SECOND-ORDER LOGIC IS LOGIC

Michèle Indira Friend

A Thesis Submitted for the Degree of PhD
at the
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By Michèle Indira Friend

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Declarations

I, Michèle Indira Friend, certify that this thesis which is approximately 70,000 words in length has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

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I was admitted as a research student in January 1992, and as a candidate for the degree of Doctor of Philosophy on the same date; the higher study for which this is a record was carried out at the University of St. Andrews between 1992 and 1997.

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"Dans une discussion, le difficile,
ce n'est pas de défendre son opinion, c'est de la connaître."

-André Maurois, Conversations

Abstract

"Second-order logic" is the name given to a formal system. Some claim that the formal system is a logical system. Others claim that it is a mathematical system. In the thesis, I examine these claims in the light of some philosophical criteria which first motivated Frege in his logicist project. The criteria are that a logic should be universal, it should reflect our intuitive notion of logical validity, and it should be analytic.
The analysis is interesting in two respects. One is conceptual: it gives us a purchase on where and how to draw a distinction between logic and other sciences. The other interest is historical: showing that second-order logic is a logical system according to the philosophical criteria mentioned above goes some way towards vindicating Frege's logicist project in a contemporary context.

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Introduction

*"Ne pas déplorer, ne pas rire, ne pas détester,
mais comprendre."*
-Spinoza

Traditionally, philosophers have accorded to logic a privileged status. Logic is what underlies all our reasoning. It underpins all rational argument. It acts as the ultimate, and therefore preferred, justification for any claim insofar as it is appropriate. That is, what we can justify by appeal to logic, we should justify by appeal to logic, since this makes for the strongest possible justification. Logic is very close to mathematics in these respects. However, *prima facie*, there seem to be differences between mathematics and logic. Mathematical theories seem to be about particular sorts of abstract objects or structures. Logic seems to be universal, and not about any objects at all.

The logicist is interested in the contrast between justifying claims by appealing to logic, and justifying claims by appealing to the "special sciences". If a claim is justified by logic, then it does not rely on particular facts about the world, or how it is we see the world. In contrast, claims which are justified by mathematics or physical assumptions, are ones which depend on assumptions about physical structures, or our sense of spatial intuition. The logicist is also interested in where the boundary lies between the two. Historically, the first guess has been that arithmetic is really part of logic, but geometry is not.

The *problématique* of the Thesis can be described in at least two ways: one historical and one conceptual. The more historically motivated question is to what extent Frege's logicism can be vindicated in the light of the tragic history of the project, and in the context of a plethora of logical and mathematical formal systems which have mushroomed since Frege. The difference that the plethora of formal systems makes to Frege's logicist project is that because the differences between these formal systems are so small, their development (or discovery) throws into doubt the very idea that a distinction can be drawn between logic and mathematics.

The more conceptual or metaphysically motivated group of questions is where and how to draw a distinction between logic and mathematics. Chapters one and three concentrate more on the historical questions. Chapters two and four are more conceptually oriented. Nevertheless, it will become clear that the two groups of questions overlap considerably, and inform each other throughout the thesis.

We begin with a little history. In the second half of the nineteenth century there was a view expressed, by Dedekind,[1] amongst others, that arithmetic is just a more sophisticated form of logic. Frege set out to make that view more precise, and indeed, to prove it true. "Herr Dedekind, like myself [Frege], is of the opinion that the theory of numbers is part of logic; but his work hardly contributes to its confirmation."[2] Frege finds fault with Dedekind's methodology. It is not rigorous enough.

According to Frege, to prove that number theory is really just logic, one has to prove that the axioms of arithmetic are derivable, as a set of theorems, from logical axioms. That is, using only logical assumptions and logical methods of deduction, we should generate the truths of arithmetic we would generate from the traditional Peano-Dedekind axioms of arithmetic (henceforth: the Peano axioms). Theorems of arithmetic should just fall out of the logic. For the purpose of showing that this was indeed possible, Frege had to set up a logical system because the existing logical systems were woefully inadequate for the task. The propositional calculus had been introduced, and a very restricted use of quantifiers was

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[1]Richard Dedekind, "The Nature and Meaning of Numbers," Richard Dedekind, Essays on the Theory of Numbers, trans. Wooster Woodruff Beman, (New York: Dover Publications, Inc., 1963), pp. 31 - 115.

[2]Gottlob Frege, The Basic Laws of Arithmetic, Exposition of the System, ed. & trans. Montgomery Furth, (Berkeley and Los Angeles: University of California Press, 1964), p. 4.

making an appearance through the work of Boole,[3] and Pierce, amongst others. Dedekind did not have an explicitly articulated formal system which he considered to be logic. For this reason, he could not formally prove that arithmetic is part of logic. He could only offer sketches of proofs, and give conceptual arguments.

Frege thoroughly articulated a conceptual leap forward in logic through his three great works: *Begriffsschrift*, a Formula Language, Modelled upon that of Arithmetic, for pure Thought,[4] (henceforth: *Begriffsschrift*), The Foundations of Arithmetic, A Logico-Mathematical Enquiry into the Concept of Number,[5] (henceforth: *Grundlagen*), and The Basic Laws of Arithmetic, Exposition of the System,[6] (henceforth: *Grundgesetze*). In his formal system, Frege lays down some axioms which he believes to be obviously logical. He also sets up a notation which traces truth preservation through rules of inference. That is, from a logical truth, only logical truths will follow.

To realise the project of proving that arithmetic follows from logic, not only did a sophisticated logical system need to be developed, but also, the philosophical notion of what a logic consists in, had to be made more precise. For, one cannot claim that arithmetic is just logic, unless one can characterise logic. Otherwise, one does not know if what one has derived arithmetic from, is really logic and not just mathematics. The reduction of arithmetic to a more "basic" formal system is a technical result. It is only exciting in

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[3]George Boole, An Investigation of The Laws of Thought on which are Founded the Mathematical Theories of Logic and Probabilities, (New York: Dover Publications, Inc., 1958).

[4]Gottlob Frege, *Begriffsschrift*, a Formula Language, Modelled upon that of Arithmetic, for Pure Thought, in From Frege to Gödel, a Source Book in Mathematical Logic, 1879 - 1931, ed. Jean van Heijenoort, (third printing; Cambridge, Massachusetts: Harvard University Press, 1976), pp. 1 - 82.

[5]Gottlob Frege, The Foundations of Arithmetic, a Logico-Mathematical Enquiry into the Concept of Number, trans. J. L. Austin, (Second edition; Evanston, Illinois: Northwestern University Press, 1980).

[6]Gottlob Frege, The Basic Laws of Arithmetic, Exposition of the System, ed. & trans. Montgomery Furth, (Berkeley and Los Angeles: University of California Press, 1964).

the light of the philosophical privilege accorded the reducing system over the reduced system.

Frege's _Grundlagen_ explores the philosophical aspect of logicism. _Begriffsschrift_ and _Grundgesetze_, respectively, set up and develop the formal aspects of logicism. Frege's project was abruptly halted with the discovery of inconsistency derivable from his fifth, purportedly logical, axiom, introduced in the first volume of his last great work: _Grundgesetze_.

The failure of Frege's project was followed by one other serious attempt at creating a logical foundation strong enough to derive the theorems of arithmetic. In fact, it was a more ambitious project: to show that, not only arithmetic, but all of mathematics is reducible to logic. The attempt was carried out by Whitehead and Russell, who developed a powerful type theory. They failed. The type theory they developed, was deemed, on philosophical grounds, to be mathematics; since it included an axiom of infinity, for example. The failure of Russell and Whitehead was followed by a search for alternative foundations for mathematics. Questions were raised as to the nature or essence of mathematics, and as to what distinguishes sound from unsound mathematical practice. Logic no longer looked like a likely candidate for providing an answer, so mathematicians turned their attention to global mathematical systems, such as set theory and type theory. The idea of founding mathematics, or even part of it, in logic had been abandoned.

Set theory was the favoured discipline of Zermelo, Fraenkel, Von Neumann, Gödel, Bernays and others. Some mathematicians advocated a tempering of mathematics through epistemological considerations. Their worry was that, in pursuing mathematics into the realm of the infinite, we were overreaching ourselves, and that some of the notions we employed were in fact incoherent. To ensure against finding a contradiction in set theory or any other theory, the constructivists, most famously represented by Brower and Heyting, advocated greater discipline in the use of the law of excluded middle, and stressed that mathematics is a building process. In mathematics, we must give methods for constructing finite or only countably large entities or structures. Hilbert suggested a less restrictive approach. He placed two constraints on mathematics: that it should be consistent and that proofs, as opposed to structures, should be finite.

Partly as a result of this search for foundations in mathematics which followed the collapse of Frege's project, and partly as a result of a concern to tailor logical and mathematical systems to applications; a number of formal systems were developed. Many of these claim, in name at least, the status of a logic. In name then, it looks as though there are many competing logical systems, and one could quickly draw the inference that "logic" is an ambiguous or relative term.

This is not to say that there was only one logical system before Frege. Frege's formal system of the _Begriffsschrift_ was accompanied by philosophical arguments to show that his system was logic, and, to my knowledge, this claim was not directly challenged by Frege's contemporaries. At the time, it was quite natural not to challenge Frege over his claim that the formal system of the _Begriffsschrift_ is a logical system since there was perceived to be an obvious gap between his formal system and mathematics. We now recognise Frege's logic to be equivalent to second-order logic,[7] and second-order logic is stronger than first-order logic. Since Frege, first-order logic has come to be regarded as logic, and broadly speaking, anything stronger has come to be thought of as being mathematics. Not everyone is adamant that first-order logic is the strongest logic. This is because there are formal systems which are just a little stronger than first-order logic, and therefore, in the spirit of a slippery slope argument: if we let first-order logic be considered logic, then we might as well consider a slightly stronger system to

[7]George Boolos, "The Consistency of Frege's *Foundations of Arithmetic*," On Being and Saying: Essays in Honor of Richard Cartwright, ed. Judith Jarvis Thompson, (Cambridge, Massachusetts: MIT Press, 1987), p. 3.

be logic as well, and so on. For example, first-order logic, with the quantifier "there are infinitely many" is considered to lie between first and second-order logic in strength. In fact, there are many formal systems which lie between first and second-order logic. There are also stronger systems than second-order logic. For example, trans-finite type theory, second-order set theory, and category theory. Interestingly, it turns out that since there is a sense in which simple type theory reduces to second-order logic;[8] third-order logic, fourth-order logic and simple type theory are often[9] not considered to be stronger formal systems than second-order logic. Some of these are powerful enough that we can do a substantial amount of mathematics in them. Thus, the very idea that there is a border between logic and mathematics becomes harder to defend. The philosophical criteria for logic have to be sharper today than they did at the turn of the century, in order to make more fine-grained distinctions between existing formal systems.

To add to the confusion, internal to a formal system there are various places where we might draw a distinction between the logical and the non-logical. For example, we can distinguish logical from non-logical terms, logical from non-logical vocabulary, sentences, rules of inference and so on.

The smallest unit is the vocabulary, sometimes referred to as an alphabet. In setting up a formal system, in the list of vocabulary, we distinguish between logical constants, non-logical constants, descriptive vocabulary, and variables. The logical constants are usually connectives like "$\vee$" and "$\neg$", operators such as "$\forall$", and sometimes relations such as "=". What distinguishes a logical constant from other vocabulary is that the interpretation of these symbols stays fixed across domains. That is, no matter what domain of objects, and possibly relations and functions, we may be

[8]This is discussed in chapter IV, § 3.
[9]Strictly speaking, this will depend on how one measures the strength of formal systems.

considering, the logical connectives work in exactly the same way. We are debarred from interpreting "$\vee$" as "$\leftrightarrow$", for example.

Non-logical constant symbols have an intended fixed interpretation across domains. The purpose of saying that they have an intended interpretation is that we can give unintended interpretations to these symbols. Sometimes this proves very fruitful. They are introduced to the language for the express purpose of studying a particular set of mathematical objects, relations or functions. We adopt the convention here of calling these "mathematical theories". For example, the mathematical constant symbol "$\in$" might be introduced in order to facilitate the study of first-order set theory. Non-logical constant symbols may stand for individuals, such as that usually designated by the symbol: "0" in arithmetic. They may stand for a relation, such as that usually designated by "$\in$", or they may stand for a function such as that which is designated by "+". It is standard practice, and we shall treat this as our default position, to regard $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$, $\exists$, $\forall$ and = as symbolising logical constants. The non-logical constants, by convention, are symbolised by other symbols. We shall also come to question the extension, if not the intension, of the distinction between logical and non-logical constants. Intensionally, the non-logical constants have no place in a logical language because understanding the use of the symbol which designates them requires some specialised knowledge based on intuition or sense perception. We shall elaborate this point in the course of the thesis.

The descriptive vocabulary is a set of predicate, relation and function symbols. A predicate is a one-place relation. The relations and functions may have any number, n, of places for variables, such that $n \in \omega$. For example, a two-place relation relates pairs of objects from the (Cartesian product of the) domain. The interpretation of the descriptive vocabulary is not independent of the domain, as it is with constants (logical or non-logical), but is fixed relative to a domain. That is, in each domain, the descriptive vocabulary receive a fixed interpretation.

Variables are not fixed at all. They range over individuals of the domain, in the case of individual variables, and over subsets of the powerset of the domain, in the case of predicate, relation and function variables. In first-order logic, we have only individual variables. To distinguish between different sorts of variables we say first-order variable for the individual variables, and second-order variables for the others.

To fix terminology, we shall say that the only non-logical vocabulary is composed of the non-logical constants. Any other combination of vocabulary can act as part of what we shall call a language of logic. This is contrasted to the language of a theory, which will include non-logical constants. "Formal languages" and "formal systems" are deliberately ambiguous between these in order not to beg any questions.

A term denotes an object. A logical term is one which has no non-logical constants. For example, "Px" is a logical term. This ensures that it can be interpreted in any domain of discourse. A non-logical term can only be interpreted in some domains, because there may not be a sensibly corresponding element in the domain. For example, the non-logical constant "0" cannot receive a sensible interpretation in the domain of objects composed of the contents of my desk drawer.

A logical sentence will also have no non-logical constants. A logical sentence is not the same as a logical truth. It is only a logical truth if it is true under any interpretation. A logical falsehood is a sentence which is false under any interpretation. A logical sentence may be neither a logical truth nor a logical falsehood. In this case, it will be true under some interpretations and not others. A logical truth is also called a valid sentence.

A logical axiom or rule of inference is one which governs the use of one or more of the logical connectives. For example, an axiom about the combination of "$\neg$" and "$\vee$" is: $\neg(p \vee q) \rightarrow (\neg p \wedge \neg q)$. A non-logical axiom or rule of inference governs the use of non-logical constants. For example, an axiom for "+" and "0" is: $0 + x = x$.

Axioms, whether logical or not, are true, in the sense that in a given theory we are obliged to choose only interpretations which make them true. A logical axiom is the limit case: it is true under all interpretations. In part, this is an artefact of the logical vocabulary receiving an interpretation which is fixed independent of any domain. In part, we think that the logical axioms reflect fundamental (pre-theoretic) truths.

At various points in the thesis it will be appropriate to discuss one or other of these different units *viz*: logical or non-logical vocabulary, logical or non-logical sentences, logical or non-logical axioms, and so on. In general, the chief aim is to distinguish a logical from a non-logical system (at times this might be done in terms of vocabulary and so on).

"Formal system" is ambiguous between a theory and a logic. In general, a formal system consists in (intended) interpretations and a formal language (vocabulary, terms, well-formed formulas), an assignment function, a satisfaction function, a set of axioms and rules of inference. A logic is such a formal system which has no intended interpretations: any interpretation may be considered. A theory has an intended interpretation. We may try to make this explicit by adding non-logical constants or special axioms to a language, which come with, or force, a particular interpretation.

While in the business of drawing distinctions; rather glibly, we have been referring to the notion of expressive strength of a formal system, as though the notion were not ambiguous. Unfortunately, this is not the case. We use different measures for the strength of a formal system: in terms of limitative results, in terms of the sorts of theorems which can be generated, and in terms of the definitional resources of the underlying formal language. The three forms of measurement are closely related but not identical. For example, consider many-sorted logic. This is a formal system made up of a second-order language, that is, there is quantification over predicate, relation and function variables, but this differs from full second-order logic in that the predicate variables only range over a fixed subset of the domain and the relation and function variables only range over a fixed subset of the powerset of the domain.[10] In contrast, in second-order logic, the second-order variables range over all subsets of the domain. Many-sorted logic and first-order logic share several limitative results. They are both compact, complete, have the upward and downward Löwenheim-Skolem properties and are semi-decidable.[11] Thus, in terms of the limitative results mentioned above, the two systems have equal expressive power. However, many-sorted logic has greater expressive power in terms of definitional resources, owing to its allowing quantification over second-order variables. Thus, in terms of definitional resources, many-sorted logic is stronger than first-order logic. Nevertheless, while there are a few examples of formal systems which show a divergence according to different measures of strength, it remains that there is more concord than disagreement over the ratings of strength of formal systems.

To return to our history; with the mushrooming of formal systems, of varying strengths, the question was raised as to whether or not second-order logic was to be considered a logic. For instance, Quine has argued that second-order logic, and *a fortiori* Frege's logical system, is part of set theory. In particular, the semantics of second-order logic is that of set theory.[12] That is, we have to interpret the syntactic part of second-order logic in terms of sets, as they are arranged in the set-theoretic hierarchy. Since the set-theoretic hierarchy is a particular interpretation, which excludes

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[10]Stewart Shapiro, Foundations Without Foundationalism: A Case for Second-Order Logic, (Oxford Logic Guides: 17; Oxford: Clarendon Press, 1991), pp. 73 - 4.
[11]A theory is decidable iff it has an effective proof procedure which can detect of any sentence that it is or is not a theorem. See George Boolos and Richard Jeffrey, Computability and Logic, (Second edition; Cambridge, Massachusetts: Cambridge University Press, 1980), p. 174. A theory is semi-decidable if there is an effective positive test for unsatisfiability. Such a test is "an effective procedure which, when applied to an arbitrary sentence S... terminates with a 'yes' iff S is unsatisfiable." George Boolos and Richard Jeffrey, Computability and Logic, (Second edition; Cambridge, Massachusetts: Cambridge University Press, 1980), p. 142. Such a test is not the same as a decision procedure. For, there is no effective test for the satisfiability of an arbitrary sentence S because some of the proofs which try to determine this will be infinite; they will not terminate.
[12]Willard Orman Van Quine, Philosophy of Logic, (Englewood Cliffs, New Jersey: Prentice Hall, 1970) pp. 66 - 7.

others, second-order logic is not general, and therefore, is not considered to be a logic.

Another question which arises in tandem with this argument is: what is the philosophical basis for characterising a logic, in contra-distinction to mathematics? This question did not have to be answered in enormous detail by Frege, mainly because there were fewer formal systems to distinguish between. As the number of formal systems increases, and the differences between them become increasingly subtle, a first question which arises is how to characterise a system at all: whether in terms of limitative (or characterising) results, or in terms of the semantic and syntactic presentation or in terms of philosophical considerations. The first approach can appear contrived or artificial. This is because it really belongs to the realm of abstract model theory, and it is not obvious, at first, why it is that some limitative results are more important than others, or whether we have anything like an exhaustive catalogue of these. The second approach holds the difficulty that not all systems are presented in readily identifiable divisions of semantic versus syntactic aspects. The third aspect: of the philosophical characterisation of logic, has received too little attention. The characterisation given by Frege is not sufficiently sharp to pick out one formal system over others. Thus, the philosophical characterisation bears sharpening. In this thesis I shall be using the first two approaches to sharpen the philosophical notions.

Discussion of the philosophical characterisation of logic belongs to the more conceptual aspect of the thesis. Logic excites philosophers for many reasons. In some sense, logic is supposed to be universal. Logical inference and logical axioms are both the starting point and the final appeal in any reasoned debate. Logical inference cuts through rhetoric. Logical systems ignore any peculiarities of a subject. A logical axiom is amongst the most basic, and may always be appealed to. A valid logical inference is always truth preserving.

This is all very well, until we start to consider rival formal systems, each with their own claims to the honorific "logic". For, one formal system will label an argument valid, where another will not. A standard definition of a valid argument is that an argument is valid if and only if (henceforth: iff) whenever the vocabulary used in the premises is interpreted in such a way as to make the premises true, so is the conclusion. As we mentioned before, we have no leeway in interpreting the logical constants. A sentence, as opposed to an argument, is logically valid, if it is true under any interpretation. Formal systems will disclose more or less information about the structure of the premises and the conclusion of an argument. How much information they disclose will depend on their expressive power. Thus, some formal systems might just miss out on what makes a given argument valid. This indicates an insufficiency in expressive power. For example, the argument:

> Mary has a sister.
>
> <u>Betty has a sister.</u>
>
> Therefore, Mary and Betty have some property in common;

is not valid in first-order logic. It is valid in second-order logic.[13]

To decide if an argument really is logically valid, independent of whether or not a particular formal system pronounces it to be so, we are left with appealing to informal notions, that is, philosophical intuitions and arguments not formally laid out. There is an informal notion of validity: to which the formal notions are ultimately responsible! Thus, it is possible to informally recognise an argument to be valid when it is not so approved by a

[13]In first-order logic we would formalise the argument as follows. Let "S" be the predicate "has a sister", let "m" stand for Mary, let "b" stand for Betty, then the premises are represented: Sm and Sb. For the conclusion, we have to invent a new two-place predicate: "H" for "have some property in common", so the conclusion is represented: "Hmb". The argument, thus represented, turns out to be invalid in first-order logic. In second-order logic, we could represent the premises in the same way: Sm and Sb. The conclusion could be written: $(\exists S)(Sm \wedge Sb)$, and is thereby shown to be valid. The conclusion follows from "$\wedge$ introduction", and second-order "$\exists$ introduction", respectively.

given formal system. This judgement will usually arise from the inadequacy of the formal language to capture the relevant logical structure of the argument. The ability of a logical system to do so, is a measure of its expressive strength. A weak logic will misrepresent more arguments than a strong logic will.

It is tempting at this stage to think that, to reflect our informal notion of logical validity, all we want is a very powerful logic: one that will find a lot of arguments valid. (Obviously, we do not want all arguments to be valid, because that defeats the purpose of argumentation, and makes disagreement trivial.) However, there must be some maximal way of expressing all valid arguments: distinguishing the valid from the invalid, and reflecting our informal notion of validity as accurately as possible.

Unfortunately, the situation is not so straightforward. Distinguish three different sorts of validity: an informal notion of logical validity, formal logical validity and validity in a theory. An argument is informally and logically valid just in case its form can be imported to any discourse, and the conclusion is deemed, on an intuitive level to follow logically from the premises. We shall say no more at this stage, since the notion is meant to be intuitive, in the sense of pre-theoretic. We shall be sharpening the notion progressively at different points in the thesis as we try to match it to validity in particular formal systems.

An argument or sentence is formally and logically valid just in case a chosen formal (purportedly) logical system pronounces it valid. The order in which we consider factors which contribute to the judgement that an argument is valid is different from the intuitive case. In the intuitive case, we are presented with an argument and asked what our intuitions say about it: whether we feel it to be logically valid or not. In the formal case, we decide first whether or not a given formal system is logic. We then express the argument in the language of the logic, and accept the verdict given by the formal system as to its logical validity.

The last sort of validity is distinct from both of these, not in the order of the factors we invoke in pronouncing an argument valid, but in what the nature of the premises is, which make that argument valid. An argument or sentence is valid in a theory just in case it appeals to some feature of the theory which that theory does not share with other theories. In other words, it is sound in that theory. What makes the premises in the argument true are assumptions peculiar to the theory. For example, a law of a theory belongs to that theory and it is inappropriate to import it to some other theories. For example, we cannot automatically take laws of physics and re-interpret them in an economic context and apply them. To do so requires argument to the effect that the analogy holds. An argument is invalid if there is an interpretation which makes the premises true, but the conclusion is false. An argument is logically invalid, but valid in a theory, or a context, just in case all falsifying interpretations are inappropriate to the theory, or context. For example, the statement that it is impossible to get Mr. Seasick on to a boat is false under the interpretation which allows us to kill Mr. Seasick, and then bring him on to a boat. In most contexts this is illegitimate. Thus, the assertion is true, and valid in a context which precludes beforehand certain types of counter-example, or certain types of interpretation. The assertion would be falsified in the context, if we consider, for example, asking Mr. Seasick to board a boat which is on land.

The logic of an argument refers to its form. The form is independent of the content. Form and content are informal notions, no pun intended. The form/ content distinction applied to arguments is meant to be drawn out by the logical vocabulary which is supposed to reflect the form of the argument, where the variables range over the content, i.e. the domain. In the case of logic, domains are perfectly general, because they are arbitrarily chosen, insofar as they are chosen at all. The non-logical constants restrict the domains we may consider, and for that reason possess content. Similarly, an axiom expressed in only logical vocabulary (logical constants, variables or descriptive vocabulary) is an axiom which governs logical form and not special content. This is the idea behind the distinction between logical vocabulary and non-logical constants or between a logic and a theory. Of course, people have different ideas as to what constitutes the form of an argument as opposed to its content. To register this, we have different formal systems. The ability of logical systems and formal languages to capture form varies accordingly. A language which is more sensitive, and so, can capture more features, is said to have greater expressive power than one which overlooks aspects of form. But for the language to be considered to be a logical language, it must not have the capacity to exploit particular features of a domain, whatever the extension of "particular" turns out to be! This is not to say that sentences written in a logical language are meaningless or that they cannot be interpreted or applied to domains of objects. Rather, the symbols of a logical language together, do not have the capacity to identify particular features had by individuals in a domain of interpretation. This is one sense in which a logic is topic neutral.

Determining which aspects of a sentence concern its form, and which its content, is not straightforward. For instance, a problem might arise if the formal validity of an argument depends on features which we can formally represent, which we can build into the formal language, and of which we have no strong intuition as to whether they (informally/ philosophically) belong to logic or not. Consider modal notions concerning time. For example, the argument:

> Everything that comes into being has a first moment.
> <u>Therefore, in particular, the universe had a first moment.</u>
> Therefore, there was a first moment of time.

Displaying the validity of the argument depends on using modal temporal notions coupled with a metaphysical theory about time (that time is dependent on the existence of the universe, for instance). To some extent, we can build the metaphysics into a formal system which will formalise the argument and have it come out valid. We simply need to make a judicious choice about the rules of inference and axioms we introduce to govern the new modal operators. It would be quite natural for some people to feel uncomfortable about calling the conclusion a logical one, because one might feel that the structure or metaphysics of time is not a matter for logic to decide. In this sense, the metaphysics underlying the system has come before the formal system. We simply designed the system to fit a preconceived idea. Thus, something's being expressed in a formal language, is not enough to guarantee logicality.

Why expressing something in a formal language is useful, is that it suppresses information contained in sentences. In particular, a formalisation will suppress (at least part of) the content of a subject matter. A logical formal language will suppress all content. It can be applied to any subject matter, it can be interpreted in any domain. In this sense, logic is universal. This is a feature the logicist looks for in a logic, as well as its ability to reflect an informal notion of validity.

Frege thought of universality as manifested by the omnipresence of logic. Frege did not employ model-theoretic notions explicitly, but he did have a conception of logic applying to an universal domain: all that there is. We know now that, on pain of contradiction, we cannot consider the universe to be an object,[14] like a set, which has already been gathered together. Thus, I propose that, in the first instance, we interpret universality as "being applicable to any domain" understood as any proper subset of "all that there is". A logic, then, is universal by virtue of its language being interpreted in any domain.

We might think that set theory complies with this criterion. The semantics of set theory includes a series of domains organised in a hierarchy called the universe (of domains). There is a sense in which we can find a domain to match any in mathematics. That is the basis upon

[14]Of course, this is an over simplification. The theory of classes would treat this as a proper class. Thus, in this theory, we do not incur contradiction by considering the whole universe. However, our discussion is restricted, because proper classes are treated differently than sets, and this seems *ad hoc*. Furthermore, there is a type theory developed by Church which allows "all that there is" to be a set. However, here again there is a *prima facie* accusation of being *ad hoc* because the unrestricted sense of "all", used in sentences which begins with this quantifier do not have one of the two truth values true and false. See Alonzo Church, Introduction to Mathematical Logic, (I; Princeton, New Jersey: Princeton University Press, 1956), p. 347 n. 577.

which it is argued that set theory is foundational to mathematics. Most of mathematics can be translated into set theory. In this sense, it is universal. What indicates that the set theoretic universe is *ad hoc* is that it contains, for instance, an axiom of regularity. This guarantees that a set, a, which is a member of a set, b, which is a member of a set, c, and so on for arbitrary number of sets n, cannot then be a member of the sets a, b, or c up to n. The purpose of the axiom is to block Russell-type paradoxes involving sets which are members of themselves, or cycle through a number of sets to become a member of an original set.[15] *Prima facie*, this seems to be an assumption made from outside logic. For, we can hold a consistent position arguing that there are such sets. That is, it is logically possible for there to exist sets which are members of themselves. For example, the set of all sets of infinite cardinality is itself an infinite set, and is therefore a member of itself. Therefore, set theory, as it is presented, cannot be a logic on pain of non-generality: the theory precludes the possibility that the universe contains "irregular" sets. To some extent, this is an artefact of presentation. In set theory we stipulate, by means of an axiom, that there are no such sets.

According the logicist, then, logic is independent of particular subject matters to which it is applied. Logic does not rely on special features of theories. Furthermore, it does not rely on special knowledge. Roughly, this aspect of logic falls under the rubric "analytic". The analytic/ synthetic distinction has been drawn in various ways. Frege drew it in such a way as to reflect his interest in disengaging logic from psychological or empirical considerations. That is, he believed that what is distinctive of logic is that it makes no appeal to empirical data, and is independent of the sense organs or the structure of the human brain. For example, it would be irrelevant to suggest that someone who is colour-blind, should not be as good a logician as someone with full colour vision; even if this were discovered to be an empirical fact. The reason it is irrelevant, is that logical inference in no way depends upon our abilities to distinguish colours. Nor should logic depend on our

[15]For a discussion of this see Patrick Suppes, Axiomatic Set Theory, (Dover Edition, New York: Dover Publications Inc., 1972), p. 53 ff.

abilities to discern various smells, textures, sounds and so on. Furthermore, logic should not depend on the brain's talent to perceive certain mathematical structures, despite the fact that our brains might affect our abilities to follow a logical argument. The inability to follow a logical argument is not one of lack of talent; it is a fault of inattention or lack of concentration. The logic itself is inviolate. It is independent of our talent or what I usually refer to as Kantian intuition.

Frege insists that logic should not depend on intuition in Kant's sense. Kant postulated that in order to do arithmetic we need temporal intuition; and in order to do geometry, we need spatial intuition. Frege disagreed with the former but agreed with the latter. Thus, he was not opposed to the doctrine of intuition as a whole. Frege also believed, along with Kant, that logic does not require intuition. Where Frege disagrees with Kant, is over the scope of logic. Frege believes that arithmetic is just a part of logic. Kant thought that the two disciplines were very much distinct, since one, and not the other, relies on intuition. Frege's desire to show that arithmetic is analytic is in part a reflection of his reaction against the Kantian doctrine. We shall refer to this as Frege's negative characterisation of analyticity.

There is a more modern conception as to what makes a judgement analytic, and this has to do with meaning. A sentence is analytically true if it is true in virtue of the meaning of the words in the sentence. For example, "a sea-worthy boat will not sink in calm waters," is analytically true, since it is in virtue of the meaning of "sea-worthiness", "sink" and "calm waters" that the sentence is true, not in virtue of some empirical fact. There are many more analytic (in the modern sense) truths than logical truths, such as the example given above. This modern conception of analyticity I shall leave aside in the thesis, since I do not think that it engages what is important in distinguishing logic from mathematics.

Frege's positive characterisation of an analytic judgement is that analytical judgements should follow from logical axioms plus definitions. This sounds like the modern conception. However, he only considers definitions which are short-hand for concepts which have already been introduced. So, what counts as analytic is what follows from logic plus logical definitions, that is, definitions for logical notions or definitions whose definiens is expressed using only logical vocabulary.

If "follows from" is taken too narrowly (and arguably how Frege meant it to be taken) then we confront another notion, again traditionally associated with logic: that of effectiveness. A formal system is effective iff it has a finitary deductive system:[16] one capable, in principle, of generating all the truths of the theory in a finite number of steps. In this sense, all truths are either axioms or theorems. Effectiveness is an idealisation on methods of proof. We realise that as humans, even with the help of computers, we can only carry out short finite proofs. If we think of proofs as a series of manipulations of symbols in a given expression, then the conclusion of a proof is the result of a series of manipulations. For a proof method to be effective, we make stipulations in two directions. One is a limitation, the other is an idealisation. The limitation is that proofs must be rigorous, in the sense that each step in an effective proof must be short. That is, we only deal with one "word" at a time.[17] Also, the instruction, for manipulation, must be unique, in that there is only one rule for the manipulation of a given word, there is no choice. For example, we might have a general rule which says to manipulate the word on the extreme left of an expression first. Thus, the expression: $\neg(\forall x)(\exists y)(Rxy \vee Ryx)$ is manipulated to: $(\exists x)\neg(\exists y)(Rxy \vee Ryx)$. This is then manipulated to: $\neg(\exists y)(Rxy \vee Ryx)$, and so on. The limitation requirement for an effective method of proof is that it be short and unique.

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[16]A formal system may be decidable while having the capacity to generate infinite proofs. For example, some formal systems which have the ω-rule will have infinite proofs since the ω-rule is infinite.

[17]For example, $(\forall x)$ is considered to be one "word", as are $\neg$, $\rightarrow$ and Px. One way of individuating "words" is by the introduction and elimination rules, or by the satisfaction function, depending on orientation. Another way is by looking at how they are individuated in the rules for constructing well-formed formulas.

The idealisation is that we suppose that we can carry out manipulations indefinitely. Nevertheless, they have to be finite. Sometimes effectiveness is expressed in terms of Turing machines. A Turing machine can only carry out short and simple manipulations, but it can do so indefinitely.

I shall not accept effectiveness as a criterion of logicality. For, I believe that this criterion answers to an epistemological concern about logic: that logical systems ought to provide us with the means of deducing all truths of logic. This is to treat logic as a tool of demonstration as opposed to its just acting as an ultimate justification in the following sense.

There is a lot more to logic than a set of effective or mechanical proofs. There are also certain dangers in thinking of logic as being exhausted by such proofs. When we do this, we forget about certain metaphysical notions, such as universality, and analyticity, which are attributed to logic. A second danger is that we blindly identify logicality with formalisation, within the constraints of effectiveness (i.e. a formal system must have an effective proof procedure and be complete). We might then allow in any formalisable notion provided it meets the constraints of effectiveness, and at the same time we lose track as to the background metaphysical importance of logic in its distinction from the rest of science. Symptomatic of this is the attitude which says that there is no clear distinction to be drawn between logic and other sciences.

In keeping with a more metaphysical orientation, showing that a sentence is justified by logic can take the form of an effective proof, but it can also take other forms as well. For example, we may examine the vocabulary in which it is written, we might be asked to justify the claim that some vocabulary is logical and some is not, we may be asked to accept infinite proofs, again, provided that they are justified by logical considerations. In other words, the stance taken in this thesis is that logic is distinguishable from other disciplines in a philosophically significant sense. The epistemological concern will be expressed in terms of the negative characterisation of analyticity.

Epistemologically, logical truths are not true or false *a posteriori*, and they do not depend on a special faculty of intuition. This does not entail that all logical truths must be the result of a mechanical procedure. There is a middle ground to be occupied by logic. In this middle ground, the types of justification allowed reach beyond an effective proof, but they fall short of resorting to appeals to intuition and to sense experience. It is in this sense that I am interested in logic in its justificatory role; not in its role as a mechanical tool. Thus, we find that the negative characterisation of analyticity Frege gives is more revealing as to what is distinctive of logic: that the justification for judgements in logic be free from intuition and sense perception.

This declaration of interest conflicts with some remarks Frege makes concerning proof. It would be consistent with some of what Frege writes, that he should think that the *Begriffsschrift* notation made for both an effective method of proof, and that it was sufficient to generate all the truths of logic and, with the addition of axiom V or Hume's principle, to generate those of arithmetic. For example, Frege writes:

> ...the fundamental truths of arithmetic should be proved... with the utmost rigor; for only if every gap in the chain of deductions is eliminated with the greatest care can we say with certainty upon what primitive truths the proof depends.[18]

The notion of gapless proof is naturally compared to that of effective proof, and Frege identified gapless proof with the manifestation of rigor. Of course, there is no effective procedure for generating all the truths of arithmetic because arithmetic is incomplete. Frege was ignorant of this because the incompleteness of arithmetic was only discovered, by Gödel, in 1936. For these reasons, we could hardly expect Frege to have been aware of the incompleteness of arithmetic. Other remarks Frege makes

[18]Gottlob Frege, The Foundations of Arithmetic, trans. J. L. Austin, (Second revised edition; Evanston, Illinois: Northwestern University Press, 1980), § 4.

concerning reducing mathematics to a series of mechanical procedures are more ambiguous. For example, when Frege criticises the formalists in *Grundgesetze*, he writes:

> This attempt at formal arithmetic must be considered a failure, since it cannot be pursued consistently. In the end numerical figures are used as signs after all. Thomae's own inventory of rules is incomplete, and we were forced to suppose that such a list could never be completed.[19]

I am not certain that we want to attribute to Frege a modern use of the words "inconsistent" and "incomplete". The point he is trying to make is that the notation of *Begriffsschrift*, and other formal deductive systems (proof theories) which are, anachronistically speaking, effective, are inadequate as a substitute for arithmetic or mathematics. Rather, we have always to bear in mind the semantics of the sentences we use. The proper practice of mathematics cannot be replaced by blindly implementing syntactical manipulation rules. In this sense, for Frege, a Turing machine does not "do" arithmetic, because it does not understand what it is proving.

In general then, while there is evidence in Frege,[20] that he would have endorsed the effectiveness criterion; in the light of the Gödel incompleteness results, he may well have chosen to reject it. Regardless, I shall grant that an effective proof method makes for a good tool of demonstration. Furthermore, an effective proof is sufficient to show, on what it is that the truth of a theorem depends, namely: an initial list of axioms. However, by itself, an effective proof method is not enough to act as an ultimate justification because citing a proof as sole justification ignores the significance of

[19]Gottlob Frege, "Frege Against the Formalists" *Grundgesetze*, Vol.II §§86 - 137, Trans. Max Black Translations from the Philosophical Writings of Gottlob Frege eds. Peter Geach & Max Black, (Third edition, Oxford: Basil Blackwell, 1980), § 137.

[20]For example, Frege rejects the possibility that there should be infinitely many axioms in logic. This poses an immediate problem. Second-order logic is unaxiomatisable in the sense that we do not and provably could not have a complete list of the axioms. This makes the set of axioms of second-order logic in some sense potentially infinite. This I take as yet another reason to give up effectiveness as a criterion for logic.

the symbols which are being manipulated, and therefore, is insufficient as an account of what the axioms themselves mean, or what status they have: whether they are logical, geometrical, and so on.

Unsurprisingly, often logical necessity is associated with logic, and a formal system which is said to be logic gives or reflects some sense of what the extension of the concept of logical necessity is. But in the context of trying to characterise logic, we must ask the question of how the notion of logical necessity is to feature in a criterion for the logicality of a formal system. Is it that whatever formal system best captures logic has, thereby, to reflect some pre-theoretic or pre-formal use of logical necessity? If so, we can readily identify a conviction, best described as logical that we ought not to contradict ourselves. This can be elaborated on both a syntactic and a semantic level. Syntactically, a statement is logically possible if it is not contradictory. Whatever it is that a set of sentences refers to, such as an event, a fact, a state of affairs, etc. is possible if the sentences do not engender contradiction. On a semantic level, possibility and necessity and are defined in terms of consistency. A contradiction takes the form of "$a \wedge \neg a$", where "a" stands for any proposition. However, it may take a few inference moves to show that a given statement contradicts another, and to show this, one has to appeal to logical laws together with rules of inference, and perhaps even some semantics. A set of sentences is mutually consistent if there is an interpretation which makes them simultaneously true. Put another way, the axioms or laws of logic are true. The rules of inference, such as *modus ponens*, are truth preserving. Logical truths are logically necessary. A valid sentence is logically necessary.

Thus, if we now want to know what the laws or axioms of logic are, we cannot turn to the notion of logical necessity, for it presupposes them, in the sense that it presupposes that they have already been chosen. Nevertheless, I shall take for granted as a criterion for a formal system to call itself a logic: that it not contain a contradiction.[21] All the formal systems we examine are consistent or equiconsistent with others.

However, there is often more associated with concepts of logical necessity and its dual: logical possibility. On a phenomenological level, for example, we might say that we are/ feel compelled to accept logically necessary truths, that it is logical necessity which compels us to accept the conclusion of a logically valid argument. We might, on a more metaphysical level, say that logical necessity is normative, in the sense that logic is normative of reasoning. Thus, anyone who fails to see or confirm a logical necessity, and is (otherwise (however that may be)) *compos mentis*, is not obeying our norms of reasoning.

Both these notions which are associated with logical necessity presuppose that we already know what logic is, or what distinguishes logic from other sciences. From the point of view adopted here, this is an illegitimate assumption. There is an intuitive sense of what logic is, but this is insufficiently precise to distinguish between formal presentations of (purported) logic(s). For this reason, the attitude adopted here is that the formal system which optimally meets our three criteria for logicality forms a sort of implicit definition for logical necessity, and therefore, appeal to logical necessity for characterising logic begs the question against the logicist.

Taking stock, we have discussed five philosophical notions involved in characterising logic: universality, logical validity, effectiveness, logical necessity and analyticity, and some of the history of these notions. We endorse validity, universality and Frege's negative characterisation of analyticity. We reject effectiveness as a necessary criterion for logicality. Logical necessity is taken to be the product of the endorsed criteria.

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[21] Even Graham Priest includes a notion of consistency in his logic, even if he does allow outright contradiction. Consistency is manifested in terms of following, or failing to follow, a rule. See for example, Graham Priest, "Can Contradictions be True?" The Aristotelian Society, Supplementary volume, (LXVII, 1993), pp. 35 - 54.

We turn now to more recent history, and how this merges with the more conceptual aspect of the thesis: the matching of the criteria of universality, analyticity and informal validity to existing formal systems. In 1983, Wright taught us in his book: Frege's Conception of Numbers as Objects,[22] that the formal system outlined in Frege's *Begriffsschrift* (second-order logic) together with Hume's principle (which introduces the notion of cardinal number in terms of one-to-one correspondence) form a consistent formal system strong enough to derive the Peano axioms! Hume's principle was introduced by Frege in the *Grundlagen* which followed *Begriffsschrift*. For philosophical reasons, Frege resisted just including Hume's principle as an axiom in his formal system. He added Basic Law V in *Grundgesetze*, to the formal system developed in *Begriffsschrift*. Basic Law V, was thought to be more obviously logical than Hume's principle since it introduces identity conditions for extensions of concepts, and Frege thought that the notion of extension of a concept was more obviously logical than that of number belonging to a concept. The purpose of adding Basic Law V, was to prove Hume's principle as a theorem, and thus to prove its logicality, and *a fortiori*, the logicality of arithmetic. Basic Law V is inconsistent. However, its removal, and the promotion of Hume's principle to the status of axiom does form a consistent system. This discovery has revived interest in the logicist project, albeit in modified form. This is because Hume's principle contains logical elements, but it does not seem to be sufficiently general to count as a logical axiom. Hume's principle is only true of infinite domains. The modifications we have to make to Frege's logicist project correspond exactly to the degree to which we can argue that second-order logic is a logic and that Hume's principle is a logical principle.

In this Thesis I propose to present a way, inspired by Frege, of characterising logic in contradistinction to some of mathematics. The project is largely programmatic in that it is impossible to survey

[22]Crispin Wright, On Frege's Conception of Numbers as Objects, (Aberdeen: Aberdeen University Press, 1983).

all formal systems. Furthermore, the philosophical criteria I use for assessing what formal systems I do examine, may not be exhaustive. The type of project it is, is one of matching some philosophical notions to some technical data in the form of particular formal systems which are all plausible candidates for the honorific "logic". Which parts of mathematics are then captured by the formal system is a largely technical result, not pursued here. The philosophical notions are universality, logical validity, and analyticity. Generality (or universality) turns out to be one of the most important, although, we shall see towards the end that all the criteria are inter-related.

The first chapter of the Thesis is introductory and historical. It gives the historical motivation for the present project and presents its conceptual roots. As has been mentioned, the prevailing thought through much of this century has been that, insofar as it makes sense to draw a line between logic and mathematics at all, it ought to be drawn at first-order logic. That is, first-order logic and any weaker system is generally accepted as being logic. Any system which is stronger, is deemed to be mathematics. The historical account begins with Leibniz who had a vision of an universal language which would serve as an uniting basis for all science, and by means of a picture script, would make all scientific truths plain. These are dim, and imperfectly related beginnings to Frege's more modest, but better argued vision: that we can show that arithmetic is just a more sophisticated logic. Frege's project is placed in the context of nineteenth century mathematical analysis. The mathematicians, who marked this tradition, aspired to bring greater rigor to the discipline of analysis. Frege applied the notion of rigor to logic. The rest of the chapter examines the chequered history of the logicist project: its demise, a related attempt by Russell and Whitehead, and the abandonment of the project along with the emergence of first-order logic as the canonical logic.

In chapter two I discuss the proposed criteria for a formal system to be called a logic. These are validity, universality and analyticity. The criterion of validity is expressed as follows: a formal system conforms to the criterion of validity if its formal notion of validity matches our informal notion. The criterion of universality has two aspects. A formal system must be applicable to any domain, and it must be topic neutral. The first aspect is fairly weak. For, any language can display this aspect provided we are allowed to consider any domain of interpretation, including ones where elements of the language have no referent. To fend from this, we insist that said language not include any non-logical constants. There is a convention governing what counts as a logical constant. Unfortunately, for the purposes of this project, we cannot simply avail ourselves of the convention without begging the question. Thus, we have to invoke the other aspect of generality: that a logic should be topic neutral. Intuitively, what this means in the context of a logic is that the language of the logic does not allow us to distinguish particular objects in a domain of interpretation. Of course, this ties in with our other two criteria: validity and analyticity. It ties in with validity in that the logical vocabulary of a sentence is supposed to reflect the logical form of the sentence. The lack of non-logical constants, should suppress the content of a sentence. Topic neutrality ties in with analyticity in that, analyticity, in our sense, is a minimal constraint on what counts as logical. None of the criteria on its own is sufficient to pick out logic from the number of formal systems which call themselves logic, because as we have already seen, the criteria are inter-related.

In chapter three, I examine the position which asserts that first-order logic is where the division lies between logic and mathematics. In particular, first-order logic is assessed according to our philosophical criteria for logicality. The arguments tend to focus on the limitative results of first-order logic, rather than on a semantic and syntactic text book presentation of the logic. This is because many of the arguments for the position which favours first-order logic as logic over second-order logic, draw on the limitative results. The particular results discussed are: compactness, the

Löwenheim-Skolem properties, completeness and decidability. There are two opposing lines of argument. The one against first-order logic being considered as exhaustive of the scope of logic is that first-order logic is expressively inadequate to the task of reflecting our informal notion of logical validity. This involves compactness and the Löwenheim-Skolem properties. The argument in favour of the thought that first-order logic exhausts the scope of logic involves compactness, completeness and decidability. It is thought that the mark of logic is that it should have an effective proof procedure.

In chapter four I entertain the possibility that some extensions of first-order logic might also be logic. In particular, having identified certain sentences or arguments as valid, or identified certain notions as logical, which are deprived of proper representation in first-order logic, we find formal systems which are able to express these properly. For obvious reasons of space and time, not all extensions of first-order logic are explored. The extensions of first-order logic we examine generalise in different ways on the notion of quantifier as it is (incompletely) presented in first-order logic. More specifically, we increase the logical vocabulary to include more quantifiers or more variables. Along with these additions, what counts as a term and a well-formed formula are suitably modified, as is the interpretation function of the formal system.

The first generalisation discussed, is one which takes seriously the task of representing, in a formal language, quantified statements in natural language. This orientation holds out promise for meeting our validity criterion. Quantifiers are analysed as a pair composed of a determiner and a set expression. Loosely, we think of quantifiers as predicates which happen to pick out a subset of the domain. Any subset will have a quantity. For example, "the green objects in the bag" counts as a quantifier, since it picks out a quantity of objects, by means of a predicate. This view treats each determiner/ set expression pair as a new symbol, requiring a separate clause in the interpretation function.

The second generalisation was suggested by Mostowski in 1957.[23] He suggested that we should think of a quantifier as a function which names a quantity. It turns out that any cardinality expression can take the quantifier position. For example, Mostowski includes "there are finitely many", "there are infinitely many", "most" and "half", as quantifiers. More broadly, we also discuss many-place quantifiers, such as, "there are more A's than B's".

We then extend Mostowski's generalisation on the notion of quantifier to include second-order quantifiers. In effect, we then have second-order logic, which is what Frege thought of as logic in the first place. All of these extensions of first-order logic are assessed in terms of our criteria: an informal notion of validity, analyticity and generality.

In the conclusion, I discuss the significance of the modern version of Frege's logicist project in the light of our characterisation of logic. The significance of the modern version of the logicist project depends, in part, on the extent to which we can argue persuasively that second-order logic is logic. Put another way, the logicist project is undermined if we find that second-order logic is really a ("special" in Frege's sense) mathematical theory, or if there is no clear distinction to be drawn between logic and the special sciences.

It might turn out, of course, that much more powerful formal systems also conform to the criteria elicited here. In that case, more of mathematics will turn out to be logic. For example, it might turn out that second-order logic is able to subsume much more than just arithmetic, but also group theory. To repeat, this would indicate that much more of mathematics is part of logic, in exactly the sense we have indicated through our criteria. Again, this is the sense in which the thesis is programmatic. The hope is that the thesis makes as

[23] Andrzej Mostowski, "On a Generalization of Quantifiers," Fundamenta Mathematicae, vol. IVIV, (1957), pp. 12 - 36.

explicit as possible the sense in which some philosophers regard logic as significant, and that it indicates at least one formal system which exhibits that significance.

Chapter I: A History of Logicism

Introduction

Our concern in this chapter is with the history of the philosophical position called logicism. A logicist will argue that there is a sense in which all (or most, or some) of mathematics is reducible to logic. The purpose of giving a history in the context of the thesis is threefold. We wish to provide a historical motivation for raising the issue of a logical foundation for mathematics, to locate the reader in the debate, so that he can better understand why it is that some questions are now being asked and not others, and to show that the neglect of the logicist project is partly due to historical accident.

All three purposes are closely interrelated. Logicism found prominence both around the turn of the century, with the work of Frege, and more recently, since 1983,[24] when it was made clear that second-order logic, together with what is referred to as Hume's principle, is sufficient to derive the Peano Axioms. Thus, insofar as the formal system which is composed of second-order logic and Hume's principle constitute a logical system, arithmetic can be said to be founded in logic. The doubt, as to whether or not this discovery vindicates logicism, concerns whether or not second-order logic is logic, and over whether or not Hume's principle is a logical principle.

Frege's logical system was what we generally recognise as second-order logic.[25] Hume's principle was also used by Frege, but

[24]Crispin Wright, Frege's Conception of Numbers as Objects, (Aberdeen: University of Aberdeen Press, 1983).

[25]Not everyone agrees with this. See for example Hintikka and Sandhu "Frege's Alleged Realism," Inquiry, vol. XX (1977), pp. 227 - 242. However, Boolos, Wright, Dummett, Clark, Hale, Heck, Parsons, Shapiro and many others agree that Frege's formal system is essentially second-order logic. Frege has quantification over second-order variables, and they can be assumed to range over the powerset of

he did not think it was obvious that it was a logical principle, because it could not be introduced into logic as a straightforward definition because it is not a definition of number (it gives the conditions under which two numbers are to be considered to be equal) since it engenders what has been referred to as the Julius Caesar problem. The Julius Caesar problem is that Hume's principle does not have the power to tell us of an arbitrary object, such as Julius Caesar, whether or not it is a number. That is, while Hume's principle gives conditions for equality between numbers, it gives us no purchase on what qualifies something as a number. Thus, Hume's principle had to be grounded in a more basic principle. Frege tried to prove that Hume's principle could be better grounded in a more general principle. Frege introduced another axiom from which he could derive Hume's principle as a theorem. Unfortunately, this axiom turned out to be inconsistent. However, if we remove the offending axiom and add Hume's principle to second-order logic as an axiom, then we have a consistent system which is powerful enough to include within it second-order arithmetic. Hume's principle is interesting for the logicist because it is not obvious whether it is logical, partially logical, or wholly mathematical. This question will be left aside in the thesis. We shall focus on the prior question, whether or not second-order logic is a logical system in a relevant sense (for the logicist).

The second purpose of giving a historical introduction is to orient and locate the reader in the present issues which are being raised. This chapter is less philosophical than others. The philosophical motivation will be made clear in subsequent chapters of the thesis. As a motivational chapter, it does not seek to give an exhaustive history of the problem, for the approach of the thesis is

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the domain of interpretation. This reading is consistent with Frege, *pace* Sanhu *et al.* Notwithstanding, the controversy over Frege's formal system does not bear on the conceptual features of this thesis. I am using Frege as inspiration to further investigation. Thus, the attitude adopted here is that correct exegesis of Frege is less important than identifying aspects of his writing which prompt fruitful discussion.

not primarily historical. Nevertheless, we do wish to indicate that, on a purely conceptual level, dropping the considerations of logicism was done too hastily.

There are six sections in this chapter. The first explores some of the earliest thoughts concerning the overall project of logicism; the sorts of thoughts which were the seed of the project. In particular, we discuss Leibniz's idea of an universal language. The language is meant to be universal in two respects. One is that it forms the basis of rational thought, and all truths emerge from it. The other respect in which the language is meant to be universal is that it is pictorially representative of these basic rational thoughts, and so cross culturally, and cross-linguistically recognisable. In section two, we discuss Frege in the context of changes in nineteenth century analysis. That is, we place Frege in the context of the tradition from which he emerged; a tradition with concerns about increasing the rigor of proof in analysis. In section three, we discuss Frege's logicism as a project in its own right. We focus on the aspects of his project which contribute to the historical analysis in this chapter, and on aspects of his work which will be discussed in subsequent chapters. In section four, we discuss Whitehead and Russell's attempt to carry out the project where Frege failed, and show that they too failed. In section five, we give some history concerning the emergence of first-order logic and the emergence of first-order set theory as the foundational discipline of mathematics. These largely superseded and precluded the logicist project. In section six, we discuss Frege's proposed foundational system of logic: second-order logic, in its present day context.

This chapter fits with the rest of the thesis in that, in contrast to the present day, it was clear to Frege that second-order logic just is logic, not only in name, (which is anachronistic anyhow) but according to certain philosophical criteria. He did not contrast it to first-order logic, since that had not yet been recognised as an independent formal system. For Frege, there was a strong philosophical basis for thinking that the formal system he proposed was entirely plausible as a candidate for founding arithmetic.

Since Frege, we have been confronted with a much larger number of logical systems. One recent train of thought which has been well received is that first-order logic is the strongest logic, and anything stronger is mathematics. Adherence to this view precludes logicism because first-order logic is not powerful enough to act as any sort of interesting foundation for mathematics, let alone arithmetic. Instead, people interested in foundational issues, find first-order set theory to be a more apt candidate. Showing that, as far as logicism is concerned, first-order logic falls short of the task of founding mathematics is the subject of the third chapter. To do this we have to define the task precisely. This is done in the second chapter, where we discuss the philosophical criteria we bring to bear on our judgement that first-order logic is inadequate.

Another recent train of thought has it that there is no clear distinction between logic and mathematics, since we have developed a number of systems one stronger than the other, but not by much. The systems overlap each other so much that the terms "logic" and "mathematics" seem vague. Thus, it is not clear how to draw a line between logic and mathematics. This blocks logicist aspirations because we cannot even identify which is the founding discipline. We cannot have grounds to favour any particular "logic", let alone identify one system as logical where others are mathematical. In this tradition, it is not clear what it is that a reduction of arithmetic, or any other branch of mathematics, to logic, would show; besides a reduction of one branch of mathematics to another. Both modern ways of thinking conspire to make the project of logicism seem hopeless.

On the other hand, having discovered that we can argue that Hume's principle is logical, at least in some respects, namely in its analyticity;[26] we revitalise the question as to whether or not

[26]Whether or not Hume's principle is analytic has been hotly debated, most prominently by Wright and Boolos. In this debate, insufficient attention has been

arithmetic can be founded in logic. To make this argument one has to argue for (1) the status of Hume's principle as a logical principle, and (2) the claim that second-order logic is in some respectable sense, logic. Arguing for (2) will be the subject of chapter four.

§ 1: Leibniz and the Beginnings of Logicism

The dim beginnings of logicism can be traced back to Leibniz who had a vision of an universal language which would:

> assign to every object its characteristic number ...[so that] people of different nations can communicate their thoughts to one another ...[and which would] embrace both the technique of discovering new propositions and their critical examination.[27]

Leibniz believed we could develop such a language and called it a *lingua characteristica universalis*. Essentially this was an universal language.

Leibniz's writing on the subject is fragmented, and he assigns different tasks to his *lingua characteristica universalis*. Among them is the task of codifying and formalising syllogistic reasoning; although he recognised later, after the age of nineteen, when he first expressed this idea, that there are valid forms of reasoning not captured by the syllogism. These are arguments whose validity depends on relations or functions, for example. Once he recognised this, he thought of ways to extend syllogistic reasoning in the hope that his *lingua characteristica univeralis* would capture other forms of reasoning as well.

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paid to the definition of analyticity, in particular, to how it is that Frege defined analytic. Frege's definition is different from the modern definition. I also think that there is a modification to Frege's definition which can accomodate Hume's principle as analytic. That is, there is a sense in which Hume's principle is analytic, and it is a sense which is in keeping with the spirit of Frege's logicist project. The relevant meaning of the term analytic, is the one explored in this thesis. However, showing that Hume's principle and no offending principles fall under this definition is a question which is left unattended in this thesis.

[27]G. W. Leibniz, "Towards a Universal Characteristic (1677)," Leibniz, Selections, ed. Philip P. Wiener (New York: Charles Scribner's sons, 1951), p. 18.

He also assigned to the *lingua characteristica universalis* the task of demonstrating correct reasoning. As such, it would produce, in an almost mechanical way, rational and indubitable arguments. He believed it would promote clarity of reasoning. This is better known under his idea of a *calculus ratiocinator*, which is a term he sometimes subsumed under the term *lingua characteristica universalis*.

Leibniz believed in a certain unity of reasoning, and that all of science could be codified by assigning numbers to the essential or primitive objects of scientific investigation. This was part of his inspiration in working on an encyclopaedia. In a sense, he wished to turn empirical science into number theory. But not only this. For, he insists that the primitive assignment of number or symbol should not be done arbitrarily but should act as a sort of picture which captures the essence of the object, and which would be universally recognised. He was inspired by the Chinese script where a word is captured in a highly abstract picture. More ambitious still, he believed that the unity of religion could be realised by means of such a language. For, he believed that all religious thinkers would be swayed by the indubitable arguments which would simply lead to the truth. Thus, misunderstanding and conflict would cease. There was a sense in which rational reasoning would lead to all truths, including empirical ones. It was the development of a *lingua characteristica universalis* which was to enable mankind to achieve this end.

Unfortunately, Leibniz did not progress very far with the technical development of the language, despite the fact that the idea occupied much of his thinking from a very early age. There were certain technical impediments to its elaboration. For example, there is a sense in which scientific knowledge would have to be complete before the basic numbers or pictures could be assigned. We would have to know what the basic stuff of matter is, and the natural laws which affect the behaviour of matter before they could be adequately represented. Leibniz's picture of the universe is that it is highly deterministic; unified by reason. Leibniz's conception was original, and was a precursor of Laplace's metaphysics. Laplace (1749 - 1827) was one of the first to really work out the implications of a fully deterministic universe. Drawing certain metaphysical conclusions from Newtonian mechanics, Laplace famously suggested that the universe worked very much like a clock. He also postulated an evil demon to whom the initial conditions of the universe are known and for whom "nothing would be uncertain and the future, as the past, would be present to its eyes."[28] In a sense, to realise his project, of a *lingua characteristica universalis,* Leibniz would have had to be in the position of the evil demon.

The idea of the *lingua characteristica universalis* is a precursor to Frege's concept script in that the concept script was meant to make logical steps evident in an almost pictorial form. Proofs were characterised by Frege, as being properly rigorous, if there was a continuous line running through the proof. We shall see this in the next section. The major difference between Leibniz and Frege lies in their view as to what constitutes logical reasoning, and what the scope of that reasoning is. Leibniz and Frege agree that there is an universal quality to reasoning, and that it is independent of human activity.

An indirect way in which Leibniz contributed to the development of logicism is by his development of the calculus. He did so independently of, but concurrently with, Newton. Their respective approaches to the subject were quite different. Newton was more pragmatic where Leibniz was more philosophical. Newton refers to the infinitesimals ambiguously, sometimes referring to them as moments, sometimes as limits. The latter was later to become standard. Newton did not engage in the metaphysical debate about the deterministic universe which was

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[28]Pierre Simon de Laplace, Essai philosophique sur les probabilités, (Paris, 1814), trans. F. W. Truscott and F. L. Emory, A Philosophical Essay on Probabilities, (London and New York, 1902). Also quoted Rom Harré, "Laplace, Pierre Simon de," Encyclopedia of Philosophy, ed. Paul Edwards, III (reprint edition, 1972), p. 392.

taking place on the continent. The role God had to play in Newton's metaphysical conception of the universe was to keep the stars from colliding and maintain the stability of the solar system against outside influences. On the continent, Leibniz's pupil, De l'Hopital, viewed the infinitesimals of the calculus as part of physical matter. This is a natural reading of the calculus if we recall that in the seventeenth century, the axiomatic method was modelled after Euclid's geometry; the principal supposition being that in proofs we begin with facts (the axioms) and conclude more facts. The facts could be empirical. Thus, when De l'Hopital writes:

> One requires that a curve may be regarded as the totality of an infinity of straight segments each infinitely small: or (which is the same) as a polygon with an infinite number of sides which determine the angle at which they meet...[29]

we are to take him literally as referring to physical entities: shapes, lines and segments. Leibniz disagreed. His conception of the infinitesimal was more subtle. He viewed them as useful fictions. For Leibniz, the concept of the infinitesimal simply resulted from extending the rules of arithmetic from finite numbers. This can lead to conceptual confusions. For example, one might ask what happens if we add an infinite number of infinitesimals together. Several answers seem possible. On the one hand, we might remain very close to 0, on the other hand an infinite number of infinitesimals might add up to the unit 1, or even beyond this to infinity!

Leibniz was interested in the metaphysical underpinnings of mathematics, and for this reason was a philosopher of mathematics and a precursor to Frege. Furthermore, the debate about infinitesimals and what they signify metaphysically is also a precursor to the nineteenth century debates about analysis. The debate about infinitesimals seems to have petered out in the eighteenth century, but started to pick up again with Bolzano in the

[29]G. F. A. De l'Hopital, *Analyze des Infiniment petites pour l'Intelligence des Linges Courbes*, (first edition; Paris, 1696, second edition; Paris, 1715), translated by and quoted from: A. Robinson, "The Metaphysics of the Calculus," Problems in the Philosophy of Mathematics, ed. Imre Lakatos, p. 32.

early nineteenth century. Bolzano gained little recognition during his life time, but was associated with Weierstrass and others. Frege saw himself as very much part of this tradition. The proposed solution to the *problématique* concerning the infinitesimals lies in the insistence on greater rigor in proof.[30]

§ 2: Frege and Nineteenth Century Analysis

Whereas we can trace the dim beginnings of logicism to Leibniz; the more clear beginnings can be found in Dedekind and Frege who were the first to articulate the philosophical position explicitly. For example, in the preface to The Nature and Meaning of Numbers, Dedekind writes:

> In speaking of arithmetic (algebra, analysis) as a part of logic I mean to imply that I consider the number-concept entirely independent of the notions of or intuitions of space and time, that I consider it an immediate result from the laws of thought.[31]

Frege shares with Dedekind the conviction that the concept of number is a logical concept. However, he registers dissatisfaction with Dedekind on account of how he goes about showing that this is indeed the case. Dedekind does not prove that arithmetic is a part of logic, despite his declared intention to the contrary:

> In science nothing capable of proof ought to be accepted without proof. Though this demand seems so reasonable yet I cannot regard it as having been met even in the most recent methods of laying the foundation of the simplest science; *viz.*, that part of logic which deals with the theory of numbers.[32]

[30]William Demopoulos, "Frege and the Rigorization of Analysis", Frege's Philosophy of Mathematics, ed. William Demopoulos, (Cambridge, Massachusetts: Harvard University Press, 1995), p. 78.
[31]Richard Dedekind, "The Nature and Meaning of Numbers", Richard Dedekind, Essays on the Theory of Numbers, Trans. Wooster Woodruff Beman, (New York: Dover Publications Inc., 1963), p. 31 (first page of the preface to the first edition).
[32]Richard Dedekind, "The Nature and Meaning of Numbers", Richard Dedekind, Essays on the Theory of Numbers, Trans. Wooster Woodruff Beman, (New York:

Dedekind came up with some axioms for arithmetic (hence the Peano-Dedekind axioms). This was not enough for Frege. Ironically for Dedekind, Frege writes:

> ...compare [to Grundgesetze] Herr Dedekind's work... the most thoroughgoing work on the foundations of arithmetic which has lately come to my notice. In much less space it pursues the laws of arithmetic much farther than is done here. To be sure, this brevity is attained only because a great deal is not really proved at all. Frequently Herr Dedekind merely says that the proof follows such and such propositions... an inventory of the logical or other laws taken by him as basic is nowhere to be found, and even if it were, there would be no way of telling whether no others were actually used; for that to be possible the proofs would have to be not merely indicated but carried out, without gaps.[33]

The dissatisfaction aired by both Dedekind and Frege was very much part of a trend in mathematics towards greater rigor in proof. This tradition belonged primarily to the mathematicians pursuing analysis in the nineteenth century.

Frege is part of the tradition because of his demand for proof, where proof can be given. In order to prove, more convincingly than Dedekind did, that numbers are an immediate result of the laws of thought, Frege had to develop a more powerful formal system of logic than that of the propositional calculus or the relatively weak innovations introduced by Pierce and Boole. One of Frege's greatest contributions to logic was to disentangle the quantifier from connectives and predicates. Frege is often portrayed as the inventor of the quantifier. But this is misleading. The explicit use of second-order quantifiers together with the logical treatment of a sentence in terms of function and argument is what separates Frege from Boole

Dover Publications Inc., 1963), preface, p. 31 (first page of the preface to the first edition).

[33]Gottlob Frege, The Basic Laws of Arithmetic, Exposition of the System, ed. and Trans. Montgomery Furth (Berkeley, California: University of California Press, 1964), p. 4.

who developed a version of propositional calculus, and from Peirce who popularized use of the second-order quantifier.[34]

The second innovation Frege made in developing his logical foundation for arithmetic was to abandon the subject/ predicate analysis of a sentence and include a function/ course-of-values analysis as part of logic. Alongside a few other innovations, this turns out to be a very powerful device which increases the capacity of the formal language to express mathematical notions. This falls directly in line with the tradition of the nineteenth century analysts. What Cauchy, Weierstrass *et al.* were trying to do, in developing an analysis of sentences in terms of function and argument, was to set analysis back on a firm epistemological foundation. What they understood was the philosophical importance of the fact that it is possible to characterise real-valued functions in terms of a many-one correspondence. A many-one correspondence can be expressed in a formal language, and this allows one to conceptually divorce analysis from geometry and kinematics. This in turn, frees analysis from the puzzles involved in trying to imagine and apply the obscure and abstract concept of infinitesimal straight lines, for

[34]In his article "Peirce the Logician", Putnam seeks to redress the balance between Pierce and Frege *vis-à-vis* their contributions to logic, and in particular, the roles they played in introducing the quantifiers to the discipline. Notwithstanding, Frege's contribution is conceptually more important. Putnam paints Frege as the inventor of the quantifiers, but he argues, because he presented them bound up (no pun intended) in a cumbersome notation coupled with a grand metaphysical theory, he cannot be credited with popularizing the notion. Putnam argues that the credit for this, must be attributed to Pierce who presented the quantifiers unencumbered with metaphysics and in a manageable notation. "Frege tried to "sell" a grand logical-metaphysical scheme with a dubious ontology, while Peirce... was busy "selling" a modest, flexible, and extremely useful notation." Hilary Putnam, "Pierce the Logician," Realism with a Human Face, ed. James Conant (Cambridge Massachusetts: Harvard University Press, 1990), p. 257.

Putnam's analysis is not obvious. For one thing, Aristotle can be said to have invented the quantifiers, since he made use of them, as we can survey in his syllogistic logic. Rather, Frege's contribution is more significant than Peirce's (despite Putnam's mercantile metaphor) because he freed the quantifier from logical connective and predicate. See Gregory H. Moore, "The Emergence of First-Order Logic," History and Philosophy of Modern Mathematics, eds. William Asprey and Philip Kitcher, (Minesota Studies in the Philosophy of Science: Vol. XI; Mineapolis: University of Mineapolis Press, 1988), p. 97.

example. For, analysis becomes more an exercise in the formal expression of ideas, and manipulations of the symbols of the formal language. When we build into a language the capacity to analyse sentences in terms of function and argument, this allows the language to exhibit the structure of relations and functions, which is crucial if one wants to demonstrate the validity of certain mathematical arguments; those on whose validity the structure of the relations and functions featuring in them, depends.

Where Frege departs from the tradition of the rigorisation of analysis is with his preoccupation with logic playing a foundational role towards arithmetic, and the philosophical privilege he confers on logic. In this, Frege takes the project of rigorisation one step further. Whereas Cauchy and Weierstrass increase rigor in analysis, to divorce it from geometry and kinematics; Frege takes the foundation of analysis: arithmetic, and makes that more rigorous. Thus again, he divorces arithmetic from geometry and kinematics. Moreover, he does so by founding arithmetic in logic and developing the philosophical implications which accrue.

§ 3: Frege's Logicism

Frege produced his very candid and explicit articulation of logicism in three great works: the *Begriffsschrift*, (1879) the *Grundlagen der Arithmetik* (1884) and the *Grundgesetze der Arithmetik* (1893, 1903). Frege's technical contribution of the concept script in the *Begriffsschrift*, is that of introducing a logical system[35] which, is equivalent to second-order logic. Besides bringing to light new technical logical innovations, Frege articulated a philosophical theory about the reduction of arithmetic to logic in his *Grundlagen*. The concept script was meant to provide the technical underpinning

[35]By "system" I mean a language together with axioms and rules of inference, or something equivalent. A system may or may not have a set of intended models or domains of interpretation.

for the philosophical logicist project of showing that numbers are really just logical objects.

In his presentation of the concept script in the *Begriffsschrift*, Frege laid the groundwork for his project of proving arithmetic to be part of logic. This is a technical work, introducing a new method of proof. The proofs are entirely formal and mechanical. There is a conceptual difference between the way in which we think of logical proofs today and how Frege thought of them. Today, there are many methods of doing proofs. For example, we might begin with assumptions (some of which may be known to be false!), use truth preserving manipulation rules and then discharge the assumptions. For Frege, we must begin with truths. Proofs take us from truths, through truth preserving manipulation rules to more truths. Frege did not have a device for discharging assumptions. One begins a proof with axioms, then one follows an unbroken line to theorems which are the result of manipulations of the symbols in the axioms. Which propositions depend on which other propositions and under what circumstances (of negation, for example) is indicated by symbols along a line. For example, a true judgement, A, is indicated by placing a turnstile symbol to its left:

$$\vdash A.$$

A simple judgement, of whose truth value we are ignorant, is indicated by a horizontal line:

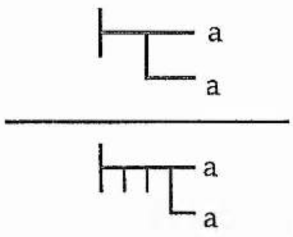
$$- A.$$

A negation is indicated by a vertical line dropping from the horizontal line leading to a judgement:

$$\top A.$$

A material conditional which for us reads: "if B then A" takes the shape:

Proofs take the form of a string of judgements dependent on one another, followed by a conclusion. The conclusion is separated from the argument by a horizontal line. For example,



In modern notation we would write this:

$P \rightarrow P,$

$\therefore\ \neg\neg(P \rightarrow P).$

That is, where "P" is a propositional letter, and can be substituted for "a" in the Begriffsschrift notation, from $P \rightarrow P$, we can conclude that $\neg\neg(P \rightarrow P)$.

A notion which becomes pivotal in Frege's attempt to show that intuition and experience/ psychology play no role in arithmetic is that of a gapless logical proof. The design of the concept script shows that this is not merely meant metaphorically. It is meant literally. A gapless proof is one in which it is possible to trace a continuous line to a concluding proposition. To manipulate a pair of propositions by using a rule of inference, such as *Modus Ponens,* there are clear "transition signs" so that inferences have to be of a type which we can recognise from an already justified inventory.

What a gapless proof indicates technically is that nothing has been used in the proof save truths, i.e. axioms, logical rules of inference and definitions. The definitions are of a special type. A definition is logical, and therefore, may be introduced in a gapless proof just in case the *definiens* is in purely logical notation. It turns out that acceptable definitions are for the most part,[36] just a short-hand for longer expressions. Philosophically, a gapless proof indicates that there has been no appeal to intuition or matters of experience. *Pace* Benacerraf,[37] the emphasis is not so much on

[36]The possible exception to this is Hume's principle, and other abstraction principles. These are sometimes regarded as definitions of a sort, namely as contextual definitions.

[37]Paul Benacerraf, "Frege: the Last Logicist," Midwest Studies in Philosophy, eds. Peter A. French, Theodore E. Uehling, Jr., & Howard K. Wettstein. (Vol. VI, Minneapolis: University of Minnesota Press. 1981).

insuring against contradiction, as it is on insuring that we have not introduced any suspect/ contingent notions into our proofs. Whatever can be proved by these means is considered to be logic by Frege. A gapless proof indicates that the conclusion of a given argument is analytic *a priori*. Thus, a proof acts as a certificate of justificatory pedigree.

The logical system of the *Begriffsschrift* was not enough to show that arithmetic is part of logic. While Frege was aware that his project was so far incomplete, he published the *Begriffsschrift* as an introduction to the concept script, confident that he could later provide a properly logical axiom from which he would be able to derive the Peano axioms.

It was suggested to Frege, by one of the reviewers of the *Begriffsschrift*, that he write a philosophical accompaniment to it in order to motivate the use of the cumbersome notation and inform philosophers of the significance he attributes to the development of logic in this direction. He did this, and produced one of the best examples of analytic philosophy in his famous: *Grundlagen der Arithmetik*. The work is a striking example of clarity, honesty and perspicuity of argument. In it, he spends the first half discrediting previous philosophical theories concerning the foundations of arithmetic. Having swept the floor and made room for his theory by showing the poor state of the foundations of arithmetic, he develops his own theory. While he entertains many possible objections, he finally indicates, in part, how one can derive the natural numbers from logical principles. To do this Frege needed the help of an additional supposition which he attributes to Hume, following Baumann. Call this: "Hume's principle".[38] Frege was dissatisfied

[38]Baumann first encountered the statement in Hume's A Treatise Concerning Human Nature. In the context in which it appears, it is clear that Hume had no idea of the power of the principle, (that, together with second-order logic, it is enough to derive the Peano axioms). For the original Hume quotation see: David Hume, A Treatise of Human Nature, (Book I, London, 1739), part iii, § 1. The statement of the principle is quoted in Gottlob Frege, The Foundations of Arithmetic, trans. J. L. Austin, (Second revised edition; Evanston, Illinois: Northwestern University Press, 1980), § 63.

with taking Hume's principle as an assumption for philosophical reasons. Essentially Hume's principle introduces the notion of cardinal number. Hume's principle is:

$$(\forall F)(\forall G)(Nx{:}Fx = Nx{:}Gx \leftrightarrow F \approx G).$$

It reads: for any concepts F and G, the number of F's is equal to the number of G's if and only if F and G can be placed into one-to-one correspondence. That is, it introduces the notion of number in terms of one-to-one correspondence. Frege does not consider it to be obvious that this is a purely logical truth, so it is not acceptable as a logical axiom. The trouble, he says, is that it does not give us the resources to tell whether or not an arbitrarily given object is a number. Hume's principle cannot really be counted as a definition of number for this reason. Furthermore, Hume's principle is not basic enough to count as an axiom either. Leaving aside his misgivings, and confident that he could provide an axiom from which he would be able to derive Hume's principle as a theorem, he strides ahead. Towards the end of the book he sketches a proof of the infinity of the natural numbers.

He was well aware of the fact that his project was still incomplete. Very candidly he writes:

> I [Frege] do not claim to have made the analytic character of arithmetical propositions more than probable, because it can still always be doubted whether they [the natural numbers] are deducible solely from purely logical laws, or whether some other type of premiss is not involved at some point in their proof without our noticing it. This misgiving will not be completely allayed even by the indications I have given of the proof of some of the propositions; it can only be removed by producing a chain of deductions with no link missing, such that no step in it is taken which does not conform to some one of a small number of principles of inference recognized as purely logical.[39]

[39]Gottlob Frege, The Foundations of Arithmetic, trans. J. L. Austin (Second revised edition; Evanston Illinois: Northwestern University Press, 1980), § 90.

To bring his logicist aims to a conclusion, Frege wrote another technical work in two volumes: the Grundgesetze der Arithmetik. The first volume was published in 1893, and the second in 1903. The aim was to give a more formal presentation of concepts which would extend the work begun in Begriffsschrift, and would prove rigorously that Hume's principle, and therefore arithmetic, are grounded in logic.

It is significant that Frege felt he had to prove Hume's principle to be logical. He was thereby indicating that he did not think it obvious that it was part of logic, but that it can be justified in terms of more basic and obviously logical principles.

> Of course the pronouncement is often made that arithmetic is merely a more highly developed logic; yet that remains disputable so long as transitions occur in proofs that are not made according to acknowledged laws of logic, but seem rather to be based upon something known by intuition.[40]

Frege wanted to derive the natural numbers from clearly logical axioms, and thus demonstrate that arithmetic is really just logic and that numbers are logical objects, requiring only basic suppositions. In the tradition of the nineteenth century analysts, he wanted to produce a rigorous proof: one from which it is clear that no "special" considerations have crept in. A conception is "special" just in case it is based on empirical observation or is based on intuition. To this end, Frege had to derive Hume's principle from more basic axioms.

In the first volume of Grundgesetze, Frege introduced a new axiom, which was not present in the Begriffsschrift. This is the infamous axiom V (otherwise known as basic law V). It governs the relationship between objects and concepts. It is:

$$(\forall F)(\forall G)([x]Fx = [x]Gx \leftrightarrow (\forall x)(Fx \leftrightarrow Gx)).$$

In the left to right direction, it says that there is a mapping from concepts to objects. In the right to left direction, it says that the

[40]Gottlob Frege, The Basic Laws of Arithmetic, Exposition of the System, ed. and trans. Montgomery Furth (Berkeley, California: University of California Press, 1964), p. 3.

mapping is one-to-one. Frege expresses some reservations about it in the introduction to Grundgesteze. There he writes:

> A dispute can arise, so far as I can see, only with regard to my Basic Law concerning courses-of-values (V), which logicians perhaps have not expressly enunciated, and yet it is what people have in mind, for example, where they speak of the extensions of concepts.[41]

Frege was right to be apprehensive. In 1902, Bertrand Russell read the first volume of Grundgesetze and indicated that a paradox is derivable from axiom V. He wrote a letter to this effect to Frege, who published the second volume anyhow, but included an acknowledgement to Russell, a derivation of paradox from axiom V, and the very candid comment: "I have never concealed from myself its [basic law V's] lack of the self-evidence which the others [laws] possess, and which must properly be demanded of a law of logic."[42] He then suggests a possible patch on axiom V in the second appendix. He called the patch: "basic law V*". The patch axiom is ineffectual. Contradiction can be derived from that as well. We shall see an example of a property which generates a contradiction from axiom V shortly.

The contradiction has come to be known as Russell's paradox despite the fact that it was first discovered by Zermelo in 1897.[43] Nevertheless, Russell seems to have discovered it independently, and its application to Frege's logical system is the most famous example of the paradox being deployed. The problem, manifested in Russell's paradox, runs deep. Once Frege realised this, he was devastated. Even before full realisation of the extent of the problem had taken hold, Frege wrote:

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[41]Gottlob Frege, The Basic Laws of Arithmetic, Exposition of the System, ed. and trans. Montgomery Furth (Berkeley, California: University of California Press, 1964), pp. 3 - 4.
[42]Gottlob Frege, The Basic Laws of Arithmetic, Exposition of the System, ed. and trans. Montgomery Furth (Berkeley, California: University of California Press, 1964), p. 127.
[43]B. Rang and W. Thomas, "Zermelo's Discovery of the 'Russell Paradox'," Historia Mathematica, vol. VIII, (1981), pp. 15 - 22.

> Hardly anything more unwelcome can befall a scientific writer than that one of the foundations of his edifice be shaken after the work is finished.
>
> I have been placed in this position by a letter of Mr. Bertrand Russell just as the printing of this (second) volume was nearing completion.[44]

After the publication of the second volume of Grundgesetze, Frege did not put pen to paper for fourteen years; and then, it was to re-found arithmetic on geometry!

The problem which Russell's paradox poses for axiom V is a conceptual one. Axiom V is an unrestricted axiom of abstraction: for all objects there is a corresponding concept, and for all concepts, there is a corresponding object. A special case of this is the axiom of extensionality: which says that for all properties, there is an extension, and for all extensions there is a corresponding property. That is, the mapping from properties to extensions is one-to-one and onto. The extension is a set: the set of objects which have the property. Unfortunately for Frege, the relationship between some properties and their extensions is not straightforward.[45] In particular, axiom V allows for the possibility of what Dummett calls indefinitely extensible concepts. These are concepts, the object corresponding to which, are not totalities in the sense that they are sets which cannot be amassed together. This is because any attempt to amass the set fails because it immediately grows. For example,

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[44]Gottlob Frege, The Basic Laws of Arithmetic, ed. and trans. Montgomery Furth, (Berkeley and Los Angeles: University of California Press, 1964), p. 127.

[45]Charles Chihara, Ontology and the Vicious Circle Principle, (Ithica & London: Cornell University Press, 1973), p.2. It does take some work to justify this reading, however, bear in mind that we are using the notion of set in a naive way, that is, we are not yet committing ourselves to a sophisticated set-theoretic semantics. Nevertheless, it should be noted that in the literature, axiom V is often confused with an axiom of extensionality: that every property has an extension and every extension has a property. It is irrelevant to the point here that historically, the paradox was presented to Frege in much the same was as above. Frege recognised that this was not the best way of presenting as a paradox derivable from his system. He re-formulated the presentation in the Appendix to Grundgesetze. For a thorough discussion of the Russell paradox as it applies to Frege's formal system see Michael Resnik, Frege and the Philosophy of Mathematics, (London: Cornell University Press, 1989), pp. 211 - 224.

consider the concept, R: "being a set which s not self-membered". The object corresponding to this is a set. We ask the question: is R a member of itself? Say it is, then by the meaning of the concept under consideration, it is not a member of itself - contradiction. Try the other tack. Say it is not. If the object corresponding to the set is not in its own set, then again by the fact that all sets which are not self-membered are members of the set, the set itself must be a member of the original set - contradiction.

There is a tension between, on the one hand, an imprecise but powerful idea of the perfect generality and universality of the relationship between objects and concepts. On the other hand, we know that there is no one-to-one correspondence between objects and concepts. For any set of objects, all the subsets of that set will have a corresponding concept. That is, the number of concepts is equal to the powerset of the number of objects. The two sets are always of different size, and therefore, there is no one-to-one correspondence between objects and their corresponding concepts. This, of course, presupposes a certain amount of set theory, and that a domain of objects can be thought of as a set of objects. The issue about exactly how much set theory has to be presupposed and to what extent this is in keeping with Frege, is a complicated one. It will not be addressed here.

All is not lost for the logicist, since, interestingly, if we ignore Basic Law V, that is, simply subtract it from second-order logic, and include Hume's principle as an axiom, then we have a consistent system, powerful enough to derive the Peano axioms as theorems! Another way of putting this is to say that we ignore *Grundgesetze*, and add Hume's principle to Frege's system as it is developed in the *Begriffsschrift*. This was first suggested by Wright. In a further technical development, Heck showed that the only essential use made of axiom V is in the derivation of Hume's principle.[46] All

[46]Richard Heck Jr., "The Development of Arithmetic in Frege's *Grundgesetze der Arithmetik*," Journal of Symbolic Logic, vol. LVIII.2 (1993), pp. 579 - 601.

other mentions of axioms V in the <u>Grundgesetze</u>, can be replaced by other axioms and definitions.

The philosophical problem left us, is to once again argue that the newly derived foundation is a logical one, i.e. argue that second-order logic is logic and that Hume's principle is logical too. The logicist has to do these tasks separately because Hume's principle is independent of the axioms of second-order logic, in the sense of not being derivable from them. The philosophical situation is complicated by the fact that second-order logic does not have a finite and complete set of axioms. This gives us explicit scope for suggesting new ones. We are not free to add anything we like. The constraints applied in practice are very rigid and largely technical, but we can also be philosophically motivated to add new axioms, as in the case of Hume's principle. Ideally, the logicist would like to justify Hume's principle in terms of logic, that is, either derive it as a theorem of logic (possibly by adding another axiom instead of axiom V), or by arguing philosophically that Hume's principle is a principle of logic. What is interesting in Frege is that while he firmly states that a logical axiom is such that it cannot admit of further justification; he believes that arithmetic is a part of logic, that Hume's principle is a logical principle, but feels that he has to prove it to be such. Frege treats Hume's principle as a theorem. Logical theorems are derived using a gapless proof from logical axioms. Thus, they are logical principles which can be justified in terms of other, more basic logical principles. In this sense, logical axioms are not chosen arbitrarily, but according to philosophical criteria. The theorems are just what is entailed by these basic truths.

Given that there is hope for the logicist, but that we now recognise that the situation is more complicated than Frege perhaps realised, let us return to Frege's earlier concerns to see what we can salvage. Frege wished not so much to fend off paradox, thus inconsistency in arithmetic; but to banish the idea, which comes partly as a legacy from Kant, that arithmetic is synthetic. Unlike the nineteenth century analysts, Frege is not so concerned with epistemology and certainty *per se*. Instead, he is interested in the type of justification which can, and ought, to be given for the truths of arithmetic. When Frege does mention certainty, he is not concerned with knowing for certain that $2 + 8 = 10$, for example. Rather, he is interested in the sort of justification needed for the assertion; so what it is the assertion depends on. In particular he wishes to show clearly that arithmetic does not depend on (Kantian) intuition. Another way to put the point is that Frege was interested in demonstrating the autonomy or even objectivity of arithmetic.

How Frege expresses what he means by logical or ultimate justification varies. It is partly meant in the sense of independence from temporal intuition and from kinematics. It is also meant in the sense of independence from psychological considerations. Frege was very concerned that we should distinguish between (a) how it is that we arrive at our knowledge of mathematics and (b) the (ultimate) justification for mathematical statements.

> It not uncommonly happens that we first discover the content of a proposition, and only later give the rigorous proof of it, on other more difficult lines; and often this same proof also reveals more precisely the conditions restricting the validity of the original proposition. In general, therefore, the question of how we arrive at the content of judgement should be kept distinct from the other question, Whence do we derive the justification for its assertion?[47]

For Frege, it is crucial we preserve the distinction between how we happen to arrive at a statement in mathematics and the conditions which justify it. A statement is justifiably asserted to be true just in case it is proved. The nature of the proof, so what assumptions or axioms have to be appealed to, form the basis of the justification.

For Frege, the confusion is made most blatantly by those who think of mathematics as an empirical science and those who identify mathematics with biological or psychological processes. Frege is concerned to discredit psychologism: the doctrine that enquiry into the nature of mathematics is ultimately empirical in the sense that it

[47]Gottlob Frege, The Foundations of Arithmetic, trans. J. L. Austin (Second revised edition; Evanston Illinois: Northwestern University Press, 1980), § 3.

should focus on psychological, biological or neurological phenomena. Psychologism seeks to trace or reduce mathematical operations to psychological states. Frege crudely characatures the psychologistic position as one which identifies mathematics with muscular movements which are then organised in the brain:

> When Stricker, for instance, calls our ideas [or images] of numbers motor phenomena and makes them dependent on muscular sensations, no mathematician can recognize his numbers in such stuff or knows what on earth to make of such a proposition.[48]

Put in terms of muscular sensations, it is easy to dismiss such an idea of mathematics, as a nineteenth century aberration which was short lived. However, the spirit of the idea lives on in the new sophisticated language of neuroscience where mathematics is somehow "hard-wired" into our brains in the form of a predisposition. The philosophical tendency which underpins this way of thinking is naturalism. Nevertheless, the diagnosis Frege offers is revealing. Rather bitingly, he drives the point home:

> An arithmetic founded on muscular sensations, would certainly turn out sensational enough,[49] but also every bit as vague as its foundation. No, sensations are absolutely no concern of arithmetic. No more are mental pictures, formed from the amalgamated traces of earlier sense-impressions. All these phases of consciousness are characteristically fluctuating and indefinite, in strong contrast to the definiteness and fixity of the concepts and objects of mathematics.[50]

Thus, as far as Frege is concerned, there is a mis-match between mathematics and psychology; something psychologism is hard put to explain. Mathematics is fixed and, we might say, timeless. The advocate of psychologism cannot account for this in any way which is satisfying to their opponents. An advocate of psychologism can describe our mathematical behaviour well enough. He can even

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[48]Gottlob Frege, The Foundations of Arithmetic, trans. J. L. Austin (Second revised edition; Evanston Illinois: Northwestern University Press, 1980), p. v.
[49]While this is a pun in the English translation, it is not in the original German.
[50]Gottlob Frege, The Foundations of Arithmetic, trans. J. L. Austin (Second revised edition; Evanston Illinois: Northwestern University Press, 1980), pp. v - vi.

predict it, by just observing as an empirical fact that we are least willing to revise mathematics when we experience a surprising result which involves mathematics. For example, when presented with a puzzling situation, we will say that we have mis-counted, rather than say that the laws of arithmetic are wrong. However, what the advocate of psychologism does not offer is an account of this phenomenon, which is convincing to someone not swayed by naturalistic thinking. As far as the opponent of psychologism is concerned, an advocate of psychologism owes an explanation as to why mathematics gives the appearance of being more fixed than other sciences. We can accumulate mathematical truths. We cannot revise them, in the same way as we do in physics, chemistry, history, economics and so on. The advocate of psychologism will search for an explanation in evolutionary theory, history, psychological disposition or the composition of the brain. But these are not enough to account for the prominence given mathematical truths. That is, even if we tell a story about its being to our evolutionary advantage to learn some mathematics, the opponent to psychologism will insist that we have then to supply some explanation as to why it is advantageous to do arithmetic this way; why, in other words, there is a right and a wrong mathematics, and how it is that we develop mathematical notions well beyond their ability to satisfy our need to eat, sleep, procreate and so on.

There are two general sources of worry for Frege. One is that philosophers like Mill seem to think that arithmetic is *a posteriori*: Mill argued that even arithmetic is an empirical science. For, he argues that arithmetic is based on counting, and counting is something we learn through sense experience. To learn to count we pick up pebbles, and organise them in a line. Counting is thereby reduced to the physical manipulation of objects. Frege very cuttingly disparaged this view by pointing out that we can only have inaccurate physical experience of very large numbers, and we cannot be said to have any experience of zero:

> But what in the world can be the observed fact, or the physical fact (to use another of Mill's expressions), which is asserted in the definition of the number 777864? Of all the whole wealth of physical facts in his apocalypse, Mill names for us only a solitary one, the one which he holds is asserted in the definition of the number 3. It consists, according to him, in this, that collections of objects exist, which while they impress the senses thus, ∵, may be separated into two parts, thus, ... What a mercy, then, that not everything in the world is nailed down; for if it were, we should not be able to bring off this separation, and 2 + 1 would not be 3! What a pity that Mill did not also illustrate the physical facts underlying the numbers 0 and 1![51]

For Frege, it is ludicrous to suggest that physical observations underpin arithmetic, since for large numbers, it is impossible to distinguish observations which differ from each other by a small number.

The second worry for Frege concerns philosophers who follow Kant in thinking of arithmetic as synthetic rather than analytic. Kant argued that mathematics is *a priori* synthetic. He needed the notion of the *a priori* synthetic for his metaphysical system, in order to account for the possibility of the application of mathematics and the grounding of metaphysics. For this purpose, then, Kant argues that there is an applied intuition which works directly with sense experience, and that there is pure intuition which relates and organises concepts. For Kant, there are two sorts of pure intuition: spatial and temporal. Geometry relies on spatial intuition and arithmetic relies on temporal intuition. We shall discuss this further in the next chapter, in section three, when we discuss analyticity. We retain analyticity as a criterion for logic, since it indicates justificatory pedigree for logical sentences.

There are two aspects to justifying an assertion. One is to show that the assertion is true, but more subtly, a proof indicates

[51]Gottlob Frege, The Foundations of Arithmetic, trans. J. L. Austin (Second revised edition; Evanston Illinois: Northwestern University Press, 1980), § 7. The text that Frege is referring to is Mill's System of Logic.

upon what the truth of the assertion depends. For the logicist, favour is extended to purely logical proofs of an assertion, because this shows that the truth of the assertion depends only on logical truths: truths of which there is none more basic. Logic, is thus given a certain conceptual priority over other disciplines. This is because logic is thought to be omnipresent, in the sense of being basic to all systems of rational enquiry, be they physics, mathematics, or even, political science. One reason for spelling out a justification in the form of a proof is to give the pedigree of the assertion, logical assertions, having the highest pedigree: logical justification is ultimate in this sense. Rather unhelpfully, Frege suggests that we know that we have traced a justification back to the most basic (and therefore logical) axioms, because they are self-evident. The relationship between how basic the logical axioms are and their self-evidence is ultimately unstable. For, while logic might be thought of as being basic and universal, and while this might suggest self-evidence, self-evidence does not imply truth. On the other hand, to declare something to be self-evident is a way of saying that the justification has ended.

The notion of self-evidence, while often associated with Frege's logicism, must be handled carefully. On the one hand, often in mathematics it has been shown that what we took to be self-evident turned out not to be so in the light of more rigorous proofs which expose assumptions as not invariably true. Take for example, Euclid's fifth postulate. It states that "if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."[52] Once we overcome the convoluted turn of phrase, this may seem self-evident. After all, it is included as a postulate, or axiom, and its seeming self-evidence is reason to count it as an

[52]Euclid, Euclid's Elements, Book 1: On Postulate 5, commentary by Proclus, trans. and published in The History of Mathematics, eds. John Fauvel and Jeremy Gray, (London: MacMillan Press, 1987), p. 101.

axiom. Proclus, who wrote a commentary on the definitions and postulates recognises the self-evidence of the postulate. However, he regards the fifth postulate essentially as a theorem, requiring proof from definitions. He writes:

> But perhaps some persons might mistakenly think this proposition deserves to be ranked among the postulates on the ground that the angle's being less than two right angles makes us at once believe in the convergence and intersection of the straight lines.[53]

He then argues that this is a mistake, but gives no proof, i.e. no derivation of the fifth postulate from the other postulates. No clear proof or disproof was given until the nineteenth century when it was shown to be independent of the other axioms, and therefore, not provable from them at all! Before that, mathematicians were interested in showing the postulate to be a theorem since they regarded it as a truth, but recognised the unease expressed in Proclus. This is reminiscent of Frege's ambivalence towards Hume's principle. He wants to treat Hume's principle as a theorem, yet he believes that it is obviously true (unless all of arithmetic is false). Moreover, Frege believes that Hume's principle is a truth of logic; similarly with Euclid's fifth postulate being a truth of geometry. For centuries, whenever mathematicians found some principle was inconsistent with Euclid's fifth postulate, they took this as a *reductio* argument, and jettisoned the suspect principle in favour of the fifth postulate. It was not until non-Euclidean geometries, such as Lobochevski's geometry on a sphere, were explicitly and rigorously developed, that the (sometimes falsity) of the fifth postulate, from the other postulates, was recognised. An example of a case where the fifth postulate is false, was developed by Poincaré in a so called

[53]Euclid, Euclid's Elements, Book 1 "On Postulate 5", commentary by Proclus, trans. and published in The History of Mathematics, eds. John Fauvel and Jeremy Gray, (London: MacMillan Press, 1987), p. 104. We might also add to the grounds for believing this that many centuries later the independence of Euclid's fifth postulate had not been proved. Its truth was a subject of debate both with the Arab mathematicians and the Western mathematicians until the nineteenth century when non-euclidean geometry was properly understood and investigated. See the same book, pp. 508 - 540.

"inner model" of Euclidean geometry. Poincaré's example is as follows: take a circle as the universe. This can have a chord and a point not on the chord through which an infinite number of straight lines can pass without intersecting with the first chord.[54] Thus, there is a model of the other four postulates in which the fifth postulate is false. In this sense, the fifth postulate is sometimes refer to as being independent of the other axioms of Euclidean Geometry.

The discovery of the so called independence of Euclid's fifth postulate from the rest of the postulates has changed our thinking. Previously we thought that axioms were fundamental truths, mathematical truths were self-evident and they formed an unified system. Frege thinks this of logical and *a fortiori*, the arithmetical axioms and Hume's principle, but not of geometrical axioms. What came to be questioned was the relationship between truth and axioms, and what the grounding of mathematical truths is. The negation of an independent axiom is consistent with the original set of axioms. This, in fact, is what led to talk of systems. For, with the independence of certain axioms, we found that we had to qualify their truth, and make it relative to a system. We say, for example, that Euclid's fifth postulate is true in Euclidean geometry. This may give the impression of undermining the unity of truth in mathematics.

The idea of an unified theory of truth is very appealing. We do not give it up easily. Indeed, it is one of the motivations for logicism. For, the thought goes, we want to review the foundations of our reasoning. The hunch is that the foundation is logic, and this is analytic and universal since the axioms of logic are self-evident, timeless and unshakeable - as were the Euclidean axioms before the discovery of the independence of Euclid's fifth postulate. This notion of self-evidence, unity and independence is what lies at the

[54]Another example is due to Riemann. In this, we take another inner model, this time the sphere. A straight line on a sphere is any line which traces a great circle. A great circle is one which bisects the sphere into two equal halves. Consider one such straight line and an arbitrary point distinct from the line. There is no straight line which can pass through the point and not intersect with the original line.

heart of Frege's insistence that logic and arithmetic are analytic, and for this reason provide a foundation for the rest of mathematics. It would have been inconceivable for Frege to think of the logical axioms as independent of each other in this sense, i.e. that there should be alternative logical systems which differ from each other by the negation of one of the axioms of the original system. This is because the logical axioms are universal truths for Frege. It is exactly in this conviction about the conceptual priority of logic over everything else, that Frege most disagreed with Hilbert.

Contrast Frege's to Hilbert's view of axioms.[55] Hilbert has the more modern view. He is more impressed, than Frege is, by the independence of certain axioms. Rather than thinking that this problem only infected a few (geometrical) axioms, he thought that so called logical and arithmetical axioms could also be thought of as essentially arbitrary. They then lose their arbitrariness through considerations of applicability and formal considerations such a mutual consistency. Thus, logic and arithmetic do not assume a privileged position over other mathematical theories. For Hilbert, the only justification for axioms are formal or mathematical criteria, and come in the form of limitative results about the body of theorems which are generated by the system which is encapsulated in the axioms and rules of inference. The most important justification for a set of axioms is that they should be mutually consistent. A system that is consistent is thereby thought to be *prima facie* legitimate. Of lesser import are considerations of completeness and finiteness; where the latter is an expression of concern for methodology, and does not imply that Hilbert was only interested in finite numbers. Our methods of investigation have to be finite, not the objects/ sets investigated. The notion that logical axioms are

[55]The clearest place to see the contrast is in the Frege/ Hilbert correspondence, published and translated into English in Frege: Philosophical and Mathematical Correspondence, tans. (of Frege) Hans Kaal, (Oxford: Basil Blackwell, 1980). For an excellent discussion, see Michael Hallett, "Hilbert's Axiomatic Method and the Laws of Thought," Mathematics and Mind, ed. Alexander George, (Oxford: Oxford University Press, 1994), pp. 158 - 200.

universally true is erased, as is any sense of self-evidence. Similarly, Hilbert cannot entertain the notion that axioms are true independently of the system which they encapsulate. Instead, one formal system might be found to be easier to apply to one situation than another. Thus, a notion of fit or applicability replaces that of truth.

Hilbert's conception of the logical axioms enjoys certain advantages over Frege's conception of logic. For, it avoids much of the metaphysical baggage which accompanies Frege's formal system, namely, Frege's philosophy concerning the conceptual priority of logic, the self-evidence of the logical axioms, and Frege's platonism towards logical objects. However, it should be noted that in order to show that a system is consistent or complete, one has make appeal to a stronger system. The question then arises as to how we regard the more powerful system we used to generate the results concerning the original system. Is the more powerful system true? Hilbert disallows this question in this form. Is the system consistent? To answer this, we have to move to a yet more powerful system, and we are faced with the threat of an infinite regress. Furthermore, the only information we are allowed to seek is technical, and defined within the confines of the practice. The metaphysical importance of logic is then sadly neglected. In contrast, Frege thought of his logic as the ultimate *point de repair* just because of the supposed self-evidence and truth of the axioms, Hilbert did not have recourse to a notion of self-evidence.

Hilbert's view spawned other philosophical attitudes towards logic. For example, first-order logic, second-order logic, propositional calculus and other formal systems were thought of only as tools. As such, the only merit one formal system could have over another concerns its suitability to a given application. Frege's formal system, or second-order logic in general are then seen to be just a formal system amongst others, as opposed to one which lies at the source of our reasoning.

A concern with applications, in turn, seeded the limiting idea that applications were only legitimate if made outside mathematics. That is, to justify a formal system, one has to do this in terms of its application to physics, or engineering, or whatever. Under this conception, logic and mathematics lose their privileged status, and worse still, are restricted to the finite and the mundane.

§ 4: Logicism after Frege: Whitehead and Russell

Despite having discovered a fatal flaw in Frege's attempt to elaborate logicism, Russell found logicism appealing. He took up the slack where Frege left off and tried, with Whitehead, to develop a more ambitious project: that of showing that all of mathematics is ultimately based on logic. He and Whitehead published the three volumes of *Principia Mathematica* from 1910 to 1913. Russell was most interested in the philosophical implications of the project; whereas Whitehead was more interested in the mathematics. Russell is often counted among the logicists, and while he was certainly inspired by Frege's work, the Russell and Whitehead project differs markedly. Their project is more ambitious than Frege's, and they found that with the technical differences between their's and Frege's system, they had to also modify the philosophical agenda. Russell and Whitehead developed a ramified type theory. This forces a conception of a typed/ stratified universe. Frege considered an elaborate type stratification briefly, but found it unacceptable because it is too elaborate, and therefore, not general enough.[56]

The criticism usually levelled against the ramified type theory *qua* vindication of logicism; is that the type theory is arguably not a theory of pure logic. This is because it includes an axiom of infinity and an axiom of reducibility which are not logically necessary. That there should exist an infinite set in the

[56]However, also see Michael Potter, Chapter two of forthcoming book.

universe cannot just be stipulated, and then taken to be a necessary truth of logic.

The failure of Russell and Whitehead's attempt should not be the last word on logicism. From it, we know that a ramified type theory is not the best basis from which to prove logicism. The existence of an infinite set has to be proven. This is what Frege tried to do, by deriving the infinity of the natural numbers as a theorem from Hume's principle, which in turn was meant to be derived from axiom V, which in turn was meant to be a logical axiom. This was the strategy Frege employed to show that the infinity of the natural numbers was a matter of logic, not of mathematics. We shall be ignoring Russell and Whitehead's attempt at vindicating logicism on two counts. One is that the project turned out to be quite different from that of Frege's version of logicism, and two, the recent revival of logicism largely concerns Frege's attempt, not Russell's.

After Russell's failed attempt, there was a long silence on the issue. In part this was due to an interest in rival foundationalist theories and a concern to avoid set theoretic paradox; in part, silence was due to the promotion of first-order logic.

§5: The Rise of First-order Logic

The promotion of first-order logic started in 1923, when Skolem argued that first-order logic exhausted the scope of logic. At the time, the proposal met with strong opposition. However, with the proof that set theory could be done in a first-order language, Gödel's incompleteness results (1931, 1932) together with a growing sensitivity to suggestions from the intuitionists to re-found mathematics on a firmer epistemological footing, Skolem's position began to dominate. Philosophers too, joined the first-order camp, and the idea that first-order logic is the most powerful logic remains entrenched in the *corpus* of philosophical myth.

Before 1923, this was not at all the case. As we have noted already, Frege's logical system, as it is presented in the Begriffsschrift, is more-or-less equivalent to second-order logic.[57] Russell and Whitehead's theory of types is sometimes considered to be a sort of higher-order logic. With the developments introduced by Frege, Peirce, Poincaré and others, what was considered logic tended to be second-order or higher. It was only in 1917 that a clear exposition of first-order logic appeared. This was due to Hilbert.[58] He called his system the calculus of functions, and developed it for the purposes of some lectures on logic in Göttingen. His philosophical comments on the logical system are revealing. He believed that the logic was sufficient for the purposes of "formalising logical deduction."[59] However, if we wish to investigate the foundations of mathematics:

[57]Exactly where the differences lie between the system of the *Begriffsschrift* and second-order logic is not entirely straightforward to determine. This is mainly because exactly what constitutes second-order logic is not a settled issue. What we do know, for example, is that in the system of the *Begriffsschrift* there is quantification over second-order variables. We also know that Frege did not have an axiom of comprehension, the need for such an axiom was not pressing until Frege wanted to ground Hume's principle. To do this Frege introduced axiom V. This is an abstaction principle, which is not quite the same as a comprehension principle. As far as I know, a clear discussion of the exact differences is absent from the literature. Furthermore, insofar as a reconception of logicism is needed in the light of the many formal systems which present themselves as logic, what is being argued for here is that second-order logic, in its modern guise, can be characterised as logic, and this constitutes a partial vindication of logicism. To what extent the vindication is loyal to Frege's original project, in part, depends on the perceived *rapport* between modern second-order logic and the formal system of the *Begriffsschrift*. For exmple, whether or not what Frege explicitly proves in the system of the *Begriffsschrift* could also be proven in a weaker system than that of second-order logic, such as many-sorted logic, is an open question, as far as I know. Many-sorted logic is an obvious choice because it also has quantifcation over second-order variables, but it is complete.
[58]Hilbert: *Prinzipien der Mathematik und Logik*. Unpublished lecture notes of a course given at Göttingen during the winter semester 1917-18 (math. Institute, Göttingen). Referred to in Gregory Moore, "The Emergence of First-order Logic," eds. William Asprey and Philip Kitcher, History and Philosophy of Modern Mathematics, (Minnesota Studies in Philosophy of Science, XI, Minneapolis: University of Minnesota Press, 1988), pp. 95 - 138.
[59] Gregory Moore, "The Emergence of First-order Logic," eds. William Asprey and Philip Kitcher, History and Philosophy of Modern Mathematics, (Minnesota Studies in Philosophy of Science, XI, Minneapolis: University of Minnesota Press, 1988), p. 114.

"to examine in what relation mathematics stands to logic and to what extent mathematics can be obtained from purely logical operations and concepts,"[60] then we need a much stronger logic. There is no hint, in Hilbert, of the scope of logic *per se* being exhausted by first-order logic. Indeed, Hilbert viewed the formal system of first-order logic as a restriction to logic.[61] Instead, how strong a logical system is needed depends on our purpose: to capture deductive reasoning or to measure the scope of logic *vis-à-vis* mathematics.

It was mentioned in the introduction that we can compare the strengths of systems according to different criteria. Similarly, there are different ways of characterising formal systems. We can characterise them in terms of the presentation of syntax and semantics: so a formal system is identified with a particular presentation; or we can characterise a formal system in terms of its characterising (or limitative) results. In some ways,[62] the latter is cleaner since it avoids problems in ambiguity of presentation, where we are not sure whether or not two sets of notation, for example, are really equivalent. When we identify a formal system by its limitative results, we might identify systems with very different looking presentations.[63]

For reasons of hygiene, and because limitative results were central to the debate about the logical status of first-order logic, we

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[60]Gregory Moore, "The Emergence of First-order Logic," eds. William Asprey and Philip Kitcher, History and Philosophy of Modern Mathematics, (Minnesota Studies in Philosophy of Science, XI, Minneapolis: University of Minnesota Press, 1988), p. 114.

[61]David Hilbert, "*Prinzipien der Mathematik und Logik*," Unpublished lecture notes of a course given at Göttingen during the winter semester 1917 - 1918, (Mathematics institute, Göttingen), pp. 222 - 3, cited in Gregory H. Moore, "The Emergence of First-order Logic," eds. William Asprey and Philip Kitcher, History and Philosophy of Modern Mathematics, (Minnesota Studies in Philosophy of Science, XI, Minneapolis: University of Minnesota Press, 1988), p. 115.

[62]In other ways, this will depend on the meta-language we use to present a system in the first place, and our ability to translate from one presentation to another.

[63]We have already mentioned the case of many-sorted logic and first-order logic. These share limitative results, but differ markedly in their semantics.

shall favour that characterisation in general. Having said that, note also that the debate between Quine and Boolos, discussed in section six of this chapter, is about the interpretation of the quantifiers, so again, about the presentation of second-order logic, as opposed to its characterising results.

In 1923, Skolem suggested in print[64] that set theory be formulated in the language of first-order logic with set-theoretic membership added in. Thus, he was implicitly viewing set theory as a first-order theory.[65] He wanted to do this in order to demonstrate clearly the relativity of set theoretic notions. This relativity is evident, for example, in what we refer to as Skolem's paradox: that first-order (Zermelo) set theory can be modelled with a countable model "even though this system implies the existence of uncountable sets."[66] The problem is to fit the uncountable model into the countable model, and this is impossible because it contradicts Cantor's theorem: that two sets have the same size if they can be placed into one-to-one correspondence. Cantor also showed that the powerset of a set is strictly greater than the original set.

Central to Skolem's understanding of the paradox is the consideration that set-theoretic membership is a non-logical relation. In particular, it is not a logical constant. This makes room for non-standard interpretations and competing set theories. Each defines (implicitly), by means of axioms, the set-theoretic membership symbol in its own way. Skolem's proposal that we view set-theoretic membership as a non-logical relation, makes it

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[64]He made the suggestion a year earlier at a conference. It took a year for the conference proceedings to be printed.

[65]Thoraf Skolem "*Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre*" in Videnskapsselskapets skrifter, I. Matematik-naturvidenskabelig klasse, no.4. Translation of section 1 in Van Heijenoort, From Frege to Godel: A Source Book in Mathematical Logic, 1879 - 1931, (Third printing, Cambridge, Massachusetts: Harvard University Press, 1967), pp. 252 - 63.

[66]Gregory H. Moore, "The Emergence of First-order Logic," eds. William Asprey and Philip Kitcher, History and Philosophy of Modern Mathematics, (Minnesota Studies in Philosophy of Science, XI, Minneapolis: University of Minnesota Press, 1988), p. 123.

obvious that it should be subject to various interpretations. This had a knock-on effect. If set-theoretic membership is only a relative notion ("ambiguous" would have been better, but Skolem uses "relative"), then so is that of set, subset, powerset and so on.

Also of significance to how we view logic is that Skolem ousted the membership relation, from its privileged position among the logical symbols, to the more common-place position of being a non-logical constant relation. Ramsay too,[67] made a similar point about Whitehead and Russell's type theory: that they treat "$\in$" as a logical constant, when really it is a variable, as far as logic was concerned. This is because the axioms governing its use are contingent (relative to logic). *Grosso modo*, Ramsay criticised them for using a mixture of logical and non-logical notions, so not founding mathematics on logic but on more mathematics.

Viewing set-theoretic membership as a non-logical constant relation, allows the Skolem paradox to arise because what characterises a set, in particular, the cardinality of a set, becomes a relative notion. Skolem believed that this revealed a feature of mathematics which runs very deep. This is a mistake if we consider that the paradox is confined to first-order set theory. Unfortunately, this is an oversimplification since there is a sense in which the paradox is ubiquitous. For, it is true, not only of mathematics, but of every discourse. The Skolem paradox only points out (in a particular instance) that any judgement we make about a formal system has to be expressed and demonstrated in another (meta-language). This other formal system can in turn come under question, particularly when it conflicts with pronouncements made in a third (meta-meta level) formal system. The Skolem paradox is a twist on this where we seem to get contradictory results about one formal system (first-order set theory). At the time, the Skolem paradox presented a very real difficulty. There was a marked lack of clarity concerning what it was that the paradox indicated. The

[67] Frank Plumpton Ramsay, The Foundations of Mathematics and Other Logical Essays, ed. Richard B. Braithwait, (London: 1931).

difference between first and second-order logic had not yet been fully appreciated. Reviews by Fraenkel (1923)[68] and von Neumann (1925)[69] of Skolem's results concerning the Löwenheim-Skolem theorems seem to have been unclear about the difference between first and second-order logic. As a result, it was assumed that the results generalised to all axiomatic systems, and that the Skolem paradox is ubiquitous. Even Gödel, in his doctoral dissertation, seemed unclear about the scope of the results with respect to a distinction between first and second-order logic.[70] The supposed ubiquity of the result was only beginning to be undermined in 1930 when Gödel realised that another limitative theorem: the Skolem-Gödel theorem is restricted to first-order logic. The Skolem-Gödel theorem is: that every "denumerably infinite set of formulas... is simultaneously satisfiable or else possesses a finite subset whose logical conjunction is refutable."[71] Gödel was aware that this theorem implies compactness, and that not all theories are compact. This shows that by 1930 Gödel had the resources to show that there is a significant difference between first and second-order logic: that they differ in their limitative results. This might have suggested to Gödel, and others, that the scope of the Skolem results, and the Skolem paradox, could not be immediately extended to any system. In fact, this is exactly what Zermelo implicitly suggested in his

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[68]Fraenkel, "Review of Skolem's *Einige Bermerkungen zur axiomatischen Begründung der Mengenlehre*," Jahrbüch uber die Fortschritte der Mathematik, XLIX (1923). Referred to in Gregory H. Moore, "The Emergence of First-order Logic," eds. William Asprey and Philip Kitcher, History and Philosophy of Modern Mathematics, (Minnesota Studies in Philosophy of Science, XI, Minneapolis: University of Minnesota Press, 1988), pp. 95 - 138.

[69]J. von Neumann, "*Eine Axiomatisierung der Mengenlehre*," Journal für die reine und angewandte Mathematik, CLIV (1925) pp. 219 - 40. Translated in Jean van Heijenoort (ed.), From Frege to Godel: A Source Book in Mathematical Logic, 1879 - 1931, (Third printing, Cambridge, Massachusetts: Harvard University Press, 1967), pp. 393 - 413.

[70]Gregory Moore, "The Emergence of First-order Logic," eds. William Asprey and Philip Kitcher, History and Philosophy of Modern Mathematics, (Minnesota Studies in Philosophy of Science, XI, Minneapolis: University of Minnesota Press, 1988), p. 125.

[71]John W. Dawson Jr., "The Compactness of First-order Logic," History and Philosophy of Logic, XIV (1993), p. 17.

letters to Skolem where he insists on clarification of the Skolem results. His letters seem to have had no effect on Skolem. This is due in part to the fact that a new set of considerations was coming to the fore in support of first-order logic over second or higher-order logic.

The considerations came from Brouwer and Heyting. They were worries about the epistemological foundations of mathematics. These worries were taken seriously by Gödel and Skolem, and interpreted in such a way that it was thought that if mathematicians confined their mathematical enquiries to first-order logic, then these epistemological worries would be alleviated.[72] Tarski and Bernays criticise Gödel on exactly this point, namely that this is not a sufficient reason to confine logic to first-order. Thus, what was emerging despite the efforts of Tarski, Zermelo and Bernays was what we might call, following Wolenski,[73] "the first-order thesis": that first-order logic exhausts the scope of logic. That is, there is no stronger logic and any stronger formal system is mathematics, or something else.

One person's proof is another person's *reductio*. Skolem interpreted the non-categoricity of first-order logic (which follows from the Löwenheim-Skolem theorem, and allows the Skolem paradox to be generated) to indicate the wholesale relativity of set theoretic notions. He then shifted to intuitionistic considerations to continue advocating first-order logic as a foundation for mathematics. In contrast, Tarski, Zermelo and Bernays interpreted the Löweheim-Skolem theorem as a *reductio*. They suggested moving to a more powerful system, such as second-order logic or second-order set theory or type theory, where there is reason to

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[72]One piece of evidence for this is that when Gödel was asked why it took so long for the mathematical community to come to grips with the compactness theorem, he said that mathematicians were not taking the relevant mathematical attitude towards mathematics. See also Wolenski, talk at St. Andrews, where he signals that one of the points where Gödel and Tarski diverged most dramatically, was in their epistemological attitude. Tarski was unmoved by such constructivist considerations.

[73]Jan Wolenski, "In Defence of the First-order Thesis," Logica '93, Proceedings of the 7th International Symposium, eds. P. Kolár & V. Svoboda, (Praha: Filosofia, 1994), pp. 1 - 11.

think that the relativity of set theoretic notions is eradicated. Of course, this is not entirely the case.[74] What is established, is that second-order arithmetic is categorical. That is, models for the natural numbers are isomorphic. Similar results occur when we move to a higher-order set theory. However, to show exactly the relationship between categoricity and the relativity of set theoretic notions requires certain assumptions which tend to be built into a meta-language, and the justification for the choice of meta-language is then an open question. An infinite regress threatens. This is the sense in which the Skolem paradox runs deep. On the other hand, it is a problem which is hardly confined to formal languages. There will be further discussion of this issue in chapter three, section two.

These arguments were then succeeded by others which centred around compactness, undecidability and completeness. By this time it was clear that first-order logic was not very powerful and that it was to be distinguished from first-order set theory.

The prevailing thought that first-order logic is both the strongest logic and yet not very powerful, combined to make the old logicist project look hopeless. Two ideas emerged. One was that first-order logic should be considered all of logic because anything stronger, like first-order set theory, is relative and mathematical. The second, is that we could not possibly consider second-order logic to be logic because if it is in any respect stronger than first-order set theory, then it violates our epistemological sensitivities. In fact, those who were sensitive to the epistemological problems of mathematics often advocated a more stringent system to fall under the title "logic", namely, intuitionistic logic.[75] However, as we shall see in subsequent chapters, there is a sense in which the neo-Fregean logicist can side-step these considerations.

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[74]For an excellent discussion see Stewart Shapiro, Foundations Without Foundationalism: A Case for Second-order Logic, (Oxford Logic Guides: 17; Oxford: Clarendon Press, 1991), pp. 204 - 8.

[75]Brouwer was not entirely happy with the idea of intuitionism being given a formal expression. However, in a sense, this was inevitable.

§ 6: The Present-Day Status of Second-Order Logic

Confusion persists. Quine has made one of the most influential contributions towards how it is that we view second-order logic. He argues that second-order logic is really set theory.[76] His argument has four steps. First, when we bind a variable with a quantifier we treat that variable as a potential name, in the sense that it stands in a place where a name could stand (if we were to instanciate). *A fortiori*, this is also true of second-order variables. Second, names are such, in virtue of the fact that they refer. Thus, a (first-order) name has an individual as its referent. A predicate may have a number of individuals in its extension. This implies that predicates refer to entities of some sort. Third, what they refer to are subsets of the domain of individuals, and the only way to understand this, is in terms of set theory. Fourth, since set theory is the study of sets, second-order logic is also covertly about sets, and therefore, makes the same ontological commitment as set theory, and therefore, in all essential respects, second-order logic is really set theory.

Why we are so slow to recognise that second-order logic is set theory, Quine diagnoses, is because we believe that there is a real distinction to be drawn between treating bound predicates as being thus and so and treating bound predicates as standing in the place where a name could stand, an so referring to entities which are thus and so. By virtue of the choice of words, the first analysis is proper to logic; the second is proper to set theory. But, Quine argues, this is a distinction without a difference since the only way to understand the first is in set theoretic terms, i.e. in terms of the second. He concludes that second-order logic is misleading since it poses as logic but is really set theory. To keep second-order logic honest, Quine suggests we translate predicate expressions into set-

[76]Wilfred Orman van Quine, Philosophy of Logic, (Cambridge Massachusetts: Harvard University Press, 1970).

membership expressions. For example, "Fx" would be written as "x ε F". Since set theory is clearly not logic, the only way we can be assured of remaining within the realm of logic proper is by sticking to honest first-order logic. His argument is this:

> Consider first some ordinary quantifications: '(∃x)(x walks)', '(∀x)(x walks)', '(∃x)(x is prime)'. The open sentence [formula] after the quantifier shows 'x' in a position where a name could stand; a name of a walker, for instance, or of a prime number. The quantifications do not mean that names walk or are prime; what are said to walk or be prime are the things that could be named *by* names in those positions. To put the predicate letter 'F' in a quantifier, then, is to treat predicate positions suddenly as name positions, and hence to treat predicates as names of entities of some sort.[77]

In "On Second-Order Logic,"[78] Boolos argues that Quine's argument is fallacious. He exposes the hidden assumption in Quine's argument, namely, that we are obliged to treat bound predicates as standing in name positions. He denies the assumption, claiming that there is no reason to think that by binding predicates we thereby treat them as potential names. Thus, no confusion need arise between treating them as being thus and so, and treating them as being in a position which could be occupied by a name.

In Quine's favour, the appeal in treating all bound variables as standing in a place where a name might stand is that, if we treat *individual* (first-order) bound variables as standing in name positions, then it seems reasonable that we should treat anything we bind with a quantifier as standing in a name position. Of course, this does not immediately follow. For, bound individual variables may stand in name positions, not in virtue of their being bound by quantifiers, but in virtue of being individual variables as opposed to predicate variables. To argue against Quine's position, we have to

[77]Wilfred Orman van Quine, Philosophy of Logic, (Cambridge Massachusetts: Harvard University Press, 1970), pp. 66 - 7.
[78]George Boolos, "On Second-order Logic," The Journal of Philosophy , LXXII (1975), pp. 509 - 527.

argue that bound predicate variables are different from bound individual variables in the respect that the one lot of variables are individual variables, and the other lot are predicate variables. Names do not engage the issue.

There are different ways of understanding second-order quantification *qua* extension of first-order quantification.[79] One of these is that the salient feature of bound variables is that we can instanciate them using names. But this is not the only way of extending first-order quantification to second-order. At the risk of labouring the point, we are not constrained to treat second-order bound variables as standing in name positions, since we can argue that this is not the relevant feature of quantifiable variables we wish to preserve in the extension from first to second-order.

What is the alternative? Boolos suggests we treat bound predicate variables, not as standing in a place for names, but as having a range. Crucially, this allows us to deny that they stand for objects.

> ...[W]e have no reason not to think that there might be a sort of variable, a predicate variable, that ranges over objects in its range (these will be extensions) but does not *name* them "indefinitely" or any other way; rather, predicate variables will *have* them "indefinitely," as (constant) predicates have their extensions definitely.[80]

In other words, the suggestion is that we treat bound predicate variables not as individual variables *qua* bound variables, but as bound predicate variables *qua* predicates. That is, when we extend first-order logic to second-order by allowing quantification over predicates, the way we should understand the extension is that predicates are treated in much the same way as they were in first-order logic, as having a different status than individuals (variables or constants/names), except that now we can bind these new sorts

[79]Notice the shift in language. This reflects the history of second-order logic. We now think of second-order logic as an extension of first-order logic, rather than thinking of first-order logic as a peculiar fragment of logic (which could be either second-order or higher-order).

[80]George Boolos, "On Second-order Logic," The Journal of Philosophy, LXXII (1975), p. 511.

of variables. This is in keeping with Frege, since he proposes a first level consisting in all the objects there are, and above that is a level of first-order concepts. For Frege, the nature of concepts is very different from that of objects. Implicit in Boolos' argument is the following characterisation of Quine's reasoning.

Quine's understanding of the extension of first-order logic to second-order, is one where we confuse the role of predicates between that of a name for some entity and as being some entity. In fact, the extension is best understood as one into set theory. For, if the predicates are treated as standing where names can stand, then the best way to understand this is that names name subsets of the domain, and the obvious way to understand the domain (since it now includes all subsets of any other original domain) is in terms of the iterative hierarchy.

In contrast, Boolos understands the extension of first-order logic to second-order as one which preserves the distinction between an individual variable and a predicate variable. The difference is that:

> '(∃F)' does not have to be taken as saying that some entities of the sort named by predicates are thus and so; it can be taken to say that some of the entities (extensions) had by predicates contain thus and such. So *some variables eligible for quantification might well belong in predicate positions and not in name positions*.[81] (Italics mine.)

The argument is that we may choose to read a variable in one of two ways. In case we do so one way, it is read as a predicate variable. If we read it in another way, it is an individual variable. Typically, we signal the two readings typographically: by the use of lower-case (for individuals) and upper-case (for predicates). The two ways of reading variables amounts to referring to individual entities in the domain (in the case of individual variables) or to the extension of the predicate to those individuals in the domain which have the

[81]George Boolos, "On Second-order Logic," The Journal of Philosophy, LXXII (1975), p. 511.

predicate. The purpose of stating the predicate variable reading in the passive is that every individual in the domain has some predicates (features). Quantifying over a predicate allows us to attempt to section off those individuals, to partition the domain into those which have F and its complement: those which do not. This does not sanction treating the sectioned off part of the domain as an entity in its own right. Concepts, which are what second-order variables refer to, are not objects. Only their extensions are composed of objects. All we know when we write the false sentence '$(\forall x)(\exists F)(\exists G)((Fx \leftrightarrow Gx) \rightarrow (Fx \wedge Gx))$' in second-order logic is that given a domain, the sentence is true when all the individuals in it have two distinct properties. We have not committed ourselves to predicates or properties being entities, since we also have said nothing about the status of the domain.

To make explicit the divergence between set theory and second-order logic, Boolos gives some examples of sentences which are valid in second-order logic but not in set theory. Quine thinks that the two sentences: "Fx" and "$x \varepsilon F$" are equivalent. However, then Quine is faced with the quandary that the sentence: $(\exists F)(\forall x)(Fx)$ is valid in second-order logic (it just says that there is a domain of individuals, and follows from an axiom of comprehension) whereas his proposed translation into set theory: $(\exists y)(\forall x)(x \varepsilon y)$ is false in set theory.[82] Quine believes the translation to be more honest. However, it cannot really be a translation since the suggested way of translating sentences produces false, from true, sentences.

This divergence in truth value is underpinned by the divergence in ontological commitment between set theory and second-order logic. As Quine puts it, set theory makes staggering ontological commitments. It assumes, as domain of quantification, all proper subsets of the iterative hierarchy. In contrast, under

[82]George Boolos, "On Second-order Logic," The Journal of Philosophy, LXXII (1975), p. 512.

Boolos' reading, second-order logic is not even committed to the existence of a two-membered set. This is because, the sentence:

$$(\exists X)(\exists x)(\exists y)(Xx \wedge Xy \wedge x \neq y),$$ [83]

expressing the existence of a property which holds of two distinct individuals is invalid. As a counter-example, consider a universe with only one member. In set theory, the property, under the Quinean translation:

$$(\exists X)(\exists x)(\exists y)(x \, \varepsilon \, X \wedge y \, \varepsilon \, X \wedge x \neq y),$$

expresses the existence of a two-membered set. We are assured of its existence by inspection of the set theoretic hierarchy. In logic, no assumption is made as to which domains exist.

To summarise, the dispute between Boolos and Quine shows that Quine's conclusion does not follow from the premises. Boolos succeeds in doing this by pointing out that there is a consistent, alternative, way of extending first to second-order logic. Furthermore, the conclusion is false *tout court*. This is demonstrated by the divergence in truth values between second-order sentences and their set theoretic counterparts (under Quine's translation).

This makes it clear why certain axioms in set theory are not considered to be logical axioms. For, they pertain to the construction of the hierarchy.

In conclusion, around the turn of the century, when logic suddenly became much stronger, the issue of the distinction between mathematics and logic did not arise. At the time, logicism was criticised for different reasons. Logicism was a project with a clear goal and clear boundaries. It seemed obvious that logic was distinct from mathematics, and that logic held a philosophically privileged place over mathematics. Now this is less clear. With the plethora of logics which confront us, there seems to be no obvious cut-off point between logic and mathematics. Moreover, there are rival logical systems which are not mutually compatible. It is not at all clear that logic has to found mathematics in any sense. On the

[83]George Boolos, "On Second-order Logic," The Journal of Philosophy, LXXII (1975), p. 513.

other hand, rival foundational theories do not fare much better. When it seemed clear which formal systems were logic, and when it was thought that there was only one way to reason and that logic captures this; the importance of the logicist project was obvious.

With the collapse of Frege's system and the emergence of the Löwenheim-Skolem results, the logicist project was abandoned. Skolem raised the question as to the relativity of powerful set theoretic notions. It emerged that what counted as a truth of logic would depend on what system was used - not in the sense of one system being more powerful than another, but in the sense of conflicting candidates for logical truth emerging. The debate over logicism did not really survive the criticisms of Ramsay levelled against Russell and Whitehead's type theory. Since there does not seem to be a clean cut off point between logic and mathematics, there seems little point in trying to revive the logicist project. The only sense in which it could be revived, according to popular opinion, is by taking a formal system which is definitely logic, such as first-order logic or propositional calculus, as foundational. But then, logicism is uninteresting, since first-order logic and propositional calculus are so very weak, and therefore, we can reduce very little of mathematics to it. But that is only the prevailing dogma, fostered by Quine.

In 1983, with the publication of Frege's Conception of Numbers as Objects, Wright suggested that if we remove axiom V from Frege's formal system and promote Hume's principle to the status of axiom, then we have a consistent theory which partly vindicates the logicist project. The extent to which second-order logic is logic, is the subject of the rest of the Thesis. In chapter two we discuss philosophical criteria which a formal system should meet in order to be considered to be a logic.

Chapter II: Philosophical Characteristics of Logic

- A distinction can be important and principled without being sharp.
-C. Peacocke "What is a Logical Constant."

Introduction

"There is no sharp border between mathematics and logic."[84] While in some respects this might be true, it does not entail that we cannot draw a principled distinction between logic and mathematics, and that such a distinction cannot serve any purpose. Furthermore, once we do this, we shall find that there are some formal systems which are clearly logic, and some which are mathematics. The philosophical project of determining which formal systems deserve the honorific title "logic" has largely been abandoned. Many mathematicians and philosophers think that no line can be intelligibly drawn between logic and mathematics. Others do not see that drawing a line could be fruitful.

In this chapter, we shall be examining some philosophical notions which characterise a logic for the logicist, and which motivated Frege in the writing of his three great works. Of course, I am not interested in resurrecting Frege's project in exactly the way he intended. Rather, I believe that his writings serve as an interesting starting point for this aspect of the philosophy of mathematics: one which can inspire a moderate logicism.

Having discussed some of the philosophical notions which inspired logicism, we shall apply them in the next chapter. We shall examine the limitative, or characterising results, of first-order logic to assess whether they engage, and if so are in conformity with, the philosophical notions discussed in this chapter. In this sense, the philosophical notions are adopted as criteria for assessing the claim

[84]Stewart Shapiro, Foundations without Foundationalism. A Case for Second-order Logic, (Oxford Logic Guides: 17; Oxford: Clarendon Press, 1991), p. vi.

that first-order logic is "logic", and that any stronger system is mathematics or science.

In section one of this chapter, we shall discuss validity. The chief idea is that logic tracks validity, in the sense of reflecting an informal notion of logical validity. There are various ways of defining validity formally. One is to say that an argument is valid just in case it has one of a particular list of forms (usually inductively defined). That is, we give some base forms and then show how to break up arguments into units which fit the forms, or not. This sort of definition obviously begs the question against the purposes here, since our task is to determine which formal systems best reflect an informal notion. If we use a list taken from an already existing formal system, then obviously that system will be the one which perfectly conforms to our criterion of validity. However, in doing this, we have not assessed the formal system, we shall simply have adopted it as our measure of validity. This ignores any pre-theoretic or intuitive notion of logical validity.

To reflect our informal notion without begging the question, it is useful to explore the distinction between the form and the content of an argument. The form of an argument is often associated with the "logic" of an argument, and the content with the subject matter, or the application of the form to a subject. This is alright as a starting point for our intuitions, however, we soon find that we have to make them sharper.

To then distinguish content from form, we need to appeal to other notions such as universality, or generality. "Generality" and "universality" are used inter-changeably. They are discussed in section two. We interpret "universality" to mean that when applying a logic, as opposed to a theory, no restriction is placed on the domain of individuals we may consider.

Careful examination of this criterion, reveals that while we have said something about the models of a given formal system, we have placed no restriction on the language. Thus, any language will do. It may be very sophisticated and abstruse. Not all notions in a language will find application within all domains, but that is alright, according to our criterion. However, under the criterion of universality, we wish to restrict the language too. Thus, we add in another aspect to the criterion of universality, and that is that the universality of logic should involve its being topic neutral. This aspect is elaborated in terms of not being able to distinguish between members of a domain, in the sense of not being able to pick out particular members. We then have to say something about what counts as a particular feature, or particular object. For this we shall need to appeal to our other criteria for logicality.

In section three, we shall discuss the idea that the sentences of logic are analytic, as opposed to synthetic. The analyticity of sentences in logic ensures their purity and independence from what Frege called "special" concepts. A concept is special just in case it belongs to a particular discipline or practice but does not belong to all discourses. One way of ensuring that no special considerations are included in a proposition is to prove it by means of what Frege referred to as a gapless proof which uses only logical axioms and definitions as premises. A gapless proof is one in which the conclusion follows from the premises using only logical rules of inference. It is natural to assume that the method of proof must be effective. That is, that it can be carried out by a Turing machine in a finite number of steps. We shall challenge this assumption, and suggest an alternative: that we relax the constraint of effective gapless proof and accept the corresponding model theoretic notion of the conclusion being satisfied in all models in which the premises are satisfied. Put in terms of analytic sentences: for Frege, a sentence is considered to be analytic if follows from logical axioms by means of a gapless proof. Gapless proofs can be generated in any science. Making a gapless proof is just a matter of being explicit about the assumptions and rules of inference used to justify the conclusion. The conclusion to a gapless proof is an analytic truth only if it follows from definitions, the assumptions are all logical axioms, and the rules of inference are all logical rules of inference. Returning to our re-formulation: a sentence will be analytically true if it is true in all models. Recognition of this or its contrary: that there are models in which the purported conclusion does not hold, should not depend on the use of sense perception or on Kantian spatial or temporal intuition. Thus, the notion of "logical proof" becomes a little broader then just meaning a finite formal proof of natural deduction, or an effective proof, the generation of which only involves the application of an algorithm, or mechanical proof procedure.

What is interesting is that while it may require ingenuity (and so possibly intuition in some sense) to generate a "proof" which relies on semantics, and not on a mechanical proof procedure or an algorithm; recognition that the proof is a gapless proof, relying only on logical assumptions does not require intuition or sense experience.

A sentence will be analytic (but not necessarily true) if it can be written in a logical formal language, that is, one which can be written without recourse to any non-logical constants. This does not preclude the possibility of sentences being analytic if they are written using non-logical constants, or if they depend on the use of definitions. The point here is that there is a core of analytic sentences which are logical sentences. It is these which we focus on. The presence, in a sentence, of non-logical constants often, but not invariably, indicates of that sentence that it is synthetic. At least, we take this as our default position, in the sense of our being suspicious of such sentences. Our suspicions can be shown to be unfounded if we then prove that we can show that the truth or falsity of the sentence does not depend on sense experience or on Kantorian intuition. We show this when we show that a given notion can be defined in terms of logical vocabulary. The reason non-logical constants are suspicious is that they have an intended interpretation. The non-logical constants are distinguished from both the logical constants and the rest of the of vocabulary. "The rest" includes variables and descriptive vocabulary such as predicate and relation symbols. Non-logical constants are distinct from the rest of the vocabulary in that there are axioms governing their use. The rest of the vocabulary have no axioms governing their use, although they figure in the axioms. They have no intended interpretation.

The importance of the intended interpretation has to do with the generality criterion. Both logical and non-logical constants are accompanied by axioms governing their use. However, whereas a logical constant is invariant across domains, a non-logical constant is interpreted by a member of the domain or by a subset of the powerset of the domain. That is, the interpretation function picks out a member, or subset of the powerset, of the domain to assign to non-logical constants. For instance, the interpretation function will pick out one member in the case of an individual constant, such as "0". The interpretation function will pick out a subset of the domain in the case of a one-place predicate constant, such as "is a number". The interpretation function will pick out a subset of the full powerset of the domain in the case of 2-place relation and function constants, such as "+" or ">". The interpretation function might get it wrong. That is, it might mis-interpret a given function symbol, and interpret "+" as "x", for example. This might, for example, be due to there not being a suitable element or subset of the powerset of the domain available, such as if we try to interpret the relation "is friends with" in the domain of natural numbers. The intended interpretation is independent of the setting up of the language. That is, we know before we set up a formal language, what is meant by the particular non-logical constants. We can tell that a constant has received an unintended interpretation when the interpretation makes the axioms for the non-logical symbol come out false. Put the other way around, the non-logical constants, together with ensuring that their axioms come out as true, constrain both the interpretation function and the domains of interpretation we are allowed to consider. The absence of non-logical constants indicates analyticity and logicality, but not necessarily truth. The following characterisation of analyticity is in keeping with Frege's. A sentence is analytic if detecting its truth or falsity does not have to depend on sense experience or temporal or spatial intuition. As we shall see in this chapter, the extension of this characterisation of analyticity is not obvious to determine.

The notions of analyticity, reflection of an informal notion of logical validity, and generality, are ambiguous, in the sense of not being sharp enough to decide between competing formal systems which lay claim to being logic. Thus, in chapter three, we shall home in on specific arguments of a technical nature to do with the limitative or characterising results which hold of first-order logic. In this sense, we use first-order logic as a test case.

§ 1: Validity

Our criterion for validity is: "to be called a logic, a formal system has to have the capacity to track validity". I am taking validity to be synonymous with semantic entailment in the sense that if an argument is valid, its conclusion is entailed by the premises. The entailment is semantic in the sense that any interpretation which makes the premises true will also make the conclusion true. An interpretation is a pair, consisting in a domain (of interpretation) and an interpretation function. The interpretation function assigns members of the domain arbitrarily to variables, subsets of the domain to one-place predicate symbols, and subsets of the powerset of the domain to n-place relation and function symbols. Constant symbols receive a fixed interpretation.

Validity is often identified with a formal requirement: that all valid arguments display one of a disjunction of forms of argument. These are already determined to be valid in the formal system. To think of validity in this way is to reverse the perspective adopted here, because which forms end up being the valid ones, is a function of the limitations of the formal system (namely its expressive power). Instead, we are trying to match an informal notion of validity to a formal system. The informal notion informs the formal notion, in the sense that when we set up a formal system, the extension of valid sentences and arguments of the formal system should mirror, as closely as possible, our informal intuitions about what counts as a logically valid argument. This is not to say, of course, that we cannot be surprised by the conclusion of a (valid) argument. That we do not always correctly anticipate the conclusion, either indicates our weakness in carrying out many logical manipulations in our head, in the cases where we reach the conclusion by means of a system of deduction, or our surprise indicates our weakness in being able to imagine what models there might be, or which models are mutually inconsistent. Thus, our being surprised by a given conclusion does not indicate a shortcoming of the formal system to match our informal notion of validity. It only indicates our limitations in time, space, ink, concentration and imagination.

Of course, things are not so simple. The very fact that the informal notion of validity is informal entails that there is scope for disagreement as to what constitutes an optimal formal representation. That is, as a matter of fact, we have divergent intuitions concerning logical validity.

In practice, when a formal system fails to reflect our informal notion, we tend to react in one of two ways. We either reject the formal system, and try to construct a better one, or we revise our notion of validity. For example, in classical logic, from a contradiction, anything follows: $A \wedge \neg A \vdash B$, where A and B are well-formed formulas. On encountering this "artefact" of classical logic, students are told to suppress their intuitions and become accustomed to this as an instance of a valid argument. However, not all logicians or philosophers have grown accustomed to this as an instance of a valid argument. Some have developed alternative logics, such as systems of relevance logic. Which is the more appropriate response, philosophically, is determined by further argument involving other features of the formal system. We leave those aside. Nevertheless, even within a classical conception of logic, there is some room for play, in deciding which formal system best reflects our informal notion of validity. For instance, there is a debate over whether weaker or stronger logics, such as first-order logic and second-order logic, better reflect our informal notion of validity.

The contrast between form and content is usefully invoked to make sense of our informal notion of validity. If our formal system seems (intuitively) to be too crude, i.e. has low expressive power, then we say that the formal system will not recognise certain aspects of the form, which an argument takes, which make the argument valid. Put another way, if one logic has lower expressive power than another, it will pick out fewer arguments as valid! For example, we say that first-order logic has greater expressive power than propositional calculus. All arguments which come out as valid in propositional calculus come out as valid in first-order logic, but not the other way around. For instance, consider the argument:

Everybody laughs at Daffy.	$(\forall x)xLd$
Therefore, even Daffy laughs at Daffy.	dLd

To show the validity of the argument formally, we need first-order logic, because in it, we have descriptive vocabulary in the form of relations and functions, we have individual variables and we have the quantifiers. In propositional calculus we cannot show the validity of the argument. For, the form of the two sentences cannot be shown to be inter-connected in propositional calculus. We would formalise the premiss by a proposition symbol: "P", and the conclusion by "Q". The argument represented as: P, therefore Q, is invalid.

Similarly, first-order logic falls short of the task of tagging, as valid, certain arguments which we often intuitively think are valid. These will be any arguments involving concepts not formalisable in first-order logic; they might be formalisable only in second-order logic. Included in such concepts are infinite cardinalities such as

"Dedekind infinite" and "uncountably many", but on a more mundane level so are: "most", "same property", "even", "is one of them" and so on. For example:

the even numbers form a proper subset of the natural numbers.

The even numbers can be placed into one-to-one correspondence with the natural numbers.

Therefore, the natural numbers are Dedekind infinite.

The notions of "one-to-one correspondence", and Dedekind infinite are only uniquely captured by second-order formulations. This is because we have to quantify over the proper subset of the set of natural numbers and over the relation of one-to-one correspondence. No attempt at a first-order formulation of these notions will represent the above arguments as valid. As a second example consider:

more people suffer from tooth decay

than get run over by a car.

Therefore, there are people who suffer from tooth decay, but are never run over by a car.

The quantifier "more" is not first-order expressible.

But now comes the question as to where to draw the distinction between form and content, because Dedekind infinite, for example, does not appear, *prima facie,* to be a logical concept: it is numerical. Thus, an argument which relies on the formalisation of Dedekind infinite to show its validity, might not be thought of intuitively as a logically valid argument. By moving to a powerful logic we seem to be able to analyse, in terms of form, notions which we intuitively deem to be non-logical: and, *a fortiori,* the conclusion follows because of the content of the notion in the argument, not because of the form of the argument. Here, "content" is being identified with particular ideas, or ideas proper to what Frege referred to as "special sciences". That is, there seems to be a point where languages become so strong as to include notions proper to a special science in the form of a non-logical a constant with its accompanying axioms. Then, what is represented as a formal property is really content. For example,

> Everything that comes into being has a first moment.
> <u>Therefore, in particular, the universe had a first moment.</u>
> Therefore, there was a first moment of time.

Displaying the validity of the argument depends on using modal temporal notions. We just add these to the language, together with appropriate axioms, and we can formally represent valid arguments with the new language. This is an example of what we shall refer to as the "representing content as form problem". The problem is that in designing formal systems, we seem to be able to represent formally what we intuitively think of as subject matters or contentful notions. Our capacity to give formal representation over reaches what we intuitively think is the boundary between what is the formal logical aspect of a given argument and what is particular to the subject matter that the argument deals with.

Before going any further, we should discuss an associated problem. Content is often associated with ontology. Thus, one way of construing the "representing content as form problem" is in terms of the ontological commitment of the formal systems. This is alright except that one of the problems associated with formal systems endowed with great expressive power is that they seem to make a proportionately (to their expressive power) great ontological commitment. Loosely, this is because content is associated with ontology, and the more content we can express, the more ontological commitment we must be making. After all, it would be a fictional science which could give such precise and detailed characteristics of objects, to whose existence it remained uncommitted. We usually do not think of logic and mathematics as fiction because of the degree of certainty which we confer on logic and mathematics. The certainty must be grounded in something. Thus, to parry the accusation that logic is a fiction, it is customary to posit entities which ground our certainty, in the sense of committing ourselves to the existence of a platonic heaven of (mathematical) entities. Of course, it is never clear how this is to ground our certainty.

In particular, strong formal systems seem to be committed to the existence of objects not deemed to be logical *per se*, such as infinite sets. That is, we might take a logic's great expressive power as evidence for the logic making ontological commitments which are outwith the business of logic. The thought would go, roughly: showing that an argument is valid, involves showing that when the premises are true, so is the conclusion. Showing that a sentence is (sometimes) true, involves showing that there are models which satisfy the sentence. The thought then goes that we must be committed to the existence of all these models which vindicate all sentences which are not self-contradictory. This makes for a lot of models. We end up with a staggering ontology. Moreover, some pairs of sentences laying claim to which models exist are mutually inconsistent. For this reason, we have to keep the models separated. We then arrange them in a hierarchy. Our formal system then begins to look more like mathematics than logic, because part of the system will be dedicated to the study of the hierarchy and its levels and members and so on. In particular, we soon find that we need an axiom asserting, for example, the existence of a Dedekind infinite set. However, we have now shown that we have strayed from logic. For, the existence of a Dedekind infinite set has nothing to do with logic, but with how the world happens to be. Therefore, when we are shown a formal system which can express such notions, we judge that it is not logic but mathematics or science, because it has content.

However, this whole train of thought misses the mark. We should remember that just because we can express something using logical vocabulary, this does not make it a logical truth. This is born out on a technical level, in that the sentence asserting the existence of a Dedekind infinite set is not valid, i.e. it is false under some (all finite) interpretations. Furthermore, the sentence asserting the existence of a two-membered set is invalid. Consider a universe consisting in a one membered set, such as, the empty set. The existence of a Dedekind infinite set might be necessary (relative to a theory), but not logically necessary. Similarly, the existence of an even prime is necessary, but not logically necessary. For, the universe (logically) might have been such that there were no primes at all, consider the universe composed of the natural numbers with all primes removed.

Driving the point home: it turns out that, arguably, the only ontological commitment made by first or second-order logic is to the empty set.[85] But even this argument relies on accepting a convention! We say that first or second-order logic is committed to the existence of the empty set because the empty set is a subset of any set. However, this reasoning rests on a particular set-theoretic understanding of domains to which we can apply the subset operation. So, all domains are thought of as composed of sets. However, in the case of logic, we do not need to think of domains as composed of sets. We are free to think of them as domains of objects. If there are no sets in a domain, then there may be no empty set figuring as a subset. Thus, even the insubstantial ontological commitment attributed to first and second-order logic to the existence of the empty set is debatable. It is a stipulation which is not logically motivated, it is methodologically motivated. Moreover, it is motivated from mathematical (set-theoretic) considerations. It only makes things technically easier. Thus, the greater ability of one logic to track validity over another, does not imply that the one makes any ontological commitment whatsoever, let alone in proportion to its expressive power.[86]

Let us return to our question about form, content and their distinction. In general, a lot of expressive power in a formal language allows us to capture more arguments as valid. Consider just the following arguments.

[85]George Boolos, "On Second-order Logic," The Journal of Philosophy, LXXII (September, 1975), p. 513.
[86]What the ontological status of objects, such as infinite sets turns out to be is a matter for mathematics or metaphysics to decide, not logic.

(i) Betty has a sister.
Mary has a sister.
Therefore, Betty and Mary have some property in common.

Or:

(ii) most of the people in the room listen to opera.
Most of the people in the room play chess.
Therefore, there is someone in the room who listens to opera and plays chess.

These all, plausibly, count as valid. However, formal representation of the validity of these arguments requires the expressive resources of second-order logic.[87] The question is whether or not these are logically valid, as opposed to valid because of the subject matter. Argument (i) does not rely on an understanding of what it is to be a sister. To recognise the validity of (i), all we have to do is recognise that the same property pertains to both Mary and Betty. Also, we do not need to know who Betty and Mary are, to recognise the validity of (i). In the language of second-order logic, we can represent the argument as:

$Sb,$
$\underline{Sm,}$
$(\exists P)(Pb \wedge Pm).$

The conclusion is read: there is a predicate P, b falls under it and m falls under it. S is an arbitrary descriptive letter, P is a variable, and this is why the expressive resources of second-order logic are required for displaying the validity of the argument. Similarly with argument (ii). This can be represented as:

$(\text{Most } x)\ Ox,$
$\underline{(\text{Most } x)\ Cx,}$
$(\exists x)(Ox \wedge Cx).$

The reason the conclusion of the argument follows is because of how we characterise "most". If "most" is construed in the usual way

[87]For other examples see: George Boolos, "Nonfirstorderizability Again", Linguistic Inquiry. XV, (1984) pp. 343 - 4.

as: "more than half", then the conclusion follows. We can either take "most" to be a primitive quantifier, or define it in terms of the second-order quantifiers and relations. Either way, we first have to give a characterisation of "most" which is sufficiently precise to be captured by a formal language. There are many ways of doing this.[88] Let us just take one example. Characterise "most" as: more than half of a finite domain. We shall leave aside the interpretation of "most" in infinite domains. (Most x)Px can be thought of as saying that there are more things in the (finite) domain that have P than things that do not. We can use P to partition the domain in to two: the P things and their complement: the $[\neg P]$ things. We know that most things are P if there is a mapping from the complement set $[\neg P]$ into the set P, and the mapping is not one-to-one. That makes the $[\neg P]$ set strictly less than the P set. We can then express in the language of second-order logic (Most x)Px as:

$$(\exists P)(\forall f)((\forall x)([\neg P]x \rightarrow Pfx) \wedge$$
$$\neg(\forall x)(\forall y)(([\neg P]x \wedge [\neg P]y \wedge fx = fy) \rightarrow x = y)).$$ [89]

Then, insofar as we accept second-order quantification as logical we have shown that we can express the notion of "more than half" in the language of second-order logic. Insofar as we think it is important to capture this notion in order to reflect our informal notion of logical validity, we have shown that second-order logic is better at tracking our notion of logical validity than is first-order logic.

As a matter of empirical fact, informally, we happen to think that these sorts of argument are logically valid. This suggests that we should adopt second-order logic as logic because, just on empirical grounds, it better represents our notion of informal logical

[88]For a scattered catalogue see Gila Sher, The Bounds of Logic, (Cambridge, Massachusetts: MIT Press, 1991).

[89]All the necessary components for this can be found in Stewart Shapiro, Foundations Without Foundationalism, (Oxford Logic Guides: 17, Oxford: Clarendon Press, 1991), p. 102. The more perspicuous way to represent this is in terms of the P partition being strictly greater than the $\neg$P partition, but then we would have to argue for accepting the symbols for inequality in the list of logical constants.

validity. However, now we have a problem. We can no longer distinguish a (merely) mathematically valid argument from a logically valid one. For example, there are arguments which include the notion of well-ordering and which are such that, accurate (second-order) representation of well-ordering is essential to displaying the validity of the argument in question. For example: the well-ordering of the set X by the relation R (WO X by R) is expressed:

$$(\exists X)((\forall x)(\neg Rxx) \wedge (\forall x)(\forall y)(\forall z)((Rxy \wedge Ryz) \rightarrow Rxz)$$
$$\wedge (\forall x)(\forall y)(x \neq y \rightarrow (Rxy \vee Ryx))$$
$$\wedge (\exists y)(Xy \wedge (\forall z)(Xz \rightarrow (y = z \vee Ryz)))).$$

Intuitively, R is usually read as the relation of "preceding". The first conjunct says that no individual member of a class (of numbers, say), X, precedes itself. The second conjunct is transitivity of the "precedes" relation, R. If one member precedes a second, and the second precedes a third, then the first precedes the third. This is enough to give us a partial ordering of X by R. Now we have to add the condition for a simple ordering: that if x is distinct from y, then either x precedes y or y precedes x. The fourth conjunct is what distinguishes a simple ordering from a well-ordering: that in a non-empty class of individuals X there is a first individual member, i.e. a member which precedes all the others.[90] The argument:

$$\underline{(\exists X)\mathrm{WO}\ X\ \mathrm{by}\ R}$$
$$(\forall x)(\forall y)(x \neq y \rightarrow (Rxy \vee Ryx))$$

is valid. Because well-ordering is *prima facie* a mathematical notion, we seem to have included too much. For, our logic is tracking not only logical validity but mathematical validity as well!

This conclusion rests on a mistake. The validity of the above argument does not rely on the content of the notion of well-ordering. It relies on the very simple rule that if a conjunction is true, then so are all of the conjuncts. In particular, one of the conjuncts is true of a true conjunction. Thus, if there is a well-

[90]Alonzo Church, Introduction to Mathematical Logic, (vol. I; Princeton, New Jersey: Princeton University Press, 1956), p. 338.

ordered set X, then it will obey one of the constraints placed upon being a well-ordered set.

However, the worry is not idle. If we simply opt for a very expressive logic, and decide that the capacity to display validity of arguments in a formal language, is what shows them to be logically valid, then any notion is a candidate for being considered to be a logical notion. For, it seems that the greater the sensitivity we can display in a given formal language, or the more notions we can accurately represent, the more arguments will be classed as valid. In particular, we will very quickly see that all of mathematics is logic, that possibly all of science is logic, metaphysics is just a branch of logic and even moral theory could be part of logic. For, we just introduce symbols to our language which represent a notion particular to a special science, and gives some axioms governing its use such that mis-interpretation is precluded. This, of course, harks back to Leibniz's idea of an unified science. The worry is one of ending up with a formal system which is too powerful for logical validity, in the sense that it has the resources to exploit the contents of particular subject matters to indicate formal validity, and the formal validity clashes with our informal notion of what counts as logical validity.

If we generalise this sort of argument, we might end up having to include not only second-order quantifiers as part of logic, but also modal operators, deontic operators, temporal operators, set-theoretic membership, addition, and all manner of symbol. In this case, our problem is to distinguish an argument which is logically valid from an argument which is legitimate in a particular discourse because of the content of the notions in question. These will be arguments which hold good of one domain or type of domain, for example, but not of another. This will be indicated by their reliance on intended interpretations of non-logical constants. Our problem now is to find some way of distinguishing logical from non-logical constants which is non-circular.

Here is a circular way of doing this. It is not entirely silly to run through the argument since it is tacit in much of the literature,[91] and the argument brings to light many presuppositions which warrant being questioned anew. In other words, this argument may be circular, but it is not viciously circular. It forms a step in our enquiries into the characterisation of logic.

A structure is a set together with possibly some non-logical constants (often called distinguished elements), functions and relations. Two structures are isomorphic if their respective sets have the same cardinality; and all of the relations, functions and constants are preserved in a one-to-one and onto mapping from one structure to the other. Put another way, there is a function which takes constants in one structure to constants in another structure, relations and functions in one structure to relations and functions in another, respectively, such that the two structures will satisfy the same set of sentences which include the constants relations and functions in question. Being able to show that two structures are isomorphic, when they are, is considered by mathematicians, to be a desirable property of a formal system. Why we want structures to be identified up to isomorphism and not beyond, is because this is considered to be the nature of the subject. We would go beyond mathematics or logic were we to identify structures on the basis of something other than their cardinality and properties associated with characterising structures. The properties which characterise structures are then certain mathematical constants, relations and functions. Expressing relations and functions in a formal language allows us to suppress aspects of certain concepts used in arguments. In particular, we suppress what the objects in a domain actually are, or what instances a relation might have, such as sisterhood. Instead,

[91] See for example, Michael Hallett, "Putnam and the Skolem Paradox," Reading Putnam, eds. Peter Clark and Bob Hale, (Oxford: Basil Blackwell, 1994) pp. 66 - 97. Rudolph Carnap, Logical Foundations of Probability, (London: Routledge and Kegan Paul, 1950), Stewart Shapiro, Foundations Without Foundationalism, (Oxford Logic Guides: 17, Oxford: Clarendon Press, 1991), and Gila Sher, The Bounds of Logic, (Cambridge, Massachusetts: MIT Press, 1991).

we are interested in the mathematical aspects of how it is that individuals are related to each other. We tend to identify content with the particular elements of a domain, and formal properties with how it is that arbitrary, but structurally similar elements are mathematically related to each other; which functions a domain is closed under, and so on.

Why it is that mathematicians are interested in identifying structures all the way up to isomorphism is that they want to examine all the relations which hold between elements in the domain. To elaborate further, take for example, a typical scenario where there is an intended domain of interpretation, say the natural numbers. There are a number of sentences which give mathematical truths (as opposed to philosophical, Cabalistic or scientific truths) about the natural numbers. We say that a theory has high (even optimal) expressive power just in case it can identify the set of natural numbers uniquely up to isomorphism.

There are two aspects to this. One is that there is an independent (pre-theoretic/ intuitive/ meta-linguistic) perspective from which it is judged that there are languages which cannot even express functions powerful enough to characterise sets uniquely up to isomorphism. For example, first-order arithmetic has non-standard interpretations. These either have a different number of members than does the set of natural numbers, or they have different constants or relations. They are judged to be non-standard from a meta-linguistic perspective. The judgement says that the non-standard structures are not isomorphic to the natural numbers. Therefore, as far as mathematicians are concerned, non-standard models of the natural numbers are different from the natural numbers. Nevertheless, the two sets will satisfy the same set of sentences (theorems) of first-order arithmetic such as: "$0 + 8 = 8$". For this reason, we find first-order arithmetic inadequate because while the axioms and theorems are true of the natural number structure, they are also true of other structures which are not (from the meta-linguistic perspective) isomorphic to the natural numbers.

First-order arithmetic cannot pick out the natural number structure uniquely up to isomorphism. This judgement has to be made from a more powerful, sensitive or fine-grained theory, such as that of second-order arithmetic. Thus, in the practice of mathematics, a formal system's incapacity to characterise structures uniquely up to isomorphism seems, *prima facie*, to be a limitation. Later it is sometimes considered to be a very fruitful limitation, because we then go on to study non-standard models in their own right. Be that as it may, in general, mathematicians are interested in a formal system's capacity to characterise structures uniquely up to isomorphism. It seems that in terms of describing mathematical practice, while many mathematicians work with first-order theories, their judgements about the adequacy of those theories tends to be made in a second-order meta-language.[92]

The other aspect of identifying structures uniquely up to isomorphism, is that the business of mathematics is often negatively characterised as that of not being interested in any further details about structures. For example, the mathematician is not interested in which is the platonic set of natural numbers, in the instantiation of the Fibonacci series in nature, in the history of numbers and so on. *Qua* mathematician, he is only interested in any sets isomorphic to the set of natural numbers.[93] Moreover, there are many philosophers who think that this is the best way to characterise the business of mathematics, and that as a result, any further questions as to the true or deeper nature of the subject matter of mathematics, are misguided.[94] Here, we do not wish to make such a strong claim, but examine (not necessarily to undermine) the basis upon which it is made: that there is something sacrosanct about the notion of isomorphic structures in mathematics as characterised in a second-

[92]This is very much one of the thrusts of Stewart Shapiro's book, Foundations Without Foundationalism, (Oxford Logic Guides: 17, Oxford: Clarendon Press, 1991).
[93]Michael Hallett, "Putnam and the Skolem Paradox," Reading Putnam, eds. Peter Clark and Bob Hale, (Oxford: Basil Blackwell, 1994) p. 72.
[94]These philosophers are called structuralists. Among them count David Hilbert, Michael Resnik, Stewart Shapiro, Paul Benaccerraf and Geoffrey Hellman.

order language.[95] If we characterise mathematics as the study of structures, then, as a consequence we shall favour branches of mathematics which can identify structures uniquely up to isomorphism. In other words, there is no principled reason for not being interested in aspects of structures which reach beyond their cardinality, constants, relations and functions. It is only descriptive that this is implicit in the practice of mathematics.

There are three perspectives adopted here. One is that what counts as isomorphic structures is just whatever satisfies a set of sentences in a theory. Thus, in first-order arithmetic, the natural numbers and any non-standard models are isomorphic (relative to first-order arithmetic). The second-perspective is that adopted at a meta-linguistic level which then judges of structures which satisfy sentences formulated in the object language, whether or not they are isomorphic. The more powerful the meta-language relative to the object language, the more inadequate the object language will seem in its capacity to characterise structures. The meta-language is a precisely delineated language, which can be formally represented. it may be a first-order language or a second-order language or the language of some theory such as set theory (again in first or second-order versions). The third perspective is that of a more intuitive conception as to what counts as isomorphic structures. What we observe is that in general, mathematicians adopt a second-order language to judge the adequacy of an object language's capacity to characterise structures uniquely up to isomorphism. Thus, if the object language is first-order, it will often be judged inadequate in its expressive capacity.

Let us shift from mathematics to logic. The languages in which mathematical theories are written, have to have the resources to pick out structures. Moreover, to have optimal expressive power, they have to be able to identify isomorphic structures. Since we are thinking of logic as a formal language together with the freedom to

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[95]Rudolf Carnap, The Logical Syntax of Language, Trans. Amethe Smeaton, (Third Impression; London: Routledge & Kegan Paul Ltd., 1951), p. 265 - 67.

consider any domain as a domain of interpretation, isomorphism of structures will enable us to sharpen what our understanding of logic amounts to. Thus, we might say that, for the mathematician, our logic is expressively optimal if it can pick out domains uniquely up to isomorphism.

That is good for the mathematician, but what of our intuitions concerning logical validity? The problem with our intuitive notion of logical validity is that the language of logic has to attain a certain level of expressive power to match the intuitions. Otherwise the logic will fail to characterise as valid certain arguments which we intuitively think of as logically valid. Relating this to our three perspectives from which we characterise structures, the intuitions seem best reflected by the expressive power of second-order languages than first-order languages. It turns out that if we insist that our logic has sufficient resources to characterise structures uniquely up to (what a second-order language would judge to be) isomorphism, this is enough to take care of arguments (i) and (ii): the Betty and Mary case and the "most" case. So, here is the proposal. The mathematician's intuitions are good ones, *prima facie*, because they accord with ours over the logicality of arguments (i) and (ii) above. We should consider a formal system to be capable of reflecting our (mathematical) informal notion of validity just in case it is capable of identifying structures/ domains uniquely up to what is deemed in a second-order language to be isomorphism.

There is a caveat. *Prima facie*, not even the resources of a second-order language are, in fact, sufficient to characterise any structures uniquely up to isomorphism. For example, no countable (in the sense of finite or denumerable) language can characterise all cardinalities, and therefore, no formal system whose language is countable can characterise all structures uniquely up to isomorphism. There simply are not enough symbols in the language. However, this does not have to worry us. For, we are trying to capture an informal notion of logical validity. Insofar as it is informal, it is probably safe to say that people's intuitions run out when faced with very large cardinals: those which need more than countably many symbols to express. Thus, it will suffice for our purposes to measure the adequacy of the expressive power of a language in terms of its being able to characterise uniquely up to isomorphism a number of models with which we feel familiar. The expressive power of first-order logic is inadequate on this score because it can only express particular finite numbers. In contrast, second-order logic can also express the notion of finite, countable, uncountable, and since we can also express the powerset operation, any powerset (iterated) of these is also expressible.

This will not quite do. The proposal is limited because it is dependent on what it is mathematicians consider to be relevant to characterising a structure. In particular, we are owed a reason, which is relevant to our project, for considering the relation of sisterhood not to be mathematical or logical, and the relation of "having a property in common" to be logical. The difference is already assumed in our practice. It is embedded in the vocabulary of the language before we even check whether or not the logic has the capacity to characterise structures uniquely up to isomorphism.

For example, the logical constants are guaranteed in advance to receive the same "interpretation" in any structure, whereas the non-logical constants are fixed anew in each domain. This is why arguing that the traditional logical constants ought to be so considered on the grounds that they are invariant across domains is a circular way of arguing for the distinction between logical and non-logical constants. It is circular because what we discover is a stipulation. One set of constants is presented as logical, the other as non-logical. So the question for us is: whence the discrepancy? On what basis do we decide, when we set up a language, which symbols are to count as the logical constants, and which are to count as the non-logical constants? The reason the question is relevant is that the term "logical constant" determines the extension of formal logical validity. The list of logical constants will also determine what is a logical axiom or rule of inference and what is not.

What lies at the heart of this sort of argument are the background assumptions for the claim that there exists a one-to-one correspondence between two sets/ domains. Technically, logical constants are fixed independently of domains. That is, the fixed part of the structure, common to all the structures are the ways in which the logical constants work, the truth of their axioms; and all this by (technical) stipulation! This is what we mean when we say that logical constants are "invariant". We shall argue in subsequent sections of this chapter that individual variables, relation and function letters, are harmless enough. We interpret those arbitrarily, given a domain. What makes the non-logical constants, non-logical, is that they get fixed in each new interpretation, and they have intended interpretations which dance attendance on them. The intended interpretations are determined independently of the language, and constrain our choice of domains of interpretation. When we mis-interpret non-logical constants, we pick the wrong domain or interpret the non-logical constant relations or functions incorrectly. Notice that, in contrast, we have no room to mis-interpret a logical constant (by convention), and there is no such thing as mis-interpreting an individual variable or an arbitrary two place relation letter because of the role these symbols play in the language.

Conventionally, the logical constants of, say, first-order logic, are: $\rightarrow$, $\leftrightarrow$, $\exists$, $\forall$, $\neg$, $\wedge$, $\vee$ and $=$. In contrast, symbols such as: x, +, $<$, $\in$, K, B, $\Box$, $\Diamond$, are not usually considered to be logical symbols. If we want to include them in our language, then we either have to introduce them as logical constants, and give a philosophical argument for this, or we have to introduce them as non-logical constants and give some (*a fortiori*) non-logical axioms governing their use (it is these which a mis-interpretation will fail to satisfy). Under this second possibility, we should also have to give an indication as to what was the intended interpretation for our theory (which is *de dicto* not a logic!). This strategy rules out the constants from being treated as logical, from the outset, since they will not be uniformly interpreted, and therefore, some mis-interpretations will make the axioms governing their use, false.

What if we were to break with convention? We can do this in two ways. We can add, what are generally considered to be, non-logical constants to our (conventional) list of logical constants. In this case, *a fortiori*, the axioms governing their use would be slotted under the heading: "logical axioms"! Or say we were to break with convention the other way and pare down our logical vocabulary, and, say, remove $\exists$ and $\forall$? Besides finding ourselves ostracised from the company of classical logicians; we would adjust what it is we look for in trying to show that two structures are isomorphic. In particular, our "logical" connectives would be fixed across interpretations/ structures/ domains. They would have clauses in the satisfaction function, but not in the interpretation function. Also, we would know in advance that we were only allowed to consider interpretations which confirmed the axioms governing them. For, the axioms would now be in the list of logical axioms. Similarly, if we were to pronounce $\exists$ and $\forall$ as non-logical, then we would be free to interpret them unconventionally, that is, mis-interpret them. That is, make the axioms for $\exists$ and $\forall$ false. These are the two ways of breaking with convention regarding logical constants, and the impending repercussions in setting up a formal system.[96]

Let us work through an example. Say we had some very persuasive and independent (philosophical) arguments to the effect that $\Box$, $\Diamond$ and $\in$ were all to be considered to be logical constants. When we present our logical system, we would include them in the list of logical constants. The axioms governing their use would be logical axioms. In setting up the relevant formal system, the new logical constants would be accompanied by satisfaction conditions. For example, where A is any sentence:

Sat($\Box$A) iff A is provable in first-order logic, say.

Sat($\Diamond$A) iff A is not provably a logical contradiction.

[96]Actual formal systems which break with convention are often called deviant logics.

$Sat(x \in y)$ iff x is a member of the set y.
Arguments in which these symbols figured in the determination of their validity or invalidity, would either be logically valid or logically invalid, so, valid under any appropriate interpretation. That is, under any interpretation in which the axioms, governing the use or meaning of the symbols, would be true.

We can generalise further. Any non-logical constant can be imported to the list of logical constants. We remove mention of them in the clauses of the interpretation function and write clauses for them in the satisfaction function, in the presentation of the formal system. We would then, by fiat, ensure the formally ascribed logicality of the constant. The extension of the notion of isomorphism would be adjusted accordingly, so that the symbol would tell us beforehand which domains are appropriate for interpretation of the variables and logical constants.

We proceed similarly with the paring down of the logical constants. We remove these from the clauses of the satisfaction function and subject them to the wiles of interpretation. In this case, their interpretation becomes domain dependent, and the extension of the notion of isomorphism undergoes suitable adjustment.

The upshot is that the reasoning is circular which tries to get a grip on characterising logic, by saying that logic is that discipline which reflects our informal notion of logical validity, when this is made precise in terms of the accepted practice as to what counts as isomorphic structures. This begs the question because it amounts to identifying form with properties used to characterise structures. In contrast, from our point of view, which properties and so on remain invariant across domains is *ad hoc*. Nevertheless, this does give us a starting point. We now know that we have to account for the choice of logical constants. We are less interested in giving a description of practice than of motivating a coherent practice. Less circular arguments will come from considerations of our other two criteria.

In conclusion, one of the virtues of logic is that it ignores the contents of arguments and concentrates on the form, and furthermore, logic can be brought to bear in any reasoning situation, i.e. it can be applied to any domain of interpretation. However, if we argue too narrowly, these are only so many words. For, what counts as "any domain" is not wholly arbitrary, it is constrained by convention, or at least by specifying a formal language. We cannot lend "any" interpretation to all the symbols in the formalisation of an argument, since the logical constants receive a fixed interpretation, and which symbols those are, is regulated in advance, in the language. Even worse, how we distinguish content from form is decided in the same way. In other words, there are alternative underlying logics to theories.

Nevertheless, the discussion of validity has helped us to gain greater precision as to the distinction between a logic and a non-logical theory. It seems that, *prima facie*, we would like expressive resources rich enough to identify domains of interpretation uniquely up to isomorphism. This would accord with mathematical practice and would optimally reflect our informal notion of validity. However, we have also discovered that the extension of what constitutes identity of domains or structures uniquely up to isomorphism, is hostage to our language. More particularly, it is hostage to which symbols we include in our list of logical constants and which we include in our list of non-logical constants. The descriptive vocabulary and the variables do not affect the extension of "isomorphic structure". To avoid begging the question we have to argue on philosophical grounds for inclusion or exclusion of particular symbols in the list of logical constants. I suggest that it is the criteria of universality and analyticity which will help to motivate a list of logical constants.

§2: Universality/ Generality

We begin with a preliminary comment. In characterising universality or generality as "interpretability in any domain" we have departed from Frege, whose logic was interpreted. What he meant by universality was that the domain of interpretation (to use vocabulary strictly not in keeping with Frege) is all that there is. In other words, Frege believed that there is a sort of universal domain, this is the one logic presides over. Logic is thereby ubiquitous. All other "special" disciplines preside only over subsets of this. This view of Frege's contributed to the contradiction in his system. This is one reason why I choose to interpret universality to mean "interpretable in any domain" or "applicable to any domain". "Any domain" does not include the whole universe on pain of contradiction. Any particular domain is a proper subset of all that there is, which itself is a proper class. However, the two conceptions of universality are not so far removed from each other. If we could take the union of "any domain", we would have, in effect, the proper class: "the universal domain".

Logic is universally applicable. This has two aspects: the universality aspect and the applicability aspect. We shall discuss applicability later. With respect to universality, the necessary but not sufficient condition for this aspect of universality is that we can give any domain of interpretation. In a sense then, one difference between logic and theory concerns whether or not there are any non-logical constants in the language. If there is a restriction on the domains of interpretation, be it either on how they must be arranged relative to each other, or a restriction as to which sets (domains) we consider, then we have wandered out of the realm of logic. For, we have violated the notion of universality. In the case of restricting the domains brought under consideration, before we apply a logical "calculation" concerning the satisfaction of sentences; we effectively restrict the sorts of counter-examples we are allowed to invoke. In logic, we are allowed to entertain any counter-example. In the case of relations restricting our choice of domains of interpretation, what we mean is that the restriction is due to a consideration which does not belong to logic. For example, relationships, such as "is friends with", "is more beautiful", or even "has a sharper curve", (unless these are subsequently shown to be logical) will not do. Exactly which relationships should be considered to be logical and which not, depends on philosophical considerations and their interpretation by formal systems. Hopefully, we shall hone our intuitions by considering particular examples.

First-order logic is generally accepted as being the quintessential logic. It has three sorts of vocabulary: logical constants, individual variables and descriptive vocabulary. The logical constants we accept, for now. Individual variables range over the domain. There being a domain of individuals is what ensures that we are able to apply the logic. That there should be variables at all, assures that the language can be applied. What those individuals are, is specified not as a matter of logic but from outside logic, since the interpretation function picks them out arbitrarily. Thus, there is no conceptual quibble with individual variables.

Previously, in this thesis we have run rough-shod over descriptive vocabulary. We shall now examine its role more closely. Descriptive vocabulary consists in predicate, relation and function symbols. They are not variables because we cannot quantify over them. When we apply first-order logic to a specific domain, we specify a key which interprets the descriptive vocabulary. We include in that key, a translation for the predicate, relation and function symbols used in a given argument. The symbol has a fixed meaning relative to a given domain. It looks as though first-order logic has within it some (other than logical) constants because the descriptive vocabulary receives a fixed interpretation which is domain dependant.

This is a false impression. For, the reference of descriptive vocabulary is not fixed until we specify a key, and there are no restrictions on the key except in terms of type. That is, an n-place relation symbol must be interpreted by n-tuples. For example, a one-place predicate letter is to be interpreted by a subset of the domain. A two-place relation letter is interpreted by ordered pairs. The first of each pair is taken from the domain, and the second is taken from a copy of the domain, thus the set of ordered pairs is said to be a subset of the square of the domain, or Cartesian product of the domain, and similarly for triples and any n-tuple. What is instructive is that in looking for a counter-example, we can shift domain and re-interpret the key. In this sense, the descriptive vocabulary only gains constancy once a domain is given and an interpretation is fixed. We have no quibble with these as vocabulary of a logical language (a language which can figure as the basis of a logic). A language passes the universality test just in case it (1) can be interpreted in any domain, (2) has no non-logical constants, and (3) has a legitimate set of logical constants. What makes a logical constant legitimate is that it has passed muster with our other philosophical criteria, to which we turn now.

As we noted in the previous section, the logical constants are fixed commonly to all structures we use to interpret a given formal language. How do these earn their status?

Among the logical constants are one and two-place connectives such as: $\rightarrow$, $\leftrightarrow$, $\neg$, $\wedge$ and $\vee$. Their meaning is fixed by the satisfaction function. They by-pass the interpretation function, that is, they are defined independently of a domain of interpretation. We need them in order to calculate the truth of sentences which include several terms. The terms are logically related to each other if they modify the truth of sentences according to the truth assignment given to the terms.

It is helpful to look at this from an algebraic perspective (that of the calculation of the Boolean combinations of any two terms). From this perspective, it is irrelevant which connectives we choose to be primitive. All we need is to be able to consider all the truth-value (Boolean) combinations between terms. This is what allows us to say that a rule of inference is truth-preserving, for us to establish the validity of an argument, or sentence, and so on. In other words, we can think of the logical connectives as fulfilling a function: that of enabling us to calculate all the two-valued (true and false (1 and 0)) Boolean combinations of terms. Call this the "logical connective function". If a term is given the value 0, we want a connective which will transform the value to 1. Negation usually fulfils this part of the function. When two terms are connected, under the four possible (truth) value assignments to the pair, we want to have available all sixteen possible (truth table) outcomes to the (truth) valuation. In more familiar terminology, for two terms connected by arbitrary logical connectives, there are sixteen possible truth table results for the whole connected proposition.

A logic must be able to fulfil this logical connective function for many reasons: in order to assess validity, to preserve the idea that in following a logical rule of inference we preserve truth, to preserve the idea that logical axioms are true in all domains, and so on. In other words, we need logic to fulfil the logical connective function in order for logic to distinguish truth from falsity at all.

Without this logical connective function, we would be unable to do even propositional calculus or syllogistic reasoning. For, we would have no basis upon which to indicate a transformation in truth value of either a proposition on its own or in combination with others. A valid sentence is one which is connected in such a way that, any truth-value assignment to the basic propositions (whichever are the smallest units in the sentence which can have a truth value assigned to them) will still yield the truth-value "true" for the whole connected sentence. This idea is extended in the obvious way for valid arguments and the preservation of truth through logical inference. Thus, the logical connective function is integral to logic.

Exactly how this is done, in terms of which connectives we chose to be primitive, and so on, is unimportant. Indeed, our choice of primitives is somewhat *ad hoc* relative to the considerations here.

In our traditional and standard list of logical constants, we also have the two quantifiers: $\exists$ and $\forall$. The universal quantifier can be interpreted in any domain. However, unlike the logical connectives, it needs a domain. Similarly, for the existential quantifier, it needs a domain, but any will do. In both cases, the understanding of the symbols does not depend on any particular domain or feature of a domain, there just has to be one. The very existence of a domain of interpretation is a precondition for the application of logic. An inapplicable language may be of technical interest, but it will not be of philosophical interest. The logicist is interested in logic functioning as a sort of ultimate justification, in the sense of its being universally applicable, analytic and conforming to our intuitive sense of what constitutes a logically valid argument. In order for a language to count as a logic, in the sense of conforming to our philosophical criteria, it has to be applicable to domains of objects. We want to be able to say of a given domain whether or not it contains any objects which have a certain property P, for example. For this, we minimally need the existential quantifier. We tend to choose the existential and universal quantifiers because we do not seem to require any special knowledge, in order to understand how they operate on a formula.

The criterion of universality, or generality, is also sometimes understood in terms of topic neutrality.[97] This is when the applicability aspect of the criterion becomes important. I propose we understand this as requiring that logical sentences can receive a meaningful interpretation in any domain. "Meaningful" here just means "has the capacity to make sense", no matter what domain it is we consider. For example, we can make sense of the claim that there is some individual in a domain which fulfils certain criteria, or all members of the domain fulfil certain criteria. That is, in invoking the quantifiers, we have not decided beforehand, of what type the

[97]See, for example, Jan Wolenski, "In Defence of the First-order Thesis," Logica '93 Proceedings of the 7th International Symposium eds. P. Kolár, V. Svoboda, (Praha, 1994), p. 4, and Christopher Peacocke, "What is a Logical Constant?" The Journal of Philosophy, vol. LXXIII.9 (1976), p. 229ff.

domain has to be, or what category of things are allowed to figure in a domain. In particular, both the universal and existential quantifier are neutral with regard to what properties the objects in the domain has, or how members of the domain are related to each other, or anything particular about them at all. We shall further develop these arguments concerning quantifiers in chapter four.

Bearing in mind the topic neutrality aspect of generality, consider the important example of the set theoretic membership relation. This too, has to be interpreted in a domain which includes sets or can be thought of as a set itself. However, there are many domains, which are simply not composed of sets, *pace* Maddy, and no logical considerations, so far (pending a definition in already accepted logical vocabulary), force us to consider domains themselves as sets in the very rich sense of being part of a set-theoretic hierarchy. Put very succinctly,

> Those who have wished to include '∈' among the logical constants have probably been influenced by the feeling that it satisfies some intuitive requirement of topic neutrality.... When, however, a genuine ontology of sets is *required* for interpretation... the topic neutrality claim for '∈' seems indefeasible (*sic!*): though it is indeed the case that any object whatsoever may be a member of a set, talk involving '∈' has a genuine subject matter - the sets.[98]

Thus, it would be up to us to show that it is legitimate to include the set theoretic membership symbol as a logical constant. Showing this involves defining set theoretic membership in terms of the other, already accepted, symbols in a logical language. We have to do this on pain of violating the topic neutrality aspect of the generality criterion for logic.

The violation of topic neutrality by "∈" shows up in a more obvious sense when we consider the axioms of set theory. Axioms, at least partially, determine the use of a symbol, and in that sense

[98]Christopher Peacocke, "What is a Logical Constant?" The Journal of Philosophy, vol. LXXIII.9 (1976), p. 237.

give it an implicit definition. Thinking now in terms of the axioms of theories, rather than logical axioms: axioms are such that they are always true in a theory (short of a contradictory theory). So, a theory can be thought of as the study of the consequences of certain axioms. Under this view, axioms testify to the content or subject matter of a theory.[99] For example, the set theoretic axioms include the axiom of regularity, which precludes the existence of sets which are members of themselves or cycles of sets which are such that the membership relation cycles through them. That is, one set will be a member of a second set which will be a member of a third set and so on, to set n, and this in turn is a member of the first set. Traditionally, this has been viewed as a non-logical axiom. Why this offends logical scruples is that it violates universality. The axiom of regularity precludes the possibility of certain sorts of set or certain relations holding between a number of sets, and this offends against a sense of logical possibility which says that we do not contradict ourselves if we imagine that the universe might contain such sets. Contradiction only occurs when such sets are included in the set theoretic hierarchy, for instance. For this reason, the axiom is *ad hoc* relative to our characterisation of logic. If "$\in$" were considered to be a logical constant, and it were taken with its full set theoretic meaning (i.e. all the axioms involving "$\in$" would then be considered to be logical truths) then the axiom of regularity would be considered to be a logical truth. It would then be inappropriate to consider "irregular" sets or cycles of sets.

Another useful way of discussing the universality of applicability of a logic is in terms of invariance under permutations of the domain.[100] Under this conception of universality, domains of

[99]In the case of logical axioms, the content will then be formal properties of reasoning. Rather confusingly, the content of logic is logical form, under this view.

[100]Alfred Tarski, "What are Logical Notions", History and Philosophy of Logic, vol. VII, (1986), pp. 143 - 154, see also Gila Sher, The Bounds of Logic, (Cambridge, Massachusetts: MIT Press, 1991), pp. 36 - 66.

interpretation are logically the same just in case they are invariant under permutations of the members of a domain onto itself.

> ...[C]onsider the class of *all* transformations [permutations] of the space, or universe of discourse, or 'world', onto itself. What will be the science which deals with the notions invariant under this widest class of transformations? Here we will have very few notions, all of a very general character. I [Tarski] suggest that they are the logical notions, that we will call a notion 'logical' if it is invariant under all possible one-one transformations of the world unto itself.[101]

The "general notions" are those of the connectives, equality and cardinality notions, such as the universal and existential quantifiers. Furthermore, any notion we can define in terms of these also turns out to be a logical notion. Thus, all of our default logical constants pass the Tarski criterion for generality. The criterion rules out non-logical individual constants, relation constants and function constants. Arbitrary expressions will conform to the criterion of topic neutrality, just in case they do not contain non-logical constants in any position which is significant for their truth-evaluation. With the interesting exception of notions of cardinality, this falls very much in step with our thinking so far. Tarski's "general notions" turn out to be the logical constants, and cardinality notions.[102] On the other hand, this should not come as a great surprise since the logical constants are just those which remain invariant across domains anyhow, and therefore, could not possibly change under permutations of a domain onto itself. A logical constant is invariant in two closely related senses. One is that axioms governing the use of logical constants are true in all domains. The second sense in which logical constants are invariant is that the satisfaction conditions for the constants is fixed independent of any domain of interpretation. For example, the truth table for $P \vee Q$ is the same regardless of what propositions we

[101] Alfred Tarski, "What are Logical Notions", History and Philosophy of Logic, vol. VII, (1986), p. 149.
[102] These will be examined explicitly in chapter four.

substitute for P and Q. In contrast, "+", for example, will receive a different interpretation in a domain consisting in partially-ordered objects than in a domain where the objects are totally ordered.

The account of how the invariance of certain symbols in a language testify to the topic neutrality of that language is what is interesting. The account runs: the truth of sentences containing non-logical constants, in a position which is significant for their truth evaluation, depends upon reference to particular objects or concepts. Particular objects are not the business of logic. Logic is meant to be general in the sense of being indifferent to what subject matter it is being applied to.

This elicits an interesting point. Sometimes it turns out that we can define, in terms of a logical language, what we previously thought was not a logical notion. By means of such a definition we might discover that particular objects or concepts are not being referred to. Rather, they are logical objects, or logical notions. This discovery depends upon finding a definition of the object or concept which is written without using non-logical constants. For example, $\in$ is *prima facie* a non-logical constant. This is because if we replace it by another two-place relation, the truth-value of many sentences which contain $\in$ will change. This, in turn, is because set-membership is a particular sort of relation. Sets are not generally defined using only logical vocabulary. Were we to define "$\in$" in terms which did not include non-logical constants, then we would learn that "$\in$" is a logical relation after all. It will have earned its place among the logical constants.

In contrast, the number seven might be thought of as a particular object. Or rather, the judgement that a set is composed of seven objects, might be considered, *prima facie*, not to be a logical judgement. However, we subsequently find out that we can define the number seven in terms of already accepted logical vocabulary: in terms of the existential quantifier, variables, negation and equality. Thus, if we accept the existential quantifier, variables, negation and equality to be logical symbols, then the number seven is a logical notion. This is not to say that the existence of seven object is a logical truth, only that discussion of the number seven is perfectly general, and when mention of it modifies the truth of a sentence, this does not require that we pay attention to particular features of a domain. More will be said about what counts as a particular feature in the next section.

We now leave aside the quantifiers and logical connectives, and turn to the last member of our list of logical constants: "=". "=" is an interesting case. Reason for thinking that "=" is not a logical relation is that we may only come to know that two things are equal *a posteriori*. For example, that Hesperus is Phosphorus, was only discovered through empirical investigation. However, to be tempted by this sort of consideration is to confuse knowing that two names refer to the same thing with two things being equal. We avoid this temptation if we look to mathematics where "=" is either thought of as a licence for substitution or as an equivalence relation. That is, a relation which is reflexive, symmetric and transitive: $x \equiv x$, $x \equiv y \rightarrow y \equiv x$, $(x \equiv y \wedge y \equiv z) \rightarrow x \equiv z$. Either way of defining equality passes the Tarski criterion of remaining invariant under permutations of the domain. What is equal in one presentation of a domain, will be equal in a permutation of that domain. Passing the Tarski criterion is not enough to warrant "=" occupying a place in our list of logical constants, so let us leave "=" with the following provisional argument.

"=" can be defined in second-order logic. Other arguments for considering "=" to be a logical constant aside, if second-order logic is logic, then we have grounds for accepting "=" as a logical connective. This is because "=" will then be given the same interpretation in any structure, because it will have been defined using only logical vocabulary which remains invariant across domains, or is arbitrarily interpreted. Of course, if second-order logic turns out not to be a logic (but mathematics) then we either have to find some other reason for accepting "=" as a logical symbol, or we have to subtract it from our list of logical constants in first-order logic.

To summarise, why it is that we do not allow non-logical constants into the language of logic, is because, for a formal system to deserve the title "logic", we have to be able to apply it to any models we like. This is our criterion of universality. The symbols in a language are interpreted by objects, concepts and various sorts of connectives. The symbols which make up a language suitable for logic must either be commonly interpreted in any domain (logical connectives) or they must be capable of being arbitrarily interpreted, as in the case of descriptive vocabulary and individual variables. These last are necessary in order to allow us to apply logic at all, but logic is applied in a topic neutral way. This is the other aspect of our criterion of universality. This is what elaborates our idea that validity, for example, is considered to be independent of content, i.e. a matter of form, i.e. universally applicable. Similarly, *prima facie*, it is not a matter of logic that the universe includes (or excludes) certain objects: provided that denial (or affirmation) of their existence does not elicit contradiction. As we have seen, the conception of universality, understood as interpretation in any domain, is still not entirely satisfactory as a means of determining which symbols should be excluded from a list of logical constants.

We still have some unfinished business in characterising a logic from philosophical considerations, and these have to do with the axioms and truths of logic. These have to be analytic.

§3: Analyticity

One feature which is identified with logic is obviousness. When a logically valid inference has been made with known premises, we are compelled to accept the conclusion. Anyone who fails to accept it is thought to be justifiably accused of not fully understanding, perversion or madness. This is because ultimately we can break up logical inferences into smaller components each of which is, in itself, obvious. Of course, we end up with a few "primitive" components which we cannot further break up. These either take the form of axioms or rules of inference. These should be obvious in the case of logic: "...we select as axioms certain laws which we feel are evident from the nature of the concept involved."[103] What signals that something is "evident" is that there is no need for further elaboration.

It might appear dogmatic to claim that there are such rules or laws. For, one may argue, that we are owed some account as to why logical laws are evident since their obviousness seems, at best, just to report a psychological fact. In this sense logical laws may be obvious to us, but may not be obvious to someone or something else. This makes the obviousness contingent upon something outside logic, such as, culture, education, psychology, physiology, and so on. For this reason, obviousness can be a misleading guide to logicality. Obviousness has too strong a phenomenological connotation, as does the idea of being compelled to accept the conclusion of an argument. Thus, we still have to supply arguments for what it is we consider to be obvious or compelling.

One way philosophers have tried to deny that there is any contingency to logic is to say that logic is *a priori*, and analytic. It is *a priori* because we do not rely on facts gained from sense experience to recognise the validity of a proof. We rely on sense experience only insofar as this enables us to read a proof from a page or do some equivalent act. Put another way, there is no particular set of sense experiences with which we have to be familiar, before we can judge an argument to be valid.

Very close to this characterisation of *a prioricity* is analyticity. If a proposition is analytic, then it should reveal something internal to the concepts invoked in the proposition. It should not bring together disparate concepts. How to draw the distinction between

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[103]Joseph R. Shoenfield, Mathematical Logic, (Addison-Wesley Series in Logic; Reading Massachusetts: Addison-Wesley Publishing Company, 1967), p. 1.

what lies internal to a concept and what lies outside, is notoriously difficult to motivate, and proportionately difficult to defend.

Logic is thought to be analytic in the sense that logical truths, and definitions of logical concepts, do not introduce anything new. They reorganise and re-acquaint us with basic concepts. We do not need the "basic" or "primitive" concepts to be defined because they "should be so simple and clear that we can understand them without precise definition."[104] Instead, it is the whole body of sentences generated by the logic which act as a collective or implicit definition. In fact, logic just is, on this view, the study of the combination of a few ill-defined concepts with which we happen to feel very familiar.

To use Quine's metaphor, logical truths are located at the centre of our web of belief. For Quine, this brings us back to obviousness. For, it is merely descriptive of the fact that when a prediction given in science, say, is not met, we are least inclined to revise our logic or mathematical practice. It seems that, this is by implicit stipulation. Ayer seems to agree with Quine when he writes that "the principles of logic and mathematics are true universally simply because we never allow them to be anything else."[105] The sentiment expressed by this sentence sounds like it could have inspired Quine's "Two Dogmas of Empiricism".[106] For, preceding it, Ayer suggests that the necessity of logic and mathematics consists in just the fact that, should we find a situation where it appears as though the laws of logic or mathematics are contradicted, the last explanation we offer ("as it happens", for a deflationist such as Quine) is that logic is at fault. Instead, we say that we have miscalculated or there have been intervening circumstances which we failed to observe. Indeed, this might be read as a report on a

[104]Joseph R. Shoenfield, Mathematical Logic, (Addison-Wesley Series in Logic; Reading Massachusetts: Addison-Wesley Publishing Company, 1967), p. 1.
[105]Alfred Jules Ayer, Language Truth and Logic, (New York: Dover Publications Inc., 1935), p. 77.
[106]W. V. Quine, "Two Dogmas of Empiricism," The Philosophy of Language, ed. A. P. Martinich, (Oxford: Oxford University Press, 1985), pp. 26 - 39.

psychological, evolutionary, anthropological or genetic fact, which locates logic at the centre of our web of belief.

Call Quine's parochial reading of the basic axioms of logic "weakly necessary", since ultimately, it is left unexplained why it is that we favour logical concepts over others. Also, we cannot have any guarantee that they will remain in their favoured position in the future. We also lose any normative force we might wish to attribute to logic, since any "rational compulsion" we feel to accept the conclusion of a valid argument, would just be an artefact of psychology or whatever. Then, someone who does not feel such a compulsion is psychotic. Worse still, we cannot rule out the possibility that such a person might be vindicated by future communities. To end our explanation of the necessity of logic here, would be to embrace a more-or-less conventionalist position in the philosophy of mathematics, one which very naturally leads to instrumentalism. This is because, we might then seek to justify logic in terms of science, as a tool of science.[107]

A stronger sort of necessity is advocated by Ayer. For, he believes that our practice, of explaining apparent contradictions, last of all by revising our logic, or arithmetic, is meant to be taken as symptomatic not diagnostic. Part of the symptom is that the truths of mathematics and logic are analytic. He continues after the above quotation:

> And the reason for this [behaviour] is that we cannot abandon them [the basic laws of logic] without contradicting ourselves, without sinning against the rules which govern the use of language, and so making our utterances self-stultifying. In other words, the truths of logic and mathematics are *analytic propositions* or tautologies.[108] (Italics, mine)

[107] See, for example, Hilary Putnam, "Mathematics Without Foundations," Mathematics Matter and Method, (Philosophical papers, vol. I, Second edition; Cambridge: Cambridge University Press, 1979), pp. 43 - 59.

[108] Alfred Jules Ayer, Language Truth and Logic, (New York: Dover Publications Inc., 1935), p. 77.

That is, logic forms not only an internally coherent system, i.e. one free from contradiction, but also, it is one of which we perforce make constant use of, in our linguistic practice. Logic is a prerequisite to language in the sense that we could not recognise, as a language, something which is not underpinned by a structure or a set of rules; and those rules must be consistent.[109] Furthermore, the truths of mathematics and logic must be analytic, since they are justified by consistent and analytic axioms. Here, "recognise" is not to be read psychologically. Rather, it is a metaphysical point. It has to do with the nature of language itself and of features which it must possess necessarily to count as a language. The argument for this point is a transcendental one about the necessary conditions for language.

At this stage in our discussion, it is worth comparing Quine to Frege. While Quine's general position is one which we are trying to oppose, Quine and Frege use very different definitions of analytic. Of course, Quine tries to show that there are no stable analytic truths. Nevertheless, to show this, he still has to have a definition of the thing he tries to show does not exist, before he can show, that it does not exist. For Quine, a sentence is analytically true if it is true in virtue of the meaning of the words in the sentence and nothing else. In the light of the differences in the target of debate, it is not relevant to pit Frege directly against Quine. Nonetheless, from a Quinean perspective, we can still discern a challenge to the Fregean: to make the distinction between analytic and synthetic sentences clear enough that we can pick out, for a significant class of sentences, whether it is analytic or synthetic. Put in the terms of a question: if logic is to play a justificatory role, and it is analytic, then what is the difference between this sort of justification and a synthetic one? One might suppose, for example, that an analytic

[109] I am aware of the fact that I am ignoring logics which allow contradictions. In these, it will be noted, the damage wrecked by any particular contradiction is limited. What we do not have in these formal systems is a purely random language, there are rules, and one can have disputes over whether or not one is adhering to them.

justification is just a trivial, or tautological, justification. To answer this, it will be useful to turn to the debate between Frege and Kant over the distinction between analytic and synthetic propositions.

Kant's discussion of the analytic/ synthetic distinction provides a good foil for Frege's definition of analytic because Frege spends some time discussing his differences with Kant. For Frege, in marked contrast to Kant, arithmetic is analytic. Frege respects Kant as a philosopher and finds that some of the formulations he gives of the analytic/ synthetic distinction fall very close to his own. Others sound very different. The difference between the two conceptions is a subtle one to locate since it has to do with the scope and role of logic, so with the terms of the debate, rather than its substance.

In his Grundlagen Frege maintains, *contra* Kant, that arithmetic is analytic. He agrees with Kant that geometry is *a priori* synthetic. The truths of geometry are not based on logic alone, "everything geometrical must be given originally in intuition."[110] Later in his life Frege thought that arithmetic too is synthetic, since he thought it is based on geometry; but this does not concern us here. From the texts, it is unclear to what extent, geometry's being synthetic, is attributed by Frege to rest on Kantian spatial intuition.[111] That is, it is not clear how well Frege understood the Kantian doctrine of intuition or to what extent he endorsed the doctrine, as opposed to just accepting it by default. On the other hand, Frege certainly thought he largely agreed with Kant. "I [Frege] consider Kant did a great service in drawing the distinction between synthetic and analytic judgements. In calling the truths of geometry synthetic and *a priori*, he revealed their true nature."[112] Frege sums up his dispute with Kant by saying that Kant sometimes defined "analytic" too narrowly.

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[110]Gottlob Frege, The Foundations of Arithmetic, trans. J. L. Austin (Second revised edition; Evanston, Illinois: Northwestern University Press, 1980), § 64.
[111]Michael Dummett, "Frege and Kant on Geometry," Inquiry, vol. XXV (1980), p. 234.
[112]Gottlob Frege, The Foundations of Arithmetic, trans. J. L. Austin (Second revised edition; Evanston, Illinois: Northwestern University Press, 1980), § 89.

We shall begin with Kant's construal of the distinction. We shall not analyse this in detail, since careful exegesis of Kant is not as important for us as a grasp of what Frege thought Kant meant.

In B10/A7 of the Critique of Pure Reason, Kant relates analytic judgements to ones "thought through identity". That is, analytic judgements expose aspects of an object or concept which figure in the form of an identity between the subject and predicate of a sentence. He writes:

> Analytic judgements are... those in which the connection of the predicate with the subject is thought through identity; those in which the connection is thought without identity should be entitled synthetic. The former, as shedding nothing through the predicate to the concept of the subject, but merely breaking it up into those constituent concepts that have all along been thought in it, although confusedly, can also be entitled explicative. The latter, on the other hand, add to the concept of the subject a predicate which has not been in any wise thought in it, and which no analysis could possibly extract from it; and they may therefore be entitled ampliative.[113]

One obvious difference between Kant and Frege which emerges from the quoted passage is that Kant is still analysing sentences in terms of subject and predicate, whereas Frege analyses them in terms of function and argument. Frege's mode of analysis is much more expressively powerful, especially with respect to mathematical concepts, and this makes for a greater number of judgements being counted as analytic.

Let us focus on the terms ampliative and explicative. Often analytic judgements are glossed as trivial. This is even the case in Kant. On this conception, they cannot provide us with new information. Nevertheless, the construal of analytic judgements as explicative indicates that we can learn from them to some degree. The information we gain is limited. It is in the form of an explanation which does not go beyond the concept in question.

[113]Immanuel Kant, Critique of Pure Reason, trans. Norman Kemp Smith (Second impression; London: MacMillan Education Ltd., 1986), B10/A7.

Nevertheless, through analysing a concept we deepen our understanding.

To use Kant's example: "bodies are extended" is an analytic judgement. Any familiarity with the concept "body" will make the truth of the statement immediately apparent. Being extended is part of what it is to be a body. Kant contrasts "bodies are extended" with the judgement "all bodies are heavy". The difference between the two may not be obvious; but this is partly due to our modern conception: that mass, body and energy are inseparable since we see them as aspects of the same thing, due to a post-Kantian relativistic physical theory. Kant is a bit clearer on the difference when he elaborates on the two examples:

> ...it is evident: 1. that through analytic judgements our knowledge is not in any way extended, and that the concept which I already have is merely set forth and made intelligible to me;[114]

This is more conservative than what he says earlier and diverges markedly from Frege. Concerning the example of a synthetic judgement, Kant continues:

> 2. that in synthetic judgements I must have besides the concept of the subject something else (X), upon which the understanding may rely, if it is to know that a predicate, not contained in this concept, nevertheless belongs to it. ...For though I do not include in the concept of a body in general the predicate 'weight', the concept none the less indicates the complete experience through one of its parts; ...By prior analysis I can apprehend the concept of a body through the characters of extension, impenetrability, etc., all of which are thought in this concept. To extend my knowledge, I then look back to the experience from which I have derived this concept of body, and find that weight is always connected with the above characters.[115]

[114]Immanuel Kant, Critique of Pure Reason, trans. Norman Kemp Smith (Second impression; London: MacMillan Education Ltd., 1986), A8.
[115]Immanuel Kant, Critique of Pure Reason, trans. Norman Kemp Smith (Second impression; London: MacMillan Education Ltd., 1986), A8.

In the above example we draw on experience to test the truth of the claim. This makes the claim *a posteriori*, and *a fortiori* for Kant, synthetic. In the case of geometry, we draw on spatial intuition. Knowledge from intuition counts as *a priori* for Kant, since spatial and temporal intuitions are part of what enable us to make judgements about the physical world. "But intuition takes place only in so far as the object is given to us. This again is only possible, to a man at least, in so far as the mind is affected in a certain way."[116] Under this conception, the propositions of geometry are *a priori* synthetic.

Frege wanted to divorce arithmetic from the stain of the synthetic. Frege did not think that numbers do not have to be given to us in such a way as to engage our temporal intuition. Instead, Frege extends Kant's notion of analyticity through identity, to anything which follows from definitions plus logic. As was mentioned in the introduction, we favour "is justified by" to "follows from" because of the restrictive connotations of having an effective procedure at our disposal.

> The problem [of discovering the ultimate justification for a mathematical truth] becomes, in fact, that of finding the proof of the proposition, and of following it right back to the primitive truths. If, in carrying out this process, we come only on general logical laws and definitions, then the truth is an analytic one, ... If, however, it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the sphere of a special science, then the proposition is a synthetic one.[117]

The issue of effectiveness gained new impetus with Gödel's incompleteness results of the 1930's. The Gödel incompleteness results made it clear that the truths of formal systems could not all

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[116] Immanuel Kant, Critique of Pure Reason, trans. Norman Kemp Smith (Second impression; London: MacMillan Education Ltd., 1986), A19. Presumably, the "certain way" is one which engages intuition, because we have to use our intuitive faculty to recognise a given object.

[117] Gottlob Frege, The Foundations of Arithmetic, trans. J. L. Austin (Second revised edition; Evanston, Illinois: Northwestern University Press, 1980), § 3.

be proved by means of an effective (mechanical) system of deduction. We can forgive Frege his lack of foresight. What incompleteness indicates *vis-à-vis* Frege's definition of analyticity is that there is space between Frege's positive and his negative characterisations of analyticity. It is possible to justify a sentence without appeal to intuition or sense experience, and yet not be able to prove it using an effective method of proof! In such instances, we have to rely on semantic, as opposed to syntactic, justifications, so justifications in terms of models. I shall be calling "analytic" any sentence whose justification does not rely on sense experience or intuition, where the justification need not take the form of a mechanical proof. What I propose to do then, is focus on logic as a sort of justification, as opposed to thinking of logic as a tool of demonstration. For, I am interested in the very nature of logic, not in its adaptability to computers or in its usefulness as a learning tool. Thus, while the deductive system of the *Begriffsschrift* is certainly sufficient to show that a proposition is analytic, it is not necessary. In the light of the Gödel results, we must make a decision as to whether to favour the negative or the positive characterisation Frege gives to analyticity. Here we choose his negative characterisation, and the complement of this is considered to be synthetic. A gapless proof is sufficient for justification, but not necessary. After all, Frege's motivation for making arithmetic more rigorous was to establish the justificatory dependence of arithmetic on logic, as opposed to temporal intuition. This is why the negative aspect of his definition of analyticity is so important.

Relevant to this conception of rigor, is Frege's concern to rid logic of what he refers to as "special" or "foreign elements".[118] A proof is rigorous if it is gapless.[119] It demonstrates what it is, that

[118]William Demopoulos "Frege and the Rigorization of Analysis," Frege's Philosophy of Mathematics, ed. William Demopoulos (London: Harvard University Press, 1995), pp. 68 - 88.
[119]William Demopoulos "Frege and the Rigorization of Analysis," Frege's Philosophy of Mathematics, ed. William Demopoulos (London: Harvard University Press, 1995), p. 73.

the conclusion depends on. Demonstration aside, a justification is logical if it contains no foreign elements. Frege negatively characterises foreign elements as extra-logical notions; these are notions requiring experience or intuition in Kant's sense. They include assumptions about kinematics, or appeal to physical properties, for example. We make such reference, for example, when we describe equality between triangles in terms of rotating one triangle and moving it across the graph paper and superimposing it on another. This description is not a logical justification for the equality of two triangles, since it appeals to foreign elements, namely to notions belonging to kinetic theory. This sort of proof makes appeal to our spatial intuition.

Later on, in the Critique of Pure Reason, Kant explains how it is that we arrive at analytic judgements from a given concept. Here he is closer to Frege. He writes:

> I have only to extract from it [a given concept], in accordance with the principle of contradiction, the required predicate, and in so doing can at the same time become conscious of the necessity of the judgement - and that is what experience [or intuition] could never have taught me.[120]

Analytic judgements are formed by breaking up a concept. This breaking up, or analysing, relies entirely on logic; expressed above in terms of the law of non-contradiction. We come to a second explicit difference between Kant and Frege. This has to do with the scope of the law of non-contradiction; or what is here considered the same thing: the scope of logic.

The formal logic available to Kant was Aristotelian syllogistic logic.[121] Frege worked with a very flexible and powerful formal system which he considered to be logic. So, we ask: would Kant recognise a justification, which might take the form of a proof, in

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[120] Immanuel Kant, Critique of Pure Reason, trans. Norman Kemp Smith (Second impression; London: MacMillan Education Ltd., 1986), B12.

[121] There is question in the literature whether Kant might not have had something broader in mind. This is not really relevant here because whatever he had in mind was always much weaker than Frege's logic. For one thing, Kant is still analysing sentences in terms of subject and predicate instead of function and argument.

second-order logic, as being analytic? Put another way, if Kant were introduced to second-order logic and were he to accept it as logic, would he then have recognised the extension of his definition to include all that can be derived from a concept using logic plus definitions, as Frege would have it? If the answer is in the affirmative, then from what we have discussed so far, the only differences in the characterisation of analytic judgements as given by Kant and Frege, are in extension, i.e. in what counts as logic, and in the analysis of propositions in terms of function and argument, as opposed to subject and predicate.

Another interesting question we might anachronistically ask Kant is whether the truths of second-order logic are synthetic, in the sense of being justified by intuition. After all, we know that second-order logic is incomplete. This implies that all the truths of second-order logic are not theorems: that is they cannot be generated by algorithms for proof procedures. This suggests that there will be cases where demonstrating[122] that a given sentence is true or false, does not only involve generating a syntactical proof. We may have to invoke models to demonstrate the truth or falsity of a sentence. Coming up with such a demonstration requires ingenuity. We might then think that this ingenuity is, to the intents and purposes here, just as bad as Kantian spatial and temporal intuition. That is, the ingenuity is only reducible to some intuitive faculty.

In good Fregean tradition, let us distinguish how it is that we come up with a proof or demonstration, with the ultimate justification for the conclusion of our proof or demonstration. The coming up with a proof may well require ingenuity, and this may well be thought of as drawing on some intuitive faculty. However, what counts as a logical truth for the logicist is one which receives an ultimate justification. This is one which we can explicitly

[122]"Demonstrate" is being used as a loose word for "proof" in that proofs have to follow a rigid procedure, and rely solely on syntactical devices. Demonstrate includes syntactical proofs and semantical proofs: ones which rely on our semantic understanding, or more precisely: on models being invoked as evidence or counter-evidence for a claim.

demonstrate only relies on logical laws, logical rules of inference and definitions. That is, we want to be able to recognise that the justification for the claim that a given sentence is true or false is a logical justification; even if we are not guaranteed in advance to be able the come up with such a justification. A demonstration can be gapless, in the sense of being explicit about all the assumptions which are made and all the definitions which are invoked. A gapless demonstration can be logical just in case all the assumptions and definitions turn out to be logical.[123] A logical and gapless demonstration can be analytic in the sense of not requiring of someone following the demonstration that they draw on sense experience or Kantian intuition. There is no reason to think that there could not be such demonstrations.

However, these are not the only differences between Kant and Frege. In other passages, the difference between Kant and Frege, has to do with the role and potentialities of logic. For Kant, analytic judgements can, at best, clear up confusions. They are merely explicative, not ampliative. Frege disagrees with this.

For Frege, analytic truths may be revelatory.[124] They include anything which can be said at a purely general level about a concept. The point is that often the truths we deduce from a given set of assumptions "cannot be inspected in advance".[125] It is because we then "extend our knowledge",[126] that Kant could argue that a given proposition is synthetic. Yet, for Frege, we can extend our

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[123]This is not logical only in the sense of being analytic, but in the sense of meeting our three criteria: that a logic should reflect an informal notion as to what constitutes logical validity, that the truths and axioms should be universal and that understanding claims in the logic should not rely on sense experience or Kantian intuition.

[124]William Kneale and Martha Kneale, The Development of Logic, (Oxford: Clarendon Press, 1988), pp. 435 - 47.

[125]Gottlob Frege, The Foundations of Arithmetic, trans. J. L. Austin (Second revised edition; Evanston, Illinois: Northwestern University Press, 1980), § 88.

[126]Gottlob Frege, The Foundations of Arithmetic, trans. J. L. Austin (Second revised edition; Evanston, Illinois: Northwestern University Press, 1980), § 88.

knowledge by purely logical means. Dummett aptly sums up Frege's insight:

> Frege is virtually the only philosopher who both recognised the power of deductive reasoning to yield knowledge that we did not previously possess and tried to explain what gave it this power.[127]

For Frege it is crucial that logic is powerful and revealing. This is why it was important, for Frege, to free logic from the "legend of sterility". The richness and power of Frege's logic is what justifies his view that logic is not sterile, and explains the power of deductive reasoning to provide new knowledge. If logic (philosophically characterised) were best exemplified by a very weak formal system, deductive reasoning would not have such power. Second-order logic has great expressive power, and yet it is based on a small vocabulary and a small number of axioms[128] and rules of inference.

What is it then, that makes these axioms analytic as opposed to the axioms of geometry, which are synthetic? As an interesting aside, we can see Frege's attempt to derive the Peano axioms from logical axioms as an attempt to remove any doubt as to whether or not they rest on temporal intuition. Hume's principle, because it presupposes the notion of number, also had to be proved, in order to show that it does not rely on temporal intuition. By means of Hume's principle, "the concept of Number has not yet been fixed."[129] This situation is reminiscent of the one we were left with at the end of the last section. If we can define a *prima facie* non-logical constant in terms of vocabulary which does not include non-

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[127]Michael Dummett, "The Justification of Deduction," The Logical Basis of Metaphysics, ed. Michael Dummett (Cambridge, Massachussetts: Harvard University Press, 1991), p. 195.

[128]Second-order logic is unaxiomatizable. That is, whatever set of axioms we propose, this will be incomplete. This does not, however, detract from the fact that to justify any particular assertion written in the language of second-order logic, we need only appeal to a finite number of axioms, and that in practice, we use very few axioms.

[129]Gottlob Frege, The Foundations of Arithmetic, trans. J. L. Austin (Second revised edition; Evanston, Illinois: Northwestern University Press, 1980), § 63.

logical constants, then we can accept the symbol as a logical symbol. Similarly, if we can justify a sentence by means of logical axioms and logical definitions then the sentence is analytic in our sense.

This may sound odd. We usually speak in terms of analytic truths, not just analytic sentences. I am allowing analytic falsehoods in the sense that "bachelors are married men" is an analytic falsehood. The sentence is false, and what makes it false are the definitions of the words, not some empirical fact. Similarly, I have at times been lax in my use of language and talked of notions as being analytic. What I mean by this is that grasp, understanding, or knowing the notion does not require appeal to sense experience or to intuition.

Returning to the analyticity of the logical axioms: one way of arguing for this, is in terms of the component parts of the axioms. First-order logic is analytic because it is written in a logical language. The language is analytic because it does not have the resources to exploit our sense perception or intuitive faculties, in Kant's sense. We break with this constraint on a language, *prima facie*, when we add mathematical constants to the language such as +, 0, ε, point, line, plane, and so forth. If something is a logical truth then it can be expressed in a logical language. This does not imply that all analytical truths are so expressible or that logical languages only express logical truths, they also express logical falsehoods and just sentences which are sometimes true and sometimes false. This is not quite enough. For, showing that the axioms of logic are expressible in a logical language shows that the logical axioms are analytic, but not that they are true. To justify them as true, we have to show that they are universally applicable. In particular, we have to argue that there is no domain in which they are false.[130] Within the classical tradition, all the axioms of first-order logic are accepted as being universally applicable. The true sentences of first-order

[130]Such arguments tend to come from constructivists, intuitionists, relevance logicians, or logicians using quantum logic. These arguments lie outside the scope of this thesis, and I shall not address them here.

logic are all either axioms or they are justified by the axioms plus logical definitions. The definitions are logical definitions in the sense that they are all a shorthand for expressions written using only logical symbols.

To summarise, Frege's conception of analyticity exhibits several features. The positive characterisation is that a proposition is analytic if it follows from (is justified by) logical axioms plus definitions expressed in a logical language. Justifying is to be distinguished from proving by means of an effective deductive procedure. Analytic judgements are explicative and informative. The negative feature of analytic propositions is that their truth is determined without appeal to either sense experience or to intuition, in Kant's sense. The propositions of a formal system display analyticity just in case they display these features.

In the next chapter, we shall examine some of the limitative results of first-order logic, since it is upon the basis of these that the first-order thesis tends to be defended. We shall judge whether or not each limitative result is salient to our characterisation of a logic: as a formal system which is universal, reflects our informal notion of logical validity, and whose truths are analytic.

Chapter III: Limitative Results of First-Order Logic

Introduction

Thanks to Lindström we have a precise characterisation of first-order logic. Lindström describes first-order logic as the most powerful logic which (a) has the downward Löwenheim-Skolem property and (b) either is compact or has the upward Löwenheim-Skolem property.[131] The last is sometimes referred to as the Tarski property. A logic, L, has the downward Löwenheim-Skolem property if every theory T (a possibly infinite set of sentences) in the language of L, if T has a model of whatever cardinality κ, where $\kappa > \aleph_0$, then T has a countable model.[132] The upward Löwenheim-Skolem property is: for any theory T, in the language of L, if T has a finite model, then T has a countable model. A logic, L, is compact when for every theory T in the language of L, if every finite subset of T is satisfiable, T is satisfiable.[133]

When Lindström's characterisation of first-order logic was published, in 1969, it was considered surprising, since first-order logic was defined in terms of its syntax and semantics, not in terms of characterising results. Due partly to the influence of abstract model theory,[134] we have grown accustomed to the Lindström characterising results acting as a stipulative definition of first-order logic. This is all very well, except that implicitly a move has been made, which is widely accepted, from the descriptive to the normative claim that not only do the Lindström results describe first-order logic but they delimit the scope of logic *tout court*.

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[131] Per Lindström, "On Extensions of Elementary Logic," Theoria, XXXV (1969), p. 8.

[132] Joseph R. Shoenfield, Mathematical Logic, (Addison-Wesley Series in Logic; Reading Massachusetts: Addison-Wesley Publishing Company, 1967), p. 79.

[133] George S. Boolos and Richard C. Jeffrey, Computability and Logic, (Second edition; Cambridge: Cambridge University Press, 1980), p. 140.

[134] This is a fairly recent development in which it is natural to compare logics by means of their characterising results.

Following Wolenski,[135] I shall refer to this as the "first-order thesis". The better arguments for the first-order thesis deal piecemeal with the limitative results we shall be investigating, and argue that these are necessary for a formal system to qualify as a logic.

There are two considerations which are salient to this thesis. One speaks in our favour, and concerns compactness and the Löwenheim-Skolem properties. These entail the expressive inadequacy of any formal language which has them, and this in turn, bears on the inability of a formal system to reflect our informal notion of logical validity.

The other consideration speaks against us, and concerns compactness, completeness and decidability. These properties speak in favour of the first-order thesis. For, compactness, completeness and decidability entail, of a logic which has them, that it has an effective proof procedure. In support of the first-order thesis, it is argued that effectiveness is more important than expressive adequacy. I shall be arguing for the opposite conclusion: that effectiveness is worth sacrificing for the advantage of expressive adequacy. Throughout this chapter we shall be applying our criteria of validity and universality to these results in order to show that the first-order thesis does not hold.

There are three sections in this chapter: (1) compactness, (2) the Löwenheim-Skolem properties, and (3) decidability and completeness.

§1: Compactness

Let us begin by saying what compactness is. A logic, L, (a logical language together with any domain) is compact when for every theory T in the language of L, if every finite

[135]Jan Wolenski, "In Defence of the First-order Thesis," Logica '93 Proceedings of the 7th International Symposium eds. P. Kolár, V. Svoboda, (Praha, 1994).

subset of T is satisfiable, T is satisfiable. Put this way, compactness is a corollary of completeness.

The proof is very simple. Contrapose compactness and we have that: for every theory T in the language of L, if T is not satisfiable, then there a finite set of sentences G which is also not satisfiable, and G is a subset of T. By completeness, if T is not satisfiable, then T contains a contradiction. By completeness again, the proof a contradiction is finite. The proof's being finite implies that it only has a finite number of premises. Thus, there is a subset of T which proves that T contains or implies a contradiction. By soundness (which is part of completeness) there is a finite subset of T which is unsatisfiable.[136]

There are formal systems which are not compact, they will also be incomplete in the sense given above. For compactness to fail, the antecedent of the conditional must hold while the consequent does not. That is, for L not to be compact, there must be a theory T in the language of L whose every finite subset is satisfiable, but T as a whole is not.

For example, second-order logic is not compact. To show that a logic is not compact, we simply have to add a sentence, G, which is only satisfiable in finite models. For example, we might add the sentence, G:

$$(\forall f)(\neg(\forall x)(\forall y)(fx = fy \rightarrow x = y) \vee (\exists x)(\forall y)(fy \neq x)).$$

The second disjunct asserts that there is, what we might call, a zero element: one which is not an f of anything. The first disjunct says that it is not always the case that if two individuals are equal under f, then they are equal. Thus, either there is a zero element, or two things identical under f are not then also always identical. Intuitively, for a set to be finite, the "ends" either have to just stop (as in a zero element) or end in a finite loop: $\neg(fx = fy \rightarrow x = y)$. The

[136]The proof is cribbed from Stewart Shapiro, Foundations Without Foundationalism, (Oxford Logic Guides: 17, Oxford: Clarendon Press, 1991), p. 79.

sentence is written in second-order because f is universally quantified; and it is this which guarantees that the sentence is only satisfied by finite sets.

We add G to an infinite set, M, of sentences asserting that the model contains at least two members, at least three members, at least four, and so on for every finite number:

$$\{(\exists x_1)(\exists x_2)(x_1 \neq x_2), (\exists x_1)(\exists x_2)(\exists x_3)(x_1 \neq x_2 \wedge x_2 \neq x_3 \wedge x_1 \neq x_3), \ldots\}$$

For every proper subset of this set of sentences together with the sentence G, there is a model: one of finite size equal to, or greater than, the greatest number of existential quantifiers listed in any member of the proper subset of sentences of M. However, no models will satisfy all the sentences M together with the sentence G! Thus, second-order logic is not compact. G can only be expressed in a language which allows quantification over higher-order variables.[137] However, some logics which allow quantification over second-order variables are compact. This will be the case, for example, when we have surreptitiously precluded conditions for non-compactness either by specifying a finite language (as opposed to a countable language), or by simply specifying that the language is only to be interpreted by finite models.

We direct our attention now to some philosophical considerations about compactness. The language of a logic, as we have defined it, confers upon the theories which use that language, the same limitative results as has the logic. To be more precise, what we mean by "using a language" is that given sentences are written in the language of the logic. What is important about this is that the limitative results of a theory are the same as those of its underlying logic.

Some of these results will only involve the language, and be syntactic; others will involve both semantics and syntax, and

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[137] Stewart Shapiro, Foundations Without Foundationalism, (Oxford Logic Guides: 17, Oxford: Clarendon Press, 1991), pp. 86 - 7.

therefore, the whole logic or theory. Examples of syntactic limitative results are decidability and the Löwenheim-Skolem properties. Examples of limitative results involving both semantics and syntax are compactness and completeness. One reason why these results are called limitative is that they indicate limitations in the capacity of the logics to either express certain notions or to detect certain features. This can be turned into an advantage.

Compactness can seem like a good property for the language of a formal system to have. The compactness of a language opens up the possibility of studying non-standard models. This is because compactness allows for the existence of non-standard models which are not isomorphic to the standard model. That the standard model and the non-standard model are not isomorphic can only be shown in a second-order language. When we study non-standard models, we learn how imprecise (relative to a meta-linguistic characterisation) our object level language is. This has proved very fruitful. The most notable examples are studies of non-standard arithmetic and non-standard analysis. However, this advantage of compact languages over non-compact languages is a mathematical advantage: one leading to interesting mathematical theories, and so, is not strictly of interest to us here. On the other hand, under exactly the same mathematical considerations, we find a disadvantage.

The very fact that compactness can be used to construct unintended (non-standard) models implies that the language has limited expressive power. As we shall see, this arguably compromises the ability of the logic to reflect our intuitive notion of logical validity. That is, it will fail to pick out some arguments as valid which we (mathematicians, in particular) intuitively think of as logically valid. For example, the first-order theory of arithmetic together with elementary inessential extensions (by the addition of certain harmless constants to the language) under-determines its models. This is considered to be a fault in first-order arithmetic.

A non-standard model is one which we did not intend or expect. It looks different from the standard model, but still manages to satisfy the axioms of the theory. For example, in the theory of first-order arithmetic, we have an intended domain of study, namely: the natural numbers. We then develop a formal language to generate the valid formulas. We intend our syntax to describe the operations of addition, multiplication and so on. In the case of first-order arithmetic, the syntax succeeds in doing this, but it does not determine its model uniquely up to isomorphism. We do not have only isomorphic copies of the natural numbers modelling the theory of first-order arithmetic.

This is a direct consequence of the compactness of first-order logic. For, using compactness, we can generate a non-standard model of first-order arithmetic which has the same cardinality as the set of natural numbers, i.e. it has $\aleph_0$ members. However, it has a different structure from the natural numbers. This is the sense in which the non-standard models look different from the standard model of the natural numbers. Note that there are also non-standard models of first-order arithmetic which have a greater cardinality. This is a consequence of the upward Löwenheim-Skolem property and compactness.

The compactness of first-order logic also entails that the notions of finitude and Dedekind infinity are not expressible in first-order logic. This is why we say that finitude, or Dedekind infinity are second-order notions. It turns out that if the language of the theory of arithmetic is second-order, that is, if we allow second-order quantification: over predicates relations and functions; we then find that the second-order theory of arithmetic, together with inessential extensions, can determine its models uniquely up to isomorphism. Thus, any two models of second-order arithmetic are isomorphic. Because it has the expressive resources to pick out the natural numbers, as the model of the theory, second-order arithmetic is considered to be a better theory than first-order arithmetic. As we shall see in the next section, when we discuss Putnam's Skolemisation argument, this is not as straightforward as it might appear. This is because recognition that a theory determines its models uniquely up to isomorphism depends directly on features of the meta-theory or meta-language.

Exploiting the *rapprochement* between a theory and its underlying logic, lies an argument which is often summoned on the side of those defending second-order logic over first-order. It is that certain notions, such as finitude, can be expressed in second-order logic but not in first. In first-order logic we can express particular finite numbers. For example, if we want to say that there are exactly three objects we write:

$$(\exists x)(\exists y)(\exists z)(x \neq y \wedge y \neq z \wedge x \neq z \wedge (\forall x')(x' = x \vee x' = y \vee x' = z))$$

That is, there exist three things, they are distinct and any fourth thing is not distinct from the three, i.e. there is no fourth. All finite numbers are similarly expressible. However, we cannot even express the more general thought that there are a finite number of objects without naming a particular finite number. In first-order logic, what we can do is write formulas which are only satisfied in an infinite domain, for example:

$$(\forall x)\neg Rxx \wedge (\forall x)(\exists y)Rxy \wedge (\forall x)(\forall y)(\forall z)(Rxy \wedge Ryz \rightarrow Rxz).$$

For this sentence to be satisfied, it requires of R that R be anti-reflexive, not have an end point and be transitive. Only an infinite domain will satisfy a relation, R, like this. Thus, there is a sense in which we can capture the notion of infinity in first-order logic. However, what we have not done is capture the notion of Dedekind infinity: that a set is infinite if it has a proper subset which can be placed into one-to-one correspondence with the original set. To do so, we need second-order quantification. We need to assert that there exists a proper subset which can be placed into one-to-one correspondence with the original set. Here is the definition of Dedekind infinite. A set U is Dedekind infinite iff:

$$(\exists U')((\forall x)(U'x \rightarrow Ux) \wedge (\exists y)(Uy \wedge \neg U'y))$$
$$\wedge (\exists f)(\forall x)(\forall y)(Ux \wedge Uy \wedge (fx = fy \rightarrow x = y))$$
$$\wedge (\exists x)(Ux \rightarrow U'fx).$$

There are three major conjuncts each on a different line. The first major conjunct says that there is a proper subset of U, call it U'. This is, a set all of whose members are in U, but there is at least one member of U which is not in U'. The second conjunct says that there exists a one-to-one mapping between the members of U and those of U'. The third conjunct says that the mapping is to a subset of U. For example, consider the natural numbers, and a proper subset, such as the set of natural numbers without the number three. These can be placed into one-to-one correspondence. Just match one to one, two to two, three to four (in the proper subset) four to five, five to six and so on. Because these can be matched one-to-one and one set is a proper subset of the other, the set of natural numbers is Dedekind infinite. If we negate the definition of Dedekind infinity, we get the general notion of finitude of the set U fully defined:

$$\begin{aligned}&\neg((\exists U')((\forall x)(U'x \rightarrow Ux) \wedge (\exists y)(Uy \wedge \neg U'y))\\&\wedge (\exists f)(\forall x)(\forall y)(Ux \wedge Uy \wedge (fx = fy \rightarrow x = y))\\&\wedge (\exists x)(Ux \rightarrow U'fx)).\end{aligned}$$

That is, finite is just the negation of Dedekind infinite which is that a set is finite if it has no proper subset which is of the same cardinality as the original set. The part which requires second-order quantification is the function whose value is the subset. The greater expressive power of second-order logic over first-order logic speaks in favour of second-order logic.

While this is a very nice consideration, it is not enough to show that second-order logic is logic, only that it has greater expressive power than first-order logic. In particular, we can express many mathematical notions using a second-order language. Thus, as it stands, the argument is of no use to us, since in our role of logicist, we want to show that some of mathematics is really logic. Given this aim, we are not entitled to claim that the validity of arguments which rely on the capturing of certain mathematical notions is logical and intuitive. This would simply beg the question. The expressibility of Dedekind infinite or finitude is not *prima facie* an advantage we want to attribute to a logic. *Prima facie*, it is only a mathematical advantage. However, we might retort that nevertheless, finitude is expressible in the language of second-order logic and, therefore, employs only logical notation. That is also not enough to make finitude a logical notion. For, we first have to show that second-order logic is logic according to philosophical criteria.

So, why is expressive power interesting to the philosopher? Before we answer this, we should first distinguish between a logical truth and a logical notion. The latter is a notion expressible in a logical language. For example, the capacity of a language to express finitude does not make the finitude of the universe into a truth of logic. For, there are infinite domains in which the sentence expressing the finitude of the universe is false. However, we should not be too quick in drawing conclusions. In a logical language we can write sentences which are true (in all domains, such as axioms), sentences which are always false, and sentences which are sometimes true and sometimes false. The latter are notions, the grasp of which, does not depend on intuition and sense experience. What would characterise such notions as logical is that they can be expressed in already approved logical vocabulary. We are then left with the task of arguing for the logicality of a vocabulary.

Similarly, the validity of arguments which make reference to non-logical notions, might be logically valid, provided their validity does not depend on the particular notions involved. For example, the notion of Dedekind infinity could figure in an argument whose validity depends on a logical manipulation of the expression rather than on a grasp of the notion of finitude. This will be realised, provided that the perceived validity is a matter of the form and not the content of the argument, and form is identified with structure and content is identified with particular domains. Consider the argument:

<u>There is a set composed of three objects,</u>

There is a Dedekind finite set.

If we accept that Dedekind finite is expressible using only logical vocabulary, then showing that the set composed of three objects is

Dedekind finite is just a matter of giving a syntactical proof where we show that the set composed of three objects complies with the definition of Dedekind finite. Thus, the question as to whether or not the validity of the argument is logical validity, depends upon our accepting that the definition of Dedekind finite can be written using only logical vocabulary. The (logical) validity of the argument does not depend on our believing the universe to be finite, or that Dedekind-finite domains exist, or that even a three membered set exists. The distinction between a logical truth and a logical notion is that a logical truth is true under any interpretation. A logical notion is one which can be expressed using only logical vocabulary. A notion can be expressible in a logical language without being always true. A logically valid argument is one which relies on the form of the argument and not the content. That is, it relies on showing that whatever models the premises will also model the conclusion, without making any statement as to the existence of said models. The way in which we have drawn the form/ content distinction is enough to eliminate from competition for the title logic, any language with non-logical constants, or distinguished elements. It is not enough to favour, say, first-order logic over second.

Nevertheless, we do want a logical language to have the capacity to discuss the models with a certain accuracy. The inability of a compact language to characterise structures uniquely up to isomorphism reveals an expressive inadequacy. Insofar as this figures in our intuitive notion of logical validity, the compactness of a language is not a desirable property for the language of a logic to have.

Reason to think that the capacity of a language to characterise structures uniquely up to isomorphism, does accord with our intuitive understanding of where the boundary lies between form and content, is that this describes mathematical practice. "The logic of a mathematical theory", for a mathematician, just is the underlying language together with some logical rules of inference. In practice, a logical language must be one which does not discriminate between members of the domain of objects of a theory. There are other conditions as well. What "discriminate" means in this context has to do with our other two criteria: universality and analyticity; and will receive further treatment in chapter four.

We can leave validity and compactness with the provisional conclusion that insofar as the language of second-order logic is a logical language, in the sense of not containing any non-logical constants, any argument expressed in this language will either be shown to be logically valid or logically invalid.

In contrast, the compactness of first-order languages plays against, at least the mathematician's, intuitive notion of logical validity, since they lack the capacity to characterise structures uniquely up to isomorphism. What we have yet to do is give a philosophical justification for the mathematician's intuitions and practice.

Compactness does not bear directly on the issues of analyticity or universality. The compactness of a logic does not restrict the domain we are allowed to consider, it just tells us something about domains relative to a language. Compactness also does not indicate of a language that it fails with regard to topic neutrality. The language of first-order logic does not include any vocabulary whose application is to particular objects or to particular domains. As for analyticity: the understanding of sentences written in the language of first-order logic does not require appeal to sense experience or to intuition. So in particular, the compactness of first-order logic does not indicate that sentences in first-order logic are synthetic. In chapter two we saw arguments for accepting some of the symbols of the language of first-order logic as symbols properly belonging to logic. The remaining symbols were left with a promissory note that arguments would be given in chapter four. If the arguments of chapter four are good, then as a corollary, we have that any notion expressed in the language of first-order logic is analytic. There may, of course be other notions which are analytic too, which are not expressible in the language of first-order logic.[138]

§2: The Löwenheim-Skolem Properties

There are two theorems associated with the names Löwenheim and Skolem. One is the downward Löwenheim-Skolem theorem the other is the upward Löwenheim-Skolem theorem. The first was announced by Löwenheim in 1915, and was later given a more elegant proof by Skolem. The second theorem looks like an inverse of the first. That is why it is associated with the names of Löwenheim and Skolem. However, it was proved by Tarski.[139] Since neither result holds of all logics, and since it is a type of meta-theorem, being formed in the meta-language of a theory, we shall refer to systems or logics as having, or not, the upward and downward Löwenheim-Skolem properties, respectively. We are using the term logic, here, just to denote a language with, *prima facie*, no non-logical constants. A logic has the downward Löwenheim-Skolem property or the upward Löwenheim-Skolem property, just in case the downward Löwenheim-Skolem theorem or the upward Löwenheim-Skolem theorem, respectively, are true of the logic.

[138]There are analytic notions which are not captured in logical language at all such as: bachelors are unmarried men. I am not concerned with these here, since the interest in analyticity is one of the interest in the analyticity of logic, in contrast to the synthetic aspect of mathematical theories which lie outside the scope of logic. These are mathematical theories such as Euclidean geometry, as it is traditionally presented. This is classed as synthetic because grasp of its truths seems to irreducibly require Kantian spatial intuition. If we were to show that there was another way of justifying Euclidean geometry: which did not rely on Kantian spatial intuition, (or empirical facts) then we would have shown that Euclidean geometry too is analytic.
[139]Helena Raisowa, An Algebraic Approach to Non-classical Logics, (Amsterdam: North Holland, 1974), p.?

> The downward Löwenheim-Skolem theorem states: if L is a logic with a countable language, then for any theory T of L having a model, T has a countable model.[140]

A logic has a countable language if its language has less than or equal to $\aleph_0$ symbols which are not logical constants. These include individual variables and predicate or function letters. In other words, if a logic has a countable language and the Löwenheim-Skolem property, then we know that if it has a model, however big, we also know that it will have a countable model. A logic L has the upward Löwenheim-Skolem property iff:

> If L is a logic with a countable language, then for any theory T of L, if T has an infinite model, then for any cardinal κ, $\kappa >$ countable, T has a model of cardinality κ.

The two properties are not co-extensional. That is, there are some logics which have the downward Löwenheim-Skolem property, but not the upward Löwenheim-Skolem property, and other logics which have the upward Löwenheim-Skolem property but not the downward Löwenheim-Skolem property. For example, weak second-order logic, is undecidable, is not compact, does not have the upward Löwenheim-Skolem property, but it does have the downward Löwenheim-Skolem property![141] Weak second-order logic is first-order logic augmented by a restricted list of relation and function variables which we can quantify over. The restriction is that these second-order variables should only have finite extensions. This is what distinguishes weak second-order logic from second-order logic. In second-order logic, all subsets of the powerset of the domain fall within the range of the predicate, relation and function variables.

What the downward and upward Löwenheim-Skolem properties show, of first-order logic is that, for example, the first-

[140] Joseph R. Shoenfield, Mathematical Logic, (Addison-Wesley Series in Logic; Reading Massachusetts: Addison-Wesley Publishing Company, 1967), p. 79.
[141] J. Donald Monk, Mathematical Logic, (New York: Springer-Verlag, 1976), pp. 488 - 9.

order theory of arithmetic, has non-standard models of cardinality greater than $\aleph_0$. That is, we can construct models with different cardinalities which all satisfy the same sets of sentences of the theory. By implication, these will then also have a different structure; think of the difference in structure between the natural numbers and the reals. This leads directly to the Skolem paradox, which reveals the inability of a logic, which has the Löwenheim-Skolem property, to characterise models for its theories uniquely up to isomorphism. Skolem took the paradox to be an argument for the relativity of set theory. I take this to be a *reductio*: concluding that a logic with the Löwenheim-Skolem properties is deficient in its expressive power.

Skolem's paradox is traditionally run on the downward Löwenheim-Skolem theorem rather than the upward Löwenheim-Skolem theorem. However, it can also be run on the latter. Skolem ran the paradox on first-order Zermelo-Fraenkel set theory, since this set theory was proposed by Skolem as a foundation to mathematics. The conclusion he drew was that the foundation of mathematics, i.e. set theory, is relative. For anyone harbouring foundational concerns, this was very dramatic.

The Skolem paradox runs as follows. The Löwenheim-Skolem theorem implies of first-order Zermelo-Fraenkel set theory (henceforth: ZF1) that if it is consistent, it has models of a size smaller than the size of sets which the theory claims exist. The puzzle is: how can a set of cardinality κ fit into a model of cardinality λ when $\kappa > \lambda$?

More specifically, ZF1 is essentially first-order logic plus the relation symbol $\in$, which we read as membership. With this relation we introduce some axioms and rules of inference governing sentences where $\in$ figures as the major relation, i.e. rules and axioms concerning set construction. Since ZF1 has the Löwenheim-Skolem property, if it is consistent, then it has a denumerable model. However, within ZF1 we can prove the existence of cardinalities greater than $\aleph_0$. In particular, we can carry out

Cantor's diagonal argument which proves that the size of the continuum is strictly greater than the size of the set of natural numbers. Since we can prove this as a theorem of ZF1, there is a consequence of the theory to the effect that the continuum exists and is strictly greater than $\aleph_0$. But this is true in a denumerable model, i.e. a model of size $\aleph_0$. The continuum has a cardinality greater than that of the natural numbers, but there are not enough elements available in the domain of the theory to instanciate this: hence the paradox.

In ZF1 we can define or characterise the notion of a one-to-one and onto mapping, and it is this definition that gives us our notion of size, or cardinal equivalence in set theory. So, when we prove that the continuum is strictly greater in size than that of a denumerable set, we show that there is no mapping which is one-to-one and onto between the set we are calling the continuum and the natural numbers. Thus, according to our definition of size, the two sets are of different cardinality. Since the cardinality of the continuum is strictly greater than that of the natural numbers, the set satisfying the sentence asserting the existence of a set of cardinality the size of the continuum, must exist outside some models of ZF1.

However, we should look closely at the evidence for the existence a continuum large set. What testifies to this is the absence of our size comparison mapping.[142] That is, a one-to-one and onto mapping provably cannot be constructed between (a) a supposed set satisfying the sentence which is only satisfied by the continuum, and (b) the smallest model which satisfies the theory (i.e. which models the theory). It is the absence of a mapping which indicates a discrepancy in cardinality between the two sets.

However, this is an artefact, and limitation in expressive power, of the first-order language of Zermelo-Fraenkel set theory. In

[142] For a perspicuous discussion of this issue see Michael Hallett, "Putnam and the Skolem Paradox," Reading Putnam, eds. Peter Clark and Bob Hale, (Oxford: Basil Blackwell Ltd., 1994), pp. 66 - 97.

second-order Zermelo-Fraenkel set theory, the paradox vanishes, since second-order logic does not have the Löwenheim-Skolem properties. Therefore, we are able to show that there is a one-to-one and onto mapping between the set satisfying the sentence expressing the existence of the continuum and a subset of the smallest model for the set of theorems of second-order Zermelo-Fraenkel set theory. Thus, the Skolem paradox can perfectly well be taken to indicate an expressive inadequacy in first-order set theory, rather than the relativity of set theory, and of the rest of mathematics.

We turn now to the expressive weakness of first-order logic, which is indicated by the Löwenheim-Skolem properties, and a very interesting twist to the notion of topic neutrality. With respect to our criterion of validity, we can again advance an argument about expressive power. Both the Löwenheim-Skolem properties limit the expressive power of a logic, such that, its theories cannot characterise models, such as the natural numbers, the reals, and so on, uniquely up to isomorphism. There are arguments which informally, we think of as valid, but which require greater resources in expressive power to reflect. We have seen many examples in the introduction and in chapter II. One example is the Mary and Betty example, where Mary has a sister, Betty has a sister, and conclude that Mary and Betty have a property in common. The obvious question to ask is why it is that uniqueness up to isomorphism is so important. The answer seems to be that if we want our logic to discuss structures at all, then it has to be able to identify these, and when we study structures in mathematics we seek to characterise them uniquely up to isomorphism. What constitutes a structure is a set of objects together with some relations which hold between them and some functions under which they are closed.

In mathematics, there is strong consensus over which objects, relations and functions are of interest. That is, some relations, such as, "is friends with" or "is sweeter than", are ignored in the determination of a mathematical structure. The mathematician is interested in mathematical properties, relations and functions, not in relations particular to social science, geography, history and so on.

However, from our perspective, the notion of structure is not determined beforehand, nor can we simply appeal to the practice of mathematicians. We cannot assume a grasp of what it is that distinguishes mathematics from other disciplines. Nor can we simply rely on mathematical practice to determine for us where the distinction lies. For, if we were to do so, we should end up with a mere description of mathematical practice. If we then were to attribute substance to the logicist claim about logic being the ultimate justification of certain parts of mathematics, we would beg the question. The logicist's claim (that some parts of mathematics are ultimately justified by logic) is merely descriptive. Any foundational pursuit which is not merely descriptive, takes with it a commitment to the possibility of some revision concerning practice. For this reason, we cannot simply rely on practice to tell us what constitute logical relations and functions.

Instead, what we wish to develop here is an intensional characterisation of structure which is relevant to the logicist project. To do this, we cannot just accept the practice of treating the standard logical constants as what remain invariant across structures. Instead, we have to assess the practice by appeal to more substantial considerations. The descriptive element of mathematical practice concerning what constitutes the logical properties of a structure, as opposed to the non-logical properties, is important. Indeed, it forms part of our criterion of validity. For, it sneaks in when we discuss the intuitive notion of logical validity. This is because the "intuition" part of the criterion is meant to elicit a description of practice. The other criteria are more prescriptive or substantive in this respect.

The programmatic nature of the present project is such that we can assume a certain modesty. In seeking to motivate what constitutes a structure, for the purposes of logic, or the distinction between form and content of an argument, we do not have to give a characterisation which determines all future cases. We only have to give a characterisation which gives a starting point for deciding which structures can be said to be logical, in as explicit a sense as possible while preserving our interests. The risk we run is of making our technical specifications of the philosophical criteria too precise to recover the philosophical significance of the criteria.

Let me suggest that the expressive adequacy of a logic is measured by its capacity to characterise (certain) structures uniquely up to isomorphism, as they are characterised in a second-order language. Of course, this will determine which symbols in the language remain invariant across structures. That is, a second-order language characterisation of structure will determine which constants are the logical constants.

The criterion of topic neutrality (an aspect of universality) is interesting in this case. The Löwenheim-Skolem properties both entail of a logic that has the properties, that it cannot show that its logical constants pass Tarski's criterion for logicality.[143] The Tarski criterion is that a notion (or constant)[144] is a logical notion (or constant) if it is invariant under any one-to-one permutation of the domain onto itself. A permutation is one-to-one and onto itself if it preserves the cardinality of the domain. Apart from that, the mapping function can shuffle around the objects in the domain. The logical constants: $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$, $\exists$, $\forall$, = and the cardinality of the domain all remain invariant.

The reason second-order logic has the resources (according to Tarski) to express the logicality of the logical connectives, the

[143]This is due to Tarski, and is nicely explained in Peter Simons, Philosophy and Logic in Central Europe from Bolzano to Tarski, (Nijhoff International Philosophy Series, Vol. 45; Dordrecht, Holland: Kluwer Academic Publishers, 1992), pp. 21 - 25. The reference to Tarski is to Alfred Tarski, "What are Logical Notions?" ed. John Corcoran, History and Philosophy of Logic, vol. 7. (1986), pp. 143 - 154 and A. Lindenbaum and Alfred Tarski, "On the Limitations of the Means of Expression of Deductive Theories," Logic, Semantics, Metamathematics, trans. J. H. Woodger, ed. J. Corcoran, (Second edition; Indianapolis: Hackett, 1983), pp. 384 - 392.
[144]Notion includes logical constant. Alfred Tarski, "What are Logical Notions," History and Philosophy of Logic, Vol. 7 (1986), p.150, n. 6.

quantifiers and cardinality notions is that a second-order language has the capacity to express the relation of satisfaction formally. In a second-order language we can express the relationship between formulas and truth because we can quantify over formulas. The formulas we need are the atomic well-formed formulas which only contain one occurrence of a given connective or quantifier. Other cardinality notions are expressed in terms of a partitioning of the domain. The relationship between cardinality notions and the partitioning of the domain will be made explicit in chapter IV, § 2. However, we have to be careful. To show the logicality of the connectives, and so on, what we show is that the satisfaction conditions are met across all domains of interpretation, e.g. the truth table for $P \vee Q$ is the same whatever propositions we substitute for P and Q. That is, we show that the satisfaction conditions for the connectives are invariant. We can show this just because we can represent satisfaction formally as a relation.

Of course, it is not quite accurate to say that second-order logic shows of its own connectives, and so on, that they are invariant. What we actually do is use a second-order language at both the object level and the meta-level. In contrast, if we want to show the logicality of the logical connectives in a first-order language, we have to do this in a second-order meta-language.

First-order logic cannot determine the cardinality of a domain unless it is finite, as was shown in the discussion of the Skolem paradox. Second-order logic can. Thus, second-order logic can show of its logical notions that they are invariant, whereas first-order logic cannot. Relations such as "less than" are *prima facie* filtered out by Tarski's criterion. Tarski calls what is left, the most general notions, and thereby, the logical notions. This is not to say that such relations ought not to be expressible in logic. On the contrary: if they can be expressed by appeal only to logical notions and constants, in Tarski's sense; then we reveal of them that they are topic neutral. The important idea is that what we accept first as a logical notion or constant is anything which passes the Tarski criterion. In this sense the Tarski criterion gives us our default list of logical notions. Later we might define new notions or new constants in terms of those in the default list. The new notions or constants then earn a place in the list of logical notions. Thus, the list is under constant revision.

What is interesting for us is that to show, of a notion, that it is invariant under permutations, one needs to appeal either to set theory or to a higher-order logic or type theory. First-order logic is inadequate in this respect.

> PL1 [First-order logic] is too weak to show of many logical constants *that* they are logical constants. ...One may employ a method of Tarski, and show that these are all logical constants, instead of doing what is normally done, simply presupposing that certain constant symbols are logical constants and treating them specially (as in the standard semantics for PL1). But to express some of the constants and to carry out all of the proofs of their logicality, it is essential that one employ higher-order bound variables.[145]

What this reveals is that knowledge of the logicality (according to Tarski's criterion) of the logical constants in first-order logic presupposes set theory or higher-order logic because to demonstrate their invariance under permutations of a given domain, one needs set theory or higher-order logic in the meta-language. Both set theory and higher-order logic are very different from first-order logic, and it is their expressive power which provides for information about the object language. This is not the case with second-order logic. Used in the meta-language, second-order logic can demonstrate of logical constants in a second-order object language, that they are invariant under permutations of the domain. We might say that in this respect second-order logic is more autonomous than first-order logic. If logic is to count as an ultimate justification, then there ought to be some sense at least, in

[145]Peter Simons, "Who is Afraid of Second-order Logic?," Relativism and Contextualism, Essays in Honour of Henri Lauener, eds. Alex Burri and Jürg Freudiger, (*Grazer Philosophische Studien*, IVIV, 1993), pp. 257, 258.

which it can justify itself to itself. (Or, some argument to show that it is legitimate for the justification to end with logic). It is only in a limited sense that we can expect logic to justify itself to itself, because otherwise we run the risk of paradox or infinite regress. Nevertheless, it would be odd to think of first-order logic as an ultimate justification if we were then to have to appeal to set theory (which would then *a fortiori* lie outside logic) to show that the logical constants are invariant under permutations of the domain onto itself. By choosing a logic which has the expressive resources to show that the logical constants remain invariant under permutations of the domain, we add greater substance to our claim that logic acts as an "ultimate" justification. Incidentally, the Tarski criterion for logicality also passes cardinality notions. The significance of this will receive attention in chapter four. The importance of this depends upon the significance we attach to the invariance of logical constants and cardinality notions.

Grosso modo, one reason for accepting the claim that compactness and the Löwenheim-Skolem properties preclude our calling a formal system, which has these properties, a logic; depends on the following. (1) We consider that invariance under permutations of a domain is necessary for considering a symbol to be a logical constant or for a notion to be a logical notion. Furthermore, (2) we have to associate the capacity to show this with justificatory dependence. That is, we have to think that if we need to appeal to set theory to show that "$\rightarrow$" in first-order logic is a logical constant, then we think that the assertion to the effect that it is a logical constant, depends on set theory. For a philosopher who endorses the above two claims, and who thinks that logic is significant and important in its differences from some of mathematics, along the lines of a logicist, the Löwenheim-Skolem properties and compactness indicate, of a formal system, that it does not deserve the honorific "logic". Such a philosopher will have to turn to an expressively richer formal system such as second-order logic.

It turns out that this is good news for the logicist, since second-order logic is very expressively powerful, and ontologically non-committal. Second-order logic will be able to show, in accordance with the invariance criterion, that many notions are logical, and so re-capture a large part of mathematics!

However, before we merrily endorse second-order logic, we must consider some reservations traditionally harboured against second-order logic. These concern decidability and completeness.

§3: Decidability and Completeness

A set of sentences is decidable if the set of Gödel numbers of its members forms a recursive set.

> "A theory is decidable iff the set θ of Gödel numbers of its theorems is recursive, iff the characteristic function of θ is recursive."[146]

Decidability concerns only syntax. The completeness theorem relates the syntax to the semantics of a theory in a language. It is:

> if an ennumerable set of sentences is unsatisfiable, there is a refutation of that set.[147]

That is, if there is no model of a set of sentences, then there is a deductive proof procedure which will discover this. This result concerns both syntax and semantics. In a sense, it guarantees that there will be a match up between the two, so that whatever we can do with the semantics we can also do with the syntax and *vice-versa*. Sometimes we say that there are two aspects to this. One is soundness: that all the theorems generated by the proof procedures of the theory are truths. The other aspect is the reverse: that all the truths can potentially be generated by the proof procedures.

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[146] George S. Boolos and Richard C. Jeffrey, Computability and Logic, (Second edition; Cambridge: Cambridge University Press, 1980), p. 174.
[147] George S. Boolos and Richard C. Jeffrey, Computability and Logic, (Second edition; Cambridge: Cambridge University Press, 1980), p. 131.

Decidability and completeness are results which carry epistemological significance. In both cases they tell us of a theory, that there will be certain obstacles to our knowing things in the theory. In this sense, a logic which is incomplete or undecidable is flawed. Often compactness, completeness and decidability are not properly distinguished in the literature, and this for the very good reasons that they are epistemologically significant and they are closely related. However, it would be a mistake to think that all undecidable theories are incomplete. For example, first-order logic is only semi-decidable, but it is complete. Decidability and completeness are neither extensionally nor intensionally equivalent. They are different results, about different aspects of a theory.

Many mathematicians restrict their study to at least semi-decidable logics. The restriction is imposed by epistemological concerns. If the logic is complete, and therefore compact, then it will have an effective method for discovering all the truths of a given theory. Constructivists, in particular, show concern for having finite proof procedures, and being in a position to know before trying to carry out a proof that in a finite number of steps we can mechanically generate a given theorem.

However, if we are interested in discovering what it is that logic consists in and its scope, we have no truck with complaints about incompleteness and undecidability. For, completeness and decidability results tell us of the extent to which logic can be applied to solving particular problems, or the extent to which it is practical to use it, or to what extent we can entrust machines with finding a solution to certain problems. That is, if we know that a formal system is complete and consistent, we know that if there is a model for a set of sentences, then we have a mechanical method of proving any further sentence which is a consequent of those sentences and which is written in the vocabulary of the other sentences. That is, there is a perfect match between the truths of the formal system and the theorems. A proof procedure is then effective if all the theorems can be mechanically generated in a finite number of steps.

"Mechanically generated" just means that there is a finite stock of rules of inference and there is an unique one to use in the analysis of a given formula.

Of course, the machine analogy is not to be taken too seriously. For, if we really want a machine to solve a problem, we usually want it to solve it quickly, so we need a guarantee, not only that the proof procedure is effective, but also that it is short! Rather, the significance of the machine analogy lies with a matching of our intuitive concept of computability with a technical definition of Turing computability. Church's Thesis is that all computable functions are Turing computable. Taken as a normative claim, this legislates the extension of the concept "computable function". Taken as a descriptive claim, Church's Thesis matches an extension: (all Turing computable functions), to an intuitive or pre-theoretic concept: (that of computable function). This is insufficient to legislate terms of acceptability of a logic. For, such terms of acceptability for a mathematical system, are pragmatically motivated, and this does not engage with our considerations.

It is exactly the same sorts of epistemological concern with completeness which motivate the constructivists. Their scruples concerning epistemology issue in an argument to the conclusion that the incompleteness of second-order logic shows the project of logicism to be untenable. The reason someone might think that the logicist project is undermined by Gödel's incompleteness results rests on a confusion. The confusion is to think of logicism as the project of showing that we can reduce all arithmetic theorems to logic. Emphasis is placed on "showing". We read "showing" to mean mechanically demonstrate. That is, before we even discuss the logic, we have to show that there is a recursive procedure which will generate all the theorems of arithmetic. However, the Gödel incompleteness results tell us that there will be some arithmetical truths which are not demonstrable, namely, a sentence saying of itself that it is true but not provable, Q. E. D.

However, this is confused. For, the argument equivocates between "show", "prove", "demonstrate", and "mechanically prove". The logicist is not forced to identify "demonstrate" with "mechanically generate". It would certainly be nice for the logicist to have a mechanical proof deriving all the truths of second-order arithmetic from logical laws. However, we already know from the Gödel incompleteness theorem, that arithmetic is incomplete. Thus, to have such a slew of mechanical proofs is not possible. However, it is not necessary to have a completable set of mechanical proofs of all the truths of arithmetic for a modified logicism to be vindicated. It is enough for logicism to trace the ultimate justification of arithmetic to logic; and justificatory dependence does not depend on the existence of a series of mechanical proofs.

Ultimate justification will take the form of a philosophical argument which develops a conception of the nature of the discipline in question, which in this case is arithmetic. The nature of a discipline may be shown to be logical if it can be shown to share salient philosophical properties with logic. The easiest way to show this is through a series of mechanical proofs, but this is not the only way. Already, Frege's project was to derive the Peano axioms from laws of logic, not to mechanically generate all the theorems of Peano arithmetic. Thus, Frege found it to be sufficient, for the purposes of meeting his criterion of analyticity that the axioms of arithmetic could be proven from logical laws.

However, to some extent even tracing the justification of arithmetic to logic through philosophical argument is moot, since even under this modification, we cannot show that arithmetic is universal, since the Peano axioms are false in finite domains. However, it remains open that the Peano axioms (and *a fortiori* arithmetic) share some philosophical properties with logic. For example, we might show that they are analytic, in the sense that we have developed here.

The resolution of the question, as to whether or not arithmetic is analytic, is beyond the scope of this thesis. Instead, I am simply interested in showing which formal systems can plausibly count as logic according to certain philosophical criteria. Since the philosophical criteria are consistent with a modified sort of logicism, it is hoped that the project indicates to some extent the limitations and the viability of the project. This particular topic will be addressed in the conclusion to the thesis.

Before we decide that completeness and decidability are irrelevant, let us explore what it is we are letting ourselves in for, by looking at what it is that an incomplete and undecidable logic is like. A logic is incomplete iff there is no single mechanical method that proves all and only the true or valid arguments. Turn this around, and what we have is that recursive methods fall short of generating all the logically true sentences of an incomplete logic. Thus, the semantic notion of true sentence diverges from the syntactic notion of theorem. At first glance, it looks as though an incomplete logic is useless. Validity is a semantic notion: a formula is valid if it is true under all interpretations. Theoremhood is a syntactic notion. A formula is a theorem if it is an instance of one of the axioms of a formal system or if it is derived from the axioms, plus maybe some definitions, by the rules of inference of the formal system. In an incomplete system, there are more true sentences than theorems. The impression that such a logic is useless, comes from dwelling on the fact that there is no guaranteed means of showing that all the truths are theorems, and concluding from this that there are no means at all of showing that a given truth is true. Insofar as incompleteness shows that a logic is useless, it is ironic that incompleteness and undecidability are features which hold of very powerful systems. That is, they are features of systems which have the power express quite subtle notions.

Some writers have suggested, that there is a clean trade-off between expressive power (or definitional resources) and completeness.

> Completeness is very nice if you can get it, but if the price is expressive inadequacy, it seems to me we should be prepared to accept incompleteness. After all,

> the primary reasons for wanting higher-order logic are semantic ones, and the standard semantics for higher-order logic engenders completeness (*sic!*) not because of its own inherent problems but because we cannot recursively enumerate the sentences valid with respect to this semantics.[148]

However, the trade is not always a neat exchange. We might gain definitional resources while retaining completeness.[149] For example, Keisler's system of first-order logic with the quantifier "there are uncountably many" added on[150] is complete and compact but the Lowenheim-Skolem property fails. This is an a-typical example. In general, when we lose completeness we gain definitional resources.[151]

Incompleteness indicates a deficiency in the deductive system relative to the semantics of the formal system. We might think that this tells against such a theory's ability to track validity. In particular, we shall think this if we identify tracking validity with generating mechanical proofs, or with giving a mechanical proof

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[148]Peter Simons, "Who's Afraid of Higher-order Logic?" Relativism and Contextualism, Essays in Honour of Henri Lauener, eds. Alex Burri and Jürg Freudiger, (*Grazer Philosophische Studien*, IVIV, 1993) p. 256.

[149]There will be more discussion of this throughout the thesis. However, for the moment note that while a system which is both finite and complete implies that it is also compact, the implication does not hold the other way around. For example, take weak second-order logic. See J. Donald Monk Mathematical Logic, (New York: Springer-Verlag, 1976), p. 488.

[150]H. Jerome Keisler, "Logic with the Quantifier "There are Uncountably Many"," Annals of Mathematical Logic, (1970), pp. 1 - 93. Also see M. Kaufmann, "The Quantifier "There Exists Uncountably Many" and some of its Relatives," Model-Theoretic Logics, eds. J. Barwise & S. Feferman, (New York: Springer Verlag, 1985), pp. 123 - 176.

[151]More specifically, consider extensions of first-order logic using generalised quantifiers. The study of these was first proposed in 1957 by Mostowski. These act syntactically just like the first-order quantifiers $\exists$ and $\forall$. Examples are: "there are finitely many", "there is an even number", "there are more than $\aleph_0$" and "most". Any extension of first-order logic by means of a generalised quantifier will be incomplete if the set of true sentences and the set of false sentences satisfying the new quantifier are denumerable. An example of a quantifier which, if added to first-order logic, makes the resulting system incomplete is the quantifier: "is even". We shall discuss generalised quantifiers in chapter four.

procedure. A failure in tracking validity will then amount to a failure in mechanically generating all valid arguments.

However, recall that validity is a semantic notion. So, our concern with validity, is a concern with semantics. The fact that the syntactic proof procedure cannot keep up, is not a problem. It just shows that we have no algorithm which is guaranteed to generate all the truths of the formal system. What we demand from our criterion of validity is that it indicates to us a formal system which can show the semantic validity of an argument, in the sense of showing that all models which satisfy the premises also satisfy the conclusion of said argument.

Let us go back a little. A theory is sound, by definition, iff all its theorems are valid. Incompleteness shows that there will be some truths which are not theorems. They are not deducible by means of an effective method. One often comes across the remark that proving soundness is trivial. This is so, for a very good reason. A system which is unsound is of little use: "Soundness would seem to be an essential requirement of a proof procedure, since there is little point in proving formulas which may turn out false under some interpretations."[152] Therefore, we develop a syntax with either a particular semantic interpretation in mind or a completely general interpretation. There are trivial ways of doing this. For example there is the null procedure.[153] This is when all valid formulas are awarded the status of an axiom. If there are an infinite number of valid sentences then there will be an infinite number of axioms. In this case, we say that the theory is not effective. The drawback to this procedure is that it is cumbersome and impractical. However, especially in mathematics, we tend to find "shortcuts" very rapidly. In other words, a sound proof procedure is one where, when we have a syntactic proof of a theorem, we can deduce (in the meta-language) that the theorem is valid.

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[152]Leslie H. Tharp, "Which Logic is the Right Logic," Synthèse, XXXI (1975), p. 5.
[153]Leslie H. Tharp, "Which Logic is the Right Logic," Synthèse, XXXI (1975), p. 5.

The completeness of a formal system carries epistemological interest. Indeed, Tharp argues that to be a logic a formal system must at least be complete. He gives us two roles which a logic might assume: "the first is as an instrument of demonstration, and the second can perhaps be described as an instrument for the characterisation of structures."[154] The former requires that the logic be effective, i.e. decidable and complete. The latter requires that a logic have enough expressive power to identify important structures uniquely up to isomorphism. To fulfil this role, a logic must not be compact or have the Lowenheim-Skolem properties. In fact, it is this which Mostowski identifies as the role of logic.

> In spite of this negative result [that the addition of most cardinality quantifiers results in a loss of completeness] we believe that some at least of the of the generalised quantifiers deserve a closer study and some deserve even to be included into systematic expositions of symbolic logic. *This belief is based on the conviction that the construction of formal calculi is not the unique and even not the most important goal of symbolic logic.* (Italics mine)[155]

In contrast, Tharp argues for the role of logic as a tool of demonstration. For this, a logic must be (at least semi-)decidable[156] and complete. Without completeness, we have no guarantee that we shall always have the ability to show, or demonstrate mechanically, the validity of an argument in any particular case. This is because there is no single algorithm, or mechanical method, that generates all the true formulas. Tharp writes:

> If completeness fails there is no algorithm to list the valid formulas; so one can expect many of the principles of the logic to be unknowable, or determinable only by means of *ad hoc* or inconclusive

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[154] Leslie H. Tharp, "Which Logic is the Right Logic," Synthèse, XXXI (1975), p. 5.
[155] Andrzej Mostowski, "On a Generalization of Quantifiers," Fundamenta Mathematicae, IVIV, (1957), p. 12.
[156] First-order logic is semi-decidable. That is, we have a mechanical procedure which will generate all the truths of the theory, but we have no mechanical procedure which will tell us in a finite number of steps if a sentence is a non-theorem. There are three types of proof: those which prove that a sentence is true, those that prove that a sentence is false, and those which are infinite.

> arguments. ...The negative evidence, together with the epistemological appeal of the completeness condition, make it seem reasonable to suppose that completeness is essential to an important sense of logic [namely, in its role as an instrument of demonstration].[157]

In its role as instrument of demonstration, it is essential that "if something follows, it can be known to follow."[158] However, at this point we should be careful. Let us paraphrase the paragraph quoted from Tharp, in such a way as to draw out the distinction between knowledge acquired by means of syntax and knowledge acquired by means of semantics. For, demonstrating validity in model-theoretic semantics is no less rigorous than proving theorems using the syntax of a system.

> If completeness fails, there is no mechanical procedure to list the valid formulas; so one can expect many of the principles of the logic to be unknowable by a Turing machine. The negative evidence together with the epistemological appeal of the completeness condition, make it seem reasonable to suppose that completeness is essential to an important sense of logic, namely the ability of machines to reproduce it.

In other words, while completeness of a logic guarantees the existence of a mechanical procedure for generating truths, this is not the only means of gaining the truth. Furthermore, it is not the only rigorous means of proving theorems. A proof can be rigorous, in some sense, without being mechanical. The motivation for adding greater rigor in proof is to make explicit, on what, certain theorems depend. In particular, the logicist is interested in proofs which are logical in the sense of not relying on intuition or sense experience.

We are not precluded from explicitness, in this sense, when we give a non-mechanical proof. Absence of a guaranteed algorithm for generating a conclusion to a proof is not synonymous with a proof having to rely on intuition or sense experience. The sort of proof which is not mechanical, but is rigorous, and does not rely on

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[157] Leslie H. Tharp, "Which Logic is the Right Logic," Synthèse, XXXI (1975), p. 7.
[158] Leslie H. Tharp, "Which Logic is the Right Logic," Synthèse, XXXI (1975), p. 7.

sense experience or intuition is a proof which invokes semantic considerations. It is the semantic considerations for which we have no guarantee that there exists an algorithm, in an incomplete system. But this is in the nature of semantics.

This is not to deny that there are mathematical models, the grasp of which, requires intuition. Geometrical models, arguably at least according to Frege, do require spatial intuition, in order to be appreciated.

Thus, the lack of a mechanical procedure in a formal system is not as tragic as it may at first seem, since we do not lose any truths we could establish in a weaker, but complete, system. Which mechanical procedures are available to us in a complete system are still available in an incomplete system. For example, in second-order logic, we are at liberty to avail ourselves of the mechanical procedures used in first-order logic. All the theorems of first-order logic, will also be theorems of second-order logic. Furthermore, it takes little to adapt the procedure to cope with sentences which include second-order quantification. Thus, not only can we generate all the truths of first-order logic, we can also generate some of the truths of second-order logic. The incompleteness and undecidability of second-order logic entails that there will be some truths which include second-order quantification, which will not be mechanically generated, and so will not show up as theorems. For example, "this sentence is true but not provable," or "if a sentence P is provable, then it is provable that it is provable".[159]

In conclusion, the argument in favour of completeness in logic relies on a conception of logic whereby it functions as a tool of demonstration because, presumably, we want to be able to demonstrate to people correct reasoning. For this, completeness is crucial.

> Completeness, after all, is not just another nice property of a system. When a deductive system of

[159]George Boolos, The Logic of Provability, (Cambridge, Massachusetts: Cambridge University Press, 1993), p. 150. Of course, any of these sentences can be added to the logic as axioms.

> whatever sort is presented, one of the most immediate questions is whether it is (in the relevant sense) complete. If all valid (or true) formulas can be proven by the rules, then apart from practical limitations such as length and complexity, they can be *known* to be valid (or true).[160]

Thus, Tharp is associating "being in possession of a mechanical method" with being able to know. Contrast this with the quotation earlier from Simons where he says that he finds that it is worth while to sacrifice completeness for the sake of expressive adequacy. For, certainly, if there are notions which are not even expressible, then we cannot know anything about them at all! If our system is complete, then we enjoy two independent statements concerning the validity of any given sentence. Truth and theoremhood will be co-extensive.

This is the point stressed by Wolenski in his defence of the first-order thesis. In a complete system the valid sentences are the theorems and *vice-versa*. A sentence is valid just in case it is true in any model. It is a theorem just in case it is generated by the formal/syntactic proof method. In first-order logic the two notions of validity and theoremhood coincide in extension. Wolenski states that "logic is universal because it is universally true."[161] He then goes on to argue that we can tell that the set of universal truths are exhausted by first-order logic since it is complete. This is because he identifies universal truth with universal applicability, and this in turn, with the extensional coincidence of syntax and semantics. But this just begs the question, because he has built into his conception of logic the co-extension of theorems with truths.

In contrast, in an incomplete system, we have independent statements concerning truth and theoremhood. They do not coincide in extension. The notions of true sentence and theorem

[160]Leslie H. Tharp, "Which Logic is the Right Logic," Synthèse, XXXI (1975), pp. 5 - 6. Compare this to the quotation from Simons, quoted earlier in this section, which begins with the words: "Completeness is very nice if you can get it..."!
[161]Jan Wolenski, "In Defence of the First-order Thesis," Logica 93, Proceedings of the 7th International Symposium eds. P. Kolář, V. Svoboda (Praha, 1994), p. 4.

diverge. The set of theorems is a subset of the true sentences. The logic can still function as a sort of justification, but it loses its status as a tool of mechanical demonstration.

Chapter IV: Extending First-Order Logic

Introduction

In chapter three, we concluded that the motivation for adopting the first-order thesis *viz*: that the scope of logic is exhausted by first-order logic, is not sustained by the considerations we have brought into play to characterise logic. First-order logic was found to be inadequate in its expressive power: it could not recognise as valid, what intuitively look like valid arguments. Thus, we are curious to know what ways there are, of extending the language of first-order logic, to make a more powerful and more expressive language. In particular, we are interested in quantifiers. In this chapter I shall examine arguments for the inclusion of some, and the exclusion of others, from the list of logical constants.

In this chapter, we shall look at several extensions of the language of first-order logic. We shall develop them technically, insofar as this is helpful, and offer an assessment along the lines adopted in chapter three: with the difference that in chapter three we used the philosophical criteria of validity, analyticity and universality to assess the limitative results of first-order logic; in this chapter, we shall apply the criteria to languages and whole formal systems which include various types of quantifier. The reason I focus on quantifiers is that I have not really justified the inclusion of even the first-order quantifiers in a list of logical constants. I believe that the best justification for their inclusion also works as a justification for the inclusion of other quantifiers.

I shall not consider all quantifier based extensions of the language of first-order logic. In this sense, the thesis is programmatic: the considerations which are brought to bear on these systems can be brought to bear on others as well.

The strategy I examine, for increasing the expressive power of first-order logic, is to generalise on the notion of quantifier. A new perspective is adopted where the quantifiers of first-order logic are thought of as instances of a more general notion of quantifier. Familiarity with the first-order universal and existential quantifiers is insufficient to give purchase on how to extend the (more general) notion of quantifier. Thus, there are divergent ways of generalising.

It might be conceptually useful to think of ourselves as asking questions about this general notion of quantifier, such as: are "all" and "at least one" the only quantities we are allowed to consider? What about other cardinal notions such as "eight", "exactly three", "$\aleph_0$", "finitely many", or even more ambiguous quantities such as "most"? In the light of such questions, this chapter can be seen as a general polemic on why we should include some quantifiers and not others in the list of logical constants. Which ones we include will depend on our criteria. However, before we assess particular quantifiers, we have to decide to some extent, what the general notion of quantifier consists in. We may do so with an eye to capturing quantitative notions in natural language, or we may do so with an eye to mathematical arguments.

Neutral between the natural language perspective and the mathematical perspective, we can list two minimal criteria for quantifiers: (1) quantifiers play the syntactic role of binding variables. They turn well-formed formulas into sentences; we say that quantifiers are variable binding sentence operators. In binding all the variables in a well-formed formula, they make the formula apt for truth evaluation. Also very obviously, (2) quantifiers have to pick out a quantity. Consistent with these criteria, is the thought that quantifiers are higher-order predicates. That is, quantifiers are predicates of predicates; they attribute a quantity to a set picked out by a predicate.

Under these few constraints, we can develop quantifiers with an eye to our natural language use of quantified expressions to solve the problem of validity: to incorporate into a formal language the capacity to show that certain arguments are logically valid when they intuitively appear to be logically valid. Bearing these considerations in mind, a natural generalisation on the notion of quantifier is to include any predicate in which we can detect a quantitative notion, for example, "more", "the", "60%" and "finitely many". This generalisation on the notion of quantifier is developed by Westerstahl[162] and Barwise and Cooper.[163] To avoid entering into unnecessary detail concerning presentation, I shall confine my comments to Barwise and Cooper's presentation of quantifiers.

At times it will be useful to distinguish a quantifier from a quantifier prefix as follows.[164] A quantifier includes the variable over which it quantifies. A quantifier prefix is the quantifier without the variable. For example, $(\exists x)$ and $(\forall x)$ are quantifiers. $\exists$ and $\forall$ are quantifier prefixes. Drawing on this distinction, we might also ask, of a general notion of quantifier prefix, if these are restricted to having only one individual variable following them. An example of a quantifier which is sometimes analysed as two-placed is "more", as in "there are more A's than B's".[165] A further question is whether

[162]Dag Westerstahl, "Quantifiers in Formal and Natural Languages," Handbook of Philosophical Logic, Eds. Dov Gabbay & F. Guenthner, (IV, Synthese Library: Studies in Logic and Methodology and the Philosophy of Science; Dordrecht, Holland: D. Reidel Publishing Company, 1989), pp. 1 - 132, and "Quantifiers in Natural Language: A Survey of some Recent Work," Quantifiers: Logics Models and Computation, eds. Michal Krynicki, Marcin Mostowski, Leslaw W. Szczerba, (I; Synthese Library: Studies in Epistemology, Logic, Methodology and Philosophy of Science; Dordrecht: Kluwer Academic Publishers, 1995), pp. 359 - 408.

[163]Jon Barwise and Robin Cooper, "Generalized Quantifiers and Natural Language," Linguistics and Philosophy, IV, (1981), pp. 159 - 219.

[164]Michal Krynicki and Marcin Mostowski, "Henkin Quantifiers," Quantifiers: Logics, Models and Computation, Michal Krynicki, Marcin Mostowski, Leslaw W. Szczerba (eds.) (I; Synthese Library: Studies in Epistemology, Logic, Methodology, and Philosophy of Science; Dordrecht: Kluwer Academic Publishers, 1995), p. 194, n. 3.

[165]Per Lindström, "Prologue," Quantifiers: Logics, Models and Computation, Michal Krynicki, Marcin Mostowski, Leslaw W. Szczerba (eds.) (I; Synthese Library: Studies in Epistemology, Logic, Methodology, and Philosophy of Science; Dordrecht: Kluwer Academic Publishers, 1995), pp. 21 - 24. We begin with a familiar example. Lindström analyses "All A's are B" also as two place: $(\forall x,y)(Ax; By)$ but notes that the traditional: $(\forall x)(Ax \rightarrow Bx)$ is usually considered adequate. The judgement of adequacy here is relative to our use of quantified expressions in

quantifiers have to quantify only over first-order variables, or can they also quantify over predicates, relations and functions? The latter possibility has lead to the more modern study of second and higher-order logic.

In section one, I shall examine the view, as it is developed by Barwise and Cooper, that quantifiers are a pair composed of a determiner and a set expression. This will solve the validity problem we were left with in chapter III: that first-order logic is inadequate in its expressive power. First-order logic will fail to identify arguments as valid which we intuitively think are logically valid. The theory of quantifier as it is developed by Barwise and Cooper will pick out a lot more arguments as valid. However, they are not logically valid since the Barwise and Cooper theory of quantifiers is not a theory of logical quantifiers. Thus, a first-order language augmented with the Barwise and Cooper quantifiers fails to meet our criteria of generality and analyticity: the inclusion of these quantifiers in sentences often renders them synthetic, and special (in Frege's sense).

In section two, I shall examine a more restrained view, that of "generalised quantifier" as it was first suggested by Mostowski, and later developed by Sher. Mostowski associates all quantifiers with a certain sort of cardinality function. Mostowski also draws distinctions among quantifiers using two (co-extensional) criteria. One distinction is between logical and non-logical quantifiers. The other is between general and non-general quantifiers. We shall find that with a little tinkering, Mostowski's criteria will pass muster under our criteria for logicality as well. However, his general and logical quantifiers do not pass without repercussion. Mostowski's criteria for quantifiers to be considered logical are necessary but not

natural language. "Most" cannot be analysed in the same way as we traditionally do the universal quantifier since the semantic interpretation for material implication is disloyal to our natural language understanding of "most". Lindström writes "most" as we first did the universal quantifier: (most x,y)(Ax; By).

sufficient, relative to our criteria for a formal system to be considered to be a logic.

In section three, we find out that from the perspective of second-order logic, we can justify, by our criteria, the inclusion of the Mostowski quantifiers. We recover the Mostowski quantifiers by means of definitions. For this, we need powerful definitional resources. The addition of the second-order universal and existential quantifiers is sufficient to provide these resources. Thus, in this final section, we shall introduce second-order quantifiers. Once we do this, we find little difference between second-order logic and simple type theory. We shall then be able to define most of the Mostowski quantifiers in terms of higher-order functions! In this sense, all the quantifiers we have analysed here, are shown to be either non-logical or strictly redundant. But then, we are faced with the question as to whether or not we are prepared to accept second-order logic as logic, according to our criteria. That is, we ask if we are entitled to view second-order logic as logic according to our criteria of validity, analyticity and universality.

§ 1: Quantifiers as Pairs Composed of a Determiner and a Set Expression

Barwise and Cooper, in their article: "Generalized Quantifiers and Natural Language"[166] are interested in giving a formal account of quantified expressions as they are used in natural language. This is important for us insofar as we are trying to capture reasoning, to some extent both in natural language and in mathematics in our characterisation of an intuitive notion of logical validity. Recall that our measure of adequacy in reflecting the intuitive notion of logical validity corresponds to a language's capacity to distinguish models uniquely up to isomorphism. This is something we have found to be

[166]Jon Barwise and Robin Cooper, "Generalized Quantifiers and Natural Language," Linguistics and Philosophy, IV, (1981), pp. 159 - 219.

appropriate for reflecting mathematical practice. We are tacitly viewing mathematicians as having more highly developed intuitions, and this is due to their training. While we are accepting the intensional characterisation of isomorphic structures, we do not accept as definitive what the practice of mathematicians count as the extension of the notion of isomorphism.[167]

As we have seen, first-order logic only uses the quantifiers "all" and "there exists". In the language of first-order logic we can construct particular finite numbers, and co-finite numbers such as "all but 14"; but that is all. Barwise and Cooper complain that, in this respect, first-order logic falls short of capturing many quantifiers found in natural language.

> The quantifiers of standard first-order logic... are inadequate to treat the quantified sentences of natural languages in at least two respects. First, there are sentences which simply cannot be symbolized in a logic which is restricted to the first order quantifiers $\exists$ and $\forall$. Second, the syntactic structure of the quantified sentences in predicate calculus is completely different from the syntactic structure of quantified sentences in natural languages.[168]

Examples of sentences whose quantified expressions cannot be captured in a first-order language with only the standard first-order quantifiers are: "there are only a finite number of stars", "more than half of John's arrows hit the target",[169] "There is an even number of letters in the English alphabet" and "More than one third of the world's population suffers from hunger".[170] For Barwise and

[167]For example, we have not yet determined that first-order quantifiers deserve to be logical constants. If they do not deserve to be logical constants, then they can be re-interpreted in every new domain, and *a fortiori*, will not be invariant across domains, as is the common practice among mathematicians. Certainly, we can envisage a case where first, but not second-order quantifiers were deemed to rightfully occupy a place among the logical constants.

[168]Jon Barwise and Robin Cooper, "Generalized Quantifiers and Natural Language," Linguistics and Philosophy, IV (1981), p. 159.

[169]Jon Barwise and Robin Cooper, "Generalized Quantifiers and Natural Language," Linguistics and Philosophy, IV (1981), pp. 160 - 1.

[170]Gila Sher, The Bounds of Logic, (Cambridge, Massachusetts: MIT Press, 1991), p. 18.

Cooper, any natural language expression in which some quantity is referred to, counts as a quantified expression. The quantity need not be a precise number. Indeed, we often use quantified expressions in which no particular number is named. "Most", "the majority" and "few" all count as elements in quantified expressions.

The project Barwise and Cooper set themselves is to give a formal account of all quantifiers used in natural language. They include quantifiers such as "infinitely many", "there are finitely many", as well as fractions such as "one third" and more complicated functions such as "60%".

Barwise and Cooper observe that the actual underlying number in an expression such as "60%" will change from context to context. For this reason, they analyse quantifiers to be partly determined by a context. 60% is 6 out of 10, 30 out of 50 and so on. For this reason, they include a set expression in their analysis of quantifier. This is what specifies the context. Their syntactic analysis of quantified sentences diverges from the standard analysis of the first-order universal and existential quantifiers.

For Barwise and Cooper, a quantifier is a pair composed of a determiner and a set expression. Quantifiers correspond to noun phrases in natural language. The set expression is a noun. The determiner is an operator which turns the noun into a noun phrase. For example, the noun "chicken" can be turned into a noun phrase by adding the determiner "four". The noun phrase is "four chickens". The quantifier is "four chickens", not "four" as we would traditionally have it.

The advantage of this analysis is that noun phrases such as "most of the people..." can now be captured formally. If we were to restrict ourselves to the traditional syntax which accompanies the first-order universal and existential quantifiers, then we would fail to represent many noun phrases accurately. For example, consider the sentence: Most people listen to Bach. If we represent this as: $(\text{most}(x))(Px \rightarrow Bx)$, then this will not work as a formal representation because what it says is: for most things, if they are people, then they listen to Bach. The same faulty analysis occurs if we substitute other connectives in the place of the material conditional. We shall see more examples shortly. For this reason, Barwise and Cooper suggest that we analyse the sentence as (most people(x))(Bx). The quantifier is "most people". The set expression is "people", and the determiner is "most".

The upshot of the analysis is that, in general, a quantifier has no meaning independent of the domain of interpretation. In order to formalise the natural language use of quantified expressions, Barwise and Cooper end up with an enormous variety of quantifiers. Since quantifiers are a pair composed of a determiner and a set expression, we not only have a new quantifier whenever the determiner changes, but also whenever the set expression changes. Thus, "the chair", "three chairs", "Hilbert's chairs",[171] all count as separate quantifiers, as do "the horse", "the tartan", "the distance" and so on.

It looks as though we are allowing into a formal language non-logical quantifiers because non-logical set expressions are now finding themselves as part of the formal representation of quantifiers. We are then treating chairs, postcards, wave lengths and butterflies as formal entities. Such sets of entities cannot be admitted to the logical part of the theory because they are not analytic in the following sense. Determining the truth of sentences, whose truth is modified by some of the Barwise and Cooper (non-logical) quantifiers, requires the use of sense perception or of intuition.

However, we might be able to circumvent the objection by allowing the set expressions into the formal language as non-logical constants. This will not do for the very obvious reason that for a formal language to be the language of a logic, it must not include

[171] Even proper names count as separate quantifiers under this view! Jon Barwise and Robin Cooper, "Generalized Quantifiers and Natural Language," Linguistics and Philosophy, IV (1981), p. 164. For an excellent discussion and critique of their view, see Gila Sher, The Bounds of Logic, (Cambridge, Massachusetts: MIT Press, 1991), pp. 17 - 28.

non-logical constants, as we have argued in chapter two, sections two and three.

Barwise and Cooper recognise that their view makes for a very large number of quantifiers, not all of which would count as logical quantifiers. To remedy this, they distinguish logical from non-logical quantifiers. For them, a quantifier is logical just in case its interpretation does not depend on a model. That is, logical quantifiers are invariant across domains of interpretation.

> ...the interpretation of quantifiers, even those like "every man", will vary from model to model since the interpretation of "man" is determined by the model. The difference between "every [x such that x is a] man" and "most men" is this. The interpretation of both "most" and "man" depend on the model whereas the interpretation of "every" is the same for every model. "Every", unlike "open", "more than half" and "most", is a logical quantifier.[172]

In other words, without loss of meaning, the logical quantifiers do not need to be analysed in terms of determiner and set expression, although they can be so analysed![173] That is, the traditional representation of the first-order universal and existential quantifiers is adequate for those quantifiers, but is inadequate for most quantifiers. The noun phrase "every person" is adequately captured by $(\forall x)(Px \rightarrow$ .... This anomaly indicates that the first-order universal quantifier is a logical quantifier. In contrast, the noun phrase "most polar bears", for example, cannot be captured in the format: (Most(x))(Px ... followed by a logical connective. If the logical connective were the material conditional, then the representation of the noun phrase would read: "most things, if they are polar bears then...". If the logical constant were a conjunction,

[172]Jon Barwise and Robin Cooper, "Generalized Quantifiers and Natural Language," Linguistics and Philosophy, IV (1981), p. 163.

[173]The distinction between logical and non-logical quantifiers is not important for Barwise and Cooper, which is why they say so little about it. They prefer to analyse the noun phrase "every man", for example, as "every" being the determiner and "man" being the set expression. So, for the sake of uniformity with the other quantifiers, "every boat", "every brigand", and "every controversy" are all different quantifiers.

then the noun phrase would read: "most things are polar bears and...". If the constant were a disjunction, then the noun phrase would read: "Most things are either polar bears or..." and so on. It turns out that, pending further investigation,[174] Barwise and Cooper believe that only the first-order universal and existential quantifiers, and any quantifier defined in terms of them, are logical quantifiers.

It is ironic that, while Frege disentangled the quantifier from other aspects of sentences, and gained great expressive resources in so doing; Barwise and Cooper restrict themselves to first-order quantifiers to gain expressive resources, and re-entangle quantifiers in noun phrases. This makes all quantifiers, *prima facie*, synthetic in the sense that determining whether or not a quantified expression is satisfied requires sense experience or Kantian spatial or temporal intuition over and above understanding of expression without quantifiers.

Recall Frege's positive characterisation of analyticity: that a sentence is analytic if it follows from logical axioms and definitions, where "follows from" means logically follows from in a gapless proof. We decided that this was too stringent a criterion, roughly because of undecidability. Compliance with the positive characterisation is sufficient, but unnecessary, to indicate analyticity.

Bearing this in mind, it turns out that one sharp way of distinguishing what Barwise and Cooper deem to be logical quantifiers from non-logical quantifiers is that the logical quantifiers have syntactic rules governing their use. For example, the natural deduction rule for elimination of the universal quantifier is:

$$\frac{(\forall x)\phi x}{\phi x.}$$

That is, replace all x's in ϕ with an arbitrary variable, or with a name, (depending on the system).

[174] Strictly speaking, whether or not the universal and existential quantifiers are the only logical quantifiers, is an open question. Nevertheless, the point still remains that there are many quantifiers studied by Barwise and Cooper, which are not logical quantifiers in their sense.

With a non-logical quantifier, such as "60% of widows". There is no syntactic rule. Any rule would have to be partly semantic. This distinction between quantifiers corresponds exactly to the distinction between quantifiers taken as logical constants and quantifiers taken as non-logical constants. Only the quantifiers which are logical constants will be "the same in every model". The rule governing the use of logical quantifiers is in this respect model independent. This implies analyticity in the sense that the syntactic rule governing the use of a logical quantifier is independent of domains or models.

This is not the case with the non-logical quantifiers as they are identified by Barwise and Cooper: i.e. those which do not have similar accompanying rules given independently of a domain. That is, the non-logical quantifiers, if they have a rule at all, have a semantic rule for every set expression/ domain. A new rule for quantifier introduction or elimination has to be devised for each determiner in each new domain.

If we think of Barwise and Cooper's suggestion, as one for including the logical quantifiers in the list of logical constants, and the non-logical quantifiers in the list of non-logical constants, then we shall find that we have not progressed very far.[175] The reason is that only the first-order universal and existential quantifiers, and any we can define in terms of them, turn out to be logical. But as far as we know, that is all! In fact, this is their point. Barwise and Cooper believe that the analysis of quantifiers in terms of determiner and set expression is optimal for the analysis of quantified expression as they are used in natural language. As we have noted, they favour analysing the logical first-order universal and existential quantifiers in this way. They mention no other quantifiers as logical.[176] Thus, in terms of capturing logical validity,

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[175]Barwise and Cooper do not discuss quantifiers in terms of their status in the language of a formal system. Thus, this is very interpretative, in the sense of fitting their conceptions to the ones outlined in this thesis.

[176]Jon Barwise and Robin Cooper, "Generalized Quantifiers and Natural Language," Linguistics and Philosophy, Vol. IV (1981), pp. 162 - 3.

and giving the ultimate (in the sense of logical and philosophical) justification for including notions such as "most" or "finite" in logic, we have not progressed at all. For, what we end up with is a formal system which is not logic in our sense.

Before we decide to exclude all but the first-order universal and existential quantifiers from our list of logical constants, let us examine an alternative analysis of quantifiers which might fare better, in responding to our intuitive notion of logical validity, and which will conform to the analyticity criterion.

§2: Extending First-Order Logic Using Mostowski's Generalised Quantifiers

In 1957,[177] Mostowski suggested generalising the notion of quantifier, from the first-order universal and existential quantifiers, to other cardinals notions, such as: "finite", "infinite", "uncountably many" and "most". As we have seen in chapter three, section one, there is no point in symbolising small finite cardinals, such as "three", since these can be captured by a judicious use of the first-order and existential quantifiers. But, as we have also seen, in chapter three, the notions of "finite", "uncountably many", "most" and so on, cannot be captured in first-order logic, only in second-order logic. It is interesting to note that because "finite", "uncountably many" and so on, can be expressed in Frege's system, to add such quantifiers to his system would be redundant. Thus, it only makes sense to suggest extending first-order logic with generalised quantifiers, in the context of distrust of second-order quantifiers.

Mostowski's proposal is to view quantifiers, as operators which turn formulas with free variables into sentences. These operators are each associated with a function t. The function takes as

[177] Andrzej Mostowski, "On a Generalization of Quantifiers," Fundamenta Mathematicae, IVIV, (1957), pp. 12 - 36.

arguments arbitrary assignments of individual variables. The values of the function are the truth values "true" and "false". Essentially, t is an interpretation function and a satisfaction function combined. The value of the t function allows the truth-value of a given sentence to be determined. Mostowski distinguishes two sorts of quantifier: the general and the non-general. A quantifier is general if its function is such that "the [truth] value assigned depends on the cardinalities of the set in question and its complement in the universe and on nothing else."[178] A non-general quantifier is also a variable binding operator. However, the truth value assigned to its function depends on something other than the cardinality of a subset of the domain or its complement. Thus, its meaning will, in some sense, depend on the content of the domain, or particular features had by members of the domain. In this respect, the Mostowski non-general quantifiers resemble the Barwise and Cooper quantifiers. Where the Mostowski non-general quantifiers differ from the Barwise and Cooper quantifiers is in their formal analysis. As we have argued in section one of this chapter, the majority of the Barwise and Cooper quantifiers are synthetic because the truth of sentences in which they feature, depends on the content of the domain of quantification. Exactly in this respect, the Mostowski non-general quantifiers will also be synthetic. For this reason, we shall focus on Mostowski's general quantifiers since, *prima facie*, they better meet our criteria for logicality than do non-general quantifiers. Henceforth, we shall simply refer to them as Mostowski quantifiers, since Mostowski himself was only interested in these and not in the non-general quantifiers.

In extension, the Mostowski quantifiers differ from the Barwise and Cooper logical quantifiers because there corresponds a Mostowski quantifier to any cardinal notion which can be interpreted in the standard semantics; even those not expressible in terms of the first-order universal and existential quantifiers.

[178]Gila Sher, The Bounds of Logic, (Cambridge, Massachusetts: MIT Press, 1991), p. 10.

Mostowski's syntactical analysis of quantifiers is closer to that of Frege, than it is to that of Barwise and Cooper. The quantifiers are not analysed as a pair consisting in a determiner and a set expression, rather, they bind individual variables, making sentences from formulas, and are treated as separable entities from formulas.

Apart from these syntactical differences between the Barwise and Cooper quantifiers and Mostowski's quantifiers, there are other differences. There turn out to be fewer Mostowski generalised quantifiers than Barwise and Cooper quantifiers. Also, the similarity between "half of the people in China" and "half of Dwarf Dahlias" is preserved, trivially, since "Chinese people" and "Dwarf Dahlias" cannot figure as parts of quantifiers, only as predicates in quantified expressions. At least in this respect then, the refinement is more in keeping with Frege's analysis. This is because, *prima facie* at least, Mostowski's generalised quantifiers look as though they will pass the negative analyticity test: that detection of the truth of sentences whose truth they modify, does not require further appeal to sense perception or to intuition. That is, understanding a quantifier free formula may require sense experience or intuition. However, ascertaining the truth of a sentence, made from a formula by binding all the variables by means of a Mostowski generalised quantifier, does not require further recourse to sense experience or intuition. For example, "books on the shelf have green bindings" is a formula, the understanding of which requires, amongst other things, that we discriminate the colour green from other colours. "Nine books on the shelf have green bindings" is a sentence whose truth can be ascertained without recourse to any more sense experience or intuition than did the original formula. This is because we can define nine in terms of the existential quantifier and non-equality, which (for the sake of argument) are already accepted as logical constants. Mostowski's generalised quantifiers also, *prima facie*, increase the expressive power of first-order logic. They, thereby, stand a better chance at reflecting our informal notion of validity than does first-order logic.

Mostowski draws a further distinction, that which lies between logical and non-logical quantifiers. Logical quantifiers do "not allow us to distinguish between different elements of the universe."[179] Intuitively, the idea is that whether there are Chinese people in a given domain, Dwarf Dahlias, or bottles of Scotch Whisky, is irrelevant to logic. Mostowski's logicality criterion is reminiscent of the topic neutrality aspect of the generality criterion we have introduced. However, what little Mostowski says of it, is not enough to match our criterion precisely. As we saw in chapter III, §1, "distinguish between different elements" is ambiguous, and not sharp enough to meet our criteria for what counts as logical. Nevertheless, already on an intuitive level, the analysis rules out the offensive quantifiers included in the Barwise and Cooper analysis, such as, "the tank" or "most of the men in the room".

We shall give a general framework within which to analyse and define particular Mostowski generalised quantifiers, and then discuss the philosophical implications of Mostowski's two distinctions between different sorts of quantifier. Finally, we shall discuss a generalisation on Mostowski's framework: from one-place quantifiers to n-place quantifiers, where $n \in \omega$. In the following section, we shall examine a further generalisation: to second-order quantifiers.

I assume familiarity with first-order logic. Let us begin with the syntax. Added to the language of first-order logic are one or several quantifiers Q. They are added to the list of logical constants. They will sometimes be distinguished from each other by superscripts explained as they are introduced. The universal and existential quantifiers may be removed.[180] Formulas are the same as for first-order logic, except that we generalise on the clause about formulas containing the universal and existential quantifiers, and

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[179] Andrzej Mostowski, "On a Generalization of Quantifiers," Fundamenta Mathematicae, IVIV, (1957), p. 13.

[180] There is no case I know of where this is done. However, because we are treating the first-order universal and existential quantifiers as instances of a broader notion, there is no reason to suppose that they have to be included.

add a clause for the new quantifier(s): (Qx)(A) is a formula when A is. Sentences are well-formed formulas with no free variables.

The semantics of a logic whose language includes Mostowski quantifiers is the same as for first-order logic except for clauses involving the new quantifiers. Q is a quantifier, t^Q is a (meta-language) (truth) function associated with the quantifier **Q** which gives the cardinality conditions under which **Q** can be satisfied. This will be some cardinality statement. Often it is stated in terms relative to the size of the domain. In accordance with our universality criterion, the size of domains is not presumed to be absolute. We may quantify over any domain and in particular, any size of domain.

As a first step to meeting the validity criterion, the expressive power of the formal system is greater than that of first-order logic, because we have more quantifier prefixes than $\exists$ and $\forall$. We can therefore express many more cardinality notions. Some of these require that we take into account the size of the domain (but not the content as in the Barwise and Cooper quantifiers). For example, we want to provide for flexibility in recognising proportional notions, such as "half", percentages, and so on.

To do this, the function t includes a pair (β, γ) where β stands for the cardinal number which has to be met (i.e. be equal to, greater than or less than) to satisfy the quantifier, and γ is the complement of β within the domain or universe. $\beta + \gamma = \alpha$, where α is the cardinality of the domain.

Notationally, we write the scheme for the satisfaction clause, or truth function, for quantifiers thus:

$$t^Q_\alpha(\beta, \gamma) = \begin{cases} \text{T if } \beta = \text{ whatever conditions are required by the quantifier,} \\ \qquad\qquad\qquad \text{F otherwise.} \end{cases}$$

Q stands in the place of a name for the quantifier. α is the size of the universe or domain. β and γ partition the universe into two subsets in keeping with the meaning of the quantifier. γ is the complement of β in α. $t^{Q}_{\alpha}(\beta, \gamma)$ can be read "the satisfaction of Q in a domain of size α requires the partitioning of the domain into two: β and γ such that ..." Satisfaction, or truth, obtains when the truth conditions after the curly bracket are met. Falsity, obtains under all other circumstances.

Let us give some examples of t-functions which accompany various quantifiers. The existential quantifier is associated with the function t:

$$t^{\exists}_{\alpha}(\beta, \gamma) = \begin{cases} \text{T if } \beta > 0, \\ \text{F otherwise.} \end{cases}$$

The t function for the universal quantifier is written:

$$t^{\forall}_{\alpha}(\beta, \gamma) = \begin{cases} \text{T if } \gamma = 0, \\ \text{F otherwise.} \end{cases}$$

In general, if we want an exact number δ, then the t function is defined:[181]

$$t^{\delta}_{\alpha}(\beta, \gamma) = \begin{cases} \text{T if } \beta = \delta, \\ \text{F otherwise.} \end{cases}$$

More specifically, if we substitute 3 for δ, above, we have:

$$t^{3}_{\alpha}(\beta, \gamma) = \begin{cases} \text{T if } \beta = 3, \\ \text{F otherwise.} \end{cases}$$

What about the quantifier "all but three"?

$$t^{-3}_{\alpha}(\beta, \gamma) = \begin{cases} \text{T if } \gamma = 3, \\ \text{F otherwise.} \end{cases}$$

where -3 is the complement of 3. Thus, we can include in this framework, complements of cardinalities too. We can generalise further and allow no exact cardinal numbers to be mentioned at all. Consider the quantifier "most" as in "most of the members of the domain are A":

$$t^{\text{most}}_{\alpha}(\beta, \gamma) = \begin{cases} \text{T if } \beta > \gamma, \\ \text{F otherwise.} \end{cases}$$

[181] Gila Sher, The Bounds of Logic, (Cambridge Massachusetts: MIT Press, 1991), p. 12.

To take another example, we have a function associated with the quantifier "is finite and even":

$$t_{\alpha}^{\text{even}}(\beta, \gamma) = \begin{cases} \text{T if } \beta \text{ is finite and even,} \\ \quad \text{F otherwise.} \end{cases}$$

We may also want to consider "manifold" cardinality functions. These are ones which change their "β conditions" with the size of the universe; that is, with the cardinality of α. For example, say our manifold function is 2 out of 3, 3 out of 6, 19 out of 19... We then index α for each new size, so α_1 is 3, α_2 is 6, α_3 is 19. Then β will be 2, 3 and 19 respectively:

$$t_{\alpha_1}^{\text{manifold}}(\beta, \gamma) = \begin{cases} \text{T if } \beta = 2, \\ \text{F otherwise;} \end{cases}$$

$$t_{\alpha_2}^{\text{manifold}}(\beta, \gamma) = \begin{cases} \text{T if } \beta = 3, \\ \text{F otherwise;} \end{cases}$$

$$t_{\alpha_3}^{\text{manifold}}(\beta, \gamma) = \begin{cases} \text{T if } \beta = 19, \\ \text{F otherwise;} \end{cases}$$

and so on. "According to Mostowski, any formula-binding operator defined by some cardinality function (simple or vacillating [what we called manifold]) as described above is a generalized quantifier."[182] Call this Mostowski's generality criterion. The set of cardinality functions just is the set of two-partitions of the domain. A two-partition is an exhaustive non-overlapping partitioning of the domain into two (a set and its complement). Explicitly, Mostowski's theorem to this effect says:

> Let A be a set. Let *T* be the set of cardinality functions on 2-partitions of $\alpha = |A|$. Let *Q* be the set of quantifiers on A. Then there exists a one-to-one function *h* from *T* onto *Q* defined as follows:
>
> For any $t \in T$, $h(t)$ = the quantifier q on A such that for any $B \subseteq A$, $q(B) = t(|B|, |A - B|)$.[183]

[182]Gila Sher, The Bounds of Logic, (Cambridge Massachusetts: MIT Press, 1991), p. 13.

[183]The theorem in this form is not to be found explicitly in Andrzej Mostowski, "On a Generalization of Quantifiers," Fundamenta Mathematicae, Vol. ILIV, (1957). It seems to be a combination of his theorem 1 on p. 13 and his lemma on p. 21. The theorem as it stands is in Gila Sher, The Bounds of Logic, (Cambridge

| A | is the cardinality of A. *Q* turns out to be the set of general quantifiers on A. The number of general quantifiers is equal to the powerset of A: it is all the ordered pairs composed of subsets of the domain together with their complements. So, any pair, consisting in a subset together with its complement will have a corresponding general quantifier on A, according to Mostowski.

Mostowski has a second criterion: that of a logical quantifier. Mostowski says of these: "that they should not allow us to distinguish between members of the universe."[184] Call this Mostowski's logicality criterion. One way of understanding this is in terms of characterising structures uniquely up to isomorphism. This would be in keeping with mathematical practice and would be consistent with what little Mostowski says about the criterion.[185]

It turns out that Mostowski thinks that his logicality criterion and his generality criterion are co-extensional. The co-extensionality of the two criteria depends upon a particular elaboration of the meaning of Mostowski's logical criterion. According to Sher, and it is fairly clear in Mostowski as well, if we assume standard conditions (semantics) for invariance under permutations of domains, then the two criteria are co-extensional. They also amount to Tarski's criterion for logicality. "Logical quantifiers are invariant under permutations of the universe in a given model for the language."[186]

We need one more assumption to show that Mostowski's two criteria are co-extensional: that the model theory we employ is the standard one. This is very important. The issue has been raised in this thesis in the form of the question as to what counts as a permutation: what are the invariant operators (logical constants)

Massachusetts: MIT Press, 1991), p. 141. Her purported proof of the co-extensionality of the generality criterion and the logicality criterion is on pp. 141 - 2.

[184] Andrzej Mostowski, "On a Generalization of Quantifiers," Fundamenta Mathematicae, Vol. ILIV, (1957), p. 13.

[185] See also the discussion in chapter III, §1.

[186] Gila Sher, The Bounds of Logic, (Cambridge, Massachusetts: MIT Press, 1991), p. 14.

and which are not (the non-logical constants)? Here we ask what is the background rule governing what counts as permutations of a set onto itself. In other words, we want to make explicit what can be changed and what cannot, and determine whether or not this is in keeping with our criteria. The problem is not that we do not know what a permutation of a set onto itself is. Rather, the problem is that it is so much taken for granted, and is so embedded and accepted in mathematical practice that it merits a re-appraisal. In particular, this is important for us because under the standard semantics, cardinality notions remain invariant, and are deemed logical because they are implicit in the formal languages we have been considering.

Contrast this with two cases of quantifiers which are logical according to Mostowski's criterion, but are not general according to his criterion. The quantifiers are:

(1). "(Qx)Φx says that the extension of Φx contains a non-empty open set."[187]

(2). The quantifier "more than half" in some contexts of infinite domains.

(1) requires a topological measure of distance to be imposed on the set. The second also sometimes requires a non-standard theory of cardinality. For, the standard theory cannot always make sense of "more than half" in infinite contexts.[188] The examples stated in Sher are: "more than half the integers are not prime" and "more than half the real numbers between 0 and 1, expressed in decimal notation, do not begin with 7".[189] For us, the importance of these examples lies

[187]Gila Sher, The Bounds of Logic, (Cambridge Massachusetts: MIT Press, 1991), p. 15.

[188]The first quantifier was studied by J. Srgo, "Completeness theorems for Topological Models," Annals of Mathematical Logic, Vol. XI, (1977), pp. 173 - 193. The second was studied by Jerome H. Keisler, "Hyperfinite Model Theory," Logic Colloquium 76, eds. Robin Gandy and M. Hyland, (Amsterdam: North Holland, 1977), pp. 5 - 110.

[189]Gila Sher, The Bounds of Logic, (Cambridge Massachusetts: MIT Press, 1991), p. 16. The reason the last example requires a non-standard interpretation is that in terms of the standard semantics, the two following sets equal: the real numbers between 0 and 1 not beginning with 7, and the real numbers between 0 and 1

with the topic neutrality aspect of our generality criterion. A logic is topic neutral if the formal language lacks the resources to discriminate between members of the domain on the basis of particular features held by those members. A standard semantics for first-order logic where cardinality notions are "standardly" conceived underpins how it is that we identify and distinguish sets. By itself, this begs the question against the logicist. For, it makes appeal to a notion of what is standard which is *ad hoc* relative to our considerations. The examples suggest that we could just as easily have adopted a non-standard semantics for our background theory. For example, we might include topological notions in the background theory. *Prima facie*, we do not want this because topology seems to require intuition. Mostowski's criteria do not give us purchase on a justification for preferring the standard semantics over non-standard semantics.

Nevertheless, we can go some way to justifying philosophically, the co-extensionality of the two criteria. Co-extensionality of the two criteria is the product of our being allowed to strictly consider cardinality notions in our determination as to what counts as a (Mostowski) generalised quantifier. That is, we have to be able to ask of a sentence with a general quantifier: $(Qx)\Phi$ (where Φ only has x as a free variable) "For the sentence to be true, how many x's have to conform to Φ, and how many not?". A "how many?" question is a question about cardinality which is defined in terms of one-to-one correspondence. This does not require that we distinguish members of the domain in terms of their properties, in the very fundamental sense that such quantifiers are applicable to any domain because they do not infringe on the analyticity of a given sentence! More precisely: if a sentence is analytic, its truth will be justifiable without appeal to sense experience, or Kantian spatial or temporal intuition.[190]

This gives us a way of sharpening the form/ content distinction, as it was discussed in chapters two and three with respect to topic neutrality. If we identify content with the particular objects in a given domain, then we can say that any sentence which exploits the particularities displayed by the objects, will not count as logical on pain of loss of generality. Now we have to say what we mean by "particularity". Intuitively, this means something like: if, having examined an arbitrary member of the domain, we have a basis on which to include it or exclude it from a subset, then we have appealed to a particular feature.

This will not do, of course, since if we are told to pick 10 things, and put the first 10 we chose into an appointed set, then we have a (cardinality) basis for excluding the eleventh object from that set.[191] What we mean by "examining in order to find a basis for discrimination" requires further elaboration. Let us start with an example.

If someone says: "Pick out ten oranges from the bag of oranges," and say, the domain is a bag of more than ten oranges: we are given no way of choosing between oranges: accepting some and rejecting others. It is because we are given no means of discriminating between members of the domain, that we can claim to satisfy the sentence purely logically. If we were told "pick 10", we could do so arbitrarily: that is, by ignoring other features.

The arbitrariness is important. For, doing something arbitrarily, is just to ignore or suppress the content of what it is we

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beginning with 7. This is because we can place the two sets into one-to-one correspondence. To say truly that more than half the real numbers between 0 and 1 do not begin with 7 we have to take a notion of order into account.

[190]Recognising a one-to-one correspondence between sets is analytic in this sense. Also, since cardinality is universal according to Tarski's criterion for logical notions, this gives us a hint on how to argue for the logicality of Hume's principle, for example.

[191] The number 10 itself is not a particular object in this context for two reasons. One is that it is not an object in a domain. The second reason is that because we can define 10 using only logical vocabulary, it is a logical notion.

do. When we pick out things arbitrarily, we do not discriminate, where discrimination involves intuition or sense perception. The word "pick" in the context of oranges evokes the image of handling, which involves tactile discrimination, if nothing else. However, this is not meant to be the salient image. We can pick abstract objects as well, such as numbers.

Consider a different sort of case where we partition a domain into a subset and its complement on considerations of cardinality alone. This is not done on the basis of what the particular features of the objects in the domain are, but on the cardinality of the domain. If the domain were a bag consisting in more than ten oranges together with some apples, the quantifier "10 oranges" would not count as a logical quantifier. For, it makes us distinguish between apples and oranges.

These examples do not constitute an argument. Rather they help us sharpen the form/ content distinction. Let us give further precision to the distinction, and then examine the implications. In particular, we have to return to the issue of the non-standard semantics forming a possible background meta-theory for the use of Mostowski's generalised quantifiers.

Returning to the syntax: in general, the constraint on t^Q has to take into account that α, β, and γ are all partitionings of the domain according to cardinality only. Within the framework above, a first attempt at a rule for ensuring the topic neutrality of quantifiers is that:

<u>Rule for topic neutrality (R. T. N.)</u>:

> we are able to define α, β, and γ as equal to, greater than, less than, precise cardinal notions, or in terms of each other, with the understanding that α, β, and γ all range over cardinal numbers and nothing else.

The "nothing else" clause is meant to rule out any appeal to sense experience or Kantian intuition. The second disjunct allows for the case where we do not always give a precise cardinal number, as in the case of the t-function associated with the quantifier "most".

However, what happens with the quantifier: "a lot"? *Prima facie*, this sounds like a cardinality notion. If we think that it can play a part in logically valid arguments, as was mentioned above with "most", we would have to make this artificially precise. In particular, we would have to develop a definition of "a lot" which conforms to the rule above. That is, we would have to define "a lot" in terms of the cardinality of the domain and some partitioning of it. Needless to say, we have little hope of both conforming to R. T. N. and of loyally capturing the use made in natural language arguments of "a lot". We interpreted "most A's" as: "there are more A's than non-A's", and this ends up being a precise cardinal notion given the (precise) cardinality of a domain. Similarly, "a lot" could be interpreted as "more than half". Of course, this is artificial with respect to how we use the term "a lot" in natural language. For example, if we say "four votes (out of several thousand) were cast for the Communist party this year," and someone replies: "Gosh that was a lot!" we do not mean that more than half the votes cast, were for the Communist party, we mean that the number is unexpectedly high. The artificiality of our interpretation of a given quantifier is a price we have to pay, to conform to our gloss on Mostowski's generality constraint.

On the other hand, "a lot" is a notion which is relative to expectations. We could say that the non-conformity of notions such as "a lot" to R. T. N. indicates that it is not a logical notion. Whether or not it is, will depend on what it is the expectation is founded on, or rather, to what extent the expectation can be expressed in terms of cardinality. For example, if "a lot of numbers between 10 and 50 are prime," can be loyally translated as "there are more than three numbers which lie between 10 and 50 and are prime", then "a lot" is a logical quantifier. If, on the other hand, no translation can be made which draws on (already accepted) cardinality properties, then the quantifier is not a logical one. If "a lot" can be loyally captured in conformity to R. T. N., then it is a logical notion; if not, then it is based on our epistemic situation, our psychology or whatever. In other words, we can take R. T. N. as normative of our criterion for topic neutrality. However, the extension of the application of the rule is not predetermined, as the case of "a lot" indicates. Thus, we can rule out "a lot" as a logical quantifier on the basis of its not being a precise cardinal notion. It changes from context to context, not in the sense of the manifold Mostowski generalised quantifiers, but the modification depends on considerations which are often sensitive to intuition and perception.

Can we generalise further? Mostowski did not consider the possibility of partitioning the domain into more than two subsets. A consequence of this is that his framework does not accommodate many-place quantifiers. However, we can accommodate this, in keeping with his generality and logicality constraints. Consider the quantifier "there are more x's than y's". In the notation above, we could represent the t-function as:

$$t_{\alpha}^{\text{more}}(\beta, \gamma, \delta) = \begin{cases} \text{T if } \beta \rangle \gamma, \\ \text{F otherwise.} \end{cases}$$

Here, the quantifier expression would be written more familiarly: "(more x, y)Φ", where Φ is a well-formed formula. The expression would be read: "there are more x's than y's, such that Φ". β, γ and δ are a 3-partition of the domain; that is, $\beta + \gamma + \delta = \alpha$. The first variable after the quantifier prefix "more" is given the cardinality β, the second is given the cardinality γ, and the quantifier is satisfied just in case $\beta > \gamma$ in α. δ is the complement of $\beta + \gamma$. In general, we can have n-place quantifiers for any $n \in \omega$, and we can modify R. T. N. in the obvious way. The simplicity of the modification makes it surprising that Mostowski did not generalise from one place to two place quantifiers. More importantly, conceptually if not historically, it is surprising that Mostowski did not generalise from the first-order quantifiers to second-order quantifiers.

Taking stock: Mostowski's criteria conform to ours, modulo a better argument for cardinality notions remaining invariant under permutations of the domain onto itself. So far, the standard semantics assumes some cardinality measures implicitly, but not others.

> Mostowski's quantifiers express measures of a particular kind, namely measures which have to do with the *cardinality* of sets, ...these are all the second-level 1-place measure predicates [first-order quantifiers] satisfying (LQ2) [Mostowski's logicality criterion] relative to standard model theory. But these are not the only second-level measures conforming to (LQ2). Other quantifier measures of first-level extensions have been developed involving more elaborate model structures. ...[For example, the] quantifier Q, where "(Qx)Φx" says that the extension of Φx contains a non-empty open set. This quantifier requires that models be enriched by some measure of distance (topology). The second [example] has to do with infinite sets [where the notion of half an infinite set makes sense].[192]

As we have seen, using Mostowski's general framework, we can include quantifiers that Mostowski himself, did not consider, such as two-place quantifiers. These conform, in spirit, to his criteria for logicality and generality. They do not conform in letter, since Mostowski explicitly only considers two-partitions of the domain. Nevertheless, this gives us a marked improvement, over first-order logic, on expressive power. In turn, this allows first-order logic augmented with the Mostowski quantifiers to show a greater number of intuitively logically valid arguments as valid. For example:

an even number of people want to play football,

therefore, two teams of equal number can play each other.

This argument relies on the quantifiers "even number" because we have to understand that any even number can be divided by two whole numbers. Particular even numbers can be expressed in first-order logic, but not the more general notion of even number.

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[192]Gila Sher, The Bounds of Logic, (Cambridge Massachusetts: MIT Press, 1991), pp. 15 - 16.

Earlier, we identified content with particular members of a domain, and form with cardinality properties of arbitrary subsets of the domain. The latter was expressed in terms of invariance under permutations of the domain. Insofar as we endorse this view, as a means of sharpening the criterion of universality in the context of generalised quantifiers, this will complement the criterion of validity. This is because logical validity is supposed, on an intuitive level, to be a matter of the formal properties of an argument, and should not rely on particular features of the subject matter, or content, of the argument.

However, including the Mostowski quantifiers in the list of logical constants presupposes purchase on a some fairly sophisticated notions concerning cardinality. For example, if we include, "there are $\aleph_2$" amongst our logical constants, then the interpretation of the quantifier remains invariant across domains; by the stipulated meaning of "logical constant". For this reason, we have to already be able to recognise cardinalities such as $\aleph_1$ and $\aleph_0$; we need to know, for example, whether or not a given domain has this number of members as a subset or not, so, if it can satisfy a sentence in which the Mostowski generalised quantifier features.

The standard theory about cardinal numbers is fairly sophisticated and abstruse. For example, it includes theorems such as, *prima facie,* $\aleph_1 + \aleph_2 = \aleph_2$, and that a set of infinite cardinality is one which has a proper subset of the same cardinality as itself. Including these cardinality notions in the form of quantifiers in a list of logical constants, amounts to a stipulation that cardinality notions are logical notions. To make this stipulation almost begs the question against the logicist, who thought he had to do some technical work to prove that the axioms of arithmetic are part of logic. The stipulation does not fully beg the question, since including the Mostowski generalised quantifiers in the list of logical constants, will give us cardinality notions for free, but it is an open question whether they give us the resources to prove the Peano axioms. Frege wanted to prove, by means of a rigorous proof starting only from logical axioms and definitions, that arithmetic is part of logic.[193]

Worse still, what the formal system, which results from adding Mostowski's generalised quantifiers to the language of first-order logic, lacks is an argument for preferring the standard theory of cardinality over the non-standard. The standard theory forms part of the standard semantics for first-order logic. But we have no basis upon which to justify our preference for the standard semantics. Thus, increasing the expressive power of first-order logic by adding Mostowski generalised quantifiers, in order to accommodate our validity criterion, proves a severe limitation on the logicist project as a whole. For, what we effectively do is sneak cardinality in by the back door. Moreover, we have no reason to exclude topological notions from the background theory, or alternative cardinality theories to accommodate ordinal notions, for example. These theories sometimes contradict the standard semantics. But we have no basis yet, upon which to select one semantics over another.

For now, then, let us register that the method of vindicating logicism, by adding Mostowski generalised quantifiers to the logical constants, to some extent, begs the question against the logicist. Mostowski's criteria comply with our criteria of analyticity and validity. However, they make a farce of generality by offering a circular justification for topic neutrality. This is because on an intuitive level we were associating cardinality notions, *prima facie*, with mathematical notions. We did this on the grounds that theories about cardinal numbers seem to be about particular objects, or particular domains of objects. Thus, we need to do some work to show that they belong to logic, if they do. The Mostowski quantifiers already include such notions in the language, so that we just point to the vocabulary and say look: "we have a quantifier:

[193] Underlying the satisfaction conditions for arithemtical notions included among the Mostowski quantifiers, (such as 0, Dedekind finite, greater than) will be some theory of arithmetic, thus, it may be possible to distill the Peano axioms out of the satisfaction conditions for an appropriate set of Mostowski quantifiers.

Dedekind infinite in the list of logical constants, and therefore, it is a logical notion. While Mostowski's demarcation of invariance and logicality may conform to present practice among mathematicians and logicians, it is not good enough for the logicist. Let us look to an alternative method of vindicating logicism: through the introduction of the second-order universal and existential quantifiers.

§3: Second-Order Quantifiers

In the same vein as in the previous sections of this chapter, we might ask the question: "is there any reason to restrict ourselves to first-order quantifiers?" That is, is there an *a priori* logical or philosophical reason why we cannot quantify over second-order variables and consider the formal system (of second-order logic) to be a logic? To answer this question, we propose to generalise on the notion of quantifier from our examples of the first-order universal and existential quantifiers to second-order universal and existential quantifiers.

This is not new. As was made clear in the first chapter, this is an a-historical question. Frege, used not only first, but also second-order quantifiers. Frege's formal system is commonly agreed to be equivalent to what we think of today as second-order logic. The structure of the system is a little more elaborate than that of first-order logic.

Frege analysed first-order quantifiers as (one-place) second order concepts, and second-order quantifiers as (one place) third-order concepts. In this construal of Frege's system we have a domain of individuals, then above those, there are first-order concepts, then the second-order concepts. First-order quantifiers quantify over members of the domain of individuals.

Distinguish orders from levels as follows. Levels are absolute. Orders are typed, in the sense that in the type "concept" there are first-order concepts, second-order concepts, and so on. In the type "quantifier", there are first-order quantifiers, second-order quantifiers and so on. The only other type we discuss is the type of individuals. There are no orders of individuals. Different type/ order combinations are located at different levels as shown in the figure below. The first level comprises the first-order individual objects. There are no second-order objects or individuals. The next level up is that of the first-order concepts. These apply to the individuals. The third level comprises the second-order concepts, that is, concepts of concepts. On the third level we find the first-order quantifiers. On the fourth level, we find the second-order quantifiers. These quantify over concepts at the second level (that of first-order concepts).

Figure

4th level:	second-order quantifiers, third-order concepts*, i.e. concepts of concepts of concepts, these range over the powerset of concepts at level 3.
3rd level:	first-order quantifiers, second-order concepts, i.e., concepts of concepts, these range over the powerset of concepts at level 2.
2nd level:	first-order concepts, i.e. concepts which pertain directly to objects, these range over the powerset of the objects which are found at level 1.
1st level:	individuals or objects. First-order variables: x, y, z... range over these.

* "Concepts" is generic for predicates, relations and functions.[194]

[194]This is not how Frege used the word "concept".

First-order quantifiers are cardinality predicates which relate individuals to first-order concepts. For example, if we write: $(\forall x)Px$, we assert that all the individuals in the domain fall under P, or have the feature P. Similarly, a second-order quantifier, makes a cardinality judgement about the first-order concepts, for example, that a concept, P, exists. This is written: $(\exists P)P$, where P is a first-order predicate.

From the first chapter, we learned that the notion of second-order quantifier in logic, as a part of logic, was abandoned early this century. However, we could still find it in second-order set theory and in most meta-languages. We also know from the first chapter, that the abandonment was too swift, since the formal system of the *Begriffsschrift* is consistent. The only offending part of Frege's formal system is axiom V of *Grundgesetze*. Interestingly, the infamous axiom V of *Grundgesetze* was shown to be dispensable in deriving all the theorems of *Grundgesetze*, save Hume's principle and any theorems of analysis.[195] Hume's principle is consistent with the formal system of the *Begriffsschrift*, in which there is no mention of axiom V in *Grundgesetze*. We can promote Hume's principle from its status as theorem in *Grundgesetze*, to the status of axiom and add it to the formal system of the *Begriffsschrift*. This makes a formal system which is equiconsistent with second-order arithmetic and is commonly accepted to be equivalent to second order logic together with Hume's principle added as an axiom. This system is powerful enough to derive the Peano axioms as theorems.

The logicist cannot rest content with this. He has to develop a case for saying the Hume's principle is a logical principle, in some sense. The problem is philosophical. There is little agreement in the literature over what the philosophical status of Hume's principle is: whether it is analytic, whether it is an abstraction principle, or a contextual definition: that is, to what extent it is logical.

[195] Richard Heck, "The Development of Arithmetic in Frege's *Grundgesetze der Arithmetik*," The Journal of Symbolic Logic, LVIII.2 (1993), pp. 579 - 601.

I do not propose to discuss this issue here. I am just flagging a motivation and a limitation concerning the present thesis. The limitation is that even if I do show that second-order logic is logic, this is not enough to vindicate logicism. Hume's principle must also be shown to be logical, at least to some degree. Furthermore, exactly in which respects Hume's principle is logical had better accord with why the logicist thinks that second-order logic is logical. Thus, apart from arguing for the logical status of Hume's principle, it is crucial for the logicist that second-order logic be shown to be a logic. If second-order logic is a piece of mathematics, then the logicist project, of showing that arithmetic is in some sense part of logic, is of technical interest, but loses its philosophical interest. The philosophical interest can only be made clear if we have reasons to think that second-order logic can be regarded as logic. That same philosophical interest will also make explicit wherein lies the interest in the logicist project in its modern guise.

In the previous section, we discussed Mostowski's criterion for a quantifier to be regarded as a logical quantifier: that it not distinguish between different members of the domain. We registered dissatisfaction with what he considered to be the outcome of conformity to the criterion because it presupposes a background theory of cardinality, which, for all we can tell, requires intuition to recognise its true sentences. This led to the problem that we cannot justify the standard semantics over the non-standard semantics. The problem with endorsing both non-standard and standard semantics is that some of the non-standard theories contradict the standard theory, and at least *prima facie*, the rival cardinality theories or theories of topology, look more like mathematical theories than logic, because choosing between them (for all we know) requires intuition. This makes cardinality notions, in full generality (standard theories and non-standard theories) seem, *prima facie*, synthetic. We have no logical basis upon which to make a choice, which we are forced to make on pain of contradiction. While this is the case *prima facie*, we shall see that many of the Mostowski logical quantifiers can be expressed using higher-order predicates. This allows us to define the notions in second-order logic. If we can define a notion in second-order logic, and we consider second-order logic to count as logic according to our criteria, then the definitions turn out to be analytic. We shall see this at work shortly. First, let us introduce second-order logic.

When we introduce second-order quantification, the syntactic difference with first-order logic is that we introduce an infinite set of first-order predicate, relation and function symbols as variables. That is, a formula is well formed, if it is of the form $(Qx)(\phi)$; where x stands for individuals, first-order predicates, first-order relations or first-order functions, Q is either the existential or the universal quantifier prefix, and ϕ is a formula. A sentence is a formula with no free variables. In particular, it cannot contain unbounded individual, predicate, relation or function variables! For example, "$(\forall x)Px$" is not a sentence in the language of second-order logic because P is free.

The remarkable thing is that we gain the expressive advantages of higher-order languages immediately if we allow variables and quantifiers at any level above levels three and four, respectively. We now find ourselves in the realm of higher-order logic which is a sort of simple type theory. Moreover, we gain the advantages at little cost. The jump from first to second-order logic is far more dramatic than the jump from second to third order or any finite order thereafter. This is so, in two senses. One is that predicates, relations and functions of higher-orders are definable in terms of non-logical constants at lower levels. Thus, there is a sense in which higher-order logic can be made to "collapse" into second-order logic, by successively making definitions which lengthen the number of places in a relation, say, but each time bring it one level down. The details are not important for us and they are well documented.[196]

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[196]Stewart Shapiro, Foundations Without Foundationalism, (Oxford Logic Guides: 17; Oxford: Clarendon Press, 1991), pp. 137 - 140.

The second sense in which higher-order logic is similar to second-order logic is that second-order logic and higher-order logics have many of the same limitative results. First-order logic is complete, semi-decidable, has the upward and downward Löwenheim-Skolem properties and is compact. In contrast, second-order logic is incomplete, undecidable, has neither Löwenheim-Skolem property and is not compact. This is also the case with third-order logic, fourth order logic, and indeed, higher-order logic.[197] As we saw in chapter three, a formal theory's having this combination of limitative results, is, by and large, an advantage, if our purpose is to pick out logic in the philosophically interesting sense I have suggested.

As a separate point, if we are willing to extend the notion of quantifier in first-order logic, to that of second-order quantifier, it is not at all clear that we should stop at second-order. Just as a point of fact, quantifiers of order higher than two, are seldom needed to express mathematical notions. Still, there are some notions whose full definition requires third order quantifiers, and some which require even higher-order quantifiers. For example: "is a two-place relation". Thus, while in practice, we rarely go very high in a higher-order language, our considerations give us no way of stopping ourselves at second-order logic. However, here, we shall simply consider second-order logic, and leave the question open as to the conformity of higher-order logics and other systems.

We should now evaluate philosophically whether or not the second-order quantifiers belong to a list of logical constants, whether this makes for a logical language, and if this language applied to any domain, makes a logic. Let us begin with universality. That is, we ask if the language, resulting from adding second-order quantifiers to the list of logical constants to the language of first-order logic, can really be applied to any domain. The semantics for second-order logic are more complicated than the

[197]Stewart Shapiro, Foundations Without Foundationalism, (Oxford Logic Guides: 17; Oxford: Clarendon Press, 1991), pp. 137 - 140.

semantics for first-order logic. We could see some of this from the figure above. Nevertheless, the domain of individuals is still arbitrary. Thus, the individual variables range over any domain we like. This was also the case with first-order logic. Unlike in first-order logic, we also have concept variables (where "concept" is generic for predicates relations and functions). The one-place predicates are variables which range over the powerset of the set of individuals, that is, over all subsets of the domain. Thus, we have maximal arbitrariness. We shall discuss topic neutrality shortly.

As far as validity is concerned, second-order logic is more satisfying than first-order logic. This is because arguments such as:

> the first sculpture in the exhibition is large,
> the last sculpture in the exhibition is large,
> the first and last sculpture are distinct,
> therefore, there are at least two sculptures which have some property in common;

turns out valid in second-order logic. Similarly, the following argument can also be shown to be valid in second-order logic, but not in first order.[198]

> Most trees are tall.
> Most trees have green foliage.
> Therefore, there are some tall green trees.

Second-order logic has much more expressive power than first-order logic. This is because in second-order logic, we can generalise (in the sense of universally quantify) over predicates, relations and functions. More levels of generality are allowed, unlike in first-order logic, where we are restricted to generalising over individuals. Thus, there is a sense in which second-order logic is more general than first-order logic: we may not only consider and quantify over any domain of individuals, we may also quantify over their properties, relations and functions.

[198]For many more examples see: George Boolos, "Nonfirstorderizability Again", Linguistic Enquiry, XV, 2, 1984 pp. 343-4, or George Boolos, "To Be is to Be a Value of a Variable (or to Be Some Values of Some Variables)", Journal of Philosophy, Vol. LXXXI 1984, pp. 430 - 449.

From the perspective of second-order logic, first-order logic seems artificial in allowing quantification over individual variables, but not over predicates, relations and functions. As logicists, we want logic to be general: we want to be able to quantify over any domain. Quantification over first-order predicates is quantification over subsets of the domain of individuals. Quantification over n-place relations, is quantification over ordered n-tuples of the domain of individuals raised to the power n. Similarly for functions. Quantification over n-place first-order functions is quantification over (n + 1)-tuples of the domain raised to the power of (n + 1). By allowing in second-order quantifiers we have allowed ourselves, not only quantifications over the domain of individuals, but also over subsets of the powerset of that set. Because we can quantify over first-order concepts, when we say that we may quantify over any domain, we are being more general than in the first-order case. Thus, even this aspect of universality is better exemplified by second-order logic than first-order logic.

The other aspect of universality is topic neutrality. Second-order logic is categorical. That is, it can pick out models for its theories uniquely up to isomorphism. In particular, second-order logic has the expressive power to express and recognise any finite cardinality. Furthermore, in the language of second-order logic, we can express Dedekind finiteness: that a set is finite if it has no proper subset which can be placed into one-to-one correspondence with it. We can express Dedekind infinity, and the notion of powerset, and therefore, many infinite cardinalities.[199] We decided earlier, in chapter two, section one, that the capacity of a logic to pick out sets uniquely up to isomorphism was to be treated as a standard of adequacy for the ability of the logic to reflect an informal notion of logical validity.

However, we seem to run into trouble. For, if we allow quantification over individuals, properties, relations and functions,

[199] Stewart Shapiro, Foundations Without Foundationalism, (Oxford Logic Guides: 17; Oxford: Clarendon Press, 1991), pp. 141 - 157.

then we presuppose that we can distinguish the different sorts of variable. This does not require sense experience, but it may seem to require intuition. For, we are asked to identify different types or sorts. However, note that quantification over a property and quantification over a relation is a matter of number of places. A property is a one-place relation. The distinction between functions and relations is greater. However, it is worth noting, that in higher-order logic or in simple type theory, a first-order n-place function can be defined as a second-order n + 1-place relation. The function is described as a relation such that it relates series of arguments to a value. As we noted earlier, technically, there is not much difference between allowing higher-order quantification or second-order quantification. Insofar as we accept this, the salient difference between sorts of variable: individual, predicate, relation or function variable, can be reduced to one between the number of places of variables and whether they are individual or concept variables. As for the number of places, this is perfectly in keeping with our conception of logic since all we want to argue for here are relations and functions of finite numbers of places, and finite numbers are already expressible in terms of logical vocabulary. As for the appreciation of the distinction between a concept variable and an individual variable, this looks as though it might require intuition of some sort. Furthermore, the intuition might be of a kind offensive to our criteria.

The price we would have to pay in rejecting a formal system which distinguishes concepts from objects is high. For, the distinction is embedded in first-order logic in the syntax, and therefore, if we were to reject a formal system as a logic on the grounds that it distinguishes concepts from objects, we should have to give up first-order logic as well. This is just a warning. However, we do not have to give up both first and second-order logic. For, we can argue for the distinction, between concept (in the generic sense) and object, being worthy of consideration as a logical distinction. Let us apply our criteria.

We might think that in making a distinction between concept and object in this way in a formal system, that we have strayed from generality, since we have started to delineate different categories. We might think that appreciation of the difference requires sense perception or intuition. However, so far, we have left open the possibility that abstract objects can feature as individuals just as much as physical objects. Thus, apprehension of a set of objects does not require sense perception particularly, although it may do, such as when our domain is composed of the red objects in the room. We are, after all, allowed to consider any domain we like. Picking out a domain as such is not the business of logic. Logic features only when we want to make arguments concerning objects in a domain.

Furthermore, we do not require Kantian spatial or temporal intuition to appreciate the difference between objects and concepts. Nevertheless, we might still require intuition in an offensive sense. Under the view of second-order logic being developed here, once we are given a domain of objects, all subsets of those objects have a concept. Thus, for an arbitrary subset, there is a concept corresponding to it. For this reason, we do not seem to violate topic neutrality. That is, there are no particular subsets of individuals which are singled out.

Moreover, a distinction between individual and concept under this very broad understanding is nothing more than what we find when we distinguish a token from a type or a genus from a species. For example, when we say "all whales are mammals" or "Socrates is a man". This distinction is crucial in appreciating the validity of an argument, and is therefore, a crucial component of a logic according to our criterion of validity.

Moreover, with respect to topic neutrality, second-order logic fares better than the extension of first-order logic with the Mostowski generalised quantifiers.

The complaint about the Mostowski generalised quantifiers was that in the case of named cardinals, such as "most" or $\aleph_0$, we had to presuppose a sophisticated (and not obviously logical) theory about cardinal numbers.[200] In the case of ambiguous cardinal properties such as "most", "few", "half", these depended on the size of the domain. Furthermore, we may need to tinker with the standard theory of cardinality, in order to accommodate, say, the notion of "half" in some contexts involving infinite sets. *Prima facie,* this suggests that we require intuition, in some sense, to determine the truth of sentences whose truth depends upon particular non-standard theories of cardinality or some theory of topology or whatever. Of course, we have run into the analyticity criterion. The relationship between topic neutrality and analyticity is this. Topic neutrality is a generality requirement. Deciding whether or not some aspect of a logic conforms to the criterion requires that attention be paid as to the analyticity of sentences which use that aspect of the logic.

In contrast, $\exists$ and $\forall$ do not require a sophisticated background theory about cardinality. As Frege put it: "affirmation of existence is in fact nothing but denial of the number nought."[201] Similarly, the universal quantifier does not depend on the size of the domain to know its satisfaction conditions. We just take all of the domain.

Thus, according to the considerations here, where Mostowski's quantifiers fail, the second-order universal and existential quantifiers pass. It turns out that accepting the second-order universal and existential quantifiers allows us to recover at least the standard theory of cardinality, in the sense that we can define finite numbers, the notion of Dedekind finite, Dedekind infinite, $\aleph_0$, $\aleph_1$, and so on. We can also recover many of the Mostowski generalised quantifiers such as "Most", "half", and so on.[202] This allows us to show, through the definitional resources of

[200] Also, this will beg the question against the logicist, since he is not allowed to presuppose a sophisticated theory about cardinality if he is trying to prove, in any respectable sense, that the natural numbers are a part of logic.
[201] Gottlob Frege, The Foundations of Arithmetic, Trans. J. L. Austin. (Second revised edition; Evanston, Illinois: Northwestern University Press, 1980), § 53.
[202] Stewart Shapiro, Foundations Without Foundationalism, (Oxford Logic Guides: 17; Oxford: Clarendon Press, 1991), pp. 96 - 109.

second-order logic, that these cardinality notions are analytic! They then earn their place amongst the logical notions. Recall that Frege's positive characterisation of analyticity was that a sentence is analytic if it follows from logical laws together with definitions.

Grosso modo, the addition, to the language of first-order logic, of the second-order universal and existential quantifiers to the list of logical constants, is consistent with our criteria for a language to be logical. Furthermore, the formal system resulting from stipulating that the language can be applied to any domain of individuals is consistent with our criteria for being a logic. Therefore, second-order logic is logic.

Conclusion

The thesis provides a meeting ground for three broad lines of thought. One is to partially contribute towards the revival of a philosophical idea which, for various historical reasons, we too hastily abandoned. The idea is that of logicism. It was championed by Frege at the turn of the century.

The second line of thought is that logic enjoys a certain philosophical privilege. We want to make this more explicit: account for it and explain what the privilege consists in.

The third line of thought is somewhat negative. It is that if we consult the technical books on the subject of logic, mathematical logic, meta-mathematics, model theory and so on, we are faced with some formal systems which are called mathematical, and others which are called logic. For any formal system with more expressive power than first-order logic, the distinction seems somewhat arbitrary; or at least, does not seem to be motivated by an unique set of considerations. Faced with this situation, we want to give clear reasons for drawing a distinction between logic and mathematics.

The reasons can be clear, without being determinate in all cases. That our reasons should not always determine, for a given formal system, whether it constitutes logic or mathematics, is somewhat inevitable since we have chosen to draw on philosophical considerations rather than on technical ones, and the match is not guaranteed beforehand to be perfect!

This choice is in keeping with the second line of thought: that logic enjoys a certain philosophical privilege over (the rest of) mathematics. Thus, fitting the second and third lines of thought together, we have the task of facing philosophical considerations with formal systems and looking for a match. Fitting in the first line of thought, we chose our philosophical considerations from those which motivated Frege when he proposed to prove that numbers are logical objects.

Both logic and mathematics have developed considerably this century. In the light of certain developments it has turned out that we cannot entirely keep intact the philosophical considerations which motivated Frege. On the one hand, they have to be made more precise. On the other hand, they have to be modified. Nevertheless, the technical discovery that the formal system of the *Begriffsschrift*, together with Hume's principle added as an axiom, is both consistent and sufficiently powerful to derive the Peano axioms, encourages us to look again at Frege's logicist project.

The technical discovery, in conjunction with some interesting philosophical arguments to the effect that the resulting formal system goes some way towards vindicating Frege's logicism, has renewed interest in the logicist project. Two debates ensued.[203] One was centred on the status of Hume's principle. The other focused on particular arguments which were meant to show that second-order logic is really set theory. In the thesis, we have largely left these debates aside: the first, because it lies outwith the scope of the thesis; the second, because it is too narrow.

The sought after conclusion of logicist arguments is that the natural numbers are logical objects. Showing this can be thought of as having two parts. One is to show that Hume's principle is a logical principle. The other is to show that the formal system, to which we are adding Hume's principle as an axiom, deserves to be called a logic. The aim of the thesis, has been to argue that second-order logic does deserve this title.

I have endeavoured to defend this claim in two respects. One defence is against specific arguments to show that anything stronger

[203] For example, see: George Boolos, "Saving Frege from Contradiction," Proceedings of the Aristotelian Society, vol. LXXXVII (1986 - 7), pp. 137 - 151, Goege Boolos, "The Standard of Equality of Numbers," Meaning and Method: Essays in Honor of Hilary Putnam, ed. George Boolos, (Cambridge: Cambridge University Press, 1990), pp. 261 - 278, Michael Dummett, Frege Philosophy of Mathematics, (London: Duckworth, 1991), Crispin Wright, Frege's Conception of Numbers as Objects, (Aberdeen: Aberdeen University Press, 1983), Richard Heck Jnr., "On the Consistency of Second-Order Definitions," Nous, vol. XXVI.4 (1992), pp. 491 - 494, Bob Hale, "Frege's Platonism," The Philosophical Quarterly, vol. XXXIV (special issue: Frege, 1984).

than first-order logic does not deserve to be called logic. The other respect in which I have tried to defend the claim is in terms of philosophical criteria which indicate the significance of the logicist project.

There is a conjunction of criteria which are intimately related. They are: an informal notion of logical validity, universality and analyticity. Their interrelation is as follows. We have an intuitive notion as to what counts as a logically valid argument. Mathematicians in particular, have cultivated these intuitions. As it turns out, to reflect this intuitive notion, we need an expressively powerful formal language. What the intuition consists in is a sense of universality. A logically valid argument is one which can be brought to bear in any situation. Universality has two aspects. One, is that a logical language should be applicable to any domain of objects. The other, is that a logic has to be topic neutral. That is, logic ignores particular features which objects in a domain possess.

The first aspect ruled out any non-logical constants being allowed in a logical language on the grounds that these effect a selection of domains, because they are accompanied by intended interpretations. Furthermore, recognition that the intended domain is manifest comes from outside logic. It has to be expressed in a meta-language. The second aspect was expressed in terms of being able to pick out structures uniquely up to isomorphism. This proved to be a good measure of the adequacy of a language to reflect our informal notion of logical validity. What shows us that isomorphism is a salient measure is that the truths of logic must be analytic.

This criterion also has into two aspects. This time we have positive and negative characterisations. The negative aspect is that a sentence is analytically true if recognition of its truth does not rely on sense experience or on intuition. That is, we have to be able to justify an analytic assertion by appeal to very basic truths. The positive aspect is meant to flesh out what "recognition" amounts to. One does not have to look very far in Frege to think that what he had in mind was an effective proof procedure, expressed in terms of his famous gapless proofs. We rejected this positive construal of analyticity on the grounds that Gödel's incompleteness results show us that requiring all truths of a formal system to be justified by means of an effective proof procedure makes for a very limited formal system. In particular, said formal systems have very low expressive power, and therefore, compromise our validity criterion too much.

The formal system of second-order logic passes muster under these criteria. The significance of second-order logic's conforming to these criteria is that it shows wherein the feasibility of the logicist project lies in the modern context. It is also, therefore, in the light of these criteria that arguments for and against the logicist project should be conducted.

"*...et j'aurai vu encore une belle journée s'émietter on ne sait comment, inutile, raccourcie, gâchée...*"
-Collette, L'entrave

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