Furthermore, since $(u(t), v(t))\delta_X(\tilde{u}, \tilde{v})\delta_X$ is accepted by $\mathcal{A}_{(a_i,a_j)_x}$, we have the following diagram in Δ:



That is,

$$d_\Delta(a_i.u(t), a_j.v(t)) \leq |\tilde{u}| + |\tilde{v}| + |x| \leq 2N + 1,$$

which proves that the fellow traveller property holds in Δ with respect to L_i, L_j and a_i, a_j. Since L_i, L_j were arbitrarily chosen, we may deduce that Δ possesses the fellow traveller property with respect to $L_1, \ldots, L_n$ and A_0. □

Corollary 3.7.6. *Let S be an automatic semigroup. Then the Cayley graph of S possesses the fellow traveller property.*

Proof. Immediately follows from Proposition 3.1.10 and Propositions 3.7.4 and 3.7.5. □

Corollary 3.7.7. *Let S be a monoid and assume that $A = S/\mathcal{R}$ is an automatic left S-act. Then $\Gamma_X(A, S)$ has the fellow traveller property.*

We will also make use of the following corollary.

Corollary 3.7.8. *Let S be a semigroup generated by a finite set X and let A be an S-act. Assume that $(X, L_1, \ldots, L_n)$ forms an automatic structure with respect to the generating set $A_0 = \{a_1, \ldots, a_n\}$ for A. Let $l \in \mathbb{N}$. Then there exists a constant $k \in \mathbb{N}$, such that whenever $a_i.uw = a_j.v$ holds for some $w \in X^+$, $|w| \leq l$, where $(u, v) \in L_i \times L_j$, then $d_\Delta(a_i.u(t), a_j.v(t)) \leq k$ holds in $\Delta = \Gamma_X(A, S)$.*

Proof. Let $L = \{w \in X^+ \mid |w| \leq l\}$. Since for all $a_i, a_j \in A_0$ and $w \in L$ the languages $L_{(a_i,a_j)_w} = \{(u, v) \mid a_i.uw = a_j.v\}\delta_X$ are regular by Lemma 3.4.8, finite state automata $\mathcal{A}_{(a_i,a_j)_w}$ can be defined accepting these languages. We can choose a constant $N \in \mathbb{N}$ that is greater than the number of states in any of the automata defined. The proof now continues along the same lines as the proof of Proposition 3.7.5. □

In preparation for the next section, we make the following observation. Let S be a regular semigroup generated by a finite set X and let R be an $\mathcal{R}$-class of S. Then, for any regular language L over X and $s \in R$, for which $s \cdot L\varphi = R^0$ holds, the graph $\Delta = \Gamma_X(R^0, S^0)$ associated to the right S^0-act R^0 possesses the fellow traveller property with respect to L and $\{s\}$. Indeed, since we have for all $s, t \in R^0$ that $d_\Delta(s \cdot u, s \cdot v) \leq 2$. This fact ruins our hopes to connect Schützenberger automaticity of R and the fellow traveller property of Δ, and urges us to introduce the following definition.

Definition 3.7.9. Let S be a semigroup and R be an $\mathcal{R}$-class of S. Let L be a regular language over X and $s \in R$ such that $s \cdot L\varphi = R$. The Schützenberger graph $\Gamma = \mathrm{S}\Gamma_X(R)$ is said to have the *fellow traveller property with respect to L and $\{s\}$*, if there exists a constant $k \in \mathbb{N}$ such that in the Schützenberger graph of R whenever $u, v \in L$ with $d_\Gamma(s \cdot u, s \cdot v) \leq 1$, then $d_\Gamma(s \cdot u(t), s \cdot v(t)) \leq k$ holds for all $t \geq 1$.

Similarly to Proposition 3.7.5 and Corollary 3.7.8 the following proposition and corollary can be proved:

Proposition 3.7.10. *Let S be a regular semigroup generated by a finite set X and let R be a Schützenberger automatic $\mathcal{R}$–class of S. Let $s \in R$. Then there exists a regular language L over X, such that $\mathrm{S}\Gamma_X(R)$ has the fellow traveller property with respect to L and $\{s\}$.*

Corollary 3.7.11. *Let S be a regular semigroup generated by a finite set X and R be an $\mathcal{R}$-class of S. Let $s \in R$ and K be a regular language over X satisfying condition $(S2)$ of Proposition 3.2.1. Let $l \in \mathbb{N}$. Then there exists a constant $k \in \mathbb{N}$, such that whenever $s \cdot uw = s \cdot v$ for some $w \in X^+$, $|w| \leq l$, where $u, v \in L$, then $d_\Gamma(s \cdot u(t), s \cdot v(t)) \leq k$ holds in the Schützenberger graph $\Gamma = \Gamma_X(R)$.*

3.8 Fellow traveller property II.

We have seen in the previous section that if A is an automatic S-act, then the fellow traveller property holds in $\Gamma_X(A, S)$. The question naturally arises whether the converse holds or not:

Problem: Let A be an S-act. Is it true that if $\Gamma_X(A, S)$ possesses the fellow traveller property then A is automatic?

We know that groups can be characterized by the fellow traveller property, while semigroups usually cannot. Hence, with Propositions 3.1.10 and 3.7.4 in mind an affirmative answer can be given if A is the right S-act S, where S is a group, but if A is the right S-act S, where S is a semigroup then the answer is expected to be negative. As a reaction to the result that fellow traveller property does not characterize automatic semigroups, investigations continued in two directions:

(a) On one hand, semigroup classes closely related to groups came into the focus of research; i.e. it was shown in [4] that completely simple semigroups can be characterized by the fellow traveller property.

(b) On the other hand, modification of the concept of automaticity has been considered in [34]. With the introduction of the notion of prefix-automatic monoids and the notion of monoids of finite geometric type the following results are verified:

 (i) Prefix-automaticity is still a valid generalization of the group theoretical notion of automaticity.

 (ii) Prefix-automatic monoids of finite geometric type are characterized by the fellow traveller property.

These considerations suggest that if we want to obtain a positive answer to our proposed problem, then we need to have structural information and a finiteness condition on the S-act A.

The main purpose of this section is to prove

Theorem 3.8.1. *Let S be a regular semigroup with finitely many idempotents. Assume that each $\mathcal{R}$-class possesses the fellow traveller property. Then S is Schützenberger automatic.*

The key to prove Theorem 3.8.1 is based on the following proposition.

Proposition 3.8.2. *Let S be a regular semigroup with finitely many idempotents. Let R be an arbitrary $\mathcal{R}$-class of S and let $e \in E(S) \cap R$. Then R is Schützenberger automatic if and only if H_e is automatic.*

Proposition 3.8.2 is proved in three steps. The first step is to find a generating set for H_e in terms of generators of S. The second step is to prove that if the Schützenberger graph of R possesses the fellow traveller property then the Cayley graph of H_e possesses the fellow traveller property. The final step is to prove that that if H_e is automatic, then R is Schützenberger automatic.

First step. Let S be a regular semigroup generated by a set X, and let G be a maximal subgroup of S. In particular, G is an $\mathcal{H}$-class of S, and the identity element e of G is an idempotent in S. Recall that e is a left identity in its $\mathcal{R}$-class. Our aim is to find a generating set for G. In doing so, we will translate the results from [32] into our setting.

The following concept plays a central role. Consider the $\mathcal{R}$-class R of e and choose representatives $e = r_0,\ r_1,\ r_2, \ldots$ of the $\mathcal{H}$-classes contained in R. For each chosen representative choose an inverse $r_0' = e,\ r_1',\ r_2',\ \ldots\ (r_i' \in V(r_i))$. Clearly $er_i = r_i$ and $r_i \,\mathcal{D}\, r_j'$. Illustrating what we have done:

R_e	$\cdot r_0 = e$	$\cdot r_1$	$\cdot r_2$	$\cdot r_3$	
		$\cdot r_3'$			
	$\cdot r_1'$			$\cdot r_2'$	

For a word w over X and a representative r_i, if $r_i \cdot w\varphi \mathrel{\mathcal{R}} e$, then we let r_{iw} denote the representative of the $\mathcal{H}$-class of $r_i \cdot w\varphi$.

R_e	$\cdot r_0 = e$		$\cdot r_i$		$\cdot r_i w\varphi$ $\cdot r_{iw}$

We verify

Proposition 3.8.3. *If X generates S, then*

$$Y = \{r_i x (r_{ix})' e \mid r_i x \mathrel{\mathcal{R}} e\}$$

generates G as a semigroup.

Proof. First we show that $Y \subseteq G$. Let $r_i x (r_{ix})' e \in Y$. Then, $r_{ix} \mathrel{\mathcal{H}} r_i x \mathrel{\mathcal{R}} e$, and hence $R_{(r_{ix})'} \cap L_{r_i x}$ contains an idempotent, namely $(r_{ix})' r_{ix}$. It follows by Proposition 2.1.5, that $r_i x (r_{ix})' \in L_{(r_{ix})'} \cap R_{r_i x} \subseteq R_e$. On the other hand, $r_i x (r_{ix})' \mathrel{\mathcal{H}} r_{ix} (r_{ix})' \mathrel{\mathcal{R}} e$, and hence $R_e \cap L_{r_i x (r_{ix})'}$ contains an idempotent, namely $r_{ix}(r_{ix})'$. Applying Proposition 2.1.5 one more time, we obtain that $r_i x (r_{ix})' e \in L_e \cap R_{r_i x} = G$. Illustrating the argument:

R_e	$\cdot r_0 = e$	$\cdot r_i x (r_{ix})'$ $\cdot r_{ix}(r_{ix})'$		$\cdot r_i x$ $\cdot r_{ix}$	
		$\cdot (r_{ix})'$		$\cdot (r_{ix})' r_{ix}$	

By Proposition 2.1.5 we also have that for $y_1, y_2 \in Y$, $y_1 \cdot y_2 \in G$ holds. Let $g \in G$, and assume that $g = x_1 \ldots x_n$, where $x_i \in X$ $(1 \leq i \leq n)$. We verify that g can be written in terms of generators in Y. Clearly $g = eg = ex_1 \ldots x_n$ and hence $r_{0x_1 \ldots x_n} = r_0 = e$. We make the following further observations.

(1) Since $e \mathrel{\mathcal{R}} ex_1 \ldots x_n$, we have for all $1 \leq i \leq n$ that

$$e \mathrel{\mathcal{R}} ex_1 \ldots x_i. \tag{3.6}$$

Moreover, since $\mathcal{R}$ is a left congruence, it follows that

$$h \ \mathcal{R} \ hx_1 \dots x_i \qquad \text{for all} \qquad h \in G, \ 1 \le i \le n. \tag{3.7}$$

(2) For all $1 \le i < n$,

$$r_{0x_1 \dots x_i} x_{i+1} \ \mathcal{R} \ e. \tag{3.8}$$

For, by definition and by (3.6), $e \ \mathcal{R} \ r_0 x_1 \dots x_i \ \mathcal{H} \ r_{0x_1 \dots x_i}$ and hence it follows by Proposition 2.1.11 that $Gr_0x_1 \dots x_i = Gr_{0x_1 \dots x_i}$. So, there exists $h \in G$ such that

$$hx_1 \dots x_i = hr_0x_1 \dots x_i = er_{0x_1 \dots x_i} = r_{0x_1 \dots x_i}.$$

Multiplying each side on the right by x_{i+1}, and making use of (3.7),we obtain that $r_{0x_1 \dots x_i} x_{i+1} \ \mathcal{R} \ e$ indeed holds.

(3) Since $r_{0x_1 \dots x_i} x_{i+1} \ \mathcal{H} \ r_{0x_1 \dots x_{i+1}} \ \mathcal{L} \ ((r_{0x_1 \dots x_{i+1}})' r_{0x_1 \dots x_{i+1}})$, and since every idempotent is a right identity in its $\mathcal{L}$-class, we have that

$$r_{0x_1 \dots x_i} x_{i+1} = r_{0x_1 \dots x_i} x_{i+1} ((r_{0x_1 \dots x_{i+1}})' r_{0x_1 \dots x_{i+1}}). \tag{3.9}$$

We are now ready to rewrite g in terms of generators of Y:

$$\begin{aligned}
g = ex_1x_2 \dots x_n &= r_0x_1x_2 \dots x_n \\
&= (r_0x_1r'_{0x_1}r_{0x_1})x_2 \dots x_n && \text{by (3.9)}, \\
&= r_0x_1r'_{0x_1}(r_{0x_1}x_2r'_{0x_1x_2}r_{0x_1x_2}) \dots x_n && \text{by (3.9)}, \\
&\ \vdots \\
&= r_0x_1r'_{0x_1}r_{0x_1}x_2r'_{0x_1x_2}r_{0x_1x_2} \dots (r_{0x_1 \dots x_{n-1}}x_n(r_{0x_1 \dots x_n})'r_{0x_1 \dots x_n}).
\end{aligned}$$

Using the fact that e is a left identity in its $\mathcal{R}$-class and that $r_{0x_1 \dots x_i} \ \mathcal{R} \ e$ for all $1 \le i \le n$ by (3.6), we obtain that

$$g = (r_0x_1r'_{0x_1}e)(r_{0x_1}x_2r'_{0x_1x_2}e) \dots (r_{0x_1 \dots x_{n-1}}x_n(r_{0x_1 \dots x_n})'e).$$

□

To make notation convenient, we write $[r_i, x]$ instead of $r_ix(r_{ix})'e \in Y$. Proposition 3.8.3 asserts that $Y = \{[r_i, x] \mid r_ix \ \mathcal{R} \ e\}$ generates G. By the proof of Proposition 3.8.3, we have that for $g \in G$,

$$g = x_1x_2 \dots x_n = [r_0, x_1][r_{0x_1}, x_2] \dots [r_{0x_1 \dots x_{n-1}}, x_n].$$

We observe here that the length of a word over X that represents an element of G does not change when we rewrite it in terms of generators of Y.

Second step. We keep the notation of the first step. We prove that if $R = R_e$ is a Schützenberger automatic $\mathcal{R}$-class, and $E(S)$ is finite, then G, our chosen maximal subgroup, is automatic. We need the following lemma and its corollary.

Lemma 3.8.4. *Let S be a regular semigroup generated by a finite set X and assume that $E(S)$ is finite. Let R be an $\mathcal{R}$-class of S. There exists a constant $c \in \mathbb{N}$ such that whenever $s, t \in R$ are such that $s = tx$ for some $x \in X$, then $t \cdot w = s$ for some word $w \in X^+$ with length less then or equal to c.*

Proof. Since $E(S)$ is finite, there exists a constant $c_1 \in \mathbb{N}$ such that each idempotent $e \in E(S)$ can be represented by a word of length less then or equal to c_1. For each $x \in X$, choose $x' \in V(x)$. Since X is finite, there exists a contant $c_2 \in \mathbb{N}$ such that each chosen element $x' \in V(x)$ can be represented by a word of length less then or equal to c_2. Let $c = 2c_1 + c_2$. Assume that $s \mathcal{R} t$ and that $sx = t$. Then, by Proposition 2.1.8, $ss' = tt'$ for some $s' \in V(s)$ and $t' \in V(t)$. Let $h \in S(s's, xx')$. Then, by Proposition 2.1.10, $t'' = x'hs' \in V(xs) = V(t)$, and so we obtain that

$$s = tt's = tt''tt's = tt''s = t \cdot x'hs's.$$

By definition $h \in E(S)$ and so we obtain that $x'hs's$ can be represented by a word of length less then or equal to c. $\square$

An immediate corollary of Lemma 3.8.4:

Corollary 3.8.5. *Let S be a regular semigroup generated by a finite set X and assume that $E(S)$ is finite. Let R be an $\mathcal{R}$-class of S and denote by Γ the Schützenberger graph of R. Let $k \in \mathbb{N}$. Then there exists a constant $c \in \mathbb{N}$ such that whenever $s, t \in R$ are such that $d_\Gamma(s, t) \leq k$, then $s \cdot w = t$ for some word $w \in X^+$ with length less then or equal to c.*

Proof. By definition, if $d_\Gamma(s, t) \leq k$, then there exists an undirected path

$$s = s_0 \overset{x_0}{\text{———}} s_1 \overset{x_1}{\text{———}} s_2 \dots s_{n-1} \overset{x_{n-1}}{\text{———}} s_n = t,$$

such that $n \leq k$. We want to verify that there exists a directed path with some bounded length from s to t. Two cases can occur:

(i) If $s_i x_i = s_{i+1}$, then our arrow point in the "right" direction.

(ii) If $s_{i+1}x_i = s_i$, then by Lemma 3.8.4, there exists a constant c_1 and a word $w \in X^+$, whose length is less then or equal to c_1 such that $s_i \cdot w = s_{i+1}$. Hence arrows that point in the "wrong" direction can be substituted by a sequence of at most c_1 arrows pointing in the "right" direction.

We may now deduce that there is a directed path from s to t in the Schützenberger graph of R, whose length is less then or equal to $c = kc_1$. $\square$

Proposition 3.8.6. *Let S be a regular semigroup generated by a finite set X and let G be a maximal subgroup of S with identity element e. Assume that $E(S)$ is finite and that the Schützenberger graph of R_e possesses the fellow traveller property. Then G is an automatic group. In particular, if R_e is Schützenberger automatic, then G is automatic.*

Proof. For the sake of convenience, we assume that $e \in X$. We let $\varphi : X^+ \to S$ denote the homomorphism extending the identity map $\iota_1 : X \to S$. Since R_e possesses the fellow traveller property, there exists a regular language L over X such that $e \cdot L\varphi = R_e$. By our assumptions $E(S)$ is finite, and so R_e contains finitely many $\mathcal{H}$-classes. Choose representatives $e = r_0, r_1, \ldots, r_n$ of the $\mathcal{H}$-classes contained in R_e and also choose inverses of the representatives $e = r'_0, r'_1, \ldots, r'_n$ $(r'_i \in V(r_i))$. According to Proposition 3.8.3 and using the notation introduced after Proposition 3.8.3, the finite set $Y = \{[r_i, x] \mid x \in X,\ r_i x \mathrel{\mathcal{R}} e\}$ generates G as a semigroup. Let $\psi : Y^+ \to G$ be the homomorphism extending the map $\iota : Y \to G;\ [r_i, x] \mapsto r_i x r'_{ix} e$.

Next, we construct a regular language over Y that is mapped onto G. For this, let

$$\widetilde{L} = \{ew \in eL \mid (ew)\varphi \in G\}.$$

It is immediate that $\widetilde{L}\varphi = G$. Consider the map

$$\varsigma : \widetilde{L} \to Y^+;\ x_1x_2\ldots x_m \mapsto [r_0, x_1][r_{0x_1}, x_2]\ldots[r_{0x_1\ldots x_{m-1}}, x_m].$$

Let $K = \widetilde{L}\varsigma$. Since $\widetilde{L}\varphi = G$, we have that $K\psi = G$. We first show that K is a regular language over Y. Let $\mathcal{A}_1 = (\Sigma_1, X, \nu, p, T)$ be a finite state automaton accepting the regular language eL. Let $\epsilon \notin Y$ and $Z = Y \cup \{\epsilon\}$. Consider the automaton $\mathcal{A} = (\Sigma, Y, \mu, q, F)$, where

$$\Sigma = (\Sigma_1 \times Y) \cup \{q\} \cup FS,\ q = (p, \epsilon),\ F = T \times Y,$$

and the transition function μ is defined as follows:

$$\mu(q, [r_j, y]) = (\nu(p, y), [r_j, y]),$$

for $(\tilde{p}, [r_i, x]) \in \Sigma_1 \times Y$ we define

$$\mu((\tilde{p}, [r_i, x]), [r_j, y]) = \begin{cases} (\nu(\tilde{p}, y), [r_j, y]) & \text{if } r_{ix} = r_j, \\ FS & \text{otherwise} \end{cases}$$

and

$$\mu(FS, [r_j, y]) = FS.$$

By the definition of K, it is straightforward, that every word $w \in K$ is accepted by $\mathcal{A}$. On the other hand, by the definition of μ, we have that every word accepted by $\mathcal{A}$ is an element of K. We may now deduce that K is a regular language for which $K\psi = G$.

We next prove that G is automatic by showing that the fellow traveller property holds in the Cayley graph Δ of G. The Schützenberger graph of R will be denoted by Γ. We will work with distances both in the Cayley graph Δ of G and in the Schützenberger graph Γ of R. The notation will always make it clear in which graph the distance is understood. First, we introduce the following upper bounds:

- Since Y is finite, there exists $c_1 \in \mathbb{N}$, such that for every generator $[r_i, x] \in Y$ there exists a word w over X of length less then or equal to c_1 such that $[r_i, x]\psi = w\varphi$.

- Since the set of representatives of the $\mathcal{H}$-classes contained in R_e is finite, there exists $c_2 \in \mathbb{N}$, such that $r_i'e$, $(r_i' \in V(r_i),\ 1 \leq i \leq n)$ can be represented by a word over X of length less then or equal to c_2. In particular $e = r_0$ can be represented by a word with length less or equal to c_2.

Let $u, v \in K$ for which $d_\Delta(u, v) \leq 1$ holds. We verify that there exists a constant $c \in \mathbb{N}$ such that $d_\Delta(u(t), v(t)) \leq c$ holds for all $t \geq 1$.

We may assume that $u \cdot [r_i, x] = v$ for some $[r_i, x] \in Y$. Consider $e\tilde{u}, e\tilde{v} \in \widetilde{L}$ for which $(e\tilde{u})\varsigma \equiv u$ and $(e\tilde{v})\varsigma \equiv v$. Let w be a word over X such that $w\varphi = [r_i, x]\psi$. We may assume that $|w| \leq c_1$. Then $e \cdot \tilde{u}w = e \cdot \tilde{v}$, and hence $d_\Gamma(e \cdot \tilde{u}, e \cdot \tilde{v}) \leq c_1$. Making use of Corollary 3.7.11, we have that there exists $k \in \mathbb{N}$ such that $d_\Gamma(e \cdot \tilde{u}(t), e \cdot \tilde{v}(t)) \leq k$ for all $t \geq 1$.

Let $t \geq 1$. We verify that in the Schützenberger graph of R, the distance between the vertices $e \cdot \tilde{u}(t)\varphi$ and $u(t)\psi$ is less then c_2. Assume that

$$u \equiv [r_0, x_1][r_{0x_1}, x_2] \ldots [r_{0x_1 \ldots x_{l-1}}, x_l].$$

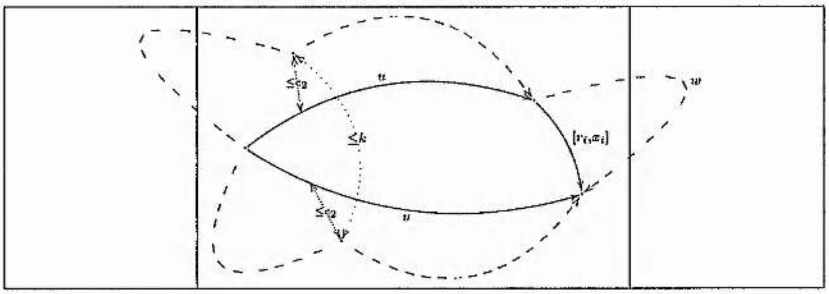
Then $e\tilde{u} \equiv x_1 \ldots x_l$ and

$$\begin{aligned} u(t) &= [r_0, x_1][r_{0x_1}, x_2] \ldots [r_{0x_1 \ldots x_{t-1}}, x_t] \\ &= r_0 x_1 r'_{0x_1}(e \cdot r_{0x_1}) x_2 r'_{0x_1x_2} e \cdot \ldots (e \cdot r_{0x_1 \ldots x_{t-1}}) x_t r'_{0x_1 \ldots x_t} e \\ &= (r_0 x_1 r'_{0x_1} r_{0x_1}) x_2 r'_{0x_1x_2} \cdot \ldots \cdot r_{0x_1 \ldots x_{t-1}} x_t r'_{0x_1 \ldots x_t} e \qquad \text{(since } r_i \mathcal{R} e) \\ &= r_0 (x_1 \ldots x_t) r'_{0x_1 \ldots x_t} e \qquad \text{(by (3.9))} \\ &= e \cdot \tilde{u}(t) \cdot r'_{0x_1 \ldots x_t} e. \end{aligned}$$

It follows that $d_\Gamma(u(t), e \cdot \tilde{u}(t)) \leq c_2$. Similarly, $d_\Gamma(v(t), e \cdot \tilde{v}(t)) \leq c_2$ and so we have that

$$\begin{aligned} d_\Gamma(u(t), v(t)) &\leq d_\Gamma(u(t), e \cdot \tilde{u}(t)) + d_\Gamma(e \cdot \tilde{u}(t), e \cdot \tilde{v}(t)) + d_\Gamma(e \cdot \tilde{v}(t), v(t)) \\ &\leq c_2 + k + c_2. \end{aligned}$$

Illustrating this, we have



By Corollary 3.8.5, there exists a constant $c_3 \in \mathbb{N}$ and a word $\overline{w} \in X^+$ such that $|\overline{w}| \leq c_3$ and

$$u(t)\psi \cdot (\overline{w}\varphi) = u(t)\psi \cdot (e\overline{w})\varphi = v(t)\psi.$$

It follows that $(e\overline{w})\varphi \in G$. Clearly $(e\overline{w})\varphi$ can be represented by a word over X with length less then or equal to $1 + c_3$. According to the remark after Proposition 3.8.3, rewriting a word over X that represents an element of G does not change the length of the word and so we have that $(e\overline{w})\varphi$ can be represented by a word over Y with length less then or equal to $1 + c_3$. We may now deduce that there exists a word $\widetilde{w} \in Y^+$, such that $|\widetilde{w}| \leq 1 + c_3$, and $u(t)\widetilde{w} = v(t)$ and so $d_\Delta(u(t), v(t)) \leq 1 + c_3$ for all $t \geq 1$. □

Third step. We prove

Proposition 3.8.7. *Let S be a regular semigroup finitely generated by $X \subseteq S$ and assume that $E(S)$ is finite. Let $e \in E(S)$ and assume that H_e is automatic. Then R_e is Schützenberger automatic.*

Proof. Assume that H_e is generated by a finite set $Y \subseteq H_e$. Let $\psi : Y^+ \to H_e$ denote the homomorphism extending the identity map $\iota_1 : Y \to H_e$. Let $K \subseteq Y^+$ be a regular language such that (Y, K) form an automatic structure for H_e with uniqueness. Since $E(S)$ is finite, D_e contains finitely many $\mathcal{L}$-classes. We choose representatives $r_0 = e, r_1, \ldots, r_n$ in each $\mathcal{H}$-class contained in R_e. For each representative r_i, we choose an inverse $r_i' \in V(r_i)$. Since S is generated by X, it is also generated by the finite set

$$Z = X \cup Y \cup \{r_0, \ldots, r_n\}.$$

Let $\varphi : Z^+ \to S$ denote the homomorphism extending the identity map $\iota : Z \to S$. Let

$$L = \bigcup_{i=1}^{n} K r_i.$$

By Green's Lemma, $L\varphi = R_e$. Moreover, since (Y, K) is an automatic structure with uniqueness, for each $s \in R_e$, there exists exactly one $w \in L$ such that $w\varphi = s$, and so $L_= = \{(u, u) | u \in L\}\delta_Z$ is a regular language. To show that (Z, L) is a Schützenberger automatic structure for R_e, we have to show that for each $z \in Z$, L_z is a regular language. We first fix notation. Let

$$I = \{(i, j) \mid 1 \leq i, j \leq n,\ r_i z \,\mathcal{H}\, r_j\}.$$

Observe that if $(i, j) \in I$, then $r_i z r_j' e \in H_e$. Indeed, since $r_j' r_j \in L_{r_i z} \cap R_{r_j'}$, we have by Proposition 2.1.5, that $r_i z r_j' \in R_{r_i z} \cap R_{r_j}$. Since $r_j r_j' \in L_{r_i z r_j'} \cap R_e$, we have $r_i z r_j' e \in R_{r_i z r_j'} \cap L_e = H_e$ by Proposition 2.1.5. Thus, for each $(i, j) \in I$ we can fix a word $w_{i,j} \in Y^+$ such that $w_{i,j}\psi = (r_i z r_j' e)\varphi$. For each $(i, j) \in I$, we let

$$L_{i,j} = L_z \cap (Y^* r_i \times Y^* r_j)\delta_Z.$$

We first verify that

$$L_z = \bigcup_{(i,j) \in I} L_{i,j}, \tag{3.10}$$

then we show that $L_{i,j}$ is a regular language for each $(i, j) \in I$. It is immediate that $L_{i,j} \subseteq L_z$ for each $(i, j) \in I$. To prove the reverse inclusion, we show the following stronger statement:

$$(\tilde{u} r_i, \tilde{v} r_j)\delta_Z \in L_z \iff r_i z \,\mathcal{H}\, r_j \text{ and } (\tilde{u}, \tilde{v})\delta_Y \in K_{w_{i,j}}. \tag{3.11}$$

First, assume that $(\tilde{u} r_i, \tilde{v} r_j)\delta_Z \in L_z$. Then $\tilde{u} r_i z = \tilde{v} r_j$ and it follows that $r_i z \,\mathcal{H}\, r_j$. On the other hand, multiplying each side on the right by $r_j' e$, we obtain that $\tilde{u} r_i z r_j' e = \tilde{v} r_j r_j' e$. Since $r_j r_j' \mathcal{R}\, e \,\mathcal{H}\, \tilde{v}$, we have $\tilde{v} r_j r_j' e = \tilde{v}$ and so $\tilde{u} r_i z r_j' e = \tilde{v}$. It follows that $(\tilde{u}, \tilde{v})\delta_Z \in K_{w_{i,j}}$.

For the converse, assume that $(\tilde{u},\tilde{v})\delta_Z \in K_{w_{i,j}}$ and that $r_iz\ \mathcal{H}\ r_j$. Then $\tilde{u}w_{i,j} = \tilde{v}$, and so $\tilde{u}r_izr'_je = \tilde{v}$ holds in S. Multiplying each side on the right by r_j, we obtain that $\tilde{u}r_izr'_jer_j = \tilde{v}r_j$. Since $e\ \mathcal{R}\ r_j$, we have $er_j = r_j$. By assumption $r_iz\ \mathcal{H}\ r_j\ \mathcal{L}\ r'_jr_j$, and so $r_izr'_jr_j = r_iz$. We may now deduce that $\tilde{u}r_izr'_jer_j = \tilde{u}r_iz = \tilde{v}r_j$, and so $(\tilde{u}r_i,\tilde{v}r_j)\delta_Z \in L_z$.

Fix $(i,j) \in I$. We construct a finite state automaton accepting $L_{i,j}$. To simplify notation, we let w denote $w_{i,j}$. Since (Y,K) is an automatic structure for H_e, we have by Proposition 2.3.14, that K_w is a regular language. Let $\mathcal{A}_1 = (\Sigma_1, Y, \mu_1, p_1, F_1)$ be a finite state automaton accepting K_w. Consider the finite state automaton $\mathcal{A}_{i,j} = (\Sigma, Y \cup \{r_i, r_j\}, \mu, p, F)$, where

$$\Sigma = (\Sigma_1 \times \{\emptyset, r_i\} \times \{\emptyset, r_j\}) \cup \{FS\}; \quad F = F_1 \times \{r_i\} \times \{r_j\}; \quad p = (p_1, \emptyset, \emptyset).$$

The transition function μ is defined in the following way.

$$\mu((q,\emptyset,\emptyset),(y_1,y_2)) = \begin{cases} (\mu_1(q,(y_1,y_2)),\emptyset,\emptyset) & \text{if } y_1,y_2 \in Y, \\ (\mu_1(q,(\$,y_2)),r_i,\emptyset) & \text{if } y_1 = r_i,\ y_2 \in Y, \\ (\mu_1(q,(y_1,\$)),\emptyset,r_j) & \text{if } y_1 \in Y,\ y_2 = r_j, \\ (q,r_i,r_j) & \text{if } y_1 = r_i,\ y_2 = r_j, \\ FS & \text{otherwise.} \end{cases}$$

$$\mu((q,r_i,\emptyset),(y_1,y_2)) = \begin{cases} (\mu_1(q,(\$,y_2)),r_i,\emptyset) & \text{if } y_1 = \$,\ y_2 \in Y, \\ (q,r_i,r_j) & \text{if } y_1 = \$,\ y_2 = r_j, \\ FS & \text{if } y_1 \neq \$. \end{cases}$$

Similarly,

$$\mu((q,\emptyset,r_j),(y_1,y_2)) = \begin{cases} (\mu_1(q,(y_1,\$)),\emptyset,r_j) & \text{if } y_1 \in Y,\ y_2 = \$, \\ (q,r_i,r_j) & \text{if } y_1 = r_i,\ y_2 = \$, \\ FS & \text{if } y_2 \neq \$, \end{cases}$$

and define

$$\mu((q,r_i,r_j),(y_1,y_2)) = FS.$$

Let $u \equiv a_1 \ldots a_k$ and $v \equiv b_1 \ldots b_m$ be words over Z and assume that $(u,v)\delta_Z$ is accepted by $\mathcal{A}_{i,j}$. Without loss of generality we may assume that $k \leq m$. Since $(u,v)\delta_Z$ is accepted by $\mathcal{A}_{i,j}$, we have that the path

$$p \xrightarrow{(a_1,b_1)} q_1 \xrightarrow{(a_2,b_2)} \cdots \xrightarrow{(a_k,b_k)} q_k \xrightarrow{(\$,b_{k+1})} \cdots \xrightarrow{(\$,b_m)} q_m$$

is successful, and so $q_m \in F$. It follows that $a_k = r_i$ and $b_m = r_j$ and that the word $(a_1 \ldots a_{k-1}, b_1 \ldots b_{m-1})\delta_Y \in K_w$. Keeping in mind (3.11), we indeed obtain that $(u,v)\delta_Z \in L_{i,j}$.

Assume that $(u,v)\delta_Z \in L_{i,j}$. Then $u \equiv a_1 \ldots a_{k-1}r_i$ and $v \equiv b_1 \ldots b_{m-1}r_j$. Moreover, by (3.11), the word $(a_1 \ldots a_{k-1}, b_1 \ldots b_{m-1})\delta_Y \in K_w$ and so it is accepted by $\mathcal{A}_1$. From the construction of $\mathcal{A}_{i,j}$, it follows htat $(u,v)\delta_Z$ is indeed accepted by $\mathcal{A}_{i,j}$.

We may now deduce for each $(i,j) \in I$, the language $L_{i,j}$ is regular and so L_z is a regular language, since by (3.10), it is a finite union of regular languages. We have thus proved that (Z, L) is a Schützenberger automatic structure for R_e. □

Combining Proposition 3.8.6 and 3.8.7, we obtain:

Corollary 3.8.8. *Let S be a regular semigroup with finitely many idempotents. Let $e \in E(S)$. Then H_e is automatic if and only if R_e is Schützenberger automatic.*

Corollary 3.8.9. *Let S be a regular semigroup with finitely many idempotents. Let R be an $\mathcal{R}$-class of S and assume that the fellow traveller property holds in the Schützenberger graph of R with respect to some regular language.Then R is Schützenberger automatic.*

Proof. Let $e \in R$. If the fellow traveller property holds in R, then H_e is automatic by Proposition 3.8.6. If H_e is automatic, then R_e is Schützenberger automatic by Proposition 3.8.7. □

3.9 Fellow traveller property III.

In the previous section, we verified that if S is a regular semigroup with finitely many idempotents, then the fellow traveller property of $S\Gamma_R$ implies that R is Schützenberger automatic. In this section, we give an example which shows that Corollary 3.8.9 does not hold without the finiteness condition.

Let $X = \{x_0, x_1, x_2, \ldots\}$ be an infinite set, and consider the symmetric inverse monoid I_X. Let $T = \{\alpha_{x_i,x_j} : x_i \mapsto x_j\} \cup \emptyset$. Clearly T is an inverse subsemigroup of I_X, moreover

$$\alpha_{x_i,x_j} \,\mathcal{R}\, \alpha_{x_n,x_m} \iff x_i = x_n \qquad \text{and} \qquad \alpha_{x_i,x_j} \,\mathcal{L}\, \alpha_{x_n,x_m} \iff x_j = x_m.$$

We may deduce that T is an aperiodic 0-bisimple inverse subsemigroup in I_X. In particular, $T \setminus \{\emptyset\}$ is a $\mathcal{D}$-class of I_X containing infinitely many idempotents. Further on we denote α_{x_0,x_0} by α.

Next, we construct a finitely generated inverse subsemigroup S of I_X containing T. For this, consider the following three elements of I_X:

$$\beta : x_i \mapsto \begin{cases} x_{i+2} & \text{if } i \text{ is even,} \\ \text{undefined} & \text{otherwise} \end{cases} \qquad \gamma : x_i \mapsto \begin{cases} x_{i+2} & \text{if } i \text{ is odd,} \\ x_1 & \text{if } i = 0, \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$\delta : x_i \mapsto \begin{cases} x_{i-1} & \text{if } i = 2n+1, \text{ where } n \text{ is not a prime,} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Let S be the inverse subsemigroup of I_X generated by $Y = T \cup \{\beta^{\pm 1}, \gamma^{\pm 1}, \delta^{\pm 1}\}$.

Lemma 3.9.1. *S is finitely generated by $\alpha, \beta^{\pm 1}, \gamma^{\pm 1}$ and $\delta^{\pm 1}$.*

Proof. We show, that any element of T can be written in terms of $\alpha, \beta^{\pm 1}, \gamma^{\pm 1}, \delta^{\pm 1}$. By the definition of α, β, γ we have that

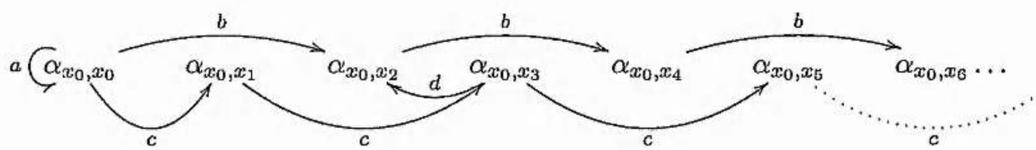
$$\alpha_{x_0, x_{2n}} = \alpha \cdot \beta^n \qquad \text{and} \qquad \alpha_{x_0, x_{2n+1}} = \alpha \cdot \gamma^{n+1}. \tag{3.12}$$

On the other hand $\alpha_{x,y} = \alpha_{x,x_0} \cdot \alpha_{x_0,y} = \alpha^{-1}_{x_0,x} \cdot \alpha_{x_0,y}$ holds. It follows now by (3.12) that S is generated by $\alpha, \beta^{\pm 1}, \gamma^{\pm 1}$ and $\delta^{\pm 1}$ as a semigroup. $\square$

For notational convenience, we introduce the alphabet

$$Z = \{a, b, b^{-1}, c, c^{-1}, d, d^{-1}\},$$

where a corresponds to α, $b^{\pm 1}$ corresponds to $\beta^{\pm 1}$, $c^{\pm 1}$ corresponds to $\gamma^{\pm 1}$ and $d^{\pm 1}$ corresponds to $\delta^{\pm 1}$. Consider now the Schützenberger graph R of α in the $\mathcal{D}$-class T of S:



(Note that we omitted the inverse arrows). Let $L = a \cup b^* \cup c^*$. Our aim is to show that $\mathrm{S}\Gamma_R$ possesses the fellow traveller property with respect to L and $\{\alpha\}$, and that R is not Schützenberger automatic.

Lemma 3.9.2. *The Schützenberger graph $\mathrm{S}\Gamma_R$ possesses the fellow traveller property with respect to L and $\{\alpha\}$.*

Proof. All distances considered in the proof will be distances in the Schützenberger graph $\Gamma = \mathrm{S}\Gamma_X(R)$. Therefore we write $d(s,t)$ instead of $d_\Gamma(s,t)$. Clearly L is a regular language, and $\alpha \cdot L\psi = R$. We verify that for all $u, v \in L$ with $d(\alpha \cdot u, \alpha \cdot v) \leq 1$, we have that $d(\alpha \cdot u(t), \alpha \cdot v(t)) \leq 4$. The following cases have to be dealt with:

(1) If $\alpha \cdot u = \alpha \cdot v$, then either $u \equiv v \equiv a$ or $u \equiv v \equiv b^n$ or $u \equiv v \equiv c^n$, $n \in \mathbb{N}$. Thus, $d(\alpha \cdot u(t), \alpha \cdot v(t)) \leq 4$ clearly follows.

(2) The case, when $\alpha \cdot u \cdot a = \alpha \cdot v$ is similar to the previous case.

(3) If $\alpha \cdot u \cdot b = \alpha \cdot v$, then either $u \equiv a$ and $v \equiv b$, or $u \equiv b^n$ and $v \equiv b^{n+1}$, $n \in \mathbb{N}$. In both cases, we clearly have that $d(\alpha \cdot u(t), \alpha \cdot v(t)) \leq 4$.

(4) If $\alpha \cdot u \cdot c = \alpha \cdot v$, then either $u \equiv a$ and $v \equiv c$, or $u \equiv c^n$ and $v \equiv c^{n+1}$, $n \in \mathbb{N}$. In both cases, we clearly have that $d(\alpha \cdot u(t), \alpha \cdot v(t)) \leq 4$.

(5) If $\alpha \cdot u \cdot d = \alpha \cdot v$, then $u \equiv c^{n+1}$ and $v \equiv b^n$, where n is not a prime. To verify that $d(\alpha \cdot u(t), \alpha \cdot v(t)) \leq 4$ holds for all $t \geq 1$, we consider the following cases:

(a) If t is even and $t \neq 2$, then

$$\alpha \cdot c^t \cdot (c \cdot d) = \alpha \cdot c^{t+1} \cdot d = \alpha_{x_0, x_{2t+1}} \cdot d = \alpha_{x_0, x_{2t}} = \alpha \cdot b^t,$$

and hence $d(\alpha \cdot u(t), \alpha \cdot v(t)) = 2 \leq 4$.

(b) If $t = 2$, then

$$\alpha \cdot c^2 \cdot d = \alpha \cdot b,$$

and hence $d(\alpha \cdot u(t), \alpha \cdot v(t)) = 1 \leq 4$.

(c) If t is odd and t is not a prime, then

$$\alpha \cdot c^t \cdot (c \cdot d) = \alpha \cdot c^{t+1} \cdot d = \alpha_{x_0, x_{2t+1}} \cdot d = \alpha_{x_0, x_{2t}} = \alpha \cdot b^t,$$

and hence $d(\alpha \cdot u(t), \alpha \cdot v(t)) = 2 \leq 4$.

(d) If t is an odd prime, then using the fact that $t + 1$ is even and making use of (a), we obtain that

$$\alpha \cdot c^t \cdot c^2 \cdot d = \alpha \cdot c^{t+1} \cdot (c \cdot d) = \alpha \cdot b^{t+1},$$

and it follows that

$$\alpha \cdot c^t \cdot (c^2 \cdot d \cdot b^{-1}) = \alpha \cdot b^t.$$

That is, $d(\alpha \cdot u(t), \alpha \cdot v(t)) = 4$. $\square$

Lemma 3.9.3. *The $\mathcal{R}$-class R of α is not Schützenberger automatic with respect to L and α.*

Proof. One can easily see, that $L_a = L_= = \{(u,u) \mid u \in L\}\delta_Z$, that

$$L_b = \{(u,v) \mid u,v \in L,\ \alpha \cdot u \cdot b = \alpha \cdot v\}\delta_Z = (a,b) \cup (b,b)^+(\$,b),$$

and that

$$L_c = \{(u,v) \mid u,v \in L,\ \alpha \cdot u \cdot c = \alpha \cdot v\}\delta_Z = (a,c) \cup (c,c)^+(\$,c).$$

Hence, these languages are regular. Clearly, we have that $(u,v) \in L_d$, if and only if $u = c^{n+1}$ and $v = b^n$, where n is not a prime. That is to say, that

$$L_d = \{(c^{n+1}, b^n) \mid n \in \mathbb{N},\ n \text{ is not prime}\}\delta_Z.$$

Using the pumping lemma one can easily verify that L_d is not regular. □

3.10 Finite presentability

Automatic groups are finitely presented. As far as automatic semigroups are concerned an example is given in [5, Example 4.4] for an automatic semigroup that is not finitely presented. On the other hand, as soon as we get closer to groups, for example consider completely simple semigroups, we see that the group theoretical result generalizes [4], that is being automatic does imply finite presentability.

Proposition 3.10.1. *Let S be a Schützenberger automatic regular semigroup having finitely many idempotents. Then S is finitely presented.*

Proof. By Proposition 3.8.6, the maximal subgroups of S are automatic, and hence they are finitely presented. According to [32, Theorem 4.1], a regular semigroup S with finitely many idempotent elements is finitely presented if and only if all maximal subgroups of S are finitely presented, and so we obtain the desired result. □

The following proposition provides a generalization of [4, Corollary 1.2].

Proposition 3.10.2. *Let S be an automatic regular semigroup with finitely many idempotents. Then S is finitely presented.*

Proof. It follows by 3.5.2 that S is Schützenberger automatic. Making use of Proposition 3.8.2, we have that the maximal subgroups of S are automatic, and hence they are finitely presented. Making use of [32, Theorem 4.1] once again, we obtain our desired result. □

We end this section by proving:

Proposition 3.10.3. *For a finitely generated completely simple semigroup S, the following are equivalent:*

(1) S is Schützenberger automatic;

(2) All maximal subgroups of S are automatic;

(3) S is automatic.

Proof. (1) $\Rightarrow$ (2) Finitely generated completely simple semigroups have finitely many $\mathcal{L}$- and $\mathcal{R}$-classes. Hence, by Proposition 3.8.6 if a completely simple semigroup S is Schützenberger automatic, then all of its maximal subgroups are automatic.

(2) $\Leftrightarrow$ (3) See [4, Theorem 1.1].

(3) $\Rightarrow$ (1) If S is automatic, then it is finitely generated, and hence it has finitely many $\mathcal{L}$- and $\mathcal{R}$-classes. Making use of Corollary 3.5.2, we thus have that S is Schützenberger automatic. □

3.11 Equality and word problems

Let S be a semigroup generated by a finite set X. The word problem is said to be solvable for S, if there exists an algorithm which decides whether or not given any two words $u, v \in X^+$ represent the same element in S or not. Automatic groups and semigroups have solvable word problem in quadratic time. In this section we will introduce the concept of the equality problem for S-acts, and show that the equality problem is solvable for the right S-act S if and only if the word problem is solvable for the semigroup S. Moreover we consider the following two problems:

Problem 1. Is it true that automatic S-acts have solvable equality problem?

Problem 2. Is it true that (strongly) Schützenberger automatic regular semigroups have solvable word problem?

Concerning Problem 1, we see that the semigroup theoretical approach to the word problem of automatic semigroups will carry over, that is, with slight modification of the proofs, we will obtain an affirmative answer to Problem 1. As far as Problem 2 is concerned we will give a partial answer. Throughout this section S will denote a semigroup, X a finite generating set for S. We assume that $X \subseteq S$ and denote by $\varphi : X^+ \to S$ the homomorphism extending the identity map $\iota : X \to S$.

Equality problem for S-acts. Let A be an S-act, generated by a finite set $A_0 \subseteq A$. If there exists an algorithm that decides whether or not for any $(a_i, a_j) \in A_0 \times A_0$ and for any two given words $u, v \in X^+$, $a_i.u = a_j.v$ holds, then we say that the equality problem is solvable for the S-act A.

Proposition 3.11.1. *Let S be a semigroup generated by a finite set X. Then the word problem is solvable for S if and only if the equality problem is solvable for the right S-act S.*

Proof. Recall that if S is generated by X, then the right S-act S is generated by X (see Proposition 3.1.10). Assume that the word problem is solvable for S. Let $x_i, x_j \in X$, $u, v \in X^+$ and $u' \equiv x_i u$, $v' \equiv x_j v$. By our assumptions, there exists an algorithm which decides whether or not $x_i \cdot u = u' = v' = x_j \cdot v$, proving that the equality problem is solvable for the right S-act S.

For the converse, assume that the equality problem is solvable for the right S-act S. Let $u, v \in X^+$, and assume that $u \equiv x_i u'$ and that $v \equiv x_j v'$. By our assumptions, there exists an algorithm that decides whether or not $u = x_i \cdot u' = x_j \cdot v' = v$ holds in S or not, proving that the word problem is indeed solvable for S. $\square$

In giving a solution to Problem 1, we follow the semigroup theoretical approach.

Proposition 3.11.2. *Let A be an automatic S-act. Let $(X, L_1, \ldots, L_n)$ be an automatic structure with respect to the generating set A_0 for A. Then, there exists a constant N such that for any $a_i \in A_0$ and $u \in L_i$ the following hold for the elements $a = a_i.u$ or $a = a_i.(u \cdot x)$ of A, where $x \in X$:*

(i) There exists $a_j \in A_0$ and $v \in L_j$ such that $|v| \leq |u| + N$ and $a = a_j.v$.

(ii) If there exists $a_j \in A_0$ and $v \in L_j$, such that $|v| > |u| + N$ and $a = a_j.v$. Then there exists infinitely many $w \in L_j$ such that $a = a_j.w$.

Proof. (i) For all $(a_j, a_k) \in A_0 \times A_0$ consider finite state automata $\mathcal{A}_{(a_j,a_k)_=}$, $\mathcal{A}_{(a_j,a_k)_x}$, $(x \in X)$ accepting the regular languages $L_{(a_j,a_k)_=}$, $L_{(a_j,a_k)_x}$ respectively. Let N be greater than the number of states in any of the automata defined.

Let $a_i \in A_0$ and $u \in L_i$. If $a = a_i.u$, then (i) is straightforward. Assume that $a = a_i.(u \cdot x)$, where $x \in X$. Since $(X, L_1, \ldots, L_n)$ is an automatic structure with respect to the generating set A_0, there exists $a_j \in A_0$ and $v \in L_j$ such that $a = a_j.v$. If $|v| \leq |u| + N$, then we are finished. Assume that $|v| > |u| + N$. Clearly $(u, v)\delta_X$ is accepted by $\mathcal{A} = \mathcal{A}_{(a_i,a_j)_x}$.

Reading through all of u, we visit a state say q in $\mathcal{A}$ at least twice. Removing the subword of v between successive visits to q, we get a shorter word v_1, moreover $(u, v_1)\delta_X$ is still accepted by $\mathcal{A}_{(a_i,a_j)_x}$. Repeating this procedure as many times as necessary, we obtain a word w, which satisfies that $|w| \leq |u| + N$ and $a = a_j.w$.

(*ii*) Assume that there exists $a_j \in A_0$ and $v \in L_j$, such that $|v| > |u|+N$ and $a = a_j.v$. In particular we have that $(u, v)\delta_X$ is accepted by one of the automata defined; say by $\mathcal{A}$. After reading through all of u, we visit a state say q in $\mathcal{A}$ more then once. Inserting the subword in between two successive visits to q in v in the appropriate place, we will get a longer word v_1 so that $(u, v_1)\delta_X$ is still accepted by $\mathcal{A}$. Repeating this process as many times as we want, we get the desired result. □

In the following proposition, we let S^1 denote the semigroup obtained by adjoining an identity element to S.

Proposition 3.11.3. *Let A be an automatic S-act. Then A is also an automatic S^1-act.*

Proof. Let $(X, L_1, \ldots, L_n)$ be an automatic structure with respect to the generating set A_0 for A. Let $Y = X \cup \{1\}$ and extend the action of S on A to an action of S^1 on A by defining $a.1 = a$ for all $a \in A$. Define $K_i = L_i \cup \{1\}$. Clearly each K_i is a regular language over Y, and $\bigcup_{j=1}^{n} a_j.K_j\varphi = A$ holds. To show that $(Y, K_1, \ldots, K_n)$ is indeed an automatic structure with respect to the generating set A_0 for A, we claim that the languages $K_{(a_i,a_j)_=}$, $K_{(a_i,a_j)_y}$, $y \in Y$ are regular. If $a_i \neq a_j$, then

$$\begin{aligned}K_{(a_i,a_j)_=} &= \{(u,v) \in K_i \times K_j \mid a_i.u = a_j.v\}\delta_X \\ &= L_{(a_i,a_j)_=} \cup \{(u,1) \mid u \in L_i,\ a_i.u = a_j\}\delta_X \\ &\quad \cup \{(1,v) \mid v \in L_j,\ a_i = a_j.v\}\delta_X,\end{aligned}$$

and hence is regular by Lemma 3.1.9 and Proposition 2.3.1. If $a_i = a_j$, then

$$\begin{aligned}K_{(a_i,a_i)_=} &= \{(u,v) \in K_i \times K_i \mid a_i.u = a_i.v\}\delta_X \\ &= L_{(a_i,a_i)_=} \cup \{(u,1) \mid u \in L_i,\ a_i.u = a_i\}\delta_X \\ &\quad \cup \{(1,v) \mid v \in L_i,\ a_i = a_i.v\}\delta_X \cup \{(1,1)\},\end{aligned}$$

hence is regular. It is straightforward that for all $(a_i, a_j) \in A_0 \times A_0$ we have $K_{(a_i,a_j)_1} = K_{(a_i,a_j)_=}$. Let $y \in X$. Then

$$\begin{aligned}K_{(a_i,a_j)_y} &= \{(u,v) \in K_i \times K_j \mid a_i.(u \cdot x) = a_j.v\}\delta_X \\ &= L_{(a_i,a_j)_y} \cup \{(u,1) \mid u \in L_i,\ a_i.(u \cdot x) = a_j\}\delta_X \\ &\quad \cup \{(1,v) \mid v \in L_j,\ a_i \cdot x = a_j.v\}\delta_X,\end{aligned}$$

and hence is regular by Lemmas 3.1.8, 3.1.9 and Propositions 2.2.3 and 2.3.1. □

Proposition 3.11.4. *Let $(X, L_1, \ldots, L_n)$ be an automatic structure with respect to the generating set A_0 for the S-act A. Let $u \in X^+$, $a_i \in A_0$, and consider $a = a_i.u$. Then we can find a generator $a_j \in A_0$ and $v \in L_j$ such that $a = a_j.v$ in time proportional to $|u|^2$.*

Proof. By Proposition 3.11.3 we can consider A as an automatic S^1-act. Let $Y = X \cup \{1\}$ and consider the automatic structure $(Y, K_1, \ldots, K_n)$ with respect to the generating set A_0 for A as defined in the proof of Proposition 3.11.3. For each $(a_i, a_j) \in A_0 \times A_0$ and $x \in X$, let $\mathcal{A}_{(a_i,a_j)_x}$ denote a finite state automaton recognizing the regular language $K_{(a_i,a_j)_x}$. Let $u \equiv x_1 \ldots x_m$ where $x_i \in X$, $(1 \le i \le m)$, and let $a = a_i.u$. We will run the following procedure:

Step 1. Consider the finite state automata $\mathcal{A}_{(a_i,a_1)_{x_1}}, \ldots, \mathcal{A}_{(a_i,a_n)_{x_1}}$ in this order. We follow a path in the automaton $\mathcal{A}_{(a_i,a_j)_{x_1}}$, in which the first component of the labels of the edges are $1\$\$\ldots$. We begin this procedure with the first automaton, that is when $j = 1$.

(a) If a final state can be reached in $\mathcal{A}_{(a_i,a_j)_{x_1}}$, then the second component of the labels of the edges give a word $\alpha_1 \in L_j$, so that $a_i.1x_1 = a_j.\alpha_1$. Continue with step 2.

(b) If a final state cannot be reached, then we consider $\mathcal{A}_{(a_i,a_{j+1})_{x_1}}$ and repeat (a) with this automaton.

We note that since $\bigcup_{j=1}^n a_j.K_j\varphi = A$, we will find a generator a_{k_1} and $\alpha_1 \in L_{k_1}$, so that $a_i.1x_1 = a_{k_1}.\alpha_1$.

Step 2. Consider the finite state automata $\mathcal{A}_{(a_{k_1},a_1)_{x_2}}, \ldots, \mathcal{A}_{(a_{k_1},a_n)_{x_2}}$ in this order. We follow a path in $\mathcal{A}_{(a_{k_1},a_j)_{x_1}}$, in which the first component of the labels of the edges are $\alpha_1\$\$\ldots$. We begin this procedure with the first automaton, that is when $j = 1$.

(a) If a final state can be reached in $\mathcal{A}_{(a_{k_1},a_j)_{x_2}}$, then the second component of the labels of the edges give a word $\alpha_2 \in L_j$, so that $a_{k_1}.\alpha_1 x_2 = a_j.\alpha_2$. Continue the procedure with step 3.

(b) If a final state cannot be reached, then we consider $\mathcal{A}_{(a_{k_1},a_{j+1})_{x_2}}$ and repeat (a) with this automaton.

Since $\bigcup_{j=1}^n a_j.K_j\varphi = A$, there exists a generator a_{k_2} and a word $\alpha_2 \in L_{k_2}$, so that $a_i.1x_1x_2 = a_{k_1}.\alpha_1 x_2 = a_{k_2}.\alpha_2$.

$\vdots$

Step m. Consider the finite state automata $\mathcal{A}_{(a_{k_{m-1}},a_1)_{x_m}}, \dots, \mathcal{A}_{(a_{k_{m-1}},a_n)_{x_m}}$ in this order. We follow a path in $\mathcal{A}_{(a_{k_{m-1}},a_j)_{x_m}}$, in which the first component of the labels of the edges are $\alpha_{m-1}\$\$\dots$. We begin this procedure with the first automaton, that is when $j = 1$.

(a) If a final state can be reached in $\mathcal{A}_{(a_{k_{m-1}},a_j)_{x_m}}$, then the second component of the labels of the edges give a word $\alpha_m \in L_j$, so that $a_{k_{m-1}}.\alpha_{m-1}x_m = a_j.\alpha_m$.

(b) If a final state cannot be reached, then we consider $\mathcal{A}_{(a_{k_{m-1}},a_{j+1})_{x_2}}$ and repeat (a) with this automaton.

As before, we note that since $\bigcup_{j=1}^{n} a_j.K_j\varphi = A$, we will eventually find a generator $a_k \in A_0$ and $\alpha \in L_k$, so that

$$a_i.1x_1x_2\dots x_m = a_{k_1}.\alpha_1x_2\dots x_m = \dots a_{k_{m-1}}.\alpha_{m-1}x_m = a_k.\alpha.$$

In the worst case, the time being taken to find α_j is $n \cdot |\alpha_j|$, since then we need to input $\alpha_{i-1}\$\$\dots$ in n automata. By Proposition 3.11.2 there exists a constant N such that $|\alpha_i| \le |\alpha_{i-1}| + N$. So the time being taken to find α is $O(\sum_{j=1}^{|u|} n(|1| + jN)) = O(|u|^2)$. $\square$

Proposition 3.11.5. *Let A be an automatic S-act, then the equality problem is solvable for A in quadratic time.*

Proof. Let $(X, L_1, \dots, L_n)$ be an automatic structure with respect to the generating set A_0 for the S-act A. Let $u, v \in X^+$ and let $a_i, a_j \in A_0$. Let $a = a_i.u$ and $b = a_j.v$. Then, according to Proposition 3.11.4 we can find generators $a_k, a_l \in A_0$ and words $\tilde{u} \in L_k, \tilde{v} \in L_j$ in quadratic time, so that $a_i.u = a_k.\tilde{u}$ and $a_j.v = a_l.\tilde{v}$. With the help of the automaton $\mathcal{A}_{(a_k,a_l)_=}$ accepting the regular language $L_{(a_k,a_l)_=}$ we will obtain an answer whether $a_i.u = a_j.v$ in A. $\square$

An immediate consequence of Proposition 3.11.5 is:

Corollary 3.11.6. *Let $S = \langle X \rangle$ be an $\mathcal{R}$-class automatic monoid, and $u, v \in X^*$. Then we can decide in quadratic time whether $u\varphi \ \mathcal{R} \ v\varphi$ in S.*

Also we have:

Corollary 3.11.7. *Let $S = \langle X \rangle$ be a regular semigroup and R be a Schützenberger automatic $\mathcal{R}$-class of S. Let $u, v \in X^+$. Then we can decide whether $u\varphi, v\varphi \in R$. Moreover if $u\varphi, v\varphi \in R$, then we can decide whether $u\varphi = v\varphi$.*

Proof. Using the notation of Proposition 3.2.1, let (X, N) be a Schützenberger automatic structure for R, let $L = N \cup \{\mathbf{0}\}$ and $e \in E(R)$. According to Proposition 3.2.1, (X^0, L) is an automatic structure with respect to the generating set $\{e\}$ for the right S^0-act R^0. Let $u \in X^+$. In the right S^0-act R^0 we have that $e.u = u$ if $u \in R$, and $e.u = \mathbf{0}$ if $u \notin R$. By Proposition 3.11.5, we can decide in quadratic time whether or not $e.u = e \cdot \mathbf{0} = \mathbf{0}$.

Now let $u, v \in X^+$. Decide first whether or not $e.u = e.\mathbf{0}$ and $e.v = e.\mathbf{0}$. If $e.u \neq e.\mathbf{0}$ and $e \cdot v \neq e.\mathbf{0}$, then $u = e \cdot u$ and $v = e \cdot v$ are elements of R. By Proposition 3.11.5 again we can decide whether or not $u = e \cdot u = e \cdot v = v$. □

Corollary 3.11.8. *Let S be a strongly Schützenberger automatic regular semigroup with finitely many $\mathcal{R}$-classes. Then the word problem is solvable for S in quadratic time.*

Proof. Let $S/\mathcal{R} = \{R_1, \ldots, R_n\}$ and let (X, N_i) be a Schützenberger automatic structure for R_i $(1 \leq i \leq n)$. Let $u, v \in X^+$. Since S is also $\mathcal{R}$-class automatic we can decide by Corollary 3.11.6 in quadratic time whether or not the elements represented by u and v are $\mathcal{R}$-related. Since S has finitely many $\mathcal{R}$-classes, we can find their $\mathcal{R}$-class in finitely many steps and then decide by Corollary 3.11.7 in quadratic time whether or not they represent the same element in S.

To be more precise we decide with the following procedure whether or not $u = v$ holds in S. As in Corollary 3.11.7, let (X^0, L_i), where $L_i = N_i \cup \{\mathbf{0}\}$ be an automatic structure with respect to a one element generating set $\{e_i\}$, where $e_i \in E(R_i)$ for the right S^0-act R_i^0. Let $u, v \in X^+$. Since we have finitely many $\mathcal{R}$-classes and hence S^0-acts we can find in finitely many steps the idempotents e_i, e_j so that $e_i.u \neq e_i.\mathbf{0}$, $e_j.v \neq e_j.\mathbf{0}$. In other words, we can find in finitely many steps the Schützenberger automatic structure assigned to the $\mathcal{R}$-class of the element represented by u and the $\mathcal{R}$-class of the element represented by v. If the $\mathcal{R}$-classes of the elements represented by u and v happens to be the same, then as seen in Corollary 3.11.7 we can also decide whether or not u and v represent the same element in quadratic time. □

Whether the word problem is solvable for strongly Schützenberger automatic regular semigroups with infinitely many $\mathcal{R}$-classes seems to be a more difficult question. We can obviously decide whether or not the elements represented by the two given words belong to the same $\mathcal{R}$-class. If they happen to be in the same $\mathcal{R}$-class, then a technique needs to be developed with the help of which one can find in finitely many steps the Schützenberger automatic structure of the $\mathcal{R}$-class of these elements, so that it can be decided whether or not the two given words represent the same element. We end this section with the following

Open question. Is it true that strongly Schützenberger automatic regular semigroups have solvable word problem?

3.12 Inverse free product of inverse semigroups.

It is verified in [3] that the group free product of automatic groups is automatic, and in [5] it is shown that the free product of automatic semigroups is automatic. In this section we seek answer to the following problem:

Problem: Is the inverse free product of Schützenberger automatic inverse semigroups Schützenberger automatic?

The results in [25] concerning the description of Schützenberger graphs of the inverse free product of two inverse semigroups will serve as a basis of the answer of the above problem. First we summarize some necessary notions and results needed for this section.

Throughout the section S_1 and S_2 denote two disjoint inverse semigroups. The inverse free product $S = S_1 *_{\text{Inv}} S_2$ is an inverse semigroup satisfying the following properties:

(i) There exists embeddings $\alpha_1 : S_1 \to S,\ \alpha_2 : S_2 \to S$;

(ii) If there exists an inverse semigroups T and homomorphisms $\beta_1 : S_1 \to T,\ \beta_2 : S_2 \to T$, then there exists a unique homomorphism $\delta : S \to T$ such that the following diagram is commutative:

$$\begin{tikzcd}
& S \arrow[dd, "\delta"] & \\
S_1 \arrow[ur, "\alpha_1"] \arrow[dr, "\beta_1"'] & & S_2 \arrow[ul, "\alpha_2"'] \arrow[dl, "\beta_2"] \\
& T &
\end{tikzcd}$$

Assume that $S_1 = \text{Inv}\langle X|P\rangle$ and $S_2 = \text{Inv}\langle Y|Q\rangle$ where $X \cap Y = \emptyset$. Then $S_1 *_{\text{Inv}} S_2 = \text{Inv}\langle X \cup Y|P \cup Q\rangle$.

The first general result concerning the structure of the inverse free product of two inverse semigroups was obtained by Jones in [24]. This structure theorem provides a set of canonical forms for the elements of the inverse free product, with the help of which Green's relations are described. A graph theoretical approach to describe the structure of the inverse free product of two inverse semigroups is introduced in [25]. This approach employs a technique introduced by Stephen [35] for obtaining Schützenberger graphs relative to an inverse semigroup presentation. As our answer to our proposed problem relies on the latter approach, we summarize the necessary notions and results of [25] regarding the Schützenberger graphs of $S_1 *_{\mathrm{Inv}} S_2$.

Let X be a finite set and let $X^{-1} = \{x^{-1} \mid x \in X\}$ denote the set of formal inverses of elements of X. Let Γ be a directed labelled graph, where the labels of the edges are elements of $X \cup X^{-1}$. We say that Γ is a *word graph* over $X \cup X^{-1}$ if it satisfies the following two conditions:

(i) The graph Γ is strongly connected, that is to say that for any two distinct vertices v_1, v_2 of Γ, there exists a directed path from v_1 to v_2.

(ii) If $v_1 \xrightarrow{x} v_2$ is an edge of Γ, then $v_2 \xrightarrow{x^{-1}} v_1$ is an edge of Γ as well.

Let X and Y be disjoint finite sets, and let Γ be a word graph over $Z = X \cup X^{-1} \cup Y \cup Y^{-1}$. The edges of Γ can be coloured with two colours in such a way that the edges of Γ labelled with elements of $X \cup X^{-1}$ get one of the colours, and the edges of Γ labelled with elements of $Y \cup Y^{-1}$ get the other one. A subgraph of Γ is said to be *monochromatic*, if all of its edges have the same colour. We say that a path in Γ is monochromatic, if all of its edges have the same colour. A *lobe* is a maximal monochromatic subgraph of Γ. A vertex of Γ is called an *intersection vertex* if it belongs to more then one lobe. A *switchpoint* of a path p in Γ is a vertex that is common to successive edges of different colour. The *switchpoint sequence* of a path p in Γ is the sequence of switchpoints p traverses, in order.

A path p in Γ is called *simple* if there are no repeated vertices in p other then perhaps its first and last vertex, in which case p is called a *simple cycle*. A word graph Γ over Z is called *cactoid*, if the following two conditions hold:

(i) Γ has finitely many lobes;

(ii) every simple cycle of Γ is monochromatic.

The following important properties of cactoid graphs over Z is proved in Lemma 2.1. and Corollary 2.2 of [25].

Proposition 3.12.1. *Let Γ be a cactoid graph over Z, and let v_1, v_2 be two distinct vertices of Γ. Then the following hold:*

(i) *Any two simple paths from v_1 to v_2 traverse the same switchpoint sequence.*

(ii) *Distinct lobes of Γ possess at most one common vertex.*

Let Γ be a cactoid graph. Fix a vertex v in Γ, and let v_1 be a vertex distinct from v. The *switchpoint sequence* of v_1 is the switchpoint sequence of a (in fact any) simple path from v to v_1. Denoting by l the length of the switchpoint sequence of an arbitrary vertex v_1 of Γ, a norm with respect to v is then defined for v_1 in the following way:

$$\|v_1\| = \begin{cases} 0 & \text{if } v_1 = v, \\ 1+l & \text{otherwise} \end{cases}$$

The following results are proved in [25, Lemma 2.3.] and [25, Corollary]:

Proposition 3.12.2. *Let Γ be a cactoid graph over Z and let v be a vertex of Γ. Then the following hold:*

(i) *Each lobe Λ of Γ possess a unique vertex called the root λ_Λ of least norm with respect to v.*

(ii) *For each lobe Λ of Γ, λ_Λ is either an intersection vertex of Γ or is the vertex v.*

(iii) *For each vertex $u \neq \lambda_\Lambda$ of the lobe Λ, $\|u\| = \|\lambda_\Lambda\| + 1$.*

(iv) *The chosen vertex v is the root of two distinct lobes if and only if it is an intersection vertex in Γ and it is the root of a unique lobe otherwise. Every other intersection vertex of Γ is the root of a unique lobe of Γ.*

One of the main results of [25] is Theorem 4.1.:

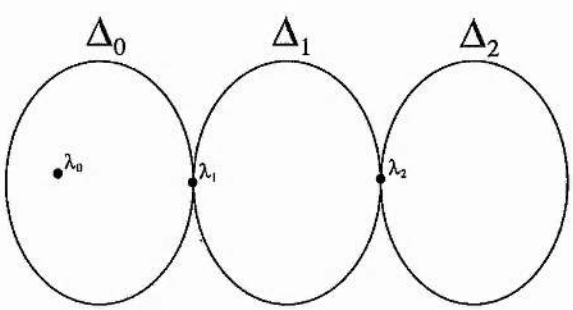
Theorem 3.12.3. *Let S_1 be an inverse semigroup generated by a set X and S_2 be an inverse semigroup generated by Y, where $X \cap Y = \emptyset$. Then the Schützenberger graphs of $S_1 *_{\mathrm{Inv}} S_2$ are precisely the cactoid graphs over $X \cup X^{-1} \cup Y \cup Y^{-1}$ each of whose lobes is isomorphic to a Schützenberger graph of S_1 or of S_2.*

We are ready to give an answer to our proposed problem.

Proposition 3.12.4. *Let S_1 and S_2 be Schützenberger automatic inverse semigroups. Then the inverse free product $S = S_1 *_{\mathrm{Inv}} S_2$ is also a Schützenberger automatic inverse semigroup.*

Step 1: notation. In the first step, we set up the notation. Assume that S_1 is generated by the finite set X, where X is closed under taking inverses. For the sake of simplicity we assume that $X \subseteq S_1$. Let $\varphi_1 : X^+ \to S_1$ denote the semigroup homomorphism extending the identity map $\iota_1 : X \to S_1$. Assume that S_2 is generated by the finite set $Y \subseteq S_2$, where Y is closed under taking inverses and $X \cap Y = \emptyset$. Let $\varphi_2 : Y^+ \to S_2$ denote the homomorphism extending the identity map $\iota_2 : Y \to S_2$. Let $Z = X \cup Y$ and let $\varphi : Z^+ \to S$ denote the homomorphism extending the identity map $\iota : Z \to S$. Let $s \in S$ and consider the Schützenberger graph $\Gamma = \mathrm{S}\Gamma_s(Z)$ of R_s. By Theorem 3.12.3, Γ is a cactoid graph over Z, and so it has finitely many lobes each of which is isomorphic to a Schützenberger graph of S_1 or of S_2. Let Λ_X denote the set of lobes, whose edges are labelled by elements of X and let Λ_Y denote the set of lobes, whose edges are labelled by the elements of Y. Let $\lambda_1, \ldots, \lambda_{n-1}$ denote the intersection vertices of Γ and let λ_0 be the vertex corresponding to ss^{-1}. For the sake of convenience, we assume that λ_0 is not an intersection vertex in Γ. (Note that since we have $n-1$ intersection vertices, Γ has n lobes.) For each $1 \leq i \leq n-1$ we fix a simple path p_i in Γ from λ_0 to λ_i. Denote by w_i the word over Z that labels the path p_i. Let $W = \{w_1, \ldots, w_{n-1}\}$. Clearly W is a regular language over Z.

According to Proposition 3.12.2, each intersection vertex λ_i is the unique root of one of the lobes of Γ. We let Δ_i denote the lobe whose root is λ_i.



For all $\Delta_i \subseteq \Lambda_X$ we choose an $\mathcal{R}$-class R_i of S_1, such that Δ_i is isomorphic to the Schützenberger graph of R_i. The set of $\mathcal{R}$-classes chosen in S_1 will be denoted by $\mathcal{R}_1$. Similarly, for all $\Delta_i \subseteq \Lambda_Y$ we choose an $\mathcal{R}$-class R_i of S_2, such that Δ_i is isomorphic to the Schützenberger graph of R_i. The set of $\mathcal{R}$-classes chosen in S_2 will be denoted by $\mathcal{R}_2$. To simplify notation, we will actually think of Δ_i as the Schützenberger graph of the chosen $\mathcal{R}$-class R_i. In particular, we will think of the vertices of Δ_i as elements of R_i. Define

$$W_X = \{w_i \in W \mid \Delta_i \in \Lambda_X\},$$

and let $W_Y = W \setminus W_X$. In other words, $w_i \in W_X$ if and only if $\lambda_0 \cdot w_i = \lambda_i$ is a root of a lobe in Λ_X, and $w_j \in W_Y$ if and only if $\lambda_0 \cdot w_j = \lambda_j$ is a root of a lobe in Λ_Y.

By assumptions, S_1 and S_2 are Schützenberger automatic inverse semigroups. In particular, for each $\mathcal{R}$-class $R_i \subseteq \mathcal{R}_1$, we may assume by Propositions 3.6.12, 3.2.1 and Proposition 3.4.4 that there exists a regular language K_i over X such that the following hold:

(1a) For each $t \in V(\Delta_i)$ there exists exactly one $u \in K_i$ with $\lambda_i \cdot u\varphi_1 = t$ and so $(K_i)_= = \{(u,w) \in K_i \times K_i \mid \lambda_i \cdot u = \lambda_i \cdot w\}\delta_X = \{(u,u) \mid u \in K_i\}\delta_X$ is a regular language.

(1b) $(K_i)_x = \{(u,w) \in K_i \times K_i \mid \lambda_i \cdot (u \cdot x) = \lambda_i \cdot w\}\delta_X$ is a regular language for each $x \in X$.

We let $\mathcal{K}_X$ denote the set of regular languages obtained. Similarly, for each $\mathcal{R}$-class $R_j \subseteq \mathcal{R}_2$ we may assume that there exists a regular language K_j over Y such that the following hold:

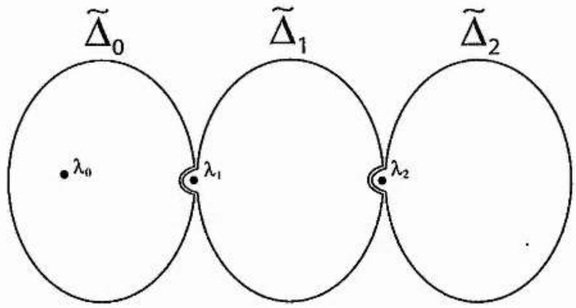
(2a) For each $t \in R_j$ there exists exactly one $u \in K_j$ with $\lambda_j \cdot u\varphi_2 = t$ and so $(K_j)_= = \{(u,w) \in K_j \times K_j \mid \lambda_j \cdot u = \lambda_j \cdot w\}\delta_Y = \{(u,u) \mid u \in K_j\}\delta_Y$ is a regular language.

(2b) $(K_j)_y = \{(u,w) \in K_j \times K_j \mid \lambda_j \cdot (u \cdot y) = \lambda_j \cdot w\}\delta_Y$ is a regular language for each $y \in Y$.

We let $\mathcal{K}_Y$ denote the set of regular languages obtained. We let $\mathcal{K} = \mathcal{K}_X \cup \mathcal{K}_Y = \{K_0, K_1, \ldots, K_{n-1}\}$.

Step 2: the regular language. In the second step, we are going to construct a regular language K with the help of the regular languages in $\mathcal{K}$, so that λ_0 and K satisfy that $\lambda_0 \cdot K\varphi = R_s$.

Let Δ_i be an arbitrary lobe of Γ with vertex set $V(\Delta_i)$ and root λ_i. Let V_i denote the intersection vertices of Γ belonging to Δ_i. Clearly V_i is a finite set. Let $\widetilde{\Delta}_i$ denote the subgraph of Δ_i, whose vertex set is $V(\widetilde{\Delta}_i) = (V(\Delta_i) \setminus V_i) \cup \{\lambda_i\}$. Illustrating this:



Clearly

$$\Gamma = \widetilde{\Delta}_0 \cup \ldots \cup \widetilde{\Delta}_{n-1} \quad \text{and} \quad \widetilde{\Delta}_i \cap \widetilde{\Delta}_j = \emptyset, \quad \text{for all } 0 \le i, j \le n-1, i \ne j. \tag{3.13}$$

For each $v \in V_i$, consider the language

$$K_{i,v} = \{u \in K_i \mid \lambda_i \cdot u = v\}. \tag{3.14}$$

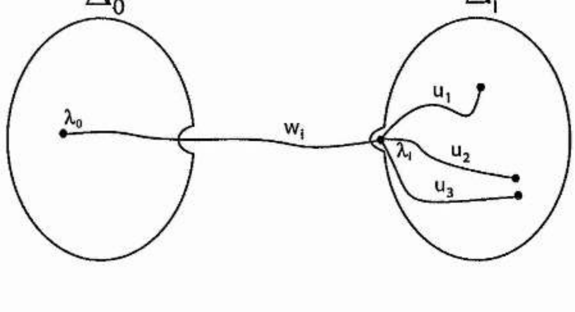
By Lemma 3.1.9, we have that $K_{i,v}$ is a regular language over X or over Y, depending on whether Δ_i is an element of Λ_X or of Λ_Y. In fact, since K_i satisfies $(1a)$ or $(2a)$, for each $v \in V_i$, $K_{i,v}$ has exactly one element. Let

$$K_{i,V_i} = \bigcup_{v \in V_i} K_{i,v} \tag{3.15}$$

The language K_{i,V_i} is regular, since it is a finite union of regular languages. Define

$$\widetilde{K}_i = K_i - K_{i,V_i} \quad \text{and} \quad L_i = \widetilde{K}_i \cup \{\epsilon\},$$

where ϵ denotes the empty word. Clearly $\widetilde{K}_i$ and L_i are regular languages. Taking into consideration that for all $1 \le i \le n-1$, w_i is path between λ_0 and λ_i, the importance of the languages defined can be illustrated in the following way in Γ:



$(u_1, u_2, u_3 \in \widetilde{K}_i)$

That is to say that

$$\lambda_0 \cdot (w_i L_i)\varphi = V(\widetilde{\Delta}_i). \tag{3.16}$$

Moreover since K_i is a language satisfying $(1a)$ or $(2a)$, we also have that for all $v \in V(\widetilde{\Delta}_i)$ there exists exactly one $w \in L_i$ so that $\lambda_0 \cdot (w_i w)\varphi = v$ holds. To have a complete picture, we note that

$$\lambda_0 \cdot (w_i K_{i,V_i})\varphi = V_i. \tag{3.17}$$

Let

$$K = L_0 \cup w_1 L_1 \cup \ldots \cup w_{n-1} L_{n-1},$$

where w_i denotes the labels of the path we fixed from λ_0 to λ_i and L_i is the regular language over X or over Y so that $\lambda_0 \cdot (w_i L_i)\varphi = V(\widetilde{\Delta}_i)$. It follows by (3.16) and (3.13) that $\lambda_0 \cdot K\varphi = R_s$.

Third step: the Schützenberger automatic structure. We verify that λ_0 and K satisfy the last two conditions of $(S2)$ of Proposition 3.2.1.

Clearly, by the definition of K and by the note after (3.16) we have that for all $t \in R_s$, there exists exactly one $w \in K$ so that $\lambda_0 \cdot w\varphi = t$ proving that

$$K_= = \{(u, w) \mid \lambda_0 \cdot u = \lambda_0 \cdot w\}\delta_Z = \{(u, u) \mid u \in K\}\delta_Z.$$

Hence $K_=$ is a regular language by Proposition 2.3.2.

Next we claim that for all $x \in X$, the language

$$K_x = \{(u, w) \in K \times K \mid \lambda_0 \cdot (u \cdot x) = \lambda_0 \cdot w\}\delta_Z$$

is regular. It is straightforward that if $(u, w) \in K_x$, then $\lambda_0 \cdot (u \cdot x)$ and $\lambda_0 \cdot w$ belong to the same lobe say to Δ_i, where $\Delta_i \in \Lambda_X$. Moreover, since the "last arrow" is an element of X and since by definition a lobe is a maximal monochromatic subgraph, $\lambda_0 \cdot u$ and $\lambda_0 \cdot w$ have to be vertices of Δ_i. Depending on what kind of vertices (intersection vertex or not) the elements $\lambda_0 \cdot u$ and $\lambda_0 \cdot w$ represent in the Schützenberger graph R_s, the following four cases can occur:

(1) Our first case is when $(u, w)\delta_Z$ is a word for which $\lambda_0 \cdot u$ and $\lambda_0 \cdot w$ are not intersection vertices of Γ. First we introduce the following notation. Let

$$(L_i)_x = (K_i)_x \setminus \{(u, w) \mid u, w \in K_{i,V_i},\ \lambda_i \cdot u \cdot x = \lambda_i \cdot w\}\delta_Z.$$

Since $(K_i)_x$ is a regular language and since the latter set is finite, $(L_i)_x$ is a regular language.

We verify that words $(u,w)\delta_Z \in K_x$ that satisfy that $\lambda_0 \cdot u$, $\lambda_0 \cdot w$ are not intersection vertices in a lobe $\Delta_i \subseteq \Lambda_X$ are exactly the words contained in the regular language

$$N_1 = \bigcup_{w_i \in W_X} (w_i, w_i)\delta_Z (L_i)_x.$$

If $(u,w) \in N_1$, then $u \equiv w_i\tilde{u}$ and $w = w_i\widetilde{w}$, where $w_i \in W_X$ and $(\tilde{u},\widetilde{w})\delta_Z \in (L_i)_x$. It follows that $\tilde{u},\widetilde{w} \notin K_{i,V_i}$, in other words $\lambda_i\tilde{u}$ and $\lambda_i\widetilde{w}$ are not intersection vertices of Γ. Moreover we have that $\lambda_i \cdot (\tilde{u} \cdot x) = \lambda_i \cdot \widetilde{w}$ holds. It follows, that $\lambda_0 \cdot (u \cdot x) = \lambda_0 \cdot (w_i \cdot \tilde{u} \cdot x) = \lambda_0 \cdot w_i \cdot \widetilde{w} = \lambda_0 \cdot w$, proving that $(u,w)\delta_Z \in K_x$.

For the converse, assume that $(u,w)\delta_Z \in K_x$ and that $\lambda_0 \cdot u$, $\lambda_0 \cdot w$ are not intersection vertices in a lobe $\Delta_i \subseteq \Lambda_X$. Then, as noted before $\lambda_0 \cdot u$ and $\lambda_0 \cdot w$ are vertices of the same lobe $\Delta_i \in \Lambda_X$. It follows that $u \equiv w_i\tilde{u}$, and $w \equiv w_i\widetilde{w}$, where $\tilde{u},\widetilde{w} \in \widetilde{K}_i$. In particular, we have that

$$\lambda_i \cdot (\tilde{u} \cdot x) = (\lambda_0 \cdot w_i) \cdot \tilde{u} \cdot x = \lambda_0 \cdot w_i\widetilde{w} = \lambda_i \cdot w.$$

In other words, $(\tilde{u},\widetilde{w}) \in (L_i)_x$.

Since $(L_i)_x$ is regular we have that $(w_i,w_i)\delta_Z(L_i)_x$ is regular and N_1 is a finite union of regular languages, proving that N_1 is regular.

(2) Our second case is when $(u,w)\delta_Z \in K_x$ is a word such that $\lambda_0 \cdot u$ is not an intersection vertex of Γ and $\lambda_0 \cdot w$ is an intersection vertices of Γ. Again, we first introduce the notation we are going to use. For each $\Delta_i \in \Lambda_X$ and for all intersection vertices $\lambda_k \in V_i$, we let $w_{i,k} \in K_{i,V_i} \subseteq K_i$ so that $\lambda_i \cdot w_{i,k} = \lambda_k$. Such an element exists, since Γ_i and Δ_i are isomorphic and since $\lambda_i \cdot K_i\varphi_1 = R_i$. We let

$$K_{(i,x,k)} = \{\overline{u} \in K_i |\ \lambda_i \cdot \overline{u} \cdot x = \lambda_k\} \setminus K_{i,V_i}.$$

By Lemma 3.1.8 and Proposition 2.2.3, $K_{(i,x,k)}$ is a regular language. Let

$$M_1 = \{(w_i,w_k) \in W_X \times W_Y \mid \lambda_0 \cdot w_k \in V_i\}.$$

We will verify that words $(u,w)\delta_Z \in K_x$ that satisfy that $\lambda_0 \cdot u$ is not an intersection vertex of $\Delta_i \subseteq \Lambda_X$ and $\lambda_0 \cdot w$ is an intersection vertex of Δ_i are exactly the words contained in the language

$$N_2 = \bigcup_{(w_i,w_k) \in M_1} (w_i,w_k)\delta_Z \odot (K_{(i,x,k)} \times \{\$\}^*)\delta_Z,$$

where $\odot$ denotes the padded product.

If $(u, w) \in N_2$, then $w \equiv w_k$ and $u \equiv w_i\tilde{u}$, where $\tilde{u} \in K_{(i,x,k)}$, $\lambda_0 \cdot w_i, \lambda_0 \cdot w_k \in V_i$. In particular we have that $\lambda_i \cdot \tilde{u}$ is not an intersection vertex and that $\lambda_i \cdot \tilde{u} \cdot x = \lambda_k = \lambda_0 \cdot w_k$. Hence $(\lambda_0 \cdot w_i) \cdot \tilde{u} \cdot x = \lambda_0 \cdot w_k$ holds, which verifies that $(u, w)\delta_Z \in K_x$.

For the converse assume that $(u, w)\delta_Z \in K_x$ such that $\lambda_0 \cdot u$ is not an intersection vertex of $\Delta_i \in \Lambda_X$ and $\lambda_0 \cdot w$ is an intersection vertex. Since $\lambda_0 \cdot w$ is an intersection vertex, we have that $w \equiv w_k$, where $\lambda_k \in V_i$. Moreover, since $\lambda_0 \cdot u$ is not an intersection vertex in Δ_i, we have that $u \equiv w_i \cdot \tilde{u}$, where $\tilde{u} \in \widetilde{K}_i$. Also, since $(u, w)\delta_Z \in K_x$, we have that

$$\begin{aligned}\lambda_k = \lambda_0 \cdot w_k &= \lambda_0 \cdot w \\ &= \lambda_0 \cdot u \cdot x \\ &= (\lambda_0 \cdot w_i) \cdot \tilde{u} \cdot x \\ &= \lambda_i \cdot \tilde{u} \cdot x,\end{aligned}$$

verifying that $\tilde{u} \in K_{(i,x,k)}$. Hence we may deduce that

$$(u, w)\delta_Z \in (w_i, w_k)\delta_Z \odot (K_{(i,x,k)} \times \{\$\}^*)\delta_Z \subseteq N_2.$$

Finally we verify that N_2 is a regular language over Z. By Proposition 2.2.3, the language $(K_{(i,x,k)} \times \epsilon\{\$\}^*)\delta_Z$ is regular. Since $M = (w_i, w_k)\delta_Z$ is finite, it satisfies the condition of Proposition 2.3.16 and hence we have that for all $(w_i, w_k) \in M_1$ the language $(w_i, w_k)\delta_Z \odot (K_{(i,x,k)} \times \{\$\}^*)\delta_Z$ is regular. We may now deduce that N_2 is regular, since it is a finite union of regular languages.

(3) This case can be considered as the dual of the previous case. We consider words $(u, w)\delta_Z \in K_x$, where $\lambda_0 \cdot u$ is an intersection vertex of Γ and $\lambda_0 \cdot w$ is not an intersection vertices of Γ. We introduce the following notations.

For each $\Delta_i \subseteq \Lambda_X$ and for all intersection vertices $\lambda_k \in V_i$, let $w_{i,k} \in K_{i,V_i} \subseteq K_i$ so that $\lambda_i \cdot w_{i,k} = \lambda_k$. Such an element exists, since Γ_i and Δ_i are isomorphic and since $\lambda_i \cdot K_i\varphi_1 = R_i$. Let

$$K_{(x,i,k)} = \{\overline{u} \in K_i | \ \lambda_i \cdot \overline{u} = \lambda_k \cdot x\} \setminus K_{i,V_i}.$$

By Lemma 3.1.9 and Proposition 2.2.3, $K_{(x,i,k)}$ is a regular language. Let

$$M_2 = \{(w_k, w_i) \in W_Y \times W_X \mid \lambda_0 \cdot w_k \in V_i\}.$$

We will verify that words $(u,w)\delta_Z \in K_x$ that satisfy that $\lambda_0 \cdot u$ is an intersection vertex of some $\Delta_i \in \Lambda_X$ and $\lambda_0 \cdot w$ is not an intersection vertex of Δ_i are exactly the words contained in the language

$$N_3 = \bigcup_{(w_k,w_i)\in M_2} (w_k, w_i)\delta_Z \odot (\{\$\}^* \times K_{(x,i,k)})\delta_Z,$$

where $\odot$ denotes the padded product.

If $(u,w) \in N_3$, then $u \equiv w_k$ and $w \equiv w_i\widetilde{w}$, where $\widetilde{w} \in K_{(x,i,k)}$, $\lambda_0 \cdot w_i, \lambda_0 \cdot w_k \in V_i$. In particular we have that $\lambda_i \cdot \widetilde{w}$ is not an intersection vertex and that $\lambda_0 \cdot w_k \cdot x = \lambda_k \cdot x = \lambda_i \cdot \widetilde{w}$. Hence $(\lambda_0 \cdot w_k) \cdot x = \lambda_0 \cdot w_i \cdot \widetilde{w}$ holds, which verifies that $(u,w)\delta_Z \in K_x$.

For the converse assume that $(u,w)\delta_Z \in K_x$ such that $\lambda_0 \cdot u$ is an intersection vertex of $\Delta_i \subseteq \Lambda_X$ and $\lambda_0 \cdot w$ is not an intersection vertex. Since $\lambda_0 \cdot u$ is an intersection vertex, we have that $u \equiv w_k$, where $\lambda_k \in V_i$. Moreover, since $\lambda_0 \cdot w$ is not an intersection vertex in Δ_i, we have that $w \equiv w_i \cdot \widetilde{w}$, where $\widetilde{w} \in \widetilde{K}_i$. Also, since $(u,w)\delta_Z \in K_x$, we have that

$$\begin{aligned}\lambda_k \cdot x = \lambda_0 \cdot u_k \cdot x &= \lambda_0 \cdot w \\ &= (\lambda_0 \cdot w_i) \cdot \widetilde{w} \\ &= \lambda_i \cdot \widetilde{w}\end{aligned}$$

verifying that $\widetilde{w} \in K_{(x,i,k)}$. Hence we may deduce that

$$(u,w)\delta_Z \in (w_k, w_i)\delta_Z \odot (\{\$\}^* \times K_{(x,i,k)})\delta_Z \subseteq N_3.$$

Finally we verify that N_3 is a regular language over Z. By Proposition 2.2.3, the languages $(K_{(x,i,k)} \times \{\$\}^*)\delta_Z$, where $\lambda_0 \cdot w_k \in V_i$ are regular. Since $M = (w_k, w_i)\delta_Z$ is finite it satisfies the condition of Proposition 2.3.16 and hence we have that for all $(w_k, w_i) \in M_2$ the language $(w_k, w_i)\delta_Z \odot (K_{(x,i,k)} \times \epsilon\{\$\}^*)\delta_Z$ is regular. We may now deduce that N_3 is regular, since it is a finite union of regular languages.

(4) Our final case is when $(u,w)\delta_Z \in K_x$ is such that the vertices $\lambda_0 \cdot u$, $\lambda_0 \cdot w$ are intersection vertices of some $\Delta_i \in \Lambda_X$. In this case $(u,w)\delta_Z \in K_x$ if and only if the word $(u,w)\delta_Z$ is an element of the following language

$$N_4 = \{(w_j, w_k) \mid w_j, w_k \in W, \lambda_0 \cdot w_j \cdot x = \lambda_0 \cdot w_k\}\delta_Z.$$

Since W is finite, N_4 is finite, and hence is a regular language.

Now we may deduce that

$$K_x = N_1 \cup N_2 \cup N_3 \cup N_4,$$

is a finite union of regular languages and hence is regular. That K_y is a regular language for all $y \in Y$ can be proved in a similar way. Since R_s was an arbitrary $\mathcal{R}$-class of S, we may deduce that every $\mathcal{R}$-class of S is Schützenberger automatic, and hence S is Schützenberger automatic. $\square$

Chapter 4

Semidirect product

Semilattices and groups are some of the handiest examples of inverse semigroups. In fact, every inverse semigroup can be described using semilattices and groups as building blocks with a semidirect product construction. Namely, every inverse semigroup is an idempotent separating homomorphic image of a subsemigroup of a semidirect product of a semilattice by a group. In addition, a dual approach tells us that every inverse semigroup is a subsemigroup in an idempotent separating homomorphic image of a semidirect product of a semilattice by a group. In this chapter we give necessary and sufficient conditions for the construction of a semidirect product of a semilattice by a group to be finitely generated, to be finitely presented and to be Schützenberger automatic.

4.1 Presentations.

Semigroup presentations.

We first summarize the basic definitions and properties concerning semigroup presentations.

Let X be a non-empty set. A *semigroup presentation* is an ordered pair $\langle X|P\rangle$, where P is a binary relation on the free semigroup X^+. Let τ denote the congruence generated by P on the free semigroup X^+. The semigroup $S = X^+/\tau$ is said to be *presented* by the generators X and relations P and we denote this by

$$S = \langle X|P\rangle.$$

If X can be chosen to be a finite set, then we say that S is *finitely generated*, and if both X and P can be chosen to be finite sets, then we say that S is *finitely presented*. Let w_1, w_2 be words over X. We write $w_1 \equiv w_2$, if w_1 and w_2 are identical as words and we write

$w_1 = w_2$, if w_1 and w_2 represent the same element of S. If $w_1 = w_2$, then we also say that S satisfies the relation $w_1 = w_2$. We say that w_2 is obtained from w_1 by an *application of a relation of* P, if $w_1 \equiv \alpha u \beta$ and $w_2 \equiv \alpha v \beta$, where $\alpha, \beta \in X^*$ and $u = v$ or $v = u$ is a relation of P. We say that $w_1 = w_2$ is a *consequence of relations in* P, if there exists a sequence of words

$$w_1 \equiv \alpha_0, \ldots, \alpha_m \equiv w_2,$$

where α_{j+1} is obtained from α_j by applying a relation of P.

We recall [33, Proposition 2.3.]:

Proposition 4.1.1. *Let S be a semigroup generated by a set X and P be a binary relation on X^+. Then $S = \langle X|P \rangle$ if and only if the following two conditions hold:*

(i) S satisfies all relations in P;

(ii) If $w_1, w_2 \in X^+$ are such that $w_1 = w_2$ holds in S, then $w_1 = w_2$ is a consequence of relations in P.

Let S be defined by the finite presentation $\langle X|P \rangle$. One can obtain an alternative semigroup presentation for S by applying the following *elementary Tietze transformations*:

(T1) adding a new generating symbol y and a new relation $y = w$, where $w \in X^+$;
(T2) if P possesses a relation of the form $y = w$, where w is a word over X that does not contain the symbol y, then deleting the generating symbol y and the relation $y = w$ and replacing all occurrences of y by w in the remaining relations;
(T3) adding a new relation $w_1 = w_2$ to P, in the case where $w_1 = w_2$ is a consequence of relations in P;
(T4) deleting a relation $w_1 = w_2$ from P, in the case where $w_1 = w_2$ is a consequence of relations in $P - \{w_1 = w_2\}$.

Proposition 2.5. of [33] tells us:

Proposition 4.1.2. *Two (finite) semigroup presentations define the same semigroup if and only if one can be obtained from the other by applying (a finite sequence of) elementary Tietze transformations.*

Inverse semigroup presentations.

In the rest of the thesis, we work frequently with inverse semigroup presentations. Therefore we recall the basic definitions and properties concerning presentations of inverse semigroups.

We consider inverse semigroups as algebras of type (2,1), where the binary operation is multiplication and the unary operation assigns to each element its unique (von Neumann) inverse.

The class of inverse semigroups forms a variety and hence free inverse semigroups exist. This fact enables us to define inverse semigroup presentations. First, we recall the description of the free inverse semigroup on a non-empty set X as a factor semigroup of the free semigroup with involution on X.

Let X be a non-empty set and $X^{-1} = \{x^{-1} \mid x \in X\}$. Consider the free semigroup $F = (X \cup X^{-1})^+$ and define a unary operation on F in the following way: For each $y \in X \cup X^{-1}$, let

$$y^{-1} = \begin{cases} x^{-1} & \text{if } y = x \in X \\ x & \text{if } y = x^{-1} \in X^{-1} \end{cases}$$

and define

$$(y_1 \dots y_n)^{-1} = y_n^{-1} \dots y_1^{-1}.$$

With the unary operation defined, $(F, \cdot, {}^{-1})$ is the free semigroup with involution on X, which we shall denote by $FSI(X)$. Define the following binary relation on $FSI(X)$:

$$\Re = \{(uu^{-1}u, u) \mid u \in FSI(X)\} \cup \{(uu^{-1}vv^{-1}, vv^{-1}uu^{-1}) \mid u, v \in FSI(X)\}.$$

The congruence generated by $\Re$ is called the *Wagner congruence* which we denote by ρ. The factor-semigroup

$$FI(X) = FSI(X)/\rho$$

is the free inverse semigroup on X. We will refer to the elements of $\Re$ as *standard inverse semigroup relations.*

An *inverse semigroup presentation* is an ordered pair $\langle X|P\rangle$, where P is a binary relation on $FSI(X)$. Let τ denote the congruence generated by $P \cup \Re$. The semigroup $S = FSI(X)/\tau$ is said to be *presented as an inverse semigroup* by the generators X and relations P and we denote this by

$$S = \mathrm{Inv}\langle X|P\rangle.$$

If X can be chosen to be a finite set, then we say that S is *finitely generated* as an inverse semigroup, and if both X and P can be chosen to be finite, then we say that S is *finitely presented* as an inverse semigroup. We note that if S is generated as an inverse semigroup by X, then S is generated by $X \cup X^{-1}$ as a semigroup. In particular S is finitely generated as a semigroup if and only if it is finitely generated as an inverse semigroup. Moreover, if S

is given by the inverse semigroup presentation $\mathrm{Inv}\langle X|P\rangle$, then it is given by the following semigroup presentation:

$$\langle X \cup X^{-1}|P \cup \Re\rangle.$$

Proposition 4.1.3. *Let S be an inverse semigroup defined by the semigroup presentation $\langle X|P\rangle$. Then S is defined by the inverse semigroup presentation* $\mathrm{Inv}\langle X|P\rangle$.

Let $S = \mathrm{Inv}\langle X|P\rangle$ and let w_1, w_2 be words over $X \cup X^{-1}$. We write $w_1 \equiv w_2$, if w_1 and w_2 are identical as words and we write $w_1 = w_2$, if w_1 and w_2 represent the same element of S. If $w_1 = w_2$, then we also say that S satisfies the relation $w_1 = w_2$. We say that w_2 is obtained from w_1 by an *application of a relation of P or of a standard inverse semigroup relation*, if $w_1 \equiv \alpha u\beta$ and $w_2 \equiv \alpha v\beta$, where $\alpha, \beta \in (X \cup X^{-1})^*$ and $u = v$ or $v = u$ is a relation of P or a standard inverse semigroup relation. We say that $w_1 = w_2$ is a consequence of relations in P and of standard inverse semigroup relations, if there exists a sequence of words

$$w_1 \equiv \alpha_0, \ldots, \alpha_m \equiv w_2,$$

where α_{j+1} is obtained from α_j by applying a relation of P or a standard inverse semigroup relation.

The following proposition is a modification of [33, Proposition 2.3.]:

Proposition 4.1.4. *Let S be an inverse semigroup generated by a set X. Let P be a relation on $FSI(X)$ and let $\Re$ denote the set of standard inverse semigroup relations on $FSI(X)$. Then $S = \mathrm{Inv}\langle X|P\rangle$ if and only if the following two conditions hold:*

(i) S satisfies all relations in $P \cup \Re$;

(ii) If $w_1, w_2 \in (X \cup X^{-1})^+$ are such that $w_1 = w_2$ holds in S, then $w_1 = w_2$ is a consequence of relations in $P \cup \Re$.

Proof. ($\Longrightarrow$) If $S = \mathrm{Inv}\langle X|P\rangle$, then $S = FSI(X)/\tau$ by definition, where τ is the congruence generated by $P \cup \Re$ and hence (i) and (ii) hold.

($\Longleftarrow$) Let F denote the free semigroup with involution on X and let $\varphi : F \to S$ be the unary homomorphism extending the identity map $\iota : X \to X$. Let τ denote the congruence generated by $P \cup \Re$. We show that $S = \mathrm{Inv}\langle X|P\rangle$ by verifying that $\ker\varphi = \tau$. Since S satisfies all relations in $P \cup \Re$ by (i), we clearly have that $\tau \subseteq \ker\varphi$. Suppose now that $(w_1, w_2) \in \ker\varphi$. Then w_1 and w_2 represent the same element of S, and hence by (ii) $\ker\varphi \subseteq \tau$. $\Box$

Let $S = \mathrm{Inv}\langle X|P\rangle$. One can obtain an alternative inverse semigroup presentation for S by applying the following *elementary Tietze transformations*:

(T1) adding a new generating symbol y and a new relation $y = w$, where $w \in (X \cup X^{-1})^+$;

(T2) if P possesses a relation of the form $y = w$, where w is a word over $X \cup X^{-1}$ that does not contain the symbols y or y^{-1}, then deleting the generating symbol y and the relation $y = w$ and replacing all occurrences of y by w and y^{-1} by w^{-1} in the remaining relations;

(T3) adding a new relation $w_1 = w_2$ to P, in the case where $w_1 = w_2$ is a consequence of relations in P and of standard inverse semigroup relations;

(T4) deleting a relation $w_1 = w_2$ from P, in the case where $w_1 = w_2$ is a consequence of relations in $P - \{w_1 = w_2\}$ and of standard inverse semigroup relations.

Similarly to Proposition 4.1.2, we have:

Proposition 4.1.5. *Two (finite) inverse semigroup presentations define the same inverse semigroup if and only if one can be obtained from the other by applying (a finite sequence of) elementary Tietze transformations.*

4.2 Finite generation with respect to an action.

In the first section of the third chapter, we introduced the notion of an S-act and among others defined finite generation for S-acts. We also showed that every semigroup S can be considered as an S-act, but in doing so we forget the structure which S possesses and only keep the underlying set. Bearing in mind that an S-act A can have an algebraic structure underlying it, we give a more sophisticated version of the definition of finite generation and introduce the concept of finite presentability with respect to an action in Section 4.

Let $(Y, \wedge)$ be a semilattice and denote by $\leq$ the natural partial order on it. Recall that $x \leq y$ if and only if $x = x \wedge y$. We say that $y \in Y$ is a *maximal element* of Y if $x \in Y$, $y \leq x$ always implies that $x = y$. We say that Y satisfies the *maximum condition* if it has finitely many maximal elements, and for all $x \in Y$ there exists a maximal element $y \in Y$ such that $x \leq y$. The following lemma is immediate from the definitions.

Lemma 4.2.1. *Let $(Y, \wedge)$ be a semilattice and let φ be an automorphism of $(Y, \wedge)$. Then $x \leq y$ if and only if $x\varphi \leq y\varphi$. In particular we have that y is a maximal element of Y if and only if $y\varphi$ is a maximal element of Y as well.*

Next we recall the notion of a left action of a semigroup S on a semigroup T. Let $(T, *)$ and $(S, \cdot)$ be semigroups. We say that S *acts on T by endomorphisms on the left*, if there exists a map $f : T \times S \to T, (t, s) \mapsto {}^{s}t$ satisfying the conditions

(A1) ${}^{(s_1 \cdot s_2)}t = {}^{s_1}({}^{s_2}t)$;

(A2) ${}^{s}(t_1 * t_2) = {}^{s}t_1 * {}^{s}t_2$

for all $t, t_1, t_2 \in T$ and $s, s_1, s_2 \in S$. If S is a monoid, then we also require the following condition to hold:

(A3) ${}^{1}t = t$ for all $t \in T$.

We say that S *acts on T by automorphisms on the left*, if besides conditions (A1), (A2) (and (A3), if S is a monoid), the following conditions are satisfied:

(A4) for all $t \in T$ and $s \in S$ there exists $\tilde{t} \in T$ such that ${}^{s}\tilde{t} = t$;

(A5) for all $t_1, t_2 \in T$ and $s \in S$, ${}^{s}t_1 = {}^{s}t_2$ implies that $t_1 = t_2$.

The reason we deviated from our notation introduced in the third chapter, and write ${}^{s}t$ instead of $t.s$, is that we would like to keep the traditional notation used in semidirect products, which will be given soon. We also mention that a right action of S on T by endomorphisms (automorphisms) can be defined similarly.

Definition 4.2.2. Let $(S, \cdot)$ and $(T, *)$ be semigroups. Assume that S acts on T on the left by endomorphisms (automorphisms). We say that T is generated by $T_0 \subseteq T$ with respect to the action of S, if $T = \langle {}^{S}T_0 \cup T_0 \rangle$, where ${}^{S}T_0 = \{{}^{s}t \mid s \in S,\ t \in T_0\}$. We say that T is *finitely generated with respect to the action of S* if T_0 can be chosen to be a finite subset of T.

Lemma 4.2.3. *Let T be a finitely generated semigroup and assume that the semigroup S acts on T on the left by endomorphisms. Then T is finitely generated with respect to the action of S as well.*

Proof. Clearly, if T is generated by a finite set Y, then T is finitely generated by Y with respect to the action of S as well. $\square$

Finitely generated semilattices are finite. We give three examples for infinite semilattices that are finitely generated with respect to a group action defined on them. These examples will be referred to throughout the fourth chapter to illustrate the results concerning finite generation and finite presentability of a semidirect product of a semilattice by a group. Our first two examples have already been introduced in the third chapter, these are Example 3.3.6 and Example 3.3.7.

Proposition 4.2.4. *Let Y_∞ be the double infinite chain with an identity adjoined on top:*

$$\ldots < e_{-1} < e_0 < e_1 < \ldots < \mathbf{1}.$$

Let $G = \langle g \rangle$ be the infinite cyclic group. Let $n, k \in \mathbb{Z}$ and consider the following map:

$$f : Y_\infty \times G \to Y_\infty; \quad (e_n, g^k) \mapsto {}^{g^k}e_n = e_{n+k}; \quad (\mathbf{1}, g^k) \mapsto {}^{g^k}\mathbf{1} = \mathbf{1}.$$

With this action, G acts on Y_∞ on the left by automorphisms, and Y_∞ is generated by $Y_0 = \{e_0, \mathbf{1}\}$ with respect to this action of G.

Proof: We first show that conditions (A1)-(A5) hold.

(A1) Let $y \in Y_\infty$ and $g_1 = g^n, g_2 = g^m,\ n, m \in \mathbb{Z}$. If $y = \mathbf{1}$, then clearly ${}^{g_1g_2}\mathbf{1} = \mathbf{1} = {}^{g_1}\mathbf{1} = {}^{g_1}({}^{g_2}\mathbf{1})$. If $y = e_k$, for some $k \in \mathbb{Z}$, then

$${}^{g_1g_2}y = {}^{g^{n+m}}e_k = e_{k+n+m} = {}^{g^n}e_{k+m} = {}^{g^n}({}^{g^m}e_k) = {}^{g_1}({}^{g_2}y),$$

proving that (A1) does indeed hold.

(A2) Let $y_1 = e_n, y_2 = e_m$ and let $h = g^k$, where $n, m, k \in \mathbb{Z}$. Without loss of generality we can assume that $n \le m$, and hence $n + k \le m + k$. Then

$$\begin{aligned} {}^h(y_1 \wedge y_2) = {}^{g^k}(e_n \wedge e_m) = {}^{g^k}e_n &= e_{n+k} \\ &= e_{n+k} \wedge e_{m+k} \\ &= {}^{g^k}e_n \wedge {}^{g^k}e_m = {}^hy_1 \wedge {}^h y_2. \end{aligned}$$

The case when y_1 and/or y_2 equals $\mathbf{1}$ can be verified similarly.

(A3) Clearly ${}^1y = {}^{g^0}y = y$ for all $y \in Y_\infty$.

(A4) Let $e_n \in Y_\infty$ and $g^m \in G$. By definition, ${}^{g^m}e_{n-m} = e_n$ and also ${}^{g^m}\mathbf{1} = \mathbf{1}$, verifying that (A4) holds.

(A5) Let $y_1 = e_n,\ y_2 = e_m$ and let $h = g^k \in G$ such that ${}^hy_1 = {}^hy_2$. That is

$$e_{n+k} = {}^{g^k}e_n = {}^{g^k}e_m = e_{m+k}$$

holds and hence we obtain that $e_n = e_m$, proving that (A5) is satisfied.

We may now deduce that G acts on Y_∞ on the left by automorphisms. By (A4), for all $n \in \mathbb{Z}$, $e_n = {}^{g^n}e_0$, and so the semilattice Y_∞ is finitely generated with respect to the action of G by $\mathbf{1}$ and e_0. $\square$

Proposition 4.2.5. *Let A be the infinite antichain*

$$\dots,\ e_{-1},\ e_0,\ e_1,\dots.$$

Adjoin an identity $\mathbf{1}$ on top and a zero $\mathbf{0}$ on bottom, that is for each $e_i \in A$, $\mathbf{0} < e_i < \mathbf{1}$ holds. Let A_∞ denote the (semi)lattice obtained. Let $G = \langle g \rangle$ be the infinite cyclic group. Let $n, k \in \mathbb{Z}$ and consider the following map:

$$f : A_\infty \times G \to A_\infty;\ (e_n, g^k) \mapsto {}^{g^k}e_n = e_{n+k};\ (\mathbf{1}, g^k) \mapsto {}^{g^k}\mathbf{1} = \mathbf{1};\ (\mathbf{0}, g^k) \mapsto {}^{g^k}\mathbf{0} = \mathbf{0}.$$

With this action, G acts on A_∞ on the left by automorphisms, and A_∞ is generated by $Y_0 = \{e_0, \mathbf{1}\}$ with respect to this action of G.

Proof. We show that conditions (A1)-(A5) hold.

(A1) Clearly if $y \in A_\infty$ equals to $\mathbf{1}$ or to $\mathbf{0}$, then for all $g_1, g_2 \in G$, ${}^{g_1g_2}y = {}^{g_1}({}^{g_2}y)$. Let $y = e_k$, $g_1 = g^n$, $g_2 = g^m$. Then

$${}^{g_1g_2}y = {}^{g^ng^m}e_k = {}^{g^{n+m}}e_k = e_{n+m+k} = {}^{g^n}e_{k+m} = {}^{g^n}({}^{g^m}e_k) = {}^{g_1}({}^{g_2}y),$$

proving that (A1) indeed holds.

(A2) Let $y_1, y_2 \in A_\infty$ and $h = g^k$. If either $y_1 = \mathbf{0}$ or $y_2 = \mathbf{0}$ (or both), then clearly ${}^h(y_1 \wedge y_2) = {}^hy_1 \wedge {}^hy_2$. Similarly, if $y_1 = \mathbf{1}$ and $y_2 \in A_\infty$ arbitrary, then ${}^h(y_1 \wedge y_2) = {}^hy_2 = \mathbf{1} \wedge {}^hy_2 = {}^hy_1 \wedge {}^hy_2$. The only case remaining to consider is when $y_1 = e_n, y_2 = e_m$ for some $n, m \in \mathbb{Z}$. If $n \neq m$, then we have

$${}^h(y_1 \wedge y_2) = {}^{g^k}(e_n \wedge e_m) = {}^{g^k}\mathbf{0} = {}^{g^k}e_n \wedge {}^{g^k}e_m = {}^hy_1 \wedge {}^hy_2.$$

If $n = m$, then

$${}^h(y_1 \wedge y_2) = {}^{g^k}(e_n \wedge e_n) = {}^{g^k}e_n = {}^{g^k}e_n \wedge {}^{g^k}e_n = {}^hy_1 \wedge {}^hy_2.$$

verifying that (A2) holds.

(A3) By definition, it is clear, that for all $y \in A_\infty$, ${}^1y = {}^{g^0}y = y$.

(A4) Let $y = e_n \in A$ and $g^m \in G$. By definition, ${}^{g^m}e_{n-m} = e_n$. Also ${}^{g^m}\mathbf{1} = \mathbf{1}$ and ${}^{g^m}\mathbf{0} = \mathbf{0}$, verifying that (A4) holds.

(A5) Let $y_1, y_2 \in A_\infty$ and let $h = g^k$ such that ${}^hy_1 = {}^hy_2$. Then the following three cases can occur:

(a) If ${}^h y_1 = {}^h y_2 = \mathbf{1}$, then $y_1 = y_2 = \mathbf{1}$.

(b) If ${}^h y_1 = {}^h y_2 = \mathbf{0}$, then $y_1 = y_2 = \mathbf{0}$.

(c) If ${}^h y_1 = {}^h y_2 = e_m$, then ${}^{h^{-1}}({}^h y_1) = {}^{h^{-1}}({}^h y_2) = {}^{h^{-1}}e_m$, from which it follows that $y_1 = y_2 = {}^{h^{-1}}e_m$.

Hence (A5) is satisfied.

We may deduce that G acts on A_∞ on the left by automorphisms. Since by (A4) for all $n \in \mathbb{Z}$, $e_n = {}^{g^n}e_0$, and since $e_n \wedge e_m = \mathbf{0}$ where $n, m \in \mathbb{Z}$ $(n \neq m)$, we also obtain that the semilattice A_∞ is finitely generated with respect to the action of G by $\mathbf{1}$ and e_0. □

Before presenting our last example, we make a few basic observation about the free semilattice generated by a set A. To be more precise, in our case A is going to be an infinite set. Let F be the free semilattice generated by $A = \{\ldots,\ e_{-1},\ e_0,\ e_1, \ldots\}$. Since F is the free semilattice, for every element $f \in F$, there exists a unique subset $\{f_1, \ldots, f_m\}$ of A, such that

$$f = f_1 \wedge \ldots \wedge f_m \tag{4.1}$$

and the following conditions hold:

(F1) $f_i \neq f_j$ for all $i \neq j, 1 \leq i, j \leq m$.

(F2) If $f = l_1 \wedge \ldots \wedge l_n$, where $l_i \neq l_j$ $(i \neq j, 1 \leq i, j \leq n)$ then $m = n$ and $\{f_1, \ldots, f_m\} = \{l_1, \ldots, l_n\}$.

In the next proposition, if we write

$$f = f_1 \wedge \ldots \wedge f_m, \tag{4.2}$$

then we assume that conditions (F1) and (F2) hold.

Proposition 4.2.6. *Let F be the free semilattice generated by infinitely many elements $A = \{\ldots,\ e_{-1},\ e_0,\ e_1, \ldots\}$. Adjoin an identity $\mathbf{1}$ on top that is for each $e_i \in A_0$, $e_i < \mathbf{1}$ holds. Denote by F_∞ the (semi)lattice obtained. Let $G = \langle g \rangle$ be the infinite cyclic group. Let $n, k \in \mathbb{Z}$ and consider the following map:*

$$\varphi : F_\infty \times G \to F_\infty; \quad (e_n, g^k) \mapsto {}^{g^k}e_n = e_{n+k}; \quad (\mathbf{1}, g^k) \mapsto {}^{g^k}\mathbf{1} = \mathbf{1},$$

where $e_n \in A$. If $e \in F$ so that $e = f_1 \wedge f_2 \wedge \ldots \wedge f_m$, then we define

$$(e, g_k) \mapsto {}^{g^k}e = {}^{g^k}f_1 \wedge {}^{g^k}f_2 \wedge \ldots \wedge {}^{g^k}f_m.$$

With this action, G acts on Y_∞ by automorphisms. Furthermore Y_∞ is generated by $Y_0 = \{e_0, \mathbf{1}\}$ with respect to this action.

Proof. That f is a well-defined map follows, since (F1) and (F2) hold. We note that for $g^n \in G$, we have by definition that

$$\begin{aligned} {}^{g^n}({}^{g^k}f_1 \wedge {}^{g^k}f_2 \wedge \ldots \wedge {}^{g^k}f_m) &= {}^{g^n}({}^{g^k}f_1) \wedge {}^{g^n}({}^{g^k}f_2) \wedge \ldots \wedge {}^{g^n}({}^{g^k}f_m) \\ &= {}^{g^{n+k}}f_1 \wedge {}^{g^{n+k}}f_2 \wedge \ldots \wedge {}^{g^{n+k}}f_m. \end{aligned}$$

We are now ready to show that conditions (A1)-(A5) hold.

(A1) Clearly if y equals to $\mathbf{1}$, then for all $g_1, g_2 \in G$, ${}^{g_1g_2}y = {}^{g_1}({}^{g_2}y)$. Let $y = f_1 \wedge \ldots \wedge f_n$, where $f_1, \ldots, f_n \in A$ and let $g_1 = g^m, g_2 = g^k$. Then on one hand

$${}^{g_1g_2}y = {}^{g^m g^k}(f_1 \wedge \ldots \wedge f_n) = {}^{g^{m+k}}(f_1 \wedge \ldots \wedge f_n) = {}^{g^{m+k}}f_1 \wedge \ldots \wedge {}^{g^{m+k}}f_n.$$

On the other hand

$${}^{g_1}({}^{g_2}y) = {}^{g^m}({}^{g^k}(f_1 \wedge \ldots \wedge f_n)) = {}^{g^m}({}^{g^k}f_1 \wedge \ldots \wedge {}^{g^k}f_n) = {}^{g^{m+k}}f_1 \wedge \ldots \wedge {}^{g^{m+k}}f_n$$

proving that (A1) indeed holds.

(A2) Let $y_1, y_2 \in F_\infty$ and $h = g^k$. If $y_1 = \mathbf{1}$ and $y_2 \in F$ is arbitrary, then ${}^h(y_1 \wedge y_2) = {}^hy_2 = {}^h\mathbf{1} \wedge {}^hy_2 = {}^hy_1 \wedge^h y_2$. The only remaining case to consider is when $y_1, y_2 \in F$. Assume that $y_1 = f_1 \wedge \ldots \wedge f_m$ and that $y_2 = l_1 \wedge \ldots \wedge l_n$, where $f_i, l_j \in A$. Then, on one hand

$${}^h(y_1 \wedge y_2) = {}^{g^k}(f_1 \wedge \ldots \wedge f_m \wedge l_1 \wedge \ldots \wedge l_n) = {}^{g^k}f_1 \wedge \ldots \wedge {}^{g^k}f_m \wedge {}^{g^k}l_1 \wedge \ldots \wedge {}^{g^k}l_n.$$

On the other hand

$${}^hy_1 \wedge {}^hy_2 = {}^{g^k}(f_1 \wedge \ldots \wedge f_m) \wedge {}^{g^k}(l_1 \wedge \ldots \wedge l_n) = {}^{g^k}f_1 \wedge \ldots \wedge {}^{g^k}f_m \wedge {}^{g^k}l_1 \wedge \ldots \wedge {}^{g^k}l_n.$$

Thus (A2) holds.

(A3) By definition, it is clear that for all $y \in F_\infty$, ${}^1y = {}^{g^0}y = y$.

(A4) Let $e = f_1 \wedge \ldots \wedge f_n \in F$ and $g^m \in G$. Let $l = {}^{g^{-m}}e$. By (A1) and (A3) we have that ${}^{g^m}l = {}^{g^m}({}^{g^{-m}}e) = {}^1e = e$. Also, if $e = \mathbf{1}$ then ${}^{g^m}\mathbf{1} = \mathbf{1}$, verifying that (A4) holds.

(A5) Let $y_1, y_2 \in F_\infty$ and let $h = g^k$ such that ${}^hy_1 = {}^hy_2$. Then either $y_1 = y_2 = \mathbf{1}$ or $y_1 = y_2 = 0$, or $y_1 = f_1 \wedge \ldots \wedge f_n$ and $y_2 = l_1 \wedge \ldots \wedge l_m$. In the latter case we have that

$${}^{g^k}y_1 = {}^{g^k}(f_1 \wedge \ldots \wedge f_n) = {}^{g^k}(l_1 \wedge \ldots \wedge l_m) = {}^{g^k}y_2.$$

That is, ${}^{g^k}f_1 \wedge \ldots \wedge {}^{g^k}f_n = {}^{g^k}l_1 \wedge \ldots \wedge {}^{g^k}l_m$. Having in mind that $(F1)$ and $(F2)$ hold, we have that

$$B_1 = \{{}^{g^k}f_1, \ldots, {}^{g^k}f_n\} = \{{}^{g^k}l_1, \ldots, {}^{g^k}l_m\} = B_2,$$

where $n = m$. In particular, we have that for each ${}^{g^k}f_i \in B_1$ there exists exactly one element ${}^{g^k}l_j \in B_2$ such that ${}^{g^k}f_i = {}^{g^k}l_j$ and vica versa. It follows that

$$\{f_1, \ldots, f_n\} = \{l_1, \ldots, l_m\},$$

and hence $y_1 = y_2$, verifying that (A5) is satisfied.

We may now deduce that G acts on F_∞ on the left by automorphisms. By definition for all $n \in \mathbb{Z}$, $e_n = {}^{g^n}e_0$. Moreover every element of F is a product of finitely many elements of A, therefore we may deduce that F_∞ is finitely generated with respect to the action of G by $\mathbf{1}$ and e_0. □

Let $(Y, \wedge)$ be a semilattice and $(G, \cdot)$ be a group. Assume that G acts on Y on the left by automorphisms. The *semidirect product* $S = Y \rtimes G$ of Y by G is the set $Y \times G$ equipped with the following multiplication

$$(e, g)(f, h) = (e \wedge {}^g f, g \cdot h).$$

With this construction an inverse semigroup is obtained. We recall some basic properties of the semidirect product $S = Y \rtimes G$.

Proposition 4.2.7. *Let $(Y, \wedge)$ be a semilattice and $(G, \cdot)$ be a group. Assume that G acts on Y on the left by automorphisms and consider the semidirect product $S = Y \rtimes G$. Then we have that*

(i) $(e, g)^{-1} = ({}^{g^{-1}}e, g^{-1})$;

(ii) $(e, g) \leq (f, h)$ *if and only if* $e \leq f$ *in* Y *and* $g = h$;

(iii) $(e, g) \in E(S)$ *if and only if* $g = 1$;

(iv) $(e, g)\ \mathcal{R}\ (f, h)$ *if and only if* $e = f$.

(v) $(e, g)\ \mathcal{L}\ (f, h)$ *if and only if* ${}^{g^{-1}}e = {}^{h^{-1}}f$.

4.3 Finite generation.

In this section, we give a necessary and sufficient condition for a semidirect product of a semilattice by a group to be finitely generated. Throughout this section, if we say that the group G acts on a semilattice Y, then it will be understood that G acts on Y on the left by automorphisms.

Proposition 4.3.1. *Let $(Y, \wedge)$ be a semilattice, and $(G, \cdot)$ be a group acting on Y. The semidirect product $S = Y \rtimes G$ is finitely generated if and only if the following conditions hold:*

(i) G is finitely generated;

(ii) Y satisfies the maximum condition;

(iii) Y is finitely generated with respect to the action of G.

Proof. ($\Longrightarrow$) Assume that $S = Y \rtimes G$ is finitely generated by the elements $A = \{(e_1, h_1), \ldots, (e_n, h_n)\}$, where $A \subseteq S$. By Proposition 4.2.7, we obtain that $A^{-1} = \{({}^{h_1^{-1}}e_1, h_1^{-1}), \ldots, ({}^{h_n^{-1}}e_n, h_n^{-1})\}$. Let

$$X = \{h_1, h_2, \ldots, h_n\} \qquad \text{and} \qquad Y_0 = \{e_1, \ldots, e_n\} \cup \{{}^{g^{-1}}e \mid (e, g) \in A\}.$$

Let $(e, g) \in S$. Write

$$(e, g) = (f_1, g_1)(f_2, g_2) \ldots (f_k, g_k)$$

where $(f_j, g_j) \in A \cup A^{-1}$ for all $1 \leq j \leq k$. Then, on one hand

$$g = g_1 \cdots g_k,$$

where $g_j \in X \cup X^{-1}$ verifying that G is generated by the finite set X. On the other hand,

$$e = f_1 \wedge {}^{g_1}f_2 \wedge \ldots \wedge {}^{g_1 \cdots g_{k-1}}f_k,$$

where $f_1, \ldots, f_k \in Y_0$. It follows that for each $e \in Y$ there exist $f_1, \ldots, f_k \in Y_0$, and $t_1, \ldots, t_{k-1} \in G$ such that

$$e = f_1 \wedge {}^{t_1}f_2 \wedge \ldots \wedge {}^{t_{k-1}}f_k, \tag{4.3}$$

proving that $Y = \langle {}^G Y_0 \rangle$. We also obtain from (4.3), that $e \leq f_1$. That is to say that every element of Y is less than or equal to an element of Y_0 with respect to the natural partial order. In particular, we have that the maximal elements of Y are the maximal elements of Y_0, and so we deduce that Y satisfies the maximum condition.

($\Longleftarrow$) For the converse, let G be a group acting on a semilattice Y, where G and Y satisfy conditions $(i)-(iii)$. More precisely, assume that the finite set X generates G as a group, and suppose that $\langle {}^G\overline{Y}_0\rangle = Y$. For the sake of simplicity we assume that $X \subseteq G$ and $\overline{Y}_0 \subseteq Y$. Let Y_m denote the set of maximal elements of Y and let $Y_0 = Y_m \cup \overline{Y}_0$. By assumption, Y_m is a finite set, and so Y_0 is a finite set as well. Moreover, we still have that $\langle {}^G Y_0\rangle = Y$. We claim that the set

$$A = \{(e,1)\colon e \in Y_0\} \cup \{(e,h)\colon e \in Y_m,\ h \in X\}$$

generates S as an inverse semigroup.

Let $(e,g) \in Y_0 \times G$. We verify that (e,g) can be written in terms of elements of $A \cup A^{-1}$. Let $\tilde{e} \in Y_m$ be such that $e \leq \tilde{e}$ and suppose that $g = g_1 g_2 \ldots g_k$ where $g_1, g_2, \ldots, g_k \in X \cup X^{-1}$. By Lemma 4.2.1, there exist $f_1, f_2, \ldots, f_{k-1} \in Y_m$ such that ${}^{g_1}f_1 = \tilde{e}$ and ${}^{g_j}f_j = f_{j-1}$ for all $2 \leq j \leq k-1$. It follows that ${}^{g_1 \cdots g_j}f_j = \tilde{e}$ for all $1 \leq j \leq k-1$, and we obtain

$$\begin{aligned}(e,1)(\tilde{e},g_1)(f_1,g_2)\ldots(f_{k-1},g_k) &= (e \wedge \tilde{e} \wedge {}^{g_1}f_1 \wedge \ldots \wedge ({}^{g_1 g_2 \cdots g_{k-1}}f_{k-1}), g) \\ &= (e \wedge \underbrace{\tilde{e} \wedge \tilde{e} \ldots \wedge \tilde{e}}_{k}, g) = (e \wedge \tilde{e}, g) \\ &= (e,g),\end{aligned}$$

verifying that any element of $Y_0 \times G$ can be written in terms of elements of $A \cup A^{-1}$.

Let $e \in Y$ be arbitrary. Since $\langle {}^G Y_0\rangle = Y$, there exist $t_1, \ldots, t_k \in G$, and $f_1, \ldots, f_k \in Y_0$ such that

$$e = {}^{t_1}f_1 \wedge \ldots \wedge {}^{t_k}f_k.$$

Since Y satisfies the maximum condition, there exists $\tilde{e} \in Y_m$ such that $e \leq \tilde{e}$, and so $e = \tilde{e} \wedge e$ holds. Let $g \in G$ and write $g = t_1 t_2 \ldots t_k u$ where $u \in G$. Then

$$(e,g) = (\tilde{e},t_1)(f_1,t_1^{-1}t_2)\ldots(f_{k-1},t_{k-1}^{-1}t_k)(f_k,t_k^{-1})(\tilde{e},u).$$

By the above argument each component of the product can be written in terms of elements of $A \cup A^{-1}$, and thus S is finitely generated as an inverse semigroup by A. $\square$

We have the following useful corollary of the above proof.

Corollary 4.3.2. *Let G be a group generated as a group by a finite set X and let 1 denote the identity element of G. Let Y be a semilattice satisfying the maximum condition and let Y_m denote the set of maximal elements of Y. Assume that G acts on Y on the* *left and that the finite set Y_0 containing Y_m generates Y with respect to the action of G. Then the semidirect product $S = Y \rtimes G$ is generated as an inverse semigroup by $A = (Y_0 \times \{1\}) \cup (Y_m \times X)$.*

We end this section by giving three examples for finitely generated semidirect products.

Example 4.3.3. Let Y_∞ be the double infinite chain with an identity element adjoined on top and let G be the infinite cyclic group. Define the action of G on Y_∞ as it was in Proposition 4.2.4. Clearly G is finitely generated and Y_∞ satisfies the maximum condition. According to Proposition 4.2.4, Y_∞ is finitely generated with respect to the action of G. Hence, by Proposition 4.3.1, the semidirect product $S = Y_\infty \rtimes G$ is finitely generated.

Example 4.3.4. Consider the semilattice A_∞ introduced in Proposition 4.2.5 and let G be the infinite cyclic group. Define the action of G on A_∞ as it was in Proposition 4.2.5. Clearly G is finitely generated and A_∞ satisfies the maximum condition. According to Proposition 4.2.5, A_∞ is finitely generated with respect to the action of G. Hence, by Proposition 4.3.1, the semidirect product $S = A_\infty \rtimes G$ is finitely generated.

Example 4.3.5. Consider the semilattice F_∞ introduced in Proposition 4.2.6 and let G be the infinite cyclic group. Define the action of G on F_∞ as it was in Proposition 4.2.6. Clearly G is finitely generated and F_∞ satisfies the maximum condition. We have also shown in Proposition 4.2.6 that F_∞ is finitely generated with respect to the action of G. Hence, by Proposition 4.3.1, the semidirect product $S = F_\infty \rtimes G$ is finitely generated.

4.4 Finite presentability with respect to an action I.

The main purpose of this section is to introduce the concept of finite presentability of a semigroup with respect to a semigroup action. We investigate basic properties regarding this new notion. Among other things, we prove that a finitely presented semigroup on which a finitely generated semigroup acts on the left by endomorphisms is also finitely presented with respect to the action of the finitely generated semigroup. We set the concept introduced in this section into an inverse semigroup theoretic context in Section 6.

Throughout this section, if we say that a semigroup S acts on a semigroup T, then it will be understood that S acts on T on the left by endomorphisms. Moreover, if $S = \langle X | P \rangle$, then we assume for the sake of convenience that $X \subseteq S$. We let λ denote the empty word.

Let X and Y be non-empty sets and let $Z = X^*$. Consider the set

$$^{Z}Y = \{^{\alpha}y \mid \alpha \in Z,\ y \in Y\}.$$

In other words, the elements of ^{Z}Y consist of symbols of the form $^{x_1 \dots x_n}y$, where $y \in Y,\ x_i \in X \cup \{\lambda\}$. Consider the map:

$$f : (^{Z}Y)^{+} \times Z \to (^{Z}Y)^{+}$$

$$(^{\beta}y, \alpha) \mapsto {}^{\alpha\beta}y,\ (y, \lambda) \mapsto y,\ (y_1 \dots y_n, \alpha) \mapsto {}^{\alpha}y_1 \dots {}^{\alpha}y_n, \tag{4.4}$$

where $\beta \in Z \setminus \{\lambda\}, \alpha \in Z$ and $y_1, \dots, y_n \in {}^{Z}Y$. Clearly, f satisfies $(A1)$ and $(A2)$ and so Z acts on $(^{Z}Y)^{+}$. As before, we write the image of $(y_1 \dots y_n, \alpha)$ under f as $^{\alpha}(y_1 \dots y_n)$. That is,

$$^{\alpha}(y_1 \dots y_n) \equiv {}^{\alpha}y_1 \dots {}^{\alpha}y_n.$$

In particular we obtain $Y \subseteq {}^{Z}Y$, since for all $y \in Y$, $^{\lambda}y \equiv y$. Throughout this section, we consider $(^{Z}Y)^{+}$ as a semigroup on which the semigroup Z acts, where the action is determined by the map f, defined in (4.4).

Definition 4.4.1. Let $\langle X|P\rangle$ be a semigroup presentation, $Z = X^*$ and Y be a non-empty set. A *semigroup presentation with respect to the action of* $\langle X|P\rangle$ is an ordered pair $\langle Y|Q\rangle$, where Q is a binary relation on $(^{Z}Y)^{+}$. Let

$$\begin{aligned} Q_A = {} & \{^{\alpha u \beta}p = {}^{\alpha v \beta}q \mid (p = q) \in Q,\ (u = v) \in P,\ \alpha, \beta \in Z\} \\ & \cup \{^{\alpha u \beta}w = {}^{\alpha v \beta}w \mid w \in (^{Z}Y)^{+},\ (u = v) \in P,\ \alpha, \beta \in Z\} \\ & \cup \{^{\alpha}p = {}^{\alpha}q \mid (p = q) \in Q,\ \alpha \in Z\}. \end{aligned}$$

We note that $Q \subseteq Q_A$. The semigroup defined by the semigroup presentation $\langle {}^{Z}Y|Q_A\rangle$ is said to be *presented with respect to the action of* $\langle X|P\rangle$ by the generators Y and relations Q and we denote this by

$$T = \mathrm{Act}_{\langle X|P\rangle}\langle Y|Q\rangle.$$

If Y and Q can be chosen to be finite sets, then we say that T is finitely presented with respect to the action of $\langle X|P\rangle$. If we fix a presentation $\langle X|P\rangle$ for a semigroup S, then we write

$$T = \mathrm{Act}_{S}\langle Y|Q\rangle$$

instead of $T = \mathrm{Act}_{\langle X|P\rangle}\langle Y|Q\rangle$, and say that T is *presented with respect to an action of* S by Y and Q.

Let $T = \mathrm{Act}_{\langle X|P\rangle}\langle Y|Q\rangle$, $Z = X^*$ and let $w_1, w_2 \in ({}^Z Y)^+$. We say that w_2 is obtained from w_1 by an *application of a relation of* Q_A, if $w_1 \equiv \alpha u \beta$ and $w_2 \equiv \alpha v \beta$, where $\alpha, \beta \in ({}^Z Y)^*$ and $u = v$ is a relation of Q_A. We say that $w_1 = w_2$ is a consequence of relations in Q_A, if there exists a sequence of words

$$w_1 \equiv \alpha_0, \ldots, \alpha_m \equiv w_2$$

such that α_{j+1} is obtained from α_j by applying a relation of Q_A.

According to Proposition 4.1.1, we obtain:

Proposition 4.4.2. *Let $\langle X|P\rangle$ be a semigroup presentation and Y be a non-empty set. Let $Z = X^*$ and Q be a binary relation on $({}^Z Y)^+$. Then $T = \mathrm{Act}_{\langle X|P\rangle}\langle Y|Q\rangle$ if and only if the following two conditions hold:*

(i) T satisfies all relations in Q_A;

(ii) If $w_1, w_2 \in ({}^Z Y)^+$ are such that $w_1 = w_2$ holds in T, then $w_1 = w_2$ is a consequence of relations in Q_A.

Every semigroup can be defined in terms of a semigroup presentation. The question naturally arises whether a semigroup T on which a semigroup S acts can be defined in terms of a presentation with respect to the action of S. We give an answer to this problem in the next proposition.

Proposition 4.4.3. *Let S and T be semigroups and assume that S acts on T. Then T can be defined in terms of a presentation with respect to the action of S.*

Proof. Fix a presentation $\langle X|P\rangle$ for S. Let $Z = X^*$ and let T be given by the presentation $\langle Y|R\rangle$. Since S acts on T, there exists a function

$$f : T \times S \to T;\ (t, s) \mapsto {}^s t$$

satisfying the following two conditions:

$$(A1) \quad {}^{(s_1 s_2)}t = {}^{s_1}({}^{s_2}t); \qquad \text{and} \qquad (A2) \quad {}^s(t_1 t_2) = {}^s t_1 {}^s t_2$$

for all $t, t_1, t_2 \in T$ and $s, s_1, s_2 \in S$. For all $y \in Y$ and $\alpha \in Z$, we fix a word $v_{\alpha,y}$ over Y, so that

$${}^\alpha y = v_{\alpha,y}$$

holds. We note that because of conditions $(A1)$ and $(A2)$ and since f is well-defined, the following hold: On one hand, if $\alpha, \beta \in Z$ are such that $\alpha = \beta$ holds in S, then for all $y \in Y$

$$v_{\alpha,y} = v_{\beta,y} \tag{4.5}$$

holds in T. More generally, if $w \equiv y_1 \dots y_n$ is a word over Y, then

$$v_{\alpha,y_1} \dots v_{\alpha,y_n} = v_{\beta,y_1} \dots v_{\beta,y_n}. \tag{4.6}$$

On the other hand, for all $y \in Y$ and $\delta \equiv \alpha\beta$, where $\alpha, \beta \in Z$, if $v_{\beta,y} \equiv y_1 \dots y_n$, then

$$v_{\delta,y} = v_{\alpha,y_1} \dots v_{\alpha,y_n} \tag{4.7}$$

holds in T. Let

$$Q = R \cup \{{}^{\alpha}y = v_{\alpha,y} \mid \alpha \in Z,\ y \in Y\}.$$

Consider the semigroup presentation $\langle {}^{Z}Y | Q_A \rangle$, where

$$\begin{aligned} Q_A = \{&{}^{\alpha u \beta}p = {}^{\alpha v \beta}q \mid (p = q) \in Q, (u = v) \in P, \alpha, \beta \in Z\} \\ \cup \{&{}^{\alpha u \beta}w = {}^{\alpha v \beta}w \mid w \in ({}^{Z}Y)^+, (u = v) \in P, \alpha, \beta \in Z\} \\ \cup \{&{}^{\alpha}p = {}^{\alpha}q \mid (p = q) \in Q, \alpha \in Z\}. \end{aligned}$$

We verify that $T = \mathrm{Act}_S\langle Y | Q \rangle$, by showing that the semigroup presentation $\langle {}^{Z}Y | Q_A \rangle$ can be obtained from the semigroup presentation $\langle Y | R \rangle$ by applying elementary Tietze transformations.

It is immediate that applying Tietze transformations of type $(T1)$, the semigroup presentation $\langle {}^{Z}Y | Q \rangle$ is obtained from $\langle Y | R \rangle$. We verify that $\langle {}^{Z}Y | Q_A \rangle$ can be obtained from $\langle {}^{Z}Y | Q \rangle$ by applying Tietze transformation of type $(T3)$. In other words, we show that every relation in $Q_A \setminus Q$ is a consequence of relations in Q.

First, consider a relation in R:

$$a_1 \dots a_n = b_1 \dots b_m.$$

Let $\alpha, \beta \in Z$ and $(u = v) \in P$. Let

$$\delta_1 \equiv \alpha u \beta \qquad \text{and} \qquad \delta_2 \equiv \alpha v \beta.$$

It is immediate that the relation $\delta_1 = \delta_2$ holds in S. Bearing in mind that f is a well-defined map and that $(A2)$ holds, we obtain that

$$v_{\delta_2,a_1} \dots v_{\delta_2,a_n} = v_{\delta_2,b_1} \dots v_{\delta_2,b_m} \tag{4.8}$$

holds in T. In particular, we have that

$$\begin{aligned}
{}^{\delta_1}(a_1 \dots a_n) &\equiv {}^{\delta_1}a_1 \dots {}^{\delta_1}a_n \\
&= v_{\delta_1,a_1} \dots v_{\delta_1,a_n} && \text{applying relations in } Q \\
&= v_{\delta_2,a_1} \dots v_{\delta_2,a_n} && \text{by (4.5)} \\
&= v_{\delta_2,b_1} \dots v_{\delta_2,b_m} && \text{by (4.8)} \\
&= {}^{\delta_2}b_1 \dots {}^{\delta_2}b_m && \text{applying relations in } Q \\
&\equiv {}^{\delta_2}(b_1 \dots b_m).
\end{aligned}$$

It can be similarly proved that for all $\alpha \in Z$,

$$^{\alpha}(a_1 \dots a_n) = {}^{\alpha}(b_1 \dots b_m)$$

is a consequence of relations in Q.

Next, let $w \in ({}^{Z}Y)^{+}$ and let $\delta_1, \delta_2 \in Z$ as defined before. Assume that $w \equiv {}^{\alpha_1}y_1 \dots {}^{\alpha_n}y_n$, where $\alpha_i \in Z$ and ${}^{\alpha_i}y_i = y_i$, if α_i is the empty word. Let $\mu_i \equiv \delta_1\alpha_i$ and $\nu_i \equiv \delta_2\alpha_i$, $(1 \leq i \leq n)$. Since $\delta_1 = \delta_2$ holds in S, we also have that $\mu_i = \nu_i$ holds for all $1 \leq i \leq n$. Making use of (4.5), we obtain that $v_{\mu_i,y_i} = v_{\nu_i,y_i}$ holds in T for all $1 \leq i \leq n$. Hence, we have that

$$\begin{aligned}
{}^{\delta_1}w &\equiv {}^{\delta_1\alpha_1}y_1 \dots {}^{\delta_1\alpha_n}y_n \\
&\equiv {}^{\mu_1}y_1 \dots {}^{\mu_n}y_n \\
&= v_{\mu_1,y_1} \dots v_{\mu_n,y_n} && \text{applying relations in } Q \\
&= v_{\nu_1,y_1} \dots v_{\nu_n,y_n} && \text{by (4.5)} \\
&= {}^{\nu_1}y_1 \dots {}^{\nu_n}y_n && \text{applying relations in } Q \\
&= {}^{\delta_2\alpha_1}y_1 \dots {}^{\delta_2\alpha_n}y_n \\
&\equiv {}^{\delta_2}w.
\end{aligned}$$

Finally, consider a relation of the form ${}^{\gamma}y = v_{\gamma,y}$ in Q. Let δ_1 and δ_2 be the words over X as defined before. Let $\mu \equiv \delta_1\gamma$ and $\nu \equiv \delta_2\gamma$. Since $\delta_1 = \delta_2$ holds in S, we have that $\mu = \nu$ also holds in S. Assume that $v_{\gamma,y} \equiv y_1 \dots y_n$. Then,

$$\begin{aligned}
{}^{\mu}y &= v_{\mu,y} && \text{applying a relation in } Q \\
&= v_{\nu,y} && \text{by (4.5)} \\
&= v_{\delta_2,y_1} \dots v_{\delta_2,y_n} && \text{by (4.7)} \\
&= {}^{\delta_2}y_1 \dots {}^{\delta_2}y_n && \text{applying relations in } Q \\
&\equiv {}^{\delta_2}(y_1 \dots y_n) \equiv {}^{\delta_2}v_{\gamma,y}.
\end{aligned}$$

It can be similarly proved that for all $\alpha \in Z$,

$$^{\alpha}({}^{\gamma}y) = {}^{\alpha}v_{\gamma,y}$$

is a consequence of relations in Q. □

From the proof of Proposition 4.4.3 we obtain the following useful corollary:

Corollary 4.4.4. *Let $S = \langle X|P\rangle$, $Z = X^*$ and $T = \langle Y|R\rangle$. Assume that S acts on T. For each $\alpha \in Z$ and $y \in Y$, let $v_{\alpha,y}$ be a word over Y representing ${}^{\alpha}y$. Let*

$$Q = R \cup \{{}^{\alpha}y = v_{\alpha,y} \mid \alpha \in Z,\ y \in Y\}$$

and

$$\begin{aligned} Q_A = &\{{}^{\alpha u\beta}p = {}^{\alpha v\beta}q \mid (p = q) \in Q,\ (u = v) \in P,\ \alpha, \beta \in Z\} \\ &\cup \{{}^{\alpha u\beta}w = {}^{\alpha v\beta}w \mid w \in ({}^{Z}Y)^{+},\ (u = v) \in P,\ \alpha, \beta \in Z\} \\ &\cup \{{}^{\alpha}p = {}^{\alpha}q \mid (p = q) \in Q,\ \alpha \in Z\}. \end{aligned}$$

Then the following hold:

(1) Every relation in $Q_A \setminus Q$ is a consequence of relations in Q.

(2) $T = \mathrm{Act}_S\langle Y|Q\rangle = \langle {}^{Z}Y|Q_A\rangle = \langle {}^{Z}Y|Q\rangle$.

Proposition 4.4.5. *Let T be a finitely presented semigroup and assume that a finitely generated semigroup S acts on T. Then T is also finitely presented with respect to the action of S.*

Proof. Fix a semigroup presentation $\langle X|P\rangle$ for S, where X is a finite set and let $Z = X^*$. Let T be defined by the semigroup presentation $\langle Y|R\rangle$, where Y is a finite set and R is a finite set of relations on Y^+. For all $y \in Y$ and $\alpha \in Z$ we fix a word $v_{\alpha,y}$ over Y so that

$${}^{\alpha}y = v_{\alpha,y}.$$

Let

$$Q = R \cup \{{}^{\alpha}y = v_{\alpha,y} \mid \alpha \in Z,\ y \in Y\},$$

and consider the semigroup presentation $\langle {}^{Z}Y|Q_A\rangle$, where

$$\begin{aligned} Q_A = &\{{}^{\alpha u\beta}p = {}^{\alpha v\beta}q \mid (p=q) \in Q,\ (u=v) \in P,\ \alpha,\beta \in Z\} \\ &\cup \{{}^{\alpha u\beta}w = {}^{\alpha v\beta}w \mid w \in ({}^{Z}Y)^{+},\ (u=v) \in P,\ \alpha,\beta \in Z\} \\ &\cup \{{}^{\alpha}p = {}^{\alpha}q \mid (p=q) \in Q, \alpha \in Z\}. \end{aligned}$$

Making use of Corollary 4.4.4, we have that $T = \langle {}^{Z}Y|Q_A\rangle = \langle {}^{Z}Y|Q\rangle$. Let

$$\widetilde{Q} = R \cup \{{}^{x}y = v_{x,y} \mid x \in X,\ y \in Y\}.$$

Since R is a finite set of relations and since X and Y are finite sets, $\widetilde{Q}$ is a finite set of relations. Let

$$\begin{aligned} \widetilde{Q}_A = &\{{}^{\alpha u\beta}p = {}^{\alpha v\beta}q \mid (p=q) \in \widetilde{Q},\ (u=v) \in P,\ \alpha,\beta \in Z\} \\ &\cup \{{}^{\alpha u\beta}w = {}^{\alpha v\beta}w \mid w \in ({}^{Z}Y)^{+},\ (u=v) \in P,\ \alpha,\beta \in Z\} \\ &\cup \{{}^{\alpha}p = {}^{\alpha}q \mid (p=q) \in \widetilde{Q},\ \alpha \in Z\}. \end{aligned}$$

We claim that T is finitely presented with respect to the action of S, by verifying that $\langle {}^{Z}Y|\widetilde{Q}_A\rangle$ can be obtained from $\langle {}^{Z}Y|Q\rangle$ by applying elementary Tietze transformations. Clearly $\widetilde{Q}_A \subseteq Q_A$. By Corollary 4.4.4, the elements of $Q_A \setminus Q$ and hence of $\widetilde{Q}_A \setminus \widetilde{Q}$ are consequences of relations in Q, and so we obtain

$$T = \langle {}^{Z}Y|Q \cup \widetilde{Q}_A\rangle.$$

Finally, we claim by induction on the length of words $\alpha \in Z$, that all relations of the form ${}^{\alpha}y = v_{\alpha,y}$, where $|\alpha| \geq 2$ are consequences of relations in $\widetilde{Q}_A$.

We recall that by (4.7), for all $y \in Y$ and $\delta \equiv \alpha\beta$, where $\alpha, \beta \in Z$, if $v_{\beta,y} \equiv y_1 \dots y_n$, then

$$v_{\delta,y} = v_{\alpha,y_1} \dots v_{\alpha,y_n}. \tag{4.9}$$

Let $\alpha \equiv x_1x_2$. Then ${}^{x_2}y = v_{x_2,y}$ is a relation in $\widetilde{Q}$, and so ${}^{x_1x_2}y = {}^{x_1}v_{x_2,y}$ is a relation in $\widetilde{Q}_A$. Assume that $v_{x_2,y} \equiv y_1 \dots y_n$. Then we obtain

$$\begin{aligned} {}^{x_1x_2}y &= {}^{x_1}v_{x_2,y} && \text{applying a relation in } \widetilde{Q}_A \\ &\equiv {}^{x_1}y_1 \dots {}^{x_1}y_n && \\ &= v_{x_1,y_1} \dots v_{x_1,y_n} && \text{applying relations in } \widetilde{Q} \\ &= v_{x_1x_2,y} && \text{by (4.9).} \end{aligned}$$

Next, assume that for all words α over X with length less then m, where $2 \leq m$, the relation ${}^{\alpha}y = v_{\alpha,y}$ is a consequence of relations in $\widetilde{Q}_A$ and let $\alpha \in Z$ such that $|\alpha| = m$.

Then $\alpha = \beta x$ for some $x \in X$ and $\beta \in Z$, where $|\beta| = m-1$. Since ${}^{x}y = v_{x,y}$ is a relation in $\widetilde{Q}$ we have that ${}^{\beta x}y = {}^{\beta}v_{x,y}$ is an element of $\widetilde{Q}_A$. Assume that $v_{x,y} \equiv y_1 \dots y_n$. Then, we obtain

$$\begin{aligned}
{}^{\alpha}y \equiv {}^{\beta x}y = {}^{\beta}v_{x,y} & & \text{applying a relation in } \widetilde{Q}_A \\
\equiv {}^{\beta}(y_1 \dots y_n) & & \\
\equiv {}^{\beta}y_1 \dots {}^{\beta}y_n & & \\
= v_{\beta,y_1} \dots v_{\beta,y_n} & & \text{by the inductive hypothesis} \\
= v_{\alpha,y} & & \text{by (4.9).}
\end{aligned}$$

Hence, applying Tietze transformations of type $(T4)$, we obtain

$$T = \langle {}^{Z}Y | \widetilde{Q}_A \rangle = \mathrm{Act}_S\langle Y | \widetilde{Q} \rangle,$$

proving that T is indeed finitely presented with respect to the action of S. $\square$

We introduce a sharper version of Corollary 4.4.4.

Corollary 4.4.6. *Let $S = \langle X|P \rangle$, $Z = X^*$ and $T = \langle Y|R \rangle$. Assume that S acts on T. For each $\alpha \in Z$ and $y \in Y$, let $v_{\alpha,y}$ be a word over Y representing ${}^{\alpha}y$. Let*

$$\widetilde{Q} = R \cup \{ {}^{\alpha}y = v_{\alpha,y} \mid \alpha \in Z \setminus X,\ y \in Y \}$$

and

$$Q = R \cup \{ {}^{x}y = v_{x,y} \mid x \in X,\ y \in Y \}.$$

Let

$$\begin{aligned}
Q_A = & \{ {}^{\alpha u \beta}p = {}^{\alpha v \beta}q \mid (p=q) \in Q,\ (u=v) \in P,\ \alpha,\beta \in Z \} \\
& \cup \{ {}^{\alpha u \beta}w = {}^{\alpha v \beta}w \mid w \in ({}^{Z}Y)^+,\ (u=v) \in P,\ \alpha,\beta \in Z \} \\
& \cup \{ {}^{\alpha}p = {}^{\alpha}q \mid (p=q) \in Q,\ \alpha \in Z \}.
\end{aligned}$$

Then the following hold:

(1) Every relation in $\widetilde{Q} \setminus Q$ is a consequence of relations in Q_A.

(2) Every relation in $Q_A \setminus Q$ is a consequence of relations in Q.

(3) $T = \mathrm{Act}_S\langle Y|Q \rangle = \mathrm{Act}_S\langle Y|\widetilde{Q} \rangle$.

Finally, we prove:

Proposition 4.4.7. *Let $S = \langle X|P\rangle$, where X is a finite set and let $Z = X^*$. Let T be a semigroup on which S acts. Assume that the finite sets Y_1 and Y_2 generate the semigroup T with respect to the action of S and that $T = \mathrm{Act}_S\langle Y_1|Q\rangle$, where Q is a finite set of relations on $({}^ZY_1)^+$. Then there exists a finite set of relations $\widetilde{Q}$ on $({}^ZY_2)^+$ such that $T = \mathrm{Act}_S\langle Y_2|\widetilde{Q}\rangle$.*

Proof. Since Y_2 generates T with respect to the action of S, for each $y \in Y_1$ there exists a word v_y over ${}^ZY_2$ such that y and v_y represent the same element of T. In particular, for all $\alpha \in Z$, we have that ${}^\alpha y$ and ${}^\alpha v_y$ represent the same element of T. The map

$$y \mapsto v_y,\ {}^\alpha y \mapsto {}^\alpha v_y$$

can be extended to a homomorphism

$$\eta : ({}^ZY_1)^+ \to ({}^ZY_2)^+.$$

We note that for each $w \in ({}^ZY_1)^+$, w and $w\eta$ represent the same element of T. Consider the word $w \equiv {}^{\alpha_1}y_1 \dots {}^{\alpha_n}y_n$, where $\alpha_i \in Z$ and $y_i \in Y_1$ for all $1 \leq i \leq n$. Let $\alpha \in Z$ and let $\delta_i \equiv \alpha\alpha_i$, where $1 \leq i \leq n$. Then

$$\begin{aligned}({}^\alpha w)\eta &\equiv ({}^{\delta_1}y_1 \dots {}^{\delta_n}y_n)\eta \\ &= {}^{\delta_1}v_{y_1} \dots {}^{\delta_n}v_{y_n} \\ &\equiv {}^\alpha({}^{\alpha_1}v_{y_1} \dots {}^{\alpha_n}v_{y_n}) \\ &= {}^\alpha(w\eta),\end{aligned}$$

and so for all $w \in ({}^ZY_1)^+$ and $\alpha \in Z$,

$$({}^\alpha w)\eta \equiv {}^\alpha(w\eta) \tag{4.10}$$

holds. Similarly, since Y_1 generates T with respect to the action of S, for each $y \in Y_2$ there exists a word $\tilde{v}_y \in ({}^ZY_1)^+$ such that y and $\tilde{v}_y$ represent the same element of T. In particular, for all $\alpha \in Z$, we have that ${}^\alpha y$ and ${}^\alpha\tilde{v}_y$ represent the same element of T. The map

$$y \mapsto \tilde{v}_y,\ {}^\alpha y \mapsto {}^\alpha\tilde{v}_y$$

can be extended to a homomorphism

$$\xi : ({}^ZY_2)^+ \to ({}^ZY_1)^+,$$

and for each $w \in ({}^{Z}Y_2)^+$, w and $w\xi$ represent the same element of T. Furthermore, for all $w \in ({}^{Z}Y_2)^+$ and $\alpha \in Z$,

$$({}^{\alpha}w)\xi \equiv {}^{\alpha}(w\xi). \tag{4.11}$$

Since $T = \mathrm{Act}_S\langle Y_1|Q\rangle$, we have that $T = \langle {}^{Z}Y_1|Q_A\rangle$, where

$$\begin{aligned} Q_A = &\{{}^{\alpha u\beta}p = {}^{\alpha v\beta}q \mid (p=q) \in Q,\ (u=v) \in P,\ \alpha,\beta \in Z\} \\ &\cup \{{}^{\alpha u\beta}w = {}^{\alpha v\beta}w \mid w \in ({}^{Z}Y_1)^+,\ (u=v) \in P,\ \alpha,\beta \in Z\} \\ &\cup \{{}^{\alpha}p = {}^{\alpha}q \mid (p=q) \in Q,\ \alpha \in Z\}. \end{aligned}$$

Let

$$\widetilde{Q} = \{p\eta = q\eta \mid (p=q) \in Q\} \cup \{y = y\xi\eta \mid y \in Y_2\}$$

and let

$$\begin{aligned} \widetilde{Q}_A = &\{{}^{\alpha u\beta}p = {}^{\alpha v\beta}q \mid (p=q) \in \widetilde{Q}, (u=v) \in P, \alpha,\beta \in Z\} \\ &\cup \{{}^{\alpha u\beta}w = {}^{\alpha v\beta}w \mid w \in ({}^{Z}Y_2)^+, (u=v) \in P, \alpha,\beta \in Z\} \\ &\cup \{{}^{\alpha}p = {}^{\alpha}q \mid (p=q) \in \widetilde{Q}, \alpha \in Z\}. \end{aligned}$$

Our aim is to show that $T = \mathrm{Act}_S\langle Y_2 \mid \widetilde{Q}\rangle$. By the definition of the homomorphisms η and ξ and by (4.10), we have that T satisfies all relations in $\widetilde{Q}$ and of $\widetilde{Q}_A$ as well.

Before we proceed and show that every relation $w_1 = w_2$, $(w_1, w_2 \in ({}^{Z}Y_2)^+)$ that holds in T is a consequence of relations in $\widetilde{Q}_A$, we verify that for all $y \in Y_2$ and $\alpha \in Z$

$${}^{\alpha}y = ({}^{\alpha}y)\xi\eta$$

holds in T. Keeping (4.10) and (4.11) in mind, we obtain for all $y \in Y_2$ and $\alpha \in Z$, that

$$({}^{\alpha}y)\xi\eta \equiv (({}^{\alpha}y)\xi)\eta \equiv ({}^{\alpha}(y\xi))\eta \equiv {}^{\alpha}(y\xi\eta) \tag{4.12}$$

holds. Let $y \in Y_2$ and $\alpha \in Z$. Since $y = y\xi\eta$ is a relation in $\widetilde{Q}$, we have that ${}^{\alpha}y = {}^{\alpha}(y\xi\eta)$ is a relation in $\widetilde{Q}_A$. Having (4.12) in mind we have that ${}^{\alpha}(y\xi\eta) \equiv ({}^{\alpha}y)\xi\eta$ and hence we obtain that

$${}^{\alpha}y = ({}^{\alpha}y)\xi\eta \tag{4.13}$$

holds in T.

Let $w_1, w_2 \in ({}^{Z}Y_2)^+$ such that w_1 and w_2 represent the same element of T. We show that $w_1 = w_2$ is a consequence of relations in $\widetilde{Q}_A$. Since $w_1 = w_2$, we have that $w_1\xi = w_2\xi$ holds in T, and hence there exists a finite sequence of words over ${}^{Z}Y_1$:

$$w_1\xi \equiv u_1, u_2, \ldots, u_m \equiv w_2\xi$$

such that u_{i+1} is obtained from $u_i, (1 \leq i \leq m-1)$ by applying a relation in Q_A. Consider now the following sequence of words over ${}^{Z}Y_2$:

$$w_1\xi\eta \equiv u_1\eta, u_2\eta, \dots, u_m\eta \equiv w_2\xi\eta.$$

If u_{i+1} is obtained from $u_i, (1 \leq i \leq m-1)$ by applying a relation in Q, then $u_{i+1}\eta$ is obtained from $u_i\eta$ by applying a relation in $\widetilde{Q}$. Assume that u_{i+1} is obtained from $u_i, (1 \leq i \leq m-1)$ by applying a relation in $Q_A \setminus Q$. Then the following three cases can occur:

(a) u_{i+1} is obtained from u_i by applying a relation of the form

$${}^{\alpha u\beta}p = {}^{\alpha v\beta}q,$$

where $\alpha, \beta \in Z,\ (p = q) \in Q$ and $(u = v) \in P$. Then $u_{i+1}\eta$ is obtained from $u_i\eta$ by substituting the subword $({}^{\alpha u\beta}p)\eta$ of $u_i\eta$ by $({}^{\alpha v\beta}q)\eta$. Keeping in mind (4.10), we have that

$$({}^{\alpha u\beta}p)\eta \equiv {}^{\alpha u\beta}(p\eta)$$

and that

$$({}^{\alpha v\beta}q)\eta \equiv {}^{\alpha v\beta}(q\eta),$$

and so we may deduce that $u_{i+1}\eta$ is obtained from $u_i\eta$ by applying a relation in $\widetilde{Q}_A$.

(b) u_{i+1} is obtained from u_i by applying a relation of the form

$${}^{\alpha u\beta}w = {}^{\alpha v\beta}w,$$

where $\alpha, \beta \in Z,\ w \in ({}^{Z}Y_1)^+$ and $(u = v) \in P$. Then $u_{i+1}\eta$ is obtained from $u_i\eta$ by substituting the subword $({}^{\alpha u\beta}w)\eta$ of $u_i\eta$ by $({}^{\alpha v\beta}w)\eta$. By (4.10),

$$({}^{\alpha u\beta}w)\eta \equiv {}^{\alpha u\beta}(w\eta)$$

and

$$({}^{\alpha v\beta}w)\eta \equiv {}^{\alpha v\beta}(w\eta),$$

and hence we may deduce that $u_{i+1}\eta$ is obtained from $u_i\eta$ by applying a relation in $\widetilde{Q}_A$.

(c) u_{i+1} is obtained from u_i by applying a relation of the form

$${}^{\alpha}p = {}^{\alpha}q,$$

where $\alpha \in Z$, $(p = q) \in Q$. Then $u_{i+1}\eta$ is obtained from $u_i\eta$ by substituting the subword $({}^{\alpha}p)\eta$ of $u_i\eta$ by $({}^{\alpha}q)\eta$. Since

$$({}^{\alpha}p)\eta \equiv {}^{\alpha}(p\eta) \quad \text{and} \quad ({}^{\alpha}q)\eta \equiv {}^{\alpha}(q\eta),$$

we have that $u_{i+1}\eta$ is obtained from $u_i\eta$ by applying a relation in $\widetilde{Q}_A$.

To finish our proof, we need to verify that $w_1 = w_1\xi\eta$ and $w_2\xi\eta = w_2$ is a consequence of relations in $\widetilde{Q}_A$. Assume that $w_1 \equiv {}^{\alpha_1}y_1 \dots {}^{\alpha_n}y_n$, where $\alpha_i \in Z$ and $y_i \in Y_2$ for all $1 \leq i \leq n$. Keeping in mind (4.13) and that ξ and η are homomorphisms we obtain

$$\begin{aligned} w_1 &\equiv {}^{\alpha_1}y_1 \dots {}^{\alpha_n}y_n \\ &= ({}^{\alpha_1}y_1)\xi\eta \dots ({}^{\alpha_n}y_n)\xi\eta \\ &= ({}^{\alpha_1}y_1 \dots {}^{\alpha_n}y_n)\xi\eta \\ &\equiv w_1\xi\eta. \end{aligned}$$

Similarly $w_2 = w_2\xi\eta$ holds, and so $w_1 = w_2$ is a consequence of relations in $\widetilde{Q}_A$. □

4.5 Examples I.

The aim of this section is to illustrate through a sequence of examples the concept of finite presentability with respect to a semigroup action. By Proposition 4.4.5, we know that if a semigroup T is finitely presented and a finitely generated semigroup S acts on it, then T is also finitely presented with respect to the action of S. To demonstrate that T does not necessarily have to be finitely presented as a semigroup in order to be finitely presented with respect to a semigroup action, we will consider infinite semilattices. The reason we choose infinite semilattices is twofold. On one hand infinite semilattices are not finitely generated and hence they are not finitely presented. On the other hand these examples serve as a preparation for considering semidirect products. We give an example of an infinite semilattice that is finitely presented as a semigroup with respect to a semigroup action and we give two examples of semilattices that are not finitely presented as a semigroup with respect to a semigroup action.

Proposition 4.5.1. *Let Y_∞ be the double infinite chain with an identity element adjoined on top, and let G be the infinite cyclic group. Define the action of G on Y_∞ as in Proposition 4.2.4. Then Y_∞ is finitely presented as a semigroup with respect to the action of G.*

Proof. Consider the semilattice Y_∞:

$$\ldots < e_{-1} < e_0 < e_1 < \ldots < \mathbf{1}.$$

Since Y_∞ is infinite, it is not finitely generated and hence it is not finitely presented. To be more precise, if we let

$$R = \{e_i \wedge e_j = e_j \wedge e_i = e_i,\ e_i \wedge \mathbf{1} = \mathbf{1} \wedge e_i = e_i,\ \mathbf{1} \wedge \mathbf{1} = \mathbf{1} \mid i \leq j,\ i, j \in \mathbb{Z}\},$$

then we have that Y_∞ is defined by the semigroup presentation $\langle Y_\infty | R\rangle$. Let $G = \langle g \rangle$ be the infinite cyclic group. To make notation clear, we let $X = \{g, g^{-1}, 1\}$ and let $Z = X^*$. Let

$$P = \{gg^{-1} = g^{-1}g = 1,\ 1g = g,\ 1g^{-1} = g^{-1},\ 11 = 1\}.$$

Clearly G is defined by the semigroup presentation

$$G = \langle X | P\rangle.$$

Define the action of G on Y_∞ as in Proposition 4.2.4. According to Proposition 4.2.4, Y_∞ is finitely generated by $\{e_0, \mathbf{1}\}$ with respect to the action of G. It follows that Y_∞ is generated by the finite set $Y_0 = \{e_0, e_1, \mathbf{1}\}$ with respect to this action of G as well. Let

$$\begin{aligned} Q = \{&e_0 \wedge e_1 = e_1 \wedge e_0 = e_0,\ e_0 \wedge e_0 = e_0,\ \mathbf{1} \wedge e_0 = e_0 \wedge \mathbf{1} = e_0, \\ &{}^{g}e_0 = e_1,\ {}^{1}e_0 = e_0\ , {}^{1}\mathbf{1} = \mathbf{1},\ {}^{g}\mathbf{1} = \mathbf{1}\} \end{aligned}$$

and

$$\begin{aligned} Q_A = &\{{}^{\alpha u \beta}p = {}^{\alpha v \beta}q \mid (p = q) \in Q, (u = v) \in P, \alpha, \beta \in Z\} \\ &\cup \{{}^{\alpha u \beta}w = {}^{\alpha v \beta}w \mid w \in ({}^{Z}Y_0)^+, (u = v) \in P, \alpha, \beta \in Z\} \\ &\cup \{{}^{\alpha}p = {}^{\alpha}q \mid (p = q) \in Q, \alpha \in Z\}. \end{aligned}$$

Let $Y = \langle {}^{Z}Y_0 | Q_A\rangle$. We claim that $Y_\infty = \mathrm{Act}_G\langle Y_0 | Q\rangle$, by giving two homomorphism $\eta : Y_\infty \to Y$ and $\xi : Y \to Y_\infty$ that are inverse to each other.

We first show that the map $e_n \mapsto {}^{g^n}e_0,\ \mathbf{1} \mapsto \mathbf{1}$ induces a homomorphism $\eta : Y_\infty \to Y$ by showing that relations in R are mapped onto relations that hold in Y. First consider a relation of the form $e_n \wedge e_m = e_n$, where $n \leq m$. Then

$$\begin{aligned}
e_n\eta \wedge e_m\eta &= {}^{g^n}e_0 \wedge {}^{g^m}e_0 \\
&= {}^{g^n}e_0 \wedge {}^{g^n}e_1 \wedge {}^{g^m}e_0 && \text{applying a relation in } Q_A \\
&= {}^{g^n}e_0 \wedge {}^{g^{n+1}}e_0 \wedge {}^{g^m}e_0 && \text{applying a relation in } Q_A \\
&\vdots \\
&= {}^{g^n}e_0 \wedge {}^{g^{n+1}}e_0 \wedge \ldots \wedge {}^{g^{m-1}}e_0 \wedge {}^{g^m}e_0 && \text{applying a relation in } Q_A \\
&= {}^{g^n}e_0 \wedge {}^{g^{n+1}}e_0 \wedge \ldots \wedge {}^{g^{m-1}}(e_0 \wedge e_1) && \text{applying a relation in } Q_A \\
&= {}^{g^n}e_0 \wedge {}^{g^{n+1}}e_0 \wedge \ldots \wedge {}^{g^{m-1}}e_0 && \text{applying a relation in } Q_A \\
&\vdots \\
&= {}^{g^n}e_0 = e_n\eta,
\end{aligned}$$

and so $(e_n \wedge e_m)\eta = e_n\eta$ holds. It can be similarly proved that all relations of the form $e_m \wedge e_n = e_n$, where $n \leq m$ are mapped onto relations that hold in Y.

It is immediate by the definition of η that $\mathbf{1} \wedge \mathbf{1} = \mathbf{1}$ is mapped onto the relation $\mathbf{1} \wedge \mathbf{1} = \mathbf{1}$.

Before we proceed, we first claim that ${}^{g^k}\mathbf{1} = \mathbf{1}$ holds in Y for all $k \geq 0$ by induction. Clearly

$$\begin{aligned}
{}^{g^2}\mathbf{1} &= {}^{g}\mathbf{1} && \text{applying a relation in } Q_A \\
&= \mathbf{1} && \text{applying a relation in } Q.
\end{aligned}$$

Assume that ${}^{g^k}\mathbf{1} = \mathbf{1}$ holds in Y for all $k < m$. Then $\mathbf{1} = {}^{g^{m-1}}\mathbf{1}$ holds. On the other hand ${}^{g^{m-1}}({}^{g}\mathbf{1}) = {}^{g^{m-1}}\mathbf{1}$ is a relation in Q_A, and so we may deduce that ${}^{g^m}\mathbf{1} = \mathbf{1}$ holds in Y.

We also note that ${}^{g^{-1}1}\mathbf{1} = {}^{g^{-1}g}\mathbf{1}$, ${}^{g^{-1}g}\mathbf{1} = {}^{1}\mathbf{1}$, ${}^{g^{-1}1}\mathbf{1} = {}^{g^{-1}}\mathbf{1}$ are relations in Q_A, and so

$$ {}^{g^{-1}}\mathbf{1} = \mathbf{1} $$

holds in Y. Furthermore, for all $k \leq 0$, ${}^{g^k}\mathbf{1} = \mathbf{1}$ holds.

Consider a relation of the form $e_n \wedge \mathbf{1} = e_n$. Then

$$\begin{aligned}
e_n\eta \wedge \mathbf{1}\eta &= {}^{g^n}e_0 \wedge \mathbf{1} \\
&= {}^{g^n}e_0 \wedge {}^{g^n}\mathbf{1} && \text{applying relations in } Q_A \\
&= {}^{g^n}e_0 && \text{applying a relations in } Q_A \\
&= e_n\eta,
\end{aligned}$$

and hence $(e_n \wedge \mathbf{1})\eta = e_n\eta$. It can be similarly proved that all relations of the form $\mathbf{1} \wedge e_n = e_n$ are mapped onto relations that hold in Y.

Before we define a homomorphism from Y to Y_∞, we introduce the following notation. Let $\alpha \in Z$, and assume that the group reduced word obtained from α is g^n. Then we let $||\alpha|| = n$.

We verify that the map ${}^\alpha e_0 \mapsto e_{||\alpha||}$, ${}^\alpha e_1 \mapsto e_{||\alpha||+1}$, ${}^\alpha\mathbf{1} \mapsto \mathbf{1}$ induces a homomorphism $\xi : Y \to Y_\infty$, by showing that relations in Q_A are mapped onto relations that hold in Y_∞.

First, we consider relations in Q. We have the following seven cases:

(i) $e_0\xi \wedge e_1\xi = e_0 \wedge e_1 = e_0 = e_0\xi = (e_0 \wedge e_1)\xi$. Similarly, $(e_0 \wedge e_1)\xi = e_0\xi$ holds.

(ii) $e_0\xi \wedge e_0\xi = e_0 \wedge e_0 = e_0 = e_0\xi = (e_0 \wedge e_0)\xi$.

(iii) $(\mathbf{1} \wedge \mathbf{1})\xi = \mathbf{1}\xi$.

(iv) $({}^g e_0)\xi = e_1 = e_1\xi$.

(v) $({}^1 e_0)\xi = e_0 = e_0\xi$.

(vi) $({}^g\mathbf{1})\xi = \mathbf{1} = (\mathbf{1})\xi$.

(vii) $({}^1\mathbf{1})\xi = \mathbf{1} = (\mathbf{1})\xi$.

Thus, we may deduce that relations of Q are mapped onto relations that hold in Y_∞.

Let $\alpha, \beta \in Z$ and $(u = v) \in P$. Let $\delta_1 \equiv \alpha u\beta$ and $\delta_2 \equiv \alpha v\beta$. Since $(u = v) \in P$, we have that $||u|| = ||v||$, and it follows that $||\delta_1|| = ||\delta_2||$. Consider the relation $e_0 \wedge e_1 = e_0$ in Q. Then

$$({}^{\delta_1}e_0)\xi \wedge ({}^{\delta_1}e_1)\xi = e_{||\delta_1||} \wedge e_{||\delta_1||+1} = e_{||\delta_1||} = e_{||\delta_2||} = ({}^{\delta_2}e_0)\xi = ({}^{\delta_1}(e_0 \wedge e_1))\xi$$

holds. Similarly, we have that

$$({}^\alpha e_0)\xi \wedge ({}^\alpha e_1)\xi = e_{||\alpha||} \wedge e_{||\alpha||+1} = e_{||\alpha||} = ({}^\alpha e_0)\xi = ({}^\alpha(e_0 \wedge e_1))\xi$$

holds. It can be proved similarly, that for any relation $(p = q) \in Q$,

$$({}^{\delta_1}p)\xi = ({}^{\delta_2}q)\xi \qquad \text{and} \qquad ({}^\alpha p)\xi = ({}^\alpha q)\xi$$

hold. Finally, let

$$w \equiv {}^{\alpha_1}y_1 \wedge \ldots \wedge {}^{\alpha_n}y_n,$$

where $\alpha_i \in Z$ and $y_i \in Y_0$. Let

$$\mu_i \equiv \delta_1\alpha_i \qquad \text{and} \qquad \nu_i \equiv \delta_2\alpha_i,$$

where $1 \leq i \leq n$. Clearly, $||\mu_i|| = ||\nu_i||$ for all $1 \leq i \leq n$. It is straightforward that $({}^{\delta_1}w)\xi = ({}^{\delta_2}w)\xi$ holds, if $y_i = \mathbf{1}$ for all $1 \leq i \leq n$. Assume that there exists $y_i \neq \mathbf{1}$. Then, applying the appropriate relations in Q_A, we may assume that in w each $y_i = e_0$. Let $\mu_i \in Z$ be such that $||\mu_i|| \leq ||\mu_j||$ for all $1 \leq j \leq m$. Then, applying the appropriate relations in Q_A, we obtain that

$$^{\delta_1}({}^{\alpha_1}y_1 \wedge \ldots {}^{\alpha_n}y_n) = {}^{\mu_i}y_i$$

and that

$$^{\delta_2}({}^{\alpha_1}y_1 \wedge \ldots {}^{\alpha_n}y_n) = {}^{\nu_i}y_i.$$

Taking into account that

$$({}^{\mu_i}y_i)\xi = e_{||\mu_i||} = e_{||\nu_i||} = ({}^{\nu_i}y_i)\xi$$

holds, we obtain that $({}^{\delta_1}w)\xi = ({}^{\delta_2}w)\xi$ holds in Y_∞.

That η and ξ are inverse to each other follows from the following facts. On one hand we have

$$(e_n)\eta\xi = ({}^{g^n}e_0)\xi = e_n \qquad \text{and} \qquad 1\eta\xi = 1.$$

On the other hand

$$({}^{\alpha}e_0)\xi\eta = (e_{||\alpha||})\eta = {}^{g^{||\alpha||}}e_0 = {}^{\alpha}e_0,$$

$$({}^{\alpha}e_1)\xi\eta = (e_{||\alpha||+1})\eta = {}^{g^{||\alpha||+1}}e_0 = {}^{\alpha}({}^{g}e_0) = {}^{\alpha}e_1$$

and

$${}^{\alpha}\mathbf{1}\xi\eta = \mathbf{1}.$$

□

Proposition 4.5.2. *Let A_∞ be the semilattice obtained by adjoining an identity and a zero element to an infinite antichain. Let G be the infinite cyclic group. Define the action of G on A_∞ as in Proposition 4.2.5. Then A_∞ is not finitely presented as a semigroup with respect to the action of G.*

Proof. Let A be the infinite antichain

$$\ldots, e_{-1}, e_0, e_1, \ldots.$$

Adjoin an identity $\mathbf{1}$ on top and a zero $\mathbf{0}$ on bottom. Then, for each $e_i \in A$, $\mathbf{0} < e_i < \mathbf{1}$ holds. Let A_∞ denote the (semi)lattice obtained. Since A_∞ is an infinite semilattice, it is not finitely generated and hence it is not finitely presented. To be more precise, if we let

$$\begin{aligned} R = \{ & e_i \wedge e_j = e_j \wedge e_i = \mathbf{0}, e_i \wedge \mathbf{1} = \mathbf{1} \wedge e_i = e_i,\ e_i \wedge \mathbf{0} = \mathbf{0} \wedge e_i = \mathbf{0}, \\ & e_i \wedge e_i = e_i,\ \mathbf{1} \wedge \mathbf{1} = \mathbf{1}, \mathbf{0} \wedge \mathbf{0} = \mathbf{0} \mid i, j \in \mathbb{Z}, i \neq j\}, \end{aligned}$$

then we have that A_∞ is defined by the semigroup presentation $\langle A_\infty | R\rangle$. Let $G = \langle g \rangle$ be the infinite cyclic group. To make notation clear, we let $X = \{g, g^{-1}, 1\}$ and let $Z = X^*$. Let

$$P = \{gg^{-1} = g^{-1}g = 1,\ 1g = g,\ 1g^{-1} = g^{-1},\ 11 = 1\}.$$

Clearly G is defined by the semigroup presentation

$$G = \langle X | P \rangle.$$

Define the action of G on A_∞ as in Proposition 4.2.5. As we verified in Proposition 4.2.5, A_∞ is finitely generated by $Y_0 = \{e_0, \mathbf{1}\}$ with respect to the action of G.

Assume that A_∞ is finitely presented with respect to the action of G. Then, according to Proposition 4.4.7, there exists a finite set of relations Q on $({}^Z Y_0)^+$ such that $A_\infty = \mathrm{Act}_G\langle Y_0 | Q\rangle$. In other words, if we let

$$\begin{aligned} Q_A = & \{{}^{\alpha u \beta}p = {}^{\alpha v \beta}q \mid (p = q) \in Q, (u = v) \in P, \alpha, \beta \in Z\} \\ & \cup \{{}^{\alpha u \beta}w = {}^{\alpha v \beta}w \mid w \in ({}^Z Y_0)^+, (u = v) \in P, \alpha, \beta \in Z\} \\ & \cup \{{}^{\alpha}p = {}^{\alpha}q \mid (p = q) \in Q, \alpha \in Z\} \end{aligned}$$

then $A_\infty = \langle {}^Z Y_0 | Q_A \rangle$.

We make some observations. Every relation in Q is of the following form:

$${}^{\alpha_1}a_1 \wedge \ldots \wedge {}^{\alpha_n}a_n = {}^{\beta_1}b_1 \wedge \ldots \wedge {}^{\beta_m}b_m,$$

where $a_i, b_j \in Y_0$ and $\alpha_i, \beta_j \in Z$ for all $1 \leq i \leq n$, $1 \leq j \leq m$. Without loss of generality we may assume that the following finite set of relations:

$$\widetilde{Q} = \{e_0 \wedge e_0 = e_0,\ e_0 \wedge \mathbf{1} = \mathbf{1} \wedge e_0 = e_0,\ \mathbf{1} \wedge \mathbf{1} = \mathbf{1},\ {}^g\mathbf{1} = {}^1\mathbf{1} = \mathbf{1}\}$$

is contained in Q. To make notation convenient, we let

$$\begin{aligned} \widetilde{Q}_A = & \{{}^{\alpha u \beta}p = {}^{\alpha v \beta}q \mid (p = q) \in \widetilde{Q}, (u = v) \in P, \alpha, \beta \in Z\} \\ & \cup \{{}^{\alpha u \beta}w = {}^{\alpha v \beta}w \mid w \in ({}^Z Y_0)^+, (u = v) \in P, \alpha, \beta \in Z\} \\ & \cup \{{}^{\alpha}p = {}^{\alpha}q \mid (p = q) \in \widetilde{Q}, \alpha \in Z\}. \end{aligned}$$

As in the proof of the previous proposition, it can be verified that for all $k \in \mathbb{Z}$, the relation

$$^{g^k}\mathbf{1} = \mathbf{1}$$

is a consequence of relations in Q_A. Since for all $k \in \mathbb{Z}$, the relations

$$^{g^k}e_0 \wedge {}^{g^k}e_0 = {}^{g^k}e_0 \qquad \text{and} \qquad {}^{g^k}e_0 \wedge {}^{g^k}\mathbf{1} = {}^{g^k}\mathbf{1} \wedge {}^{g^k}e_0 = {}^{g^k}e_0$$

are relations in Q_A, we may assume that for all relations

$$^{\alpha_1}a_1 \wedge \ldots \wedge {}^{\alpha_n}a_n = {}^{\beta_1}b_1 \wedge \ldots \wedge {}^{\beta_m}b_m,$$

in $Q \setminus \widetilde{Q}$ the following hold:

(a) $a_i = b_j = e_0$ for all $1 \leq i \leq n,\ 1 \leq j \leq m$;

(b) there is no subword of p and of q of the form $^{\alpha}e_0 \wedge {}^{\alpha}e_0$;

(c) α_i, β_j are group reduced words for all $1 \leq i \leq n,\ 1 \leq j \leq m$.

We also note that $n = 1$ if and only if $m = 1$, in which case the relation in $Q \setminus \widetilde{Q}$ reads $^{\alpha}e_0 = {}^{\alpha}e_0$, and hence is redundant. Because of this fact, we assume that for all relations

$$^{\alpha_1}a_1 \wedge \ldots \wedge {}^{\alpha_n}a_n = {}^{\beta_1}b_1 \wedge \ldots \wedge {}^{\beta_m}b_m$$

in $Q \setminus \widetilde{Q}$,

$$n, m \geq 2$$

also holds. Let

$$I = \{\alpha \in Z \mid {}^{\alpha}e_0 \text{ occurs in one of the relations in } Q\}.$$

By our assumptions, for every $\alpha \in I$, we have $\alpha = g^k$ for some $k \in \mathbb{Z}$. Let $i, j \in \mathbb{Z}$ such that $g^i, g^j \in I$, and if $g^k \in I$, then $j \leq k \leq i$. Let $d = i - j$.

Clearly, the relation

$$^{g^{i+1}}e_0 \wedge {}^{g^{j-1}}e_0 = {}^{g^{j-1}}e_0 \wedge {}^{g^{i+1}}e_0$$

holds in A_∞. We claim that the above relation is not a consequence of the relations in Q_A. For this, assume that there exists a finite sequence of words

$$^{g^{i+1}}e_0 \wedge {}^{g^{j-1}}e_0 \equiv u_0, u_1, \ldots, u_n \equiv {}^{g^{j-1}}e_0 \wedge {}^{g^{i+1}}e_0,$$

such that u_{t+1} is obtained from u_t, $(0 \leq t \leq n-1)$ by applying a relation in Q_A.

Let us take a closer look how the word u_1 can be obtained:.

(i) It is straightforward to see that we cannot apply any of the relations in Q to the word u_0, since ${}^{g^{i+1}}e_0$ and ${}^{g^{j-1}}e_0$ do not occur in any of the relations in Q, and so to obtain a different word from u_0, we need to apply a relation in $Q_A \setminus Q$.

(ii) It is also obvious that by applying a relation in $\widetilde{Q}_A$, the order of the elements ${}^{g^{i+1}}e_0$ and ${}^{g^{j-1}}e_0$ in the word p cannot be changed.

(iii) If a relation in $Q_A \setminus \widetilde{Q}_A$ can be applied, then it is either of the form

$$^{\alpha}p = {}^{\alpha}q,$$

where $(p = q) \in Q \setminus \widetilde{Q}$, and $\alpha \in Z \setminus \{\lambda\}$, where λ denotes the empty word, or of the form

$$^{\alpha u \beta}p = {}^{\alpha v \beta}q,$$

where $(p = q) \in Q \setminus \widetilde{Q}$, and $\alpha, \beta \in Z$. Considering both cases, it follows that there exists a relation $p = q$ in $Q \setminus \widetilde{Q}$, such that $p \equiv {}^{g^l}e_0 \wedge {}^{g^k}e_0$, and a word $g^m \in Z$, such that

$$^{g^{i+1}}e_0 \wedge {}^{g^{j-1}}e_0 \equiv {}^{g^{m+l}}e_0 \wedge {}^{g^{m+k}}e_0.$$

It follows that $l - k = m + l - m - k = i + 1 - j + 1 = d + 2$, which contradicts our choice of the number d. We may now deduce that relations in $Q_A \setminus \widetilde{Q}_A$ cannot be applied to the word u_0.

To summarize the above argument, we have that only relations in $\widetilde{Q}_A \setminus \widetilde{Q}$ can be applied to the word u_0 to obtain a different word u_1, but applying relations in $\widetilde{Q}_A \setminus \widetilde{Q}$, the order of the elements ${}^{g^{i+1}}e_0$ and ${}^{g^{j-1}}e_0$ in the word p cannot be changed. Inductively, it can be seen that for all $1 \leq t \leq n-1$, u_{t+1} is obtained from u_t, $(1 \leq t \leq n-1)$ by applying a relation in $\widetilde{Q}_A \setminus \widetilde{Q}$, but doing so the order of the elements ${}^{g^{i+1}}e_0$ and ${}^{g^{j-1}}e_0$ in the word u_t cannot be changed. We may now deduce that the relation

$$^{g^{i+1}}e_0 \wedge {}^{g^{j-1}}e_0 \equiv u_0, u_1, \ldots, u_n \equiv {}^{g^{j-1}}e_0 \wedge {}^{g^{i+1}}e_0,$$

is not a consequence of relation in Q_A, and so A_∞ is not finitely presented with respect to the action of G. $\square$

Proposition 4.5.3. *Let F_∞ be the semilattice obtained by adjoining an identity element to the free semilattice generated by infinitely many elements. Let G be the infinite cyclic group. Define the action of G on F_∞ as in Proposition 4.2.6. Then F_∞ is not finitely presented as a semigroup with respect to the action of G.*

Proof. Let F be the free semilattice generated by infinitely many elements $A = \{\ldots, e_{-1}, e_0, e_1, \ldots\}$. Adjoin an identity $\mathbf{1}$ on top. Then, for each $e_i \in A$, $e_i < \mathbf{1}$ holds. Denote by F_∞ the (semi)lattice obtained. Clearly F_∞ is not finitely presented, since it is not finitely generated. To be more precise, if we let

$$R = \{e_i \wedge e_j = e_j \wedge e_i,\ e_i \wedge \mathbf{1} = \mathbf{1} \wedge e_i = e_i,\ e_i \wedge e_i = e_i,\ \mathbf{1} \wedge \mathbf{1} = \mathbf{1} \mid i, j \in \mathbb{Z}\},$$

then we have that F_∞ is defined by the semigroup presentation $\langle A, \mathbf{1} | R\rangle$. Let $G = \langle g \rangle$ be the infinite cyclic group. To make notation clear, we let $X = \{g, g^{-1}, 1\}$ and let $Z = X^*$. Let

$$P = \{gg^{-1} = g^{-1}g = 1,\ 1g = g,\ 1g^{-1} = g^{-1},\ 11 = 1\}.$$

Clearly G is defined by the semigroup presentation

$$G = \langle X | P \rangle.$$

Define the action of G on F_∞ as in Proposition 4.2.6. As we verified in Proposition 4.2.6, F_∞ is finitely generated by $Y_0 = \{e_0, \mathbf{1}\}$ with respect to the action of G.

Assume that F_∞ is finitely presented with respect to the action of G. Then, according to Proposition 4.4.7, there exists a finite set of relations Q on $({}^Z Y_0)^+$ such that $F_\infty = \mathrm{Act}_G\langle Y_0 | Q \rangle$. In other words, if we let

$$\begin{aligned} Q_A = &\{{}^{\alpha u \beta}p = {}^{\alpha v \beta}q \mid (p = q) \in Q, (u = v) \in P, \alpha, \beta \in Z\} \\ &\cup \{{}^{\alpha u \beta}w = {}^{\alpha v \beta}w \mid w \in ({}^Z Y_0)^+, (u = v) \in P, \alpha, \beta \in Z\} \\ &\cup \{{}^{\alpha}p = {}^{\alpha}q \mid (p = q) \in Q, \alpha \in Z\}, \end{aligned}$$

then $F_\infty = \langle {}^Z Y_0 | Q_A \rangle$.

As in the previous proposition, we now make some observations. Every relation in Q is of the following form:

$${}^{\alpha_1}a_1 \wedge \ldots \wedge {}^{\alpha_n}a_n = {}^{\beta_1}b_1 \wedge \ldots \wedge {}^{\beta_m}b_m,$$

where $a_i, b_j \in Y_0$ and $\alpha_i, \beta_j \in Z$ for all $1 \le i \le n,\ 1 \le j \le m$. Without loss of generality we may assume that the following finite set of relations:

$$\widetilde{Q} = \{e_0 \wedge e_0 = e_0,\ e_0 \wedge \mathbf{1} = \mathbf{1} \wedge e_0 = e_0,\ \mathbf{1} \wedge \mathbf{1} = \mathbf{1},\ {}^g\mathbf{1} = {}^1\mathbf{1} = \mathbf{1}\}$$

is contained in Q. To make notation convenient, we let

$$\begin{aligned} \widetilde{Q}_A = &\{{}^{\alpha u \beta}p = {}^{\alpha v \beta}q \mid (p = q) \in \widetilde{Q}, (u = v) \in P, \alpha, \beta \in Z\} \\ &\cup \{{}^{\alpha u \beta}w = {}^{\alpha v \beta}w \mid w \in ({}^Z Y_0)^+, (u = v) \in P, \alpha, \beta \in Z\} \\ &\cup \{{}^{\alpha}p = {}^{\alpha}q \mid (p = q) \in \widetilde{Q}, \alpha \in Z\}. \end{aligned}$$

As in the previous propositions, it can be verified that

$$^{g^k}\mathbf{1} = \mathbf{1}$$

is a consequence of relations in Q_A. Since for all $k \in \mathbb{Z}$, the relations

$$^{g^k}e_0 \wedge {}^{g^k}e_0 = {}^{g^k}e_0 \qquad \text{and} \qquad {}^{g^k}e_0 \wedge {}^{g^k}\mathbf{1} = {}^{g^k}\mathbf{1} \wedge {}^{g^k}e_0 = {}^{g^k}e_0$$

are contained in Q_A, we may assume that for all relations

$$p \equiv {}^{\alpha_1}a_1 \wedge \ldots \wedge {}^{\alpha_n}a_n = {}^{\beta_1}b_1 \wedge \ldots \wedge {}^{\beta_m}b_m \equiv q$$

in $Q \setminus \widetilde{Q}$ the following hold:

(a) $a_i = b_j = e_0$ for all $1 \leq i \leq n,\ 1 \leq j \leq m$;

(b) there is no subword of p and of q of the form ${}^{\alpha}e_0 \wedge {}^{\alpha}e_0$;

(c) α_i, β_j are group reduced words for all $1 \leq i \leq n,\ 1 \leq j \leq m$.

We also note that $n = 1$ if and only if $m = 1$, in which case the relation in $Q \setminus \widetilde{Q}$ reads ${}^{\alpha}e_0 = {}^{\alpha}e_0$, and hence is redundant. Because of this fact, we assume that for all relations

$$^{\alpha_1}a_1 \wedge \ldots \wedge {}^{\alpha_n}a_n = {}^{\beta_1}b_1 \wedge \ldots \wedge {}^{\beta_m}b_m$$

in $Q \setminus \widetilde{Q}$,

$$n, m \geq 2$$

also holds. Let

$$I = \{\alpha \in Z \mid {}^{\alpha}e_0 \text{ occurs in one of the relations in } Q\}.$$

By our assumptions, for every $\alpha \in I$, we have $\alpha = g^k$ for some $k \in \mathbb{Z}$. Let $i, j \in \mathbb{Z}$ such that $g^i, g^j \in I$, and if $g^k \in I$, then $j \leq k \leq i$. Let $d = i - j$.

Clearly, the relation

$$^{g^{i+1}}e_0 \wedge {}^{g^{j-1}}e_0 = {}^{g^{j-1}}e_0 \wedge {}^{g^{i+1}}e_0$$

holds in F_∞. We claim that the above relation is not a consequence of the relations in Q_A. For this, assume that there exists a finite sequence of words

$$^{g^{i+1}}e_0 \wedge {}^{g^{j-1}}e_0 \equiv u_0, u_1, \ldots, u_n \equiv {}^{g^{j-1}}e_0 \wedge {}^{g^{i+1}}e_0,$$

such that u_{t+1} is obtained from u_t, $(0 \leq t \leq n-1)$ by applying a relation in Q_A.

Let us take a closer look how the word u_1 can be obtained:

(i) It is straightforward, that we cannot apply any of the relations in Q to the word u_0, since ${}^{g^{i+1}}e_0$ and ${}^{g^{j-1}}e_0$ do not occur in any of the relations in Q, and so to obtain a different word from u_0, we need to apply a relation in $Q_A \setminus Q$.

(ii) It is also obvious, that by applying a relation in $\widetilde{Q}_A$, the order of the elements ${}^{g^{i+1}}e_0$ and ${}^{g^{j-1}}e_0$ in the word p cannot be changed.

(iii) If a relation in $Q_A \setminus \widetilde{Q}_A$ can be applied, then it is either of the form

$$^{\alpha}p = {}^{\alpha}q,$$

where $(p = q) \in Q \setminus \widetilde{Q}$, and $\alpha \in Z \setminus \{\lambda\}$, where λ denotes the empty word, or of the form

$$^{\alpha u\beta}p = {}^{\alpha v\beta}q,$$

where $(p = q) \in Q \setminus \widetilde{Q}$, and $\alpha, \beta \in Z$. Considering both cases, it follows that there exists a relation $p = q$ in $Q \setminus \widetilde{Q}$, such that $p = {}^{g^l}e_0 \wedge {}^{g^k}e_0$, and a word $g^m \in Z$, such that

$${}^{g^{i+1}}e_0 \wedge {}^{g^{j-1}}e_0 \equiv {}^{g^{m+l}}e_0 \wedge {}^{g^{m+k}}e_0.$$

It follows that $l - k = m + l - m - k = i + 1 - j + 1 = d + 2$, which contradicts our choice of the number d. We may now deduce that relations in $Q_A \setminus \widetilde{Q}_A$ cannot be applied to the word u_0.

To summarize the above argument, we have that only relations in $\widetilde{Q}_A \setminus \widetilde{Q}$ can be applied to the word u_0 to obtain a different word u_1, but applying relations in $\widetilde{Q}_A \setminus \widetilde{Q}$, the order of the elements ${}^{g^{i+1}}e_0$ and ${}^{g^{j-1}}e_0$ in the word p cannot be changed. Inductively, it can be seen that for all $1 \leq t \leq n-1$, u_{t+1} is obtained from u_t, $(1 \leq t \leq n-1)$ by applying a relation in $\widetilde{Q}_A \setminus \widetilde{Q}$, but doing so the order of the elements ${}^{g^{i+1}}e_0$ and ${}^{g^{j-1}}e_0$ in the word u_t cannot be changed. We may now deduce that the relation

$${}^{g^{i+1}}e_0 \wedge {}^{g^{j-1}}e_0 \equiv u_0, u_1, \ldots, u_n \equiv {}^{g^{j-1}}e_0 \wedge {}^{g^{i+1}}e_0,$$

is not a consequence of relation in Q_A, and so F_∞ is not finitely presented with respect to the action of G. □

4.6 Finite presentability with respect to an action II.

In Section 4, we introduced the concept of a semigroup presentation with respect to an action of a semigroup and discussed basic properties. In this section we introduce the concept of an inverse semigroup presentation with respect to an action of a semigroup. The new notion serves as a key point in giving a necessary and sufficient condition for finite presentability of a semidirect product of a semilattice by a group.

As in the previous sections, if we say that a semigroup S acts on a semigroup T, then it will be understood that S acts on T on the left by endomorphisms. Moreover, if $S = \langle X|P \rangle$, then we assume for the sake of convenience that $X \subseteq S$. We let λ denote the empty word.

As in Section 4, we use the following notations. Let X and Y be non-empty sets, and let $Z = X^*$. Consider the set

$$^ZY = \{^\alpha y \mid \alpha \in Z,\ y \in Y\}.$$

In other words, the elements of ^ZY consist of symbols of the form $^{x_1 \dots x_n}y$, where $y \in Y$, $x_i \in X \cup \{\lambda\}$, where λ denotes the empty word. Let $(^ZY)^{-1}$ denote the set of formal inverses of elements of ^ZY. Let

$$B = {^ZY} \cup (^ZY)^{-1}.$$

Consider the following map:

$$f : B^+ \times Z \to B^+;$$

$$(^\beta y, \alpha) \mapsto {^{\alpha\beta}y},\ ((^\beta y)^{-1}, \alpha) \mapsto (^{\alpha\beta}y)^{-1},\ (y, \lambda) \mapsto y,\ (y^{-1}, \lambda) \mapsto y^{-1}, \tag{4.14}$$

where $\beta \in Z \setminus \{\lambda\}, \alpha \in Z$ and $y \in Y$. For any word $y_1 \dots y_n$ over B, we define

$$(y_1 \dots y_n, \alpha) \mapsto {^\alpha y_1} \dots {^\alpha y_n}.$$

Clearly, f satisfies $(A1)$ and $(A2)$ and so Z acts on B^+ on the left by endomorphisms. As before, we write the image of $(\alpha, y_1 \dots y_n)$ under f as $^\alpha(y_1 \dots y_n)$. That is,

$$^\alpha(y_1 \dots y_n) \equiv {^\alpha y_1} \dots {^\alpha y_n}.$$

In particular, we have that $Y \cup Y^{-1} \subseteq B$. Throughout this section, we consider B^+ as a semigroup on which the semigroup Z acts on the left by endomorphisms, where the action is determined by the map f defined in (4.14).

Definition 4.6.1. Let $\langle X|P \rangle$ be a semigroup presentation, Y be a non-empty set and $Z = X^*$. Let $B = {^ZY} \cup (^ZY)^{-1}$, where $(^ZY)^{-1}$ is the set of formal inverses of elements of elements ^ZY. An *inverse semigroup presentation with respect to the action of* $\langle X|P \rangle$ is an ordered pair $\langle Y|Q\rangle$, where Q is a binary relation on B^+. Let $\Re$ denote the standard inverse semigroup relations on $FSI({}^ZY)$, and let

$$\begin{aligned} Q_R = &\{{}^{\alpha u\beta}p = {}^{\alpha v\beta}q \mid (p=q) \in Q \cup \Re, (u=v) \in P, \alpha, \beta \in Z\} \\ &\cup \{{}^{\alpha u\beta}w = {}^{\alpha v\beta}w \mid w \in B^+, (u=v) \in P, \alpha, \beta \in Z\} \\ &\cup \{{}^{\alpha}p = {}^{\alpha}q \mid (p=q) \in Q \cup \Re, \alpha \in Z\}. \end{aligned}$$

We note that $Q \cup \Re \subseteq Q_R$. Let $Q_A = Q_R \setminus \Re$. The semigroup defined by the inverse semigroup presentation $\mathrm{Inv}\langle {}^ZY|Q_A\rangle$ is said to be *presented as an inverse semigroup with respect to the action of* $\langle X|P\rangle$ by the generators Y and relations Q and we denote this by

$$T = \mathrm{InvAct}_{\langle X|P\rangle}\langle Y|Q\rangle.$$

If Y and Q can be chosen to be finite sets, then we say that T is finitely presented as an inverse semigroup with respect to the action of $\langle X|P\rangle$. If we fix a presentation $\langle X|P\rangle$ for a semigroup S, then we write

$$T = \mathrm{InvAct}_S\langle Y|Q\rangle$$

instead of $T = \mathrm{InvAct}_{\langle X|P\rangle}\langle Y|Q\rangle$, and say that T is *presented as an inverse semigroup with respect to an action of* S by Y and Q.

Let $T = \mathrm{InvAct}_{\langle X|P\rangle}\langle Y|Q\rangle$ and $Z = X^*$. Let $B = {}^ZY \cup ({}^ZY)^{-1}$, where $({}^ZY)^{-1}$ is the set of formal inverses of elements ${}^ZY$. Let $w_1, w_2 \in B^+$. We say that w_2 is obtained from w_1 by an *application of a relation of* Q_R if $w_1 \equiv \alpha u\beta$ and $w_2 \equiv \alpha v\beta$, where $\alpha, \beta \in B^*$ and $u = v$ is a relation in Q_R. We say that $w_1 = w_2$ is a *consequence of relations* in Q_R, if there exists a sequence of words

$$w_1 \equiv \alpha_0, \ldots, \alpha_m \equiv w_2$$

such that α_{j+1} is obtained from α_j by applying a relation of Q_R.

According to Proposition 4.1.4, we have:

Proposition 4.6.2. *Let $\langle X|P\rangle$ be a semigroup presentation and Y be a non-empty set. Let $Z = X^*$, $B = {}^ZY \cup ({}^ZY)^{-1}$ and Q be a binary relation on B^+. Then $T = \mathrm{InvAct}_{\langle X|P\rangle}\langle Y|Q\rangle$ if and only if the following two conditions hold:*

(i) T satisfies all relations in Q_R;

(ii) If $w_1, w_2 \in B^+$ are such that $w_1 = w_2$ holds in T, then $w_1 = w_2$ is a consequence of relations in Q_R.

Next, we set Propositions 4.4.3, 4.4.5 and 4.4.7 into an inverse semigroup theoretic context. Propositions 4.6.3, 4.6.4 and 4.6.5 can be verified similarly to Propositions 4.4.3, 4.4.5 and 4.4.7. To be more accurate, one needs to consider inverse semigroup presentations, Tietze transformations regarding inverse semigroups and inverse semigroup homomorphisms instead of semigroup presentations, Tietze transformations of semigroups and semigroup homomorphisms.

Proposition 4.6.3. *Let S be a semigroup and T be an inverse semigroup. Assume that S acts on T. Then T can be defined in terms of an inverse semigroup presentation with respect to the action of S.*

Proposition 4.6.4. *Let T be a finitely presented inverse semigroup and assume that S is a finitely generated semigroup that acts on T. Then T is also finitely presented as an inverse semigroup with respect to the action of S.*

Proposition 4.6.5. *Let $S = \langle X|P\rangle$ and let T be an inverse semigroup on which S acts. Assume that the finite sets Y_1 and Y_2 generate T with respect to the action of S and that T can be defined by a finite inverse semigroup presentation with respect to the action of S in terms of Y_1. Then T can be defined by a finite inverse semigroup presentation with respect to the action of S in terms of Y_2 as well.*

Finally, we prove the following useful proposition.

Proposition 4.6.6. *Let S be a semigroup defined by the semigroup presentation $\langle X|P\rangle$, and let T be an inverse semigroup. Assume that $T = \mathrm{Act}_S\langle Y|Q\rangle$. Then we have $T = \mathrm{InvAct}_S\langle Y|Q\rangle$. In particular, we have that if T is finitely presented as a semigroup with respect to an action of S, then it is also finitely presented as an inverse semigroup with respect to the same action of S.*

Proof. Let $Z = X^*$. If $T = \mathrm{Act}_S\langle Y|Q\rangle$, then T is defined by the semigroup presentation $\langle {}^ZY|Q_A\rangle$, where

$$\begin{aligned} Q_A = \{{}^{\alpha u\beta}p = {}^{\alpha v\beta}q \mid (p=q) \in Q, (u=v) \in P, \alpha, \beta \in Z\} \\ \cup \{{}^{\alpha u\beta}w = {}^{\alpha v\beta}w \mid w \in ({}^ZY)^+, (u=v) \in P, \alpha, \beta \in Z\} \\ \cup \{{}^{\alpha}p = {}^{\alpha}q \mid (p=q) \in Q, \alpha \in Z\}. \end{aligned}$$

Making use of Proposition 4.1.3, we have that $T = \text{Inv}\langle {}^{Z}Y | Q_A \rangle$. Let $\mathfrak{R}$ denote the standard inverse semigroup relations on $FSI({}^{Z}Y)$. To verify that $T = \text{InvAct}_S\langle Y|Q\rangle$, we need to show that

$$\begin{aligned}\mathfrak{R}_A = \{ {}^{\alpha u \beta}p = {}^{\alpha v \beta}q \mid (p=q) \in \mathfrak{R}, (u=v) \in P,\ \alpha, \beta \in Z\} \\ \cup \{ {}^{\alpha}p = {}^{\alpha}q \mid (p=q) \in \mathfrak{R},\ \alpha \in Z\}.\end{aligned}$$

is a consequence of relations in Q_A. Since T is an inverse semigroup, the relations in $\mathfrak{R}$ are consequences of relations in Q_A, and hence the relations in $\mathfrak{R}_A$ are also consequences of relations in Q_A. $\square$

4.7 Examples II.

Examples introduced in Section 5 involved inverse semigroups. To deepen the notion of an inverse semigroup presentation with respect to a semigroup action, we continue working on these examples. We investigate whether the semilattices Y_∞, A_∞ and F_∞ are finitely presented as inverse semigroups with respect to the group actions defined on them.

Proposition 4.7.1. *Let Y_∞ be the double infinite chain with an identity element adjoined on top, and let G be the infinite cyclic group. Define the action of G on Y_∞ as in Proposition 4.2.4. Then Y_∞ is finitely presented as an inverse semigroup with respect to the action of G.*

Proof. According to Proposition 4.5.1, Y_∞ is finitely presented as a semigroup with respect to the action of G. Making use of Proposition 4.6.6, we may deduce that Y_∞ is finitely presented as an inverse semigroup with respect to the action of G as well. $\square$

Proposition 4.7.2. *Let F_∞ be the semilattice obtained by adjoining an identity element to the free semilattice generated by infinitely many elements. Let G be the infinite cyclic group. Define the action of G on F_∞ as in Proposition 4.2.6. Then F_∞ is finitely presented as an inverse semigroup with respect to the action of G.*

Proof. Let F be the free semilattice generated by infinitely many elements $A = \{\ldots,\ e_{-1},\ e_0,\ e_1, \ldots\}$. Adjoin an identity $\mathbf{1}$ on top. Then, for each $e_i \in A_0,\ e_i < \mathbf{1}$ holds. Denote by F_∞ the semilattice obtained. Clearly F_∞ is not finitely presented as an inverse semigroup, since it is not finitely generated. To be more precise, if we let

$$R = \{e_i \wedge e_i = e_i,\ e_i \wedge \mathbf{1} = e_i,\ \mathbf{1} \wedge \mathbf{1} = \mathbf{1} \mid i \in \mathbb{Z}\},$$

then we have that F_∞ is defined by the inverse semigroup presentation $\mathrm{Inv}\langle A, \mathbf{1}|R\rangle$. Let $G = \langle g\rangle$ be the infinite cyclic group. To make notation clear, we let $X = \{g, g^{-1}, 1\}$ and let $Z = X^*$. Let

$$P = \{gg^{-1} = g^{-1}g = 1,\ 1g = g,\ 1g^{-1} = g^{-1},\ 11 = 1\}.$$

Clearly G is defined by the semigroup presentation

$$G = \langle X|P\rangle.$$

Define the action of G on F_∞ as in Proposition 4.2.6. As we verified in Proposition 4.2.6, F_∞ is finitely generated by $Y_0 = \{e_0, \mathbf{1}\}$ with respect to the action of G. Let $B = {}^ZY_0 \cup ({}^ZY_0)^{-1}$, and let

$$Q = \{e_0 \wedge e_0 = e_0,\ e_0 \wedge \mathbf{1} = e_0,\ \mathbf{1} \wedge \mathbf{1} = \mathbf{1},\ {}^g\mathbf{1} = \mathbf{1},\ {}^1e_0 = e_0,\ {}^1\mathbf{1} = \mathbf{1}\}.$$

Let

$$\begin{aligned} Q_R = &\{{}^{\alpha u\beta}p = {}^{\alpha v\beta}q \mid (p = q) \in Q \cup \Re,\ (u = v) \in P,\ \alpha, \beta \in Z\} \\ &\cup \{{}^{\alpha u\beta}w = {}^{\alpha v\beta}w \mid w \in B^+, (u = v) \in P,\ \alpha, \beta \in Z\} \\ &\cup \{{}^{\alpha}p = {}^{\alpha}q \mid (p = q) \in Q \cup \Re,\ \alpha \in Z\}, \end{aligned}$$

where $\Re$ denotes the standard inverse semigroup relations on $FSI({}^ZY_0)$. Consider $Y \doteq \mathrm{InvAct}_G\langle Y_0|Q\rangle$. We claim that $F_\infty = \mathrm{InvAct}_G\langle Y_0|Q\rangle$, by giving two inverse semigroup homomorphism $\eta : F_\infty \to Y$ and $\xi : Y \to F_\infty$ that are inverse to each other.

We first show that the map $e_n \mapsto {}^{g^n}e_0,\ \mathbf{1} \mapsto \mathbf{1}$ induces an inverse semigroup homomorphism $\eta : F_\infty \to Y$ by showing that relations in R are mapped onto relations that hold in Y. As in Proposition 4.5.3, we have that for all $k \in \mathbb{Z}$, the relation

$${}^{g^k}\mathbf{1} = \mathbf{1}$$

is a consequence of relations in Q_R.

The following three cases have to be considered.

(i) $e_i\eta \wedge e_i\eta = {}^{g^i}e_0 \wedge {}^{g^i}e_0 = {}^{g^i}e_0 = e_i\eta = (e_i \wedge e_i)\eta$.

(ii) $e_i\eta \wedge \mathbf{1}\eta = {}^{g^i}e_0 \wedge \mathbf{1} = {}^{g^i}e_0 \wedge {}^{g^i}\mathbf{1} = {}^{g^i}e_0 = e_0\eta = (e_0 \wedge \mathbf{1})\eta$.

(iii) $\mathbf{1}\eta \wedge \mathbf{1}\eta = \mathbf{1} \wedge \mathbf{1} = \mathbf{1} = \mathbf{1}\eta = (\mathbf{1} \wedge \mathbf{1})\eta$.

We may hence deduce that every relation in R is mapped onto a relation that holds in Y.

Before we define a homomorphism from Y to F_∞, we introduce the following notation. Let $\alpha \in Z$, and assume that the group reduced word obtained from α is g^n. Then we let $||\alpha|| = n$.

We verify that the map

$$^{\alpha_1}e_0 \wedge \ldots {}^{\alpha_n}e_0 \mapsto e_{||\alpha_1||} \wedge \ldots \wedge e_{||\alpha_n||}, \quad {}^{\alpha}\mathbf{1} \mapsto \mathbf{1}$$

induces a homomorphism $\xi : Y \to F_\infty$, by showing that relations in Q_R are mapped onto relations that hold in F_∞. It is immediate by the definition of ξ that relations of Q are mapped onto relations that hold in F_∞.

Let $\alpha, \beta \in Z$ and $(u = v) \in P$. Let $\delta_1 \equiv \alpha u \beta$ and $\delta_2 \equiv \alpha v \beta$. Since $(u = v) \in P$, we have that $||u|| = ||v||$, and it follows that $||\delta_1|| = ||\delta_2||$. Consider the relation $e_0 \wedge e_0 = e_0$ in Q. Then,

$$({}^{\delta_1}(e_0 \wedge e_0))\xi = e_{||\delta_1||} \wedge e_{||\delta_1||} = e_{||\delta_1||} = e_{||\delta_2||} = ({}^{\delta_2}e_0)\xi$$

holds. Moreover, we have that

$$({}^{\alpha}(e_0 \wedge e_0))\xi = e_{||\alpha||} \wedge e_{||\alpha||} = e_{||\alpha||} = ({}^{\alpha}e_0)\xi$$

holds. It can be proved similarly, that for any relations $(p = q) \in Q \cup \Re$,

$$({}^{\delta_1}p)\xi = ({}^{\delta_2}q)\xi \qquad \text{and} \qquad ({}^{\alpha}p)\xi = ({}^{\alpha}q)\xi$$

holds.

That η and ξ are inverse to each other follows from the following observations. On one hand

$$e_n\eta\xi = ({}^{g^n}e_0)\xi = e_n \qquad \text{and} \qquad \mathbf{1}\eta\xi = \mathbf{1}.$$

On the other hand

$$({}^{\alpha}e_0)\xi\eta = e_{||\alpha||}\eta = {}^{g^{||\alpha||}}e_0 = {}^{\alpha}e_0 \qquad \text{and} \qquad \mathbf{1}\xi\eta = \mathbf{1}.$$

□

Proposition 4.7.3. *Let A_∞ be the semilattice obtained by adjoining an identity and a zero element to an infinite antichain. Let G be the infinite cyclic group. Define the action of G on A_∞ as in Proposition 4.2.5. Then A_∞ is not finitely presented as an inverse semigroup with respect to the action of G.*

Proof. Let A be the infinite antichain

$$\ldots,\ e_{-1},\ e_0,\ e_1, \ldots.$$

Adjoin an identity $\mathbf{1}$ on top and a zero $\mathbf{0}$ on bottom, that is for each $e_i \in A$, $\mathbf{0} < e_i < \mathbf{1}$ holds. Let A_∞ denote the semilattice obtained. Clearly A_∞ is not finitely presented as an inverse semigroup. To be more precise, if we let

$$\begin{aligned} R = \{& e_i \wedge e_j = \mathbf{0}, e_i \wedge \mathbf{1} = e_i,\ e_i \wedge \mathbf{0} = \mathbf{0}, \\ & e_i \wedge e_i = e_i,\ \mathbf{1} \wedge \mathbf{1} = \mathbf{1}, \mathbf{0} \wedge \mathbf{0} = \mathbf{0} \mid i, j \in \mathbb{Z}, i \neq j\}, \end{aligned}$$

then we have that A_∞ is defined by the inverse semigroup presentation $\mathrm{Inv}\langle A_\infty | R\rangle$. Let $G = \langle g \rangle$ be the infinite cyclic group. To make notation clear, we let $X = \{g, g^{-1}, 1\}$ and let $Z = X^*$. Let

$$P = \{gg^{-1} = g^{-1}g = 1,\ 1g = g,\ 1g^{-1} = g^{-1},\ 11 = 1\}.$$

Clearly G is defined by the semigroup presentation

$$G = \langle X | P \rangle.$$

Define the action of G on A_∞ as in Proposition 4.2.5. As we verified in Proposition 4.2.5, A_∞ is finitely generated as a semigroup and hence as an inverse semigroup by $Y_0 = \{e_0, \mathbf{1}\}$ with respect to the action of G.

Let $B = {}^Z Y_0 \cup ({}^Z Y_0)^{-1}$ and let $\Re$ denote the standard inverse semigroup relations on $FSI({}^Z Y_0)$. Assume that A_∞ is finitely presented as an inverse semigroup with respect to the action of G. Then, according to Proposition 4.6.5, there exists a finite set of relations Q on B^+ such that $A_\infty = \mathrm{InvAct}_G\langle Y_0 | Q\rangle$. In other words, if we let

$$\begin{aligned} Q_R = &\{{}^{\alpha u \beta} p = {}^{\alpha v \beta} q \mid (p = q) \in Q \cup \Re, (u = v) \in P, \alpha, \beta \in Z\} \\ &\cup \{{}^{\alpha u \beta} w = {}^{\alpha v \beta} w \mid w \in B^+, (u = v) \in P, \alpha, \beta \in Z\} \\ &\cup \{{}^{\alpha} p = {}^{\alpha} q \mid (p = q) \in Q \cup \Re, \alpha \in Z\}. \end{aligned}$$

and $Q_A = Q_R \setminus \Re$, then $A_\infty = \mathrm{Inv}\langle {}^Z Y_0 | Q_A \rangle$. Without loss of generality we may assume that the following set of relations

$$\widetilde{Q} = \{e_0 \wedge e_0 = e_0,\ e_0 \wedge \mathbf{1} = e_0,\ \mathbf{1} \wedge \mathbf{1} = \mathbf{1},\ {}^{g}\mathbf{1} = {}^{1}\mathbf{1} = \mathbf{1}\}$$

is a subset of Q. For further use, we introduce the following notations. We let

$$\begin{aligned} \widetilde{Q}_A = &\{{}^{\alpha u \beta} p = {}^{\alpha v \beta} q \mid (p = q) \in \widetilde{Q}, (u = v) \in P, \alpha, \beta \in Z\} \\ &\cup \{{}^{\alpha} p = {}^{\alpha} q \mid (p = q) \in \widetilde{Q}, \alpha \in Z\} \end{aligned}$$

and

$$\Re_A = \{{}^{\alpha u\beta}p = {}^{\alpha v\beta}q \mid (p = q) \in \Re, (u = v) \in P, \alpha, \beta \in Z\}$$
$$\cup \{{}^{\alpha}p = {}^{\alpha}q \mid (p = q) \in \Re, \alpha \in Z\}.$$

It is immediate that for all $\alpha \in Z$, ${}^{\alpha}e_0 \wedge {}^{\alpha}e_0 = {}^{\alpha}e_0$ is a relation in $\widetilde{Q}_A$, and hence by applying relations in $\Re$ we obtain that for all $\alpha \in Z$,

$${}^{\alpha}e_0 = ({}^{\alpha}e_0)^{-1}.$$

In particular, every relation in $\Re$ is a consequence of relations in

$$\mathcal{P} = \{{}^{\alpha}a \wedge {}^{\alpha}a = {}^{\alpha}a,\ {}^{\alpha}a \wedge {}^{\beta}b = {}^{\beta}b \wedge {}^{\alpha}a \mid a, b \in Y_0,\ \alpha, \beta \in Z\}.$$

We let

$$\mathcal{P}_A = \{{}^{\alpha u\beta}p = {}^{\alpha v\beta}q \mid (p = q) \in \mathcal{P}, (u = v) \in P, \alpha, \beta \in Z\}$$
$$\cup \{{}^{\alpha}p = {}^{\alpha}q \mid (p = q) \in \mathcal{P}, \alpha \in Z\}.$$

In a similar way as claimed in Proposition 4.5.3, we have that for any $k \in \mathbb{Z}$

$${}^{g^k}\mathbf{1} = \mathbf{1}$$

is a consequence of relations in Q_R. Having these observations in mind, we assume that for all relations

$$p \equiv a_1 \wedge \ldots \wedge a_n = b_1 \wedge \ldots \wedge b_m \equiv q$$

in $Q \setminus \widetilde{Q}$, the following conditions hold:

(i) $a_j, b_i \in {}^{Z}Y_0$ for all $1 \leq j \leq n,\ 1 \leq i \leq m$;

(ii) $a_j = {}^{\alpha}e_0$ for all $1 \leq j \leq n$, where α is a group reduced word over Z;

(iii) $b_i = {}^{\alpha}e_0$ for all $1 \leq i \leq m$, where α is a group reduced word over Z;

(iv) there is no subword of p and of q of the form ${}^{\alpha}e_0 \wedge {}^{\alpha}e_0$.

With these conditions, we also have that $n = 1$ if and only if $m = 1$, in which case our relation $p = q$ reads ${}^{\alpha}e_0 = {}^{\alpha}e_0$, and hence is redundant. We thus assume, that for all relations in $Q \setminus \widetilde{Q}$:

$$p \equiv a_1 \wedge \ldots \wedge a_n = b_1 \wedge \ldots \wedge b_m \equiv q,$$

$n, m \geq 2$ also holds.

We recall that for all $\alpha \in Z$, if $\alpha = g^k$ in G, then we let $||\alpha|| = k$. We need the following notion, to verify that A_∞ is not finitely presented as an inverse semigroup with respect to the action of G. Consider a word

$$w \equiv {}^{\alpha_1}y_1 \wedge \ldots \wedge {}^{\alpha_n}y_n$$

over ${}^ZY$, and assume α_i, α_j, $(1 \leq i, j \leq n)$ are such that $||\alpha_i|| \leq ||\alpha_k|| \leq ||\alpha_j||$ for all $1 \leq k \leq n$. Define

$$d(w) = ||\alpha_j|| - ||\alpha_i||,$$

and say that $d(w)$ is the distance in w. Let

$$I = \{\alpha \in Z \mid {}^{\alpha}e_0 \text{ occurs in one of the relations in } Q\}.$$

By our assumptions, for every $\alpha \in I$, we have $\alpha = g^k$ for some $k \in \mathbb{Z}$. Let $i, j \in \mathbb{Z}$ such that $g^i, g^j \in I$, and if $g^k \in I$, then $j \leq k \leq i$. Let $d = i - j$. It is straightforward that the relation

$$p \equiv {}^{g^{i+1}}e_0 \wedge {}^{g^{j-1}}e_0 = {}^{g^{i+2}}e_0 \wedge {}^{g^{j-2}}e_0 \equiv q$$

holds in A_∞, and that

$$d(p) \neq d(q).$$

To show that $A_\infty = \mathrm{InvAct}_G\langle Y_0 \mid Q\rangle$ does not hold, we need to verify that the above relation is not a consequence of relations in $(Q_R \setminus \Re_A) \cup \mathcal{P}_A$. Assume that there exists a finite sequence of words

$$p \equiv u_0, u_1, \ldots, u_m \equiv q,$$

so that u_{t+1} is obtained from u_t applying a relation in $(Q_R \setminus \Re_A) \cup \mathcal{P}_A$.

Let us take a closer look how the word u_1 can be obtained from u_0.

(a) It is immediate that we cannot apply any of the relation in Q, since ${}^{g^{i+1}}e_0$ and ${}^{g^{j-1}}e_0$ do not occur in any of the relations in Q.

(b) Applying a relation in $\widetilde{Q}_A \setminus \widetilde{Q}$, the following words can be obtained:

(i) $u_1 \equiv {}^{\alpha}e_0 \wedge {}^{\alpha}e_0 \wedge {}^{g^{j-1}}e_0$, where $\alpha = g^{i+1}$ in G;

(ii) $u_1 \equiv {}^{g^{i+1}}e_0 \wedge {}^{\beta}e_0 \wedge {}^{\beta}e_0$, where $\beta = g^{j-1}$ in G;

(iii) $u_1 \equiv {}^{\alpha}e_0 \wedge {}^{\alpha}1 \wedge {}^{g^{j-1}}e_0$, where $\alpha = g^{i+1}$ in G;

(iv) $u_1 \equiv {}^{g^{i+1}}e_0 \wedge {}^{\beta}e_0 \wedge {}^{\beta}1$, where $\beta = g^{j-1}$ in G;

We note that with the application of relations in $\widetilde{Q}_A \setminus \widetilde{Q}$, the distance in the obtained word u_1 equals to the distance in u_0.

(c) When applying a relation in $\mathcal{P}_A$, we may have the following situations occur:

(i) $u_1 \equiv {}^{\beta}e_0 \wedge {}^{\alpha}e_0$, where $\alpha = g^{i+1}$ and $\beta = g^{j-1}$ in G;

(ii) $u_1 \equiv {}^{\alpha}e_0 \wedge {}^{\alpha}e_0 \wedge {}^{g^{j-1}}e_0$, where $\alpha = g^{i+1}$ in G;

(iii) $u_1 \equiv {}^{g^{i+1}}e_0 \wedge {}^{\beta}e_0 \wedge {}^{\beta}e_0$, where $\beta = g^{j-1}$ in G.

We note that in all three cases, $d(u_1) = d(u_0)$.

(d) With the application of a relation in $Q_R \setminus (Q \cup \widetilde{Q}_A \cup \mathcal{P}_A \cup \mathcal{R}_A)$, the we obtain a word of the form

$$u_1 \equiv {}^{g^{\alpha}}e_0 \wedge {}^{\beta}e_0,$$

where $\alpha = g^{i+1}$, $\beta = g^{j-1}$ in G. That relations in the set

$$\begin{aligned}\overline{Q}_A = \{{}^{\alpha u \beta}p = {}^{\alpha v \beta}q \mid (p = q) \in Q \setminus \widetilde{Q}, (u = v) \in P, \alpha, \beta \in Z\} \\ \cup \{{}^{\alpha}p = {}^{\alpha}q \mid (p = q) \in Q \setminus \widetilde{Q}, \alpha \in Z \setminus \{\lambda\}\}.\end{aligned}$$

cannot be applied to u_0 follows from the following remarks. Assume that a relation of $\overline{Q}_A$ can be applied to u_0. Then there exists a word $g^m \in Z$ and a relation relation $p = q$ in $Q \setminus \widetilde{Q}$, such that $p \equiv {}^{g^l}e_0 \wedge {}^{g^k}e_0$ and

$${}^{g^{i+1}}e_0 \wedge {}^{g^{j-1}}e_0 \equiv {}^{g^{m+l}}e_0 \wedge {}^{g^{m+k}}e_0.$$

It follows that $l - k = m + l - m - k = i + 1 - j + 1 = d + 2$, which contradicts our choice of the number d. We may now deduce that relations in $\overline{Q}_A$ cannot be applied to the word u_0. To highlight the key point of the fourth case, we note that applying a relation in $Q_R \setminus (Q \cup \widetilde{Q}_A \cup \Re_A \cup \mathcal{P}_A)$ results a word u_1 so that $d(u_1) = d(u_0)$.

Thus we may deduce, that u_1 is a word so that $d(u_1) = d(u_0)$. We can make the same observation, if we take a closer look how the word u_2 can be obtained from u_1. With an inductive argument we thus arrive to the conclusion, that

$$d(p) = d(u_0) = d(u_1) = \ldots = d(u_m) = d(r)$$

leading to a contradiction. We may hence deduce that A_∞ is not finitely presented as an inverse semigroup with respect to the action of G. □

4.8 Matched and conjugate words

We introduce the concepts of matched and conjugate words inspired by the proof of Proposition 4.3.1. The new notions and related notations will be frequently used in the next two sections. As in the previous sections, if we say that a semigroup S acts on a semigroup T, then it will be understood that S acts on T on the left by endomorphisms.

Matched words

We first fix the notation for this section. Let Y be a semilattice and G be a group acting on Y. Assume that the semidirect product $S = Y \rtimes G$ is generated as an inverse semigroup by the finite set A. For simplicity we suppose that $A \subseteq S$, and we let $B = A \cup A^{-1}$. Let

$$X = \{g \in G \mid (e, g) \in A \text{ for some } e \in Y\}$$

and let

$$Y_0 = \{e \in Y \mid (e, g) \in B \text{ for some } g \in X \cup X^{-1}\}.$$

By Proposition 4.3.1 we may assume that $Y_m \subseteq Y_0$, where Y_m denotes the set of maximal elements of Y. Let $Z = X \cup \{1\}$ and define the following map:

$$\mathbf{n} : Y_m \times (Z \cup Z^{-1})^+ \to B^+;\ (f_1, g_1g_2 \ldots g_k) \mapsto (f_1, g_1)(f_2, g_2) \ldots (f_k, g_k),$$

where ${}^{g_j}f_{j+1} = f_j$, $(1 \le j+1 \le k)$. By Lemma 4.2.1, $f_j \in Y_m$ for all $1 \le j \le k$. Moreover,

$$f_1 = {}^{g_1}f_2 = {}^{g_1}({}^{g_2}f_3) = \ldots = {}^{g_1}({}^{g_2}(\ldots({}^{g_{k-1}}f_k)\ldots)) = {}^{g_1\cdots g_{k-1}}f_k \tag{4.15}$$

holds, and thus the word $\mathbf{n}(f_1, g_1g_2 \ldots g_k)$ represents the element $(f_1, g_1g_2 \ldots g_k)$ in S. We say that the word $\mathbf{n}(e, g_1 \ldots g_k)$ is *matched to the word* $\mathbf{n}(f, h_1 \ldots h_m)$, if $e = {}^{g_1\cdots g_k}f$. We introduce two lemmas regarding matched words.

Lemma 4.8.1. *The word $\mathbf{n}(e, g_1 \ldots g_k)$ is matched to the word $\mathbf{n}(f, h_1 \ldots h_m)$ if and only if $\mathbf{n}(e, g_1 \ldots g_k)\mathbf{n}(f, h_1 \ldots h_m) \equiv \mathbf{n}(e, g_1 \ldots g_kh_1 \ldots h_m)$.*

Proof. ($\Rightarrow$) Assume that $\mathbf{n}(e, g_1 \ldots g_k)$ is matched to the word $\mathbf{n}(f, h_1 \ldots h_m)$. Then we know that

$$e = {}^{g_1\cdots g_k}f. \tag{4.16}$$

On one hand, we have that

$$\mathbf{n}(e, g_1 \ldots g_k)\mathbf{n}(f, h_1 \ldots h_m) \equiv (e, g_1) \ldots (e_k, g_k)(f, h_1) \ldots (f_m, h_m),$$

where ${}^{g_i}e_{i+1} = e_i$, $(1 \leq i \leq k-1,\ e_1 = e)$ and ${}^{h_j}f_{j+1} = f_j$, $(1 \leq j \leq m-1,\ f_1 = f)$. On the other hand, we have that

$$\mathbf{n}(e, g_1 \dots g_k h_1 \dots h_m) \equiv (e, g_1) \dots (e_k, g_k)(l_1, h_1) \dots (l_m, h_m),$$

where ${}^{g_i}e_{i+1} = e_i$, $(1 \leq i \leq k-1,\ e_1 = e)$, ${}^{g_k}l_1 = e_k$ and ${}^{h_j}l_{j+1} = l_j$, $(1 \leq j \leq m-1)$. Making use of (4.15) we obtain that

$$e = {}^{g_1 \cdots g_k}l_1.$$

Bearing in mind (4.16), we thus obtain that

$${}^{g_1 \cdots g_k}l_1 = {}^{g_1 \cdots g_k}f_1$$

holds. It follows that $l_j = f_j$ for all $1 \leq j \leq k$ verifying that

$$\mathbf{n}(e, g_1 \dots g_k)\mathbf{n}(f, h_1 \dots h_m) \equiv \mathbf{n}(e, g_1 \dots g_k h_1 \dots h_m).$$

($\Leftarrow$) Assume that

$$\mathbf{n}(e, g_1 \dots g_k)\mathbf{n}(f, h_1 \dots h_m) \equiv \mathbf{n}(e, g_1 \dots g_k h_1 \dots h_m).$$

Hence, if

$$\begin{aligned} \mathbf{n}(e, g_1 \dots g_k) &\equiv (e, g_1) \dots (e_k, g_k) \\ \mathbf{n}(f, h_1 \dots h_m) &\equiv (f, h_1) \dots (f_m, h_m) \\ \mathbf{n}(e, g_1 \dots g_k h_1 \dots h_m) &\equiv (e, g_1) \dots (e_k, g_k)(l_1, h_1) \dots (l_m, h_m), \end{aligned}$$

then $l_j = f_j$, $(1 \leq j \leq m, f_1 = f)$ and

$$e = {}^{g_1 \cdots g_k}l_1 = {}^{g_1 \cdots g_k}f_1,$$

proving that the word $\mathbf{n}(e, g_1 \dots g_k)$ is indeed matched to the word $\mathbf{n}(f, h_1 \dots h_m)$. $\square$

The following lemma can be proved using similar techniques.

Lemma 4.8.2. *Assume that $\mathbf{n}(e, g_1 \dots g_k)$ is matched to $\mathbf{n}(f, h_1 \dots h_m)$. Then*

$$\mathbf{n}(e, g_1 \dots g_k)(f, g) \equiv \mathbf{n}(e, g_1 \dots g_k g)$$

holds for all $g \in Z \cup Z^{-1}$.

The following lemma is immediate from the definitions. Before asserting the lemma, we recall that by Lemma 4.2.1, if $l \in Y_m$, then for any word $g_1 \ldots g_k$ over $Z \cup Z^{-1}$, ${}^{g_1 \cdots g_k} l \in Y_m$.

Lemma 4.8.3. *Consider the words* $u \equiv \mathbf{n}(e, g_1 \ldots g_k)$ *and* $v \equiv \mathbf{n}(\tilde{e}, g_k^{-1} \ldots g_1^{-1})$, *where* $e = {}^{g_1 \cdots g_k}\tilde{e}$. *Then the following hold:*

(i) u *is matched to* v*;*

(ii) v *is matched to* u*;*

(iii) $uvu = v$ *and* $vuv = v$ *hold in* S.

(iv) If $w = \mathbf{n}(f, g_k^{-1} \ldots g_1^{-1})$ *such that* $uwu = u$ *and* $wuw = w$ *hold in* S, *then* $w \equiv v$.

We note that considering u and v as elements of S, $v = u^{-1}$. Therefore, if $u \equiv \mathbf{n}(e, g_1 \ldots g_k)$, then we let u^{-1} denote the word $\mathbf{n}(\tilde{e}, g_k^{-1} \ldots g_1^{-1})$, where $e = {}^{g_1 \cdots g_k}\tilde{e}$.

Conjugate words of idempotents

We keep the notation introduced so far. Before we introduce the notion of conjugate words, we prove the following lemma.

Lemma 4.8.4. *Let* $f \in Y$ *and assume that*

$$f = {}^{g_1 \cdots g_k} l,$$

where $l \in Y_0$ *and* $g_1, \ldots, g_k \in Z \cup Z^{-1}$. *Let* $e \in Y_m$ *such that* $f \leq e$. *Let* $u \equiv \mathbf{n}(e, g_1 \ldots g_k)$. *Then the word*

$$u(l, 1)u^{-1}$$

represents the element $(f, 1)$ *in* S.

Proof. Indeed, we have

$$\begin{aligned} u \cdot (l, 1) \cdot u^{-1} &= (e, g_1 \ldots g_k) \cdot (l, 1) \cdot (\tilde{e}, g_k^{-1} \ldots g_1^{-1}) \\ &= (e \wedge {}^{g_1 \cdots g_k} l \wedge {}^{g_1 \cdots g_k}\tilde{e}, 1) \\ &= (e \wedge f \wedge e, 1) \\ &= (f, 1). \end{aligned}$$

□

Keeping the notation of Lemma 4.8.4, we say that

$$u \cdot (l, 1) \cdot u^{-1}$$

is a *conjugate word of the idempotent* $({}^{g_1 \cdots g_k}l, 1)$. Clearly, any idempotent $({}^{g_1 \cdots g_k}l, 1)$ in S can have several conjugate words representing it, depending on how many maximal idempotents there are above it with respect to the natural partial order.

Notation. The set of all conjugate words of $({}^{g_1 \cdots g_k}l, 1)$ will be denoted by

$$c({}^{g_1 \cdots g_k}l).$$

Clearly, if Y satisfies the maximum condition, then $c({}^{g_1 \cdots g_k}l)$ is a finite set for all $l \in Y_0$ and $w \equiv g_1 \dots g_k \in (Z \cup Z^{-1})^+$.

4.9 Finite presentability I.

The main result of this chapter is

Theorem 4.9.1. *Let $(Y, \wedge)$ be a semilattice, and G be a group acting on Y on the left by automorphisms. The semidirect product $S = Y \rtimes G$ is finitely presented as an inverse semigroup if and only if the following conditions hold:*

(i) G is finitely presented as a group;

(ii) Y satisfies the maximum condition;

(iii) Y is finitely presented as an inverse semigroup with respect to the action of G.

We prove direct implication in this section. As in the previous sections, if we say that a group G acts on a semilattice Y, then it will be understood that G acts on Y on the left by automorphisms. We first fix notation for this and the following section.

Notation

Let Y be a semilattice and G be a group acting on Y. Let $S = Y \rtimes G$ and let A be a finite inverse semigroup generating set for S. We assume that $A \subseteq S$. We let

$$X = \{g \in G \mid (e, g) \in A \text{ for some } e \in Y\}$$

and let

$$Y_0 = \{e \in Y \mid (e, g) \in A \cup A^{-1} \text{ for some } g \in X \cup X^{-1}\}.$$

Because of Proposition 4.3.1 we assume that Y_m, the finite set of maximal elements of Y, is contained in Y_0. We let $Z = X \cup X^{-1} \cup \{1\}$. Since we are going to work with words over three different alphabets, namely over $A \cup A^{-1}$, $X \cup X^{-1}$ and over $B = {}^Z Y_0 \cup ({}^Z Y_0)^{-1}$, we agree that words over $A \cup A^{-1}$ and $X \cup X^{-1}$ will be written in the usual way, i.e. we write the letters consecutively. From the context it will be clear over which alphabet the word being considered is taken. For the sake of legibility words over B will be written in the following way:

$$a_1 \wedge \ldots \wedge a_k.$$

As in Section 6, we consider the semigroup B^+ as a semigroup on which Z acts, where the action is defined by the rule (4.14) introduced in Section 6. As introduced in the previous section, we define

$$\mathbf{n} : Y_m \times (Z \cup Z^{-1})^+ \to B^+;\ (f_1, g_1 g_2 \ldots g_k) \mapsto (f_1, g_1)(f_2, g_2) \ldots (f_k, g_k),$$

where ${}^{g_j} f_{j+1} = f_j$, $(1 \leq j + 1 \leq k)$.

We also adopt the following notation. For a word $w \equiv (e_1, g_1) \ldots (e_k, g_k) \in (A \cup A^{-1})^+$, we let

$$\widetilde{w} \equiv e_1 \wedge {}^{g_1} e_2 \wedge \ldots \wedge {}^{g_1 \cdots g_{k-1}} e_k \quad \text{and} \quad \widehat{w} \equiv g_1 \ldots g_k.$$

Note that the element of S represented by w is $(\widetilde{w}, \widehat{w})$. It is immediate that if $w \equiv uv$, then

$$\widetilde{w} \equiv \widetilde{u} \wedge {}^{\widehat{u}} \widetilde{v}.$$

We recall that $FSI(A)$ denotes the free semigroup with involution on the set A. We first prove

Lemma 4.9.2. *Let $(Y, \wedge)$ be a semilattice and G be a group acting on Y. Assume that the semidirect product $S = Y \rtimes G$ is generated by A. Then for any $w \in FSI(A)$,*

$$\widetilde{ww^{-1}} = \widetilde{w}$$

holds in Y.

Proof. We prove the lemma by induction on the length of w. If $|w| = 1$, then $w \equiv (e, g) \in A \cup A^{-1}$ and so $(e, g)^{-1} = ({}^{g^{-1}} e, g^{-1})$ by Proposition 4.2.7. To simplify notation, let $u \equiv ww^{-1}$. Then

$$\widetilde{u} \equiv e \wedge {}^{g}({}^{g^{-1}} e) = e \wedge e = e$$

verifying that $\widetilde{u} = \widetilde{w}$, when $|w| = 1$. Assume that

$$\widetilde{ww^{-1}} = \widetilde{w}$$

holds in Y for all $w \in FSI(A)$, whose length is less then $m, (m > 1)$. Let $w \in FSI(A)$ so that the length of w is m. Then $w \equiv (e, g)w_1$, where w_1 is a word of length $m-1$ and $(e, g) \in A \cup A^{-1}$. To simplify notation, let $u \equiv ww^{-1}$ and $v \equiv w_1 w_1^{-1}$. Then

$$\begin{aligned}
\widetilde{u} &\equiv e \wedge {}^{g}\widetilde{v} \wedge {}^{g\hat{v}}({}^{g^{-1}}e) \\
&= e \wedge {}^{g}\widetilde{w_1} \wedge {}^{g\hat{v}}({}^{g^{-1}}e) && \text{by the induction hypothesis} \\
&= e \wedge {}^{g}\widetilde{w_1} \wedge e && \text{since } g\hat{v}g^{-1} = 1 \\
&= e \wedge {}^{g}\widetilde{w_1} && \text{standard inverse semigroup relation} \\
&\equiv \widetilde{w}
\end{aligned}$$

proving that $\widetilde{u} = \widetilde{w}$. □

The necessary condition

We prove

Proposition 4.9.3. *Let $(Y, \wedge)$ be a semilattice, and G be a group acting on Y. If the semidirect product $S = Y \rtimes G$ is finitely presented as an inverse semigroup then the following conditions hold:*

(i) G is finitely presented as a group;

(ii) Y satisfies the maximum condition;

(iii) Y is finitely presented as an inverse semigroup with respect to the action.

Proof. Assume that the semidirect product $S = Y \rtimes G$ is given by the presentation $S = \mathrm{Inv}\langle A|P\rangle$, where $A \subseteq S$ is a finite set and P is a finite set of relations. As introduced, we let

$$X = \{g \in G \mid (e, g) \in A \text{ for some } e \in Y\}$$

and let

$$Y_0 = \{e \in Y \mid (e, g) \in A \cup A^{-1} \text{ for some } g \in G\}.$$

By Proposition 4.3.1 we may assume that $Y_m \subseteq Y_0$. That condition *(ii)* holds, follows by Proposition 4.3.1, and thus it remains to show that G is finitely presented as a group and that Y is finitely presented with respect to the action as an inverse semigroup. Let

$$\widehat{P} = \{\hat{r} = \hat{p} \mid (r = p) \in P\}.$$

We show that

$$G = \mathrm{Grp}\langle X | \widehat{P} \rangle.$$

We already know by Proposition 4.3.1 that G is generated by X. It is straightforward that the relations of $\widehat{P}$ hold in G. We verify that any relation that holds in G is a consequence of the relations in $\widehat{P}$. For this, assume that the relation

$$g_1 g_2 \dots g_m = h_1 h_2 \dots h_k,$$

$(g_i, h_j \in X \cup X^{-1})$ holds in G. Let $f \in Y_m$. Then

$$\mathbf{n}(f, g_1 g_2 \dots g_m) = \mathbf{n}(f, h_1 h_2 \dots h_k),$$

holds in S. Thus, there exists a finite sequence of words

$$\mathbf{n}(f, g_1 g_2 \dots g_m) \equiv q_0, q_1, \dots, q_t \equiv \mathbf{n}(f, h_1 h_2 \dots h_k)$$

where q_{j+1} is obtained from q_j by applying a relation in P or a standard inverse semigroup relation. In particular we have that:

(i) If q_{j+1} is obtained from q_j by applying a relation in P, then $\hat{q}_{j+1}$ is obtained from $\hat{q}_j$ by applying a relation in $\widehat{P}$.

(ii) If q_{j+1} is obtained from q_j by applying a standard inverse semigroup relation, then $\hat{q}_{j+1}$ is obtained from $\hat{q}_j$ by applying a sequence of standard group relations.

It follows that there exists a finite sequence of words

$$g_1 g_2 \dots g_m \equiv \alpha_0, \alpha_1, \dots, \alpha_l \equiv h_1 h_2 \dots h_k,$$

such that α_{j+1} is obtained from α_j by applying a relation in $\widehat{P}$ or a standard group relation, verifying that $G = \mathrm{Grp}\langle X | \widehat{P} \rangle$.

We let $Z = (X \cup X^{-1} \cup \{1\})^*$ and

$$\overline{P} = \widehat{P} \cup \{xx^{-1} = x^{-1}x = 1,\ 1x = x,\ 1x^{-1} = x^{-1},\ 11 = 1 \mid x \in X\}.$$

Clearly $G = \langle X \cup X^{-1} \cup \{1\} \mid \overline{P} \rangle$.

Next, we claim that Y is finitely presented as an inverse semigroup with respect to the action of G. We already know that Y satisfies the maximum condition and that Y is finitely generated as a semigroup by Y_0 with respect to the action. We recall that $({}^{Z}Y_0)^{-1}$

denotes the set of formal inverses of elements of ${}^{Z}Y_0$. We let $B = {}^{Z}Y_0 \cup ({}^{Z}Y_0)^{-1}$. We denote by $\Re$ the set of standard inverse semigroup relations on $FSI({}^{Z}Y_0)$. Let

$$Q = \{\tilde{r} = \tilde{p} \mid (r = p) \in P\} \cup \{e = f \wedge e \mid e, f \in Y_0,\ e \leq f\}.$$

We show that

$$Y = \mathrm{InvAct}_G\langle Y_0 | Q\rangle$$

by proving that $Y = \mathrm{Inv}\langle {}^{Z}Y_0 | Q_A\rangle$, where $Q_A = Q_R \setminus \Re$ and

$$\begin{aligned} Q_R = Q &\cup \{{}^{\alpha u \beta}p = {}^{\alpha v \beta}q \mid (p = q) \in Q \cup \Re, (u = v) \in \overline{P}, \alpha, \beta \in Z\} \\ &\cup \{{}^{\alpha u \beta}w = {}^{\alpha v \beta}w \mid w \in B^+, (u = v) \in \overline{P}, \alpha, \beta \in Z\} \\ &\cup \{{}^{\alpha}p = {}^{\alpha}q \mid (p = q) \in Q \cup \Re, \alpha \in Z\}. \end{aligned}$$

It is clear that every relation of Q and hence of Q_R holds in Y. Furthermore, for all $\alpha \in Z$, the relation ${}^{\alpha}e \wedge {}^{\alpha}e = {}^{\alpha}e$ is an element of Q_R, and so it follows that

$$({}^{\alpha}e)^{-1} = {}^{\alpha}e. \tag{4.17}$$

Making use of this observation, we may assume that any relation that holds in Y is of the form:

$${}^{\alpha_1}e_1 \wedge {}^{\alpha_2}e_2 \wedge \ldots \wedge {}^{\alpha_m}e_m = {}^{\beta_1}f_1 \wedge {}^{\beta_2}f_2 \wedge \ldots \wedge {}^{\beta_k}f_k, \tag{4.18}$$

where $e_i, f_j \in Y_0$ and $\alpha_i, \beta_j \in Z$. Assume now that the above relation holds in Y. Let $\tilde{e}_1$ be a maximal element of Y, such that $e_1 \leq \tilde{e}_1$. Since Y satisfies the maximum condition, such an element exists. In particular, the relation

$${}^{\alpha_1}\tilde{e}_1 \wedge {}^{\alpha_1}e_1 = {}^{\alpha_1}(\tilde{e}_1 \wedge e_1) = {}^{\alpha_1}e_1,$$

is an element of Q_R. Moreover, ${}^{\alpha_1}\tilde{e}_1 \in Y_m \subseteq Y_0$ by Lemma 4.2.1. Let $e_0 = {}^{\alpha_1}\tilde{e}_1$. Then we obtain

$${}^{\alpha_1}e_1 \wedge {}^{\alpha_2}e_2 \wedge \ldots \wedge {}^{\alpha_m}e_m = e_0 \wedge {}^{\alpha_1}e_1 \wedge {}^{\alpha_2}e_2 \wedge \ldots \wedge {}^{\alpha_m}e_m.$$

With a similar argument, we also obtain that

$${}^{\beta_1}f_1 \wedge {}^{\beta_2}f_2 \wedge \ldots \wedge {}^{\beta_k}f_k = f_0 \wedge {}^{\beta_1}f_1 \wedge {}^{\beta_2}f_2 \wedge \ldots \wedge {}^{\beta_k}f_k$$

holds for some $f_0 \in Y_m \subseteq Y_0$. We verify that (4.18) is a consequence of relations in Q_R by verifying that

$$e_0 \wedge {}^{\alpha_1}e_1 \wedge {}^{\alpha_2}e_2 \wedge \ldots \wedge {}^{\alpha_m}e_m = f_0 \wedge {}^{\beta_1}f_1 \wedge {}^{\beta_2}f_2 \wedge \ldots \wedge {}^{\beta_k}f_k \tag{4.19}$$

is a consequence of the relations in Q_R in three steps.

Step 1. Consider the element $(e_0 \wedge {}^{\alpha_1}e_1 \wedge {}^{\alpha_2}e_2 \wedge \ldots \wedge {}^{\alpha_m}e_m, 1)$ of S. Let

$$\gamma_j \equiv \begin{cases} \alpha_j^{-1}\alpha_{j+1} & 1 \leq j \leq m-1 \\ \alpha_{j+1} & j = 0 \\ \alpha_j^{-1} & j = m. \end{cases}$$

Define words $w_{1,0}, \ldots, w_{1,m}$ over $A \cup A^{-1}$ in the following way:

$$w_{1,j} \equiv \begin{cases} \mathbf{n}(e_j, \gamma_j) & \text{if } e_j \in Y_m \\ (e_j, 1)\mathbf{n}(\tilde{e}_j, \gamma_j) & \text{otherwise,} \end{cases}$$

where $\tilde{e}_j \in Y_m$ is such that $e_j \leq \tilde{e}_j$. Clearly, the word

$$w_1 \equiv w_{1,0}w_{1,1} \ldots w_{1,m}$$

represents $(e_0 \wedge {}^{\alpha_1}e_1 \wedge {}^{\alpha_2}e_2 \wedge \ldots \wedge {}^{\alpha_m}e_m, 1)$. On the other hand, by the definition of the words $w_{1,0}, \ldots, w_{1,m}$ we obtain that

$$\widetilde{w}_{1,j} = \begin{cases} e_j \wedge \ldots \wedge e_j & \text{if } e_j \in Y_m \\ e_j \wedge \tilde{e}_j \wedge \ldots \wedge \tilde{e}_j & \text{otherwise,} \end{cases}$$

and hence $e_j = \widetilde{w}_{1,j}$ is a consequence of the relations in Q_R. It follows that

$$e_0 \wedge {}^{\alpha_1}e_1 \wedge {}^{\alpha_2}e_2 \wedge \ldots \wedge {}^{\alpha_m}e_m = \widetilde{w}_1 \tag{4.20}$$

is also a consequence of the relations in Q_R.

Step 2. We consider the element $(f_0 \wedge {}^{\beta_1}f_1 \wedge {}^{\beta_2}f_2 \wedge \ldots \wedge {}^{\beta_k}f_k, 1)$ of S and repeat the first step. We let

$$\delta_j \equiv \begin{cases} \beta_j^{-1}\beta_{j+1} & 1 \leq j \leq k-1 \\ \beta_{j+1} & j = 0 \\ \beta_j^{-1} & j = k. \end{cases}$$

Define words $w_{2,0}, \ldots, w_{2,k}$ over $A \cup A^{-1}$ in the following way:

$$w_{2,j} \equiv \begin{cases} \mathbf{n}(f_j, \delta_j) & \text{if } f_j \in Y_m \\ (f_j, 1)\mathbf{n}(\tilde{f}_j, \delta_j) & \text{otherwise,} \end{cases}$$

where $\tilde{f}_j \in Y_m$ is such that $f_j \leq \tilde{f}_j$. The word

$$w_2 \equiv w_{2,0}w_{2,1} \ldots w_{2,k}$$

represents $(f_0 \wedge {}^{\beta_1}f_1 \wedge {}^{\beta_2}f_2 \wedge \ldots \wedge {}^{\beta_k}f_k, 1)$. On the other hand, by the definition of the words $w_{2,0}, \ldots, w_{2,k}$ we obtain that

$$\widetilde{w}_{2,j} = \begin{cases} f_j \wedge \ldots \wedge f_j & \text{if } f_j \in Y_m \\ f_j \wedge \tilde{f}_j \wedge \ldots \wedge \tilde{f}_j & \text{otherwise,} \end{cases}$$

and hence $f_j = \widetilde{w}_{2,j}$ is a consequence of the relations in Q_R. It follows that

$$\widetilde{w}_2 = f_0 \wedge {}^{\beta_1} f_1 \wedge {}^{\beta_2} f_2 \wedge \ldots \wedge {}^{\beta_k} f_k \tag{4.21}$$

is also a consequence of the relations in Q_R.

Step 3. Since by (4.20)

$$e_0 \wedge {}^{\alpha_1} e_1 \wedge {}^{\alpha_2} e_2 \wedge \ldots \wedge {}^{\alpha_m} e_m = \widetilde{w}_1$$

and by (4.21)

$$\widetilde{w}_2 = f_0 \wedge {}^{\beta_1} f_1 \wedge {}^{\beta_2} f_2 \wedge \ldots \wedge {}^{\beta_k} f_k$$

hold, to show that

$$e_0 \wedge {}^{\alpha_1} e_1 \wedge {}^{\alpha_2} e_2 \wedge \ldots \wedge {}^{\alpha_m} e_m = f_0 \wedge {}^{\beta_1} f_1 \wedge {}^{\beta_2} f_2 \wedge \ldots \wedge {}^{\beta_k} f_k$$

is a consequence of relations in Q_R, it is enough to verify that

$$\widetilde{w}_1 = \widetilde{w}_2$$

is a consequence of relations in Q_R. Since

$$e_0 \wedge {}^{\alpha_1} e_1 \wedge {}^{\alpha_2} e_2 \wedge \ldots \wedge {}^{\alpha_m} e_m = f_0 \wedge {}^{\beta_1} f_1 \wedge {}^{\beta_2} f_2 \wedge \ldots \wedge {}^{\beta_k} f_k$$

holds in Y, the relation $w_1 = w_2$ holds in S, and hence there exists a finite sequence of words

$$w_1 \equiv q_0, q_1, \ldots, q_t \equiv w_2$$

such that q_{j+1} is obtained from q_j by applying a relation in P or a standard inverse semigroup relation. We consider the following three cases:

(1) If q_{j+1} is obtained from q_j by applying a relation in P, then we may write $q_j \equiv t_1 s t_2$, $q_{j+1} \equiv t_1 z t_2$, where $(s = z) \in P$. It follows that $(\tilde{s} = \tilde{z}) \in \widetilde{P}$ and $(\hat{s} = \hat{z}) \in \widehat{P}$. In particular $\widehat{t_1 s} = \widehat{t_1 z}$ holds in G and we obtain that

$$\begin{aligned} \tilde{q}_j &\equiv \tilde{t}_1 \wedge {}^{\hat{t}_1} \tilde{s} \wedge {}^{\widehat{t_1 s}} \tilde{t}_2 \\ &= \tilde{t}_1 \wedge {}^{\hat{t}_1} \tilde{z} \wedge {}^{\widehat{t_1 s}} \tilde{t}_2 \\ &= \tilde{t}_1 \wedge {}^{\hat{t}_1} \tilde{z} \wedge {}^{\widehat{t_1 z}} \tilde{t}_2 \\ &\equiv \tilde{q}_{j+1} \end{aligned}$$

verifying that $\tilde{q}_{j+1}$ is obtained from $\tilde{q}_j$ by applying relations in Q_R.

(2) If q_{j+1} is obtained from q_j by applying a relation of the form $ww^{-1}w = w$, then we may write $q_j \equiv t_1ww^{-1}wt_2$, $q_{j+1} \equiv t_1wt_2$. Let $u \equiv ww^{-1}$, and $v \equiv ww^{-1}w$. Clearly $\widehat{v} = \widehat{w}$ holds in G and we have that

$$\begin{aligned}
\tilde{q}_j &\equiv \tilde{t}_1 \wedge {}^{\hat{t}_1}\widetilde{u} \wedge {}^{\widehat{t_1u}}\widetilde{w} \wedge {}^{\widehat{t_1v}}\tilde{t}_2 \\
&= \tilde{t}_1 \wedge {}^{\hat{t}_1}\widetilde{w} \wedge {}^{\widehat{t_1u}}\widetilde{w} \wedge {}^{\widehat{t_1v}}\tilde{t}_2 && \text{since } \widetilde{u} = \widetilde{w} \text{ by Lemma 4.9.2} \\
&= \tilde{t}_1 \wedge {}^{\hat{t}_1}\widetilde{w} \wedge {}^{\hat{t}_1}\widetilde{w} \wedge {}^{\widehat{t_1v}}\tilde{t}_2 && \text{since } \widehat{t_1u} = \widehat{t}_1 \text{ in } G \\
&= \tilde{t}_1 \wedge {}^{\hat{t}_1}\widetilde{w} \wedge {}^{\widehat{t_1v}}\tilde{t}_2 && \text{by applying a relation in } Q_R \\
&= \tilde{t}_1 \wedge {}^{\hat{t}_1}\widetilde{w} \wedge {}^{\widehat{t_1w}}\tilde{t}_2 && \text{since } \widehat{w} = \widehat{v} \text{ in } G \\
&\equiv \tilde{q}_{j+1}
\end{aligned}$$

proving that $\tilde{q}_j = \tilde{q}_{j+1}$ is a consequence of relations in Q_R.

(3) If q_{j+1} is obtained from q_j by applying a relation of the form $w_1w_1^{-1}w_2w_2^{-1} = w_2w_2^{-1}w_1w_1^{-1}$, then we may write

$$q_j \equiv t_1w_1w_1^{-1}w_2w_2^{-1}t_2,\ q_{j+1} \equiv t_1w_2w_2^{-1}w_1w_1^{-1}t_2.$$

Let $u \equiv w_1w_1^{-1}$, $v \equiv w_2w_2^{-1}$. Then

$$\begin{aligned}
\tilde{q}_j &\equiv \tilde{t}_1 \wedge {}^{\hat{t}_1}\widetilde{u} \wedge {}^{\widehat{t_1u}}\widetilde{v} \wedge {}^{\widehat{t_1uv}}\tilde{t}_2 \\
&= \tilde{t}_1 \wedge {}^{\hat{t}_1}\widetilde{u} \wedge {}^{\hat{t}_1}\widetilde{v} \wedge {}^{\hat{t}_1}\tilde{t}_2 && \text{since } \widehat{u} = \widehat{uv} = 1 \text{ in } G \\
&= \tilde{t}_1 \wedge {}^{\hat{t}_1}\widetilde{v} \wedge {}^{\hat{t}_1}\widetilde{u} \wedge {}^{\hat{t}_1}\tilde{t}_2 && \text{standard inverse semigroup relation} \\
&= \tilde{t}_1 \wedge {}^{\hat{t}_1}\widetilde{v} \wedge {}^{\widehat{t_1v}}\widetilde{u} \wedge {}^{\widehat{t_1vu}}\tilde{t}_2 && \text{since } \widehat{v} = \widehat{v}u = 1 \text{ in } G \\
&\equiv \tilde{q}_{j+1}
\end{aligned}$$

proving that $\tilde{q}_j = \tilde{q}_{j+1}$ is a consequence of relations in Q_R.

It follows that a finite sequence of words can be given $\widetilde{w}_1 \equiv u_0, u_1, \ldots, u_l \equiv \widetilde{w}_2$ such that u_{j+1} is obtained from u_j by applying a relation of Q_R and we may deduce that $Y = \mathrm{InvAct_G}\langle Y_0 | Q\rangle$. $\square$

4.10 Finite presentability II.

We keep the notation introduced at the beginning of Section 9. In addition, we recall that for any element ${}^{g_1\cdots g_k}e \in Y$

$$c({}^{g_1\cdots g_k}e)$$

denotes the set of conjugate words of the idempotent $({}^{g_1 \cdots g_k}e, 1) \in E(S)$. The main purpose of this section is to prove

Proposition 4.10.1. *Let $(Y, \wedge)$ be a semilattice, and G be a group acting on Y. Assume that the following conditions hold:*

(i) G is finitely presented as a group;

(ii) Y satisfies the maximum condition;

(iii) Y is finitely presented as an inverse semigroup with respect to the action.

Then the semidirect product $S = Y \rtimes G$ is finitely presented as an inverse semigroup.

Since the proof of Proposition 4.10.1 is rather long, we first summarize the main steps of the proof.

1. We give an inverse semigroup generating set A for $S = Y \rtimes G$ and define a set of relations $\mathbf{R}$ on $(A \cup A^{-1})^+$ that hold in S.
2. In Proposition 4.10.2, we give a normal form for the elements of S. Proposition 4.10.2 is proved with the help of two technical lemmas: Lemma 4.10.3 and Lemma 4.10.4.
3. Making use of Proposition 4.10.2, we prove that any relation $w_1 = w_2$ that holds in S is a consequence of relations in $\mathbf{R}$.

Proof. Let G be a group acting on a semilattice Y and assume that G and Y satisfy conditions $(i) - (iii)$. More precisely, let $G = \mathrm{Grp}\langle X | P \rangle$, where X is a finite set and P is a finite set of relations. Let Y be a semilattice satisfying the maximum condition and let Y_m denote the set of maximal elements of Y. Assume that $Y = \mathrm{InvAct}_G\langle Y_0 | Q \rangle$, where Y_0 is a finite subset of Y and Q is a finite set of relations. By Proposition 4.6.5 we may assume that $Y_m \subseteq Y_0$. Consider the semidirect product $S = Y \rtimes G$. Making use of Corollary 4.3.2, S is finitely generated as an inverse semigroup by

$$A = (Y_0 \times \{1\}) \cup (Y_m \times X).$$

By Proposition 4.2.7, we have

$$A^{-1} = (Y_0 \times \{1\}) \cup (Y_m \times X^{-1}).$$

Define the following relations on $(A \cup A^{-1})^+$:

(R1) $(e,1)(f,g) = (f,1)(e,g)$, where $e, f \in Y_m$;

(R2) $(e,g) = (e,1)(e,g)$, where $e \in Y_m$;

(R3) $(e,g) = (e,g)(f,1)$, where $f \in Y_0$, ${}^g f \leq e \in Y_m$;

(R4) $(e,g)(f,1) = (\tilde{f},1)(e,g)(f,1)$, where $f \in Y_0, e, \tilde{f} \in Y_m, {}^g f \leq \tilde{f}$;

(R5) $(e,1)(f,g) = (e,1)(f,g)(\tilde{e},1)$, where $e, \tilde{e}, f \in Y_m$, ${}^g \tilde{e} = e$;

(R6) $(e,g)(f,h) = (e,1)(\tilde{f},g)(f,h)$, where $e, f, \tilde{f} \in Y_m$, ${}^g f = \tilde{f}$;

(R7) $(e,g)(f,h) = (e,g)(f,1)(\tilde{e},h)$, where $e, \tilde{e}, f \in Y_m$, ${}^g \tilde{e} = e$.

We denote by R the relations obtained.

For each relation $g_1 \dots g_m = h_1 \dots h_k$ in P and $e \in Y_m$ consider the following relation

$$\mathbf{n}(e, g_1 \dots g_m) = \mathbf{n}(e, h_1 \dots h_k)$$

and denote the set of relations obtained by R_G. Since Y_m and P are finite sets, we have that R_G is a finite set of relations. For each $(e,g) \in Y_m \times X$ consider the relations

$$(e,g)(f,g^{-1}) = (e,1) \qquad \text{and} \qquad (f,g^{-1})(e,g) = (f,1),$$

where f is the maximal element of Y for which ${}^g f = e$. Denote the set of relations obtained by R_M.

For each relation

$${}^{\alpha_1}e_1 \wedge \dots \wedge {}^{\alpha_m}e_m = {}^{\beta_1}f_1 \wedge \dots \wedge {}^{\beta_k}f_k$$

in Q we define the following set of relations

$$\{c_{11} \dots c_{1m} = c_{21} \dots c_{2k} \mid c_{1j} \in c({}^{\alpha_j}e_j),\ c_{2j} \in c({}^{\beta_j}f_j)\}.$$

Since Y satisfies the maximum condition, each of the sets $c({}^{\alpha_j}e_j)$ and $c({}^{\beta_j}f_j)$ is finite and hence

$$\{c_{11} \dots c_{1m} = c_{21} \dots c_{2k} \mid c_{1j} \in c({}^{\alpha_j}e_j),\ c_{2j} \in c({}^{\beta_j}f_j)\}.$$

is a finite set. Denote the set of all relations obtained in this way by R_Y.

For each $e, f \in Y_0$ such that $e \leq f$ consider the relation

$$(e,1) = (f,1)(e,1)$$

and denote the set of all relations of this form by R_L.

Let

$$\mathbf{R} = R \cup R_G \cup R_M \cup R_Y \cup R_L.$$

Our aim is to show that $S = \mathrm{Inv}\langle A|\mathbf{R}\rangle$. Clearly $\mathbf{R}$ is a finite set of relations and all relations in $\mathbf{R}$ hold in S. According to Proposition 4.1.4 we need to verify that any relation $w_1 = w_2$ that holds in S is a consequence of relations in $\mathbf{R}$ and of standard inverse semigroup relations. In order to prove this, we need the following proposition.

Proposition 4.10.2. *Let $w \equiv (e_1, g_1) \dots (e_{k-1}, g_{k-1})(e_k, 1) \in (A \cup A^{-1})^+$ such that $|w| \geq 2$. Then the relation*

$$w = (e_1, 1)c_2 \dots c_{k-1}v$$

where $c_j \in c({}^{g_1 \cdots g_j}e_{j+1})$, $v \equiv \mathbf{n}(f, g_1 \dots g_{k-1})$ for some $f \in Y_m$ and $c_{k-1} \equiv v(e_k, 1)v^{-1}$, is a consequence of relations in $\mathbf{R}$.

We need the following two technical lemmas:

Lemma 4.10.3. *Let $w \equiv \mathbf{n}(e, g_1 g_2 \dots g_k)(f, 1)$, where $f \in Y_0$. Then there exists $l \in Y_m$ such that the relation*

$$w = (e, 1)\mathbf{n}(l, g_1 g_2 \dots g_k)$$

is a consequence of the relations in R. Moreover, if we let $u \equiv \mathbf{n}(l, g_1 g_2 \dots g_k)$, then $u(f, 1)u^{-1} \in c({}^{g_1 g_2 \cdots g_k}f)$.

Proof. Let $\tilde{f} \in Y_m$ so that $f \leq \tilde{f}$. Let $l_k = {}^{g_k}\tilde{f}$. Let $l_1, l_2, \dots, l_{k-1} \in Y_m$ such that $l_j = {}^{g_j}l_{j+1}$ holds. Such elements exist by Lemma 4.2.1. Since $f \leq \tilde{f}$, we also obtain that

$$ {}^{g_1 \cdots g_k}f \leq l_1. \tag{4.22}$$

Assume that

$$\mathbf{n}(e, g_1 \dots g_k) \equiv (e, g_1) \dots (e_{k-1}, g_{k-1})(e_k, g_k).$$

Applying the appropriate relations in R we obtain

$$\begin{aligned}
w &\equiv (e,g_1)\dots(e_{k-1},g_{k-1})((e_k,g_k)(f,1)) \\
&= (e,g_1)\dots(e_{k-1},g_{k-1})((l_k,1)(e_k,g_k)(f,1)) && \text{by } (R4) \\
&= (e,g_1)\dots(l_{k-1},1)(e_{k-1},g_{k-1})(l_k,1)(e_k,g_k)(f,1) && \text{by } (R4) \\
&\ \vdots && \vdots \\
&= (l_1,1)(e,g_1)(l_2,1)\dots(l_{k-1},1)(e_{k-1},g_{k-1})(l_k,1)(e_k,g_k)(f,1) && \text{by } (R4) \\
&= (e,1)(l_1,g_1)(e_2,1)\dots(e_{k-1},1)(l_{k-1},g_{k-1})(e_k,1)((l_k,g_k)(f,1)) && \text{by } (R1) \\
&= (e,1)(l_1,g_1)(e_2,1)\dots((e_{k-1},1)(l_{k-1},g_{k-1})(e_k,1))(l_k,g_k) && \text{by } (R3) \\
&= (e,1)(l_1,g_1)(e_2,1)\dots(e_{k-1},1)(l_{k-1},g_{k-1})(l_k,g_k) && \text{by } (R5) \\
&\ \vdots && \vdots \\
&= (e,1)(l_1,g_1)\dots(l_{k-1},g_{k-1})(l_k,g_k) && \text{by } (R5) \\
&\equiv (e,1)(\mathbf{n}(l_1,g_1\dots g_k)).
\end{aligned}$$

It is immediate from (4.22) that if $u \equiv \mathbf{n}(l_1, g_1g_2\dots g_k)$, then $u(f,1)u^{-1} \in c({}^{g_1g_2\cdots g_k}f)$. □

Lemma 4.10.4. *Let $w \equiv (f,1)\mathbf{n}(e,g_1g_2\dots g_k)$, where $f \in Y_m$. Then*

$$w = (e,1)\mathbf{n}(f,g_1g_2\dots g_k)$$

is a consequence of the relations in R.

Proof. Let $f_1 = f$ and let $f_2,\dots,f_k \in Y_m$ such that ${}^{g_j}f_{j+1} = f_j$ holds. Assume that

$$\mathbf{n}(e,g_1g_2\dots g_k) \equiv (e,g_1)(e_2,g_2)\dots(e_{k-1},g_{k-1})(e_k,g_k).$$

Then

$$\begin{aligned}
w &\equiv (f,1)(e,g_1)(e_2,g_2)\dots(e_{k-1},g_{k-1})(e_k,g_k) \\
&= (e,1)((f,g_1)(e_2,g_2))\dots(e_{k-1},g_{k-1})(e_k,g_k) && \text{by } (R1) \\
&= (e,1)(f,g_1)(e_2,1)(f_2,g_2)\dots(e_{k-1},g_{k-1})(e_k,g_k) && \text{by } (R7) \\
&\ \vdots && \vdots \\
&= (e,1)(f,g_1)(e_2,1)(f_2,g_2)\dots((e_{k-1},1)(f_{k-1},g_{k-1})(e_k,1))(f_k,g_k) && \text{by } (R7) \\
&= (e,1)(f,g_1)(e_2,1)(f_2,g_2)\dots((e_{k-1},1)(f_{k-1},g_{k-1}))(f_k,g_k) && \text{by } (R5) \\
&\ \vdots && \vdots \\
&= (e,1)(f,g_1)(f_2,g_2)\dots(f_{k-1},g_{k-1})(f_k,g_k) && \text{by } (R5) \\
&\equiv (e,1)\mathbf{n}(f,g_1\dots g_k),
\end{aligned}$$

verifying the lemma. □

Proof of Proposition 4.10.2. We proceed by induction on the length of w. If $|w| = 2$, then $w \equiv (f,g)(e,1)$, where $(f,g) \in A \cup A^{-1}$. We assume that $(f,g) \in Y_m \times (X \cup X^{-1})$. The case when $(f,g) \in A \cup A^{-1}$ is such that $g = 1$ can be verified similarly. Let $\tilde{e} \in Y_m$ such that ${}^{g}e \leq \tilde{e}$ and let ${}^{g^{-1}}\tilde{e} = l$. By Lemma 4.2.1, $l \in Y_m$. Moreover,

$$\begin{aligned}
w &\equiv (f,g)(e,1) & \\
&= (f,1)((f,g)(e,1)) & \text{by } (R2)\\
&= (f,1)((\tilde{e},1)(f,g))(e,1) & \text{by } (R4)\\
&= (f,1)(f,1)(\tilde{e},g)(e,1) & \text{by } (R1)\\
&= (f,1)(\tilde{e},g)(e,1) & \text{by } R_L\\
&= (f,1)(\tilde{e},g)((l,g^{-1})(\tilde{e},g)(e,1)) & \text{standard inverse sg. relation}\\
&= (f,1)(\tilde{e},g)(e,1)(l,g^{-1})(\tilde{e},g) & \text{standard inverse sg. relation}
\end{aligned}$$

Clearly, $(\tilde{e},g)(e,1)(l,g^{-1}) \in c({}^{g}e)$, verifying that the proposition holds for all words of the form $(f,g)(e,1)$.

Assume that the proposition is true for all words over $A \cup A^{-1}$ that satisfy the condition stated in the proposition and whose length is less then m, where $m \geq 2$. We consider the following two cases:

Case 1. $w \equiv (e_1,g_1)\ldots(e_{k-1},1)(e_k,1)$;

Case 2. $w \equiv (e_1,g_1)\ldots(e_{k-1},g_{k-1})(e_k,1)$, where $g_{k-1} \neq 1$.

Case 1. The inductive hypothesis can be applied to the subword

$$(e_1,g_1)\ldots(e_{k-1},1),$$

and hence we obtain that

$$w = (e_1,1)c_2\ldots c_{k-2}u(e_k,1),$$

where $c_j \in c({}^{g_1\cdots g_j}e_{j+1})$, $u \equiv \mathbf{n}(f,g_1\ldots g_{k-2})$ for some $f \in Y_m$ and $c_{k-2} \equiv u(e_{k-1},1)u^{-1}$. By Lemma 4.8.3, we know that u^{-1} is matched to u, and so applying a relation of type (R3) we obtain that $u^{-1}(f,1) = u^{-1}$. In particular, we have

$$c_{k-2}(f,1) = c_{k-2}. \tag{4.23}$$

On the other hand, by Lemma 4.10.3, there exists $l \in Y_m$ so that

$$\mathbf{n}(f,g_1\ldots g_{k-2})(e_k,1) = (f,1)\mathbf{n}(l,g_1\ldots g_{k-2}) \tag{4.24}$$

holds, and if we let v denote the word $\mathbf{n}(l, g_1 \dots g_{k-2})$, then

$$c_{k-1} \equiv v(e_k, 1)v^{-1} \in c({}^{g_1 \cdots g_{k-2}}e_k). \tag{4.25}$$

Summarizing the above facts, we obtain that

$$\begin{aligned}
w &= (e_1, 1)c_2 \dots c_{k-2}u(e_k, 1) \\
&= (e_1, 1)c_2 \dots c_{k-2}u(e_k, 1)(e_k, 1) \\
&= (e_1, 1)c_2 \dots c_{k-2}(f, 1)v(e_k, 1) && \text{by (4.24)} \\
&= (e_1, 1)c_2 \dots c_{k-2}v(e_k, 1) && \text{by (4.23)} \\
&= (e_1, 1)c_2 \dots c_{k-2}vv^{-1}v(e_k, 1) && \text{standard inverse sg. relation} \\
&= (e_1, 1)c_2 \dots c_{k-2}v(e_k, 1)v^{-1}v && \text{standard inverse sg. relation} \\
&= (e_1, 1)c_2 \dots c_{k-2}c_{k-1}v
\end{aligned}$$

verifying that the proposition indeed holds in the first case.

Case 2. If $w \equiv (e_1, g_1) \dots (e_{k-1}, g_{k-1})(e_k, 1)$ where $g_{k-1} \not\equiv 1$, then $e_{k-1} \in Y_m$, since all generators of the form (e, g), $g \neq 1$ have first component in Y_m. Applying (R2) we obtain that

$$\begin{aligned}
w &\equiv (e_1, g_1) \dots (e_{k-1}, g_{k-1})(e_k, 1) \\
&= (e_1, g_1) \dots (e_{k-1}, 1)(e_{k-1}, g_{k-1})(e_k, 1).
\end{aligned}$$

The inductive hypothesis can be applied to the subword

$$(e_1, g_1) \dots (e_{k-1}, 1),$$

and so we obtain

$$w = (e_1, 1)c_2 \dots c_{k-2}u(e_{k-1}, g_{k-1})(e_k, 1),$$

where $c_j \in c({}^{g_1 \cdots g_j}e_{j+1})$, $u \equiv \mathbf{n}(f, g_1 \dots g_{k-2})$ for some $f \in Y_m$ and $c_{k-2} \equiv u(e_{k-1}, 1)u^{-1}$. To unravel the information, we have that

$$u(e_{k-1}, 1)u^{-1} = ({}^{g_1 \cdots g_{k-2}}e_{k-1}, 1),$$

and so

$$(f \wedge {}^{g_1 \cdots g_{k-2}}e_{k-1}, 1) = ({}^{g_1 \cdots g_{k-2}}e_{k-1}, 1) \tag{4.26}$$

holds. Taking into account that $e_{k-1} \in Y_m$, we thus obtain from (4.26) that $f = {}^{g_1 \cdots g_{k-2}}e_{k-1}$. In particular, we have that $u \equiv \mathbf{n}(f, g_1 \dots g_{k-2})$ is matched to (e_{k-1}, g_{k-1}), and so by Lemma 4.8.1 we obtain

$$u(e_{k-1}, g_{k-1}) = \mathbf{n}(f, g_1 \dots g_{k-2}g_{k-1}). \tag{4.27}$$

By Lemma 4.8.3, we know that u^{-1} is matched to u, and so applying a relation of type (R3), it follows that $u^{-1}(f,1) = u^{-1}$. In particular, we have that

$$c_{k-2}(f,1) = c_{k-2}. \tag{4.28}$$

To summarize the above observations,

$$\begin{aligned} w &= (e_1,1)c_2 \dots c_{k-2}u(e_{k-1},g_{k-1})(e_k,1) \\ &= (e_1,1)c_2 \dots c_{k-2}\mathbf{n}(f,g_1 \dots g_{k-2}g_{k-1})(e_k,1). \end{aligned}$$

We need one more observation to finish the argument. Making use of Lemma 4.10.3, we have that there exists $l \in Y_m$ so that

$$\mathbf{n}(f,g_1 \dots g_{k-2}g_{k-1})(e_k,1) = (f,1)\mathbf{n}(l,g_1 \dots g_{k-2}g_{k-1}) \tag{4.29}$$

holds and if we let $v \equiv \mathbf{n}(l,g_1 \dots g_{k-2}g_{k-1})$, then

$$c_{k-1} \equiv v(e_k,1)v^{-1} \in c(^{g_1 \cdots g_{k-1}}e_k). \tag{4.30}$$

Summarizing the above facts:

$$\begin{aligned} w &= (e_1,1)c_2 \dots c_{k-2}\mathbf{n}(f,g_1 \dots g_{k-2}g_{k-1})(e_k,1) \\ &= (e_1,1)c_2 \dots c_{k-2}\mathbf{n}(f,g_1 \dots g_{k-2}g_{k-1})(e_k,1)(e_k,1) \\ &= (e_1,1)c_2 \dots c_{k-2}(f,1)v(e_k,1) && \text{by (4.29)} \\ &= (e_1,1)c_2 \dots c_{k-2}v(e_k,1) && \text{by (4.28)} \\ &= (e_1,1)c_2 \dots c_{k-2}vv^{-1}v(e_k,1) && \text{standard inverse sg. relation} \\ &= (e_1,1)c_2 \dots c_{k-2}v(e_k,1)v^{-1}v && \text{standard inverse sg. relation} \\ &= (e_1,1)c_2 \dots c_{k-2}c_{k-1}v \end{aligned}$$

proving Proposition 4.10.2. □

We are now ready to prove that $S = \mathrm{Inv}\langle A|\mathbf{R}\rangle$. Recall that $G = \mathrm{Grp}\langle X|P\rangle$ and $Y = \mathrm{InvAct}_G\langle Y_0|Q\rangle$.

Proof of Proposition 4.10.1. Assume that the relation $w_1 = w_2$ holds in S. By applying a relation of type (R3), we may assume that

$$w_1 \equiv (e_1,g_1)\dots(e_m,1) \qquad \text{and that} \qquad w_2 \equiv (f_1,h_1)\dots(f_k,1).$$

By Proposition 4.10.2, we have that

$$w_1 = (e_1, 1)c_2 \dots c_{m-1}u, \tag{4.31}$$

where $c_j \in c({}^{g_1\dots g_j}e_{j+1})$, $u \equiv \mathbf{n}(e, g_1 \dots g_{m-1})$ for some $e \in Y_m$ and $c_{m-1} \equiv u(e_m, 1)u^{-1}$. We write

$$(w_1)_Y \equiv (e_1, 1)c_2 \dots c_{m-1}$$

and recall the notation introduced in Section 9:

$$\widetilde{w}_1 \equiv e_1 \wedge {}^{g_1}e_2 \wedge \dots \wedge {}^{g_1\dots g_{m-1}}e_m \quad \text{and} \quad \widehat{w}_1 \equiv g_1 \dots g_{m-1}.$$

Similarly we have that

$$w_2 = (f_1, 1)d_2 \dots d_{k-1}v,$$

where $d_j \in c({}^{h_1\dots h_j}f_{j+1})$, $v \equiv \mathbf{n}(f, h_1 \dots h_{k-1})$ for some $f \in Y_m$ and $d_{k-1} \equiv v(f_k, 1)v^{-1}$. We write

$$(w_2)_Y \equiv (f_1, 1)d_2 \dots d_{k-1},$$

$$\widetilde{w}_2 \equiv f_1 \wedge {}^{h_1}f_2 \wedge \dots \wedge {}^{h_1\dots h_{k-1}}f_k \quad \text{and} \quad \widehat{w}_2 \equiv h_1 \dots h_{k-1}.$$

We prove in four steps that $w_1 = w_2$ is a consequence of relations in $\mathbf{R}$ and of standard inverse semigroup relations.

Step 1. Since $w_1 = w_2$ in S, we have that $\widehat{w}_1 = \widehat{w}_2$ holds in G, and hence there exists a finite sequence of words

$$\widehat{w}_1 \equiv \alpha_1, \alpha_2, \dots, \alpha_t \equiv \widehat{w}_2$$

such that α_{j+1} is obtained from α_j by applying a relation in P or a standard group relation. It follows that there exists a finite sequence of words

$$\mathbf{n}(e, g_1 \dots g_{m-1}) \equiv \beta_1, \dots, \beta_t \equiv \mathbf{n}(e, h_1 \dots h_{k-1})$$

such that β_{j+1} is obtained from β_j by applying a relation in $R_G \cup R_M$. In particular we have a finite sequence of words

$$(w_1)_Y(\mathbf{n}(e, g_1 \dots g_{m-1})) \equiv (w_1)_Y\beta_1, \dots, (w_1)_Y\beta_t \equiv (w_1)_Y(\mathbf{n}(e, h_1 \dots h_{k-1})). \tag{4.32}$$

Step 2. Since $w_1 = w_2$ in S, we have that $\widetilde{w}_1 = \widetilde{w}_2$ holds in Y, and hence there exists a finite sequence of words

$$\widetilde{w}_1 \equiv \gamma_1, \gamma_2, \dots \gamma_q \equiv \widetilde{w}_2$$

such that γ_{j+1} is obtained from γ_j by applying a relation in Q_R. It follows that there exists a finite sequence

$$(w_1)_Y \equiv \delta_1, \delta_2, \ldots, \delta_q \equiv (w_2)_Y$$

such that δ_{j+1} is obtained from δ_j by applying a relation in $R_Y \cup R_L \cup R_G \cup R_M$ or a standard inverse semigroup relation. In particular, we have a finite sequence of words

$$(w_1)_Y(\mathbf{n}(e, h_1 \ldots h_{k-1})) \equiv \delta_1(\mathbf{n}(e, h_1 \ldots h_{k-1})), \ldots$$
$$\ldots, \delta_q(\mathbf{n}(e, h_1 \ldots h_{k-1})) \equiv (w_2)_Y(\mathbf{n}(e, h_1 \ldots h_{k-1})).$$

Bearing in mind (4.32), we thus obtain that

$$(w_1)_Y(\mathbf{n}(e, g_1 \ldots g_m)) = (w_2)_Y(\mathbf{n}(e, h_1 \ldots h_{k-1})) \tag{4.33}$$

is a consequence of relations in $\mathbf{R}$ and of standard inverse semigroup relations. In the following two steps, we verify that

$$(w_2)_Y(\mathbf{n}(e, h_1 \ldots h_{k-1})) = (w_2)_Y(\mathbf{n}(f, h_1 \ldots h_{k-1}))$$

is a consequence of relations in $\mathbf{R}$, and thus verifying that $w_1 = w_2$ is a consequence of relations in $\mathbf{R}$.

Step 3. By (4.31), we have that

$$\widetilde{w_1} = \widetilde{(w_1)_Y} \wedge e$$

and so we obtain

$$\widetilde{w}_1 = \widetilde{w}_1 \wedge e.$$

On the other hand we know that $\widetilde{w}_1 = \widetilde{w}_2$, and hence

$$\widetilde{w}_2 = \widetilde{w}_2 \wedge e. \tag{4.34}$$

Step 4. We give a finite sequence of words

$$(w_2)_Y(\mathbf{n}(e, h_1 \ldots h_{k-1})) \equiv \sigma_0, \ldots, \sigma_n \equiv (w_2)_Y(\mathbf{n}(f, h_1 \ldots h_{k-1}))$$

where σ_{j+1} is obtained from σ_j, using relations in $\mathbf{R}$. Recall that

$$(w_2)_Y \equiv (f_1, 1)d_2 \ldots d_{k-1},$$

where $d_{k-1} \equiv v(f_k, 1)v^{-1}$ and $v \equiv \mathbf{n}(f, h_1 \ldots h_{k-1})$. Applying (R3), we obtain that $v^{-1}(f, 1) = v^{-1}$ and hence we get that

$$(w_2)_Y = (w_2)_Y(f, 1).$$

It follows that

$$\begin{aligned}(w_2)_Y(\mathbf{n}(e,h_1\dots h_{k-1})) &= (w_2)_Y(f,1)(\mathbf{n}(e,h_1\dots h_{k-1}))\\ &= (w_2)_Y(e,1)(\mathbf{n}(f,h_1\dots h_{k-1})) \qquad \text{by Lemma 4.10.4.}\end{aligned}$$

Bearing in mind that $\widetilde{w}_2 = \widetilde{w}_2 \wedge e$ by (4.34), there exists a finite sequence of words

$$(w_2)_Y(e,1) \equiv \zeta_1, \dots, \zeta_l = (w_2)_Y,$$

such that ζ_{j+1} is obtained from ζ_j using relations in $R_Y \cup R_L \cup R_G \cup R_M$. It follows that

$$(w_2)_Y(e,1)v \equiv \zeta_1 v, \dots, \zeta_l v = (w_2)_Y v,$$

verifying that $w_1 = w_2$ is indeed a consequence of relations in $\mathbf{R}$ and of inverse semigroup relations. □

We end this section by returning to our examples.

Example 4.10.5. Let Y_∞ be the double infinite chain with an identity element adjoined on top and let G be the infinite cyclic group. Define the action of G on Y_∞ as it was in Proposition 4.2.4. Clearly G is finitely presented and Y_∞ satisfies the maximum condition. According to Proposition 4.7.1, Y_∞ is finitely presented as an inverse semigroup with respect to the action of G. Hence, by Theorem 4.9.1, the semidirect product $S = Y_\infty \rtimes G$ is finitely presented as an inverse semigroup.

Example 4.10.6. Consider the semilattice F_∞ introduced in Proposition 4.2.6 and let G be the infinite cyclic group. Define the action of G on F_∞ as it was in Proposition 4.2.6. Clearly G is finitely presented and F_∞ satisfies the maximum condition. We have also shown in Proposition 4.7.2 that F_∞ is finitely presented as an inverse semigroup with respect to the action of G. Hence, by Theorem 4.9.1, the semidirect product $S = F_\infty \rtimes G$ is finitely presented as an inverse semigroup.

Example 4.10.7. Consider the semilattice A_∞ introduced in Proposition 4.2.5 and let G be the infinite cyclic group. Define the action of G on A_∞ as it was in Proposition 4.2.5. We verified in Proposition 4.7.3, that A_∞ is not finitely presented as an inverse semigroup with respect to the action of G. Hence, by Theorem 4.9.1, the semidirect product $S = A_\infty \rtimes G$ is not finitely presented.

4.11 Schützenberger automaticity

In this section, we give a necessary and sufficient condition for a semidirect product of a semilattice by a group to be Schützenberger automatic.

As in the previous sections, if we say that a group G acts on a semilattice Y, then it will be understood that G acts on Y on the left by automorphisms. We recall the notation introduced in Section 9: Let A be a semigroup generating set for the semidirect product $S = Y \rtimes G$ of a semilattice by a group and assume that $A \subseteq S$. For a word $w \equiv (e_1, g_1) \ldots (e_m, g_m) \in A^+$, we let

$$\widehat{w} \equiv g_1 \ldots g_m.$$

Proposition 4.11.1. *Let Y be a semilattice and G be a group acting on Y. Then the semidirect product $S = Y \rtimes G$ is Schützenberger automatic if and only if the following conditions hold:*

$(SA1)$ *G is automatic;*

$(SA2)$ *Y satisfies the maximum condition and is generated by a finite set Y_0 with respect to the action of G;*

$(SA3)$ *There exists an automatic structure (X, K) for G such that for all $e \in Y$ and $f \in Y_0$ the language*

$$K_e^f = \{w \in K |\ e \leq {}^w f\}$$

is regular.

Proof. $(\Longrightarrow)$ If $S = Y \rtimes G$ is Schützenberger automatic, then S is finitely generated. It follows by Proposition 4.3.1, that G is generated by a finite set X, that Y satisfies the maximum condition and is generated by a finite set Y_0 with respect to the action of G, and hence $(SA2)$ holds. Let Y_m denote the set of maximal elements of Y. For the sake of convenience we assume that X is closed under taking inverses, that $1 \in X \subseteq G$ and that $Y_m \subseteq Y_0 \subseteq Y$. In particular we may assume that S is generated by the finite set

$$A = (Y_0 \times \{1\}) \cup (Y_m \times X).$$

Since X is closed under taking inverses, for each element $(e, h) \in A$, $(e, h)^{-1} \in A$ holds by Lemma 4.2.1 and Proposition 4.2.7. Thus A is a semigroup generating set for S. Let $\varphi : A^+ \to S$ denote the homomorphism extending the identity map $\iota_1 : A \to S$ and let $\psi : X^+ \to G$ denote the homomorphism extending the identity map $\iota_2 : X \to G$. Let $e \in Y$ and consider $s = (e, 1) \in S$. Let R denote the $\mathcal{R}$-class R of s. By Proposition 4.2.7, we have that

$$R = \{(e, g) | g \in G\}. \tag{4.35}$$

Making use of Proposition 3.2.1, and of Corollaries 3.4.6 and 3.6.6 in Chapter 1, we have that there exists a regular language L over A such that the following conditions hold:

(c1) $s \cdot L\varphi = R$;

(c2) $L_= = \{(u,u) | u \in L\}\delta_A$ and hence is a regular language;

(c3) $L_a = \{(u,v) | u, v \in L,\ s \cdot u \cdot a = s \cdot v\}\delta_A$ is a regular language for all $a \in A$.

We claim that G is automatic by verifying that the fellow traveller property holds in the Cayley graph Δ of G. Consider the map

$$\widehat{\ }: L \to X^+; w \mapsto \widehat{w},$$

and let $K = \widehat{L}$. It follows by (4.35) and (c1), that $K\psi = G$.

The language K is regular, since if $\mathcal{A} = (\Sigma, A, \mu, p, F)$ is a finite state automaton accepting L, then the automaton obtained from $\mathcal{A}$ by changing the labels $(e,h) \in A$ to h accepts K. Let w_1 and $w_2 \in K$ such that

$$d_\Delta(w_1, w_2) \leq 1.$$

Then $w_1 \cdot x = w_2$ for some $x \in X$. Consider the elements u_1, u_2 of L for which $\widehat{u}_1 \equiv w_1$ and $\widehat{u}_2 \equiv w_2$. Clearly, for all $t \geq 1$

$$\widehat{u_1(t)} \equiv w_1(t) \qquad \text{and} \qquad \widehat{u_2(t)} \equiv w_2(t). \tag{4.36}$$

Let $f \in Y_m$ such that $e \leq {}^{w_1}f = {}^{\widehat{u}_1}f$. Since Y satisfies the maximum condition, such an element of Y_m exists by Lemma 4.2.1. Then $s \cdot u_1 \cdot (f, x) = s \cdot u_2$, and hence

$$d_\Gamma(s \cdot u_1, s \cdot u_2) \leq 1.$$

Since R is Schützenberger automatic, the fellow traveller property holds in the Schützenberger graph Γ of R by Proposition 3.7.10. Hence, there exists a constant $k \in \mathbb{N}$ such that whenever $u, v \in L$ with $d_\Gamma(s \cdot u, s \cdot v) \leq 1$, then $d_\Gamma(s \cdot u(t), s \cdot v(t)) \leq k$ holds for all $t \geq 1$. It follows that for all $t \geq 1$,

$$d_\Gamma(s \cdot u_1(t), s \cdot u_2(t)) \leq k.$$

Because of the observations made in Example 3.7.2 of Chapter 1, there exists a word α over A, whose length is less then or equal to k and

$$s \cdot u_1(t) \cdot \alpha = s \cdot u_2(t).$$

Bearing in mind (4.36) and that $s = (e, 1)$, we obtain that $w_1(t)\widehat{\alpha} = w_2(t)$, and hence

$$d_\Delta(w_1(t), w_2(t)) \leq k,$$

proving that the fellow traveller property holds in the Cayley graph of G.

Let $w_1, w_2 \in K$. Before we show that $(SA3)$ holds, we verify that

$$w_1 = w_2 \quad \text{if and only if} \quad w_1 \equiv w_2. \tag{4.37}$$

Clearly, if $w_1 \equiv w_2$, then $w_1 = w_2$. Assume that $w_1 = w_2$ and let $u_1, u_2 \in L$ such that $\hat{u}_1 \equiv w_1$ and $\hat{u}_2 \equiv w_2$. In particular we have that $s \cdot u_1 = (e, w_1)$ and $s \cdot u_2 = (e, w_2)$ and hence we may deduce that $(u_1, u_2) \in L_=$. It follows by condition (c2) that $u_1 \equiv u_2$ and so $w_1 \equiv w_2$, verifying (4.37). In particular we have obtained that

$$w_1 \equiv w_2 \quad \text{if and only if} \quad u_1 \equiv u_2. \tag{4.38}$$

To show that $(SA3)$ holds, let $f \in Y_0$ and consider the generator $(f, 1) \in A$. Our aim is to show that

$$K_e^f = \{w \in K | e \leq {}^w f\}$$

is a regular language. Note that

$$\begin{aligned} (u,v)\delta_A \in L_{(f,1)} &\iff s \cdot u \cdot (f,1) = s \cdot v \\ &\iff (e, \hat{u})(f,1) = (e, \hat{v}) \\ &\iff (\hat{u}, \hat{v}) \in K_=, \ \hat{u} \in K_e^f. \end{aligned}$$

Thus $\hat{u} \equiv \hat{v}$ by (4.37) and hence bearing in mind (4.38) we may deduce that

$$(u, v) \in L_{(f,1)} \quad \text{if and only if} \quad u \equiv v \text{ and } \hat{u} \in K_e^f. \tag{4.39}$$

By Proposition 2.3.1, we have that the language

$$L_e^f = \{u \in L \mid (u,v)\delta_A \in L_{(f,1)} \text{ for some } v \in A^\star\}$$

is regular. It follows that the language

$$\widehat{L}_e^f = \{\hat{u} \in K \mid u \in L_e^f\}$$

is regular as well. We finish our proof by showing that

$$\widehat{L}_e^f = K_e^f.$$

Let $w \in \widehat{L}_e^f$. Then $w \equiv \hat{u}$ for some $u \in L_e^f$, and hence $e \leq {}^{\hat{u}}f$ holds, verifying that $w \in K_e^f$. Now let $w \in K_e^f$. Then $e \leq {}^{w}f$. Let $u \in L$ such that $\hat{u} \equiv w$. Then by (4.39), $(u, u) \in L_{(f,1)}$, and so $u \in L_e^f$, proving that $w \in \widehat{L}_e^f$.

($\Longleftarrow$) For the converse, assume that G is a group acting on a semilattice Y, where G and Y satisfy conditions $(SA1) - (SA3)$. More precisely, assume that (X, K) is an automatic structure with uniqueness for G, where X is closed under taking inverses. To simplify notation, we assume that $1 \subseteq X \subseteq G$. Let $\psi : X^+ \to G$ denote the homomorphism extending the identity map $\iota_1 : X \to G$. Suppose that Y is generated by the finite set Y_0 with respect to the action of G. We assume that the set of maximal elements Y_m is contained in Y_0 and that for all $e \in Y$ and $f \in Y_0$, the language $K_e^f = \{w \in K | e \leq {}^{w}f\}$ is regular. Let $S = Y \rtimes G$. By Corollary 4.3.2, the finite set

$$A = (Y_0 \times \{1\}) \cup (Y_m \times X)$$

generates S as a semigroup. Let $\varphi : A^+ \to S$ be the homomorphism extending the identity map $\iota_2 : A \to S$.

Let R be an arbitrary $\mathcal{R}$-class of S and assume that $(e, 1) \in R$. To simplify notation, denote $(e, 1)$ by s. Choose $\tilde{e} \in Y_m$ such that $e \leq \tilde{e}$. Since Y satisfies the maximum condition, such a maximal element of Y exists. Define

$$\xi : K \to A^+; \ w \mapsto \mathbf{n}(\tilde{e}, w)$$

where $\mathbf{n}$ is the function defined in Section 8.

Let $L = K\xi$. We show that L is a regular language, by constructing a finite state automaton accepting it. Let $\mathcal{A}_1 = (\Sigma_1, X, \mu, p, T)$ be a finite state automaton accepting K. Such an automaton exists, since K is a regular language. To simplify notation, let $B = (Y_m \times X)$ and consider the automaton $\mathcal{A} = (\Sigma, B, \nu, (p, (\hat{e}, 1)), T \times B)$, where $\Sigma = (\Sigma_1 \times B) \cup \{FS\}$ and

$$\nu : ((r, (l, g)), (f, h)) \mapsto \begin{cases} (\mu(r, h), (f, h)) & \text{if } {}^{g}f = l, \\ FS & \text{otherwise.} \end{cases}$$

By the definition of the function $\mathbf{n}$, and since $\mathcal{A}_1$ is an automaton accepting K, we have that $\mathcal{A}$ is a finite state automaton accepting L.

By the definition of the function $\mathbf{n}$ and since K is an automatic structure with uniqueness, we have that

$$\mathbf{n}(\tilde{e}, w_1) = \mathbf{n}(\tilde{e}, w_2) \quad \text{if and only if} \quad w_1 = w_2. \tag{4.40}$$

Since $K\psi = G$ and since $R = \{(e, g) | g \in G\}$ by Proposition 4.2.7, $s \cdot L\varphi = R$ holds.

Let $u, v \in L$ and $w_1, w_2 \in K$ such that $w_1\xi \equiv u$ and $w_2\xi \equiv v$. Then

$$\begin{aligned}(u,v)\delta_A \in L_= &\iff s \cdot u = s \cdot v \\ &\iff (e,1)n(\tilde{e}, w_1) = (e,1)n(\tilde{e}, w_2) \\ &\iff (e, w_1) = (e, w_2) \\ &\iff (w_1, w_2)\delta_X \in K_=.\end{aligned}$$

It follows by (4.40) that $(u,v)\delta_A \in L_=$ if and only if $u \equiv v$, verifying that

$$L_= = \{(u,u) \mid u \in L\}\delta_A.$$

Let $(f,x) \in A$ and consider the language $K_e^f = \{u \in K | e \leq {}^u f\}$. By our assumptions, K_e^f is a regular language. Let $u, v \in L$ and $w_1, w_2 \in K$ such that $w_1\xi \equiv u$ and $w_2\xi \equiv v$. Then

$$\begin{aligned}(u,v)\delta_A \in L_{(f,x)} &\iff s \cdot u \cdot (f,x) = s \cdot v \\ &\iff (e,1)n(\tilde{e}, w_1)(f,x) = (e,1)n(\tilde{e}, w_2) \\ &\iff (e, w_1)(f,x) = (e, w_2) \\ &\iff (w_1, w_2)\delta_X \in K_x,\ w_1 \in K_e^f.\end{aligned}$$

It follows that

$$L_{(f,x)} = K_x\delta_X^{-1}(\xi \times \xi)\delta_A \cap (K_e^f \times K)(\xi \times \xi)\delta_A.$$

Since K, K_x and K_e^f are regular languages we have that $K_x\delta_X^{-1}(\xi\times\xi)\delta_A$ and $(K_e^f \times K)(\xi \times \xi)\delta_A$ are regular languages as well. Thus we may deduce that $L_{(f,x)}$ is a regular language. $\square$

Chapter 5

HNN extensions

In the 1950s, G. Higman, B. Neumann and H. Neumann investigated embeddability questions of groups. Among others, they considered the following problem. Let G be a group and U, V be subgroups of G that are isomorphic via $\varphi : U \to V$. They asked whether it is possible to embed G into a group H, so that H possesses an element t for which $t^{-1}ut = u\varphi$ holds for all $u \in U$. They showed that this is always possible; the construction they gave for H is called an HNN extension of G.

The above problem was first introduced for semigroups by Howie [20] in the following settings. Let S be a semigroup, $e \in E(S)$. Let $U, V \subseteq eSe$ be subsemigroups of S that are isomorphic via $\varphi : U \to V$. Does there exists a semigroup H and $t, t' \in H$, where $t' \in V(t)$ so that S embeds in H, $t'ut = u\varphi$ and $tt' = t't = e$. Howie proved that if U and V are unitary subsemigroups of eSe, then this is possible.

HNN extension of inverse semigroups was considered in recent years. There are two alternative definitions for an HNN extension of an inverse semigroup. One is introduced by Yamamura [36], the other by Gilbert [14]. In this chapter, we first discuss group HNN extensions from finite generation and finite presentability point of view. Regarding finite generation, a sufficient condition will be straightforward. Moreover, we shall see that Britton's lemma is the key in giving a necessary and sufficient condition for finite presentability of an HNN extension of a group. Considering Yamamura's HNN extension, we verify that the group theoretic results generalize. The role of Britton's lemma is played by the strong HNN property in this context. Gilbert's HNN extension is more delicate and the $\mathcal{J}$-preorder is the key notion in understanding finite generation and presentability. Regarding finite presentability, a variation of the strong HNN property will help us in giving a necessary and sufficient condition. Results of sections 3-8 are the result of a joint work [11] with N.Gilbert and N.Ruškuc. We give more detailed proofs for these results.

5.1 Group HNN extensions

The construction of an HNN extension of a group arose while Higman, Neumann and Neumann studied embeddability problems related to groups. The construction is employed in solving algorithmic problems, e.g. undecidability of Markov properties of finitely presented groups.

Definition 5.1.1. Let $G = \mathrm{Grp}\langle X|P\rangle$. Let U and V be subgroups of G that are isomorphic via $\varphi : U \to V$. Let $t \notin X$. The group presented by

$$H = \mathrm{Grp}\langle X, t \ |P,\ t^{-1}ut = u\varphi\ (u \in U)\rangle$$

is called the *HNN extension of G associated with $\varphi : U \to V$*. The element t is called the *stable letter* of H.

In what follows, we assume that $G = \mathrm{Grp}\langle X|P\rangle$, that U and V are subgroups in G that are isomorphic via $\varphi : U \to V$. To simplify notation, instead of

$$H = \mathrm{Grp}\langle X, t \ |P,\ t^{-1}ut = u\varphi\ (u \in U)\rangle,$$

we write

$$H = \mathrm{Grp}\langle G, t \ |t^{-1}Ut = V\rangle.$$

The following theorem is due to Higman, Neumann and Neumann.

Theorem 5.1.2. *The group G embeds in $H = \mathrm{Grp}\langle G, t \ |t^{-1}Ut = V\rangle$.*

Proof. See [2, Chapter IV].□

Clearly, if G is finitely generated, then the HNN extension H is also finitely generated. However, the converse is not necessarily true. To demonstrate this fact, we give an example. Let $C_2 = \langle x\rangle$ be the cyclic group of order two and let

$$G = \ldots \times C_2 \times C_2 \times C_2 \times \ldots.$$

Clearly G is not finitely generated. In particular we have that G is generated by the elements of the form

$$\mathbf{x}_i = (\ldots, 1, 1, x, 1, 1, \ldots),$$

where x appears exactly at the ith position. Denote the set of these elements by X. Let $P = \{\mathbf{x}^2 = 1_G, \mathbf{xy} = \mathbf{yx} | \mathbf{x}, \mathbf{y} \in X\}$. Then $G = \mathrm{Grp}\langle X|P\rangle$. Let $U = V = G$. We consider the following automorphism of G:

$$\varphi : G \to G;\ \mathbf{x}_{i_1}\mathbf{x}_{i_2}\ldots\mathbf{x}_{i_n} \mapsto \mathbf{x}_{i_1+1}\mathbf{x}_{i_2+1}\ldots\mathbf{x}_{i_n+1}.$$

Consider the HNN extension of G associated with $\varphi : G \to G$:

$$H = \mathrm{Grp}\langle X, t \mid P,\ t^{-1}ut = u\varphi,\ u \in G\rangle.$$

Observe that for every $i \in \mathbb{Z}$, we have $\mathbf{x}_i = (t^{-1})^i \mathbf{x}_0 t^i$. It follows that H is generated by the finite set $\{\mathbf{x}_0, t\}$ as a group.

Next, we are going to discuss when H is finitely presented. For this, we need the following notion.

Definition 5.1.3. Consider the HNN extension $H = \mathrm{Grp}\langle G, t \mid t^{-1}Ut = V\rangle$. A *pinch* is a word over $G \cup \{t, t^{-1}\}$ of the form $t^{-1}ut$, where $u \in U$ or of the form tvt^{-1}, where $v \in V$.

Lemma 5.1.4 (Britton). *Consider $H = \mathrm{Grp}\langle G, t \mid t^{-1}Ut = V\rangle$. Let*

$$w \equiv g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \dots t^{\epsilon_n} g_n,$$

where $g_j \in G$ $(0 \le j \le n)$ and $\epsilon_j \in \{1, -1\}$ $(1 \le i \le n)$. If w represents the identity element of H, then it contains a pinch.

Proof. See [2, Chapter IV].□

Proposition 5.1.5. *Let G be defined by the finite group presentation $\langle X \mid P\rangle$. Let U and V be subgroups in G that are isomorphic via $\varphi : U \to V$. Then the HNN extension*

$$H = \mathrm{Grp}\langle X, t \mid P,\ t^{-1}ut = u\varphi\ (u \in U)\rangle$$

is finitely presented if and only if U is finitely generated.

Proof. Assume that the HNN extension

$$H = \mathrm{Grp}\langle X, t \mid P,\ t^{-1}ut = u\varphi\ (u \in U)\rangle$$

is finitely presented. Then there exists a finite subset $Y \subseteq U$ such that H is isomorphic to

$$H_0 = \mathrm{Grp}\langle G, t \mid t^{-1}ut = u\varphi,\ (u \in Y)\rangle.$$

Let U_0 denote the subgroup of U generated by Y. In particular, we have that H_0 is the HNN extension of G associated with $\hat{\varphi} : U_0 \mapsto U_0\varphi$. By assumption U is not finitely generated, and so there exists $u \in U \setminus U_0$. Then $t^{-1}ut(u\varphi)^{-1} = 1$ holds in H and hence in H_0. Making use of Britton's lemma it follows that $t^{-1}ut(u\varphi)^{-1}$ contains a pinch and so $u \in U_0$, leading to a contradiction.

For the converse, assume that U is finitely generated by Y. Then, it is immediate that

$$H = \mathrm{Grp}\langle X, t \mid P,\ t^{-1}yt = y\varphi\ (y \in Y)\rangle.$$

□

5.2 Yamamura's HNN extension

Yamamura generalized the notion of an HNN extension to inverse semigroups in such a way that the construction possesses similar properties as in the case of groups. In [36], it is shown that several important inverse semigroups, e.g. free inverse semigroups and the bicyclic monoid arise as Yamamura HNN extensions of certain semilattices. The Yamamura HNN extension is as powerful as in the case of groups for discussing algorithmic problems. Among others, undecidability of Markov properties of finitely presented inverse semigroups is proved in [36].

Definition 5.2.1. Let $S = \mathrm{Inv}\langle X|P\rangle$ be an inverse semigroup. Let $e, f \in E(S)$. Let U and V be inverse subsemigroups of S such that $e \in U \subseteq eSe$ and $f \in V \subseteq fSf$. Assume that U and V are isomorphic via $\varphi : U \to V$. The inverse semigroup S^* defined by the presentation

$$S^* = \mathrm{Inv}\langle X, t \mid P,\ tt^{-1} = e,\ t^{-1}t = f,\ t^{-1}ut = u\varphi\ (u \in U)\rangle$$

is called the *Yamamura HNN extension of S associated with $\varphi : U \to V$*. The element t is called the *stable letter* of S^*.

In what follows, we assume that $S = \mathrm{Inv}\langle X|P\rangle$, that $e, f \in E(S)$ and that $e \in U \subseteq eSe$ and $f \in V \subseteq fSf$ are isomorphic subsemigroups in S via $\varphi : U \to V$. To simplify notation, instead of

$$S^* = \mathrm{Inv}\langle X, t \mid P,\ tt^{-1} = e,\ t^{-1}t = f,\ t^{-1}ut = u\varphi\ (u \in U)\rangle$$

we write

$$S^* = \mathrm{Inv}\langle S, t \mid tt^{-1} = e, t^{-1}t = f, t^{-1}Ut = V\rangle.$$

The following theorem is proved by Yamamura in [36].

Theorem 5.2.2. *The inverse semigroup S embeds in the Yamamura HNN extension $S^* = \mathrm{Inv}\langle S, t \mid tt^{-1} = e,\ t^{-1}t = f,\ t^{-1}Ut = V\rangle$.*

If S is finitely generated, then the HNN extension S^* is also finitely generated. As in the case of groups, the converse is not true. The same example as the one introduced in the previous section demonstrates this fact.

The main purpose of this section is to discuss finite presentability of the Yamamura HNN extension. For this, we need the following theorem proved by Yamamura [36, Theorem 12].

Theorem 5.2.3. *Let S be an inverse semigroup and consider the Yamamura HNN extension $S^* = \mathrm{Inv}\langle S, t \mid tt^{-1} = e,\ t^{-1}t = f,\ t^{-1}Ut = V\rangle$. Then $t^{-1}St \cap S = t^{-1}Ut = V$.*

The above theorem can be considered as a weaker version of Britton's lemma and is called the strong HNN property by Yamamura. Whether Britton's lemma can be generalized for Yamamura's construction is still unknown. Yamamura's conjecture is that a fully analogous result does not hold. We see that concerning finite presentability of Yamamura's HNN extension, the group theoretic result generalizes.

Proposition 5.2.4. *Let S be defined by the finite inverse semigroup presentation $\langle X|P\rangle$. Let $e, f \in E(S)$ and let $e \in U \subseteq eSe$ and $f \in V \subseteq fSf$ be inverse subsemigroups that are isomorphic via $\varphi : U \to V$. Then the Yamamura HNN extension*

$$S^* = \mathrm{Inv}\langle X, t \mid P,\ tt^{-1} = e,\ t^{-1}t = f,\ t^{-1}ut = u\varphi\ (u \in U)\rangle$$

is finitely presented if and only if U is finitely generated.

Proof. Assume that the Yamamura HNN extension

$$S^* = \mathrm{Inv}\langle X, t \mid P,\ tt^{-1} = e,\ t^{-1}t = f,\ t^{-1}ut = u\varphi\ (u \in U)\rangle$$

is finitely presented. Then there exists a finite subset $Y \subseteq U$ such that S^* is isomorphic to

$$S_0^* = \mathrm{Inv}\langle X, t \mid P,\ tt^{-1} = e,\ t^{-1}t = f,\ t^{-1}ut = u\varphi\ (u \in Y)\rangle.$$

Without loss of generality we may assume that $e \in Y$. Let U_0 be the inverse subsemigroup of U generated by Y. Since e is the identity element of U_0, we have that for all $u \in U_0$, the relation $t^{-1}ut = u\varphi$ is a consequence of relations of the form $t^{-1}yt = y\varphi$, where $y \in Y$. It follows that S_0^* is the Yamamura HNN extension of S associated with $\hat{\varphi} : U_0 \to U_0\varphi$. Making use of Theorem 5.2.3, we obtain that

$$t^{-1}St \cap S = t^{-1}U_0t = U_0\hat{\varphi} = U_0\varphi. \tag{5.1}$$

By assumption U is not finitely generated, and so there exists $u \in U \setminus U_0$. Clearly $t^{-1}ut = u\varphi$ holds in S^*, and so in S_0^*. By (5.1), there exists $u_0 \in U_0$ such that $t^{-1}ut = u_0\varphi$, and hence $u\varphi = u_0\varphi$. Since φ is an isomorphism it is injective, and so we obtain the contradiction that $u = u_0$. Thus U is indeed finitely generated.

For the converse, assume that U is generated by a finite set Y. Since e is the identity element of U, we obtain that the Yamamura HNN extension is defined by the finite presentation

$$S^* = \mathrm{Inv}\langle X, t \mid P,\ tt^{-1} = e,\ t^{-1}t = f,\ t^{-1}ut = u\varphi\ (u \in Y)\rangle.$$

□

5.3 Gilbert's HNN extension

Making use of the relationship between inductive groupoids and inverse semigroups, Gilbert introduced an alternative notion for an HNN extension of an inverse semigroup in [14]. The relationship between the two notions of HNN extensions is discussed, moreover maximal subgroups and the maximal group homomorphic image of the construction is described [14]. In [13], Gilbert shows that free inverse semigroups, and the bicyclic monoid arise as Gilbert's HNN extension of certain semilattices. To be more accurate, the latter case is set in a more general context. Namely, it is explained how Bruck-Reilly extensions of monoids arise as Gilbert's HNN extensions. The purpose of this section is to introduce Gilbert's notion of an HNN extension of an inverse semigroup, and to prove some useful results.

First, we recall some definitions and results regarding inverse semigroups. We have already mentioned, that the natural partial order $\leq$ has an important role in inverse semigroup theory.

Definition 5.3.1. Let S be an inverse semigroup and $s, t \in S$. We define

$$s \leq t \iff s = et$$

for some $e \in E(S)$.

It is easy to see, that

$$\begin{aligned} s \leq t &\iff s = ft \quad \text{for some } f \in E(S) \\ &\iff s^{-1} \leq t^{-1}. \end{aligned}$$

Definition 5.3.2. Let S be an inverse semigroup and U be a subset of S. We say that U is an *order ideal* of S, if whenever $s \leq u$, where $u \in U$, then $s \in U$ holds.

There is another relation which will be important in our considerations, namely the $\leq_{\mathcal{J}}$ preorder.

Definition 5.3.3. Let S be an inverse semigroup and $s, t \in S$. We define

$$s \leq_{\mathcal{J}} t \iff s \in StS.$$

Note that $s \leq_{\mathcal{J}} t$ and $t \leq_{\mathcal{J}} s$ if and only if $s \mathrel{\mathcal{J}} t$. Moreover, if $s \leq t$, then $s \leq_{\mathcal{J}} t$. With the help of the natural partial order and the $\mathcal{D}$-relation the $\leq_{\mathcal{J}}$ preorder can be described in the following way [27, Proposition 8]:

Lemma 5.3.4. *Let S be an inverse semigroup and $s, t \in S$. Then $s \leq_{\mathcal{J}} t$ if and only if there exists $u \in S$ such that $s\mathcal{D}u \leq t$.*

Definition 5.3.5. Let S be an inverse semigroup. A set $F \subseteq E(S)$ is said to *$\mathcal{J}$-dominate* S, if for every $s \in S$ there exists $f \in F$ such that $s \leq_{\mathcal{J}} f$.

Notation. Let U be an inverse subsemigroup of S. Then for any $F \subseteq E(U)$ we let

$$U_F = \{u \in U \mid \exists e, f \in F \text{ such that } uu^{-1} \leq e,\ u^{-1}u \leq f\};$$

this is easily seen to be an inverse subsemigroup of S.

Definition 5.3.6. Let $S = \mathrm{Inv}\langle X|P\rangle$. Let U and V be inverse subsemigroups of S that are order ideals. Assume that U and V are isomorphic via $\varphi : U \to V$. The inverse semigroup $S*_{U,\varphi}$ defined by the presentation

$$S*_{U,\varphi} = \mathrm{Inv}\langle X \cup \{t_e \mid e \in E(U)\} \mid P,\ t_e t_f^{-1} = ef,\ t_e^{-1}t_f = (ef)\varphi, \\ t_{uu^{-1}}^{-1} u t_{u^{-1}u} = u\varphi\ (u \in U)\rangle$$

is called the *Gilbert HNN extension of S associated with $\varphi : U \to V$*. The elements t_e are called the *stable letters* of $S*_{U,\varphi}$.

To make reference easier to certain relations, we fix the following notations for this section. Let $S = \mathrm{Inv}\langle X|P\rangle$. We let U and V denote inverse subsemigroups of S that are order ideals. We assume that U and V are isomorphic via $\varphi : U \to V$. For a subset $L \subseteq E(U)$ we let

$$T_L = \{t_e t_f^{-1} = ef,\ t_e^{-1}t_f = (ef)\varphi \mid e, f \in L\}$$

and for brevity we denote $T_{E(U)}$ by T. For a subset $L \subseteq E(U)$ and a subset K of U we let

$$W_{L,K} = \{t_e^{-1} a t_f = a\varphi \mid a \in K,\ e, f \in L,\ aa^{-1} \leq e,\ a^{-1}a \leq f\}$$

and for brevity we denote $W_{E(U),U}$ by W. The following proposition will prove useful.

Proposition 5.3.7. *(a) If $e \in E(S)$ and $f \in E(U)$, then $et_f = t_{ef}$. In particular, if $e \leq f$, then $et_f = t_e$.*
(b) Let $e, f \in E(U)$ and $u \in U$ with $uu^{-1} \leq e$ and $u^{-1}u \leq f$. Then the relation $t_e^{-1}ut_f = u\varphi$ holds in $S*_{U,\varphi}$, and so

$$S*_{U,\varphi} = \mathrm{Inv}\langle X \cup \{t_e | e \in E(U)\} | P, T, W\rangle.$$

(c) Fix a subset $F \subseteq E(U)$ and suppose that U_F is generated by a subset $Y \subseteq U_F$. Then for all $u \in U_F$ with $uu^{-1} \leq e$ and $u^{-1}u \leq f$, $e, f \in F$, the relation $t_e^{-1}ut_f = u\varphi$ is a consequence of the relations in $P \cup T_F \cup W_{F,Y}$.

Proof. (a) Let $e \in E(S)$ and $f \in E(U)$. We verify that $et_f = t_{ef}$ is a consequence of the defining relations of $S*_{U,\varphi}$ and of standard inverse semigroup relations. Indeed,

$$\begin{aligned}
et_f &= et_f t_f^{-1} t_f \\
&= eft_f && \text{by applying a relation in } T \\
&= t_{ef} t_{ef}^{-1} t_f && \text{by applying a relation in } T \\
&= t_{ef}(ef)\varphi && \text{by applying a relation in } T \\
&= t_{ef}(ef)\varphi t_{ef}^{-1} t_{ef} && \text{by applying standard inv. sg. relations} \\
&= (ef)t_{ef} && \text{since } t_{ef}(ef)\varphi t_{ef}^{-1} = ef \\
&= t_{ef} t_{ef}^{-1} t_{ef} && \text{by applying a relation in } T \\
&= t_{ef}.
\end{aligned}$$

(b) Since $uu^{-1} \leq e$, we have that $euu^{-1} = uu^{-1}$. Since $u^{-1}u \leq f$, we have that $u^{-1}uf = u^{-1}u$. Thus, with the previous part of the proposition in mind, we obtain that

$$t_e^{-1}ut_f = t_e^{-1}uu^{-1}uu^{-1}ut_f = t_{uu^{-1}}^{-1}ut_{u^{-1}u} = u\varphi.$$

It follows that all relations in W hold in $S*_{U,\varphi}$. On the other hand

$$\{t_{uu^{-1}}^{-1}ut_{u^{-1}u} = u\varphi \mid u \in U\} \subseteq W,$$

and so we obtain that

$$S*_{U,\varphi} = \mathrm{Inv}\langle X \cup \{t_e \mid e \in E(U)\} | \; P, T, W\rangle.$$

(c) We proceed by induction on the number of elements of Y required to express $u \in U$ as a product. The base case is dealt with by the relations with $y \in Y$. Let $u = u_1u_2$, with $u_1, u_2 \in U_F$ and assume that $u_1u_1^{-1} \leq i = t_it_i^{-1}$, $u_1^{-1}u_1 \leq g = t_gt_g^{-1}$ and $u_2u_2^{-1} \leq h = t_ht_h^{-1}$, $u_2^{-1}u_2 \leq j = t_jt_j^{-1}$ for some $i, j, g, h \in F$. Then $u_1 = iu_1g$ and $u_2 = hu_2j$ hold as a consequence of P, and hence

$$\begin{aligned}
t_e^{-1}ut_f &= t_e^{-1}u_1u_2t_f \\
&= t_e^{-1}(iu_1g) \cdot (hu_2j)t_f \\
&= t_e^{-1}t_i(t_i^{-1}u_1t_g)(t_g^{-1} \cdot t_h)(t_h^{-1}u_2t_j)t_j^{-1}t_f \\
&= t_e^{-1}t_i \; (u_1\varphi) \; t_g^{-1}t_h \; (u_2\varphi) \; t_j^{-1}t_f \\
&= (ei)\varphi \; (u_1\varphi) \; (gh)\varphi \; (u_2\varphi) \; (jf)\varphi = (e(iu_1g)(hu_2j)f)\varphi \\
&= (eu_1u_2f)\varphi = u\varphi.
\end{aligned}$$

□

Definition 5.3.8. Let S be an inverse semigroup. A product $s_1s_2\dots s_n$ of elements of S is called a *trace product* if $s_i^{-1}s_i = s_{i+1}s_{i+1}^{-1}$ for all $1 \le i \le n-1$.

Lemma 5.3.9. *Let S be an inverse semigroup and let $s_1, s_2 \in S$. Then there exists $t_1, t_2 \in S$ such that the product of s_1s_2 equals to the trace product t_1t_2 and $t_1 \le s_1, t_2 \le s_2$ holds.*

Proof. Consider the idempotent $e = s_1^{-1}s_1s_2s_2^{-1}$ and let $t_1 = s_1e$ and $t_2 = es_2$. Clearly $s_1 \le t_1$ and $s_2 \le t_2$. Moreover an easy calculation shows that $t_1^{-1}t_1 = t_2t_2^{-1}$ and that $t_1t_2 = s_1es_2 = s_1s_2$. □

By an easy induction the following lemma can be verified.

Lemma 5.3.10. *Let S be an inverse semigroup. Then a product of elements $s_1s_2\dots s_n$ of S equals to a trace product $t_1t_2\dots t_n$ with $t_i \le s_i$ for all $1 \le i \le n$. Moreover, if $s = t_1\dots t_n$ is a trace product, then $ss^{-1} = t_1t_1^{-1}$ and $s^{-1}s = t_n^{-1}t_n$.*

In the HNN extension $S*_{U,\varphi}$, we shall consider products that are words on the alphabet $S \cup \{t_e | e \in E(U)\}$: for such words, we can form trace products that preserve the structure of the word. Combining Lemma 5.3.10 with Proposition 5.3.7(a) we obtain the following:

Lemma 5.3.11. *An element $h = s_1t_{f_1}^{\epsilon_1}s_2t_{f_2}^{\epsilon_2}\cdots s_nt_{f_n}^{\epsilon_n}s_{n+1} \in S*_{U,\varphi}$, where $\epsilon_i = \pm 1$ for $1 \le i \le n$ and $s_i \in S$ for $1 \le i \le n+1$ is equal to a trace product $r_1t_{e_1}^{\epsilon_1}r_2t_{e_2}^{\epsilon_2}\cdots r_nt_{e_n}^{\epsilon_n}r_{n+1}$ where $e_i \le f_i$ for $1 \le i \le n$ and $r_i \le s_i$, $r_i \in S$ for $1 \le i \le n+1$.*

Definition 5.3.12. Let S be an inverse semigroup. Let U and V be inverse subsemigroups of S that are order ideals. Assume that U and V are isomorphic via $\varphi : U \to V$. Consider the Gilbert HNN extension $S*_{U,\varphi}$. A *pinch* is a word over $S \cup \{t_e, t_e^{-1} | \ e \in E(U)\}$ of the form $t_{uu^{-1}}^{-1}ut_{u^{-1}u}$, where $u \in U$ or of the form $t_{uu^{-1}}^{-1}(u\varphi)t_{u^{-1}u}$.

The following proposition will prove useful when proving Proposition 5.5.1.

Proposition 5.3.13. [18, 29] *Let $h = s_1t_{e_1}^{\epsilon_1}s_2t_{e_2}^{\epsilon_2}\cdots s_nt_{e_n}^{\epsilon_n}s_{n+1}$ be a trace product in $S*_{U,\varphi}$, with $n > 0$. If h is an idempotent then it contains a pinch.*

5.4 An alternative presentation for Gilbert's HNN extension

In preparation for discussing finite presentability of Gilbert's HNN extension, we give an alternative presentation for the HNN extension $S*_{U,\varphi}$. Throughout this section, we use the notation introduced in the previous section.

Proposition 5.4.1. *Let $S = \mathrm{Inv}\langle X|P\rangle$ be an inverse semigroup and let U, V be inverse subsemigroups of S that are order ideals. Assume that U and V are isomorphic via $\varphi : U \to V$. Let F be any $\mathcal{J}$-dominant subset of U. Then*

$$S*_{U,\varphi} = \mathrm{Inv}\langle X \cup \{t_e : e \in F\}|P,\ T_F,\ W_{F,U_F}\rangle.$$

Proof. Let $X_F = X \cup \{t_e : e \in F\}$ and $H_F = \mathrm{Inv}\langle X_F|P,\ T_F,\ W_{F,U_F}\rangle$. By Proposition 5.3.7, $S*_{U,\varphi} = \mathrm{Inv}\langle X \cup \{t_e : e \in E(U)\}|P,\ T,\ W\rangle$. We show that $S*_{U,\varphi}$ and H_F are isomorphic inverse semigroups by giving two inverse semigroup homomorphisms $\eta : H_F \to S*_{U,\varphi}$ and $\xi : S*_{U,\varphi} \to H_F$ that are inverse to each other.

Since $T_F \subseteq T$ and $W_{F,U_F} \subseteq W$, we clearly have that the identity map on X_F induces an inverse semigroup homomorphism $\eta : H_F \to S*_{U,\varphi}$.

For each $e \in E(U)$, choose and fix $u_e \in U$ and $\overline{e} \in F$ such that

$$e = u_e u_e^{-1},\ u_e^{-1}u_e \leq \overline{e};$$
$$u_e = \overline{e} = e\ (e \in F).$$

We show that the mapping $t_e \mapsto u_e t_{\overline{e}}(u_e\varphi)^{-1}$ and the identity map on X together induce an inverse semigroup homomorphism $\xi : S*_{U,\varphi} \to H_F$. That is we show that ξ maps relations in $S*_{U,\varphi}$ to relations that hold in H_F.

Let $e_1, e_2 \in E(U)$. Clearly $u_{e_1}^{-1}u_{e_2}u_{e_2}^{-1}u_{e_1} \leq u_{e_1}^{-1}u_{e_1} \leq \overline{e}_1$ and $u_{e_2}^{-1}u_{e_1}u_{e_1}^{-1}u_{e_2} \leq u_{e_2}^{-1}u_{e_2} \leq \overline{e}_2$, and we have

$$\begin{aligned} t_{e_1}\xi t_{e_2}^{-1}\xi &= u_{e_1}t_{\overline{e}_1}(u_{e_1}\varphi)^{-1}(u_{e_2}\varphi)t_{\overline{e}_2}^{-1}u_{e_2}^{-1} \\ &= u_{e_1}t_{\overline{e}_1}(u_{e_1}^{-1}u_{e_2})\varphi t_{\overline{e}_2}^{-1}u_{e_2}^{-1} \\ &= u_{e_1}u_{e_1}^{-1}u_{e_2}u_{e_2}^{-1} = e_1e_2 = (e_1e_2)\xi \end{aligned}$$

using the relation $t_{\overline{e}_1}(u_{e_1}^{-1}u_{e_2})\varphi t_{\overline{e}_2}^{-1} = u_{e_1}^{-1}u_{e_2}$ from W_{F,U_F} and it follows that $(t_{e_1}t_{e_2}^{-1})\xi = (e_1e_2)\xi$ indeed holds in H_F. Similarly,

$$\begin{aligned} t_{e_1}^{-1}\xi t_{e_2}\xi &= (u_{e_1}\varphi)t_{\overline{e}_1}^{-1}u_{e_1}^{-1}u_{e_2}t_{\overline{e}_2}(u_{e_2}\varphi)^{-1} \\ &= (u_{e_1}\varphi)(u_{e_1}^{-1}u_{e_2})\varphi(u_{e_2}\varphi)^{-1} \\ &= (u_{e_1}u_{e_1}^{-1}u_{e_2}u_{e_2}^{-1})\varphi = (e_1e_2)\varphi = ((e_1e_2)\varphi)\xi. \end{aligned}$$

and it follows that $(t_{e_1}^{-1}t_{e_2})\xi = ((e_1e_2)\varphi)\xi$. These considerations show that all the relations in T, once rewritten in terms of X_F, follow from T_U and W_{F,U_F} in H_F.

Now consider a relation $t_{e_1}^{-1}ut_{e_2} = u\varphi$ from W, where $uu^{-1} \leq e_1$ and $u^{-1}u \leq e_2,\ u \in U,\ e_1, e_2 \in E(U)$. Consider $t_{e_1}\xi = u_{e_1}t_{\overline{e}_1}(u_{e_1}\varphi)^{-1}$ and $t_{e_2}\xi = u_{e_2}t_{\overline{e}_2}(u_{e_2}\varphi)^{-1}$. Note that

$$(u_{e_1}^{-1}uu_{e_2})(u_{e_2}^{-1}u^{-1}u_{e_1}) = u_{e_1}^{-1}ue_2u^{-1}u_{e_1} = u_{e_1}^{-1}uu^{-1}u_{e_1} = u_{e_1}^{-1}u_{e_1} \leq \overline{e}_1$$

and similarly $(u_{e_2}^{-1}u^{-1}u_{e_1})(u_{e_1}^{-1}uu_{e_2}) \leq \overline{e}_2$, hence $u_{e_1}^{-1}uu_{e_2} \in U_F$. In particular, we have that $t_{\overline{e}_1}^{-1}u_{e_1}^{-1}uu_{e_2}t_{\overline{e}_2} = (u_{e_1}^{-1}uu_{e_2})\varphi$ is a relation in W_{F,U_F} and we obtain

$$\begin{aligned} t_{e_1}^{-1}\xi u\xi t_{e_2}\xi &= (u_{e_1}\varphi)t_{\overline{e}_1}^{-1}u_{e_1}^{-1}uu_{e_2}t_{\overline{e}_2}(u_{e_2}\varphi)^{-1} \\ &= (u_{e_1}\varphi)(u_{e_1}^{-1}uu_{e_2})\varphi(u_{e_2}\varphi)^{-1} \\ &= (u_{e_1}u_{e_1}^{-1}uu_{e_2}u_{e_2}^{-1})\varphi \\ &= (e_1ue_2)\varphi = u\varphi = (u\varphi)\xi. \end{aligned}$$

It clearly follows that $(t_{e_1}^{-1}ut_{e_2})\xi = (u\varphi)\xi$ indeed holds in H_F. This completes the check that ξ is a homomorphism.

Let us now check that η and ξ are inverse to each other. Indeed, for $x \in X$ we clearly have $x\eta\xi = x$ and $x\xi\eta = x$. Likewise, for $e \in F$ we have $t_e\eta\xi = t_e$. Now consider any $e \in E(U)$; we have

$$t_e\xi\eta = (u_et_{\overline{e}}(u_e\varphi)^{-1})\eta = u_et_{\overline{e}}(u_e\varphi)^{-1} = u_eu_e^{-1}t_e = et_e = t_e,$$

using the relations $t_e^{-1}u_et_{\overline{e}} = u_e\varphi$ and $et_e = t_e$, which hold in $S*_{U,\varphi}$. This completes the proof that η and ξ are mutually inverse. □

5.5 Finite generation of Gilbert's HNN extension

In this section, a necessary and sufficient condition will be given for the Gilbert HNN extension of inverse semigroups to be finitely generated.

As, in the previous section, we use the following notation. We let S denote an inverse semigroup. We let U and V denote inverse subsemigroups of S that are order ideals. We assume that U and V are isomorphic via $\varphi : U \to V$. For a subset $L \subseteq E(U)$ we let

$$T_L = \{t_et_f^{-1} = ef,\ t_e^{-1}t_f = (ef)\varphi \mid e, f \in L\}$$

and for brevity we denote $T_{E(U)}$ by T. For a subset $L \subseteq E(U)$ and a subset K of U we let

$$W_{L,K} = \{t_e^{-1}at_f = a\varphi \mid a \in K,\ e, f \in L,\ aa^{-1} \leq e,\ a^{-1}a \leq f\}$$

and for brevity we denote $W_{E(U),U}$ by W.

Proposition 5.5.1. *Let S be a finitely generated inverse semigroup. Let U and V be inverse subsemigroups of S that are order ideals. Assume that U and V are isomorphic via $\varphi : U \to V$. Then the Gilbert HNN extension $S*_{U,\varphi}$ is finitely generated if and only if U is finitely $\mathcal{J}$-dominated.*

Proof. Suppose that S is finitely generated by X and that $S*_{U,\varphi}$ is finitely generated. Then there exists a finite subset $F \subseteq E(U)$, such that $S*_{U,\varphi}$ is generated by $X \cup \{t_f | f \in F\}$, and hence by $S \cup \{t_f | f \in F\}$. Let $e \in E(U) \setminus F$. Then, in the Gilbert HNN extension $S*_{U,\varphi}$, the stable letter t_e is equal to some word

$$w \equiv s_1 t_{f_1}^{\epsilon_1} s_2 t_{f_2}^{\epsilon_2} \cdots s_n t_{f_n}^{\epsilon_n} s_{n+1}$$

on the generating set $S \cup \{t_f | f \in F\}$, where $\epsilon_i = \pm 1, (1 \leq i \leq n)$. By Lemma 5.3.11, t_e may also be represented by a trace product

$$w' = r_1 t_{e_1}^{\epsilon_1} r_2 t_{e_2}^{\epsilon_2} \cdots r_n t_{e_n}^{\epsilon_n} r_{n+1}$$

where for each $i, (1 \leq i \leq n)$, $e_i \leq f_i$ and $r_i \leq s_i$ for each $i, (1 \leq i \leq n+1)$. By application of W–relations we may assume that w' contains no pinch. It is possible that w' is the trace product t_e, in which case $e \leq f$ for some $f \in F$, in particular $e \leq_{\mathcal{J}} f$. Assume now that w' is not identical to t_e. Note that by Lemma 5.3.10 $w' t_e^{-1}$ is a trace product and by Proposition 5.3.13 the trace product $w' t_e^{-1}$ contains a pinch. This must be the final segment $t_{e_n}^{\epsilon_n} r_{n+1} t_e^{-1}$, where $r_{n+1} = u\varphi$ for some $u \in U$ and $e = u^{-1}u$. Hence $e\mathcal{D}uu^{-1}$ and $uu^{-1} \leq f_n \in F$: that is, $e \leq_{\mathcal{J}} f_n$. Therefore, U is finitely $\mathcal{J}$–dominated by the set F.

Conversely, suppose that U is finitely $\mathcal{J}$–dominated by F. Given $e \in E(U)$ suppose that $e \leq_{\mathcal{J}} f$ with $e\mathcal{D}h$ and $h \leq f$. Take $u \in U$ with $uu^{-1} = e$ and $u^{-1}u = h$. Then

$$\begin{aligned}
t_e^{-1} &= t_e^{-1} t_e t_e^{-1} && \text{by applying standard inv. sg. relations} \\
&= t_e^{-1} uu^{-1} && \text{by applying relations in } T \\
&= t_e^{-1} uu^{-1}uu^{-1} && \text{by applying standard inv. sg. relations} \\
&= t_e^{-1} uhu^{-1} && \\
&= t_e^{-1} ut_h t_h^{-1} u^{-1} && \text{by applying relations in } T \\
&= u\varphi t_h^{-1} u^{-1} && \text{by applying relations in } T,
\end{aligned}$$

and so $t_e = ut_h(u\varphi)^{-1}$ holds. In particular we have that the following hold in $S*_{U,\varphi}$:

$$\begin{aligned}
t_e = ut_h(u\varphi)^{-1} &= ut_h t_h^{-1} t_h (u\varphi)^{-1} && \\
&= uht_h(u\varphi)^{-1} && \text{by applying relations in } T \\
&= uht_f(u\varphi)^{-1} && \text{by Proposition 5.3.7} \\
&= ut_f(u\varphi)^{-1}. &&
\end{aligned}$$

Thus we may deduce that t_e is expressible as a word on the generators $X \cup \{t_f | f \in F\}$, and $S*_{U,\varphi}$ is finitely generated. $\square$

5.6 Finite presentability of Gilbert's HNN extension

The main purpose of this section is to give a necessary and sufficient condition for $S*_{U,\varphi}$ to be finitely presented.

Proposition 5.6.1. *Let S be a finitely presented inverse semigroup and let U, V be inverse subsemigroups of S, that are order ideals. Assume that U and V are isomorphic via $\varphi : U \to V$. Then the Gilbert HNN extension $S*_{U,\varphi}$ is finitely presented if and only if U is finitely $\mathcal{J}$-dominated by a subset $F \subseteq E(U)$ such that the inverse subsemigroup U_F is finitely generated.*

Proof. Assume that the inverse semigroup S is defined by the finite inverse semigroup presentation $\langle X|P\rangle$. Assume that U is $\mathcal{J}$-dominated by a finite subset $F \subseteq E(U)$, and that U_F is generated by a finite set Y of words on X. As before, let $X_F = X \cup \{t_e | e \in F\}$. Then by Proposition 5.4.1,

$$S*_{U,\varphi} = \mathrm{Inv}\langle X_F | P,\ T_F,\ W_{F,U_F}\rangle.$$

On the other hand, since $W_{F,Y} \subseteq W_{F,U_F}$ and since all relations in W_{F,U_F} are consequences of relations in $R \cup T_F \cup W_{F,Y}$ by Proposition 5.3.7 (c), we have that

$$\mathrm{Inv}\langle X_F | P,\ T_F,\ W_{F,U_F}\rangle = \mathrm{Inv}\langle X_F | P,\ T_F,\ W_{F,Y}\rangle.$$

Since F and Y are finite it is clear that the latter presentation is finite, hence $S*_{U,\varphi}$ is indeed finitely presented.

For the converse, suppose that $S*_{U,\varphi}$ is a finitely presented inverse semigroup. In particular, $S*_{U,\varphi}$ is finitely generated and so by Proposition 5.5.1, there exists a finite $\mathcal{J}$-dominant subset $F \subseteq E(U)$ of U. Then, by Proposition 5.4.1,

$$S*_{U,\varphi} = \mathrm{Inv}\langle X_F | P,\ T_F,\ W_{F,U_F}\rangle,$$

where P and T_F are finite sets of relations, but W_{F,U_F} is possibly infinite. Since $S*_{U,\varphi}$ is finitely presented by assumption, there exists a finite subset $W_0 \subseteq W_{F,U_F}$, such that

$$S*_{U,\varphi} = \mathrm{Inv}\langle X_F | P,\ T_F,\ W_0\rangle.$$

Consider the finite set

$$Y = \{u \in U | \exists e, f \in F \ \text{ such that } (t_e^{-1} u t_f = u\varphi) \in W_0\} \cup F,$$

and let U_0 be the inverse subsemigroup of U generated by Y. Our aim is to show that U_F is finitely generated by showing that $U_0 = U_F$. By definition of U_0, we clearly have $U_0 \subseteq U_F$. For the reverse inclusion we need the following variation of the strong HNN embeddability.

Lemma 5.6.2. *Let $e, f \in F$ and $U_{e,f} = \{u \in U_0 | uu^{-1} \leq e,\ u^{-1}u \leq f\}$. Then, in $S{*}_{U,\varphi}$, we have*

$$t_e^{-1}St_f \cap S = t_e^{-1}U_{e,f}t_f = U_{e,f}\varphi.$$

Proof. Relations in W_{F,U_F} imply immediately that $t_e^{-1}U_{e,f}t_f = U_{e,f}\varphi$ and that

$$t_e^{-1}U_{e,f}t_f \subseteq t_e^{-1}St_f \cap S.$$

To prove the reverse inclusion we employ a technique used by Yamamura [36]. Let $U_0\varphi = V_0$ and S' be a copy of S with $\iota : S \to S'$ an isomorphism, let $U_0' = U_0\iota$. Since U_0 is an inverse subsemigroup of S, the amalgam $K = S *_{V_0=U_0'} S'$ can be considered. By a theorem of Hall [16] (see also [30]), S and S' are canonically embedded in K, that is we have embeddings $\iota_1 \colon S \to K$, $\iota_2 \colon S' \to K$. Moreover by the strong amalgamation property of inverse semigroups [16] we have that $S\iota_1 \cap S'\iota_2 = U_0' = V_0$. Let $\psi = \iota_1^{-1}\iota\iota_2$. Then it is clear that $\psi \colon S\iota_1 \to S'\iota_2$ is an isomorphism; furthermore for all $u \in U_0$, $(u\iota_1)\psi = (u\iota)\iota_2 = (u\varphi)\iota_1$. It follows by identifying $S\iota_1$ with S, that $\psi|_{U_0} = \varphi|_{U_0}$. Let K^1 be obtained from K by adjoining an identity to it, and form the Yamamura HNN extension

$$K^* = \mathrm{Inv}\langle K^1, t | tt^{-1} = 1 = t^{-1}t, t^{-1}st = s\psi\,, (s \in S)\rangle$$

associating the subsemigroups $(S\iota_1)^1$ and $(S'\iota_2)^1$.

Next we show that the mapping $t_e \mapsto et$ and the identity map on S together induce an inverse semigroup homomorphism $\nu : S{*}_{U,\varphi} \to K^*$, that is we verify that ν maps relations in $S{*}_{U,\varphi}$ to relations that hold in K^*. Let $e, f \in F$ and $u \in U_{e,f}$ such that $uu^{-1} \leq e$ and $u^{-1}u \leq f$:

$$\begin{aligned}
&(t_e\nu)(t_f^{-1}\nu) = (et)(ft)^{-1} = ett^{-1}f = ef = (ef)\nu,\\
&(t_e^{-1}\nu)(t_f\nu) = (et)^{-1}(ft) = t^{-1}eft = (ef)\psi = (ef)\varphi = ((ef)\varphi)\nu,\\
&(t_e^{-1}\nu)(u\nu)(t_f\nu) = (et)^{-1}u(ft) = t^{-1}euft = t^{-1}ut = u\psi = u\varphi = (u\varphi)\nu.
\end{aligned}$$

Therefore relations in $P \cup T \cup W$ are indeed mapped onto relations that hold in K^*.

Let us now consider an arbitrary $s \in t_e^{-1}St_f \cap S$, and write $s = t_e^{-1}zt_f$ for some $z \in S$. Then in K^* we have

$$s = s\nu = t^{-1}ezft \in t^{-1}eSft \cap S.$$

By the above mentioned strong amalgamation property, we have that

$$(et)^{-1}S(ft) \cap S = t^{-1}eSft \cap S = (eSf)\psi \cap S \subseteq S\psi \cap S \subseteq S'\iota_2 \cap S\iota_1 = U_0' = V_0.$$

Therefore there exists $u_0 \in U_0$ such that the relation $t^{-1}ezft = u_0\varphi$ holds in K^*. Let $g, h \in F$ be such that $u_0u_0^{-1} \leq g$ and $u_0^{-1}u_0 \leq h$, so that the relation $u_0\varphi = (gt)^{-1}u_0(ht) = t^{-1}u_0t$ holds in K^*. Now from $t^{-1}ezft = t^{-1}u_0t$ and $tt^{-1} = t^{-1}t = 1$ it follows that $ezf = u_0$. This, in turn, implies that $u_0u_0^{-1} \leq e$ and $u_0^{-1}u_0 \leq f$; in other words $u_0 \in U_{e,f}$. Also, notice that $ezf = u_0$ is a relation between elements of S, and hence it also holds in $S*_{U,\varphi}$. So, returning to $S*_{U,\varphi}$, we have

$$s = t_e^{-1}zt_f = t_e^{-1}ezft_f = t_e^{-1}u_0t_f \in t_e^{-1}U_{e,f}t_f,$$

completing the proof. □

We are now in the position to prove the remaining inclusion $U_F \subseteq U_0$. Let $u \in U_F$. Then there exist $e, f \in F$ such that $uu^{-1} \leq e$ and $u^{-1}u \leq f$. It follows that $t_e^{-1}ut_f = u\varphi$ holds in $S*_{U,\varphi}$. Applying Lemma 5.6.2, we have that there exists $u_0 \in U_0$ such that $u\varphi = u_0\varphi$, and so we obtain that $u = u_0 \in U_0$, proving that $U_F = U_0$ is indeed finitely generated. □

We conclude this section by formulating Proposition 5.6.1 in the special case when $E(U)$ satisfies the maximum condition. Recall that we say that $E(U)$ satisfies the maximum condition if U has finitely many maximal idempotents $F = \{f_1, f_2, \ldots, f_m\}$ and for every idempotent $e \in E(U)$ there exists $f_i \in F$ such that $e \leq f_i$. Note that in this case F is also a finite $\mathcal{J}$–dominant subset and $U_F = U$ holds, and so we may conclude:

Corollary 5.6.3. *Let S be a finitely presented inverse semigroup and let U, V be inverse subsemigroups of S that are order ideals. Assume that U and V are isomorphic via $\varphi : U \to V$. Assume that $E(U)$ satisfies the maximum condition. Then the HNN extension $S*_{U,\varphi}$ is finitely presented if and only if U is finitely generated.*

5.7 A compelling consequence

Proposition 5.6.1 and its proof have the following intriguing consequence. Assume that S, U, V and φ are as in Proposition 5.6.1 and suppose that $S*_{U,\varphi}$ is finitely presented. Taking any finite $\mathcal{J}$-dominant set $F \subseteq E(U)$, we have

$$S*_{U,\varphi} = \mathrm{Inv}\langle X_F | R, T_F, W_0\rangle,$$

where W_0 is a finite subset of W_{F,U_F}. But then the converse part of the proof of Proposition 5.6.1 shows how this presentation yields a finite generating set for U_F. In other words if U_F is finitely generated for some finite $\mathcal{J}$-dominant subset F then U_G is finitely generated for every finite $\mathcal{J}$-dominant subset G. In fact, we can prove this in full generality removing the requirements that S be finitely presented, that U be an order ideal, and any mention of V.

Proposition 5.7.1. *Let S be an inverse semigroup and assume that the inverse subsemigroup U is finitely $\mathcal{J}$-dominated. Then U_F is finitely generated for some finite $\mathcal{J}$-dominant set F if and only if U_G is finitely generated for every finite $\mathcal{J}$-dominant set G.*

The proof is divided into the following two lemmas.

Lemma 5.7.2. *Let U_F be finitely generated and g be an arbitrary idempotent of U. Then $U_{F\cup\{g\}}$ is finitely generated.*

Proof. Assume that U_F is generated by the finite set A. Let $e \in F$ be such that $g \leq_{\mathcal{J}} e$. Then there exist $k, l \in U$, such that $g = kel = kell^{-1}ek^{-1} = kll^{-1}ell^{-1}k^{-1}$, hence there exists $v \in U$ such that $g = vev^{-1}$. From this we have that $gv = ve = gve$. We claim that $U_{F\cup\{g\}}$ is generated by $A \cup \{gve\}$. Let $s \in U_{F\cup\{g\}}$. The following four cases are to be considered.

(i) The case when $s \in U_F$ is straightforward.

(ii) Assume that $ss^{-1}, s^{-1}s \leq g$. Then $s = gsg = v(ev^{-1}\cdot s\cdot ve)v^{-1}$. Since $ev^{-1}sve \in U_F$, there exist $a_1, \ldots, a_k \in A \cup A^{-1}$ such that $ev^{-1}sve = a_1 \ldots a_k$. From this we have that $s = ve \cdot a_1 \ldots a_k \cdot ev^{-1} = gve \cdot a_1 \ldots a_k \cdot (gve)^{-1}$.

(iii) Assume that $ss^{-1} \leq g$ and $s^{-1}s \leq f$ for some $f \in F$. Then $s = gsf = v(ev^{-1} \cdot sf)$. Since $ev^{-1}sf \in U_F$, there exist $a_1, \ldots, a_k \in A \cup A^{-1}$ such that $ev^{-1}sf = a_1 \ldots a_k$. From this we have that $s = ve \cdot a_1 \ldots a_k = gve \cdot a_1 \ldots a_k$.

(iv) The case when $s^{-1}s \leq g$ and $ss^{-1} \leq f$ for some $f \in F$ can be proved similarly as (iii).

□

Lemma 5.7.3. *Assume that U_F is finitely generated and that for some $g \in F$, $G = F\setminus\{g\}$ is also $\mathcal{J}$-dominant in U. Then U_G is finitely generated.*

Proof. Let U_F be generated by the finite subset A which we may assume, without loss of generality, is closed under taking inverses. There exists $e \in F \setminus \{g\}$ for which $g \leq_{\mathcal{J}} e$, and hence $g = vev^{-1}$ for some $v \in U$. Let A' denote the set of elements of A for which $aa^{-1} \leq g$ and for which there is no other $j \in F \setminus \{g\}$ such that $aa^{-1} \leq j$; also let $A'' = (A')^{-1}$. Notice that if $a \in A'$ then $a = ga$, while if $a \in A''$, then $a = ag$. Let $\hat{B} = A \setminus (A' \cup A'')$. We show that U_G is generated by

$$B = \hat{B} \cup \{a_i ve\colon\ a_i \in A \setminus A'\} \cup \{ev^{-1}a_i\colon\ a_i \in A \setminus A''\} \cup \{ev^{-1}a_i ve\colon a_i \in A\}.$$

Let $s \in U_G \subseteq U_F$. Then $s = a_1 a_2 \dots a_k$ for some $a_1, \dots, a_k \in A$. If $a_i \in A'$, then substitute a_i by ga_i; if $a_i \in A''$, then substitute a_i by $a_i g$; and if $a_i \in A' \cap A''$, then substitute a_i by $ga_i g$. Consider now all subwords w of the form $a_{i-1}a_i g a_{i+1} g \dots g a_{m-1} g a_m a_{m+1}$. These subwords can be written in terms of B, since

$$w = a_{i-1} \cdot a_i ve \cdot ev^{-1} a_i ve \cdot ev^{-1} a_{i+1} \dots \cdot ev^{-1} a_{m-1} ve \cdot ev^{-1} a_m \cdot a_{m+1}.$$

It follows that s can be written in terms of B, hence U_G is generated by B. □

5.8 Examples

Here we provide four examples of HNN extension of inverse semigroups.

Example 1. Let B be the bicyclic monoid, considered as the set $\mathbb{N} \times \mathbb{N}$ with binary operation $(m,n)(p,q) = (m - n + \max(n,p), q - p + \max(n,p))$. Now B is presented as an inverse monoid by $B = \mathrm{Inv}\langle a | a = a^2 a^{-1}\rangle$, where $a = (0,1)$. Let $U = B$ and $V = \{(m,n) | m,n > 0\}$ with $\varphi : U \to V$ the shift map $(p,q) \mapsto (p+1, q+1)$, so that $a\varphi = a^{-1}a^2$. It is clear that $U = B$ is $\mathcal{J}$–dominated by the identity element $1 = (0,0)$, and that $U_{\{1\}} = B$ is finitely generated. Hence the HNN extension $B*_{B,\varphi}$ is finitely presented. It is easy to check that

$$\mathrm{Inv}\langle a, t | a = a^2 a^{-1}, tt^{-1} = aa^{-1}, t^{-1}t = a^{-1}a, t^{-1}at = a^{-1}a^2\rangle$$

gives a presentation.

Example 2. With B as before, suppose now that $U = E(B) = \{(m,m) | m \in \mathbb{N}\}$ with $\varphi : (m,m) \mapsto (m+1, m+1)$. It is clear that U is finitely $\mathcal{J}$–dominated by the identity element $1 = (0,0)$ of U, but $U_{\{1\}} = U$ is not finitely generated. Hence we may conclude that the HNN extension $B*_{U,\varphi}$ is finitely generated but not finitely presented.

Example 3. In this example we construct a finitely presented HNN extension $S*_{U,\varphi}$, with U not finitely generated. Let $S = \mathrm{Inv}\langle a, e | aa^{-1} = 1, ae = 0, e^2 = e\rangle$ considered as a semigroup with 0. It can be easily seen that

$$\{a^{-i}a^j, a^{-i}ea^j | i, j \in \mathbb{N}\} \cup \{0\}$$

is a set of normal forms for S. Consider the inverse subsemigroup

$$U = \{a^{-i}ea^j | i, j \in \mathbb{N}\} \cup \{0\}$$

of S , together with the identity isomorphism $\iota \colon U \to U$. We observe that U is isomorphic to the infinite aperiodic Brandt semigroup, and hence U is $\mathcal{J}$–dominated by any of its idempotents. Let $F = \{a^{-i}ea^i\}$. Then $U_F = \{a^{-i}ea^i, 0\}$ is finite, so certainly finitely generated, and it follows that the HNN extension $S*_{U,\iota}$ is finitely presented. However, the associated inverse subsemigroup U is not finitely generated.

Example 4. Our final example shows that the maximum condition on $E(U)$ is essential in Corollary 5.6.3. We give an example of a finitely presented HNN extension of a finitely presented inverse semigroup S, where the associated subsemigroups contain finitely many maximal idempotents, but do not satisfy the maximum condition and are not finitely generated. Consider the semidirect product $S = Y_\infty \rtimes G$, introduced in Example 4.10.5. We verified that S is finitely presented as an inverse semigroup. In particular, it is generated by the set $\{(e_0, 1), (\mathbf{1}, g), (\mathbf{1}, g^{-1})\}$ and if we correspond a to $(\mathbf{1}, g)$, b to $(\mathbf{1}, g^{-1})$ and c to $(e_0, 1)$, then we have that

$$S = \mathrm{Inv}\langle a, b, c | ab = ba = 1, c^2 = c, cbc = bc, cac = ca\rangle.$$

Then it can be easily seen that

(i) $\{a^i \colon i \in \mathbb{N}\} \cup \{b^i \colon i \in \mathbb{N}\} \cup \{ab\} \cup \{x^icy^j \colon x, y \in \{a, b\}\ i, j \in \mathbb{N}\}$ is a set of normal forms for S,

(ii) $E(T)$ is an infinite chain with an identity element adjoined on top:

$$\ldots < b^2ca^2 < bca < c < acb < a^2cb^2 < \ldots < 1,$$

(iii) the $\mathcal{H}$–class H_1 of 1 is isomorphic to the infinite cyclic group,

(iv) $K = T \setminus H_1$ is a bisimple aperiodic inverse subsemigroup of S, which is not finitely generated.

It follows from (ii) and (iv) that, for any $e \in E(S) \setminus \{\mathbf{1}\}$, the subsemigroup

$$U_e = \{u \in U \colon uu^{-1}, u^{-1}u \le e\}$$

is finitely generated. To be more accurate, if $e = (e_n, 1)$, then U_e is generated by the finite set $\{(e_n, 1), (e_n, g), ({}^{g^{-1}}e_n, g^{-1})\}$.

Take two copies $S^{(1)}$ and $S^{(2)}$ of S and let T be their 0–direct union. It is straightforward that T is finitely presented. Let $f \in E(S^{(2)})$ and let U be the 0-direct union of K with U_f. Then U has one maximal idempotent f, and is not finitely generated since the subsemigroup K contains an infinite ascending chain of idempotents. Moreover since K is bisimple, for any $e \in E(K)$, the idempotents e and f are $\mathcal{J}$–maximal in U and

$$U_{\{e,f\}} = \{u \in U \colon uu^{-1}, u^{-1}u \leq e \text{ or } uu^{-1}, u^{-1}u \leq f\}$$

is isomorphic to the 0–direct union of U_e and U_f, and hence is finitely generated. It follows by Proposition 5.6.1, that the HNN extension $H = S*_{U,\iota}$, where $\iota \colon U \to U$ is the identity isomorphism, is finitely presented.

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