THE DYNAMIC TOPOLOGY OF THE SOLAR CORONA: MAPPING THE SUN'S THREE DIMENSIONAL MAGNETIC SKELETON

Benjamin Matthew Williams

A Thesis Submitted for the Degree of PhD at the University of St Andrews

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The Dynamic Topology of the Solar Corona:
Mapping the Sun’s Three Dimensional Magnetic Skeleton

Benjamin Matthew Williams

University of St Andrews

This thesis is submitted in partial fulfilment for the degree of
Doctor of Philosophy (PhD)
at the
University of St Andrews

26th March 2018
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Abstract
Observations of the surface of the Sun reveal multi-scaled, mixed magnetic features that carpet the entire solar surface. Not surprisingly, the global magnetic fields extrapolated from these observations are highly complex. This thesis explores the topology of the Sun’s global coronal magnetic fields. The magnetic skeleton of a magnetic field provides us with a way of examining the magnetic field and quantifying its complexity.

Using specialised codes to find the magnetic skeletons which were written during the course of this work, we first examine potential field extrapolations of the global solar coronal magnetic field determined from observed synoptic magnetograms from the Heliospheric Magnetic Imager on the Solar Dynamics Observatory. The resolution of the PFSS models is found to be very important for discovering the true nature of the global magnetic skeleton. By increasing the maximum number of harmonics used in the potential field extrapolations and, therefore, the grid resolution, 60 times more null points may be found in the coronal magnetic field. These high resolution fields also have a large global separator network which connects the coronal magnetic field over large distances and involves between 40% and 60% of all the null points in the solar atmosphere. This global separator network exists at both solar minimum and solar maximum and has separators that reach high into the solar atmosphere (> 1R⊙) even though they connect null points close to the solar surface.

These potential field extrapolations are then compared with magnetohydrostatic (MHS) extrapolations of the coronal magnetic field which also provide us with information about the plasma in the corona. With a small component of electric current density in the direction perpendicular to the radial direction, these MHS fields are found to have a plasma beta and pressure typical of the corona. As this small component of electric current density grows, the heliospheric current sheet is warped significantly and the magnetic field, plasma beta and pressure become unphysical.

Torsional spine reconnection is also studied local to a single null point. First using a dynamical relaxation of a spiral null point under non-resistive magnetohydrodynamics (MHD) to a MHS equilibrium is form in which a current layer has built up around the spine lines. Then the reconnection under resistive MHD in this current sheet is studied. The current about the spine lines is dissipated and the magnetic energy is mainly converted into heat directly as the field lines untwist about the spine line.
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Chapter 1

Introduction

1.1 The Sun

The Sun (see figure 1.1) forms the central point of our solar system and is our nearest star at a mere \(1.496 \times 10^8\) km from Earth. Its radius is about 109 times that of Earth (and measures \(R_\odot = 6.957 \times 10^8\) m = 695.7 Mm) and its mass about 330000 times that of Earth \(M_\odot = 1.988 \times 10^{30}\) kg).

The Sun is very hot and its lowest temperature is around a few thousand Kelvin although it reaches temperatures of over ten million Kelvin in its core. Since it is so hot, the Sun is essentially a ball of plasma. Plasma is a fourth state of matter similar to a gas but with electrons and atomic nuclei (ions) free of each other and so not forming to the standard atomic model. This ionisation means plasmas are highly electrically conductive and so contain electric and magnetic fields that can play a major role in determining the behaviour of the matter. In particular, the Sun’s magnetic field forms an important underlying property of the Sun which is key to understanding its dynamics.

1.1.1 The Solar Interior

The solar radius \(R_\odot = 695.7\) Mm) is defined by the radius to the layer in the Sun’s photosphere at which its optical depth equals 2/3 (Haberreiter, Schmutz and Kosovichev 2008). Essentially, this is the layer in the photosphere where visible light is now able to escape from the solar interior. The solar interior is made up of three separate regions: the core, the radiative zone and the convective zone.

The energy the Sun radiates is the most important source for life on Earth even though Earth receives only a fraction of the total amount of energy emitted by the
1. Introduction

Figure 1.1
An image of the Sun in visible light from 26th September 2014. It shows several sunspots on the photospheric surface. Courtesy of NASA/SDO and the AIA, EVE, and HMI science teams.
1.1. *The Sun*

Sun. The Sun’s energy comes from the nuclear fusion in its core. This nuclear fusion creates Helium and other heavier elements by fusing together, principally two hydrogen atoms, but also Hydrogen with other elements. The Sun is now made up of 92% hydrogen, 8% helium and all the other elements make up the other 0.1%. By around 30% of the Sun’s radius, fusion has almost entirely stopped. The Sun is currently in an approximate equilibrium: a higher rate of fusion would cause the core to heat up and expand which in turn would lower the pressure reducing the amount of fusion and conversely for a lower rate of fusion. The fusion process generates significant amounts of energy. This is the reason for huge temperatures found in the core which is modelled to be of the order 15 MK. The core makes up about 20% to 25% of the radius of the Sun’s interior but 99% of the energy emitted by the Sun is produced there (Priest 2014).

Surrounding the core, there is the radiative zone extending up to approximately 70% of the radius of the Sun. Here the energy generated in the core continually leaks outward by radiative diffusion. This radiation can take thousands to millions of years to pass through the radiative zone due to the density of the plasma (Mitalas and Sills 1992) as the photons bounce around being scattered and emitted by the plasma particles. In the final, and outer region, of the solar interior, the convective zone, there is a transition from radiation to convection being the dominant method of heat transfer.
1. Introduction

The temperature, density and pressure in the lower solar atmosphere with the heights of the photosphere, chromosphere and the corona according to the VAL (Vernazza-Avrett-Loeser) model. (Data from Avrett and Loeser 2008)

Figure 1.3

transport. Radiation does not stop in this region and there is still a large amount of radiative heat transfer. At the transition between the radiative and convective zones, there is a strong shear layer which is called the tachocline (Spiegel and Zahn 1992) where it is expected that much of the Sun’s large-scale magnetic field is generated. The core and the radiation zone are so dense that they rigidly rotate whereas, in the convection zone, differential rotation becomes important which creates this shear layer.

1.1.2 The Solar Atmosphere

The exterior of the Sun is known as the solar atmosphere. The atmosphere can be thought of as four different regions: the photosphere, the chromosphere, the transition region and the corona. Although these regions can be thought of as spherical shells like in figure 1.2, this is far from the real picture because of the highly turbulent plasma at the solar surface.

The underlying driving mechanism of this turbulence is the convective zone below the solar surface which creates a granulation effect across almost the entire surface of the Sun (Leighton 1963). There are millions of irregularly shaped granules continually forming and disappearing in a turbulent manner making the surface look like a
1.1. *The Sun*

![Sunspot and Granulation](image)

Figure 1.4

A sunspot surrounded by the granulation on the solar surface.

boiling liquid with each granule lasting only 20 minutes or so. These granules are due to the rising and falling of plasma as it is heated within the Sun. The plasma rises to the surface where it cools and falls again under convection. The centre of each granule is the location of the hot rising plasma with cool and falling plasma at the granule edges. Features such as sunspots can stop granulation due to the intense magnetic fields which develop. Figure 1.4 shows an example of a sunspot and the surrounding granulation on the solar surface. A typical granule has a diameter on the order of $1.5 \text{ Mm}$. The size of sunspots can vary massively.

The photosphere is a comparatively very thin layer of the solar atmosphere at only several hundred kilometres thick. It is relatively dense compared with the rest of the solar atmosphere (figure 1.3) but it is cooler than the majority of the solar atmosphere. It emits the majority of the solar radiation in all different wavelengths which are able to travel through the layers above into the solar system. As the deepest layer of the Sun that we can see, this can be thought of as the solar surface, although, only to give the Sun this planet-like property. There is no physical surface as such. The photospheric magnetic field is very complex. It is made up of many patches of positive (outward) and negative (inward) magnetic field patches of varying size and strength. In the quiet Sun, these patches are very small but in active regions of the Sun, these patches of magnetic field can be strong and grow very large. Figure 1.8 shows magnetograms of the photospheric magnetic field. On the left, we see mainly small scale quiet Sun magnetic field but on the right, there are active regions
and large scale patches of magnetic field have formed. Above the photosphere is the chromosphere. At the boundary between these two layers is where the lowest temperature is obtained. The chromosphere is slightly warmer than the photosphere, but throughout it, the density also continues to fall off with height above the solar surface (figure 1.3).

Above the chromosphere, there is a very narrow layer called the transition region where there is a very rapid rise in the temperature and fall off in the density (figure 1.3. The density falls off by around a factor 100 while the temperature rises by a factor of around 150 to 200. After this region of rapid change, the density and temperature begin to stabilise and just gradually fall off for the rest of the corona. The plasma of the corona extends out into deep into the solar system. There is a flow of plasma out from the Sun called the solar wind. This can be detected out at the edges of the solar system. This stream of charged particles causes some of the aurorae we see on Earth.

The significant temperatures in the corona of over 1 MK are several hundred times hotter than the solar surface but the exact heating mechanism remains unknown (Parnell and De Moortel 2012). This is called the coronal heating problem and is a topic of active research around the world. It was not until 1940 that it was realised this was the case. Before then it was thought that the temperature in the solar atmosphere just decreased with height. What is known, however, is that the energy to heat the whole solar atmosphere comes from the Sun’s magnetic field.

1.2 The Magnetic Field in the Solar Atmosphere

The Sun has a magnetic field which governs much of the dynamics in the solar atmosphere. For the majority of the Sun, the magnetic field is frozen-in to the plasma (for further details see section 1.5 on MHD). This means the plasma and magnetic field move in unison. The turbulent nature of the solar plasma at the solar surface injects energy into the magnetic field as it moves around causing twisting and winding of the magnetic field throughout the solar atmosphere. The speed of these turbulent motions either cause waves or, if it is slow enough, move the field about which stresses it.

Another property of the Sun is that it differentially rotates. This means that, in the
outer layers of the solar interior, the plasma at the equator rotates faster than at the poles causing a shearing along lines of constant latitude that can also help inject more energy and twist up the field further. These effects when coupled with the frozen-in nature of the field mean that the Sun ends up having a highly complex magnetic field.

The Sun’s magnetic field can form many structures, one example of which is a solar prominence (also called a filament depending on how it is viewed). Prominences are anchored at the solar surface and can extend thousands of kilometres out into the solar atmosphere. A typical length of a solar prominence is 10 Mm to 100 Mm. The dense plasma structure is supported by the solar magnetic field against falling under gravity as magnetic forces dominate over the hydrodynamic forces (Hansen and Bellan 2001). However, it is not yet known exactly how and why they initially form. These structures can last over long time periods, perhaps months, and usually erupt giving rise to a coronal mass ejection which ejects plasma far into space.

1.3 The Solar Cycle

The Sun’s magnetic field and activity varies, on average, over an 11 year cycle known as the solar cycle, although each cycle can vary between nine and fifteen years. The solar cycle can be seen in both sunspot data, shown in figure 1.6, and in mean polar magnetic field data (figure 1.7). Each cycle goes through what are known as a period of solar minimum and a period solar maximum. At a solar minimum, the Sun’s large scale global magnetic field is almost dipolar-like and during this period, the Sun has...
1. Introduction

Figure 1.6
Yearly averaged sunspot data since 1700. Source: SILSO World Data Center (1700-2017)

Figure 1.7
1.4. Observing the Sun

Over the last twenty five years, the Sun has been observed in ever greater detail due to the advances in imaging technology. There are now many ground and space based...
1. Introduction

telescopes observing the Sun. Ground based telescopes can be built without worrying about restrictions of weight and size but are hindered by observing conditions such as night time and the clouds and haze which obscure the Sun. Although their cost is much higher, this gives a huge advantage to space observatories which can observe the Sun without interruption if they are in a polar orbit about the Earth or a Sun synchronous orbit.

One of the more recent satellites is the Solar Dynamics Observatory (SDO) which was launched by NASA in 2010 and contains, on board, many different instruments like the Helioseismic and Magnetic Imager (HMI) and Atmospheric Imaging Assembly (AIA) (United States. National Aeronautics and Space Administration 2009). It is in a geosynchronous orbit about the Earth and provides us with a continuous view of the Sun. Data from HMI is used extensively in this thesis as it produces full-disk, high resolution magnetograms. These are used to make synoptic maps by taking strips of data from the full-disk magnetograms (see figure 1.9d for example) and stitching them together to provide a global view of the magnetic field on the solar surface. Either daily or Carrington rotation (one solar rotation) synoptic maps are available. Synoptic maps of the magnetic field of the solar surface have been available for more than thirty years from the ground based National Solar Observatory and fifteen years from the space based Solar and Heliospheric Observatory but these data are at a much lower resolution than those produced using HMI.

AIA provides high resolution, full-disk views of the Sun over many wavelengths of the electromagnetic spectrum. Each wavelength representing a different range of temperatures which allow observers to see the Sun in great detail. Each different observed wavelength is typically represented by a unique colour so many wavelengths may be overplotted on the same image which helps to build up “layers” of the plasma on what would otherwise appear like the flat disk of the Sun. These colours are not the real colours of the light from the Sun since the wavelengths observed are in the UV (ultraviolet), EUV (extreme ultraviolet) and X-ray part of the electromagnetic spectrum.

Figure 1.9 shows three AIA images of these different wavelengths. It is possible to make out the hot coronal loop structures in these images and the so-called active regions above sunspots in AIA 94 Å (figure 1.9c) and the granulation in the AIA 304 Å image (figure 1.9b). The regions with a lot of hot plasma, such as the active regions, are where there are strong areas of magnetic field as seen in the HMI magnetogram
1.4. Observing the Sun

(a) AIA 171 Å (6.3 × 10^5 K, quiet corona and the upper transition region)

(b) AIA 304 Å (5 × 10^5 K, Chromosphere and the transition region)

(c) AIA 94 Å (6.3 × 10^6 K, flaring regions)

(d) HMI Magnetogram

Figure 1.9
A sample of images from SDO from 16th November 2014 between 15:10 and 15:15 UTC. Courtesy of NASA/SDO and the AIA, EVE, and HMI science teams.
1. Introduction

Coronal plasma prefers to travel along coronal loops which makes them easy to see in the images. From the coronal images, loop-like structures are clearly visible. These are magnetic structures and they are highlighted in these images because the influence of thermal conduction acts $10^{13}$ times greater along field lines than across.

1.5 Magnetohydrodynamic (MHD) Equations

There are three main approaches used for modelling the behaviour of plasmas. One is a kinetic approach which models the dynamics of the plasma at a microscopic level. The second approach involves modelling the behaviour of similar types of particles as separate fluids. For instance in a two fluid approach, the equations for the electrons and the ions are solved separately. The third approach considers the plasma as a single, large scale, macroscopic fluid. It is called MagnetoHydroDynamics (MHD).

The MHD equations are a combination of the fluid equations (involving the Navier-Stokes equation from fluid dynamics) and Maxwell’s equations from electromagnetism. The model assumes a single large scale fluid which is continuous in space. It is possible to derive the MHD equations from the particle equations by taking different order moments with respect to velocity space. However, here we will not discuss the kinetic model but simply state the MHD equations. Much of the work in this thesis is over large length scales which is one of the central assumptions of MHD.

The assumptions of MHD are

- the plasma is quasi-neutral, which means the plasma acts as if the overall charge is zero and the number of positive and negative charges is equal in each volume element
- the speed of the plasma is much smaller than the speed of light
- the pressure can be written as a scalar quantity
- the length and time scales of the system are much longer than the equivalent kinetic scales.

These equations form a remarkably successful model for modelling the dynamics of a plasma despite their seemingly unreasonable assumptions about the plasma. The equations of MHD are now discussed.
1.5 Magnetohydrodynamic (MHD) Equations

1.5.1 Maxwell’s Equations

Maxwell’s equations describe the fundamental properties of electromagnetism. They are named after James Clark Maxwell who published them in 1865 (Maxwell 1865). There are two different versions of the equations for both microscopic and macroscopic scales, but only the macroscopic equations are discussed here. They are given by the following four equations:

1. Ampère’s Law:
   \[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}, \tag{1.1} \]

2. Faraday’s Law:
   \[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.2} \]

3. Gauss’ Law:
   \[ \nabla \cdot \mathbf{E} = \frac{\rho_c}{\varepsilon_0}, \tag{1.3} \]

4. Solenoidal Condition:
   \[ \nabla \cdot \mathbf{B} = 0, \tag{1.4} \]

where \( \mathbf{B} \) is the magnetic induction, \( \mathbf{J} \) is the current density, \( \mathbf{E} \) is the electric field and \( \rho_c \) is the charge density which all vary with space and time. The speed of light, \( c \), is a constant and given in terms of two constants: magnetic permeability, \( \mu_0 = 4\pi \times 10^{-7} \text{ H m}^{-1} \), and permittivity of free space, \( \varepsilon_0 \approx 8.854 \times 10^{-12} \text{ F m}^{-1} \). They are related by

\[ c^2 = \frac{1}{\mu_0 \varepsilon_0} \]

such that \( c \) is given exactly as 299 792 458 m s\(^{-1} \). The quasi-neutral approximation of MHD implies \( \rho_c = 0 \). However, Gauss’ law does not form part of the MHD equations.

Maxwell originally derived his equations in terms of the electric displacement, \( \mathbf{D} \), and the magnetic field, \( \mathbf{H} \), which are related to \( \mathbf{E} \) and \( \mathbf{B} \) in a vacuum by

\[ \mathbf{B} = \mu_0 \mathbf{H} \]

\[ \mathbf{D} = \varepsilon_0 \mathbf{E}. \]

However, Maxwell’s equations are now usually given in the above way in terms of \( \mathbf{B} \) and \( \mathbf{E} \). Although \( \mathbf{B} \) is actually the magnetic induction, it generally referred to as the
magnetic field in solar and astronomical contexts. In this thesis, \( \mathbf{B} \) will be called the magnetic field from now on.

Ampère’s law (1.1) implies that a temporally changing electric field or an electric current density produce a magnetic fields where as Faraday’s law (1.2) and Gauss’ Law (1.3) imply that that a temporally changing magnetic field or a non-zero electric charge density creates an electric field. The solenoidal condition (1.4) then implies that there are no magnetic monopoles.

The solenoidal condition can also be thought of an initial condition. Using Faraday’s Law, if the initial magnetic field is divergence free, it will stay divergence free for all time:

\[
\frac{\partial}{\partial t} (\nabla \cdot \mathbf{B}) = \nabla \cdot \frac{\partial \mathbf{B}}{\partial t} = - \nabla \cdot (\nabla \times \mathbf{E}) = 0.
\]

### 1.5.2 The MHD Approximation to Ampère’s Law

To derive the MHD equations, an approximation is made that affects Ampère’s law. A fundamental assumption of MHD is that there are no relativistic effects. That is, a typical speed, \( v_0 \), in the system, is much smaller than the speed of light:

\[ v_0 \ll c. \]

\( v_0 \) is the characteristic speed of the bulk plasma. It is related to the typical length, \( L_0 \), and time scales, \( t_0 \), by \( L_0 = v_0 t_0 \).

Looking at the characteristic scales using Faraday’s Law (1.2), we form the following relationship,

\[
\begin{align*}
\left[ \frac{\partial \mathbf{B}}{\partial t} \right] &= [\nabla \times \mathbf{E}], \\
\frac{B_0}{t_0} &= \frac{E_0}{L_0}, \\
E_0 &= v_0 B_0.
\end{align*}
\]

Using this relationship between the characteristic scales, we find that,

\[
\left[ \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \right] = \frac{E_0}{c^2 t_0} = \frac{v_0 B_0}{c^2 t_0} = \frac{v_0^2 B_0}{c^2 L_0} = \frac{v_0^2}{c^2} [\nabla \times \mathbf{B}].
\]
1.5. Magnetohydrodynamic (MHD) Equations

By the assumption earlier that \( v_0 \ll c \),

\[
\frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \ll |\nabla \times \mathbf{B}|,
\]

and so in MHD the third term in Ampère’s Law is neglected and equation 1.1 becomes

\[
\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{1.5}
\]

1.5.3 Ohm’s Law

Georg Ohm came up with this equation in 1827 in a much simpler form (Ohm 1827). However the term Ohm’s law is used to refer to any generalisation of his original relationship between the electric field and the current. Ohm’s Law is the equation that links the electromagnetic fields in Maxwell’s equations to the plasma that is governed by the fluid equations. In its classical form, Ohm’s law is given by

\[
\mathbf{J} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{1.6}
\]

where \( \mathbf{v} \) is the plasma velocity and \( \sigma \) is the electrical conductivity. This is a generalisation of the simpler, more well known version of Ohm’s law given by \( V = IR \). Ohm’s law is where extra electromagnetic effects may be applied, one example of which is the Hall effect.

1.5.4 Fluid Equations

In the 18\textsuperscript{th} century, Leonard Euler published his equation of motion to describe incompressible, frictionless fluids. But in 1822, Claude-Louis Navier generalised the equation by adding viscosity. Sir George Gabriel Stokes then improved on Navier’s work throughout the middle of the 19\textsuperscript{th} century. Their equation is still used today and is known as the Navier-Stokes equation which is the equation of motion below with viscosity. We use it here to describe the motion of the plasma in terms of the forces that act on it.

The plasma is coupled to the magnetic field through the following four fluid equations

1. Continuity of Mass:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1.7}
\]
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2. Energy Equation:

\[
\frac{\rho^\gamma}{\gamma - 1} \frac{D}{Dt} \left( \frac{p}{\rho^\gamma} \right) = -\mathcal{L},
\]

(1.8)

3. Equation of Motion:

\[
\frac{Dv}{Dt} = J \times B - \nabla p + \rho g + F,
\]

(1.9)

4. Ideal Gas Law:

\[
p = \frac{1}{\mu} \rho RT,
\]

(1.10)

where \( \rho \) is the plasma density, \( p \) is the plasma pressure, \( T \) is the plasma temperature, \( g \) is the gravitational acceleration, \( F \) is any other force on the system, \( R \) is the specific gas constant, \( \mu \) is the mean atomic weight, \( \gamma \) is the ratio of specific heats and \( \mathcal{L} \) is the energy loss function. Here we will take \( \mu = 1 \) and \( \gamma = 5/3 \). \( R \) is related to the gas constant \((8.31 \text{ J K}^{-1} \text{ mol}^{-1})\) and here is taken to be \( R \approx 8.31 \times 10^3 \text{ J K}^{-1} \text{ kg}^{-1} \).

The continuity of mass equation (1.7) describes the conservation of mass i.e. mass cannot be created or destroyed and the energy equation (1.8) simply describes the conservation of energy. If \( \mathcal{L} = 0 \), this describes an adiabatic plasma where there are no thermal effects between the plasma and its surroundings and so entropy is conserved. However, accounting for all the energy changes in MHD, \( \mathcal{L} \) is generally written as

\[
\mathcal{L} = \nabla \cdot q + L_r - \frac{J \cdot J}{\sigma} - H_\nu
\]

where \( q \) is the heat flux due to conduction, \( L_r \) is the net radiation, the term involving \( J \) represents ohmic heating and \( H_\nu \) represents viscous heating associated with the viscous force described later.

The equation of motion (1.9) is the Navier-Stokes equation with the added Lorentz force due to the magnetic field. It is essentially Newton’s second law for a fluid rather than one particle and describes each of the forces on the plasma:

1. \( -\nabla p \) is the pressure force
2. \( J \times B \) is the Lorentz force
3. \( \rho g \) is the force due to gravity where \( g \) is generally either in the \( e_r \) or \( k \) direction depending on the coordinate system used in the model.
4. \( F \) is used to denote any other forces on the plasma such as viscous forces (\( F_\nu \)).
The usual form in a compressible plasma for the viscous force is

\[ F_\nu = \rho \nu \left( \nabla^2 \mathbf{v} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{v}) \right) \]

This simplifies to the usual form of \( F_\nu = \rho \nu \nabla^2 \mathbf{v} \) used in the Navier-Stokes equations when the fluid is incompressible \((\nabla \cdot \mathbf{v} = 0)\).

Finally, the ideal gas law (1.10) describes the relationship between the plasma pressure, density and temperature and closes the system of equations in MHD. It assumes there are no interactions between individual particles i.e. the plasma behaves like a perfect gas.

Equation 1.2 and equations 1.5 to 1.10 altogether form the MHD equations. The solenoidal condition (1.4) is not part of the MHD equations but it is instead an initial condition. Sometimes the MHD equations are referred to as non-resistive and resistive MHD and each of these imply the terms included in the equations. Resistive and non-resistive MHD simply imply whether magnetic resistivity is included. In non-resistive MHD, \( \eta = 0 \) and Ohm’s law becomes \( \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \).

### 1.5.5 The Induction Equation

Although the equations of MHD are now complete, there is one more major equation in MHD. It is formed from the curl of Ohm’s Law, Faraday’s Law and the MHD approximation of Ampère’s Law and is given by

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \nabla \times (\eta \nabla \times \mathbf{B}) \]

where \( \eta = 1/\mu_0 \sigma \) is the magnetic diffusivity. This is the induction equation and which provides an insight into the important processes given by the two terms on the right hand side of the induction equation. These processes are called advection and diffusion.

Looking at the ratio of the typical sizes of the two terms on the right hand side of the induction equation, we can define the dimensionless magnetic Reynolds number

\[ R_m = \frac{[\nabla \times (\mathbf{v} \times \mathbf{B})]}{[\eta \nabla^2 \mathbf{B}]} = \frac{L_0 v_0}{\eta} \]

\( R_m \) determines whether advection or diffusion dominates the magnetic evolution.
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For \( R_m \gg 1 \), advection dominates and the induction equation reduces to

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}).
\]

In this limit, the magnetic field is frozen into the plasma and leads to the frozen-in-flux theorem which says that

\[
\frac{d}{dt} \left( \iint_S \mathbf{B} \cdot d\mathbf{S} \right) = 0,
\]

for a time-evolving surface \( S \) that moves with the plasma. This means that as the plasma moves, so does the magnetic field in the same way. The magnetic field is advected with the plasma. If at some time, \( t \), two fluid elements lie upon a magnetic field line then they will lie upon the same magnetic field line for all time. The other consequence is that magnetic flux is always conserved.

In the opposite limit, when \( R_m \ll 1 \), instead of advection, diffusion dominates and

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\eta \nabla \times \mathbf{B}) = \eta \nabla^2 \mathbf{B}
\]

(if \( \eta \) is uniform). This type of differential equation is known as a diffusion equation. The diffusion term will attempt to smooth out the magnetic field. When diffusion dominates, the magnetic field is no longer frozen-into the plasma but can slip through the plasma, i.e. plasma elements are no longer tied to the field lines.

For the majority of the Sun, \( R_m \) is large since \( L_0 \) and \( v_0 \) (typical lengths and speeds respectively) are large, therefore advection dominates. Diffusion only dominates in very localised regions such as current sheets where there are large gradients in the magnetic field and therefore very small length scales.

1.5.6 The Lorentz Force

The Lorentz force is the force felt by a fluid particle element due to the magnetic field. The Lorentz force can itself be divided up into two different forces. Using vector identities, the Lorentz force can be written as

\[
\mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} - \nabla \left( \frac{\mathbf{B} \cdot \mathbf{B}}{2\mu_0} \right).
\]
1.6. Magnetic Null Points

The two terms here are the magnetic tension and magnetic pressure forces. The tension force acts to straighten field lines, while the pressure forces acts to spread the field out to form a uniform magnetic field strength everywhere. So for a field with a region of high $|\mathbf{B}|$ and a region of low $|\mathbf{B}|$, the pressure force will act in the direction from high $|\mathbf{B}|$ to low $|\mathbf{B}|$ and in order to eradicate the gradient in the field strength.

1.5.7 MHD Wave Speeds

Through linearisation of the MHD equations, it can be shown that there are three different types of waves in MHD (Priest 2014), namely, Alfvén waves and two types of magnetoacoustic waves (fast and slow). These speeds are used later in this thesis. Alfvén waves are incompressible waves whose energy is carried along the field lines and have a characteristic speed of

$$c_A^2 = \frac{\mathbf{B} \cdot \mathbf{B}}{\mu_0 \rho}.$$  

The magnetoacoustic waves are compressible waves. Fast magnetoacoustic waves can propagate perpendicular to the magnetic field and they have a characteristic speed of

$$c_{mf}^2 = c_A^2 + c_s^2$$

where

$$c_s^2 = \frac{\gamma p}{\rho}$$

is the sound speed in the plasma. Slow magnetoacoustic waves cannot travel perpendicular to the magnetic field and they propagate fastest parallel to the field like Alfvén waves. However they travel much slower than Alfvén waves as their characteristic speed is given by

$$c_{ms}^2 = \frac{c_s^2 c_A^2}{c_s^2 + c_A^2}$$

1.6 Magnetic Null Points

Magnetic null points are points where all three components of the magnetic field are zero. Null points can exist in any field. However because of the solenoidal condition, magnetic null points have a limited number of possible structural forms. They are
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studied extensively throughout this thesis.

To study the magnetic field structure at null points we make use of the local linear field about a null point. The magnetic field at any point, \( \mathbf{r} = (x, y, z) \), may be written using a Taylor series in 3D expanded about a point \( \mathbf{n} \) as

\[
\mathbf{B}(\mathbf{r}) = \mathbf{B}(\mathbf{n}) + \nabla \mathbf{B}(\mathbf{n}) \cdot (\mathbf{r} - \mathbf{n}) + \cdots.
\]

The local linear field about a 3D null point at \( \mathbf{n} \) (which, without loss of generality, may be assumed to be at \((0, 0, 0)\) so \( \mathbf{n} = (0, 0, 0) \)), simplifies to

\[
\mathbf{B}(\mathbf{r}) = \left| \begin{array}{ccc}
\frac{\partial B_x}{\partial x} & \frac{\partial B_y}{\partial x} & \frac{\partial B_z}{\partial x} \\
\frac{\partial B_x}{\partial y} & \frac{\partial B_y}{\partial y} & \frac{\partial B_z}{\partial y} \\
\frac{\partial B_x}{\partial z} & \frac{\partial B_y}{\partial z} & \frac{\partial B_z}{\partial z}
\end{array} \right| \mathbf{n} \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\]

since \( \mathbf{B}(\mathbf{n}) = 0 \).

This form for the local linear field about a null point is made use of throughout this section. First we look at null points in two dimensions.

1.6.1 2D

In 2D, there are only two different types of null points. Applying equation 1.11 in 2D, the local field to a 2D null point can be written as

\[
\mathbf{B}(x, y) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\]

However, since \( \nabla \cdot \mathbf{B} = 0 \), this implies \( a = -d \). This means in general the local field is given by

\[
\mathbf{B}(x, y) = \begin{pmatrix} ax + by \\ cx - ay \end{pmatrix},
\]

for \( a, b, c \in \mathbb{R} \).
1.6. Magnetic Null Points

Finding the field lines using a parameter $s$ gives us a differential equation in $x$ and $y$

\[ \frac{1}{y} \frac{d^2 y}{ds^2} = a^2 + bc = \frac{1}{x} \frac{d^2 x}{ds^2} \]

So for $a^2 + bc > 0$ we get hyperbolic field lines and for $a^2 + bc < 0$ we get elliptic field lines. In the case when $a^2 + bc = 0$, we get a degenerate null line where there are simply anti-parallel field lines so this is not considered. These two cases are called X-type ($a^2 + bc > 0$) and O-type ($a^2 + bc < 0$) null points and are illustrated in figure 1.10. In both cases the field strength is simply decreasing to zero as the null is approaching. The significant feature of the X-type null points is their asymptotes which are known as separatrices. These separate the magnetic field domain into four topologically different regions as shown in figure 1.10.

1.6.2  3D

In three dimensions, $\nabla \cdot \mathbf{B} = 0$ implies the three dimensional magnetic null points have a surface of field lines (fan plane) and two lines (spines) extending into/out of them, instead of just the two lines seen in the case of X-type 2D null points. A 3D magnetic null point is characterised by three different mathematical properties: its
1. Introduction

sign, the inclination of its spine line to its fan plane and the nature of the field in the fan plane. These can be characterised by analysing at the local field about a magnetic null point (equation 1.11). Here, we will only consider first order 3D null points.

Using the constraint that \( \nabla \cdot \mathbf{B} = 0 \) implies the trace of the Jacobian matrix \( \mathbf{M} = \nabla \mathbf{B}(\mathbf{n}) \) in the local linear field must sum to zero:

\[
\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0.
\]

Defining the three eigenvalues of \( \mathbf{M} \) to be \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) and the corresponding eigenvectors to be \( \mathbf{e}_1, \mathbf{e}_2 \) and \( \mathbf{e}_3 \) respectively, then the solenoidal constraint requires that the eigenvalues of \( \mathbf{M} \) must sum to zero:

\[
\lambda_1 + \lambda_2 + \lambda_3 = 0.
\]

Following the approach of Parnell et al. (1996), the equation of a general magnetic field line in the local linear field about a null point at \((0,0,0)\) dependent on some parameter \( s \) along the field lines may be written in the form

\[
\frac{d\mathbf{r}}{ds} = \mathbf{M}(s) = \mathbf{B}(\mathbf{r}) .
\]

Using the substitution \( \mathbf{r}(s) = \mathbf{P} \mathbf{u}(s) \), where \( \mathbf{P} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \) is the matrix of eigenvectors of \( \mathbf{M} \), this equation becomes

\[
\frac{d\mathbf{u}}{ds} = \mathbf{P}^{-1} \mathbf{M} \mathbf{u}(s) .
\]

We now have two cases depending on whether \( \mathbf{M} \) is diagonalisable or not. Here, the case where \( \mathbf{M} \) is diagonalisable is discussed to show how the theory works. A more detailed derivation of both cases can be found in Parnell et al. (1996).

If \( \mathbf{M} \) is diagonalisable then

\[
\mathbf{A} = \mathbf{P}^{-1} \mathbf{M},
\]

where \( \mathbf{A} \) is the matrix of eigenvalues of \( \mathbf{M} \) given by

\[
\mathbf{A} = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}.
\]
1.6. Magnetic Null Points

Then we can write the solution as

\[ u = e^{\Lambda s} a = \begin{pmatrix} e^{\lambda_1 s} & 0 & 0 \\ 0 & e^{\lambda_2 s} & 0 \\ 0 & 0 & e^{\lambda_3 s} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 e^{\lambda_1 s} \\ a_2 e^{\lambda_2 s} \\ a_3 e^{\lambda_3 s} \end{pmatrix}, \]

where \( a = (a_1, a_2, a_3) \) is the vector constant of integration.

Since \( r(s) = Pu(s) \), we may write the magnetic field lines now as a linear sum of the three eigenvectors of \( M \)

\[ r(s) = Pe^{\Lambda s} a, \]

\[ = a_1 e^{\lambda_1 s} e_1 + a_2 e^{\lambda_2 s} e_2 + a_3 e^{\lambda_3 s} e_3. \]  \hspace{1cm} (1.12)

Since the eigenvalues must sum to zero, we must have two eigenvalues of one sign and one of the other. Here, taking \( \Re(\lambda_1) > 0 \) and \( \Re(\lambda_2), \Re(\lambda_3) < 0 \) (a negative null point) means that as \( s \to \infty \) (i.e. we travel forwards along a field line close to the null), we get

\[ r(s) \to a_1 e^{\lambda_1 s} e_1. \]

That is, in one direction along the field lines, all field lines converge to one of the eigenvectors. This eigenvector \( e_1 \) is called the spine since it is associated with the eigenvalue \( \lambda_1 \) which is of opposite sign to the other two eigenvalues \( \lambda_2 \) and \( \lambda_3 \).

Oppositely, taking \( s \to -\infty \) we get

\[ r(s) \to a_2 e^{\lambda_2 s} e_2 + a_3 e^{\lambda_3 s} e_3, \]

since \( \lambda_2, \lambda_3 < 0 \) and the two eigenvectors \( e_2 \) and \( e_3 \) form what is called the fan plane. The field lines come down the spine close to a 3D null point and spread over this plane. This known as a negative null point. If these two eigenvalues are both real and one eigenvalue dominates over the other, say \( |\lambda_3| > |\lambda_2| \), the field lines in the fan plane tend to just one eigenvector, \( e_3 \). This situation, which is very common, implies that the local field lines about the null point behave as if they are following another spine rather than lying in a plane. This causes the null point to look more like a 2D X-type null point rather than a 3D null point.

If the signs of the eigenvalues are reversed to that above, we would have a positive null point and the argument above simply reverses. So the signs of the eigenvalues
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A potential, radial 3D null point (red dot) surrounded by magnetic field lines in black. The fan plane and the spine line are denoted by the red dotted lines.

Figure 1.11
A potential, radial 3D null point (red dot) surrounded by magnetic field lines in black. The fan plane and the spine line are denoted by the red dotted lines.

characterises the sign of the null point and in which direction the field lines are directed locally about the null point.

In the most simple case, where we have a purely radial, potential null point, the fan plane and spine line are perpendicular to each other. A positive, potential null point is illustrated in figure 1.11.

The Jacobian matrix, $M$, may be reduced to a much simpler form as shown in Parnell et al. (1996). Without loss of generality, the magnetic field of the null point may be rotated twice: first such that the spine line is directed along the $z$-axis and then it may be rotated again about the $z$-axis so that any component of current perpendicular to the spine line is directed along the $x$-axis. This means that the current associated with the null point is now of the form $J = (J_\perp, 0, J_\parallel)$ with parallel ($||$) and perpendicular ($\perp$) being relative to the spine line.

The magnetic field of such a null point, after applying a scaling factor, can be written in the form

$$B = \begin{pmatrix}
1 & \frac{1}{2}(a - J_\parallel) & 0 \\
\frac{1}{2}(a + J_\parallel) & b & 0 \\
0 & J_\perp & -(b + 1)
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}$$
1.6. Magnetic Null Points

with \( a, b \in \mathbb{R} \) being two parameters corresponding to the potential field. The parameter \( a \) can be thought of as rotation parameter and \( b \) controls the relative strength of the potential field eigenvalues. However, these parameters are restricted. The configuration here corresponds to a positive null. If the scaling factor is negative, the signs of the terms would switch and this would reveal a negative null. As will be seen later, the final \(- (b + 1)\) element is the eigenvalue of the eigenvector corresponding to the spine line. The sign of this must be negative for a positive null and the other two eigenvalues must be positive to correspond to the fan. This requires \( b \geq -1 \) and \( a^2 \leq J_\parallel + 4b \).

The eigenvalues of the matrix are given as

\[
\lambda_1 = \frac{b + 1 - \sqrt{(b - 1)^2 + a^2 - J_\parallel^2}}{2},
\]

\[
\lambda_2 = \frac{b + 1 + \sqrt{(b - 1)^2 + a^2 - J_\parallel^2}}{2},
\]

\[
\lambda_3 = -(b + 1)
\]

The first two eigenvalues are associated with the fan plane and the third is associated with the spine line.

By considering a potential field where \( J = 0 \) and without loss of generality taking \( a = 0 \) (the restriction on the current perpendicular to the fan plane no longer holds), the Jacobian matrix of this field may be reduced further to

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & b & 0 \\
0 & 0 & -(b + 1)
\end{pmatrix}.
\]

Since it is a diagonal matrix, the entries along the diagonal are the eigenvalues: the first two corresponding to the fan and the third corresponding the spine.

Essentially the eigenvalues and associated eigenvectors define the geometry of the null. When the two eigenvalues in the fan plane are equal in a potential field, the field lines in the fan plane radiate outwards with no preferred direction. However, when one eigenvalue is stronger than the other, the field lines tend to group towards the eigenvector with the stronger eigenvalue.

The eigenvalue associated with the spine line is always real, as seen above, however
1. Introduction

the eigenvalues in the fan can be complex. They are complex when

$$(b - 1)^2 + a^2 - J_\parallel^2 < 0.$$  \hspace{1cm} (1.13)

This causes a spiralling of the field lines (in and on either side of the fan plane) about
the spine of the null point. These null points are called spiral null points. The
condition for spiralling is entirely based on $J_\parallel$ and does not depend on $J_\perp$. In
expression 1.13, $\sqrt{(b - 1)^2 + a^2}$ can be thought of as some threshold current which
causes spiralling in the fan plane.

$J_\perp$ has a different effect on the null point. $J_\perp$ causes the fan plane to no longer be
perpendicular to the spine line and tilts the fan plane away from its potential field
position. The larger $J_\perp$ becomes, the greater the tilt becomes. However the tilt angle
also depends on $J_\parallel$. As $J_\parallel$ gets larger, this causes the fan plane to get closer to being
perpendicular again. the normal vector to the fan plane in general is

$$n_f = \left( \frac{2J_\perp (a + J_\parallel)}{-4J_\perp (b + 2)} \right).$$

Therefore using the dot product of this with the spine vector ((0, 0, 1) as the spine
vector is aligned with the $z$ axis) gives an expression for the tilt angle, $\theta$ of the fan
plane:

$$\cos \theta = \frac{9 (b + 1)^2 - (b - 1)^2 - a^2 + J_\parallel^2}{|n_f|}.$$ 

The treatment of individual cases and combinations of the four parameters $a$, $b$, $J_\parallel$
and $J_\perp$ can be found in Parnell et al. (1996).

A major part of this thesis is focussed on finding and determining the nature of 3D
null points. Some examples of different null points for a range of different parameters
sets are shown later in figure 2.9.

1.7 Magnetic Skeleton

The topological features of a magnetic field together form a magnetic skeleton. One
of the topological features is a null point, another is the spine of a null point and a
third is the fan plane of a null point. As the distance from the null point increases,
1.7. Magnetic Skeleton

the linear approximation of the magnetic field breaks down as non-linear effects take over and the idea of a flat fan plane and straight spine line become warped. The field lines that make up these features continue. Since there are no sources and sinks in the magnetic field. These field lines can only end when they reach the domain’s boundary or reach another null point. The special field lines which start at one null point and end at another null point are called separators. Separators are a fourth topological feature of a magnetic field’s skeleton.

In the more global sense of a magnetic field, the fan plane becomes a separatrix surface. The infinite number of field lines coming from the null point form a surface through the domain which no field lines may cross. Thus these separatrix surfaces separate two different magnetic topologies, hence their name, similar to the separatrix lines of a 2D X-type null point.

Something even more unique about these separatrix surfaces is that two different separatrix surfaces from oppositely signed nulls may intersect with each other once or multiple times. This creates multiple different topological regions in the magnetic field since field lines cannot cross these surfaces. This line of intersection between the two surfaces must be a field line. These are the field lines called separators discussed earlier. Figure 1.12 illustrates the topological features that form the magnetic skeleton. There are two red and blue spheres representing positive and negative null points respectively with their respective separatrix surfaces (translucent surfaces) and spines (thick lines) again in red and blue. The intersection of the two surfaces (thick green line) is the separator field line between the two null points.

A separator at the intersection of two separatrix surfaces is not the only type of separator. A separator may also be formed from the connection of two spine lines from two oppositely signed null points or the “intersection” of a spine line and a separatrix surface from two null points of the same sign. However these two other types of separator are topologically unstable. A small ideal perturbation on the field could cause the misalignment of the features and thus a loss of the separator, whereas ideal perturbations of the intersection of two separatrix surfaces will not prevent them intersecting. So separators formed from the intersection of two separatrix surfaces are the only type of separator considered here.

The local field line structure projected onto a plane perpendicular to a separator is analogous to the structure of a 2D null point (Parnell, Haynes and Galsgaard 2010). When two separatrix surfaces intersect, the surfaces divide the region into four
1. Introduction

An illustration of a magnetic skeleton of a simple magnetic field involving a single separator. The separatrix surfaces (shaded surface) from oppositely signed null points (spheres) intersect forming the separator line (green) between the two null points. Field lines laying in the separatrix surfaces and the spine lines are also drawn.
1.7. Magnetic Skeleton

topologically different domains but the field lines may either rotate about the separator creating elliptic type field lines like in a 2D O-type null point (if there is a current along the separator) or may form hyperbolic field lines like in a 2D X-type null point, although the local ‘X’ of the projected field lines will not necessarily coincide with the global ‘X’ formed by a cut through the separatrix surfaces.

Together, these magnetic features – null points, separatrix surfaces, spine lines and separators – build a structure in a magnetic field which is called the magnetic skeleton which is globally significant despite the fact that both null points and separators are local features.

1.7.1 Topological Structures in Global Solar Corona

Due to the geometry of models of the coronal magnetic fields, new magnetic topological structures form due to the separator networks and the associated null points’ separatrix surfaces. These are discussed in this section together with a discussion on the heliospheric current sheet.

1.7.1.1 Heliospheric Current Sheet

The heliospheric current sheet (HCS) is a surface in space where the polarity of the radial component of the solar magnetic field changes in its polar open field (Smith 2001). The solar wind is continually flowing outward from the Sun. This advects the magnetic field with it and drags it outward from the solar surface causing the Sun’s magnetic field to be essentially radial after a certain radius. Using observations of the Sun, this radius is usually taken to be around $2.5 R_\odot$ (Jon Todd Hoeksema 1984). There must be a change in sign in the radial component of the field and this anti-parallel magnetic field creates a current sheet (figure 1.13a). This current sheet reaches far out into the solar system. The rotation of the Sun causes its magnetic field and this current sheet to twist into what is known as the Parker spiral (figure 1.13b).

Potential field source surface (PFSS) models (chapter 3) are used to extrapolate the observed magnetic field on the surface of the Sun to create a global potential field throughout the solar corona. The outer boundary of these finite models is restricted to only have a radial component of the magnetic field. At the boundary between the positive (outward) and negative (inward) directed magnetic field, a line with zero radial magnetic field exists which represents the start of the HCS. Since, on this
1. Introduction

(a) The Heliospheric Current Sheet

(b) The Parker Spiral

Figure 1.13
Features of the outer corona magnetic field.
1.7. Magnetic Skeleton

surface, there is only a radial component of magnetic field, when this vanishes, a null line is created.

Although there are no individual null points in the line, the base of the HCS forms part of the global solar magnetic skeleton. Field lines may be traced from this line, both forwards and backwards. Together the sets of forwards traced and backwards traced field lines form large surfaces which extend down to the solar surface and are called the heliospheric current sheet curtains. They are a type of separatrix surface and form a boundary between the open and closed magnetic field from the Sun. Additionally, the field lines in these special separatrix surfaces may form separators as the HCS curtains intersect with separatrix surfaces from null points in the magnetic field.

1.7.1.2 Magnetic Topological Structures and Their Separator Networks

As the magnetic skeleton of the global models becomes more complex, particular types of magnetic structures may be identified. These structures are made up of different numbers and arrangements of the elements of the magnetic skeleton. These structures are important in the solar atmosphere since they can, for instance, create closed field regions in what would otherwise be an open field region. These separatrix surfaces can create dome-like structures over the solar surface which magnetic flux may only cross during reconnection. A separatrix surface must also either be bounded by a spine line from an oppositely signed null point or the heliospheric current sheet curtains, or the surface continues into the sun’s interior or out into the solar system. When a separatrix surface is bounded by the spine line of an oppositely signed null point, there must also be a separator associated with the two null points. In these networks of separators, a complex configuration of separatrix surfaces from different null points can form many complex structures such as tunnel and cave like structures. These structures can be characterised by the networks of null points and separators that form them.

Work on the characterisation of these separator networks was started by Platten et al. (2014) and Edwards (2014). Two examples of these structures found by them are given in figures 1.14 and 1.15. These two examples are created using the PFSS model described in chapter 3 using custom synoptic maps designed specifically for creating these magnetic fields. The synoptic maps are extrapolated to form a 3D magnetic field between $r = 1R_\odot$ (the solar surface) and $r = 2.5R_\odot$. 
1. Introduction

A simple illustration of a two null point cave using two 3D models, a cut through the magnetic skeleton and a simple node network graph. The two 3D models are simply two different views on the magnetic skeleton structure. The left hand image only plots the null points (red and blue spheres), their spines (red and blue thick lines), their separatrix surfaces (red and blue thin lines) and their separators (yellow lines). The right hand image also shows the HCS base (thick green line) and its curtains (thin green lines) and the null-HCS separators (orange lines). The cuts through the magnetic skeleton at $r = 1.1R_\odot$ show the spines (red and blue dots), the separatrix surfaces (red and blue lines), the heliospheric current sheet curtains (green dotted lines) and the radial component of the magnetic field is contoured in the background of the plot. The node network graph simply shows the null points (red and blue dots) and their null-null separators (black lines). Red and blue is used throughout to denote positive and negative respectively.
A simple illustration of a three null point (negative) tunnel using two 3D models, a cut through the magnetic skeleton and a simple node network graph. See figure 1.14 for a description of the features seen in the figure.
1. Introduction

Figure 1.14 shows a so-called separatrix cave (Platten et al. 2014) formed from the separatrix surfaces of two oppositely signed null points connected by a separator. Figure 1.14a shows a 3D plot in which most of the separatrix surface from the positive red null point comes down to the solar surface except for one part which is bounded by the spine from the negative blue null point. This creates a dome-like structure with an entrance and so it is called a cave. The separatrix surface of the negative null point simply acts like a vertical wall which is bounded by the spine line of the positive null, the solar surface, and also the HCS. It also reaches the outer boundary of the magnetic field where it forms a semi-closed separatrix curtain.

Figure 1.14b shows a cut at a constant radius of $r = 1.1 R_\odot$ through the global 3D magnetic skeleton structures. Here the separatrix surface that forms the cave-like structure with the open entrance cuts the radial surface at the red line, the separatrix surface that forms the wall cuts the radial surface forming the blue line, whereas the spines that bound these surfaces cut the radial surface in the red and blue dots shown. The green dashed line shows where the HCS curtains cut the radial surface.

Figure 1.14c shows the key elements of the separator network; two null points connected by one separator. Separatrix caves are formed from a chain of separators where the signs of the null points at either end of the separator chain are of opposite sign. This means there must be an even number of null points ($2n$) in the chain with $n$ positive null points and $n$ negative null points. Here we illustrated a chain of just two null points but $n$ can be any number. The cave would become longer if there were more null points in the chain.

Figure 1.15 shows a so-called separatrix tunnel (Platten et al. 2014). For an even number of null points connected by separators in a linear chain, a separatrix cave is formed where one end of a dome-like magnetic structure must be open. With three null points (or any odd number of null points), there are two options: either both ends of the dome-like structure are open or both ends are closed. This creates either a separatrix dome or a separatrix tunnel. Figure 1.15 shows a separatrix tunnel.

Here, the spines of the two negative null points bound the positive null point’s separatrix surface. While the two separatrix surfaces from the negative null points are bounded on one side by the spine line of the positive null point and on the other by the HCS. This separatrix surface also reaches the solar surface and the outer boundary. The tunnel entrances are formed by the spine lines from the two negative null points on the end of the chain which prevent the positive separatrix surface reaching the solar surface everywhere.
1.7. Magnetic Skeleton

If, in figure 1.15, the positive null point’s separatrix surface formed a wall and the two negative null points’ separatrix surfaces went down to the solar surface with the positive null point’s spine line bounding both in the centre, there would be a separatrix dome instead. There would be a completely enclosed region of field within the dome. Each separatrix surface from the negative null points would form cave-like structures. But the two entrances would now be bounded to each other and therefore closing any exits to the dome for the field lines (see Edwards 2014, for more discussion).

These networks of null points and separators can get more and more complex, as will be seen in this thesis, and may contain thousands of nulls with many different connectivities. In order to try to better understand the structures found in global solar magnetic skeletons, we classify a number of the structures via their separator networks in this thesis. Figure 1.16 shows the six classifications we use by plotting node graphs of their separator networks. The more complex of these networks are more difficult to visualise in 3D so we just show them as the node null point networks. Anything with a higher complexity than these six types of separator network will be simply called “more complex”. There are examples of each of the six categories although each can contain many more nulls than the simple networks that are shown in the examples in figure 1.16.

A chain is any linear connection of any number of null points as shown in the first three graphs of figure 1.16. A cave (figure 1.16a) has an even number of null points, thus it has oppositely signed null points at either end of the chain. Positive (figure 1.16c) and negative (figure 1.16b) domes/tunnels have both positive or both negative null points, respectively, at both ends of the chain which must have an odd number of null points. To distinguish between a dome and a tunnel, the locations of the end points of the spine lines must be analysed. A ring is defined to be any linearly closed circular connection made up of any number of null points (figure 1.16d). Due to the requirement for positive to negative connectivity, there must be an even number of null points in a ring and an even number of separators. Next, in figure 1.16e, we plot the node graph for a chain containing a ring. This may be any chain structure which has a ring structure in the middle of it or on its end. The final classified separator network (figure 1.16f) has three linear chains all connected at a single null point creating a so-called tri-junction network.

These classifications of separator networks will be used throughout the thesis.
1. *Introduction*

Examples of the different separator networks in this thesis. The red and blue dots represent positive and negative null points respectively. The black lines represent null-null separators. Since the linear chains of null points and null-null separators representing the caves and the domes/tunnels may be of any length (subject to their specific configurations), the dotted lines represent the possible continuation of the linear chains which can be of any length.

**Figure 1.16**

(a) Cave

(b) Negative dome/tunnel

(c) Positive dome/tunnel

(d) Ring

(e) Linear chain containing a ring

(f) Three chains connected at a tri-junction
1.8 Global Solar Magnetic Field Models

The MHD equations do a great job at modelling the Sun. However, solving them is highly computationally expensive and it is not easy to couple experiments using the MHD equations with observations. One compromise is to use the force-free approximation which essentially means setting \( \mathbf{J} \times \mathbf{B} = 0 \). A derivation of which follows.

The equation of motion for an inviscid plasma is given by

\[
\frac{\rho \mathbf{Dv}}{Dt} = \mathbf{J} \times \mathbf{B} - \nabla p + \rho \mathbf{g}.
\]

Much like the MHD approximation, we consider typical scales of each of the terms to argue their importance.

First, the acceleration term on the left hand side is compared to each of the forces on the right hand side of the equation of motion. Using typical scales we get

\[
\left[ \rho \frac{\mathbf{Dv}}{Dt} \right] = \left[ \mathbf{J} \times \mathbf{B} \right], \left[ -\nabla p \right], \left[ \rho \mathbf{g} \right] \implies \frac{\rho_0 v_0}{t_0} = \frac{B_0^2}{\mu_0 L_0}, \frac{p_0}{L_0}, \rho_0 g
\]

In order to neglect the acceleration in favour of the three forces, we require

\[
\frac{\rho_0 v_0}{t_0} \ll \frac{B_0^2}{\mu_0 L_0}, \frac{p_0}{L_0}, \rho_0 g
\]

Rearranging and using \( v_0 = L_0/t_0 \) gives

\[
v_0^2 \ll \frac{B_0^2}{\mu_0 \rho_0} = c_A^2, \quad v_0^2 \ll \frac{p_0}{L_0} = \gamma c_s^2, \quad v_0^2 \ll g L_0 = v_g^2
\]

where \( v_g \) is the free fall speed. In the corona, the typical Alfvén speed is \( 10^6 \text{ m s}^{-1} \), the typical sound speed is \( 10^5 \text{ m s}^{-1} \) and the typical free fall speed is \( 2 \times 10^4 \text{ m s}^{-1} \) to \( 3 \times 10^4 \text{ m s}^{-1} \) where as typical speeds in the corona are \( 10^4 \text{ m s}^{-1} \). This means that plasma is evolving sufficiently slowly it may be assumed to be in equilibrium.

Now comparing the pressure force with the gravitational force:

\[
\left[ -\nabla p \right] = \left[ \rho \mathbf{g} \right] \implies \frac{p_0}{L_0} = \rho_0 g
\]
1. Introduction

In order to neglect the gravitational force in favour of the pressure force, we require

$$\rho_0 g \ll \frac{p_0}{L_0} \implies L_0 \ll \frac{p_0}{\rho_0 g} = \frac{RT_0}{\mu g} = H$$

where $H$ is the pressure scale height. A typical value of the pressure scale height is $6 \times 10^7$ m in the corona and a typical length scale is $10^6$ m hence the gravitational force may be neglected.

Finally, comparing the pressure force with the Lorentz force:

$$[-\nabla p] = [J \times B] \implies \frac{p_0}{L_0} = \frac{B_0^2}{\mu_0 L_0}$$

In order to neglect the pressure gradient in favour of the Lorentz force, we require

$$\frac{p_0}{L_0} \ll \frac{B_0^2}{\mu_0 L_0} \implies 1 \ll \frac{B_0^2}{\mu_0 p_0} = \frac{2}{\beta}$$

where $\beta$ is the plasma beta. A typical value of $\beta$ is $10^{-3}$ in the corona hence the pressure gradient term can be neglected. This leaves

$$J \times B = 0$$

under the assumption that $v_0 \ll c_A, c_s, v_g, L_0 \ll H$ and $\beta \ll 1$.

There are three different ways for this to be satisfied non-trivially (i.e. $B \neq 0$). First $J = 0$ and we have a potential field. The final two require that $J \parallel B$ since $J \times B = 0$. This means we can write

$$J = \alpha(r) B.$$ 

These final two are called linear and non-linear force free fields depending on the properties of $\alpha$. If $\alpha$ is constant, then the field is linear force-free, otherwise, if $\alpha$ is spatially dependent then the field is non-linear force-free. Using the parallelism of $J$ and $B$, Ampère’s law becomes:

$$\nabla \times B = \alpha B$$

and since the divergence of a curl is always zero,

$$\nabla \cdot (\alpha B) = 0 \implies B \cdot \nabla \alpha = 0.$$
1.8. *Global Solar Magnetic Field Models*

This means that $\alpha$ must be constant along field lines. Obviously this is trivially satisfied if $\alpha$ is constant, but in a non-linear force-free field, each field line may be associated with a different value of $\alpha$.

Potential field models of the Sun’s global magnetic field have been widely studied in the past and their magnetic skeletons are studied in detail in thesis using potential field source surface models (chapter 3). Potential fields are unique within a closed volume given that the value of the potential function or the normal component of the magnetic field is imposed on the boundary (Priest 2014). Spherical PFSS magnetic fields are relatively easy to calculate since there are no parameters worry about. Potential fields unfortunately do not replicate all features found in the solar corona since there is no current. So there are no twisted features such as may be found in prominences. They also have the minimum energy of any magnetic field with the same boundary conditions so no ‘free’ energy exists that can be released from them such as would be the case during reconnection events such as solar flares or coronal bright points. Linear and non-linear force free fields have also been used for studying coronal magnetic fields to try to mitigate the weaknesses of potential fields. They have been studied by, for example, Mackay and van Ballegooijen (2006), Mackay and Yeates (2012), Wiegelmann (2007), Tadesse, Wiegelmann, MacNeice et al. (2014) and Tadesse, Wiegelmann, Gosain et al. (2014). The use of force-free models for the global solar magnetic field is relatively new compared with potential fields and the techniques used to calculate these fields are still being developed (Mackay and Yeates 2012). There are several other models that extend the global potential models and are discussed in Wilcox Solar Observatory (2016).

Another way to improve upon a potential field is to consider a particular MHS model which is presented in chapter 6 and first derived by Bogdan and Low (1986) and extended by Neukirch (1995). Such a model includes both a plasma from which a pressure, density and temperature can be derived and also electric currents. There is a component of current parallel to the magnetic field in the same form as above where $\mathbf{J} = \alpha \mathbf{B}$ where $\alpha$ is constant and also a component perpendicular to the magnetic field to help support the plasma against the effect of gravity. Such magnetohydrostatic models allow for even more complex and possibly more realistic magnetic fields, while they are still relatively simple to compute.
1. Introduction

Figure 1.17
A simple illustration of the process of 2D reconnection at a 2D X-type null point (red dot)

1.9 Magnetic Reconnection

Magnetic reconnection is a process within magnetic fields where the field lines may reconfigure themselves. The process can be thought of as the field lines “breaking” and forming new, different connections in order to lower their energy state. Magnetic reconnection is at the core of the conversion of magnetic energy to other forms in energetic events in the solar corona such as solar flares. It is one of the primary mechanisms through which the corona can be heated (Parnell and De Moortel 2012).

In 2D the process can be visualised quite nicely. In 2D, magnetic reconnection can only occur at X type null points (Priest and Forbes 2000): at the unique point located between four topologically different regions formed by the separatrices. Reconnection allows the magnetic field to transfer flux between the four topological domains around the null point.
1.9. Magnetic Reconnection

Figure 1.17 illustrates the reconnection process in 2D. A plasma flow into the null point advects the field lines towards the null point since they are frozen into the plasma (figure 1.17a). This builds up current at the null point and the magnetic energy in the magnetic field begins to increase. Increasingly small scales form within the current as it forms a layer (figure 1.17b). These small scales cause micro-instabilities that create a localised region in which $R_m \lesssim 1$ and diffusion becomes important. Now that locally the field lines are no longer frozen-in, they start to “break” and “reconnect” with each other at the null point and magnetic flux can cross the separatrices from one topological region to another (figure 1.17c). Hence, the field lines can change their connectivities and the process induces a strong plasma flow away from the null point. The energy built up in the magnetic field due to the initial advection is released as kinetic, thermal energy or particle acceleration.

3D separators are similar to 2D null points. 2D reconnection can be generalised to 2.5D magnetic fields where there is a component in the third dimension which is invariant. If a component is added to the third dimension to the 2D field involving a null point configuration, then where the null point was a field line is created which is equivalent to a 3D magnetic separator with its four topologically distinct domains about the separator.

However, the actual process of reconnection in 3D is not so simply extended from 2D reconnection and has only been studied extensively for the past 15 to 20 years. The third dimension creates an added complexity which makes it much more difficult to visualise than 2D reconnection as, for instance, individual field lines do not reconnect in a pairwise fashion. 3D reconnection also does not require a null point to occur. The key requirement for 3D reconnection is that

$$\int_C \mathbf{E} \cdot \mathrm{d}\mathbf{r} \neq 0$$

in a localised diffusive region of the magnetic field where the curve, $C$, is a magnetic field line (Schindler, Hesse and Birn 1988; Hesse and Schindler 1988). 3D reconnection occurs continuously throughout the diffusive region and does not happen at a single point. The field lines continuously change their connectivity and do not pair up in the reconnection process like in 2D.

Essentially, for 3D reconnection to occur, there needs to be a non-zero component of the electric field along the magnetic field line. Using Ohm’s law (equation 1.6), the component of electric field parallel to the magnetic field is proportional to the
component of current parallel to the magnetic field since \((v \times B) \cdot B = 0\) allowing localised regions of high current to reconnect.

3D reconnection can occur at null points. One type of reconnection local to a single null point is studied in chapter 7 of this thesis. Reconnection at magnetic null points has been studied by, for example, Pontin, Hornig and Priest (2004a), Pontin, Hornig and Priest (2004b), Pontin, Hornig and Priest (2005), Masson et al. (2009) and Priest and Pontin (2009). Current layers tend to accumulate around much of the magnetic skeleton (Parnell 2018; Longcope and Cowley 1996) which creates localised regions in which magnetic reconnection has a tendency to occur, including separators. This makes knowledge of the magnetic skeleton useful in the study of reconnection. Separator reconnection has been studied by Haynes et al. (2007), Parnell, Haynes and Galsgaard (2010), Parnell et al. (2011), Longcope et al. (2005) Wilmot-Smith and Hornig (2011), Stevenson et al. (2015), Stevenson and Parnell (2015a), Stevenson and Parnell (2015b) and Threlfall et al. (2016).

1.10 Outline

In this thesis, chapter 2 describes fully the methods and details of a new suite of codes called the Magnetic Skeleton Analysis Tools which have been written for finding and analysing the skeleton of a magnetic field. These are then used throughout the rest of the thesis to analyse the topology of the magnetic fields from all models studied.

Chapter 3 is another method style chapter in which the PFSS model is described that is used throughout the majority of this thesis to create global magnetic fields of the global solar corona. Again a new code has been developed. This code is based on existing techniques and methods but, to extend the PFSS models to higher numbers of harmonics (greater resolution), a new code had to be developed.

Chapter 4 discusses the validity of using low resolution PFSS models and the quality of the magnetic skeleton that is findable as these PFSS models are increased in resolution.

Chapter 5 uses high resolution PFSS models to examine the magnetic fields and their skeletons at both solar minimum and solar maximum. The separator networks of these global magnetic field models are identified.

Chapter 6 starts by introducing a derivation of the MHS model derived by Neukirch
1.10. *Outline*

(1995) with a more detailed derivation of some elements in appendix B. The model is then used to calculate global MHS magnetic fields that are compared with the PFSS model magnetic fields.

Chapter 7 studies the dynamics of a magnetic null point in 3D. The 3D magnetic field containing a null point is relaxed under non-resistive MHD and then its torsional spine reconnection is studied under resistive MHD.

Finally, chapter 8 summarises the findings in this thesis and discusses how the work could be extended in future.
Chapter 2
Magnetic Skeleton Analysis Tools

Finding the magnetic skeleton is not an easy task. In cases where the analytic form of the field is simple, null points together with their spine lines and fan planes in the local vicinity to the null point may be found analytically. However, the real difficulty is finding an analytic expression for separators or the location of the separatrix surfaces far from the null point. So this task must usually be done numerically. There is nothing unique about a separator field line except that it connects two null points which makes its analytical expression difficult to find and although we can find an expression for the local fan plane, the global separatrix surface can be very complex.

In general, a given magnetic field does not have a nice analytic form, but instead comes from a numerical experiment thus it is only known at a discrete set of points. In such cases, the whole magnetic skeleton needs to be found numerically. Here we describe a new package of codes written specifically for this purpose.

The Magnetic Skeleton Analysis Tools (MSAT) are a set of Fortran codes written to find the topological skeleton given a vector field, \( \mathbf{F} \), defined on a discrete grid. Although the only restriction on \( \mathbf{F} \) is that it is divergence free (i.e. \( \nabla \cdot \mathbf{F} = 0 \)), we will concentrate on vector fields that are magnetic fields, \( \mathbf{F} = \mathbf{B} \). A common assumption in fluid dynamics modelling is incompressibility such that \( \nabla \cdot \mathbf{v} = 0 \) so MSAT could also analyse the fluid velocity skeleton in this case. Additionally, the electric current density \( \mathbf{J} \) and vorticity \( \omega \) is automatically divergence free so their topological skeletons could also be found by MSAT.

The source code is available at https://github.com/benmatwil/msat and information on the codes is split between both the manual provided with the codes and here in this chapter of my thesis. The manual provides instructions on how to use MSAT with descriptions of any changeable parameters. The manual is a LaTeX document and can be compiled with \texttt{pdflatex} using the command \texttt{make doc} and can be found in \texttt{doc/manual.pdf} within the code directory. In this thesis, there is a
2. Magnetic Skeleton Analysis Tools

description of the specific mathematical and numerical methods used in MSAT for a
deeper understanding of how they work.

There were a set of skeleton finding codes written by A. Haynes several years ago.
The MSAT have been written from scratch using similar methods in places to those
described in Haynes and Parnell (2007) and Haynes and Parnell (2010). They have
been re-written for several reasons.

1. The original codes were written in C and the language was not well understood
by the majority of previous users.

2. The majority of the source code had been hidden or was missing and there was
basically no documentation available. So it was not possible to know exactly
what Haynes' codes were doing and they could not be edited or recompiled for
different systems. We only had access to the executables which were difficult to
use and set up. We knew the general methods being used, but no specifics, and
one of the main indications that Haynes' codes were running correctly was an
error message suggesting they were not!

3. The spherical coordinate system used by Haynes' codes was not the standard
one used by mathematicians globally. Rather than using \((r, \theta, \phi)\) with
\(0 \leq \theta \leq \pi\) as is widely used, fields needed to be of the form \((r, \phi, \theta)\) with
\(-\pi/2 \leq \theta \leq \pi/2\). Hence, most fields such as those from the PFSS (potential
field source surface) model in chapter 3 needed to be “translated” before their
magnetic skeleton could be analysed.

4. The original code would produce tens to hundreds more separators than those
that actually existed. This occurred even in simple situations where only a few
should exist and this was a problem with most separators identified not actually
being real. When analysed properly, most separators were clearly just multiple
copies of the same separator which were identified multiple times by the code.
This made it difficult to do large scale separator analysis since users were
required to do further analysis to identify the real and unique separators.

5. The setup of the code was not easy to understand and it was not known what
some of the initial parameters actually did. It was only known what “good”
parameter values were and how best to adjust them was a bit of a mystery.

6. The Separatrix Surface Finder would fail when analysing magnetic fields not
considered to be unusual and it was not known why. The error messages
Haynes' skeleton finding codes and methods (Haynes and Parnell 2007; Haynes and Parnell 2010) have been used by several researchers around the world, most notably members of the Solar and Magnetospheric Group in St Andrews where Haynes was a member (e.g. Edwards 2014; Edwards and Parnell 2015; Stevenson et al. 2015) and Marc De Rosa at the Lockheed Martin Solar and Astrophysics Laboratory in California (e.g. Driel-Gesztesy et al. 2012; Harra et al. 2017).

Without the skeleton codes, it is extremely difficult to find the full separatrix surface of a null point due to the usual imbalance of the eigenvalues in the fan plane causing the field lines to converge to a line. Thus, the approach taken by many (e.g. Longcope, Brown and Priest 2003; Régnier, Parnell and Haynes 2008; Longcope and Parnell 2009; Longcope, Parnell and DeForest 2009; Cook, Mackay and Nandy 2009; Edwards and Parnell 2015; Freed, Longcope and McKenzie 2015) has been to only find the null points in the magnetic field. The null points are generally found by identifying those grid cells that may contain a null point using either the same method as step 1 of the Null Finder discussed later in section 2.2 or Greene’s method (Greene 1992). This is then followed by either a Newton-Raphson-like scheme also implemented by Greene or the Newton-Raphson iteration scheme itself to find the null point’s exact location within the grid cell. Greene’s method uses the Poincaré–Hopf index theorem to determine the existence of a null point in a grid cell. The Poincaré–Hopf index method is discussed in Haynes and Parnell (2007) where they show that using this method for finding null points can be inaccurate and can give rise to both false-positive and false-negative null points. Additionally, the Newton-Raphson method may also fail depending on the start point(s) of the scheme meaning that a failure to converge to a point in the grid cell may mean there is no null point in the grid or maybe the start points(s) chosen are poor and there really is a null point in the grid cell.

Development of the MSAT set of codes is now largely complete and we believe that the majority of problems that were found with Haynes’ codes have been eradicated. Changes to MSAT do continue to be made if issues arise in rare cases. The MSAT suite of codes

1. are written in modern Fortran using features from the Fortran 2008 standard,
2. the source code is publicly available to allow it to be easily improved and edited
2. Magnetic Skeleton Analysis Tools

in future,

3. use globally standard mathematical coordinate systems for Cartesian, cylindrical and spherical coordinates,

4. only identify unique separators, as far as we have tested,

5. are well documented.

The MSAT suite is compiled to analyse data given in either Cartesian, spherical or cylindrical geometries. All the internal calculations in the codes are done in “generic” grid coordinates, with conversions to the specific coordinate systems in specific parts, to provide generality for analysing data in any coordinate system. However, throughout this chapter, Cartesian grid coordinate notation is used to demonstrate the mathematics of the codes.

The input magnetic fields, $\mathbf{B}$, are assumed to have sufficiently high resolution such that they may be considered approximately trilinear between grid points. This means the field is assumed to vary linearly parallel to each of the three dimensions of the field in between grid points but in any direction the field can be non-linear up to an $xyz$ term. This allows the magnetic field to be interpolated anywhere in the domain using trilinear interpolation which is discussed in detail in section 2.1.1 below.

MSAT consists of three main codes which are used successively to fully analyse the skeleton of a magnetic field:

**Null Finder** *(nf.f90, section 2.2):*

Finds the null points themselves.

**Sign Finder** *(sf.f90, section 2.3):*

Finds the sign of each null point found by the Null Finder and the local vectors associated with the spine line and the fan plane of each null point

**Separatrix Surface Finder** *(ssf.f90, section 2.4):*

Traces out the separatrix surfaces of each null point and identifies the separators connecting the null points in the magnetic field. The Separatrix Surface Finder also traces out the spine lines to aid later analysis and ensure they are traced under the same integration scheme as that used to find the separatrix surfaces.

A file of parameters used in the running of MSAT is called *params.f90*. The parameters included are mentioned throughout this chapter. The parameters are also discussed in detail in the MSAT manual.
There are also several auxiliary routines provided in the package in addition to those above:

**Magnetic Field Data Writer (writedata.f90):**
Provides an example of writing an analytic magnetic field expression as a discretised set of magnetic field points to file in preparation for use in the main analysis codes. This can be edited to use your own analytical field expression on a grid of your choice. More details on using this file is provided in the manual. Also there are instructions in the manual provided on writing equivalent magnetic fields to file in IDL and Python in the appropriate format.

**Heliospheric Current Sheet Finder (hcs.f90, section 2.5):**
Specific to global spherical magnetic fields such as those produced by the PFSS model, this calculates the location of the line(s) on the upper radial boundary defining the base of the heliospheric current sheet (HCS). It also traces this line down to the solar surface, mapping out the HCS curtains which are the surfaces separating open and closed field and find the separators between the HCS and null points in the magnetic field. There is also a bald patch finder as well which does a similar job to the HCS finder but with bald patches on the solar surface. This is detailed further however in section 2.9.

**Cut Maker (make_cut.f90, section 2.6):**
Takes a cut on a surface defined by constant coordinate and locates where all the magnetic skeleton features cut this surface creating a 2D visualisation which aids visualisation and analysis of the fields and the magnetic skeleton.

Additionally, there are a set of tools written in both Python and IDL for reading, visualising and analysing the features of the magnetic skeleton outputted by the MSAT package. The specifics of these and their usage are again detailed fully in the MSAT manual. There is a small amount of discussion of these tools in section 2.8.

## 2.1 Numerical Interpolation

Before the main codes in MSAT are discussed, we present some of the numerical interpolation methods used throughout the MSAT package.
2. Magnetic Skeleton Analysis Tools

2.1.1 Trilinear

To find the features of the magnetic skeleton, the magnetic field must be found at all points in the domain, not just at the grid points. To allow MSAT to find the magnetic field values at any point, trilinear interpolation is used. It is an extension of linear interpolation in 1D, with linear interpolation taken in all three directions. That is the field varies linearly as you move though a grid cell parallel to an axis, but as you move in any general direction, not parallel to an axis, the field is non-linear, as will be shown below. Trilinear interpolation is the typical form of interpolation used to draw field lines or stream lines by packages such as VisIt etc.

First examining the case of a 1D function \( f(x) \), we can formulate linear interpolation as

\[
f(x) = f(x_1) + \frac{x - x_1}{x_2 - x_1} (f(x_2) - f(x_1)),
\]

where \( x_1 \) and \( x_2 \) are the locations of two known values of \( f \) (e.g. two grid points) and \( x \) is a point lying between \( x_1 \) and \( x_2 \) \((x_1 < x < x_2)\) where we wish to know \( f \). This process may be visualised as finding the straight line between the two known points \( f(x_1) \) and \( f(x_2) \) and using this straight line to find value of the required point as shown in figure 2.1.

Since we will be working in grid coordinates, the expression for linear interpolation may be simplified. The known values of our function \( f \) are on the edges of the grid cells and the positions of these edges are given by the indices. So relabelling the known function values as \( f_0 \) and \( f_1 \) and letting \( x \) be the fractional distance across a
2.1. Numerical Interpolation

The magnetic field components on the eight vertices of each grid cell with distances $x$, $y$ and $z$ in each direction respectively as a fraction of 1, the whole grid cell. $\mathbf{B}(x, y, z)$ is the interpolated magnetic field value.

grid cell ($0 \leq x \leq 1$), the expression for linear interpolation becomes

$$f(x) = f_0 + x(f_1 - f_0) = (1 - x)f_0 + xf_1,$$

since the distance across a grid cell is $x_2 - x_1 = 1$.

This technique can be extended to 3D with linearity in all three directions. Given that MSAT works in grid coordinates (i.e. each grid point is labelled by an integer), the equations for trilinear interpolation will be given here in grid coordinates relative to an individual cell such that the distance across a cell in each direction is 1. For simplicity, we use Cartesian coordinates with $0 \leq x, y, z \leq 1$ (figure 2.2) i.e. $x, y, z$ are the fractional distances in each direction of the required point within the cell.

The trilinear interpolation approximation of a 3D magnetic field $\mathbf{B}(x, y, z)$ at a point inside a cell at a point $(x, y, z)$ (Haynes and Parnell 2007) given as a vector
formulation rather than component wise is

\[
B(x, y, z) = B_{0,0,0} + x(B_{1,0,0} - B_{0,0,0}) + y(B_{0,1,0} - B_{0,0,0}) + z(B_{0,0,1} - B_{0,0,0}) + xy(B_{1,1,0} - B_{1,0,0} - B_{0,1,0} + B_{0,0,0}) + xz(B_{1,0,1} - B_{1,0,0} - B_{0,0,1} + B_{0,0,0}) + yz(B_{0,1,1} - B_{0,1,0} - B_{0,0,1} + B_{0,0,0}) + xyz(B_{1,1,1} - B_{1,1,0} - B_{1,0,1} - B_{0,1,1} + B_{1,0,0} + B_{0,1,0} + B_{0,0,1} - B_{0,0,0})
\]

This can be thought of as three separate equations combined using vector notation.

### 2.1.2 Bilinear

Bilinear interpolation is also used on the faces of the grid cells in the Null Finder. It is simply a 2D version of linear interpolation like trilinear is for 3D. Similar to expression 2.1 for trilinear interpolation, the bilinear interpolation approximation of a 2D magnetic field is given by

\[
B(x, y) = a + bx + cy + dxy,
\]

with

\[
a = B_{0,0} \\
b = B_{1,0} - B_{0,0} \\
c = B_{0,1} - B_{0,0} \\
d = B_{1,1} - B_{1,0} - B_{0,1} + B_{0,0}
\]

as vector coefficients where \(x\) and \(y\) are the fractional distances within each grid cell to the desired point.
2.1. Numerical Interpolation

2.1.3 Analytical Solution to the Pair of Bilinear Equations

For any pair of bilinear equations (using expression 2.2) in component form

\[ f_i(x, y) = a_i + b_i x + c_i y + d_i x y, \]

with \( i \in \{1, 2\} \), the solution to \( f_1(x, y) = f_2(x, y) = 0 \) may be found by solving either of the following quadratic equations

\[
\begin{vmatrix}
  a_1 & a_2 \\
  c_1 & c_2
\end{vmatrix}
+ \left( \begin{vmatrix}
  a_1 & a_2 \\
  d_1 & d_2
\end{vmatrix}
+ \begin{vmatrix}
  b_1 & b_2 \\
  c_1 & c_2
\end{vmatrix} \right) x
+ \begin{vmatrix}
  b_1 & b_2 \\
  d_1 & d_2
\end{vmatrix} x^2 = 0,
\]

or

\[
\begin{vmatrix}
  a_1 & a_2 \\
  b_1 & b_2
\end{vmatrix}
+ \left( \begin{vmatrix}
  a_1 & a_2 \\
  d_1 & d_2
\end{vmatrix}
- \begin{vmatrix}
  b_1 & b_2 \\
  c_1 & c_2
\end{vmatrix} \right) y
+ \begin{vmatrix}
  c_1 & c_2 \\
  d_1 & d_2
\end{vmatrix} y^2 = 0.
\]

Since these are quadratic equations, analytical solutions can easily be found. Once \( x \) or \( y \) has been determined from one of the above quadratics, then analytically, the other coordinate (which was not solved using the quadratic) can be found using either

\[ y = -\frac{a_i + b_i x}{c_i + d_i x} \tag{2.3} \]

or

\[ x = -\frac{a_i + c_i y}{b_i + d_i y} \tag{2.4} \]

with either \( i = 1 \) or \( i = 2 \).

These solutions are implemented by the Null Finder (discussed later in section 2.2) for analytically finding zero points in 2D on cell faces.

2.1.4 Runge-Kutta-Fehlberg (RKF45) Integration Scheme

To trace the field lines in the Separatrix Surface Finder, an RKF45 Integration Scheme is used. The RKF45 integration scheme is a 4\(^{th}\) order accurate integration scheme that uses a 5\(^{th}\) order estimate to calculate an error which is used to improve the accuracy at each step. In particular, the error is used to dynamically reduce the numerical step size in order to reduce the error compared with the standard 4\(^{th}\) order Runge-Kutta scheme (Fehlberg 1970). It allows the magnetic field lines in the MSAT package to be traced with good accuracy.
2. Magnetic Skeleton Analysis Tools

2.1.4.1 Hadamard Product

In order to explain concisely how the RKF45 integration works, I first define the binary operator $\circ$ known as the Hadamard product, Schur product or the entrywise product. It operates on the vectors in such a way that multiplication is done element by element and keeps matrices and vectors the same size. So for example

$$
\begin{pmatrix}
a_1 \\
a_2 \\
a_3 
\end{pmatrix} \circ 
\begin{pmatrix}
b_1 \\
b_2 \\
b_3 
\end{pmatrix} 
\equiv 
\begin{pmatrix}
a_1b_1 \\
a_2b_2 \\
a_3b_3 
\end{pmatrix}.
$$

It can be thought of as being equivalent to array multiplication and is similar to a dot product without the summation.

2.1.4.2 RKF45 Scheme

MSAT uses grid coordinates for generality, so we calculate a vector $\mathbf{h}$ which is based on the step size, $h$, in the appropriate coordinate system you are working in. The infinitesimal line element vector $d\mathbf{r}$ is defined for the coordinate system being used. For Cartesian coordinates this is simply $d\mathbf{r} = (dx, dy, dz)$ and in cylindrical $(R, \phi, z)$ and spherical coordinates $(r, \theta, \phi)$, it is

$$d\mathbf{r} = (dR, r d\phi, dz),$$

and

$$d\mathbf{r} = (dr, r d\theta, r \sin \theta d\phi)$$

respectively. From this we define

$$\mathbf{h} = \frac{h \min(d\mathbf{r})}{d\mathbf{r}}$$

so for example in spherical coordinates

$$\mathbf{h} = h \min(dr, r d\theta, r \sin \theta d\phi) \left(\frac{1}{dr}, \frac{1}{r d\theta}, \frac{1}{r \sin \theta d\phi}\right).$$

So $\min(d\mathbf{r})$ is the minimum of the three vector components of $d\mathbf{r}$ and division by $d\mathbf{r}$ is the same the multiplication by a vector containing of the reciprocal components of $d\mathbf{r}$. This vector $\mathbf{h}$ is required to give the correct scaling when converting from grid...
2.1. Numerical Interpolation

coordinates to the original coordinate system.

Starting from a point in the domain, \( r_i \), the coefficients may be calculated, with \( s = 1 \) to begin with, using the Hadamard product:

\[
k_1 = s \mathbf{h} \circ \hat{\mathbf{B}}(r)
\]
(2.5)

\[
k_2 = s \mathbf{h} \circ \hat{\mathbf{B}}\left( r + \frac{1}{4} k_1 \right)
\]
(2.6)

\[
k_3 = s \mathbf{h} \circ \hat{\mathbf{B}}\left( r + \frac{3}{32} k_1 + \frac{9}{32} k_2 \right)
\]
(2.7)

\[
k_4 = s \mathbf{h} \circ \hat{\mathbf{B}}\left( r + \frac{1932}{2197} k_1 - \frac{7200}{2197} k_2 + \frac{7296}{2197} k_3 \right)
\]
(2.8)

\[
k_5 = s \mathbf{h} \circ \hat{\mathbf{B}}\left( r + \frac{439}{216} k_1 - 8 k_2 + \frac{3680}{513} k_3 - \frac{845}{4104} k_4 \right)
\]
(2.9)

\[
k_6 = s \mathbf{h} \circ \hat{\mathbf{B}}\left( r - \frac{8}{27} k_1 + 2 k_2 - \frac{3544}{2565} k_3 + \frac{1859}{4104} k_4 - \frac{11}{40} k_5 \right)
\],
(2.10)

where all values of the magnetic field in the six equations above are normalised.

The 4\(^{\text{th}}\) and 5\(^{\text{th}}\) order estimates, \( e_4 \) and \( e_5 \), are then be calculated using the following:

\[
e_4 = \frac{25}{216} k_1 + \frac{1408}{2565} k_3 + \frac{2197}{4101} k_4 - \frac{1}{5} k_5,
\]
(2.11)

\[
e_5 = \frac{16}{135} k_1 + \frac{6656}{12825} k_3 + \frac{28561}{56430} k_4 - \frac{9}{50} k_5 + \frac{2}{59} k_6.
\]

These are used to determine a new coefficient \( s \) based on their difference and some error tolerance \( t \) which can be set by the user:

\[
s = \left( \frac{t}{2 |e_5 - e_4|} \right)^{\frac{1}{4}}.
\]

This coefficient \( s \) is used to recalculate improved \( k_1 \) to \( k_5 \) vectors using equations 2.5 to 2.9 again. This can be seen as reducing the stepsize where necessary. These new vector coefficients are used to calculate \( e_4 \) again (equation 2.11) to give the next interpolated point along a field line as

\[ r_{i+1} = r_i + e_4. \]
2. Magnetic Skeleton Analysis Tools

2.2 Null Finder

The job of the Null Finder is to simply locate the magnetic null points in a 3D magnetic field (points such that \( \mathbf{B}(x, y, z) = 0 \)). In each grid cell of a numerical vector field, the three trilinear equations for each component of the magnetic field could be solved simultaneously to obtain three cubic equations, one equation for each of \( x, y \) and \( z \). However, finding an analytic solution to cubic equations is not trivial so the Null Finder uses similar techniques to those in Haynes and Parnell (2007) for finding null points based around trilinear interpolation. The major difference is that a deca-section method is used rather than a Newton-Raphson method for converging to the location of each null point. This was chosen since Newton-Raphson has the possibility of failing and the decreased computational time of Newton-Raphson is not necessary. Generally, the current computational time constraint of the code comes from reading in the data file containing the full magnetic field, which cannot be avoided. There are three steps to the analysis which are described.

**Step 1:**

First there is a reduction step as used by Haynes and Parnell (2007) which checks each grid cell for the possibility of containing a null point and tries to reduce the number of cells that require checking in the subsequent steps of analysis. The code considers the eight vertices of each cell and checks for a change in sign in all three components of the magnetic field. Using the vertex indexing notation in figure 2.2 and letting a single component of the magnetic field (say \( B_x \)) be denoted by \( B_j \) then we can calculate whether there is a change in sign within the cell of each component using

\[
S_j = \sum_{i_x=0}^{1} \sum_{i_y=0}^{1} \sum_{i_z=0}^{1} \frac{(B_j)_{i_x,i_y,i_z}}{|(B_j)_{i_x,i_y,i_z}|}.
\]

If \( S_j = \pm 8 \), then the magnetic field component at all eight vertices has the same sign. So we look for cells which have \( S_j \neq \pm 8 \) in all three components of the magnetic field. If a cell has \( S_j \neq \pm 8 \) for all three \( j \) then the three surfaces \( B_x = 0, B_y = 0 \) and \( B_z = 0 \) must pass through that cell, an example of which is illustrated in figure 2.3. Cells are removed from the analysis if any of the three components of the field have the same sign at all eight vertices since such cells cannot contain any null points under the trilinear assumption.
Step 2:

In the second step, each of the remaining cells is checked to see if it really does contain a null point by determining if the surfaces $B_x = 0$, $B_y = 0$ and $B_z = 0$ all intersect each other. Since there is a change in sign in the cell in all three components, the three zero surfaces that pass through the cell must intersect with the boundary of the cell as shown by the thick lines in figure 2.3 and in figure 2.4 where we have now merged the three cubes drawn in figure 2.3 and folded out the faces of the cube. For there to be a null point, these bi-zero lines must all intersect each other in pairs (or a triplet if the null point is on the face). These intersection points on the faces are given by the coloured dots in figure 2.4. These intersection points are where the three bi-zero lines located inside the cell formed from the intersections of the three surfaces $B_x = B_y = 0$, $B_y = B_z = 0$ and $B_x = B_z = 0$ hit the cell walls. If a null point exists then it must lie on all of these three bi-zero lines.

It is possible that these bi-zero lines exist solely within the grid cell and therefore their path is a circuit. A circuit within the cell implies that the grid resolution is not fine enough to identify all null points and it is likely that the trilinear assumption does not hold locally in this region of the field. The Null Finder therefore concentrates on those cells whose bi-zero curves have endpoints on the faces of the cell.

Given the trilinear magnetic field is bilinear on the faces, this allows us to find these intersection points using the analytical expressions from bilinear interpolation. The bilinear equations for $B_x = 0$, $B_y = 0$ and $B_z = 0$ can be solved simultaneously in pairs to find their intersections on each of the 6 cell faces.
A cube’s faces folded out to show the intersection of the three zero surfaces in figure 2.3 with the faces of the grid cell and the intersections of these lines given by the dots coloured by a mix of the two colours intersecting. The shading represents $B_x < 0$, $B_y < 0$ and $B_z < 0$.

Using figures 2.3 and 2.4, the change in sign of the third magnetic field component (here $B_z$) on the faces of the cell connected by the line defined by the intersection of the other two magnetic field components (here $B_x$ and $B_y$) equal to zero.
2.2. Null Finder

using the formulation given in section 2.1.3. Analytically one would imagine that only one equation (2.3 or 2.4) needs to be solved, but numerically it is found that both equations must be solved in order to find all solutions. Due to numerical errors, it is possible to miss a solution by only solving one set of equations.

These bi-zero curves which intersect the boundary of the cell must intersect in pairs for there to be a null point, as in figure 2.4. The sign of the “unused” field component (i.e. $B_z$ at the intersection of the $B_x = B_y = 0$ lines) must be of opposite sign at the pair of intersection points such that a change in sign occurs along the bi-zero line as in figure 2.5. If this is true, there must be a null point somewhere along the line and therefore inside the cell. Or, if the null point is on the cell face, the “unused” component may be zero at one of the intersection points.

There are also several other tests done to ensure all null points are found. The code checks for null points on each of the cell edges and cell vertices just in case these are missed by the bilinear tests. This just requires solving a linear equation along each edge of the cell.

If the tests for a null point indicate that a null point exists within the grid cell then it is marked. The same analysis is repeated on all cells in the domain that have not been removed following the reduction phase.

Step 3:

The third step of the Null Finder considers each marked cell in turn to locate, to a given accuracy, the null point in the cell. For all cells which have a confirmed null point inside, the cell is split into a trilinear grid of $10 \times 10 \times 10$ smaller grid cells (as in figure 2.6) and the analysis from steps one and two are repeated on each of these smaller grid cells. First the reduction phase finding changes in sign of the field components and then the bilinear check on the cell faces. If again a null point is confirmed, the subcell is marked as a positive and the code moves onto analysing the next cell. This process of going to subgrid resolution is repeated until the required numerical accuracy for the null point location is achieved. The accuracy in the location is restricted by the numerical precision of the data.

Finally, some final checks are done for complete results. The code can sometimes find it difficult to find null points on the vertices of the cells so these are checked individually. Each vertex of the original grid is checked for a null point by simply checking the value of the three components of $\mathbf{B}$. Also, any null points that are
A grid cell (black) split into $10 \times 10 \times 10$ grid cells (red) when going to sub grid resolution. The magnetic field is calculated at each sub grid cell corner using trilinear interpolation.

It is possible that the null points which the code identifies on the domain boundary are actually part of a null line. This, for example, is certainly the case in the PFSS model. On the outer radial surface where both $B_\theta$ and $B_\phi$ are zero everywhere, there is a line along which $B_r = 0$ (i.e. indicating a change of sign in $B_r$) which is known as the base of the heliospheric current sheet (HCS). The Null Finder finds as many null points as there are grid cells crossed by this line none of which are proper 3D null points: they are all part of the HCS null line. These boundary HCS null points are removed from the final set of null points. However if proper 3D null points can occur on the boundaries, they may be retained for verification.

Theoretically, under the trilinear assumption more than one null point can arise in each grid cell. However, the null finder currently only finds one null point per grid cell. This restriction may be lifted in the future with a reformulation of the algorithm. However for magnetic fields such as those in the PFSS analysis where there is a null line on the boundary, being able to find multiple null points per grid cell can make run times prohibitively long depending on the final null point location accuracy required. Additionally, the magnetic field in each cell is trilinear, this restricts the number of null points possible in a grid cell. So far testing shows that with this restriction, only one null point at most exists per grid cell in the PFSS fields.

Finally, the locations of all null points found are then saved to file in both grid coordinates (to be used by the Sign Finder and Separatrix Surface Finder) and the coordinate system of the original data for visualisation and other uses.
2.3 Sign Finder

After analysis by the Null Finder, the Sign Finder can be invoked to find the local magnetic structure of each null point. It attempts to find the sign of each null point and the vectors representing the directions of the spine lines and the normal to the fan surface close to the null point. For the vector normal to the fan surface, we find any two vectors defining the fan plane and take the cross product. These do not need to be the two specific eigenvectors associated with the eigenvalues of the linear magnetic field about the null.

The approach used to find the sign of the null, the spine vector and the fan normal vector is a convergence type method. This convergence method exploits the properties of the field lines in the local vicinity of a 3D null point. Here we now describe the theory behind our convergence method. This method varies from that described in Haynes and Parnell (2010).

2.3.1 The Theory of the Convergence Algorithm

The mathematics of magnetic null points introduced in section 1.6 gives rise to the equation that governs the field lines around the null point (equation 1.12) given by

\[ \mathbf{r}(s) = a_1 e^{\lambda_1 s} \mathbf{e}_1 + a_2 e^{\lambda_2 s} \mathbf{e}_2 + a_3 e^{\lambda_3 s} \mathbf{e}_3 \]

recalling that \( \lambda_i \) are the eigenvalues associated with the eigenvectors \( \mathbf{e}_i \) of the magnetic field local to the null point and \( \mathbf{r} \) is the position of the field line relative to the null point parameterised by \( s \). We will use this to determine the sign of the null point. This is done by the repeated application of the Jacobian matrix, \( \mathbf{M} \), to \( \mathbf{r} \). The first application of \( \mathbf{M} \) to any vector directed out from the null, \( \mathbf{r} \) gives

\[ \mathbf{M} \mathbf{r} = \lambda_1 a_1 e^{\lambda_1 s} \mathbf{e}_1 + \lambda_2 a_2 e^{\lambda_2 s} \mathbf{e}_2 + \lambda_3 a_3 e^{\lambda_3 s} \mathbf{e}_3. \]

By repeatedly applying \( \mathbf{M} \), we obtain

\[ \mathbf{M}^n \mathbf{r} = \lambda_1^n a_1 e^{\lambda_1 s} \mathbf{e}_1 + \lambda_2^n a_2 e^{\lambda_2 s} \mathbf{e}_2 + \lambda_3^n a_3 e^{\lambda_3 s} \mathbf{e}_3. \]
2. Magnetic Skeleton Analysis Tools

So as \( n \to \infty \), \(|\lambda_1^n|\) will become dominant since \(|\lambda_1| > |\lambda_2|, |\lambda_3|\) and the terms involving the fan eigenvalues and eigenvectors become negligible:

\[
M^n r \to \lambda_1^n a_1 e^{\lambda_1 s} e_1.
\]

From this we obtain the eigenvector of the magnetic field corresponding to the largest eigenvalue which is the spine eigenvector \( e_1 \).

This convergence technique is employed as the main spine line and fan plane finding algorithm in the Sign Finder code. The algorithm is described in the following section.

2.3.2 Main Algorithm

Initially, two sets of the same points (explained in more detail below) are placed uniformly in \( \theta \) and \( \phi \) (from spherical coordinates) on the surface of a sphere a small distance, \( r_0 \), away from the null point. The number of points in each direction is given by \( n_\theta \) and \( n_\phi \) and they can be changed in the parameters file. An example of the arrangement of the starting points is given in figure 2.7, however, the default in the code is to use many more start points than have been illustrated to give a denser grid. The radius \( r_0 \) must be chosen based on the accuracy to which the null points are found in the Null Finder in order to account for any possible inaccuracies in the null location and to ensure there are no other null points within the sphere: \( r_0 \) is set in the parameters file.

There is one difference between the two sets of points which are initially co-located. One of the two sets of points will be iterated forwards along field lines and the other backwards. The idea being that one set of points will iterate towards the spine line and the other towards the fan plane using the theory in section 2.3.1. This is demonstrated graphically in figure 2.8. Since in the initial configuration there are two points at each initial location, each of these two points will now be referred to as a “pair of points” as they will be iterated along a field line simultaneously.

Once the initial points are placed, the code iterates through each pair of points in turn and uses a two step operation iteratively in order to try and converge the points.

1. First, the code uses the magnetic field at the location of the points and moves one of the pair of points a small distance forward in the direction of the magnetic field and the other a small distance backwards. The location of a
2.3. Sign Finder

The locations of the start points (green dots) on a sphere of radius $r_0$ centred on the null point (red dot). Here $n_\theta = 10$ and $n_\phi = 20$.

Point after this operation, $q_{i+1}$, in terms of its current position relative to the null, $p_i$, is given by

$$q_{i+1} = p_i + d \frac{B(p_i + n)}{|B(p_i + n)|},$$

where $d$ is a small parameter set in the code that is positive or negative depending on whether the point is being traced forwards or backwards along a field line.

2. The two $q_{i+1}$ points are now mapped directly back onto the sphere in the radial direction. In effect, the vectors are renormalised such that their magnitude become the radius of the original sphere, $r_0$, again so

$$p_{i+1} = r_0 \frac{q_{i+1}}{|q_{i+1}|}.$$

This process of moving the points with the field and renormalisation back to the sphere is then successively repeated until one of the pair of points moves only a very tiny distance (i.e. $|p_{i+1} - p_i| \ll d \ll r_0$). At this point the pair of points are considered to have converged to their final positions. This occurs when the points get
A 2D illustration of the convergence of the points around a null point (red dot) in the Sign Finder. In the first iteration, there are a pair of points in their initial position, $p_0$ (the black point) and the field line passing through this initial point (green line). One of the points is traced in the direction of the field forwards (in red) and the other is traced in the direction of the field backward (in blue) a small distance creating $q_1$ and $q_1$. These two points are then remapped back to the sphere (grey line) with red and blue dots denoting their final position after the first iteration, $p_1$ and $p_1$. The second iteration illustrates the same process but now the pair of points are in different positions initially and the pair of points move further apart as they move closer to their final positions on the spine (orange line) and in the fan plane (blue line). This second iteration creates $p_2$ and $p_2$. 

Figure 2.8
close to either the spine line or fan plane, that is where the magnetic field becomes essentially perpendicular to the surface of the sphere around the null point and thus the points will hardly move with each iteration.

The convergence to the spine is almost always the fastest given that this eigenvalue is the sum of the other two eigenvalues thus the field there is stronger. The point converging to the spine therefore should have stopped the algorithm. This may mean that the points converging to the fan plane are not fully converged; a fact which helps later in the algorithm to decide upon the spine and fan eigenvectors. This is because the points converging towards the fan converge to a plane rather than a single line, then it is easier to discriminate between the fan and the spine, especially in cases where one of the fan eigenvalues is dominant.

In the event of one of the initial pairs of points lying directly on an eigenvector, the code would stop immediately as neither point would move (under equations 2.12 and 2.13) since \( B(p_0 + n) \) is parallel to \( p_0 \). This pair of points would be mapped perpendicular to the surface of the sphere and remapped back onto the original point. One of the points would be at the correct eigenvector and the other is unable to escape. In this case, the algorithm forces the pairs of points through the above algorithm a minimum number of iterations before it stops to ensure this pair of points are truly stuck on an eigenvector initially. If this is the case, the code resets all the initial pairs of points by rotating them a small angle in both \( \theta \) and \( \phi \) and starts again. This allows all pairs of points that would have initially been on an eigenvector to move away from their initial locations and to converge properly.

After all pairs of points are deemed to have converged, there should be equal numbers of points that have moved forwards as have moved backwards. These should now be grouped into distinctive bunches: one set lying on or around the spine vector and the other lying in the fan plane (see figure 2.9 for examples). At the spine, there are two scenarios to be considered.

1. The points appear at two very localised spots: one spot positioned on one spine and the other spot on the spine in the opposite direction. Obviously, which direction of the spine eigenvector is classed as positive is arbitrary. Something similar would also happen in the fan plane if one eigenvalue is very dominant over the other. The case of two very localised spots will occur for many null points (figures 2.9a to 2.9g).

2. When the value of \( J_\parallel \) is large, two rings around the spine vector are formed.
instead of spots like the null point in figure 2.9h. The large $J_\parallel$ produces a large component of magnetic field perpendicular to the spine vector. The points originally located close the spine vector are pushed away until they reach an equilibrium point between this extra outward push from the current and the inward direction of the field running towards spine vector.

The algorithm has two different options on how to determine the sign of the null point which depends on which of the two cases arises. To distinguish between the two cases above, the code identifies how many of each of the forward and backwards iterated sets of points caused the algorithm to stop. In the case where there are two tightly bunched points for the spine vector, then one of the forwards or backwards iterated sets of points should cause the algorithm to stop as they converge to the spine vector. The other set of points in this case (the one corresponding to the fan plane) is likely to continuously rotate about the spine for spiralling null points in the case where $J_\parallel$ is not too large (like in figure 2.9d). However in the case when $J_\parallel$ is large, neither set of points converge; they continue to rotate with the field lines around the spine vector and in the fan plane for both the forwards and backwards sets of iterated points (like in figure 2.9h). The two different options are now discussed separately.

**Low $J_\parallel$:**

First the code chooses which of the iterated sets of points corresponds to the spine vector and fan plane. It does this by looking at how many points from each iterated set of points converged causing the algorithm to stop. The set which caused the algorithm to stop more frequently (and therefore converged most) should correspond to the strongest eigenvalue, that is the spine vector. If both sets of points of points converge the same amount, then the code calculates the dot products of each of the vectors in the final iterative sets with the magnetic field at each corresponding point. This gives an estimation of the eigenvalues associated with each set of points. Whichever set contains the larger eigenvalue is then selected to correspond to the spine vector. The two sets of iterated points each now correspond to the spine vector and fan plane.

The spine vector and fan plane sets of points are now reduced separately by comparing the distances between each of the points. Each point is picked in turn and within a small radius of it, all other points are removed. This should reduce the number of points in both sets of points to unique points. Ideally, the spine vector set of points will reduce to only two points. One point on either side of the
2.3. Sign Finder

sphere indicating each direction of the spine vector. If both sets of points are reduced to just two points, the code checks the eigenvalues of all four points and will switch which set corresponds to the spine vector if the original fan plane set’s eigenvalue is larger. Since there should be only two spine vector points left, if there are fewer points in the fan plane set of points than in the spine vector set, checks are made to see if the identification should be switched.

Once the spine vector has been selected, the code then moves onto finding the two fan plane vectors. The first fan vector is selected by using the point which was originally most densely packed before the number of points in each set is reduced. Here we do not call the two fan plane vectors the major and minor fan vectors but instead we refer to them as the first and second fan vectors. Although for some null point configurations, the Sign Finder will identify the major and minor fan vectors, this is not always the case. The Separatrix Surface Finder only requires the normal to the fan plane to trace out the separatrix surfaces so we do not worry here about trying to identify the actual major and minor fan vectors. The Sign Finder will certainly not identify the major and minor fan vectors in the case that the field lines are spiralling about the null point.

The algorithm for finding the second fan vector depends on the spread of the points in the final iterated set of fan plane points. The two possibilities are that the points are roughly equally distributed around the whole circle (as seen in the null points in figures 2.9a, 2.9d and 2.9g) or they have gathered around the stronger eigenvalue (as seen in the null points in figures 2.9b, 2.9c, 2.9e and 2.9f). The code distinguishes between these two possibilities by taking the first fan vector selected earlier and taking the dot product of it with all the others. In the case when the points are roughly equally distributed around the whole circle, the minimum dot product should be close to zero since a vector almost perpendicular to the first fan vector should exist. Otherwise the minimum dot product will be larger than zero and closer to one in magnitude. In the situation where the points are generally equally distributed around the whole circle, the second fan vector is chosen as the vector most perpendicular to the first fan vector (i.e. the vector with the dot product nearest zero).

In the situation where the points have gathered around the first fan eigenvector, a different approach is used since no points exist which will be perpendicular to the chosen first fan vector. Here the convergence scheme from the start of the Sign
2. Magnetic Skeleton Analysis Tools

Finding algorithm is used again but this time a modified magnetic field is used which given by

\[ \mathbf{B} - (\mathbf{B} \cdot \hat{e}_f) \hat{e}_f \]

where \( e_f \) is the first identified fan vector and \( \mathbf{B} \) is the original magnetic field. This new magnetic field, which still has a null point at the same location, represents the original magnetic field with its component in the direction of the first fan vector removed. The convergence method is applied again, but only with one set of points located initially in the same positions as before. The points are moved in the direction away from the spine. The points should converge to two spots in the fan plane perpendicular to the first fan vector chosen before. After a reduction of the number of points, this finally gives us a second fan vector as required.

**High \( J_\parallel \):**

Some of the steps in the algorithm for high \( J_\parallel \) are similar to those for low \( J_\parallel \). The major difference is due to the rings that have formed around the spine vector (as seen for the null point in figure 2.9h).

First the code needs to identify which iterated set of points correspond to the fan plane and spine vector. The set corresponding to the spine vector has split into the two rings whereas the set corresponding to the fan has formed a single ring around the equator. Because of the spiralling field lines due to the large \( J_\parallel \), the points in the fan plane will always be distributed around the whole circle. These are identified by calculating the dot products between one of the points in each set with all the other points in the same set. The set of points corresponding to the fan vector should have a minimum dot product close to zero in magnitude while the set corresponding to the spine will have a much larger minimum dot product given they are more tightly packed, even in the ring formation. This allows the algorithm to determine which of the two sets of points correspond to the spine vector and fan plane. Two fan vectors are then identified simply by picking two perpendicular vectors in the fan plane.

The spine vector should be at the centre of the two rings of points corresponding to the spine vector. First the two separate rings are identified by again using the dot products calculated earlier. The points in the same ring as the point chosen to take all the dot products between should have a dot product greater than some number, \( c \), given by the maximum angle between the vector chosen and all those vectors in the same ring relative to the null point. Whereas in the other ring, the
2.3. Sign Finder

dot products will all be less than $-c$. Once one of the rings has been identified, the vectors relative to the null point are averaged to give their centre and therefore the spine vector of the null point.

Although the main analysis done by the Sign Finder is discussed above, the first thing the Sign Finder actually does is identifying whether the null point in question is a zero signed null point. Null points with zero sign may arise if there is perhaps not enough resolution to resolve the behaviour of the field lines in one part of the field leading to an apparent local magnetic field with $\nabla \cdot \mathbf{B} \neq 0$, i.e. the local field about the null points look like either a source or sink (see later in figure 2.15 for examples). These zero signed null points are not a different type of null point. To identify these, field lines are traced about the null point using the RKF45 algorithm discussed in section 2.1.4. A ball of start points are placed around the null point and the field lines are traced in both directions until they reach a relatively large distance from the null point or the number of points reaches a maximum number of iterations. If all field lines reach the relatively large distance, there is a proper null point. Otherwise the field lines will have all headed into the null point and the code identifies this as a zero signed null point. If the code identifies a zero signed null point, it will just move onto the next null point and the convergence method will not be used.

The Separatrix Surface Finder does not perform any analysis on zero signed null points. However, it does check for field lines which may get stuck close to zero signed null points so it is important they are identified here. It has been shown using analytically formed magnetic fields placed on a numerical grid that as the grid resolution is increased, these source or sink null points can become proper divergence free null points (see section 2.10). There is a Sign Finder Checker included in the visualisation package which will plot local field lines around a null point with the sets of converged points so the user may check the validity of the null point and its sign, especially those identified as having a zero sign. More details are given in section 2.8 and the user manual.

After each of these steps has been completed, the Sign Finder has finished its analysis on the null point and moves onto considering the next null point. Once the analysis has been performed on all the null points found by the Null Finder, the spine vector, the normal to the fan plane (the cross product of the two fan vectors), the signs of the null points and any warning flags for each null point are written to file ready for the Separatrix Surface Finder.
2.3.3 Example Results from the Sign Finder

Examples of the results of the Sign Finder convergence method for eight different null point configurations are given in figure 2.9. In the figure, first the field lines local to the null point are given on the left and on the right, where the points converge to after the reduction. The large red cross and dots indicate what the Sign Finder chose as the spine vector and the blue and green crosses are what were chosen as the two fan vectors.

Figure 2.9a is a radial potential null point with the two eigenvalues associated with fan plane equal. This means that the field is symmetric about the spine and all three eigenvectors should be at right angles to each other. The converged points in the fan plane are equally spaced around the null point. What is perhaps most noticeable in figure 2.9a is how sparse the final iterated points in the fan plane are compared to say figure 2.9d. This is because the null point in figure 2.9a has been chosen to be perfectly radial thus the starting points cannot move in the $\phi$ direction from their original positions (see figure 2.7), they can only move in the $\theta$ direction and so the points simply gravitate towards the fan plane within planes of constant $\phi$.

The fan eigenvalues of the null point in figure 2.9d are complex conjugates due to a current parallel to the spine causing a rotation of the field lines about the spine. This causes the final iterated points in the fan plane cover the whole circle because the field lines cause the convergence points to continually rotate about the null point. This is also the case for the null points in figures 2.9g and 2.9h which are also spiral null points (even for very small amounts of spiralling in the field lines).

As the two eigenvalues that are associated with the fan plane begin to differ (as seen in figures 2.9b and 2.9f) and become significantly different (as in figures 2.9c and 2.9e), the points converging to the fan plane converge more and more to just two spots, similarly to the two spots on the spines. The magnetic field in figure 2.9c is an example of where the field about the null point starts to look like a 2D X-type null point. However, even in cases such as that of figure 2.9e where the null point field is clearly not 2D null point like, the convergence in the fan plane is still very strong. When such strong convergence happens in the fan plane, the second fan vector is found using the additional convergence method with the modified magnetic field.

The final example of null points in figure 2.9h is an example where the spine vector points form two rings. One of the rings is averaged to find the final spine vector as
2.3. Sign Finder

(a) $a = 1, b = 0, J_{\parallel} = 0, J_{\perp} = 0$

(b) $a = 1.5, b = 0, J_{\parallel} = 0, J_{\perp} = 0$
2. Magnetic Skeleton Analysis Tools

(c) \( a = 4, b = 0, J_\parallel = 0, J_\perp = 0 \)

(d) \( a = 1, b = 0, J_\parallel = 4, J_\perp = 0 \)
2.3. Sign Finder

(e) $a = 2.5$, $b = 0.5$, $J_{\parallel} = 0$, $J_{\perp} = 0.2$

(f) $a = 1.3$, $b = 0$, $J_{\parallel} = 0$, $J_{\perp} = 8$
2. Magnetic Skeleton Analysis Tools

Figure 2.9
Eight examples of null points (orange dots) with local field lines drawn in red on the left and the convergence points (green, blue and red dots) from the Sign Finder on the right. The actual spine line and the fan plane are indicated by the black dashed line and circle respectively. The converged points identified as the spine vector and the two fan vectors are marked by the red, blue and green dots respectively. The final vectors chosen are then indicated by pluses in the same respective colours. The parameters in the subcaptions refer to the null point configurations of the magnetic fields as described in section 1.6.

(g) \( a = 2, b = 1, J_\parallel = 2, J_\perp = 3 \)

(h) \( a = 5, b = 1, J_\parallel = 350, J_\perp = 50 \)
2.4 Separatrix Surface Finder

Finally, now that we have all the information about the local field of each of the null points, the Separatrix Surface Finder can be used to trace out the separatrix surfaces for each null point and to find the separators connecting the null points (intersections of the separatrix surfaces). The Separatrix Surface Finder does this by tracing field lines out from the local fan plane while trying to maintain a good distribution of field lines across the whole separatrix surface. It also traces out the spine lines from each null point.

The Separatrix Surface Finder starts by placing a number of start points, equally spaced, in a circle, here called a ring, around the null point lying in the fan plane using the fan normal vector calculated by the Sign Finder. The number of start points on this ring is adjustable in the parameters file. The code ensures that these points lie within the domain if the null point is located close to the boundary.

Each point on the ring is integrated forward using the RKF45 scheme of integration (detailed in section 2.1.4) by again assuming that the field is trilinear within each grid cell. Each point on the ring is moved approximately the same distance, $h$, in grid coordinates along the field line. The value of $h$ can be changed in the parameters file. These new points then form the next ring. For the first 50 rings created, the value of $h$ is 5 times smaller than is set in the parameters file (as shown in figure 2.10) to ensure small changes in the field close to the null point are represented accurately.

Two equivalent but different copies of the locations of each point on the rings are stored during the calculations because of the use of grid coordinates in MSAT. One set of locations takes into account if any point on a ring cross a periodic boundary with the point’s new locations then recorded with the periodicity of the grid taken into account. In the other set, the point’s new location continues to be recorded as if the boundary was never there. As an example, consider a 1D domain with periodic boundaries at grid indices 1 and 10. A point initially located at 1.5 that moves 0.8 of a grid cell to the left would then be located at 0.7. In the case of the second location copy, the location would be recorded unaltered, i.e. would remain at 0.7, while for the
first copy, the location would be moved to 9.7. This system makes it possible to check the distances between adjacent points later in the code even when points have crossed a periodic boundary in all coordinate systems.

As well as the two sets of locations of the points on a ring, several other pieces of information about each point on a ring are stored.

1. The associations which link each point on the ring to a point on the previous ring. This allows the field lines to be re-traced later.

2. The break points which indicate a split in the ring. A break point flags to the code that two adjacent points in the list of points are not technically adjacent anymore. This allows them to move apart freely without having points added in between them such as after a ring has hit a separator or when part of a ring leaves the domain.

3. Points, if any, on the current ring that are in the vicinity of any oppositely signed null points to the null point they originated from. This flags to the code that it needs to check for separators.

Once all the points on the ring have been interpolated forwards and a new ring has been formed, the new ring goes through four steps of analysis. Here, a null point of opposite sign to the null point whose separatrix surface is being mapped out will be called an oppositely signed null point.

**Vicinity of Oppositely Signed Null Points:**

If any points on the new ring are close to an oppositely signed null point, the step size, $h$, with which each ring is integrated forward is halved to improve accuracy. This closeness is determined in the code and the distance is dependent on the value of $h$ set by the user in the parameters file. The value of $h$ will return to its original value set in the parameters file as soon as all points on the ring are sufficiently far from any oppositely signed null points again. This flags to the Separatrix Surface Finder whether it will need to check for separators and make the resolution of points higher near the null point to ensure the detection of any separator that may exist.

**Removal of Points:**

Here the Separatrix Surface Finder determines whether any point on the ring needs to be removed. Such a point arises because it has exited the domain, it has just been flagged as a separator or it is too close to an adjacent point. Points too
2.4. Separatrix Surface Finder

Figure 2.10
A plot of the first 60+ rings lying in the separatrix surface traced out from a null point (central red dot). After 50 rings, the step size, $h$, is increased. The magnetic field considered here is the same as that used in figure 2.9a where the field is completely radially symmetric.
2. Magnetic Skeleton Analysis Tools

Figure 2.11
A plot demonstrating how the points (coloured dots) on consecutive rings are associated. There is a small section of three consecutive rings illustrated here. The first of the three rings is coloured red with the next two in green and cyan. There are no points added or removed between the red and green rings and so the associations between points of these two rings, given by the black lines, have a one-to-one correspondence. Then between the green and cyan rings, some of the points on the cyan ring have now become too far apart. So a point is added in between each pair of points which have become too far apart and these new points are associated with the same point on the green ring as the previous point on the cyan ring. So we get a subset of the points on the green ring associated to two points each on the cyan ring. Those points which are still within the maximum distance apart on the cyan ring (i.e. did not get points added between them) are then not close enough on the next ring which is off the plot. It can be seen that there is a splitting in the black lines going off the plot indicating points being added on the next ring. This magnetic field is the same as the magnetic field in figure 2.9c.
close to each other are removed to reduce computational time. If the code removes any points due to having exited the domain or being a separator, a break in the ring is flagged. This prevents any points being added between two points that naturally need to move apart.

**Addition of Points:**

Next, the new ring is checked to see whether any points need to be added to keep a good resolution and spacing on the ring. If two adjacent points on the ring have become too far apart, a point half way between them is added unless a break point has been flagged in the removal step. The new point is associated with the same point on the previous ring as the first of the two points it was added in between. If any points have become close to an oppositely signed null point, the maximum distance in between adjacent points is halved for those points on the ring which are close to the other null points. This causes a temporary higher density of points only in areas of the ring that are close to other null points. This improves the resolution of the ring close to null points in preparation for the check for any separators.

**Separator Check:**

Finally, the code checks whether any separators end on the ring by considering each oppositely signed null point in turn (or zero-signed null point for error checking – see later). Each point on the ring is again flagged for its closeness to the oppositely signed null point with a different distance criteria to that above and the shortest chain of points is formed from the ring which includes all sufficiently close points to the oppositely signed null point with a few extra added onto each end of the chain to try to ensure the separator is not missed.

To find a separator, the code traces out field lines from those in the chain of points until the field lines are sufficiently far away. If a separator lies amongst the points in the chain, there will be a change in the direction of travel of the field lines, since points on one side of the separatrix surface of the oppositely signed null point will travel along the spine of the oppositely signed null point in one direction whilst the rest of the points will lie on the other side of the separatrix surface and travel along the other spine in the opposite direction. In order to determine if there is a split in the direction travelled by the traced field lines, the dot product of the position of each final field line location, \( \mathbf{r} \), relative to the oppositely signed null point, \( \mathbf{n} \), with the spine vector, \( \mathbf{e}_s \), of the oppositely signed null point.
null point

\[(r - n) \cdot e_s\]

is calculated. If there is a change in sign of the dot product between two adjacent field lines, this indicates a split in the direction of travel of two field lines from two of the rings’ adjacent points indicating that a separator exists. The separator is flagged on the first of the two adjacent points which is the last point before the dot product of the field line changes. A break point is also inserted on the same point. Since the two adjacent points on the ring will move apart as they travel along different spines from the oppositely signed null point, we do not want the code to keep trying to add in points (possibly very close to the oppositely signed null point). It will also flag this separator point for removal next time the code removes points, as described earlier. After this step is finished, those points which were traced forward to check for a separator are returned to their original positions on the ring before this step.

During this stage the code also checks for any points on the ring which may have become stuck near a null point (which is likely to be of zero sign). These are also flagged for removal and a break is inserted to stop any more points being added nearby which may also get stuck. These points are found by checking how many steps are required for the RKF45 integration scheme to trace the field lines sufficiently far from the null point. Those points that are stuck will exceed a certain threshold in the number of iterations.

An illustration of the removal and addition of new points (steps 2 and 3) and how this affects the associativity of points is given in figure 2.11. If there were a removal of a point, a black line that ends would be seen. An illustration of how the code checks for separators (step 4) is given in figure 2.12.

Finally, the code checks for any reason it might need to finish. These include:

1. the existence of more points on a ring than the given maximum threshold set in the parameters file,
2. the number of rings created has exceeded the maximum number set in the parameters file,
3. all points on the ring have left the domain, and
4. the RKF45 integration scheme is struggling to trace a point and creates an
An illustration of how the Separatrix Surface Finder checks for a separator. First the code decides which set of points on the ring to trace along the field lines. All points within a certain distance of the null point (here the black points inside the inner, dotted circle) are selected and then a number of points outside this distance are also selected from the green points on either side. All selected points are then traced along field lines until they are sufficiently far away (illustrated by the dashed, outer circle) from the null point again. If there is a separator, these will divide into two sets of points as they diverge and they follow the spines in either direction. In this case, we have a separator. The point coloured in blue will be flagged with a break point to stop the code from adding in extra points as it and the adjacent point diverge from each other on either side of the fan plane. In reality the outer, dotted circle is much further from the null than illustrated here.
2. Magnetic Skeleton Analysis Tools

If none of these criteria are triggered, the code continues and interpolates all the points on the current ring forward to create a new ring of points.

After the code has finished analysing the final ring and the complete separatrix surface in the domain from the null point has been found, each separator from this null point which has been flagged is then traced back from the flagged point by recording the locations of the associated points on each of the previous rings saved to file during the separatrix surface analysis.

The two spine lines are then traced in each direction for this null point. The spines are traced by taking the two starting points as

\[ r_0 = n \pm \varepsilon \hat{e}_s \]

where \( \varepsilon \) is some small value and then tracing the field lines out using the RKF45 method again. The points are then saved to file. This technique to find the separatrix surfaces, spines and separators is then repeated for all null points.

Changing \( h \) in the parameters file affects how far the points move (and therefore how many rings are required) and how many points are on each ring. Although the value of \( h \) does change temporarily during the analysis, the other effects of this are small. This change is approximately linear in both cases so the run time is roughly proportional to \( 1/h^2 \) i.e. halving \( h \) will approximately quadruple the run time.

The following are the issues we feel have been solved or improved in MSAT in comparison to Haynes’ original version of the Separatrix Surface Finder (originally called ssfind).

1. To overcome ssfind’s problem of identifying the same separator a multitude of times, a condition has been added such that a separator can only be detected if the point has not already been flagged as a break point. This reduced the detection rate to only include unique separators.

2. Parameters are defined in terms of grid coordinates rather than their real lengths and they are listed in a single file called parameters.f90 making them easy to adjust as required.

3. Originally there was one file per null point. There is now one file per type of magnetic structure e.g. separators, spines, etc. So there are now a total of 7
2.5. Heliospheric Current Sheet Base Finder

outputted files from the new Separatrix Surface Finder. This reduces the requirement for thousands of files in cases when there are thousands of null points. More details on the output files can be found in the manual.

2.5 Heliospheric Current Sheet Base Finder

The Heliospheric Current Sheet Base Finder is specialised code that is used when analysing global solar magnetic fields which have an outer boundary such as those produced by the PFSS model in chapter 3. On the upper radial boundary of global solar magnetic field models, the two angular components of the spherical magnetic field are zero and the radial component of the magnetic field must have a region of positive (outwards directed) and a region of negative (inwards directed) field otherwise the solenoidal constraint would not be satisfied. At the boundary between these two oppositely signed regions, null lines exist (lines along which \( B = 0 \)). These lines denote the base of the heliospheric current sheet(s). Identifying these null lines is the first step of this algorithm.

Once the null lines (the base of the heliospheric current sheet) are found, its so-called heliospheric current sheet curtains are found. These “curtains” represent the boundary between open and closed field lines originating from the solar surface and are found by tracing field lines from the null line on the upper boundary to the solar surface both forwards (for one curtain) and backwards (for the other curtain). This is important for measuring how much of the solar magnetic field is open field. Next we discuss the specific methods used to find these features.

The null lines on the top boundary are found by looking for zeros in the radial component of \( \mathbf{B} \). Zeros are searched for along lines of constant \( \theta \) and \( \phi \) with small separations to find a dense set of points representing the line. First the code looks for possible zeros between the grid points from the original grid and then uses linear interpolation under the original trilinear assumption to find the exact location of the zero. Once the points have been found, the code sorts these into a line by using an algorithm which is detailed later in section 2.7. The zero finding mentioned above could be done using the Null Finder, as mentioned in section 2.2. The null points it finds on the upper boundary and removes from the list of null points are the zero points on the null line found here. However, the points produced are too dispersed with only one null point per grid cell so a different algorithm is used.
2. **Magnetic Skeleton Analysis Tools**

Once the points on the null line are sorted, the heliospheric current sheet curtains can be traced. This is done by a very similar algorithm to the Separatrix Surface Finder in section 2.4. However, since at each point on the null line on the outer radial boundary, the magnetic field is entirely radial, all the points on the null line are moved by a tiny fraction of a grid cell inside the domain. Once this is done, this line is equivalent to a ring in the Separatrix Surface Finder and the field lines are traced in both directions down to the solar surface, adding and removing points as required and checking for separators if it gets close to a null point.

### 2.6 Cut Finder

The Cut Finder is an aid to help visualise the final magnetic skeleton and can only be run after all the other tools in MSAT have been run. It looks for where any of the magnetic skeleton features (e.g. separatrix surfaces, spines, separators, heliospheric current sheet curtains etc.) cross a pre-defined plane, for example a plane of constant $r$ in the case of a spherical magnetic field. No calculations to find the magnetic skeleton are done here, this algorithm simply reads in all the data output from the other main codes from MSAT. By considering the points making up each magnetic structure, the algorithm checks where the structure crosses the pre-defined surface.

In its current form, this algorithm only works for pre-defined surfaces that involve one coordinate being held constant (e.g. surfaces of constant $r$ or constant $x$). In future, we aim to generalise this to accept any linearly defined surface of the form

$$ax_1 + bx_2 + cx_3 + d = 0$$

where $(a, b, c)$ is the vector normal to the required surface, $d$ is a constant and $x_1$, $x_2$ and $x_3$ are the three coordinates of the grid.

Structures that are just lines, such as the spines or separators, are quite simple to check. The algorithm just determines if the line between any two adjacent points in the line crosses the given surface and if it does linear interpolation is used to find the exact point it crosses.

Finding where the separatrix surfaces cross the given surface is more complex since they are made up of many rings with many points. Starting from the final ring of each null point, the algorithm reads in each ring of points in turn with their
associativity data and the previous ring of points. By using the associativity data, all
the points in the rings are matched up to create the original field lines and these are
used to determine which field lines cross the surface. Again linear interpolation is
used to find their exact crossing points.

### 2.7 Line Sorting Algorithm

Several codes in the MSAT package require what can be a randomly ordered set of
points to be organised into a line. How this is done is detailed briefly here.

This is done by first identifying the two points in the set that are closest together.
These two points form the start of the line and points are added onto this initial line
segment. Starting from one end of this line segment the next point is added according
to two main criteria:

1. what is the next closest unallocated point available in the set, and
2. the deviation (angle) of the line segment to the next point relative to the
   previous line segment.

If the new point is deemed reasonable under a set of criteria (i.e. it is suitably close
and has a small enough change in angle), it is added onto the line and the process is
repeated. Once there are no longer any more points that are close enough or the
change in direction of the line is too great, the algorithm then starts to consider the
other end of the initial line segment. The line is then traced in the other direction
from the initial line segment using the same process as above to add points to the line
until once more there are no more suitable points available to be added. At this point
the line is taken as complete and written to file.

If all the points in the set have been allocated to the line then the algorithm will
stop. If they have not, the closest pair of points in the remaining set are identified
and another line is traced out. This continues until all the points in the set are
allocated to a line.
2. Magnetic Skeleton Analysis Tools

(a) Cartesian magnetic field involving two null points joined by two separators derived by D. Chambers

(b) Spherical global magnetic field generated by the PFSS model using an HMI synoptic map

Figure 2.13

Two examples of magnetic skeletons using the 3D visualisation tools. Null points are represented as small spheres with red corresponding to a positive null point and blue corresponding to a negative null point. The thick coloured red and blue lines are the spines from positive and negative null points respectively. The thin, lighter coloured, red and blue lines form the separatrix surfaces from positive and negative null points respectively. The thick yellow lines represent separators between two null points and thick orange lines represent separators between a null point and the heliospheric current sheet base. The thick green line represents the heliospheric current sheet base with its curtains being outlined by the thin green lines representing field lines.
2.8 Data Analysis and Visualisation Tools

Also included in MSAT are several analysis tools for further analysis and visualisation of the magnetic skeletons found by MSAT. They are, in the main, written in Python and IDL, with basically the same features available in both.

The following is a short summary of what is available:

- a field line tracer and trilinear interpolator
- a 3D skeleton viewer
- a null point structure checker
- a magnetic skeleton cut plotter
- read routines for almost all data created by MSAT

All these routines are detailed more fully in the MSAT manual.

The null point structure checker is provided to allow the sign of any null point to be individually verified if the Sign Finder has struggled to identify its properties. It plots the null point in 3D with a set of local field lines about it and the convergence points which the code uses to define the properties of the null point. Each of the null points in figure 2.9 were plotted with a modified version of this code.

The magnetic skeleton cut plotter provides a 2D visualisation of the cuts created by the Cut Maker.

Two examples of magnetic skeletons that have been analysed by MSAT and drawn using the 3D visualisation tools are given in figure 2.13. The first has a Cartesian grid and the second has a spherical grid.

2.9 Future Improvements to the Algorithms

The points on the rings in the Separatrix Surface Finder are stored in arrays. The addition and removal of the points requires the allocation and deallocation of these arrays which could be quite time consuming for the algorithm. It has been suggested that linked lists, which are not a native Fortran data structure, could be used instead for a more efficient way to add and remove points from the rings. This may be investigated in the future.
2. Magnetic Skeleton Analysis Tools

There is a code provided as part of the tools for finding bald patches (bp.f90) on the solar surface and their associated separatrix surfaces. It is similar to the other separatrix surface finding codes and the main outline of the code is there. Although it is currently perhaps a little rough around the edges, it may be updated in the future. It is not used for any analysis in this thesis.

The algorithm for sorting points into lines described in section 2.7 could perhaps be improved in future. It may not be especially efficient in general and it could be inaccurate in some of the more extreme cases. I am not sure if there are any better ways or publicly available algorithms available to do this. After a search, nothing has surfaced.

2.10 The Importance of Trilinearity

Finally there is a short discussion on the importance of having enough resolution in a magnetic field for the trilinear assumption to hold. There is a longer discussion on this with regards to the PFSS model in chapter 4.

In order to demonstrate the importance of having sufficient resolution, a magnetic field was created which was analytically defined and then written to a numerical grid at different resolutions. The analytical expression for the field was designed by D. Pontin (University of Dundee) and is not listed here due to its complexity. It was designed to contain 14 magnetic null points of a range of different types. During the testing of the new Null Finder and Sign Finder in MSAT, this magnetic field defined on different grids highlighted the importance of resolution. The magnetic field was written onto 5 different $n \times n \times n$ grid resolutions for $n \in \{20, 30, 40, 60, 80\}$. Here, we only consider the results from the resolutions for $n \in \{20, 30, 40\}$ which is where the issues occur. MSAT analyses the magnetic fields for $n \in \{60, 80\}$ with no issues.

Figure 2.14 shows the null points and spine lines in the magnetic fields for $n = 20$ and $n = 40$. When $n = 20$, the Null Finder finds 12 null points, 6 of which are negative and 5 are positive but the sign of the final null point is ambiguous. The local field lines to this ambiguous null point are plotted on the left of figure 2.15a and it is the green null point in figure 2.14a. It is clear from the field lines in figure 2.15a that this is actually a source or a sink as all the field lines head into the centre of the plot where the null point is located. Additionally, in figure 2.14a, the spine from a nearby negative null point, below it, is sucked in – indicating that $\nabla \cdot \mathbf{B} \neq 0$. 

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2.10. The Importance of Trilinearity

![Diagram](image)

(a) $20 \times 20 \times 20$ magnetic field

(b) $40 \times 40 \times 40$ magnetic field

Figure 2.14

The null points (spheres) and the spine lines plotted for two resolutions of the test magnetic field. The red and blue colours represent positive and negative respectively. The green denotes that the Sign Finder has identified the null as a source or a sink.
(a) The green null point in the $20 \times 20 \times 20$ magnetic field in figure 2.14 which is correctly identified in the $30 \times 30 \times 30$ magnetic field.

(b) A rogue null point in the $30 \times 30 \times 30$ magnetic field which is correctly identified in the $40 \times 40 \times 40$ magnetic field.

Figure 2.15

Two examples of the same null point changing from a source or a sink at low resolution to a proper divergence free null point at higher resolution. The red and orange lines are magnetic field lines both traced the same set of starting points placed on a sphere about the null point in the forwards and backwards direction respectively.
2.10. The Importance of Trilinearity

When \( n = 30 \), the Sign Finder now correctly identifies the sign of this ambiguous null point which was flagged as a source or a sink when \( n = 20 \). At the higher resolution of \( n = 30 \), the field lines around the null point now behave appropriately for a divergence free null point (figure 2.15a). Furthermore, when \( n = 30 \), the Null Finder also finds the two final null points in the set of 14 null points which missing at the \( n = 20 \) resolution. These two null points are a positive and negative null point pair located close together in the top left quarter of figure 2.14b. Since this pair of null points includes both a positive and a negative null point, together they have a topological degree of zero. So the Null Finder was not be able to find either of them at the lower \( n = 20 \) resolution as they were both in the same grid cell. However, at \( n = 30 \), not all is fine. There is still one ambiguous null point whose sign cannot be determined. At this resolution the field local to the null point is not divergence free and all the field lines local to the zero point head out from the zero point creating a source.

At the higher \( n = 40 \) resolution, the \( n = 30 \) ambiguous null point is found to be a negative spiral null point in figure 2.15b. In the magnetic field for \( n = 40 \), all the null points are located correctly and their signs are all identified correctly. For the \( n = 40 \) and higher resolution fields, MSAT is able to analyse the magnetic fields fully and properly.

This example shows the importance of the resolution for finding the magnetic skeleton. Not having enough resolution can significantly affect the ability to determine the type of the null point and can also affect whether they can be found at all. This, of course, hinders how well a magnetic field can be analysed.
2. Magnetic Skeleton Analysis Tools
Chapter 3

The PFSS Model

The Potential Field Source Surface (PFSS) model is a simple global coronal field model which creates an analytic magnetic field in spherical coordinates. It uses photospheric magnetic field observations at the solar surface and assumes a purely radial magnetic field at some outer radius.

The model was first independently derived by Altschuler and Newkirk (1969) and Schatten, Wilcox and Ness (1969). Their derivations used line-of-sight magnetic field observations which at the solar surface which were matched to the $B_r$ and $B_\theta$ components of the magnetic field. However, it was shown by Wang and Sheeley (1992) that by converting the line-of-sight observed magnetic field to only its radial component and matching this to the radial component of the PFSS model at the lower boundary, this creates better agreement with the observed fields. This method of only using the radial component is used in calculating the PFSS fields in this thesis. The PFSS model is widely used in research and has been used to model coronal holes (Wang and Sheeley 1990), coronal null points (Edwards and Parnell 2015) and the coronal magnetic fields of other stars (Jardine, Collier Cameron and Donati 2002), for example.

The PFSS model essentially requires solving Laplace’s equation with the required boundary conditions. This can be done with the associated Legendre polynomials and the spherical harmonics which are now introduced.
3. Spherical Harmonics and Associated Legendre Polynomials

Legendre’s differential equation is given by

\[(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \left( l(l + 1) - \frac{m^2}{1 - x^2} \right) y = 0, \tag{3.1} \]

with \(l, m \in \mathbb{C}\) and its solutions are called the associated Legendre functions, \(P_l^m(x)\). However in this thesis, we only require the solutions for \(l, m \in \mathbb{Z}\). This subset of solutions to the Legendre equation are called the associated Legendre polynomials. They only exist for \(0 \leq m \leq l\) and they are non-singular and orthogonal over the range \(x \in (-1, 1)\) (see figure 3.1 for examples of the associated Legendre polynomials). When \(m = 0\), the solutions to Legendre’s differential equation are simply the Legendre functions \(P_l(x)\).

There are a number of different ways of calculating the associated Legendre Polynomials. One example of such is

\[P_l^m(x) = \frac{(-1)^m}{2^l l!} (1 - x^2)^{\frac{m}{2}} \frac{d^{l+m}}{dx^{l+m}} \left( (x^2 - 1)^l \right). \tag{3.2} \]

This gives an analytic expression for the associated Legendre polynomials for all \(l, m \in \mathbb{Z}\). The solutions are not technically polynomials when \(m\) is odd since there is a term involving \((1 - x^2)^{\frac{m}{2}}\), as seen in equation 3.2, and if \(|m| > l\) then \(P_l^m = 0\) since
3.1. Spherical Harmonics and Associated Legendre Polynomials

the derivative term is zero.

There is another useful way to calculate the associated Legendre polynomials. Due to the recursive nature of the associated Legendre polynomials, if \( P^0_0(x) \) is known then the other polynomials may be calculated for all \( l, m \in \mathbb{Z} \) with \( l \geq |m| \). Starting with

\[
P^0_0 = 1,
\]

the recursive relationships are given by

\[
(P) P^l_l(x) = -\sqrt{1-x^2} \left(2l - 1\right) P^{l-1}_{l-1}(x),
\]

\[
P^{l+1}_l(x) = x (2l + 1) P^l_l(x),
\]

\[
P^m_l(x) = \frac{x (2l - 1) P^m_{l-1}(x) - (l - 1 + m) P^m_{l-2}(x)}{l - m},
\]

\[
P^{-m}_l(x) = (-1)^m \frac{(l - m)!}{(l + m)!} P^m_l(x),
\]

\[
P^m_l(-x) = (-1)^{l+m} P^m_l(x),
\]

with the restriction \( l \neq m \) in equation 3.5.

These five equations are much more useful for calculating the associated Legendre polynomials numerically than equation 3.2 since this would require a symbolic computation package and cannot be calculated as quickly. The routes to be taken using these recursive relationships from \( P^0_0 \) to any \( P^m_l \) with \( m \geq 0 \) are demonstrated graphically in figure 3.2. It shows how, starting with \( P^0_0 \), all the higher order associated Legendre polynomials can be calculated. Equation 3.6 can then be used to simply calculate all the associated Legendre Polynomials for \( m < 0 \).

Solving Laplace’s equation in spherical coordinates using separation of variables reveals the \( \theta \)-dependent component to be Legendre’s differential equation under a transformation of \( x = \cos \theta \). The associated Legendre polynomials are therefore part of the solution to Laplace’s equation. Under the transformation of \( x = \cos \theta \), the associated Legendre polynomials become over the range \( \theta \in [0, \pi] \) as required for the polar angle, \( \theta \), in spherical coordinates.

The associated Legendre polynomials are usually normalised and denoted by, \( Q^m_l \), and they are still solutions to the Legendre equation (3.1). They are defined as

\[
Q^m_l(x) \equiv N^m_l P^m_l(x),
\]
3. The PFSS Model

Figure 3.2
Graphical illustration of how the recursive relationships (equations 3.3, 3.4 and 3.5 which are labelled 1, 2 and 3) can be used to calculate all of the associated Legendre polynomials for \( m \geq 0 \).
3.1. Spherical Harmonics and Associated Legendre Polynomials

with $N_l^m$ as the normalising factor. These normalised associated Legendre polynomials are used in spherical harmonics to create a special function called a spherical harmonic which is defined as

$$Y_l^m(\theta, \phi) \equiv Q_l^m(\cos \theta) e^{i m \phi}.$$  

We say that $Y_l^m$ is the spherical harmonic of degree $l$ and order $m$ and they are used in the solutions of many partial differential equations in spherical coordinates including Laplace’s equation.

There are various forms of the normalising factors that are chosen, even between different fields in physics. But here the standard normalising factor is chosen which is

$$N_l^m = \sqrt{\frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!}}. \quad (3.8)$$

This normalising factor ensures the simplest form for their orthogonality:

$$\int_\theta=0^{\pi} \int_\phi=0^{2\pi} Y_{l_1}^{m_1}(\theta, \phi) \overline{Y_{l_2}^{m_2}}(\theta, \phi) \, dS = \delta_{l_1, l_2} \delta_{m_1, m_2}, \quad (3.9)$$

where $dS$ is the infinitesimal spherical surface area element (i.e. $dS = \sin \theta d\theta d\phi$), the overbar denotes conjugation (i.e. $\overline{z}$ is the complex conjugate of $z$) and

$$\delta_{i,j} = \begin{cases} 
0 & i \neq j \\
1 & i = j \end{cases}$$

is the Kronecker delta. This means that this integral is only non-zero when $l_1 = l_2$ and $m_1 = m_2$. Note that if a different form of $N_l^m$ to that given above in equation 3.8 was chosen then this would only change the definitions given later in the derivation of the model. For example, it would change the definition of $B_l^m$ which is defined in a similar way using this orthogonality property later in section 3.2.2 in equation 3.27.

In particular, a different normalisation factor would cause extra factors to appear in the orthogonality result itself of equation 3.9 and other equations relying on the orthogonality of $Y_l^m$, but it would not affect the final values of the magnetic field components. The standard normalising factor used here is only chosen for simplicity.

Using this normalisation factor, the recursive relationships between each $Q_l^m$ may be
3. The PFSS Model

derived from equations 3.3 to 3.7 and are

\[ Q_l^l(x) = -\sqrt{1 - x^2} \sqrt{\frac{2l+1}{2l}} Q_{l-1}^{l-1}(x), \]
\[ Q_{l+1}(x) = x \sqrt{2l+3} Q_l^l(x), \]
\[ Q_l^m(x) = x \sqrt{\frac{4l^2-1}{l^2-m^2}} Q_{l-1}^{m-1}(x) - \sqrt{\frac{2l+1}{2l-3}} \frac{(l-1)^2-m^2}{l^2-m^2} Q_{l-2}^{m-2}(x), \]
\[ Q_l^{-m}(x) = (-1)^m Q_l^m(x), \]
\[ Q_l^m(-x) = (-1)^{l+m} Q_l^m(x), \]

with

\[ Q_0^0 = \sqrt{\frac{1}{4\pi}}. \]

3.2 Derivation of the Model

Here we discuss how to derive the PFSS model. We extrapolate a global potential magnetic field in spherical geometry from an observed radial synoptic magnetogram and assuming that the field above a certain height is purely radial. A synoptic map provides the magnetic field data over the entire solar surface compared with a magnetogram which provides the magnetic field data only for the current view of the observer. Data from magnetograms over a period of time are stitched together to make a synoptic map. The synoptic map forms the base of our model at \( r = R_\odot \) and the top outer surface at \( r = R_{\text{max}} \) (where \( R_{\text{max}} \) can be any height provided that \( R_{\text{max}} > R_\odot \)) is constrained to have purely radial field. The top boundary is usually set to be \( R_{\text{max}} = 2.5R_\odot \) since, in observations, the field is generally seen to be radial from this point onwards. This value for \( R_{\text{max}} \) has been widely accepted since Jon Todd Hoeksema (1984) although a value as low as \( R_{\text{max}} = 1.3R_\odot \) has been suggested by Levine et al. (1977). This value of \( R_{\text{max}} = 2.5R_\odot \) will be used throughout the work in this thesis. In most PFSS models, \( R_{\text{max}} \) is normally denoted as \( R_{SS} \), the height of the source surface. I have decided to change the notation since I find this original terminology confusing since the bottom boundary is also a source surface with many more sources on!

The derivation of the model is as follows. Firstly the magnetic field in spherical
3.2. Derivation of the Model

coordinates

\[ \mathbf{B}(r, \theta, \phi) = B_r \mathbf{e}_r + B_\theta \mathbf{e}_\theta + B_\phi \mathbf{e}_\phi \]

is written in the usual form for a scalar potential field,

\[ \mathbf{B} = -\nabla f, \quad (3.15) \]

where \( f = f(r, \theta, \phi) \) is some scalar function. This form assumes the magnetic field is automatically current free such that

\[ \mathbf{J} = \frac{\nabla \times \mathbf{B}}{\mu_0} \equiv 0. \]

Then, using the solenoidal condition for magnetic fields,

\[ \nabla \cdot \mathbf{B} \equiv 0, \]

we obtain Laplace’s equation

\[ \nabla \cdot \nabla f \equiv \nabla^2 f \equiv 0, \]

which is given in spherical coordinates \((r, \theta, \phi)\) by

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} = 0. \]

3.2.1 Solving Laplace’s Equation

Laplace’s equation can be solved using separation of variables as follows. Assuming a separable solution of the form

\[ f(r, \theta, \phi) = R(r) T(\theta) P(\phi), \]

using the three separation functions \( R(r), T(\theta) \) and \( P(\phi) \) (this \( P \) is different to the associated Legendre polynomials), we can rewrite Laplace’s equation as

\[ \frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{T \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dT}{d\theta} \right) + \frac{1}{P \sin^2 \theta} \frac{d^2P}{d\phi^2} = 0. \]

This may be split into three independent equations. The first term is only
3. The PFSS Model

$r$-dependent, therefore using a separation constant $\lambda$, we may write

\[
\frac{1}{T \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dT}{d\theta} \right) + \frac{1}{P \sin^2 \theta} \frac{d^2P}{d\phi^2} = -\lambda, \tag{3.16}
\]

\[
\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = \lambda. \tag{3.17}
\]

The first of these two equations (3.16) may be rearranged and separated again using a constant $m^2$ (chosen to make the equation involving $P$ easier to solve) to form the final two equations which are each only $\theta$- and $\phi$-dependent, respectively,

\[
\frac{\sin \theta}{T} \frac{d}{d\theta} \left( \sin \theta \frac{dT}{d\theta} \right) - \lambda \sin^2 \theta = m^2, \tag{3.18}
\]

\[
\frac{1}{P} \frac{d^2P}{d\phi^2} = -m^2. \tag{3.19}
\]

Here, $\lambda, m \in \mathbb{C}$, although it will be shown they can only be integers.

The equation in $P$ (3.19) can be solved simply to be a linear combination of complex exponentials $e^{\pm im\phi}$. Since the solutions must be periodic over $0 \leq \phi \leq 2\pi$, this restricts the second separation constant $m$ to the integers.

Next, using $x = \cos \theta$ transforms the equation for $T$ (3.18) into Legendre’s differential equation (3.1) and imposing the regularity of the solution at the poles $\theta = 0$ and $\theta = \pi$ constrains the first separation constant to be $\lambda = l(l+1)$ where $l$ is some non-negative integer such that $l \geq |m|$. This means the solution of the equation for $T$ produces the associated Legendre polynomials, $P_l^m(\cos \theta)$. Given that some multiple of the associated Legendre polynomials is also a solution to equation 3.18, we actually use the normalised associated Legendre polynomials, $Q_l^m$, (discussed in section 3.1) in order to use the spherical harmonics. For each integer $l$, there are $2l+1$ independent solutions to the Legendre equation: one for each $m$ ranging from $-l$ to $l$.

Finally we can solve the differential equation for $R(r)$ (3.17). By a substitution of $r = e^u$, the differential equation is transformed into a simple second order homogeneous linear differential equation

\[
\frac{d^2R}{du^2} + \frac{dR}{du} - l(l+1)R = 0
\]

which is easily solved. This reveals the solution $R(r)$ to be a linear combination of $r^l$ and $r^{-(l+1)}$. 

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3.2. Derivation of the Model

The three separation functions can now be defined as

\[ R(r) = a_l^m r^l + b_l^m r^{-(l+1)}, \]
\[ T(\theta) = Q_l^m(\cos \theta), \]
\[ P(\phi) = e^{im\phi}, \]

with the integration constants for \( T \) and \( P \) combined into \( a_l^m \) and \( b_l^m \) such that the final solution to Laplace’s equation in spherical coordinates is

\[ f(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( a_l^m r^l + b_l^m r^{-(l+1)} \right) Q_l^m(\cos \theta) e^{im\phi}. \]

The coefficients \( a_l^m \) and \( b_l^m \) are determined by the boundary conditions.

We may now use equation 3.15 to find the three components of the potential magnetic field \( \mathbf{B} \). First we divide \( R(r) \) by \( r \) and denote this as

\[ A_l^m(r) = \frac{a_l^m r^l + b_l^m r^{-(l+1)}}{r} = a_l^{m,l-1} + b_l^{m,-l+2}. \] (3.20)

Then from this define \( B_l^m(r) \) as

\[ B_l^m(r) = -\frac{d(rA_l^m)}{dr} = (l+1) b_l^m r^{-(l+2)} - l a_l^m r^{l-1}. \] (3.21)

So the three magnetic field components become

\[ B_r(r, \theta, \phi) = -\frac{\partial f}{\partial r} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_l^m(r) Q_l^m(\cos \theta) e^{im\phi}, \] (3.22)
\[ B_\theta(r, \theta, \phi) = -\frac{1}{r} \frac{\partial f}{\partial \theta} = -\sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l^m(r) \frac{dQ_l^m}{d\theta} e^{im\phi}, \] (3.23)
\[ B_\phi(r, \theta, \phi) = -\frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} = -\sum_{l=0}^{\infty} \sum_{m=-l}^{l} imA_l^m(r) \frac{Q_l^m(\cos \theta)}{\sin \theta} e^{im\phi}. \] (3.24)

Note, here \( i \) is the imaginary unit: the \( im \) term does not refer to taking the imaginary part. Also, although this final form for \( B_\phi \) is explicitly dependent on \( i \), its final value is purely real. From now on, \( Q_l^m(\cos \theta) \) will be denoted \( Q_l^m(\theta) \) since we will always be working in terms of the \( \theta \) coordinate in spherical coordinates.
3. The PFSS Model

3.2.2 Applying the boundary conditions

There are two boundary conditions for the PFSS model. These are used to eliminate the unknowns, $a^m_l$ and $b^m_l$, to find $A^m_l$ (and implicitly $B^m_l$) in terms of known quantities.

1. An observed radial synoptic map $B_\odot(\theta, \phi)$ (or some other base field defining the radial field component at the solar surface) is the radial magnetic field component at the base of the potential field

$$B_r(R_\odot, \theta, \phi) = B_\odot(\theta, \phi).$$  \hspace{1cm} (3.25)

2. The magnetic field at the upper boundary $r = R_{\text{max}}$ must be completely radial i.e.

$$B_\theta(R_{\text{max}}, \theta, \phi) = B_\phi(R_{\text{max}}, \theta, \phi) = 0.$$ \hspace{1cm} (3.26)

Using the expression for $B_r$ (3.22) and the orthogonality of the spherical harmonics $Y^m_l$ (3.9), we can get an expression for $B^m_l(r)$ by taking the integral of $B_r$ multiplied by the conjugate of $Y^m_l$ over any surface of constant $r$

\[
\int_0^{2\pi} \int_0^{\pi} B_r(r, \theta, \phi) \overline{Y^m_l}(\theta, \phi) \, dS \equiv \int_0^{2\pi} \int_0^{\pi} \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} B^m_{l'}(r) Y^{m'}_{l'}(\theta, \phi) \overline{Y_l^m}(\theta, \phi) \, dS,
\]

\[
\equiv \int_0^{\infty} \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} \left( B^m_{l'}(r) \int_0^{2\pi} \int_0^{\pi} Y^{m'}_{l'}(\theta, \phi) \overline{Y_l^m}(\theta, \phi) \, dS \right),
\]

\[
\equiv B^m_l(r).
\]

By applying the first boundary condition (3.25), we have the $B_r$ component on the base boundary, $r = R_\odot$, i.e. the observed synoptic map $B_\odot(\theta, \phi)$. Applying this integral on the base gives us

$$B^m_l(R_\odot) \equiv \int_0^{2\pi} \int_0^{\pi} B_\odot(\theta, \phi) \overline{Y_l^m}(\theta, \phi) \, dS,$$ \hspace{1cm} (3.27)

allowing us to define all the $B^m_l(R_\odot)$ using the synoptic map magnetic field $B_\odot$. This definition for $B^m_l(R_\odot)$, together with the second boundary condition, can be used to find $B^m_l(r)$ for all $r$ in terms of the radial base boundary field harmonics $B^m_l(R_\odot)$ as
3.2. Derivation of the Model

follows. $B_l^m(R_\odot)$ will be used to denote the integral in equation 3.27 for each $l$ and $m$. These integrals are all calculable from the synoptic map and therefore known constants.

First using the original definition for $B_l^m(r)$ (3.21) and taking the ratio with $B_l^m(R_\odot)$ gives

$$\frac{B_l^m(r)}{B_l^m(R_\odot)} = \frac{la_l^m r^{-l-1} - (l+1) b_l^m r^{-(l+2)}}{la_l^m R_\odot^{-l-1} - (l+1) b_l^m R_\odot^{-(l+2)}},$$

and rearranging gives

$$B_l^m(r) = B_l^m(R_\odot) \frac{la_l^m r^{-l-1} - (l+1) b_l^m r^{-(l+2)}}{la_l^m R_\odot^{-l-1} - (l+1) b_l^m R_\odot^{-(l+2)}}, \quad (3.28)$$

The second boundary condition (3.26) requires that all $A_l^m(R_{\text{max}}) = 0$ to automatically make $B_\theta$ and $B_\phi$ equal to zero on the top boundary. This implies that

$$a_l^m = -\frac{b_l^m}{R_\text{max}^{2l+1}},$$

which can be used to eliminate all the unknown $a_l^m$ and $b_l^m$ constants from the ratio in equation 3.28.

After this elimination of the constants, equation 3.28 becomes

$$B_l^m(r) = B_l^m(R_\odot) \frac{(l+1) \left(\frac{r}{R_\odot}\right)^{-(l+2)} + l \left(\frac{r}{R_\odot}\right)^{l-1} \left(\frac{R_\odot}{R_{\text{max}}}ight)^{2l+1}}{l + 1 + l \left(\frac{R_\odot}{R_{\text{max}}}ight)^{2l+1}}. \quad (3.29)$$

Hence, this produces a final equation for $B_l^m(r)$ in terms of $r$, $R_\odot$, $R_{\text{max}}$, $l$ and $B_l^m(R_\odot)$ which are all known. This equation can be rearranged using the radial ratio terms to find an equivalent form given by

$$B_l^m(r) = B_l^m(R_\odot) \left(\frac{r}{R_\odot}\right)^{-(l+2)} \frac{l + 1 + l \left(\frac{r}{R_{\text{max}}}ight)^{2l+1}}{l + 1 + l \left(\frac{R_\odot}{R_{\text{max}}}ight)^{2l+1}}. \quad (3.30)$$

Finally using these two expressions for $B_l^m$ (3.29 and 3.30), the two equivalent forms
for $A_l^m$ are calculated using equation 3.20 to be

\[
A_l^m(r) = B_l^m(R_\odot) \left( \frac{r}{R_\odot} \right)^{-(l+2)} - \left( \frac{r}{R_\odot} \right)^{l-1} \left( \frac{R_\odot}{R_{\text{max}}} \right)^{2l+1},
\]

\[
A_l^m(r) = B_l^m(R_\odot) \left( \frac{r}{R_\odot} \right)^{-(l+2)} \left( \frac{r}{R_{\text{max}}} \right)^{2l+1},
\]

(3.31)

The second forms of these expressions for $A_l^m$ (3.32) and $B_l^m$ (3.30) are better for numerical calculations since the $(r/R_\odot)^{l-1}$ term in the first forms (3.31 and 3.29) can become larger than the maximum floating point number (computationally infinite) for large $l$ (see section 3.3.3 for a further discussion of this).

Substituting the definitions for $A_l^m(r)$ (3.32) and $B_l^m(r)$ (3.30) into equations 3.22 to 3.24 yields the final equations for each of the magnetic field components with no unknowns,

\[
B_r(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_l^m(\theta, \phi) B_l^m(R_\odot)}{l+1 + l \left( \frac{r}{R_{\text{max}}} \right)^{2l+1}} \left( \frac{r}{R_\odot} \right)^{-(l+2)} \left( \frac{r}{R_{\text{max}}} \right)^{2l+1},
\]

(3.33)

\[
B_\theta(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{dQ_l^m}{d\theta} e^{im\phi} B_l^m(R_\odot) \left( \frac{r}{R_\odot} \right)^{-(l+2)} \left( \frac{r}{R_{\text{max}}} \right)^{2l+1},
\]

(3.34)

\[
B_\phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{im Y_l^m(\theta, \phi) B_l^m(R_\odot)}{\sin \theta} \left( \frac{r}{R_\odot} \right)^{-(l+2)} \left( \frac{r}{R_{\text{max}}} \right)^{2l+1}.
\]

(3.35)

So given some base radial field at $r = R_\odot$ and the value of $R_{\text{max}}$, these give the potential magnetic field at any $(r, \theta, \phi)$ with $R_\odot \leq r \leq R_{\text{max}}$ in standard spherical coordinates. These are the equations solved in the PFSS model code, I have written, as described next in section 3.3.
3.3 A New PFSS code

Although there are several PFSS codes, I have written my own which will be referred to as BMW2016. It is a Fortran code based on A. van Ballegooijen’s PFSS field code written in 1997 referred to as AVB1997. It is thought that van Ballegooijen wrote this code in preparation for his work done on magnetic flux transport models (van Ballegooijen, Cartledge and Priest 1998). I have written my own version for two main reasons.

Speed:

AVB1997 was primarily written in IDL (with small parts in the older Fortran 77), but we suspected it would be much faster to have the model completely written in modern Fortran. Firstly, for the speed up that is achievable through both a compiled language over an interpreted language like IDL and for Fortran’s ability to be parallelised onto multiple processing units. Secondly, having to keep switching between using two different languages and environments, several times, was time consuming. Now there is only one switch to be made. Pre-processing of the radial synoptic map data is done in Python (see section 3.3.1) and then the final magnetic field is calculated entirely in Fortran.

Accuracy:

As more and more harmonics were considered in AVB1997, it was not clear that the results were scaling as expected. We wanted to check if this was due to numerical effects or was real. To do this we needed to pull the previous code apart and check exactly what it was doing. It turned out that AVB1997 had implemented some interpolation to create more points in the final grid and several other odd aspects were identified, detailed further later in section 3.3.3. We wanted to check whether there were effects on the final results.

BMW2016 was written to fix these issues and implement parallelisation using OpenMP. The source code can be downloaded or forked from GitHub at https://github.com/benmatwil/PFSS.

3.3.1 Pre-processing the Synoptic Maps

Before BMW2016 can be run, the observed radial magnetic field data at the solar surface must be prepared: the radial synoptic map must be transformed and
3. The PFSS Model

processed. This data processing is done in Python:

1. for the ease of reading data from the FITS files and
2. the added flexibility of routines already available.

First we need to extract the data from the FITS file. Earth orbiting satellites do not have a full view from pole to pole of the Sun which causes missing values in the data files. This is due to the relative tilt and position of the Sun with respect to the telescope (here HMI on SDO). Any missing data values (usually at the poles) must be filled so that we have finite values everywhere in the grid. Initially they are set in the FITS files to be NaN (Not a Number). We fill the missing values at each of the poles with smoothed mean polar field data provided by a separate data product from HMI which gives mean values for the poles at several different latitudes. We have chosen to use the mean value for latitudes greater than $70^\circ$. The mean polar field values are averaged over a 15 day period either side of the synoptic map date for daily synoptic maps and averaged over the Carrington rotation for Carrington rotation synoptic maps. Then for those synoptic maps with data at one of the poles (i.e. a full view of one of the polar regions), the data at either $\theta = 0$ or $\theta = \pi$ of the synoptic map is smoothed to ensure that the final row of data at the pole is uniform to prevent discontinuities over the pole.

Then we subtract the mean average net magnetic field value of the whole synoptic map,

$$B_\odot = \frac{\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} B_\odot(\theta, \phi) \, dS}{\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} dS},$$

from all the pixels, to ensure that the solenoidal constraint is satisfied (i.e. the total net magnetic flux is zero). A typical value of $B_\odot$ is about $0.015 \text{ G}$, which is smaller than approximately $97.7\%$ of the magnitude of all pixel values (figure 3.3).

The synoptic map is now ready to be used as the base boundary condition for the PFSS model (given in equation 3.25) to find the $B_m(R_\odot)$ (given by equation 3.27). This 2D magnetic field is written to a binary file for input to the main Fortran code. Alternatively it is possible to write a custom field in the same format for input to BMW2016. This was done by Edwards (2014) to create simple PFSS fields to show different magnetic structures in the magnetic skeleton without the complexity of real
3.3. A New PFSS code

Figure 3.3
The absolute values of all the pixels in a HMI daily synoptic map with the value of $\bar{\mathbf{B}}_\odot$ indicated by the black vertical line

3.3.2 The Main Calculations

BMW2016 calculates the associated Legendre polynomials $Q_{l}^{m}$ on a discretised grid in $\theta$ for all required positive $l$ and $m$ using the first three recursive equations given by 3.10 to 3.12 and

$$Q_0^0 = N_0^0 P_0^0 = \sqrt{\frac{1}{4\pi}}.$$ 

The polynomial range can then be extended to negative $m$ and $x$ using the final two recursive relationships for $Q_{l}^{m}$ given in equations 3.13 and 3.14.

Note that BMW2016 refers to arrays named $plm$ and $qlm$ differently to the mathematical notation defined above. These arrays both contain the values of the normalised associated Legendre polynomials. However, both array values are normalised. So both $plm$ and $qlm$ in the code are actually “$Q_{l}^{m}$”. The $plm$ array refers to normalised associated Legendre polynomials when calculated on a grid linear in $\cos \theta$ and the $qlm$ array refers to those on a grid linear in $\theta$. This is the same notation used by AVB1997.
Two different sections of code are used to calculate the two different $p_l m$ and $q_l m$ arrays since the magnetic field data from the synoptic maps are linear in $\cos \theta$ while the final grids are linear in $\theta$. Since we need to know the derivatives of $Q^m_l$ for the $B_\theta$ component of the magnetic field, it is easier to use the derivatives of the recursive relationships in equations 3.10 to 3.14 in different forms depending on the grid. As an example, consider equation 3.10

$$Q^l_l(x) = -\sqrt{1-x^2} \sqrt{\frac{2l+1}{2l}} Q^{l-1}_{l-1}(x).$$

This can be differentiated to find

$$\frac{dQ^l_l}{dx} = \sqrt{\frac{2l+1}{2l}} \left( -\sqrt{1-x^2} \frac{dQ^{l-1}_{l-1}}{dx} + \frac{x}{\sqrt{1-x^2}} Q^{l-1}_{l-1}(x) \right).$$

However, under the transformation of $x = \cos \theta$, noting that $\sqrt{1-x^2} = \sin \theta,$

$$Q^l_l(\cos \theta) = -\sin \theta \sqrt{\frac{2l+1}{2l}} Q^{l-1}_{l-1}(\cos \theta),$$

and

$$\frac{dQ^l_l}{d\theta} = -\sqrt{\frac{2l+1}{2l}} \left( \sin \theta \frac{dQ^{l-1}_{l-1}}{d\theta} + \cos \theta Q^{l-1}_{l-1}(\cos \theta) \right).$$

The form of these relationships is quite different and so there are two different subroutines to calculate them. Note, the derivatives calculated are also exact; they are not calculated using any sort of finite difference method.

The final potential field is calculated on a discretised grid in $r$, $\theta$ and $\phi$. The size of the grid is denoted by $n_r \times n_\theta \times n_\phi$. Unfortunately, computationally we cannot add together an infinite number of terms so we must truncate the series and pick a maximum value of $l$ which here will be called $L$. This means, for example, our expression for $B_r$ (3.33) (for example) becomes

$$B_r(r, \theta, \phi) = \sum_{l=0}^{L} \sum_{m=-l}^{l} B^m_l(r) Q^m_l(\theta) e^{im\phi}.$$  

(3.36)

The infinite summation over $l$ is replaced by a finite sum. The value of $L$ is limited by

1. the resolution of the original observed synoptic map, $N_\phi \times N_\theta$ and,
2. the computing resources available.
3.3. A New PFSS code

So as not to over-extrapolate the original data, the resolution of the extrapolated final grid at the base boundary, \( n_\theta \times n_\phi \), must not exceed the resolution of the original observed synoptic map. Therefore \( n_\theta < N_\theta \) and \( n_\phi < N_\phi \). As \( L \) is increased, the simulated base field more accurately replicates the observed base boundary condition and more small scale magnetic features appear. This is not due to an increase in the resolution available, but due to the summation of more small scale Fourier components.

When \( L \) is being picked, one other effect which must be taken into account is a “ringing” effect which becomes more significant as the number of harmonics \( L \) is increased and is known as Gibb’s phenomenon. This ringing was actually discovered 50 years earlier by Henry Wilbraham (Wilbraham 1848) and rediscovered by Gibb in 1898 (Gibbs 1898). Fourier series struggle to represent their original function at discontinuities and in numerical data, this is what happens at large changes in the magnetic field: an oscillating pattern appears in the field near large changes. In PFSS models, this can be very near the poles or about strong sources of magnetic field, such as sunspots. It is due to numerical errors in the small, high ordered, harmonics. This is affected greatly by the numerical precision of the floating point numbers used, but it can be dampened by applying a Gaussian filter to the observed data.

This filter is applied during the calculation of the \( B^m_l(R_\odot) \) from the observed synoptic map. From our expression for \( B^m_l(R_\odot) \) (3.27), we actually define

\[
I^m_l = \iint_S B_\odot(\theta, \phi) Y^m_l(\theta, \phi) \, dS,
\]

and the Gaussian filter is applied in the code as

\[
B^m_l(R_\odot) = I^m_l e^{-\frac{a^2(l+1)}{4a^2}} ,
\]  

with \( a \) being an adjustable free parameter. This reduces the effect of the high \( l \) harmonics. This is exactly the same filter as used by AVB1997. From experience of using AVB1997 which also includes this type of filter, \( a \) is chosen in BMW2016 such that the Gaussian/exponential factor \( e^{-\frac{a^2(l+1)}{4a^2}} \) is approximately equal to 0.1 when \( l = L \). To achieve this, we set \( a \approx L \) as this is the default behaviour of the code. The parameter, \( a \), can also be varied to find an optimal field and is a command line option for input to the code. In the end, the optimal number of harmonics, \( L \), and the parameter \( a \) can only be found by trial and error as they are dependent on the base
boundary condition.

BMW2016 has two options to calculate the final field, either by direct summation or fast Fourier transform (FFT). If the extrapolation is done by direct summation, any arbitrary coordinate grid in $r$, $\theta$ and $\phi$ may be chosen. Since we know the field analytically everywhere, the only restriction is not to over-extrapolate the original data from the synoptic map which restricts the number of points in the $\theta$ and $\phi$ dimensions. On the other hand, if the FFT approach is used, we have a restriction on the grid in the $\phi$ direction. This is due to the FFT and how many points it is able to calculate. The $r$ and $\theta$ grids in the FFT case are still arbitrary, within the restrictions above. This means there is a trade off between resolution in the $\phi$ direction and speed of the full calculation. The FFT method of calculation is significantly faster than the direct summation method and is discussed later in this section with a short description of FFTs.

### 3.3.2.1 Default Coordinate Grids

We now describe the default grids used in BMW2016 for each coordinate which comply with the restriction in the $\phi$ direction when using the FFT method. These default grids are similar to those used in AVB1997 except without any interpolation (discussed later in section 3.3.3).

The number of points available in the $\phi$ direction directly depends on $L$ due to the FFT. The other two directions also depend on $L$ by default, although this is not necessary as they are theoretically arbitrary. They are picked such that as $L$ increases, the resolution of the output magnetic field also increases.

**The number of points in the $\phi$ direction, $n_\phi$:**

The default $n_\phi$ depends directly on $L$ as

$$n_\phi = 2(L + 1) + 1 = 2L + 3,$$

with $\phi_1 = 0$ and $\phi_{n_\phi} = 2\pi$. This number is the maximum number of points the FFT is able to calculate based on the number of harmonics calculated for other terms in the summation. The FFT is able to calculate $2(L + 1)$ points and then the periodicity of the grid means we have one more point such that $\phi = 0$ is the same as $\phi = 2\pi$. This restriction is discussed later when Fourier transforms are introduced.
3.3. A New PFSS code

The number of points in the $\theta$ direction, $n_\theta$:

The default $n_\theta$ is picked to be approximately half the number of points in the $\phi$ direction

$$n_\theta = (L + 1) + 1 = L + 2,$$

with $\theta_1 = 0$ and $\theta_{n_\theta} = \pi$ so that $\theta$ and $\phi$ are equally spaced in the domain.

The number of points in the $r$ direction, $n_r$:

The values of the default grid in $r$ are based on an exponential grid with more points close to the solar surface. The number of grid points is given by

$$n_r = \left\lceil \frac{2(L + 1)}{\pi} \log \frac{R_{\text{max}}}{R_\odot} \right\rceil.$$

In the PFSS equations (3.33 to 3.35), $r$ never appears without being divided by $R_\odot$. So the grid in $r$ is normalised by $R_\odot$ such that the first point is $r_1 = 1$ (i.e. $r_1 = 1 \times R_\odot$). The following expression then gives the value of each of the grid points in $r$

$$r_k = \frac{e^{\alpha(k,L)} - 1}{e^{\alpha(n_r,L)} - 1} \left( \frac{R_{\text{max}}}{R_\odot} - 1 \right) + 1 \text{ for } k \in \{1, \ldots, n_r\}$$

(3.38)

with

$$\alpha(k, L) = \frac{\pi (k - 1)}{2(L + 1)}$$

and $e^{\alpha(k,L)}$ being the main exponential distribution of the points in $r$. The value of the last point in the exponential grid given by $e^{\alpha(n_r,L)}$ is just below $r = R_{\text{max}}$. So, in effect, the above expression (equation 3.38) is rescaling this grid to be exactly from $r_1 = 1$ to $r_{n_r} = R_{\text{max}}$. By default a value of $R_{\text{max}} = 2.5 (R_\odot)$ is used although this can be changed using a command line input.

The default dimensions $n_r \times n_\theta \times n_\phi$ for the model are indicated in figure 3.4. As an example, in the case of $L = 701$, a grid of $410 \times 703 \times 1405$ points is created if the field is extrapolated using the default grid for the FFT method. The grid for both $r$ and $\theta$ could be manually changed to whatever is required by the user (taking into the account the over-extrapolation constraint) in the correct section of the calc_grids subroutine.

Note, if we have an observed synoptic map of resolution $N_\phi \times N_\theta$, $L$ must be picked
3. The PFSS Model

Figure 3.4
Default number of points on the numerical PFSS grid when using the FFT approach.
### 3.3. A New PFSS code

#### Table 3.5

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Figure 3.5

Required terms for the summations

![Figure 3.5](image-url)

![Table 3.5](table-url)

to satisfy

\[ L < \min \left( \frac{N_\phi}{2} - 1, N_\theta - 1 \right) \]

Furthermore, the maximum number of harmonics, \( L \), should be odd due to the way the FFT is currently implemented. If \( L \) is chosen to be even, a different implementation would be required.

#### 3.3.2.2 Final Summation and Fast Fourier Transforms

In BMW2016, as in AVB1997, the actual summation order of the harmonics in the potential field is reversed to permit the FFT to be used to improve efficiency and speed of the code. At first glance this can be confusing.

In figure 3.5, the terms used in the summation are ticked. We see that undertaking the summation in the order stated using, for example, in our expression for \( B_r \) (3.33),

\[
B_r(r, \theta, \phi) = \sum_{l=0}^{L} \sum_{|m|=l}^{l} B_l^m(r) Q_l^m(\theta) e^{im\phi},
\]

we would first sum along each row in figure 3.5 for \(-l \leq m \leq l\) and then sum over all these \( l \) subtotals for \( 0 \leq l \leq L \). However, swapping this summation order around allows us to use an FFT since the complex exponential term in the summation only depends on \( m \). This switch means that we sum over each column for \(|m| \leq l \leq L\) and then over the whole range of \( m \) for \(-L \leq m \leq L\). Thus the expression for \( B_r \), when
3. The PFSS Model

the order of the summation is switched becomes

\[ B_r(r, \theta, \phi) = \sum_{m=-L}^{L} \sum_{l=|m|}^{L} B_l^m(r) Q_l^m(\theta) e^{im\phi}, \]

\[ = \sum_{m=-L}^{L} e^{im\phi} \left( \sum_{l=|m|}^{L} B_l^m(r) Q_l^m(\theta) \right). \] (3.39)

This is now in a form to which a discrete Fourier transform may be applied.

A Fourier transform for a continuous function \( f \) is given by

\[ g(y) = \int_{-\infty}^{\infty} e^{-2\pi ixy} f(x) \, dx \]

and the equivalent discrete Fourier transform is of the form

\[ y_k = \sum_{n=0}^{N-1} e^{-\frac{2\pi ink}{N}} x_n \]

where \( x \) and \( y \) are both one-dimensional sets of points of length \( N \) and \( x_k \) and \( y_k \) are the \( k^{th} \) points respectively. Here, \( x \) and \( y \) are the input and output values of the transform, respectively. The inverse transform is then given by

\[ x_k = \frac{1}{N} \sum_{n=0}^{N-1} e^{\frac{2\pi ink}{N}} y_n \] (3.40)

(the location and inclusion of the \( 1/N \) factor varies between definitions and FFT algorithms – be careful!) Taking the inverse Fourier transform of Fourier transformed data \( y \) will return the original input data \( x \).

The expression for \( B_r \) in 3.39 for \( m > 0 \) is in the form of the inverse discrete Fourier transform given in equation 3.40. By comparing equation 3.39 and 3.40, it is possible to see how the Fourier transform works here. The exponential term in equation 3.40 can be written as

\[ e^{i\times m \times \phi} = e^{i\times n \times k \frac{2\pi}{N}}. \]

In the exponent, \( n \) is equivalent to \( m \) in equation 3.39 and each multiple of \( 2\pi/N \) (using \( k \)) represents each point in the \( \phi \) direction. Therefore the number of divisions in the \( \phi \) direction is dependent on \( N \) which is in turn dependent on the number of
3.3. A New PFSS code

terms in the summation. In the PFSS model equations we are summing from $-L$ to $L$ which restricts the maximum value of $N$. Larger $L$ means that the $N$ can be larger. This is why the number of points in the $\phi$ direction is restricted by the Fast Fourier transform.

The summation in equation 3.40 is only over positive values of $n$ (or $m$ as has just been shown). However we require the summation over negative $m$ as well. When $m < 0$, the terms are calculated using conjugation which exploits the mathematical properties of the functions here. From equation 3.13

$$Q_{l}^{-m}(x) = (-1)^{m} Q_{l}^{m}(x),$$

and the equivalent relationships for both $B_{l}^{m}$ and $Y_{l}^{m}$ (derived in Appendix A)

$$B_{l}^{-m}(x) = (-1)^{m} \overline{B_{l}^{m}(x)},$$
$$Y_{l}^{-m}(x) = (-1)^{m} \overline{Y_{l}^{m}(x)},$$

such that when we take the product of the two terms $B_{l}^{-m}(x) Y_{l}^{-m}(x)$ (for example, in our expression for $B_{r}$), a term $(-1)^{2m} = 1$ is produced simplifying the calculation. Thus it is not necessary to calculate the terms for negative $m$ explicitly and they are not calculated explicitly in the code: only a conjugation of the final results for $m > 0$ is required.

$$B_{l}^{-m}(x) Y_{l}^{-m}(x) \equiv \overline{B_{l}^{m}(x) Y_{l}^{m}(x)}.$$  

A fast Fourier transform is a very efficient algorithm for calculating these discrete Fourier transforms. It improves the efficiency of calculating the sum from $O(n^2)$ to $O(n \log n)$ therefore significantly speeding up the calculation. Some FFT algorithms have also been shown to be numerically more accurate than the direct summations themselves (Schatzman 1996).

Using an FFT rather than direct computation of the outer sum improves computation time significantly and possibly accuracy too for large enough $L$. The computational cost of attempting to calculate this summation directly is prohibitive and only very small maximum harmonic numbers, $L$, can be calculated in a reasonable time. As an example, for $L = 161$, calculating the PFSS magnetic field at the default grid resolution using the FFT method takes approximately 2 seconds whereas calculating the same magnetic field using the direct summation method would require over 3.5 minutes – over 100 times longer! The time taken to calculate magnetic fields under
3. The PFSS Model

the PFSS using the direction summation method is discussed further in chapter 4. The actual FFT calculations are done by an external piece of software called FFTW (“Fastest Fourier Transform in the West”) and can be found at www.fftw.org. It prides itself as being one of the fastest FFT software packages available and is one of the few FFT pieces of software that is written in a modern programming language and does not require specific sized grids. We have used version 3 since it has Fortran 2003 linking libraries (although it is itself written in C) and has the option to be parallelised. On initialisation of the library, it self-optimises to try to maximise the speed of the specific transform being done.

3.3.3 Differences Between The Revised Code (BMW2016) and van Ballegooijen’s code (AVB1997)

Here we list, for completeness, the differences between AVB1997 and BMW2016 even though some of the changes do not have a significant effect on the results.

Numerical Precision:

AVB1997 was written in IDL. It was also written using only single precision (32 bit) floating point numbers. BMW2016 has been written in Fortran in double precision (64 bit) although this could be increased to any numerical accuracy required. BMW2016 has also been parallelised.

Output Magnetic Field Grids:

AVB1997 calculates the magnetic field using a FFT with a grid of the same size in the $\phi$ direction as BMW2016 for a given $L$. Then AVB1997 uses a cubic spline interpolation to produce additional points on a new, finer grid, i.e. it produces a new grid point halfway between each original grid point. This essentially doubles the number of grid points in the $\phi$ direction and produces a new grid with $n_\phi = 4(L + 1) + 1$ grid points with only half being analytical solutions to Laplace’s Equation. In the $\theta$ direction in AVB1997, to ensure the resolutions of the $\theta$ and $\phi$ grids are the same, no interpolation is required, simply more values $\theta$ are calculated in the first place which is possible since their number if not limited by the FFT. So, again, the $\theta$ grid is doubled compared with BMW2016 and has $n_\theta = 2(L + 1) + 1$ grid points. The grid in the $r$ direction in AVB1997 ranges from 1 to just under $R_{\text{max}}$ based on the unnormalised exponential function ($e^{\alpha (k, L)}$ from equation 3.38) for the radii mentioned earlier in the default grids.
section. Instead in BMW2016, all the radii are scaled to end exactly at \( R_{\text{max}} \).

For example, for \( L = 81 \), AVB1997 produces a grid of \( 48 \times 165 \times 329 \) whereas in BMW2016 the default grid is \( 48 \times 83 \times 325 \). To get a similar grid resolution in \( \theta \) and \( \phi \), BMW2016 requires \( L = 161 \) which produces a grid of \( 95 \times 163 \times 325 \).

Analytic Form of the Model and Terms at Higher Order Harmonics:

During the calculation of the \( B_l^m \) and \( A_l^m \), AVB1997 uses the first forms for \( B_l^m \) and \( A_l^m \) given earlier by equations 3.29 and 3.31 whereas BMW2016 uses the second, mathematically equivalent forms in equations 3.30 and 3.32. An issue arises with the first forms of \( B_l^m \) and \( A_l^m \) (discovered when pushing the value of \( L \) in the code above approximately 775 for 64 bit floating point numbers and approximately 100 for 32-bit floating point numbers) which means that the \((r/R_{\odot})^{l-1}\) term grows larger than the maximum value of the numerical floating point number and causes infinities in the final magnetic field. This issue is fixed using the forms of \( B_l^m \) and \( A_l^m \) given in equations 3.30 and 3.32 where all radial terms are now decreasing with \( l \). Given that in 1997 the available synoptic magnetogram data would have permitted \( L \approx 31 \) at most with the AVB1997 default grid, this would not have been an issue at the time. But with current high resolution synoptic observations, this is a significant problem.

Another issue with AVB1997 is that the calculation of some of the individual terms is not done. In particular, the individual terms in the radially varying fraction within the first expression for \( B_l^m \) earlier in equation 3.29,

\[
(l + 1) \left( \frac{r}{R_{\odot}} \right)^{-(l+2)} + l \left( \frac{r}{R_{\odot}} \right)^{-l-1} \left( \frac{R_{\odot}}{R_{\text{max}}} \right)^{2l+1}
\]

\[
\frac{l}{l + 1 + l} \left( \frac{R_{\odot}}{R_{\text{max}}} \right)^{2l+1}
\]

are not all calculated. Each of these radial ratio terms are calculated separately for each \( l \) within the code. The majority of these terms are decreasing with increasing \( l \) and tend to zero. AVB1997 sets all the ratio terms to zero for \( l \) above a threshold given by

\[
\left\lfloor \frac{10}{\log_{10} R_{\text{max}}} \right\rfloor.
\]

For \( R_{\text{max}} = 2.5 \), this threshold is equal to 25. Clearly, this is not ideal when \( L \) is large since all terms for \( l > 25 \) will be ignored.

I do not completely understand why this constraint was applied, but my best
3. The PFSS Model

explanation is that terms such as \((R_\odot / R_{\text{max}})^{2l+1}\) are very small for large enough \(l\) that they are negligible in comparison to the terms for small \(l\), so they are ignored to save computational time. BMW2016 calculates these terms for all \(l\). Ignoring the terms for large \(l\) may also have been done to avoid the earlier issue of infinities occurring when \(l\) is about 100 for 32-bit floating point numbers.

Reversal of Summation:
In BMW2016, we have decided to reversed the order in which the summation over all the \(l\) subsums is done. The summation of floating point numbers of different orders of magnitude is more accurate if they are added in order of their absolute value and starting with the smallest. This ensures any truncation of the result due to numerical precision following the summation of terms is reduced as much as possible at each step of the summation. Given that the harmonics are small for large \(l\), BMW2016 reverses the order of the summation in \(l\) to maximise accuracy of the final result. The inner summation is now done starting from \(l = L\) and ending at \(l = 0\). This summation cannot be reversed for \(m\) because of the FFT approach which is believed to be more accurate than direction summation anyway.

Automation of the Gaussian Filter:
In AVB1997, the parameter \(a\) used in the calculation of the Gaussian filter (equation 3.37) has to be manually chosen and any change must be hard coded when \(L\) is changed. In BMW2016, it is now automated as described in section 3.3.2. However, if required, it can be chosen by input of a value on the command line when the code is executed.

Alteration in the Calculation of the Associated Legendre Polynomials:
AVB1997 did not take into account the negative sign in the first recursive relationship for \(Q_l^m\) given in equation 3.10. This is fine due to the relationship for \(B_l^m\) depending on \(Q_l^m\),

\[
B_l^m(r) = \int \int_S Q_l^m(\theta) e^{-im\phi} B_r(r, \theta, \phi) \, dS,
\]

because when \(Q_l^m\) switches sign, then so will \(B_l^m\). Thus the product of the two in the final summations, is always positive. Hence for calculating the PFSS model, the negative sign in equation 3.10 is not required and the correct result is obtained whether it is included in PFSS model code or not. However, the
### 3.4 Comparison of the Results from AVB1997 and BMW2016

<table>
<thead>
<tr>
<th>$L$</th>
<th>AVB1997</th>
<th>BMW2016 (Serial)</th>
<th>BMW2016 (4 Cores)</th>
</tr>
</thead>
<tbody>
<tr>
<td>41</td>
<td>0.368</td>
<td>0.0171</td>
<td>0.00886</td>
</tr>
<tr>
<td>81</td>
<td>2.96</td>
<td>0.201</td>
<td>0.0869</td>
</tr>
<tr>
<td>161</td>
<td>32.6</td>
<td>2.75</td>
<td>1.19</td>
</tr>
<tr>
<td>321</td>
<td>529</td>
<td>80.3</td>
<td>23.5</td>
</tr>
<tr>
<td>641</td>
<td>7580</td>
<td>2240</td>
<td>1010</td>
</tr>
</tbody>
</table>

Table 3.1
The computational times in seconds for each of the codes for different numbers of harmonics.

The negative sign in BMW2016 is kept for correctness and for re-usability of the harmonic calculation parts of the code.

### 3.4 Comparison of the Results from AVB1997 and BMW2016

Here the two codes are compared on their run times and their final output magnetic field results. The aim is to have a short comparison showing the significant speed up magnetic field calculation of BMW2016 compared with AVB1997 while obtaining the same resolution magnetic fields at a higher accuracy.

#### 3.4.1 Performance of the Codes

First, AVB1997 and BMW2016 are run for different numbers of harmonics and the run time of each is calculated. BMW2016 was run with just one processor and OpenMP off (serial) and with four processors using OpenMP (4 cores). All other parameters and the base synoptic map are kept the same for all runs. The two codes were run for the same value of $L$: the time just for the final summation calculation is taken. Table 3.1 presents the results in seconds.

Clearly, the new code calculates the magnetic fields much more quickly. For the lower numbers of harmonics, the speed up between the serial version of BMW2016 and AVB1997 is at least a factor of 10. This speed up is still significant for higher numbers of harmonics but less so. The speed up for 4 processors is then at a factor of two better compared to the single processor version of BMW2016. This means that
calculating the magnetic field for \( L = 641 \) using BMW2016 could be as much as 8 times faster than AVB1997.

### 3.4.2 The Magnetic Fields Produced

Finally, the solar surface and extrapolated magnetic fields from the two codes are compared. Figure 3.6 shows the relative absolute difference of the base of the two fields. The interpolation of AVB1997 has been removed to make the two fields as similar as possible and make their sizes equivalent. That is if \( \mathbf{B}_1 \) denotes the field from BMW2016 and \( \mathbf{B}_2 \) denotes the field from AVB1997 then

\[ |\mathbf{B}_1(R_\odot, \theta, \phi) - \mathbf{B}_2(R_\odot, \theta, \phi)| \]

is plotted. The maximum difference for the majority of the domain is very small. There will be some errors due to the two different FFT algorithms and zeroing of the terms for higher \( l \) and \( m \) described in section 3.3.3. This shows the output from the codes is very similar and no major changes have occurred.

### 3.5 Final Simulated Radial Fields at \( r = R_\odot \)

In figure 3.7, the simulated radial fields at the base \( r = R_\odot \) are plotted (figures 3.7b to 3.7e) with the corresponding original synoptic map (figure 3.7a) for comparison. The synoptic map used is a daily HMI synoptic map from 1\(^{st}\) June 2010 which has
3.5. Final Simulated Radial Fields at \( r = R_\odot \)

dimensions of 3600 pixels in the longitudinal direction and 1440 pixels in the latitudinal direction. Although the pixels in the latitudinal direction in the original FITS file are originally distributed as sine latitude, the data is plotted here linearly in latitude to match the output from the PFSS model.

We see that as the number of harmonics increases, the detail of the original synoptic map is slowly revealed. The small scale features are gradually replicated as more harmonics are summed, i.e. more high frequency sinusoidal terms are summed. In figure 3.8, the same zoomed in section for a subset of the plots in figure 3.7 is shown highlighting the increasing small scale features more clearly as \( L \) increases.

The HMI daily magnetic radial synoptic maps all have a noticeable break in the data at the 120° line. This can be clearly seen in figure 3.7a. There are much more sudden changes in sign around this line and it prompted us to check whether this was causing any degradation in the final results.

After testing the data on several different dates of potential fields, we do not find there to be any extra nulls or out of place features. The simulated data is smoothed by the fact that the series are truncated at \( L \) and not all harmonics up to infinity are summed. This anomaly in the original data is certainly not noticeable at low harmonic numbers and is difficult to see even for high numbers of harmonics.
3. The PFSS Model

(a) HMI daily radial synoptic map with grid size of 3600 \times 1440 which has been preprocessed as described in section 3.3.1.

(b) $L = 81$

(c) $L = 161$
3.5. Final Simulated Radial Fields at $r = R_\odot$

Figure 3.7
The HMI daily synoptic map in (a) with the calculated PFSS magnetic fields in (b) to (e) on the solar surface for different numbers of harmonics on 1st June 2010.
3. The PFSS Model

![Synoptic Maps]

Figure 3.8
Same fields as figure 3.7 but zoomed in to show detail.
Chapter 4

Applicability of the PFSS Model

The PFSS model is widely used in research because of its simplicity and there are several codes available to calculate it, such as that by van Ballegooijen (AVB1997) compared here in this thesis with BMW2016 in chapter 3. In much of the previous research, PFSS models have generally only been calculated for very low numbers of harmonics. This is due to a combination of

1. a lack computational resources and,
2. low resolution observed synoptic maps.

Work done using the PFSS model has been done by, for example, Cook, Mackay and Nandy (2009), Platten et al. (2014) and Freed, Longcope and McKenzie (2015) using a maximum number of harmonics of 63, 81 and 30 respectively. Even though new high resolution data resources such as HMI and much more powerful computers are available, these recent pieces of research have not taken advantage of these advances in technology. Even the Solar Software IDL PFSS code, which is widely available to scientists, still uses what we think to be a maximum number of harmonics of around 192. In this chapter we consider whether these small maximum numbers of harmonics are sufficient to fully model the global magnetic skeleton.

As described in chapter 3, when codes use an FFT for calculations, the number of computable points of the grid in the extrapolated global magnetic field is restricted by the number of harmonics. Although this restriction is only on the $\phi$ direction (the longitudinal component of the field), authors have typically chosen a similar resolution in the $\theta$ direction to that in the $\phi$ direction and often choose the grid spacing in the radial direction as some fraction of that in the $\phi$ direction. Given these restrictions (real and self-imposed) that are used by many in research, the question arises as to whether the restricted, default resolution of AVB1997 is sufficient to capture the key features of an observed synoptic magnetogram and its extrapolated global field. In order to measure this, we consider how the resolution can affect the
number of magnetic skeleton features that can be found.

Thus having written BMW2016 to calculate the PFSS model on the same default resolution grid used by van Ballegooijen (see chapter 3), we have also added the ability to calculate the magnetic field analytically on any arbitrarily sized grid so we can test how good the default grids are. Questions that we will ask are:

- What detail are we missing, if any, by using the default grid?
- Is the assumption (used by most to draw the field lines) that the field within each grid cell is trilinear valid?
- If detail is missing and the trilinear assumption is found to be poor, is there anything that can be done to improve the situation?

In this chapter, to compare the magnetic fields, we look at the results from the Magnetic Skeleton Analysis Tools (described in chapter 2). Different magnetic fields are extrapolated from the same synoptic magnetogram by changing the resolution of the numerical grid and different maximum harmonic numbers in the PFSS model. From this we can investigate what grid resolution is required in order to find the full magnetic skeleton for different specific maximum harmonic numbers and, if the full magnetic skeleton cannot be found, we consider ways to improve the situation. We also compare the results as the number of harmonics is increased to see how much more detail we see.

### 4.1 Set-Up

To investigate the effect of the resolution of the PFSS model on the extrapolated global magnetic field, we have used a daily synoptic map from HMI from 1\textsuperscript{st} June 2010 which is shown in figure 3.7a. Originally, it was planned to do separator network analysis on daily synoptic map data for both solar minimum and maximum. As the direction of work changed to the work in this chapter, 1\textsuperscript{st} June 2010 was used for this work as it was the first date in the data downloaded for the original work and the first date available from HMI.

The synoptic map has a resolution of 3600 pixels in the $\phi$ direction and 1440 pixels in the $\theta$ direction and undergoes the preprocessing as described in section 3.3.1. This modified synoptic map is then used as the base boundary condition for all the PFSS
4.1. Set-Up

The addition of points onto the grid illustrated in 2D. The black points representing the grid points on the default resolution grid. The red points then being the first addition of extra points for the $2 \times (n_r, n_\theta, n_\phi)$ grid halfway between the original black points. The green points being the second addition of extra points halfway between the red points to form the $4 \times (n_r, n_\theta, n_\phi)$ grid. This pattern keeps repeating for higher resolutions.

models in this chapter. As explained in chapter 3, the default grid has dimensions given by the following

$$n_r = \left\lceil \frac{2(L + 1)}{\pi} \log \frac{R_{\text{max}}}{R_\odot} \right\rceil,$$

$$n_\phi = 2L + 3$$

$$n_\theta = L + 2$$

where $L$ is the maximum harmonic number and $R_{\text{max}} = 2.5R_\odot$. The points in the $\theta$ and $\phi$ directions are linear while the points in the radial direction are exponentially distributed giving a higher resolution grid closer to the solar surface. The default grids are the lowest resolutions of the extrapolated magnetic fields for a given $L$. These are the same FFT fields as those described in detail in section 3.3.2. Examples of the radial component of the magnetic field from the preprocessed synoptic map and different values of $L$ can be found in figure 3.7.

To find higher resolution magnetic field grids for a given $L$, an analytical calculation of each grid point is used. We simply double the number of points in each grid direction from the defaults above. For each higher resolution grid, a new point is placed exactly halfway between each original grid point and then the magnetic field is calculated completely analytically at each grid point as the FFT approach can no longer be used. Figure 4.1 illustrates the scheme used to add additional points.
4. Applicability of the PFSS Model

<table>
<thead>
<tr>
<th>Multiple</th>
<th>Grid Resolution ((n_r \times n_\theta \times n_\phi))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Default: (1 \times (n_r, n_\theta, n_\phi))</td>
<td>(48 \times 83 \times 165)</td>
</tr>
<tr>
<td>2 ((n_r, n_\theta, n_\phi))</td>
<td>(95 \times 165 \times 329)</td>
</tr>
<tr>
<td>4 ((n_r, n_\theta, n_\phi))</td>
<td>(189 \times 329 \times 657)</td>
</tr>
<tr>
<td>8 ((n_r, n_\theta, n_\phi))</td>
<td>(377 \times 657 \times 1313)</td>
</tr>
<tr>
<td>16 ((n_r, n_\theta, n_\phi))</td>
<td>(753 \times 1313 \times 2625)</td>
</tr>
</tbody>
</table>

Table 4.1

The resolution doubling for \(L = 81\) and \(L = 161\) and the notation for denoting the resolution multiple for each field.

As an example, for \(L = 81\), the default number of grid points in each direction is \(48 \times 83 \times 165\) when the FFT is used. The next higher resolution after doubling is \(95 \times 165 \times 329\). This continual doubling is illustrated in table 4.1. The amount of doubling for each grid resolution is repeated up to memory, computational and time constraints or the limit of the synoptic map dimensions. The resolution of each dimension is given by

\[
2d(n_i - 1) + 1
\]

where \(n_i\) is the number of points in one of the dimensions in the default case and \(d\) is the multiple the grid has been multiplied by.

This doubling in resolution from the default for a fixed maximum harmonic number \(L\) has approximately the same effect on the resolution as doubling \(L\) and using the default grid resolution for the new \(L\). This can be seen when comparing the doubling resolution in the two columns of table 4.1 which show the resolution doubling \(L = 81\) and \(L = 161\). The notation used in the column labelled multiple will be used throughout this chapter. The maximum resolution considered here in this chapter is equivalent to the \(16 \times (n_r, n_\theta, n_\phi)\) resolution listed in table 4.1 for \(L = 81\). However this is still smaller than the actual resolution of the observed HMI synoptic map, \(3600 \times 1440\) \((n_\phi \times n_\theta)\). Calculating PFSS models by using the default resolution and maximising the number of harmonics \((L \sim 1440)\) produces a magnetic field which would require over 75 GB to store, much more memory than this during the computation and several hours to compute. This may be impractical for most to compute for a large scale study with current computational resources.

All the different PFSS models, considered and analysed, are listed in table 4.3. There are also additional models run with a maximum number of harmonics of \(L = 41\).
4.2. Increasing the Number of Harmonics

Increasing the Number of Harmonics

<table>
<thead>
<tr>
<th>$L$</th>
<th>Default Grid Resolution ($n_r \times n_\theta \times n_\phi$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>41</td>
<td>$25 \times 43 \times 85$</td>
</tr>
<tr>
<td>81</td>
<td>$48 \times 83 \times 165$</td>
</tr>
<tr>
<td>161</td>
<td>$95 \times 163 \times 325$</td>
</tr>
<tr>
<td>321</td>
<td>$188 \times 323 \times 645$</td>
</tr>
<tr>
<td>641</td>
<td>$375 \times 643 \times 1285$</td>
</tr>
</tbody>
</table>

The default resolutions for each $L$

However these are considered separately due to the low numbers of harmonics (see section 4.6). In each case, the BMW2016 PFSS model is run to create a global magnetic field and then the MSAT package is run on each global magnetic field to find its full magnetic skeleton. The magnetic skeleton for each grid resolution is then compared with various different measures.

4.2 Increasing the Number of Harmonics

First we consider the changes to the magnetic skeleton as the maximum number of harmonics, $L$, is doubled. Table 4.3 summarises the quantitative data on the magnetic skeleton for each PFSS field calculated.

The first thing to notice is how few magnetic features exist for small maximum numbers of harmonics. For $L = 81$, only 10s of null points are found compared with the thousands found for high numbers of harmonics and similarly for the numbers of separators. Clearly the low maximum harmonic number models produce relatively simple global magnetic fields which are associated with simple magnetic skeletons – even when the grid resolution is high.

The number of nulls increases rapidly as the maximum number of harmonics is increased. There are 5-6 times more nulls in the case for $L = 161$ compared with $L = 81$ while there are only 2-3 times more nulls between $L = 641$ and $L = 321$. Although the numbers of nulls continues to increase with $L$, the increase is slowing down as $L$ increases.

As the resolution of each individual set is increased for constant $L$, we see an increase in the number of nulls found. For each $L$, this number appears to increase up to a limit. This can certainly be seen for $L = 81$ and $L = 161$. For $L = 81$, there are 63
4. Applicability of the PFSS Model

<table>
<thead>
<tr>
<th>Resolution</th>
<th>( n_r )</th>
<th>( n_\theta )</th>
<th>( n_\phi )</th>
<th>Null Points</th>
<th>Separators</th>
<th>Time</th>
<th>File Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L = 81 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Default</td>
<td>48</td>
<td>83</td>
<td>165</td>
<td>63</td>
<td>20</td>
<td>( 32 % )</td>
<td>( 44 % )</td>
</tr>
<tr>
<td>2( \times )</td>
<td>95</td>
<td>165</td>
<td>329</td>
<td>84</td>
<td>35%</td>
<td>27</td>
<td>( 32 % )</td>
</tr>
<tr>
<td>4( \times )</td>
<td>189</td>
<td>329</td>
<td>657</td>
<td>88</td>
<td>40%</td>
<td>29</td>
<td>( 33 % )</td>
</tr>
<tr>
<td>8( \times )*</td>
<td>377</td>
<td>657</td>
<td>1313</td>
<td>91</td>
<td>44%</td>
<td>31</td>
<td>( 34 % )</td>
</tr>
<tr>
<td>16( \times )</td>
<td>753</td>
<td>1313</td>
<td>2625</td>
<td>92</td>
<td>46%</td>
<td>31</td>
<td>( 34 % )</td>
</tr>
<tr>
<td>( L = 161 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Default</td>
<td>95</td>
<td>163</td>
<td>325</td>
<td>360</td>
<td>186</td>
<td>52%</td>
<td>59%</td>
</tr>
<tr>
<td>2( \times )</td>
<td>189</td>
<td>325</td>
<td>649</td>
<td>426</td>
<td>18%</td>
<td>212</td>
<td>50%</td>
</tr>
<tr>
<td>4( \times )*</td>
<td>377</td>
<td>649</td>
<td>1297</td>
<td>461</td>
<td>28%</td>
<td>224</td>
<td>49%</td>
</tr>
<tr>
<td>8( \times )</td>
<td>753</td>
<td>1297</td>
<td>2593</td>
<td>468</td>
<td>30%</td>
<td>231</td>
<td>49%</td>
</tr>
<tr>
<td>( L = 321 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Default</td>
<td>188</td>
<td>323</td>
<td>645</td>
<td>1700</td>
<td>1293</td>
<td>76%</td>
<td>68%</td>
</tr>
<tr>
<td>2( \times )*</td>
<td>375</td>
<td>645</td>
<td>1289</td>
<td>1952</td>
<td>15%</td>
<td>1658</td>
<td>85%</td>
</tr>
<tr>
<td>4( \times )</td>
<td>749</td>
<td>1289</td>
<td>2577</td>
<td>2027</td>
<td>19%</td>
<td>1739</td>
<td>86%</td>
</tr>
<tr>
<td>( L = 641 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Default*</td>
<td>375</td>
<td>643</td>
<td>1285</td>
<td>4940</td>
<td>5512</td>
<td>112%</td>
<td>73%</td>
</tr>
<tr>
<td>2( \times )</td>
<td>749</td>
<td>1285</td>
<td>2569</td>
<td>5450</td>
<td>10%</td>
<td>6251</td>
<td>114%</td>
</tr>
</tbody>
</table>

Table 4.3

A table compiling many quantitative details about all the magnetic fields and their magnetic skeleton analysed here for different resolutions and number of harmonics \( L \). The number (no.) of nulls, the number of separators together with file sizes, calculation times and the following three percentages: (1): The percentage increase in the nulls compared to the default resolution, (2): The percentage of separators compared to the number of nulls and (3): The percentage of nulls connected by separators. The timings given are the “typical” times to calculate the magnetic fields themselves after all harmonics have been calculated. *These are the (approximately equal) grid resolutions which are compared in section 4.3. **This time is estimated as it was run on a different computer to the others.
null points found at the default resolution while there is almost a 150% increase in the number of null points found when the grid resolution is increased by a factor of 16. The 92 null points at this higher resolution appear to be close to the limit on the number of nulls that exist for $L = 81$ and suggests that the field must finally be close to being truly trilinear within each grid cell. This same pattern can be seen for $L = 161$. However, rather than the 150% increase, there is only an 130% increase.

Due to computational resources, the $16 \times (n_r, n_\theta, n_\phi)$ grid cannot be achieved for $L = 161$ but in this case the $8 \times (n_r, n_\theta, n_\phi)$ grid seems to have reached convergence in the number of nulls and being truly trilinear. Higher than $4 \times (n_r, n_\theta, n_\phi)$ cannot be achieved for any of the higher resolutions either, due to computational resources, so a convergence in the number of nulls cannot be established. Nonetheless, from the results of the high $L$ fields, it would seem that again the increase in the number of nulls becomes smaller as the grid is increased from the default resolution for larger $L$, as can be seen in the column labelled (1) in table 4.3.

The number of separators found also appears to be related to the maximum harmonic number $L$. However the increase in the number of separators is much quicker than the increase in the number of null points found (see column labelled (2) in table 4.3). Even though the number of separators is much lower than the number of null points for low $L$, their number increases very quickly and overtakes the number of null points when $L = 641$. For this number of harmonics, there are 12% more separators than null points while for $L = 81$, there are 68% fewer separators than there are null points. For low numbers of harmonics, the ratio of the number of null points to the number of separators stays relatively constant as the grid resolution is increased. The high $L$ models create fields with considerably greater complexity giving rise to more and more separators.

We can also determine the percentage of the null points actually connected by separators and this increases with $L$ also. Less than 50% of null points are connected for the low resolution $L = 81$ field while the number connected starts to approach 75% for $L = 641$. Combined with the ratio of the number of separators to the number of nulls, this implies many more multiply-connected nulls and more complex separator networks in the global magnetic field created from large numbers of harmonics.
4. Applicability of the PFSS Model

(a) $L = 161, 4 \times (n_r, n_\theta, n_\phi)$

(b) $L = 641, 1 \times (n_r, n_\theta, n_\phi)$

Figure 4.2
The global magnetic skeletons for two different $L$ with similar grid resolutions. The nulls (red and blue spheres), spines (red and blue lines), separators between null points (yellow lines), separators between a null point and the HCS base (orange lines) and the HCS curtains (green lines). The separatrix surfaces are not plotted.
4.3 Varying L With A Fixed Grid Resolution

Here we look at the effect of changing L. However, to ensure no grid resolution effects influence the results, the grid resolutions have been chosen to be approximately equal. The grid resolutions and different L which are compared here are those which have been indicated with an asterix in table 4.3. Two examples of the magnetic skeletons in 3D are shown in figure 4.2.

4.3.1 Radial Differences through the Magnetic Skeleton

In figures 4.3 to 4.7, cuts through the magnetic skeletons of these models are plotted showing the locations of the different topological magnetic structures. It is difficult to fully study and comprehend the magnetic skeleton using the 3D models in figure 4.2, especially for high L, so we use the cuts instead. These cuts can be thought of as where the features in the 3D models cross a sphere of constant radius. In each figure, there are four graphs, one for each model at a different maximum number of harmonics; \( L = 81 \), \( L = 161 \), \( L = 321 \) and \( L = 641 \). Each model has an approximate grid resolution of \( 375 \times 643 \times 1285 \), see table 4.3 for the exact grid resolutions. Figure 4.3 shows cuts at \( r = 2.495R_\odot \) (the top boundary at the source surface), figure 4.6 shows cuts at \( r = 1.1R_\odot \) and figure 4.7 shows cuts at \( r = 1.01R_\odot \) (the bottom boundary at the solar surface). In each of the cuts, the radial component of the magnetic field at the given height is plotted in the background in greyscale with white corresponding to positive radial field and black corresponding to negative radial field. On top of the radial component of the field, cuts through the magnetic skeleton features are plotted. The dots in tones of red and blue indicate where spine lines from positive and negative null points respectively cross the cut and the lines in tones of red and blue indicate the cut through separatrix surfaces from positive and negative null points respectively. The yellow and orange stars both indicate separators; yellow indicating a cut through separators linking two nulls points and orange linking a null point and the base of the heliospheric current sheet. Finally, the green dotted lines are where the heliospheric current sheet curtains cross the cut. The tones of red and blue are randomly distributed among all the null points in a single cut with the aim of making features associated with different null points that are located close by to each other different tones of the appropriate colour to help the visualisation of the cuts. The same tone of red or blue corresponds to features from the same null point.
4. Applicability of the PFSS Model

We start with the cuts at \( r = 2.495R_\odot \) corresponding to the top boundary of the model (source surface) given in figure 4.3. These plots show that the location of the heliospheric current sheet base is almost identical for all four different maximum numbers of harmonics suggesting that the maximum number of harmonics does not seem to have a significant effect on the location of the heliospheric current sheet base. However, the maximum number of harmonics seems to have an effect on the number of features of the magnetic skeleton on the top boundary.

Open, semi-closed and closed are the three possible types of separatrix surfaces on the top boundary and depend on their connectivity with the HCS in the cut. A separatrix surface with both ends bounded by the base of the HCS is closed and creates an open field region within the boundary of it and the heliospheric current sheet. A separatrix surface is then open if both ends are bounded by spine lines instead. Semi-open then refers to the combination of open and closed where one end of the separatrix surface is bounded by the HCS and the other is bounded by a spine line. The network of these separatrix surfaces and the heliospheric current sheet can create many open field regions. Table 4.4 lists the numbers of open, semi-closed and closed separatrix surfaces in the four cuts in figure 4.3. For \( L = 81 \), there is only one closed separatrix curtain and one semi-open separatrix curtain but no open separatrix curtains. Contrasting this with \( L = 161 \), there are now many separatrix curtains of all three types. The closed separatrix curtain found at \( L = 81 \) still exists at \( L = 161 \), however, the semi-open separatrix curtain has now become fully closed. As \( L \) increases further, we again see that the features from the previous lower \( L \) field still existing at higher \( L \), but sometimes their type changes. However what is more significant is the increasing number of separatrix curtains and the number of spines. This means that at the top boundary, a complex magnetic field structure begins to form for high values of \( L \). At \( L = 641 \), there are now a significant number, a total of 221, of separatrix curtains compared to just 2 at \( L = 81 \). There has been an increase from one closed separatrix surface at \( L = 81 \) to 28 closed separatrix surfaces at \( L = 641 \) and an even bigger increase in the number of open and semi-closed separatrix surfaces forming many different open field regions. These open field regions are conjectured to be important for the solar wind with closed separatrix curtains know to be associated with pseudo-streamers (Antiochos et al. 2011; Wang et al. 2012) which are believed to be sites associated with the fast solar wind.

Figure 4.4 shows the number of null points whose separatrix surfaces reach the top boundary at \( r = R_{\text{max}} \) with the heights of the null points above the solar surface. As
4.3. Varying L With A Fixed Grid Resolution

<table>
<thead>
<tr>
<th>$L$</th>
<th>Separatrix Curtains</th>
<th>Spines</th>
<th>Open Field Regions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Open</td>
<td>Semi-closed</td>
<td>Closed</td>
</tr>
<tr>
<td>81</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>161</td>
<td>10</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>321</td>
<td>41</td>
<td>25</td>
<td>18</td>
</tr>
<tr>
<td>641</td>
<td>130</td>
<td>63</td>
<td>28</td>
</tr>
</tbody>
</table>

Table 4.4

The number of separatrix curtains, spines and open field regions of each type on the top boundary $r = R_{\text{max}}$ in the four subfigures of figure 4.3

$L$ increases, the minimum height of the null points with separatrix surfaces which reach the top boundary gets smaller. This shows that null points closer to the solar surface may give rise to large scale separatrix surfaces and therefore become important to the large scale field modelling. There are also many more null points found lower down with separatrix surfaces which reach the top boundary.

Figure 4.5 shows the absolute flux through each radial surface in the PFSS fields given by

$$F(r) = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} |B_r(r, \theta, \phi)| r^2 \sin \theta d\theta d\phi.$$  \hspace{1cm} (4.1)

Although the fields for high $L$ have many more open field regions than those for low $L$, figure 4.5 shows that the absolute flux on the top boundary is the same. However the absolute flux through the solar surface is much larger for high $L$. The difference in the absolute flux at the solar surface between the $L = 641$ and $L = 81$ fields is $1.54 \times 10^{15}$ Wb whereas at the outer boundary, the difference is $1.46 \times 10^{11}$ Wb. The difference has decreased by 4 orders of magnitude. At the outer boundary, the values for the absolute flux starts to converge for all $L$. The larger absolute flux at the solar surface for higher $L$ is about 2.5 times larger due to much more small scale field being replicated on the solar surface for high $L$ which creates much more complexity in the magnetic field. The absolute flux of the original synoptic map (the maximum absolute flux available to be replicated) is $3.57 \times 10^{15}$ Wb. So $L = 641$ is missing 25% of the absolute flux compared with 70% when $L = 81$. The absolute flux for all four PFSS fields however becomes very similar after about 20 Mm.

Now we look at the cuts lower down closer to the solar surface at $r = 1.1 R_\odot$ in figure 4.6. Although the cut for $L = 81$ gives the impression that the field is still quite simple at this height, comparing this with the $L = 641$ field, we get a very different
4. Applicability of the PFSS Model

- Negative Spines
- Positive Spines
- Negative Separatrix Surfaces
- Positive Separatrix Surfaces
- HCS Curtains
- Null-Null Separators
- Null-HCS Separators

![Graph showing radial field strength and geodesics](image)

(a) $L = 81, 8 \times (n_r, n_\theta, n_\phi)$

(b) $L = 161, 4 \times (n_r, n_\theta, n_\phi)$
4.3. Varying $L$ With A Fixed Grid Resolution

Figure 4.3

Cuts at the outer boundary, $r = 2.495 R_\odot$, showing the magnetic skeleton features which cut this radial surface. The grid resolutions of the four different models which have different maximum number of harmonics are approximately equivalent. The green dotted line is the location of the heliospheric current sheet, the red and blue lines are cuts through the separatrix surfaces from the positive and negative nulls respectively, the red and blue dots are the locations of spines from positive and negative nulls respectively, yellow stars are cuts through separators between nulls and orange stars are through separators between a null and the base of the heliospheric current sheet. The tones of red and blue are unique for each null point with each null point given a random tone of the applicable colour.
4. Applicability of the PFSS Model

Figure 4.4
The number of null points at each radial height for each $L$ whose separatrix surfaces reach the top boundary. The plot has been truncated between 65 and 165 null points.

Figure 4.5
The absolute flux (equation 4.1) through each surface of constant $r$ for each PFSS field with different $L$ and constant grid resolutions. The $x$- and $y$-axis titles of the subplot are the same as the main plot.
4.3. Varying $L$ With A Fixed Grid Resolution

<table>
<thead>
<tr>
<th>$L$</th>
<th>Spines</th>
<th>Separators</th>
<th>Separatrix Surfaces</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Positive</td>
<td>Negative</td>
<td>Nulls</td>
</tr>
<tr>
<td>81</td>
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<td>9</td>
</tr>
<tr>
<td>161</td>
<td>175</td>
<td>160</td>
<td>65</td>
</tr>
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<td>321</td>
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<td>507</td>
<td>665</td>
</tr>
<tr>
<td>641</td>
<td>874</td>
<td>932</td>
<td>2572</td>
</tr>
</tbody>
</table>

(a) $r = 1.1R_{\odot}$

(b) $r = 1.01R_{\odot}$

Table 4.5

Number of magnetic skeleton features for each $L$ and radial heights in figures 4.6 and 4.7

picture. Although the original structures found at $L = 81$ are still evident at $L = 641$, there are now also a multitude of other structures that almost obscure these features.

Again like the top boundary cut, the location of the heliospheric current sheet curtains is almost exactly the same for all four maximum numbers of harmonics and the magnetic features seen at lower resolution continue to remain at higher maximum numbers of harmonics. What becomes evident, however, is that some separatrix surfaces are not bounded by spines (as they should be) so the field seems improper or the surfaces appear disconnected and incomplete for the lower $L$ fields. In figure 4.6a, three examples of this can be seen for the $L = 81$ field where there are separatrix surfaces (blue) from three different negative nulls around $(\theta, \phi) = (\pi/2, \pi/2)$ (labelled (A)) and $(\theta, \phi) = (\pi/2, 5\pi/4)$ (labelled (B)) with incomplete separatrix surfaces. However when $L = 161$ in figure 4.7b, these three improper separatrix surfaces are fixed: one does not exist any more and the other two have now closed over. Thus, there is clearly an issue with resolution for the $L = 81$ field. It is possible that the $L = 81$ field is not trilinear in some regions. So the null points which would provide the bounding spines are not found and in other cases, false null points are found creating erroneous separatrix surfaces.

Although, the field for $L = 161$ is better than that for $L = 81$, it is still not perfect.
4. Applicability of the PFSS Model

At around \((\theta, \phi) = (\pi/2, 7\pi/12)\) (labelled \((\mathbb{C})\)) and \((\theta, \phi) = (3\pi/4, \pi)\) (labelled \((\mathbb{D})\)), there are two more examples of incomplete separatrix surfaces – one from a negative null (blue) and one from a positive null (red). However, these problems are again fixed in the \(L = 321\) field; spines now bound the negative separatrix surface and the positive separatrix surface has closed over. This pattern continues as \(L\) increases.

Finally we look at cuts close to the solar surface at \(r = 1.01R_{\odot}\) in figure 4.7. At this height, most of the structure of the magnetic skeleton can be seen and as \(L\) increases the complexity of the magnetic skeleton appears to increase exponentially with \(L\) with the errors in the skeleton at lower resolution fixed by going to higher resolution. At this height, there is a small difference in the location of the heliospheric sheet curtains with two major differences noticeable in the graphs of figure 4.7. Both differences occur between the cuts at \(L = 81\) (figure 4.7a) and \(L = 161\) (figure 4.7b). First there is an island-like feature created in the \(L = 161\) field at the end of the channel at approximately \((\theta, \phi) = (3\pi/8, 3\pi/2)\) (labelled \((\mathbb{E})\)). The second difference is that the heliospheric current sheet curtains double back on themselves in the northern hemisphere at \(\phi = 3\pi/2\) in the \(L = 161\) field (labelled \((\mathbb{F})\)) but not in the \(L = 81\) field. This is certainly due to the additional variation in the magnetic field which is created from the extra number of harmonics. These features continue to persist for higher \(L\).

Table 4.5 presents the number of spines, separators and separatrix surfaces found in each of the cuts in figures 4.6 and 4.7. This quantitative data shows clearly what we see qualitatively in the corresponding figures. There is a large increase in the number of all of the magnetic skeleton features as we increase \(L\) for both \(r = 1.1R_{\odot}\) and \(r = 1.01R_{\odot}\) as expected. Although the dominant type of some features changes as \(L\) increases which again suggests that there is missing detail in low maximum harmonic number models. For example, at \(L = 81\), there are about 60\% more negative separatrix surfaces than positive separatrix surfaces which cut both radial surfaces. However, as \(L\) increases, these two values start to equalise and then the number of positive separatrix surfaces becomes greater. In fact, when \(L = 641\), there are more than 40\% more positive separatrix surfaces. This switch over in the type of dominant feature also occurs for the spine lines, although less significantly. This means that using models with low maximum harmonic numbers, it is possible to come to the wrong conclusions about the magnetic skeleton. It is also worth noting that, although as expected, there is a significant increase in the number of magnetic skeleton features close to the solar surface, there is also a significant increase in the feature numbers at \(r = 1.1R_{\odot}\). This shows that the higher harmonics not only affect the
4.3. Varying $L$ With A Fixed Grid Resolution

(a) $L = 81, 8 \times (n_r, n_\theta, n_\phi)$

(b) $L = 161, 4 \times (n_r, n_\theta, n_\phi)$
4. Applicability of the PFSS Model

Figure 4.6
Same as figure 4.3 but for $r = 1.1R_\odot$.

(c) $L = 321, 2 \times (n_r, n_\theta, n_\phi)$

(d) $L = 641, 1 \times (n_r, n_\theta, n_\phi)$
magnetic field close to the solar surface but also affects the whole global magnetic skeleton and thus the field far from the solar surface.

### 4.3.2 Null Points and Separators

Figure 4.8 shows two plots of the differences in the number and heights of the null points between models with different maximum harmonic numbers. Figure 4.8a shows the difference between $L = 81$ and $L = 161$ and figure 4.8b shows the locations of all the null points found in each of the PFSS fields for different $L$. The main thing that can be seen (although more difficult at higher $L$) in both figures is that the additional null points found in the models with high maximum number of harmonics lie close to the solar surface. This is expected, since this is where the complexity of the field is greatest and is also where we increase the grid resolution most. The majority of nulls high above the solar surface are found in all fields, although there are also some extra null points found high above the solar surface at $r = 1.1R_{\odot}$. However as we turn our attention to the locations of separators, we see a different picture.

Figure 4.9 shows the locations of all separators between null points coloured by their height above the solar surface for each $L$ at fixed resolution. Those separators connected to the heliospheric current sheet base are not plotted since these obviously reach the top boundary at $r = R_{\text{max}}$. As $L$ is increases, not only do the number of separators increase as shown earlier, but we find more separators extending to high heights as indicated by the yellow segments of the separator lines in figure 4.9. These separators are new and are connected to the additional low lying null points found in the high $L$ field. At least part of many separators at $L = 641$ (and also at $L = 321$) reach at least above $r = 1.4R_{\odot}$. It was thought that higher harmonic numbers would only increase the complexity of the field close to the solar surface and not affect the large scale (global) magnetic features. Clearly from figure 4.8, the increase in $L$ increases the number of null points close to the solar surface. Here we see that as $L$ increases, there is a large increase in the number of separators high above the solar surface.

Figure 4.10 is similar to figure 4.9 but now each separator is coloured by the type of the separator network it is part of. The different types of separator networks are discussed in section 1.7.1.2. For low numbers of harmonics, the individual separator networks are not very complex and each one only contains a few null points and separators. As $L$ is increased, the simple separator networks grow and become more...
4. Applicability of the PFSS Model

(a) $L = 81, 8 \times (n_r, n_\theta, n_\phi)$

(b) $L = 161, 4 \times (n_r, n_\theta, n_\phi)$
4.3. Varying $L$ With A Fixed Grid Resolution

(c) $L = 321, 2 \times (n_r, n_\theta, n_\phi)$

(d) $L = 641, 1 \times (n_r, n_\theta, n_\phi)$

Figure 4.7
Same as figure 4.3 but for $r = 1.01R_\odot$. 

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4. Applicability of the PFSS Model

Figure 4.8

The null points, ordered by their heights, in the PFSS magnetic fields for different $L$ and constant grid resolutions.

(a) The change in the heights of the null points in the $L = 81$ and $L = 161$ PFSS fields.

(b) The heights of the null points for each of the fields.
4.4. Varying the Grid Resolution For A Fixed $L$

complex. Over half of the networks found at $L = 81$ are now part of large, complex networks with additional complex networks formed at $L = 161$. There are also a couple of new simple networks at $L = 161$. As $L$ increases, this pattern of behaviour continues. What is most significant at high $L$ is the size of some of the separator networks. To show this even more clearly, the 10 biggest individual separator networks are plotted in the $L = 321$ and $L = 641$ models in figure 4.11. The separator networks for low $L$ are not re-plotted as each individual separator network in these fields is easily visible in figure 4.10. For both $L = 641$ and $L = 321$, the biggest separator network wraps around the entire Sun (covering 360° longitude). The fields for low $L$ are significantly different. For $L = 81$, the largest separator network covers only 30° longitude and for $L = 161$, the largest network covers less than 180° longitude. This is significant as a reconnection event on one separator in a network could cause a chain reaction effect throughout its whole separator network effects on the magnetic field on the other side of the Sun: such a chain reaction is now possible in the high $L$ magnetic fields.

Changing the maximum number of the harmonics used to model the fields but keeping the grid resolution approximately the same clearly has a major effect on the magnetic structure at all radial heights. To find a more complete magnetic skeleton, we require the extra detail given by the high harmonics which is lost if only low values of $L$ are used. Magnetic fields formed using only the low harmonics do not reveal the true or full magnetic skeleton. The high harmonics are also required to fix the improper or incomplete magnetic skeleton features found with low $L$ magnetic fields.

4.4 Varying the Grid Resolution For A Fixed $L$

Here we now compare models with different grid resolutions but the same maximum number of harmonics $L$. Rather than seeing a significant increase in the complexity of the magnetic skeleton, changes in the grid resolution reveals what is missed at low grid resolutions. Figures 4.12 and 4.13 show cuts at $r = 1.1R_\odot$ for both $L = 81$ and $L = 161$, respectively, for different grid resolutions. This radial height is chosen since it gives a good balance between the extremely dense complexity of the cut through the magnetic skeleton at the base boundary and the simplicity of the magnetic skeleton at the top boundary.

For the $L = 81$ fields, the cuts through the magnetic skeleton are similar in all four
4. Applicability of the PFSS Model

(a) $L = 81, 8 \times (n_r, n_\theta, n_\phi)$

(b) $L = 161, 4 \times (n_r, n_\theta, n_\phi)$
4.4. Varying the Grid Resolution For A Fixed $L$

Figure 4.9

The locations of all separators except those connected to the heliospheric current sheet base each coloured showing the height at each point for each of the PFSS fields for different $L$ and similar grid resolutions. The red and blue dots are the $(\phi, \theta)$ locations of positive and negative nulls respectively ignoring their heights above the solar surface. The contours in the background are the radial magnetic field strength on the solar surface at $r = R_\odot$ similar to figures 4.3 to 4.7 with the minimum and maximum values set at $\pm 30 \text{G}$. These contours are the same as in figure 4.10.
4. Applicability of the PFSS Model

- Positive Null Point
- Negative Null Point
- Cave
- Positive Dome/Tunnel
- Negative Dome/Tunnel
- Ring
- Chain Containing a Ring
- Three Chains Connected (Tri-Junction)
- Connection to HCS
- More complex

(a) $L = 81$, $8 \times (n_r, n_\theta, n_\phi)$

(b) $L = 161$, $4 \times (n_r, n_\theta, n_\phi)$
4.4. Varying the Grid Resolution For A Fixed $L$

The locations of all separators each coloured by the type of the separator network they belong to for each of the PFSS fields for different $L$ with similar grid resolutions. The colours corresponding to the type of each of the separator network is given in the key at the start. The dashed dark green lines (in the same colour as the key) still denote a separator connecting to the HCS but they are dashed so any separators underneath are still visible. The lighter green dotted line (similar to the cut figures) is the heliospheric current sheet base. The red and blue dots are the $(\phi, \theta)$ locations of positive and negative nulls respectively ignoring their heights above the solar surface. The greyscale contours represent the radial magnetic field strength on the solar surface.

Figure 4.10

(c) $L = 321, 2 \times (n_r, n_\theta, n_\phi)$

(d) $L = 641, 1 \times (n_r, n_\theta, n_\phi)$
4. Applicability of the PFSS Model

The ten biggest separator networks in the magnetic skeleton for the two highest values of the maximum numbers of harmonics each given a different colour with the radial magnetic field strength on the solar surface in the background.
4.4. Varying the Grid Resolution For A Fixed $L$

(a) $1 \times (n_r, n_\theta, n_\phi)$

(b) $2 \times (n_r, n_\theta, n_\phi)$
4. Applicability of the PFSS Model

Figure 4.12
Same as figure 4.3 but for the varying grid resolutions of the $L = 81$ fields at $r = 1.1R_{\odot}$. 
4.4. Varying the Grid Resolution For A Fixed $L$

cases (figure 4.12). The only major differences are new magnetic skeleton features
associated with the extra nulls found when the grid resolution is higher. There are
two noticeable magnetic skeleton changes as the grid resolution increases. The first is
around $(\theta, \phi) = (\pi/2, \pi/2)$ (labelled $\mathcal{G}$). At the default resolution (figure 4.12a), no
negative separatrix surfaces exist here. Two small separatrix surface sections appear
at $2 \times (n_r, n_\theta, n_\phi)$ resolution in this location and grow in size as the resolution
increases, but always remain incomplete. We know this separatrix surface exists and
what it should look like from the $L = 161$ cuts where it forms a continuous line in the
cut (figure 4.13a). The continual increase in the grid resolution at $L = 81$ reveals the
separatrix surface but cannot complete it. The second noticeable change in the
magnetic skeleton is in the southern hemisphere at around $\phi = 5\pi/4$ (labelled $\mathcal{H}$).
For this change in figure 4.12, there is a similar situation where a separatrix surface
exists at $2 \times (n_r, n_\theta, n_\phi)$ resolution but not at the default resolution. Here though,
the cut through the separatrix surface which appears does not change after
$2 \times (n_r, n_\theta, n_\phi)$ resolution. Again it is fully formed in the $L = 161$ fields.

As the resolution of the grid is increased, the field becomes more resolved leading not
only to additional null points being found, but also to some null points disappearing!
This implies that the low grid resolution models were not approximately trilinear
leading to spurious null points being created. In figure 4.13, there are several
separatrix surfaces which disappear going from the default resolution (figure 4.13a) to
$2 \times (n_r, n_\theta, n_\phi)$ resolution (figure 4.13b, labelled $\mathcal{I}$).

The high resolution fields are clearly important for finding the full magnetic skeleton.
But they come at a major cost – both computational time and file storage size are
excessive (see the final two columns of table 4.3). The computational time is the time
to calculate the final magnetic field grid only (i.e. the summation part of the
equations) and not the individual terms and harmonics such as the $B_l^m(R_\odot)$. Also the
times are approximate since they were done on a computer which is shared so other
jobs may have had a small effect on the times. The time for the $L = 641$,
$2 \times (n_r, n_\theta, n_\phi)$ grid to run on the same machine as the others is estimated since it
had to be run on a HPC multi-node machine. The file sizes are for the final file
created by BMW2016 containing the 64-bit magnetic field with the grid coordinates
and grid sizes.

The FFT method used to calculate the default resolution ($1 \times (n_r, n_\theta, n_\phi)$) magnetic
field grids cannot be used on the other non-default ($k \times (n_r, n_\theta, n_\phi)$ with $k > 1$)
4. Applicability of the PFSS Model

Figure 4.13
Same as figure 4.3 but for the varying grid resolutions of the $L = 161$ fields at $r = 1.1R_{\odot}$. 

(a) $1 \times (n_r, n_{\theta}, n_{\phi})$

(b) $2 \times (n_r, n_{\theta}, n_{\phi})$

(c) $4 \times (n_r, n_{\theta}, n_{\phi})$
4.5. Attempting To Improve The Model Keeping The FFT

resolution magnetic field grids. The first thing to notice (in table 4.3) is the significant increase in the computational time between the $1 \times (n_r, n_\theta, n_\phi)$ and $2 \times (n_r, n_\theta, n_\phi)$ resolution grids. The time to calculate the $2 \times (n_r, n_\theta, n_\phi)$ resolution grid is around a factor of 1000 bigger than the $1 \times (n_r, n_\theta, n_\phi)$ resolution grid. This huge increase in computational time is due to the lack of FFT for higher resolutions. There may be some optimisations in the code available for the non-FFT-able grids, but the extra efficiency achievable will not produce anywhere near enough of a time saving to level with the FFT timings. After this there is a simple octupling in time when the grid is doubled which is obvious from the grid sizings. This means that running the PFSS model without the FFT is prohibitively long, especially for cases with high numbers of harmonics. The computation time for the $L = 641, 2 \times (n_r, n_\theta, n_\phi)$ magnetic field is over 3 weeks compared with just half an hour for the default resolution case using the FFT method. These computational times must then be combined with the analysis time of the MSAT routines which also increase in computation time with an increasing of both grid size and $L$. This is because of the grid size itself and the number of null points found. Thus these high resolution grids are not really feasible.

The other issue is the amount of memory required for the calculations and the file size itself afterwards. The file size for the magnetic field grids at the highest resolution used here is around 60 GB and more memory than that is required during the calculation of the magnetic field. This means high performance machines are required which can again be quite prohibitive.

Given that ideally we need to calculate the PFSS model for large numbers of harmonics, we really need to use the FFT approach to make any progress quickly. So in the next section we discuss what can be done to aid the situation.

4.5 Attempting To Improve The Model Keeping The FFT

Back in chapter 3, during the discussion of the PFSS model, we consider the possibilities of the grid resolutions when using an FFT to calculate the PFSS magnetic field. Although the grid points in the $\phi$ direction are fixed by the FFT, the grid resolutions in both the $r$ and $\theta$ directions are still free to be picked. So to mitigate the issue of long computational time, we can still increase the grid resolutions in $r$ and $\theta$ without adjusting the $\phi$ grid. Although this will cause the grid
4. Applicability of the PFSS Model

spacing to become highly irregular (certainly in the angular directions, where originally they are approximately uniform), this will allow us to try and increase the grid resolution while keeping computational time to a minimum.

To investigate the effect of this change in the grid resolution on the magnetic skeleton, we have used several combinations of quadrupling the grid resolution in the $r$ and $\theta$ direction for $L = 81$ and $L = 161$ models. These are listed in table 4.6 which provides the same data as in table 4.3 for each of these new different magnetic fields.

Firstly, looking at the computational time for each of the new magnetic fields, the only effect the quadrupling of the grid resolution in the $r$ and/or $\theta$ direction is a quadrupling of the computational time. This means there are no significant computational penalties with each magnetic field being computed quickly even at high resolutions.

Now we turn our attention to the quantitative magnetic skeleton data in table 4.6. The same analysis has been done on each of the fields as those previously. We see that for both values of $L$ the number of null points found in the fields, in general, increases with increasing grid resolution. However, an increase in $n_r$ has a much smaller effect than increasing $n_\theta$. In fact in the $4 \times n_r$ case for $L = 161$, the number of null points has decreased: we suspect that this is due to the removal of false-positives when the higher radial grid resolution is used. It was thought that much of the complexity is at low $r$ and that increasing the grid resolution in the $r$ direction close to the solar surface would increase the number of null points. However this seems to have had the opposite effect. It has instead refined the field close to the base removing some spurious null points. This would suggest that we actually require both a higher radial and higher angular resolution than the default in the PFSS model.

Changing the $\theta$ grid has the largest effect and increases the magnetic skeleton in the same qualitative way as increasing all three dimensions. For $L = 81$, the number of null points increases by 25% when only $n_\theta$ is increased and 20% when both $n_r$ and $n_\theta$ are increased (table 4.6). This is about half the increase found when all three dimensions were increased equally. This halving in the increase is also seen for the case when $L = 161$.

Increasing only $n_r$ increases the number by 5% for $L = 81$ and decreases the number of nulls by 3% for $L = 161$. These changes are much smaller than the increases due to $n_\theta$. So comparing with the $L = 81$ models given in table 4.3, there is a 50% total increase in the number of null points possible and half of these can be found by
4.6. The Magnetic Skeletons for $L = 41$

increasing $n_\theta$ and the remaining other half found by increasing $n_\phi$, with the increase in the radial resolution removing some of the original false-positives. This shows this method is very good for finding the full magnetic skeleton. The only extra increase available is something that cannot be changed.

Figures 4.14 and 4.15 show the cuts at $r = 1.1 R_\odot$ in the new $4 \times (n_r, n_\theta)$ fields compared with the original resolutions and the $4 \times (n_r, n_\theta, n_\phi)$ resolutions. For $L = 81$, we compare the three subfigures 4.14a to 4.14c, we see when using the $4 \times (n_r, n_\theta)$ grid that we found several separatrix surfaces, and therefore null points, which appear in the $4 \times (n_r, n_\theta, n_\phi)$ resolution field that are missing in the default resolution field (e.g. label J). Although we have also lost some of the magnetic skeleton; the most notable being the horseshoe shaped separatrix surface at around $\phi = 5\pi/4$ (labelled K) in the default resolution grid which becomes closed over at $4 \times (n_r, n_\theta, n_\phi)$ resolution. This null point and separatrix surface persist at higher resolutions suggesting that this is possibly an issue of the irregular grid resolution. However the separatrix surface which is found just below it is not complete at $4 \times (n_r, n_\theta, n_\phi)$ resolution but is fully closed over at the $4 \times (n_r, n_\theta)$ resolution and not found at the default resolution. So we clearly have trade offs here using this irregular resolution. We gain much of the magnetic skeleton that would exist in higher resolution (and possibly in better resolution/more properly formed/complete) but there is also the possibility of losing some detail in the magnetic skeleton. It is worth noting that the horseshoe-like structure disappears again at the irregular resolution for $L = 161$ in figure 4.15c. This suggests that around the horseshoe-like structure, there is an issue with the resolution of the field when changes in only $r$ and $\theta$ are implemented.

### 4.6 The Magnetic Skeletons for $L = 41$

Finally we turn our attention to the very low harmonic case of $L = 41$. This is around the maximum number of harmonics that is used in much research when using the PFSS model. However, we have separated it from the rest of the results because of the significant differences found in the behaviour of the numbers of magnetic skeleton features as grid resolution is increased.

Table 4.7 lists the numerical results of the magnetic skeleton analysis for different grid resolutions for $L = 41$ (similar to tables 4.3 and 4.6). For all 6 different grid
### 4. Applicability of the PFSS Model

#### Table 4.6

<table>
<thead>
<tr>
<th>Resolution</th>
<th>Null Points</th>
<th>Separators</th>
<th>Time</th>
<th>File Size</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$n_r$</td>
<td>$n_\theta$</td>
<td>$n_\phi$</td>
<td>No.</td>
</tr>
<tr>
<td>$L = 81$</td>
<td></td>
<td></td>
<td></td>
<td>No.</td>
</tr>
<tr>
<td>Default</td>
<td>48</td>
<td>83</td>
<td>165</td>
<td>63</td>
</tr>
<tr>
<td>$4 \times n_r$</td>
<td>189</td>
<td>83</td>
<td>165</td>
<td>66</td>
</tr>
<tr>
<td>$4 \times n_\theta$</td>
<td>48</td>
<td>329</td>
<td>165</td>
<td>79</td>
</tr>
<tr>
<td>$4 \times (n_r, n_\theta)$</td>
<td>189</td>
<td>329</td>
<td>165</td>
<td>76</td>
</tr>
</tbody>
</table>

#### Table 4.7

The number of null points found at the new different resolutions with file sizes and calculation times. ①: The percentage increase in the nulls compared to the default resolution. ②: The percentage of separators compared to the number of nulls. ③: The percentage of nulls connected by separators.

<table>
<thead>
<tr>
<th>Resolution</th>
<th>Null Points</th>
<th>Separators</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L = 41$</td>
<td>$n_r$</td>
<td>$n_\theta$</td>
</tr>
<tr>
<td>Default: $1 \times (n_r, n_\theta, n_\phi)$</td>
<td>25</td>
<td>43</td>
</tr>
<tr>
<td>$2 \times (n_r, n_\theta, n_\phi)$</td>
<td>49</td>
<td>85</td>
</tr>
<tr>
<td>$4 \times (n_r, n_\theta, n_\phi)$</td>
<td>97</td>
<td>169</td>
</tr>
<tr>
<td>$8 \times (n_r, n_\theta, n_\phi)$</td>
<td>193</td>
<td>337</td>
</tr>
<tr>
<td>$16 \times (n_r, n_\theta, n_\phi)$</td>
<td>385</td>
<td>673</td>
</tr>
<tr>
<td>$32 \times (n_r, n_\theta, n_\phi)$</td>
<td>769</td>
<td>1345</td>
</tr>
</tbody>
</table>

The number of nulls and separators found at different resolutions for $L = 41$.
4.6. *The Magnetic Skeletons for \( L = 41 \)*

Figure 4.14

Same as figure 4.3 but for \( r = 1.1R_\odot, L = 81 \) and a difference variance in the grid resolutions.
4. Applicability of the PFSS Model

Figure 4.15
Same as figure 4.3 but for $r = 1.1R_\odot$, $L = 161$ and a difference variance in the grid resolutions.
4.6. The Magnetic Skeletons for $L = 41$

![Figure 4.16](image)

The HMI radial synoptic map and the radial component of the calculated PFSS field for $L = 41$ on the solar surface.
resolution magnetic fields, we find the exact same number of nulls and separators. There is no difference in the magnetic skeleton as the grid resolution is increased.

The maximum number of harmonics used to create the PFSS global fields determines how much small-scale field is replicated on the model solar surface from the observed synoptic map as seen in figure 3.7. The more harmonics there are, the greater the number and the smaller the scale of magnetic field structures reproduced on the model solar surface of the numerical PFSS models. Clearly, \( L = 81 \) is such a low number of harmonics that it fails to produce the small-scale features and only creates a large scale field which has little to no changes in sign over significant regions of the solar surface. An increase in grid resolution has no effect on the magnetic skeleton and it appears to be fully resolved for \( L = 41 \). It is expected this would be the same for fields at even lower numbers of harmonics.

Figure 4.17 shows three radial cuts for the \( L = 41 \) at default resolution. We see only large-scale features and no significant detail even at the base with such a low maximum number of harmonics. Unsurprisingly, there also appears to be issues with the incomplete separatrix surfaces at the solar surface indicating that the skeleton is incomplete.

### 4.7 Conclusions

From this work, we see that the magnetic fields extrapolated using the PFSS model with a low maximum number of harmonics at almost any grid resolution give unsatisfactory results revealing incomplete separatrix surfaces, missing structures and false-positives. It’s only for models with a high maximum number of harmonics that we start to create magnetic fields which reveal the true nature of the magnetic skeleton. Additionally it seems that the models using default resolutions and high maximum numbers of harmonics to create magnetic fields are better resolved than those for low maximum numbers of harmonics. For high \( L \), we see much smaller increases in the number the magnetic skeleton features found as the grid resolution is increased, compared with low \( L \). So ideally we need to use a high \( L \) to create models since these fields provide a much more complete and reliable magnetic skeleton, computed quickly using the FFT method.

The magnetic skeletons for magnetic fields created using low \( L \) can lead to wrong conclusions. These models erroneously suggest that there are no large complex
4.7. Conclusions

Figure 4.17
Cuts for the $L = 41$ fields at the three radial heights as before.
4. Applicability of the PFSS Model

separator networks, very few magnetic skeleton features high above the solar surface and the wrong ratio of positive and negative magnetic features. All of these findings are corrected by using higher $L$ models. Only a tiny fraction of the true skeleton is available at very low maximum harmonic numbers, especially in fields with $L = 41$, because there is simply no small scale field replicated on the model solar surface hence very few null points are found.

There are also issues with the default grid resolution used for the low $L$ models. To try to mitigate the resolution issues at low $L$ without sacrificing time, the grid resolutions in both the $r$ and $\theta$ directions can be increased. This can give a substantial increase in the number of magnetic skeleton features found while keeping the computational time of the PFSS magnetic fields to a minimum. Although not all available new magnetic skeleton features can be found using this method, it certainly is a step in the right direction.
Chapter 5
Separator Networks at Solar Minimum and Maximum

The abundance and location of null points in the solar corona have been studied using many different magnetograms taken by different instruments (e.g. Longcope and Parnell 2009; Edwards and Parnell 2015; Cook, Mackay and Nandy 2009) with several studies comparing the results at solar minimum and solar maximum (e.g. Cook, Mackay and Nandy 2009; Platten et al. 2014). However the magnetic networks created by null points and connected by separators have not really been explored. These networks were first explored by Close, Parnell and Priest (2004). A comparison of null points and global features of solar minimum and solar maximum have also been done through the magnetic skeleton analysis of global magnetic fields produced using PFSS models with a maximum number of harmonics of $L = 81$ by Platten et al. (2014) and $L = 351$ by Edwards and Parnell (2015) who only considered null points. The low maximum numbers of harmonics used in Platten et al. (2014) is likely to produce differences in the magnetic skeletons to those with high maximum numbers of harmonics as shown in chapter 4. So, combined with the opportunity here to study the skeletons of global coronal magnetic fields created using PFSS models with high numbers of harmonics, we also look in detail at the networks of null points connected by separators (which we call separator networks) in these fields with a particular focus on what differences there may be between solar minimum and solar maximum.

5.1 Set-Up and the HMI Data Used

In order to create potential global magnetic fields with as complete separator networks as possible HMI data must be used. Since only HMI data has enough resolution to be able to produce PFSS model at the high maximum number of
harmonics required for the work in this chapter. Rather than use HMI daily synoptic maps, like those used in chapter 4, here we use Carrington rotation synoptic maps to more effectively study longer periods of time at solar minimum and solar maximum. We have chosen five synoptic maps from solar minimum and five from solar maximum. However, HMI did not start operation until 2010 with the first Carrington rotation synoptic map covering Carrington rotation number 2097. Carrington rotation 2097 started in May 2010 while solar cycle 24 started in December 2008, one and a half years earlier, so the data used here is as close to solar minimum as possible. Carrington rotation numbers 2097 to 2101 and 2148 to 2152 are used to compare the separator networks at solar minimum and solar maximum. Figure 5.1 shows the time periods for each five Carrington rotations relative to the sunspot number and the operational period of SDO.

In this chapter, we use $L = 641$ for all the PFSS field extrapolations at its default resolution and all other default options for $L = 641$ given in chapter 3 including $R_{\text{max}} = 2.5R_\odot$. $L = 641$ is picked since its default resolution gives as close to the full available synoptic map resolution as possible without creating too large a file size. Additionally the magnetic skeleton of the default FFT field at $L = 641$ appears to be close to the full magnetic skeleton available for $L = 641$ as shown chapter in 4. The PFSS fields at $L = 641$ all have a resolution of $375 \times 643 \times 1285$ ($n_r \times n_\theta \times n_\phi$).
5.2. Results

<table>
<thead>
<tr>
<th>Carrington Rotation</th>
<th>Start Date</th>
<th>Null Points</th>
<th>Null-Null Separators Number</th>
<th>1</th>
<th>2</th>
<th>Null-HCS Separators</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Close to) Solar Minimum</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2097 20/5/10</td>
<td>4700</td>
<td>5261</td>
<td>112 %</td>
<td>72 %</td>
<td>113</td>
<td></td>
</tr>
<tr>
<td>2098 16/6/10</td>
<td>4807</td>
<td>4898</td>
<td>102 %</td>
<td>72 %</td>
<td>87</td>
<td></td>
</tr>
<tr>
<td>2099 13/7/10</td>
<td>4533</td>
<td>4718</td>
<td>104 %</td>
<td>74 %</td>
<td>112</td>
<td></td>
</tr>
<tr>
<td>2100 9/8/10</td>
<td>4457</td>
<td>4700</td>
<td>105 %</td>
<td>73 %</td>
<td>99</td>
<td></td>
</tr>
<tr>
<td>2101 5/9/10</td>
<td>4167</td>
<td>4140</td>
<td>99 %</td>
<td>72 %</td>
<td>141</td>
<td></td>
</tr>
<tr>
<td>Solar Maximum</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2148 11/3/14</td>
<td>3118</td>
<td>2660</td>
<td>85 %</td>
<td>69 %</td>
<td>108</td>
<td></td>
</tr>
<tr>
<td>2149 7/4/14</td>
<td>3257</td>
<td>2876</td>
<td>88 %</td>
<td>69 %</td>
<td>108</td>
<td></td>
</tr>
<tr>
<td>2150 4/5/14</td>
<td>3334</td>
<td>3232</td>
<td>97 %</td>
<td>71 %</td>
<td>88</td>
<td></td>
</tr>
<tr>
<td>2151 31/5/14</td>
<td>3732</td>
<td>3810</td>
<td>102 %</td>
<td>72 %</td>
<td>127</td>
<td></td>
</tr>
<tr>
<td>2152 28/6/14</td>
<td>3742</td>
<td>3790</td>
<td>101 %</td>
<td>73 %</td>
<td>103</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.1

Table compiling many quantitative details about the magnetic fields and their magnetic skeleton for each Carrington rotation. The number of null points, the number of null-null separators together with the following two percentages: ①: The percentage of null-null separators compared to the number of null points and ②: The percentage of nulls connected by separators. The number of HCS-null separators are also given.

5.2 Results

The PFSS \((L = 641)\) magnetic fields of the HMI Carrington rotation synoptic maps are calculated and then their magnetic skeletons are determined using MSAT. The analyses of these skeletons are presented in this section.

5.2.1 Abundance of Nulls and Separators

First we look at the total numbers of null points and null-null separators found in the derived PFSS fields. Table 5.1 compiles the numbers of null points and separators for each of the Carrington rotations. We see almost 50% more null points are found at solar minimum compared with solar maximum. This is in agreement with what was seen by Edwards and Parnell (2015) where there are significantly more null points (at least 50%) at solar minimum than at solar maximum. This is due to much more small-scale quiet Sun magnetic field found at solar minimum than at solar maximum which causes many more changes in sign in the photospheric magnetic field. It can be seen how the number of null points starts to significantly fall off in Carrington rotations 2099 to 2101 as the solar cycle gets closer to solar maximum.
5. Separator Networks at Solar Minimum and Maximum

The number of null-null separators then follows a similar trend. There are more of these separators at solar minimum than at solar maximum, with almost a doubling in the number at Carrington rotation 2148 compared to 2097. However the percentage of null points connected to a separator does not change significantly while the ratio of the number of null-null separators to the number of null points is slightly higher on average at solar minimum than at solar maximum. Although the numbers of null-null separators decrease in line with the decrease in the number of null points, there are about the same number of null-HCS separators at both solar minimum and solar maximum.

Figure 5.2 shows the distribution of null points density versus colatitude. Null points are much more uniformly distributed at solar minimum compared with solar maximum (figure 5.2). There is a clear bimodal distribution at solar maximum. In fact, it would seem that null points are about equally densely populated in the higher latitudes for both solar minimum and solar maximum. The null points which are lost at solar maximum are only those in the equatorial regions where there are strong active regions and the field is less mixed in comparison to the quiet Sun at solar minimum. The angular locations of the null points can be seen later in figure 5.13.
5.2. Results

5.2.2 3D Magnetic Skeleton Plots

Examples plots of the 3D magnetic skeletons are shown in figures 5.3 and 5.4 for Carrington rotations 2097 (solar minimum) and 2148 (solar maximum). The Sun’s rotation axis (black line), null points (red and blue spheres), spine lines (red and blue lines), null-null separators (yellow lines), null-HCS separators (orange lines) and the heliospheric current sheet base and its curtains (thick and thin green lines respectively) are plotted. Through these 3D plots, significant differences can be seen in the magnetic skeletons of the two Carrington rotations. At solar minimum (figure 5.3), the heliospheric current sheet base roughly circles above the equator. However at solar maximum (figure 5.4), the HCS base is much more complex: it has split into two different components and has a large deviation from the equator. From the open
Figure 5.4

The global magnetic skeleton for Carrington rotation 2148 (solar maximum). The base of the heliospheric current sheet base (thick green lines) has split into two components. See figure 5.3 for the key to this plot.
5.2. Results

Figure 5.5
The surface radial magnetic field at the northern pole of Carrington Rotation 2097.

field regions, there appears to be significantly fewer open spine lines at solar maximum than at solar minimum and the open field regions, at solar maximum, have warped significantly so they are no longer quite at the poles and stretch down towards the equator in narrow strips. Additionally, the reversal in the sign of the field at the poles from solar minimum to solar maximum is clear from the colour of the open spines. At solar minimum, all the open spines in the northern hemisphere (red) come from positive null points (indicating that the field below is negative) whereas at solar maximum they are blue in this hemisphere so come from negative null points (indicating that the field below is positive).

At solar maximum, there are many separators which lie close to the poles of the Sun (seen more clearly in figures 5.13 and 5.16 later). Due to the obvious observational constraints, the magnetic field close to the solar surface at the poles has been filled in during preprocessing of the synoptic maps (see section 3.3.1) and so the magnetic field in the polar regions may be considered less “believable”. This creates a region that lacks any spatial detail, as shown by the zoom in on a 3D plot of the surface magnetic field of the north pole from Carrington rotation 2097 in figure 5.5. The lack of spatial detail means we are probably missing null points and maybe separators that could occur near the poles, especially at solar maximum when the polarity of the poles reverses. Although the signs of the null points that are formed within this region will be correct, they quantity cannot be relied upon to be accurate.

Although these 3D plots are great for a general view of the magnetic skeleton, it is difficult to see any of the detail in the skeleton because there are so many magnetic skeleton features, especially at solar minimum. So, like chapter 4, cuts through the
### 5. Separator Networks at Solar Minimum and Maximum

#### 5.2.3 Radial Changes through the Magnetic Skeleton

Figures 5.6, 5.8 and 5.9 each show cuts through the 3D magnetic skeletons for two example Carrington rotations from solar minimum and two from solar maximum at three different radial heights. Tables 5.2, 5.3 and 5.4 summarise the numbers of magnetic features in each of these plots respectively.

#### 5.2.3.1 Cuts at $r = 2.495R_\odot$

Figure 5.6 shows the cuts through the magnetic skeletons at the top boundary with table 5.2 quantifying the features observed. The base of the heliospheric current sheet (HCS) is generally more warped at solar maximum than at solar minimum (as seen in figures 5.3 and 5.4) as expected (J. Todd Hoeksema, Wilcox and Scherrer 1983).

Hence, the deviation of the HCS base from the equator (known as the tilt) is generally greater at solar maximum than at solar minimum (table 5.2). At solar maximum the HCS tilt reaches $62^\circ$ in Carrington rotation 2148 compared with just $44^\circ$ at solar minimum.
5.2. Results

- Negative Spines
- Positive Spines
- Negative Separatrix Surfaces
- Positive Separatrix Surfaces
- HCS Curtains
- Null-Null Separators
- Null-HCS Separators

(a) Carrington Rotation 2097

(b) Carrington Rotation 2098
5. Separator Networks at Solar Minimum and Maximum

Figure 5.6
The cuts through the magnetic skeleton at $r = 2.495R_\odot$ (almost the top boundary). The colourings match those for the radial cuts in chapter 4. The green dotted line is the location of the heliospheric current sheet, the red and blue lines are cuts through the separatrix surfaces from the positive and negative nulls respectively, the red and blue dots are the locations of spines from positive and negative nulls respectively, yellow stars are cuts through separators between nulls (although none are visible here on the top boundary) and orange stars are through separators between a null and the base of the heliospheric current sheet. The tones of red and blue are different for each null point with each null point given a random tone of the applicable colour. The legend at the top of the figure summarises these. The radial component of the magnetic field at $r = 2.495R_\odot$ is also plotted in the background.
5.2. Results

The HCS tilts calculated by WSO using their standard PFSS model (see source for more information) with the HCS tilts found here using HMI synoptic maps in PFSS models at \( L = 641 \) overplotted. Source: The Wilcox Solar Observatory (1976-2017)

At solar minimum, Platten et al. (2014) found the HCS tilts from PFSS models using SOLIS (on Kitt Peak) data and \( L = 81 \), for the same Carrington rotations studied here. From their plot (figure 16) of the HCS tilts which shows a 12 Carrington rotation running mean, the solar minimum tilts appear to be larger than those found here. Platten et al. (2014) compared their HCS tilts to the maximum HCS tilts found by Wilcox Solar Observatory (WSO). They found that their HCS tilts were generally higher than those by WSO. In figure 5.7, we compare our HCS tilts with the WSO data values derived using \( R_{\text{max}} = 2.5R_{\odot} \). At solar minimum, our values for the HCS tilts are very similar to those found by WSO. During the previous two solar minima, the tilt values found by WSO and Platten et al. (2014) (SOLIS) are around 10° to 25° respectively which are much smaller than the 40° to 50° tilts in Carrington rotations 2097 to 2101 found here by our PFSS models and WSO. Platten et al.
5. Separator Networks at Solar Minimum and Maximum

(2014) explained that the large tilts found at this minimum are a consequence of the weak unipolar fields at the poles during the minimum which mean that the usual dipolar field is not as dominant. At solar maximum, the maximum HCS tilts we find are much lower than those found by WSO. Whereas, Platten et al. (2014) found higher tilts than WSO at solar maximum. These differences in the tilt values are likely, in part, to be a consequence of how the missing polar data in the synoptic maps have been filled.

The separatrix surfaces reaching the top boundary (called separatrix curtains) are also pushed towards the poles at solar maximum. This is due to the strong active regions in the equatorial region of the solar surface that give rise, in several cases, to open field regions above whilst the polar regions, which are no longer unipolar, are now closed. Hence, the separatrix surfaces are pushed towards and over the poles.

The abundance of closed separatrix curtains at both solar minimum and solar maximum as a percentage of the total number of null points in the field are approximately the same (table 5.2). However, there are generally more semi-closed separatrix curtains and less open separatrix curtains at solar minimum than at solar maximum. There are generally more open field regions at solar maximum than at solar minimum (table 5.2). The solar maximum open field regions are created from the joining together of the extra semi-closed and open separatrix curtains to form networks that create a closed separatrix curtain (figure 5.6c and 5.6d). This does not happen anywhere near as much at solar minimum (figure 5.6a and 5.6b). At solar maximum, these open separatrix curtain networks even create extra open field regions towards the poles which are completely disconnected from the HCS (e.g. in the bottom right hand corner of figure 5.6c). Indeed, there are fewer open separatrix curtains at solar maximum that are not part of a closed network than at solar minimum.

5.2.3.2 Cuts at $r = 1.1R_\odot$ and $r = 1.01R_\odot$

Figures 5.8 and 5.9 show cuts through the magnetic skeleton at $r = 1.1R_\odot$ and $r = 1.01R_\odot$ respectively for the same Carrington rotations shown in figure 5.6.

Similar to the cuts at the top boundary, the biggest difference between solar minimum and solar maximum is the location of the features in the cuts. The strong active regions in the equatorial region at solar maximum mean that there are far fewer small-scale magnetic field features around the equator compared with solar
5.2. Results

(a) Carrington Rotation 2097

(b) Carrington Rotation 2098
Figure 5.8
Same as figure 5.6 but for $r = 1.1R_\odot$. 

(c) Carrington Rotation 2150

(d) Carrington Rotation 2151
5.2. Results

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Table 5.3
The number of features of each type which cross $r = 1.1R_\odot$ in each Carrington rotation PFSS field.

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<th>Carrington Rotation</th>
<th>Types of Magnetic Structure Crossing $r = 1.01R_\odot$</th>
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</thead>
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<td>2152</td>
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</tbody>
</table>

Table 5.4
The number of features of each type which cross $r = 1.01R_\odot$ in each Carrington rotation PFSS field.
5. Separator Networks at Solar Minimum and Maximum

(a) Carrington Rotation 2097

(b) Carrington Rotation 2098
5.2. Results

Figure 5.9
Same as figure 5.6 but for \( r = 1.01R_\odot \).
5. Separator Networks at Solar Minimum and Maximum

Figure 5.10
The distribution of separators through the cut of the magnetic skeleton at \( r = 1.01 \) for each band of colatitude at solar minimum and solar maximum.

minimum. This means there are significantly fewer null points around the equator in the PFSS fields at solar maximum (as seem in figure 5.2) and thus at this time, the majority of the magnetic skeleton features are away from the equator and instead lie closer to the two poles. At \( r = 1.1R_\odot \), the locations of the separators (yellow stars) clearly follows this pattern, but there are still many separatrix surfaces which lie close to or cross over the equatorial region (figure 5.8c and 5.8d). At \( r = 1.01R_\odot \) just above the solar surface, the distinction between solar maximum and solar minimum is also clear with far fewer features near the equator at solar maximum than at solar minimum (figure 5.9).

Figure 5.10 shows the number of separators in each colatitude band crossing \( r = 1.01R_\odot \) for Carrington rotations 2097 and 2148. The bimodal distribution of the separators either side of the equator close to the solar surface at solar maximum (Carrington rotation 2150) is clear compared with the unimodal distribution at solar minimum (Carrington rotation 2097) where the separators are more concentrated around the equator. This perhaps suggests that, close to the solar surface, the equatorial regions are more likely to be associated with small-scale magnetic reconnection activity in the quiet Sun at solar minimum whereas at solar maximum this type of behaviour is more likely to be found further from the equator on either
side. These distributions are very similar to those of the null points at solar minimum and solar maximum seen in figure 5.2.

At solar maximum (figures 5.8 and 5.9), the heliospheric current sheet curtains are also more complex at these lower heights since at solar maximum there are usually more open field regions (table 5.2). As the heliospheric current sheet curtains travel down to the solar surface they encounter null points which create separators between the base of the HCS and the null point. At solar maximum, there are generally just as many of these separators as there are at solar minimum even though the total number of null points has decreased (table 5.1).

Another consequence of the HCS curtains encountering null points is that the HCS curtains become bounded by the spine of the null points and do not reach the solar surface. In the cuts through the magnetic skeleton close to the base, there are splits in the line where the HCS curtains are. At solar maximum, these splits are more numerous and occur higher in the solar atmosphere causing a much more complex structure to the HCS curtains than at solar minimum. There are three splits that can be seen in 3D in figure 5.11 which are labelled $S$ where the HCS curtains are bounded by spines in Carrington rotation 2151. The cuts through the magnetic skeleton at $r = 1.1R_\odot$ (figure 5.8) show that generally the HCS curtains are quite continuous at this height at solar minimum but at solar maximum, there are several splits (although it is difficult to see).

Another common feature of the HCS curtains at solar maximum is that they cover the polar regions. In figure 5.11, there is an example of the heliospheric current sheet covering the polar region which is located at the base of the straight black line on the top left of the figure. This is difficult to recognise in the cuts through the magnetic skeleton, especially due to the number of splits in the line.

In tables 5.3 and 5.4, the number of each of the magnetic skeleton features which cross the radial cuts through the field at $r = 1.1R_\odot$ and $r = 1.01R_\odot$ (figures 5.8 and 5.9) are listed. For each Carrington rotation, the number of spine lines, null-null separators, HCS-null separators and separatrix surfaces which cross these two surfaces are given. Looking at the numbers of spines and separatrix surfaces of each sign, at solar minimum the majority of spines are from negative null points and the majority of separatrix surfaces are from positive null points, both of which involve field lines heading away from the null point. Whereas at solar maximum there appears to be a switch to a majority of positive spines and negative separatrix...
Figure 5.11
A 3D plot of the PFSS global field for Carrington rotation 2151 zoomed in on the north pole of the Sun. The heliospheric current sheet curtains (green) are going over the north pole located at the base of the straight black line. The null points are denoted by red and blue spheres, the spines by the red and blue lines, the separators between null points and the separators between the HCS and null points in yellow and orange respectively and the heliospheric sheet curtain rings by green lines. S denotes a spine bounding the edge of the curtains of the heliospheric current sheet.
5.2. Results

![Graph showing the variation of absolute flux with radial height above the solar surface for each Carrington rotation PFSS field.](image)

Figure 5.12

The absolute flux varying with radial height above the solar surface for each Carrington rotation PFSS field.

surfaces. The polar field has switched or is in the process of switching between solar minimum and solar maximum so it would seem there are more magnetic skeleton features in the southern hemisphere. This may also be a result of the tilt angle of the Sun’s axis during these Carrington rotations.

5.2.4 Absolute Magnetic Flux

Figure 5.12 shows the variation of absolute flux with radial height (calculated using equation 4.1) for each Carrington rotation. At solar minimum (dashed lines), the absolute flux on the solar surface is approximately $2.6 \times 10^{15}$ Wb ($2.6 \times 10^{23}$ Mx) whereas at solar maximum it is about $5.2 \times 10^{15}$ Wb ($5.2 \times 10^{23}$ Mx). So there is almost double the amount of absolute flux on the solar surface at solar maximum than solar minimum. This is due to the large-scale active regions that exist at solar maximum which have strong unipolar magnetic fields containing significant amounts of magnetic flux. The initial fall off in the absolute flux as the height above the solar surface is increased is more rapid at solar minimum than at solar maximum. However, at solar minimum, the fall off in absolute flux decreases such that from
around 600 Mm \((2R_\odot)\) above the surface to more than 1000 Mm \((2.5R_\odot)\) the flux remains approximately constant (figure 5.12). At solar maximum the absolute flux continues to decrease and these curves fall below the solar minimum curves. Indeed, there is generally more absolute flux at the top boundary \((2.5R_\odot)\) at solar minimum \((2.0 \times 10^{14} \text{ Wb})\) compared with solar maximum \((1.6 \times 10^{14} \text{ Wb})\). We note, however, that as already discussed, the solar minimum between cycles 23 and 24 is atypical and so it is not clear if the absolute flux at \(r = 2.5R_\odot\) is always greater at solar minimum than at solar maximum. Mackay, Priest and Lockwood (2002) showed this was the case but they used a maximum number of harmonics of 60 so they may have very little small scale flux. This conclusion may also be different using a non-potential model.

5.2.5 Heights of the Separator Networks

The plots in figure 5.13 show the spatial distribution of all the null-null separators together with the angular locations of the null points for the same four Carrington rotations shown in figures 5.6, 5.8 and 5.9. Those separators at solar maximum that cross the equator generally do so high in the solar corona but there are only a few such sets of these such separators. They are high lying because the strong magnetic fields below push the separator field lines up (as indicated by the yellow colouring of the separator lines in figure 5.13). There are fewer separators in the equatorial region at solar maximum than at solar minimum, as already mentioned but there are more separators over the poles at solar maximum than at solar minimum. At solar minimum, the distribution of high lying separators appears greater than at solar maximum where many of the high separators appear to stem from the same few null points.

In order to quantify the number of high lying separators in figure 5.13, figure 5.14 shows the maximum heights reached by each null-null separator (blue) and at the same location on the \(x\)-axis, the heights of the two null points it connects (red) are plotted. Since a single null point may be connected to several different separators, the same null point will be plotted as many times as it has separators connecting to it leading, in places, to a horizontal line of red dots. There is very little difference between solar minimum and solar maximum in the distributions of the maximum heights of the separators. The maximum heights of all separators at both solar maximum and solar minimum are approximately equal in both cases. Separators in
5.2. Results

(a) Carrington Rotation 2097

(b) Carrington Rotation 2098
5. Separator Networks at Solar Minimum and Maximum

Figure 5.13

The heights of each null-null separator and the angular locations of each null point (randomly coloured with a tone of red and blue respectively if the null point is positive and negative, similar to the spines in the cuts in figures 5.6, 5.8 and 5.9).
5.2. Results

(a) Carrington Rotation 2007

(b) Carrington Rotation 2008
5. Separator Networks at Solar Minimum and Maximum

Figure 5.14
The maximum height reached of each separator together with the heights of the two null points which connect them plotted directly below.
5.2 Results

The maximum height reached by each separator for all minimum and maximum Carrington rotations.

Figure 5.15 shows all the maximum height curves for all Carrington rotations on one plot. The largest maximum separator heights are generally higher at solar maximum than at solar minimum, however the solar maximum curves (solid lines) fall off less quickly than the solar minimum curves. This implies that there are generally more separators reaching maximum heights of over 0.6\(R_\odot\) above the solar surface at solar maximum than solar minimum (at least for the maximum and minimum studied here).
5. Separator Networks at Solar Minimum and Maximum

- Biggest Network
- Cave
- Positive Dome/Tunnel
- Negative Dome/Tunnel
- Ring
- Chain Containing a Ring
- Three Chains Connected (Tri-Junction)
- More complex

(a) Carrington Rotation 2097

(b) Carrington Rotation 2098
5.2. Results

The angular locations of the all networks found in the PFSS fields. The biggest network is plotted in blue and the rest of the networks are plotted in the colours corresponding to their type given in the key above. The separators to the HCS are not plotted here.
5. Separator Networks at Solar Minimum and Maximum

Figure 5.17
Same as figure 5.16 but for Carrington rotation 2149 where there are two separate polar networks.

5.2.6 Large-Scale Magnetic Skeleton Networks

Figure 5.16 shows plots of the types of all the separator networks (similar to figure 4.10) at solar minimum and solar maximum for the same four Carrington rotations as shown in figure 5.13. This plot enables us to see the distribution and the nature of groups of null points that are connected together by strings of separators (i.e. separator networks). In these plots the biggest network is plotted in blue.

As seen in chapter 4, the majority of the null-null separators in each Carrington rotation are linked together such that they form just one single large complex network. However, there are also quite a few much smaller networks found at both solar minimum and solar maximum. At both solar minimum and solar maximum for each Carrington rotation (except Carrington rotation 2149, figure 5.17), there is one separator network, coloured blue, which goes all the way around the Sun. The biggest network in each Carrington rotation at solar minimum is clearly one network which lies mainly over the equator (figure 5.16a and 5.16b). While, in each Carrington rotation at solar maximum, the biggest network appears more like two separate networks lying over each pole away from the equator. For example, in Carrington rotation 2150, the largest network would be two separate separator networks around either pole if it was not for the two separators forming connections over the equator and creating one large network. However, in Carrington rotation 2149 (figure 5.17), there really are two separator networks lying close to each pole, with no separators crossing the equator and connecting these networks.
### 5.2. Results

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<th>Number of other networks</th>
<th>Isolated Null Points</th>
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Table 5.5

The number of null points and separators in the largest separator network, the numbers of each type of separator network, the number of null points in the second largest (which is the largest of all networks counted in the complex category) separator network, the number of isolated null points (no connected separators) and the percentage of the isolated null points which have an open spine (spine which reaches the top boundary).
5. Separator Networks at Solar Minimum and Maximum

Table 5.5 lists the number of each type of separator network for each Carrington rotation studied in this chapter including details on the largest network and the number of isolated null points. The largest separator network found in Carrington rotation 2149 is significantly smaller than in all other Carrington rotations because, as already mentioned, there are two separate polar networks rather than a single large network (figure 5.17). It is possible that as $L$ is increased, more null points and separators may be found which have not been found at $L = 641$. One or two of these separators may provide a connection over the equatorial region between these two large global networks.

The number of null points in the biggest network is approximately proportional to the total number of null points for all Carrington rotations, although the connection to the number of separators is less clear. There is almost half the number of null points compared to the number of separators in the biggest network at solar minimum. Yet at solar maximum, this ratio of the number of null points to the number of separators is closer to two thirds. This indicates that there are more null points connected to multiple separators in these large networks at solar minimum than at solar maximum.

The number of other separator networks generally decreases from solar minimum to solar maximum approximately in line with the decrease seen in the total number of null points in each Carrington rotation. The only increase is in the number of linear chain networks containing a ring although the overall numbers of these features is so small that the significance of the change is hard to say. These small networks are generally pretty uniformly distributed in longitude and latitude at solar minimum (figures 5.16a and 5.16b), but, as seen earlier, they are away from the equator at solar maximum (figures 5.16c and 5.16d). Networks that form caves (i.e. have the same number of positive and negative null points in a linear chain) are generally uniformly distributed. However the positive and negative domes or tunnels (i.e. the networks which have an odd number of null points in a linear chain, which, respectively, have one more positive (or negative) null point than they have negative (or positive)) have a tendency to be either in the northern or southern hemisphere respectively. This is very clearly defined at solar minimum (figures 5.16a and 5.16b) but much less so at solar maximum. The distribution of positive and negative domes switches when the polar magnetic field switches.

What is very clear from table 5.5 is that the most common types of small separator networks are linear networks that either form caves or domes and tunnels with
5.3. Conclusions

typically double the number of caves found compared with the number of domes and tunnels. Approximately, 180 caves and 90 domes and tunnels were found at solar minimum and around 140 and 69 of each network at solar maximum. There are actually very few other types of small networks (a total of just 20 to 40 of all other types of networks were found). These networks involved tri-junctions, formed rings or were even more complex. Those that involved a ring were the rarest type of networks at both solar minimum and solar maximum. There are a significant number of isolated null points found in the global magnetic fields with approximately 1200 identified at solar minimum and around 1000 at solar maximum. The separatrix surfaces from these isolated null points create domes that enclose usually small regions of magnetic field. There is a clear decrease in the number of isolated null points with open spines (spines which reach the top boundary) between solar minimum to solar maximum. These isolated null points are typically low lying and exist in open field regions. They are likely to be associated with solar plumes and may contribute to the solar wind. The spines from these null points can clearly be seen in the 3D magnetic field plots (figures 5.3 and 5.4) extending out from the open field regions.

5.3 Conclusions

The main difference between the separator networks at solar minimum and solar maximum is simply their angular distribution in the solar atmosphere. There are no significant differences in their radial locations. They both have separators which reach the outer boundary and also null-null separators that extend high, over $0.5R_{\odot}$ ($\geq 350$ Mm), above the solar surface. The main difference is that at solar minimum, the majority of separators are found near the equator away from the poles while at solar maximum, the reverse is true, most separators are found away from the equator and lie close to or even over the poles.

There is generally a dominant separator network that covers the entire Sun at both solar minimum and solar maximum involving more than 50% of the total number of null points and 75% of all the separators. There are also many linear chains of separators at both solar minimum and solar maximum. Other, more complex, networks are actually not so common. There are also still many null points not connected by separators in the global magnetic field. The percentage of null points that are isolated is approximately 25% at solar minimum and 30% at solar
5. Separator Networks at Solar Minimum and Maximum

maximum. Only a small portion of these isolated null points, 12% at solar minimum and 6% at solar maximum, reside within open field regions and therefore might be associated with plumes. It is likely that many of the isolated null points, caves, domes and tunnels are associated with X-ray bright points, but this something that needs further work to test.
Chapter 6
Global Magnetohydrostatic Field Model

The PFSS model produces a simple potential magnetic field whose physical realism may be argued. As discussed in the Introduction, a plasma may be assumed to be force-free if $|v| \ll v_g, v_A, v_s, L \ll H$ and $\beta \ll 1$. Slow foot point motions could provide enough time for any currents in the system to accumulate about topological features (van Ballegooijen, Asgari-Targhi and Berger 2014) in local current layers, leaving the surrounding field potential. However, for a global field, the typical length scales are almost certainly going to be bigger than the pressure scale height and so the plasma cannot be ignored, hence a magnetohydrostatic (MHS) field model is probably more realistic than a simple potential field. There have been several derivations of MHS models (e.g. Bogdan and Low 1986; Low 1991; Low 1992). Neukirch (1995) extends and derives a generalised MHS model and concentrates on spherical geometry; he also provides an equivalent Cartesian model though this is not required here.

Here we will use his approach in order to compare and see what effect non-potentiality may have on the magnetic skeleton of global magnetic fields derived from the same synoptic magnetogram. The global MHS model that is calculated here is essentially that of Neukirch (1995) but with constraints at $r = R_{\text{max}}$ to approximate a radial source surface in the same way as implemented in the global potential field source surface models in chapter 3.

First, the Bessel functions are introduced as they are required for the calculation of the MHS models considered in this thesis. Then a detailed derivation of the MHS model in the same way as Neukirch (1995) follows. We then introduce the boundary conditions which we have tried to implement into the model and a short discussion of the code we use to calculate fields.
6.1 Bessel Functions

The Bessel functions are solutions to the differential equation

\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - z^2) y = 0, \tag{6.1} \]

for any complex number \( z \) and \( x \), although only real \( z \) and \( x \) are used in this thesis. Equation 6.1 is known as Bessel’s differential equation.

Bessel functions of the first kind, \( J_z(x) \), are solutions to Bessel’s differential equation where \( z \) is known as the order of the Bessel function. They can be defined by a series expansion given by

\[ J_z(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+z+1)} \left( \frac{x}{2} \right)^{2m+z}. \]

There are many other ways of defining the Bessel functions but these are not listed here in this thesis, as we will only be dealing with the Bessel functions of half integer order.

Both \( J_z(x) \) and \( J_{-z}(x) \) are solutions to Bessel’s differential equation (6.1). So for non-integer \( z \), these are the two solutions to the second order differential equation since they are linearly independent. However, for integer \( z \), i.e. \( z = n \) where \( n \in \mathbb{Z} \),

\[ J_{-n}(x) = (-1)^n J_n(x), \]

so the solutions are not linearly independent and so a second linearly independent solution is required. This additional solution to equation 6.1 is known as the Bessel function of the second kind. Commonly these are denoted \( Y_z(x) \) however, since in this thesis, \( Y_l^m \) denotes the spherical harmonics, we will denote these Bessel functions by their less common notation of \( N_z(x) \) to avoid any confusion.

For non-integer values of \( z \), Bessel functions of the second kind are related to those of the first kind by

\[ N_z(x) = \frac{J_z(x) \cos(\pi z) - J_{-z}(x)}{\sin(\pi z)} \]

and for integer values of \( z = n \),

\[ N_n(x) = \lim_{z \to n} N_z(x). \]
6.1. Bessel Functions

The half-integer Bessel Functions are a specific subset of the Bessel functions and are the only solutions to the form of the Bessel differential equation solved in this thesis. They are denoted by $J_{n+\frac{1}{2}}$ and $N_{n+\frac{1}{2}}$ where $n \in \mathbb{Z}$. They occur when solving Poisson’s equation in spherical coordinates and are closely related to the spherical Bessel functions. Figure 6.1 contains plots of both the half-integer Bessel functions for $n \in 0, 1, 2, 3, 4$. Clearly for large $x$, these functions are oscillatory in nature.

The half-integer order Bessel functions have closed analytic forms given by

$$J_{n+\frac{1}{2}}(x) = (-1)^n \sqrt{\frac{2}{\pi x^{n+\frac{1}{2}}}} \left( \frac{1}{x} \frac{d}{dx} \right)^n \frac{\sin x}{x},$$
6. Global Magnetohydrostatic Field Model

\[ N_{n+\frac{1}{2}}(x) = (-1)^n \sqrt{\frac{2}{\pi x}} x^{n+\frac{1}{2}} \left( \frac{1}{x} \frac{d}{dx} \right)^n \cos \frac{x}{x}, \]

for positive integer \( n \). However since we will be calculating the Bessel functions numerically these will not be used except for getting analytic solutions for \( n = 0, 1 \). The recurrence relations given next in section 6.1.2 are used instead.

The solutions we will require are \( J_{n+\frac{1}{2}} \) and \( J_{-(n+\frac{1}{2})} \). However, for integer \( n \), the first and second kind Bessel functions are related by

\[ J_{-(n+\frac{1}{2})}(x) = (-1)^{n+1} N_{n+\frac{1}{2}}(x) \]

allowing us to find the solution to Bessel’s differential equation as a linear combination of \( J_{n+\frac{1}{2}} \) and \( N_{n+\frac{1}{2}} \), negating the requirement to find the negative order Bessel functions.

### 6.1.2 Recurrence Relations

The first two Bessel functions of half-integer order (\( z = \frac{1}{2} \) and \( z = \frac{3}{2} \)) of both the first and second kind are given analytically by

\[ J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \]
\[ J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left( \frac{\sin x}{x} - \cos x \right), \]
\[ N_{\frac{1}{2}}(x) = -\sqrt{\frac{2}{\pi x}} \cos x, \]
\[ N_{\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}} \left( \frac{\cos x}{x} + \sin x \right). \]

These can be used to start the recurrence relations to find the Bessel functions and their derivatives for all higher orders. These are given by

\[ Z_z(x) = \frac{2(z-1)}{x} Z_{z-1}(x) - Z_{z-2}(x), \quad (6.2) \]
\[ \frac{dZ_z}{dx} = Z_{z-1}(x) - \frac{z}{x} Z_z(x), \quad (6.3) \]

where \( Z \) is the Bessel function of either first or second kind and \( z \in \mathbb{C} \). They are especially useful for finding the values of the Bessel functions numerically in
6.2. Calculating the Global MHS Magnetic Field

Combination with their analytic forms.

Equations 6.2 and 6.3 can be used to calculate the Bessel functions of half-integer
order recursively. Ordinarily you can start with \( J_{\frac{1}{2}} \) and \( J_{\frac{3}{2}} \) and iterate upwards using
equation 6.2 in their analytic forms. However, the Bessel functions of the first kind
are numerically unstable under this recursive scheme, because each higher order term
decreases rapidly in size and inaccuracies in numerical precision cause significant
problems. To combat this effect, the Bessel functions of the first kind must be
calculated in reverse order. By setting the two Bessel functions with the highest order
(the first two term in the iterative scheme) to zero and one respectively then iterating
from high order to low order using

\[
Z_z(x) = \frac{2(z + 1)}{x} Z_{z+1}(x) - Z_{z+2}(x),
\]

the numerical scheme is now numerically stable since each term is growing larger.

Then, since \( J_{\frac{1}{2}} \) is known analytically, the value of the final term in the iterative
scheme, every term is then be renormalised to the correct value which allows the
Bessel functions of the first kind to be calculated.

The Bessel functions of the second kind grow rapidly in magnitude with increasing
order so they can be calculated by iterating forwards using equation 6.2 as it is
numerically stable.

6.2 Calculating the Global MHS Magnetic Field

This section is simply a derivation of the MHS model which is an expanded version of
that given in Neukirch (1995). The derivation of the MHS model starts with the
MHS equations. These are found by taking the MHD equations with \( \mathbf{v} = 0 \)

\[
\begin{align*}
\mathbf{J} \times \mathbf{B} - \nabla p - \rho \nabla \psi &= 0 \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \\
\nabla \cdot \mathbf{B} &= 0
\end{align*}
\]

where \( \psi \) is the gravitational potential given by

\[
\psi = -\frac{GM_{\odot}}{r},
\]
6. Global Magnetohydrostatic Field Model

and therefore

\[ \nabla \psi = \frac{GM_\odot}{r^3} \mathbf{r}, \]

where \( G \approx 6.674 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2} \) is the gravitational constant and \( M_\odot = 1.989 \times 10^{30} \text{ kg} \) is the mass of the sun.

Following the work of Neukirch (1995), a particular form for the electric current density, \( \mathbf{J} \), is assumed to make the MHS equations solvable. The form chosen for \( \mathbf{J} \) is

\[ \mathbf{J} = \alpha \mathbf{B} + \nabla \times (\nabla \times \psi), \quad (6.7) \]

where \( \nabla \alpha = 0 \) and \( F = F(\nabla \psi \cdot \mathbf{B}, \psi) \) is some free function dependent on its arguments. The two terms in this form for \( \mathbf{J} \) are a component of current parallel to the magnetic field (\( \alpha \mathbf{B} \)) and a component perpendicular to the radial direction (\( \nabla \times (\nabla \times \psi) \)) which will provide a component of current perpendicular to the magnetic field as well as an additional component parallel to it depending on the direction of the \( \mathbf{B} \). This form reduces Ampère’s Law (6.5) to an analytically tractable form.

Substituting this form for the current, \( \mathbf{J} \), into equation 6.5 simply yields

\[ \nabla \times \mathbf{B} = \mu_0 \left( \alpha \mathbf{B} + \nabla \times (\nabla \times \psi) \right), \]

and taking the curl and then dotting the result with \( \mathbf{r} = r \hat{e}_r \) yields (derivation in appendix B.3, equation B.14)

\[ \mathbf{r} \cdot \nabla \times (\nabla \times \mathbf{B}) = \alpha^2 \mu_0^2 \mathbf{r} \cdot \mathbf{B} + \mu_0 \mathbf{r} \cdot \nabla \times (\nabla \times (\nabla \times \psi)). \quad (6.8) \]

The left hand side of equation 6.8 can be simplified somewhat. Applying the properties of vector calculus, we find that (derivation in appendix B.3, equation B.15)

\[ \mathbf{r} \cdot \nabla \times (\nabla \times \mathbf{B}) \equiv -\nabla^2 (\mathbf{r} \cdot \mathbf{B}). \quad (6.9) \]

This means we can write equation 6.8 as

\[ \nabla^2 (\mathbf{r} \cdot \mathbf{B}) + \alpha^2 \mu_0^2 \mathbf{r} \cdot \mathbf{B} + \mu_0 \mathbf{r} \cdot \nabla \times (\nabla \times (\nabla \times \psi)) = 0. \quad (6.10) \]

Since the first two terms in equation 6.10 depend on \( \mathbf{r} \cdot \mathbf{B} \) explicitly, \( F \) is chosen in
6.2. Calculating the Global MHS Magnetic Field

order to write the final term also in terms of $\mathbf{r} \cdot \mathbf{B}$. Neukirch (1995) chose $F$ to be

$$F(\nabla \psi \cdot \mathbf{B}, \psi) = K(\psi) \nabla \psi \cdot \mathbf{B} = K(\psi) \frac{GM_\odot}{r^3} \mathbf{r} \cdot \mathbf{B},$$

where $K(\psi)$ is a function to be chosen. Now, we define a new function $H(r)$ in terms of $K(\psi)$ to be

$$H(r) = \left( \frac{GM_\odot}{r^3} \right)^2 K(\psi).$$

This is chosen in order to enable us to find a form for the final term of equation 6.10 in terms of $\mathbf{r} \cdot \mathbf{B}$. This form of $H(r)$ leads to (derivation in appendix B.3, equation B.16)

$$\nabla F \times \nabla \psi = H(r) \nabla (\mathbf{r} \cdot \mathbf{B}) \times \mathbf{r}. \quad (6.11)$$

Now we make use of the the angular momentum operator $\mathbf{L}$ from quantum mechanics to form a new differential operator similar to the Laplacian (a more detailed description of the operator and its derivation is in appendix B.2). The angular momentum operator $\mathbf{L}$ with $\hbar$ neglected is

$$\mathbf{L} \equiv \frac{1}{i} \mathbf{r} \times \nabla \equiv \frac{1}{i} \left( 0, -\frac{1}{\sin \theta} \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta} \right)$$

and from this, dotting $\mathbf{L}$ with itself, we can form

$$\mathbf{L}^2 = \mathbf{L} \cdot \mathbf{L} = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \phi} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}.$$

By using this operator, it can be shown that

$$\mathbf{r} \cdot \nabla \times (\nabla F \times \nabla \psi) = H(r) \mathbf{L}^2 (\mathbf{r} \cdot \mathbf{B}). \quad (6.12)$$

(derivation in appendix B.3, equation B.17) and so finally, we may write equation 6.10 entirely in terms of $\mathbf{r} \cdot \mathbf{B}$:

$$\nabla^2 (\mathbf{r} \cdot \mathbf{B}) + \mu_0 H(r) \mathbf{L}^2 (\mathbf{r} \cdot \mathbf{B}) + \alpha^2 \mu_0^2 \mathbf{r} \cdot \mathbf{B} = 0.$$

To make this equation slightly simpler, we define

$$\bar{H}(r) = \mu_0 H(r),$$
and

\[ \bar{\alpha} = \mu_0 \alpha, \]

so that the differential equation becomes

\[ \nabla^2 (r \cdot B) + \bar{H}(r) L^2 (r \cdot B) + \bar{\alpha}^2 r \cdot B = 0 \] (6.13)

where \( \bar{H}(r) \) and \( \bar{\alpha} \) are to be chosen.

This differential equation in terms of \( r \cdot B \) (6.13) can be solved in the same way as Laplace’s equation in chapter 3 using separation of variables. Similarly we will use the three separation functions \( R(r) \), \( T(\theta) \) and \( P(\phi) \) and assume a separable solution of the form

\[ r \cdot B = R(r) T(\theta) P(\phi). \]

Substituting this into equation 6.13 and a rearrangement gives

\[ \frac{1}{r^2 R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \left( \frac{1}{r^2} - \bar{H}(r) \right) \left( \frac{1}{T \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dT}{d\theta} \right) + \frac{1}{P \sin^2 \theta} \frac{d^2 P}{d\phi^2} \right) + \bar{\alpha}^2 = 0. \]

Using a separation constant \( \lambda \), we get

\[ \frac{1}{r^2 R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \lambda \left( \frac{1}{r^2} - \bar{H}(r) \right) + \bar{\alpha}^2 = 0, \] (6.14)

\[ \frac{1}{T \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dT}{d\theta} \right) + \frac{1}{P \sin^2 \theta} \frac{d^2 P}{d\phi^2} = -\lambda. \] (6.15)

The second of these two equations (6.15) is identical to the \( \theta \) and \( \phi \) separable equation found when solving Laplace’s equation for the PFSS model and so we already know that the solutions to this differential equation are the spherical harmonics, \( Y_l^m(\theta, \phi) \), with \( \lambda = l(l+1) \) and \( -l \leq m \leq l \).

The \( r \)-dependent equation (6.14) has solutions which are dependent on \( \bar{H}(r) \) and thus \( \bar{H}(r) \) will be chosen to make the differential equation solvable. Theoretically any \( \bar{H}(r) \) can be chosen but, like Neukirch (1995), a form is chosen that permits analytic solutions to equation 6.14.

First, we note that equation 6.14 is a second order linear differential equation and so, in general, the solution is a linear sum of two functions of \( r \). This solution will be called \( B_l^m(r) \). It is dependent on both \( l \) and \( m \) and is similar to \( B_l^m(r) \) in the PFSS
model. This means we can write the solution for \( \mathbf{r} \cdot \mathbf{B} \) as

\[
\mathbf{r} \cdot \mathbf{B} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_l^m(r) Y_l^m(\theta, \phi).
\] (6.16)

Firstly, \( B_r \) is easily calculated to be

\[
B_r = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_l^m(r) Y_l^m(\theta, \phi)
\]

Now \( B_\theta \) and \( B_\phi \) need to be determined.

Using the properties of \( \mathbf{L} \) and defining \( \mathbf{B}_t = B_\theta \mathbf{e}_\theta + B_\phi \mathbf{e}_\phi \), we find that (derivation in appendix B.3, equation B.18)

\[
\mathbf{L} \cdot \mathbf{B}_t = \frac{1}{i} \mathbf{r} \cdot \mathbf{B}.
\] (6.17)

So we require a representation of \( \mathbf{L} \cdot \mathbf{B}_t \) that conforms with the summation of \( \mathbf{r} \cdot \mathbf{B} \) which we have. Neukirch (1995) states that it can be readily shown by Jackson (1975) that this is fulfilled by

\[
\mathbf{B}_t = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} v_l^m(r) \mathbf{L} Y_l^m(\theta, \phi) + w_l^m(r) \nabla Y_l^m(\theta, \phi).
\]

By looking at a single-order element of the summation, it can be shown that

\[
(L \cdot B_t)_l^m = l(l+1) v_l^m(r) Y_l^m(\theta, \phi)
\]

and using equality 6.17 above and the general form for \( \mathbf{r} \cdot \mathbf{B} \) (6.16), we obtain the following form for \( v_l^m(r) \)

\[
v_l^m(r) = \frac{1}{l} \frac{B_l^m(r)}{l(l+1)}.
\]

Then finally the \( w_l^m \) can be found using \( \nabla \cdot \mathbf{B} = 0 \)

\[
\nabla \cdot \mathbf{B} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{B_l^m(r)}{r} Y_l^m(\theta, \phi) \right) + \nabla \cdot (v_l^m \mathbf{L} Y_l^m) + \nabla \cdot (w_l^m \nabla Y_l^m) = 0.
\]

Calculating this and rearranging gives

\[
w_l^m = \frac{1}{l(l+1)} \frac{d}{dr} \left( r B_l^m \right).
\]
6. Global Magnetohydrostatic Field Model

Both \( v_l^m \) and \( w_l^m \) are derived in appendix B.3, equations B.19 and B.20.

So the final magnetic field, \( \mathbf{B}(r, \theta, \phi) \), is given by

\[
\mathbf{B} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( r \frac{B_l^m(r)}{r^2} Y_l^m(\theta, \phi) + \frac{\alpha}{l(l+1)} L Y_l^m(\theta, \phi) + \frac{d}{dr}(rB_l^m(r)) \frac{\nabla Y_l^m(\theta, \phi)}{l(l+1)} \right),
\]

which can be written component wise as

\[
B_r(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{B_l^m(r)}{r} Y_l^m(\theta, \phi), \tag{6.18}
\]

\[
B_\theta(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{l(l+1)} \left( \tilde{\alpha} B_l^m(r) i m \frac{Q_m^m(\theta)}{\sin \theta} e^{im\phi} + \frac{1}{r} \frac{d}{dr}(rB_l^m(r)) \frac{dQ_m^m}{d\theta} e^{im\phi} \right),
\]

\[
B_\phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{l(l+1)} \left( -\tilde{\alpha} B_l^m(r) \frac{dQ_m^m}{d\theta} e^{im\phi} + \frac{1}{r} \frac{d}{dr}(rB_l^m(r)) i m \frac{Q_m^m(\theta)}{\sin \theta} e^{im\phi} \right).
\]

Thus, provided \( \tilde{H}(r) \) is known and equation 6.14 can be solved to give an analytical form for \( B_l^m(r) \) then the magnetic field components may be found for a given set of boundary conditions. The approach in Neukirch and Rastätter (1999) may also be used to find a coronal MHS model.

In section 6.4, we discuss the application of the boundary conditions. First though, in section 6.3, we derive the pressure and density in the MHS equilibrium.

### 6.3 Solving for the Plasma Pressure and Density

Now that the magnetic field has been derived, we can now find the plasma pressure and density. Expanding the cross product of \( \mathbf{J} \times \mathbf{B} \) in equation 6.4, we get

\[
\nabla p + \rho \nabla \psi + (\mathbf{B} \cdot \nabla \psi) \nabla F - (\mathbf{B} \cdot \nabla F) \nabla \psi = 0
\]

To get a non-trivial current flow, \( \nabla \psi \) and \( \nabla F \) must be linearly independent. Thus we must have \( p = p(F, \psi) \) and \( \nabla p \) can be expanded as

\[
\frac{\partial p}{\partial F} \nabla F + \frac{\partial p}{\partial \psi} \nabla \psi + \rho \nabla \psi + (\mathbf{B} \cdot \nabla \psi) \nabla F - (\mathbf{B} \cdot \nabla F) \nabla \psi = 0.
\]
6.4. Applying the Boundary Conditions

Using the linear independence, the equation can be split into

$$\frac{\partial p}{\partial F} + B \cdot \nabla \psi = \frac{\partial p}{\partial F} + \frac{F}{K'(\psi)} = 0,$$

(6.19)

and

$$\frac{\partial p}{\partial \psi} + \rho - B \cdot \nabla F = 0.$$

(6.20)

The first of these equations (6.19) can be immediately integrated to give

$$p(r, \theta, \phi) = p_0(\psi) - \frac{F^2}{2K'(\psi)} = p_0(r) - \frac{H(r)}{2} (r \cdot B)^2 = p_0(r) - \frac{\bar{H}(r)}{2\mu_0} (r \cdot B)^2$$

(6.21)

where $p_0(r)$ is a free function that represents a background radially dependent pressure.

The second of the two equations (6.20) can then be solved to give the density

$$\rho(r, \theta, \phi) = \frac{r^2}{\mu_0 GM_\odot} \left( \frac{1}{2} \frac{d\bar{H}}{dr} (r \cdot B)^2 + r \bar{H} B \cdot \nabla (r \cdot B) - \mu_0 \frac{dp_0}{dr} \right).$$

(6.22)

The final term can be thought of as a background radially dependent density. A short discussion on finding this expression for the density is given at the end of appendix B.

The pressure and density can then be used to find other quantities like temperature. Temperature can be found using the ideal gas law from chapter 1.

We now go on to define the boundary conditions such that the functions $B_l^m(r)$ can be found.

6.4 Applying the Boundary Conditions

The initial aim of this work was to apply a source surface to the MHS model. This would then allow a direct comparison between the magnetic skeletons of the PFSS model with the magnetic fields from the MHS model which, in theory, are more realistic.

Ruan et al. (2008) claims to have used a source surface in his work on the MHS model, however the implementation of this is so far unknown and the method he used has not been documented. After our investigations, we find it unlikely that they implemented a true source surface. Rudenko (2001) claims that a source surface is
only possible in the case that \( \alpha = 0 \) but no proof was given. So most MHS models using the equations from Neukirch (1995) implement a zero magnetic field at infinity including Rudenko (2001).

After many attempts to implement a source surface for non-zero \( \alpha \), a possible iterative scheme for its calculation was found. However, it essentially required solving three different equations for the two unknown constants. Hence the problem was over-specified and it was not possible to implement a true source surface. A summary of the method is detailed in appendix C.

Given the lack of a true source surface MHS field for non-zero \( \alpha \), we instead tried to minimise the angular components of the magnetic field on the outer boundary instead. Three different boundary conditions are derived here:

1. a true source surface model when \( \alpha = 0 \),
2. a partial source-surface model for \( \alpha \neq 0 \) and,
3. zero magnetic field at infinity.

First we need to pick a form for \( \bar{H}(r) \). Here we pick Neukirch (1995)’s case II (also used by Bogdan and Low 1986) where

\[
\bar{H}(r) = \frac{1}{r^2} - \frac{1}{(r + d)^2}. \quad (6.23)
\]

where \( d \) is an arbitrary constant which can be thought of as controlling the amount of current perpendicular to the radial direction. This form for \( \bar{H}(r) \) gives analytic solutions to equation 6.14. Equation 6.14 now becomes

\[
\frac{d^2}{dr^2}(rR) + \left( \frac{\alpha^2}{(r + d)^2} - \frac{l(l+1)}{(r + d)^2} \right) rR = 0, \quad (6.24)
\]

where the following identity is used,

\[
\frac{1}{r} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) \equiv \frac{d^2}{dr^2}(rR).
\]

When \( \alpha = 0 \), equation 6.24 gives solutions similar to the PFSS model. Substituting \( \alpha = 0 \) gives

\[
\frac{d^2}{dr^2}(rR) - \frac{l(l+1)}{(r + d)^2} rR = 0.
\]
6.4. Applying the Boundary Conditions

By defining two new parameters $u(r) = rR(r)$ and $v(r) = \log(r + d)$, this differential equation may be transformed into the following linear second order differential equation

$$\frac{d^2 u}{dv^2} - \frac{du}{dv} - l(l + 1)u = 0$$

which may be easily solved to obtain $u(v)$. This can be transformed to give $R(r)$ as a linear combination of powers of $r$

$$R(r) = a\frac{(r + d)^{l+1}}{r} + b\frac{(r + d)^{-l}}{r}$$

where $a$ and $b$ are integration constants. This is equal to $B_l^m(r)$, hence

$$B_l^m(r) = a_l^m\frac{(r + d)^{l+1}}{r} + b_l^m\frac{(r + d)^{-l}}{r} \quad (6.25)$$

where $a_l^m$ and $b_l^m$ are now constants dependent on $l$ and $m$.

When $\alpha \neq 0$, equation 6.24 instead leads to solutions in terms of Bessel functions. Using a substitution of the form $rR = (r + d)^{\frac{1}{2}}u(v)$ with $v = \bar{\alpha}(r + d)$, it transforms the differential equation into

$$v^2 \frac{d^2 u}{dv^2} + v \frac{du}{dv} + \left(v^2 - \left(l + \frac{1}{2}\right)^2\right)u = 0.$$ 

Recalling Bessel’s differential equation (6.1) and setting $z = l + \frac{1}{2}$, the solutions for $u(v)$ are Bessel functions and so our solution for $B_l^m(r)$ is

$$B_l^m(r) = \sqrt{\frac{r + d}{r}} \left(a_l^m J_{l+\frac{1}{2}}(\bar{\alpha}(r + d)) + b_l^m N_{l+\frac{1}{2}}(\bar{\alpha}(r + d))\right),$$

where $J_z$ and $N_z$ are Bessel functions of the first and second kind respectively of order $z \in \mathbb{C}$ (see section 6.1 for details). These are the solutions to the MHS equations which were derived in Neukirch (1995). To avoid repetitive use of the $\bar{\alpha}(r + d)$ in the function argument, I now define two new Bessel functions to simplify the expressions to follow

$$\bar{J}_z(r) = J_z(\bar{\alpha}(r + d)), \quad (6.26)$$

$$\bar{N}_z(r) = N_z(\bar{\alpha}(r + d)). \quad (6.27)$$

The ‘ operator is used to denote differentiation with respect to the function argument.
such that
\[ J_z'(r) = \frac{dJ_z(\bar{\alpha}(r + d))}{d(\bar{\alpha}(r + d))} = \frac{1}{\bar{\alpha}} \frac{d}{dr}(J_z(\bar{\alpha}(r + d))). \]

Finally, \( B_l^m(r) \) can be written as
\[ B_l^m(r) = \frac{\sqrt{r + d}}{r} \left( a_l^m \bar{J}_{l+\frac{1}{2}}(r) + b_l^m \bar{N}_{l+\frac{1}{2}}(r) \right). \] (6.28)

The boundary conditions are now applied to these forms for \( B_l^m \) in equation 6.25 and 6.28 to find the unknown \( a_l^m \) and \( b_l^m \). The techniques used to find them are identical to those used in the derivation of the PFSS model.

The first boundary condition is common to each of the three sets of boundary conditions considered here. As in the PFSS model (equation 3.25), we require the radial component of the base boundary of the magnetic field to be equal to the radial synoptic map i.e.
\[ B_r(R_\odot, \theta, \phi) = B_\odot(\theta, \phi). \] (6.29)

There is only one small difference between this and the PFSS model in how \( B_l^m \) is defined in terms of the synoptic map after the first boundary condition is used. Since \( B_l^m \) is now divided by \( r \) in the \( B_r \) component of the magnetic field (equation 6.18), we get an extra \( R_\odot \) in the definition of \( B_l^m \) compared with its definition in the PFSS model (equation 3.27):
\[ B_l^m(R_\odot) \equiv R_\odot \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} B_\odot(\theta, \phi) Y_{l,m}(\theta, \phi) \, dS \] (6.30)

If, as will be done to calculate the MHS model numerically, \( r \) is normalised with respect to \( R_\odot \), then this extra factor on \( B_l^m(R_\odot) \) will disappear.

6.4.1 Finite Boundary Conditions

The two outer boundary conditions depend on whether we have a finite boundary at \( r = 2.5R_\odot \) with a true source surface (\( \bar{\alpha} = 0 \)) or an approximate source surface (\( \bar{\alpha} \neq 0 \)). In the case when \( \bar{\alpha} = 0 \), we require
\[ B_\theta(R_{\text{max}}, \theta, \phi) = B_\phi(R_{\text{max}}, \theta, \phi) = 0. \]
6.4. Applying the Boundary Conditions

Since $\bar{\alpha} = 0$, there is now only one term in the summation of $B_{\theta}$ and $B_{\phi}$ for each $l$ and $m$ so a source surface requires

$$\frac{d}{dr} \left( r B_{l}^{m} \right) \bigg|_{R_{\text{max}}} = 0. \quad (6.31)$$

When $\bar{\alpha} \neq 0$, we instead attempt to minimise the $B_{\theta}$ and $B_{\phi}$ components on the outer boundary to create an approximate source surface. $B_{l}^{m}$ cannot be set to zero on the outer boundary as there would then be no radial component of the magnetic field on the outer boundary – exactly the opposite of what we want. So instead, we set the radial derivative term to zero and the boundary condition is the same as the true source surface (equation 6.31).

Applying this to the solution for $B_{l}^{m}(r)$ when $\bar{\alpha} = 0$ gives

$$a_{l}^{m} = -\frac{l + 1}{l} \left( R_{\text{max}} + d \right)^{2l+1} b_{l}^{m}.$$ 

Hence, by using the same technique as the PFSS model applying the lower boundary condition reveals the final form for $B_{l}^{m}(r)$ and the required derivative term as

$$B_{l}^{m}(r) = B_{l}^{m}(R_{\odot}) \frac{R_{\odot}^2}{r^2} \left( \frac{R_{\odot} + d}{r + d} \right)^l l + 1 + l \left( \frac{r + d}{R_{\text{max}} + d} \right)^{2l+1} \frac{l + 1 + l}{l + 1} \left( \frac{R_{\odot} + d}{R_{\text{max}} + d} \right)^{2l+1} \quad (6.32)$$

and

$$\frac{1}{r} \frac{d}{dr} \left( r B_{l}^{m} \right) = l \left( l + 1 \right) B_{l}^{m}(R_{\odot}) \frac{R_{\odot}^2}{r} \left( \frac{R_{\odot} + d}{r + d} \right)^{l} l + 1 + l \left( \frac{r + d}{R_{\text{max}} + d} \right)^{2l+1} - 1 \quad \left( \frac{r + d}{R_{\text{max}} + d} \right)^{2l+1}.$$ 

Similarly when $\bar{\alpha} \neq 0$, applying the outer boundary condition gives

$$a_{l}^{m} = -\frac{2\bar{\alpha} \left( R_{\text{max}} + d \right) N_{l+\frac{1}{2}}^{'}(R_{\text{max}}) + \bar{N}_{l+\frac{1}{2}}(R_{\text{max}})}{2\bar{\alpha} \left( R_{\text{max}} + d \right) J_{l+\frac{1}{2}}^{'}(R_{\text{max}}) + J_{l+\frac{1}{2}}(R_{\text{max}})} b_{l}^{m}.$$ 

Defining

$$C_R = \frac{2\bar{\alpha} \left( R_{\text{max}} + d \right) N_{l+\frac{1}{2}}^{'}(R_{\text{max}}) + \bar{N}_{l+\frac{1}{2}}(R_{\text{max}})}{2\bar{\alpha} \left( R_{\text{max}} + d \right) J_{l+\frac{1}{2}}^{'}(R_{\text{max}}) + J_{l+\frac{1}{2}}(R_{\text{max}})},$$
6. Global Magnetohydrostatic Field Model

6.4.2 Infinite Boundary Condition

Although work using an infinite boundary condition is not used in this thesis, we briefly discuss how to find such boundary conditions. It requires that all three magnetic field components tend to zero at infinity:

$$\lim_{r \to \infty} B(r, \theta, \phi) = 0$$

Applying this condition requires any term that can increase to infinity must be set to zero. It is not possible to apply this boundary the same way we did for the finite boundary condition since the behaviour of the two linearly independent functions in each of the solutions needs to be known.

First taking a more general approach, let us consider the solution for $B^m_l(r)$ to be

$$B^m_l(r) = a^m_l R_l(r) + b^m_l R_{l+1}(r),$$

and after applying the infinite boundary condition to be

$$B^m_l(r) = c^m_l R_l(r)$$

where $R_l(r)$ represents either of the two functions from the general solution and either $a^m_l$ or $b^m_l$ has been set to zero. If we then apply the first boundary condition on
the base, we obtain a final solution of

\[ B_{l}^{m}(r) = B_{l}^{m}(R_{\odot}) \frac{R_{l}(r)}{R_{l}(R_{\odot})}. \]

To determine which of two linearly independent functions \( R_{l}(r) \) represents,

\[ \lim_{l \to \infty} \frac{R_{l,i}(r)}{R_{l,i}(R_{\odot})} \]

must be determined for \( i \in 1, 2 \). The limit which becomes infinite are the set of solutions which must be set to zero. Although these limits can depend on the values of the parameters \( \bar{\alpha} \) and \( d \) so these unknown constants must be calculated on a case by case basis.

Hence, for \( d > 0 \) and \( \bar{\alpha} > 0 \), after applying the first boundary condition we have

\[ B_{l}^{m}(r) = c_{l}^{m} \frac{\sqrt{r + d}}{r} Z_{l+\frac{1}{2}}(\bar{\alpha}(r + d)) \]

where \( Z_{l+\frac{1}{2}} \) represents either of the two terms in the actual solution. If we then apply the boundary condition on the base, we obtain a solution of

\[ B_{l}^{m}(r) = B_{l}^{m}(R_{\odot}) \frac{R_{\odot}}{R_{l}(R_{\odot})} \sqrt{\frac{r + d}{R_{\odot} + d}} Z_{l+\frac{1}{2}}(\bar{\alpha}(R_{\odot} + d)) \]

It turns out that as \( l \to \infty \),

\[ \frac{J_{l+\frac{1}{2}}(\bar{\alpha}(r + d))}{J_{l+\frac{1}{2}}(\bar{\alpha}(R_{\odot} + d))} \to \infty \]

while

\[ \frac{N_{l+\frac{1}{2}}(\bar{\alpha}(r + d))}{N_{l+\frac{1}{2}}(\bar{\alpha}(R_{\odot} + d))} \to 0. \]

Hence, to keep \( B_{l}^{m}(R_{\odot}) \) bounded, we require \( c_{l}^{m} = 0 \) and so discard all terms involving the Bessel functions of the first kind. This is surprising since \( J_{l}(r) \) themselves are bounded. Hence the final solution for infinite boundary conditions is

\[ \frac{B_{l}^{m}(r)}{r^{2}} = B_{l}^{m}(R_{\odot}) \frac{R_{\odot}^{2}}{R_{l}(R_{\odot})} \sqrt{\frac{r + d}{R_{\odot} + d}} \frac{N_{l+\frac{1}{2}}(r)}{N_{l+\frac{1}{2}}(R_{\odot})} \]
6. Global Magnetohydrostatic Field Model

\[
\frac{1}{r} \frac{d}{dr} (r B^m_l) = \frac{B^m_l (R_\odot)}{\sqrt{r + d \sqrt{R_\odot} + d}} \bar{\alpha} (r + d) \frac{\bar{N}'_{l+\frac{1}{2}} (r) + \frac{1}{2} \bar{N}_{l+\frac{1}{2}} (r)}{\bar{N}_{l+\frac{1}{2}} (R_\odot)}
\]

again using the notation given in equations 6.26 and 6.27.

6.5 An MHS Code

The code used to calculate the MHS model is very similar to the PFSS code described chapter 3 since the MHS equations are similar to the PFSS equations. In the MHS code, simply the radially dependent parts of the PFSS code have been changed and the final summations have been updated to the MHS equations. This code is available to download from GitHub at \url{https://github.com/benmatwil/MHS}. Again, like the PFSS code, FFTW3 is used to calculate the fast Fourier transforms in the code.

The biggest issue that occurs when calculating the Bessel functions is that they decrease and increase in size very quickly as their order increases for the first and second kind respectively. As \( l \to \infty \),

\[
J_{l+\frac{1}{2}} (x) \to 0
\]

\[
\bar{N}_{l+\frac{1}{2}} (x) \to -\infty.
\]

This can be seen even for small \( l \) in figure 6.1. So for large orders, the Bessel functions of the second kind increase past the largest exponent of 64-bit floating point numbers and those of the first kind decreases past the smallest exponent. For example, when \( l = 200 \),

\[
J_{l+\frac{1}{2}} (1.5) \approx 7.9498 \times 10^{-402}
\]

\[
\bar{N}_{l+\frac{1}{2}} (1.5) \approx -1.9971 \times 10^{308}.
\]

These are both out of the maximum exponent range for 64-bit floating point numbers of \( 10^{\pm 308} \). To alleviate this issue, two changes are made from the PFSS model code. Firstly, 128-bit floating point numbers are used which has a maximum exponent of \( 10^{\pm 4032} \) and thus allows higher orders of Bessel functions to be calculated. Secondly, the equations for \( B^m_l (r) \) involving Bessel functions can all be written such that all Bessel functions are divided by another Bessel function of the same kind. This means that the majority of terms involving Bessel functions become normalised to a value around 1 again.
The three different boundary condition cases are implemented and the code creates two different executables to compensate for this. One is called `mhs_finite` which calculates cases 1 and 2 and the other is called `mhs_infinite` which calculates case 3. The parameters $\bar{\alpha}$ and $d$ are customisable in the code using command line arguments on execution using the options `-a` or `-alpha` and `-d` respectively. Like the code for the PFSS model, all lengths are normalised by $R_{\odot}$.

6.6 Connection To The PFSS Model

On inspection of the MHS magnetic field equations, they look quite similar to the PFSS model equations. In fact, if you take $d = 0$ and $\alpha = 0$, this is equivalent to setting $J = 0$. Setting $\alpha = 0$ removes the component of current in equation 6.7 parallel to the magnetic field and setting $d = 0$ makes the second, free-function term in equation 6.7 equal to zero removing any coupling between the plasma and the magnetic field. Therefore, in this case, we have a potential field. By using the true source surface boundary condition in case 1 (in which $\alpha = 0$), the MHS equations simplify to the PFSS equations when $d = 0$.

On the solar surface, the radial component of the magnetic field in the MHS model will also always be equal to the radial component of the magnetic field in the PFSS model. However, the $B_\theta$ and $B_\phi$ components will not be the same and will depend on the given set of parameters.

6.7 Comparison of the Global MHS fields with the PFSS Model Fields

In this section, we compare the global MHS model fields with the PFSS model fields. Here, we present a comparison of the magnetic skeletons of the global MHS magnetic fields for various values of $d$ and $\alpha = 0$ including $d = 0$, the PFSS model magnetic field.

The same synoptic map, Carrington rotation 2097, is used to extrapolate all the 3D MHS magnetic fields with a range of values for $d$. Each extrapolation uses a maximum number of harmonics of $L = 321$. This value of $L$ gives a good balance between the time to create and analyse each of the magnetic fields and the quality of
the magnetic skeleton. Like in chapter 5, all the default (FFT) options for $L = 321$ are used to calculate the magnetic fields and the upper source surface boundary is situated at $R_{\text{max}} = 2.5R_\odot$. This creates magnetic fields of resolution $188 \times 323 \times 645$ ($n_r \times n_\theta \times n_\phi$).

Recalling the definition of the current in the global MHS model, the parameter $d$ controls the amount of current in the direction perpendicular to the radial direction as it is defined in the function

$$\bar{H}(r) = \frac{1}{r^2} - \frac{1}{(r + d)^2}.$$  

When the models are calculated, the radial distance is normalised by $R_\odot$ such that the distance in the radial direction varies from 1 to 2.5. Thus the parameter $d$ has also been normalised by $R_\odot$ (it ordinarily has units of metres) and it is always listed here using its normalised value. This would also be applied to $\alpha$ but it is not used here.

The values which are possible for $d$ must be restricted in order to avoid $\bar{H}(r)$ becoming infinite. $\bar{H}(r)$ becomes infinite when $r + d = 0$ in the range $1R_\odot \leq r \leq 2.5R_\odot$ for any value of $d$ satisfying $-2.5 \leq d \leq -1$. So these values are not considered as they would create an infinite current and therefore an infinite pressure and density in the domain.

First, we look at values of $d > -1$ before a short of comparison of these results with values of $d < -2.5$ in section 6.7.7.

### 6.7.1 Global MHS Models

Figure 6.2 shows the 3D models of the magnetic skeleton for four MHS fields with $d = -0.5$, $d = 0$ (PFSS), $d = 1$ and $d = 5$. What is immediately obvious is the large differences in location of the base of the heliospheric current sheet as $d$ is increased. Carrington rotation 2097 is on the rising phase, about 18 months after solar minimum. So, although the base of the heliospheric current sheet should not vary a large amount, according to the PFSS model, it will not be flat around the equator. In chapter 5, we showed that the HCS tilt determined using a PFSS model for this Carrington rotation was larger than is usual at this time in the solar cycle and this was thought to be due to the unusual start to solar cycle 24.

In the cases where $d = -0.5$ and $d = 0$ (the PFSS model), (figures 6.2a and 6.2b),
6.7. Comparison of the Global MHS fields with the PFSS Model Fields

(a) $d = -0.5$

(b) $d = 0$ (PFSS)
Figure 6.2
3D models of the magnetic skeleton of the MHS fields with varying values of \( d \). The null points are denoted by small red and blue spheres, the spines by the red and blue lines, the separators between null points by yellow lines, the separators between null points and the heliospheric current sheet by orange lines and the heliospheric sheet curtain rings by green lines. The thick green line is the base of the heliospheric current sheet. The separatrix surfaces from each null point are not drawn otherwise they obscure many of the other features plotted.
6.7. Comparison of the Global MHS fields with the PFSS Model Fields

Figure 6.3

The location of the base of the heliospheric current sheet for each value of $d$. The range of colours used in the colourbar here will be used throughout this chapter to show changes in $d$.

there is a reasonably significant tilt in the base of the heliospheric current sheet, however, it generally follows the equator. As $d$ is increased, the base of the heliospheric current sheet becomes much more warped in nature such that when $d = 1$, it starts to look like the sort of HCS that may be found at solar maximum in the PFSS model. The HCS for $d = 1$ has two components to it with the component that has broken off on the left of figure 6.2c. As $d$ increases still further to $d = 5$ (figure 6.2d), the HCS has split into many different components and its maximum tilt is now approximately $60^\circ$. The regions of closed magnetic field enclosed by the heliospheric current sheet curtains are now very thin as the HCS curtains drop almost radially from the base of the HCS. Indeed much of the global field is now completely radial as evidenced by the spine lines that appear to be essentially radially, straight lines. The extreme amount of current in the direction perpendicular to the radial direction when $d = 10$ has changed the global magnetic field significantly. The magnetic skeleton in the MHS field when $d = 10$ looks less physical than when $d$ is between $-1$ and $1$.

Figure 6.3 shows the locations of the base of the heliospheric current sheet on the outer boundary for 9 values of $d$ ranging from $-0.999$ up to 10 (ranging in colour
6. Global Magnetohydrostatic Field Model

### Table 6.1

<table>
<thead>
<tr>
<th>$d$</th>
<th>HCS Components</th>
<th>Open Field Regions</th>
<th>Spines</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Positive</td>
<td>Negative</td>
</tr>
<tr>
<td>−0.75</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>−0.5</td>
<td>1</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>−0.25</td>
<td>1</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>0.005</td>
<td>1</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>0.05</td>
<td>1</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>0.5</td>
<td>2</td>
<td>12</td>
<td>11</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>16</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>15</td>
<td>25</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>27</td>
<td>30</td>
</tr>
<tr>
<td>10</td>
<td>8</td>
<td>42</td>
<td>40</td>
</tr>
</tbody>
</table>

The numbers of different features on the top boundary.

from blue to green to red). This shows how much variance there is of the base of the HCS. The simplest HCS line follows a sinusoidal pattern and as discussed above is found for values of $d$ close to $−1$ (blue lines). This underlying pattern becomes distorted as $d$ increases. For $d \lesssim 0$, there is just a single warped HCS but for values of $d \geq 0.5$, there are at least two HCS components. Table 6.1 lists the number of components the HCS has split into. It can be seen that the number of HCS components increases as the current increases. When $d = 10$, there are as many as 8 components of the HCS starts as it breaks up into many different islands. This occurs because as $d$ increases, the closed magnetic field structures expand and eventually reach the top boundary and become open field regions for large enough values of $d$. As $d$ increases, the radial component of the field on the top boundary becomes more and more like the radial component of the field on the solar surface. The small scale field from the solar surface is increasingly replicated on the top boundary creating many more small islands in the HCS at the zero magnetic field isolines.

#### 6.7.2 Features on the Top Boundary

Table 6.1 also lists the number of open field regions on the top boundary and the number of spines which reach the top boundary. The increasingly radial magnetic field as $d$ increases can be seen to be causing many more spine lines to head to the
6.7. Comparison of the Global MHS fields with the PFSS Model Fields

The absolute flux varying with $r$ for each different MHS field as $d$ varies

Figure 6.4

The number of open field regions also increases with $d$ ranging from 2 open field regions when $d = -0.75$ to 82 when $d = 10$. These are formed by many more separatrix surfaces reaching the top boundary i.e. the existence of many separatrix curtains. Only 8 of these 82 are created through the break up of the base of the HCS.

### 6.7.3 Absolute Flux

Figure 6.4 shows how the absolute magnetic flux varies with height for each value of $d$. As $d \to \infty$, less and less absolute flux is lost with height. In fact, the radial component of the magnetic field on the source surface is almost identical to the radial component on the base for $d = 999$. This confirms the results seen earlier which showed that as $d$ increases, there is an increase in the amount of open magnetic field. Conversely as $d \to -1$, the flux is lost almost immediately with radial height. Here, what is happening is that there are many low lying closed magnetic field structures.

At the outer boundary, the absolute flux when $d = -0.999$ is approximately 1000 times smaller (of the order $10^{12}$ Wb or $10^{20}$ Mx) than when $d = 999$ where it is
6. Global Magnetohydrostatic Field Model

approximately $10^{15}$ Wb or $10^{23}$ Mx. Clearly these extreme values of $d$ both give physically unrealistic flux fall off and field line behaviour, but values of $d$ in the mid-range give magnetic fields with possibly realistic magnetic skeletons.

6.7.4 Cuts Through the Magnetic Skeleton

Figure 6.5 shows cuts through the magnetic skeleton at $r = 1.2 R_\odot$. What can be seen in the cuts is that generally the magnetic skeleton features are the same, except the field is being “lifted” upwards as $d$ increases, as mentioned above. The closed magnetic structures are effectively stretched upwards. When $d = 5$ (figure 6.5d), the cut through the magnetic skeleton looks very like a cut of the potential field close to the solar surface, in that it has a large number of separators and spines, plus very complex HCS and separatrix surface curves (see cuts through the magnetic skeleton in previous chapters). By comparing the strengths of the radial components of the magnetic field in each of the cuts, it can be seen that the field strength is very small for $d = -0.5$ in figure 6.5a while it is much larger when $d = 5$ in figure 6.5d. In fact, the radial component when $d = 5$ has clear evidence of active regions and is starting have a similar complexity to the solar surface magnetic field.

6.7.5 Abundance of Null Points and Separators

Table 6.2 lists the number of null points and separators for each different MHS magnetic skeleton. As $|d|$ increases from zero (the PFSS field), the number of null points increases. We have seen through the previous figures that the magnetic field is being stretched as $d$ increases. This stretching of the magnetic field probably raises more of the extremely low down null points. This enables them to be detected. It is also possible that there are genuinely more null points in the fields when $d$ is large.

However, in contrast to the increase in the number of null points with increasing $d$, the number of separators decreases. This is likely to be because even for a potential field many separators reach up high into the solar atmosphere. As $d$ increases, some of the separators which extend high up in the solar atmosphere are stretched such that they hit the outer boundary, at which point the connection between the end of the field lines is lost and hence some separators are lost. This may not explain the loss of all the separators however. The percentage of null points which are connected by a separator drops off slowly as $d$ is increased but the actual number of nulls
6.7. Comparison of the Global MHS fields with the PFSS Model Fields

- Negative Spines
- Positive Spines
- Negative Separatrix Surfaces
- Positive Separatrix Surfaces
- HCS Curtains
- Null-Null Separators
- Null-HCS Separators

![Graph (a) $d = -0.5$](image)

![Graph (b) $d = 0$ (PFSS)](image)
6. Global Magnetohydrostatic Field Model

Figure 6.5

A radial cut at \( r = 1.2R_\odot \) through the magnetic skeleton of the MHS fields for \( d = -0.5, d = 0, d = 1 \) and \( d = 10 \). The colourings match those for the radial cuts in chapter 4. The green dotted line is the location of the heliospheric current sheet, the red and blue lines are cuts through the separatrix surfaces from the positive and negative nulls respectively, the red and blue dots are the locations of spines from positive and negative nulls respectively, yellow stars are cuts through separators between nulls (although none are visible here on the top boundary) and orange stars are through separators between a null and the base of the heliospheric current sheet. The tones of red and blue are different for each null point with each null point given a random tone of the applicable colour. The legend at the top of the figure summarises these. The radial component of the magnetic field at \( r = 1.2R_\odot \) is also plotted in the background.
### Table 6.2

A table compiling features of the magnetic skeleton for each of the MHS fields with different values of $d$. The number of nulls, the number of null-null separators and the number of null-HCS separators together with the following two percentages: (1): The percentage of null-null separators compared to the number of nulls and (2): The percentage of nulls connected by null-null separators. It was not possible to determine the number of HCS-null separators for $d = 10$.

<table>
<thead>
<tr>
<th>$d$</th>
<th>Null Points</th>
<th>Null-Null Separators</th>
<th>Null-HCS Separators</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.75</td>
<td>1777</td>
<td>1454</td>
<td>82% 71%</td>
</tr>
<tr>
<td>-0.5</td>
<td>1617</td>
<td>1232</td>
<td>76% 67%</td>
</tr>
<tr>
<td>-0.25</td>
<td>1596</td>
<td>1239</td>
<td>78% 66%</td>
</tr>
<tr>
<td>0</td>
<td>1600</td>
<td>1240</td>
<td>78% 66%</td>
</tr>
<tr>
<td>0.005</td>
<td>1601</td>
<td>1241</td>
<td>78% 66%</td>
</tr>
<tr>
<td>0.05</td>
<td>1602</td>
<td>1246</td>
<td>78% 66%</td>
</tr>
<tr>
<td>0.5</td>
<td>1610</td>
<td>1211</td>
<td>75% 66%</td>
</tr>
<tr>
<td>1</td>
<td>1640</td>
<td>1146</td>
<td>70% 66%</td>
</tr>
<tr>
<td>2</td>
<td>1683</td>
<td>1157</td>
<td>69% 67%</td>
</tr>
<tr>
<td>5</td>
<td>1741</td>
<td>1067</td>
<td>61% 65%</td>
</tr>
<tr>
<td>10</td>
<td>1914</td>
<td>1147</td>
<td>57% 61%</td>
</tr>
</tbody>
</table>
connected remains reasonable constant. So it is possible that fewer multiply connected null points are formed and also it indicates that many of the new null points are not connected by separators. The number of null-HCS separators increases considerably as \( d \) increases. This is similar to solar maximum like the position of the base of the heliospheric current sheet.

Figure 6.6 shows the heights of each null point in order of their radial heights from highest to lowest as \( d \) is varied. It is clear from this plot that the null points are lifted up as \( d \) is increased. As \( d \) is increased from zero to \(-1\), the null points are restricted from being too high in the solar atmosphere but also restricted from being too close to the solar surface. Indeed, when \( d = -0.999 \), all the null points seemed to be pushed to a single radial height quite low down such that 97% of all null points are between 1 Mm and 4 Mm above the solar surface with 51% lying between the highest null point at 3.425 Mm and 3.3 Mm above the solar surface.

Figures 6.7a to 6.7c show the maximum heights of each null-null separator found in the MHS models for \( d = -0.5, d = 0.5 \) and \( d = 5 \) (blue) with the heights of the corresponding null points plotted directly below in red (similar to figure 5.13). In figure 6.7d, the maximum heights of the separators found in all the MHS magnetic
6.7. Comparison of the Global MHS fields with the PFSS Model Fields

(a) $d = -0.5$

(b) $d = 0.5$
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Figure 6.7

The maximum height reached by each separator together with the heights of the two connected null points plotted directly below in figures (a) to (c) and the maximum height reached by each separator for all values of $d$ in (d).
Comparison of the Global MHS fields with the PFSS Model Fields

Table 6.3

<table>
<thead>
<tr>
<th>$d$</th>
<th>Largest Network</th>
<th>Number of other networks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Null Points</td>
<td>Separators</td>
</tr>
<tr>
<td>−0.75</td>
<td>711</td>
<td>1081</td>
</tr>
<tr>
<td>−0.5</td>
<td>634</td>
<td>945</td>
</tr>
<tr>
<td>−0.25</td>
<td>646</td>
<td>963</td>
</tr>
<tr>
<td>0</td>
<td>594</td>
<td>913</td>
</tr>
<tr>
<td>0.005</td>
<td>594</td>
<td>912</td>
</tr>
<tr>
<td>0.05</td>
<td>594</td>
<td>913</td>
</tr>
<tr>
<td>0.5</td>
<td>490</td>
<td>765</td>
</tr>
<tr>
<td>1</td>
<td>455</td>
<td>650</td>
</tr>
<tr>
<td>2</td>
<td>175</td>
<td>244</td>
</tr>
<tr>
<td>5</td>
<td>99</td>
<td>118</td>
</tr>
<tr>
<td>10</td>
<td>60</td>
<td>71</td>
</tr>
</tbody>
</table>

The numbers of each type of separator network in the global magnetic fields for each $d$ and the number of isolated null points with the number of null points and separators in the largest network.

fields as $d$ varies are plotted. As $d$ increases, it can be seen how the maximum heights of the separators gets higher in general (figure 6.7d) although for $d$ between 0.5 and 2, there appears to be a slight difference in this trend. In figures 6.7a to 6.7c, the null points connecting the separators can also be seen to get higher as $d$ increases. When $d = −0.5$, the majority of null points are above 0.1 Mm whereas when $d = 5$, the majority of null points are now above 10 Mm.

6.7.6 Separator Networks

Figure 6.8 shows the biggest separator network (in blue) as well as the other separator networks (various colours) in the MHS fields for four example values of $d$ ($d = −0.75, 0, 2, 10$). The biggest separator network when $d = −0.75$ is very large and is clearly the dominant separator network involving 40% of all the null points in the global field. However, as $d$ increases, this large separator network decreases in size. This is probably due to higher separators in the separator network hitting the outer boundary as $d$ increases and ceasing to exist. This causes the biggest network to start
6. Global Magnetohydrostatic Field Model

- Biggest Network
- Cave
- Positive Dome/Tunnel
- Negative Dome/Tunnel
- Ring
- Chain Containing a Ring
- Three Chains Connected (Tri-Junction)
- More complex

(a) $d = -0.75$

(b) $d = 0$
6.7. Comparison of the Global MHS fields with the PFSS Model Fields

![Figure 6.8](image)

The angular locations of all networks found in the MHS fields. The biggest network is plotted in blue and the rest of the networks are plotted in the colours corresponding to their type given in the key above. The separators connecting to the HCS are not plotted here.
to break up into much smaller networks. When \( d = 2 \), the biggest separator network is no longer much bigger than all the other networks and involves just 10\% of the null points in the magnetic field. It is actually almost the same size as the next biggest complex network. In the \( d = 10 \) case, the largest separator network is now insignificant containing just 3\% of the null points.

Table 6.3 summarises the number of each separator network of a given type in the MHS magnetic skeletons for each \( d \). The size of the biggest network varying with \( d \) is clear from this table. The biggest network starts at its biggest with 711 null points and 1081 separators (almost 75\% of all separators) when \( d = -0.75 \) but only contains 60 null points and 71 separators (6\% of all separators) when \( d = 10 \). The increase in the number of other separator networks is likely a result of the break up of the biggest network, especially the number of complex separator networks. For \( d = 10 \) in figure 6.8, it can be seen that the biggest network is barely distinguishable from the other complex networks.

6.7.7 MHS fields for \( d < -2.5 \)

Now we look at the MHS fields created on the other side of the infinite asymptote of \( \bar{H}(r) \) i.e. values of \( d < -2.5 \). Due to limitations in the numerical scheme used to calculate the magnetic fields, it has not been possible to study MHS fields with \(-5 < d < -2.5\), but we have extrapolated MHS fields with \( d = -5 \) and \( d = -10 \). The MHS fields for \( d \leq -5 \) are similar to those fields with large positive \( d \) (i.e. \( d > 2 \)). In fact, the values of \( \bar{H}(r) \) are equal as \( d \to \pm\infty \). The only difference is the radial variance of \( \bar{H}(r) \) for large \( d \). It gives small differences in the magnetic fields created but the general trend is exactly the same and they still look quite unphysical. When \( d < -2.5 \), \( \bar{H}(r) \) may now be both positive and negative for \( R_\odot \leq r \leq 2.5R_\odot \). This can create a more complex pressure and Lorentz force in the domain, however this only occurs for \(-5 < d < -2.5\) which have not been calculated in this section. Here is a short discussion on the differences.

Table 6.4 shows the number of null points and separators in the MHS magnetic fields for \( d = -5 \) and \( d = -10 \), with the same quantities repeated for \( d = 1 \), \( d = 2 \), \( d = 5 \) and \( d = 10 \). It can be seen that the numbers, for the \( d = -5 \) and \( d = -10 \) cases, are similar to those for larger positive \( d \). The number of null points and the number of null-HCS separators increase as \(|d|\) increases, but the number of separators basically decreases for the same reasons as described earlier. Figure 6.9 shows the radial
6.7. Comparison of the Global MHS fields with the PFSS Model Fields

<table>
<thead>
<tr>
<th>$d$</th>
<th>Null Points</th>
<th>Null-Null Separators</th>
<th>Null-HCS Separators</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number</td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>−10</td>
<td>1708</td>
<td>1010</td>
<td>59 %</td>
</tr>
<tr>
<td>−5</td>
<td>1473</td>
<td>1020</td>
<td>69 %</td>
</tr>
<tr>
<td>1</td>
<td>1640</td>
<td>1146</td>
<td>70 %</td>
</tr>
<tr>
<td>2</td>
<td>1683</td>
<td>1157</td>
<td>69 %</td>
</tr>
<tr>
<td>5</td>
<td>1741</td>
<td>1067</td>
<td>61 %</td>
</tr>
<tr>
<td>10</td>
<td>1914</td>
<td>1147</td>
<td>57 %</td>
</tr>
</tbody>
</table>

Table 6.4

A table compiling quantitative details about some of the magnetic skeleton features for MHS fields with $d = −10$ and $d = −5$ and, for comparison, those with $d = 1$, $d = 2$, $d = 5$ and $d = 10$. The number of nulls, the number of separators together with the following two percentages: (1): The percentage of separators compared to the number of nulls and (2): The percentage of nulls connected by separators. The number of HCS-null separators for $d = 10$ has not been able to be determined.

Figure 6.9

The heights of all the null points ordered by their height for each value of $d$. 237
distribution of the null points for similar values of \(d\) to table 6.4. The distribution of null points for \(d < 2.5\) is similar to the large positive \(d\). However, there appears to be an early fall off in the height of the null points closer to the solar surface compared to the large positive \(d\) values which is probably due to the different variation in \(\bar{H}(r)\).

Figure 6.10 shows the absolute flux as it varies with radial height. The absolute flux for \(d \leq -5\) is similar to those for \(d \geq 2\) at the outer boundary but the fall off in height is initially more rapid for the \(d \geq 2\) cases. This is due to the difference in the radial variance of \(H(r)\).

Table 6.5 shows the number of different types of separator networks in the MHS fields for the same six cases as in table 6.4 and figure 6.11 shows the locations of these different networks for the \(d = -10\) field. The biggest separator networks in the \(d = -5\) and \(d = -10\) cases involve just 13\% and 5\% of the null points and only 27\% and 10\% of all the separators respectively. Thus as \(|d|\) increases, the largest separator network breaks just as for large positive values of \(d\) due the field being stretched upwards. The numbers of other networks start to increase when \(d = -10\) similarly to when \(d = 10\).
6.7. Comparison of the Global MHS fields with the PFSS Model Fields

<table>
<thead>
<tr>
<th>$d$</th>
<th>Largest Network</th>
<th>Number of other networks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Null Points</td>
<td>Separators</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Caves</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Domes/Tunnels</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Rings</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Tri-Junction with ring</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Complex</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Isolated Null Points</td>
</tr>
<tr>
<td>−10</td>
<td>82</td>
<td>105</td>
</tr>
<tr>
<td></td>
<td></td>
<td>94</td>
</tr>
<tr>
<td></td>
<td></td>
<td>67</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>21</td>
</tr>
<tr>
<td></td>
<td></td>
<td>583</td>
</tr>
<tr>
<td>−5</td>
<td>198</td>
<td>273</td>
</tr>
<tr>
<td></td>
<td></td>
<td>68</td>
</tr>
<tr>
<td></td>
<td></td>
<td>46</td>
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<td></td>
<td>2</td>
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<tr>
<td></td>
<td></td>
<td>9</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>11</td>
</tr>
<tr>
<td></td>
<td></td>
<td>494</td>
</tr>
<tr>
<td>2</td>
<td>175</td>
<td>244</td>
</tr>
<tr>
<td></td>
<td></td>
<td>92</td>
</tr>
<tr>
<td></td>
<td></td>
<td>59</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>7</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>17</td>
</tr>
<tr>
<td></td>
<td></td>
<td>551</td>
</tr>
<tr>
<td>5</td>
<td>99</td>
<td>118</td>
</tr>
<tr>
<td></td>
<td></td>
<td>96</td>
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<tr>
<td></td>
<td></td>
<td>57</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>24</td>
</tr>
<tr>
<td></td>
<td></td>
<td>592</td>
</tr>
<tr>
<td>10</td>
<td>60</td>
<td>71</td>
</tr>
<tr>
<td></td>
<td></td>
<td>101</td>
</tr>
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<td></td>
<td></td>
<td>70</td>
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<tr>
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<td></td>
<td>4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>13</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>31</td>
</tr>
<tr>
<td></td>
<td></td>
<td>752</td>
</tr>
</tbody>
</table>

Table 6.5
The numbers of each type of separator network in the global magnetic fields for each $d$ and the number of isolated null points with the number of null points and separators in the largest network.

Figure 6.11
The angular locations of the biggest separator network and the types of the other smaller networks. See figure 6.8 for the key.
6.7.8 Pressure and Plasma Beta in the MHS model

What the MHS models have that the PFSS models do not is information about the plasma in the global field. We can now associate a pressure, density and temperature with the global MHS magnetic fields. A plasma can be associated with the magnetic fields produced by PFSS models but there is no spatial dependence with the magnetic field.

First we look at how the pressure varies. As we are trying to model the coronal magnetic field which is believed to be approximately low beta, it would be ideal if we could find a pressure such that the plasma beta,

$$\beta = \frac{2\mu_0 p}{B \cdot B}$$

is low ($\beta \ll 1$) for the majority of the corona. Since $B$ is already set, we need a small $p$ to achieve this.

Recalling the form for pressure (equation 6.21),

$$p(r, \theta, \phi) = p_0(r) - \frac{\vec{H}(r)}{2\mu_0} (r \cdot B)^2$$

we would like to find a function for $p_0(r)$ which produces a low plasma beta. The other restriction is that $p(r, \theta, \phi)$ must be positive everywhere, otherwise the pressure would be unphysical. The form of $p_0(r)$ also affects the density so it must also be chosen such that the density is positive everywhere too.

The pressure may be minimised by selecting $p_0(r)$ to be just larger than the maximum of

$$\frac{\vec{H}(r)}{2\mu_0} (r \cdot B)^2,$$

at each radial height $r$. This ensures that $p(r, \theta, \phi) > 0$ while minimising $p(r, \theta, \phi)$.

Using this form for $p_0(r)$ defines a pressure everywhere in the domain.

Figure 6.12a shows a plot of how the mean $p(r, \theta, \phi)$ varies with height for the different MHS models. Clearly, the pressure for small $|d|$ (light blue and green curves) is about in the correct range for the corona ($10^{-1}$ Pa to $10^{-3}$ Pa) (Priest 2014). There is even a rapid fall off in the pressure close to the solar surface, although not quick enough to match the fall off which occurs in the photosphere and the chromosphere according to the VAL model (see figure 1.3). The pressure for very large $|d|$ are
Comparison of the Global MHS fields with the PFSS Model Fields

\begin{align*}
\text{d} = -10.0 & \quad \text{d} = -5.0 & \quad \text{d} = -0.999 & \quad \text{d} = -0.95 \\
d = -0.75 & \quad \text{d} = -0.5 & \quad \text{d} = 0.05 & \quad \text{d} = 0.0 \\
\text{d} = 0.005 & \quad \text{d} = 0.05 & \quad \text{d} = 1.0 & \quad \text{d} = 2.0 \\
\text{d} = 2.0 & \quad \text{d} = 5.0 & \quad \text{d} = 10.0 & \quad \text{d} = 100.0 \\
\text{d} = 100.0 & \quad \text{d} = 999.0
\end{align*}

Figure 6.12
The mean pressure and plasma beta at each radial height.
6. Global Magnetohydrostatic Field Model

The percentage of grid points with $\beta$ less than 0.1 and 1 and the median value of $\beta$ for each value of $d$ and the minimising background pressure function.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\beta &lt; 0.1$</th>
<th>$\beta &lt; 1$</th>
<th>Median</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td>0.05%</td>
<td>2.55%</td>
<td>354.57</td>
</tr>
<tr>
<td>-5</td>
<td>3.83%</td>
<td>15.52%</td>
<td>23.59</td>
</tr>
<tr>
<td>-0.999</td>
<td>8.83%</td>
<td>28.35%</td>
<td>2.00</td>
</tr>
<tr>
<td>-0.95</td>
<td>9.47%</td>
<td>30.68%</td>
<td>1.77</td>
</tr>
<tr>
<td>-0.75</td>
<td>11.45%</td>
<td>41.28%</td>
<td>1.11</td>
</tr>
<tr>
<td>-0.5</td>
<td>14.00%</td>
<td>84.35%</td>
<td>0.59</td>
</tr>
<tr>
<td>-0.25</td>
<td>20.55%</td>
<td>100.00%</td>
<td>0.25</td>
</tr>
<tr>
<td>0</td>
<td>100.00%</td>
<td>100.00%</td>
<td>0.00</td>
</tr>
<tr>
<td>0.005</td>
<td>61.61%</td>
<td>89.75%</td>
<td>0.05</td>
</tr>
<tr>
<td>0.05</td>
<td>26.49%</td>
<td>61.07%</td>
<td>0.51</td>
</tr>
<tr>
<td>0.5</td>
<td>5.13%</td>
<td>26.37%</td>
<td>6.16</td>
</tr>
<tr>
<td>1</td>
<td>3.35%</td>
<td>18.15%</td>
<td>13.43</td>
</tr>
<tr>
<td>2</td>
<td>1.95%</td>
<td>10.74%</td>
<td>30.21</td>
</tr>
<tr>
<td>5</td>
<td>0.27%</td>
<td>4.30%</td>
<td>133.03</td>
</tr>
<tr>
<td>10</td>
<td>0.01%</td>
<td>0.13%</td>
<td>1022.30</td>
</tr>
<tr>
<td>100</td>
<td>0.00%</td>
<td>0.00%</td>
<td>98319.18</td>
</tr>
<tr>
<td>999</td>
<td>0.00%</td>
<td>0.00%</td>
<td>191657.30</td>
</tr>
</tbody>
</table>

Table 6.6

It is now possible to use our $p(r, \theta, \phi)$ and $B(r, \theta, \phi)$ from the model to calculate the plasma beta everywhere in the domain.

On the outer boundary, the plasma beta becomes very large around the location of the base of the heliospheric current sheet where the magnetic field is zero, as seen in figure 6.13. These values ($> 100$) dominate over much of the rest of the domain (figure 6.12b) and makes finding the mean value of $\beta$ in the whole domain a poor measure of the overall physical validity of the MHS fields. So the median is used instead. Gary (2001) showed that in the high corona the plasma pressure starts to dominate over the magnetic pressure due to the diminishing magnetic field strength giving rise to a $\beta > 1$ high above the solar surface. They also showed that it is possible to have regions of $\beta \sim 1$ at relatively low coronal heights ($r \approx 1.2R_\odot$) too.

Table 6.6 shows the percentage of all the grid points which have a plasma beta less

completely incorrect.
We have studied MHS models with varying $d$ but $\alpha = 0$. For large values of $|d|$, the magnetic fields and their skeletons look quite unphysical as the magnetic field lines become more and more radially oriented. The large tilt angles of the HCS and its break up into multiple components as $|d|$ increases, and the large plasma betas are much greater than 0.1 and 1 and the median $\beta$ for all the grid points. For large $|d|$, clearly it is not possible to have a low $\beta$ magnetic field with the majority of the grid points having a $\beta > 1$. It is only for small $|d|$, that a low beta at the majority of grid points is actually obtainable. From looking at both the pressure and plasma beta obtainable and the magnetic skeletons from each MHS field, clearly MHS fields with high $|d|$ are completely unphysical for the solar corona. Figure 6.12b shows the mean value of the plasma beta varying with the radial height. For values for small $|d|$ (green and light blue curves), the plasma beta is less than one for the majority of the radial heights. Clearly this is not the case at the outer boundary as discussed. Also, like the pressure, there is a rapid fall off close to the solar surface for small values of $|d|$. The plasma beta for large $d$ are much greater than 1 for the whole domain.

The MHS magnetic field when $d = 0$ (i.e. the PFSS field) is completely decoupled from the pressure and therefore can be made as small as required. It is no surprise that this minimising pressure causes a $\beta = 0$.

### 6.8 Conclusions

We have studied MHS models with varying $d$ but $\alpha = 0$. For large values of $|d|$, the magnetic fields and their skeletons look quite unphysical as the magnetic field lines become more and more radially oriented. The large tilt angles of the HCS and its break up into multiple components as $|d|$ increases, and the large plasma betas are much greater than 0.1 and 1 and the median $\beta$ for all the grid points. For large $|d|$, clearly it is not possible to have a low $\beta$ magnetic field with the majority of the grid points having a $\beta > 1$. It is only for small $|d|$, that a low beta at the majority of grid points is actually obtainable. From looking at both the pressure and plasma beta obtainable and the magnetic skeletons from each MHS field, clearly MHS fields with high $|d|$ are completely unphysical for the solar corona. Figure 6.12b shows the mean value of the plasma beta varying with the radial height. For values for small $|d|$ (green and light blue curves), the plasma beta is less than one for the majority of the radial heights. Clearly this is not the case at the outer boundary as discussed. Also, like the pressure, there is a rapid fall off close to the solar surface for small values of $|d|$. The plasma beta for large $d$ are much greater than 1 for the whole domain.

The MHS magnetic field when $d = 0$ (i.e. the PFSS field) is completely decoupled from the pressure and therefore can be made as small as required. It is no surprise that this minimising pressure causes a $\beta = 0$.
6. *Global Magnetohydrostatic Field Model*

completely atypical of the solar corona. However, the MHS models involving small
values of $|d|$ may have small plasma beta values and their magnetic fields could be
better for modelling the global coronal magnetic field compared with the magnetic
fields created by the PFSS model. Although, the magnetic fields produced are very
fairly similar. It is important though that $d$ remains not too close to $-1$ (and
presumably $-2.5$) since in these models, all the magnetic field complexity is limited
to a very narrow region of the atmosphere.

It is unfortunate that we have not been able to do a parameter study in which MHS
models with a varying $\alpha$ are studied. These MHS models would have a better
comparison with global linear force-free models where there is only a component of
current parallel to the magnetic field in the same way as $\alpha$ here.
Chapter 7
Magnetic Reconnection at Spiral Null Points

Much of this thesis considers global solar magnetic fields extrapolated from observations and investigates the nature of their magnetic skeletons. As has been shown, thousands of null points exist at any one time in the solar corona. Here we discuss the dynamics local to a single null point instead. This work continues from Fuentes-Fernández and Parnell (2012) where they look at the dynamical relaxation of a spiral null point to form a current accumulation. This relaxation of the magnetic field under non-resistive MHD creates an enhanced region of electric current density local to the spines of the null point and the whole system reaches an equilibrium. A spiral null point embedded in an MHS equilibrium is the starting point for the experiments investigated here to study the reconnection at these null points. They also look at the relaxation of tilted null points as in Fuentes-Fernández and Parnell (2013).

Magnetic null points have been associated with magnetic reconnection ever since the process was first conceived (Priest and Forbes 2000). In two dimensions, reconnection can only occur at X-type null points but in three dimensions, it is not so restricted (Priest, Hornig and Pontin 2003). However, null points are still key sites for both current accumulation (Syrovatski 1971; Fuentes-Fernández and Parnell 2012; Fuentes-Fernández and Parnell 2013) and resistive energy release Priest and Pontin (2009) and so are still important in the study of reconnection. The energy release during magnetic reconnection is important for solar flares (Aulanier et al. 2000) and many energetic events on the sun and with these, there are associated null points. Galsgaard et al. (2017) constructed the magnetic topology of coronal bright points and found that null points were associated with 9 out of 10 of these energetic events. Masson et al. (2017) found that null points are in the magnetic topology of the ribbon flares they studied.
Magnetic reconnection has been studied both analytically and numerically at null points and it has been found that there are three different reconnection regimes that can occur. These depend on the flows and the boundary disturbances (Priest and Pontin 2009). The three possibilities are: torsional spine reconnection, torsional fan reconnection and fan-spine reconnection. The difference between torsional spine and torsional fan reconnection is the direction the field lines are twisted above and below the fan. This difference in the twisting of the field lines causes a current to localise around different parts of the null point. The field lines twist in the same direction above and below the fan plane for torsional spine reconnection and this twist causes a current to localise around the spine line. Oppositely, when the field lines twist in opposite directions above and below the fan plane for torsional fan reconnection, the current instead localises around the fan plane due to the anti-parallel field. Spine-fan reconnection occurs when there is a shearing of the magnetic field at the null point and the current is localised in the weak field region in the vicinity of the null point.

Here in our experiments we will look at a case of torsional spine reconnection. We start with a 3D null point with a field which is not in equilibrium and has a uniform current throughout the domain. During the formation of the MHS equilibrium, the rotation (or un-rotation in our case) of the fan field lines leading to a rotation of the field lines about the spine localises the current about the spine. Within these regions of strong localised current, it is assumed that micro-instabilities may occur giving rise to an enhanced value of resistivity, \( \eta \), which, coupled with small length scales in the strong current region give rise to a small magnetic Reynolds number \( R_m \leq 1 \) meaning diffusion becomes as important as advection, permitting reconnection to occur. The reconnection causes slippage of the field lines through the plasma dissipating the current density. The spine and the fan of the null point remain perpendicular because no component of current is produced perpendicular to the spine.

Most studies of 3D null point reconnection use a forcing to build up current about the null point in specific locations dependent on the nature of the forcing. Reconnection can then occur if the current is large enough and/or is located in an appropriate region in which the resistivity is enhanced. However we wanted a more spontaneous reconnection that may occur naturally in domains of high current density. Parnell et al. (1997) showed that the local linear field of a non-potential three dimensional null point has a Lorentz force than cannot be balanced by a plasma pressure force. So a linearly defined, non-potential null point in equilibrium does not exist and if the null point is perturbed and left to relax, it must evolve to a locally
7.1. **LARE3D**

potential null or will develop a current singularity. In these experiments, we relax a spiral null point to an MHS equilibrium and allow the electric current density build up to form a genuine current layer as was done in Fuentes-Fernández and Parnell (2012). We continue on their work done to look at the reconnection that occurs in such a current layer. The MHS equilibrium allows the waves caused by the reconnection to be studied with no interference from other waves which may be present in the domain if the fluid was not static.

First, the code used to do the numerical MHD experiments (relaxation and resistive) is discussed (section 7.1) and then I go on to describe the relaxation (section 7.3) and reconnection (section 7.4) experiments, with the results from the experiments with a range of different parameters.

### 7.1 LARE3D

LARE is a LAgrangian-REmap code that solves the full normalised resistive magnetohydrodynamical (MHD) equations written by Arber et al. (2001). The version used for the work here is 3.0.3. These equations solved by LARE, given in normalised Lagrangian form, are given by

\[
\rho \frac{Dv}{Dt} = J \times B - \nabla p + \rho g + F_v,
\]

\[
\rho \frac{De}{Dt} = -p \nabla \cdot v + \eta J \cdot J + H_v,
\]

\[
\frac{DB}{Dt} = (B \cdot \nabla) v - (\nabla \cdot v) B - \nabla \times (\eta \nabla \times B),
\]

\[
J = \nabla \times B,
\]

\[
p = \rho c (\gamma - 1),
\]

\[
\nabla \cdot B = 0,
\]

where these quantities have been defined in section 1.5 except for \( c \) which is the internal energy. The internal energy is proportional to temperature using the ideal gas law. Note that the continuity of mass equation from the MHD equations is not actually used by LARE but mass is conserved via a combination of the Lagrangian and remap processes (see section 7.1.2 for further discussion).

The variables and parameters in the usual MHD equations (here denoted by a hat)
have been normalised by constants (denoted by a subscript 0) for the magnetic field \( B_0 \), density \( \rho_0 \) and lengths \( L_0 \) so are related to the original variables by

\[
\hat{B} = B_0 B, \\
\hat{\rho} = \rho_0 \rho, \\
\hat{r} = L_0 r.
\]

From these, the normalising constants for the pressure \( p_0 \), internal energy \( \varepsilon_0 \), velocity \( v_0 \) and magnetic diffusivity \( \eta_0 \) are defined as

\[
p_0 = \frac{B_0^2}{\mu_0} \\
\varepsilon_0 = \frac{B_0^2}{\mu_0 \rho_0} \\
v_0 = \sqrt{\varepsilon_0} \\
\eta_0 = \mu_0 v_0 L_0.
\]

The plasma beta also can also be calculated and equals

\[
\beta = \frac{2p}{B \cdot B} = \frac{2\mu_0 \hat{\rho}}{\hat{B} \cdot \hat{B}}.
\]

This normalisation means that \( \mu_0 = 1 \) in the LARE code and hence does not appear in Ampère’s law or the plasma beta.

There are two different versions of the code, one in 2D and the other 3D and the codes are second-order accurate in both space and time. LARE is parallelised via MPI and scales linearly up to 1000 cores on a HPC cluster. There are also many options that may be turned on and off such as the Hall term (only in 2D) and the type of ionisation of the plasma. Thermal conduction and radiation are also available in LARE but much improved algorithms are currently in development (Johnston et al. 2017).

LARE is named as such because of its dynamical update scheme. At each time step, there is a Lagrangian step and then a remap step. In the Lagrangian step, the grid is deformed as each grid cell is advected with the plasma, then in the remap step this deformed grid is remapped back to its original position. All the physics is contained in the Lagrangian step while the remap step is purely for geometrical purposes. Any additional physics may be easily added into LARE by just editing the Lagrangian...
The arrangement of scalars and vector components in each grid cell in LARE3D.

step. LARE’s grid uses a staggered Cartesian grid where each of the scalars and vectors are defined at different points on the grid. This staggering in the grid prevents the checkerboard instability and builds conservation laws into the finite difference scheme used to solve the MHD equations.

7.1.1 The Staggered LARE grid

LARE’s computational grid is staggered such that the scalar quantities (pressure, internal energy and density) are all defined at the centres of each grid cell. The velocity components are defined on the vertices of the grid cell and the magnetic field components are defined at each of the cell faces as illustrated in figure 7.1.

This arrangement of the different physical quantities helps to reduce errors in codes. Taking the finite difference derivative of the magnetic field components means that the location of all three derivatives required to compute $\nabla \cdot \mathbf{B}$ are at the centre of the cell. If the locations of the three magnetic field components were at the same point, these three derivatives would be in different locations and would need to be interpolated introducing extra error. Thus one way LARE minimises errors is through the staggering of its grid and so it can better maintain $\nabla \cdot \mathbf{B} = 0$. 

Figure 7.1

The arrangement of scalars and vector components in each grid cell in LARE3D.
7. Magnetic Reconnection at Spiral Null Points

7.1.2 The Lagrangian and Remap Step

LARE’s two step algorithm contains a Lagrangian and a remap step. The Lagrangian step advects the grid with the plasma flow which ensures conservation of mass. The mass in each grid cell will be the same at the start and the end of the Lagrangian step, but the size of the grid cell may have changed under the deformation. Hence, the continuity of mass equation is not needed in LARE. The remap step is simply a remap of the grid back to its original position. The overlap of each of the deformed cells with the original grid is calculated so the density can be mapped back to the original grid in an accurate way. LARE uses the Evans and Hawley’s constrained transport method (Evans and Hawley 1988) in the remap step to try and ensure that $\nabla \cdot B = 0$.

The Lagrangian step is where LARE calculates all physics of the system. The core solver is written assuming ideal MHD (no resistive terms including resistivity and viscosity) and additional modules are called when optional terms such as resistivity are required so any new physics can also be added easily. LARE uses a predictor-corrector scheme and it is second-order accurate in time and space.

7.1.3 Viscous Forces

In the equation of motion, $F_\nu$ is the viscous force on the plasma with its associated viscous heating $H_\nu$ in the energy equation. These are calculated using the stress tensor $\sigma_{i,j}$ and strain rate tensor $\epsilon_{i,j}$ given by

$$\sigma_{i,j} = 2\nu_r \left( \epsilon_{i,j} - \frac{1}{3} \delta_{i,j} \nabla \cdot \mathbf{v} \right),$$

$$\epsilon_{i,j} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

These give each component of the viscous force and viscous heating as

$$(F_\nu)_i = \frac{\partial \sigma_{i,j}}{\partial x_j},$$

$$H_\nu = \sum_{i,j} \sigma_{i,j} \epsilon_{i,j},$$

where $\nu_r$ is defined as the real viscosity. This may be converted to the kinematic viscosity which is defined as $\nu = \nu_r/\rho$. The variable parameter in LARE named
visc3 is the normalised value of \( \nu_r \), given by

\[
\hat{\nu}_r = \rho_0 L_0 v_0 \nu_r.
\]

There are also two artificial shock viscosities in LARE to deal with shocks which are named \texttt{visc1} and \texttt{visc2}. In the experiments presented in this chapter, \texttt{visc1} and \texttt{visc2} are both set to zero.

### 7.1.4 Resistive terms

The resistive terms involving \( \eta \) in the equations of MHD are calculated in LARE separately from the main code as they can be turned on and off. Resistivity does not necessarily have to be uniform over the whole grid. It is in fact calculated as a function of the current density so it is turned on for points where the magnitude of the current is higher than a certain value. This allows for reconnection only in specific regions of the domain. A uniform background resistivity can also be set. More complex spatial variations in the resistivity can be defined but they are not available by default.

When resistive diffusion is calculated in the Lagrangian step magnetic flux is no longer conserved. So we cannot simply remap the magnetic flux back after the Lagrangian step. Instead, it must be calculated explicitly. This is done by adding the resistive term into the predictor step. This also adds ohmic heating which can then be calculated and corrected in the energy update.

### 7.1.5 Stability

Given, as usual, we are conducting numerical experiments on a finite discretised grid using a finite difference scheme which is never exact, we must impose a restriction on the time step size used by the code which will depend on the spatial step size to try to bring stability to the numerical scheme. This is done using the CFL condition (Courant, Friedrichs and Lewy 1928).

The CFL condition can be derived using von Neumann stability analysis. Looking at a 1D diffusion equation

\[
\frac{\partial Q}{\partial t} = k \frac{\partial^2 Q}{\partial x^2},
\]
the CFL condition becomes

\[ \delta_t \leq \frac{(\delta_s)^2}{k} , \]

where \( \delta_t \) is the numerical time step and \( \delta_s \) is the smallest spatial grid spacing. This can be thought of ensuring that the numerical diffusion is not bigger than the physical diffusion

\[ k \leq \frac{(\delta_s)^2}{\delta_t} . \]

Our initial experiments studying reconnection at 3D null points failed due to large gradients developing at the domain boundaries. It was discovered this was due to the viscosity and that the time step calculated by LARE was not restrictive enough to ensure stability of the experiments. As standard, LARE restricts the time step by

\[ \delta_t \leq \min \left( \frac{\delta_s}{c_s^2 + c_A^2 + v_s^2}, \frac{1}{5} \frac{(\delta_s)^2}{\eta} \right) \]

where

\[ (\delta_s)^2 = \frac{1}{\frac{1}{(\delta_s)^2} + \frac{1}{(\delta_y)^2} + \frac{1}{(\delta_z)^2}} , \]

\( \eta \) is the magnetic diffusivity, \( c_s \) is the sound speed, \( c_A \) is the Alfvén speed and \( v_s \) is a velocity associated with shocks. The factor of 1/5 is simply chosen to ensure stability of the code.

To prevent this issue with the viscosity from happening, a new extra restriction on the time step has been implemented for our experiments based on the value of the viscosity and it is of the same form as the diffusive time step. We have simply added an additional condition based on the viscosity, \( \nu \), of the same form as the resistive time step such that the time step must now satisfy

\[ \delta_t \leq \min \left( \frac{\delta_s}{c_s^2 + c_A^2 + v_s^2}, \frac{1}{5} \frac{(\delta_s)^2}{\eta}, \frac{1}{5} \frac{(\delta_s)^2}{\nu} \right) \]

and when \( \eta = 0 \) or \( \nu = 0 \), LARE does not use the respective individual conditions on the time step. As far as I am aware, this viscosity time step restriction has now been fully implemented into newer versions of LARE because of this.
Spiral null points only have a component of current, $J$, parallel to the spine line and in the initial condition, the current density is uniform throughout the domain so the local field lines spiral throughout the domain. This is the starting point for the numerical relaxation experiments here and is the same as that used by Fuentes-Fernández and Parnell (2012). The initial local field is linear and is given by

$$B(r) = \begin{pmatrix} 1 & -\frac{J_0}{2} & 0 \\ \frac{J_0}{2} & b & 0 \\ 0 & 0 & -(b + 1) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - \frac{J_0}{2}y \\ \frac{J_0}{2}x + by \\ -(b + 1)z \end{pmatrix}.$$ 

This is the same as the form given in section 1.6 with $J_\perp = 0$ and $a = 0$ with the null point located at the origin. The spine line of the null point runs along the $z$-axis and the fan plane lies entirely in the $z = 0$ plane. The magnetic field has a uniform electric current everywhere of $J(r, 0) = J_0 \hat{k}$. This magnetic field is placed onto a uniformly distributed numerical grid of size $512 \times 512 \times 512$ within the box with dimensions $-1 \leq x \leq 1, -1 \leq y \leq 1$ and $-1 \leq z \leq 1$. The null point and the field lines about the null point for $b = 1$ and $J_0 = 1$, including the field lines representing the spine line and fan plane are illustrated in figure 7.2a. When $b = 1$, the field lines about the spine line are rotationally symmetric. The two eigenvalues in the fan plane are equal. This symmetry is lost for $b \neq 1$. In the numerical experiments, different values of $b$ and $J_0$ will be picked in order to determine the effect of varying the initial condition on the formation of the MHS equilibrium and the subsequent reconnection.

The density, internal energy and velocities are then initially set to be

$$\rho(r, 0) = 1$$

$$\varepsilon(r, 0) = \frac{3}{2}$$

$$v(r, 0) = 0$$

which means that there is an initial uniform pressure of

$$p(r, 0) = 1$$

using the ideal gas law. The average plasma beta is 2.00 in the case when $J_0 = 1$ and
Configuration of the initial magnetic field in (a) and at MHS equilibrium in (b) for \( b = 1 \) and \( J_0 = 1 \). The spine line is coloured blue, the field lines in the fan plane are green and the red sphere indicates the location of the null point. Example field lines above the fan are orange and below are purple.
7.2. Numerical Experiment Setup

$b = 1$. An isosurface of $\beta = 1$ in the domain is an ellipsoid with a constant radius of 1.26 in the $z = 0$ plane (where it is circular) and reaching $\pm 0.707$ in the $z$-direction so we have $\beta < 1$ for more extreme values of $z$ and the corners of the domain but not locally to the null point where the magnetic field is very small.

Gravity is set to zero throughout these experiments. Since we are investigating only the local region about a null point, the length scale is much much less than the pressure scale height and so gravity is neglected in the momentum equation.

The viscosity ($\text{visc3}$) is set to be $\nu = 0.005$ for both non-resistive and resistive experiments while $\eta$ is changed. For the relaxation experiment, $\eta = 0$ while for the reconnection experiments, $\eta$ is non-zero and spatially dependent (see section 7.4 for further details).

The boundary conditions of the domain are set in such way that the velocity is zero everywhere on the boundaries (meaning the field is line-tied at the boundary during the non-resistive MHD experiments) and the magnetic field and all scalars have local maxima or minima on the boundaries.

Any times throughout this chapter are normalised by the shortest time for the fast wave mode to travel from the null at the centre of the domain along any one of the axes to the boundary in the MHS equilibrium field formed. This is calculated by

$$t_f = \int_C \frac{ds}{c_f(s)}$$

where $c_f(s)$ is the local magnetoacoustic fast mode speed along at a distance $s$ from the null point along any one of the axis lines, $C$. In the case when $b = 1$ and $J_0 = 1$, $t_f = 0.3679$. The values of $t_f$ for each different relaxation experiment are listed in table 7.1.

First these non-force-free spiral null points are relaxed via non-resistive MHD using LARE to create a magnetohydrostatic equilibrium involving an enhanced current layer (section 7.3). The current layers created via this type of relaxation are studied by Fuentes-Fernández and Parnell (2012). These MHS equilibria are then used as the initial states of the reconnection experiments (section 7.4).
7. Magnetic Reconnection at Spiral Null Points

<table>
<thead>
<tr>
<th>(b)</th>
<th>(J_0)</th>
<th>(t_f)</th>
</tr>
</thead>
<tbody>
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<td>0.4362</td>
</tr>
<tr>
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<td>0.3308</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.3679</td>
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<tr>
<td>2</td>
<td>1</td>
<td>0.2870</td>
</tr>
</tbody>
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Table 7.1
The different values of \(b\) and \(J_0\) used for the initial magnetic fields in the relaxation phase together with the normalisation time for each experiment at equilibrium.

7.3 Relaxation Phase

We start the experiments by relaxing the local field about the non-force-free null points to create an equilibrium for different values of \(b\) and \(J_0\). A number of different null point configurations for the relaxation experiments are studied and they are listed in table 7.1. They all start with exactly the same set up as discussed earlier except for their varying values of \(b\) and \(J_0\) which produce different initial magnetic fields.

The initial state of the magnetic field is not in equilibrium. The initial uniform pressure means that \(\nabla p = 0\), but the Lorentz force is non-zero. The initial Lorentz force is calculated to be

\[
\mathbf{J} \times \mathbf{B} = -J_0 \left( \frac{J_0}{2} x + by, \frac{J_0}{2} y - x, 0 \right)
\]

The magnetic tension force acts to straighten out the field lines

\[
(\mathbf{B} \cdot \nabla) \mathbf{B} = \left( \left( 1 - \frac{J_0^2}{4} \right) x - \frac{J_0}{2} (b + 1) y, \right.
\]

\[
\left. \frac{J_0}{2} (b + 1) x + \left( b^2 - \frac{J_0^2}{4} \right) y, (b + 1)^2 z \right)
\]
7.3. Relaxation Phase

while the magnetic pressure force acts to spread the field lines out

\[- \nabla \left( \frac{B \cdot B}{2} \right) = - \left( 1 + \frac{J_0^2}{4} \right) x + \frac{J_0}{2} (b - 1) y,\]

\[\frac{J_0}{2} (b - 1) x + \left( b^2 + \frac{J_0^2}{4} \right) y, (b + 1)^2 z.\]

There is no \( z \) component of the Lorentz force and so the Lorentz force acts to untwist the spiralling field lines in the fan plane under non-resistive MHD.

The dynamics of the relaxation are now discussed for the case when \( b = 1 \) and \( J_0 = 1 \). Figure 7.3a and figure 7.3b shows the field lines in the fan plane in the initial magnetic field and at the MHS equilibrium. The Lorentz force acts to make the field more potential in the fan plane i.e. the field lines there try to become more radial. At equilibrium in the fan plane, the field cannot become completely radial since the field lines are line tied at the boundary. So out to a radius of just less than 0.5, the field lines are basically radial and they bend towards their original position on the boundary. The new radial nature of the field lines in the fan plane can be seen in figure 7.3b.

This untwisting of the field in the fan plane leads to a twisting of the field above and below the fan plane about the spines as can be seen in figures 7.3c and 7.3d which shows a projection of the field onto the \( y = 0 \) plane. This transfer of twist is due to the conservation of helicity. The new localised field line twist leads to larger currents above and below the fan plane than the initial uniform \( J_0 \) current. The nature of these currents at equilibrium can be seen later in figure 7.5g.

Figure 7.4 shows the energies as they evolve during the relaxation experiment. During the relaxation, the total energy is conserved changing by a maximum 0.0002% of its initial value. The magnetic energy is converted into internal energy via kinetic energy. This process is dominated by viscous heating with only a small contribution from adiabatic heating. As the Lorentz force moves the magnetic field and the plasma, this converts magnetic energy into kinetic energy. This kinetic energy is then dissipated by the viscous forces and converted into internal energy.

The magnetic energy does not monotonically decrease over the relaxation experiment (dark blue line in figure 7.4). As the field relaxes, the field lines move under the Lorentz force towards their equilibrium location. However, they overshoot which causes a rotation in the opposite direction in the fan plane. This causes the field lines
7. Magnetic Reconnection at Spiral Null Points

Figure 7.3
The magnitude of the forces $|J \times B - \nabla p|$ initially in (a) and (c) and at equilibrium in (b) and (d). (a) and (b) show the fan plane $z = 0$ and (c) and (d) are the plane $y = 0$. The field lines are overplotted onto the fan plane ($z = 0$) in black and the 3D projected field lines are overplotted onto the $y = 0$ plane in the same colours as figure 7.2.
to undergo a damped oscillation about their final equilibrium position and hence the magnetic energy to oscillate. No ohmic heating occurs as expected since \( \eta = 0 \).

7.3.1 Verifying the MHS Equilibrium

Once the system has evolved under non-resistive MHD from the initial state, it reaches a state where the velocities are very small and thus the system is in (or near) equilibrium. The results here are for the \( b = 1 \) and \( J_0 = 1 \) case. The other cases are very similar.

We check that an MHS equilibrium has been achieved by examining the forces in the system (figure 7.3) and confirming that the velocity in the domain is (essentially) zero, which it is since the kinetic energy drops by almost 10 orders of magnitude from its maximum over the relaxation period.

Given that at equilibrium \( \text{D}v/\text{D}t = 0 \) and we have \( \mathbf{F}_v = 0 \) since \( v = 0 \), we check that the total forces in the system \( |\mathbf{J} \times \mathbf{B} - \nabla p| \) are zero. In figures 7.3b and 7.3d, we see that at equilibrium there is no overall force except very close to the corners and edges of the domain. This is again due to the boundary conditions which prevent the field lines from completely relaxing and straightening. Fuentes-Fernández and Parnell (2012) confirmed the existence of their equilibria everywhere by also calculating \( \mathbf{B} \cdot \nabla p \). We also confirm that \( \mathbf{B} \cdot \nabla p = 0 \) everywhere except along the larger currents (although still very small compared to the maximum current at equilibrium) in the fan plane in figure 7.5a similarly to Fuentes-Fernández and Parnell (2012).

7.4 Reconnection Phase

Now that MHS equilibria have been created, these seven different 3D null point MHS equilibria are used as the initial conditions for different reconnection experiments. The current and the pressure for \( b = 1 \) and \( J_0 = 1 \) are shown in subfigures a, d and g of figures 7.5 and 7.6 respectively. Both the current and the pressure are very weak over the whole fan plane (figures 7.5a and 7.6a). In figure 7.5d and 7.5g, the current accumulation about the spine lines can be seen. It exists everywhere along the spines away from the null point in figure 7.5g. The current is strongest along the spine lines themselves, increasing in strength away from the null points. A very small current
The evolutions of all the energies and dissipation terms over time. Each energy (except kinetic energy) has been tripled and shifted down so that it can be shown on the same plot and show energy conservation over the total relaxation period. The initial magnetic energy is shifted to 0.9 and the initial internal energy is shifted to the final value of the magnetic energy. The heating values are then shifted so the viscous heating starts at the initial value of internal energy and the adiabatic heating starts at the same value as the final viscous heating. The ohmic heating is not shifted since it is zero throughout the relaxation experiment. This means that the magnetic energy values have been shifted by $-25.10$, the internal energy values have been shifted by $-35.45$, the viscous heating values have been shifted by 0.55 and the adiabatic heating values have been shifted by 0.894 after the tripling. The horizontal dotted lines mark the initial internal energy/initial viscous heating/final magnetic energy value, the initial adiabatic heating/final viscous heating value and the initial magnetic energy/final internal energy/final adiabatic heating value. The total energy in the system has been normalised by its initial value.
7.4. Reconnection Phase

![Diagram showing evolution of $|\mathbf{J}|$ for different planar cuts for $b = 1$ and $J_0 = 1$.](image)

(a)* (b) (c)

(d) (e) (f)

(g) (h) (i)

MHS Equilibrium

$|\mathbf{J}(r,0)|$

$t = 1.359 t_f$

$t = 5.436 t_f$

Evolution of $|\mathbf{J}|$ for different planar cuts for $b = 1$ and $J_0 = 1$: (a), (b) and (c) show $z = 0$, (d), (e) and (f) show $z = 0.8$ and (g), (h) and (i) show $y = 0$. In the first column, the magnitude of the initial MHS equilibrium current ($|\mathbf{J}(r,0)|$) is plotted. In the second two columns, the perturbed current ($|\mathbf{J}(r,t)| - |\mathbf{J}(r,0)|$) at $t = 1.359 t_f$ in (b), (e) and (h) and $t = 5.436 t_f$ in (c), (f) and (i) is plotted.

*The colour bar for (a) ranges over $[0, 1]$ instead of over $[0, 10]$ which is the range for (d) and (g).
Evolution of $p$ for different planar cuts for $b = 1$ and $J_0 = 1$: (a), (b) and (c) show $z = 0$, (d), (e) and (f) show $z = 0.8$ and (g), (h) and (i) show $y = 0$. In the first column, the initial MHS equilibrium pressure ($p(r,0)$) is plotted. In the second two columns, the perturbed pressure ($p(r,t) - p(r,0)$) at $t = 1.359t_f$ in (b), (e) and (h) and $t = 5.436t_f$ in (c), (f) and (i) is plotted. (Same as figure 7.5 but for pressure)

*The colourbar for (a) ranges over $[1.06, 1.07]$ instead of $[0.95, 1.07]$ which is the range for (d) and (g).
7.4. Reconnection Phase

The box containing the domain $|x|, |y| < 0.5, |z| < 1$ showing the region in which $\eta$ may become non zero (outlined by the box), the field lines at equilibrium and an isosurface of $|J| = 3$ in green for the case $b = 1$ and $J_0 = 1$. The colours of the field lines are the same as those used in figure 7.2.

<table>
<thead>
<tr>
<th>$b$</th>
<th>$J_0$</th>
<th>$J_e$</th>
<th>$\eta_0$</th>
</tr>
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</table>

Table 7.2

The different values of $b$, $J_0$, $J_e$ and $\eta_0$ used for the initial magnetic fields in the reconnection phase.
has also accumulated above and below the fan plane near the boundaries due to the line-tied boundary condition (figure 7.5g).

For the reconnection experiments, we now introduce a non-uniform, non-zero $\eta$. As discussed in section 7.1, LARE allows $\eta$ to be defined by some critical current value, $J_c$. This is made use of here in order to only allow reconnection to occur in the main current accumulation around the spine line. There are also some current accumulations near the boundaries local to the fan plane where the field lines are unable to relax fully due to line-tying. We do not want any reconnection caused by this current, so there is a further restriction such that $\eta$ is zero on the domain boundary where this current has accumulated. Thus, $\eta$ is given by

$$
\eta = \begin{cases} 
\eta_0 & |J| > J_c \text{ and } |x|, |y| < 0.5 \\
0 & \text{otherwise} 
\end{cases}
$$

where $\eta_0$ is the resistivity parameter and $J_c$ is the critical value of current above which, reconnection can occur. These parameters are varied in different experiments to study the reconnection. Figure 7.7 shows the field lines at equilibrium and the restricted box $|x|, |y| < 0.5$ in which $\eta$ may become non-zero. The isosurface of $|J| = 3$ is also plotted. If $J_c = 3$, the diffusive region where reconnection would be able to occur lies within both the box and the volume given by the green surface which surrounds the spine lines.

Table 7.2 shows the thirteen different parameters sets that will be considered here with $\eta_0$ and $J_c$ now being varied for the various relaxation experiments with different $b$ and $J_0$.

Below we discuss in detail the results for the $J_0 = 1$ and $b = 1$ experiment with a comparison of the reconnection rates for each of the different parameter runs later.

### 7.4.1 Energies

The effect of allowing reconnection is to enable the field lines around the spine line to untwist via diffusion. At the $z = \pm 1$ boundaries where $|J| > J_c$, the field lines are no longer line-tied to the boundary and may move due the to reconnection. This leads to a conversion of magnetic energy into both kinetic and internal energy.

Figure 7.8 shows the changes in energy during the reconnection experiment. The
7.4. Reconnection Phase

(a) The total energies: The energies have been shifted down to show conservation like in figure 7.4 and the total normalised energy shifted to be just above. The magnetic energy values have been shifted by \(-25.65\) and the internal energies values have been shifted by \(-36.35\).

(b) The instantaneous heating rates: The heating values have not been shifted and no scaling has occurred.

Figure 7.8
The energy evolution during the reconnection phase.
energies are well conserved with an overall error of 0.0001% and the rate of change of the magnetic and internal energies are almost constant. We see a monotonic decrease in the magnetic energy while the internal energy monotonically increases. These curves are very close to but not quite linear. There is a very small increase in kinetic energy, however it is over 200 times smaller than the changes in magnetic or internal energy. So it is not surprising that we find that the ohmic heating is significantly bigger than the viscous and adiabatic heating and is the dominant heating mechanism during the reconnection.

7.4.2 Waves and Evolution

At the instant the reconnection starts and the MHS equilibrium is lost, waves spread out from the reconnection region and the field lines about the spine in the reconnection region are able to untwist. In figures 7.5 and 7.6, we show snapshots of the evolution of the system for the perturbed current (|\(J(\mathbf{r}, t)\)\(-|J(\mathbf{r}, 0)|\)) and pressure (\(p(\mathbf{r}, t)\)-\(p(\mathbf{r}, 0)\)) respectively. These figures show growth in time of the enhancements and deficits in the current and pressure relative to the MHS values. The first column in figure 7.5 shows the MHS current in three different planes at time \(t = 0\) while the second and third column shows the perturbed current in the same planes at \(t = 1.359t_f\) and \(t = 5.436t_f\) respectively. The final time \(t = 5.436t_f\) is when the experiment was stopped. Figures 7.5a, 7.5b and 7.5c show plots of the fan, \(z = 0\) plane and here the current is below the critical current for reconnection. So this plane is still in a non-resistive region. Early on in the experiment, this is no real change in the current in this plane, but later, in response to changes elsewhere in the domain, the equilibrium across this plane is lost and the field relaxes to try and again form an equilibrium causing perturbations on the current. The changes may also be a cause of the waves from the reconnection region passing through the fan plane later in the experiments.

The two other planes \(z = 0.8\) (in figures 7.5 and 7.6, d, e and f) and \(y = 0\) (in figures 7.5 and 7.6, g, h and i) cut through the reconnection region(s) themselves. In the plane \(z = 0.8\), the distinct twisted square shape of the current and pressure are clearly visible initially, but, as time evolves, the twisted shape gets smoothed and becomes more regular. The spiralling arms are truncated in the difference plots because the deficits only occur in the reconnection region.

We see waves of enhanced current and pressure travel from the reconnection region
7.4. Reconnection Phase

The integral of $E_\parallel$ along the field line at each point in the plane $z = 0.95$ within the reconnection region.

along the field lines down towards the fan plane and turning as they continue along the field lines running away from the null point, parallel to the fan plane. The waves do not cross the fan plane.

We found that these waves travel at approximately the local Alfvén speed $c_A = \sqrt{\mathbf{B} \cdot \mathbf{B}/\rho}$, starting out from the edge of the reconnection region (approximately the edge of the enhanced region of green in figure 7.5g). Below the fan plane in figure 7.5h, we plotted crosses which travel at the local Alfvén wave speed along the field lines. These crosses follow the perturbation fronts of both the current and pressure perturbations as they travel along the field lines both out from and into the reconnection region. The same happens in the upper half of the system but these crosses are not plotted there. Waves travelling at the sound or fast magnetoacoustic speed hit the boundary of the system much earlier as for much of the domain, $\beta > 1$. The null point gives small magnetic field values throughout the domain.

7.4.3 Reconnection Rate

At a particular instance in time, the reconnection on all the field lines in the upper half of the domain can be seen by looking at a contour plot of the integral of the
electric field parallel to a magnetic field line, in a plane which cuts through the
reconnection region (figure 7.9). A dense grid of points is chosen in the plane and
from each of these points, we calculated

\[ R = \int_C \mathbf{E} \cdot d\mathbf{s} = \int_C \eta J_\parallel ds \]

along the field line \( C \) which passes through that point. Note that in LARE, \( E_\parallel = \eta J_\parallel \)
where parallel means the component parallel to the magnetic field. From figure 7.9,
we see the maximum reconnection rate occurs on the spine in the centre of the plane
with a spiralling decrease towards the edge of the reconnection region where the rate
is zero by the definition of \( \eta \). No field lines there pass through the \( \eta \neq 0 \) region. The
same reconnection on all field lines is determined by doing the same calculation for
the equivalent grid of points in the plane \( z = -0.95 \) but the result is exactly the same
but of opposite sign. Over time, this spiralling gets dissipated.

Since the spine is the location of the maximum reconnection rate, we then plotted the
evolution of this rate in time in figure 7.10. \(|R|\) is calculated for the spine field line for
positive \( z \) over time for five different combinations of the varying parameters. The
five combinations are varying \( b \), \( J_0 \), \( J_0 \& J_c \) and \( \eta \). The other parameters are kept
constant for each of the five combinations. We do not see the same result as Stevenson
and Parnell (2015a) who studied spontaneous reconnection at a separator and found
two stages to the reconnection: a short rapid reconnection phase followed by a longer
impulsive bursts of weak reconnection. However, there is evidence of this behaviour
in figure 7.10d in the case when \( J_c = 6 \). It is possible that many of the reconnection
experiments here would exhibit the behaviour seen in Stevenson and Parnell (2015a)
if the null points were able to reconnect further. Stevenson and Parnell (2015a) found
that the change in behaviour happened at \( t = t_f \) which does not happen here.

Varying \( b \) from the symmetric case with radial field of \( b = 1 \) causes the absolute
reconnection rate to increase (figure 7.10a). Doubling \( b \) causes the reconnection rate
to increase by about 60\% and halving \( b \) causes the reconnection rate to increase by
just more than a third. As the system evolves, the three reconnection rates start to
converge but this is after \( t = t_f \) where waves caused by the loss of equilibrium have
reflected from the boundaries and thus may start to affect the reconnection.

As \( J_0 \), which may be thought of as a parameter that determines the twist about the
spine, is varied, the reconnection rate naturally increases with \( J_0 \) as there is a higher
component of \( J_\parallel \) along the spine when the amount of twist is greater (figure 7.10b).
7.4. Reconnection Phase

Figure 7.10
The absolute reconnection rates, $|R|$, over time for different experiments where, unless otherwise stated, the non-changing constant parameters are $b = 1$, $J_0 = 1$, $J_c = 3$ and $\eta = 0.001$. 

(a) Varying $b$
(b) Varying $J_0$ with $b = 0.5$
(c) Varying $J_0$ and $J_c$
(d) Varying $J_c$ with $b = 0.5$
(e) Varying $\eta$
The higher $J_0$ not only reconnects more quickly over time but also reconnects for longer. When $J_0 = 0.5$, the reconnection the current in the domain reduces to below $J_c$ and so reconnection switches off. However, the field still desires to be in equilibrium and so the current is built up again and a balance between the reconnection rate and the build up in current is reached.

The reconnection rate essentially depends on the the size and strength of the reconnection region which is defined by $J_c$. As $J_c$ becomes larger, the amount of current greater than $J_c$ becomes smaller i.e. the volume of $|J| > J_c$ reduces and therefore the amount of flux that is available to reconnect is smaller. In the case where $J_0$ is fixed but only $J_c$ is varied (figure 7.10d), this trend is clearly visible. For the case when $J_c = 6$ in figure 7.10d, the reconnection rate slows until it again reaches the “reconnection equilibrium point” where the build up in current is approximately equal to the reconnection rate similar to Stevenson and Parnell (2015a). When both $J_0$ and $J_c$ are varied in the same way (figure 7.10c), although the $J_c$ is large, the volume of $|J| > J_c$ does not change as both are increased/decreased by the same factor. However, the strength of the current in the reconnection region increases as $J_0$ increases so the reconnection rate is greater for greater $J_0$ and $J_c$. Furthermore, evolutions of the reconnection rates are similar except their relative size such that if $J_0$ and $J_c$ are both increased by a factor $n$, the reconnection rate is scaled by $n$.

Finally varying $\eta$ initially just scales reconnection rate (figure 7.10e). As $\eta$ doubles so does the initial reconnection rate but as $t$ increases, the larger $\eta$ cases decrease more rapidly as the current is used up more quickly and it is not replaced by the relaxation quick enough. As in the cases where $b$ is varied (figure 7.10a), the slight change in the reconnection rate at around $t = t_i$ is possibly due to waves reflecting from the boundary.

7.4.4 Other Quantities Along the Spine

Given that the spine is the strongest reconnection site in each of the experiments, we also look at how some other quantities vary along the spine in order to understand the reconnection process better. Specifically, we determine the nature of the magnetic field and the velocity close to the spine which involves taking the discriminant of the perpendicular components with respect to the spine of these vector fields.

That is for some 2D vector field $\mathbf{A}(x, y) = (A_x, A_y)$, we may calculate the Jacobian
7.4. Reconnection Phase

![Graphical representation](image)

(a) $E_\parallel$ (filled contour plot) and the discriminant of $B_\perp$ (contours)

(b) the discriminant of $v_\perp$ (filled contour plot) and $E_\parallel$ (contours)

(c) the $z$-component of vorticity $\omega_z$ (filled contour plot) and discriminant of $v_\perp$ (contours)

Figure 7.11
The evolutions in time of different quantities along the spine.
matrix evaluated at the spine. The characteristic equation of the Jacobian matrix is

\[
\begin{vmatrix}
\nabla A - \lambda I
\end{vmatrix} = \left| \begin{pmatrix}
\frac{\partial A_x}{\partial x} - \lambda & \frac{\partial A_x}{\partial y} \\
\frac{\partial A_y}{\partial x} & \frac{\partial A_y}{\partial y} - \lambda
\end{pmatrix} \right| = \lambda^2 - \lambda \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) + \frac{\partial A_x}{\partial x} \frac{\partial A_y}{\partial y} - \frac{\partial A_y}{\partial x} \frac{\partial A_x}{\partial y}.
\]

The discriminant \((b^2 - 4ac)\) of the characteristic equation, is given by

\[
discr(A) = \left( \frac{\partial A_x}{\partial x} - \frac{\partial A_y}{\partial y} \right)^2 + 4 \frac{\partial A_x}{\partial y} \frac{\partial A_y}{\partial x}.
\]

The sign of the discriminant determines the nature of field lines in the plane about the point it is evaluated at. In particular it determines whether the solutions of \(\nabla A - \lambda I = 0\) are real or complex. If the discriminant is positive then there are X-type field lines and if it is negative then there are O-type field lines.

In figure 7.11, the evolution in time of the quantities along the spine are plotted: figure 7.11a shows \(E_\parallel\) in order to identify the locations of the reconnection overplotted with the discriminant of \(B_\perp\) which shows whether the magnetic field at the reconnection site is elliptic or hyperbolic, figure 7.11b shows \(E_\parallel\) and the discriminant of \(v_\perp\) to identify if the fluid flow is elliptic or hyperbolic about the spine and figure 7.11c shows the discriminant of \(v_\perp\) and the perpendicular vorticity to locate regions of counter rotating fields.

In figure 7.11a, we see that the length of the reconnection region along the spine decreases in size over time (reduction of the coloured regions) but also the magnitude of \(E_\parallel\) gets smaller with both factors causing the reconnection rate to decrease. The discriminant of \(B_\perp\) shows that the perpendicular field around the spine is always O-type in 2D and also the field lines are twisted around the spine. But outside the reconnection region, the current is not dissipated leading to an increase in the discriminant of \(v_\perp\) due to larger velocities here. These are left behind as the reconnection region recedes.

In figure 7.11c, the vorticity changes sign at the boundary between the \(\eta = 0\) region and the reconnection region. Across this boundary, there is a counter rotating flow which is a signature of reconnection in 3D (Hornig and Priest 2003). After one fast
crossing time, the counter rotating flow is confined to the very top and bottom boundary $|z| \approx 1$, but a strong shear flow remains at the edge of the reconnection region.

7.5 Conclusions

The torsional spine reconnection studied here is actually quite an straightforward reconnection case. It is expected that using the tilted null points from Fuentes-Fernández and Parnell (2013) and studying the reconnection of those would be a more interesting case of null point reconnection.

The reconnection at spiral null points is very similar no matter how $b$ and $J_0$ are varied. The two parameters only affect the rate of reconnection but the general evolution is similar. The perturbations caused by the reconnection travel at the Alfvén speed along the field lines away from the reconnection site. The reconnection induces a rotational flow about the spine lies as the field lines are able to untwist in the diffusive region but the majority of the energy released (over 99\%) during the reconnection is converted into heat.
7. Magnetic Reconnection at Spiral Null Points
Chapter 8

Summary and Future Work

In this thesis, I have studied how to calculate the skeletons of magnetic fields, two different global magnetic field models, the magnetic skeletons of these global magnetic fields and finished by looking at the torsional spine reconnection local to a magnetic null point.

In Chapter 2, the new MSAT package for calculating the magnetic skeletons of magnetic fields defined on a discrete grid is presented. The package includes methods for locating and characterising null points and finding their associated spine lines, separatrix surfaces and separators. I have been told by fellow PhD students and members of the SMTG group in St Andrews, the first users of the package, that this package of skeleton finding tools is more robust and more user friendly than the codes previously available. We hope that making this package publicly available will be of great help to many researchers investigating the topology of magnetic fields or other divergence free vector fields.

In Chapter 3, a new code for calculating PFSS fields has been presented (BMW2016). This code was required in order to try and mitigate the issues found with the previous version by van Ballegooijen (AVB1997). A detailed description of this PFSS model which is used in this thesis is given along with the assumptions made and any constraints.

Chapter 4 showed the importance of using an appropriate grid resolution and using a large maximum number of harmonics, $L$, when implementing the PFSS model numerically. It was shown that for magnetic fields derived from a PFSS model with a small maximum number of harmonics ($L \leq 160$), the identifiable magnetic skeleton is sparse compared with the skeleton derived from magnetic fields determined using a large maximum number of harmonics ($L \geq 641$). The number of null points identified in a PFSS magnetic field calculated using a large number of harmonics ($L = 641$) is almost 60 times greater than the number found in a PFSS field with only a small
number of harmonics ($L = 81$). The separator networks which are found in PFSS fields with a small maximum number of harmonics are simple in the number and complexity of the networks found whereas in PFSS fields with a large maximum number of harmonics, the separator networks can be highly complex: indeed there typically exists a single, very large separator network which circles the entire Sun giving rise to connectivity throughout almost the entire global solar magnetic field. As well as this single, large, complex separator network, there are also many smaller, simpler separator networks forming domes, tunnels and caves with linear chains of null points and separators in global magnetic fields determined from PFSS models with a large maximum number of harmonics. Some of the new separators found in PFSS models with high numbers of harmonics are low down in the solar atmosphere, as expected, but many (around 20%) can reach high ($\geq 0.5R_\odot$) into the solar atmosphere. The increase in the maximum number of harmonics clearly has consequences which affect both the small and larger scales of the global solar magnetic field.

This work also showed how the grid resolution of the magnetic fields affects the number of identifiable magnetic skeleton features. Although there is a computational time penalty to calculating the magnetic fields without the fast Fourier transform, increasing the grid resolution of the PFSS fields can significantly increase the number and accuracy of characterisation of the magnetic skeleton features. This increase can be up to 50% when the grid resolution of PFSS models with small maximum numbers of harmonics is increased. In chapter 4, it is explained that the resolution of the grid in the $\phi$ direction is restricted because of the fast Fourier transform. I have had a new idea as to how to increase the grid resolution in the $\phi$ direction while still using the fast Fourier transform which I hope to test in the near future so that the PFSS model may be calculated at much higher resolutions without the time penalty. This hopefully will allow the magnetic skeletons of these PFSS fields to be more reliable in future without the time penalties of calculating the magnetic fields without using a fast Fourier transform. It would also be interesting to also see what effect increasing the maximum number of harmonics further has on the global magnetic fields. As yet we still are not sampling the HMI data at the maximum resolution (using the highest $L$) possible. Inevitably more magnetic skeleton features are likely to be found.

Chapter 5 discussed the differences in the global magnetic skeletons and the separator networks found at solar minimum and solar maximum using magnetic fields extrapolated from HMI synoptic maps using the PFSS model with a large maximum
number of harmonics \((L = 641)\). The main differences between the magnetic skeletons found at solar minimum and solar maximum are the number and the distribution of the null points and separators. There are significantly more null points and separators at solar minimum than solar maximum with what appears to be a loss of null points from the equatorial regions at solar maximum. This loss happens because of the large scale magnetic fields in active regions which restrict the amount of variation in the field. This also means there are significantly less separators in the equatorial region at solar maximum except those which cross the equator high in the solar atmosphere. Both solar minimum and solar maximum have a single, highly complex, large scale separator network which circles the entire Sun. This large, complex separator network at solar minimum contains many more null points and separators than at solar maximum. Also, at solar minimum the null points are more multiply connected by separators. Apart from this large complex separator network, there are very few other complex separator networks. These other separator networks are mainly linear chains of null points forming caves, domes and tunnels. Since this study was only done for 5 Carrington rotations at solar minimum and 5 Carrington rotations at solar maximum, this study should probably be extended by studying the magnetic skeletons found in global PFSS magnetic fields derived with this maximum number of harmonics over a much longer time period. This would hopefully validate the effects seen here at solar minimum and solar maximum but also reveal how the separator networks evolve during the periods between solar minimum and solar maximum during which the number of sunspots rises and falls. The separatrix curtains and open field regions could also be compared with observations of the solar wind and the S-web model Antiochos et al. (2011).

Chapter 6 discussed the global MHS model derived by Neukirch (1995) and implemented this with a boundary condition approximating a source surface after \(r = 2.5R_\odot\). Global MHS magnetic fields were produced for different values of the parameter \(d\) which governs the perpendicular currents in the model. The magnetic skeletons of these magnetic fields were compared with the PFSS model for the same synoptic magnetogram and maximum number of harmonics. The associated plasma in the MHS model was also discussed along with its validity for different values of \(d\). It was shown that the magnetic skeletons produced from these global MHS models for very large \(d\) are quite unphysical with their increasingly radial field lines and their associated pressure and plasma beta atypical of the solar corona. However, for small values of \(|d|\), magnetic fields are produced by the MHS model which are more
8. Summary and Future Work

physical and their associated pressures and plasma beta are typically in the range of the real values found in the solar atmosphere. It is unfortunate that MHS models for $\alpha \neq 0$ have not been able to be calculated. It would be interesting to also be able to investigate the effect of varying $\alpha$ on the magnetic fields and associated plasma properties produced by the model. Once this has been done, the MHS model may be able to be used in future as an improvement to the PFSS model to study the magnetic field and the plasma of the global solar magnetic field.

It would also be interesting to compare both the PFSS model and MHS model with other global coronal magnetic field models which are an extension of the PFSS model. These other models, often extending further out from the Sun and used for modelling the solar wind, are discussed in Wilcox Solar Observatory (2016) and are simple to compute like the PFSS and MHS model.

Finally in chapter 7, the dynamics of torsional spine reconnection local to a single null point is studied. The reconnection occurs away from the null point itself, along the spines of the null point. The magnetic energy released is converted into kinetic and ohmic heating with the majority going directly into heating. The onset of reconnection causes perturbations to travel at the Alfvén speed away from the reconnection site where they are subject to viscous heating. It would also be more interesting to investigate the reconnection at tilted null points which would give rise to fan-spine reconnection and continue the experiments from Fuentes-Fernández and Parnell (2013) where the reconnection event may be significantly different more eventful.
Appendix A

Relationships of $Y_l^m$ and $B_l^m$ for negative $m$

We start with the definition of $Y_l^m$

$$Y_l^m(\theta, \phi) = Q_l^m(\theta) e^{im\phi}$$

and from this we find that

$$Y_l^{-m}(\theta, \phi) = Q_l^{-m}(\theta) e^{-im\phi}$$

$$= (-1)^m Q_l^m(\theta) e^{-im\phi}$$

$$= (-1)^m Q_l^m(\theta) e^{im\phi}$$

$$= (-1)^m \overline{Q_l^m}(\theta) e^{im\phi}$$

$$= (-1)^m \overline{Y_l^m}(\theta, \phi).$$

Hence, $Y_l^{-m}$ is related to $Y_l^m$ through complex conjugation.

We also find a similar relation for $B_l^m$ by starting with the definition

$$B_l^m(r) = \int_S \overline{Y_l^m}(\theta, \phi) B_r(r, \theta, \phi) \, dS$$
A. Relationships of $Y_i^m$ and $B_i^m$ for negative $m$

and using the relationship for $Y_i^m$ above, we find that

$$B_i^{-m}(r) = \int \int_S \overline{Y_i^{-m}(\theta, \phi)} B_r(r, \theta, \phi) \, dS$$

$$= \int \int_S (-1)^m Y_i^m(\theta, \phi) B_r(r, \theta, \phi) \, dS$$

$$= (-1)^m \int \int_S Q_i^m(\theta) e^{im\phi} B_r(r, \theta, \phi) \, dS$$

$$= (-1)^m \int \int_S Q_i^m(\theta) e^{-im\phi} B_r(r, \theta, \phi) \, dS$$

$$= (-1)^m \int \int_S Q_i^m(\theta) e^{-im\phi} B_r(r, \theta, \phi) \, dS$$

$$= (-1)^m \int \int_S Q_i^m(\theta) \overline{B_r(r, \theta, \phi)} \, dS$$

$$= (-1)^m \int \int_S \overline{Y_i^m(\theta, \phi)} B_r(r, \theta, \phi) \, dS$$

$$= (-1)^m \overline{B_i^m(r)}.$$
Given some of the lack of some steps in the derivation of the MHS model available in other documentation. I have listed some of the derivations in more detail here with some vector identities. First we start with the vector identities and then introduce the angular momentum operator and finally many steps in the derivation of the MHS model are given in more detail.

### B.1 Some Vector Identities

First the derivatives of the spherical coordinate unit vectors are listed with some cross products with \( \mathbf{r} = r \mathbf{e}_r \). They are required in the derivations of the angular momentum operator in section B.2.

\[
\begin{align*}
\frac{\partial \mathbf{e}_r}{\partial r} &= 0 \\
\frac{\partial \mathbf{e}_r}{\partial \theta} &= \mathbf{e}_\theta \\
\frac{\partial \mathbf{e}_r}{\partial \phi} &= \mathbf{e}_\phi \sin \theta \\
\mathbf{r} \times \mathbf{e}_r &= 0
\end{align*}
\]

\[
\begin{align*}
\frac{\partial \mathbf{e}_\theta}{\partial r} &= 0 \\
\frac{\partial \mathbf{e}_\theta}{\partial \theta} &= -\mathbf{e}_r \\
\frac{\partial \mathbf{e}_\theta}{\partial \phi} &= \mathbf{e}_\phi \cos \theta \\
\mathbf{r} \times \mathbf{e}_\theta &= r \mathbf{e}_\phi
\end{align*}
\]

\[
\begin{align*}
\frac{\partial \mathbf{e}_\phi}{\partial r} &= 0 \\
\frac{\partial \mathbf{e}_\phi}{\partial \theta} &= 0 \\
\frac{\partial \mathbf{e}_\phi}{\partial \phi} &= -\mathbf{e}_r \sin \theta - \mathbf{e}_\theta \cos \theta \\
\mathbf{r} \times \mathbf{e}_\phi &= -r \mathbf{e}_\theta
\end{align*}
\]

Next, a few vector identities are listed which are required for the derivations of the MHS model in section B.3. For any \( f = f(r) \) and \( \mathbf{F} = \mathbf{F}(r, \theta, \phi) \),

\[
\nabla \cdot \mathbf{r} = 3 \tag{B.1}
\]

\[
\nabla \times \mathbf{r} = 0 \tag{B.2}
\]
B. Mathematics of the Global MHS Model

\[ \nabla^2 r = 0 \quad (B.3) \]
\[ \nabla f \times r = 0 \quad (B.4) \]
\[ (\mathbf{F} \cdot \nabla) r = \mathbf{F} \quad (B.5) \]

B.2 The Angular Momentum Operator and Its Properties

During the derivation of the MHS model, we obtain a differential equation which can be nicely simplified using the angular momentum operator from quantum mechanics. However here it used in its common form but divided by \( \hbar \). Using the vector identities listed in section B.1, it allows us to find the angular momentum operator using standard cross products and the definition of \( \nabla \):

\[
\mathbf{L} = \frac{1}{i} \mathbf{r} \times \nabla
\]
\[
= \frac{1}{i} \left( \mathbf{r} \times \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{r} \times \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{r} \times \mathbf{e}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right)
\]
\[
= \frac{1}{i} \left( \mathbf{e}_\phi \frac{\partial}{\partial \theta} - \frac{\mathbf{e}_\theta}{\sin \theta} \frac{\partial}{\partial \phi} \right) \quad (B.6)
\]

Then we can use the definition of \( \mathbf{L} \) to find \( \mathbf{L}^2 \). By taking the dot product of \( \mathbf{L} \) with itself, taking dot products element-wise while noting the order of the derivative operations and using the unit vector derivatives before, we can derive \( \mathbf{L}^2 \):

\[
\mathbf{L}^2 = \mathbf{L} \cdot \mathbf{L}
\]
\[
= - \left( \mathbf{e}_\phi \frac{\partial}{\partial \theta} - \frac{\mathbf{e}_\theta}{\sin \theta} \frac{\partial}{\partial \phi} \right) \cdot \left( \mathbf{e}_\phi \frac{\partial}{\partial \theta} - \frac{\mathbf{e}_\theta}{\sin \theta} \frac{\partial}{\partial \phi} \right)
\]
\[
= \frac{\mathbf{e}_\theta}{\sin \theta} \cdot \frac{\partial}{\partial \phi} \left( \mathbf{e}_\phi \frac{\partial}{\partial \theta} - \frac{\mathbf{e}_\theta}{\sin \theta} \frac{\partial}{\partial \phi} \right) - \mathbf{e}_\phi \cdot \frac{\partial}{\partial \theta} \left( \mathbf{e}_\phi \frac{\partial}{\partial \theta} - \frac{\mathbf{e}_\theta}{\sin \theta} \frac{\partial}{\partial \phi} \right)
\]
\[
= \frac{\mathbf{e}_\theta}{\sin \theta} \left( -\mathbf{e}_r \sin \theta - \mathbf{e}_\theta \cos \theta \frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{\partial^2}{\partial \theta \partial \phi} - \frac{\mathbf{e}_\phi \cos \theta \frac{\partial}{\partial \phi} - \frac{\mathbf{e}_\theta}{\sin \theta} \frac{\partial^2}{\partial \phi^2} \right)
\]
\[
- \mathbf{e}_\phi \left( \frac{\partial^2}{\partial \theta^2} + \left( \frac{\mathbf{e}_r}{\sin \theta} + \frac{\mathbf{e}_\theta \cos \theta}{\sin^2 \theta} \right) \frac{\partial}{\partial \phi} - \frac{\mathbf{e}_\theta}{\sin \theta} \frac{\partial^2}{\partial \theta \partial \phi} \right)
\]

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B.2. The Angular Momentum Operator and Its Properties

\[ -\cot \theta \frac{\partial}{\partial \theta} - \frac{\partial^2}{\partial \theta^2} - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \]
\[ = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \]  
(B.7)

This is identical to part of the Laplacian in spherical coordinates. Comparing this to the definition of the Laplacian in spherical coordinates gives

\[ L^2 \equiv -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \equiv \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) - r^2 \nabla^2 \]  
(B.8)

There are also other useful properties of \( L \). For any vector function \( \mathbf{F} = \mathbf{F}(r, \theta, \phi) \):

\[ iL \cdot \mathbf{F} = (\mathbf{r} \times \nabla) \cdot \mathbf{F} = \mathbf{r} \cdot \nabla \times \mathbf{F} \]  
(B.9)

It allows the switch of the two binary vector operations. The dot product with the gradient operator is also zero:

\[ \nabla \cdot L = 0 \]  
(B.10)

\[ L \cdot \nabla = 0 \]  
(B.11)

\( Y_l^m \) is an eigenfunction of \( L^2 \). It is easy to show this by substituting \( Y_l^m \) into the expression for \( L^2 \) and comparing it to the Legendre’s differential equation. Legendre’s differential equation (3.1) with the substitution of \( x = \cos \theta \) is

\[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial y}{\partial \theta} \right) + l(l + 1) y - \frac{m^2}{\sin^2 \theta} y = 0 \]

Comparing this to the expression for \( L^2 Y_l^m \):

\[ L^2 Y_l^m = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y_l^m}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2 Y_l^m}{\partial \phi^2} \]
\[ = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y_l^m}{\partial \theta} \right) + \frac{m^2}{\sin^2 \theta} Y_l^m \]
\[ = l(l + 1) Y_l^m \]  
(B.12)

Given \( Y_l^m \) has no dependence on \( r \), \( Y_l^m \) is also similarly an eigenfunction of \( \nabla^2 \)

\[ -r^2 \nabla^2 Y_l^m = L^2 Y_l^m - \frac{\partial}{\partial r} \left( r^2 \frac{\partial Y_l^m}{\partial r} \right) = l(l + 1) Y_l^m \]  
(B.13)
B.3 Detailed Derivations of the MHS Model

Now we move onto the model derivation itself. Equation 6.8 is derived by first taking the curl of the chosen form for the current:

$$\nabla \times B = \mu_0 (\alpha B + \nabla F \times \nabla \psi).$$

This gives

$$\nabla \times \nabla \times B = \mu_0 (\nabla \times (\alpha B) + \nabla \times (\nabla F \times \nabla \psi))$$

$$= \mu_0 (\alpha \nabla \times B + B \times \nabla \alpha + \nabla \times (\nabla F \times \nabla \psi))$$

$$= \mu_0 (\alpha \mu_0 (\alpha B + \nabla F \times \nabla \psi) + \nabla \times (\nabla F \times \nabla \psi))$$

$$= \alpha^2 \mu_0^2 B + \mu_0 \nabla \times (\nabla F \times \nabla \psi) + \alpha \mu_0 \nabla F \times \nabla \psi.$$

Then finally dotting this result with \(r = r_\text{e}_r\) gives

$$r \cdot \nabla \times \nabla \times B = \alpha^2 \mu_0^2 r \cdot B + \mu_0 r \cdot \nabla \times (\nabla F \times \nabla \psi) \quad (B.14)$$

since \(r \cdot \nabla F \times \nabla \psi = 0\). The double curl term on the left hand side can be simplified somewhat by applying the properties of vector calculus, we find that

$$r \cdot \nabla \times \nabla \times B \equiv r \cdot (\nabla (\nabla \cdot B) - \nabla^2 B) \equiv -r \cdot \nabla^2 B$$

and similarly using equations B.2, B.3 and B.5

$$\nabla^2 (r \cdot B) \equiv r \cdot \nabla^2 B - B \cdot \nabla^2 r + 2 \nabla \cdot ((B \cdot \nabla) r + B \times \nabla \times r) \equiv r \cdot \nabla^2 B.$$

Putting these two together gives

$$r \cdot \nabla \times \nabla \times B \equiv -r \cdot \nabla^2 B \equiv -\nabla^2 (r \cdot B). \quad (B.15)$$

Equation 6.11 is derived as follows

$$\nabla F \times \nabla \psi = \nabla \left( K(\psi) \frac{GM}{r^3} r \cdot B \right) \times \frac{GM}{r^3} r$$

$$= \frac{GM}{r^3} \left( \frac{GM}{r^3} K(\psi) \nabla (r \cdot B) + \frac{GM}{r^3} r \cdot B \nabla K + K(\psi) r \cdot B \nabla \left( \frac{GM}{r^3} \right) \right) \times r$$
\[ \left( \frac{GM}{r^3} \right)^2 K(\psi) \nabla(r \cdot B) \times r = H(r) \nabla(r \cdot B) \times r \]  \hspace{1cm} (B.16)

using equation B.4.

Using equation B.1 and B.5, equation 6.12 can be derived as the following

\[ \nabla \times (\nabla F \times \nabla \psi) = \nabla \times (H \nabla(r \cdot B) \times r) \]
\[ = H \nabla(r \cdot B) \nabla \cdot r - r \nabla \cdot (H \nabla(r \cdot B)) + (r \cdot \nabla)(H \nabla(r \cdot B)) \]
\[ - r \nabla \cdot (H \nabla(r \cdot B)) \]
\[ = 2H \nabla(r \cdot B) + (r \cdot \nabla)(H \nabla(r \cdot B)) \]
\[ - r \nabla \cdot (H \nabla(r \cdot B)) \]
\[ = 2H \nabla(r \cdot B) + \frac{\partial}{\partial r}(H \nabla(r \cdot B)) \]
\[ - r \nabla H \cdot \nabla(r \cdot B) - rH \nabla^2(r \cdot B) \]
\[ = 2H \nabla(r \cdot B) + r \frac{\partial}{\partial r}(H \nabla(r \cdot B)) \]
\[ - r \nabla H \cdot \nabla(r \cdot B) - rH \nabla^2(r \cdot B) \]

Dotting with \( r \) allows us to cancel terms and simplify to the final result

\[ r \cdot \nabla \times (\nabla F \times \nabla \psi) = 2rH \frac{\partial}{\partial r}(r \cdot B) + r^2 \frac{\partial}{\partial r} \frac{\partial}{\partial r}(r \cdot B) + r^2 H \frac{\partial^2}{\partial r^2}(r \cdot B) \]
\[ - r^2 \frac{\partial}{\partial r} \frac{\partial}{\partial r}(r \cdot B) - r^2 H \nabla^2(r \cdot B) \]
\[ = H(r) \left( 2r \frac{\partial}{\partial r}(r \cdot B) + r^2 \frac{\partial^2}{\partial r^2}(r \cdot B) - r^2 \nabla^2(r \cdot B) \right) \]
\[ = H(r) \left( \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r}(r \cdot B) \right) - r^2 \nabla^2(r \cdot B) \right) \]
\[ = H(r) L^2(r \cdot B) \]  \hspace{1cm} (B.17)

Using equation B.9, there is an alternative derivation of equation 6.12:

\[ r \cdot \nabla \times (\nabla F \times \nabla \psi) = r \cdot \nabla \times (H(r) \nabla(r \cdot B) \times r) \]
\[ = r \times \nabla \cdot (H(r) \nabla(r \cdot B) \times r) \]
\[ = H(r) r \times \nabla \cdot (\nabla(r \cdot B) \times r) \]
\[ = - H(r) r \times \nabla \cdot (r \times \nabla(r \cdot B)) \]
\[ = H(r) \mathbf{L} \cdot \mathbf{L}(r \cdot \mathbf{B}) \]
\[ = H(r) \mathbf{L}^2(r \cdot \mathbf{B}) \]

Equation 6.17 gives a link between the radial and angular components of the field and can be shown by dotting Ampère’s law with \( r \) with our form for \( \mathbf{J} \) and using the properties of \( \mathbf{L} \). Starting with equation 6.7

\[ \nabla \times \mathbf{B} = \mu_0 \left( \alpha \mathbf{B} + \nabla F \times \nabla \psi \right) \]
\[ r \cdot \nabla \times \mathbf{B} = \mu_0 \alpha r \cdot \mathbf{B} \]
\[ r \times \nabla \cdot \mathbf{B} = \mu_0 \alpha r \cdot \mathbf{B} \]
\[ i \mathbf{L} \cdot \mathbf{B} = \mu_0 \alpha r \cdot \mathbf{B} \]

Since \( \mathbf{L} \) has no radial component, \( \mathbf{L} \cdot \mathbf{B} = \mathbf{L} \cdot \mathbf{B}_t \) meaning

\[ \mathbf{L} \cdot \mathbf{B}_t = \frac{\bar{\alpha}}{i} r \cdot \mathbf{B} \quad \text{(B.18)} \]

Finding the \( \theta \) and \( \phi \) components of \( \mathbf{B} \) requires using equations B.10 and B.11. Looking at a single order element

\[ (\mathbf{L} \cdot \mathbf{B}_t)_l^m = \frac{\bar{\alpha}}{i} (r \cdot \mathbf{B})_l^m \]
\[ \mathbf{L} \cdot (v_{l}^{m}(r) \mathbf{L} \mathbf{Y}_l^m + w_{l}^{m}(r) \nabla \mathbf{Y}_l^m) = \frac{\bar{\alpha}}{i} B_l^m(r) \mathbf{Y}_l^m \]
\[ v_{l}^{m}(r) \mathbf{L}^2 \mathbf{V}_l^m + w_{l}^{m}(r) \mathbf{L} \cdot \nabla \mathbf{Y}_l^m = \frac{\bar{\alpha}}{i} B_l^m(r) \mathbf{Y}_l^m \]
\[ v_{l}^{m}(r) l (l + 1) \mathbf{Y}_l^m = \frac{\bar{\alpha}}{i} B_l^m(r) \mathbf{Y}_l^m \]
\[ v_{l}^{m}(r) = \frac{\bar{\alpha}}{i} \frac{B_l^m(r)}{l (l + 1)} \quad \text{(B.19)} \]

Then finally the \( w_{l}^{m} \) can be found using \( \nabla \cdot \mathbf{B} = 0 \). Again, in terms of a single order element:

\[ \nabla \cdot \mathbf{B} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 B_l^m(r) \right) \frac{Y_l^m}{r} + \nabla \cdot (v_{l}^{m} \mathbf{L} \mathbf{Y}_l^m) + \nabla \cdot (w_{l}^{m} \nabla \mathbf{Y}_l^m) \]
\[ 0 = \frac{1}{r^2} \frac{d}{dr} (r B_l^m(r)) Y_l^m + v_{l}^{m} \nabla \cdot \mathbf{L} \mathbf{Y}_l^m + \nabla v_{l}^{m} \cdot \mathbf{L} \mathbf{Y}_l^m + w_{l}^{m} \nabla^2 \mathbf{Y}_l^m + \nabla w_{l}^{m} \cdot \nabla \mathbf{Y}_l^m \]
\[ 0 = \frac{1}{r^2} \frac{d}{dr} (r B_l^m(r)) Y_l^m + \frac{w_{l}^{m} l (l + 1) Y_l^m}{r^2} \quad \text{(B.20)} \]
and rearranging this gives
\[
\frac{d}{dr} \left( rB_l^m \right).
\]

Finally there is a short discussion on finding the density in the MHS model. Given the differential equation for pressure (6.19) has been solved to give
\[
p(r, \theta, \phi) = p_0(r) - \frac{\bar{H}(r)}{2\mu_0} \left( r \cdot B \right)^2,
\]
the density in the MHS model can now be found using
\[
\rho = B \cdot \nabla F - \frac{\partial p}{\partial \psi}
\]
to give equation 6.22. \( B \cdot \nabla F \) can be found using
\[
F(r, \theta, \phi) = H(r) \frac{r^3}{GM_\odot} \cdot B
\]
and the derivative of \( p \) can be found using the chain rule:
\[
\frac{\partial p}{\partial r} = \frac{\partial p}{\partial \psi} \frac{\partial \psi}{\partial r} + \frac{\partial p}{\partial F} \frac{\partial F}{\partial r}
\]
since all the other terms are known including
\[
\frac{\partial p}{\partial F} = -B \cdot \nabla \psi
\]
from the differential equation for pressure (6.19).
B. Mathematics of the Global MHS Model
Appendix C

An Iterative Scheme for Calculating a Source Surface in the MHS Model

To implement a source surface in the MHS model, \( B_\theta \) and \( B_\phi \) must equal zero at \( r = R_{max} \). We attempted to implement this element wise i.e. \( (B_\theta)_l^m|_{R_{max}} = 0 \) and \( (B_\phi)_l^m|_{R_{max}} = 0 \)

The two equations to be solved are

\[
\left( \alpha B_l^m Q^m_l \sin \theta + \frac{1}{r} \frac{d}{dr}(r B_l^m) \frac{dQ_l^m}{d\theta} \right) e^{im\phi} = 0 \tag{C.1}
\]

\[
\left( -\alpha B_l^m \frac{dQ_l^m}{d\theta} + \frac{1}{r} \frac{d}{dr}(r B_l^m) i m Q_l^m \sin \theta \right) e^{im\phi} = 0 \tag{C.2}
\]

at \( r = R_{max} \). To attempt to solve these, we need to be able to write only the \( r \)
dependent terms equal to zero. The \( \phi \) dependent term \( (e^{im\phi}) \) is already separable in
both expressions, so we just need to be able to separate out the \( \theta \) dependent terms.

We can do this using a recursive relationship of the associated Legendre polynomials.
The one we require is

\[
(1 - x^2) \frac{dP_l^m}{dx} = \frac{1}{2l+1} \left( (l+1)(l+m) P_{l-1}^m - l(l-m+1) P_{l+1}^m \right)
\]

By converting this to \( \theta \) using the usual transformation of \( x = \cos \theta \), we are able to
write all \( \theta \) dependent terms in both expressions above in terms of \( P_l^m / \sin \theta \). Then
using the normalisation of the associated Legendre polynomials given by equation 3.8,
we obtain

\[
\frac{dQ_l^m}{d\theta} = l \sqrt{\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}} \frac{Q_{l+1}^m}{\sin \theta} - (l+1) \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}} \frac{Q_{l-1}^m}{\sin \theta}
\]
C. An Iterative Scheme for Calculating a Source Surface in the MHS Model

This means equations C.1 and C.2 at \( r = R_{\text{max}} \) become

\[
\text{i} m \alpha B_l^m \frac{Q_l^m}{\sin \theta} + \frac{1}{r} \frac{d}{dr} (r B_l^m) \left( l \sqrt{\frac{(l + m + 1)(l - m + 1)}{(2l + 1)(2l + 3)}} \frac{Q_{l+1}^m}{\sin \theta} \right) - \frac{1}{r} \frac{d}{dr} (r B_l^m) \left( l + 1 \right) \sqrt{\frac{(l + m)(l - m)}{(2l + 1)(2l - 1)}} \frac{Q_{l-1}^m}{\sin \theta} = 0 \quad (C.3)
\]

\[
\text{i} m \frac{d}{dr} (r B_l^m) \frac{Q_l^m}{\sin \theta} - \alpha B_l^m \left( l \sqrt{\frac{(l + m + 1)(l - m + 1)}{(2l + 1)(2l + 3)}} \frac{Q_l^m}{\sin \theta} \right) + \alpha B_l^m \left( l + 1 \right) \sqrt{\frac{(l + m)(l - m)}{(2l + 1)(2l - 1)}} \frac{Q_{l-1}^m}{\sin \theta} = 0 \quad (C.4)
\]

In order to make \( Q_l^m / \sin \theta \) common to all three terms in both \( B_\theta \) and \( B_\phi \), each term is adjusted to make the orders of the associated Legendre polynomials equal in every term:

\[
\text{i} m \frac{d}{dr} (r B_l^m) \frac{Q_l^m}{\sin \theta} + \frac{1}{r} \frac{d}{dr} (r B_l^m) \left( l - 1 \right) \sqrt{\frac{(l + m)(l - m)}{(2l - 1)(2l + 1)}} \frac{Q_l^m}{\sin \theta} - \frac{1}{r} \frac{d}{dr} (r B_{l+1}^m) \left( l + 2 \right) \sqrt{\frac{(l + m + 1)(l - m + 1)}{(2l + 3)(2l + 1)}} \frac{Q_l^m}{\sin \theta} = 0 \quad (C.5)
\]

\[
\text{i} m \frac{d}{dr} (r B_l^m) \frac{Q_l^m}{\sin \theta} - \alpha B_{l-1}^m \left( l - 1 \right) \sqrt{\frac{(l + m)(l - m)}{(2l - 1)(2l + 1)}} \frac{Q_l^m}{\sin \theta} + \alpha B_{l+1}^m \left( l + 2 \right) \sqrt{\frac{(l + m + 1)(l - m + 1)}{(2l + 3)(2l + 1)}} \frac{Q_l^m}{\sin \theta} = 0 \quad (C.6)
\]

\( Q_l^m / \sin \theta \) is now common to every term and they can be cancelled out. By defining

\[
k_1 = (l - 1) \sqrt{\frac{(l + m)(l - m)}{(2l - 1)(2l + 1)}}
\]

\[
k_2 = (l + 2) \sqrt{\frac{(l + m + 1)(l - m + 1)}{(2l + 3)(2l + 1)}}
\]

this leaves:

\[
\text{i} m \alpha B_l^m + k_1 \frac{d}{dr} (r B_{l-1}^m) - k_2 \frac{d}{dr} (r B_{l+1}^m) = 0 \quad (C.7)
\]
\[
\frac{im}{r} \frac{d}{dr} (r B_i^m) - \alpha k_1 B_{i-1}^m + k_2 \alpha B_{i+1}^m = 0 \tag{C.8}
\]

These two equations are now simply the \( r \) dependent terms as required. They form an iterative scheme for \( B_i^m \) and essentially its coefficients \( a_i^m \) and \( b_i^m \).

\[
B_i^m (r) = \sqrt{\frac{r + d}{r}} \left( a_i^m \tilde{J}_{i+\frac{1}{2}} (r) + b_i^m \tilde{N}_{i+\frac{1}{2}} (r) \right)
\]

The base boundary condition can be used to define one in terms of the other such that the iterative scheme above only needs to solve for one of them.

\[
b_i^m = \frac{R_\odot B_i^m (R_\odot) - a_i^m \tilde{J}_{i+\frac{1}{2}} (R_\odot)}{\tilde{N}_{i+\frac{1}{2}} (R_\odot)} \tag{C.9}
\]

Next, by defining (similarly to chapter 6)

\[
\tilde{J}_i^0 = J_{i+\frac{1}{2}} (\bar{\alpha} (R_\odot + d)) \quad \tilde{J}_i^\infty = J_{i+\frac{1}{2}} (\bar{\alpha} (R_{\max} + d))
\]

\[
\tilde{N}_i^0 = N_{i+\frac{1}{2}} (\bar{\alpha} (R_\odot + d)) \quad \tilde{N}_i^\infty = N_{i+\frac{1}{2}} (\bar{\alpha} (R_{\max} + d))
\]

\[
\alpha_1 = \bar{\alpha} (R_{\max} + d)
\]

the following two final forms for the recursive relations are obtained as

\[
\frac{im a_i^m}{2} \left( \tilde{J}_i^\infty - \frac{\tilde{J}_i^0 \tilde{N}_i^\infty}{\tilde{N}_i^0} + \alpha_1 (\tilde{J}_i^\infty - \tilde{J}_{i-1}^\infty) - \alpha_1 \frac{\tilde{J}_i^0}{\tilde{N}_i^0} (\tilde{N}_{i-1}^\infty - \tilde{N}_i^\infty) \right) - k_1 \alpha_1 a_{i-1}^m \left( \tilde{J}_{i-1}^\infty - \frac{\tilde{J}_{i-1}^0 \tilde{N}_{i-1}^\infty}{\tilde{N}_{i-1}^0} \right) + k_2 \alpha_1 a_{i+1}^m \left( \tilde{J}_{i+1}^\infty - \frac{\tilde{J}_{i+1}^0 \tilde{N}_{i+1}^\infty}{\tilde{N}_{i+1}^0} \right) = 0 \tag{C.10}
\]

\[
\frac{im}{2} \left( \frac{B_{i-1}^m (R_\odot) \tilde{N}_i^\infty}{\tilde{N}_i^0} + \alpha_1 \frac{B_i^m (R_\odot)}{\tilde{N}_i^0} (\tilde{N}_{i-1}^\infty - \tilde{N}_i^\infty) \right) - k_1 \alpha_1 \frac{B_{i-1}^m (R_\odot) \tilde{N}_{i-1}^\infty}{\tilde{N}_{i-1}^0} + k_2 \alpha_1 \frac{B_{i+1}^m (R_\odot) \tilde{N}_{i+1}^\infty}{\tilde{N}_{i+1}^0} = 0
\]
\[ \text{im} \alpha_1 a^m_l \left( \bar{J}^\infty_l - \frac{\bar{J}_l^0}{\bar{N}_l^0} \right) + a^m_{l-1} \frac{k_1}{2} \left( \bar{J}^{\infty}_{l-1} - \frac{\bar{J}^0_{l-1}}{\bar{N}^{l-1}_{l-1}} \right) + \alpha_1 \left( \bar{J}^{\infty}_{l-2} - \bar{J}_l^\infty \right) - \alpha_1 \frac{\bar{J}^0_{l-1}}{\bar{N}^{l-1}_{l-1}} \left( \bar{N}^{\infty}_{l-2} - \bar{N}^\infty_l \right) \\
- a^m_{l+1} \frac{k_2}{2} \left( \bar{J}^{\infty}_{l+1} - \frac{\bar{J}^0_{l+1}}{\bar{N}^{l+1}_{l+1}} \right) + \alpha_1 \left( \bar{J}^{\infty}_{l+2} - \bar{J}_l^{\infty} \right) - \alpha_1 \frac{\bar{J}^0_{l+1}}{\bar{N}^{l+1}_{l+1}} \left( \bar{N}^{\infty}_{l+2} - \bar{N}^\infty_l \right) \\
+ \text{im} \alpha_1 \frac{B^m_l(R_\odot)}{\bar{N}^0_l} \left( \bar{N}_l^{\infty} \right) + \frac{k_1}{2} \left( \frac{B^m_{l-1}(R_\odot)}{\bar{N}^{l-1}_{l-1}} \right) + \alpha_1 \frac{B^m_{l-1}(R_\odot)}{\bar{N}^{l-1}_{l-1}} \left( \bar{N}^{\infty}_{l-2} - \bar{N}^\infty_l \right) \\
- \frac{k_2}{2} \left( \frac{B^m_{l+1}(R_\odot)}{\bar{N}^{l+1}_{l+1}} \right) + \alpha_1 \frac{B^m_{l+1}(R_\odot)}{\bar{N}^{l+1}_{l+1}} \left( \bar{N}^{\infty}_{l+2} - \bar{N}^\infty_l \right) = 0 \quad (C.11) \]

These give two recursive relationships for \( a^m_{l-1}, a^m_l \) and \( a^m_{l+1} \) which can be solved simultaneously to find \( a^m_{l+1} \) in terms of \( a^m_l \), for example. Given two recursive relationships of the form:

\[ p_{-1} a^m_{l-1} + p_0 a^m_l + p_1 a^m_{l+1} + p_2 = 0 \]
\[ q_{-1} a^m_{l-1} + q_0 a^m_l + q_1 a^m_{l+1} + q_2 = 0 \]

the \( a^m_{l-1} \) can be eliminated to find

\[ a^m_{l+1} = \frac{(q_{-1} p_0 - p_{-1} q_0) a^m_l + p_2 q_{-1} - p_2 q_{-1}}{p_{-1} q_1 - q_{-1} p_1} \]

Using this, it should now be possible to start at \( a^m_m \) for each \( m \) and iterate upwards through each \( l \) finding all coefficients. The equivalent \( b^m_l \) can then be found using equation C.9 above. This should then give all \( a^m_m \) and \( b^m_l \) to find the MHS model with a source surface. However, as mentioned, as this is a overspecification of the unknown constants, it doesn’t not work.
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