A THEORY OF GRAVITATION INCORPORATING THE QUADRATIC ACTION PRINCIPLE OF RELATIVITY

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A Theory of Gravitation incorporating the Quadratic Action Principle of Relativity

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Declaration

I hereby declare that the accompanying Thesis is my own composition, that it is based upon research carried out by me. and that no part of it has previously been presented in application for a Higher Degree.

Certificate

I certify that Valerie A. Wynne has spent nine terms as a research student in the Department of Theoretical Physics of the United College of St. Salvator and St. Leonard in the University of St. Andrews, that she has fulfilled the conditions of Ordinance No. 16 of the University Court of St. Andrews and that she is qualified to submit the accompanying Thesis in application for the Degree of Doctor of Philosophy.

Research Supervisor.

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Career

In 1967, I graduated from the University of Leeds with a class II, division (i), honours degree in Mathematics. The following year, 1968, I graduated with the degree of M.Sc. from the University of Leeds. In October, 1968 I was admitted by the Senatus Academicus of the University of St. Andrews as a Research Student.

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Into this Universe, and *Why* not knowing,
Nor *Whence*, like Water willy-nilly flowing;
    And out of it, as Wind along the Waste,
I know not *Whither*, willy-nilly blowing.

Omar Khayyám

Abstract

The hypothesis adopted in this work is that any permissible metric field whatsoever must satisfy the field equations deduced from an action principle in which the Lagrangian is quadratic in the components of the Riemann curvature tensor. The adoption of such a hypothesis is motivated by the precariousness of the general relativistic interpretation of Mach's principle, which is often used to justify a phenomenological approach to the theory. The quadratic action principle is chosen to provide the fundamental equations of the gravitational field because it is logically and aesthetically appealing, and causes us to lose nothing of the standard relativity theory based on Einstein's vacuum equations. The set of relationships, $R_{\rho\sigma} - \frac{1}{2}g_{\rho\sigma}R = -\kappa T_{\rho\sigma}$, is retained as a definition of the matter tensor $T_{\rho\sigma}$.

Attention is concentrated on the solutions of the (generally fourth order) fundamental field equations in the static, spherically symmetric case. Sets of exact, series and numerical solutions are obtained corresponding to certain boundary conditions, or with certain properties in common. Study of the geometrical, topological and physical properties of several of the universes obtained as a result of our hypothesis leads us to believe that our theory is not implausible. We conclude by considering the further possibilities of the theory.

INTRODUCTION

The theory of gravitation that will be described is not in any sense an alternative to Einstein's theory. Of all modern physical theories, the latter must surely be the most beautiful, both in terms of the underlying physical and philosophical ideas and in its mathematical elegance. It is suggested that what we describe be regarded as a tentative extension of the theory, in which matter is represented in a more natural way, arising clearly as an aspect of geometry.

Let us consider the framework within which matter is represented in standard relativity theory. Space-time is a four dimensional Riemannian manifold of normal hyperbolic type. The Einstein gravitational equations for free space, $R_{\rho\sigma} = 0$ (in which the components of the Ricci tensor $R_{\rho\sigma}$ are obtained from a differential combination of the components of the metric tensor $g_{\rho\sigma}$), describe the coupling of the gravitational field to the space-time geometry. These are generalised, in order to express the idea that all forms of energy also act gravitationally, to the form $R_{\rho\sigma}-\frac{1}{2}g_{\rho\sigma}R = -\kappa T_{\rho\sigma}$, where the stress-energy (matter) tensor $T_{\rho\sigma}$ represents the contribution from all force fields other than gravitation. Thus, two fields — the matter field and the metric field — together satisfy Einstein's gravitational field equations for non-empty space.

Mathematically, since there are ten unknown $g_{\rho\sigma}$ and ten unknown $T_{\rho\sigma}$, the set of ten field equations is highly underdetermined for solution. The problem of solving them becomes trivial if it is assumed that the metric tensor is given arbitrarily, for then we have only to calculate the $R_{\rho\sigma}$ from the $g_{\rho\sigma}$ and read off the components of the matter tensor $T_{\rho\sigma}$. In most cases the $T_{\rho\sigma}$ obtained in this manner will be unphysical, hence this procedure cannot be justified and is not used. An improvement on this approach is to suppose that the metric and matter tensors are in some way of "equal significance", with the Einstein equations representing a set of constraints on the twenty unknown quantities. The latter are further restricted by the imposition of conditions of signature and smoothness on the $g_{\rho\sigma}$ and reality of eigenvalues and positivity of energy density on the tensor $T_{\rho\sigma}$. This is not really satisfactory as the equations are still indeterminate. The most frequently adopted means of dealing with the equations is to remove their indeterminacy by forming a phenomenological description of matter through the stress-energy tensor so that, with appropriate boundary conditions, the differential equations may be solved to determine the metric field corresponding to the particular matter distribution. Of the different approaches to the field equations, the last one seems the most reasonable.

Viewed on its own, however, this procedure is disquieting. It is an ad hoc method, in which matter is necessarily accorded an *a priori* privileged position. Our conceptions of matter are limited, and we must further limit the matter distributions we consider to those for which the solution of the differential equations presents a tractable problem. If experimental verification of the predictions of general relativity theory is required we need only turn to the solutions of the vacuum equations; and as Einstein pointed out repeatedly in later years, it is the theory of the limiting case of the pure gravitational field and its relation to the metric structure of space-time which can possibly make a claim to final significance.

There are two obstacles in the way of abandonment of the phenomenological approach. Firstly, it may be justified by appeal to the validity of Mach's principle; in this way its defects are camouflaged. Secondly, it is convenient; alternatives generally require that the basic structure of the theory be supplemented or modified and some complications are unavoidably introduced.

The aim of this work is to overcome these two objections. Mach's principle is elusively vague; its status within the confines of general relativity is examined at the outset and seen to be low. We shall give plausibility arguments for the introduction into the theory of a sourceless set of equations for the $g_{\rho\sigma}$, to be regarded as the fundamental equations of the gravitational field. Complications *do* arise, but these are associated with the mathematical problem of solving differential equations; the basic framework of the theory is retained.

It is, of course, essential that the choice of the equations to serve as the fundamental equations of the field does not involve the destruction of that which stands firm in the theory: the solutions of Einstein's vacuum equations. These are retained if we take as our basic equations those deduced from an action principle in which the Lagrangian is quadratic in the components of the Riemann curvature tensor; as we shall see in chapter 2, this choice has a certain justification on the grounds of logic and of aesthetic appeal. The set of relationships, $R_{\rho\sigma}-\frac{1}{2}g_{\rho\sigma}R=-\kappa T_{\rho\sigma}$, cease to be regarded as field equations, but rather as a definition of the matter tensor $T_{\rho\sigma}$. We make the hypothesis that any permissible metric field whatsoever must satisfy the field equations of the quadratic action principle, and that the associated matter distribution is determined from $T_{\rho\sigma}=-\frac{1}{\kappa}(R_{\rho\sigma}-\frac{1}{2}g_{\rho\sigma}R)$. In this way, matter is represented in the theory, though it would be naïve to expect to obtain a realistic model of elementary particles therefrom.

There have been various interesting attempts to obtain realistic descriptions of matter from purely field theoretic considerations, without modifying Riemannian geometry. The Rainich-Misner-Wheeler "already-unified" theory[1,2] of the gravitational-electromagnetic field operates with equations which involve only the contracted

Riemann tensor $R_{\rho\sigma}$, and eliminates the singularities of the field by geometrical construction. Das and Coffman[3] have worked with the fully determinate system of the combined Klein-Gordon-Maxwell-Einstein equations obtained by Das[3] as a result of introducing the complex scalar field into general relativity to replace the standard phenomenological approach. Lanczos[4,5] has used the quadratic action principle in general relativity in order to construct some static, non-singular models of particles through the assumption that material particles represent only a weak superstructure on a very strong metrical substructure.

Our main interest will be in studying the solutions of the exact, non-linearised field equations obtained from the quadratic action principle in its most general form suggested by Lanczos. Due to the complexity of these equations, an algebraic classification of admissible space-times by their Weyl tensor, or by the continuous groups of motions that they admit, though desirable, seems out of the question. Instead we concentrate attention on the solutions in the static, spherically symmetric case for which the field equations reduce to a pair of coupled differential equations, one of fourth and the other of third order. Our method of approach is to impose certain boundary conditions at the outset and search for solutions with specific properties; these properties are, however, those that would be expected of reasonable space-times. The results of chapter 3 reduce the problem — and the order of the differential equations — to a considerable extent and sets of series solutions exhausting the various possibilities are obtained in chapter 4. In addition, we obtain several exact solutions, including the complete set of space-times Minkowskian at spatial infinity. The geometrical, topological and physical properties of some of the exact solutions are studied in chapter 5. The cases studied in chapter 3 for which we have been unable to obtain exact solutions are dealt with numerically in chapter 6, the work being based on the series expansion results of chapter 4. Buchdahl[6] has studied the field equations generated by a quadratic Lagrangian that is a component of the complete Lagrangian which we shall use. The application of his methods to the complete field equations is investigated in chapter 7 and a further set of exact solutions obtained. We conclude in chapter 8 by considering the further possibilities of the theory and by examining whether or not our objectives have been attained. First, we provide the details of the conventions to be followed and notation to be employed throughout this thesis. Then we return to the task of motivating this work by entering into a brief discussion of Mach's principle.

NOTATION

The conventions and notation that will be employed throughout this thesis are as follows:

Greek indices take the values $1 \rightarrow 4$, Latin indices $1 \rightarrow 3$ and the summation convention is followed.

Space-time, represented by a four-dimensional Riemannian manifold V_4 with metric form

$$ds^2 = g_{\rho\sigma}\, dx^\rho\, dx^\sigma$$

is of normal hyperbolic type with signature - 2.

The metric tensor $g_{\rho\sigma}$ has determinant $g = |g_{\rho\sigma}|$.

The full Riemann-Christoffel (Riemann) curvature tensor is given by:

$$R_{\alpha\beta\gamma\delta} = \frac{\partial}{\partial x^\gamma}[\beta\delta,\alpha] - \frac{\partial}{\partial x^\delta}[\beta\gamma,\alpha] + \left\{{\rho \atop \beta\gamma}\right\}[\alpha\delta,\rho] - \left\{{\rho \atop \beta\delta}\right\}[\alpha\gamma,\rho],$$

where $[\alpha\beta,\gamma]$, $\left\{{\alpha \atop \beta\gamma}\right\}$ represent the Christoffel symbols of first and second kind respectively, defined according to

$$[\alpha\beta,\gamma] = \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \right),$$

$$\left\{{\alpha \atop \beta\gamma}\right\} \equiv \Gamma^\alpha_{\beta\gamma} = g^{\alpha\delta}\,[\beta\gamma,\delta].$$

The Ricci tensor is defined in terms of the Riemann tensor:

$$R_{\beta\gamma} = R^{\alpha}_{\cdot\beta\gamma\alpha}$$

and the curvature invariant (Riemann scalar) is given by:

$$R = g^{\beta\gamma} R_{\beta\gamma} .$$

The partial and covariant derivatives of a tensor component will be indicated by a comma and a semicolon subscript respectively.

The fundamental alternating tensor density in V_4 is denoted by $\varepsilon^{\alpha\beta\gamma\delta}$, which has the value + 1 when the indices form an even permutation of the natural order 1234, and - 1 when they form an odd permutation, and the value 0 if any two (or more) indices are the same.

Einstein's constant of gravitation, $\kappa = 2.073 \times 10^{-48}$ cm^{-1} gm^{-1} sec^{2} is related to the Newtonian constant $G = 6.667 \times 10^{-8}$ dyne cm^{2} gm^{-2} by

$$\kappa = \frac{8\pi G}{c^4}$$

where $c = 3.00 \times 10^{10}$ cm sec^{-1} is the velocity of light in vacuo. We shall generally use units such that the constant c has the value unity.

Appendices are labelled according to the chapter to which they refer. For example, appendix A2.3 refers to section 3 of chapter 2.

1. MACH'S PRINCIPLE: A DISCUSSION

Synge[7] has referred to Mach's principle and the Equivalence principle as "a pair of decaying and dangerous ruins." Whether or not this is an overstatement, the relationship of Mach's principle to general relativity is still a subject for debate. Although Einstein was led towards his formulation of general relativity as a result of Mach's ideas, his view was ultimately quite different from that of Mach: that the world is understandable entirely in terms of geometry, which exists before any physical experiment is carried out. He came to regard all attempts to represent matter in the theory as unsatisfactory and, in order to free it from any particular choice of matter tensor, dealt eventually only with the gravitational equations of free space.[8]

It seems that Mach's principle in the sense in which it is usually understood is illusory; it has never found expression in a satisfactory mathematical form, nor can it be verified experimentally. Mach's principle is not basic to the theory of relativity, nor does that theory imply it. Let us review how this is so.

Mach's positivistic view was that only experiments have a physical meaning and any theory should be concerned and built up only with observable material objects. He believed that the principles of science offer an economic description of a great diversity of sense observations and expressed the idea — Mach's principle — that the familiar inertial effects observed terrestially are "reducible to the comportment of the earth with respect to the remote heavenly bodies. If we were to assert we know more of moving objects than this their last mentioned, experimentally-given comportment with respect to the celestial bodies we should render ourselves culpable of a falsity".[9] This is a rather vague, untestable statement that inertial properties are determined by the actual contents of the universe in the large. The mechanism by means of which the local behaviour is influenced is not made clear, but the principle implies that it is not just coincidence that bodies rotating with respect to the stars experience centrifugal force, while others do not. Mach's principle usually finds its interpretation in general relativity in a form equivalent to the statement that the geometry of space-time is not given *a priori* but is only determined by the matter present.[10]

The Einstein equations ($R_{\rho\sigma}-\frac{1}{2}g_{\rho\sigma}R = -\kappa T_{\rho\sigma}$) may be regarded as expressing the dependence of the space-time geometry on the matter distribution. Einstein's interpretation[11] of Mach's principle required that the inertial field (defined by the geometrical quantities $g_{\rho\sigma}$) should be not merely dependent upon, but indeed be entirely determined by the matter present (represented by the $T_{\rho\sigma}$). In order to ensure this the Einstein differential equations must be supplemented by boundary conditions to change the *dependency* of the $g_{\rho\sigma}$ on the $T_{\rho\sigma}$ into the stronger relationship of *determinacy*. It was demonstrated by Einstein that it is impossible to choose boundary conditions so that the inertial field is fully determined from the field equations, since a neutral test particle in the Schwarzschild field will have inertial properties as nearly Newtonian as required at indefinitely large distances. It cannot be maintained that these are due to the central mass.[12] In view of this, Einstein introduced the cosmological term into the field equations, for in this way all the difficulties at infinity were avoided.

Implicit in Mach's philosophy is the idea that a world without matter is inconceivable. Accordingly, Einstein required not only that the inertial field should be completely determined by the matter present, but also that it should be completely indeterminate in the absence of matter. However, the cosmological equations

($R_{\rho\sigma} - \frac{1}{2} g_{\rho\sigma} R + \Lambda g_{\rho\sigma} = -\kappa T_{\rho\sigma}$) which he now assumed admit as a solution the de Sitter space-time for which $T_{\rho\sigma} = 0$ and the inertial properties of neutral test particles are well-defined (they accelerate away from an observer at the spatial origin). That is, although there is no matter in the universe, the inertial field is completely determined. Einstein could not reconcile this with his interpretation of Mach's principle.

A further nail was put in the Machian coffin with the discovery of the Gödel solution in 1949.[13] For even if we do not accept Einstein's postulate that the $g_{\rho\sigma}$ should be indeterminate when $T_{\rho\sigma} = 0$, from the point of view of almost any interpretation of Mach's principle we should expect that the matter of the universe should *uniquely* determine the geometry of the universe. Gödel's solution meant that there are two, essentially different solutions of Einstein's cosmological equations in which $T_{\rho\sigma}$ represents an incoherent matter distribution: the Gödel cosmos and Einstein's static universe. Furthermore, according to Mach's principle it might be expected that if the bulk matter is at rest in a particular coordinate system then the path of a single test particle (given an initial radial velocity) should not rotate relative to it; the system should be inertial.[14] But this is not the case for the Gödel universe. Given an initial radial velocity in the direction of distant matter, a test particle will spiral outwards instead of travelling in the straight line that one would expect on the basis of Mach's principle. The Gödel universe is a consequence of general relativity but is not consistent with Mach's principle; the latter is in no way built into general relativity.

Then if Mach's principle is to have any real meaning in the context of general relativity it must be built in by the addition of some suitable boundary condition. It was mentioned previously that it may be argued that the Schwarzschild universe is non-Machian. As a result the required boundary condition is often taken to be that only those solutions of Einstein's equations (the cosmological term having been removed) are allowed which are such that the spatial geometry is closed and singularity free. Even so, problems arise. In 1951 Taub[15] took the general relativistic interpretation of Mach's principle to be the statement that the nature of space-time is determined by the matter present. The latter is to be described by the stress-energy tensor $T_{\rho\sigma}$ or by singularities in the $g_{\rho\sigma}$. Taub points out that on the basis of this interpretation one could conclude that, in the absence of matter ($T_{\rho\sigma} = 0$) and singularities in the $g_{\rho\sigma}$, the field equations ($R_{\rho\sigma} = 0$) should imply that space is flat ($R^{\mu}_{\cdot\rho\sigma\nu} = 0$). He gives counter-examples in the form of space-times admitting transitive three parameter groups of motions which are not flat and for which the $g_{\rho\sigma}$ are finite for all finite values of the time. This would tend to imply that the above conclusion, a consequence of Mach's principle, is incorrect. Taub reminds us, however, that the lack of a suitable definition in the theory of a real singularity — corresponding to matter — means that the implication cannot be firmly made. More recently, McVittie[16] has pointed out that although the objection has been made to Taub's solution that it must contain a singularity of the energy tensor, the physical system represented by the singularity has not been indicated. A corresponding objection to the solution of Ozsváth and Shücking[17] (similar in that $R_{\rho\sigma} = 0 \not\Rightarrow R^{\mu}_{\cdot\rho\sigma\nu} = 0$) has not been raised.

The Taub universe apparently satisfies the conditions that it is spatially closed and singularity free. It is also an example of a pure gravitational field. Then, if Mach's principle is still to have any meaning, it is necessary to specify the distribution of gravitational radiation in addition to the matter distribution.

This leads to the well known "circularity" argument[18], and it seems that this may be avoided only by formulating the principle precisely and putting it into mathematical form.

However, Mach's principle strongly resists its mathematization.[19] In his work on the problem, Lynden-Bell[20] has given a convincing argument (based on the fact that the null cones are fundamentally invariant structures of space-time) that if the inertia of a body is attributed to the influence of distant matter, then the local space-time in which the body is situated must be attributed to the same cause. His mathematical formulation (in which Einstein's equations are written as explicitly covariant integral equations involving retarded bi-tensor Green's functions) leads to a scheme for determining which universes are Machian, but it seems that this may be restrictive and difficult to apply. Lynden-Bell's conclusion is that any Machian universe must be such that the influence of matter propagates out to make space and it is that space over which later influences propagate out to make the space at a later time. In recent work, McCrea[21] interprets this as saying that the model must be caused by the model. He comments "It is hard to see what this could mean"; furthermore, he reaffirms "... the discussion of Mach's principle in the context of general relativity is given some significance only by retaining concepts of pre-general relativity physics. I consider that Mach's principle has never been formulated strictly within the concepts of general relativity."

From the preceding discussion, it has become clear that we shall violate little or nothing by not seeking to adhere to Mach's principle. With reference to McCrea's comment, it seems to us that perhaps one should not be surprised at drawing a meaningless conclusion from the meaningless premis that there exists a causal relationship between matter and geometry, related as in Einstein's theory. Schrödinger too has rejected such a causal notion.[22] Drawing a parallel with Poisson's equation in electrostatics, he tells us not to regard the set of relationships $R_{\rho\sigma} - \frac{1}{2} g_{\rho\sigma} R = -\kappa T_{\rho\sigma}$ as field equations, in which matter *causes* the geometrical quantity on the left hand side to be other than zero, but as a *definition* of the matter tensor $T_{\rho\sigma}$. It is not an uncommon view that matter is in some way a secondary phenomenon to the underlying reality of space-time. This was Eddington's opinion, for example.[23] He regarded energy, stress and momentum as belonging to the world and not to some extraneous substance in the world. But care must be taken that one philosophically unsatisfactory attitude is not replaced by another, in which we regard, consciously or unconsciously, matter to be *caused* by geometry. If we are to take the view that there is one fundamental reality — space-time — and that what we familiarly call matter is an aspect of it — the certain quantity defined by Einstein — then we must find a means within the concepts of general relativity of deciding which space-time geometries are possible. Thus, we look for a sourceless set of field equations for the $g_{\rho\sigma}$.

2. THE FIELD EQUATIONS

2.1 Introduction

Einstein said, "It is my conviction that pure mathematical construction enables us to discover the concepts and the laws connecting them which gives us the key to the understanding of the phenomena of nature".[24] In this spirit, let us check what our construction involves.[25]

Geometry is of Riemannian type. The means of deciding which particular space-time geometries are permissible is provided by a self-contained set of field equations for the metric gensor $g_{\rho\sigma}$. Satisfaction of these fundamental gravitational field equations does not mean that the gravitational field is necessarily "pure"; the matter distribution is given by substitution of the determined $g_{\rho\sigma}$ into Einstein's equations for non-empty space, $T_{\rho\sigma} = -\frac{1}{\kappa}(R_{\rho\sigma} - \frac{1}{2} g_{\rho\sigma} R)$, which are now regarded as a definition of the matter tensor. For consistency with standard relativity theory, all space-times which satisfy Einstein's gravitational equations for free space, $R_{\rho\sigma} = 0$ (equivalently, $R_{\rho\sigma} - \frac{1}{2} g_{\rho\sigma} R = 0$), must be solutions of our field equations. In view of the universal appearance of action principles in all branches of theoretical physics, and in particular, since Einstein's gravitational equations for free space are deducible from one, we expect an action principle to provide the field equations that we require. Since Riemannian geometry is retained, the basic Lagrangian must be composed of the components of the Riemann-Christoffel tensor, and in order that the field laws be independent of the accidental choice of coordinate system to be used, the action integral must be invariant with respect to arbitrary coordinate transformations. We now consider the choice of Lagrangian, the field equations derived from it, and their first consequences.

2.2 The Action Lagrangian

It is well known that the simplest non-trivial Lagrangian density that can be formed algebraically from the $g_{\rho\sigma}$ and the components of the Riemann-Christoffel tensor is based on the Riemann scalar R; and that the Hamiltonian derivative $\delta(R\sqrt{-g})/\delta g^{\rho\sigma}$ of this Lagrangian density is equal to the Einstein tensor density $(R_{\rho\sigma}-\frac{1}{2}g_{\rho\sigma}R)\sqrt{-g}$.[22] Since it is necessary that the Lagrangian density of the action principle is a scalar density in order to make the action integral an invariant, the next most simple Lagrangian density that can be formed will be based on scalars which are quadratic in the components of the curvature tensor. Due to an algebraic identity, the set of such scalars may be reduced to[26]

$$L_1 = R_{\rho\sigma}R^{\rho\sigma}, \quad L_2 = R^2, \quad L_3 = R_{\rho\sigma\alpha\tau}R^{\rho\sigma\alpha\tau},$$
$$L_4 = \frac{1}{\sqrt{-g}}\,\varepsilon^{\alpha\tau\gamma\delta}R_{\alpha\tau\rho\sigma}R_{\gamma\delta}{}^{\rho\sigma}{}_{\cdot\cdot} \quad .$$

Now the Hamiltonian derivatives of $(L_2-4L_1+L_3)\sqrt{-g}$ and $L_4\sqrt{-g}$ vanish identically[27] so that the only quadratic terms that we shall need to consider are L_1 and L_2.

Various motivating factors have influenced the investigation, from time to time, of Lagrangians incorporating $L_1, \ldots, L_4$. Sometimes a term representing matter is included in addition. Weyl[28] introduced such a Lagrangian when he proposed a generalisation of Riemannian geometry in order to embed into it the theory of the electromagnetic field. This proposal was taken up principally by Pauli[29] and Eddington.[30] Lanczos, convinced of the necessity for the fundamental action integral of general relativity to be independent of the arbitrary units employed in measuring lengths in space-time,[27] i.e., gauge invariant, has studied purely quadratic Lagrangians.[4,5] As mentioned in the introduction, he has used the field equations derived therefrom in order to construct some static, non-singular particle models. Buchdahl[6,31] has investigated the field equations arising from quadratic Lagrangians, in particular, from L_2, regarding them as equations of the pure gravitational field to replace those derivable from Einstein's linear R . Gregory[32] has considered the effect of adding the Lagrangian L_2 to R insofar as the problem of motion is concerned. A similar combination is employed by Pechlaner and Sexl[33] in a phenomenological approach following that of standard relativity theory; they suggest that such a theory would yield unrealistic results.

Let us consider the choice of our basic Lagrangian L . As implied previously, any quadratic Lagrangian in V_4 is such that the action integral is not only invariant under arbitrary coordinate transformations but also invariant under changes in gauge. This is because the four-dimensional element of volume, $dv = \sqrt{-g}\, d^4x$, has the dimensions of the fourth power of a length and the Ricci tensor $R_{\rho\sigma}$ (obtained from the second order, and first order – second degree derivatives of the $g_{\rho\sigma}$) has the dimensions of the inverse of a length squared. If the action integral is other than gauge invariant, it can (formally) be made arbitrarily small by a suitable choice of the units in which lengths are measured. Lanczos[27] contends that the minimising procedure is consequently without meaning. Such an argument, however, loses validity in view of the fact that although the variational procedure ensures that the action is stationary, the action is rarely a true minimum. Thus, unlike Lanczos', our choice is not governed by any need for gauge invariance. Our Lagrangian could be composed of L_1 , L_2 and the Riemann scalar R , i.e.,

$$L = \alpha L_1 + \beta L_2 + \gamma R$$

in which the constants α and β are pure numbers while the constant γ must have the dimensions of the inverse of a length squared. In this way a fundamental unit of length would be introduced into the theory. The view could be taken that at least one fundamental constant with the dimensions of a length should be present in order to fix the size of particles — Nature does not allow particles of arbitrary radii. However, the presence of the γ-term would considerably weaken the field equations. Its inclusion would mean that there is no *a priori* reason for rejecting an additional "cosmological" type of Lagrangian term, Λ, with the dimensions of the inverse of a length to the fourth power. It seems logically persuasive that there should be as few terms as possible in the Lagrangian. Consequently, in the present work, we put γ equal to zero, bearing in mind that inclusion of the linear term may well yield a more realistic model. The pure number α may be chosen to have the value unity (for the field equations arising when the Lagrangian L_2 is present alone, i.e., the case $\alpha = 0$, has been studied already by Buchdahl.[6] We shall consider the application of his methods to our Lagrangian in Chapter 9). Accordingly we shall henceforth consider the purely quadratic Lagrangian

$$\begin{aligned} L &= L_1 + \beta L_2 \\ &= R_{\rho\sigma} R^{\rho\sigma} + \beta R^2 \;. \end{aligned} \qquad 2.1$$

The Lagrangian (2.1) is similar to the Lagrangian introduced by Weyl in his modification of Riemannian geometry, mentioned earlier in this section. It was essential for Weyl's theory that only the ratios between the components of the metric tensor should be fully determined from the field equations. This was because he hypothesised that not only is the transference of the direction of a vector from one point to another path dependent, as in Riemannian geometry, but there is also a corresponding path dependence for the transference of length. Lanczos[34] points out the similarities and differences between a theory based on the Lagrangian (2.1) and Weyl's theory; Einstein's criticism of Weyl's theory on the grounds of the indeterminacy of the $g_{\rho\sigma}$ is inapplicable to ours because in our case the $g_{\rho\sigma}$

are fully determined. There is a point of tangency[34] between the field equations obtained from (2.1) and those of Weyl; this fact will be very useful to us in obtaining solutions of the field equations.

2.3 The Field Equations and their First Consequences

Our hypothesis is that any allowable metric field whatsoever must satisfy the field equations obtained from the vanishing of the Hamiltonian derivatives of $L\sqrt{-g}$ where L is given by (2.1). The action integral is

$$I = \int_{D_4} \left(R_{\rho\sigma} R^{\rho\sigma} + \beta R^2 \right) \sqrt{-g}\, d^4x \quad , \qquad 2.2$$

and the field equations obtained by setting the Hamiltonian derivatives equal to zero are[35]

$$G_{\rho\sigma} = 0 \quad ,$$

where 2.3

$$\begin{aligned} G_{\rho\sigma} = \; & g_{\rho\sigma} R^{\alpha\tau}{}_{;\alpha\tau} + g^{\alpha\tau} R_{\rho\sigma;\alpha\tau} - R^{\alpha}{}_{\cdot\sigma;\rho\alpha} - R^{\alpha}{}_{\cdot\rho;\sigma\alpha} \\ & + \tfrac{1}{2} g_{\rho\sigma} R_{\alpha\tau} R^{\alpha\tau} - 2 R_{\alpha\rho} R^{\alpha}{}_{\cdot\sigma} \\ & + \beta \left(2 g_{\rho\sigma} g^{\alpha\tau} R_{;\alpha\tau} - R_{;\rho\sigma} - R_{;\sigma\rho} + \tfrac{1}{2} g_{\rho\sigma} R^2 - 2 R R_{\rho\sigma} \right) \quad . \end{aligned}$$

Provided that the metric tensor satisfies (2.3) we may insert it into Einstein's equations

$$T_{\rho\sigma} = -\frac{1}{\kappa} \left(R_{\rho\sigma} - \tfrac{1}{2} R g_{\rho\sigma} \right) \qquad 2.4$$

to determine $T_{\rho\sigma}$. The set of all solutions of (2.3) will thus indicate what distributions of energy-momentum-stress may be permitted in Nature, within the limitations of our hypothesis.

We now look for the immediate consequences of (2.3). At the beginning of this chapter it was specified that all space-times which satisfy Einstein's gravitational equations for free space should be solutions of our field equations. It is easy to see[36] that a first consequence of (2.3) is that any Einstein space, given by

$$R_{\rho\sigma} = \Lambda g_{\rho\sigma} \quad , \tag{2.5}$$

where Λ is an arbitrary constant, satisfies (2.3) for all values of the constant β. In particular, since Λ may be chosen to have the value zero, there is no conflict with any of the standard relativity theory based on the vacuum equations, as required. A theory based on the field equations obtained from the Lagrangian (2.1) thus includes the standard relativity theory described by (2.5); the set of solutions of (2.3) augments those of (2.5).

A second consequence is obtained by transvection of (2.3) with $g^{\rho\sigma}$. This gives:

$$G \equiv G^{\cdot\rho}_{\cdot\rho} = 2(1+3\beta)\, g^{\alpha\tau} R_{;\alpha\tau} = 0 \quad , \tag{2.6}$$

so that any solution of (2.3) must satisfy

$$g^{\alpha\tau} R_{;\alpha\tau} = 0 \tag{2.7}$$

unless $\beta = -\frac{1}{3}$ in which case $G^{\cdot\rho}_{\cdot\rho}$ is identically zero and Lanczos' equations are underdetermined for solution since an equation is lost. Now Buchdahl[31] has shown that the field equations generated by the Lagrangian (2.1) in which $\beta = -\frac{1}{3}$ are satisfied by all spaces conformal to an Einstein space. This may be seen from the fact, mentioned in the preceding section, that there is a point of tangency between the theory considered here and Weyl's theory. This occurs when the Lagrangian of Weyl's theory is that considered by Pauli[29] and for the specific value $\beta = -\frac{1}{3}$. Now for all values of β, a subset S_1 of the solution set S of Lanczos' equations is the set of Einstein spaces, given by (2.5). When $\beta = -\frac{1}{3}$, S is the same as the solution set of Weyl's theory for which only the ratios of the $g_{\rho\sigma}$ are determined, that is, a subset S_2, $(S_1 \subset S_2)$, of S is the set of space-times arbitrarily conformal to spaces given by (2.5).

2.4 Specialisation to the Static, Spherically Symmetric Case

It does not seem possible to obtain readily further consequences of the field equations (2.3) without the imposition of symmetry requirements. In view of the comparative simplicity of spherical symmetry it is natural to consider this first in any detailed examination of the differential equations that may arise in general relativity.

There are many equivalent ways of writing the metric in this case. Following Tolman[37] the metric form for a spherically symmetric space-time may be written:

$$ds^2 = -e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) + e^{\nu} dt^2 \quad , \qquad 2.8$$

where the functions λ and ν are generally dependent upon the radial coordinate r and the time coordinate t ; (2.8) has the advantage of being simple and conceptually convenient. It is well known (Birkhoff's theorem[38]) that for Einstein spaces, given by (2.5), the time dependence of the components of the metric tensor in the spherically symmetric case may be removed by a coordinate transformation, but we cannot say that this is true for spherically symmetric space-times satisfying (2.3). Thus we make the additional assumption that the field is static; i.e. that the g_{i4} vanish and the remaining $g_{\rho\sigma}$ are independent of the time coordinate t , taking λ and ν to be functions of r only,

$$\lambda = \lambda(r) \ , \ \ \nu = \nu(r) \ .$$

Writing $x^1 \equiv r, x^2 \equiv \theta, x^3 \equiv \phi, x^4 \equiv t$, and with a dash denoting differentiation with respect to r , the $g_{\rho\sigma}$, $g^{\rho\sigma}$ for the metric (2.8) are given in appendix A2.4 by (A2.19), the Christoffel symbols of the second kind $\Gamma^{\alpha}_{\rho\sigma}$ by the set (A2.20), the components of the Ricci tensor (in mixed form) $R^{\rho}_{.\sigma}$ and the curvature invariant R by (A2.21), (A2.22) respectively.

One of the advantages of a variational formulation of a problem is that it allows information about the symmetry of the situation under examination to be inserted before carrying out the variation. Thus, we may form the Lagrangian density of the action integral:

$$(R_{\rho\sigma} R^{\rho\sigma} + \beta R^2)\sqrt{-g}$$

for the metric (2.8) and derive the field equations directly from the action principle. More laboriously, the various terms on the right hand side of (2.3) may be calculated from (A2.19)$\longrightarrow$(A2.22) and combined to form the $G_{\rho\sigma}$ for the spherically symmetric, static field. Although these two methods are equivalent in the present case, a check on accuracy is thereby provided which is not only useful but also essential in view of the lengthy and tedious nature of either calculation. Furthermore, it is well known[39] that the Hamiltonian derivatives of any invariant density that depends only on the $g_{\rho\sigma}$ and their partial derivatives with respect to the coordinates up to any finite order have an identically vanishing divergence. This means that whether or not the $G_{\rho\sigma}$ given by the latter of (2.3) vanish, they must satisfy the identity

$$G^{\rho}_{.\sigma;\rho} \equiv 0 \quad . \qquad 2.9$$

In consequence, for our metric (2.8), G^2_2 (= G^3_3) may be identically expressed in terms of G^1_1 and G^4_4 :

$$G^2_2 \equiv \frac{1}{2}\left[r\left\{\left(\frac{1}{2}\frac{d\nu}{dr} - \frac{1}{r}\right)(G^1_1 - G^4_4) + \frac{dG^1_1}{dr}\right\} + 3G^1_1 - G^4_4\right] \qquad 2.10$$

so that there are only two independent field equations as would be expected, and (2.10) serves as an additional check on our accuracy in obtaining the $G_{\rho\sigma}$ from (2.3). Since the calculation of the $G_{\rho\sigma}$ —

by either means — is long but straightforward it is omitted, as is the expression for G^2_2 since knowledge of it is not required as a result of the identity (2.10).

The system to be solved is thus (in mixed form):

$$\left.\begin{aligned} G^1_1 &= 0 \\ G^4_4 &= 0 \end{aligned}\right\} \qquad 2.11$$

and these two equations are given by (A2.23), (A2.24). Since G^2_2 is expressed in terms of G^1_1 and G^4_4 according to (2.10), the invariant G may be written in terms of G^1_1 and G^4_4 :

$$G = r\left\{\left(\frac{1}{2}\frac{d\nu}{dr} - \frac{1}{r}\right)(G^1_1 - G^4_4) + \frac{dG^1_1}{dr}\right\} + 4G^1_1 \quad .$$

Then a system of equations equivalent to (2.11) is

$$\left.\begin{aligned} G^1_1 &= 0 \\ G &= 0 \end{aligned}\right\} \quad , \qquad 2.12$$

unless $d\nu/dr = 2/r$ in which case the vanishing of G^1_1 is sufficient in itself to ensure the vanishing of G and it is necessary to demand the additional vanishing of G^4_4 . When this case, $d\nu/dr = 2/r$, is investigated it is found that there is no solution as the pair of equations (2.11) are inconsistent. Thus it is sufficient to solve (2.12). A further reduction is possible, since G is given by (2.6):

$$G = 2(1+3\beta)\, g^{\alpha\tau} R_{;\alpha\tau} \quad ,$$

so that, excluding consideration of the special case $\beta = -\frac{1}{3}$, it is sufficient to solve the system

$$\left.\begin{aligned} G^1_1 &= 0 \qquad \text{(a)} \\ g^{\alpha\tau} R_{;\alpha\tau} &= 0 \qquad \text{(b)} \end{aligned}\right\} \qquad 2.13$$

We shall consider the case $\beta = -\frac{1}{3}$ at the end of this section.

The pair (2.13) is more convenient to use than (2.11) because

(2.13b) is a condition on the scalar curvature which does not depend on the value of the constant β . It reads

$$\frac{d^2R}{dr^2} + \left(\frac{1}{2}\frac{d\nu}{dr} - \frac{1}{2}\frac{d\lambda}{dr} + \frac{2}{r}\right)\frac{dR}{dr} = 0 \,, \tag{2.14}$$

which has the first integral

$$\frac{dR}{dr} = \frac{C}{r^2}\, e^{-\frac{1}{2}\nu + \frac{1}{2}\lambda} \,, \tag{2.15}$$

where C is an arbitrary constant. Explicitly (2.13b) is fourth order in ν and third order in λ ; it is not necessary to write it out in full as it may always be conveniently replaced by (2.14). The differential equation (2.13a) is third order in ν and second order in λ ; it is sometimes useful to have it written in terms of λ , ν , their derivatives up to second order only, the curvature invariant R and its first derivative. In this form, $G^1_1 = 0$ is given by (A2.25). All spherically symmetric, static solutions of Lanczos' equations therefore satisfy (A2.25) and the condition (2.14), the Riemann scalar being given by (A2.22).

From (2.15) — more generally, from (2.7) — it can be expected that some solutions of the field equations will be such that the Riemann scalar has a constant value, i.e.

$$R = K \tag{2.16}$$

(K= constant), corresponding to $C = 0$ in (2.15). Explicitly this is:

$$e^{-\lambda}\left\{\left(-\frac{1}{2}\nu'^2 - \nu'' + \frac{1}{2}\nu'\lambda'\right) + \frac{1}{r}(-2\nu' + 2\lambda') - \frac{2}{r^2}\right\} + \frac{2}{r^2} - K = 0 \,. \tag{2.17}$$

There is no *a priori* reason to suppose that (2.16) is not only a possible, but also a necessary consequence of the field equations, as is assumed for his purposes by Lanczos.[40] We shall find, however, in the next chapter, that on the basis of a "regularity condition" this will indeed be the case.

We now turn to the solution of the equations in the underdetermined case $\beta = -\frac{1}{3}$. In the preceding section the result was given that a subset of the complete set of solutions of Lanczos' equations when $\beta = -\frac{1}{3}$ is the set of all space-times conformal to an arbitrary Einstein space. Buchdahl[31] has shown, by employing a simple coordinate transformation, that when the space-time is static and spherically symmetric, solutions of this kind are the only solutions; that is, any member of the set of $\beta = -\frac{1}{3}$ solutions must be reducible to the form:

$$ds^2 = D^2(\rho)\left\{\frac{-d\rho^2}{(1-2m/\rho-\Lambda\rho^2/3)} - \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2) + (1 - \frac{2m}{\rho} - \frac{\Lambda}{3}\rho^2)dt^2\right\} \quad 2.18$$

where m and Λ are arbitrary constants and D is an arbitrary function of ρ. This strong result would be of no further interest, as $\beta = -\frac{1}{3}$ is a degenerate case, if it were not for the fact that contained in (2.18) is a set of well determined solutions of Lanczos' equations; in particular, the set of all solutions Minkowskian at spatial infinity. This latter result depends upon the theorem obtained in section 2 of chapter 3; the set of well determined solutions referred to will be obtained in section 5 of chapter 4.

3. BOUNDARY CONDITIONS AND EXISTENCE THEOREMS FOR THE STATIC, SPHERICALLY SYMMETRIC CASE

3.1 Introduction

The field equations to be solved, (2.13), are differential equations of high order, coupled and non-linear, and there is no hope of obtaining the general solution with the full number of arbitrary constants. Our method will be a simple one: to look for solutions which have certain properties, imitating in some respect the static, spherically symmetric space-times that arise in standard relativity theory. Our interest will be only in space-times which satisfy the field equations for values of β other than $-\frac{1}{3}$; those that satisfy the equations only when $\beta = -\frac{1}{3}$ will not be considered to be true solutions. It will be useful to list those space-times that arise in the spherically symmetric, static case in standard theory.[42]

a) The most reliable verification of general relativity is based on the Schwarzschild line element:

$$ds^2 = \frac{-dr^2}{1-2m/r} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \left(1-\frac{2m}{r}\right)c^2dt^2, \qquad 3.1$$

which satisfies Einstein's vacuum equations

$$R_{\rho\sigma} = 0.$$

The gravitational field represented by (3.1) is regarded as generated by a point mass M situated at the spatial origin. The mass M is related to the constant of integration m by $M = 8\pi m/\kappa c^2$.

b) A second important space-time is the Riessner-Nordström solution, which corresponds to the gravitational field due to the electromagnetic field energy of a point charge at the spatial origin:

$$ds^2 = \frac{-dr^2}{1-\frac{2m}{r}+\frac{\kappa}{2}\frac{e^2}{r^2}} - r^2(d\theta^2+\sin^2\theta\, d\phi^2) + \left(1-\frac{2m}{r}+\frac{\kappa e^2}{2r^2}\right)c^2dt^2. \qquad 3.2$$

This satisfies Einstein's equations

$$R_{\rho\sigma} - \tfrac{1}{2} R g_{\rho\sigma} = -\kappa T_{\rho\sigma} \qquad 3.3$$

in which the energy-momentum tensor $T_{\rho\sigma}$ is due to the electric field of the point charge. The constant of integration m is related to the mass of the particle as in the Schwarzschild solution (case (a)), while the constant of integration e is the charge of the particle.

Both the Schwarzschild solution (3.1) and the Riessner-Nordström solution (3.2) are asymptotically flat, i.e. in the limit as the spatial coordinate r increases without bound the metric approaches the Minkowskian form:

$$ds^2 = -dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + c^2 dt^2 \,. \qquad 3.4$$

Neither (3.1) nor (3.2) is regular at the spatial origin. Three space-times which exhibit regular behaviour are de Sitter space-time, the "interior" Schwarzschild solution and the Einstein universe.

c) <u>De Sitter</u> space-time is unique in having constant space-time curvature everywhere. It has the metric form:

$$ds^2 = \frac{-dr^2}{1-\Lambda r^2/3} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \left(1 - \Lambda\frac{r^2}{3}\right)c^2 dt^2 \,, \qquad 3.5$$

which satisfies the field equations (2.5):

$$R_{\rho\sigma} = \Lambda g_{\rho\sigma} \,.$$

d) If, in (3.3), the energy-momentum tensor $T_{\rho\sigma}$ is taken to be that of a perfect incompressible fluid with constant proper density μ_0, the <u>"interior" Schwarzschild</u> solution is obtained:

$$ds^2 = \frac{-dr^2}{1-\kappa\mu_0 r^2/3} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \left(\frac{3}{2}\frac{\alpha}{\mu_0} + C\sqrt{1 - \tfrac{1}{3}\kappa\mu_0 r^2}\right)^2 c^2 dt^2 . \qquad 3.6$$

The constants of integration α and C are related to the physical properties of the fluid sphere.

e) The Einstein universe,

$$ds^2 = \frac{-dr^2}{1 - r^2/r_0^2} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + c^2 dt^2 , \qquad 3.7$$

is a solution of the cosmological equations

$$R_{\rho\sigma} - \tfrac{1}{2} g_{\rho\sigma} R + \Lambda g_{\rho\sigma} = -\kappa T_{\rho\sigma} . \qquad 3.8$$

The energy-momentum tensor $T_{\rho\sigma}$ is that for an incompressible perfect fluid. The constant of integration r_0, which must be real for a realistic model, is related to the cosmological constant Λ, the constant pressure p and constant proper density μ_0 of the fluid according to

$$\frac{1}{r_0^2} = \Lambda - \kappa p = \frac{\kappa\mu_0}{3} + \frac{\Lambda}{3} .$$

When the pressure p is put equal to zero the line element (3.7) represents the gravitational field of an incoherent matter distribution.

Of the most important spherically symmetric, static space-times of standard relativity theory, (a),....,(e), only the Schwarzschild solution and de Sitter space-time are also solutions of our field equations (2.13). The Riessner-Nordström solution, the "interior" Schwarzschild solution and the Einstein universe (which satisfies the field equations only when $\beta = -\frac{1}{3}$) are lost. We wish to know what solutions arise in our theory to replace them.

Our investigations begin in section 3.2 with the search for solutions to replace the Riessner-Nordström solution, or extend the Schwarzschild solution. Such space-times are flat at spatial infinity and have corresponding matter distribution of only finite influence. We cannot be certain that it is in any way meaningful to talk of "infinity" but such solutions — especially the Schwarzschild solution — play a role in standard theory that is important on historical and psychological grounds at the very least. Thus, it is natural that our attention is turned first in this direction.

It is difficult to conceptualise "infinity" but it is easier to give its meaning mathematically. On the other hand, we feel that we know what is meant by "regularity" but it is difficult to reach a satisfactory definition in general relativity. What we mean by "regular" will be made precise in section 3.3. We shall look for solutions which share this property with de Sitter space-time, the "interior" Schwarzschild solution and the Einstein universe, and which replace the latter two space-times. It is found in section 3.4 that there is no solution, regular or not, which is asymptotically like the Einstein universe. We cannot say, however, that de Sitter space-time is the unique solution with asymptotically constant curvature; solutions which have this property are examined in section 3.5.

It was stated earlier that those solutions in standard theory which are Minkowskian at spatial infinity are not regular at the spatial origin. It is also true that those solutions in standard theory which are regular at the spatial origin are not asymptotically flat. It is thus a question of considerable interest whether or not our field equations permit a solution (other than, of course, the trivial Minkowski solution) which has both properties. In section 3.6 the results of sections 3.2 and 3.3 are combined to provide the information needed to answer this question.

In our investigations in this chapter it is necessary to assume that the unknown functions possess appropriate series expansions, but it is recognised that in using this technique certain solutions could be missed. For example, the function $f(x) = \exp(-x^{-2})$ is such that f and all its derivatives at $x = 0$ vanish, and consequently $f(x)$ has no series expansion in positive powers of x. Our technique is justified, however, by the fact that we are interested in solutions with certain asymptotic properties. This is not our exclusive interest; it is an essential line of attack in dealing with what could otherwise amount to an intractable problem. Solutions without the properties mentioned may arise; their physical interpretation may, however, be obscure.

3.2 The Non-Existence of Asymptotically Flat Space-Times with Non-Vanishing Curvature Invariant

In a preliminary investigation of Lanczos' equations to look for spherically symmetric, static, asymptotically flat space-times the following approach was taken. The unknown functions e^{λ}, e^{ν} of the metric form (2.8) were assumed to have series expansions in terms of decreasing powers of the radial coordinate r with leading term unity. These were substituted into the field equations (2.13). The coefficients of successive powers of $1/r$ were then equated to zero and the equations solved in order to find expressions for the first few unknown coefficients of the negative powers of r in the series expansions of e^{λ}, e^{ν}. It was found that the first coefficients are algebraic functions of only two unknown constants, and do not depend on the value of β. This latter fact is highly suggestive. For, if the series solutions are completely independent of β then they must simultaneously satisfy the equations obtained from the separate vanishing of the Hamiltonian derivatives both of the Lagrangian $R_{\rho\sigma}R^{\rho\sigma}$ and of the Lagrangian R^2. Now Buchdahl[6] has shown that the field equations obtained from the vanishing of the Hamiltonian derivatives of the Lagrangian R^2 have no asymptotically flat, spherically symmetric, static solutions such that the Riemann scalar R is not zero everywhere. It is thus possible that Buchdahl's result is true for the equations obtained from the complete quadratic Lagrangian $R_{\rho\sigma}R^{\rho\sigma}+\beta R^2$. That this is so is demonstrated by the following theorem:

Any spherically symmetric, static metric that is a solution of Lanczos' equations and tends towards the Minkowski form for large values of the radial coordinate must be such that the curvature invariant vanishes everywhere.

For large r we suppose that the unknown functions e^{λ}, e^{ν} of the metric form (2.8) may be expanded:

$$e^{\lambda} = 1 + \frac{a_1}{r} + \frac{a_2}{r^2} + \dots \quad ,$$
$$e^{\nu} = 1 + \frac{b_1}{r} + \frac{b_2}{r^2} + \dots \quad , \qquad 3.9$$

for some constants $a_1, a_2, \dots, b_1, b_2, \dots$.

Then from the expression for the curvature invariant R given by (A2.22) the highest power of r that can occur in its expansion is clearly r^{-3}. Thus:

$$R = r^{-n}\left(C_0 + \frac{C_1}{r} + \frac{C_2}{r^2} + \dots\right), \quad C_0 \neq 0 , \qquad 3.10$$

for some constants $C_0, C_1, C_2, \dots$, and for some integer n :

$$n \geq 3 . \qquad 3.11$$

Now since it is required that the field equations be satisfied, (2.14) (equivalently, (2.15)) is true. With the expressions (3.9) for e^{λ}, e^{ν} , and (3.10) for R , we obtain from (2.14) the condition on n :

$$C_0 n(n-1) = 0$$

which is a contradiction, from (3.11).

Then, in (3.10), C_0 and all successive coefficients must vanish. Thus $R \equiv 0$ to any order in $1/r$ in the region of spatial infinity. Hence $R = 0$ everywhere.

If we suppose more generally that the expansions for e^{λ}, e^{ν} are

$$e^{\lambda} = a_0 + \frac{a_1}{r} + \frac{a_2}{r^2} + \dots \quad , \quad a_0 \neq 0 ,$$
$$e^{\nu} = b_0 + \frac{b_1}{r} + \frac{b_2}{r^2} + \dots \quad , \quad b_0 \neq 0 ,$$

then the proof follows in identical fashion, with the condition (3.11) replaced by $n \geqslant 2$. The additional condition $a_0 = 1$ is obtained; b_0 may be chosen to have the value unity by a transformation of the time coordinate.

As a further generalisation, we may assume that the expansions for e^{λ}, e^{ν} are in powers of $1/r^m$, where m is not necessarily an integer:

$$\begin{aligned} e^{\lambda} &= a_0 + \frac{a_1}{r^m} + \frac{a_2}{r^{2m}} + \dots \quad , \quad a_0 \neq 0 \ , \\ e^{\nu} &= b_0 + \frac{b_1}{r^m} + \frac{b_2}{r^{2m}} + \dots \quad , \quad b_0 \neq 0 \ . \end{aligned} \qquad 3.9'$$

If the field equations are satisfied, it is found that m must have the value unity. This result is proved as a lemma in the appendix to this section, A3.2. The theorem then follows as above.

The theorem that has been proved in this section concerning the boundary condition of asymptotic flatness is of some interest, despite the condition of spherical symmetry on the static field. Buchdahl[6] has shown that when the field equations obtained from the Lagrangian R^2 are being considered, his result may be generalised by abandoning the condition of spherical symmetry; that is, there exist no static (sufficiently often differentiable) solutions of the field equations generated by R^2 which are asymptotically flat but do not satisfy $R = 0$. His methods are discussed in chapter 7; the generalised result may not readily be shown to hold for the field equations generated by $R_{\rho\sigma}R^{\rho\sigma} + \beta R^2$. Buchdahl's result is unfortunately of no great interest since the field equations generated by R^2 are very weak: they are satisfied by *all* space-times with vanishing curvature invariant. However, the field equations generated by the complete quadratic Lagrangian are much stronger; only a finite subset of space-times with $R = 0$ will satisfy them. In view of this, our theorem is extremely useful. As will be shown in the next chapter it has the consequence that we are able to obtain the complete exact solution in the static, spherically symmetric case under the imposition of the boundary condition of asymptotic flatness.

3.3 The Non-Existence of Regular Space-Times with Non-Constant Curvature Invariant

The problem of singularities in general relativity is a crucial one and is only partly solved at the present time. In this section we shall see how it appears in the context of Lanczos' equations.

3.3.1 The Singularity Problem in General Relativity

The task of defining a real, or physical singularity in general relativity is made very difficult as a result of the general covariance of the theory; an apparent singularity of the space-time may not be physical but may exist merely as a result of an unfortunate choice of coordinate system used to describe the space-time.

Intuitive notions as to what should constitute a real singularity fail because we are not considering the behaviour of certain quantities referred to a background metric, as in other field theories, but rather the background metric itself. Geroch[43] has examined the arguments which lead to a definition of a physical singularity based on the idea of geodesic incompleteness, but he shows that no definition is entirely satisfactory due to the existence of geodesically complete space-times containing timelike curves with bounded acceleration and finite total proper length. Hawking and Penrose[44] have carried through a programme the aim of which is to find conditions to predict whether a solution has singularities, the nature of any predicted singularities and the consequences of their existence for general relativity theory. Timelike or null geodesic incompleteness is taken as the indication of the presence of a singularity and it is shown in a "corollary" that any space-time satisfying a set of four conditions, together with Einstein's equations (3.3), cannot satisfy causal geodesic completeness. They say, "We may interpret failure of the causal geodesic completeness condition in our corollary as *virtually* a statement that any space-time satisfying (these four conditions) 'possesses a singularity'...." (our italics).

3.3.2 The Singularity Problem and Lanczos' Equations

In order to apply the Hawking-Penrose corollary[44] to indicate the presence or otherwise of singularities it is necessary to know more about the space-time than the field equations that it satisfies. Thus, no direct application may be made to our field equations to tell us whether or not they admit non-singular solutions. Furthermore, the Hawking-Penrose result is in general inapplicable even with full knowledge of the metric form and corresponding matter distribution. This is because one of the four conditions required by the corollary is an "energy condition": the energy density must be nowhere less than minus each principal pressure nor less than minus the sum of the three principal pressures. This is a completely reasonable condition in view of the aim of the authors of the corollary to relate their results to the known universe in order that these results be experimentally verifiable. However, it is also rather strong and restrictive; the solutions of our field equations will not usually satisfy such a condition.

Thus there is hope that Lanczos' equations may admit non-singular solutions, on the somewhat negative grounds that if the Hawking-Penrose energy conditions are not satisfied there is very little that we can say about the presence or otherwise of singularities. More positively, however, we know that these energy conditions are also violated by the C-field of Hoyle and Narlikar,[45] while the presence of the C-field in the gravitational equations of standard theory can prevent, in a very obvious way, singularities from occuring. It will be generally necessary for us to consider the singular nature of each solution as it occurs, as far as this is possible.

In looking for solutions of our field equations which are regular, we must use a very unsophisticated working definition of "regularity". We shall follow Bondi[46] in saying that a space is non-singular or regular at a point if a local Minkowski tangent space exists at that point. If it does not then, at the point under consideration, the space is singular. We now examine what this means for our metric (2.8).

3.3.3 The Regularity Definition

The point that is problematic when using a pseudo-polar coordinate system r, θ, ϕ, t, as in (2.8), is the spatial origin, $r = 0$. This is because the determinant of the metric tensor

$$g = \det g_{\rho\sigma} = -e^{\lambda+\nu} r^4 \sin^2\theta$$

is either undefined at $r = 0$, since there is no preferred radial direction, or zero there. In either case, the tensor congruent to $g_{\rho\sigma}$, i.e. $g^{\rho\sigma}$, does not have meaning and the field equations break down.

In terms of the regularity of the space-times that satisfy the equations (2.13), our interest will thus be focussed on the spatial origin. Since it is easy to detect when a space-time is not well behaved at points other than $r = 0$ we choose to apply Bondi's definition of regularity specifically to $r = 0$. Then the condition for regularity that we impose on any space-time whose metric is exhibited in (2.8) is that

$$\begin{aligned} &e^{\lambda(r)} \longrightarrow 1 \text{ as } r \longrightarrow 0, \text{ and:} \\ &e^{\nu(r)} \text{ is continuous and bounded away from the origin} \\ &\qquad \text{as } r \longrightarrow 0. \end{aligned} \tag{3.12}$$

This condition, (3.12), is not by itself sufficient to ensure that the determinant of the metric tensor, after transformation to a set of coordinates meaningful at the spatial origin, does not vanish there. For, if a coordinate transformation is applied to (2.8) to put it in the form described by pseudo-Cartesian coordinates, which have validity at the spatial origin, then g is given in terms of e^{λ}, e^{ν} in the new coordinates by

$$g = -e^{\lambda+\nu} \tag{3.13}$$

and it is clear that the condition (3.12) does not stop this from vanishing at $r = 0$. (The formulae of transformation from pseudo-polar coordinates to pseudo-Cartesian coordinates, together with the components of the metric tensor in such coordinates are given in the appendix to this section, A3.3, by equations (A3.31), (A3.32), (A3.33).)

The fact that g may vanish in a particular coordinate system for some value of the coordinates does not mean that there necessarily exists a real singularity at the point represented by the coordinates. For example, when a synchronous coordinate system ($g_{4i} = 0$, $g_{44} = 1$) is used in cosmological problems, g vanishes inevitably for some finite value of the time coordinate, whatever the matter distribution assumed. Belinskii, Khalatnikov and Lifshitz[47] have discussed this matter in their analysis of the problem as to whether or not singularities occur in fully general solutions of Einstein's equations. They show that in this case the singularity is fictitious – it disappears in other coordinate systems. Consequently, we need not be concerned that the definition allows the possibility that g might vanish in some coordinate system.

We shall find, however, that g does not vanish in pseudo-Cartesian coordinates at the spatial origin for solutions of (2.13) that are regular in the sense of (3.12). In addition, all components of the metric tensor $g_{\rho\sigma}$ are well behaved. Then the components of the conjugate tensor are well defined and well behaved at the spatial origin. Any space-time satisfying (2.13) and (3.12) cannot have a physical singularity at $r = 0$.

3.3.4 The Application of the Regularity Condition to the Field Equations

The aim of this section is to demonstrate that there is no solution of the static, spherically symmetric field equations satisfying the regularity condition (3.12) that does not have constant curvature invariant everywhere.

It is first assumed that near the spatial origin e^{λ}, e^{ν} possess series expansions in terms of rising powers of r. In accordance with the condition for regularity (3.12) these may be written

$$
\begin{aligned}
e^{\lambda} &= 1 + a_1 r + a_2 r^2 + a_3 r^3 + \cdots \quad , \\
e^{\nu} &= d_0 r^A \left(1 + c_1 r + c_2 r^2 + c_3 r^3 + \cdots\right), \quad d_0 \neq 0
\end{aligned}
\qquad 3.14
$$

for some constants A, a_1, a_2, a_3, ... , c_1, c_2, c_3, ... , d_0. There are no constraints on A to take non-negative or integer values.

It is now supposed that e^{λ}, e^{ν} given by (3.14) satisfy Lanczos' equations (2.13). The following results are proved as lemmas I, II and III in the appendix to this section, A3.3:

I. The constant A may take only the values 2, -2, 0.

II. The values A = 2, A = -2 are excluded.

III. The expansions (3.14) are in terms of even, non-negative powers of the radial coordinate only.

We may now state and prove the theorem:

Any solution of the static, spherically symmetric field equations that is regular in the sense (3.12) must satisfy, for some constant K :

$$R = K$$

everywhere.

For small r, e^{λ}, e^{ν} may be expanded, according to the results of lemmas I, II and III:

$$e^{\lambda} = 1 + a_2 r^2 + a_4 r^4 + \dots\dots ,$$
$$e^{\nu} = d_0 (1 + c_2 r^2 + c_4 r^4 + \dots) , \quad d_0 \neq 0 , \tag{3.15}$$

for some constants $a_2, a_4, \dots, c_2, c_4, \dots, d_0$.

Then from the expression (A2.22) for the curvature invariant R, with e^{λ}, e^{ν} given by (3.15) we must have

$$R = 6(a_2 - c_2) + D_2 r^2 + D_4 r^4 + \dots , \tag{3.16}$$

for some constants $D_2, D_4, \dots\dots$.

Now since it is required that the field equations be satisfied, (2.14) (equivalently, (2.15)) is true. Clearly, by (2.14), the successive coefficients $D_2, D_4, \dots$ in (3.16) must be equated to zero. Then $R \equiv \text{constant} = K$ to any order in r near the origin. Hence

$$R = K$$

everywhere.

It is easily seen, from the formulae (A3.32), (A3.33) for the metric tensor in pseudo-Cartesian coordinates, that, with e^{λ}, e^{ν} given by (3.15), $g_{\rho\sigma}$ is well-behaved, g non-zero and $g^{\rho\sigma}$ well-defined and well-behaved at the spatial origin. The invariants that may be constructed from such quantities cannot be singular. We can be sure that the imposition of the fairly weak regularity condition (3.12) on Lanczos' equations in the spherically symmetric, static case results in space-times that have no physical singularity at the spatial origin.

3.4 The Non-Existence of Space-Times Asymptotically of the Form of the Einstein Universe

It was mentioned in section 3.1 that the Einstein universe satisfies Lanczos' equations only when $\beta = -\frac{1}{3}$ and thus does not constitute a true solution. It is reasonable to enquire whether or not there are any solutions that are asymptotic to the Einstein universe. That is, do there exist solutions of the form

$$e^{\lambda} \to \frac{1}{1 - r^2/r_0^2}, \qquad e^{\nu} \to 1 \tag{3.17}$$

as $r \to r_0$? From purely formal considerations, r_0^2 may take positive or negative values.

In order to find if (3.17) is a sensible form to try it is necessary to find what formal solutions exist for large r of the type:

$$e^{-\lambda} = u_0 r^2 + u_1 r + u_2 + \frac{u_3}{r} + \cdots, \quad u_0 \neq 0, \qquad e^{\nu} = v_0 + \frac{v_1}{r} + \frac{v_2}{r^2} + \frac{v_3}{r^3} + \cdots, \quad v_0 \neq 0, \tag{3.18}$$

for some constants $u_0, u_1, u_2, \ldots, v_0, v_1, v_2, \ldots$.

It is found that there are no solutions of this type for $u_0 \neq 0$; furthermore, there are none of the type (3.18) with $u_0 = 0$ and $u_1 \neq 0$. This result is proved as a lemma in the appendix to this section, A3.4. When u_0 and u_1 are both zero we arrive at the asymptotically Minkowskian form analysed in section 3.2.

Then there are no solutions, regular or otherwise, that are asymptotically of the form of the Einstein universe.

3.5 The Non-Existence of Space-Times Asymptotically of Constant Curvature with Non-Constant Curvature Invariant

We know that there exists at least one solution of Lanczos' equations that has asymptotically constant space-time curvature. This is the complete solution of the field equations (2.5) in the spherically symmetric case, being given by the metric:

$$ds^2 = \frac{-dr^2}{1-2m/r-(\Lambda/3)r^2} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + (1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2)\,dt^2.$$

There is no reason to believe that there are not others with asymptotically constant space-time curvature; indeed, numerical results seem to indicate there are. We show in this section that solutions with this property must have constant curvature invariant.

For large r we suppose that the unknown functions $e^{-\lambda}$, e^{ν} of the metric form (2.8) have the formal expansions

$$\begin{aligned} e^{-\lambda} &= b_0 r^2(1 + \frac{b_1}{r} + \frac{b_2}{r^2} + \frac{b_3}{r^3} + \dots)\;;\; b_0, b_2 \neq 0, \\ e^{\nu} &= a_0 r^2(1 + \frac{a_1}{r} + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \dots)\;;\; a_0, a_2 \neq 0, \end{aligned} \qquad 3.19$$

for some constants $a_0, a_1, a_2, \dots, b_0, b_1, b_2, \dots$.

(The form assumed, (3.19) is more general than that required to represent a space-time with asymptotically constant curvature, i.e., one which is asymptotic to the de Sitter form (3.5). It transpires, however, that an application of the field equations reduces (3.19) to the latter). The curvature invariant R is found from (A2.22) to be given by

$$R = \sum_{i=0} \frac{A_i}{r^i} \qquad 3.20$$

and the combination $(\frac{1}{2}\frac{d\nu}{dr} - \frac{1}{2}\frac{d\lambda}{dr} + \frac{2}{r})$ is given by

$$\frac{1}{2}\frac{d\nu}{dr} - \frac{1}{2}\frac{d\lambda}{dr} + \frac{2}{r} = \frac{4}{r} + \sum_{i=1} \frac{B_i}{r^{i+1}} \qquad 3.21$$

where, in (3.20) and (3.21), A_i and B_i are constants such that $A_i = A_i(a_j, b_j)$, $B_i = B_i(a_j, b_j)$ for $j \leq i$ and each $i \geq 0$. As usual, e^{λ} and e^{ν}, given by (3.19), must satisfy the differential equation (2.14). Substitution into the latter from (3.20), (3.21) yields a succession of relationships, the first two of which are the equations $A_1 = 0$ and $A_2 = 0$. In place of an algebraic equation for A_3, however, an identity is obtained. It is clear that $A_n = 0$ for all $r > 3$ if $A_3 = 0$, and to show that the latter is true we must employ in addition the second field equation (A2.25). That A_3 vanishes is proved as a lemma in appendix A3.5. Then, as in previous sections, $R = A_0$ = constant everywhere.

3.6 Conclusions

As stated in section 3.1, it is pertinent to ask if there are any solutions which are both asymptotically flat and regular, apart from the Minkowski metric. To answer this question, we may draw a corollary from the theorems of sections 3.2, 3.3:

Any spherically symmetric, static, regular metric that is a solution of Lanczos' equations and tends towards the Minkowski form for large values of the radial coordinate must be such that the curvature invariant vanishes everywhere.

In chapter 4, section 5 we shall find the set of solutions that satisfy $R = 0$ and see that there is no such regular, asymptotically Minkowskian solution.

The fact that in the spherically symmetric, static case all regular solutions of Lanczos' equations satisfy R = constant is interesting in itself. In addition, it ratifies (at least under these symmetry conditions) Lanczos' claim that R = constant is not only a possible, but also a necessary consequence of the field equations.[40] Einstein's requirement that the gravitational field equations should be quasi-linear and of second order only in the components of the metric tensor is to some extent heuristic. However, the condition of regularity on the solutions of the field equations is in this case sufficient to ensure that it is completely fulfilled. For the field equation (2.13a) is now given by (A2.25) with R replaced by the constant K (we henceforth denote this equation

$(A2.25)_K$), while (2.13b) is replaced by (2.17). It is clear that the highest derivatives λ'', ν'' appear only linearly; knowledge of λ, λ' and ν' is sufficient to determine them uniquely. Any objection to Lanczos' equations as the fundamental gravitational field equations on account of their high differential order is thus diminished.

The system of field equations to be solved when $R = K$, consisting of $(A2.25)_K$ and (2.17), may be reduced to a single integro-differential equation. Making the change of variable

$$r = e^{z}$$

and writing

$$\gamma = \frac{d\nu}{dz} ,$$

(with a dash here denoting differentiation with respect to z) the integro-differential equation is

$$\begin{aligned} &F^2(\gamma)\{(\gamma+1)^2(\gamma-2)^2-\gamma'(\gamma-2)(\gamma^2+8)+3\gamma'^2(1-\gamma)+\gamma''(\gamma+4)(\gamma-2)\} \\ &+F(\gamma)\{-2\gamma^2+8\gamma-8+4\gamma\gamma'-8\gamma' \\ &\qquad -Ke^{2z}[(1+2\beta)\gamma^3+(5+18\beta)\gamma^2+16(1+3\beta)\gamma+4(3+8\beta)+2\gamma'(\gamma-2)]\} \\ &+(2-Ke^{2z})^2(\gamma+1)+Ke^{2z}(1-Ke^{2z}/4)(\gamma+4)^2(1+2\beta) = 0 \end{aligned} \tag{3.22}$$

in which

$$F(\gamma) = \frac{1}{G(\gamma)}\left\{\int \frac{2(2-Ke^{2z})}{(\gamma+4)}\, G(\gamma)\, dz + C\right\},$$

where C is an arbitrary constant and

$$G(\gamma) = e^{-6z}(\gamma+4)^2 \exp\left\{\int\left[(\gamma+4)+\frac{12}{(\gamma+4)}\right]dz\right\} ,$$

(except when $\gamma = -4$. This case must be treated separately but it may be shown that there are no solutions of this type).

The unknown quantities e^{λ} and e^{ν} are given by

$$e^{\lambda} = 1/F(\gamma) ,$$
$$e^{\nu} = \exp\left\{\int \gamma\, dz\right\} .$$

Analytic solution of (3.22), however, presents an intractable problem, though numerical solution by iterative techniques may be possible.

4. SERIES SOLUTIONS AND EXACT SOLUTIONS

4.1 Introduction

Our aim is now to develop series solutions, and, where possible, to obtain exact solutions, corresponding to each of the boundary conditions studied in chapter 3. The first few terms of a series solution satisfying the regularity condition (3.12) are given in section 4.2 and lead to two new exact solutions. Series solutions consistent with the boundary condition of asymptotically constant space-time curvature are investigated in section 4.3. In the following section 4.4 a family of exact solutions is obtained which includes those of section 4.2. We then return, in section 4.5, to consideration of perhaps the most interesting boundary condition - that of asymptotic flatness - and derive, amongst other solutions without the property of asymptotic flatness, a solution corresponding to a line element that contains the Schwarzschild metric (3.1) as a special case. The properties of the universes obtained as solutions in this chapter will be discussed in chapter 5.

4.2 Series Solutions corresponding to Section 3.3 for Small Values of the Radial Coordinate

Regular solutions are such that e^{λ} and e^{ν} have expansions of the form (3.15), in terms of even, non-negative powers of the radial coordinate and satisfy $R = K$. For small values of r we expand:

$$e^{-\lambda} = 1 + V_2 r^2 + V_4 r^4 + V_6 r^6 + \ldots \quad ,$$

$$e^{\nu} = d_0 (1 + u_2 r^2 + u_4 r^4 + u_6 r^6 + \ldots), \; d_0 \neq 0, \qquad 4.1$$

for some constants V_2 , V_4 , V_6 ,..., u_2 , u_4 , u_6 ,..., d_0 , and substitute these expressions together with those for the derivatives of e^{λ} and e^{ν} into the pair of equations $(A2.25)_K$, (2.17).

Equating the coefficients of successive powers of r to zero we obtain equations for the constants v_4, v_6, ..., u_4, u_6, ... which may be solved and substituted into (4.1) to give

$$\begin{aligned} e^{-\lambda} &= 1 + v_2 r^2 + \tfrac{1}{5}(u_2 - v_2)[-v_2 - 3\beta(u_2+v_2)]r^4 \\ &+ \frac{(u_2-v_2)}{2\cdot 5\cdot 7}[(-2u_2^2 + 2u_2 v_2 + v_2^2) + 3\beta(u_2+v_2)(2u_2 - v_2) - 2(3\beta)^2(u_2+v_2)^2]r^6 + \cdots, \\ e^{\nu}/d_0 &= 1 + u_2 r^2 + \tfrac{1}{10}(u_2 - v_2)[(4u_2 + v_2) + 3\beta(u_2+v_2)]r^4 \\ &+ \frac{(u_2-v_2)}{2\cdot 3\cdot 5\cdot 7}[(20u_2^2 - 23u_2 v_2 - 13v_2^2) + 3\beta(u_2+v_2)(25u_2 - 11v_2) + 2(3\beta)^2(u_2+v_2)^2]r^6 + \cdots, \end{aligned} \qquad 4.2$$

with

$$K = -6(u_2 + v_2). \qquad 4.3$$

In addition to the parameter β, two arbitrary constants, u_2 and v_2, are present in (4.2). The two situations $u_2 = \pm v_2$ are of immediate interest. Firstly the case $u_2 = v_2$ corresponds to de Sitter space-time, which we had expected. Secondly, when $u_2 = -v_2$, $K = 0$ and there is no dependence upon β. The series expansions (4.2) become:

$$e^{-\lambda} = 1 - u_2 r^2 + \frac{2}{5}u_2^2 r^4 - \frac{3}{5\cdot 7}u_2^3 r^6 + \cdots, \qquad (a)$$
$$e^{\nu}/d_0 = 1 + u_2 r^2 + \frac{3}{5}u_2^2 r^4 + \frac{2}{7}u_2^3 r^6 + \cdots. \qquad (b) \qquad 4.4$$

Up to terms in r^6, $e^{\nu}e^{-\lambda} = d_0$, and the right hand side of (4.4b) is the expansion of

$$-\frac{1}{3u_2 r^2}\sinh^2(\sqrt{-3u_2}\,r) \quad \text{or} \quad \frac{1}{3u_2 r^2}\sin^2(\sqrt{3u_2}\,r). \qquad 4.5$$

A check shows that

$$e^{\lambda} = \frac{e^{\nu}}{d_0} = \left\{\frac{Cr}{\sinh(Cr)}\right\}^2 \text{ and } e^{\lambda} = \frac{e^{\nu}}{d_0} = \left\{\frac{Cr}{\sin(Cr)}\right\}^2, \qquad 4.6$$

where C is an arbitrary constant, do indeed constitute two *exact* solutions of the field equations. Then, from (2.8), the metrics corresponding to these solutions are

$$ds^2 = \frac{-C^2r^2}{\sinh^2(Cr)}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{C^2r^2}{\sinh^2(Cr)}dt^2, \qquad 4.7$$

$$ds^2 = \frac{-C^2r^2}{\sin^2(Cr)}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{C^2r^2}{\sin^2(Cr)}dt^2. \qquad 4.8$$

That a second arbitrary constant may be introduced into the metrics (4.7), (4.8) will be demonstrated in section 4.4 .

4.3 Series Solutions corresponding to Section 3.5 for Large Values of the Radial Coordinate

It was shown in section 3.5 that any spherically symmetric, static solutions of the field equations representing space-times asymptotically of constant curvature must have constant curvature invariant. The formal series expansions, given by (3.19), are to be substituted into the pair of equations $(A2.25)_K$, (2.17) in order that relationships between the constants a_i, b_i may be obtained by equating to zero the coefficients of successive powers of $1/r$. We denote by $\mathcal{C}_n$ the pair of coefficients which involve terms of the form $c_{r_1} c_{r_2} \cdots c_{r_p}$ where each c_{r_i} represents either a_{r_i} or b_{r_i} and $\sum_{i=1}^{p} r_i = n$. Equating $\mathcal{C}_1, \ldots, \mathcal{C}_5$ to zero and solving the resultant equations we obtain (as in appendix A3.5):

$$\begin{aligned} b_1 &= a_1 = 0 , \\ b_2 &= a_2 = 1/b_0 , \\ b_3 &= a_3 , \\ b_4 &= \tfrac{4}{3}a_4 , \\ b_5 &= \tfrac{5}{3}a_5 . \end{aligned} \qquad 4.9$$

The set (4.9) suggests the exact relationship

$$e^{-\lambda} = \frac{1}{3} + b_0 r^2 - \frac{b_0}{3a_0} e^{\nu}\left(r\frac{d\nu}{dr} - 2\right) \qquad 4.10$$

between e^{λ} and e^{ν} . It has not proved possible to demonstrate the general consistency or otherwise of the equations $(A2.25)_K$ and (2.17) under the substitution (4.10); that the latter cannot hold in general for this type of solution is shown by considering higher terms in the expansions.

As each successive pair $\mathcal{C}_n$ is equated to zero two possibilities are generally obtained. One is that the constant β takes a specific numerical value, the other that a pair a_i , b_i vanish but β is unrestricted. Considering values of $n \leq 10$ the following series expansions of the form (3.19) are obtained:

Case A: Special β values

For the particular values $\beta = -\frac{1}{12}, \frac{1}{6}, \frac{1}{2}, \frac{11}{12}, \frac{17}{12}, \ldots$ certain pairs of coefficients a_i, b_i vanish and higher coefficients are given in terms of *three* arbitrary constants. (To determine the coefficients a_9, b_9 and higher the expansion would have to be taken to greater order than $n = 10$).

(i) $\beta = -\frac{1}{12}$: $a_5 = 0$; $b_5 = 0$. a_2 , a_3 , a_4 arbitrary,

$$e^{-\lambda} = \frac{r^2}{a_2}\left(1 + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \frac{4}{3}a_4\cdot\frac{1}{r^4} + 0\cdot\frac{1}{r^5} - \frac{10}{21}a_2a_4\cdot\frac{1}{r^6}\right.$$
$$\left. - \frac{5}{18}a_3a_4\cdot\frac{1}{r^7} + a_4\left(\frac{2}{7}a_2^2 - \frac{2}{9}a_4\right)\frac{1}{r^8} + \frac{b_9}{r^9} + \frac{b_{10}}{r^{10}} + \ldots\right),$$
$$e^{\nu} = a_0 r^2\left(1 + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \frac{a_4}{r^4} + 0\cdot\frac{1}{r^5} - \frac{5}{21}a_2a_4\cdot\frac{1}{r^6}\right.$$
$$\left. - \frac{1}{6}a_3a_4\cdot\frac{1}{r^7} + a_4\left(\frac{1}{7}a_2^2 - \frac{1}{6}a_4\right)\cdot\frac{1}{r^8} + \frac{a_9}{r^9} + \frac{a_{10}}{r^{10}} + \ldots\right).$$

(ii) $\beta = \frac{1}{6}$: $a_4 = a_6 = 0$; a_2, a_3 , a_5 arbitrary,
$b_4 = b_6 = 0$.

$$e^{-\lambda} = \frac{r^2}{a_2}\left(1 + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \frac{0}{r^4} + \frac{5}{3}\frac{a_5}{r^5} + \frac{0}{r^6} - \frac{85}{3^4}\frac{a_2a_5}{r^7}\right.$$
$$\left. - \frac{23}{4\cdot 3^2}\frac{a_3a_5}{r^8} + \frac{b_9}{r^9} + \frac{b_{10}}{r^{10}} + \ldots\right),$$
$$e^{\nu} = a_0 r^2\left(1 + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \frac{0}{r^4} + \frac{a_5}{r^5} + \frac{0}{r^6} - \frac{13}{3^3}\frac{a_2a_5}{r^7}\right.$$
$$\left. - \frac{1}{3}\frac{a_3a_5}{r^8} + \frac{a_9}{r^9} + \frac{a_{10}}{r^{10}} + \ldots\right).$$

(iii) $\beta = \frac{1}{2}$: $a_4 = a_5 = a_7 = 0$; a_2, a_3, a_6 arbitrary,

$b_4 = b_5 = b_7 = 0$.

$$e^{-\lambda} = \frac{r^2}{a_2}\left(1 + \frac{a_2}{r^2} + \frac{a_3}{r^3} + 0 + \frac{2a_6}{r^6} + \frac{0}{r^7} - \frac{39}{22}\frac{a_2 a_6}{r^8} + \frac{b_9}{r^9} + \frac{b_{10}}{r^{10}} + \ldots\right),$$

$$e^{\nu} = a_0 r^2\left(1 + \frac{a_2}{r^2} + \frac{a_3}{r^3} + 0 + \frac{a_6}{r^6} + \frac{0}{r^7} - \frac{8}{11}\frac{a_2 a_6}{r^8} + \frac{a_9}{r^9} + \frac{a_{10}}{r^{10}} + \ldots\right).$$

(iv) $\beta = \frac{11}{12}$: $a_4 = a_5 = a_6 = a_8 = 0$; a_2, a_3, a_7 arbitrary,

$b_4 = b_5 = b_6 = b_8 = 0$.

$$e^{-\lambda} = \frac{r^2}{a_2}\left(1 + \frac{a_2}{r^2} + \frac{a_3}{r^3} + 0 + \frac{7}{3}\frac{a_7}{r^7} + \frac{0}{r^8} + \frac{b_9}{r^9} + \frac{b_{10}}{r^{10}} + \ldots\right),$$

$$e^{\nu} = a_0 r^2\left(1 + \frac{a_2}{r^2} + \frac{a_3}{r^3} + 0 + \frac{a_7}{r^7} + \frac{0}{r^8} + \frac{a_9}{r^9} + \frac{a_{10}}{r^{10}} + \ldots\right).$$

(v) $\beta = \frac{17}{12}$: $a_4 = a_5 = a_6 = a_7 = 0$; a_2, a_3, a_8 arbitrary,

$b_4 = b_5 = b_6 = b_7 = 0$.

$$e^{-\lambda} = \frac{r^2}{a_2}\left(1 + \frac{a_2}{r^2} + \frac{a_3}{r^3} + 0 + \frac{8}{3}\frac{a_8}{r^8} + \frac{b_9}{r^9} + \frac{b_{10}}{r^{10}} + \ldots\right)$$

$$= a_0 r^2\left(1 + \frac{a_2}{r^2} + \frac{a_3}{r^3} + 0 + \frac{a_8}{r^8} + \frac{a_9}{r^9} + \frac{a_{10}}{r^{10}} + \ldots\right)$$

Presumably in this case a_9 and b_9 will vanish, following the pattern in (i), (ii), (iii), (iv) above. In all cases the curvature invariant is given by

$$R = \frac{-12}{a_2}.$$

The above expansions (i),...,(v) support the following unproven conjecture: if β assumes one of the particular values $(n^2+5n-2)/24$, $n = 0, 1, 2, ..$ then the solution may be written in terms of three arbitrary constants a_2, a_3, a_{n+4} ($a_0 = 1/a_2$).

Case B: β unrestricted

$$a_4 = a_5 = a_6 = a_7 = a_8 = a_9 = 0,$$

$$b_4 = b_5 = b_6 = b_7 = b_8 = b_9 = 0,$$

$$a_2 e^{-\lambda} = \frac{e^{\nu}}{a_0} = r^2\left(1 + \frac{a_2}{r^2} + \frac{a_3}{r^3} + O\left[\frac{1}{r^{10}}\right]\right).$$

We speculate that either the higher coefficients a_{10}, b_{10}, ... vanish, or that they are functionally independent of β. If the latter is true, any solutions that exist for unrestricted β in this case are solutions which are independent of β. Such solutions must satisfy the field equations obtained from the vanishing of the Hamiltonian derivatives of the Lagrangian R^2. Now the only $R = K \neq 0$ solutions of the latter field equations must satisfy[6] $R_{\rho\sigma} - \frac{1}{4} R g_{\rho\sigma} = 0$. Thus the most general solution of Lanczos' equations satisfying $R = K \neq 0$ and independent of β is given by the set of Einstein spaces $R_{\rho\sigma} = \Lambda g_{\rho\sigma}$. In view of our choice of the metric form (2.8), this means that the coefficients a_{10}, b_{10}, ... must vanish.

A numerical search for solutions which are regular and have the asymptotic form described under case A has been carried out; this work is discussed in section 6.4. However, the series solutions that we have obtained in this section do not enable us to obtain exact solutions and are not instructive; they are of little more than formal interest.

4.4 Space-Times I $\longrightarrow$ IV

In section 4.2, as a result of seeking solutions regular at $r = 0$, two exact solutions, (4.6), were obtained such that $e^{\lambda} = e^{\nu}$. Making no assumptions about the constancy of the curvature invariant we now look for further solutions of the field equations when $e^{\lambda} = e^{\nu}$.

The field equations to be satisfied are (2.15) and (A2.25), with R given by (A2.22). When $e^{\lambda} = e^{\nu}$, (2.15) may be integrated to give

$$R = \frac{A}{r} + D \;, \tag{4.11}$$

for some constants A and D. It is convenient to make the change of variable

$$r = e^{z}$$

and to define

$$P = P(z) = R e^{2z} \,. \tag{4.12}$$

Then (4.11) is replaced by

$$P = A e^{z} + D e^{2z} \;, \tag{4.13}$$

and the expression (A2.22) for R by

$$P = e^{-\lambda}(-\lambda'' + \lambda' - 2) + 2 \,, \tag{4.14}$$

where we have substituted $e^{\nu} = e^{\lambda}$ and denoted by a dash derivatives with respect to z. Now employing (4.14) as the defining relation for λ'', we obtain in the place of the second field equation (A2.25) a first order differential equation for e^{λ}, which is

$$(1+2\beta)[-P^2 + 4P + 8Pe^{-\lambda} + P'e^{-\lambda}(-2\lambda' - 4)] + 8\beta(P - P')e^{-\lambda} = 0 \;, \tag{4.15}$$

where P is regarded as a function of the independent variable z, given by (4.13). (4.15) is now differentiated with respect to z and substitution made into this expression for P'' in terms of P and P' from (4.13) and for λ'' in terms of λ' and P from (4.14). We obtain

$$(2-\lambda')e^{-\lambda}\{(1+2\beta)(-P'\lambda' + 2P) + 4\beta(P - P')\} = 0 \;. \tag{4.16}$$

Since no solution is obtained from $\lambda' = 2$, i.e. $d\lambda/dr = 2/r$, (4.16) may be replaced by

$$(1+2\beta)(-P'\lambda' + 2P) + 4\beta(P - P') = 0 \quad . \qquad 4.17$$

We now consider the case when $P \neq 0$ and $\beta \neq -\frac{1}{2}$. Equation (4.17) may be integrated without difficulty. However, substitution of the resulting form for e^{λ} into (4.14) and (4.15) where P is given by (4.13) yields a contradiction which may not be resolved by assigning specific values to the arbitrary constants of the solution. The cases $P = 0$ and $\beta = -\frac{1}{2}$ are now considered.

(i) $P = 0$. The differential equation (4.15) is identically satisfied. Any solution of (4.14) with the left hand side zero automatically satisfies the field equations. The complete set of solutions of this equation, i.e. of

$$0 = e^{-\lambda}(-\lambda'' + \lambda' - 2) + 2$$

is given (substituting $r = e^{z}$) by

$$e^{\lambda} = \left(1 + \frac{a}{r}\right)^{-2}, \qquad \text{(a)}$$

$$e^{\lambda} = \left\{\frac{Cr}{\sinh(C(r-B))}\right\}^{2}, \qquad 4.18\text{(b)}$$

$$e^{\lambda} = \left\{\frac{Cr}{\sin(C(r-B))}\right\}^{2}, \qquad \text{(c)}$$

where a, B and C are arbitrary constants. We note that the pair of solutions (4.6) obtained in section 4.2 are special cases of (4.18b), (4.18c). The $e^{\lambda} = e^{\nu}$ solutions (4.18) correspond to three space-times given, from (2.8), by the metrics

$$ds^2 = \frac{-dr^2}{(1+a/r)^2} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{dt^2}{(1+a/r)^2} \;, \qquad \text{(a)}$$

$$ds^2 = \frac{-C^2r^2dr^2}{\sinh^2(C(r-B))} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{C^2r^2dt^2}{\sinh^2(C(r-B))} \;; \qquad 4.19\text{(b)}$$

$$ds^2 = \frac{-C^2r^2dr^2}{\sin^2(C(r-B))} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{C^2r^2dt^2}{\sin^2(C(r-B))} \;, \qquad \text{(c)}$$

which we shall denote <u>Space-Time I</u>, <u>Space-Time II</u> and <u>Space-Time III</u> respectively.

(ii) $\beta = -\frac{1}{2}$. The differential equation (4.15) is satisfied provided that $P' = P$. Accordingly the constant D in (4.13) must vanish. Then any solution of (4.14) with the left hand side replaced by Ae^{z} automatically satisfies the field equations. It has been possible to find only a particular solution of

$$Ae^{z} = e^{-\lambda}(-\lambda'' + \lambda' - 2) + 2 , \qquad 4.20$$

which is, in terms of $r = e^{z}$:

$$e^{\lambda} = \frac{3A^2r^2}{(2-Ar)^3} .$$

This solution corresponds to <u>Space-Time IV</u>:

$$ds^2 = \frac{-3A^2r^2dr^2}{(2-Ar)^3} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{3A^2r^2dt^2}{(2-Ar)^3} \qquad 4.21$$

($\beta = -\frac{1}{2}$ only). More generally, any solution of (4.20) for e^{λ} constitutes an $e^{\lambda} = e^{\nu}$ solution of Lanczos' equations. The family of exact $e^{\lambda} = e^{\nu}$ solutions obtained in this section consists of Space-Times I $\longrightarrow$ IV.

<u>4.5 A Set of Exact Solutions including the Extended Schwarzschild Solution</u>

Our investigations in this section are motivated by the boundary condition of asymptotic flatness. It was shown in section 3.2 that any spherically symmetric, static solution of Lanczos' equations satisfying the boundary condition of asymptotic flatness has zero curvature invariant. Now, regardless of any symmetry conditions, the field equations generated by the Lagrangian R^2 are satisfied by *all* space-times with $R = 0$. Consequently, an $R = 0$ solution will satisfy the field equations generated by $R_{\rho\sigma}R^{\rho\sigma} + \beta R^2$ for all values of β , and, in particular, when $\beta = -\frac{1}{3}$. But Buchdahl's result[31], given in section 2.4, is that under static, spherically symmetric conditions the *complete* set of $\beta = -\frac{1}{3}$ solutions of Lanczos' equations is the set of space-times given by (2.18):

$$ds^2 = \mathcal{D}^2(\rho)\left\{ \frac{-d\rho^2}{(1-\frac{2m}{\rho}-\frac{\Lambda}{3}\rho^2)} - \rho^2(d\theta^2+\sin^2\theta\, d\phi^2) + (1-\frac{2m}{\rho}-\frac{\Lambda}{3}\rho^2)dt^2\right\}, \quad 2.18$$

where m and Λ are arbitrary constants and $\mathcal{D}$ is a completely arbitrary function of the radial coordinate ρ . Then, under our symmetry conditions, any $R = 0$ solution must be reducible to the form (2.18), i.e. the set of space-times (2.18) includes all $R = 0$ solutions and, in particular, all solutions that satisfy the boundary condition of asymptotic flatness.

In order to generate the set of $R = 0$ solutions of the spherically symmetric, static field equations, we need only find the functional forms for $\mathcal{D}(\rho)$ which are such that the Riemann scalar of the metric (2.18) vanishes. To do this we proceed as follows:

For any metric

$$ds^2 = -e^{\alpha(\rho)}d\rho^2 - \rho^2 e^{\beta(\rho)}(d\theta^2+\sin^2\theta\, d\phi^2) + e^{\gamma(\rho)}dt^2 \qquad 4.22$$

the curvature invariant is given by[49]

$$\begin{aligned} R = {} & e^{-\alpha}\Big(-\gamma'' - \tfrac{1}{2}\gamma'^2 - \beta'\gamma' + \tfrac{1}{2}\alpha'\gamma' - \frac{2\gamma'}{\rho} - 2\beta'' - \tfrac{3}{2}\beta'^2 \\ & \quad + \alpha'\beta' - 6\frac{\beta'}{\rho} + \frac{2\alpha'}{\rho} - \frac{2}{\rho^2}\Big) \\ & + e^{-\beta}\left(\frac{2}{\rho^2}\right) . \end{aligned} \qquad 4.23$$

(a dash denoting differentiation with respect to ρ). Using (4.23), the curvature invariant R of any metric

$$ds^2 = \mathcal{D}^2 d\sigma^2, \quad \mathcal{D} = \mathcal{D}(\rho), \qquad 4.24$$

is related to the curvature invariant R_1 of $d\sigma^2$, where

$$d\sigma^2 = -e^{\lambda_1(\rho)}d\rho^2 - \rho^2(d\theta^2+\sin^2\theta\, d\phi^2) + e^{\nu_1(\rho)}dt^2, \qquad 4.25$$

by

$$R = \frac{1}{D^2} R_1 - \frac{6e^{-\lambda_1}}{D^3}\left\{D'' + D'\left(\frac{\nu_1'}{2} - \frac{\lambda_1'}{2} + \frac{2}{\rho}\right)\right\} \quad . \qquad 4.26$$

Then the curvature invariant R of the metric (2.18) is related to the curvature invariant R_1 (= 4Λ) of the Einstein space

$$d\sigma^2 = \frac{-d\rho^2}{(1-\frac{2m}{\rho}-\frac{\Lambda}{3}\rho^2)} - \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2) + (1-\frac{2m}{\rho}-\frac{\Lambda}{3}\rho^2)\, dt^2 \qquad 4.27$$

by (4.26), in which $e^{-\lambda_1} = e^{\nu_1} = 1-\frac{2m}{\rho}-\frac{\Lambda}{3}\rho^2$. We seek functional forms of D such that $R = 0$. Thus, setting $R = 0$ and substituting for R_1 , λ_1 and ν_1 , (4.26) yields the second order linear differential equation for D :

$$D''\left(\frac{\Lambda}{3}\rho^4 - \rho^2 + 2m\rho\right) + D'\left(4\frac{\Lambda}{3}\rho^3 - 2\rho + 2m\right) + 2\frac{\Lambda}{3}D\rho^2 = 0 . \qquad 4.28$$

We consider the solution of (4.28) in the four separate cases (i),...,(iv).

(i) $\Lambda = 0$, $m = 0$. The solution of (4.28) is

$$D = \frac{A\rho - a}{\rho} , \qquad 4.29$$

(A , a arbitrary) corresponding to a solution of Lanczos' equations given, from (2.18), by the conformally flat metric

$$ds^2 = \frac{(A\rho - a)^2}{\rho^2}\left\{-d\rho^2 - \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2) + dt^2\right\} . \qquad 4.30$$

We examine two non-trivial cases of the metric (4.30), (a) and (b):
(a) $A \neq 0$. Then let $A = 1$ in order that (4.30) may tend to the Minkowski metric as $\rho \to \infty$. This may be done without loss in generality since if Lanczos' equations are satisfied by $d\sigma^2 = g_{\rho\sigma} dx^\rho dx^\sigma$ they are also satisfied, for any constant C , by $ds^2 = C^2 d\sigma^2$ (as a result of the gauge invariance of the action principle). The metric (4.30) becomes

$$ds^2 = \frac{(\rho-a)^2}{\rho^2}\left\{-d\rho^2 - \rho^2(d\theta^2+\sin^2\theta\, d\phi^2) + dt^2\right\} . \qquad 4.31$$

However, (4.31) is simply the metric of space-time I, given by (4.19a), after the radial coordinate has been transformed according to

$$r = \rho - a .$$

(b) $A = 0$. The metric (4.30) becomes (setting $a = 1$)

$$ds^2 = \frac{1}{\rho^2}\left\{-d\rho^2 - \rho^2(d\theta^2+\sin^2\theta\, d\phi^2) + dt^2\right\} . \qquad 4.32$$

The coefficient of the angular term in this metric is unity. (4.32) represents a "class 2" space-time to be discussed further in chapter 7. We shall refer to (4.32) as Space-Time V.

(ii) $\Lambda = 0$, $m \neq 0$. The solution of (4.28) is

$$D = A + B\ln\left(1-\frac{2m}{\rho}\right) , \qquad 4.33$$

(A , B arbitrary), corresponding to a solution of Lanczos' equations given, from (2.18), by the metric

$$ds^2 = \left\{A + B\ln\left(1-\frac{2m}{\rho}\right)\right\}^2\left\{\frac{-d\rho^2}{1-\frac{2m}{\rho}} - \rho^2(d\theta^2+\sin^2\theta\, d\phi^2) + \left(1-\frac{2m}{\rho}\right)dt^2\right\} . \qquad 4.34$$

(4.34) is conformal to the Schwarzschild line element (3.1). As in case (i), there are two separate cases to examine, (a) and (b):

(a) $A \neq 0$. As before, A may be set equal to unity so that (4.34) tends to the Minkowski form as $\rho \to \infty$. The metric

$$ds^2 = \left\{1 + B\ln\left(1-\frac{2m}{\rho}\right)\right\}^2\left\{\frac{-d\rho^2}{1-\frac{2m}{\rho}} - \rho^2(d\theta^2+\sin^2\theta\, d\phi^2) + \left(1-\frac{2m}{\rho}\right)dt^2\right\} \qquad 4.35$$

will be referred to as the Extended Schwarzschild Solution or Space Time VI.

(b) $A = 0$. The metric (4.34) becomes (setting $B = 1$):

$$ds^2 = \left\{\ln\left(1-\frac{2m}{\rho}\right)\right\}^2 \left\{\frac{-d\rho^2}{1-\frac{2m}{\rho}} - \rho^2(d\theta^2+\sin^2\theta\, d\phi^2) + \left(1-\frac{2m}{\rho}\right)dt^2\right\}. \qquad 4.36$$

The properties of the space-time represented by (4.36) are very different from those of space-time VI. We refer to (4.36) as Space-Time VII.

(iii) $m = 0, \Lambda \neq 0$. Two solutions of (4.28) are

$$D = \frac{1}{\sqrt{\Lambda/3}} \frac{\left[A \tanh^{-1}\sqrt{\Lambda/3}\,\rho + B\sqrt{\Lambda/3}\right]}{\rho}, \quad \Lambda > 0,$$

$$D = \frac{1}{\sqrt{\Lambda'/3}} \frac{\left[A \tan^{-1}\sqrt{\Lambda'/3}\,\rho + B\sqrt{\Lambda'/3}\right]}{\rho}, \quad \Lambda' = -\Lambda > 0$$

(A, B arbitrary), corresponding to two solutions of Lanczos' equations given, from (2.18), by the metrics

$$ds^2 = \frac{3}{\Lambda} \frac{\left[A \tanh^{-1}\sqrt{\Lambda/3}\,\rho + B\sqrt{\Lambda/3}\right]^2}{\rho^2} \left\{\frac{-d\rho^2}{(1-\frac{\Lambda}{3}\rho^2)} - \rho^2(d\theta^2+\sin^2\theta\, d\phi^2) + \left(1-\frac{\Lambda}{3}\rho^2\right)dt^2\right\}, \qquad \text{(a)}$$

$$ds^2 = \frac{3}{\Lambda'} \frac{\left[A \tan^{-1}\sqrt{\Lambda'/3}\,\rho + B\sqrt{\Lambda'/3}\right]^2}{\rho^2} \left\{\frac{-d\rho^2}{(1+\frac{\Lambda'}{3}\rho^2)} - \rho^2(d\theta^2+\sin^2\theta\, d\phi^2) + \left(1+\frac{\Lambda'}{3}\rho^2\right)dt^2\right\}. \qquad 4.37 \text{ (b)}$$

Again, there are two cases, (a) and (b), to examine.

(a) $A \neq 0$. Then A may be set equal to unity and the metrics (4.37a), (4.37b) become respectively

$$ds^2 = \frac{3}{\Lambda} \frac{\left[\tanh^{-1}\sqrt{\Lambda/3}\,\rho + B\sqrt{\Lambda/3}\right]^2}{\rho^2} \left\{-\frac{d\rho^2}{(1-\frac{\Lambda}{3}\rho^2)} - \rho^2(d\theta^2+\sin^2\theta\, d\phi^2) + \left(1-\frac{\Lambda}{3}\rho^2\right)dt^2\right\}, \qquad \text{(a)}$$

$$ds^2 = \frac{3}{\Lambda'} \frac{\left[\tan^{-1}\sqrt{\Lambda'/3}\,\rho + B\sqrt{\Lambda'/3}\right]^2}{\rho^2} \left\{\frac{-d\rho^2}{(1+\frac{\Lambda'}{3}\rho^2)} - \rho^2(d\theta^2+\sin^2\theta\, d\phi^2) + \left(1+\frac{\Lambda'}{3}\rho^2\right)dt^2\right\}. \qquad 4.38 \text{ (b)}$$

However, we are not surprised to find that the space-times represented by (4.38) are simply space-times II and III, given by (4.19b), (4.19c), in a different guise. Transforming the radial coordinate in the latter metrics according to

$$r = \frac{1}{C}\tanh^{-1}(C\rho) + B, \quad \text{(a)}$$
4.39
$$r = \frac{1}{C}\tan^{-1}(C\rho) + B \quad \text{(b)}$$

respectively, and setting

$$C = \sqrt{\Lambda/3}\ , \quad \text{(a)}$$
4.40
$$C = \sqrt{\Lambda'/3}\ , \quad \text{(b)}$$

the forms (4.19b), (4.19c) become identical to (4.38a), (4.38b). Nothing new is obtained by setting the constant B equal to zero.

(b) $A = 0$. The metrics (4.37) become (setting $B = 1$):

$$ds^2 = \frac{-d\rho^2}{\rho^2(1-\frac{\Lambda}{3}\rho^2)} - (d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{(1-\frac{\Lambda}{3}\rho^2)}{\rho^2}\, dt^2, \ (\Lambda>0), \quad \text{(a)}$$
4.41
$$ds^2 = \frac{-d\rho^2}{\rho^2(1+\frac{\Lambda'}{3}\rho^2)} - (d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{(1+\frac{\Lambda'}{3}\rho^2)}{\rho^2}\, dt^2, \ (\Lambda'>0). \quad \text{(b)}$$

The pair of metrics (4.41), like the metric (4.32), represent "class 2" space-times, which we shall meet again in chapter 7. We refer to (4.41a), (4.41b) as Space-Time VIII and Space-Time IX respectively.

(iv) $m \neq 0$, $\Lambda \neq 0$. It has not been possible to obtain any solutions of the differential equation (4.28) in this case. The latter may be reduced to the Riccati form

$$\frac{du}{d\rho} = \frac{1}{\frac{\Lambda}{3}\rho^4 - \rho^2 + 2m\rho}\, u^2 + 2\frac{\Lambda}{3}\rho^2 \qquad 4.42$$

through the substitution

$$\frac{D'(\rho)}{D(\rho)} = \frac{1}{-\frac{\Lambda}{3}\rho^4 + \rho^2 - 2m\rho}\, u(\rho)\ , \qquad 4.43$$

but the solution of the Riccati equation in general requires infinite series closely connected with Bessel functions. From the metric form (2.18), such a solution clearly could not be asymptotically Minkowskian.

The complete solution in the static, spherically symmetric case under the imposition of the boundary condition of asymptotic flatness is represented by the two exact metric forms (4.31) (space-time I) and (4.35) (space-time VI). In section 3.1 we asked whether or not our field equations permit a non-trivial solution which is both asymptotically flat and regular. Neither space-time I nor space-time VI is regular. Thus there are no static, spherically symmetric, regular, asymptotically flat solutions of Lanczos' equations.

4.6 Summary

It is interesting and instructive that the use of straightforward, and somewhat naïve, series expansion techniques has led, directly or indirectly, to each of the exact solutions obtained in this chapter. In addition, we have seen in section 4.5 how an apparently redundant result concerning the set of "solutions" in the underdetermined case may be employed to obtain well determined solutions of the field equations. It is reasonable to suppose that the use of such methods may be efficacious in obtaining solutions of Lanczos' equations under different symmetry conditions.

For future convenience we collect together all the exact solutions found in this chapter:

Space-Time I

$$ds^2 = \frac{-dr^2}{\left(1+\frac{a}{r}\right)^2} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + \frac{dt^2}{\left(1+\frac{a}{r}\right)^2} \quad . \qquad 4.44(a)$$

Space-Time II

$$ds^2 = \frac{-C^2r^2dr^2}{\sinh^2(C(r-B))} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + \frac{C^2r^2dt^2}{\sinh^2(C(r-B))} \quad , \qquad 4.44(b)$$

Space-Time III

$$ds^2 = \frac{-C^2r^2dr^2}{\sin^2(C(r-B))} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{C^2r^2dt^2}{\sin^2(C(r-B))} \quad . \quad 4.44(c)$$

Space-Time IV ($\beta = -\frac{1}{2}$ only).

$$ds^2 = \frac{-3A^2r^2dr^2}{(2-Ar)^3} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{3A^2r^2dt^2}{(2-Ar)^2} \quad . \quad 4.44(d)$$

Space-Time V

$$ds^2 = \frac{1}{r^2}\left\{-dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + dt^2\right\} \quad . \quad 4.44(e)$$

Space-Time VI

$$ds^2 = \left\{1 + B\ln\left(1-\frac{2m}{r}\right)\right\}^2 \left\{\frac{-dr^2}{(1-2m/r)} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \left(1-\frac{2m}{r}\right)dt^2\right\} . \quad 4.44(f)$$

Space-Time VII

$$ds^2 = \left\{\ln\left(1-\frac{2m}{r}\right)\right\}^2 \left\{\frac{-dr^2}{(1-2m/r)} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \left(1-\frac{2m}{r}\right)dt^2\right\} . \quad 4.44(g)$$

Space-Time VIII

$$ds^2 = \frac{-dr^2}{r^2\left(1-\frac{\Lambda}{3}r^2\right)} - (d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{\left(1-\frac{\Lambda}{3}r^2\right)dt^2}{r^2} \quad . \quad 4.44(h)$$

Space-Time IX

$$ds^2 = \frac{-dr^2}{r^2\left(1+\frac{\Lambda'}{3}r^2\right)} - (d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{\left(1+\frac{\Lambda'}{3}r^2\right)dt^2}{r^2} \quad , \quad 4.44(i)$$

5. PROPERTIES OF THE EXACT SOLUTIONS

5.1 Introduction

Our original hypothesis was that any reasonable metric field whatsoever must satisfy Lanczos' equations. Provided that the latter are satisfied, the metric tensor may be inserted into Einstein's equations (2.4):

$$T_{\rho\sigma} = \frac{-1}{\kappa}\left(R_{\rho\sigma} - \frac{1}{2} R\, g_{\rho\sigma}\right)$$

to determine the stress-energy tensor $T_{\rho\sigma}$.

The problem of interpreting the stress-energy tensors obtained by the above procedure is not, however, a trivial one. Making the assumption that the cosmological constant Λ is excessively large - the reciprocal square of a length of subatomic dimensions - Lanczos[50] has performed a "practical linearisation" of his original Lagrangian (2.1). Although the terms of the new Lagrangian obtained in this way include the standard cosmological and Maxwellian invariants, the interpretation of the additional invariants is obscure. Furthermore, the assumption concerning the cosmological constant has no justification in the present work.

Thus, in general, the interpretation of $T_{\rho\sigma}$ in the light of our present knowledge is uncertain or incomplete. In addition the information contained in the stress-energy tensor provides only a partial description of a universe; some discussion of other properties is required. In order to avoid a repetitive detailing of the properties for each of the universes obtained in chapter 4, we shall concentrate attention on four that seem to be of particular interest. These are space-times I and VI, which are the only spherically symmetric, static, asymptotically flat space-times permitted by Lanczos' equations, and space-times II and III which are regular.

Space-time I is discussed in some detail and the major properties of space-times VI, II and III are considered.

5.2 Properties of Space-Time I

Space-time I has the very simple metric form (4.44a):

$$ds^2 = \frac{-dr^2}{(1+\frac{a}{r})^2} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{dt^2}{(1+\frac{a}{r})^2} \quad ,$$

which is Minkowskian at spatial infinity. It is a non-empty universe, with the T_4^4 component of the stress-energy tensor non-negative everywhere, but is not easily interpreted physically. In standard relativity theory the physical situation is known *a priori*, and correspondingly the Newtonian gravitational potential. Thus one can identify the signs of the constants of integration, as in the Schwarzschild case. The same is not true in our theory and we can at this point say nothing definite about the sign of the constant of integration a in the metric form (4.44a). As we shall see, geodesic behaviour is more interesting for $a > 0$ so the following discussion of the properties of the universe are mainly concerned with this case. It will be convenient to categorise properties as geometrical, topological or physical, following Das and Coffman.[3]

5.2.1 Geometrical Properties

We consider first a t-constant hypersurface V_3 (for $a > 0$) followed by a discussion of the universe V_4.

V_3 has the metric form

$$d\sigma^2 = \frac{-dr^2}{(1+\frac{a}{r})^2} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad . \tag{5.1}$$

Then the square root of minus the determinant of the metric tensor in V_3 is everywhere positive for $0 < r < \infty$, $0 < \theta < \pi$. Both the radial distance $R(r)$ and the volume $V(r)$ are positive in this range and approach their Euclidian values as $r \to \infty$, being given respectively by

$$\mathcal{R}(r) = r - a\ln\left(\frac{r+a}{a}\right) ,$$

$$V(r) = \frac{4\pi r^3}{3} + 4\pi\left\{-\frac{ar^2}{2} + a^2 r + a^3\ln\left(\frac{a}{r+a}\right)\right\} .$$

The length of the circumference of a circle, and the area of a sphere, at the radial distance $\mathcal{R}(r)$ have the usual Euclidian values. Both the ratio of the circumference of a circle at $\mathcal{R}(r)$ to the radial length, and the solid angle subtended by a spherical surface at $r = 0$, are infinite at $r = 0$ and tend to the Euclidian values for these quantities as r increases, being given respectively by

$$\frac{2\pi r}{\{r - a\ln(1+\frac{r}{a})\}}$$

and

$$\frac{4\pi r^2}{\{r - a\ln(1+\frac{r}{a})\}^2} .$$

V_3 is a totally geodesic hypersurface of V_4 - that is all the geodesics of V_3 are also geodesics of V_4 - due to the general property that the gravitational field under consideration is static.[51] Lines of constant θ and ϕ (r-lines) form a subset of the set of geodesics in V_3 and therefore are also geodesics in V_4 . The geodesic deviation between two adjacent radial geodesics has the usual Euclidian value.

In V_4 , with the metric given by (4.44a),

$$(-g)^{\frac{1}{2}} = r^4 \sin\theta/(r+a)^2 \geqslant 0 \text{ in } 0 < r < \infty, 0 < \theta < \pi ,$$

$$V(r,t) = 4\pi\int_0^t \left\{\frac{1}{3}r^3 - ar^2 + 3a^2 r + \frac{a^3 r}{(r+a)} + 4a^3\ln\left(\frac{a}{r+a}\right)\right\} dt .$$

Lines of constant r , θ and ϕ (t-lines) are not geodesics in V_4 , other than the one at spatial infinity. The proper time along a t-line between two t-constant hypersurfaces ,

$$\Delta\tau = \frac{r}{(r+a)}\Delta t \quad ,$$

increases with the radial coordinate r of the t-line and at spatial infinity is equal to the coordinate time. The slope dr/dt of the radial null curve is everywhere ± 1, discounting the singular hypersphere $r = 0$. The non-vanishing components of the Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$ are given by[41]

$$R_{2323} = a\sin^2\theta\,(2r+a) \;,$$

$$R_{1212} = R_{2424} = R_{3434}/\sin^2\theta = R_{3131}/\sin^2\theta = \frac{-a}{(r+a)} \;,$$

$$R_{1414} = a(2r+a)/(r+a)^4,$$

and those of the Ricci tensor $R^{\alpha}_{\cdot\beta}$ in mixed form, by

$$\begin{aligned} R^1_1 &= \frac{4a}{r^3} + \frac{3a^2}{r^4} \;, \\ R^2_2 = R^3_3 &= -\frac{2a}{r^3} - \frac{a^2}{r^4} \;, \\ R^4_4 &= \frac{-a^2}{r^4} \;. \end{aligned} \qquad 5.2$$

5.2.2 Topological Properties

Clearly we cannot cast V_4 into a form which would satisfy the regularity condition (3.12). There is little doubt that the singularity at $r = 0$ is not a function of the observer, but of the physical space-time itself. The singularity at $r = -a$ (supposing that the constant a may take negative values) is, however, only of a coordinate nature. This may be seen by obtaining the invariant components of the Riemann tensor $R_{\alpha\beta\gamma\delta}$ by transforming to the Petrov canonical form.[52] Choosing unit tetrad vectors along the coordinate axes and letting the index pairs obey the rule:

$23 \longrightarrow 1,\ 31 \longrightarrow 2,\ 12 \longrightarrow 3,\ 14 \longrightarrow 4,\ 24 \longrightarrow 5,\ 34 \longrightarrow 6,$

the bitensor (symmetric 6-tensor) R_{IJ} ($I, J = 1,\dots,6$) is obtained in canonical form:

$$R_{IJ} = \begin{pmatrix} \alpha & & & & & 0 \\ & \beta & & & & \\ & & \beta & & & \\ & & & \alpha & & \\ & & & & \beta & \\ 0 & & & & & \beta \end{pmatrix} \qquad 5.3$$

where

$$\alpha = \frac{a}{r^4}(2r + a) \;, \quad \beta = -\frac{a}{r^4}(r + a) \,.$$

Whether a is positive or negative the scalar invariants α and β of the Petrov canonical form diverge only as the value of the radial coordinate decreases to zero, indicating that the singularity at $r = -a$ is due to an improper choice of coordinates for that hypersphere. (We note that the structure of the bitensor R_{IJ} (5.3) is identical for each of the space-times I$\longrightarrow$III due to the existence of certain algebraic relations between the components of the Riemann tensor).

When a is positive there is no coordinate singularity. Now in the case of the multiply connected Schwarzschild universe (3.1), it is, in effect, the $r = 2m$ coordinate singularity which prevents a breakdown in causality via the "pinch-off" effect,[53] but there would be no such similar circumstance to prevent causality breakdown if the topology of V_3 (with a positive) of space-time I were to be other than simply connected. Thus we seek a single non-singular coordinate system by means of which we may embed V_3 (without singularity) in four dimensional pseudo-Euclidian space E_4. For V_3 of space-time I the embedding is given by

$$d\sigma^2 = dX^{1\,2} - dX^{2\,2} - dX^{3\,2} - dX^{4\,2}$$

with

$$X^1 = 2a\left\{\frac{\sqrt{2ar+a^2}}{a} - \tan^{-1}\frac{\sqrt{2ar+a^2}}{a}\right\},$$
$$X^2 = r\sin\theta\cos\phi,\ X^3 = r\sin\theta\sin\phi,\ X^4 = r\cos\theta .$$

Through elimination of the coordinate r, V_3 may be represented as a hypersurface in E_4 and is topologically Euclidian. When $a < 0$, two coordinate patches are required to cover V_3, which, like the spatial hypersurface of the Schwarzschild universe, is not topologically Euclidian.

Clearly, V_4 cannot be embedded in a five dimensional pseudo-Euclidian space E_5; Takeno[54] has given the conditions necessary for a spherically symmetric V_4 to be embedded in a five dimensional space of constant curvature k, and space-time I does not satisfy these whether or not $k = 0$. It is, however, possible to embed space-time I in a pseudo-Euclidian E_6; this may be done by following Fronsdal's method[55] for obtaining the analytic extension of the Schwarzschild manifold. The embedding of V_4 in E_6 is given by

$$\begin{aligned}
ds^2 &= dZ^{1^2} - dZ^{2^2} + dZ^{3^2} - dZ^{4^2} - dZ^{5^2} - dZ^{6^2},\\
Z^1 &= \frac{2}{(1+a/r)}\sinh\left(\frac{t}{2}\right),\\
Z^2 &= \frac{2}{(1+a/r)}\cosh\left(\frac{t}{2}\right),\\
Z^3 &= \int\left[\left\{(r+a)^2(2ar+a^2)+4a^2\right\}^{\frac{1}{2}}/(r+a)^2\right]dr,\\
Z^4 &= r\sin\theta\cos\phi,\ Z^5 = r\sin\theta\sin\phi,\ Z^6 = r\cos\theta .
\end{aligned} \qquad 5.4$$

For $a > 0$ the embedding (5.4) is real and non-singular in the entire coordinate range. When $a < 0$ a suitable choice of units ensures that Z^3 is always real.[56] Formally, the r, θ, ϕ and t coordinates may be eliminated from (5.4), but since Z^3 is a monotonically increasing function of r, r may be retained as a parameter. Then V_4 may be represented as the surface

$$Z^{2^2} - Z^{1^2} = 4/(1+\tfrac{a}{r})^2,\ Z^3 = \int\left[\left\{(r+a)^2(2ar+a^2)+4a^2\right\}^{\frac{1}{2}}/(r+a)^2\right]dr,$$
$$Z^{4^2} + Z^{5^2} + Z^{6^2} = r^2 . \qquad 5.5$$

The translation and rotation properties of the surface (5.5) are similar to those of the analytic extension of the Schwarzschild manifold. It is interesting to note that the embedding of space-time I in this way requires the employment of four space-like and two time-like coordinates, while the Schwarzschild embedding is of signature (-4).

5.2.3 Physical Properties

In the preceding section we stated that there is little doubt that the $r = 0$ singularity is of a physical nature. Any lingering doubt is dispelled by consideration of the accessibility of this region of space-time. For, as emphasised by Geroch[43], we would not wish to call a space-time singular if an affine parameter on every time-like half-geodesic (geodesic curve which has one endpoint and which has been extended as far as possible in some direction from that endpoint) attained arbitrarily large values. As Geroch suggests, in a non-singular space-time, observers who follow "reasonable" (in some sense) world lines should have an infinite total proper time. Let us consider then an idealised observer who falls freely from $r = r_1$. His path must be an r-line, and r-lines are geodesics in V_4. The geodesic equations may be integrated to give the expression for the proper time taken in travelling from $r = r_1$ to $r = 0$:

$$\tau = \int_{r_1}^{0} \frac{\ell \, dr}{\{\ell^2(1+a/r)^2 - 1\}^{1/2}} , \qquad 5.6$$

where ℓ is a positive constant related to the energy/unit mass of our observer. Clearly, for suitable values of ℓ^2 (and, in addition, for suitable values of r_1 if $a < 0$), the integrand in (5.6) is an essentially finite quantity. This means that $r = 0$ is an accessible region of space-time, and since a radial geodesic would not seem to be other than a "reasonable" path, we can conclude that space-time I has a physical singularity at $r = 0$.

Perhaps the most vivid description of space-time I is provided by its geodesics. Pirani[57] has studied the approximate perihelion motion corresponding to the metric (4.44a). This was in connection with Littlewood's[58] suggestion that Einstein's vacuum equations should be replaced by the single equation $R = 0$ together with the assumption that space-time is conformally flat. In Littlewood's theory, space-time I (obtained in the form (4.31)) would replace the Schwarzschild solution as the fundamental vacuum solution. Interpreting the constant of integration a as a Schwarzschild-type mass, Pirani found the approximate perihelion advance to be one sixth of the value obtained from the Schwarzschild solution and in the opposite direction. Since this was a result in distinct contradiction to observation, Littlewood's theory was rejected. In our theory, however, space-time I does not represent a vacuum solution and there is no *a priori* reason to interpret the constant a as a central mass. We now discuss the geodesic behaviour more fully.

For any static, spherically symmetric metric

$$ds^2 = -e^{\alpha(r)}\,dr^2 - e^{\beta(r)}\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + e^{\gamma(r)}\,dt^2 \qquad 5.7$$

the Euler-Lagrange equations lead to the differential equation of the geodesics:

$$\left(\frac{du}{d\phi}\right)^2 + u^4 e^{-\alpha+\beta} - \frac{\ell^2}{h^2} u^4 e^{-\gamma-\alpha+2\beta} = -\frac{\epsilon}{h^2} u^4 e^{-\alpha+2\beta} \,. \qquad 5.8$$

For timelike geodesics, which represent the motion of neutral test particles, the constant ϵ takes the value $+1$, while for null geodesics, representing the path of light rays, ϵ is zero. The variable u employed in (5.8) is the inverse of r. For particle geodesics the constants h and ℓ represent, per unit mass of the particle, the angular momentum about the origin and the energy respectively, being given by

$$h = e^{\beta} \frac{d\phi}{ds} ,$$

$$\ell = e^{\gamma} \frac{dt}{ds} .$$

In deriving the equation (5.8) the $\theta = 0$ axis was chosen so that at some point on the geodesic $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$. The Euler-Lagrange equations then imply that $\ddot{\theta} = 0$ and all higher derivatives of θ vanish, so that geodesic motion lies entirely within the plane $\theta = \frac{\pi}{2}$. Particle geodesics are now discussed, followed by a description of the null geodesics.

For space-time I, with metric form (4.44a), the differential equation of particle geodesics is given, from (5.7) and (5.8), by

$$\left(\frac{du}{d\phi}\right)^2 = \frac{(1+au)^2}{h^2}\left\{u^2(\ell^2a^2-h^2) + 2a\ell^2u + (\ell^2-1)\right\} . \qquad 5.9$$

We shall consider in any detail only those orbits which possess perihelia. These are of the type normally associated with an attractive force and arise only when the constant a is positive. The condition that there should be real motion between the two real roots of

$$u^2(\ell^2a^2-h^2) + 2a\ell^2u + (\ell^2-1) = 0 \qquad 5.10$$

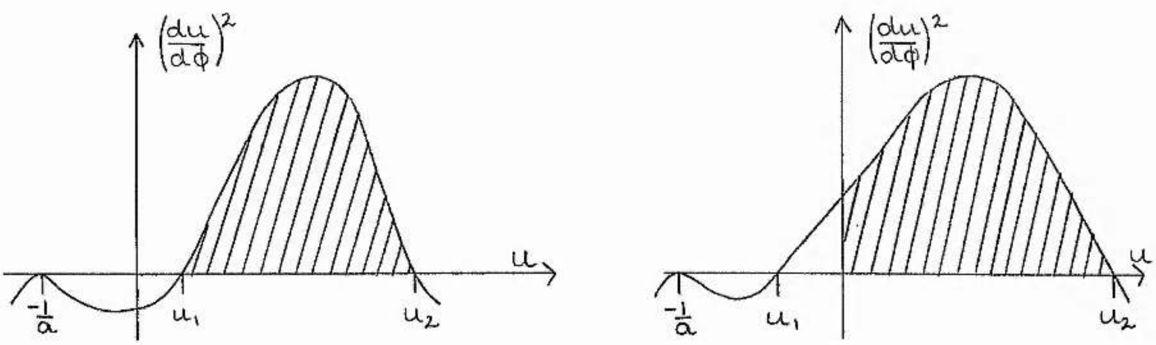
reduces to a conditional relationship between the energy and the angular momentum of the particle, for it is given by

$$\frac{1}{1+A} \leq \ell^2 < \frac{1}{A} , \qquad 5.11$$

where

$$A = \frac{a^2}{h^2} . \qquad 5.12$$

(The only behaviour available to a test particle for positive a other than perihelic motion occurs when $\ell^2 > \frac{1}{A}$ and is to fall in to the origin). The orbits described by (5.9) with the condition (5.11) fall into the expected categories. Denoting the (real) roots of (5.10) by u_1 and u_2, with $u_1 < u_2$, the motion may be illustrated by the following schematic graphs of $(du/d\phi)^2$:



Elliptic type ($u_1 > 0$, $\ell^2 < 1$). Hyperbolic type ($u_1 < 0$, $\ell^2 > 1$).

(Motion takes place within the hatched regions)

In addition, the usual special cases of circular motion ($u_1 = u_2$, $\ell^2 < 1$) and parabolic motion ($u_1 = 0$, $\ell^2 = 1$) are obtained.

The differential equation (5.9) is easily integrated, using

$$\int \frac{h\,du}{(1+au)\sqrt{u^2(\ell^2a^2-h^2)+2a\ell^2u+(\ell^2-1)}} \qquad 5.13$$

$$= \frac{h}{\sqrt{h^2+a^2}} \sin^{-1}\left\{\frac{h^2u-a}{(1+au)\sqrt{\ell^2(a^2+h^2)-h^2}}\right\},$$

to obtain the equation which describes all geodesic orbits with perihelia:

$$r = \frac{a}{A}\left\{\frac{1+AD\cos(\sqrt{1+A}\,\phi)}{1-D\cos(\sqrt{1+A}\,\phi)}\right\} \qquad 5.14$$

where

$$D = \sqrt{\ell^2(1+1/A)-1/A}$$

(putting $r = 1/u$). The similarities between (5.14) and the classical orbit equation are immediately apparent. The periodic recession is independent of the energy of the particle and is equal to

$$2\pi\left\{1 - 1/\sqrt{1+A}\right\} \qquad 5.15$$

per orbit. Some quasi-elliptical orbits of space-time I for different values of ℓ^2 and A are shown in Figs. 1$\longrightarrow$11. It should be observed that classically values of ℓ^2 much less than unity would correspond to the motion of particles in the vicinity of very massive objects.

In order to compare particle motion in space-time I with that expected classically we differentiate (5.9) with respect to ϕ to obtain

$$\frac{d^2u}{d\phi^2} + u = \frac{a}{h^2}(2\ell^2-1) + \frac{a^2u}{h^2}(6\ell^2-1) + 3au^2\left(2a^2\frac{\ell^2}{h^2}-1\right) + 2a^2u^3\left(\frac{\ell^2a^2}{h^2}-1\right) . \qquad 5.16$$

Now the classical Binet equation for the motion of a particle with potential energy $f(r)$ per unit mass is

$$\frac{d^2u}{d\phi^2} + u = \frac{1}{H^2}\frac{f'(r)}{u^2} , \qquad 5.17$$

where $H = r^2\frac{d\phi}{dt}$ = constant.

For slowly moving bodies in weak fields we may take $H \simeq h$ and compare (5.16) with (5.17). In addition to a term representing an inverse square force (attractive or repulsive as $a(2\ell^2 - 1) > 0$ or < 0) there are terms present in (5.16) representing forces of higher orders.

The discussion of the geodesics seems to indicate that the constant a is positive. In the case that a is negative, there is never any stable periodic motion, but hyperbolic orbits of the type normally associated with a repulsive force may be executed for all values of ℓ^2 outside the coordinate singularity at $r = -a$. For $\ell^2 > \frac{1}{A}$, a test particle which starts its existence at any value of r less than $-a$ will terminate its motion by falling in to the origin. No real particle can cross the "boundary" at $r = -a$.

The differential equation governing the null geodesics representing light rays in space-time I is given by (5.8) where ϵ takes the value zero and α, β and γ are obtained by comparison of the general metric form (5.7) with that for space-time I, (4.44a). This may be integrated to give

$$r = \frac{h}{\ell}\left\{-\frac{a\ell}{h} + \sec\phi\right\} .$$

For positive a, real motion is possible only if $a < \frac{h}{\ell}$, and the light ray ultimately resumes its original course after moving towards the spatial origin. When a is negative any light ray originally outside the coordinate singularity has a path which remains outside, while if it is inside $r = -a$, it is trapped and never emerges. Some typical null geodesics in space-time I are shown in Figs. 12 and 13.

The gravitational shift of spectral lines in space-time I is given, for large r, by

$$\nu_2 - \nu_1 = \nu_0\, a\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

where ν_0 is the characteristic frequency of a spectral line and ν_1, ν_2 are the coordinate frequencies at r_1, r_2 respectively. This result is identical to the Schwarzschild formula if a is replaced by the constant m; the shift is thus in the same direction as the Schwarzschild shift.

It was emphasised in section 5.1 that the problem of the interpretation of the stress-energy tensor is far from trivial. With the components of the Ricci tensor given by (5.2), the non-vanishing components of the stress-energy tensor are given from the definition (2.4) by

$$\begin{aligned} T^1_1 &= \frac{1}{\kappa}\left(-\frac{4a}{r^3} - \frac{3a^2}{r^4}\right), \\ T^2_2 = T^3_3 &= \frac{1}{\kappa}\left(\frac{2a}{r^3} + \frac{a^2}{r^4}\right), \\ T^4_4 &= \frac{1}{\kappa}\frac{a^2}{r^4} \quad . \end{aligned} \tag{5.18}$$

It is tempting to split the stress-energy tensor (5.18) into a material and an electrostatic part, in which the usual algebraic relations for the spherically symmetric case hold, with the familiar expression for the energy density due to the electromagnetic field created by a charged particle at the spatial origin. Then

$$T^{\rho}_{.\sigma} = M^{\rho}_{.\sigma} + \mathcal{E}^{\rho}_{.\sigma} ,$$

with

$$\begin{aligned} \mathcal{E}^1_1 = -\mathcal{E}^2_2 = -\mathcal{E}^3_3 = \mathcal{E}^4_4 &= \frac{1}{\kappa}\frac{a^2}{r^4} , \\ -\tfrac{1}{2}M^1_1 = M^2_2 = M^3_3 &= \frac{2}{\kappa}\left(\frac{a}{r^3} + \frac{a^2}{r^4}\right), \\ M^4_4 &= 0 \quad . \end{aligned}$$

The problem of the interpretation of the material stress tensor $M^{\rho}_{.\sigma}$, however, remains; it is clearly not due to any of the more frequently studied physical situations.

The stress-energy tensor represents all fields other than gravitation. In a general Riemannian space we can therefore expect that only some combination of $T_{\rho\sigma}$ and $g_{\rho\sigma}$ represents a conserved quantity. There is no uniqueness about the choice of such a combination, and although the various expressions give consistent results, they are intrinsically non-covariant. Furthermore, the concept of total energy is ill-defined unless the coordinate system employed is Lorenzian at the spatial infinity of each coordinate. However, we may represent space-time I by a coordinate system of this type. Transforming the spherical polars ρ , θ and ϕ of the metric (4.31) into Cartesian coordinates:

$$x = \rho \sin\theta \cos\phi \ , \ y = \rho \sin\theta \sin\phi \ , \ z = \rho \cos\theta \ ,$$

we obtain

$$ds^2 = \left(1 - \frac{a}{\rho}\right)^2 \left(-dx^2 - dy^2 - dz^2 + dt^2\right) \ . \qquad 5.19$$

We may conveniently use the expression obtained by Adler, Bazin and Schiffer[59] for the total "energy" P_4 of any finite V_3 bounded by surface S for a metric tensor which is time-independent, diagonal and spatially isotropic :

$$P_4 = \frac{c^2}{8\pi\kappa} \int_S -\sqrt{-g} \ \frac{\partial g^{11}}{\partial x^j} n_j \, dS \ , \qquad 5.20$$

where n_j is the unit outward normal to dS . Then in order to obtain the total energy in space-time I, exhibited in the metric (5.19), we may perform the integration in (5.20) over a sphere of radius R and let R tend to infinity. We obtain

$$P_4 = -\frac{ac^2}{\kappa} \ .$$

The Schwarzschild result is $P_4 = +mc^2/\kappa$. Thus it is clear that, despite some superficial similarities between the Schwarzschild universe and space-time I, they are fundamentally of a very different nature.

5.3 Properties of Space-Time VI

Space-time VI (the extended Schwarzschild solution) is, like space-time I, Minkowskian at spatial infinity. It is represented by the metric (4.44f):

$$ds^2 = \left\{1 + B\ln\left(1-\frac{2m}{r}\right)\right\}^2\left\{\frac{-dr^2}{1-2m/r} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \left(1-\frac{2m}{r}\right)dt^2\right\},$$

which reduces to the Schwarzschild metric (3.1) when B is zero. We wish to know how the properties of the Schwarzschild universe are modified by the presence in the metric (4.44f) of the non-vanishing constant of integration B .

It is of immediate interest to determine whether or not the $r = 2m$ singularity is still of a coordinate nature. This will be indicated by the behaviour of the scalar invariants of the Petrov canonical form; to obtain these the components of the Riemann tensor are first required:

$$R_{2323} = -2mr\sin^2\theta\left\{D^2 - \frac{2B^2m}{(r-2m)} - 2BD\right\} \quad ,$$

$$R_{1212} = R_{3131}/\sin^2\theta = \frac{m}{(r-2m)}\left\{D^2 - \frac{4B^2m}{(r-2m)} + 2BD\frac{(m-r)}{(r-2m)}\right\} \quad ,$$

$$R_{1414} = \frac{2m}{r^3}\left\{D^2 + \frac{2B^2m}{(r-2m)} + 2BD\right\} \quad ,$$

$$R_{2424} = R_{3434}/\sin^2\theta = -\frac{m(r-2m)}{r^2}\left\{D^2 + \frac{4B^2m}{(r-2m)} - 2BD\frac{(m-r)}{(r-2m)}\right\},$$

where

$$D \equiv 1 + B\ln\left(1-\frac{2m}{r}\right).$$

For future use, we also give the components of the Ricci tensor in mixed form:

$$R^1_1 = \frac{4m^2B^2}{r^3(r-2m)D^4}\left\{3 - \frac{D}{mB}(3m-2r)\right\},$$

$$R^2_2 = R^3_3 = \frac{4m^2B^2}{r^3(r-2m)D^4}\left\{-1 + \frac{D}{mB}(-r+2m)\right\}, \qquad 5.21$$

$$R^4_4 = \frac{4m^2B^2}{r^3(r-2m)D^4}\left\{-1 - \frac{D}{B}\right\} \quad .$$

Choosing unit tetrad vectors along the coordinate axes the bitensor R_{IJ} (I, J = 1,...,6) is obtained in canonical form

$$R_{IJ} = \begin{bmatrix} \alpha & & & & & 0 \\ & \beta & & & & \\ & & \beta & & & \\ & & & \gamma & & \\ & & & & \delta & \\ 0 & & & & & \delta \end{bmatrix} ,$$

where

$$\alpha = \frac{-2m}{r^3}\left\{\frac{1}{D^2} - \frac{2mB^2}{(r-2m)D^4} - \frac{2B}{D^3}\right\},$$

$$\beta = \frac{m}{r^3}\left\{\frac{1}{D^2} - \frac{4mB^2}{(r-2m)D^4} + \frac{2B(m-r)}{(r-2m)D^3}\right\},$$

$$\gamma = -\alpha + \frac{8mB}{r^3}\left\{\frac{1}{D^3} + \frac{mB}{(r-2m)D^4}\right\},$$

$$\delta = -\beta + \frac{4mB}{r^3(r-2m)}\left\{\frac{(m-r)}{D^3} - \frac{2mB}{D^4}\right\}.$$

For $B = 0$ the Petrov scalars α , β , γ and δ are finite at $r = 2m$ and the singularity in the metric (4.44f) is of a coordinate nature only. However for $B \neq 0$ the scalars diverge as $r \longrightarrow 2m$. This indicates that the singularity is a property of the space-time itself and not of the coordinate system used to describe it. In this respect at least, space-time VI differs radically from the Schwarzschild universe.

The spatial hypersurface V_3 of the Schwarzschild universe is not topologically Euclidian: it cannot be covered without singularity by a single non-singular coordinate system. It is not surprising to find that there is no value of the constant B for which this is possible for the V_3 of space-time VI. The Schwarzschild V_4 can be embedded in a pseudo-Euclidian E_6 as shown originally by Kasner[60] and later, amongst others, by Fronsdal[55], but this does not seem possible for space-time VI. Following Fronsdal's method (as for space-time I) we look for an embedding for which the metric is

$$ds^2 = dZ^{1^2} - dZ^{2^2} - dZ^{3^2} - dZ^{4^2} - dZ^{5^2} - dZ^{6^2}$$

with[56]

$$Z^1 = 2D(1-\tfrac{1}{r})^{\frac{1}{2}} \sinh(\tfrac{t}{2}) ,$$
$$Z^2 = 2D(1-\tfrac{1}{r})^{\frac{1}{2}} \cosh(\tfrac{t}{2}) ,$$
$$Z^3 = g(r) ,$$
$$Z^4 = rD\sin\theta\cos\phi ,$$
$$Z^5 = rD\sin\theta\sin\phi ,$$
$$Z^6 = rD\cos\theta .$$

where $g(r)$ is determined from

$$\left(\frac{dg}{dr}\right)^2 = D^2\frac{(r^2+r+1)}{r^3} - 2BD\frac{(r^3+2)}{r^3(r-1)} - B^2\frac{(r^3+4r-4)}{r^3(r-1)^2} .$$

This is a suitable embedding if it is real and exists within the entire coordinate range. This, however, is not the case since for values of r close to unity Z^3 ceases to be a real coordinate as dg/dr becomes imaginary (unless, of course, $B = 0$ when the embedding exists in the whole interval $0 < r < \infty$).

Comparing the general static, spherically symmetric metric form (5.7) with (4.44f) and substituting into (5.8) we obtain the differential equation of the geodesics in space-time VI:

$$\left(\frac{du}{d\phi}\right)^2 + u^2(1-2mu) - \frac{\ell^2}{h^2} = -\frac{\epsilon}{h^2}(1-2mu)(1+B\ln(1-2mu))^2 , \qquad 5.22$$

in which $u = 1/r$ and

$$h = r^2D^2\frac{d\phi}{ds} ,$$
$$\ell = (1-\tfrac{2m}{r})D^2\frac{dt}{ds} .$$

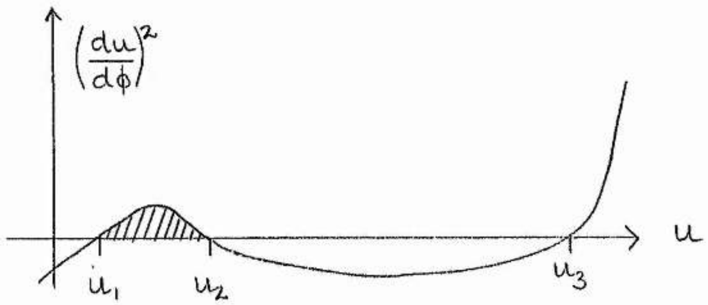
Differentiating (5.22) we obtain

$$\frac{d^2u}{d\phi^2} - 3mu^2 + u$$
$$= \frac{\epsilon}{h^2}m\{(1+2B)+B\ln(1-2mu)\}\{1+B\ln(1-2mu)\} . \qquad 5.23$$

Comparing (5.23) with the classical Binet equation (5.17), the classical inverse square force upon a particle for large r is modified by the factor $(1+2B)$. For orbits which possess perihelia the approximate perihelion advance is also modified by this factor. For large r we may write in place of (5.22) (putting $\epsilon = 1$):

$$\left(\frac{du}{d\phi}\right)^2 = 2mu^3 - u^2 + \frac{2mu}{h^2}(1+2B) + \frac{(\ell^2-1)}{h^2} \quad , \tag{5.24}$$

where we have retained only terms of order unity in (mu), which would be reasonable for planetary orbits.[61] We suppose that $(du/d\phi)^2$ has three real roots u_1, u_2, u_3 such that u_1, u_2 are small in comparison with u_3 and are positive (i.e. considering orbits of quasi-elliptical type only). Periodic motion takes place (i.e. $du/d\phi$ is real) between u_1 and u_2:



We may write

$$\left(\frac{du}{d\phi}\right)^2 = 2m(u-u_1)(u-u_2)(u-u_3) \quad , \tag{5.25}$$

where, by comparison of (5.25) with (5.24):

$$u_1 + u_2 + u_3 = \frac{1}{2m} \quad . \tag{5.26}$$

Then

$$\left.\begin{matrix} 2mu_1 \\ 2mu_2 \end{matrix}\right\} << 1 \quad . \tag{5.27}$$

The angle between successive apsides is given, from (5.25), by

$$\Delta\phi = \int_{u_1}^{u_2} \frac{du}{\sqrt{\{2mu_3(u-u_1)(u_2-u)(1-u/u_3)\}}} \quad .$$

Using (5.26), (5.27) and the approximation that during the motion u/u_3 is small the expression for $\Delta\phi$ becomes

$$\Delta\phi = \int_{u_1}^{u_2} \frac{\{1 + m(u_1+u_2+u) + O(m^2u^2)\}}{\sqrt{\{(u-u_1)(u_2-u)\}}}\, du$$
$$\simeq \pi\{1 + \tfrac{3m}{2}(u_1+u_2)\} .$$

The perihelion advance is thus

$$2\Delta\phi - 2\pi \simeq 3m\pi(u_1+u_2) \qquad 5.28$$

Again, by comparison of (5.25) with (5.24), the sum of the products (taken pairwise) of the roots is given by

$$u_1u_2 + u_2u_3 + u_3u_1 = \frac{(1+2B)}{h^2} . \qquad 5.29$$

Multiplying (5.29) by $2m$ and using (5.26) and (5.27) we obtain to the required order

$$u_1 + u_2 = \frac{2m(1+2B)}{h^2}$$

so that, from (5.28), the Schwarzschild perihelion advance is modified by the factor $(1+2B)$. The behaviour of the null geodesics in space-time VI is identical to the Schwarzschild behaviour, as is obvious from putting $\epsilon = 0$ in the geodesic equation (5.22).

From (5.21), the non-vanishing components of the stress-energy tensor are given, from the definition (2.4), by

$$T^1_1 = \frac{4m^2B^2}{\kappa r^3(r-2m)D^4}\left\{-3 + \frac{D}{mB}(3m-2r)\right\} ,$$

$$T^2_2 = T^3_3 = \frac{4m^2B^2}{\kappa r^3(r-2m)D^4}\left\{1 + \frac{D}{mB}(r-2m)\right\} , \qquad 5.30$$

$$T^4_4 = \frac{4m^2B^2}{\kappa r^3(r-2m)D^4}\left\{1 + \frac{D}{B}\right\} .$$

T^4_4 is positive only in the region $r > r_1$ where r_1 depends on B according to the relation

$$r_1 = \frac{2m}{1 - e^{-(1+1/B)}} ,$$

which approaches $2m$ as $B \longrightarrow 0$ from the right. The interpretation of the matter tensor (5.30) poses considerable problems for it does not seem to represent realistically any simple physical situation. We must leave discussion of this problem open.

In order to calculate the total "energy" P_4 in space-time VI we cast the metric (4.44f) into the spatially isotropic form

$$ds^2 = D^2(\rho)\left\{\left(1+\frac{m}{2\rho}\right)^4(-dx^2-dy^2-dz^2)+\frac{\left(1-\frac{m}{2\rho}\right)^2}{\left(1+\frac{m}{2\rho}\right)^2}dt^2\right\}, \quad 5.31$$

where

$$D(\rho) = 1 + 2B\ln\left[\frac{\left(1-\frac{m}{2\rho}\right)}{\left(1+\frac{m}{2\rho}\right)}\right],$$

through the usual transformations for the Schwarzschild universe:

$$r = \rho\left(1+\frac{m}{2\rho}\right)^2,$$

$$x = \rho\sin\theta\cos\phi, \quad y = \rho\sin\theta\sin\phi, \; z = \rho\cos\theta.$$

The coordinate system employed in (5.31) is Lorenzian at the spatial infinity of each coordinate and we may proceed in calculating P_4 exactly as for space-time I. It transpires that the total energy in space-time VI is given by

$$P_4 = \frac{mc^2}{\kappa}(1-2B)$$

so that the Schwarzschild result is in this case modified by the factor $(1 - 2B)$.

5.4 Properties of Space-Time II

Space-time II is represented by the metric (4.44b):

$$ds^2 = \frac{-C^2r^2dr^2}{\sinh^2(C(r-B))} - r^2(d\theta^2+\sin^2\theta\, d\phi^2) + \frac{C^2r^2dt^2}{\sinh^2(C(r-B))}.$$

As demonstrated in section 4.5, (4.44b) may be recast in a form conformal to the closed de Sitter model by setting

$$r = \frac{1}{C}\tanh^{-1}(C\rho) + B$$

so that the metric becomes

$$ds^2 = \frac{[\tanh^{-1}(C\rho) + BC]^2}{(C\rho)^2}\left\{\frac{-d\rho^2}{1-C^2\rho^2} - \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2) + (1-C^2\rho^2)\,dt^2\right\}. \qquad 5.32$$

Retaining the meaning of ρ as some sort of radial coordinate, the extension of the physical space in the space-time exhibited in (5.32) is

$$0 \leq \rho < 1/C \;, \; 0 \leq \theta \leq \pi \;, \; 0 \leq \phi \leq 2\pi \, .$$

Thus, since $(1/C)\tanh^{-1}(C\rho) \geq 0$, the metric (4.44b) is meaningful in

$$B \leq r < \infty \;, \; 0 \leq \theta \leq \pi \;, \; 0 \leq \phi \leq 2\pi \, .$$

Now when $B = 0$ the metric (4.44b) is regular at $r = 0$, from the regularity condition (3.12), but when $B \neq 0$ it contains an apparent singularity at $r = B$. The scalar invariants α and β of the Petrov canonical form (5.3) are

$$\alpha = \frac{1}{r^2}\left\{-1 + \frac{\sinh^2(C(r-B))}{C^2 r^2}\right\},$$

$$\beta = \frac{\sinh^2(C(r-B))}{C^2 r^4}\left\{-1 + Cr\coth(C(r-B))\right\}.$$

For $B \neq 0$, at $r = B$, α and β have the values

$$\alpha = \frac{-1}{B^2} \;, \; \beta = 0,$$

confirming that the singularity is only of a coordinate nature. As $r \longrightarrow \infty$, α and β become infinite, but we should not wish to call this circumstance a singularity.[43] We note the components of the Riemann and Ricci tensors:

$$R_{2323} = r^2 \sin^2\theta \left\{ -1 + \frac{\sinh^2(C(r-B))}{C^2 r^2} \right\} ,$$

$$R_{1212} = R_{2424} = R_{3131}/\sin^2\theta = R_{3434}/\sin^2\theta = -1 + Cr \coth(C(r-B)),$$

$$R_{1414} = \frac{C^2}{\sinh^2(C(r-B))} \left\{ 1 - \frac{C^2 r^2}{\sinh^2(C(r-B))} \right\} .$$

$$R^1_1 = \frac{3\sinh^2(C(r-B))}{C^2 r^4} - \frac{1}{r^2} - \frac{2}{Cr^3} \sinh(C(r-B)) \cosh(C(r-B)) ,$$

$$R^2_2 = R^3_3 = -\frac{\sinh^2(C(r-B))}{C^2 r^4} + \frac{1}{r^2} , \qquad 5.33$$

$$R^4_4 = -\frac{\sinh^2(C(r-B))}{C^2 r^4} - \frac{1}{r^2} + \frac{2}{Cr^3} \sinh(C(r-B)) \cosh(C(r-B)) .$$

Can we embed the spatial hypersurface V_3 of space-time II in a pseudo-Euclidian E_4 ? A suitable embedding may be given by

$$ds^2 = dX^{1\,2} - dX^{2\,2} - dX^{3\,2} - dX^{4\,2} ,$$

$$X^1 = \int \left(1 - \frac{C^2 r^2}{\sinh^2(C(r-B))}\right)^{\frac{1}{2}} dr, \qquad 5.34$$

$$X^2 = r\sin\theta \cos\phi,\; X^3 = r\sin\theta \sin\phi,\; X^4 = r\cos\theta .$$

When $B \leq 0$ the embedding (5.34) is real and non-singular so that V_3 may be represented as a hypersurface in E_4 and is topologically Euclidian (like space-time I but unlike space-time VI). When $B > 0$, X^1 is not real everywhere within the coordinate range and a second coordinate patch is required; in this case V_3 is not topologically Euclidian. As for space-time I it is easy to show, using Takeno's conditions[54] that V_4 cannot be embedded in a five dimensional space of constant curvature nor in a pseudo-Euclidian E_5. However, V_4 can be embedded in an E_6. This may be demonstrated using Fronsdal's method[55] which we employed for the embedding of space-time I. A different embedding exists which depends upon the fact that the metric of space-time II may be written in the form (5.32), conformal to the closed de Sitter model. The latter is represented by the metric

$$d\sigma^2 = \frac{-d\rho^2}{(1-C^2\rho^2)} - \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2) + (1-C^2\rho^2)dt^2 ,$$

$(0 \leq \rho < 1/C\ ,\ 0 \leq \theta \leq \pi\ ,\ 0 \leq \phi \leq 2\pi)$.

Following Tolman[62] we introduce the new (real) coordinates r and $\tilde{t}$ by

$$r = \rho\,(1 - C^2\rho^2)^{-\frac{1}{2}}\, e^{-Ct},$$

$$\tilde{t} = t + \frac{1}{2C}\ln(1 - C^2\rho^2),$$

which leads to the expression

$$d\sigma^2 = -e^{2\tilde{t}C}\left(dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right) + d\tilde{t}^2, \qquad 5.35$$

$(0 \leq r < \infty)$.

As is well known, the metric (5.35) may be cast into conformally Minkowskian form by the introduction of the new variable

$$\gamma = \frac{1}{C}\, e^{-C\tilde{t}}\ .$$

Then the de Sitter metric (5.35) becomes

$$d\sigma^2 = \frac{1}{(C\gamma)^2}\left(-dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + d\gamma^2\right).$$

Thus employing a similar sequence of transformations, space-time II, given by (5.32), is represented by the metric form

$$ds^2 = \frac{\left[\tanh^{-1}\left(\frac{r}{\gamma}\right) + BC\right]^2}{C^2 r^2}\left\{-dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + d\gamma^2\right\}. \qquad 5.36$$

With the final transformation of coordinates

$$x^1 = r\cos\theta\sin\phi\ ,\ x^2 = r\sin\theta\sin\phi\ ,\ x^3 = r\cos\theta\ ,\ x^4 = \gamma\ ,$$

the metric (5.36) of space-time II (suspending the summation convention) is

$$ds^2 = \psi^2 \sum_{\beta=1}^{4} c_\beta (dx^\beta)^2,$$

where $c_1 = c_2 = c_3 = -1$, $c_4 = +1$ and

$$\psi = \frac{\left[\tanh^{-1}\left\{\frac{1}{x^4}(x^{1^2}+x^{2^2}+x^{3^2})^{\frac{1}{2}}\right\} + BC\right]}{C(x^{1^2}+x^{2^2}+x^{3^2})^{\frac{1}{2}}} \quad .$$

Now Eisenhart[63] has given the result that any V_n conformal to an E_n may be embedded in an E_{n+2} and described the embedding. In our case it will be

$$ds^2 = \sum_{A=1}^{6} c_A \, (dz^A)^2 \quad , \tag{5.37}$$

where $c_1 = c_2 = c_3 = c_6 = -1$, $c_4 = c_5 = +1$ and

$$\begin{aligned} z^\beta &= \psi x^\beta \quad , \quad \beta = 1, \ldots, 4 \; , \\ z^5 &= \psi\left(\sum_{\beta=1}^{4} c_\beta (x^\beta)^2 - \frac{1}{4}\right) , \\ z^6 &= \psi\left(\sum_{\beta=1}^{4} c_\beta (x^\beta)^2 + \frac{1}{4}\right) . \end{aligned} \tag{5.38}$$

Then V_4 may be embedded in E_6 as given by (5.37) with (5.38) and is a hypersurface of the hypercone

$$\sum_{A=1}^{6} c_A \, (z^A)^2 = 0 \quad ,$$

with the fundamental form (5.37). We note that we could have embedded space-time I by this method since it may be written in the conformally flat form (4.31).

We now consider some of the physical properties of space-time II. A neutral test particle will have a timelike geodesic as world line;

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\rho\sigma} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} = 0 \quad . \tag{5.39}$$

If the particle is initially at rest ($dx^a/ds = 0$), the initial acceleration is given, for our static, spherically symmetric metric (2.8) by

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{44}\left(\frac{dt}{ds}\right)^2 = 0 \quad .$$

From the formulae for the Christoffel symbols (A2.20), the only Γ^{α}_{44} is $\Gamma^{1}_{44} = \frac{1}{2}\nu' e^{\nu-\lambda}$ so that, initially,

$$\frac{d^2r}{ds^2} + \frac{1}{2}\nu' e^{\nu-\lambda}\left(\frac{dt}{ds}\right)^2 = 0 \quad ,$$

and since $(dr/dt) = 0$ initially, this gives

$$\frac{d^2r}{dt^2} + \frac{1}{2}\nu' e^{\nu-\lambda} = 0 \quad . \qquad 5.40$$

Now for the metric (4.44b),

$$e^{\nu} = e^{\lambda} = \frac{C^2 r^2}{\sinh^2(C(r-B))} \quad .$$

Substituting for $e^{\nu} = e^{\lambda}$ into the geodesic equation (5.40) we find that if $B \neq 0$, $(d^2r/dt^2)_{r=B}$ is infinite. Thus it is meaningless to suppose that at observer may be situated at $r = B \neq 0$. However, when $B = 0$, (5.40) gives, for small r,

$$\frac{d^2r}{dt^2} \simeq \frac{1}{3}C^2 r$$

so that the radial acceleration at $r = 0$ is zero and there is no contradiction implicit in considering an observer stationary at that point. Now suppose that a spectral line emanating from a distant source at rest has frequency ν_S. For an atom at $r = r_1$ the coordinate frequency is given by $\nu_1 = \nu_S(\sqrt{g_{44}})_{r=r_1}$ and for an identical atom at $r = 0$, $\nu_0 = \nu_S(\sqrt{g_{44}})_{r=0}$. Then an observer at $r = 0$ will observe a gravitational shift to the blue or red end of the spectrum according as

$$\frac{(\sqrt{g_{44}})_{r=r_1}}{(\sqrt{g_{44}})_{r=0}}$$

is greater or less than unity. Putting $B = 0$, for space-time II

$$g_{44} = \frac{C^2 r^2}{\sinh^2(Cr)} \quad ,$$

so that the spectral lines emanating from distant sources at rest will appear to the observer at the origin to be displaced towards the red end of the spectrum.

Finally we consider the stress-energy tensor. From (5.33) and the definition (2.4) the non-vanishing components of $T^{\rho}_{\cdot\sigma}$ are given by

$$T^1_1 = \frac{1}{\kappa}\left\{\frac{-3\sinh^2(C(r-B))}{C^2r^4} + \frac{1}{r^2} + \frac{2}{Cr^3}\sinh(C(r-B))\cosh(C(r-B))\right\},$$

$$T^2_2 = T^3_3 = \frac{1}{\kappa}\left\{\frac{\sinh^2(C(r-B))}{C^2r^4} - \frac{1}{r^2}\right\}, \tag{5.41}$$

$$T^4_4 = \frac{1}{\kappa}\left\{\frac{\sinh^2(C(r-B))}{C^2r^4} + \frac{1}{r^2} - \frac{2}{Cr^3}\sinh(C(r-B))\cosh(C(r-B))\right\}.$$

For $B \neq 0$, at $r = B$, $T^1_1 = T^4_4 = -T^2_2$ $(= 1/B^2)$, which are the usual algebraic relations in the spherically symmetric case for a purely electromagnetic field. It is thus tempting to try to interpret the stress-energy tensor (5.41) in terms of the superposition of the field of a perfect fluid and an electromagnetic field, satisfying Maxwell's equations. Under such an interpretation the matter density and isotropic pressure would take values ≤ 0 throughout the entire coordinate range. For $B = 0$, at $r = 0$, $T^1_1 = T^2_2 = -T^4_4/3$ $(= C^2/3\kappa)$ and such an interpretation must clearly fail. It has not been possible to find a realistic interpretation of (5.41) in terms of any simple superposition of fields.

5.5 Properties of Space-Time III

Space-time III is represented by the metric (4.44c):

$$ds^2 = \frac{-C^2r^2\,dr^2}{\sin^2(C(r-B))} - r^2(d\theta^2 + \sin^2\theta\,d\phi^2) + \frac{C^2r^2dt^2}{\sin^2(C(r-B))},$$

$$\left(B \leq r \leq \frac{\pi}{2C} + B,\ 0 \leq \theta \leq \pi,\ 0 \leq \phi \leq 2\pi\right).$$

As demonstrated in section 4.5, (4.44c) may be recast into a form conformal to the open de Sitter model by setting

$$r = \frac{1}{C}\tan^{-1}(C\rho) + B$$

so that the metric becomes

$$ds^2 = \frac{[\tan^{-1}(C\rho) + BC]^2}{(C\rho)^2}\left\{\frac{-d\rho^2}{1+C^2\rho^2} - \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2) + (1+C^2\rho^2)dt^2\right\}, \qquad 5.42$$

$(0 \leq \rho < \infty)$.

For the metric (4.44c) the non-zero components of the Riemann tensor are given by

$$R_{2323} = r^2\sin^2\theta\left\{-1 + \frac{\sin^2(C(r-B))}{C^2r^2}\right\},$$

$$R_{1212} = R_{2424} = R_{3131}/\sin^2\theta = R_{3434}/\sin^2\theta = -1 + Cr\cot(C(r-B)),$$

$$R_{1414} = \frac{C^2}{\sin^2(C(r-B))}\left\{1 - \frac{C^2r^2}{\sin^2(C(r-B))}\right\}.$$

and those of the Ricci tensor by

$$R^1_1 = \frac{3\sin^2(C(r-B))}{C^2r^4} - \frac{1}{r^2} - \frac{2}{Cr^3}\sin(C(r-B))\cos(C(r-B)),$$

$$R^2_2 = R^3_3 = \frac{-\sin^2(C(r-B))}{C^2r^4} + \frac{1}{r^2}, \qquad 5.43$$

$$R^4_4 = \frac{-\sin^2(C(r-B))}{C^2r^4} - \frac{1}{r^2} + \frac{2}{Cr^3}\sin(C(r-B))\cos(C(r-B)).$$

The scalar invariants of the Petrov canonical form (5.3) are

$$\alpha = \frac{1}{r^2}\left\{-1 + \frac{\sin^2(C(r-B))}{C^2r^2}\right\},$$

$$\beta = \frac{\sin^2(C(r-B))}{C^2r^4}\left\{-1 + Cr\cot(C(r-B))\right\},$$

so that the singularity at $r = B$ is of a coordinate nature only.

The spatial hypersurface V_3 of the metric (4.44c) can be covered without singularity by a single non-singular real coordinate system which embeds it into a four dimensional pseudo-Euclidian space E_4 :

$$ds^2 = -dX^{1^2} - dX^{2^2} - dX^{3^2} - dX^{4^2},$$

$$X^1 = \int\left\{\frac{C^2r^2}{\sin^2(C(r-B))} - 1\right\}^{\frac{1}{2}} dr, \qquad 5.44$$

$$X^2 = r\sin\theta\cos\phi, \quad X^3 = r\sin\theta\sin\phi, \quad X^4 = r\cos\theta.$$

The coordinate r may be eliminated from (5.44) and V_3 may be represented as a hypersurface in E_4 ; thus it is topologically Euclidian. V_4 cannot be embedded in a five dimensional space of constant curvature nor can it be embedded in an E_6 by either of the two methods described for the other space-times studied in this chapter. It can, however, be embedded in an E_7 . The embedding is easily obtained and is given by

$$ds^2 = -dX^{1\,2} - dX^{2\,2} - dX^{3\,2} - dX^{4\,2} + dX^{5\,2} - dX^{6\,2} + dX^{7\,2},$$

where X^1 , X^2 , X^3 and X^4 are the coordinates defined above in (5.44) and

$$\begin{aligned} X^5 &= \frac{2Cr}{\sin(C(r-B))}\sinh\left(\frac{t}{2}\right) , \\ X^6 &= \frac{2Cr}{\sin(C(r-B))}\cosh\left(\frac{t}{2}\right) , \\ X^7 &= \frac{2Cr}{\sin(C(r-B))} . \end{aligned}$$

Retaining r as a parameter, V_4 may be represented as the surface in E_7 :

$$X^{2\,2} + X^{3\,2} + X^{4\,2} = r^2 , \quad X^{6\,2} - X^{5\,2} = X^{7\,2},$$

$$X^1 = \int \left\{ \frac{C^2 r^2}{\sin^2(C(r-B))} - 1 \right\}^{\frac{1}{2}} dr$$

We now consider briefly the main physical properties of space-time III. To discuss the behaviour of neutral test particles in V_4 we use an identical argument to that employed for space-time II. This again shows that in order to examine how the universe appears to some centrally situated observer the constant B must be put equal to zero. Then from the equation (5.40) the radial geodesic equation governing the behaviour of a small test particle is, for small r ,

$$\frac{d^2 r}{dt^2} \simeq -\frac{1}{3} C^2 r ,$$

where we have used, in (5.40):

$$e^{\nu} = e^{\lambda} = \frac{C^2 r^2}{\sin^2(Cr)} \quad .$$

Then, as for space-time II, the radial acceleration and force at the origin of this spherically symmetric universe is zero. Unlike space-time II, however, test particles near $r = 0$ will move in to $r = 0$, since $(d^2r / dt^2)_{r\to 0} < 0$, and spectral lines emanating from distant sources will appear to the observer at the origin to be displaced towards the blue end of the spectrum.

The feature that distinguishes space-time III from those examined earlier is that we *can* find a realistic interpretation for its stress-energy tensor. From (5.43) and the definition (2.4), $T^{\rho}_{\cdot\sigma}$ has non-vanishing components

$$\begin{aligned} T^1_1 &= \frac{1}{\kappa}\left\{ \frac{-3\sin^2(C(r-B))}{C^2r^4} + \frac{1}{r^2} + \frac{2}{Cr^3}\sin(C(r-B))\cos(C(r-B))\right\}, \\ T^2_2 = T^3_3 &= \frac{1}{\kappa}\left\{ \frac{\sin^2(C(r-B))}{C^2r^4} - \frac{1}{r^2}\right\}, \qquad 5.45 \\ T^4_4 &= \frac{1}{\kappa}\left\{ \frac{\sin^2(C(r-B))}{C^2r^4} + \frac{1}{r^2} - \frac{2}{Cr^3}\sin(C(r-B))\cos(C(r-B)) \right. . \end{aligned}$$

Now the stress-energy tensor corresponding to the superposition of the field of a perfect fluid and an electromagnetic field is

$$T_{\alpha\beta} = (p+\rho)u_\alpha u_\beta - p g_{\alpha\beta} - \frac{1}{4\pi}\left[F_{\alpha\tau}F_{\beta\cdot}^{\ \tau} - \frac{1}{4}g_{\alpha\beta}F_{\sigma\tau}F^{\sigma\tau}\right], \quad 5.46$$

where p , ρ and u^α are respectively the pressure, matter density and four velocity of the fluid ($u^\alpha u_\alpha = 1$) and $F_{\alpha\beta}$ is the electromagnetic field tensor satisfying Maxwell's equations

$$\begin{aligned} F^{\alpha\beta}{}_{;\beta} &= 4\pi j^\alpha, \\ F_{\alpha\beta} &= \phi_{\alpha,\beta} - \phi_{\beta,\alpha} \ , \end{aligned} \qquad 5.47$$

where ϕ_α is the four potential and j^α is the current density four-vector. Under our symmetry conditions, the surviving components of (5.46) are (since $\lambda^a = 0, \lambda^4\lambda_4 = 1$):

$$T^1_1 = -p - \frac{1}{8\pi}F^{14}F_{14}\ , \tag{a}$$

$$T^2_2 = T^3_3 = -p + \frac{1}{8\pi}F^{14}F_{14}\ , \qquad 5.48\text{(b)}$$

$$T^4_4 = \rho - \frac{1}{8\pi}F^{14}F_{14}\ . \tag{c}$$

Comparing (5.48) with (5.45), we must have

$$p = \frac{1}{\kappa}\left\{\frac{\sin^2(C(r-B))}{C^2r^4} - \frac{1}{Cr^3}\sin(C(r-B))\cos(C(r-B))\right\}, \tag{a}$$

$$\rho = 3p, \qquad 5.49\text{(b)}$$

$$\frac{\kappa}{8\pi}F^{14}F_{14} = \frac{2\sin^2(C(r-B))}{C^2r^4} - \frac{1}{r^2} - \frac{1}{Cr^3}\sin(C(r-B))\cos(C(r-B)) \tag{c}$$

It is now necessary to check that $p \geqslant 0$ and that $F^{14}F_{14} \leqslant 0$ so that the radial electric field F_{14} is real. Clearly, from (5.49a), p is non-negative only if

$$\tan(C(r-B)) \geqslant Cr\ . \qquad 5.50$$

However, this inequality is always violated within the coordinate range $B \leqslant r < B+\pi/2C$ unless $B = 0$, in which case (5.50) holds everywhere. We henceforth take $B = 0$ (although it should be noted that even when $B \neq 0$ the negative pressure range is bounded below and could be ascribed to the necessity for the presence of a constant of the "cosmological" variety in the definition of $T^{\rho}_{\cdot\sigma}$). To show that $F^{14}F_{14}$ is everywhere $\leqslant 0$ we must show, from (5.49c) with $B = 0$, that

$$-2\sin^2 x + x^2 + x\sin x\cos x \geqslant 0\ , \qquad 5.51$$

where we have written $x \equiv Cr$. Employing series expansions for $\sin(2x)$ and $\cos(2x)$, it may be shown without difficulty that (5.51) is true everywhere within the stated range. Then, from (5.49), the pressure, matter density and radial electric field are given by

$$p = \frac{1}{\kappa}\,\frac{\sin(Cr)}{C^2 r^4}\left\{\sin(Cr) - Cr\cos(Cr)\right\} ,$$

$\rho = 3p$, (the equation of state for pure radiation),

$$F_{14} = \sqrt{\frac{8\pi}{\kappa}}\,\frac{C^2}{\sin^2(Cr)}\left\{1 + \frac{1}{Cr}\sin(Cr)\cos(Cr) - \frac{2}{C^2r^2}\sin^2(Cr)\right\}^{\frac{1}{2}}.$$

We give the following limiting values of p , ρ and F_{14} :

(i) $r = 0$: $p = \frac{C^2}{3\kappa}$, $\rho = \frac{C^2}{\kappa}$, $F_{14} = 0$.

(ii) $r = \frac{\pi}{2C}$: $p = \frac{C^2}{\kappa}\left(\frac{2}{\pi}\right)^4$, $\rho = \frac{3C^2}{\kappa}\left(\frac{2}{\pi}\right)^4$, $F_{14} = \sqrt{\frac{8\pi}{\kappa}}C^2\left\{1 - \frac{8}{\pi^2}\right\}^{\frac{1}{2}}$.

It should be noted that if we do not demand strict adherence to the concept that the variable ρ in the metric form (5.42) should represent a radial coordinate and allow its range to be $-\infty < \rho < \infty$, the range of r in (4.44c) is extended to cover $A \leq r < \frac{\pi}{C} + B$. In this case our previous comments concerning the topology of space-time III are unaffected, as are the statements concerning the non-negative character of p , ρ and F_{14} . We addend the values of p , ρ and F_{14} at $r = \pi/C$:

(iii) $r = \frac{\pi}{C}$: $p = 0$, $\rho = 0$, $F_{14} = \infty$.

5.6 Conclusion

Only four of the space-times arising as solutions of Lanczos' equations have been examined in this chapter. A detailed study of the properties of all the universes obtained could well prove interesting. For example, the "class 2" space-time V, given by (4.44e), is a known solution of the Maxwell-Einstein equations, found by Robinson.[64] Space-time V represents, in the weak-field approximation, a constant electric field, or a constant magnetic field, or a superposition of the two. The investigations of this chapter lead us to expect that our theory may well prove fruitful in predicting space-times which are of physical interest.

Figs. $1 \longrightarrow 9$

Timelike geodesics in space-time I arising when $a > 0$. The scale is in units of a/A and two successive orbits are shown in each case.

Figs. 10 and 11

As for figs. $1 \longrightarrow 9$, but showing sets of timelike geodesics for different particle angular momentum values corresponding to fixed energies. One orbit is shown for each pair of values (ℓ^2 , A) considered.

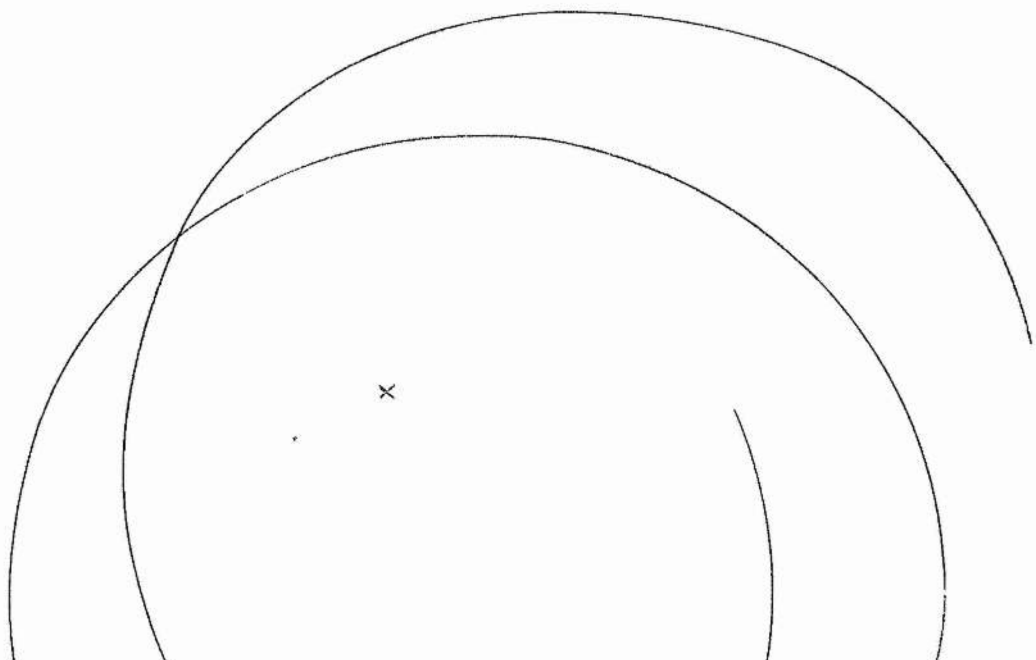


Fig. 1 Timelike geodesics in space-time I

$\ell^2 = 0.80$, $A = 0.30$

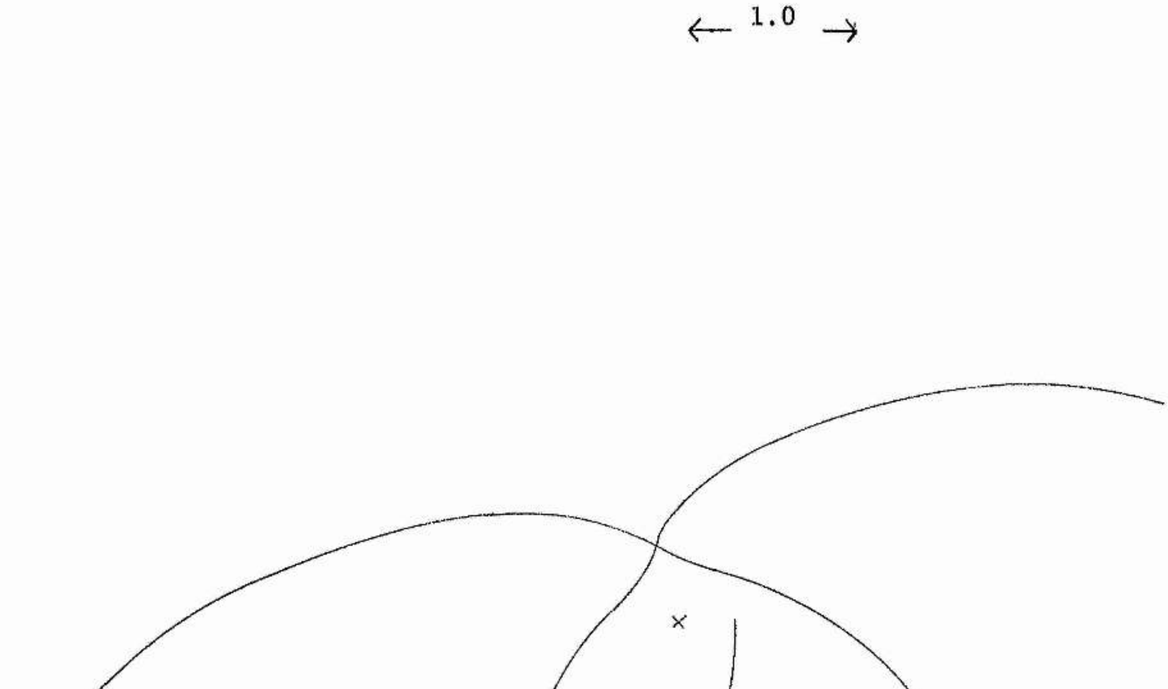


Fig. 2 Timelike geodesics in space-time I

$\ell^2 = 0.80$, $A = 0.60$

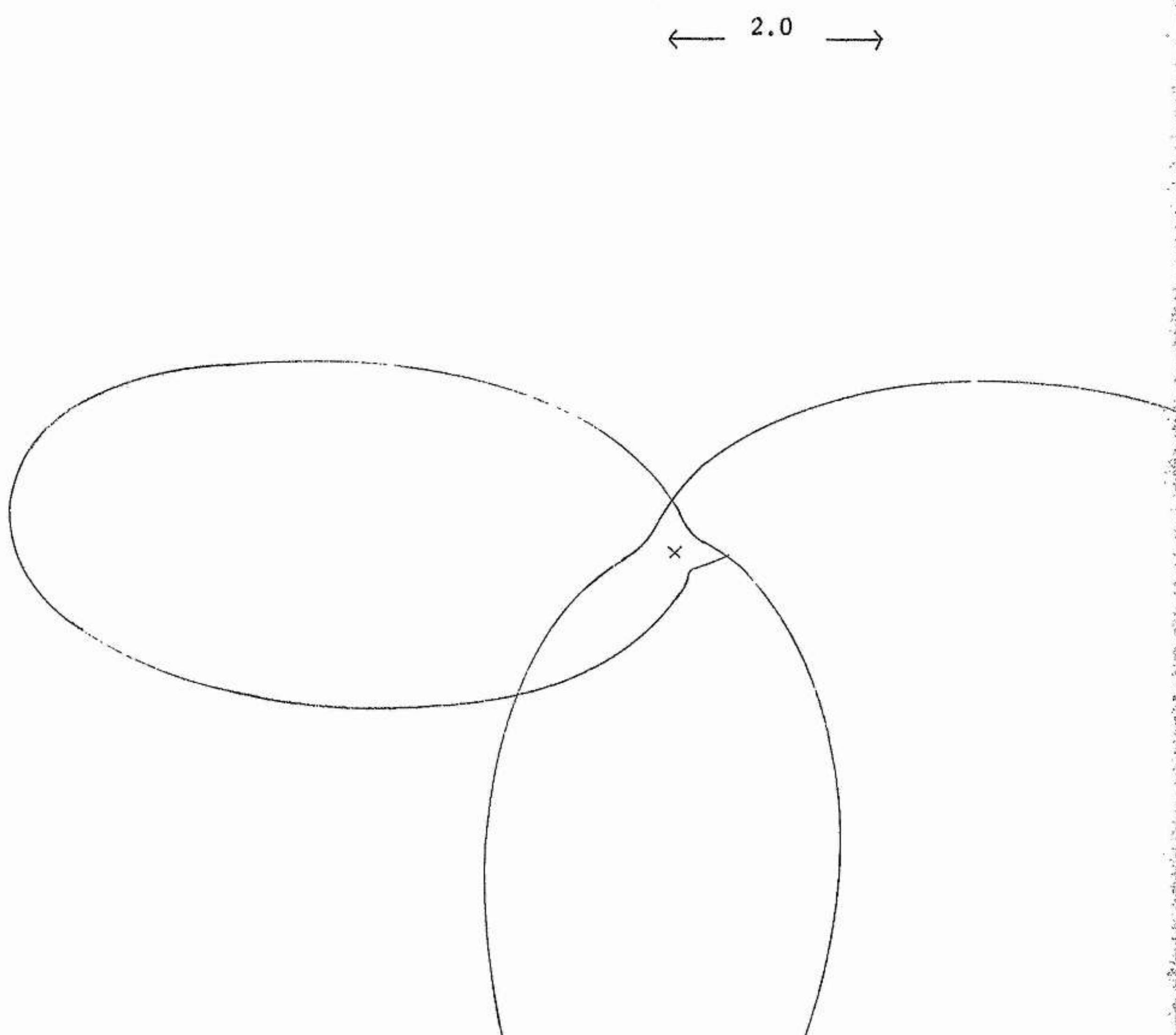


Fig. 3 Timelike geodesics in space-time I

$\ell^2 = 0.80$, $A = 0.80$

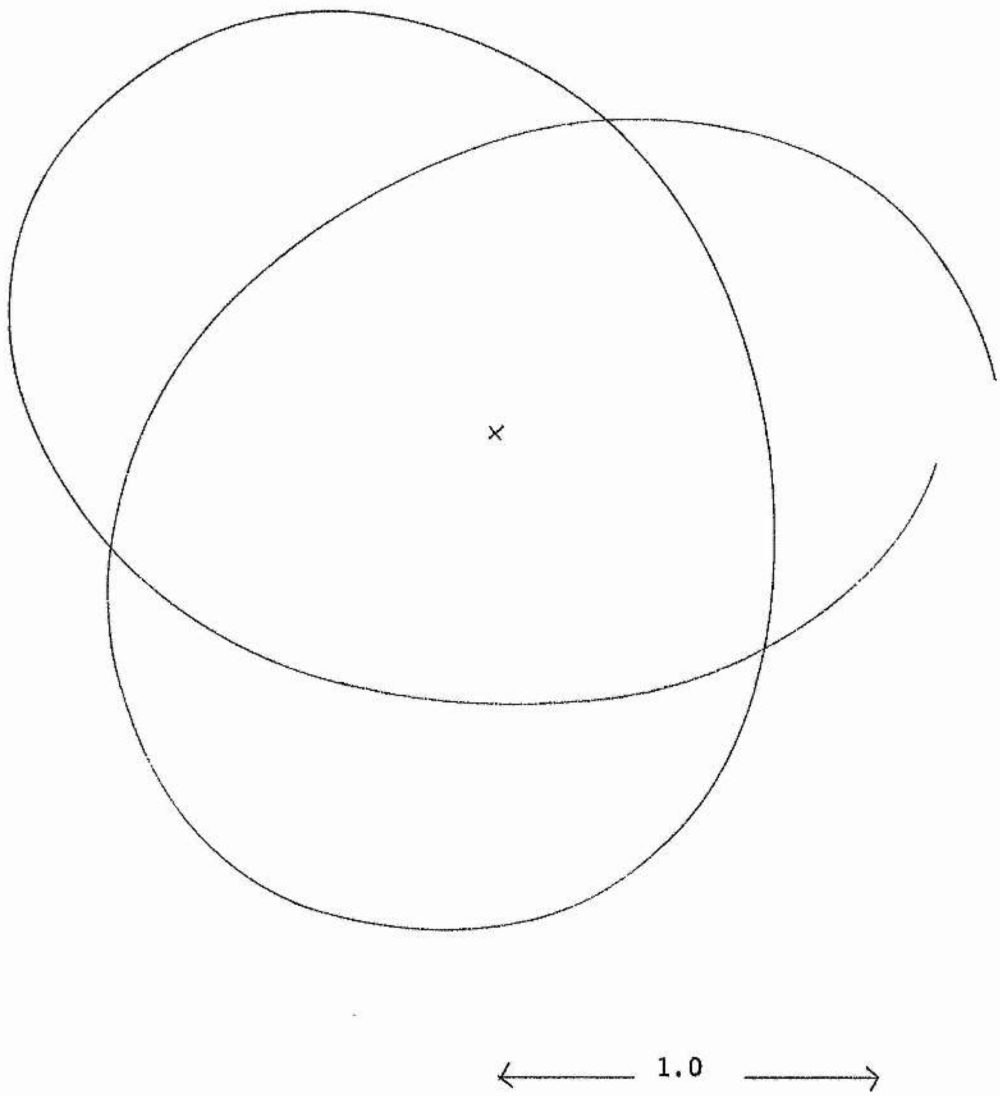


Fig. 4 Timelike geodesics in space-time I

$\ell^2 = 0.50, \ A = 1.05$

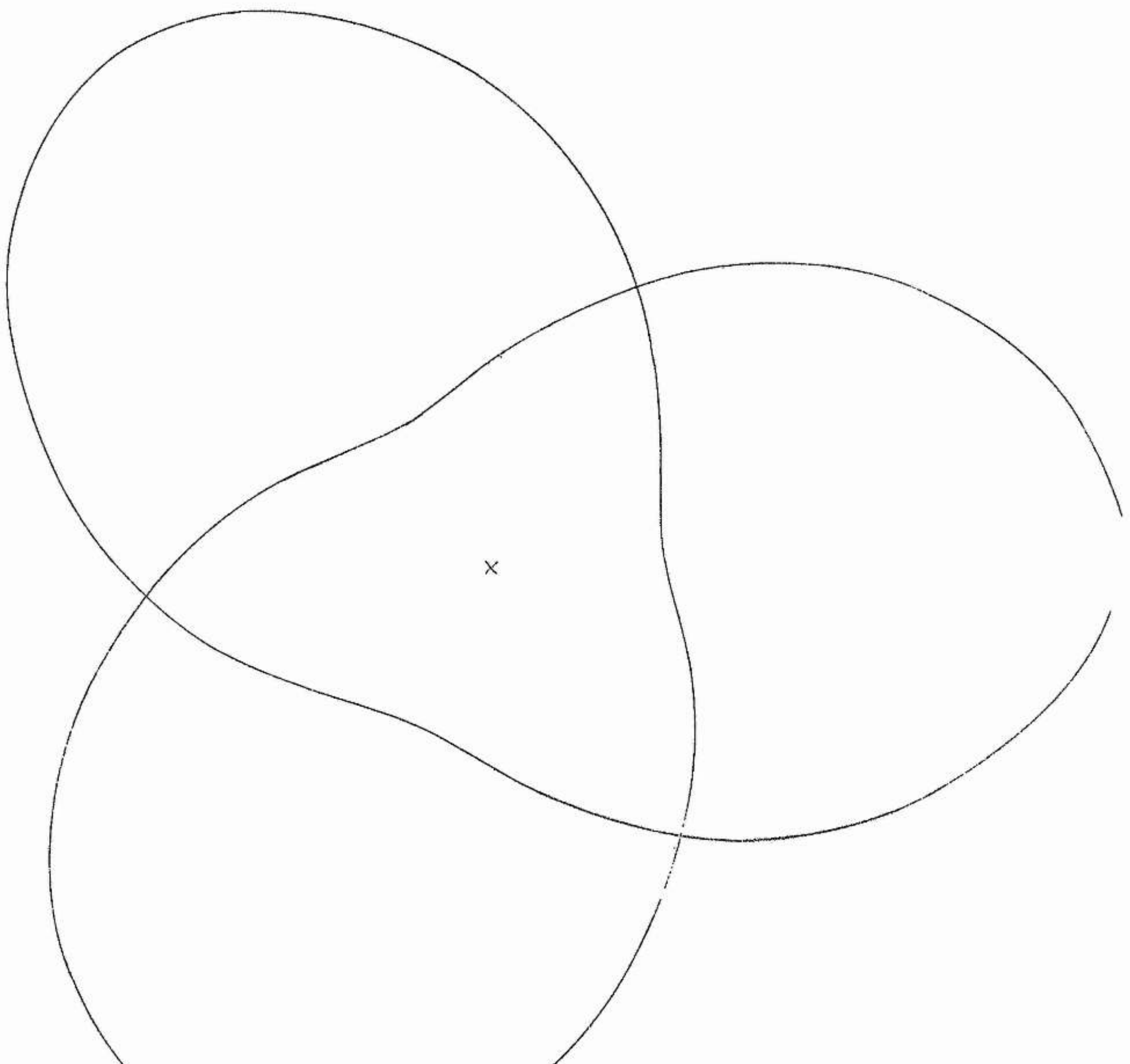


Fig. 5 Timelike geodesics in space-time I

$\ell^2 = 0.50$, $A = 1.20$

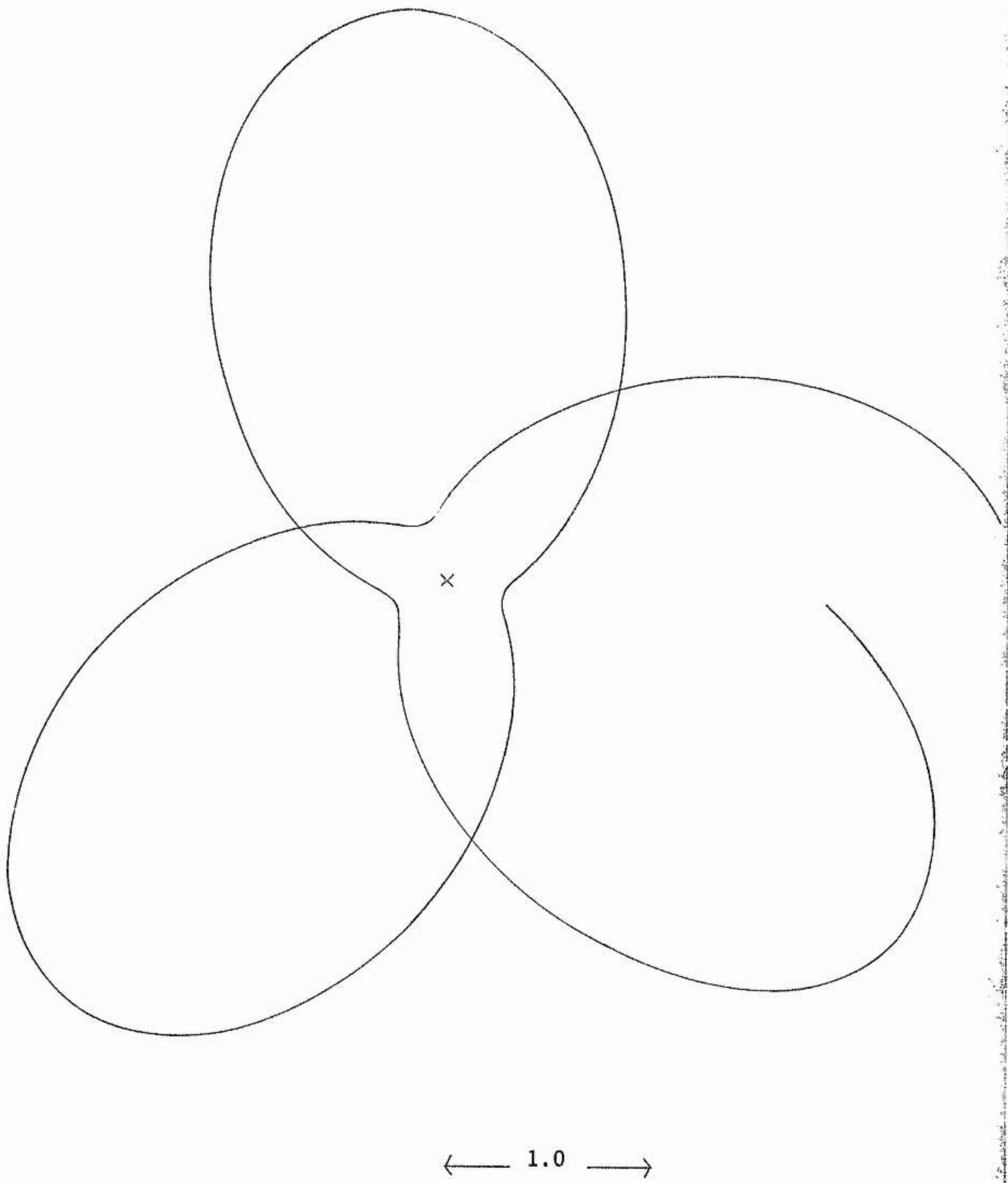


Fig. 6 Timelike geodesics in space-time I

$\ell^2 = 0.50$, $A = 1.50$

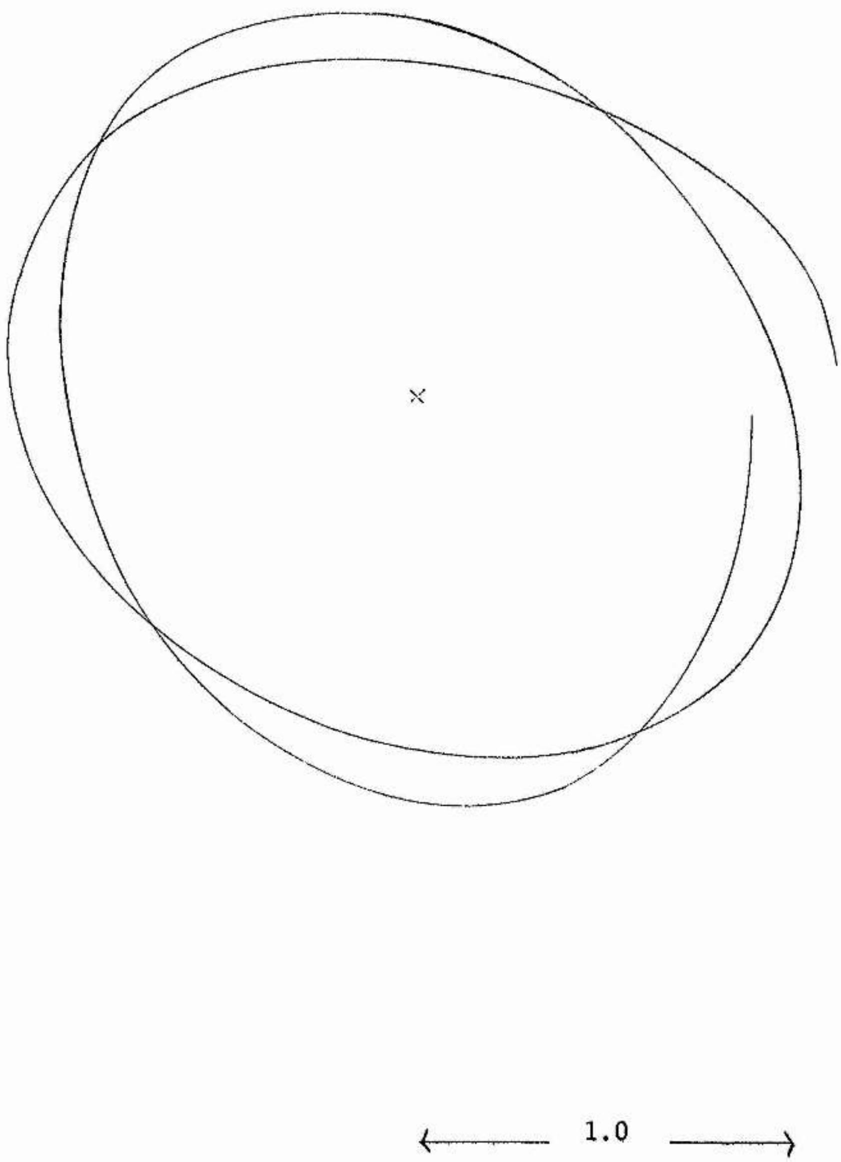


Fig. 7 Timelike geodesics in space-time I

$\ell^2 = 0.20, \ \mathcal{A} = 4.01$

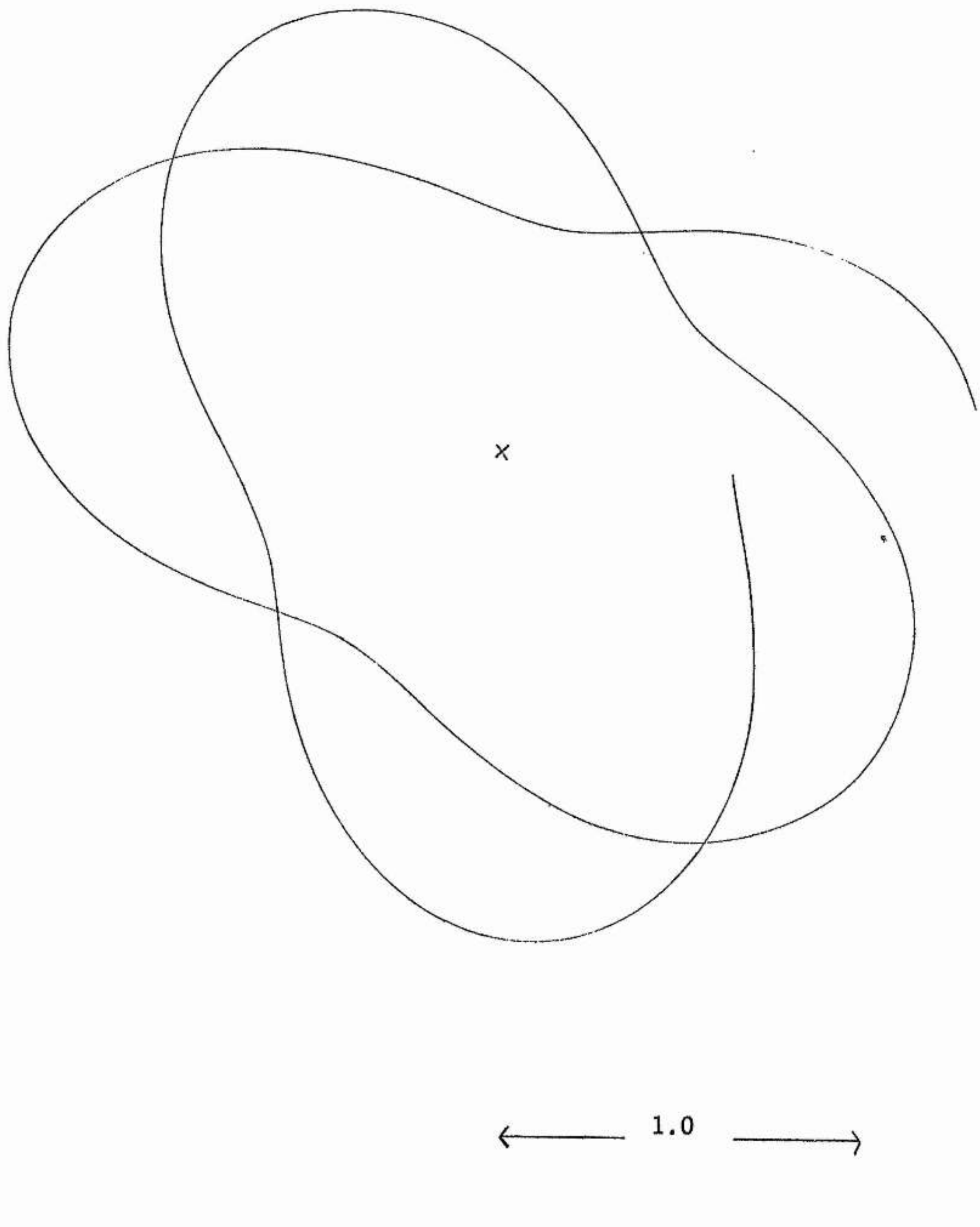


Fig. 8 Timelike geodesics in space-time I

$\ell^2 = 0.20, \quad A = 4.10$

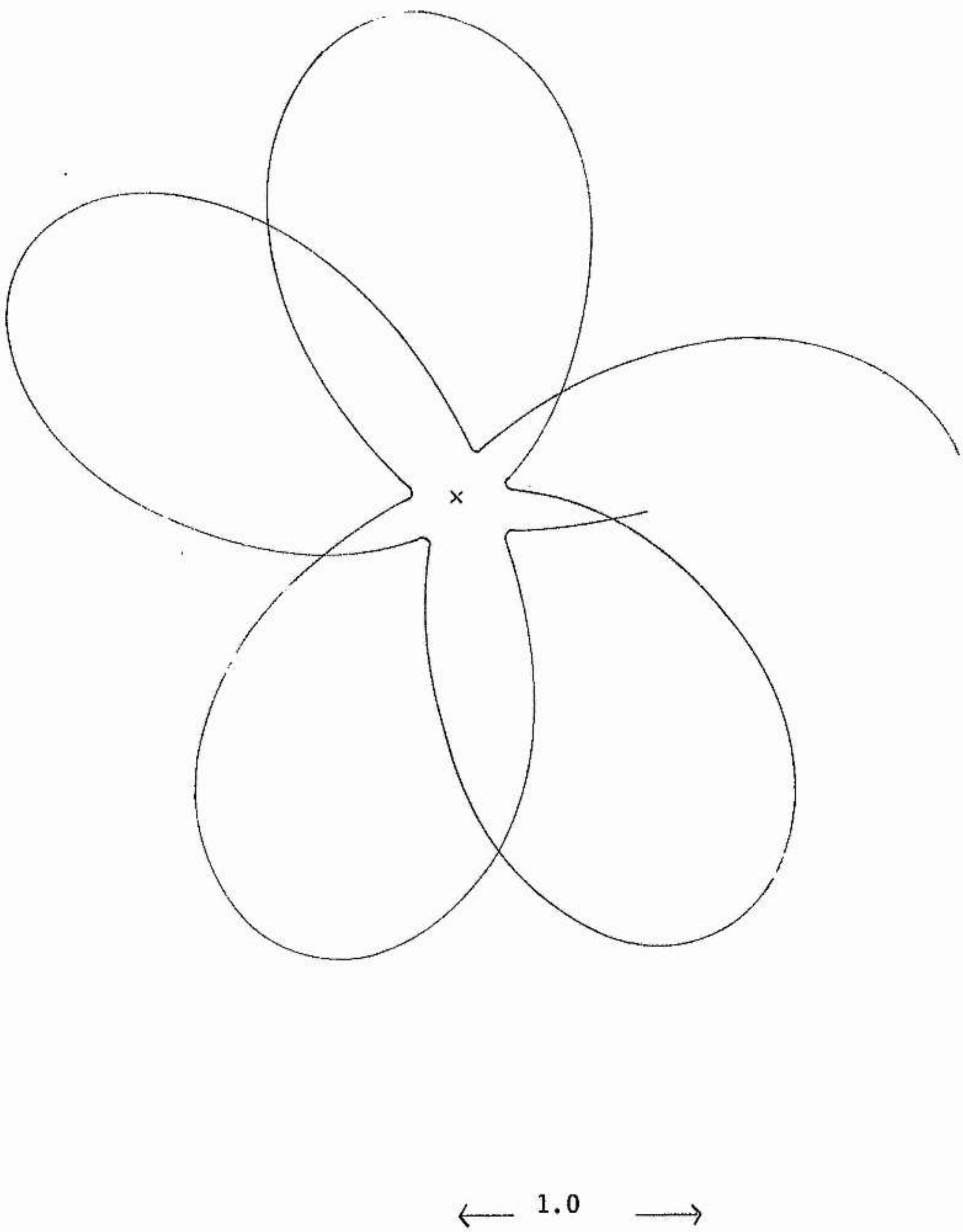


Fig. 9 Timelike geodesics in space-time I

$\ell^2 = 0.20$, $\mathcal{A} = 4.60$

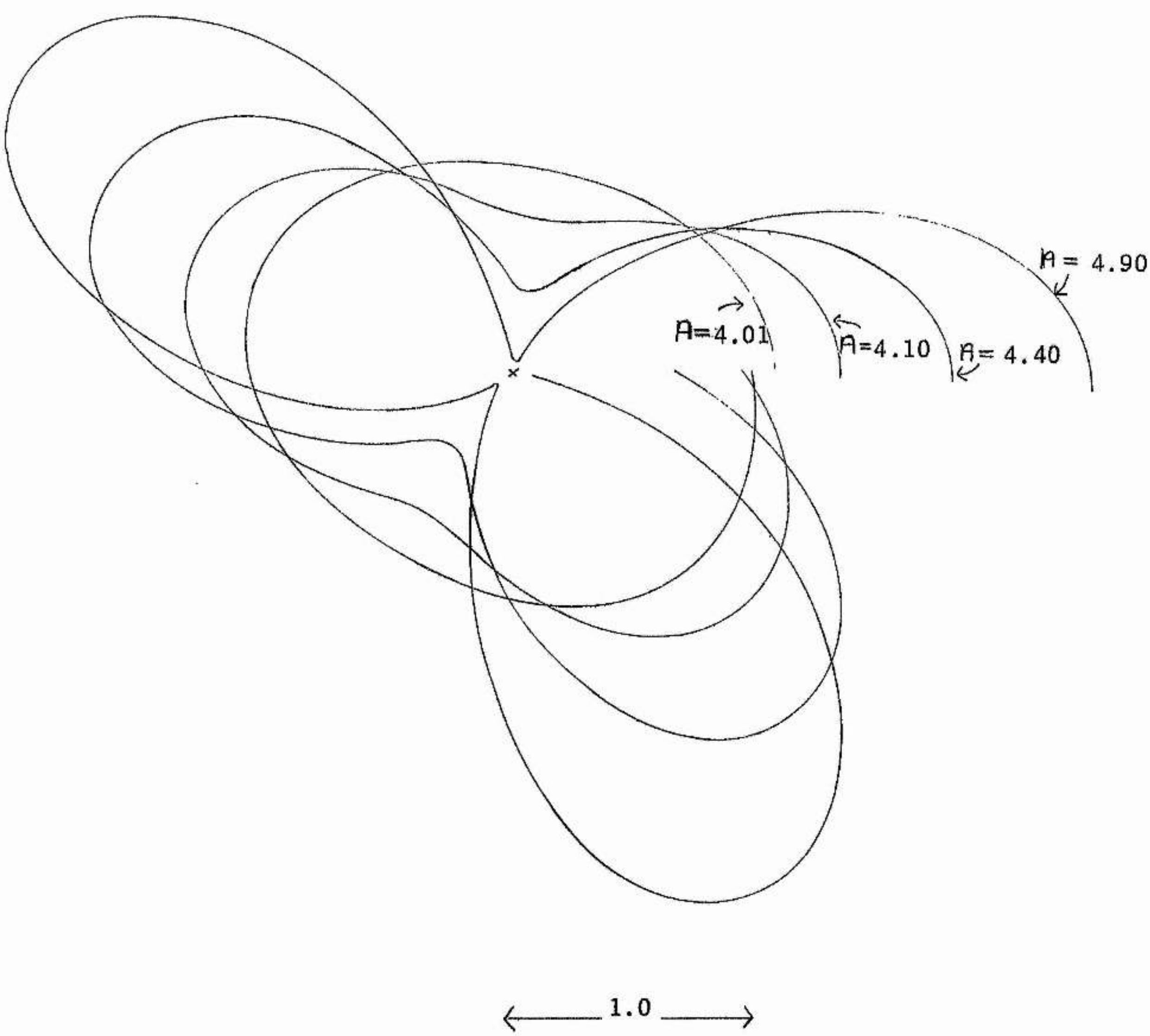


Fig. 10 Timelike geodesics in space-time I

$\ell^2 = 0.20$. $A = 4.01, 4.10, 4.40, 4.90$

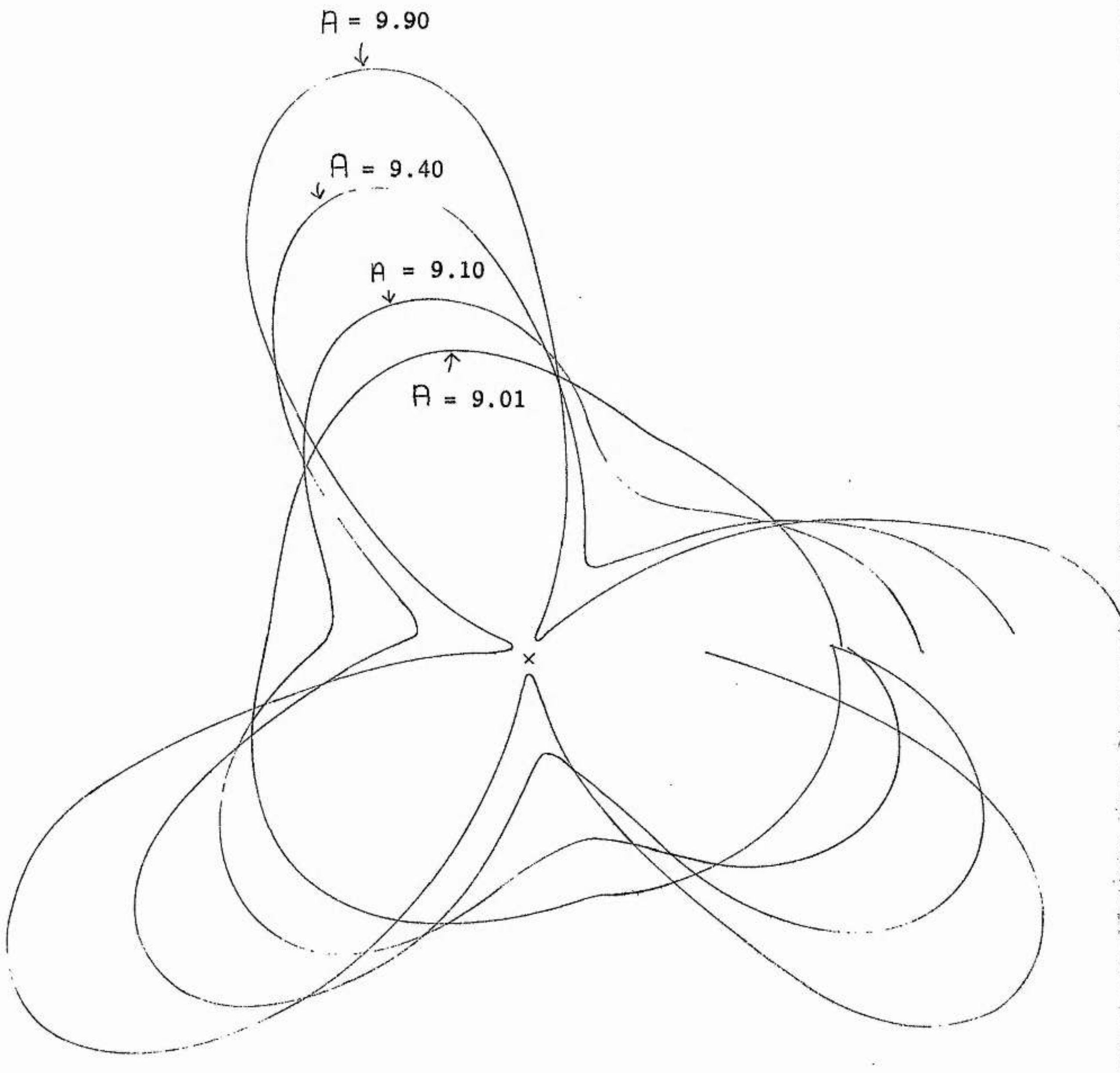


Fig. 11 Timelike geodesics in space-time I

$\ell^2 = 0.10$. $A = 9.01, 9.10, 9.40, 9.90$

Figs. 12 and 13

Null geodesics in space-time I arising when $\alpha \geqslant 0$, $\alpha < 0$ respectively. The x-scale is in units of h/ℓ and the y-scale is indicated in each case.

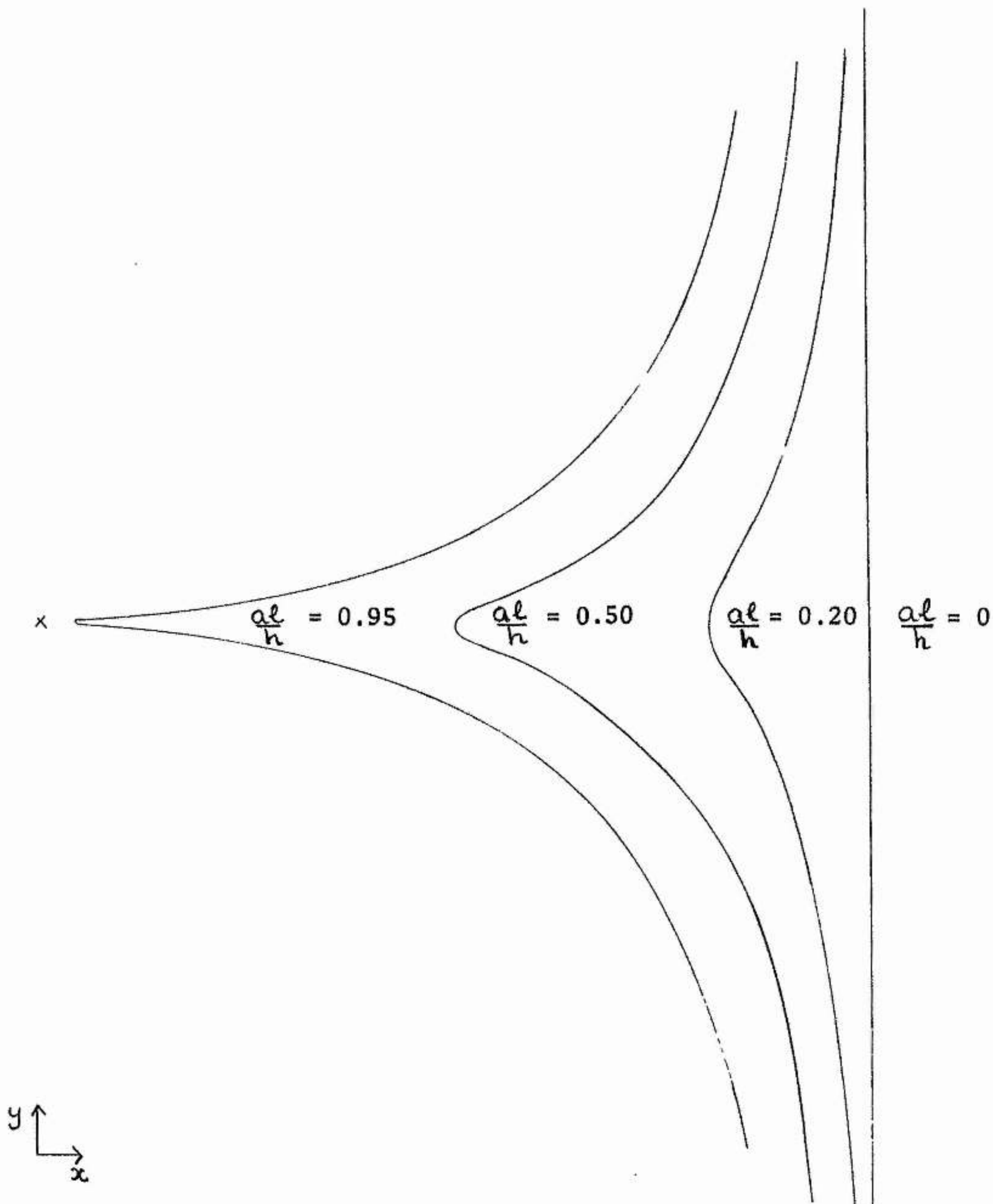


Fig. 12 Null geodesics in space-time I, $a \geqslant 0$

(y-scale) = (x-scale)/8

$$\frac{a\ell}{h} = 0,\ 0.20,\ 0.50,\ 0.95$$

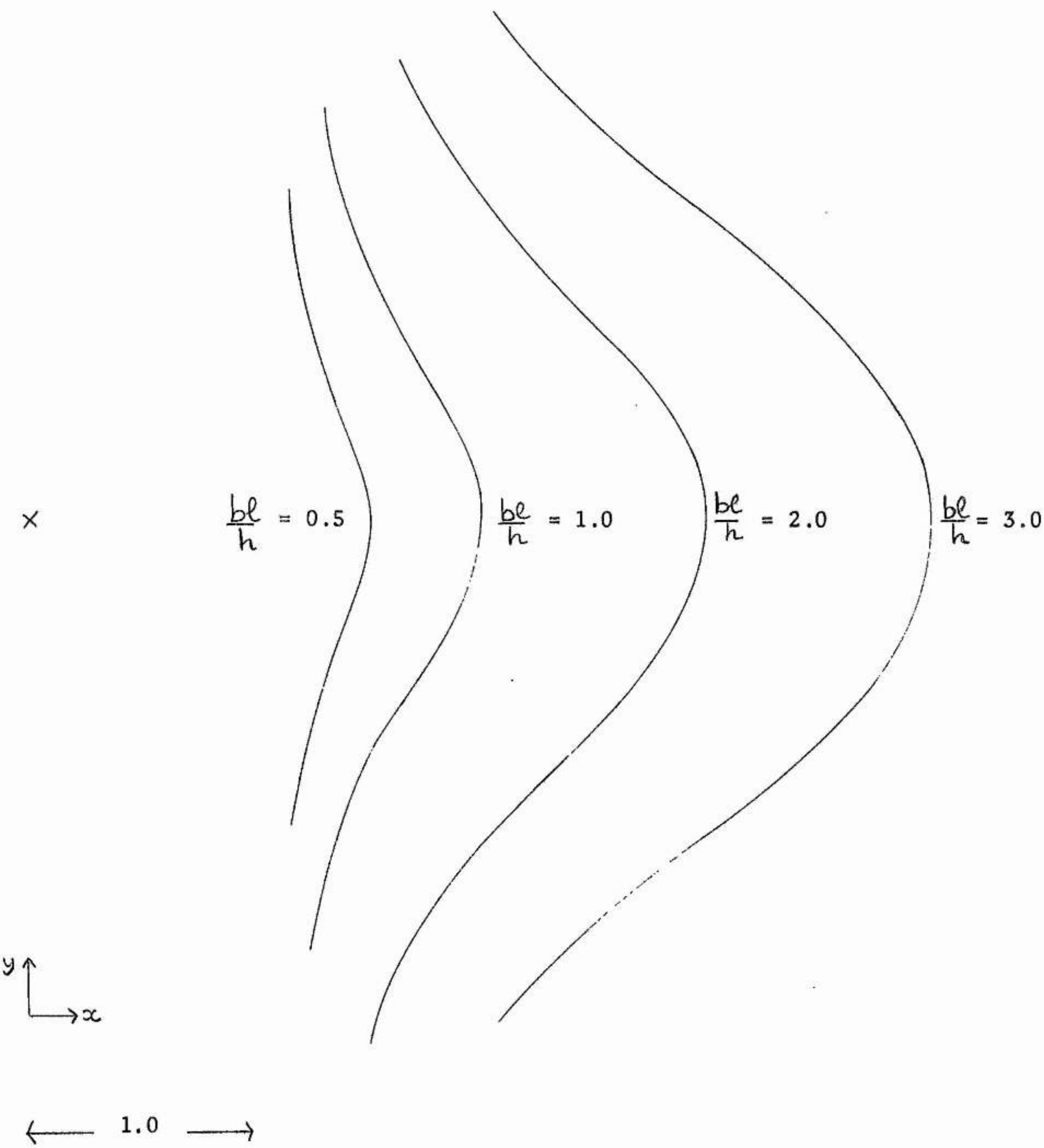


Fig. 13 Null geodesics in space-time I, $-a = b > 0$

(y-scale) = (x-scale)/3

$\frac{b\ell}{h}$ = 0.5, 1.0, 2.0, 3.0

6. NUMERICAL SOLUTIONS

6.1 Introduction

In chapter 3 we considered the imposition of several boundary conditions on the spherically symmetric, static field. The various boundary conditions, chosen according to certain reasonable requirements discussed in section 3.1, are the following:

(i) Space-time is asymptotically flat,

(ii) Space-time is regular,

(iii) Space-time is regular and asymptotically flat,

(iv) Space-time is asymptotically of the form of the Einstein universe,

(v) Space-time is asymptotically of constant curvature.

The results that we have obtained concerning the compatibility of the conditions (i), (iii) and (iv) with the field equations are conclusive. We have now to investigate the complete form of the solutions which are regular (condition (ii)) and determine whether or not there are regular space-times that also satisfy condition (v) — that they have asymptotically constant curvature. To do this we shall use numerical methods.

It was shown in section 3.3 that any regular solution of the static, spherically symmetric field equations has constant curvature invariant. As a consequence, the field equations based on the metric (2.8) become quasi-linear and of second order in the components of the metric tensor (section (3.6)). The problem may then be reduced to finding the solution of a single integro-differential equation which, however, we have not been able to accomplish. Consequently, in section 4.2 we investigated the form of regular solutions near the spatial origin by obtaining the first few coefficients of the rising powers of r in the assumed series expansions of the unknown functions e^{λ}, e^{ν}. These series expansions will be used as the basis for our numerical work.

6.2 The Numerical Method

Our aim is the numerical integration of the pair of coupled second-order differential equations (2.17), (A2.25)$_K$, with given initial values. From section 4.5, it is known that no non-trivial static, spherically symmetric, regular solution is asymptotically Minkowskian; thus we may expect that the behaviour of e^{λ}, e^{ν} will show considerable deviations from Minkowski form. It is therefore very important that a numerical integration procedure is adopted that is especially stable. Such a procedure is Hamming's modified predictor-corrector method for the solution of general initial value problems, which is conveniently incorporated in an IBM subroutine.[65]

The initial values that we provide must be sufficient to ensure the uniqueness of the corresponding numerical solution. Thus, it is not sufficient to give the initial values

$$\left.\begin{array}{l} e^{\lambda} = 1 \ , \ e^{\nu} = 1 \\ \dfrac{de^{\lambda}}{dr} = 0 \ , \ \dfrac{de^{\nu}}{dr} = 0 \end{array}\right\} \text{at } r = 0 \ , \qquad 6.1$$

as *every* regular solution must satisfy (6.1). From the series expansions (4.2), however, we know that, for given β, regular solutions depend upon two arbitrary constants, u_2 and v_2. In order to ensure uniqueness we must first assign u_2 and v_2. Then, from (4.2), we choose initial values for e^{λ} and e^{ν} and their first derivatives at some point $r = h$:

$$e^{\lambda} = \Big\{1 + v_2 h^2 + \tfrac{1}{5}(u_2 - v_2)[-v_2 - 3\beta(u_2+v_2)]h^4$$
$$+ \frac{(u_2-v_2)}{2\cdot 5\cdot 7}\Big[(-2u_2^2 + 2u_2v_2 + v_2^2) + 3\beta(u_2+v_2)(2u_2-v_2)$$
$$-2(3\beta)^2(u_2+v_2)^2\Big]h^6\Big\}^{-1}, \qquad 6.2$$

$$e^{\nu} = 1 + u_2h^2 + \frac{1}{2\cdot 5}(u_2-v_2)[(4u_2+v_2) + 3\beta(u_2+v_2)]h^4$$
$$+ \frac{(u_2-v_2)}{2\cdot 3\cdot 5\cdot 7}\Big[(20u_2^2 - 23u_2v_2 - 13v_2^2) + 3\beta(u_2+v_2)(25u_2 - 11v_2)$$
$$+2(3\beta)^2(u_2+v_2)^2\Big]h^6 .$$

together with corresponding formulae for their first derivatives. (We have transformed the time coordinate in the usual way to make $e^{\nu} = 1$ at the origin). The value of h must be chosen such that the validity of the series expansions (4.2) in representing the functions e^{λ}, e^{ν} extends to the point $r = h$. In practice this is no problem due to the presence in the IBM subroutine of devices to inform the user of any choice of inappropriate initial values.

So far, it has appeared necessary to assign numerical values independently to u_2 and v_2, so that the numerical solution set N(β, u_2, v_2) is triply infinite. However, the *essential* solution set (which contains all the information required) is only doubly infinite. For a consequence of the gauge invariance of the action principle is that if the static, spherically symmetric field equations based on the metric (2.8) are satisfied by $e^{\lambda} = e^{\lambda}(r)$, $e^{\nu} = e^{\nu}(r)$, then they are also satisfied by $e^{\lambda}(Dr)$, $e^{\nu}(Dr)$ for any non-zero constant D. Without losing any information about the form or the properties of the solutions of the equations, we have thus the free choice of *one* of the constants u_2, v_2, though we must clearly take into account the two possibilities as to the sign of these quantities. We shall find it convenient for the purposes of comparison of the numerical solutions if the choice is made such that the constant value of the curvature invariant, $K = -6(u_2 + v_2)$, is absolute. Since the case when K vanishes has already been dealt with in section 4.2, our interest lies in the study of the following two cases, A and B:

Case A

$$v_2 = -\frac{1}{3} - u_2 ,$$
$$K = +2 . \qquad 6.3\ (a)$$

Case B

$$v_2 = \frac{1}{3} - u_2 ,$$
$$K = -2 . \qquad 6.3\ (b)$$

Then the essential solution set is the union of $N(\beta, u_2, -\frac{1}{3} - u_2)$ and $N(\beta, u_2, \frac{1}{3} - u_2)$, where β and u_2 vary over all negative and positive real values. The initial values of e^{λ} and e^{ν} are given by (6.2) where v_2 is determined by (6.3a), (6.3b) for cases A, B respectively; the formulae for the initial values of the derivatives de^{λ}/dr, de^{ν}/dr are obtained similarly.

The task of setting up the integration procedure was simplified by the fact that the de Sitter solution occurs when $u_2 = v_2$ ($= -\frac{1}{6}, +\frac{1}{6}$, for the two cases), so that comparison could be made between the numerical values obtained as a result of numerical integration and those obtained from the exact solution. The de Sitter solution was thus used as a monitor, whereby the efficacy of the input parameters (step size, error bounds, etc.) in producing accurate and non-spurious results could be optimised. The high accuracy obtained for the de Sitter solution enables us to place confidence in the numerical solutions obtained.

6.3 Properties of the Numerical Solutions

Numerical solutions were obtained for a wide range of values of u_2

and β ; a representative selection is shown schematically in Figs. 14 and 15, which correspond to cases A, B respectively. To classify the solutions according to the values of u_2 and β would be somewhat artificial, as in general the boundaries between classes are to some extent arbitrary and we cannot attach any significance to the results. It is more instructive to attempt to describe the properties of the corresponding universes. Our interest lies principally in "realistic" models: those for which the energy density T^4_4 is nowhere negative, but we shall not restrict the values of the pressures given by T^1_1 and T^2_2 (= T^3_3). During the following discussion of the solutions included in cases A and B it will be helpful if reference is made to the figures.

Case A

$$v_2 = -\tfrac{1}{3} - u_2 \ , \ K = +2 \, .$$

These numerical solutions include the closed de Sitter model $\left(u_2 = v_2 = -\frac{1}{6} \ , \ T^1_1 = T^2_2 = T^3_3 = T^4_4 = \frac{1}{2\kappa} \right)$:

$$ds^2 = \frac{-dr^2}{1-r^2/6} - r^2(d\theta^2 + \sin^2\theta \, d\phi^2) + \left(1 - \frac{r^2}{6}\right) dt^2 \, .$$

For increasingly negative (positive) values of u_2, the behaviour of the numerical solutions approaches that of the limit of the special case of space-time II (space-time III):

$$e^{\nu} = e^{\lambda} = \lim_{u_2 \to -\infty} \left\{ \frac{-3u_2 r^2}{\sinh^2(\sqrt{-3u_2}\, r)} \right\} \ , \qquad \text{(space-time II)},$$

$$e^{\nu} = e^{\lambda} = \lim_{u_2 \to \infty} \left\{ \frac{3u_2 r^2}{\sin^2(\sqrt{3u_2}\, r)} \right\} \ , \qquad \text{(space-time III)}.$$

We are especially interested in solutions such that $T^4_4 \geqslant 0$ everywhere. From the definition of the stress-energy tensor (2.4), using the formulae (A2.21), (A2.22) for the Ricci tensor and the curvature invariant,

$$T^4_4 = \frac{1}{\kappa}\left\{\frac{1}{r^2}(1-e^{-\lambda}) - \frac{1}{r}\frac{de^{-\lambda}}{dr}\right\}. \qquad 6.4$$

Then T^4_4 is dependent upon e^λ only and is non-negative if and only if

$$r\frac{de^\lambda}{dr} + e^\lambda(e^\lambda - 1) \geqslant 0 \ . \qquad 6.5$$

Using the first of the series expansions (4.2) for e^λ it is easily shown that (6.5) is true near r = 0 only if (i) or (ii) is true:

(i) $\left(\frac{d^2e^\lambda}{dr^2}\right)_{r=0} > 0$ (i.e. $V_2 < 0$),

(ii) $\left(\frac{d^2e^\lambda}{dr^2}\right)_{r=0} = 0$ (i.e. $V_2 = 0$) and $\left(\frac{d^4e^\lambda}{dr^4}\right)_{r=0} > 0$ (i.e. $\beta > 0$).

Then for a numerical solution to be realistic it must satisfy either (i) or (ii) in order that T^4_4 is positive near r = 0. Clearly, from (6.4), the energy density decreases monotonically throughout any region in which $de^\lambda/dr < 0$. It seems that for case A space-times which have $T^4_4 \geqslant 0$ everywhere, e^λ diverges to infinity at some finite point r_0. The rate of divergence enables us to make the conjecture that the radial length

$$\mathcal{R}(r) = \int_0^r e^{\lambda/2}\,dr,$$

and the volume

$$V(r) = 4\pi\int_0^r r^2 e^{\lambda/2}\,dr \quad ,$$

both of which are monotonically inreasing functions of r, have finite limiting values as $r \to r_0$ (V_3 is closed). Then both the ratio of the circumference of a circle at $\mathcal{R}(r)$ to the radial length, and the solid angle subtended by a spherical surface at r = 0 have their Euclidian values at r = 0 and decrease monotonically to non-zero limiting values at $r = r_0$ being given by

$$2\pi r \Big/ \int_0^r e^{\lambda/2}\, dr \quad ,$$

$$4\pi r^2 \Big/ \left\{ \int_0^r e^{\lambda/2}\, dr \right\}^2 ,$$

respectively.

In V_4 , $\sqrt{-g} \geqslant 0$ in $0 \leqslant r < r_0$. The proper time along a t-line between two t-constant hypersurfaces is given by

$$\Delta\tau = e^{\nu/2}\Delta t$$

Now the initial behaviour of e^{ν} is determined by the given initial conditions; its behaviour for values of $r > 0$ is apparently as follows (for all case A solutions):

$u_2 < 0$	e^{ν} decreases monotonically to zero.
$u_2 = 0$	e^{ν} decreases monotonically to zero if $\beta > -\frac{1}{3}$,
	e^{ν} increases monotonically if $\beta < -\frac{1}{3}$.
$u_2 > 0$	e^{ν} initially increases but for small values of u_2 and for some values of β reaches a maximum and decreases again.

The proper time along a t-line thus depends upon r in a similar way.

The radial null geodesics satisfy the equation

$$0 = -e^{\lambda} dr^2 + e^{\nu} dt^2 .$$

The slope of the null curve is $dr/dt = (e^{\nu}/e^{\lambda})^{\frac{1}{2}}$. Near $r = 0$, $dr/dt < 1$ for all case A solutions; for physical solutions $\lim\limits_{r \to r_0} (dr/dt) = 0$. This implies, as for the closed de Sitter model, that an infinite amount of time as measured by an observer at $r = 0$ would be required for light to travel between $r = 0$ and $r = r_0$. For the observer at $r = 0$, the universe has a horizon at $r = r_0$.

Spectral lines emanating from distant sources at rest will appear to the observer at $r = 0$ to be displaced to the blue end or the red end of the spectrum according as

$$\frac{(\sqrt{g_{44}})_{r=r_1}}{(\sqrt{g_{44}})_{r=0}}$$

is greater or less than unity. Then the apparent displacement of spectral lines is determined by the behaviour of $e^{\nu/2}$, which has already been examined. The displacement may be always to the blue, always to the red, or sometimes blue and sometimes red, for all the universes (physical or not). When both red and blue shifts are present the lines from *less* distant sources appear to be displaced to the blue, while those from *more* distant sources appear to be displaced towards the red. The reverse situation does not, apparently, arise.

The initial acceleration of test particles initially at rest is given for our metric by (5.40):

$$\frac{d^2r}{dt^2} + \frac{1}{2}\nu' e^{\nu-\lambda} = 0$$

Since (de^{ν}/dr) vanishes at $r = 0$, the acceleration and force at the origin of the universe is zero. If de^{ν}/dr is negative for $r > 0$, test particles accelerate away from the observer at $r = 0$, as in de Sitter space-time. Then test particles accelerate away from $r = 0$ only if $u_2 < 0$ or $u_2 = 0$ and $\beta > -\frac{1}{3}$. An observer at $r = 0$ may see both a local contraction and a large scale expansion present in his universe, but it seems that he will not observe the reverse.

Case B

$$v_2 = \frac{1}{3} - u_2 \,, \quad K = -2 \,.$$

These numerical solutions include the open de Sitter model

$(u_2 = v_2 = \frac{1}{6}\ ,\ T^1_1 = T^2_2 = T^3_3 = T^4_4 = -\frac{1}{2\kappa})$:

$$ds^2 = \frac{-dr^2}{(1+r^2/6)} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + (1 + \frac{r^2}{6})\, dt^2.$$

The limiting behaviour of the solution set as $u_2 \to -\infty\ (+\infty)$ is the same as for case A. Again it seems that all the case B solutions which have $T^4_4 \geqslant 0$ everywhere are such that e^λ diverges to infinity at some point (but in this case, e^λ diverges less rapidly than e^ν). It has not been possible to conclude either that these universes are closed or that they are open.

In V_4 the proper time along a t-line between two t-constant hypersurfaces is given, as before, by

$$\Delta\tau = e^{\nu/2}\,\Delta t\ .$$

The behaviour of e^ν is apparently as follows (for all case B solutions)

$u_2 < 0$ — e^ν initially decreases but may reach a non-zero minimum and increase again.

$u_2 = 0$ — e^ν decreases monotonically to zero if $\beta > -\frac{1}{3}$ and increases monotonically if $\beta < -\frac{1}{3}$.

$u_2 > 0$ — e^ν increases monotonically.

The slope of the null curve, $dr/dt = (e^\nu/e^\lambda)^{\frac{1}{2}}$, is > 1 near $r = 0$ for all case B solutions but this is not necessarily the case for all $r > 0$.

For solutions such that $T^4_4 \geqslant 0$, the displacement of spectral lines appears to be always to the blue end of the spectrum (for, as mentioned earlier it is necessary that $v_2 \leqslant 0$ for $T^4_4 \geqslant 0$ near $r = 0$, and this in turn implies, for case B solutions, that $u_2 \geqslant \frac{1}{3}$). A test particle initially at rest in such a universe will move in to the spatial origin (at which the acceleration and force is zero, as before).

6.4 Regular Solutions Asymptotically of Constant Curvature

In section 4.3 we investigated the existence of series solutions for large values of the radial coordinate consistent with the boundary condition of asymptotically constant space-time curvature. The formal series expansions (3.19) were substituted into the field equations in order to obtain relationships between the constants a_i , b_i . We made the conjecture that the only solution of this form for unrestricted β is the Einstein space (4.27). In addition, we obtained the result that such solutions may exist for certain discrete values of β ($\beta = \frac{-1}{12}$, $\frac{1}{6}$, $\frac{1}{2}$, $\frac{11}{12}$, $\frac{17}{12}$,...), each solution having the form

$$e^{-\lambda} = 1 + \frac{r^2}{a_2} + \frac{a_3}{a_2}\cdot\frac{1}{r} + \frac{1}{a_2}\sum_{n\geq 3}\frac{a_n}{r^{n-2}} ,$$
$$e^{\nu} = a_0 a_2\left(1 + \frac{r^2}{a_2} + \frac{a_3}{a_2}\cdot\frac{1}{r} + \frac{1}{a_2}\sum_{n\geq 3}\frac{b_n}{r^{n-2}}\right) , \qquad 6.6$$

where n is an integer and at least one pair a_p, b_p ($p >$ 3) is non-zero, higher coefficients being expressed in terms of a_2 , a_3 and a_p . For all such solutions the curvature invariant is a constant ($R = -12/a_2$); this fact is not inconsistent with their form at the origin being regular. We thus search among the numerical solutions (the derivation and properties of which have been examined earlier in the chapter) for any which are asymptotically of de Sitter form. The de Sitter form is given by (3.5) (putting $c = 1$):

$$ds^2 = \frac{-dr^2}{1-\Lambda r^2/3} - r^2(d\theta^2+\sin^2\theta\, d\phi^2) + (1-\frac{\Lambda}{3}r^2)\,dt^2 \qquad 6.7$$

We now deal with cases A and B separately.

<u>Case A</u>

Numerical solutions are based on the series expansions (4.2) for small r, for which the de Sitter solution is $e^{-\lambda} = e^{\nu}/d_0 = 1 + u_2 r^2$, where $u_2 = v_2 = -R/12$. We have set, for reasons given previously, $d_0 = 1$ and, for case A solutions, $R = 2$. Then $u_2 = v_2 = -\Lambda/3 = -\frac{1}{6}$. Now any regular case A solution that is asymptotically of constant space-time curvature tends to the form (6.7) with $-\Lambda/3 = -\frac{1}{6}$. This is so because for the large r expansions of section 4.3, $R = -12/a_2$. Setting this equal to 2 we obtain $a_2 = -6$. Now since the de Sitter solution itself (with initial values $e^{-\lambda} = e^{\nu} = 1$) must arise as a member of the numerical solution set, we obtain from (6.6) that $a_0 = 1/a_2$. Then the asymptotic form (6.6) becomes

$$
\begin{aligned}
e^{-\lambda} &= 1 - \frac{r^2}{6} + \frac{a_3}{6}\cdot\frac{1}{r} + \frac{1}{6}\sum_{n\geq 3}\frac{a_n}{r^{n-2}}, \\
e^{\nu} &= 1 - \frac{r^2}{6} + \frac{a_3}{6}\cdot\frac{1}{r} + \frac{1}{6}\sum_{n\geq 3}\frac{b_n}{r^{n-2}},
\end{aligned}
\qquad 6.8
$$

which has the correct limiting form (6.7) with $-\Lambda/3 = -\frac{1}{6}$, as required. Thus we can make specific the asymptotic values sought for the numerical solutions.

The convergence of $e^{-\lambda}$, e^{ν} given by (6.8) to the de Sitter values may be slow, but that of the components of the stress-energy tensor will be much more rapid. From the formulae (A2.21), (A2.22) for the Ricci tensor and the curvature invariant, and the expansions (6.8) for $e^{-\lambda}$, e^{ν}, we have, asymptotically (putting $\kappa = 1$):

$$
\begin{aligned}
T_1^1 &= \frac{1}{2} + O\left(\frac{1}{r^n}\right) + O\left(\frac{1}{r^6}\right), \\
T_2^2 &= \frac{1}{2} + O\left(\frac{1}{r^n}\right) + O\left(\frac{1}{r^4}\right), \\
T_4^4 &= \frac{1}{2} + O\left(\frac{1}{r^n}\right) + O\left(\frac{1}{r^4}\right).
\end{aligned}
\qquad 6.9
$$

A thorough search was made for solutions of the required form for each of the values $\beta = -\frac{1}{12}, \frac{1}{6}, \frac{1}{2}, \frac{11}{12}$ and $\frac{17}{12}$. For the two largest values of β it was found that there seem to be values of u_2

for which the numerical solutions bear some quantitative resemblance to the appropriate de Sitter form. However, it was also found that this is true for any value of β exceeding the approximate value $\frac{2}{3}$. The closeness between the numerical and desired values is insufficient to enable us to draw any positive conclusions; it is difficult to know what to make of these results.

Case B

By a similar argument as for case A the asymptotic behaviour sought is

$$e^{-\lambda} = 1 + \frac{r^2}{6} + \frac{a_3}{6}\cdot\frac{1}{r} + \frac{1}{6}\sum_{n>3}\frac{a_n}{r^{n-2}} \quad ,$$
$$e^{\nu} = 1 + \frac{r^2}{6} + \frac{a_3}{6}\cdot\frac{1}{r} + \frac{1}{6}\sum_{n>3}\frac{b_n}{r^{n-2}} \quad ; \qquad 6.10$$

$$T^1_1 = \frac{-1}{2} + O\left(\frac{1}{r^n}\right) + O\left(\frac{1}{r^6}\right) \quad ,$$
$$T^2_2 = \frac{-1}{2} + O\left(\frac{1}{r^n}\right) + O\left(\frac{1}{r^4}\right) \quad , \qquad 6.11$$
$$T^4_4 = \frac{-1}{2} + O\left(\frac{1}{r^n}\right) + O\left(\frac{1}{r^4}\right) \quad .$$

No solutions corresponding to values of β given by $\beta = -\frac{1}{12}$, $\frac{1}{6}$, $\frac{1}{2}$, $\frac{11}{12}$ or $\frac{17}{12}$ were found with the asymptotic form (6.10), (6.11). However, it was found that for values of β not exceeding the approximate value $-\frac{3}{10}$, there exist numerical solutions for a wide range of u_2 such that T^1_1 , T^2_2 and T^4_4 approach the value $-\frac{1}{2}$. The rate of convergence is dependent upon u_2 and, as we would expect, is unobtainable for $|u_2| \longrightarrow \infty$ (since the solutions must approach the limiting forms of space-times II and III). The convergence of T^1_1 is observed to be more rapid than that of T^2_2 and T^4_4 , which supports the conjecture that the value of n , previously taken to be greater than 3, is in fact greater than 6. The convergence of $e^{-\lambda}$, e^{ν} , however, was very slow and over the range considered these functions were only qualitatively of the required form. Some such solutions are indicated in Fig. 15.

The numerical results described in this section are not in accord with our expectations. We make the following suggestions:

(i) The possible values of β are not given by a monotonically increasing sequence as conjectured in section 4.3. This, however, seems unlikely as experience with the series expansions indicates that by working to higher orders increasingly positive values for β must be obtained.

(ii) The series solutions for the specific values of β considered do not correspond to solutions regular at $r = 0$.

(iii) Those solutions that are regular at $r = 0$ and also asymptotically of constant space-time curvature do not possess series expansions which include integral powers only of $1/r$. An investigation of more general series expansions forms than those studied in section 4.3 is required. (We note in this regard that space-time II has exponential terms in its large-r expansion).

6.5 Conclusion

The numerical solutions obtained give some indication as to the nature of the static, spherically symmetric solutions permitted by Lanczos' equations, but are in no way as instructive as the exact solutions. However, it has been seen, in section 6.4, how numerical methods may be helpful in deciding the suitability of certain assumed asymptotic forms for the solutions.

Fig. 14

Numerical solutions of Lanczos' equations in the static, spherically symmetric case, corresponding to the metric (2.8), under the imposition of the boundary condition of regularity at the spatial origin. Fig. 14 consists of Fig. 14(i) and Fig. 14 (ii) and covers case A solutions:

$$V_2 = -\frac{1}{3} - u_2 \ , \ K = +2 \ .$$

A succession of solutions is shown for $u_2 \in (-1, +1)$ and $\beta \in (-0.5, 2.0)$. Increasing values of u_2 and of β are shown in successive rows and columns respectively. For increasingly negative (positive) values of u_2, the behaviour of the numerical solutions for all values of β approaches that of the limit of the special case of space-time II (space-time III). The de Sitter solution occurs when $u_2 = V_2 = -\frac{1}{6}$. The behaviour of solutions in all cases for values of β more negative than - 0.5 or more positive than 2.0 does not differ substantially from the behaviour for the latter two values.

The range of integration was generally $r \in (0, 20)$, but it is not necessary to depict this in full. A broken line indicates that the curve is completed outside the vertical range shown.

Using subscripts to identify particular solutions, so that Fig. $14(i)_{2,3}$ denotes the numerical solution for $u_2 = -\frac{1}{2}$, $\beta = 0.5$, we make the following observations:

Examples of solutions with $T^4_4 \geqslant 0$ everywhere are Figs. $14(i)_{3,3}$, $(i)_{3,4}$; $(i)_{4,\alpha}$, $(ii)_{\alpha,1}$ ($\alpha = 1,..,4$); $(ii)_{\alpha,4}$ ($\alpha = 1,2,3$); $(ii)_{1,2}$, $(ii)_{1,3}$.

The region within which $T^4_4 \geqslant 0$ (or < 0) for each solution is distinguished by a green (or red) line along that part of the horizontal axis.

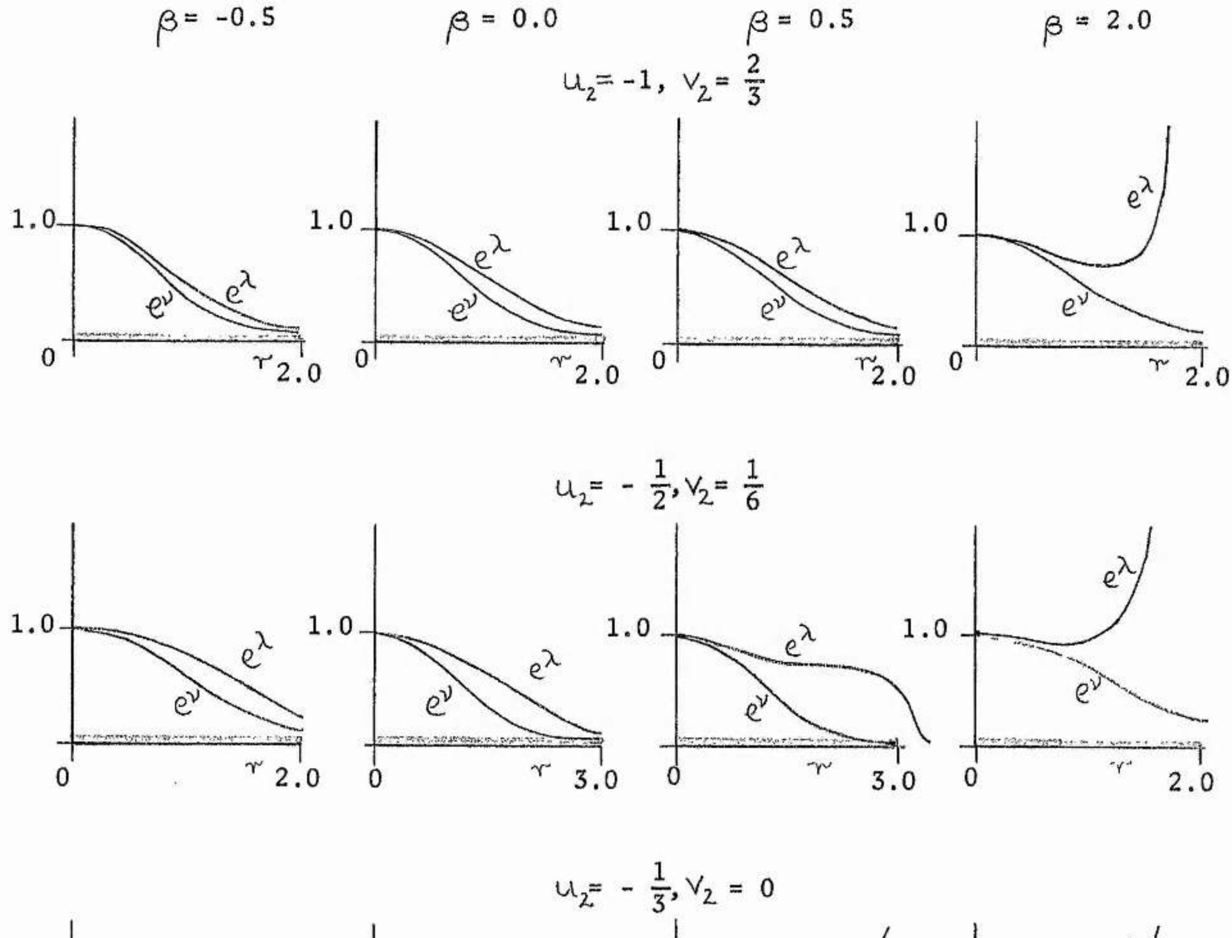
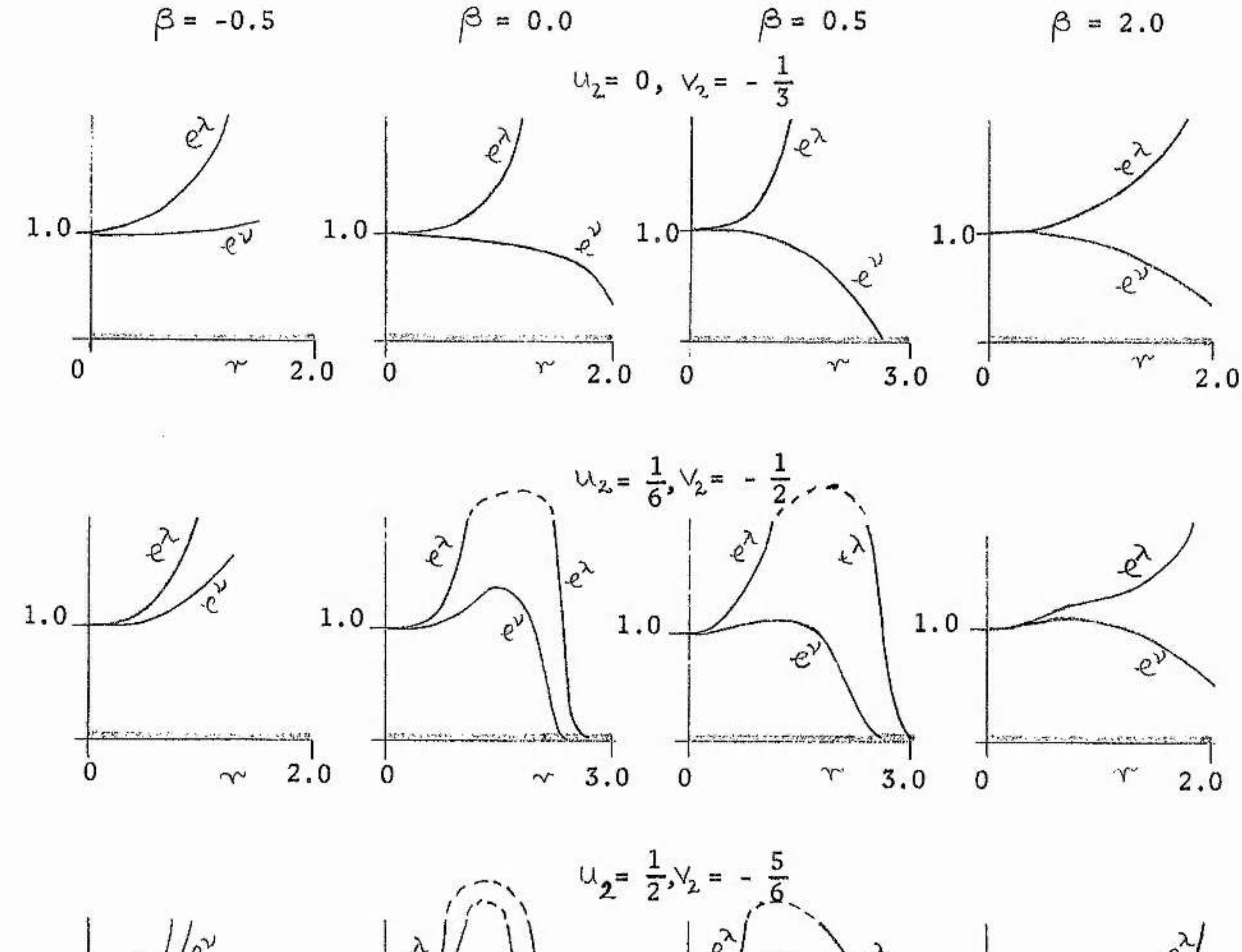

β = -0.5
β = 0.0
β = 0.5
β = 2.0
$u_2 = -1,\ v_2 = \frac{2}{3}$
$u_2 = -\frac{1}{2},\ v_2 = \frac{1}{6}$
$u_2 = -\frac{1}{3},\ v_2 = 0$
(de Sitter space-time)
$u_2 = -\frac{1}{6},\ v_2 = -\frac{1}{6}$
1.0
0
2.0
3.0
r
e^{λ}
e^{ν}

Fig. 14(i)



$\beta = -0.5$
$\beta = 0.0$
$\beta = 0.5$
$\beta = 2.0$
$u_2 = 0,\ v_2 = -\frac{1}{3}$
$u_2 = \frac{1}{6},\ v_2 = -\frac{1}{2}$
$u_2 = \frac{1}{2},\ v_2 = -\frac{5}{6}$
$u_2 = 1,\ v_2 = -\frac{4}{3}$
e^λ
e^ν
1.0
0
r
2.0
3.0

Fig. 14(ii)

Fig. 15

Numerical solutions of Lanczos' equations in the static, spherically symmetric case, corresponding to the metric (2.8), under the imposition of the boundary condition of regularity at the spatial origin. Fig. 15 consists of Fig. 15(i) and Fig. 15(ii) and covers case B solutions:

$$v_2 = \frac{1}{3} - u_2 \ , \ K = -2 \ .$$

A succession of solutions is shown for $u_2 \in (-1, +1)$ and $\beta \in (-2.0, 0.5)$. Increasing values of u_2 and of β are shown in successive rows and columns respectively. For increasingly negative (positive) values of u_2, the behaviour of the numerical solutions for all values of β approaches that of the limit of the special case of space-time II (space-time III). The de Sitter solution occurs when $u_2 = v_2 = \frac{1}{6}$. The behaviour of solutions in all cases for values of β more negative than -2.0 or more positive than 0.5 does not differ substantially from the behaviour for the latter two values.

The range of integration was generally $r \in (0, 20)$.

With similar notation as for Fig. 14, we make the following observations:

Examples of solutions with $T^4_4 \geq 0$ everywhere are:
Figs. 15(ii)$_{2,3}$, (ii)$_{2,4}$, (ii)$_{3,3}$, (ii)$_{3,4}$, (ii)$_{4,3}$, (ii)$_{4,4}$.
The region within which $T^4_4 \geq 0$ (or < 0) is distinguished as in Fig. 14 by a green (or red) line along that part of the horizontal axis.

Examples of solutions which are asymptotically of de Sitter form are:
Figs. 15(i)$_{\alpha,1}$, (ii)$_{\alpha,1}$, (ii)$_{\alpha,2}$, (α = 1,...,4)
and are distinguished by a yellow line along the vertical axis.

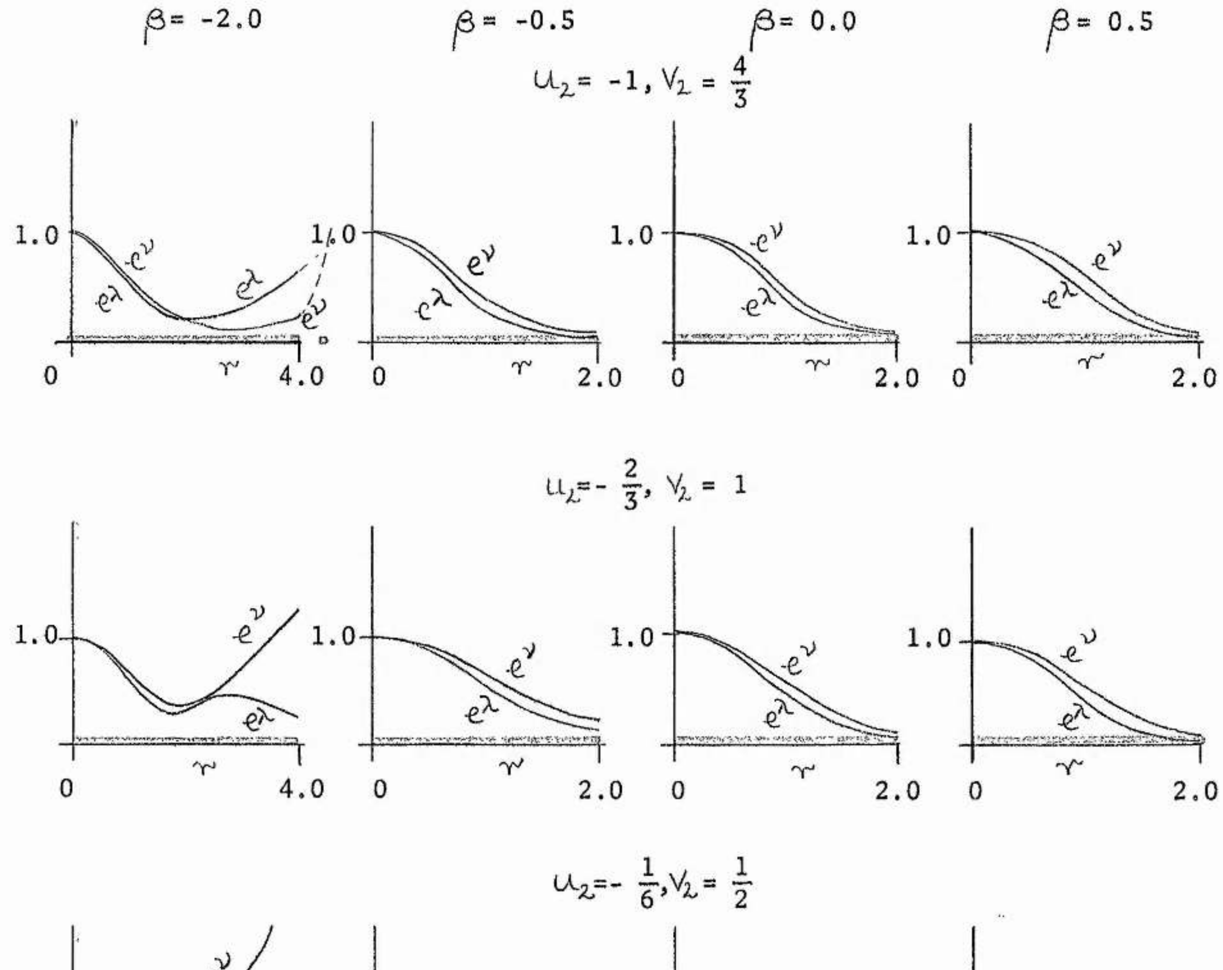
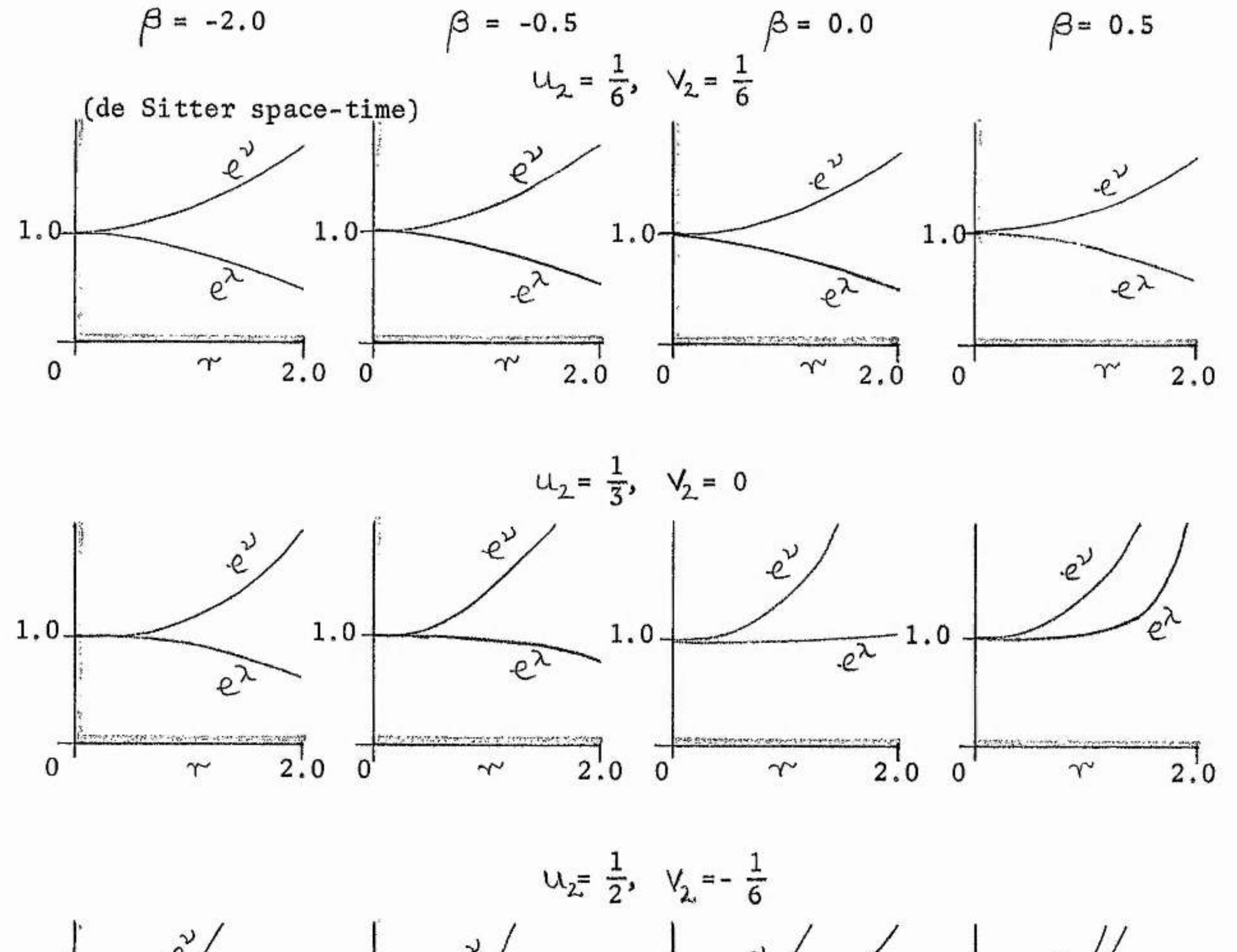

β= -2.0
β= -0.5
β= 0.0
β= 0.5
$U_2 = -1, V_2 = \frac{4}{3}$
$U_2 = -\frac{2}{3}, V_2 = 1$
$U_2 = -\frac{1}{6}, V_2 = \frac{1}{2}$
$U_2 = 0, V_2 = \frac{1}{3}$
1.0
e^ν
e^λ
r
0
4.0
2.0

Fig. 15(i)



$\beta = -2.0$
$\beta = -0.5$
$\beta = 0.0$
$\beta = 0.5$
$u_2 = \frac{1}{6}, \quad V_2 = \frac{1}{6}$
(de Sitter space-time)
e^{ν}
e^{λ}
1.0
0
r
2.0
$u_2 = \frac{1}{3}, \quad V_2 = 0$
$u_2 = \frac{1}{2}, \quad V_2 = -\frac{1}{6}$
$u_2 - 1, \; V_2 = -\frac{2}{3}$

Fig. 15(ii)

7. THE APPLICATION OF BUCHDAHL'S METHODS AND THE SET OF "CLASS 2" SPHERICALLY SYMMETRIC SPACE-TIMES

7.1 Introduction

In order to consider the field equations under the imposition of spherical symmetry we assumed in section 2.4 the usual, and conceptually convenient, metric form (2.8). It has become clear that this form is sometimes inappropriate and, in certain cases, completely inadequate for the exhibition of certain spherically symmetric solutions of Lanczos' equations. For example, we are not able to solve the equations of the coordinate transformation that would enable us to exhibit space-times VI and VII explicitly in the form (2.8), and there is no transformation whatsoever that will take the metric forms of space-times V, VIII and IX into (2.8). It may thus be advantageous to consider the field equations obtained from a different spherically symmetric metric form.

The set of all spherically symmetric space-times is divisible into two disjoint classes. Following Takeno,[54] the line element of any class 1 spherically symmetric space-time, S_1, may be written:

$$ds^2 = -e^{\lambda(r,t)}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{\nu(r,t)}dt^2, \qquad 7.1$$

and that of any class 2 spherically symmetric space-time, S_2 :

$$ds^2 = -e^{\lambda(r,t)}dr^2 - E^2(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{\nu(r,t)}dt^2, \qquad 7.2$$

where E is a constant. Thus space-times I$\longrightarrow$IV, VI and VII are class 1 while V, VIII and IX are class 2.

Class 2 spherically symmetric space-times are sometimes disregarded. In establishing a generalised form of Birkhoff's theorem, Bonnor[66] dismisses S_2's as of no physical significance, at any rate for a spherically symmetric field. This is because the surface area of a sphere in an S_2 is independent of the radial coordinate and has the constant value $4\pi E^2$. Bonnor's generalisation states: *Every physically significant spherically symmetric solution of the field equations* $R_{\rho\sigma} = \Lambda g_{\rho\sigma}$ *may be reduced, by a coordinate transformation, to the static Einstein space* (4.27). If S_2's are not regarded as without physical significance then this generalisation does not hold, since $R_{\rho\sigma} = \Lambda g_{\rho\sigma}$ admits a time-dependent solution distinct from (4.27). Birkhoff's theorem in its original form[38] ($\Lambda = 0$) is still true, as there are no solutions of $R_{\rho\sigma} = 0$ representing S_2's.

To eliminate S_2's from discussion on the aforementioned basis does not seem justifiable, since the geometrical properties of any non-trivial space-time will, to some extent, be at variance with our experience. It is noticeable that solutions of Lanczos' equations arise in families, the members of which share certain properties. We thus seek the completion of the set of static S_2 solutions of Lanczos' equations containing space-times V, VIII and IX. In order to do this, we must adopt a suitable form for the line element; preferably one which may also lead to further static S_1 solutions.

In seeking a suitable form for the line element we take into account Buchdahl's examination[6] of the field equations arising from the Lagrangian R^2. The metric which he employs and the field equations obtained from the Lagrangian $R_{\rho\sigma}R^{\rho\sigma}+\beta R^2$ for such a metric are given in section 7.3. In section 7.4 we achieve our objective in obtaining the class of static S_2 solutions of Lanczos' equations. First we briefly describe, in section 7.2, the method employed by Buchdahl to obtain the theorem (given in section 3.2) concerning static, asymptotically flat solutions of the field equations generated by the Lagrangian R^2, indicating the complications that arise when R^2 is replaced by the combination $R_{\rho\sigma}R^{\rho\sigma}+\beta R^2$.

7.2 Buchdahl's Theorem

The method used by Buchdahl to demonstrate the non-existence of static asymptotically flat solutions with $R \neq 0$ of the field equations generated by the Lagrangian R^2 is applied, as far as possible, to the field equations (2.3) generated by $R_{\rho\sigma}R^{\rho\sigma}+\beta R^2$. We proceed as follows:

An Einstein space $R_{\rho\sigma} = \Lambda g_{\rho\sigma}$ for which $\Lambda = 0$ is called a *special* Einstein space V_E and is distinguished by the subscript 0 . A (postulated) family of solutions of the equations (2.3) represents spaces V_A neighbouring to V_E , at least within some region D . The metric of any V_A is

$$g_{\rho\sigma} = \underset{\circ}{g}_{\rho\sigma} + \varepsilon h_{\rho\sigma}$$

where ε is a sufficiently small constant parameter which can tend to zero and the $h_{\rho\sigma}$ (and their derivatives to a sufficiently high order) are finite and continuous in D . Retaining only terms linear in ε , we may write

$$R_{\rho\sigma} = \varepsilon P_{\rho\sigma} \ , \quad R = \varepsilon P \tag{7.3}$$

($P = \underset{\circ}{g}^{\rho\sigma} P_{\rho\sigma}$). Denoting throughout this section covariant derivation in V_A and V_E by a colon and a semicolon respectively, the field equations (2.3) are

$$\begin{aligned} g_{\rho\sigma} R^{\alpha\gamma}{}_{:\alpha\gamma} + g^{\alpha\gamma} R_{\rho\sigma:\alpha\gamma} - R^{\alpha}{}_{\cdot\sigma:\rho\alpha} - R^{\alpha}{}_{\cdot\rho:\sigma\alpha} \\ +\tfrac{1}{2} g_{\rho\sigma} R_{\alpha\gamma} R^{\alpha\gamma} - 2R_{\alpha\rho} R^{\alpha}{}_{\cdot\sigma} \\ +\beta(2g_{\rho\sigma} g^{\alpha\gamma} R_{:\alpha\gamma} - R_{;\rho\sigma} - R_{:\sigma\rho} + \tfrac{1}{2} g_{\rho\sigma} R^2 - 2RR_{\rho\sigma}) = 0 \ . \end{aligned} \tag{7.4}$$

To the required order

$$\begin{aligned} R_{\rho\sigma:\alpha\gamma} &= \varepsilon P_{\rho\sigma;\alpha\gamma} \ , \\ R_{:\rho\sigma} &= \varepsilon P_{;\rho\sigma} \ , \end{aligned} \tag{7.5}$$

so that, using (7.3) and (7.5), equations (7.4) reduce to

$$\overset{\circ}{g}_{\rho\sigma} P^{\alpha\tau}{}_{;\alpha\tau} + \overset{\circ}{g}{}^{\alpha\tau} P_{\rho\sigma;\alpha\tau} - P^{\alpha}_{\cdot\sigma;\rho\alpha} - P^{\alpha}_{\cdot\rho;\sigma\alpha} + \beta\,(2\overset{\circ}{g}_{\rho\sigma}\overset{\circ}{g}{}^{\alpha\tau}P_{;\alpha\tau} - P_{;\rho\sigma} - P_{;\sigma\rho}) = 0 \ . \qquad 7.6$$

Transvecting (7.6) with $\overset{\circ}{g}{}^{\rho\sigma}$ and using

$$(P^{\rho\sigma} - \tfrac{1}{2}\overset{\circ}{g}{}^{\rho\sigma}P)_{;\rho\sigma} \equiv 0 \ , \qquad 7.7$$

we obtain, if $\beta \neq -\frac{1}{3}$,

$$\overset{\circ}{g}{}^{\alpha\tau} P_{;\alpha\tau} = 0 \ . \qquad 7.8$$

Using (7.7) and (7.8) in (7.6), we obtain

$$\overset{\circ}{g}{}^{\alpha\tau} P_{\rho\sigma;\alpha\tau} - P^{\alpha}_{\cdot\sigma;\rho\alpha} - P^{\alpha}_{\cdot\rho;\sigma\alpha} - 2\beta P_{;\rho\sigma} = 0 . \qquad 7.9$$

The set of equations (7.9) replaces the simpler set of Buchdahl:

$$P_{;\rho\sigma} = 0 \ . \qquad 7.10$$

We note that the first consequence, (7.8), of (7.6) contains very much less information than (7.10).

Buchdahl now proceeds by differentiating (in V_E) (7.10) and permuting the last two indices to obtain

$$P_{;\rho\sigma\tau} - P_{;\rho\tau\sigma} = 0 = \overset{\circ}{R}{}^{\alpha}_{\cdot\rho\sigma\tau} P_{;\alpha} \ .$$

By relating the components of $\overset{\circ}{R}{}^{\alpha}_{\cdot\rho\sigma\tau}$ in the four-space V_E to the components $\underset{1}{R}{}^{a}_{\cdot rst}$ in the x^4constant hypersurface of V_E , it is only necessary to apply straightforward tensor calculus to reach the conclusion that P must be a constant. Since V_A is to be asymptotically flat, this constant must have the value zero. Under the conditions of continuity and differentiability imposed earlier upon the metric, any V_A is neighbouring to V_E in a certain region so that the theorem follows.

We see that the manipulation of the set (7.9) is a very different matter. It has not been possible to proceed with the analysis in a similar fashion, and although P = constant is sufficient to satisfy (7.8) (which holds only if $\beta \neq -\frac{1}{3}$), it may not be necessary. We cannot say that Buchdahl's theorem is also true for the equations of the Lagrangian $R_{\rho\sigma}R^{\rho\sigma} + \beta R^2$.

7.3 The Field Equations in Buchdahl's Coordinates

The field equations arising from both R^2 and $R_{\rho\sigma}R^{\rho\sigma} + \beta R^2$ ($\beta \neq -\frac{1}{3}$) have the first consequence that

$$g^{\alpha\tau} R_{;\alpha\tau} = 0 . \qquad 2.7$$

In terms of the generic form of the general static spherically symmetric metric

$$ds^2 = -e^{\lambda(r)}dr^2 - e^{\mu(r)}(d\theta^2 + \sin^2\theta \, d\phi^2) + e^{\nu(r)}dt^2 \qquad 7.11$$

considered by Buchdahl, (2.7) reads

$$\left\{\exp\left[\tfrac{1}{2}(\nu - \lambda) + \mu\right] R'\right\}' = 0 , \qquad 7.12$$

(a dash denoting differentiation with respect to r). Now one of the functions λ , μ , ν may be prescribed arbitrarily subject to the condition that neither μ nor ν must be taken as constant (for taking μ = constant restricts us to S_2's only and we do not want an *a priori* distinguished time coordinate, which would follow from ν = constant.) Buchdahl makes the choice

$$\mu = \tfrac{1}{2}(\lambda - \nu) , \qquad 7.13$$

so that (7.12) becomes

$$R'' = 0 ,$$

whence

$$R = A + Br , \tag{7.14}$$

where A and B are arbitrary constants. We follow Buchdahl in the choice (7.13) so that the metric form now to be employed is

$$ds^2 = -e^{\lambda}\,dr^2 - e^{\frac{1}{2}(\lambda-\nu)}(d\theta^2+\sin^2\theta\,d\phi^2) + e^{\nu}\,dt^2 , \tag{7.15}$$

in which $\lambda = \lambda(r)$, $\nu = \nu(r)$. We know in advance that *every solution must satisfy* (7.14).

Writing $x^1 \equiv r$, $x^2 \equiv \theta$, $x^3 \equiv \phi$, $x^4 \equiv t$, the $g_{\rho\sigma}$, $g^{\rho\sigma}$ for the metric (7.15) are given in appendix A7.3 by (A7.1), the Christoffel symbols of the second kind $\Gamma^{\alpha}_{\rho\sigma}$ by the set (A7.2), the components of the Ricci tensor $R_{\rho\sigma}$ and the curvature invariant R by (A7.3), (A7.4) respectively.

As explained in section (2.4), the $G_{\rho\sigma}$ given by the latter of (2.3) must satisfy the identity (2.9):

$$G^{\rho}_{\cdot\sigma;\rho} \equiv 0 ,$$

whether or not the field equations $G_{\rho\sigma} = 0$ are satisfied. In consequence the following identity is obtained for the metric (7.15):

$$\left(\frac{d\nu}{dr} - \frac{d\lambda}{dr}\right) G^2_2 \equiv -2\frac{dG^1_1}{dr} - G^1_1\frac{d\lambda}{dr} + G^4_4\frac{d\nu}{dr} . \tag{7.16}$$

In addition, if the field equations (2.3) are satisfied,

$$G = G^1_1 + 2G^2_2 + G^4_4 = 0 . \tag{7.17}$$

From (7.16) and (7.17), excluding the two possibilities (i) $\lambda' = \nu'$, (ii) $\lambda' = 3\nu'$, the field equations are satisfied if both $G = 0$ (i.e. if $R = A + Br$ since, from (2.6), $G = 2(1+3\beta)g^{\alpha\tau}R_{;\alpha\tau}$) and $G^1_1 = 0$. We now deal with the cases (i) and (ii):

(i) $\lambda' = \nu'$. In this case we must ensure that $G^1_1 = 0$ *and* $G^2_2 = 0$.
(ii) $\lambda' = 3\nu'$. Inserting $e^{\lambda} = C^2 e^{3\nu}$ into (7.15) we arrive at the metric form

$$ds^2 = -C^2 e^{3\nu} dr^2 - Ce^{\nu}(d\theta^2 + \sin^2\theta \, d\phi^2) + e^{\nu} dt^2. \qquad 7.18$$

Transforming the radial coordinate through

$$Ce^{\nu(r)} = \rho^2 ,$$

the metric form (7.18) becomes

$$ds^2 = -\frac{\rho^6}{C}\{dr(\rho)\}^2 - \rho^2(d\theta^2 + \sin^2\theta \, d\phi^2) + \frac{\rho^2}{C} dt^2 . \qquad 7.19$$

Now (7.19) is of the form (2.8) in which $d\nu/d\rho = 2/\rho$; as mentioned in section 2.4 there are no solutions of Lanczos' equations of this type.

We thus require the equations $G_{11} = 0$, $G_{22} = 0$ for the metric (7.15); these are given in appendix A7.3 by (A7.5), (A7.6) respectively. The $G_{44} = 0$ equation, used for checking purposes, is redundant and is not given. It should be noted that we have expressed the coefficients of β (denoted $\mathscr{C}_\beta$) in (A7.5), (A7.6) in terms of λ, ν, their derivatives up to second order only, the curvature invariant $R = A + Br$ and its first derivative. By examining the $\mathscr{C}_\beta$ it is clear that Buchdahl's choice of coordinates does indeed formally simplify the problem of the solution of the field equations generated by R^2. Employing the substitution

$$\lambda = 3u + v - 3Bz , \; \nu = -u + v - Bz ,$$

where

$$A + Br = 4e^{Bz} \qquad (B \neq 0) ,$$

in the Lagrangian R^2 (it is equivalent, but more laborious, to make the substitution directly into the field equations $\mathcal{G}_{\beta} = 0$), Buchdahl reduces the problem to the solution of a single, second order, non-linear differential equation, the appearance of which is "deceptively simple". Little overall simplification is obtained by the use of a similar substitution in our field equations.

The field equations based on the metric (7.15) are thus no less complicated than those based on (2.8). The metric (7.15) has the disadvantage that it is unfamiliar: in these coordinates the limit of spatial infinity corresponds to $r \longrightarrow 0$ and the Schwarzschild line element, as given by Buchdahl, is

$$ds^2 = -\frac{16m^4 e^{-2mr}}{(1-e^{-2mr})^4}\, dr^2 - \frac{4m^2}{(1-e^{-2mr})^2}(d\theta^2+\sin^2\theta\, d\phi^2) + e^{-2mr}dt^2,$$

(which is regular in $0 < r < \infty$). However, (7.15) has the required property that it is valid for the representation of any static S_2, as is easily demonstrated. An S_2 is generally given by (7.2). Coordinates $\tilde{r}$ and $\tilde{t}$ may always be introduced such that (7.2) becomes[66]

$$ds^2 = -e^{\nu(\tilde{r},\tilde{t})}d\tilde{r}^2 - E^2(d\theta^2+\sin^2\theta\, d\phi^2) + e^{\nu(\tilde{r},\tilde{t})}d\tilde{t}^2;$$

(the tilde over the r and t coordinates is henceforth omitted). Furthermore, as mentioned in section 4.5, satisfaction of Lanczos' equations (2.3) by $d\sigma^2 = g_{\rho\sigma}dx^\rho dx^\sigma$ ensures that they are also satisfied by $ds^2 = E^2 d\sigma^2$ for any constant E ; thus E may be put equal to unity with no loss in generality. Then any static S_2 solution of Lanczos' equations may be cast into the form

$$ds^2 = -e^{\nu(r)}dr^2 - (d\theta^2 + \sin^2\theta\, d\phi^2) + e^{\nu(r)}dt^2. \qquad 7.20$$

We see that this is precisely the metric form (7.15) where $\lambda(r) = \nu(r)$.

Thus the form of the line element adopted by Buchdahl is suitable for the determination of all static S_2 solutions of Lanczos' equations.

7.4 The Set of "Class 2" Static, Spherically Symmetric Solutions

The metric form to be used is (7.15) with $\lambda = \nu$, and, as mentioned in the preceding section, we must ensure that both $G_{11} = 0$ and $G_{22} = 0$ are satisfied. Since $R = A + Br$ for all solutions, we may use this substitution freely.

When $\lambda = \nu$ the curvature invariant, from (A7.4), is given by

$$R = A + Br = -\nu'' e^{-\nu} + 2 ,$$

so that

$$\nu'' e^{-\nu} = (2 - A) - Br . \qquad 7.21$$

Putting $\lambda = \nu$ in the $G_{11} = 0$ equation (A7.5) and substituting from (7.21) for $\nu'' e^{-\nu}$ (and its first derivative), (7.5) reduces to

$$e^{\nu}\left\{\frac{B}{2}\nu' e^{-\nu} + \frac{1}{4}(B^2r^2 + (2AB - 4B)r + (A^2 - 4A)\right\}(1 + 2\beta) = 0 . \qquad 7.22$$

By a similar procedure, the second field equation (A7.6) becomes

$$-\frac{1}{4}(B^2r^2 + (2AB - 4B)r + (A^2 - 4A))(1 + 2\beta) = 0 . \qquad 7.23$$

It may be shown without difficulty that both (7.22) and (7.23) are satisfied, and are consistent with (7.21) in the following three cases only:

1. $\beta = -\frac{1}{2}$; $\nu'' e^{-\nu} = (2 - A) - Br$,
2. All values of β : $B = 0$, $A = 0$, $\nu'' e^{-\nu} = 2$,
3. All values of β : $B = 0$, $A = 4$, $\nu'' e^{-\nu} = -2$.

Case 1. $\beta = -\frac{1}{2}$.

$$\nu'' e^{-\nu} = (2-A) - Br . \qquad 7.21$$

We consider the two cases $B \neq 0$, $B = 0$, separately.

(i) $B \neq 0$. The general solution of (7.21) in terms of known functions appears to be unknown (see appendix A7.4). A particular solution of (7.21) is

$$e^{-\nu} = \frac{1}{3B^2}\{(2-A) - Br\}^3 ,$$

corresponding to the S_2 solution of Lanczos' equations given, from (7.20), by

$$ds^2 = \frac{-3B^2 dr^2}{\{(2-A)-Br\}^3} - (d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{3B^2\, dt^2}{\{(2-A)-Br\}^3} , \qquad 7.24$$

($R = A + Br$). We refer to (7.24) as Space-Time X. When the arbitrary constant A in (7.24) is put equal to zero, space-time X is conformal to the S_1 solution of Lanczos' equations for $\beta = -\frac{1}{2}$, space-time IV.

(ii) $B = 0$. Then (7.21) becomes

$$\nu'' e^{-\nu} = 2 - A . \qquad 7.25$$

When $2 - A > 0$ the following R = constant = A solutions of (7.25), (a), (b) and (c), exist:

(a) $$e^{\nu} = \frac{2}{(2-A)} C^2 \operatorname{cosech}^2(C(r-D)) ,$$

where C and D are additional arbitrary constants. This corresponds to the S_2 solution of Lanczos' equations given by

$$ds^2 = \frac{-2C^2 dr^2}{(2-A)\sinh^2(C(r-D))} - (d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{2C^2 dt^2}{(2-A)\sinh^2(C(r-D))}, \qquad 7.26$$

which will be referred to as Space-Time XI. We note that by putting the arbitrary constant A equal to zero, space-time XI becomes conformal to the β-independent S_1 solution of Lanczos' equations, space-time II.

(b) $$e^{\nu} = \frac{2}{(2-A)} \cdot \frac{1}{(r-a)^2} ,$$

where a is an additional arbitrary constant. This corresponds to the solution of Lanczos' equations, Space-Time XII:

$$ds^2 = \frac{-2\,dr^2}{(2-A)(r-a)^2} - (d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{2\,dt^2}{(2-A)(r-a)^2}. \qquad 7.27$$

Again, by putting A equal to zero, space-time XII becomes conformal to the β-independent S_1 solution, space-time I.

(c) $$e^{\nu} = \frac{2}{(2-A)}\, C^2 \operatorname{cosec}^2 (C(r-D)) ,$$

where C and D are additional arbitrary constants. This corresponds to the S_2 solution, Space-Time XIII:

$$ds^2 = \frac{-2C^2\,dr^2}{(2-A)\sin^2(C(r-D))} - (d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{2C^2\,dt^2}{(2-A)\sin^2(C(r-D))}. \qquad 7.28$$

Putting $A = 0$, space-time XIII becomes conformal to the β-independent S_1 solution, space-time III.

When $2 - A < 0$ the following R = constant = A solution of (7.25) exists:

(d) $$e^{\nu} = \frac{2}{(A-2)}\, C^2 \operatorname{sech}^2 (C(r-D)) ,$$

where C and D are additional arbitrary constants. This corresponds to the S_2 solution, Space-Time XIV:

$$ds^2 = \frac{-2C^2\,dr^2}{(A-2)\cosh^2(C(r-D))} - (d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{2C^2\,dt^2}{(A-2)\cosh^2(C(r-D))}. \qquad 7.29$$

This cannot be conformal to an S_1 solution, since all S_1 solutions with $-g_{11} = g_{44}$ have been found and comprise the set of space-times

I $\longrightarrow$ IV.

When $2 - A = 0$ the following R = constant = 2 solution of (7.25) exists:

$$e^{\nu} = e^{Cr+D}$$

where C and D are arbitrary constants. This corresponds to the S_2 solution, Space-Time XV:

$$ds^2 = -e^{Cr+D}\,dr^2 - (d\theta^2 + \sin^2\theta\, d\phi^2) + e^{Cr+D}\,dt^2 . \qquad 7.30$$

Clearly the arbitrary constant D is superfluous as it may be removed by a simple coordinate transformation.

The study of Case 1 is concluded.

Case 2. Solutions for all values of β. $A = 0$, $B = 0$, $R = 0$,

$$\nu'' e^{-\nu} = 2 . \qquad 7.31$$

The solution of (7.31) is described by Case 1 (a), (b) and (c) in which A is put equal to zero. Now all $R = 0$ solutions of Lanczos' equations have been found so that $(a)_{A=0}$, $(b)_{A=0}$ and $(c)_{A=0}$ must reduce to three of the space-times obtained in section 4.4.

$(a)_{A=0}$ The metric (7.26) becomes (writing ρ in the place of r):

$$ds^2 = \frac{C^2}{\sinh^2(C(\rho - D))}(dt^2 - d\rho^2) - (d\theta^2 + \sin^2\theta\, d\phi^2) \qquad 7.32$$

In the metric form (4.44h) representing space-time VIII, put

$$r = \frac{1}{C}\tanh(C(\rho - D)) ,$$

and set

$$\sqrt{\frac{\Lambda}{3}} = C ,$$

to obtain the form (7.32).

(b)$_{A=0}$ The metric (7.27) becomes (writing ρ in the place of r):

$$ds^2 = \frac{1}{(\rho - a)^2}\,(dt^2 - d\rho^2) - (d\theta^2 + \sin^2\theta\, d\phi^2) \quad . \qquad 7.33$$

In the metric form (4.44e), representing space-time V, put $r = \rho - a$ to obtain the form (7.33).

(c)$_{A=0}$ The metric (7.28) becomes (writing ρ in the place of r):

$$ds^2 = \frac{C^2}{\sin^2(C(\rho - D))}\,(dt^2 - d\rho^2) - (d\theta^2 + \sin^2\theta\, d\phi^2) \quad . \qquad 7.34$$

In the metric form (4.44i), representing space-time IX, put

$$r = \frac{1}{C}\tan\,(C(\rho - D))$$

and set

$$\sqrt{\frac{\Lambda'}{3}} = C$$

to obtain the form (7.34).

Thus the solutions covered by Case 2 are space-times V, VIII and IX.

<u>Case 3</u>. Solutions for all β . $A = 4$, $B = 0$, $R = 4$,

$$\nu'' e^{-\nu} = -2 \quad . \qquad 7.35$$

As stated in section (4.3), the most general solution of Lanczos' field equations satisfying $R = K \neq 0$ and independent of β is given by the set of Einstein spaces $R_{\rho\sigma} = \Lambda g_{\rho\sigma}$. For our choice of metric (7.20) we must therefore expect the solution of (7.35) to be the static S_2 solution of $R_{\rho\sigma} = \Lambda g_{\rho\sigma}$. That this is so is effectively shown by Bonnor.[67] The solution of (7.35) is

$$e^{\nu} = C^2\,\mathrm{sech}^2\,(C(r - D)) \quad ,$$

where C and D are arbitrary constants. This corresponds to the S_2 solution

$$ds^2 = \frac{-C^2}{\cosh^2(C(r-D))}\,dr^2 - (d\theta^2 + \sin^2\theta\,d\phi^2) + \frac{C^2\,dt^2}{\cosh^2(C(r-D))}\,. \qquad 7.36$$

This β-independent solution may, of course, be obtained from the metric (7.29) by setting $A = 4$. We refer to (7.36) as Space-Time XVI, although since it is a solution of $R_{\rho\sigma} = \Lambda\, g_{\rho\sigma}$ it cannot be regarded as a new solution of Lanczos' equations.

7.5 Summary

In this chapter we have combined an examination of Buchdahl's methods, as applied to the field equations generated by the Lagrangian R^2, with the search for the set of static class 2 spherically symmetric solutions of Lanczos' equations. The complete set of such solutions is given by space-times V, VIII and IX, together with:

Space-Time X ($\beta = -\frac{1}{2}$)

$$ds^2 = \frac{3B^2}{\{(2-A) - Br\}^3}(dt^2 - dr^2) - (d\theta^2 + \sin^2\theta\,d\phi^2)\,. \qquad 7.37(a)$$

(A more general metric is given by solution of the differential equation (7.21).

Space-Time XI ($\beta = -\frac{1}{2}$)

$$ds^2 = \frac{2C^2}{(2-A)\sinh^2(C(r-D))}(dt^2 - dr^2) - (d\theta^2 + \sin^2\theta\,d\phi^2)\,, \qquad 7.37(b)$$

($2 - A > 0$). When $A = 0$, (7.37b) is a solution for all β and may be transformed into the metric form of space-time VIII.

Space-Time XII ($\beta = -\frac{1}{2}$)

$$ds^2 = \frac{2}{(2-A)(r-a)^2}(dt^2 - dr^2) - (d\theta^2 + \sin^2\theta\,d\phi^2)\,, \qquad 7.37(c)$$

$(2 - A > 0)$. When $A = 0$, (7.37c) is a solution for all β and may be transformed into the metric form of space-time V.

Space-Time XIII $(\beta = -\frac{1}{2})$

$$ds^2 = \frac{2C^2}{(2-A)\sin^2(C(r-D))}(dt^2 - dr^2) - (d\theta^2 + \sin^2\theta\, d\phi^2), \quad 7.37(d)$$

$(2 - A > 0)$. When $A = 0$, (7.37d) is a solution for all β and may be transformed into the metric form of space-time IX.

Space-Time XIV $(\beta = -\frac{1}{2})$

$$ds^2 = \frac{2C^2}{(A-2)\cosh^2(C(r-D))}(dt^2 - dr^2) - (d\theta^2 + \sin^2\theta\, d\phi^2), \quad 7.37(e)$$

$(A - 2 > 0)$. When $A = 4$, (7.37e) is a solution for all β satisfying $R_{\rho\sigma} = \Lambda g_{\rho\sigma}$ and represents the metric form of space-time XVI.

Space-Time XV $(\beta = -\frac{1}{2})$

$$ds^2 = e^{Cr}(dt^2 - dr^2) - (d\theta^2 + \sin^2\theta\, d\phi^2). \quad 7.37(f)$$

Space-Time XVI

$$ds^2 = \frac{C^2}{\cosh^2(C(r-D))}(dt^2 - dr^2) - (d\theta^2 + \sin^2\theta\, d\phi^2). \quad 7.37(g)$$

8. CONCLUDING REMARKS

It is possible, as mentioned in section 2.2, that the inclusion in the action Lagrangian of the linear term R would yield a more realistic model. By introducing an extra dimensioned parameter in this way we break the gauge invariance of the theory, and thereby open up the possibility of fixing the "sizes" of the solutions, which in the present theory allow arbitrary stretching or contraction. Such a generalization, however, would considerably weaken the field equations which, we have seen, allow a fairly large manifold of solutions, at least in the spherically symmetric, static case. In order to obtain some indication of the desirability of weakening the field equations, a detailed study of the properties of the universes corresponding to each of the exact solutions obtained in chapters 4 and 7 is imperative. We have already seen (in chapter 5) that Lanczos' equations do indeed permit physically interesting space-times; for example space-time III, which corresponds to the superposition of an electromagnetic field and a perfect fluid, and space-time V, which represents a pure electromagnetic field. If further solutions are physically satisfactory a weakening of the field equations would, although interesting, not seem necessary.

The possibilities for further study are, of course, numerous. We have mentioned that the properties of the universe corresponding to each exact solution must be examined in detail. In addition, we may ask what would happen if one space-time were to impinge in some way upon another. Our discussion of space-time I indicates that the constant of integration a may represent some central mass. The geodesic equations of motion will tell us how a neutral test particle will behave in a Schwarzschild field, or in space-time I, but they cannot tell us, for example, how a Schwarzschild mass will behave in space-time I. A study of this behaviour, perhaps by examining the equations of motion of the system in the weak field approximation, could be very interesting.

Apart from the study of the universes already obtained, there is the possibility of obtaining further solutions. One feels almost certain that there exists a class of static, spherically symmetric solutions with non-zero, constant curvature invariant, perhaps with β present as an arbitrary constant. The field equations in the form based on the metric (7.15) have not been studied in any detail; it is possible that since they have led to several $R = K \neq 0$ solutions (space-times XI $\longrightarrow$ XVI), the metric form (7.15) may well be more appropriate for use in the derivation of additional solutions of this nature.

Does there exist a form of Birkhoff's theorem for Lanczos' equations? That is, suppose we remove the condition that the spherically symmetric field should be static. Then is it possible to show that the time dependence of any solution may be eliminated by reducing that solution to one of the set of static solutions? Only one form of symmetry has been considered in our present work. In examining, for example, cylindrical symmetry it may again be profitable to look for the set of $\beta = -\frac{1}{3}$ solutions, as these (at any rate in the spherically symmetric, static case) may be obtained with little difficulty. Then those solutions that satisfy the field equations for other values of β in addition to $\beta = -\frac{1}{3}$ may be obtained. In seeking the families which contain the latter solutions, the solution set may be built up.

In explaining how he arrived at the field equations

$$R_{\rho\sigma} - \frac{1}{2} g_{\rho\sigma} R = -\varkappa T_{\rho\sigma} ,$$

Einstein said "Not for a moment, of course, did I doubt that this formulation was merely a makeshift in order to give the general principle of relativity a preliminary closed expression." The theory of gravitation that has been presented is a tentative extension of a tentative, but beautiful theory. Its fruitfulness must, as in many theoretical studies, be subjective.

A2.3

Derivation of the Field Equations (2.3)

We consider the set of equations deduced from the action principle:

$$\delta I = 0, \qquad \text{A2.1}$$

where

$$I = \int_{D_4} (R_{\rho\sigma} R^{\rho\sigma} + \beta R^2)\sqrt{-g}\; d^4x \quad , \qquad \text{A2.2}$$

under the assumptions that D_4 is a bounded region in space-time with boundary S , where $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$ such that $\delta g_{\mu\nu}$ and its first derivatives vanish on S . Performing the variation, with I given by (A2.2) we obtain:

$$\delta I = \int_{D_4} \Big\{ \sqrt{-g}\,(R_{\rho\sigma}\delta R^{\rho\sigma} + R^{\rho\sigma}\delta R_{\rho\sigma} + 2\beta R \delta R)$$

$$+ (R_{\rho\sigma} R^{\rho\sigma} + \beta R^2)\,\delta\sqrt{-g} \Big\}\, d^4x \quad . \qquad \text{A2.3}$$

We define $\gamma_{\mu\nu} = \delta g_{\mu\nu}$ and $\gamma^{\mu\nu} = -\,\delta g^{\mu\nu}$.
Using $g_{\mu\rho}\, g^{\rho\nu} = \delta^{\nu}_{\mu}$, we have:

$$\gamma^{\mu\nu} = \gamma_{\rho\sigma}\, g^{\mu\rho}\, g^{\nu\sigma} \quad , \qquad \text{A2.4}$$

noting that the variations $\delta g_{\mu\nu}$ form a tensor, but that the variational operator δ and the operation of raising and lowering indices do not commute. Since $R^{\rho\sigma} = g^{\rho\mu} g^{\sigma\nu} R_{\mu\nu}$, the variations of the contravariant form of the Ricci tensor and of the curvature invariant are given by:

$$\delta R^{\rho\sigma} = g^{\rho\mu} g^{\sigma\nu} \delta R_{\mu\nu} + R_{\mu\nu}(g^{\rho\mu}\delta g^{\sigma\nu} + g^{\sigma\nu}\delta g^{\rho\mu})$$

$$= g^{\rho\mu} g^{\sigma\nu} \delta R_{\mu\nu} - g^{\sigma\mu} R^{\rho\nu} \gamma_{\mu\nu} - g^{\rho\mu} R^{\sigma\nu} \gamma_{\mu\nu} \quad , \qquad \text{A2.5}$$

$$\delta R = \delta(g^{\mu\nu} R_{\mu\nu}) = g^{\mu\nu}\delta R_{\mu\nu} - \gamma_{\mu\nu} R^{\mu\nu}.$$

Using the formulae (A2.5) the sum contained within the first round brackets under the integral sign in (A2.3) is found to be:

$$\delta R_{\mu\nu}(2R^{\mu\nu}+2\beta R g^{\mu\nu})+\gamma_{\mu\nu}(-2R_{\alpha}{}^{\mu}R^{\alpha\nu}-2\beta R R^{\mu\nu}) . \qquad \text{A2.6}$$

Our aim is to express the entire integrand in (A2.2) in the form

$$H_{\mu\nu}\gamma^{\mu\nu} ,$$

where $H_{\mu\nu}$ is a function of the $g_{\mu\nu}$ and $R_{\mu\nu}$ and their covariant derivatives and contractions. With this in mind, we express the contents of the second bracket in (A2.3) in this form, using

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\; g^{\mu\nu}\delta g_{\mu\nu} , \qquad \text{A2.7}$$

so that, substituting from (A2.6), δI becomes

$$\delta I = \int_{D_4}\Big\{2\,\delta R_{\mu\nu}(R^{\mu\nu}+\beta R g^{\mu\nu}) + \gamma_{\mu\nu}\Big[(R_{\alpha\tau}R^{\alpha\tau}+\beta R^2)\cdot\frac{1}{2}g^{\mu\nu}-2(R_{\alpha}{}^{\mu}R^{\alpha\nu}+\beta R R^{\mu\nu})\Big]\Big\}\cdot\sqrt{-g}\;d^4x . \qquad \text{A2.8}$$

It is now necessary to obtain $\delta R_{\mu\nu}$ in terms of $\delta g_{\mu\nu}$ and their derivatives. At any selected event P we may choose coordinates such that the Christoffel symbols $\Gamma^{\alpha}_{\rho\sigma}$ vanish there (locally geodesic coordinates). Thus, with the Ricci tensor defined by

$$R_{\mu\nu} = -\Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\rho}_{\alpha\nu} - \Gamma^{\rho}_{\alpha\rho}\Gamma^{\alpha}_{\mu\nu}$$

the variation of the Ricci tensor reduces at P to:

$$\delta R_{\mu\nu} = -\delta\Gamma^{\alpha}_{\mu\nu,\alpha} + \delta\Gamma^{\alpha}_{\mu\alpha,\nu} . \qquad \text{A2.9}$$

In such a coordinate system the first partial derivatives of the metric tensor vanish. Then, since

$$\Gamma^{\alpha}_{\mu\nu,\alpha} = \frac{1}{2}g^{\alpha\tau}\left(\frac{\partial^2 g_{\mu\tau}}{\partial x^{\alpha}\partial x^{\nu}} + \frac{\partial^2 g_{\nu\tau}}{\partial x^{\alpha}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha}\partial x^{\tau}}\right)$$

and

$$\Gamma^{\alpha}_{\mu\alpha,\nu} = \frac{\partial^2}{\partial x^\nu \partial x^\mu}\left(\ln\sqrt{-g}\right),$$

the quantities $\delta\Gamma^{\alpha}_{\mu\nu,\alpha}$ and $\delta\Gamma^{\alpha}_{\mu\alpha,\nu}$ required on the right hand side of (A2.9) are given by

$$\begin{aligned}\delta\Gamma^{\alpha}_{\mu\nu,\alpha} &= -\tfrac{1}{2}\, g^{\alpha\rho} g^{\tau\sigma}\gamma_{\rho\sigma}\left(\frac{\partial^2 g_{\mu\tau}}{\partial x^\alpha \partial x^\nu} + \frac{\partial^2 g_{\nu\tau}}{\partial x^\alpha \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\tau}\right)\\ &\quad + \tfrac{1}{2}\, g^{\alpha\tau}\left(\frac{\partial^2 \gamma_{\mu\tau}}{\partial x^\alpha \partial x^\nu} + \frac{\partial^2 \gamma_{\nu\tau}}{\partial x^\alpha \partial x^\mu} - \frac{\partial^2 \gamma_{\mu\nu}}{\partial x^\alpha \partial x^\tau}\right), \qquad \text{A2.10}\\ \delta\Gamma^{\alpha}_{\mu\alpha,\nu} &= \frac{\partial^2}{\partial x^\nu \partial x^\mu}\left(\tfrac{1}{2}\, g^{\mu\nu}\gamma_{\mu\nu}\right) = \frac{\partial^2}{\partial x^\nu \partial x^\mu}\left(\tfrac{1}{2}\gamma\right),\end{aligned}$$

where (A2.7) was employed in obtaining the latter of (A2.10). Thus, at the point P,

$$\begin{aligned}\delta R_{\mu\nu} &= \tfrac{1}{2} g^{\alpha\rho} g^{\tau\sigma}\gamma_{\rho\sigma}\left(\frac{\partial^2 g_{\mu\tau}}{\partial x^\alpha \partial x^\nu} + \frac{\partial^2 g_{\nu\tau}}{\partial x^\alpha \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\tau}\right)\\ &\quad - \tfrac{1}{2} g^{\alpha\tau}\left(\frac{\partial^2 \gamma_{\mu\tau}}{\partial x^\alpha \partial x^\nu} + \frac{\partial^2 \gamma_{\nu\tau}}{\partial x^\alpha \partial x^\mu} - \frac{\partial^2 \gamma_{\mu\nu}}{\partial x^\alpha \partial x^\tau}\right) \qquad \text{A2.11}\\ &\quad + \tfrac{1}{2}\frac{\partial^2 \gamma}{\partial x^\nu \partial x^\mu}.\end{aligned}$$

Consider the terms on the right hand side of (A2.11). At P, the terms in the second bracket are given by

$$\begin{aligned}\frac{\partial^2 \gamma_{\mu\tau}}{\partial x^\alpha \partial x^\nu} &= \gamma_{\mu\tau;\alpha\nu} + \gamma_{\rho\tau}\frac{\partial}{\partial x^\nu}\left(\Gamma^{\rho}_{\mu\alpha}\right) + \gamma_{\mu\rho}\frac{\partial}{\partial x^\nu}\left(\Gamma^{\rho}_{\tau\alpha}\right),\\ \frac{\partial^2 \gamma_{\nu\tau}}{\partial x^\alpha \partial x^\mu} &= \gamma_{\nu\tau;\alpha\mu} + \gamma_{\rho\tau}\frac{\partial}{\partial x^\mu}\left(\Gamma^{\rho}_{\nu\alpha}\right) + \gamma_{\nu\rho}\frac{\partial}{\partial x^\mu}\left(\Gamma^{\rho}_{\tau\alpha}\right),\\ \frac{\partial^2 \gamma_{\mu\nu}}{\partial x^\alpha \partial x^\tau} &= \gamma_{\mu\nu;\alpha\tau} + \gamma_{\rho\nu}\frac{\partial}{\partial x^\tau}\left(\Gamma^{\rho}_{\mu\alpha}\right) + \gamma_{\mu\rho}\frac{\partial}{\partial x^\tau}\left(\Gamma^{\rho}_{\nu\alpha}\right),\end{aligned}$$

so that the second term on the right hand side of (A2.11) becomes

$$\begin{aligned}&\tfrac{1}{2} g^{\alpha\tau}\gamma_{\mu\nu;\alpha\tau} - \tfrac{1}{2}\gamma^{\alpha}_{\cdot\mu;\alpha\nu} - \tfrac{1}{2}\gamma^{\alpha}_{\cdot\nu;\alpha\mu}\\ &\quad - \tfrac{1}{2}\gamma^{\alpha}_{\cdot\rho}\left[\frac{\partial}{\partial x^\nu}\left(\Gamma^{\rho}_{\mu\alpha}\right) + \frac{\partial}{\partial x^\mu}\left(\Gamma^{\rho}_{\nu\alpha}\right)\right]\\ &\quad - \tfrac{1}{2}\gamma_{\mu\rho}\, g^{\alpha\tau}\left[\frac{\partial}{\partial x^\nu}\left(\Gamma^{\rho}_{\tau\alpha}\right) - \frac{\partial}{\partial x^\tau}\left(\Gamma^{\rho}_{\nu\alpha}\right)\right]\\ &\quad - \tfrac{1}{2}\gamma_{\rho\nu}\, g^{\alpha\tau}\left[\frac{\partial}{\partial x^\mu}\left(\Gamma^{\rho}_{\tau\alpha}\right) - \frac{\partial}{\partial x^\tau}\left(\Gamma^{\rho}_{\mu\alpha}\right)\right],\end{aligned}$$

which is equal, at P, to:

$$\begin{aligned}&\tfrac{1}{2} g^{\alpha\tau}\gamma_{\mu\nu;\alpha\tau} - \tfrac{1}{2}\gamma^{\alpha}_{\cdot\mu;\alpha\nu} - \tfrac{1}{2}\gamma^{\alpha}_{\cdot\nu;\alpha\mu}\\ &\quad + \tfrac{1}{2}\gamma_{\mu\rho} R^{\rho}_{\cdot\nu} + \tfrac{1}{2}\gamma_{\nu\rho} R^{\rho}_{\cdot\mu}\\ &\quad - \tfrac{1}{2}\gamma^{\alpha\tau}\frac{\partial^2 g_{\alpha\tau}}{\partial x^\mu \partial x^\nu}, \qquad \text{A2.12}\end{aligned}$$

while the first term may be written

$$\frac{1}{2}\,\gamma^{\alpha\tau}\frac{\partial^2 g_{\alpha\tau}}{\partial x^\mu\partial x^\nu} + \frac{1}{2}\gamma^{\alpha\tau}\left[-R_{\mu\alpha\tau\nu} - R_{\nu\alpha\tau\mu}\right] \quad . \qquad \text{A2.13}$$

Then, from (A2.11), using (A2.12), (A2.13) we obtain for $\delta R_{\mu\nu}$:

$$\begin{aligned}\delta R_{\mu\nu} = \tfrac{1}{2}\,g^{\alpha\tau}\gamma_{\mu\nu;\alpha\tau} &+ \tfrac{1}{2}\gamma_{;\mu\nu} - \tfrac{1}{2}\gamma^{\alpha}_{\cdot\mu;\alpha\nu} - \tfrac{1}{2}\gamma^{\alpha}_{\cdot\nu;\alpha\mu} \\ &+\tfrac{1}{2}\gamma_{\mu\alpha}R^{\alpha}_{\cdot\nu} + \tfrac{1}{2}\gamma_{\nu\alpha}R^{\alpha}_{\cdot\mu} \\ &-\tfrac{1}{2}\gamma^{\alpha\tau}R_{\mu\alpha\tau\nu} - \tfrac{1}{2}\gamma^{\alpha\tau}R_{\nu\alpha\tau\mu} \quad .\end{aligned} \qquad \text{A2.14}$$

Using the formula for changing the order of double covariant differentiation in the third and fourth terms on the right hand side of (A2.14) this reduces to:

$$\delta R_{\mu\nu} = \tfrac{1}{2}g^{\alpha\tau}\gamma_{\mu\nu;\alpha\tau} + \tfrac{1}{2}\gamma_{;\mu\nu} - \tfrac{1}{2}\gamma^{\alpha}_{\cdot\mu;\nu\alpha} - \tfrac{1}{2}\gamma^{\alpha}_{\cdot\nu;\mu\alpha} \quad . \qquad \text{A2.15}$$

Now (A2.15) is a tensor equation; hence it is true at the point P in *any* coordinate system, and since P is arbitrary, (A2.15) is true everywhere and in all coordinate systems.

Examining (A2.8), it is the quantity $2\delta R_{\mu\nu}p^{\mu\nu}$, where $p^{\mu\nu}$ = $R^{\mu\nu} + \beta R g^{\mu\nu}$ is a symmetric tensor, that is of interest. Let us consider the terms in the expression $2\delta R_{\mu\nu}p^{\mu\nu}$ with $\delta R_{\mu\nu}$ given, quite generally, by (A2.15):

$$\begin{aligned}
&(1)\quad p^{\mu\nu}g^{\alpha\tau}\gamma_{\mu\nu;\alpha\tau} &&= (p^{\mu\nu}g^{\alpha\tau}\gamma_{\mu\nu})_{;\alpha\tau} - g^{\alpha\tau}\gamma^{\mu\nu}p_{\mu\nu;\alpha\tau} - 2g^{\alpha\tau}\gamma_{\mu\nu;\alpha}\,p^{\mu\nu}{}_{;\tau} \;,\\
&(2)\quad p^{\mu\nu}\gamma_{;\mu\nu} &&= (p^{\mu\nu}\gamma)_{;\mu\nu} - 2p^{\mu\nu}{}_{;\nu}\gamma_{;\mu} - g_{\mu\nu}p^{\alpha\tau}{}_{;\alpha\tau}\gamma^{\mu\nu} \;,\\
&(3)\quad p^{\mu\nu}\gamma^{\alpha}_{\cdot\mu;\nu\alpha} &&= (p^{\nu}_{\cdot\tau}\gamma^{\alpha\tau})_{;\nu\alpha} - \gamma^{\mu\nu}p^{\tau}_{\cdot\nu;\tau\mu} - p^{\nu}_{\cdot\tau;\nu}\gamma^{\alpha\tau}{}_{;\alpha} - p^{\nu}_{\cdot\tau;\alpha}\gamma^{\alpha\tau}{}_{;\nu} \;,\\
&(4)\quad p^{\mu\nu}\gamma^{\alpha}_{\cdot\nu;\mu\alpha} &&= (p^{\mu}_{\cdot\tau}\gamma^{\alpha\tau})_{;\mu\alpha} - \gamma^{\mu\nu}p^{\tau}_{\cdot\mu;\tau\nu} - p^{\mu}_{\cdot\tau;\mu}\gamma^{\alpha\tau}{}_{,\alpha} - p^{\mu}_{\cdot\tau;\alpha}\gamma^{\alpha\tau}{}_{;\mu} \;.
\end{aligned} \qquad \text{A2.16}$$

Straightforward manipulation of each of the terms (1)$\longrightarrow$(4) in (A2.16), (equivalently, by the properties of adjoint bilinear forms) yields the result:

$$\int_{D_4}(2\,\delta R_{\mu\nu}p^{\mu\nu} - F_{\mu\nu}\gamma^{\mu\nu})\sqrt{-g}\;d^4x = \text{Surface Integral} \;,$$

where

$$F_{\mu\nu} = g^{\alpha\tau} p_{\mu\nu;\alpha\tau} + g_{\mu\nu} p^{\alpha\tau}{}_{;\alpha\tau} - p^{\alpha}_{\cdot\mu;\nu\alpha} - p^{\alpha}_{\cdot\nu;\mu\alpha} \quad .$$

The surface integral vanishes by virtue of the assumed conditions on $\delta g_{\mu\nu}$ and the first term under the integral sign in (A2.8) may thus be replaced by

$$\left\{ g^{\alpha\tau} p_{\mu\nu;\alpha\tau} + g_{\mu\nu} p^{\alpha\tau}{}_{;\alpha\tau} - p^{\alpha}_{\cdot\mu;\nu\alpha} - p^{\alpha}_{\cdot\nu;\mu\alpha} \right\} \gamma^{\mu\nu} \sqrt{-g}$$

and we have achieved our aim to express the integrand in (A2.2) in the form $H_{\mu\nu}\gamma^{\mu\nu}$, for (A2.8) then becomes:

$$\begin{aligned} \delta I = \int_{D_4} \gamma^{\mu\nu} \Big\{ & g^{\alpha\tau} p_{\mu\nu;\alpha\tau} + g_{\mu\nu} p^{\alpha\tau}{}_{;\alpha\tau} - p^{\alpha}_{\cdot\mu;\nu\alpha} - p^{\alpha}_{\cdot\nu;\mu\alpha} \\ & + \tfrac{1}{2} g_{\mu\nu} (R_{\alpha\tau} R^{\alpha\tau} + \beta R^2) - 2(R_{\alpha\mu} R^{\alpha}_{\cdot\nu} + \beta R R_{\mu\nu}) \Big\} \cdot \\ & \cdot \sqrt{-g}\, d^4x \ , \end{aligned} \qquad \text{A2.17}$$

which must vanish, by (A2.1), for any $\delta g_{\mu\nu}$ such that it and its derivatives vanish on the boundary S. Then from (A2.17) we have:

$$\begin{aligned} & g^{\alpha\tau} p_{\mu\nu;\alpha\tau} + g_{\mu\nu} p^{\alpha\tau}{}_{;\alpha\tau} - p^{\alpha}_{\cdot\mu;\nu\alpha} - p^{\alpha}_{\cdot\nu;\mu\alpha} \\ & + \tfrac{1}{2} g_{\mu\nu} (R_{\alpha\tau} R^{\alpha\tau} + \beta R^2) - 2(R_{\alpha\mu} R^{\alpha}_{\cdot\nu} + \beta R R_{\mu\nu}) = 0 \ , \end{aligned}$$

where A2.18

$$p^{\mu\nu} = R^{\mu\nu} + \beta R g^{\mu\nu} \quad .$$

(A2.18) may also be written in the form (2.3). This completes the derivation of the field equations.

A2.4

Formulae for Spherical Symmetry

The formulae (A2.19) $\longrightarrow$ (A2.22) corresponding to the metric form (2.8) may be obtained from the general formulae for spherical symmetry given by Synge.[36] Note, however, that the conventions used here are in accordance with those specified in "Notation". Only surviving components of the various quantities are listed.

Components of the metric tensor:

$$g_{11} = -e^{\lambda},\ g_{22} = -r^2,\ g_{33} = -r^2\sin^2\theta,\ g_{44} = e^{\nu}.$$
$$g^{11} = -e^{-\lambda},\ g^{22} = \frac{-1}{r^2},\ g^{33} = \frac{-1}{r^2\sin^2\theta},\ g^{44} = e^{-\nu}. \qquad \text{A2.19}$$

Christoffel symbols of the second kind:

$$\Gamma^1_{11} = \tfrac{1}{2}\lambda',\ \Gamma^1_{22} = -re^{-\lambda},\ \Gamma^1_{33} = -r\sin^2\theta e^{-\lambda},\ \Gamma^1_{44} = \tfrac{1}{2}\nu' e^{\nu-\lambda},$$
$$\Gamma^2_{12} = \frac{1}{r},\ \Gamma^2_{33} = -\sin\theta\cos\theta, \qquad \text{A2.20}$$
$$\Gamma^3_{13} = \frac{1}{r},\ \Gamma^3_{23} = \cot\theta,$$
$$\Gamma^4_{14} = \tfrac{1}{2}\nu'.$$

Components of the Ricci tensor:

$$R^1_1 = e^{-\lambda}\left\{-\tfrac{1}{2}\nu'' - \tfrac{1}{4}\nu'^2 + \tfrac{1}{4}\nu'\lambda' + \frac{\lambda'}{r}\right\},$$
$$R^2_2 = e^{-\lambda}\left\{\frac{1}{r}\left(\frac{\lambda'}{2} - \frac{\nu'}{2}\right) - \frac{1}{r^2}\right\} + \frac{1}{r^2}, \qquad \text{A2.21}$$
$$R^3_3 = R^2_2,$$
$$R^4_4 = e^{-\lambda}\left\{-\tfrac{1}{2}\nu'' - \tfrac{1}{4}\nu'^2 + \tfrac{1}{4}\nu'\lambda' - \frac{\nu'}{r}\right\}.$$

The curvature invariant:

$$R = e^{-\lambda}\left\{\left(-\tfrac{1}{2}\nu'^2 - \nu'' + \tfrac{1}{2}\lambda'\nu'\right) + \frac{1}{r}\left(-2\nu' + 2\lambda'\right) - \frac{2}{r^2}\right\} + \frac{2}{r^2}. \qquad \text{A2.22}$$

Lanczos' Equations (in which $\alpha \equiv (1+2\beta)$, $\gamma \equiv (1+4\beta)$):

$$\begin{aligned}
G^1_1 = \frac{e^{-2\lambda}}{16}\Big[&\alpha\left\{8\nu'''\nu' + 4\nu''\nu'^2 - 4\nu''^2 - \nu'^4 - 8\nu''\nu'\lambda' - 4\nu'^2\lambda'' + 3\nu'^2\lambda'^2 - 2\nu'^3\lambda'\right\}\\
&+ \frac{1}{r}\left\{16\nu'''\gamma + 32\nu''\nu'(1+3\beta) + 4\nu'^3 - 16\nu''\lambda'\gamma - 16\nu'\lambda''\gamma + 12\nu'\lambda'^2\gamma - 16\nu'^2\lambda'(1+3\beta)\right\}\\
&+ \frac{1}{r^2}\left\{32\nu''\gamma - 12\nu'^2 - 8\nu'\lambda'(5+16\beta) - 16\lambda''(3+8\beta) + 12\lambda'^2(3+8\beta)\right\}\\
&+ \frac{1}{r^3}\left\{-32\nu'\gamma\right\}\\
&+ \frac{1}{r^4}\left\{-16(5+14\beta)\right\}\Big]\\
&+ e^{-\lambda}\left[\frac{4}{r^4}(1+3\beta)\right] + \left[\frac{\alpha}{r^4}\right] = 0.
\end{aligned}$$

A2.23

$$
\begin{aligned}
G^4_4 = \frac{e^{-2\lambda}}{16} \Big[& \alpha \{ 16\nu^{iv} + 16\nu'''\nu' - 4\nu''\nu'^2 + 12\nu''^2 - \nu'^4 - 48\nu'''\lambda' \\
& \quad - 32\nu''\lambda'' + 44\nu''\lambda'^2 - 36\nu''\nu'\lambda' - 8\nu'\lambda''' \\
& \quad + 28\nu'\lambda''\lambda' - 12\nu'\lambda'^3 - 8\nu'^2\lambda'' + 11\nu'^2\lambda'^2 \\
& \quad + 2\nu'^3\lambda' \} \\
& + \frac{1}{r} \{ 64\nu'''\alpha + 8\nu''\nu'(5+4\beta) - 4\nu'^3\gamma - 8\nu''\lambda'(15+32\beta) \\
& \quad - 8\nu'\lambda''(5+12\beta) + 4\nu'\lambda'^2(13+32\beta) - 8\nu'^2\lambda'(3+2\beta) \\
& \quad - 16\lambda'''\gamma + 56\lambda''\lambda'\gamma - 24\lambda'^3\gamma \} \\
& + \frac{1}{r^2} \{ -4\nu'^2(1+8\beta) + 8\nu'\lambda'(3+4\beta) - 16\lambda''\gamma + 4\lambda'^2(7+24\beta) \} \\
& + \frac{1}{r^3} \{ 32\lambda'\gamma \} \\
& + \frac{1}{r^4} \{ 16(3+10\beta) \} \Big] \\
+ e^{-\lambda} & \left[\frac{-4}{r^4}(1+3\beta) \right] + \left[\frac{\alpha}{r^4} \right] \\
= 0 & .
\end{aligned}
$$

A2.24

$$
\begin{aligned}
G^1_1 = \frac{e^{-2\lambda}}{4} \Big[& \frac{1}{r} (\nu'^3 + 2\nu'\lambda'' - 2\nu'\lambda'^2 + \nu'^2\lambda') \\
& + \frac{1}{r^2} (4\nu'' - 4\lambda'' - \nu'^2 + \lambda'^2) \\
& + \frac{1}{r^3} (8\nu' - 4\lambda') \Big] \\
+ e^{-\lambda} & \left[\frac{1}{r^3} (-\lambda') \right] \\
- \Big(& \frac{\nu'}{2} + \frac{1}{r} \Big) e^{-\lambda} \frac{dR}{dr} - \frac{1}{4} R^2 - \frac{1}{r} \Big(\nu' - \frac{\lambda'}{2} \Big) R e^{-\lambda} + \frac{R}{r^2} \\
+ \beta & \Big\{ -\Big(\nu' + \frac{4}{r}\Big) e^{-\lambda} \frac{dR}{dr} - \frac{1}{2} R^2 - \frac{2}{r} \Big(\nu' + \frac{1}{r} \Big) R e^{-\lambda} + \frac{2R}{r^2} \Big\} \\
= 0 & .
\end{aligned}
$$

A2.25

A3

A3.2

Lemma

The value that may be taken by the constant m in the series expansions (3.9') for the functions e^{λ}, e^{ν} is restricted to unity.

The constant b_0 in the expansion for e^{ν} may be set equal to unity by a transformation of the time coordinate. It is clear from the field equations that the constant a_0 in the expansion for e^{λ} must be unity. Then, from (3.9') we assume that e^{λ}, e^{ν} may be expanded:

$$e^{\lambda} = 1 + \frac{a_1}{r^m} + \frac{a_2}{r^{2m}} + \ldots \quad ,$$
$$e^{\nu} = 1 + \frac{b_1}{r^m} + \frac{b_2}{r^{2m}} + \ldots \quad , \qquad \text{A3.1}$$

for some constants a_1, a_2, ..., b_1, b_2, ... and for some positive number m such that at least one of a_1, b_1 is non-zero. We shall see that, according to the field equations (2.13), this assumption has validity only when m has the value unity.

The series expansions (A3.1) are to be subjected to the field equations (2.13). In order to do this the necessary terms in the expansions for the derivatives λ', λ'', ν', ν'' and for the curvature invariant are obtained from (A3.1). These expressions are then substituted into the determinate system of equations (2.14), (A2.25) and the coefficients of successive powers of $1/r$ are equated to zero.

It is necessary to retain only a very few terms in the various expansions. The notation $\sim$ is employed to indicate that higher order, non-contributing terms have been omitted. Then, from (A3.1):

$$e^{\lambda} \sim 1 + \frac{a_1}{r^m} \quad , \quad e^{\nu} \sim 1 + \frac{b_1}{r^m} \quad ,$$
$$\lambda' \sim -\frac{a_1 m}{r^{m+1}} \quad , \quad \nu' \sim -\frac{b_1 m}{r^{m+1}} \quad , \qquad \text{A3.2}$$
$$\lambda'' \sim \frac{a_1 m(m+1)}{r^{m+2}} \quad , \quad \nu'' \sim \frac{b_1 m(m+1)}{r^{m+2}} \quad .$$

From (A2.22) the curvature invariant is given by:

$$R \sim \frac{b_1 m(1-m)}{r^{m+2}} \quad . \tag{A3.3}$$

Substituting from (A3.2) and (A3.3) into the expression on the left hand side of (2.14) and equating the resulting coefficient of $r^{-(m+4)}$ to zero we obtain:

$$b_1 m(m+2)(1-m)(m+1) = 0 \quad . \tag{A3.4}$$

Since m is positive, (A3.4) allows the two possibilities: either $b_1 = 0$ or $m = 1$. Assuming now that $m \neq 1$ and proceeding in a similar manner for the equation (A2.25) we have:

$$a_1 m(m-1) = 0 \ ,$$

which is a contradiction unless $a_1 = 0$.

There is thus no positive value for m, other than unity, such that at least one of a_1 , b_1 is non-zero. This is the required result.

A3.3

Three Lemmas

For the sake of clarity and with no loss in generality the constant d_o in (3.14) is throughout this section set equal to unity corresponding to a transformation of the time coordinate.

The proceedure to be followed is the same in each lemma, and follows that of the lemma of the preceding section, A3.2. The series expansions assumed in each lemma are to be subjected to the field equations (2.13). In order to do this the necessary terms in the expansions for the derivatives λ' , λ'' , ν' , ν'' and for the curvature invariant are obtained from the assumed series expansions. These expressions are then substituted into the determinate system of equations (2.14), (A2.25) and the coefficients of successive powers of r are equated to zero.

As before, the notation $\sim$ is employed to indicate that terms in the expansions that will make no contribution have been omitted.

Lemma I

The values that may be taken by the constant A in the series expansions (3.14) of the functions e^{λ}, e^{ν} are restricted to some or all of the numbers 0, 2, - 2.

For small r, e^{λ}, e^{ν} may be expanded, from (3.14):

$$\begin{aligned} e^{\lambda} &= 1 + a_1 r + a_2 r^2 + \dots , \\ e^{\nu} &= r^{A}\left(1 + c_1 r + c_2 r^2 + \dots\right) ; \end{aligned} \tag{A3.5}$$

for some constants A, a_1, a_2, ..., c_1, c_2, ...

We need only calculate the leading term in the expansion for R. From (A2.22) this is found to be:

$$R \sim \frac{1}{r^2}\left(-\tfrac{1}{2}A^2 - A\right) .$$

Substituting into the left hand side of (2.14), the condition on A is obtained:

$$A(A+2)(A-2) = 0 . \tag{A3.6}$$

Then, from (A3.6), A can take only the values 0, 2, - 2.

Lemma II

The values that may be taken by the constant A in the series expansions (3.14) of the functions e^{λ}, e^{ν} are restricted to the single value 0.

The cases A = 2, A = - 2, will be considered separately.

Case (i) A = 2

From (3.14), with A = 2, e^{λ} and e^{ν} are assumed to have expansions:

$$e^{\lambda} = 1 + a_1 r + a_2 r^2 + \dots ,$$
$$e^{\nu} = r^2(1 + c_1 r + c_2 r^2 + \dots), \qquad \text{A3.7}$$

for some constants a_1, a_2, ..., c_1, c_2, ...

A preliminary investigation indicates that the coefficients a_1, c_1 in (A3.7) must vanish. In view of this the series expansions (A3.7) are rewritten

$$e^{\lambda} = 1 + a_n r^n + a_{n+1} r^{n+1} + \dots , \qquad \text{A3.8}$$
$$e^{\nu} = r^2(1 + c_n r^n + c_{n+1} r^{n+1} + \dots),$$

for some constants a_n, a_{n+1}, ..., c_n, c_{n+1}, ... and for some integer n. It is assumed that there exists n such that at least one of the leading pair of coefficients a_n, c_n in (A3.8) is non-vanishing. We shall see that the equations (2.14), (A2.25) are inconsistent under this assumption.

Retaining only necessary terms in the expansions near $r = 0$, we obtain, from (A3.8):

$$e^{\lambda} \sim 1 + a_n r^n \quad , \quad e^{\nu} \sim r^2(1 + c_n r^n) \quad ,$$
$$\lambda' \sim n a_n r^{n-1} \quad , \quad \nu' \sim \frac{2}{r} + n c_n r^{n-1} \quad , \qquad \text{A3.9}$$
$$\lambda'' \sim n(n-1) a_n r^{n-2} \quad , \quad \nu'' \sim -\frac{2}{r^2} + n(n-1) c_n r^{n-2}.$$

Then from (A2.22):

$$R \sim -\frac{4}{r^2} + r^{n-2}\left\{3(n+2)a_n - n(n+3)c_n\right\}. \qquad \text{A3.10}$$

Substituting from (A3.9), (A3.10) into the expression on the left hand side of (2.14) and equating the resulting coefficient of r^{n-4} to zero (that of r^{-4} vanishes identically because A = 2):

$$n\{(-n^3 - n^2 + 6n + 4)c_n + (3n^2 - 16)a_n\} = 0. \qquad \text{A3.11}$$

Proceeding in a similar fashion for the equation (A2.25) and equating the resulting coefficients of r^{-4}, r^{n-4} to zero we obtain, respectively:

$$\beta = -13/40 \,, \qquad \text{A3.12}$$

$$c_n(n^3 + n^2 - 6n) + a_n(-3n^2 + 16) = 0, \qquad \text{A3.13}$$

where, in (A3.13), we have substituted for the value of β from the condition (A3.12).

Since n cannot be zero, (A3.11) and (A3.13) are contradictory unless $c_n = 0$ and either $3n^2 = 16$ or $a_n = 0$. Since, by hypothesis, n is an integer,

$$a_n = 0 = c_n \,.$$

Then there is no value of n for which at least one of a_n, c_n is non-vanishing, which is the required result.

It should be noted that although the fact that all the a_n and c_n vanish suggests that

$$e^{\lambda} = 1, \quad e^{\nu} = r^2$$

constitutes an exact solution, this is not the case. For it was mentioned in section 2.4 that when $d\nu/dr = 2/r$, i.e., e^{ν} = constant $\times r^2$, it is not sufficient to solve the pair (2.13); the pair (2.11) must be used and these inconsistent under the substitution $d\nu/dr = 2/r$.

We may thus dismiss the value A = 2, as it is forbidden by the field equations.

Case (ii) A = - 2

From (3.14), with A = - 2, e^{λ} and e^{ν} are assumed to have expansions:

$$e^{\lambda} = 1 + a_1 r + a_2 r^2 + \dots \quad ,$$
$$e^{\nu} = \frac{1}{r^2}\left(1 + c_1 r + c_2 r^2 + \dots\right) , \qquad \text{A3.14}$$

for some constants a_1 , a_2 , ... , c_1 , c_2 , ...
We shall show that the equations (2.14), (A2.25) are inconsistent under this assumption.

Retaining only necessary terms in the expansions near $r = 0$, we obtain from (A3.14)

$$\begin{aligned} e^{\lambda} &\sim 1 + a_1 r \ , & e^{\nu} &\sim r^{-2}(1 + c_1 r) \ , \\ \lambda' &\sim a_1 \ , & \nu' &\sim -\frac{2}{r} + c_1 \ , \\ \lambda'' &\sim 0 \ , & \nu'' &\sim \frac{2}{r^2} \ , \end{aligned} \qquad \text{A3.15}$$

Then, from (A2.22):

$$R \sim \frac{0}{r^2} + \frac{3a_1}{r} + \text{Constant} \ . \qquad \text{A3.16}$$

Substituting from (A3.15), (A3.16) into the expression on the left hand side of (2.14) and equating the coefficient of $1/r^3$ to zero we have

$$a_1 = 0 \ .$$

Then, from (A3.16),

$$R \sim \text{Constant} \ . \qquad \text{A3.17}$$

Substituting from (A3.15) into the expression on the left hand side of equation (A2.25) using (A3.17) the contradiction

$$-20 = 0$$

is obtained. Then the field equations do not allow the value A = - 2. The lemma is thus proved.

Lemma III

The series expansions for the unknown functions e^{λ}, e^{ν} of the metric (2.8) for small values of the radial coordinate r include only even powers of r.

From (3.14), with A = 0, e^{λ}, e^{ν} are assumed to have expansions which may be written as the sum of an expansion in terms of even powers of r and an expansion in terms of odd powers of r:

$$e^{\lambda} = \sum_{m=0} a_{2m} r^{2m} + r^{n} \sum_{m=0} u_{2m} r^{2m} ,$$
$$e^{\nu} = \sum_{m=0} c_{2m} r^{2m} + r^{n} \sum_{m=0} w_{2m} r^{2m} , \qquad \text{A3.18}$$

in which n is an odd, positive integer, m takes all non-negative integer values, a_{2m}, u_{2m}, c_{2m}, w_{2m}, are constants with $a_0 = 1$ (according to (3.12)) and $c_0 = 1$ (by a transformation of the time coordinate). It is assumed that there exists n such that at least one of u_0, w_0 is non-vanishing. Our aim is to show that this assumption is contradicted, which will prove the result.

It is first shown that the only possible value of n such that the last mentioned assumption may not be contradicted is $n = 1$. We shall equate to zero the coefficient of the lowest odd power of r appearing on the left hand side of each of the equations (2.14), (A2.25). Retaining only necessary terms in the expansions, we obtain, from (A3.18):

$$e^{\lambda} \sim 1 + u_0 r^{n} , \qquad e^{\nu} \sim 1 + w_0 r^{n} ,$$
$$\lambda' \sim n u_0 r^{n-1} , \qquad \nu' \sim n w_0 r^{n-1} , \qquad \text{A3.19}$$
$$\lambda'' \sim n(n-1) u_0 r^{n-2}, \qquad \nu'' \sim n(n-1) w_0 r^{n-2}.$$

Then, from (A2.22):

$$R \sim \frac{Q}{r^2} + r^{n-2}(n+1)(2u_0 - n\omega_0) + \text{Constant} \quad . \qquad \text{A3.20}$$

Substituting from (A3.19), (A3.20) into the expression on the left hand side of (2.14) and equating the resulting coefficient of r^{n-4} to zero, we obtain:

$$(n-2)(n-1)(n+1)(2u_0 - n\omega_0) = 0 \, , \qquad \text{A3.21}$$

and from (A2.25) we obtain similarly:

$$(n+1)\left\{n(-u_0+w_0) + (2u_0 - nw_0)\left[(3-n) - 4\beta(n-2)\right]\right\} = 0 \, . \qquad \text{A3.22}$$

From (A3.21), we must consider four cases, (i),...,(iv).

Cases (i), (ii) $n = 2$, $n = -1$.

These may be dismissed immediately as contrary to the assumed nature of n .

Case (iii) $2u_0 = nw_0$.

Since $n \neq -1$, the condition (A3.22) may be replaced by

$$n(-u_0+w_0) + (2u_0 - nw_0)\left[(3-n) - 4\beta(n-2)\right] = 0 \, . \qquad \text{A3.23}$$

Substituting $2u_0 = nw_0$ into (A3.23) yields, since $n \neq 0$,

$$u_0 = w_0 \, ,$$

which implies again that $n = 2$, which we have already dismissed.

Case (iv) $n = 1$.

Substituting this value of n into (A3.23) gives:

$$u_0(3 + 8\beta) + w_0(-1 - 4\beta) = 0 \, . \qquad \text{A3.24}$$

This case requires a more detailed examination. The expansions (A3.19) are replaced by

$$e^{\lambda} \sim 1 + u_0 r + u_2 r^2 \;, \quad e^{\nu} \sim 1 + w_0 r + w_2 r^2 \;,$$
$$\lambda' \sim u_0 + r(2u_2 - u_0^2) \;, \quad \nu' \sim w_0 + r(2w_2 - w_0^2) \;, \qquad \text{A3.25}$$
$$\lambda'' \sim 2u_2 - u_0^2 \;, \quad \nu'' \sim 2w_2 - w_0^2 \;.$$

Then, from (A2.22)

$$R \sim \frac{1}{r}(4u_0 - 2w_0) + (6u_2 - 6w_2 - 6u_0^2 + \frac{5}{2}u_0 w_0 + \frac{5}{2}w_0^2) \;. \qquad \text{A3.26}$$

Substituting into the left hand side of (2.14) from (A3.25), (A3.26) and equating the coefficient of r^{-2} to zero (that of r^{-3} now vanishes identically, of course) we obtain:

$$(w_0 - u_0)(-2u_0 + w_0) = 0 \;. \qquad \text{A3.27}$$

Using the equation (A2.25) likewise (though it should be noted that for ease of manipulation the equivalent equation (A2.23) is to be preferred):

$$-23u_0^2 + 6u_0 w_0 - 3w_0^2 + 32\beta(u_0 w_0 - 2u_0^2) = 0 \qquad \text{A3.28}$$

(Equating the coefficient of r^{-3} to zero gives (A3.24) again). Equation (A3.27) yields the possibilities:

(a) $w_0 = 2u_0$,

(b) $w_0 = u_0$.

We consider these two cases separately.

Case (a). Substituting $w_0 = 2u_0$ into (A3.24) we obtain

$$u_0 = 0 \;.$$

Then w_0 also is equal to zero and we have the required contradiction.

Case (b). Substituting $w_0 = u_0$ into (A3.24) we obtain:

$$u_0(1 + 2\beta) = 0 \;, \qquad \text{A3.29}$$

while (A3.28) gives:

$$u_0^2(5+8\beta)=0 \ . \qquad \text{A3.30}$$

Clearly, (A3.29) and (A3.30) cannot hold simultaneously unless $u_0 = 0 (= w_0)$. Again we have the required contradiction. The expansions (A3.18) are thus in terms of even powers of r only.

Transformation of the Metric (2.8) from Pseudo-Polar Coordinates to Pseudo-Cartesian Coordinates

The transformation from pseudo-polar coordinates r, θ, ϕ, t to pseudo-Cartesian coordinates x^1, x^2, x^3, x^4, is accomplished by writing[48]

$$\begin{aligned} x^1 &= r\sin\theta\cos\phi , \\ x^2 &= r\sin\theta\sin\phi , \\ x^3 &= r\cos\theta , \\ x^4 &= t . \end{aligned} \qquad \text{A3.31}$$

Then

$$\begin{aligned} x^i x^i &= r^2 , \\ x^i dx^i &= r\,dr , \\ dx^i dx^i &= dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \end{aligned}$$

and (2.8) becomes

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta ,$$

where

$$\begin{aligned} g_{ij} &= (1-e^\lambda)\frac{x^i x^j}{r^2} - \delta_{ij} , \\ g_{4i} &= 0 , \\ g_{44} &= e^\nu . \end{aligned} \qquad \text{A3.32}$$

Then the determinant of the metric tensor, and the conjugate tensor to $g_{\mu\nu}$ are

$$g = -e^{\lambda+\nu},$$
$$g^{ij} = (1-e^{-\lambda})\frac{x^i x^j}{r^2} - \delta^{ij},$$
$$g^{4i} = 0,$$
$$g^{44} = e^{-\nu}. \qquad \text{A3.33}$$

A3.4

Lemma

There are no solutions of the field equations (2.13) that are asymptotically of the form of the Einstein Universe.

The constant V_0 in the series expansions (3.18) is set equal to unity, as usual, and the method employed is that of previous lemmas.

From (3.18), with $V_0 = 1$, the necessary terms in the series expansions of $e^{-\lambda}$, e^{ν} and their derivatives are:

$$e^{-\lambda} \sim u_0 r^2 + u_1 r, \qquad e^{\nu} \sim 1 + \frac{V_1}{r},$$
$$\lambda' \sim -\frac{2}{r} + \frac{1}{r^2}\left(\frac{u_1}{u_0}\right), \qquad \nu' \sim \frac{-V_1}{r^2},$$
$$\lambda'' \sim \frac{2}{r^2} + \frac{1}{r^3}\left(\frac{-2u_1}{u_0}\right), \qquad \nu'' \sim \frac{2V_1}{r^3}. \qquad \text{A3.34}$$

Substituting into (A2.25) from (A3.34) and equating the coefficient of the highest power of r (i.e., that of r^0) to zero gives:

$$u_0^2(1+3\beta) = 0. \qquad \text{A3.35}$$

Since the value $\beta = -\frac{1}{3}$ is excluded, from (A3.35) we have $u_0 = 0$, which proves our result.

Now, supposing that $u_0 = 0$ but $u_1 \neq 0$ in (3.18) we have, in place of (A3.34):

$$e^{-\lambda} \sim u_1 r, \qquad e^{\nu} \sim 1 + \frac{V_1}{r},$$
$$\lambda' \sim -\frac{1}{r}, \qquad \nu' \sim -\frac{V_1}{r^2},$$
$$\lambda'' \sim \frac{1}{r^2}, \qquad \nu'' \sim \frac{2V_1}{r^3}. \qquad \text{A3.36}$$

Substituting into (A2.22) from (A3.36) we obtain:

$$R \sim -\frac{4u_1}{r}. \tag{A3.37}$$

Then substituting into the left hand side of (2.14) from (A3.36) and (A3.37) and equating the coefficient of the highest power of r (that of r^{-3}) to zero we have:

$$u_1 = 0.$$

We thus arrive at the asymptotically Minkowskian form.

A3.5

Lemma

The constant A_3 in the expression (3.20) for the curvature invariant vanishes.

In order to obtain this result it is necessary to retain terms which involve a_i, b_i for $i \leq 5$ in the assumed series expansions (3.19). However, manipulation is facilitated if we initially retain terms which involve a_i, b_i for $i \leq 3$ in order to obtain simplifying relationships on the coefficients.

From (3.19) the necessary terms in the series expansions of $e^{-\lambda}$, e^{ν} and their derivatives are

$$\begin{aligned}
e^{-\lambda} &\sim b_0 r^2\left(1+\frac{b_1}{r}+\frac{b_2}{r^2}+\frac{b_3}{r^3}\right), \\
\lambda' &\sim -\frac{2}{r}+\frac{b_1}{r^2}+\frac{(2b_2-b_1^2)}{r^3}+\frac{(3b_3-3b_1b_2+b_1^3)}{r^4}, \\
\lambda'' &\sim \frac{2}{r^2}-\frac{2b_1}{r^3}-3\frac{(2b_2-b_1^2)}{r^4}-4\frac{(3b_3-3b_1b_2+b_1^3)}{r^5}, \\
e^{\nu} &\sim a_0 r^2\left(1+\frac{a_1}{r}+\frac{a_2}{r^2}+\frac{a_3}{r^3}\right), \\
\nu' &\sim \frac{2}{r}-\frac{a_1}{r^2}+\frac{(-2a_2+a_1^2)}{r^3}+\frac{(-3a_3+3a_1a_2-a_1^3)}{r^4}, \\
\nu'' &\sim -\frac{2}{r^2}+\frac{2a_1}{r^3}-3\frac{(-2a_2+a_1^2)}{r^4}-4\frac{(-3a_3+3a_1a_2-a_1^3)}{r^5}.
\end{aligned} \tag{A3.38}$$

Substituting from (A3.38) into the expression (A2.22) for R , we obtain:

$$R \sim b_0\Big\{-12 + \frac{1}{r}(3a_1 - 9b_1) + \frac{1}{r^2}\Big(4a_2 - 6b_2 - \frac{5}{2}a_1^2 + \frac{5}{2}a_1b_1 + \frac{2}{b_0}\Big)$$
$$+ \frac{1}{r^3}\big(3a_3 - 3b_3 - 5a_1a_2 + 2a_1b_2 + 3a_2b_1 + 2a_1^3 - 2a_1^2b_1\big)\Big\} . \qquad \text{A3.39}$$

Then the coefficient of interest, A_3 , is

$$A_3 = b_0\,(3a_3 - 3b_3 - 5a_1a_2 + 2a_1b_2 + 3a_2b_1 + 2a_1^3 - 2a_1^2b_1) . \qquad \text{A3.40}$$

The first and second derivatives of R are obtained from (A3.39) and, together with the expression (3.21) for the combination $(\frac{1}{2}\frac{d\nu}{dr} - \frac{1}{2}\frac{d\lambda}{dr} + \frac{2}{r})$, substituted into the field equation (2.14). Equating the coefficients of $1/r^3$, $1/r^4$, and $1/r^5$ successively to zero, we obtain equations (a), (b) and (c):

(a) $a_1 - 3b_1 = 0$,

(b) $4a_2 - 6b_2 - \frac{5}{2}a_1^2 + \frac{5}{2}a_1b_1 + \frac{2}{b_0} = 0$,

(c) $0 = 0$.

Substituting into the second field equation (A2.25) from (A3.38), (A3.39) and equating the coefficients of $1/r$, $1/r^2$, $1/r^3$ successively to zero, we obtain equations (d), (e) and (f):

(d) $(a_1 - 3b_1)(11 + 24\beta) = 0$,

which is satisfied, by (a),

(e) $-10(8a_2 - 12b_2) + 43a_1^2 + 9b_1^2 - 48a_1b_1 - 40/b_0$

$+\beta\{-24(8a_2 - 12b_2) + 96a_1^2 - 96a_1b_1 - 96/b_0\} = 0$,

which, on substitution from (b), yields

$$a_1^2(1 + 3\beta) = 0 .$$

Since $\beta \neq -\frac{1}{3}$, $a_1 = b_1 = 0$. Then, from (b),

$$4a_2 - 6b_2 = -2/b_0 . \qquad \text{A3.41}$$

(f) $0 = 0$.

Now, with the simplification that $a_1 = b_1 = 0$, we repeat the procedure retaining terms including a_4 and b_4 . From (2.14), equating the coefficient of $1/r^6$ to zero, we obtain

$$a_2(b_2 - a_2) = 0 \quad ,$$

which, since $a_2 \neq 0$, and by (A3.41), gives

$$a_2 = b_2 = 1/b_0 \ . \qquad \text{A3.42}$$

From (A2.25), equating the coefficient of $1/r^4$ to zero:

$$(-4a_4 + 3b_4)(1 + 3\beta) = 0 \ ,$$

so that, since $\beta \neq -1/3$,

$$b_4 = \frac{4}{3}a_4 \ .$$

With the additional simplification (A3.42), retaining terms including a_5 and b_5 , we obtain from (2.14):

$$10(-5a_5 + 3b_5) + 8a_2(a_3 - b_3) = 0 \qquad \text{A3.43}$$

while (A2.25) gives:

$$(1+3\beta)\{10(-5a_5 + 3b_5) + 23a_2(a_3 - b_3)\} = 0 \ . \qquad \text{A3.44}$$

Since $\beta \neq -1/3$ and $a_2 \neq 0$, (A3.44) and (A3.43) are consistent only if

$$a_3 = b_3 \ .$$

Then, from (A3.40), $A_3 = 0$ as required.

A7

A7.3

Formulae for the Spherically Symmetric Metric (7.15)

The formulae (A7.1)⟶(A7.4) corresponding to the metric form (7.15) may be obtained from the general formulae for spherical symmetry given by Synge.[41] Observe, however, that the conventions used here are in accordance with those specified in "Notation". Only surviving components of the various quantities are listed.

Components of the metric tensor:

$$g_{11} = -e^{\lambda}\,,\; g_{22} = -e^{\frac{1}{2}(\lambda-\nu)}\,,\; g_{33} = -e^{\frac{1}{2}(\lambda-\nu)}\sin^2\theta\,,\; g_{44} = e^{\nu}\,,$$

$$g^{11} = -e^{-\lambda}\,,\; g^{22} = -e^{-\frac{1}{2}(\lambda-\nu)}\,,\; g^{33} = \frac{-e^{-\frac{1}{2}(\lambda-\nu)}}{\sin^2\theta}\,,\; g^{44} = e^{-\nu}. \qquad \text{A7.1}$$

Christoffel symbols of the second kind:

$$\Gamma^1_{11} = \tfrac{1}{2}\lambda'\,,\; \Gamma^1_{22} = \tfrac{1}{4}(-\lambda'+\nu')e^{-(\frac{\lambda}{2}+\frac{\nu}{2})}\,,\; \Gamma^1_{33} = \sin^2\theta\,\Gamma^1_{22}\,,\; \Gamma^1_{44} = \tfrac{1}{2}\nu' e^{\nu-\lambda}\,,$$
$$\Gamma^2_{12} = \tfrac{1}{4}(\lambda'-\nu')\,,\; \Gamma^2_{33} = -\sin\theta\cos\theta\,,$$
$$\Gamma^3_{13} = \Gamma^2_{12}\,,\; \Gamma^3_{23} = \cot\theta\,,$$
$$\Gamma^4_{14} = \tfrac{1}{2}\nu'. \qquad \text{A7.2}$$

Components of the Ricci Tensor:

$$R_{11} = \tfrac{1}{2}\lambda'' - \tfrac{1}{8}\lambda'^2 + \tfrac{3}{8}\nu'^2 - \tfrac{1}{4}\lambda'\nu'\,,$$
$$R_{22} = -1 + e^{-\frac{1}{2}\nu-\frac{1}{2}\lambda}\left(\tfrac{1}{4}\lambda'' - \tfrac{1}{4}\nu''\right),$$
$$R_{33} = \sin^2\theta\, R_{22}\,, \qquad \text{A7.3}$$
$$R_{44} = e^{\nu-\lambda}\left(-\tfrac{1}{2}\nu''\right).$$

The curvature invariant:

$$R = e^{-\lambda}\left(-\lambda'' + \tfrac{1}{8}\lambda'^2 - \tfrac{3}{8}\nu'^2 + \tfrac{1}{4}\lambda'\nu'\right) + 2e^{\frac{\nu}{2}-\frac{\lambda}{2}}\,. \qquad \text{A7.4}$$

Lanczos' equations for the metric (7.15):

$$\begin{aligned}
G_{11} = \frac{e^{-\lambda}}{16}\Big\{&-6\lambda'''\lambda' + 6\lambda''\lambda'^2 + 3\lambda''^2 - \tfrac{5}{8}\lambda'^4 + 2\lambda'''\nu' - 2\lambda''\nu'' \\
&+6\lambda''\nu'^2 - 4\lambda''\lambda'\nu' + 2\lambda'\nu''' - \tfrac{9}{2}\lambda'\nu'^3 + \tfrac{9}{4}\lambda'^2\nu'^2 \\
&-\tfrac{1}{2}\lambda'^3\nu' - 6\nu'''\nu' + 3\nu''^2 + \tfrac{27}{8}\nu'^4\Big\} \\
&+\frac{e^{\frac{\nu}{2}-\frac{\lambda}{2}}}{4}\left\{-\lambda'^2 + 2\lambda'\nu' - \nu'^2\right\} \\
&+e^{\nu}\{-1\} \\
&+\beta\left\{\lambda' R' - \tfrac{1}{2}e^{\lambda}R^2 - R\left(\lambda'' - \tfrac{1}{4}\lambda'^2 + \tfrac{3}{4}\nu'^2 - \tfrac{1}{2}\lambda'\nu'\right)\right\} \\
&= 0 .
\end{aligned} \qquad \text{A7.5}$$

$$\begin{aligned}
G_{22} = \frac{e^{-\frac{3\lambda}{2}-\frac{\nu}{2}}}{16}\Big\{&-12\lambda^{iv} + 30\lambda'''\lambda' - 23\lambda''\lambda'^2 + 15\lambda''^2 + \tfrac{15}{8}\lambda'^4 - 2\lambda'''\nu' \\
&-2\lambda''\nu'' - 2\lambda''\lambda'\nu' + 5\lambda''\nu'^2 - 6\lambda'\nu''' + 22\lambda'\nu''\nu' \\
&+\tfrac{9}{2}\lambda'\nu'^3 - \lambda'^2\nu'' - \tfrac{31}{4}\lambda'^2\nu'^2 + \tfrac{5}{2}\lambda'^3\nu' + 4\nu^{iv} \\
&-6\nu'''\nu' - 9\nu''\nu'^2 - 9\nu''^2 - \tfrac{9}{8}\nu'^4\Big\} \\
&+\frac{e^{-\lambda}}{2}\left\{-2\lambda'' + \lambda'^2 - 2\lambda'\nu' + 2\nu'' + \nu'^2\right\} \\
&+e^{\frac{\nu}{2}-\frac{\lambda}{2}}\{1\} \\
&+\beta\Big\{-\frac{R'}{2}(\lambda'-\nu')e^{-\frac{\lambda}{2}-\frac{\nu}{2}} - \frac{1}{2}e^{\frac{\lambda}{2}-\frac{\nu}{2}}R^2 \\
&\qquad -2R\left[-1 + \frac{e^{-\frac{1}{2}\nu-\frac{1}{2}\lambda}}{4}(\lambda''-\nu'')\right]\Big\} \\
&= 0 .
\end{aligned} \qquad \text{A7.6}$$

A7.4

Solution of the Differential Equation (7.21):

$$\nu'' e^{-\nu} = (2-A) - Br , \qquad (B \neq 0) .$$

Putting $x = \frac{1}{2}[(2 - A) - Br]$, $e^{\frac{-\nu}{2}} = -2y/B$, (7.21) becomes

$$yy'' - y'^2 + x = 0 , \qquad (' \equiv d/dx) . \qquad \text{A7.7}$$

This has the particular solution y_1 :

$$y_1 = \sqrt{\frac{2}{3}}\, x^{\frac{3}{2}} \quad .$$

Write $y = y_1(x) \times v(x)$ and put $x = e^t$ in (A7.7) to obtain

$$vv'' - vv' - v'^2 - \frac{3}{2}(v^2 - 1) = 0 \quad , \qquad ('\equiv d/dt) \quad . \qquad \text{A7.8}$$

Now write $v^2 = z$ and, since $z \neq 1$, substitute $dz/dt = p$ to reduce the order of the differential equation (A7.8):

$$pp' = \frac{1}{z}p^2 + p - 3(1-z) \quad , \qquad ('\equiv d/dz) \quad . \qquad \text{A7.9}$$

Substituting $q(z) = \sqrt{2}z/p$ into (A7.9) we obtain

$$q' = \frac{-1}{\sqrt{2}z}q^2 + \frac{3(1-z)}{2z^2}q^3 \quad . \qquad \text{A7.10}$$

Making the change of variable $z = e^{-\sqrt{2}\,\xi}$, (A7.10) gives

$$\frac{dq}{d\xi} = q^2 - \frac{3}{\sqrt{2}}\left(e^{\sqrt{2}\,\xi} - 1\right)q^3 \quad . \qquad \text{A7.11}$$

Write $\xi'(t) = -1/tq(\xi)$ and $\zeta = \sqrt{2}\,\xi$ to obtain from (A7.11):

$$t^2\frac{d^2\zeta}{dt^2} - \frac{3}{\sqrt{2}}\left(e^{\zeta} - 1\right) = 0 \quad . \qquad \text{A7.12}$$

One further substitution

$$u(s) = \zeta(t) + \ln\left(\frac{-4\sqrt{2}}{3bt^{3/2}}\right), \; t = s^{2\sqrt{2}/3} \qquad \text{, some } b \neq 0,$$

casts (A7.12) into the form

$$su'' + au' + bse^u = 0 \, , \quad ('\equiv d/ds) \, , \qquad \text{A7.13}$$

where

$$a = 1 - \frac{2\sqrt{2}}{3} \quad . \qquad \text{A7.14}$$

Now Kamke[68] has discussed differential equations of the form (A7.13) and makes the observation that when the constant $\alpha \neq 1$ or 0 the behaviour of the solution of this differential equation in the neighbourhood of $S = 0$ has been described using a series expansion. Thus there appears to be no general solution of (A7.13) (and hence, no general solution of (7.21)) in terms of known functions. We must be content with the particular solution y_1 , which leads to

$$e^{-\nu} = \frac{1}{3B^2}\left[(2-A) - Br\right]^3 \quad .$$

NOTES AND REFERENCES

1. Rainich, G.Y.: Trans. Am. Math. Soc. 27, 106 (1925).
2. Misner, Charles W. and Wheeler, John A.: Ann. Phys. 2, 525 (1957). This description of "already unified" theory includes an account of the history of the idea of a purely geometrical description of Nature.
3. Das, A. and Coffman, C.V.: J. Math. Phys. 8, 1720 (1967). Das, A.: J. Math. Phys. 4, 45 (1963).
4. Lanczos, Cornelius: Rev. Mod. Phys. 29, 337 (1957).
5. Lanczos, C.: J. Math. Phys. 10, 1057 (1969).
6. Buchdahl, H.A.: Nuovo Cimento, 23, 141 (1962).
7. Synge, J.L.: "What is Einstein's theory of Gravitation?" in "Perspectives in Geometry and Relativity", ed. Hoffman (Indiana University Press, Bloomington and London, 1966).
8. Einstein, A. and Infeld, L.: Canad. Journ. Math., 1, 209 (1949).
9. Mach, E.: "Die Mechanik in ihrer Entwickelung historisch kritisch dargestellt (Leipzig, 1883); English translation (6th. edition) "The Science of Mechanics" (Open Court Co., La Salle, Illinois, 1960), p. 285.
10. A comprehensive list of different formulations of Mach's principle is given by: Heller, M.: Acta Physica Polonica, B1, 131 (1970).
11. Einstein, A: Sitzungsber. Preuss. Akad. Wiss., 1, 142 (1917); Ann. Phys., 55, 241 (1918).
12. Wheeler, John Archibald: "Mach's Principle as Boundary Condition for Einstein's Field Equations" in "Lectures in Theoretical Physics", Vol.V, ed. Brittin, Downs and Downs (Interscience, New York-London, 1963).
13. Gödel, K.: Rev. Mod. Phys. 21, 447 (1949).

14. Adler, R.A.; Bazin, M.; Schiffer, M.: "Introduction to General Relativity" (McGraw-Hill, New York, 1965), p. 377.

15. Taub, A.H.: Ann. of Math. 53, 472 (1951).

16. McVittie, G.C.: "General Relativity and Cosmology" (2nd. edition), (Chapman and Hall, London, 1965), p. 77.

17. Ozsváth, I. and Shücking, E.: "An anti-Mach-metric" in "Recent Developments in General Relativity" (Pergamon Press, London, 1962), p. 339.

18. See, for example, reference 12, p. 539. This is also what is meant by Synge, J.L.: loc. cit. (7), p. 9.

19. A critical account of the theory of R.H. Dicke and C. Brans, and of J.A. Wheeler's incorporation of Mach's principle into general relativity is given by: Heller, M.: Acta Physica Polonica, B1, 123 (1970). This author suggests (loc. cit. (10)) that it would be necessary to change some of the concepts of differential geometry in order to incorporate Mach's ideas into theoretical physics.

20. Lynden-Bell, D.: Mon. Not. R. astr. Soc., 135, 413 (1967).

21. McCrea, W.H.: Nature, 230, 95 (1971).

22. Schrödinger, E.: "Space-Time Structure" (Cambridge University Press, Cambridge, 1960), p. 99.

23. Eddington, A.S.: "Space Time and Gravitation" (Cambridge University Press, Cambridge 1921), p. 164.

24. Einstein, A.: Herbert Spencer lecture, "On the Method of Theoretical Physics", delivered at Oxford, June 10, 1933. (Clarendon Press, Oxford, 1933).

25. The programme that follows was (in part) specified by Lanczos in 1938 (see reference 27), though his reason for studying quadratic Lagrangians was purely his belief in the necessity for gauge invariance.

26. Lanczos, Cornelius: Phys. Rev. 61, 713 (1942), p. 714.

27. Lanczos, Cornelius: Ann. of Math. 39, 842 (1938).

28. Weyl, H.: Math. Z. 2, 384 (1918); Ann. Physik. 59, 101 (1919); Physik. Z. 22, 473 (1921); "Space, Time, Matter" (Methuen, London, 1922).

29. Pauli, W.: Physik. Z. 20, 457 (1919).

30. Eddington, A.S.: Proc. Roy. Soc. Lond. A, 99, 104 (1921); "The Mathematical Theory of Relativity" (Cambridge University Press, Cambridge, 1923), §90.

31. Buchdahl, H.A.: Proc. Edin. Math. Soc. 10, 16 (1953).

32. Gregory, R.C.: Phys. Rev. 72, 72 (1947).

33. Pechlaner, E. and Sexl, R.: Commun. Math. Phys. 2, 165 (1966).

34. Lanczos, Cornelius: loc. cit. (4), p. 343.

35. Derived by Cornelius Lanczos, loc. cit. (4) and in this thesis frequently referred to as "Lanczos' equations". Since the derivation is effectively scattered amongst several papers, and partly in German, this is given in full in the appendix A2.3.

36. Lanczos, Cornelius: loc. cit. (4), p. 341.

37. Tolman, R.C.: "Relativity, Thermodynamics and Cosmology" (Clarendon Press, Oxford, 1934), p. 250.

38. Birkhoff, G.D.: "Relativity and Modern Physics" (Harvard University Press, Cambridge, 1923), p. 253. See also reference 66.

39. Schrödinger, E.: loc. cit. (22), p. 96.

40. Lanczos, C.: loc. cit. (5), p. 1060.

41. Synge, J.L.: "Relativity: The General Theory" (North Holland Publishing Co., Amsterdam, 1960), p. 270. N.B. The first of Synge's formulae (80) (p. 272) is in error and should read $R_{2323} = s^2 r^2 (1-e^{-\alpha})$.

42. The solutions listed are all well known and may be found in standard texts. For example see Adler, R.A.; Bazin, M.; Schiffer, M.: loc. cit. (14).

43. Geroch, Robert: Ann. Phys. 48, 526 (1968).

44. Hawking, S.W. and Penrose, R.: Proc. Roy. Soc. Lond. A, 314, 529 (1970). See also the papers cited in this work.

45. Hoyle, F. and Narlikar, J.V.: Proc. Roy. Soc. Lond. A, 278, 465 (1964).

46. Bondi, H.: "Some Special Solutions of Einstein's Equations" in "Lectures on General Relativity", Volume One, Brandeis Summer Institute in Theoretical Physics 1964, ed. Deser and Ford (Prentice-Hall, Inc., New Jersey, 1965).

47. Belinskii, V.A.; Khalatnikov, I.M.; Lifshitz, E.M.: Adv. in Phys. 19, 525 (1970).

48. Synge, J.L.: loc. cit. (41), p. 267.

49. Calculated from the formulae given by Synge, J.L.: loc. cit. (41).

50. Lanczos, Cornelius: loc. cit. (4), p. 346.

51. Eisenhart, L.P.: "Riemannian Geometry" (Princeton University Press, Princeton, New Jersey, 1926), p. 183.

52. Petrov, A.Z.: "Einstein Spaces" (Pergamon Press, Oxford, 1969), chapter 3.

53. Fuller, Robert W. and Wheeler, John A.: Phys. Rev. 128, 919 (1962).

54. Takeno, Hyoitiro: "On the Imbedding Problem of Spherically Symmetric Space Times" in "Perspectives in Geometry and Relativity", ed. Hoffman (Indiana University Press, Bloomington and London, 1966), p. 353.

55. Fronsdal, C.: Phys. Rev. 116, 778 (1959).

56. For the embedding of the Schwarzschild universe, Fronsdal (loc. cit. (55)) chooses units such that $2m = 1$.

57. Pirani, F.A.E.: Proc. Camb. Phil. Soc. 51, 535 (1955).

58. Littlewood, D.E.: Proc. Camb. Phil. Soc. 49, 90 (1953).

59. Adler, R.A.; Bazin, M.; Schiffer, M.: loc. cit. (14), p. 320.

60. Kasner, E.: Am. J. Math. 43, 130 (1921).

61. Møller, C.: "The Theory of Relativity" (Clarendon Press, Oxford, 1952), p. 348. The discussion of geodesics in space-time VI is based substantially upon Møller's account of geodesics in the Schwarzschild universe.

62. Tolman, R.C.: loc. cit. (37), p. 347.

63. Eisenhart, L.P.: loc. cit. (51), p. 215.

64. Robinson, I.: Bull. Acad. Polon. Sci. VII, 351 (1959).

65. IBM System/360 Scientific Subroutine Package, Version III, Programmer's Manual (Program Number 360A-CM-03X), p. 337.

66. Bonnor, W.B.: "On Birkhoff's Theorem" in "Recent Developments in General Relativity" (Pergamon Press, London, 1962), p. 167.

67. Bonnor, W.B.: loc. cit. (66) p. 169, below formula (9). For our purposes, μ is a function of r only and $\Lambda = 1$.

68. Kamke, E.: "Differentialgleichungen lösungsmethoden und lösungen" (3rd. edition), (Chelsea Publishing Company, New York, 1959), p. 563.

69. Einstein, A.: "Autobiographical Notes" in "Albert Einstein: Philosopher - Scientist", ed. Schilpp (The Library of Living Philosophers, Inc., Evanston, Illinois, 1949), p. 75.