QUANTIZATION OF SOME GENERALLY COVARIANT MODEL FIELD THEORIES

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Doctor of Philosophy

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DECLARATION

I hereby declare that the accompanying thesis is my own composition, that it is based upon research carried out by me, and that no part of it has previously been presented in application for a Higher Degree.

CERTIFICATE

I certify that in October 1966 Kay-Kong Wan was admitted as a Research Student under Ordinance General No.12 in the Department of Theoretical Physics of the University of St. Andrews; that he was admitted in the following year as a candidate for the Degree of Doctor of Philosophy under Ordinance No.16; that he has fulfilled the conditions of Ordinance No.16 and the supplementary Senate regulations; and that he is qualified to submit the following Thesis in application for the Degree of Doctor of Philosophy.

Research Supervisor

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CAREER

I first matriculated in the University of St. Andrews in 1963. I graduated from the University with a first class honours B.Sc. degree in Physics with Theoretical Physics in July, 1966. In October of that year I was admitted by the Senatus Academicus of the University of St. Andrews as a research student. During the first two years of my research, I was supported by a scholarship from the Carnegie Trust for the Universities of Scotland. In October 1968 I was appointed Assistant Lecturer in Theoretical Physics in the University of St. Andrews.

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ABSTRACT

This thesis reports a study of the quantization of generally covariant and nonlinear field theories.

It begins by reviewing some existing general theories in Chapter 2 and Chapter 3. Chapter 2 deals with general classical theories while Chapter 3 examines various quantization schemes. The model field derived from the Lagrangian density

$$\mathcal{L} = \tfrac{1}{4}\, \varepsilon^{\tau\kappa\mu\lambda}\, (A_{\kappa,\tau} - A_{\tau,\kappa})(A_{\mu,\lambda} - A_{\lambda,\mu}),$$

is proposed in Chapter 4 especially for the study of general covariance. It is demonstrated that for this field general covariance alone does not appear to bring in anything physically new. A discussion is given on the differences between general covariance and Lorentz covariance. In subsequent chapters a generally covariant and nonlinear model field, a 4-surface of stationary 4-volume embedded in a 5-dimensional Pseudo-Euclidean space, is investigated. Firstly a manifestly covariant quantization programme is carried out. The model field is then examined in a special coordinate frame for the study of its nonlinearity. Various treatments of the intrinsic nonlinearity are examined starting with conventional perturbation theory in Chapter 6. The usual divergence problem in quantum field theory appears, in particular in the self-energy calculation of a

one-particle state. A new variational method is proposed in Chapter 8 which is able to lead to finite results for one-particle states.

The thesis is concluded with a chapter discussing some general problems involved and a chapter containing suggestions for further work.

CHAPTER I

1.1. Introduction

It is well known that Einstein, who himself played a pioneering role in the development of quantum physics and who was a revolutionary in modern physics, persistently criticized quantum theory. One of the reasons for his attitude towards quantum theory is that he did not see how quantum theory could be made compatible with his theory of General Relativity.[A1] It did not seem possible to him to quantize the gravitational field in the curved space. If this is indeed the case, the conflict between General Relativity and quantum theory is serious. We shall give an example[A2] which will suffice to show this. Consider a microscopic particle together with the gravitational field produced by the particle. If the gravitational field is essentially classical, then by measuring all components of the field simultaneously one can determine both the position and velocity of the particle simultaneously with any accuracy, thus violating the uncertainty principle. Hence the gravitational field must not be classical in principle. There are other arguments in favour of quantizing the gravitational field. Some of them will be discussed in the next section.

Various attempts were made, mainly in the past twenty years, to establish a quantum theory of General Relativity. Systematic attempts started with a series of papers by Bergmann[A3] and his Syracuse school.[A4]

There are many others[A5] who have done a great deal of work in this field, notably Dirac[A6], Anderson, Arnowitt, Deser, Dewitt, Gupta, Komar, Misner. It was Dirac who formulated the Hamiltonian theory of General Relativity in its present form. He also put forward a systematic quantization scheme. However, due to the tremendous complexity of the problem both mathematical and physical, there is still no completely satisfactory quantum theory for General Relativity.

1.2 Objectives

Einstein's theory of gravitational fields is fundamentally different from other field theories such as electrodynamics. Quite apart from the sheer mathematical complexity of Einstein's field equations, the theory presents the following three major problems when we try to quantize it.

(1) General Covariance.

Einstein's theory is invariant with respect to arbitrary coordinate transformations. This gives rise to mathematical and physical problems. The mathematical difficulties in effecting quantization lie in the existence of constraints in the Hamiltonian formulation, i.e. not all the canonical variables are independent of one another for reasons which will be discussed in Chapters 2 and 3. Physically one has great difficulties in relating the results obtained in an arbitrary coordinate frame to the results of measurements made by physical observers in some special coordinate frames. There are as yet no generally accepted coordinate conditions for the determination of a set of special coordinate frames to be associated with physical observers on earth, say. Indeed there does not seem to be any way by which one can experimentally test the validity of various choices of coordinate conditions. Furthermore one wants to know whether general covariance of the field equations will necessarily bring in anything physically new in quantum theory.

(2) Intrinsic Nonlinearity.

In conventional field theories such as electrodynamics, one starts with linear field equations and nonlinearity of field equations appears only after one introduces interactions, these usually arising from nonlinear coupling with an external field. The separation of the resulting theory into a linear part which corresponds to free fields and a nonlinear part which represents interaction is unambiguous and unique. In Einstein's theory of gravitational fields, the field equations are nonlinear even without coupling to any other field. There is no unique way by which one can separate such fields into linear and nonlinear parts. One can see this clearly in our discussion of a model field theory in Chapters 6 and 7. Thus one is not absolutely sure that the concepts and methods developed in conventional field theories are meaningful in such a case.

It is customary in the theory of elementary particles to assume that due to the extraordinary weakness of the gravitational field, about 10^{-40} of the strength of the electromagnetic field, one may completely ignore the gravitational field in the microscopic world. Thus, while it is intellectually satisfying to have a quantum theory of gravitation, it will apparently not make any practical contribution to elementary particle theories. However one should not be so certain about this. It is known that in the theory of differential equations, there exist solutions in a nonlinear system that cannot be reached by linear approximations. Hence it is quite possible[A7] that one should not use a perturbation treatment in the case of the intrinsically nonlinear gravitational fields. In a rigorous and fully nonlinear theory, there may be something fundamentally new appearing, corresponding to the physical situation that very close to an elementary particle, the gravitational field becomes so large that the nonlinearity may play an important role. Even qualitative changes may occur. Perhaps it is possible to combat the divergence problem in conventional quantum field theories by incorporating the nonlinear gravitational fields. One may even go one step further and try to formulate a theory of elementary particles using intrinsically nonlinear field equations in the hope that some of the major difficulties arising from essentially linear field theories may be solved in such a new theory.

(3) Conceptual Problems[A8]

The challenge is formidable on this score. The field variables $g_{\mu\nu}$ of General Relativity play a dual role in the theory. To start with one may treat $g_{\mu\nu}$ as field variables for the description of gravitational fields in the same way as the electromagnetic field tensor is used for the description of electromagnetic fields. The situation at the outset then appears to be fine and one may try to carry out a programme of quantization. Trouble starts when it is remembered that the same variables $g_{\mu\nu}$ are used to determine the metric of the curved space-time. It is extremely difficult to see what is meant by a quantized space-time. There are also great difficulties in formulating a quantum theory of measurements in a quantized space-time. Some investigations on this even lead to contradictions.[A9]

Each of these three problems arising from the quantization of General Relativity is important on its own and is well worth studying. We shall adopt the attitude that one should tackle the problems of general covariance and of nonlinearity first to try to achieve a technically complete theory. Only after this can one possibly have a rigorous solution to those conceptual problems. Hence the present arguments and difficulties should not be regarded as final. Therefore in this thesis, we shall confine our studies to the problems of general covariance and of nonlinearity. Although the ultimate goal of the whole exercise is to achieve a quantum theory of gravitation, we shall not examine the above problems directly for Einstein's theory. Instead some comparatively simple model field theories will be studied. Our present aim is to gain some qualitative understanding of the problems as well as some experience for treating such problems, while avoiding the sheer mathematical complexity of Einstein's theory. In Chapter 4, a model field theory which is generally covariant and exactly soluble is set up for the studies of the effects of general covariance in quantum theory. Another model field theory which is generally covariant and intrinsically nonlinear is formulated in subsequent chapters. Various treatments of nonlinearity are discussed. There is scope for a tremendous amount of further work to be done on these problems. Some of these problems most directly related to our present work are suggested for further research in the concluding Chapter of this thesis.

CHAPTER 2

A REVIEW OF GENERALLY COVARIANT FIELD THEORIES

CLASSICAL THEORIES

2.1 Space-Time and the Group of General Coordinate Transformations

First of all we assume that space-time is locally Euclidean and that it is possible to assign space-time coordinates. In general the space-time is assumed to be Riemannian with a symmetric metric tensor whose signature is conventionally taken to be -2. An example of space-time is the Minkowski flat space with the metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

where $\mu,\nu = 0,1,2,3$. One may use different procedures for the assignment of coordinates leading to different coordinates x^μ, x'^μ for the specification of the same space-time point. We shall only consider procedures which lead to different coordinates related in a one-one and bicontinuous manner. We may write

$$x'^\mu = x'^\mu(x) \; ; \; x^\mu = x^\mu(x') \; , \text{ and } J = \left|\frac{\partial x'^\mu}{\partial x^\nu}\right| \neq 0.$$

Only transformations with $J > 0$ will be considered. It is an intricate problem to aim at complete mathematical rigour in studying the set of coordinates and the transformations among themselves. Our subsequent discussion does not require complete mathematical rigour at this stage. Therefore we shall speak loosely and regard the set of such coordinate transformations as a connected continuous group to be called the group of general coordinate transformations.

We shall mainly concern ourselves with some necessary properties of the set in order to form such a group. That is why we can afford to speak loosely. Firstly by continuous connectivity it is meant that finite transformations may be built up by infinitesimal transformations which may be written as

$$\acute{x}^\mu = x^\mu + \varepsilon^\mu(x) \;\; ; \;\; x^\mu = \acute{x}^\mu - \varepsilon^\mu(\acute{x}) \;\; ,$$

where $\varepsilon^\mu(x)$ are called the descriptors of the transformation.
Now consider the effect of two successive infinitesimal transformations in different order, i.e.

$$\acute{x}^\mu = x^\mu + \varepsilon_1^\mu(x) \;\; ; \;\; \overset{\prime\prime}{x}^\mu = \acute{x}^\mu + \varepsilon_2^\mu(\acute{x}) = x^\mu + \varepsilon_1^\mu(x) + \varepsilon_2^\mu(x) + \varepsilon_{2,\nu}^\mu(x) \; \varepsilon_1^\nu(x) \;\; ,$$

$$\bar{\acute{x}}^\mu = x^\mu + \varepsilon_2^\mu(x) \;\; ; \;\; \bar{\overset{\prime\prime}{x}}^\mu = \bar{\acute{x}}^\mu + \varepsilon_1^\mu(\bar{x}) = x^\mu + \varepsilon_2^\mu(x) + \varepsilon_1^\mu(x) + \varepsilon_{1,\nu}^\mu(x) \; \varepsilon_2^\nu(x) \;\; .$$

Then

$$\bar{\overset{\prime\prime}{x}}^\mu - \overset{\prime\prime}{x}^\mu = \varepsilon_{1,\nu}^\mu \; \varepsilon_2^\nu - \varepsilon_{2,\nu}^\mu \; \varepsilon_{,1}^\nu = \varepsilon_3^\mu \;\; . \qquad (2.1\text{-}1)$$

The ε_3^μ are then the descriptors effecting the infinitesimal coordinate transformation from $\overset{\prime\prime}{x}^\mu$ to $\bar{\overset{\prime\prime}{x}}^\mu$. A necessary condition for a set of infinitesimal coordinate transformations with descriptors of a particular type to form a group is that ε_3^μ so obtained for any pair ε_1^μ , ε_2^μ must be descriptors of the same type, i.e. we require the infinitesimal coordinate transformation from $\overset{\prime\prime}{x}^\mu$ to $\bar{\overset{\prime\prime}{x}}^\mu$ to belong to the set. The proof is obvious. Let us adopt a symbolic notation for the infinitesimal coordinate transformations, e.g.

$$\overset{\prime}{x} = T_1\, x \;;\; \overset{\prime\prime}{x} = T_2\, \overset{\prime}{x} = T_2 T_1\, x \;,$$

$$\overset{\prime}{\bar{x}} = T_2\, x \;;\; \overset{\prime\prime}{\bar{x}} = T_1\, \overset{\prime}{x} = T_1 T_2\, x \;.$$

Then

$$\overset{\prime\prime}{\bar{x}} = T_1 T_2 T_1^{-1}\, T_2^{-1}\, \overset{\prime\prime}{x} = T_3\, \overset{\prime\prime}{x} \,,$$

$$T_3 = T_1 T_2 T_1^{-1}\, T_2^{-1} \;.$$

The infinitesimal coordinate transformations being a group implies T_3 is again in the set by the closure property of the group.

In the case of a Lie group, this becomes the necessary and sufficient condition for the set of infinitesimal coordinate transformations to be able to be integrated to obtain finite coordinate transformations independent of the path of integration. The group of general coordinate transformations is however not a Lie group.

2.2 Geometric Objects and the Realization of the group of general coordinate transformations

Geometric objects[B1] are defined to be objects which constitute the basis of a realization of the group of general coordinate transformations. The central requirement is that:

if a geometric object y is transformed to y' as coordinates $x \to \overset{\prime}{x}(x)$ and y' further goes to y'' if $\overset{\prime}{x} \to \overset{\prime\prime}{x}(\overset{\prime}{x})$, then y will be transformed to y'' by the coordinate transformation $x \to \overset{\prime\prime}{x}(\overset{\prime}{x}(x))$.

Suppose we have a geometric object $y(x)$ defined at a point x^μ. On a transformation $x \to \overset{\prime}{x}$ we have

$$y(x) \to y'(x') = F(y(x)\ ;\ x'(x)) .$$

For the identity transformation we require

$$F(y(x)\ ;\ x) = y(x) .$$

For a further transformation $x' \to x''$

$$y'(x') \to y''(x'') = F(y'(x')\ ;\ x''(x')) .$$

Then the above requirement on the transformation properties of y is equivalent to the demand that

$$F(y'(x')\ ;\ x''(x')) = F(y(x)\ ;\ x''(x'(x))) .$$

If a geometric object has more than one component, we write it as y_A. One can verify that scalars, and tensors defined in the usual way are geometric objects satisfying the above requirement on transformation law. However a finite set of such objects defined over a finite points in space-time cannot form the basis of a faithful representation of the group of general coordinate transformations. We need to have a field of objects defined at each point in the space-time. In what follows we consider only such fields which form faithful representations of the group of general coordinate transformations. Infinitesimal transformations take the form of the usual δ-variation

$$y_A(x) \to y'_A(x') = y_A(x) + \delta y_A(x)\ ,\ \text{where}\ x'^{\mu} = x^{\mu} + \varepsilon^{\mu}\ \text{and}\ \delta y_A(x)$$
depends both on $y_A(x)$ and ξ^{μ}.

Another variation $\bar{\delta}$ is often used and is defined to be

$$\bar{\delta} y_A(x) = y'_A(x) - y_A(x) = y'_A(x'-\varepsilon) - y_A(x) = y'_A(x') - y_A(x) - y'_{A,\mu}(x')\ \varepsilon^{\mu}$$
$$= \delta y_A(x) - y_{A,\mu}(x)\ \varepsilon^{\mu}\ ,\quad \because\ y'_{A,\mu}(x')\ \varepsilon^{\mu} = y_{A,\mu}(x)\ \varepsilon^{\mu}\ \text{ to first order.}$$

This deals with variation at the same value of variable x^μ and has therefore many advantages. An example is that $\bar{\delta}$ and differentiation commute.

$$\delta y_{A,\mu}(x) = y'_{A,\mu}(x) - y_{A,\mu}(x) = \frac{\partial}{\partial x^\mu}(y'_A(x) - y_A(x)) = \frac{\partial}{\partial x^\mu}\bar{\delta} y_A(x) .$$

Hence for any function $F(y_A) = F(y_A(x))$,

$$\bar{\delta} F \equiv F'(x) - F(x) = \frac{\partial F(x)}{\partial y_A(x)} \bar{\delta} y_A(x) + \frac{\partial F(x)}{\partial y_{A,\mu}(x)} (\bar{\delta} y_A(x))_{,\mu} + \dots\dots$$

One also notes that if $y'_A(x)$ is related to $y_A(x)$ by an infinitesimal coordinate transformation then

$$\bar{\delta}_1 y_A(x) = \bar{\delta}_1 y'_A(x) ,$$

to the first order of the descriptors ε_1^μ which cause the $\bar{\delta}_1$ variation. For field theory, it is more convenient to express group properties in terms of $\bar{\delta}$-variation, i.e. change of the field variable $y_A(x)$ at the same numerical values of the old and new coordinates. For $y_A(x)$ to constitute a realization of the group of general coordinate transformations, the central requirement is again as before, i.e. for any two successive coordinate transformations $x \to x' \to x''$ we have $y_A(x) \to y'_A(x) \to y''_A(x)$, then we require $y_A(x)$ to transform to $y''_A(x)$ as $x \to x''$ directly. It is of particular interest to consider infinitesimal coordinate transformations for field. Let ε_1^μ, ε_2^μ, ε_3^μ be three descriptors related by equation (2.1-1) and $\bar{\delta}_1$, $\bar{\delta}_2$, $\bar{\delta}_3$ be respectively the corresponding $\bar{\delta}$-variation in field.

We show[B2] that

$$\bar{\delta}_3 y_A(x) = \bar{\delta}_1(\bar{\delta}_2 y_A(x)) - \bar{\delta}_2(\bar{\delta}_1 y_A(x)) \ ,$$

is a necessary condition for $y_A(x)$ to be geometric objects. We use the same notation as that used in obtaining equation (2.1-1). Let $y_A(x) \to y'_A(x) \to y''_A(x)$ as $x^\mu \to x'^\mu \to x''^\mu$ and that $y_A(x) \to \bar{y}'_A(x) \to \bar{y}''_A(x)$ as $x^\mu \to \bar{x}'^\mu \to \bar{x}''^\mu$. Then

$$y'_A(x) = \bar{\delta}_1 y_A(x) + y_A(x) \ ,$$

$$y''_A(x) = \bar{\delta}_2 y'(x) + y'(x) = \bar{\delta}_2(\bar{\delta}_1 y_A(x)) + \bar{\delta}_2 y_A(x) + \bar{\delta}_1 y_A(x) + y_A(x) .$$

Similarly

$$\bar{y}''_A(x) = \bar{\delta}_1(\bar{\delta}_2 y_A(x)) + \bar{\delta}_1 y_A(x) + \bar{\delta}_2 y_A(x) + y_A(x)$$

$$\Longrightarrow \qquad \bar{y}''_A(x) - y''_A(x) = \bar{\delta}_1(\bar{\delta}_2 y_A(x)) - \bar{\delta}_2(\bar{\delta}_1 y_A(x)) \ . \qquad (2.2\text{-}1)$$

It is now obvious that $\bar{y}''_A(x) - y''_A(x)$ must equal $\bar{\delta}_3 y''_A(x)$ if $y_A(x)$ are to be geometric objects. To the second order of descriptors we have $\bar{\delta}_3 y''_A(x) = \bar{\delta}_3 y_A(x)$. This is seen by operating $\bar{\delta}_3$ on the $y''_A(x)$ expression bearing in mind that a $\bar{\delta}_3$ variation is already of 2nd order. So the statement is established.

2.3 Covariant Field Theories in Lagrangian Formulation

We shall follow the treatment by Anderson[B3] in this section, though the formulation in its present form was first studied by Bergmann[B4].

2.3.1 Covariance Properties of Physical Theories

In a physical theory, one tries to associate mathematical quantities of some kind with physical quantities. These mathematical quantities may take on a wide range of values and a set of possible values of these quantities is called a kinematically possible trajectory. Not all kinematically possible trajectories may be realized by the actual physical system. A kinematically possible trajectory which can in principle be realized by the physical system is called a dynamically possible trajectory. The set of dynamically possible trajectories is then a subset of the set of kinematically possible trajectories. Dynamical laws or equations of motion are conditions for the determination of this subset. A physical theory is said to be covariant with respect to a group of transformations if two conditions are satisfied. Firstly the kinematically possible trajectory must constitute the basis of a faithful realization of the group. Secondly we require the realization associates dynamically possible trajectory with dynamically possible trajectory. For field theory, kinematically possible trajectories are described by field variables $y_A(x)$. Then covariance with respect to a transformation group is equivalent to the requirements that $y_A(x)$ form a faithful realization of the group and that equations of motion for $y_A(x)$ are unchanged with respect to any transformation of the group. In practice, the first condition is satisfied by choosing $y_A(x)$ to be a field of geometric objects. We are left with only the second condition to examine. We shall mainly be concerned with the group of general coordinate transformations in what follows.

2.3.2 Determination of Equations of Motion

Assume the equations of motion for field $y_A(x)$ are derivable from a variational principle with a Lagrangian density which is explicitly a function of $y_A(x)$ and $y_{A,\mu}(x) \equiv \frac{\partial}{\partial x^\mu} y_A(x)$ only. The action integral is

$$I = \int_R \mathcal{L}(y_A, y_{A,\mu}) d^4x .$$

Our assumption is that I be at a stationary value with respect to arbitrary variation of field variables subject only to the condition that the variation vanishes on the boundary surface S of the region R. Consider such a variation $\bar{\delta} y_A(x) = y'_A(x) - y_A(x)$.

$$\bar{\delta} I = I(y+\bar{\delta}y) - I(y) ,$$

$$I(y+\bar{\delta}y) = \int_R \mathcal{L}(y+\delta y, y_{,\mu} + \bar{\delta}y_{,\mu}) d^4x$$

$$= \int_R \mathcal{L}(y, y_{,\mu}) d^4x + \int_R \left(\frac{\partial \mathcal{L}}{\partial y_A} \bar{\delta} y_A + \frac{\partial \mathcal{L}}{\partial y_{A,\mu}} \bar{\delta} y_{A,\mu}\right) d^4x ,$$

$$\int_R \frac{\partial \mathcal{L}}{\partial y_{A,\mu}} \bar{\delta} y_{A,\mu} \, d^4x = \int_R \left(\frac{\partial \mathcal{L}}{\partial y_{A,\mu}} \bar{\delta} y_A\right)_{,\mu} d^4x - \int_R \left(\frac{\partial \mathcal{L}}{\partial y_{A,\mu}}\right)_{,\mu} \bar{\delta} y_A \, d^4x .$$

The first term vanishes on converting to a surface integral over S. Hence we get

$$\bar{\delta} I = \int_R \frac{\delta \mathcal{L}}{\delta y_A} \bar{\delta} y_A \, d^4x , \quad \text{where} \quad \frac{\delta \mathcal{L}}{\delta y_A} \equiv \frac{\partial \mathcal{L}}{\partial y_A} - \left(\frac{\partial \mathcal{L}}{\partial y_{A,\mu}}\right)_{,\mu} .$$

$$\bar{\delta} I = 0 \iff \frac{\delta \mathcal{L}}{\delta y_A} = 0$$

on account of the arbitrary nature of $\bar{\delta} y_A$.

So the equations of motion are

$$L^A \equiv \frac{\delta \mathcal{L}}{\delta y_A} = 0 \; .$$

The next obvious question is under what condition the variational principle will ensure the invariance of these equations of motion with respect to the group of general coordinate transformations. A sufficient condition is that $\mathcal{L}$ should transform as a scalar density of weight +1. This implies the action integral I has the same value before and after a coordinate transformation. Therefore a kinematically possible trajectory in one coordinate system that corresponds to a stationary value of I will be transformed to a new kinematically possible trajectory in another coordinate system which will again correspond to a stationary value of I. This implies dynamically possible trajectories will always be transformed to dynamically possible trajectories and hence equations of motion must remain unchanged. Notice that above condition is a sufficient one but not a necessary one. Furthermore as addition of a complete divergence $Q^\mu{}_{,\mu}$ will not change the equations of motion, it is sufficient to require $\mathcal{L} + Q^\mu{}_{,\mu}$ to transform as a scalar density of weight +1. The freedom to add a perfect divergence may be used to simplify equations of motion.

2.3.3 The Nöther Identity and the Bianchi Identities

Let us examine the effect of an infinitesimal coordinate transformation on the Lagrangian density which is a scalar density of weight +1. Let J be the Jacobian of the infinitesimal coordinate transformation. We have

$$J = \left|\frac{\partial x^{\mu}}{\partial x'^{\nu}}\right| = 1 - \varepsilon^{\mu}_{,\mu}$$

$$\Longrightarrow \quad \delta\mathcal{L}(x) = \mathcal{L}'(x') - \mathcal{L}(x) = -\mathcal{L}\varepsilon^{\mu}_{,\mu} ,$$

$$\bar{\delta}\mathcal{L}(x) = \delta\mathcal{L}(x) - \mathcal{L}_{,\mu}\, \varepsilon^{\mu} = -\mathcal{L}\varepsilon^{\mu}_{,\mu} - \mathcal{L}_{,\mu}\, \varepsilon^{\mu}$$

$$= -(\mathcal{L}\varepsilon^{\mu})_{,\mu} .$$

We can also calculate $\bar{\delta}\mathcal{L}$ through the change in the field variable y_A

$$\bar{\delta}\mathcal{L} = \frac{\partial\mathcal{L}}{\partial y_A}\ \bar{\delta}y_A + \frac{\partial\mathcal{L}}{\partial y_{A,\mu}}\ \bar{\delta}y_{A,\mu}$$

$$= \frac{\delta\mathcal{L}}{\delta y_A}\ \bar{\delta}y_A + \left[\frac{\partial\mathcal{L}}{\partial y_{A,\mu}}\ \bar{\delta}y_A\right]_{,\mu} ,$$

where $\bar{\delta}y_A = \delta y_A - y_{A,\mu}\, \varepsilon^{\mu}$ and δy_A is the change in field y_A due to the infinitesimal coordinate transformation.

Thus $\bar{\delta}t^{\mu}_{,\mu} \equiv \frac{\delta\mathcal{L}}{\delta y_A}\ \bar{\delta}y_A$, where $\bar{\delta}t^{\mu} = -\mathcal{L}\varepsilon^{\mu} - \frac{\partial\mathcal{L}}{\partial y_{A,\mu}}\ \bar{\delta}y_A$.

This is an identity called the Nöther identity.

We now want to show that not all field equations are independent of one another and certain identities, called Bianchi Identities, exist among them.

Assume the field y_A transforms according to

$$\bar{\delta}y_A(x) = D^{\mu}_{A\nu}\ \varepsilon^{\nu}_{,\mu} + E_{A\mu}\varepsilon^{\mu} ,$$

where $D^{\mu}_{A\nu}$, $E_{A\mu}$ are functions of y_A and ε^{μ} are descriptors of the infinitesimal coordinate transformations.

For the usual tensor field, $D^{\mu}_{A\nu}$ is of the form

$$D^{\mu}_{A\nu} = F_{A\nu}{}^{B\mu} \, y_B \ , \text{ with } F_{A\nu}{}^{B\mu} \text{ being numbers independent of } x^{\mu} .$$

Substituting the expression for $\bar{\delta} y_A$ into the Nöther identity we get

$$(\bar{\delta} t^{\mu} - D^{\mu}_{A\nu} \, L^A \, \varepsilon^{\nu})_{,\mu} \equiv \{E_{A\nu} \, L^A - (D^{\mu}_{A\nu} \, L^A)_{,\mu}\} \, \varepsilon^{\nu} \ ,$$

$$\text{where } L^A \equiv \frac{\delta \mathcal{L}}{\delta y_A} .$$

Integrate this over an arbitrary region in space-time. One can convert the left hand volume integral to a surface integral over the boundary of the space-time region. The arbitrary nature of ε^{μ} allows us to choose ε^{μ} such that they vanish on the boundary while remaining arbitrary inside. So the surface integral vanishes and we obtain

$$\int d^4x \ \{E_{A\nu} \, L^A - (D^{\mu}_{A\nu} \, L^A)_{,\mu}\} \, \varepsilon^{\mu} \equiv 0 .$$

ε^{μ} being arbitrary implies

$$E_{A\nu} \, L^A - (D^{\mu}_{A\nu} \, L^A)_{,\mu} \equiv 0 \ , \text{ valid irrespective of field equations.}$$

Since the region of integration is arbitrary we conclude that these identities hold everywhere. They are called the Bianchi identities. One sees that indeed not all field equations are independent as one would expect for covariant field theories. There are four of these identities for a theory admitting the group of general coordinate transformations as a covariance group. This means that the number of independent equations is four less than the number of variables y_A. As a result it is impossible to fix y_A from the field equations alone. We can impose 4 further conditions on y_A in order to fix y_A. These are called coordinate conditions as one physically assumes that y_A should satisfy some special equations in a particular type of coordinate system.

We can draw more conclusions from the Bianchi identities. Let the second order field equations be of the form

$$L^A = K^{AB\rho\sigma} y_{B,\rho\sigma} + \ldots\ldots\ldots ,$$

where $K^{AB\mu\nu}$ and the terms denoted by dots depend on lower order derivatives of y_B.

One sees that $K^{AB\rho\sigma} = -\dfrac{\partial^2 \mathcal{L}}{\partial y_{A,\rho}\, \partial y_{A,\sigma}}$ which is symmetric with respect to A,B , the reason being that

B ?

$$L^A \equiv \frac{\partial \mathcal{L}}{\partial y_A} - \left[\frac{\partial \mathcal{L}}{\partial y_{A,\rho}}\right]_{,\rho} = \frac{\partial \mathcal{L}}{\partial y_A} - \left\{\left(\frac{\partial}{\partial y_{B,\sigma}}\left[\frac{\partial \mathcal{L}}{\partial y_{A,\rho}}\right]\right) y_{B,\sigma\rho} + \left(\frac{\partial}{\partial y_B}\left[\frac{\partial \mathcal{L}}{\partial y_{A,\rho}}\right]\right) y_{B,\rho}\right\}.$$

Substituting this into the Bianchi identities we get

$$D^{\mu}_{A\nu} K^{AB\rho\sigma}\, y_{B,\rho\sigma\mu} + \text{terms containing lower order derivatives of}$$

$$y_A \equiv 0.$$

Recall that these identities are satisfied for any y_A even those not satisfying the field equations.

This implies that all the derivatives of y_A appearing in the identities are arbitrary. Hence it is necessary that

$$D^{\mu}_{A\nu} K^{AB\rho\sigma} \equiv 0 \quad \text{for each } \nu \text{ .}$$

Of particular interest are the following 4 identities

$$D^{\circ}_{A\nu} K^{AB00} \equiv 0 \text{ , where } K^{AB00} = - \frac{\partial^2 L}{\partial y_{A,0} \, \partial y_{B,0}} \text{ .}$$

If we treat K^{AB00} as a matrix, then these identities imply that K^{AB00} has 4 null eigenvectors $D^{\circ}_{A\nu}$. K^{AB00} has therefore a vanishing determinant and no inverse. We cannot solve for $y_{A,00}$ in terms of lower order derivatives from the field equations. As a result it is impossible to obtain unique solution to the usual initial value problem, that is, one cannot give sufficient initial data on any 3-dimensional hyper-surface x° = constant for the complete determination of the field variables elsewhere. To see this one can try to determine the field variables in a neighbouring constant x° surface by a Taylor series. One then attempts to express the coefficients of the Taylor series $\frac{\partial^n y_A}{\partial x^{\circ n}}$ in terms of the initial data y_A and $y_{A,0}$ on the initial data surface. With the help of the field equations one can usually express $y_{A,00}$ in terms of y_A, $y_{A,0}$. Higher derivatives $\frac{\partial^n y_A}{\partial x^{\circ n}}$ may be obtained by differentiating the field equations with respect to x°. One can demonstrate this procedure easily with the Klein-Gordon Scalar Meson field.

However this procedure breaks down in our present case because $y_{A,00}$ cannot be expressed in terms of Y_A, $y_{A,0}$ in the first place. So one just cannot get the Taylor series to obtain the field variables off the original hyper-surface. This result is of course expected. The Bianchi identities take on a special form when we have a field ξ_A whose components are all scalars.

$$\bar{\delta}\xi_A = - \xi_{A,\mu} \varepsilon^{\mu}, \text{ i.e. } D^{\mu}_{A\nu} = 0 \; ; \; E_{A,\mu} = -\xi_{A,\mu} .$$

So we have

$$\xi_{A,\mu} L^A \equiv 0 ,$$

which again lead to

$$\xi_{A,\mu} K^{AB00} = 0 .$$

2.4 Hamiltonian Formulation

We shall follow Dirac's approach.[B5] Bergmann and his school[B6] at Syracuse University also made significant contributions to the formulation. The problems relating to constraints, in particular those having a direct bearing on quantization, were analysed in detail by Anderson.[B7]

2.4.1 Hamiltonian Formulation I

Let us work in a particular type of coordinate system for which surfaces with x° = constant are all space-like. A vector A_μ is called space-like if $g^{\mu\nu}A_\mu A_\nu < 0$. It is called time-like if $g^{\mu\nu}A_\mu A_\nu > 0$. The name light-like or null vector is used for A_μ if $g^{\mu\nu}A_\mu A_\nu = 0$.

By a space-like surface one means a surface whose normal is everywhere time-like. As a result, the tangent to the surface at every point must be space-like.

2.4.1.1 The primary Constraints

Define the canonical momentum density by

$$\pi^A \equiv \frac{\partial \mathcal{L}}{\partial y_{A,0}} .$$

One then hopes that the set of canonical variables y_A, π^A will be independent and that $y_{A,0}$ may be written in terms of π^B, y_C and eliminated. If this were true, one could proceed to define a Hamiltonian density and everything would be the same as for field theory in special relativity. However this cannot be the case since it would lead to a well-behaved initial value problem for the solution of the canonical equations of motion, which is contrary to the results obtained from Lagrange theory. Therefore one may anticipate the fact that not all y_A, π^A are independent and that $y_{A,0}$ may not all be expressed in terms of y_A, π^A. The reason is obvious. Firstly we have

$$\frac{\partial \pi^A}{\partial y_{B,0}} = \frac{\partial^2 \mathcal{L}}{\partial y_{A,0}\, \partial y_{B,0}} \implies \text{determinent } \left|\frac{\partial \pi^A}{\partial y_{B,0}}\right| = 0$$

using the Bianchi identities.

Thus $y_{B,0}$ cannot be expressed in terms of π^A.

Secondly we can show the existence of 4 equations relating y_A and π^A without involving $y_{A,0}$. Such equations are called the primary constraints. Recall the four Bianchi identities

$$D^{\circ}_{A,\mu}\, K^{AB00} \equiv 0 \ ,$$

where
$$K^{AB00} = -\frac{\partial^2 \mathcal{L}}{\partial y_{A,0}\,\partial y_{B,0}} = -\frac{\partial \pi^B}{\partial y_{A,0}} \equiv -\frac{\partial \pi^A}{\partial y_{B,0}} \ ,$$

and
$$D^{\circ}_{A,\mu} = F_{A\mu}{}^{B0}\, y_B \ ,$$ with $F^{B0}_{A\nu}$ being numbers for usual tensor transformation law.

Hence we have $D^{\circ}_{A\mu}\,\dfrac{\partial \pi^A}{\partial y_{B,0}} \equiv 0$.

These four equations may be integrated with respect to $y_{B,0}$ and we immediately obtain the four primary constraints. For a scalar field, we have

$$\xi_{A,\mu}\,\frac{\partial \pi^A}{\partial \xi_{B,0}} \equiv 0$$

$$\Longrightarrow \quad \xi_{A,j}\,\pi^A \equiv 0 \quad \text{and} \quad \xi_{A,0}\,\frac{\partial \pi^A}{\partial \xi_{B,0}} \equiv 0 \ .$$

The first three are primary constraints already while the fourth one will lead to the fourth constraint.

2.4.1.2 The Hamiltonian and the Equations of Motion

The Hamiltonian density $\mathcal{H}$ is defined to be

$$\mathcal{H} = \pi^A y_{A,0} - \mathcal{L} \ ,$$ where summation over A is implied.

The total Hamiltonian is

$$H = \int d^3x\, \mathcal{H} .$$

For arbitrary variations of $y_{A,0}$, π^A, y_A.

$$\delta\mathcal{H} = \pi^A\, \delta y_{A,0} + y_{A,0}\, \delta\pi^A - \frac{\partial\mathcal{L}}{\partial y_A}\, \delta y_A - \frac{\partial\mathcal{L}}{\partial y_{A,0}}\, \delta y_{A,0} - \frac{\partial\mathcal{L}}{\partial y_{A,j}}\, \delta y_{A,j}$$

$$= y_{A,0}\, \delta\pi^A - \left(\frac{\partial\mathcal{L}}{\partial y_A}\, \delta y_A + \frac{\partial\mathcal{L}}{\partial y_{A,j}}\, \delta y_{A,j}\right) .$$

This implies that $\mathcal{H}$ is expressible as a function of y_A, π^A only not involving $y_{A,0}$ explicitly, since $\delta\mathcal{H} = 0$ for arbitrary $\delta y_{A,0}$ keeping π^A, y_A fixed.

$$\delta H = \int d^3x\, \delta\mathcal{H} = \int d^3x \left(y_{A,0}\, \delta\pi^A - \left[\frac{\partial\mathcal{L}}{\partial y_A} - \left(\frac{\partial\mathcal{L}}{\partial y_{A,j}}\right)_{,j}\right] \delta y_A \right)$$

$$= \int d^3x (y_{A,0}\, \delta\pi^A - \pi^A_{\ ,0}\, \delta y_A) \text{ , using the Lagrange field equations.}$$

Equations of motion for y_A, π^A are obtained by comparing this variation in H with the variation got by considering $\mathcal{H}$ as a function of y_A and π^A. However, not all π^A, y_A are independent, so a method of Lagrange multipliers is used. The equations of motion are then

$$y_{A,0} = \frac{\partial\mathcal{H}}{\partial\pi^A} - \left(\frac{\partial\mathcal{H}}{\partial\pi^A_{\ ,j}}\right)_{,j} + U^\mu\left[\frac{\partial\phi_\mu}{\partial\pi^A} - \left(\frac{\partial\phi_\mu}{\partial\pi^A_{\ ,j}}\right)_{,j}\right] ,$$

$$\pi^A_{\ ,0} = - \left\{\left[\frac{\partial\mathcal{H}}{\partial y_A} - \left(\frac{\partial\mathcal{H}}{\partial y_{A,j}}\right)_{,j}\right] + U^\mu\left[\frac{\partial\phi_\mu}{\partial y_A} - \left(\frac{\partial\phi_\mu}{\partial y_{A,j}}\right)_{,j}\right]\right\} ,$$

where U^μ are the Lagrange multipliers and $\phi_\mu = 0$ are the primary constraints.

Using Poisson Bracket Notation, we may write

$$y_{A,0}(x) = [y_A(x), \int d^3x' \mathcal{H}(x')] + \int d^3x' \, U^\mu(x')[y_A(x), \phi_\mu(x')],$$

$$\pi^A_{,0}(x) = [\pi^A(x), \int d^3x' \mathcal{H}(x')] + \int d^3x' \, U^\mu(x')[\pi^A(x), \phi_\mu(x')].$$

Define a quantity

$$H_T = \int d^3x \, \mathcal{H}_T \ , \quad \text{where } \mathcal{H}_T = \mathcal{H} + U^\mu \phi_\mu \ ,$$

then the equations of motion may be written as

$$y_{A,0} = [y_A, H_T] \ ; \ \pi^A_{,0} = [\pi^A, H_T] \ ,$$

and for a general functional F of y_A and π^A we have

$$F_{,0} = [F, H_T] \ ,$$

provided certain rules are followed in calculating the Poisson Brackets. They are:

(1) $\phi_\mu = 0$ may not be used before evaluating the Poisson Brackets. To emphasize this point, they are sometimes written as $\phi_\mu \approx 0$ which are called weak equations while equations valid without making use of these weak equations are called strong equations.

(2) The rule [A,BC] = [A,B]C + B[A,C] is to be extended to include non-canonical variables such as U^μ, i.e.

$$[y_A(x), \int d^3x' \, U^\mu(x') \phi_\mu(x')] = \int d^3x' \{ [y_A(x), U^\mu(x')] \phi_\mu(x') + U^\mu(x') [y_A(x), \phi_\mu(x')] \}$$
$$= \int d^3x \, U^\mu(x') [y_A(x), \phi_\mu(x')] \ .$$

There is no need to consider the meaning of $[y_A(x), U^\mu(x')]$.

(3) y_A, π^A are to be treated as if they are independent in evaluating these Poisson Brackets.

From now on we shall adopt these rules.

2.4.1.3 The Secondary Constraints

There are a number of consistency equations to be satisfied. We must require those primary constraints be maintained weakly vanishing in the course of time. Therefore we want

$$[\phi_\mu, H_T] \approx 0 \quad \text{for each } \mu \ . \qquad (2.4.1.3\text{-}1)$$

If these are true for arbitrary values of U^μ, then the theory is already consistent and no further consistency equations are necessary. Such will be the case if and only if

$$[\phi_\mu(x) \ , \ \phi_\nu(x')] \approx 0 \ ; \ [\phi_\mu \ , \ H] \approx 0 \ .$$

This is the simplest case.

In general we do not have the above. Then the consistency equations (2.4.1.3-1) may lead to

(1) outright contradiction,

or (2) a second set of equations of form

$$\chi(y_A, \pi^A) \approx 0 \text{ , not involving } U^\mu \text{ .}$$

These are called secondary constraints. In this case we have to start all over again to examine consistency equations for these secondary constraints.

or (3) a set of equations involving y_A, π^A, U^μ.

In the fields we are going to study, case (1) and case (3) do not appear, so we will only consider the simplest case and case (2). In such cases, the four quantities U^μ remain arbitrary. The appearance of four arbitrary quantities in the Hamiltonian theory is expected and they correspond to the freedom of coordinate transformations in the 4-Dimensional space-time.

A constraint commuting with all the rest of the constraints is called first class, otherwise it is called second class. It can be shown that the number of independent arbitrary quantities U is equal to the number of first class constraints. So for a covariant theory, there are at least four first class constraints corresponding to the four arbitrary U^μ.

2.4.2 Hamiltonian Formulation II - the Curved Surface Formulation of Dirac[B8]

The previous Hamiltonian Theory has one drawback for theoretical discussion in that it is awkward to study arbitrary coordinate transformations as the formulation is explicitly coordinate-dependent.

Let us consider an arbitrary space-like surface $\mathcal{S}$ which extends to spatial infinity. Then in a more general theory, one should set oneself the following problem. Supposing we know the maximum compatible set of information of a field on $\mathcal{S}$, we want to establish a Hamiltonian theory which can tell us how the field varies as we go from $\mathcal{S}$ to another such space-like surface.

2.4.2.1 Geometry of a Curved Surface

To tackle the problem, the first thing is to describe such surface $\mathcal{S}$. Any surface in space-time may be fixed by specifying the coordinates x^μ of any point on the surface as functions of three parameters ν^r, i.e. it is fixed by four functions $x^\mu = y^\mu(\nu)$. A surface is an invariant concept independent of the choice of x^μ. A coordinate transformation $x^\mu \to x'^\mu$ only means that in the new coordinates, the same surface $\mathcal{S}$ is specified by

$$x'^\mu = y'^\mu(\nu) \neq y^\mu(\nu) \quad \text{in general,}$$

and the space-like nature of $\mathcal{S}$ is preserved. The important point now is to note that one can always define a new surface $\mathcal{S}'$ by $x^\mu = y'^\mu(\nu)$, which is also space-like. This means that a theory covariant with respect to the group of general coordinate transformations must also be covariant with respect to arbitrary deformation of the surface $\mathcal{S}$ provided the deformed surface S' is again space-like and vice versa. So it is sufficient for us to study surface deformation instead. Each such space-like surface is to be labelled by a parameter τ.

So a surface is completely fixed by (τ, ν^r) and we may write

$$x^\mu = y^\mu(\tau, \nu^r) = y^\mu(\tau, \nu) \ .$$

The set of 4 parameters (τ, ν) may be used to specify points in space-time. A set of 3 linearly independent tangent vectors at a point on the surface will be

$$\frac{\partial y^\mu}{\partial \nu_r} \equiv y_{,r}^{\ \mu} \ .$$

Note that r,s were used as indices for the parameters ν^r to avoid confusion with the i,j used for x^j.

Three unit tangents may be defined as

$$t_r^\mu = y_{,r}^{\ \mu} \Big/ \sqrt{|g_{\rho\nu} \, y_{,r}^{\ \rho} \, y_{,r}^{\ \nu}|} \qquad \text{(no summation over r)},$$

so that $|t_{r\mu} \, t_r^{\ \mu}| = 1$ (no summation over r).

The unit normal n_μ is defined by

$$n_\mu \, t_r^\mu = 0 \ , \text{ for each r and } |n_\mu \, n^\mu| = 1 \ .$$

For a space-like surface we require

$$n_\mu \, n^\mu = 1 \iff t_{r\mu} \, t_r^{\ \mu} = -1 \text{ (no summation over r)}.$$

The parameters ν^r form a system of coordinates on the surface. The surface metric is

$$\gamma_{rs} = g_{\mu\nu} \, y_{,r}^{\ \mu} \, y_{,s}^{\ \nu} \ ,$$

since

$$d^2s = g_{\mu\nu}\, dx^\mu\, dx^\nu = g_{\mu\nu}\, y^{\mu}_{,r}\, y^{\nu}_{,s}\, d\nu^r\, d\nu^s \ .$$

Any vector A^μ located at a point on the surface $\mathcal{S}$ may be written as

$$A^\mu = A_\perp\, n^\mu + A^r_{/\!/}\, y^{\mu}_{,r} \ ,$$

since n^μ, $y^{\mu}_{,r}$ are a set of four linearly independent vectors at the same point on $\mathcal{S}$. The coefficients are

$$A_\perp = A^\mu\, n_\mu \ ,$$

$$A^r_{/\!/} = \gamma^{rs}\, A_{/\!/ s} \quad ; \quad A_{/\!/ s} = g_{\mu\nu}\, A^\mu\, y^{\nu}_{,s} \ .$$

The scalar product of two vectors can now be expressed as

$$A^\mu\, B_\mu = A_\perp\, B_\perp + \gamma_{rs}\, A^r_{/\!/}\, B^s_{/\!/}$$

$$= A_\perp\, B_\perp + A_{/\!/ s}\, B^s_{/\!/} \ , \text{ where } A_{/\!/ r} = \gamma_{rs}\, A^s_{/\!/} \ .$$

An arbitrary infinitesimal surface deformation may be described by specifying δx^μ on every point on $\mathcal{S}$, i.e. δx^μ as functions of (τ,ν). In particular we may write

$$\delta x^\mu = \delta x_\perp\, n^\mu + \delta x^r_{/\!/}\, y^{\mu}_{,r} \ .$$

The first term is seen to be due to deformation normal to $\mathcal{S}$ while the second term represents a tangential deformation.

Before we leave this section, we want to express the four-volume element d^4x in terms of the parameters (τ, ν). Our $\mathcal{S}$ being space-like implies that ds^2 is negative. Hence the determinent of γ_{rs} is negative. Let it be $-\Gamma^2$ with Γ positive. Then a surface element is given by

$$\Gamma \, d^3\nu \, .$$

A 4-volume element may be written as

$$d^4\chi = (\Gamma \, d^3\nu) \cdot (\text{an element normal to the surface})$$

$$= \Gamma \, d^3\nu \, \delta x_\perp \, ,$$

$$\delta x_\perp = \delta x^\mu \, n_\mu \; ; \; \delta x^\mu = \frac{\partial y^\mu}{\partial \tau} \, d\tau \, .$$

Finally we obtain

$$d^4\chi = \Gamma \, \frac{\partial y^\mu}{\partial \tau} \, n_\mu \, d^3\nu \, d\tau$$

$$= \Gamma \, \dot{y}_\perp \, d^3\nu \, d\tau \, , \text{ where } \dot{y}^\mu = \frac{\partial y^\mu}{\partial \tau} \; ; \; \dot{y}_\perp = n_\mu \cdot \dot{y}^\mu \, .$$

2.4.2.2 Hamiltonian Theory

To achieve a field theory based on such curved surfaces, we may proceed in analogy with the so-called parameterized formulism[B9] in the special relativistic Hamiltonian formulation of particle mechanics. In this theory, one uses the invariant proper time as independent variable while the time coordinate is promoted to the status of a canonical coordinate. This treatment enables us to treat space and time on an equal footing as is pertinent to the spirit of Special Relativity. What we can do for field theory is to use the parameters (τ,ν) as independent variables for the specification of space-time points and treat $\chi^\mu = y^\mu(\tau,\nu)$ as field variables in addition to the original field variables.

Let us consider the case in which there is only one scalar field variable ξ in a certain coordinate system χ^μ. The problem now is to set up a theory based on states defined on arbitrary space-like surface. Firstly examine the action integral

$$I = \int \mathcal{L}(\xi,\xi,_\mu) d^4x = \int \bar{\mathcal{L}} d^3\nu \, d\tau \ ,$$

$$\bar{\mathcal{L}} = \mathcal{L}(\xi,\xi,_\mu) \mathring{y}_\perp \Gamma \ .$$

$\bar{\mathcal{L}}$ involves

$$\xi, \quad \xi,_\mu = \frac{\partial \xi}{\partial x^\mu} \ , \ n_\mu \ , \ \mathring{y}^\mu = \frac{\partial y^\mu}{\partial \tau} \ , \ y^\mu_{\ ,r} = \frac{\partial y^\mu}{\partial \nu^r} \ .$$

The field variables are taken as ξ, y^μ while $\xi,_r = \frac{\partial \xi}{\partial \nu^r}$, $y^\mu_{\ ,r}$, n_μ are functions of ξ, y^μ only. The velocity variables are $\mathring{y}^\mu$ and $\mathring{\xi} = \frac{\partial \xi}{\partial \tau}$. We want to express $\bar{\mathcal{L}}$ in terms of field variables and velocity variables only. The only troublesome term is $\xi,_\mu$. However we have

$$\xi,_\mu = \xi_\perp n_\mu + \xi^r_{/\!/} \, y^\nu_{\ ,r} \, g_{\mu\nu} \ ,$$

$$\xi^r_{/\!/} = \xi,_\rho \, y^\rho_{\ ,r} = \xi,_r \quad \text{which are functions of field variables.}$$

We now show that

$$\xi_\perp = \frac{\mathring{\xi} - \xi_{/\!/}^r\, y^\nu_{,r}\, g_{\nu\mu}\, \mathring{y}^\mu}{\mathring{y}_\perp}.$$

Proof:

$$\mathring{\xi} = \xi,_\mu \mathring{y}^\mu = (\xi_\perp n_\mu + \xi_{/\!/}^r\, y^\nu_{,r}\, g_{\nu\mu})\, \mathring{y}^\mu = \xi_\perp \mathring{y}_\perp + \xi_{/\!/}^r\, y^\nu_{,r}\, g_{\nu\mu}\, \mathring{y}^\mu ,$$

$$\xi_\perp = \frac{\mathring{\xi} - \xi_{/\!/}^r\, y^\nu_{,r}\, g_{\nu\mu}\, \mathring{y}^\mu}{\mathring{y}_\perp}.$$

Hence

$$\xi,_\mu = \left(\frac{\mathring{\xi} - \xi_{/\!/}^r\, y^\nu_{,r}\, g_{\nu\rho}\, \mathring{y}^\rho}{\mathring{y}_\perp}\right) n_\mu + \xi_{/\!/}^r\, y^\nu_{,r}\, g_{\nu\mu} ,$$

which are functions of field variables and velocity variables.

Having obtained such a Lagrangian, we are in a position to proceed with the Hamiltonian theory.

Define the canonical momentum conjugate to ξ by

$$\pi = \frac{\partial \bar{\mathcal{L}}}{\partial \mathring{\xi}} = \frac{\partial \mathcal{L}}{\partial \xi,_\mu}\, n_\mu\, \Gamma. \qquad (2.4.2.2\text{-}1)$$

Define canonical momenta conjugate to y^μ by

$$W_\mu = \frac{\partial \bar{\mathcal{L}}}{\partial \mathring{y}^\mu} = \mathcal{L}\, n_\mu\, \Gamma - \pi\, \xi,_\mu . \qquad (2.4.2.2\text{-}2)$$

Since we have introduced redundant variables into the theory, we expect constraints to appear.

Split equation (2.4.2.2-2) into a normal and a tangential part.

$$W_{\perp} = \mathcal{L}\Gamma - \pi \xi_{\perp} , \quad (2.4.2.2\text{-}3)$$

$$W_{/\!/}^{r} = - \pi \xi_{/\!/}^{r} . \quad (2.4.2.2\text{-}4)$$

Equations (2.4.2.2-4) are three primary constraints already, while equation (2.4.2.2-3) together with equation (2.4.2.2-1) may produce another primary constraint of the form

$$W_{\perp} = f(y^{\mu}, \xi, \pi) ,$$

on eliminating the velocity variables on the right hand side of (2.4.2.2-3) with the help of equation (2.4.2.2-1). There may be cases where more than one primary constraint appear. However we are interested in the standard case as mentioned above.

The Hamiltonian density is defined in the usual way by

$$\begin{aligned} \mathcal{H} &= \pi\dot{\xi} + W_{\mu} \dot{y}^{\mu} - \bar{\mathcal{L}} \\ &= \pi(\xi_{\perp} \dot{y}_{\perp} + \xi_{/\!/}^{r} g_{\nu\mu} y^{\nu}_{,r} \dot{y}^{\mu}) + W_{\perp} \dot{y}_{\perp} + \gamma_{rs} W_{/\!/}^{r} \dot{y}_{/\!/}^{s} - \mathcal{L}\dot{y}_{\perp} \Gamma \\ &= \dot{y}_{\perp}[W_{\perp} + \pi \xi_{\perp} - \mathcal{L}\Gamma] + \gamma_{rs} \dot{y}_{/\!/}^{s}[W_{/\!/}^{r} + \pi \xi_{/\!/}^{r}] . \end{aligned}$$

Let $\phi_{\mu} = 0$ denote the four primary constraints, then

$$\mathcal{H} = \dot{y}^{\mu} \phi_{\mu} ,$$

$$H = \int \dot{y}^{\mu} \phi_{\mu} d^{3}\nu = \int (\dot{y}_{\perp} \phi_{\perp} + \dot{y}_{/\!/ r} \phi^{r}) d^{3}\nu .$$

The most striking feature is that H and $\mathcal{H}$ vanish weakly independent of $\mathring{y}^{\mu}$.

In analogy with our previous Hamiltonian Formulation, we need to define a

$$\mathcal{H}_T = \mathcal{H} + U'^{\mu} \phi_{\mu} ,$$

$$H_T = \int \mathcal{H}_T d^3\nu .$$

Since U'^{μ} are arbitrary we can let $U'^{\mu} = U^{\mu} - \mathring{y}^{\mu}$. Hence

$$\mathcal{H}_T = U^{\mu} \phi_{\mu} \ ; \ H_T \approx \int U^{\mu} \phi_{\mu} d^3\nu .$$

For simplicity, we only examine here a field which does not lead to secondary constraints. This means that all the four ϕ_{μ} are first class and so is H_T.

Development of any functional F of canonical variables along different space-like surfaces is given by

$$\mathring{F} = [F,H] .$$

In particular, if we try $F = y^{\mu}$. We immediately get

$$\mathring{y}^{\mu} = U^{\mu} .$$

So we may write

$$H_T = H = \int d^3\nu (\mathring{y}^{\mu} \phi_{\mu}) = \int d^3\nu (\mathring{y}_{\perp} \phi_{\perp} + \mathring{y}_{\parallel r} \phi^{r}) .$$

The rules given in section (2.4.1.2) for treating ϕ_μ, $U^\mu = \dot{y}^\mu$ in the evaluation of Poisson Bracket must be obeyed.

Now the change in the functional F as we go from one surface $\mathcal{S}$ to a neighbouring one $\mathcal{S}'$ is

$$\delta F = \dot{F}\ \delta\tau = [F,H]\ \delta\tau = [F, \int d^3\nu\ \delta y^\mu\ \phi_\mu]$$

$$= [F, \int d^3\nu (\delta y_\perp\ \phi_\perp + \delta y_{/\!/ r}\ \phi_{/\!/}^r)] \ , \qquad (2.4.2.2\text{-}5)$$

where δy^μ are the change in y^μ as one moves to the neighbouring surface. The first term in H may be regarded as corresponding to a perpendicular displacement of the surface $\mathcal{S}'$ while the second term corresponds to a tangential displacement. Again by letting $F = y^\mu$ we have

$$\delta y^\mu = [y^\mu, \int d^3\nu\ \delta y^\rho\ \phi_\rho] \ . \qquad (2.4.2.2\text{-}6)$$

In the sense of equations (2.4.2.2-5), (2.4.2.2-6), ϕ_μ may be regarded as the generators of the infinitesimal transformation that changes the surface $\mathcal{S}$. As mentioned before, a change of surface may be regarded as a change of coordinate system, so ϕ_μ are essentially the generators for the infinitesimal canonical transformation which corresponds to a coordinate transformation.

2.4.2.3 The Problem of Covariance

The Hamiltonian theory discussed above is manifestly covariant with respect to arbitrary surface deformations which leave the surface space-like. Hence it is covariant to arbitrary coordinate transformations as well. The generators for infinitesimal transformations are obtained.

We know that our field variables are geometric objects which constitute faithful realizations of the group of general coordinate transformations. In this connection, we want to see whether equation (2.2-1) would lead to any condition to be satisfied by the generators. We have

$$\bar{\delta}_1\, y^\mu = [y^\mu, \int d^3\nu\, \bar{\delta}_1\, y^\mu\, \phi_\mu]$$

$$= [y^\mu, H_1], \text{ where } H_1 = \int d^3\nu\, \bar{\delta}_1 y^\mu\, \phi_\mu \ ,$$

$$\bar{\delta}_2(\bar{\delta}_1\, y^\mu) = [\bar{\delta}_1\, y^\mu, H_2] = [[y^\mu, H_1], H_2] \ ,$$

$$\bar{\delta}_1(\bar{\delta}_2\, y^\mu) = [\bar{\delta}_2\, y^\mu, H_1] = [[y^\mu, H_2], H_1] \ ,$$

$$\bar{\delta}_1(\bar{\delta}_2\, y^\mu) - \bar{\delta}_2(\bar{\delta}_1\, y^\mu) = [y^\mu, [H_1, H_2]] \ .$$

The Jacobi identity for Poisson Bracket was used.

So we require

$$[H_1, H_2] = \int d^3\nu\, d^3\nu'\, \bar{\delta}_1\, y^\mu\, \bar{\delta}_2\, y^\nu\, [\phi_\mu(\tau,\nu), \phi_\nu(\tau,\nu')] \ ,$$

be again generators.

Hence it is necessary that

$$[\phi_\mu(\tau,\nu), \phi_\nu(\tau,\nu')] \ , \qquad (2.4.2.3\text{-}1)$$

be weakly vanishing.

These are certainly satisfied in theories where only first class constraints appear. However things are not so trivial in quantum theory.

CHAPTER 3

A REVIEW OF GENERALLY COVARIANT FIELD THEORIES QUANTUM THEORIES

3.1 Introduction

One of the main difficulties in effecting the quantization of a covariant theory lies in the existence of constraints which imply that not all the canonical variables are independent. Different procedures to tackle this problem lead to different quantization schemes. In what follows we shall only consider fields which involve first class primary constraints and which do not demand any other consistency conditions in the classical theory.

3.2 Quantization I

3.2.1 The Quantization Scheme

One would like to investigate the possibility of a manifestly covariant quantization scheme. Therefore all the canonical variables should be treated on an equal footing. The scheme for quantization is well known.[C1] The state is to be described by a vector of a certain linear vector space while the canonical variables are represented by operators in the linear vector space. We cannot achieve complete mathematical rigour because the precise mathematical nature of the linear vector space and the operators are not known at present. Nevertheless we can still proceed to build up the theory by making assumptions that appear reasonable and that are necessary.

Following Dirac, who gave a thorough and systematic treatment of the whole problem in his monograph "Lectures on Quantum Field Theory"[C1] published in 1966, we assume the linear vector space is one endowed with non-negative metric. In analogy with the Hilbert space case, we also assume that it is possible to define Hermitian conjugates of a certain class of operators, Hermitian operators and unitary operators possessing the usual meaning and obeying the usual algebraic rules of manipulation, e.g. $(\Omega_1\Omega_2)^\dagger = \Omega_2^\dagger\Omega_1^\dagger$, $(\Omega_1 + \Omega_2)^\dagger = \Omega_1^\dagger + \Omega_2^\dagger$. This is not as trivial as it seems. We shall see presently that Hermitian conjugates are not definable for a wide class of operators because of the treatment of constraints used.

Now canonical variables are assumed to be represented by Hermitian operators obeying the usual commutation relations. The constraint equations are taken as subsidiary conditions to be satisfied by vectors which describe the states of the system. The meaning of the last statement is as follows. Let $\phi_\mu(x) \approx 0$ denote the four first class primary constraints. We then assume that vectors $|\Phi\rangle$ exist such that

$$\phi_\mu(x^\mu) \; |\Psi\rangle = 0 \; . \qquad (3.2.1\text{-}1)$$

Such vectors are called physical vectors.[C2] The statement above implies only physical vectors can be used to describe the states of the field. An operator Ω is called physical[C2] if on operating on any physical vector it gives another physical vector. The set of all physical vectors form a subspace and the physical operators operate in the subspace. The necessary and sufficient condition for Ω to be physical is that

$$[\phi_\mu(x^\mu), \Omega] \, |\Psi\rangle = 0, \text{ for all physical vectors } |\Psi\rangle ,$$

$$\text{i.e. } [\phi_\mu, \Omega] \approx 0 .$$

Recalling the fact that ϕ_μ are the infinitesimal generators for the classical canonical transformations that correspond to infinitesimal coordinate transformations, one may anticipate the significance of the physical vectors and operators. As far as operations in the subspace are concerned, they are invariants with respect to coordinate transformations, a fact that will be explicitly demonstrated when we study model field theories later. Note that there are not just four subsidiary conditions on the physical vectors, but a four-fold infinity of conditions. Consider (3.2.1-1) as applied to a constant x° space-like surface. The conditions will also imply

$$\phi_\nu(\underline{x}') \; \phi_\mu(\underline{x}) \; |\Phi\rangle = 0 \; ,$$

$$\phi_\mu(\underline{x}) \; \phi_\nu(\underline{x}') \; |\Psi\rangle = 0 \; ,$$

hence

$$[\phi_\mu(\underline{x}), \; \phi_\nu(\underline{x}')] \; |\Psi\rangle = 0 \; . \qquad (3.2.1\text{-}2)$$

Classically, the first class nature of ϕ_μ automatically ensures that the Poisson Brackets between themselves are first class constraints again, therefore no new condition arises. In quantum theory however, equations (3.2.1-1) do not automatically give equations (3.2.1-2). Failure to satisfy (3.2.1-2) would mean an inconsistency and we could not then quantize the field by this scheme. The conditions (3.2.1-2) for two different constant x^0 surfaces cannot be tested without solving the Heisenberg equations of motion first. Therefore we shall assume that they would be satisfied as well for a sensible model field theory once they are all right in the same constant x^0 surface. All these are assumed to be the case in what follows.

We now want to study some more properties of physical operators. First of all we see that Ω_1, Ω_2 being physical implies $\Omega_1 + \Omega_2$, $\Omega_1\Omega_2$, $\Omega_2\Omega_1$ hence $[\Omega_1, \Omega_2]$ to be physical. In a sensible theory the Hamiltonian H_T must be physical i.e.

$$[\phi_\mu, H_T] \, |\Psi\rangle = 0 \ ,$$

since we only consider theories with first class primary constraints alone. Therefore if Ω is physical, so is $\dot{\Omega} = \frac{[\Omega, H_T]}{i\hbar}$. Thus a physical operator always remains physical. We now come to the problem of Hermiticity of operators. By now it is clear that we need only to work in the subspace of physical vectors for the study of a physical system. So only the properties of operators which are relevant when applying to physical vectors will be important to us. It is therefore sufficient to define many properties of operators by their action on physical vectors only. A scheme by Dirac[C2] will be followed here. We take the constraints ϕ_μ to be Hermitian, or perhaps weakly Hermitian, because on operating on any physical vectors, they always give zero. We write $\phi_\mu \approx \phi_\mu^\dagger$. Let Ω be any operator. We shall again have $\Omega\phi_\mu$ weakly Hermitian. Suppose Ω is a non-physical Hermitian operator, then

$$\Omega\phi_\mu \approx (\Omega\phi_\mu)^\dagger \quad ; \quad (\Omega\phi_\mu)^\dagger \approx \phi_\mu\Omega$$

$$\Longrightarrow \qquad \Omega\phi_\mu \approx \phi_\mu\Omega \ .$$

This last weak equality cannot be true. This means that if we keep the usual algebraic rules for manipulating Hermitian operators as postulated, the Hermiticity of a non-physical operator is not definable in general. Only physical operators may be assumed to have Hermitian conjugates or to be Hermitian. Note that although Ω itself may not be physical, $(\Omega\phi_\mu)$ as a whole is physical and Hermitian since

$$[\phi_\nu(\underset{\sim}{x}'),\Omega\phi_\mu(\underset{\sim}{x})] = [\phi_\nu',\Omega]\ \phi_\mu + \Omega\ [\phi_\nu',\phi_\mu] \approx 0 \ .$$

Therefore the total Hamiltonian H_T is Hermitian.

3.2.2 The Problem of Covariance

As we have shown in classical theory, the ϕ_μ are the infinitesimal generators for canonical transformations which correspond to coordinate transformations. It has also been shown that equations (2.4.2.3-1) are necessary in order that the classical field variables constitute faithfull realizations of the group of general coordinate transformations. The corresponding situation must exist in quantum theory and equations (3.2.1-2) are necessary if the canonical operators and the linear vector space are to form a faithful realization of the group of general coordinate transformations. Therefore a theory which is covariant classically does not possess a covariant quantum theory if equations (3.2.1-2) are not satisfied. Anderson[C3] has shown that equations (3.2.1-2) are not satisfied for Einstein's equations in General Relativity. If this is correct, it would lead to the serious consequence that one may establish preferred coordinate systems from the set of inequivalent quantum theories obtained by using different coordinate conditions. The spirit of the principle of general covariance would be contradicted. This was emphatically pointed out by Anderson.

Although equations (3.2.1-2) are only necessary, one hopes that in a sensible theory, one would indeed obtain a realization of the group of general coordinate transformations. The explicit form of the quantum unitary transformations corresponding to classical coordinate transformations will be demonstrated when we consider model field theories.

3.3 Quantization II

3.3.1 Introduction

Suppose for a particular theory Quantization I may be successfully carried out. Then we are in a position to do some more analysis of the quantum theory as applied to a particular type of coordinate system. The selection of coordinate system is carried out by imposing coordinate conditions. This does not spoil the covariant nature of the theory as quantities in different coordinate systems are related by unitary transformations. Although this step of selection of coordinate system appears to add nothing fundamental to the general theory of Quantization I, it is important and is a very difficult problem in its own right, both from a physical point of view and a technical point of view. A general theory dealing with coordinate conditions will be given below. However it should be pointed out that as far as this problem is concerned, any particular field theory will have its own individuality and should be treated separately.

3.3.2 The Dirac Bracket

Before going into the problem of coordinate conditions, let us consider the problem of second class constraints in quantum theory. Obviously they cannot be treated as subsidiary conditions on physical state vectors as this immediately leads to inconsistency because the Poisson Brackets between themselves do not vanish weakly. Dirac[C4] proposed the following treatment. For definiteness, let us consider a theory of classical particle mechanics which leads to a set of first class constraints $\phi_m \approx 0$; m=1,...M and a set of second class constraints $\chi_s \approx 0$; s=1,...N. Firstly we define $C_{ss'}$ by

$$C_{ss'} \; [\chi_{s'}, \; \chi_{s''}] = \delta_{ss''} .$$

This is possible as it may be shown[C4] that the determinant of the matrix $[\chi_{s'}, \; \chi_{s''}]$ does not vanish. Secondly we define a new type of Bracket to be called the Dirac Bracket between two quantities F,G by

$$[F,G]_D = [F,G] - [F,\chi_s] \; C_{ss'} \, [\chi_{s'},G] \; .$$

The Db's may be shown to satisfy the usual properties of the Pb's. Another two important properties of the Db's are

(1) A second class constraint always has strongly vanishing Db with any quantity,

(2) The equations of motion are valid for Db's i.e.

$$\dot{G} \approx [G,H_T] \approx [G,H_T]_D \; ,$$

$$\because \; [G,H_T]_D = [G,H_T] - [G,\chi_s] \; C_{ss'} \, [\chi_s ,H_T]$$

and $[\chi_s ,H_T] \approx 0$ by consistency equations.

These properties of the Db's imply two things. We may replace Pb's by Db's in equations of motion and χ_s may be set to zero even before evaluating Db's. Therefore we may treat $\chi_s \approx 0$ as strong equations provided we replace Pb's by Db's. The introduction of the Db's sheds a new light on the significance of the second class constraints. They signify the existence of redundant variables which can be eliminated using the $\chi_s = 0$ as strong equations.

We can now pass over to quantum theory by taking the quantum commutation relations to correspond to the Db's. The $\chi_s = 0$ are taken as strong operator equations while the treatment of the first class constraints remains as before. We can eliminate the redundant variables as in classical theory.

3.3.3 Coordinate Conditions

The general idea is to treat coordinate conditions as an additional set of four constraints. Since the purpose of coordinate conditions is to eliminate four redundant degrees of freedom, i.e. eight canonical variables, the coordinate conditions should be such that they may be treated as secondary constraints and they also cause four of the original first class constraints to become second class. One then passes over to quantum theory using Db's. We therefore have eight strong equations to eliminate 8 redundant variables.

3.4 Quantization III

There is another approach due to Bergmann and Komar.[C5] It is based on the concept of observable, a name borrowed from quantum terminology. Firstly we examine how physical states of a field may be defined. Consider an infinite space-like surface $\mathcal{S}$. We may specify a complete set P of values of the field variables y_A and π^A on $\mathcal{S}$ and assume that this defines a physical state of the field on $\mathcal{S}$.

The set of values of field variables on a next such surface $\mathcal{S}'$ are obtainable from P by the solutions to the equations of motion for y_A and π^A. For covariant theory, this set will not be unique as there are arbitrary functions coming in the solutions of the equations of motion. Let P' be the totality of all these sets of values on $\mathcal{S}'$. We must consider all the seemingly different members of P' to correspond to the same physical state on $\mathcal{S}'$ which evolves from the same state on $\mathcal{S}$. In general, a physical state corresponds to many different sets of values of field variables. Supposing there are quantities which may be predicted uniquely from the initial data on $\mathcal{S}$, it is obvious that these quantities must be intrinsic to the physical state of the system and there may be a one-one correspondence between a complete set of these quantities and the physical state of the field. Such quantities are called observables. An invariant is an obvious observable. One may then build up a mode of description of the field using purely observables. The formulation will be covariant as well as being free from the arbitrariness arising from coordinate transformations. Suppose we can find four scalars which may be used to specify points in space-time uniquely. We may formally regard these four scalars as a special set of coordinates. Any quantity expressed as a function of these special coordinates alone will be an observable and a theory may be conveniently formulated.

CHAPTER 4

MODEL FIELD THEORY I

4.1 Introduction

To begin with, one would like to have an exactly soluble model field in order to be able to carry through the theories outlined before and see what the quantum theory looks like. In particular one wants to study the effect of general covariance and examine whether general covariance by itself might bring in something physically new into quantum theory. An extreme example of a generally covariant theory consisting of four field variables A_μ is taken. In such a theory the field equations will be empty in the sense that all the four A_μ will remain arbitrary and unrestricted by the field equations. In our particular example, it is possible to introduce some more essentially redundant variables into the theory to enable it to look more "respectable".

4.2 Model Field I

4.2.1 Lagrange Formalism

Consider a general four-dimensional space in which a metric is not defined but well-behaved coordinate systems may be defined.

Let A_μ be a covariant vector so that $A_{\mu,\nu} - A_{\nu,\mu}$ is a covariant anti-symmetric tensor. Then

$$\varepsilon^{\tau\kappa\mu\lambda}(A_{\kappa,\tau} - A_{\tau,\kappa})(A_{\mu,\lambda} - A_{\lambda,\mu})$$

is a scalar density of weight +1, where $\varepsilon^{\tau\kappa\lambda\mu}$ is the Levi-Civita symbol which is a tensor density of weight +1. We may therefore use this as a Lagrangian density to obtain a variational principle which will lead to a set of covariant field equations.

Now consider the variational principle

$$\delta \int \mathcal{L}\ d^4x = 0, \text{ with } \mathcal{L} = \tfrac{1}{4}\ \varepsilon^{\tau\kappa\mu\lambda}(A_{\kappa,\tau} - A_{\tau,\kappa})(A_{\mu,\lambda} - A_{\lambda,\mu}) \ .$$

The field equations for A_μ are the Euler-Lagrange equations which turn out to be

$$\varepsilon^{\alpha\beta\tau\kappa}\ A_{\kappa,\tau\beta} = 0 \ . \qquad (4.2.1\text{-}1)$$

The left hand side is identically zero, a fact which could have been anticipated from the fact that $\mathcal{L}$ may be written as the divergence

$$(\varepsilon^{\tau\kappa\lambda\mu}\ A_{\kappa,\tau}\ A_\mu)_{,\lambda} \ .$$

The general "solutions" are then just four arbitrary functions. We have a field theory with no genuine field equations in the usual sense. Still we shall formally carry on with the Hamiltonian formalism to see what sort of quantum theory turns out at the end.

4.2.2 Hamiltonian Formalism

Define the canonical momentum conjugate to A_μ by

$$P^\mu = \frac{\partial \mathcal{L}}{\partial A_{\mu,0}} = 2\,\varepsilon^{0\mu\rho\sigma}\,A_{\rho,\sigma}\ .$$

One immediately obtains four primary constraints $\phi^\beta \approx 0$ with

$$\phi^o = P^o ,$$

$$\phi^j = P^j - 2\,\varepsilon^{ojmn}\,A_{m,n}\ .$$

The total Hamiltonian density is

$$\mathcal{H}_T = \varepsilon^{j\rho i\sigma}\,A_{\rho,j}\,A_{\sigma,i} + U_\beta\,\phi^\beta\ ,$$

where U_β are arbitrary apart from possible restrictions arising from the consistency equations.

The constraints are all first class and there are no further constraints or consistency equations involving U_β. The canonical equations of motion for A_μ and P^μ give

$A_{\mu,0} = U_\mu$ which are left to be arbitrary, and

$$P^\mu{}_{,0} = (2\,\varepsilon^{o\mu\rho\sigma}\,A_{\rho,\sigma o})\ .$$

Again there are no genuine equations of motion as expected.

We may go on formally to discuss physical states of the field. Consider an infinite constant x^0 surface $\mathcal{S}$ in the four-dimensional space. A specification of the set of values of A_μ and P^μ on the surface should define a physical state of the field on that surface. However as we move away from this initial surface, the values of A_μ become completely arbitrary. We must conclude, by an argument mentioned before, that all the different sets of A_μ, P^μ on another constant x^0 surface $\mathcal{S}'$ correspond to the same physical state on that surface. We now have the situation that completely arbitrary A_μ correspond to the same physical state on $\mathcal{S}'$. By reversing the procedure, one can say that the same situation exists on the original surface $\mathcal{S}$. Therefore we end up with a single physical state possible for such a model field theory.

4.2.3 Quantization

This is straightforward. We assume the existence of a linear vector space such that $A_\mu(x), P^\nu(x)$ are operators in this space with commutation relations

$$[A_\mu(\underline{x}), P^\nu(\underline{x}')] = i\hbar c\,\delta_\mu^\nu\,\delta(\underline{x}-\underline{x}') \;;\; [A_\mu(\underline{x}), A_\nu(\underline{x}')] = 0 \;;\; [P^\mu(\underline{x}), P^\nu(\underline{x}')] = 0.$$

The state vectors $|\Psi\rangle$ are those vectors which satisfy the subsidiary conditions

$$\phi^\mu(\underline{x})\ |\Psi\rangle = 0 \tag{4.2.3-1}$$

This procedure works since the first class nature of the constraints as operators is preserved in our case.

One can solve the subsidiary conditions for $|\Psi\rangle$ in the Schrödinger Picture using the functional representation in which $P^{\mu}(\underline{x})$ goes to $-i\hbar c \frac{\delta}{\delta A_{\mu}(\underline{x})}$ and $|\Psi\rangle$ becomes a functional of $A_{\mu}(\underline{x})$. The operator $\frac{\delta}{\delta A_{\mu}(\underline{x})}$ operating on a functional $F(A_{\mu})$ of $A_{\mu}(x)$ is defined by

$$\delta F(A_{\mu}) = \int [\frac{\delta}{\delta A_{\mu}(\underline{x})} F(A_{\mu})] \; \delta A_{\mu}(x) \; d^3x$$

Now equation (4.2.3-1) becomes

$$(i\hbar c \frac{\delta}{\delta A_{\mu}} + 2 \; \varepsilon^{0\mu\rho\sigma} \; A_{\rho,\sigma}) \; |\Psi\rangle = 0 \; .$$

This gives

$$|\Psi\rangle = \text{constant } e^{\frac{i}{\hbar c}\int \varepsilon^{0jmn} \; A_j \; A_{m,n} \; d^3x} \; .$$

This is rather a surprising result showing that there is only one physical state, although it agrees with the previous analysis that only a single physical state should exist in classical theory. The total Hamiltonian is

$$H_T = \int d^3x \, \mathcal{H}_T = \int d^3x \; U_{\beta}\phi^{\beta} \; ,$$ the first term in $\mathcal{H}_T$ being zero on integration with the boundary condition that $A_{\mu,j} = 0$ at infinity. Hence

$$H_T \; |\Psi\rangle = 0 \; ,$$

which implies that $|\Psi>$ is a time-independent state vector corresponding to zero eigenvalue of the total Hamiltonian. All this is most reasonable since we actually started with an "empty" field. The transformation properties of the theory can be demonstrated without much trouble. Consider an infinitesimal coordinate transformation from x^μ to $\bar{x}^\mu = x^\mu + \varepsilon^\mu$. As a result $A_\lambda(x)$, $P^\lambda(x)$ go to $\bar{A}_\lambda(x)$, $\bar{P}^\lambda(x)$. The corresponding transformation in the classical theory is given by the generating functional

$$G = \int d^3x \quad \bar{\delta}A_\mu (P^\mu - 2\,\varepsilon^{o\mu mn} A_{m,n})$$

where $\bar{\delta}A_\mu = \bar{A}_\mu(x) - A_\mu(x) = \delta A_\mu - A_{\mu,\nu}\,\varepsilon^\nu$

$$\delta A_\mu = \bar{A}_\mu(\bar{x}) - A_\mu(x) = - A_\nu\, \varepsilon^\nu_{,\mu}$$

At first sight it seems surprising that the generating functional should contain the velocity variables $A_{\mu,o}$ which cannot be expressed in terms of the canonical variables. The involvement of $A_{\mu,o}$ is unavoidable because the transformation law for A_μ contains such quantities explicitly. However there are in fact no real difficulties in calculations as the $\bar{\delta}A_\mu$ are multiplied by the weakly vanishing constraints.

We have

$$\bar{\delta}A_\lambda(x) = \frac{\partial G}{\partial P^\lambda(x)}$$, the right hand side giving back $\bar{\delta}A_\lambda(x)$ as expected

while

$$\bar{\delta}P^\lambda(x) = - \frac{\partial G}{\partial A_\lambda(x)} = 2\,\varepsilon^{o\lambda mn}\,\bar{\delta}A_{m,n} .$$

We see that

$$(P^\lambda - 2\varepsilon^{o\lambda mn} A_{m,n}) \approx 0 \Longrightarrow \bar{\delta}(P^\lambda - 2\varepsilon^{o\lambda mn} A_{m,n}) \approx 0$$

which implies that the constraint equations are preserved as they should be.

In quantum theory, the corresponding infinitesimal unitary transformation is effected by

$$U = e^{\frac{i}{\hbar}G} = (1 + \frac{i}{\hbar}G)$$

where $G = \int d^3x\, \bar{\delta}A_\mu (P^\mu - 2\varepsilon^{o\mu mn} A_{m,n})$, the variables occurring being operators of course.

This generator G is Hermitian in the sense defined in the previous chapter. Let Ω be a physical operator and $|\Psi\rangle$ be a physical vector. Then

$$|\Psi'\rangle = U|\Psi\rangle \approx |\Psi\rangle$$

$$\Omega' = U\Omega U^\dagger = \Omega + \frac{i}{\hbar}[G,\Omega] \approx \Omega$$

These confirm our previous anticipation that physical quantities so defined are invariants with respect to coordinate transformations. One may observe that this situation holds even for a Lorentz transformation which is surprising since this is not true for the usual Lorentz invariant quantum field theory. We shall give a thorough examination on this later on.

4.2.4 Coordinate conditions

Since the solutions of the field equations consist of four arbitrary functions, we are free to impose four conditions on the field variables. An example will be the requirements

$$\eta^{\rho\sigma} A_{\mu,\rho\sigma} = 0 , \qquad (4.2.4\text{-}1)$$

where $\eta^{\rho\sigma} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \eta_{\rho\sigma}$.

These equations certainly limit the possible A_μ considerably. The common practice is to interpret such A_μ as being the field viewed by an observer in a particular type of coordinate system. In our case, we can actually manufacture all sorts of arbitrary conditions leading to totally different effective field equations for different observers. A paradoxical situation appears. An observer with conditions (4.2.4-1) may treat the field as a vector meson field. He then solves his effective field equations (4.2.4-1). It will appear to him that different physical states exist defined by the various plane wave solutions. He may further formulate a quantum theory. All this contradicts our previous conclusion that only one single physical state is possible for the field. Let us examine how this particular observer arises at his different physical states. He will specify a set P of initial data on a constant x^0 surface $\mathcal{S}$ and the field variables on another such surface $\mathcal{S}'$ are obtainable from the set P by solutions of (4.2.4-1). The values on $\mathcal{S}'$ may not be unique. Let P' be the totality of the different sets of values obtained from the set P on $\mathcal{S}$. With equations (4.2.4-1) it is possible for him to find another set of initial data $\bar{P}$ on $\mathcal{S}$ such that the corresponding set $\bar{P}'$ on $\mathcal{S}'$ is totally distinct from P'. He may then proclaim that the sets P and P' do represent different physical states because they lead to distinct subsequent motion. However imposition of conditions (4.2.4-1) merely reduces the completely arbitrary nature of the field variables. The fact that all the different field variables compatible with (4.2.4-1) still represent the same physical state cannot be changed. Therefore all these seemingly different solutions must correspond to the same physical state. Let us go a step further to see how exactly this must be the case and indeed to see exactly how the paradox arises in the first place. In a generally covariant theory, it is usually assumed that all coordinate systems are equivalent for the description of the physical system concerned and that the equations of motion must be generally covariant. What may be usually forgotten is that the coordinate variables x^μ appearing in the covariant equations of motion are not and cannot be meant to be the actual coordinates of any definite coordinate frame of a particular observer. Given two different solutions of the covariant equations as functions of x^μ, e.g. in our present case for equations (4.2.1-1) $A_\mu(x) = f_\mu(x)$, $A'_\mu(x) = F_\mu(x)$ where f_μ, F_μ are arbitrary functions. One cannot at once assert that they represent two different states for the physical system as seen by an observer in the coordinate system x^μ. In order to relate the x^μ appearing in the covariant equations to the actual coordinates of a particular observer, it is necessary to impose non-covariant coordinate conditions on the field variables. Furthermore the number of such conditions must be such that they exclude all but one coordinate system, that is, there is only one coordinate system in which the field variables satisfy all these coordinate conditions in addition to the equations of motion. Only after all this has been done can we identify x^μ as the actual coordinates of an observer. The observer can now proceed in the usual way to define physical states for the system concerned. Two distinct solutions to all those conditions and the field equations will now mean two distinct physical states. Now the coordinate conditions (4.2.4-1), though four in number, do not satisfy the above requirement. These conditions only restrict coordinate systems to a special set related by Lorentz transformations. Even at this stage it is still not permissible to identify the x^μ appearing in (4.2.4-1) with the actual coordinates of a member of the set. Consider two solutions to (4.2.4-1): $a_\mu e^{i\eta_{\alpha\beta}k^\alpha x^\beta}$ and $a'_\mu e^{i\eta_{\alpha\beta}k'^\alpha x^\beta}$ where a_μ, a'_μ are constants and $\eta_{\alpha\beta}k^\alpha k^\beta = 0 = \eta_{\alpha\beta}k'^\alpha k'^\beta$. We cannot claim that these two represent two different physical states as viewed by an observer attached to a coordinate frame belonging to the set. In order to do this, we have to impose further conditions which may take the form of initial data on $x^o = 0$ surface in such a way as to exclude any Lorentz transformation. These initial data will then lead to a unique solution to (4.2.4-1). So finally we see that for a particular observer indeed there is only one unique set of values for the field variables leading to only a single physical state. The different solutions as given above are now seen to correspond to the same state as viewed by different observers! One must therefore be extremely careful in handling coordinate conditions and in the subsequent interpretations. It is not sufficient just to count the number of coordinate conditions.

A similar situation exists in quantum theory. One should not take (4.2.4-1) as field equations to carry out quantization <u>and</u> interpret the results in the usual manner. There is nothing to prevent one from blindly quantizing the theory with arbitrary coordinate conditions such as (4.2.4-1), but having done so one may have to identify physical states with subspaces rather than single vectors in the linear vector space. The vectors in the subspaces are related by unitary transformations as allowed by the coordinate conditions[D1].

4.3 General Covariance Versus Lorentz Covariance

From the discussion in the previous section, a fundamental difference between a generally covariant theory and a Lorentz covariant theory emerges. In the latter case the equations of motion are Lorentz covariant. The important point now is that the coordinates x^μ appearing in the equations of motion in a Lorentz covariant theory are meant to be the actual coordinates of an inertial observer (measured on standard metre sticks and clocks at rest relative to himself). As a result, two distinct solutions to the equations of motion imply two distinct physical states. In a Lorentz covariant theory this is not an arbitrary assumption and the above conclusion may be tested by physical measurements made by the observer.

It is well known that a Lorentz covariant theory may be extended to become formally generally covariant by the introduction of more variables such as the non-Minkowskian metric $g_{\mu\nu}$. The best known example[D2] is the extension of Maxwell's theory of electromagnetic field to a formally generally covariant theory. Let us examine the situation step by step in detail.

(1) The original Lorentz covariant theory of electromagnetic field: The space-time is assumed to be flat. We have a group of inertial frames of reference which are related by Lorentz transformations and in which the metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} \quad .$$

The electromagnetic field is described by the antisymmetric electromagnetic field tensor $F_{\mu\nu}$ satisfying the field equations

$$\begin{cases} \frac{\partial}{\partial x^\nu} F^{\mu\nu}(x) = 0 \ , \\ \frac{\partial}{\partial x^\alpha} F^{\beta\gamma}(x) + \frac{\partial}{\partial x^\beta} F^{\gamma\alpha}(x) + \frac{\partial}{\partial x^\gamma} F^{\alpha\beta}(x) = 0 \ . \end{cases} \tag{4.3-1}$$

The coordinates x^μ are the actual coordinates of a particular inertial observer who can then count and determine different physical states of the field by means of distinct solutions to (4.3-1) in the usual way. A Lorentz transformation form x^μ to x'^μ will lead to another set of field equations with x'^μ as coordinate variables. The new equations will be identical with (4.3-1) in form. The new coordinate variables x'^μ are to be interpreted as the actual coordinates of another inertial observer who may count physical states in exactly the same way as the first observer does. It is in this sense we say that all inertial observers are physically equivalent.

(2) Extension to a "formally" generally covariant theory:

One can establish a new set of field equations which are generally covariant and which reduces to (4.3-1) in an inertial frame. To do this we introduce an arbitrary coordinate variable x^μ in which the metric tensor $g_{\mu\nu} \neq \eta_{\mu\nu}$ in general even though the flatness nature of the space-time has not been changed. The electromagnetic field is again described by an antisymmetric tensor $F^{\mu\nu}(x)$ which satisfies the generally covariant equations[D2]

$$\begin{cases} F^{\mu\nu}(x)_{;\nu} = 0 \\ (\varepsilon^{\alpha\beta\gamma\delta}\sqrt{-g}\, F_{\alpha\beta})_{;\delta} = 0, \end{cases} \tag{4.3-2}$$

where $g = |g_{\mu\nu}|$, and the semi-colon denotes covariant differentiation. In this extended theory, we have brought in 10 new variables $g_{\mu\nu}$ which satisfy 20 equations expressing the flatness of the space-time

$$R_{\alpha\beta\gamma\delta} = 0 \ ,$$

where $R_{\alpha\beta\gamma\delta}$ is the Riemann curvature tensor. Since we are given that the original Lorentz covariant theory is the correct one, the extension to a formally generally covariant theory leads to nothing physically new at all. There can be no ambiguity in fixing various physical states, as we can refer things back to an inertial observer. It should be clear from our previous analysis that the arbitrary coordinate variables x^{μ} in (4.3-2) cannot be identified with the actual coordinates of a particular (non-inertial) observer for the purpose of state determination. In this extended theory, a whole set of different solutions to (4.3-2) may correspond to the same physical state.

(3) Reduction of the extended theory to the original Lorentz Covariant theory:

The reduction process is trivial. All we need to do is to impose the coordinate conditions

$$g_{\mu\nu} = \eta_{\mu\nu} .$$

(4) A fundamental question:

Now consider a completely new situation. Suppose we do not know the original Lorentz covariant theory of electromagnetic field, and suppose we are given a theory formulated in terms of arbitrary coordinate variables x^μ in a generally covariant manner with the field being described by an antisymmetric tensor $F^{\mu\nu}(x)$ satisfying the field equations

$$\begin{cases} F^{\mu\nu}(x)_{;\nu} = 0 , \\ (\varepsilon^{\alpha\beta\gamma\delta}\sqrt{-g}\, F_{\alpha\beta})_{;\delta} = 0, \end{cases}$$

where $g = |g_{\mu\nu}|$ and the metric tensor $g_{\mu\nu}$ satisfies the flatness conditions

$$R_{\alpha\beta\gamma\delta} = 0 .$$

Now how does one determine and count physical states?

The analysis given in the previous section tells us that there is no unique answer to this question as it stands. One can attempt to answer the question in one of the following two ways

(a) Although we are not given any preferred coordinate systems, we may still make the assumption that there are preferred reference frames defined by certain coordinate conditions, say,

$$g_{\mu\nu} = \eta_{\mu\nu} .$$

Furthermore one assumes that we can identify the coordinate variables x^μ in the field equations after the imposition of the coordinate conditions with the actual coordinates of a particular observer who can then count physical states in the usual way. It is in this sense these coordinate frames are termed "preferred". It is not because that the field equations become simpler using them. An important observation must be emphatically pointed out here, that is, the above are new physical assumptions not contained in the original theory. These assumptions imply that the theory given is only formally generally covariant and is extended from a Lorentz covariant theory.

(b) One considers the given theory as a "genuinely" generally covariant theory despite the fact that field equations become simpler in a certain set of coordinate frames. There are therefore no preferred set of frames in which one can count physical states in the usual way. One has to impose sufficient numbers of coordinate conditions (e.g. more than $g_{\mu\nu} = \eta_{\mu\nu}$) to single out a unique coordinate frame and only in such a unique frame can we start to distinguish physical states in the conventional way.

Note that one is again making new physical assumptions here. We see clearly now that additional physical assumptions are necessary in order to answer the fundamental question raised. There is nothing in the given theory which tells us definitely which is the correct answer. The final test must lie in the actual physical experiments for the determination of states. By comparing experimental results with the predictions of (a) and (b) respectively we can find out which of them is the correct one.

More examples may be given. An extreme one would be to consider the relationship between our Model Field I and a Lorentz invariant vector meson field theory in which the field variables B_μ satisfy the equations

$$\eta^{\rho\sigma} B_{\mu,\rho\sigma} = 0 \quad \text{for each } \mu .$$

The above arguments may be repeated step by step. The final conclusions will of course be the same. In quantum theory a similar situation exists. In a "genuinely" generally covariant theory, physical states are described only by physical vectors which are invariant with respect to arbitrary transformations of the coordinate variables x^μ, in particular Lorentz transformations. Vectors which are not Lorentz invariant cannot be used to describe states. It is therefore clear that two such vectors related by a Lorentz transformation cannot be regarded as representing two physical states. All this is fundamentally different from a Lorentz covariant quantum theory (or a "formally" generally covariant quantum theory). Another striking feature of a "genuinely" generally covariant quantum theory is that the physical system concerned seems to be "dead" in that the state vectors in Schrödinger picture are "time"-independent. We shall go into this problem in greater detail later in Chapter 9. For the moment it suffices to say that all this does not really mean anything just as in the classical theory a vanishing Hamiltonian does not imply zero energy of the system concerned.

4.4 Harmonic Coordinates in General Relativity

This is a subject of great arguments among people like Fock[D3] and J.L. Anderson[D4]. To begin with we recall that Fock[D5] has suggested that the harmonic coordinate conditions

$$\{\sqrt{-g}\ \ g^{\mu\nu}\}_{,\mu} = 0$$

together with certain conditions at infinity lead to a preferred set of coordinate systems. In particular he[D5] has shown, though in a not very mathematically rigorous way, that in the case of an isolated system of masses the harmonic conditions together with suitable supplementary conditions determine the coordinate system uniquely apart from Lorentz transformations. We shall not go into the philosophical argument as to what is meant by preferred coordinate systems. Instead we set ourselves the following definite question. Is it possible to tell by physical means whether the theory of General Relativity is "genuinely" generally covariant or whether it is just "formally" generally covariant and is extended from a theory covariant to a more restricted set of coordinate transformations such as the set of harmonic coordinates? According to our previous analysis, the answer should be affirmative. In case of harmonic coordinates we can do one of the two things. We can identify the coordinate variables with the actual coordinates of an "harmonic" observer for the purposes of determining physical states as Fock apparently did. The results can then be put to physical test. An experimental confirmation on one way or the other will answer our question which may not be conclusively solved by theoretical argument alone.

4.5 Some Remarks

As mentioned in section 4.1, we may modify our present Model Field I by introducing more variables. One choice is to consider the model theory derived from the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\,\varepsilon^{\tau\kappa\lambda\mu}\,F_{\tau\kappa}(A_{\mu,\lambda} - A_{\lambda,\mu} - \tfrac{1}{2}F_{\lambda\mu}) \ ,$$

where $F_{\tau\kappa}$ is an anti-symmetric tensor. The Euler-Lagrange equations of motion are

$$F_{\tau\kappa} = A_{\kappa,\tau} - A_{\tau,\kappa} \ ,$$

$$F_{\tau\kappa,\lambda} + F_{\kappa\lambda,\tau} + F_{\lambda\tau,\kappa} = 0 \ .$$

The general solutions are

$$A_\tau = f_\tau(x) \;,$$

$$F_{\tau\kappa} = f_{\kappa,\tau} - f_{\tau,\kappa} \;,$$

where $f_\tau(x)$ are four arbitrary functions of x^μ.
In the Hamiltonian theory, there are 10 primary constraints plus secondary consistency equations. There are second class constraints appearing, so Dirac's treatment has to be carried out in full including the use of Dirac Brackets. The calculations are somewhat long-winded and will not be reproduced here. However the results are essentially the same. There is again only a single physical state.
An alternative approach to the Dirac's one would be to eliminate those redundant variables $F_{\tau\kappa}$ at the early stage in the Lagrange formulation.
The results are of course the same.

CHAPTER 5

MODEL FIELD II

TREATMENT I

5.1 Introduction

In this chapter, we study a generally covariant and intrinsically non-linear model field theory.

Consider a 5-dimensional Pseudo-Euclidean space with coordinates ξ^A and metric

$$\eta_{AB} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

the indices A,B taking the values 0 to 4. Any 4-dimensional surface may be fixed by specifying the 5 coordinates ξ^A as functions of 4 parameters x^μ. In general the 4-surface is a 4-dimensional Riemannian space with a metric

$$g_{\mu\nu}(x) = \eta_{AB} \frac{\partial \xi^A}{\partial x^\mu} \frac{\partial \xi^B}{\partial x^\nu} = \eta_{AB}\, \xi^A_{,\mu}\, \xi^B_{,\nu} .$$

Let $\underline{\xi}$ denotes a column vector with components ξ^A and

$\overline{\underline{\xi}}$ denotes a row vector with components $\xi_A = \eta_{AB}\, \xi^B$.

Then we may write

$$g_{\tau\kappa} = \overline{\underline{\xi}}_{,\tau}\ \underline{\xi}_{,\kappa}\ .$$

The Christoffel symbols of the first kind and second kind may be shown to be

$$[\tau\kappa,\lambda] = \tfrac{1}{2}(g_{\tau\lambda,\kappa} + g_{\kappa\lambda,\tau} - g_{\tau\kappa,\lambda}) = \overline{\underline{\xi}}_{,\tau\kappa}\ \underline{\xi}_{,\lambda} = \overline{\underline{\xi}}_{,\lambda}\ \underline{\xi}_{,\tau\kappa}\ ;$$

$$\Gamma^{\lambda}_{\tau\kappa} = g^{\lambda\mu}[\tau\kappa,\mu] = g^{\lambda\mu}\ \overline{\underline{\xi}}_{,\mu}\ \underline{\xi}_{,\tau\kappa} = \overline{\underline{\xi}}^{\lambda}\ \underline{\xi}_{,\tau\kappa}\ ,$$

where $\overline{\underline{\xi}}^{\lambda} = g^{\lambda\mu}\ \underline{\xi}_{,\mu}$.

Note that $\underline{\xi}$ is a scalar with respect to arbitrary coordinate transformations in the 4-surface.

Covariant derivatives are defined in the usual way.

$$\underline{\xi}_{;\tau} = \underline{\xi}_{,\tau}, \text{ since } \underline{\xi} \text{ is a scalar.}$$

$$\begin{aligned}\underline{\xi}_{;\tau\kappa} &= \underline{\xi}_{,\tau\kappa} - \underline{\xi}_{,\lambda}\ \Gamma^{\lambda}_{\tau\kappa} \\ &= \underline{\xi}_{,\tau\kappa} - \underline{\xi}_{,\lambda}\ \overline{\underline{\xi}}^{\lambda}\ \underline{\xi}_{,\tau\kappa} \\ &= \underline{\xi}_{;\kappa\tau}.\end{aligned}$$

The curvature tensors may be shown to be

$$R_{\tau\kappa\lambda\mu} = \overline{\underline{\xi}}_{;\lambda\tau}\ \underline{\xi}_{;\mu\kappa} - \overline{\underline{\xi}}_{;\lambda\kappa}\ \underline{\xi}_{;\mu\tau}\ ;$$

$$R_{\kappa\lambda} = g^{\tau\mu}\ R_{\tau\kappa\lambda\mu}\ .$$

5.2 Definition of a 4-surface of Stationary 4-volume.

Consider these 4-surfaces whose metric satisfies the following conditions. The metric $g_{\mu\nu}$ is nonsingular, i.e. $g = |g_{\mu\nu}| \neq 0$.

Therefore the sign of g is the same throughout the surface and it is an invariant. The contravariant metric tensor $g^{\alpha\beta}$ exists such that

$$g^{\mu\beta} g_{\beta\nu} = \delta^{\mu}_{\nu}$$

We consider only surfaces with $g < 0$ so that the Pseudo-Euclidean flat surfaces are included as special cases.

The volume of a domain $x^{\mu} \in D$ of the 4-surface is

$$V = \int_D \sqrt{-g}\, d^4x \ ,$$

which is an invariant.

A 4-surface of stationary volume is defined to be a surface whose volume V is at a stationary value with respect to small arbitrary deformation of the surface. The deformation is realized mathematically by variation of the coordinates ξ^A. The variation $\delta\xi^A$ is to be taken as zero at the boundary of the domain D. For a variation $\delta\xi^A$,

$$\delta g = \frac{\partial g}{\partial g_{\tau\kappa}} \delta g_{\tau\kappa} = \frac{\partial g}{\partial g_{\tau\kappa}} 2 \eta_{AB}\, \xi^A_{,\tau}\, \delta\, \xi^B_{,\kappa}$$

$$= 2g\, g^{\tau\kappa} \eta_{AB}\, \xi^A_{,\tau}\, \delta\, \xi^B_{,\kappa} \quad ;$$

$$\delta V = \int \sqrt{-g}\;\; g^{\tau\kappa} \eta_{AB}\, \xi^A_{,\tau}\, \delta\, \xi^B_{,\kappa}\, d^4x$$

$$= - \int \delta\xi^B (\sqrt{-g}\;\; g^{\tau\kappa} \eta_{AB}\, \xi^A_{,\tau})_{,\kappa}\, d^4x \ .$$

Hence $(\sqrt{-g}\ \ g^{\tau\kappa}\ \ \eta_{AB}\ \xi^{A}_{,\tau})_{,\kappa} = 0$, if $\delta V = 0$ for arbitrary $\delta\xi^{A}$. So a 4-surface of stationary volume is defined by equations

$$(\sqrt{-g}\ \ g^{\tau\kappa}\ \xi^{A}_{,\tau})_{,\kappa} = 0, \quad \text{which are equivalent to}$$

$$g^{\tau\kappa}\ \xi^{A}_{;\tau\kappa} = 0. \tag{5.2-1}$$

The above equations are not all independent and certain identities, the Bianchi identities, exist among them. Rewriting (5.2-1) as

$$(1 - \underline{\xi}_{,\lambda}\underline{\bar{\xi}}^{\lambda})\ g^{\tau\kappa}\ \underline{\xi}_{,\tau\kappa} = 0,$$

we see that the Bianchi identities are

$$\xi_{A,\rho}(\delta^{A}_{B} - \xi^{A}_{,\mu}\ g^{\mu\nu}\ \xi_{B,\nu})g^{\tau\kappa}\ \xi^{B}_{,\tau\kappa} = 0,$$

$\xi_{A,\rho}$ being the four linearly independent null eigenvectors of $(1 - \underline{\xi}_{,\lambda}\underline{\bar{\xi}}^{\lambda})$. Let n^{A} be the unit vector normal to $\xi_{A,\rho}$, that is,

$$n^{A} = \acute{n}^{A}/|\acute{n}^{A}|, \quad \acute{n}^{A} = \eta^{AB}\ \epsilon_{BCDEF}\ \xi^{C}_{,0}\ \xi^{D}_{,1}\ \xi^{E}_{,2}\ \xi^{F}_{,3},$$

where ϵ_{BCDEF} is the 5-dimensional permutation symbol. Then the five field equations (5.2-1) are equivalent to the single equation

$$g^{\tau\kappa}\ n_{A}\ \xi^{A}_{,\tau\kappa} = 0.$$

5.3 A special set of coordinate conditions

A natural set of coordinate conditions would be

$$\xi^{\mu} = x^{\mu} \implies \xi^{\mu}_{,\nu} = \delta^{\mu}_{\nu},$$

that is, we just choose the first four of the original 5-dimensional pseudo-Euclidean coordinates as our coordinates in the four-surface.

In this special coordinate system, we have

$$g_{\mu\nu} = \eta_{\mu\nu} - \xi_{,\mu}\, \xi_{,\nu}, \quad \text{where } \xi \text{ denotes } \xi^A \text{ with } A = 4;$$

$$g^{\mu\nu} = \eta^{\mu\nu} + (\eta^{\mu\alpha}\, \eta^{\nu\beta}\, \xi_{,\alpha}\, \xi_{,\beta})/(1 - \eta^{\rho\sigma}\, \xi_{,\rho}\, \xi_{,\sigma});$$

$$g = \det|g_{\mu\nu}| = -(1 - \eta^{\rho\sigma}\, \xi_{,\rho}\, \xi_{,\sigma}).$$

The field equation is

$$g^{\mu\nu}\, \xi_{,\mu\nu} = 0, \quad \text{which may be written as}$$

$$\frac{\partial}{\partial x^\mu}\left(\frac{\eta^{\mu\nu}\, \xi_{,\nu}}{\sqrt{1 - \eta^{\rho\sigma}\, \xi_{,\rho}\, \xi_{,\sigma}}}\right) = 0. \tag{5.3-1}$$

Our variational principle becomes

$$\delta \int \sqrt{1 - \eta^{\rho\sigma}\, \xi_{,\rho}\, \xi_{,\sigma}}\; d^4x = 0.$$

These coordinate conditions will be discussed in greater detail in the next chapter. Meanwhile we shall return to formulate the general Hamiltonian theory for the field ξ^A. Although we shall not restrict ourselves to special coordinate systems, we shall only use coordinate systems in which x^0 = constant is always a space-like surface. We may now assume that the 4-surface extends to infinity by letting the coordinates x^μ to take all values from $-\infty$ to $+\infty$. The total Lagrangian will then diverge. This does not matter as the divergence may be eliminated by subtracting 1 from the integrand.

5.4 The Classical Hamiltonian Formulation

In a physical theory we want the Lagrangian density to have the dimensions of energy density and that the Hamiltonian should be positive. Therefore we adopt the Lagrangian density

$$\mathcal{L} = -Q\sqrt{-g} ,$$

where Q is a positive number of dimension $[\frac{\text{energy}}{\text{volume}}]$ and the minus sign will lead to a positive Hamiltonian as will be seen in next chapter. Define canonical momenta conjugate to ξ^A by

$$\pi_A = \frac{\partial\mathcal{L}}{\partial\xi^A_{,o}} = \eta_{AB}\,\mathcal{L}\,\xi^B_{,\beta}\,g^{o\beta} .$$

Define π^A by

$$\pi^A = \eta^{AB}\,\pi_B = \mathcal{L}\,\xi^A_{,\beta}\,g^{o\beta} .$$

Using the general theory one can obtain the primary constraints which may be easily seen to be

$$\phi_\mu \approx 0 ,$$

With $\phi_o = \pi^A\pi_A + Q^2\Delta^{oo}$, where $\Delta^{oo} = g\,g^{oo} = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}$;

$\phi_j = \xi^A_{,j}\,\pi_A$.

The Hamiltonian density is

$$\mathcal{H} = \pi_A\,\xi^A_{,o} - \mathcal{L} .$$

$\mathcal{L}$ is a homogeneous function in $\xi^A_{,o}$ of degree one, so $\mathcal{H} = 0$.

The total Hamiltonian density is therefore

$\mathcal{H}_T = U^\mu \phi_\mu$, where U^μ are arbitrary functions of x^μ.

Equation of motion for any functional F of ξ^A, π_A is

$$\frac{\partial}{\partial x^o} F = [F, \int d^3x\, U^\mu \phi_\mu] = \int d^3x\, U^\mu\, [F, \phi_\mu] \ .$$

In particular

$$\xi^A_{,o} = U^o 2\pi^A + U^j \xi^A_j \ ;$$

$$\pi_{A,0} = (U^i \pi_A + U^o\, Q^2\, \frac{\partial \Delta^{oo}}{\partial \xi^A_{,i}})_{,i} \ .$$

From the constraint equations we can also show that

$$\pi^A \frac{\partial \Delta^{oo}}{\partial \xi^A_{,j}} \approx 0 \quad ; \tag{5.4-1}$$

$$\xi^A_{,i}\, \pi_{A,j} - \xi^A_{,j}\, \pi_{A,i} \approx 0 \ . \tag{5.4-2}$$

Another useful expression is

$$\xi^A_{,j} \frac{\partial \Delta^{oo}}{\partial \xi^A_{,i}} = 2\Delta^{oo}\, \delta^j_i \ . \tag{5.4-3}$$

With the help of these, we can show that $\phi_\mu \approx 0$ are all first class and there are no further consistency equations necessary.

5.5 Quantization

We are in a position to attempt the scheme of Quantization I. We assume the existence of a linear vector space for the description of the field. The ξ^A, π_A are operators in this space satisfying the usual equal-time commutation relations

$$[\xi^A(\underline{x}), \pi_B(\underline{x}')] = i\hbar c\delta^A_B \, \delta(\underline{x}-\underline{x}') .$$

The physical state vectors $|\psi\rangle$ are those which satisfy the subsidiary conditions

$$\phi_\mu(x)|\psi\rangle = 0 .$$

The first thing is to check the consistency of these conditions on the same constant χ^o surface, that is, to see if

$$[\phi_\mu(\underline{x}), \phi_\nu(\underline{x}')] \approx 0 ,$$

as operators acting on physical vectors. The somewhat lengthy calculations are given in the Appendix 5.1 at the end of this chapter. The results show that these consistency conditions are indeed satisfied. Therefore we may conclude that the scheme of Quantization I may be consistently carried out.

Let us examine the subsidiary conditions

$$\xi^A_{,j}\pi_A \;|\Psi\rangle = 0 ; \tag{5.5-1}$$

$$(\pi^A\pi_A + Q^2\Delta^{oo})|\Psi\rangle = 0 , \tag{5.5-2}$$

in greater detail. From the general theory given in Chapter 2, we may expect the constraints to be the generators for infinitesimal canonical transformations which correspond to arbitrary deformations of the constant x^o surface. Three of them should be the generators for tangential surface deformation while the remaining one should be the generator for deformation normal to the surface. Suppose $\phi_j = \xi^A_{,j}\pi_A$ are the three generators for tangential surface deformation. Then the subsidiary conditions (5.5-1) express that $|\Psi>$ must be invariant under arbitrary tangential surface deformation. This is equivalent to the requirement that $|\Psi>$ be invariant under 3-dimensional coordinate transformations $x^j \rightarrow x'^j$ = function (x^1, x^2, x^3). We now show explicitly that this is indeed the case. Consider an infinitesimal coordinate transformation

$$x'^j = x^j + \varepsilon^j(\underset{\sim}{x}) \;;\quad x'^o = x^o.$$

Then

$$\xi'^A(x') = \xi^A(x) = \left(1 - \varepsilon^j(x') \frac{\partial}{\partial x'^j}\right)\xi^A(x'),$$

and

$$\bar{\delta}\xi^A(x') = \xi'^A(x') - \xi^A(x') = -\varepsilon^j(x') \frac{\partial \xi^A(x')}{\partial x'^j}.$$

The change in $|\Psi(\xi^A)>$ is

$$\begin{aligned}\delta|\Psi(\xi^A)> &= |\Psi(\xi'^A)> - |\Psi(\xi^A)> = \int d^3x' \; \bar{\delta}\xi^A(x') \frac{\delta}{\delta\xi^A(x')} |\Psi(\xi^A)> \\ &= -\int d^3x' \, \varepsilon^j(x') \; \xi^A_{,j}(x') \frac{\delta}{\delta\xi^A(x')} |\Psi(\xi^A)> \\ &= 0,\end{aligned}$$

using (5.5-1) and the functional representation

$$\xi^A \to \xi^A ;$$

$$\pi_A \to -i\hbar c \frac{\delta}{\delta \xi^A} .$$

In the next section, we shall discuss these transformation properties in a systematic way. Now that we know the significance of conditions (5.5-1), we can solve (5.5-1) for $|\Psi\rangle$ without too much trouble. The solutions will be invariants in 3-dimensional tensor analysis. An example is

$$|\Psi\rangle = \Phi\left(\int d^3x \; \zeta_{ABC} \; (\xi^D) J^{ABC}\right),$$

where ζ_{ABC} are arbitrary functions of ξ^D and

$$J^{ABC} = \begin{vmatrix} \xi^A_{,1} & \xi^A_{,2} & \xi^A_{,3} \\ \xi^B_{,1} & \xi^B_{,2} & \xi^B_{,3} \\ \xi^C_{,1} & \xi^C_{,2} & \xi^C_{,3} \end{vmatrix}$$ which are scalar density fields

of weight +1, and Φ is an arbitrary function of the integral. It may be readily verified that this $|\Psi\rangle$ satisfies (5.5-1). More complicated solutions may be constructed out of invariant 3n dimensional integrals involving the fields $\xi^{A_1}_{(\underset{\sim}{x}_1)}$, $\xi^{A_2}_{(\underset{\sim}{x}_2)}$, ... at n different points $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$... . The last subsidiary condition (5.5-2) however presents difficulties. As will be explicitly shown in the next section, it essentially expresses the requirement that $|\Psi\rangle$ be invariant under arbitrary normal surface deformation. One needs to compare the values of $|\Psi\rangle$ at different constant χ^{0} surfaces, so to obtain the solutions of (5.5-2) is at least as difficult as solving the equations of motion.[E1] Note that it is not just four simultaneous functional equations, but a 4-fold infinity of equations. To get a better qualitative idea of the constraint equations we consider the much simpler case of a 2-surface of stationary volume embedded in a 3-dimensional Pseudo-Euclidean space. Everything is formally the same as before except we now have only one spatial dimension χ and three field variables ξ^{a}, $a = 0,1,2$. There are only two constraints

$$\phi_1 = \xi^{a}_{,x}\pi_a \approx 0$$

$$\phi_0 = \pi^{a}\pi_a + Q^2 \eta_{ab}\, \xi^{a}_{,x}\xi^{b}_{,x} \approx 0.$$

Written out in full in quantum theory they become

$$\left(\xi^{0}_{,x}\frac{\delta}{\delta\xi^{0}} + \xi^{1}_{,x}\frac{\delta}{\delta\xi^{1}} + \xi^{2}_{,x}\frac{\delta}{\delta\xi^{2}}\right)|\Psi\rangle = 0 \qquad (5.5\text{-}3)$$

$$\Big\{\Big[(\hbar c)^2\frac{\delta^2}{\delta^2\xi^0} + Q^2(\xi^{0}_{,x})^2\Big] - \Big[-(\hbar c)^2\frac{\delta^2}{\delta^2\xi^1} + Q^2(\xi^{1}_{,x})^2\Big] - \Big[(\hbar c)^2\frac{\delta^2}{\delta^2\xi^2} + Q^2(\xi^{2}_{,x})^2\Big]\Big\}|\Psi\rangle = 0. \qquad (5.5\text{-}4)$$

Solutions of (5.5-3) may be readily found. Some examples are

$$|\Psi\rangle_{01} = \exp\int\frac{iQ}{\hbar c}\,\xi^{0}\xi^{1}_{,x}dx\ ;\quad |\Psi\rangle_{012} = \exp\int\frac{iQ}{\hbar c}\,\xi^{0}\xi^{1}\xi^{2}_{,x}dx\ ;$$

or more generally

$$|\Psi\rangle = \Phi\left(\int \zeta_a(\xi^0,\xi^1,\xi^2)\,\xi^a_{,x}dx\right).$$

Now let

$$\mathcal{P}^{(a)} = -\xi^a_{,x}\pi_a, \qquad \text{(no summation over a)}$$

$$\mathcal{H}^{(a)} = \frac{1}{2Q}\left[-(\hbar c)^2 \frac{\delta^2}{\delta \xi^a} + Q^2(\xi^a_{,x})\right]. \qquad \text{(no summation over a)}$$

It is obvious that $\mathcal{P}^{(a)}$, $\mathcal{H}^{(a)}$ are formally the same as the linear momentum and Hamiltonian densities of a massless Klein-Gordon field. Hence the solution of (5.5-3), (5.5-4) is equivalent to the following problem:

Given 3 independent and real scalar meson field $\xi^{(0)}, \xi^{(1)}, \xi^{(2)}$ find the set of simultaneous null eigenvectors of

$$\phi_0^{(x)} = \mathcal{P}^{(0)}_{(x)} + \mathcal{P}^{(1)}_{(x)} + \mathcal{P}^{(2)}_{(x)} \text{ and } \phi_1^{(x)} = \mathcal{H}^{(0)}_{(x)} - \mathcal{H}^{(1)}_{(x)} - \mathcal{H}^{(2)}_{(x)} .$$

Let us denote the integrated momenta and energies $\int \mathcal{P}^{(a)}(x)dx$, $\int \mathcal{H}^{(a)}(x)dx$ by $P^{(a)}$, $H^{(a)}$ respectively. We have

$$\left\{\begin{array}{l} \phi_0(x)|\Psi\rangle = 0 \\ \phi_1(x)|\Psi\rangle = 0 \end{array}\right. \Longrightarrow \left.\begin{array}{l} [P^{(0)} + P^{(1)} + P^{(2)}]|\Psi\rangle = 0 \\ [H^{(0)} - H^{(1)} - H^{(2)}]|\Psi\rangle = 0 . \end{array}\right\} \qquad (5.5\text{-}5)$$

=> the necessary (yet not sufficient) conditions for $|\Psi\rangle$ to satisfy the constraint equations are that the states must have zero total linear momentum and that the total energy of the field $\xi^{(0)}$ is equal to the sum of the total energy of $\xi^{(1)}$ and $\xi^{(2)}$.

Solutions of (5.5-5) are easily found in the usual Fock space representation of quantum field theory. Only a subset of these solutions could satisfy (5.5-3), (5.5-4). This subset may be found in the following way. Firstly we can express $\mathfrak{p}^{(a)}(x)$, $\mathcal{H}^{(a)}(x)$ in terms of creation and annihilation operators in the Fock space representation. We know that the set of known eigenvectors of $H^{(0)}$, $H^{(1)}$, $H^{(2)}$ forms a complete set. Hence we can express $|\Psi\rangle$ as a linear combination of vectors in this complete set. Now substitute the expression into (5.5-3), (5.5-4) and equate the appropriate coefficients of the resulting expressions to zero. A set of recursion relations may be found which serve to determine $|\Psi\rangle$. The actual calculation involved is rather messy and will not be presented here.

5.6 Transformation Properties

We now demonstrate explicitly the infinitesimal unitary transformation which corresponds to an infinitesimal coordinate transformation. Firstly one wants to find the corresponding classical canonical transformation. The field ξ^A are scalars and on a coordinate transformation $\overline{\delta}\xi^A = -\xi^A_{,\mu}\varepsilon^\mu$ where ε^μ are the descriptors of the coordinate transformation $\xi^A_{,\mu}$ is a covariant vector. We may decompose both $\xi^A_{,\mu}$ and ε^μ into normal and tangential components to the constant x^o surface on which the field is defined before the transformation. Any contravariant vector defined at a point on $\mathcal{S}$ and tangent to $\mathcal{S}$ is proportional to

$$t^\mu_{//} = (0,\ dx^j) \ .$$

This implies that the unit normal is

$$n^\mu = \frac{g^{o\mu}}{\sqrt{g^{oo}}}\ , \because\ \ n_\nu = g_{\nu\mu}\, n^\mu = (\frac{1}{\sqrt{g^{oo}}}\ , 0,0,0),\ \text{hence}\ n_\mu t^\nu = 0\ ;$$

$$\text{also}\ n_\mu n^\mu = 1\ .$$

A set of three linearly independent contravariant tangents is $t^\mu_{//r}$ where $t^\mu_{//1} = (0,1,0,0)$; $t^\mu_{//2} = (0,0,1,0)$; $t^\mu_{//3} = (0,0,0,1)$. Any vector T^μ may be written as

$$T^\mu = T_\perp\, n^\mu + T^r_{//}\, t^\mu_{//r}\ .$$

We see that

$$T_{\perp} = T^{\mu} n_{\mu} = \frac{1}{\sqrt{g^{oo}}} T^{o} ;$$

$$T^{r}_{/\!/} = T^{r} - \frac{1}{g^{oo}} T^{o} g^{or} .$$

The decomposition of a covariant vector may be effected through its contravariant form.

Consider the scalar product of two vectors T^{μ}, P_{μ}

$$\begin{aligned} T^{\mu} P_{\mu} &= g_{\mu\nu} T^{\mu} P^{\nu} \\ &= T_{\perp} P_{\perp} + g_{\mu\nu} t^{\mu}_{/\!/ r} t^{\nu}_{/\!/ s} T^{r}_{/\!/} P^{s}_{/\!/} \\ &= T_{\perp} P_{\perp} + g_{rs} T^{r}_{/\!/} P^{s}_{/\!/} , \because t^{\mu}_{/\!/ r} = \delta^{\mu}_{r} ; \end{aligned}$$

$$P^{s}_{/\!/} = P^{s} - \frac{1}{g^{oo}} P^{o} g^{os} = (g^{s\mu} - \frac{1}{g^{oo}} g^{o\mu} g^{os})P_{\mu} = (g^{s\ell} - \frac{1}{g^{oo}} g^{o\ell} g^{os})P_{\ell} ;$$

$$g_{rs} P^{s}_{/\!/} = g_{rs}(g^{s\ell} - \frac{1}{g^{oo}} g^{o\ell} g^{os})P_{\ell} = \delta^{\ell}_{r} P_{\ell} = P_{\ell} .$$

Hence

$$T^{\mu} P_{\mu} = T_{\perp} P_{\perp} + T^{r}_{/\!/} P_{r} ,$$

so we have

$$\varepsilon^{\mu}\xi^{A}_{,\mu} = \varepsilon_{\perp}\xi_{\perp}{}^{A} + \varepsilon^{r}_{/\!/}\xi^{A}_{,r} ,$$

with $\quad \varepsilon_{\perp} = \dfrac{\varepsilon^{o}}{\sqrt{g^{oo}}} \; ; \; \xi_{\perp}{}^{A} = \dfrac{\xi^{A}_{,o}}{\sqrt{g^{oo}}} \; ; \; \varepsilon^{r}_{/\!/} = \varepsilon^{r} - \dfrac{1}{g^{oo}} \varepsilon^{o} g^{or} .$

Therefore

$$\overline{\delta}\xi^{A} = - (\varepsilon_{\perp}\xi_{\perp}{}^{A} + \varepsilon^{r}_{/\!/}\xi^{A}_{,r}) .$$

We now assert that the classical infinitesimal canonical transformation which corresponds to the above coordinate transformation is generated by the generating functional

$$\bar{\mathcal{G}} = -\int d^3x(\varepsilon_\perp \phi_\perp + \varepsilon_{/\!/}^r \phi_r) ,$$

where
$$\phi_\perp = \frac{1}{2\sqrt{-Q^2\Delta^{oo}}} (\mathfrak{J}_{AB}\pi^A\pi^B + Q^2\Delta^{oo}) \approx 0 ;$$

$$\phi_r = \xi^A_{,r}\,\pi_A \approx 0 ,$$

are essentially the primary first class constraints.

To justify this statement, we compute

$$[\xi^A, \bar{\mathcal{G}}] .$$

$$[\xi^A, \bar{\mathcal{G}}] = \int d^3x \left\{ \underline{[\xi^A, \varepsilon_\perp]}_{o}\phi_o + \varepsilon_\perp[\xi^A, \phi_o] + \underline{[\xi^A, \varepsilon_{/\!/}^r]}_{o}\phi_r + \varepsilon_{/\!/}^r[\xi^A, \phi_r] \right\}$$

$$= \int d^3x \, \{\varepsilon_\perp[\xi^A, \phi_o] + \varepsilon_{/\!/}^r[\xi^A, \phi_r]\}$$

$$= -(\varepsilon_\perp\, \xi_\perp^{\;A} + \varepsilon_{/\!/}^r\, \xi^A_{,r})$$

$$= -\varepsilon^\mu\, \xi^A_{,\mu}$$

$$= \bar{\delta}\xi^A .$$

This shows $\bar{\mathcal{G}}$ is indeed the required generating functional. This also shows the physical meaning of the constraints. ϕ_o is essentially the generator for deformation of the constant x^o surface in the normal direction while ϕ_j are the generators for tangential surface deformation. It should be noted that the variations of field variables on a tangential surface deformation have no dynamic significance in that they are determined by the appropriate geometric transformation laws. It is the variations due to normal surface deformation which has to be determined by the Hamiltonian through the equations of motion. We have

$$\xi^A_{,o} = [\xi^A, H_T] = [\xi^A, \int d^3x\, \mathcal{H}_T] \ ;$$

$$\xi^A_{,o}\, \varepsilon^o = [\xi^A, \int d^3x\, \varepsilon^o \mathcal{H}_T] \ .$$

Hence the canonical transformation may be written as

$$\bar{\delta}\xi^A = [\xi^A, \mathcal{G}] \ ,$$

where

$$\mathcal{G} = - \int d^3x (\varepsilon^o \mathcal{H}_T + \varepsilon^j\, \phi_j) \ .$$

In quantum theory the corresponding infinitesimal unitary transformation is effected by

$$U = e^{\frac{i}{\hbar}\mathcal{G}} = (1 + \frac{i}{\hbar}\mathcal{G}) \ ,$$

where $\mathcal{G} = - \int d^3x(\varepsilon^o \mathcal{H}_T + \varepsilon^j \phi_j)$ is Hermitian.

So a covariant quantum theory for our Model Field II is established. The general features of generally covariant quantum theory again manifest themselves explicitly. We see that physical vectors are invariants and that they are x^o-independent. In the next chapter we are going to study this Model Field II in greater detail using specific coordinate system.

5.7 Appendix 5.1 Consistency in Quantum Theory.

First of all we need to prove some auxiliary formulae

(1) $$\xi^A_{,j}\pi_A \approx 0 \iff \pi_A\xi^A_{,j} \approx 0 \ . \qquad \text{(A5.1-1)}$$

This is obvious as $\xi^A_{j}(\underline{x})$ commutes with $\pi^A(\underline{x})$.

(2) $$g^\alpha(\underline{x})h_\alpha(\underline{x}) \approx 0$$

$$\Longrightarrow \ J(\underline{x},\underline{y}) = \left(\frac{\partial}{\partial x^j}\,\delta(\underline{x}-\underline{y})\right)g^\alpha(\underline{x})h_\alpha(\underline{y}) - \left(\frac{\partial}{\partial y^j}\,\delta(\underline{x}-\underline{y})\right)g^\alpha(\underline{y})h_\alpha(\underline{x}) \approx 0.$$

(Note: $x^o = y^o$) (A5.1-2)

Proof: integrate $J(\underline{x},\underline{y})$ with respect to d^3x for an arbitrary region enclosing $\underline{y}$.

$$\int d^3x\ J = -\left[\left(\frac{\partial}{\partial y^j}\,g^\alpha(\underline{y})\right)h_\alpha(\underline{y}) + g^\alpha(\underline{y})\left(\frac{\partial}{\partial y^j}\,h_\alpha(\underline{y})\right)\right]$$

$$= -\frac{\partial}{\partial y^j}(g^\alpha(\underline{y})h_\alpha(\underline{y})) \approx 0 \ .$$

Similarly

$$\int d^3y\ J \approx 0 \ .$$

Hence $J \approx 0$ on account of the arbitrary nature of the region of integration. Also we have

$$\bar{J}(\underline{x},\underline{y}) = \left(\frac{\partial}{\partial y^j}\,\delta(\underline{x}-\underline{y})\right)g^\alpha(\underline{x})h_\alpha(\underline{y}) - \left(\frac{\partial}{\partial x^j}\,\delta(\underline{x}-\underline{y})\right)g^\alpha(\underline{y})h_\alpha(\underline{x}) \approx 0 \ .$$

(A5.1-3)

$$(3) \qquad \xi^A_{,j}\pi_A \approx 0 \quad \Longrightarrow \quad \frac{\partial\Delta^{oo}}{\partial\xi^A_{,j}}\pi^A \approx 0 \ . \qquad \text{(A5.1-4)}$$

This is the same as our previous expression (A5.1-1)

$$\pi^A \frac{\partial\Delta^{oo}}{\partial\xi^A_{,j}} \approx 0 \ .$$

Its validity is ensured by (1).

We now consider the consistency conditions with the help of these auxiliary expressions.

Firstly

$$[\phi_i(\underline{x}),\phi_j(\underline{x}')] = \xi^A_{,i}\pi_A\xi'^B_{,j'}\pi'_B - \xi'^B_{,j'}\pi'_B\xi^A_{,i}\pi_A$$

$$\text{where} \quad \xi'^B_{,j'} = \frac{\partial}{\partial x'^j}\xi^B(\underline{x}')$$

$$= -ci\hbar\left(\delta(\underline{x}-\underline{x}')_{,j'}\xi^A_{,i}\pi'_A - \delta(\underline{x}-\underline{x}')_{,i}\xi'^A_{,j'}\pi_A\right)$$

$$\approx 0 \ .$$

Secondly

$$[\phi_o(\underline{x}),\phi_j(\underline{x}')] = [(\pi_A\pi^A + Q^2\Delta^{oo}),\ \xi'^B_{,j'}\pi'_B]$$

$$= ① + ② \ ,$$

where

$$① = [\pi_A\pi^A,\ \xi'^B_{,j'}\pi'_B] = [\pi_A\pi^A,\ \xi'^B_{,j'}]\pi'_B$$

$$= -2i\hbar c\, \eta^{AB}\,\pi_A\,\pi'_B\,\delta(\underline{x}-\underline{x}')_{,j'} = -2i\hbar c\,\pi_A\,\pi'^A\,\delta(\underline{x}-\underline{x}')_{,j'} .$$

$$(2) = Q^2[\Delta^{oo}, \xi'^B_{,j'} \pi'_B] = Q^2 \, \xi'^B_{,j'} [\Delta^{oo}, \pi'_B] .$$

$[\Delta^{oo}, \pi'_B]$ may be evaluated using the representation $\xi^A \equiv \xi^A$;

$$\pi_A \equiv - i\hbar c \frac{\partial}{\partial \xi^A} .$$

$$[\Delta^{oo}, \pi'_B] = + ci\hbar \left(\frac{\partial \Delta^{oo}}{\partial \xi'^B}\right).$$

$$\left(\frac{\partial \Delta^{oo}}{\partial \xi'^B}\right) = \frac{\partial}{\partial \xi'^B} \int d^3x' \; \Delta^{oo}(\underline{x}') \; \delta(\underline{x}-\underline{x}') = - \left[\frac{\partial}{\partial \xi'^B_{,\ell'}} \; \Delta^{oo}(\underline{x}') \; \delta(\underline{x}-\underline{x}')\right]_{,\ell'}$$

$$= - \left[\frac{\partial \Delta'^{oo}}{\partial \xi'^B_{,\ell'}}\right]_{,\ell'} \delta(\underline{x}-\underline{x}') - \frac{\partial \Delta'^{oo}}{\partial \xi'^B_{,\ell'}} \; \delta(\underline{x}-\underline{x}')_{,\ell'} .$$

Hence

$$(2) = - i\hbar Q^2 \left(\xi'^B_{,j'} \frac{\partial \Delta'^{oo}}{\partial \xi'^B_{,\ell'}} \; \delta(\underline{x}-\underline{x}')_{,\ell'} + \xi'^B_{,j'} \left[\frac{\partial \Delta'^{oo}}{\partial \xi'^B_{,\ell'}}\right]_{,\ell'} \delta(\underline{x}-\underline{x}')\right) .$$

$$\xi'^B_{,j'} \frac{\partial \Delta'^{oo}}{\partial \xi'^B_{,\ell'}} = 2\Delta'^{oo} \; \delta^{j'}_{\ell'} \qquad \text{by (A5.1-1)}$$

$$\xi'^B_{,j'} \left(\frac{\partial \Delta'^{oo}}{\partial \xi'^B_{,\ell'}}\right)_{,\ell'} = \left(\xi'^B_{,j'} \frac{\partial \Delta'^{oo}}{\partial \xi'^B_{,\ell'}}\right)_{,\ell'} - \xi'^B_{,j'\ell'} \frac{\partial \Delta'^{oo}}{\partial \xi'^B_{,\ell'}} = 2\Delta'^{oo}_{,j'} - \Delta'^{oo}_{,j'} = \Delta'^{oo}_{,j'} .$$

$$\therefore \quad (2) = - i\hbar Q^2 (2\Delta'^{oo} \; \delta(\underline{x}-\underline{x}')_{,j'} + \Delta'^{oo}_{,j'} \; \delta(\underline{x}-\underline{x}')) .$$

$$\therefore \quad [\phi_o(\underline{x}), \phi_j(\underline{x}')] = - 2i\hbar[\pi_A \pi'^A + Q^2 \Delta'^{oo}] \delta(\underline{x}-\underline{x}')_{,j'} - i\hbar Q^2 \Delta'^{oo}_{,j'} \; \delta(\underline{x}-\underline{x}')$$

$$= - 2i\hbar \frac{\partial}{\partial x'^j} [(\pi_A \pi'^A + Q^2 \Delta'^{oo}) \delta(\underline{x}-\underline{x}')] + 2i\hbar \; \delta(\underline{x}-\underline{x}') (\pi_A \pi'^A_{,j'} + Q^2 \Delta'^{oo}_{,j'})$$

$$- i\hbar Q^2 \Delta'^{oo}_{,j'} \; \delta(\underline{x}-\underline{x}')$$

$$= -2i\hbar \frac{\partial}{\partial x'^j} (\phi_o(\underline{x})\delta(\underline{x}-\underline{x}')) + i\hbar\; \delta(\underline{x}-\underline{x}')(2\; \pi_A \pi^A_{,j} + Q^2 \Delta^{oo}_{,j})$$

$$= -2i\hbar \frac{\partial}{\partial x'^j} (\delta(\underline{x}-\underline{x}')\phi_o(\underline{x})) + i\hbar\; \delta(\underline{x}-\underline{x}')\phi_{o,j}$$

$$\approx 0 \text{ , using } \frac{\partial}{\partial x}(\delta(x-y)f(x)) = \frac{\partial}{\partial x}(\delta(x-y)f(y)) \; .$$

Lastly

$$[\phi_o(\underline{x}),\phi_o(\underline{x}')] = [\pi_A\pi^A + Q^2\Delta^{oo},\; \pi'_B\pi'^B + Q^2\Delta'^{oo}]$$

$$= [\pi_A\pi^A \; , \; Q^2\Delta'^{oo}] + [Q^2\Delta^{oo} \; , \; \pi'_B\pi'^B] \; .$$

$$[\Delta^{oo},\; \pi'_B] = -i\hbar c \left(\frac{\partial\Delta'^{oo}}{\partial\xi'^B_{,\ell'}} \; \delta(\underline{x}-\underline{x}')\right)_{,\ell'} = -i\hbar c\; \frac{\partial\Delta^{oo}}{\partial\xi^B_{,\ell}} \; \delta(\underline{x}-\underline{x}')_{,\ell}.$$

$$[\phi_o(\underline{x}),\phi_o(\underline{x}')] = i\hbar c Q^2 \left(\pi'^B \; \frac{\partial\Delta^{oo}}{\partial\xi^B_{,\ell}} \; \delta(\underline{x}-\underline{x}')_{,\ell} + \frac{\partial\Delta^{oo}}{\partial\xi^B_{,\ell}} \; \pi'^B \; \delta(\underline{x}-\underline{x}')_{,\ell}\right)$$

$$-i\hbar c Q^2 \left(\pi^A \; \frac{\partial\Delta'^{oo}}{\partial\xi'^B_{,\ell}} \; \delta(\underline{x}-\underline{x}')_{,\ell'} + \frac{\partial\Delta'^{oo}}{\partial\xi'^B_{,\ell}} \; \pi^A \; \delta(\underline{x}-\underline{x}')_{,\ell'}\right)$$

$$\approx 0. \quad \text{by (A5.1-1), (A5.1-2), (A5.1-3), (A5.1-4).}$$

So we have shown that the consistency equations are all satisfied in quantum theory.

CHAPTER 6

MODEL FIELD THEORY II

TREATMENT II

6.1 Introduction

We continue to study the field introduced in the preceding Chapter. This time, we shall examine it from the point of view of a specific coordinate frame. The fundamental reason for working in a specific coordinate frame is a subject of great controversy. We shall not go into such a controversial subject here. The choice of a specific coordinate frame is somewhat arbitrary in general. However some justification will be given to our particular choice. The conventional perturbation theory is applied to our field in the weak field approximation case. As may be expected, divergent results are obtained.

6.2 A Special Set of Coordinate Conditions and its Justification

6.2.1 A Special set of Coordinate Conditions

Consider the special coordinate conditions mentioned in the preceding chapter, that is, $\xi^\mu = x^\mu$. The variational principle becomes

$$\delta\int \mathcal{L} d^4 x = 0, \quad \text{with } \mathcal{L} = \sqrt{1-\eta^{\rho\sigma}\xi_{,\rho}\xi_{,\sigma}}\ .$$

The corresponding Lagrange equation is given by equation (5.3-1) in section 5.3 of the preceding chapter. A Hamiltonian formulation is easily set up. To obtain a positive Hamiltonian density of the right dimensions, we shall take for the Lagrangian density

$$\mathcal{L} = -Q\sqrt{1-\eta^{\rho\sigma}\xi_{,\rho}\xi_{,\sigma}} = -Q\sqrt{1-(\frac{\dot{\xi}^2}{c^2} - (\nabla\xi)^2)}\ ,$$

where Q = a positive number of the dimensions of energy density,

$x^0 = ct$ and $\dot{\xi} = \frac{\partial\xi}{\partial t}$,

c = speed of light.

Define the canonical momentum

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\xi}} = \frac{Q}{c^2}\ \frac{\dot{\xi}}{\sqrt{1-\frac{\dot{\xi}^2}{c^2}+(\nabla\xi)^2}}$$

$$\Rightarrow \dot{\xi} = \frac{c^2}{Q}\pi\sqrt{\frac{1+(\nabla\xi)^2}{1+\frac{c^2}{Q^2}\pi^2}} \quad \Rightarrow \text{ no constraint.}$$

The Hamiltonian density

$$\mathcal{H} = \pi\dot{\xi} - \mathcal{L} = Q\sqrt{(1+(\nabla\xi)^2)(1+\frac{c^2}{Q^2}\pi^2)} \ .$$

The equation of motion of any functional F of ξ,π is given by

$$\overset{\circ}{F} = [F, \int d^3x\,\mathcal{H}] \ .$$

In particular

$$\overset{\circ}{\xi} = [\xi, \int d^3x\,\mathcal{H}] = \frac{c^2\pi}{Q}\sqrt{\frac{1+(\nabla\xi)^2}{1+\frac{c^2}{Q^2}\pi^2}} \ ,$$

$$\overset{\circ}{\pi} = [\pi, \int d^3x\,\mathcal{H}] = -Q\ \nabla\cdot\left[\nabla\xi\Bigg/\sqrt{\frac{1+\frac{c^2}{Q^2}\pi^2}{1+(\nabla\xi)^2}}\right] \ .$$

One may have noticed that both the total Lagrangian and the total Hamiltonian are divergent integrals because of the additive 1 inside the square roots. However this causes no trouble as the divergence may be eliminated by subtracting 1 from the integrand.

The corresponding Lagrange equation is

$$\frac{\partial}{\partial x^\mu}\left(\frac{\eta^{\mu\nu}\ \xi_{,\nu}}{\sqrt{1-\eta^{\rho\sigma}\ \xi_\rho\xi_\sigma}}\right) = 0 \ , \qquad (6.2.1\text{-}1)$$

A general solution to this highly nonlinear equation is not available. However some exact particular solutions may be found. The most interesting ones in this latter class are those of the form

$$\xi = \phi_\kappa(x) = A\,e^{i\kappa_\mu x^\mu} + A^*e^{-i\kappa_\mu x^\mu} \ ,$$

where A, κ_μ are constants and the κ_μ satisfy

$$\eta^{\rho\sigma} \kappa_\rho \kappa_\sigma = 0 \ .$$

The metric tensors $g_{\mu\nu}$, $g^{\mu\nu}$ and the Christoffel Symbols $[\alpha\beta,\lambda]$, $\Gamma^\mu_{\alpha\beta}$ will then be

$$g_{\mu\nu} = \eta_{\mu\nu} - \kappa_\mu \kappa_\nu \phi^2_\kappa \ ,$$

$$g^{\mu\nu} = \eta^{\mu\nu} + \kappa^\mu \kappa^\nu \phi^2_\kappa \ ,$$

$$-g = \det g_{\mu\nu} = 1 \ ,$$

$$[\alpha\beta,\lambda] = -\tfrac{1}{2} \kappa_\alpha \kappa_\beta \kappa_\lambda \phi^3_\kappa \ ,$$

$$\Gamma^\mu_{\alpha\beta} = g^{\mu\lambda} [\alpha\beta,\lambda] = -\tfrac{1}{2} \kappa_\alpha \kappa_\beta \kappa^\mu \phi^3_\kappa \ ,$$

where $\phi_\kappa = i(A\, e^{i\kappa_\mu x^\mu} - A^* e^{-i\kappa_\mu x^\mu})$ and $\kappa^\mu = \eta^{\mu\alpha} \kappa_\alpha$.

One may now easily verify that the "straight" line defined by

$$x^\mu = \kappa^\mu u \ , \quad \text{where } u \text{ is some parameter,}$$

is a null geodesic in the above space-time. The solution $\xi = \phi_\kappa(x)$ has the form of a wave propagating along this geodesic.

Slightly more general solutions may be obtained by superimposing all these waves travelling in the same direction, that is,

$$\xi = \int_{-\infty}^{\infty} du\ B(u)\ \phi_{u\kappa}(x) \ ,$$

where $B(u)$ = constants,

and $\phi_{u\kappa} = A_\kappa(u) e^{iu\kappa_\mu x^\mu} + A^*_\kappa(u) e^{-iu\kappa_\mu x^\mu}$, is again an exact solution. However, superposition of waves travelling in exactly the opposite direction does not lead to another solution, nor does superposition of waves moving in different directions. The nonlinear interference effect starts to set in for these waves not propagating in the same direction, i.e., these with $\kappa'^{\mu} \neq u\kappa^{\mu}$.

One observes that

$$\xi = f(\kappa_{\mu}x^{\mu}) \;, \quad \text{where } \eta^{\rho\sigma}\kappa_{\rho}\kappa_{\sigma} = 0 \text{ and } f \text{ is an arbitrary function,}$$

is again an exact solution. One may construct "localized" solutions. Some examples of these are

$$\xi \propto \cos \kappa_{\rho}x^{\rho}\, e^{-(\kappa_{\mu}x^{\mu})^2} \;,$$

$$\xi \propto \cos \kappa_{\rho}x^{\rho}\, \text{sech}\, \kappa_{\rho}x^{\rho} \;.$$

Other types of solutions which do not lead to constant $\sqrt{-g}$ may also be found. Some simple ones are

$$\xi = \cos(x^0 + x^1) + \ell n \cos x^2 - \ell n \cos x^3 \;,$$

or more generally

$$\xi = F(x^0 + x^1) + G(x^2, x^3) \;,$$

where $F(x^0 + x^1)$ is an arbitrary function and $G(x^2, x^3)$ is any solution of $\nabla \cdot \left(\frac{\nabla G}{\sqrt{1+(\nabla G)^2}}\right) = 0$ in two dimensions. The equation for G is the same as the equation of minimal surfaces whose general solution is known.[F1] of the Coordinate Conditions

are not chosen completely
Arnowith-Deser-Misner[F3] (ADM for
case of General Relativity will

l from a variational principle

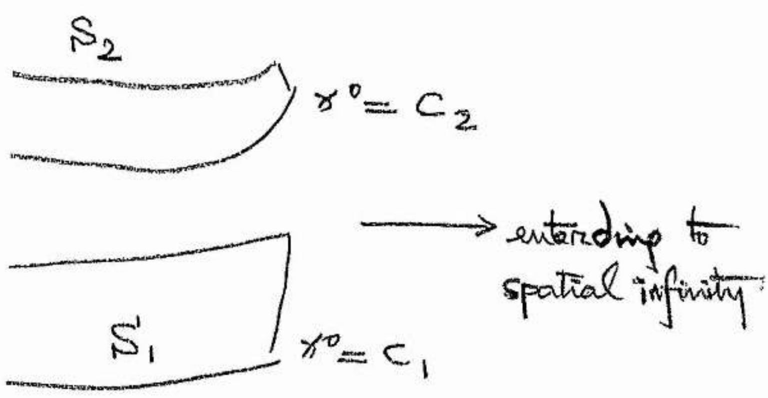
$\phi^A_{,\mu}) = 0,$

n with boundary surface S^*. Our
onstant is always a space-like
ty. Let the domain D be bounded
ne a total variation by allowing

surfaces S_1, S_2,

lary surfaces $(\bar{\delta}\phi^A = \phi'^A(x) - \phi^A(x))$,

$\bar{\delta}\phi^A$ inside D.



Under such a total variation,

$$\delta I = \int_D d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi^A} - \frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial\phi^A_{,\mu}}\right)\bar{\delta}\phi^A + G(S_2) - G(S_1),$$

where

$$G(S_2) = \int d^3x \left(\frac{\partial\mathcal{L}}{\partial\phi^A_{,0}}\delta\phi^A - T_\mu\delta x^\mu\right);$$

$$\delta\phi^A = \bar{\delta}\phi^A + \phi^A_{,\mu}\delta x^\mu$$

$$= \phi'^A(x') - \phi^A(x),$$

$$T_0 = \left(\frac{\partial\mathcal{L}}{\partial\phi^A_{,0}}\phi^A_{,0} - \mathcal{L}\right); \quad T_j = \frac{\partial\mathcal{L}}{\partial\phi^A_{,0}}\phi^A_{,j}.$$

The postulate[F2] that δI be equal to the difference of the two surface integrals leads to the usual Euler-Lagrange equations for the field variables. In the Hamiltonian formulation, the surface integral may be regarded as a generator of infinitesimal canonical transformations. Now we may write

$$G = \int d^3x (\pi_A\delta\phi^A - T_\mu\delta x^\mu), \qquad \text{where } \pi_A = \frac{\partial\mathcal{L}}{\partial\xi^A_{,0}}.$$

Consider the case of a pure translation of coordinates,

$$\text{i.e. } x'^\mu = x^\mu + \varepsilon^\mu, \qquad \varepsilon^\mu = \text{constants}.$$

Then $\delta\phi^A = 0$ and

$$G = -\int d^3x\, T_\mu\, \varepsilon^\mu.$$

One can see in this case that G serves as the generator for canonical transformations corresponding to a pure translation of coordinates. The property of G which concerns us here is that it has a fixed form. This may be utilized to identify the canonical conjugate variables. Now consider a scalar field theory formulated in a particular coordinate frame x^μ. In this frame the generator is

$$G = \int d^3x(\pi\delta\phi - T_\mu \delta x^\mu) \qquad \text{and } T_\mu \not\approx 0.$$

To build up a theory for the field in an arbitrary coordinate frame, we can use Dirac's curved surface formulation. In such formulation, x^μ are treated as additional field variables while introducing a new set of variables v^μ to specify points in space-time. Thus four more field variables plus their conjugate momenta appear. Let all the field variables including the original ϕ and their momenta be labelled respectively as ϕ^A, $\bar{\pi}_A$, $A = 0,1,2,3,4$ with $\phi^4 = \phi$. We again obtain the generator G in such formulism to be

$$G = \int d^3v \left[\bar{\pi}_A \delta\phi^A - \bar{T}_\mu \delta v^\mu \right],$$

with $$\bar{T}_\mu = (\bar{\mathcal{H}}, \ \bar{\pi}_A \phi^A_{,r}),$$

where $\phi^A_{,r} = \dfrac{\partial \phi^A}{\partial v^r}$ and the barred quantities are those as given by Dirac's formulism described in section 2.4.2 in chapter 2. We showed then that $\bar{\mathcal{H}} \approx 0$ due to the constraint equations. We now want to show $\bar{\pi}_A \phi^A_{,r} \approx 0$ as well. From equations (2.4.2.2-2) we obtain

$$\bar{\pi}_\mu x^\mu_{,r} = - \bar{\pi}_4 \phi^4_{,\mu} x^\mu_{,r} \quad ,$$

simply by multiplying (2.4.2.2-2) by $x^\mu_{,r}$. Note that x^μ is equivalent to y^μ used in chapter 2 and that

$$n_\mu \; x^\mu_{,r} = 0 \; .$$

Now we have

$$\bar{\pi}_\mu x^\mu_{,r} + \bar{\pi}_4 \phi^4_{,\mu} x^\mu_{,r} = 0 \; ,$$

which imply

$$\bar{\pi}_A \phi^A_{,r} = 0 \; .$$

Hence

$$G = \int d^3 v \; \bar{\pi}_A \delta \phi^A \; .$$

Now consider the problem the other way around, that is, given this theory formulated on a general space-like surface, how we can get back to the theory in the original special coordinate frame. The answer is obvious. We can simply impose the coordinate conditions

$$\phi^\mu = \mathbf{X}^\mu = v^\mu \; .$$

It may be readily seen that indeed the original theory is obtained and in particular the generator gets back to

$$G = \int d^3 x (\pi \delta \phi - T_\mu \delta x^\mu) \; .$$

Returning to our ξ-field now, we regard our Hamiltonian formulation before imposing coordination conditions as being essentially a theory on the curved surface in the Dirac's sense. Our task is to obtain the theory in the special coordinate frame (more than one frame in general) from which we may in turn get back the curved surface theory. This is an idea due to ADM[F3] although details of their treatment as applied to General Relativity are quite different from our present ones. Thus we see that this provides a mean of establishing a special coordinate frame. We will demonstrate this explicitly for our ξ-field.

For our ξ-field

$$G = \int d^3x \; \bar{\pi}_A \delta\xi^A \;, \quad \text{since} \quad T_\mu = (\bar{\mathcal{H}}, \bar{\pi}_A \xi^A_{,j}) \approx 0 \;.$$

So it is in the right form, and the coordinate conditions to impose are then

$$\xi^\mu = x^\mu \;.$$

Recall the constraint equations

$$\bar{\pi}_A \; \xi^A_{,j} \approx 0 \;,$$

$$\bar{\pi}_A \bar{\pi}^A + Q\Delta^{oo} \approx 0 \;.$$

Using the coordinate conditions we have

$$\bar{\pi}_4 \xi_{,j} + \bar{\pi}_j = 0 \;, \quad \text{or}$$

$$\pi\xi_{,j} + \bar{\pi}_j = 0 \ , \quad \text{where } \pi = \bar{\pi}_4 \ ;$$

$$(\bar{\pi}_o)^2 = \sum_{j=1}^{3} (\bar{\pi}_j)^2 + \pi^2 - Q^2\Delta^{oo} = \pi^2(1+(\nabla\xi)^2) - Q^2\Delta^{oo} \ .$$

Since $\quad \Delta^{oo} = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = -(1+(\nabla\xi)^2) \ ,$

we obtain

$$(\bar{\pi}_o) = -Q\sqrt{(1+(\nabla\xi)^2)(1+\frac{\pi^2}{Q^2})} \ , \quad \text{taking negative root.}$$

Therefore

$$\bar{\pi}_A \ \delta\xi^A = - T_\mu \ \delta x^\mu + \pi\delta\xi \ ,$$

where

$$T_\mu = \bar{\pi}_\mu = (Q\sqrt{(1+[\nabla\xi]^2)(1+\frac{\pi^2}{Q^2})} \ , \quad \pi\xi_{,j}) \ .$$

The generator becomes

$$G = \int d^3x(\pi\delta\xi - T_\mu \ \delta x^\mu) \ ,$$

which is exactly the one obtained in the special frame we have chosen. Hence this procedure offers some justification for our choice.

6.2.2.2 Justification II

In order to be able to study the physical states in the usual way, it is necessary to impose sufficient coordinate conditions as to single out a unique coordinate system. Our present conditions $\xi^\mu = x^\mu$ satisfy this requirement. Furthermore we note that ξ^μ are four scalars as far as arbitrary coordinate transformations in the 4-surface are concerned. Therefore the coordinate conditions are of the Bergmann-Komar type. Consequently a theory using these coordinate conditions will be covariant.

6.3. Weak Field Approximation

Due to the highly nonlinear nature of the field equation, approximation methods appear to be inevitable. Different approximation treatments are employed to study the field. In this chapter we examine an everywhere-weak field approach. We assume that the field is everywhere-weak, that is, $\xi_{,j}$ and $\frac{c}{Q}\pi$ are of the order of $\lambda^{\frac{1}{2}} \ll 1$ for all x^{μ}. Note that these are dimensionless quantities. The exact numerical value of λ is a physical assumption. Having made this definite assumption, we may carry out a binomial expansion of the Hamiltonian density regarding

1 to be of the order of λ^{0},

$\nabla\xi$, $\frac{c\pi}{Q}$ to be of the order of $\lambda^{\frac{1}{2}}$,

$(\nabla\xi)^2$, $(\nabla\xi)\frac{c\pi}{Q}$, $(\frac{c\pi}{Q})^2$ to be of the order of λ and so on.

Therefore we have

$$\mathcal{H} = Q\sqrt{1 + (\nabla\xi)^2}\sqrt{1 + (\frac{c\pi}{Q})^2}$$

$$= Q[1 + \tfrac{1}{2}(\nabla\xi)^2 - \frac{1}{8}(\nabla\xi)^4 + \frac{1}{16}(\nabla\xi)^6 + \ldots][1 + \tfrac{1}{2}(\frac{c\pi}{Q})^2 - \frac{1}{8}(\frac{c\pi}{Q})^4 + \frac{1}{16}(\frac{c\pi}{Q})^6 + \ldots]$$

$$= \mathcal{H}_o + \mathcal{H}_I , \qquad (6.3\text{-}1)$$

where $\mathcal{H}_o = Q + \frac{Q}{2}[(\nabla\xi)^2 + (\frac{c\pi}{Q})^2]$,

$$\mathcal{H}_I = -\frac{Q}{8}[(\nabla\xi)^4 + (\frac{c\pi}{Q})^4 - 2(\nabla\xi)^2(\frac{c\pi}{Q})^2] + \frac{Q}{16}[(\nabla\xi)^6 + (\frac{c\pi}{Q})^6 - (\nabla\xi)^2(\frac{c\pi}{Q})^4 - (\nabla\xi)^4(\frac{c\pi}{Q})^2]$$

$$+ \ldots\ldots .$$

To the second order, we may take

$$\mathcal{H}_0 = \frac{Q}{2}[(\nabla\xi)^2 + (\frac{c\pi}{Q})^2], \text{ ignoring the additive constant } Q,$$

$$\mathcal{H}_I = -\frac{Q}{8}[(\nabla\xi)^4 + (\frac{c\pi}{Q})^4 - 2(\nabla\xi)^2(\frac{c\pi}{Q})^2].$$

In the usual perturbation approach, $\mathcal{H}_0$ is regarded as representing the free field while $\mathcal{H}_I$ is considered as a small perturbation on the otherwise free field. The free field derived is seen to resemble the Klein-Gordon field for scalar mesons except that there is no mass term in our present case. We may call it the massless real scalar meson field. The free field equation is

$$\mathfrak{g}^{\rho\sigma}\xi_{,\rho\sigma} = 0, \quad \text{i.e.} \quad \frac{1}{c^2}\frac{\partial^2\xi}{\partial t^2} - \nabla^2\xi = 0.$$

The perturbed equations of motion are

$$\dot{\xi} = \frac{c^2\pi}{Q}(1 + \tfrac{1}{2}[(\nabla\xi)^2 - (\frac{c\pi}{Q})^2]), \tag{6.3-2}$$

$$\dot{\pi} = Q\,\nabla^2\xi + \frac{Q}{2}\,\nabla.(\nabla\xi[(\frac{c\pi}{Q})^2 - (\nabla\xi)^2]). \tag{6.3-3}$$

Equation (6.3-2) also serves to define π in terms of $\xi_{,\mu}$. One can solve for π to obtain

$$\pi = \frac{Q}{c^2}\,\dot{\xi}(1 + \tfrac{1}{2}\mathfrak{g}^{\rho\sigma}\xi_{,\rho}\xi_{,\sigma}),$$

keeping terms to the second order only.

The perturbed field equation is

$$g^{\alpha\beta}\xi_{,\alpha\beta} - \tfrac{1}{2}\left[g^{\alpha\beta}\xi_{,\alpha\beta}\, g^{\rho\sigma}\xi_{,\rho}\xi_{,\sigma} + g^{\alpha\beta}\xi_{,\alpha}\left(g^{\rho\sigma}\xi_{,\rho}\xi_{,\sigma}\right)_{,\beta}\right] = 0, \qquad (6.3\text{-}4)$$

which may be worked out from the corresponding Lagrangian density

$$\begin{aligned} \mathcal{L} &= \pi\dot{\xi} - \mathcal{H} \\ &= -Q + \tfrac{1}{2}Q\, g^{\rho\sigma}\xi_{,\rho}\xi_{,\sigma} + \tfrac{1}{8}Q\left(g^{\rho\sigma}\xi_{,\rho}\xi_{,\sigma}\right)^2 . \end{aligned}$$

Many exact solutions to the field equation (6.3-4) may be found. Of particular interest is the set of exact solutions

$$\phi_K(x) = \text{constant} * e^{iK_\mu x^\mu},$$

where the constants K_μ satisfy $g^{\rho\sigma} K_\rho K_\sigma = 0$. Solutions which resemble interference effect between these plane wave solutions may be obtained by perturbation.

6.4. Quantization

The quantization of the field may be effected in the usual way. The ξ, π are taken as operators in a linear vector space satisfying the standard equal-time commutation relations for bosons and the states of the field are represented by vectors in the linear vector space. The first problem we like to do is to find the eigenvalues of the Hamiltonian The usual time-independent perturbation method will be used. The problem of scattering will also be looked into by the S-matrix formulation.

6.4.1. The Quantization in the Schrödinger Picture

Decompose $\xi(\underline{x})$, $\pi(\underline{x})$ by Fourier Integrals.

$$\xi(\underline{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3K \, q_{\underline{K}} \, e^{i\underline{K}\cdot\underline{x}} \; ; \; \pi(\underline{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3K \, p_{\underline{K}} \, e^{i\underline{K}\cdot\underline{x}} .$$

$\Longrightarrow$

$$q_{\underline{K}} = \frac{1}{(2\pi)^{3/2}} \int d^3x \, \xi(\underline{x}) \, e^{-i\underline{K}\cdot\underline{x}} \; ; \; p_{\underline{K}} = \frac{1}{(2\pi)^{3/2}} \int d^3x \, \pi(\underline{x}) \, e^{-i\underline{K}\cdot\underline{x}} .$$

Define $a_{\underline{K}}$, $a^{\dagger}_{\underline{K}}$ by

$$q_{\underline{K}} = \sqrt{\frac{\hbar c^2}{2W_K Q}} \, (a_{\underline{K}} + a^{\dagger}_{-\underline{K}}) \; ; \; p^{\dagger}_{\underline{K}} = i\sqrt{\frac{Q\hbar W_K}{2c^2}} \, (a^{\dagger}_{\underline{K}} - a_{-\underline{K}}),$$

where $W_K^2 = c^2 \underline{K}^2$. Then the commutation relations for $\xi(\underline{x})$, $\pi(\underline{x})$ imply that

$$[a_{\underline{K}}, a_{\underline{K}'}] = 0 \;;\; [a^{\dagger}_{\underline{K}}, a^{\dagger}_{\underline{K}'}] = 0 \;;\; [a_{\underline{K}}, a^{\dagger}_{\underline{K}'}] = \delta(\underline{K}-\underline{K}'),$$

which are the standard commutation relations for boson creation and annihilation operators. Note that these relations are independent of the Hamiltonian. In the perturbation theory, our unperturbed Hamiltonian is

$$H_o = \int d^3x\, \mathcal{H}_o \quad , \quad \mathcal{H}_o = \frac{Q}{2}[(\nabla\xi)^2 + (\frac{c\pi}{Q})^2].$$

$$\Longrightarrow \qquad H_o = \int d^3k\, \hbar W_K\, a^{\dagger}_{\underline{K}}\, a_{\underline{K}}.$$

We have ignored the infinite zero point term by a normal ordering of the operators $a_{\underline{K}}$, $a^{\dagger}_{\underline{K}}$. With this expression for the Hamiltonian of the free field, we can define and give particle interpretation to the various eigenstates of H_o in the usual way. Denote the vacuum state by $|0\rangle$, then the various eigenstates representing different number of particles are

$$|\underline{K}\rangle = a^{\dagger}_{\underline{K}}|0\rangle \text{ ------------------- one-particle state,}$$

$$|\underline{K}_1\underline{K}_2\rangle = |\underline{K}_2\underline{K}_1\rangle = a^{\dagger}_{\underline{K}_1} a^{\dagger}_{\underline{K}_2}|0\rangle \text{ ------ two-particle state,}$$

and so on.

6.4.2. Time-Independent Perturbation Theory in the Schrödinger Picture

The problem now is to calculate the perturbation energy $\Delta E_{\underline{K}}$ of a one-particle state $|\underline{K}\rangle$ due to the interaction Hamiltonian

$$H_I = \int d^3x\, \mathcal{H}_I \;, \quad \mathcal{H}_I = -\frac{Q}{8}[(\nabla\xi)^4 + (\frac{c\pi}{Q})^4 - 2(\nabla\xi)^2(\frac{c\pi}{Q})^2] .$$

The usual time-independent perturbation theory for a discrete eigenvalue spectrum needs to be modified in the case of a continuous spectrum. Details of the modifications together with the derivation are given in Appendix 6.1.

In order to compute various matrix elements in the perturbation theory H_I should be expressed in terms of creation and annihilation operators $a_{\underline{K}}$, $a^{\dagger}_{\underline{K}}$ defined by

$$\xi(\underline{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \sqrt{\frac{\hbar c^2}{2W_K Q}}\, (a_{\underline{K}} + a^{\dagger}_{-\underline{K}})\, e^{i\underline{K}.\underline{x}} ,$$

$$\pi(\underline{x}) = \frac{-i}{(2\pi)^{3/2}} \int d^3k \sqrt{\frac{Q\hbar W_K}{2c^2}}\, (a_{\underline{K}} - a^{\dagger}_{-\underline{K}})\, e^{i\underline{K}.\underline{x}} .$$

Note that in quantum theory H_I is meant to be its normal product, namely, creation operators $a^{\dagger}_{\underline{K}}$ are written to the left of annihilation operators. H_I contains terms involving fourth order products of $a^{\dagger}_{\underline{K}}$ and $a_{\underline{K}}$. As will be seen later, only terms with factors

$$a^{\dagger}_{\underline{K}}\, a_{\underline{K}'}\, a_{\underline{\bar{K}}}\, a_{\underline{\bar{K}}'} \;, \quad a_{\underline{K}}\, a_{\underline{K}'}\, a_{\underline{\bar{K}}}\, a_{\underline{\bar{K}}'}$$

have nonvanishing contributation to the relevant matrix elements. Therefore only these need to be worked out explicitly. The result is

$$H_I = \int d^3K\, d^3K'\, d^3\bar{K}\, d^3\bar{K}'\; \Omega_{\underline{K}\,\underline{K}'\,\underline{\bar{K}}\,\underline{\bar{K}}'} [\delta(\underline{K}+\underline{K}'+\underline{\bar{K}}+\underline{\bar{K}}')\, a_{\underline{K}}\, a_{\underline{K}'}\, a_{\underline{\bar{K}}}\, a_{\underline{\bar{K}}'} -$$

$$4\delta(-\underline{K}+\underline{K}'+\bar{\underline{K}}+\bar{\underline{K}}')a^{\dagger}_{\underline{K}}\, a_{\underline{K}'} a_{\bar{\underline{K}}}\, a_{\bar{\underline{K}}'}] + \text{irrelevant terms}, \qquad (6.4.2\text{-}1)$$

where $\Omega_{\underline{K}\,\underline{K}'\,\bar{\underline{K}}\,\bar{\underline{K}}'} = -\dfrac{\hbar^2}{32(2\pi)^3 Q}\sqrt{W_K\, W_{K'} W_{\bar{K}}\, W_{\bar{K}'}}\,[1-(\underline{K}^o\cdot\underline{K}'^o+\bar{\underline{K}}^o\cdot\bar{\underline{K}}'^o)+(\underline{K}^o\cdot\underline{K}'^o)(\bar{\underline{K}}^o\cdot\bar{\underline{K}}'^o)]$

$$= -\frac{\hbar^2}{32(2\pi)^3 Q}\sqrt{W_K\, W_{K'} W_{\bar{K}}\, W_{\bar{K}'}}\,[(1-\underline{K}^o\cdot\underline{K}'^o)(1-\bar{\underline{K}}^o\cdot\bar{\underline{K}}'^o)] \leqslant 0.$$

and $\underline{K}^o = \dfrac{K}{|\underline{K}|}$, $W_K = c|\underline{K}| = cK.$

Let us firstly consider the perturbation on the vacuum state $|0>$ which is non-degenerate. Appendix 6.1 gives

$$\Delta E_o = \Delta E_o^{(1)} + \Delta E_o^{(2)} + ---- ,$$

where $\Delta E_o^{(1)} = <0|H_I|0>$, $\Delta E_o^{(2)} = \sum\limits_{n\neq o} \dfrac{|<0|H_I|n>|^2}{-E_n}$.

The first order term $\Delta E_o^{(1)}$ is zero. There is only one type of non-zero terms in $\Delta E_o^{(2)}$ coming from $|n> = |\underline{K}_1\,\underline{K}_2\,\underline{K}_3\,\underline{K}_4>$.

$$<0|H_I|\underline{K}_1\,\underline{K}_2\,\underline{K}_3\,\underline{K}_4>$$

$$= \int d^3K\, d^3K' d^3\bar{K}\, d^3\bar{K}'\ \Omega_{\underline{K}\,\underline{K}'\,\bar{\underline{K}}\,\bar{\underline{K}}'}\delta(\underline{K}+\underline{K}'+\bar{\underline{K}}+\bar{\underline{K}}') \sum_{p1234} \delta(\underline{K}-\underline{K}_1)\delta(\underline{K}'-\underline{K}_2)\times \delta(\bar{\underline{K}}-\underline{K}_3)\delta(\bar{\underline{K}}'-\underline{K}_4)$$

$$= \sum_{p1234} \delta(\underline{K}_1+\underline{K}_2+\underline{K}_3+\underline{K}_4)\ \Omega_{\underline{K}_1\underline{K}_2\underline{K}_3\underline{K}_4},$$

where $\sum\limits_{p1234}$ is over all the permutations of $\underline{K}_1$, $\underline{K}_2$, $\underline{K}_3$, $\underline{K}_4$.

Hence

$$\Delta E_o^{(2)} = \int \frac{d^3K_1\, d^3K_2\, d^3K_3\, d^3K_4}{4!} \frac{|<0|H_I|\underline{K}_1\underline{K}_2\underline{K}_3\underline{K}_4>|^2}{-\hbar c(K_1+K_2+K_3+K_4)}$$

$$= \int \frac{d^3K_1\, d^3K_2\, d^3K_3\, d^3K_4}{4!} \frac{[\sum_{p1234} \delta(\underline{K}_1+\underline{K}_2+\underline{K}_3+\underline{K}_4)\Omega_{\underline{K}_1\underline{K}_2\underline{K}_3\underline{K}_4}]^2}{-\hbar c(K_1+K_2+K_3+K_4)}$$

(6.4.2-2)

This expression contains the square of a δ-function which appears originally from H_I. Hence it is not very meaningful as it stands. However, as remarked in Appendix 6.1, this difficulty may be bypassed. The usefulness of $\Delta E_o^{(2)}$ as given by (6.4.2-2) will be seen in connection with the expression for $\Delta E_{\underline{K}}^{(2)}$ presently.

Let us now consider the perturbation on a one-particle state $|\underline{k}>$. The energy eigenvalue associated with $|\underline{k}>$ is degenerate, e.g., $|\underline{k}'\underline{k}''>$ with $\underline{k}' = \underline{k}'' = \frac{1}{2}\underline{k}$ also belongs to the same eigenvalue. Let $|m>$ denote an eigenvector of H_o belonging to the eigenvalue $\hbar W_K$. Then one can readily show that

$$<\underline{k}|H_I|m> = 0.$$

Therefore the degeneracy does not present any difficulty in the perturbation calculation. From Appendix 6.1, we have

$$\Delta E_{\underline{k}}^{(1)}\ \delta(\underline{K}-\underline{k}) = <\underline{K}|H_I|\underline{k}>,$$

$$\Delta E_{\underline{k}}^{(2)}\ \delta(\underline{K}-\underline{k}) = \sum_n{}' \frac{<\underline{K}|H_I|n><n|H_I|\underline{k}>}{E_k-E_n},$$

where $\sum_{n}'$ means a sum over all the eigenstates of H_o except the one-particle eigenstates and those eigenstates with $E_n = E_k$. The first order correction term

$$\Delta E_{\underline{k}}^{(1)} = 0 \qquad \because \; < \underline{K}|H_I|\underline{k} > = 0.$$

The second order expression $\Delta E_{\underline{k}}^{(2)}$ consists of two types of non-zero terms.

$$\Delta E_{\underline{k}}^{(2)} \; \delta(\underline{K}-\underline{k}) = A + B,$$

where

$$A = \int \frac{d^3K_1 \, d^3K_2 \, d^3K_3}{3!} \; \frac{< \underline{K}|H_I|\underline{K}_1\underline{K}_2\underline{K}_3 > < \underline{K}_1\underline{K}_2\underline{K}_3|H_I|\underline{k} >}{E_k - E_{K_1} - E_{K_2} - E_{K_3}} ,$$

$$B = \int \frac{d^3K_1 \, d^3K_2 \, d^3K_3 \, d^3K_4 \, d^3K_5}{5!} \; \frac{< \underline{K}|H_I|\underline{K}_1\underline{K}_2\underline{K}_3\underline{K}_4\underline{K}_5 > < \underline{K}_1\underline{K}_2\underline{K}_3\underline{K}_4\underline{K}_5|H_I|\underline{k} >}{E_k - E_{K_1} - E_{K_2} - E_{K_3} - E_{K_4} - E_{K_5}}$$

Details of the calculations for A and B are given in Appendix 6.2. The results are

$$A = \int \frac{d^3K_1 \, d^3K_2 \, d^3K_3}{3!} \; \frac{(J_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3})^2 \, \delta(\underline{K}_1+\underline{K}_2+\underline{K}_3-\underline{k}) \; \delta(\underline{K}-\underline{k})}{\hbar c[k-K_1-K_2-K_3]}$$

where $J_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3} = -4 \sum_{p123} \Omega_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3}$.

$$B = B_1 + B_2 ,$$

where

$$B_1 = \int \frac{d^3K_1\, d^3K_2\, d^3K_3\, d^3K_4}{4!} \; \frac{\left[\sum_{p1234} \Omega_{\underline{K}_1 \underline{K}_2 \underline{K}_3 \underline{K}_4}\, \delta(\underline{K}_1+\underline{K}_2+\underline{K}_3+\underline{K}_4) \right]^2}{-\hbar c(K_1+K_2+K_3+K_4)}\, \delta(\underline{K}-\underline{k}),$$

$$B_2 = \frac{20}{5!} \int d^3K_1\, d^3K_2\, d^3K_3 \; \frac{\left(\sum_{pk123} \Omega_{\underline{k}\underline{K}_1 \underline{K}_2 \underline{K}_3} \right)^2}{-\hbar c(k+K_1+K_2+K_3)}\, \delta(\underline{k}+\underline{K}_1+\underline{K}_2+\underline{K}_3)\, \delta(\underline{K}-\underline{k}).$$

As pointed in Appendix 6.1 we should regard

$$\Delta E_{\underline{k}}^{(2)} = \Delta E_{\underline{k}}^{(2)} - \Delta E_o^{(2)} = A + B_2,$$

as the true perturbed energy of the original one-particle state $|\underline{k}>$. Hence terms involving the square of the δ-function arising from H_I are cancelled out and cause no trouble. The real trouble comes from the actual structure of H_I. The integral A is shown to diverge towards negative infinity at least like

$$-\int d^3K_1\, d^3K_2\; K_1 K_2 .$$

B_2 diverges similarly.

All this is expected from the forms of H_I which involves the derivatives of the field variable. This type of interaction is known to be nonrenormalizable.[F5] As one goes to a higher order in the perturbation calculation, one gets higher order products of the derivatives. Hence more divergent factors W_K turn up in the numerators leading to a higher order divergence. Therefore the perturbation treatment of our present model field theory is again plagued by infinite quantities.

6.4.3. The Problem of Scattering in the Interaction Picture

The usual S-matrix approach will be used and the Interaction picture will be employed throughout this section.

6.4.3.1. The First Order S-Matrix Processes

The first order S-matrix is given by

$$S^{(1)} = \frac{1}{i\hbar} \int_{-\infty}^{\infty} H_I \, dt.$$

A number of processes can happen, notably the what we may call the "shower process", that is, an incoming particle is annihilated and three outgoing particles are created. However we shall consider a more conventional two-particle scattering process. The initial state will be $|i\rangle = |\underline{k}, \underline{k}'\rangle$ with $\underline{k} \neq \underline{k}'$. The transition to final states of the form $|f\rangle = |\underline{K}, \underline{K}'\rangle$ will now be studied. Let us consider the simplest case in which $\underline{k} + \underline{k}' = 0$. In other words we have a head-on collision of two quanta. This is not such a restricted case as it may appear because any two-particle collision would appear to be a head-on one in the centre-of-momentum frame of reference. The total scattering cross section is given in Appendix 6.3 to be

$$\sigma = \frac{(2\pi)^4}{2\hbar c} \int |T_{fi}|^2 \, \delta(\underline{K}+\underline{K}') \, \delta(E_f - E_i) \, d^3K \, d^3K',$$

where $T_{fi} = 8[\Omega_{\underline{kk}'\underline{KK}'} + \Omega_{\underline{kKk}'\underline{K}'} + \Omega_{\underline{kK}'\underline{k}'\underline{K}}]$ and $\underline{k}' = -\underline{k}$.

The integration may be explicitly worked out giving

$$\sigma = \frac{7E_k^6}{10\pi(\hbar c)^4 Q^2} \qquad E_k = \hbar c\ k.$$

To estimate σ, we take Q to be of the order of the energy density of the universe which lies in the range 10^{-8} to 10^{-6} ergs cm^{-3}, that is, we take

$$Q = 10^{-7} \text{ ergs cm}^{-3}$$

$$\Longrightarrow \qquad \sigma \approx 10^{79}\ E_k^6\ \text{cm}^2,$$

where E_k is to be expressed in the C.G.S. units. One can put in some typical energies for gravitons to work out σ. A typical graviton associated with the gravitational wave which may conceivably be generated in a laboratory as envisaged by Weber[F6] would have an energy of the order of 10^{-19} ergs. The total scattering cross section is then

$$\sigma = 10^{-35}\ \text{cm}^2.$$

This is very small. But it should be measurable since in some experiments on neutrinos a cross section well below 10^{-40} cm^2 is not unknown.[F7]

Let us now consider a typical graviton associated with interstellar gravitational radiation. According to J. Wheeler the density of such gravitational radiation could be as high as 10^3 ergs cm^{-3} and that its wavelength λ would be of the order of 10^{24} cm.[F8] The energy associated with a graviton may then be taken as

$$\frac{\hbar c}{\lambda} \approx 10^{-40} \text{ ergs.}$$

The corresponding scattering cross section is

$$\sigma \approx 10^{-161} \text{ cm}^2,$$

which is far too small to be measurable.

6.4.3.2. The Second Order S-Matrix

The first order S-matrix elements are all finite. We may go on to examine the second order S-matrix elements. However the results are not rewarding. The calculation is very tedious leading to divergences. Therefore we shall not pursue it any further.

6.5. Appendix 6.1 A Perturbation Theory in Quantum Field Theory

Suppose a boson field is characterized by a Hamiltonian of the form

$$H = H_o + \lambda H',$$

where H_o = the free boson field Hamiltonian and $\lambda \ll 1$. The eigenvectors are H_o are therefore

$$|0>, \ |\underline{K}> = a^{\dagger}_{\underline{K}}\ |0>, \ |\underline{K}_1\underline{K}_2> = |\underline{K}_2\underline{K}_1> = a^{\dagger}_{\underline{K}_1} a^{\dagger}_{\underline{K}_2}\ |0>, \ldots .$$

We shall not confine the field in a finite box. Hence H_o has a continuous eigenvalue spectrum, apart from the zero eigenvalue of $|0>$ which may be considered as a discrete eigenvalue especially for massive fields. The normalization conditions are

$$<0|0> = 1 \ ; \ <\underline{K}_1\underline{K}_2\underline{K}_3\ldots\underline{K}_n|\underline{K}'_1\underline{K}'_2\ldots\underline{K}'_n> = \sum_{p12..n} \delta(\underline{K}_1-\underline{K}'_1)\ldots\times \delta(\underline{K}_n-\underline{K}'_n),$$

where $\sum_{p12..n}$ is over all the permutations of $\underline{K}'_1, \underline{K}'_2\ldots\underline{K}'_n$. These vectors are assumed to form a complete set. In other words we have

$$I = |0><0| + \int d^3K\ |\underline{K}><\underline{K}| + \int \frac{d^3K_1\ d^3K_2}{2!}\ |\underline{K}_1\underline{K}_2><\underline{K}_1\underline{K}_2| + \ldots .$$

Any vector $|\Psi>$ is expressible as

$$|\Psi> = <0|\Psi>|0> + \int d^3K <\underline{K}|\Psi>|\underline{K}> + \int \frac{d^3K_1\ d^3K_2}{2!} <\underline{K}_1\underline{K}_2|\Psi>|\underline{K}_1\underline{K}_2> + \ldots$$

This may be symbolically written as

$|\Psi> = \sum_n c_n |n>$, where $|n>$ stands for an eigenvector of H_o.

<u>Perturbation Theory Proper</u>

The problem is to find approximation eigenvalues of H.

$$H_o|n> = E_n|n>,$$

$$H|\Psi_n> = \varepsilon|\Psi_n>.$$

Assume

$$|\Psi_n> = |n> + \lambda|\phi_n^{(1)}> + \lambda^2|\phi_n^{(2)}> + \dots ,$$

$$\varepsilon = E_n + \lambda\Delta E_n^{(1)} + \lambda^2 \Delta E_n^{(2)} + \dots .$$

$\Longrightarrow$

$$H_o|n> = E_n|n>,$$

$$H_o|\phi_n^{(1)}> + H_1|n> = E_n|\phi_n^{(1)}> + \Delta E_n^{(1)}|n>,$$

$$H_o|\phi_n^{(2)}> + H_1|\phi_n^{(1)}> = E_n|\phi_n^{(2)}> + \Delta E_n^{(1)}|\phi_n^{(1)}> + \Delta E_n^{(2)}|n>,$$

. . . .
. . . .

<u>(1) Perturbation on the Vacuum State</u>

$$|\Psi_o> = |0> + \lambda|\phi^{(1)}> + \lambda^2|\phi^{(2)}> + \dots ,$$

$$\varepsilon = 0 + \lambda\Delta E_o^{(1)} + \lambda^2\Delta E_o^{(2)} + \dots .$$

Following the usual procedure of the perturbation theory for discrete spectrum[F9] we obtain

$$\Delta E_o^{(1)} = <0|H'|0>, \quad \Delta E_o^{(2)} = \sum_n \frac{<0|H'|n><n|H'|0>}{-E_n} \quad \text{and so on.}$$

(2) Perturbation on an One-Particle State

$$|\Psi(\underline{k})> = |\underline{k}> + \lambda|\phi(\underline{k})^{(1)}> + \ldots ,$$

$$\varepsilon = E_k + \lambda\Delta E_{\underline{k}}^{(1)} + \ldots .$$

$\Longrightarrow$

$$H_o|\underline{k}> = E_k|\underline{k}>,$$

$$H'|\underline{k}> + H_o|\phi(\underline{k})^{(1)}> = \Delta E_{\underline{k}}^{(1)}|\underline{k}> + E_k|\phi(\underline{k})^{(1)}>, \qquad \text{(A6.1-1)}$$

$$H_o|\phi(\underline{k})^{(2)}> + H'|\phi(\underline{k})^{(1)}> = \Delta E_{\underline{k}}^{(2)}|\underline{k}> + \Delta E_{\underline{k}}^{(1)}|\phi(\underline{k})^{(1)}> + E_k|\phi(\underline{k})^{(2)}>, \qquad \text{(A6.1-2)}$$

$\vdots$

Let $|\phi(\underline{k})^{(1)}> = \sum_n c_n|n>$. Assume that we may impose the following conditions on $|\Psi(\underline{k})>$

$$<\underline{k}'|\phi(\underline{k})^{(1)}> = 0, \quad <\underline{k}'|\phi(\underline{k})^{(2)}> = 0, \ \ldots \quad \text{so on.}$$

In other words, we assume that there is no one-particle term in $|\Psi(\underline{k})>$ apart from the unperturbed $|\underline{k}>$ in the zeroth order approximation. These assumptions appear necessary in order that one can proceed with the perturbation theory in analogy with the discrete spectrum case. These conditions may also be compared with the similar conditions imposed on the perturbed vector in the discrete spectrum case.[F9] Then (A6.1-1) gives

$$\Delta E_{\underline{k}}^{(1)}\ \delta(\underline{K}-\underline{k}) = <\underline{K}|H'|\underline{k}>,$$ [F10]

implying that $\Delta E_{\underline{k}}^{(1)}$ is zero unless $< \underline{K}|H'|\underline{k} >$ itself contains a factor $\delta(\underline{K}-\underline{k})$. If H' commutes with the total linear momentum, then $< \underline{K}|H'|\underline{k} >$ must contain such a factor.

The second order expression $\Delta E_{\underline{k}}^{(2)}$ is given by (A6.1-2) to be

$$\Delta E_{\underline{k}}^{(2)}\ \delta(\underline{K}-\underline{k}) = < \underline{K}|H'|\phi(\underline{k})^{(1)} >.$$

(A6.1-1) $\Longrightarrow$

$$|\phi(\underline{k})^{(1)} > = \sum_n{}' \frac{< n|H'|\underline{k} >}{E_k - E_n} |n >.$$

The degeneracy of E_k should not cause trouble since in practice we often have $E_n = E_k \Rightarrow < n|H'|\underline{k} > = 0$. We shall confine ourselves to such cases. The summation $\sum_n{}'$ means a sum over all the eigen-sectors of H_o except the one-particle eigenstates and these eigen-states with $E_n = E_k$.

Finally we get

$$\Delta E_{\underline{k}}^{(2)}\ \delta(\underline{K}-\underline{k}) = \sum_n{}' \frac{< \underline{K}|H'|n >< n|H'|\underline{k} >}{E_k - E_n}.$$

Once again conservation of linear momentum implies that the right hand side should also contain the factor $\delta(\underline{K}-\underline{k})$.

Some Remarks

(1) The essential difference between these formulae and those for the case of discrete spectrum is the appearance of the δ-function.

The necessity of the δ-function is most clearly seen by considering a somewhat trivial example in which

$$H' = H_o^2 .$$

Our first order expression then leads to the exact result.

(2) In many practical cases, H' contains a δ-function. Then $\Delta E_o^{(2)}$ may involve the square of the δ-function as in our present ξ-field theory. This means that the expression for $\Delta E_o^{(2)}$ itself is not a meaningful quantity as it stands. However this difficulty may be bypassed. Let us consider the present ξ-field theory. Firstly one can confine the field in a finite yet large box of volume V and perform the perturbation calculation. For a large V we obtain

$$\Delta E_o^{(2)} = \frac{V}{8\pi^3} \int \frac{d^3K_1\, d^3K_2\, d^3K_3\, d^3K_4}{4!} \frac{\left| \sum_{p1234} \Omega_{\underline{K}_1 \underline{K}_2 \underline{K}_3 \underline{K}_4} \right|^2 \delta(\underline{K}_1 + \underline{K}_2 + \underline{K}_3 + \underline{K}_4)}{-\hbar c(K_1 + K_2 + K_3 + K_4)} .$$

Now if we regard the result given in equation (6.4.2-2) as the limiting case of large V, we may interpret $\delta^2(\underline{K}_1 + \underline{K}_2 + \underline{K}_3 + \underline{K}_4)$ appearing in (6.4.2-2) as the limiting case of $\frac{V}{8\pi^3}\delta(\underline{K}_1 + \underline{K}_2 + \underline{K}_3 + \underline{K}_4)$. This procedure therefore gives some meaning to (6.4.2-2). Secondly, as seen in the calculations given in Appendix 6.2, one observes that

$$\Delta E_{\underline{K}}^{(2)} = \Delta E_o^{(2)} + \Delta E_{\underline{K}}'^{(2)} ,$$

where $\Delta E'^{(2)}_{\underline{K}}$ does not involve the square of a δ-function. Hence we may consider $\Delta E'^{(2)}_{\underline{K}}$ as the genuine perturbed energy[F11] of the original one-particle state. The difficulty caused by a (δ-function)^2 therefore disappears at least as far as our ξ-field is concerned.

6.6. Appendix 6.2 Calculation of A and B.

(1) Calculation of A

$$< \underline{K}|H_I|\underline{K}_1\underline{K}_2\underline{K}_3> = J_{\underline{K}\underline{K}_1\underline{K}_2\underline{K}_3}\ \delta(\underline{K}_1+\underline{K}_2+\underline{K}_3-\underline{K}),$$

where $J_{\underline{K}\underline{K}_1\underline{K}_2\underline{K}_3} = -4 \sum_{p123} \Omega_{\underline{K}\underline{K}_1\underline{K}_2\underline{K}_3}$ and $\sum_{p123}$ is over all the permutations of $\underline{K}_1$, $\underline{K}_2$, $\underline{K}_3$.

$$A = \int \frac{d^3K_1\, d^3K_2\, d^3K_3}{3!} \frac{J_{\underline{K}\underline{K}_1\underline{K}_2\underline{K}_3}\, J_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3}\, \delta(\underline{K}_1+\underline{K}_2+\underline{K}_3-\underline{K})\,\delta(\underline{K}_1+\underline{K}_2+\underline{K}_3-\underline{k})}{\hbar c[k-(K_1+K_2+K_3)]}$$

$$= \int \frac{d^3K_1\, d^3K_2\, d^3K_3}{3!} \frac{(J_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3})^2\, \delta(\underline{K}_1+\underline{K}_2+\underline{K}_3-\underline{k})\,\delta(\underline{K}-\underline{k})}{\hbar c[k-(K_1+K_2+K_3)]}.$$

$$J_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3} = -4 \sum_{p123} \Omega_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3}$$

$$= -8[\Omega_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3} + \Omega_{\underline{k}\underline{K}_3\underline{K}_1\underline{K}_2} + \Omega_{\underline{k}\underline{K}_2\underline{K}_1\underline{K}_3}].$$

$$(J_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3})^2 = 64[J^{(1)}_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3} + J^{(2)}_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3}).$$

$$J^{(1)}_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3} = (\Omega_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3})^2 + (\Omega_{\underline{k}\underline{K}_3\underline{K}_1\underline{K}_2})^2 + (\Omega_{\underline{k}\underline{K}_2\underline{K}_3\underline{K}_1})^2.$$

$$J^{(2)}_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3} = 2\Omega_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3}\,\Omega_{\underline{k}\underline{K}_3\underline{K}_1\underline{K}_2} + 2\Omega_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3}\,\Omega_{\underline{k}\underline{K}_2\underline{K}_3\underline{K}_1} + 2\Omega_{\underline{k}\underline{K}_3\underline{K}_1\underline{K}_2}\,\Omega_{\underline{k}\underline{K}_2\underline{K}_3\underline{K}_1}.$$

Since $\underline{K}_1$, $\underline{K}_2$, $\underline{K}_3$ are dummy variables and the rest of the factors in the integrand are symmetric with respect to $\underline{K}_1$, $\underline{K}_2$, $\underline{K}_3$, we may substitute

$$3(\Omega_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3})^2 \quad \text{for } J^{(1)}_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3} \quad \text{and}$$

$$6\Omega_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3}\,\Omega_{\underline{K}\underline{K}_3\underline{K}_1\underline{K}_2} \quad \text{for } J^{(2)}_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3}$$

in A.

$\Longrightarrow$

$$A = \int \frac{d^3K_1\, d^3K_2\, d^3K_3}{3!}\; \frac{[3(\Omega_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3})^2 + 6\Omega_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3}\,\Omega_{\underline{k}\underline{K}_3\underline{K}_1\underline{K}_2}]\,\delta(\underline{K}_1+\underline{K}_2+\underline{K}_3-\underline{k})\,\delta(\underline{K}-\underline{k})}{\hbar c(k-K_1-K_2-K_3)}$$

$$= \int \frac{d^3K_1\, d^3K_2}{3!}\; \frac{[3(\Omega_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}'})^2 + 6\Omega_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}'}\,\Omega_{\underline{k}\underline{K}'\underline{K}_1\underline{K}_2}]\,\delta(\underline{K}-\underline{k})}{\hbar c(k-K_1-K_2-K')}\;,$$

where $\underline{K}' = \underline{k}-\underline{K}_1-\underline{K}_2$.

$$(\Omega_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}'})^2 + 2\Omega_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}'}\,\Omega_{\underline{k}\underline{K}'\underline{K}_1\underline{K}_2}$$

$$= \frac{(\hbar c)^4}{(32)^2\,(2\pi)^6\,Q^2}\; kK_1K_2K'\; f_{\underline{k}\underline{K}_1\underline{K}_2}.$$

$f_{\underline{k}\underline{K}_1\underline{K}_2}$ depends only on the relative orientations between $\underline{k}$, $\underline{K}_1$, $\underline{K}_2$ and is equal to

$$f_{\underline{k}\underline{K}_1\underline{K}_2} = [1-(\underline{k}^o\!\cdot\underline{K}_1^o+\underline{K}_2^o\!\cdot\underline{K}'^o) + (\underline{k}^o\!\cdot\underline{K}_1^o)(\underline{K}_2^o\!\cdot\underline{K}'^o)]^2 +$$

$$2[1-(\underline{k}^o.\underline{K}_1^o+\underline{K}_2^o.\underline{k}'^o) + (\underline{k}^o.\underline{K}_1^o)(\underline{K}_2^o.\underline{k}'^o)][1-(\underline{k}^o.\underline{k}'^o+\underline{K}_1^o.\underline{K}_2^o) + (\underline{k}^o.\underline{k}'^o)(\underline{K}_1^o.\underline{K}_2^o)],$$

where $\underline{k}'^o = \underline{k}^o-\underline{K}_1^o-\underline{K}_2^o$. Not much simplification of this expression can be achieved.

We have

$$A = \frac{(\hbar c)^3}{2(32)^2(2\pi)^6Q^2}\int d^3K_1\, d^3K_2\, \frac{kK_1K_2K'}{k-K_1-K_2-K'}\, f_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}'}.$$

Now for fixed K_1 and K_2,

$$(K')_{max.} = k+K_1+K_2, \quad (K')_{min.} = |k-K_1-K_2|.$$

Let us consider the behaviour of the integrand of A for large values of K_1 and K_2. Then the integrand is negative and we have

$$\frac{K'}{k-K_1-K_2-K'} \leqslant \frac{K'_{min}}{k-K_1-K_2-K'} \leqslant \frac{K'_{min}}{k-K_1-K_2-K'_{max}} = \frac{K_1+K_2-k}{-2(K_1+K_2)} =$$

$$-\tfrac{1}{2} + \frac{k}{2(K_1+K_2)}.$$

The integral

$$\int d^3K_1\, d^3K_2\, K_1K_2\, f^2_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}'}(-\tfrac{1}{2}+\frac{k}{2(K_1+K_2)}),$$

clearly diverges in the upper limits as K_1, K_2 tend to infinity. This implies A must diverge towards negative infinity at least like

$$-\int d^3K_1\, d^3K_2\, K_1K_2.$$

(2) Calculation of B

$$< \underline{K}|H_I|\underline{K}_1 \underline{K}_2 \underline{K}_3 \underline{K}_4 \underline{K}_5 > = \sum_{p12345} \Omega_{\underline{K}_1 \underline{K}_2 \underline{K}_3 \underline{K}_4} \delta(\underline{K}_1 + \underline{K}_2 + \underline{K}_3 + \underline{K}_4) \, \delta(\underline{K} - \underline{K}_5),$$

$$< \underline{K}|H_I|\underline{K}_1 \underline{K}_2 \underline{K}_3 \underline{K}_4 \underline{K}_5 >$$

$$= \sum_{p12345} \Omega_{\underline{K}_1 \underline{K}_2 \underline{K}_3 \underline{K}_4} \delta(\underline{K}_1 + \underline{K}_2 + \underline{K}_3 + \underline{K}_4) \, \delta(\underline{K} - \underline{K}_5)$$

$$= \sum_{p1234} \Omega_{\underline{K}_1 \underline{K}_2 \underline{K}_3 \underline{K}_4} \delta(\underline{K}_1 + \underline{K}_2 + \underline{K}_3 + \underline{K}_4) \, \delta(\underline{K} - \underline{K}_5) + \sum_{p5123} \Omega_{\underline{K}_5 \underline{K}_1 \underline{K}_2 \underline{K}_3} \times$$

$$\delta(\underline{K}_5 + \underline{K}_1 + \underline{K}_2 + \underline{K}_3) \, \delta(\underline{K} - \underline{K}_4) +$$

$$\sum_{p4512} \Omega_{\underline{K}_4 \underline{K}_5 \underline{K}_3 \underline{K}_2} \delta(\underline{K}_4 + \underline{K}_5 + \underline{K}_1 + \underline{K}_2) \, \delta(\underline{K} - \underline{K}_3) + \sum_{p3451} \Omega_{\underline{K}_3 \underline{K}_4 \underline{K}_5 \underline{K}_1} \times$$

$$\delta(\underline{K}_3 + \underline{K}_4 + \underline{K}_5 + \underline{K}_1) \, \delta(\underline{K} - \underline{K}_2) +$$

$$\sum_{p2345} \Omega_{\underline{K}_2 \underline{K}_3 \underline{K}_4 \underline{K}_5} \delta(\underline{K}_2 + \underline{K}_3 + \underline{K}_4 + \underline{K}_5) \, \delta(\underline{K} - \underline{K}_1).$$

Let $\Omega_{1234} = \sum_{p1234} \Omega_{\underline{K}_1 \underline{K}_2 \underline{K}_3 \underline{K}_4} \delta(\underline{K}_1 + \underline{K}_2 + \underline{K}_3 + \underline{K}_4)$. Then

$$< \underline{K}|H_I|\underline{K}_1 \underline{K}_2 \underline{K}_3 \underline{K}_4 \underline{K}_5 > = \sum_{c12345} \Omega_{1234} \, \delta(\underline{K} - \underline{K}_5),$$

where $\sum_{c12345}$ means a summation over all the cyclic permutations of

$\underline{K}_1, \underline{K}_2, \underline{K}_3, \underline{K}_4, \underline{K}_5$.

$$< \underline{K}|H_I|\underline{K}_1 \underline{K}_2 \underline{K}_3 \underline{K}_4 \underline{K}_5 > < \underline{K}_1 \underline{K}_2 \underline{K}_3 \underline{K}_4 \underline{K}_5 |H_I| \underline{k} >$$

$$= [\sum_{c12345} \Omega_{1234}\, \delta(\underline{K}-\underline{K}_5)][\sum_{c12345} \Omega_{1234}\, \delta(\underline{k}-\underline{K}_5)]$$

$$= [\Omega_{1234}\, \delta(\underline{K}-\underline{K}_5) + \Omega_{5123}\, \delta(\underline{K}-\underline{K}_4) + \Omega_{4512}\, \delta(\underline{K}-\underline{K}_3) + \Omega_{3451}\, \delta(\underline{K}-\underline{K}_2) + \Omega_{2345}\, \delta(\underline{K}-\underline{K}_1)] \times$$

$$[\Omega_{1234}\, \delta(\underline{k}-\underline{K}_5) + \Omega_{5123}\, \delta(\underline{k}-\underline{K}_4) + \Omega_{4512}\, \delta(\underline{k}-\underline{K}_3) + \Omega_{3451}\, \delta(\underline{k}-\underline{K}_2) + \Omega_{2345}\, \delta(\underline{k}-\underline{K}_1)]$$

$$= (\Omega_{1234})^2\, \delta(\underline{K}-\underline{K}_5)\ \delta(\underline{k}-\underline{K}_5) + (\Omega_{5123})^2\, \delta(\underline{K}-\underline{K}_4)\delta(\underline{k}-\underline{K}_4) + (\Omega_{4512})^2\, \delta(\underline{K}-\underline{K}_3)\times \delta(\underline{k}-\underline{K}_3)$$

$$+ (\Omega_{3451})^2\, \delta(\underline{K}-\underline{K}_2)\ \delta(\underline{k}-\underline{K}_2) + (\Omega_{2345})^2\, \delta(\underline{K}-\underline{K}_1)\ \delta(\underline{k}-\underline{K}_1) + R,$$

where R are the 20 remaining cross terms.

$\Longrightarrow$

$$B = B_1 + B_2,$$

where
$$B_1 = \int \frac{d^3K_1\, d^3K_2\, d^3K_3\, d^3K_4}{4!}\ \frac{(\Omega_{1234})^2\, \delta(\underline{K}-\underline{k})}{-\hbar c(K_1+K_2+K_3+K_4)}, \qquad \text{(A6.2-1)}$$

$$B_2 = \int \frac{d^3K_1\, d^3K_2\, d^3K_3\, d^3K_4\, d^3K_5}{5!}\ \frac{R}{\hbar c(k-K_1-K_2-K_3-K_4-K_5)}.$$

Compare B_1 with $\Delta E_o^{(2)}$ in (6.4.2-2) we see that the contribution to $\Delta E_{\underline{k}}^{(2)}$ from B_1 is just equal to $\Delta E_o^{(2)}$.

Calculation shows that each of the twenty terms in R contributes equally to B_2. The final result is

$$B_2 = \frac{20}{5!} \int d^3K_1\, d^3K_2\, d^3K_3 \; \frac{\left(\sum_{pk123} \Omega_{\underline{k}\underline{K}_1\underline{K}_2\underline{K}_3} \right)^2}{-\hbar c(k+K_1+K_2+K_3)} \; \delta(\underline{k}+\underline{K}_1+\underline{K}_2+\underline{K}_3)\; \delta(\underline{K}-\underline{k}) \; .$$

It can be seen that this is a divergent integral which diverges in the same way as A.

6.7. Appendix 6.3 Scattering Cross Section[F12]

Define $$S^{(1)}_{V\tau} = \frac{1}{i\hbar} \int_V d^3x \int_\tau dt\, \mathcal{H}_I, \quad [S^{(1)}_{fi}]_{V\tau} = \langle i | S^{(1)}_{V\tau} | f \rangle$$

Hence $$S^{(1)} = \lim_{\substack{V\to\infty \\ \tau\to\infty}} S^{(1)}_{V\tau}, \quad \langle i | S^{(1)} | f \rangle = \lim_{\substack{V\to\infty \\ \tau\to\infty}} [S^{(1)}_{fi}]_{V\tau} .$$

The transition probability per unit time per unit volume is

$$W_{fi} = \lim_{\substack{V\to\infty \\ \tau\to\infty}} [W_{fi}]_{V\tau} ,$$

where $$[W_{fi}]_{V\tau} = \frac{|[S^{(1)}_{fi}]_{V\tau}|^2}{V\tau} .$$

Now

$$S^{(1)}_{V\tau} = \frac{(2\pi)^3}{i\hbar V} \sum_{\mathbf{K}_1 \mathbf{K}_2 \mathbf{K}_3 \mathbf{K}_4} \Omega_{\mathbf{K}_1 \mathbf{K}_2 \mathbf{K}_3 \mathbf{K}_4} [P_{\mathbf{K}_1 \mathbf{K}_2 \mathbf{K}_3 \mathbf{K}_4} + P'_{\mathbf{K}_1 \mathbf{K}_2 \mathbf{K}_3 \mathbf{K}_4}],$$

where

$$P_{\mathbf{K}_1 \mathbf{K}_2 \mathbf{K}_3 \mathbf{K}_4} = 2a^\dagger_{\mathbf{K}_1} a^\dagger_{\mathbf{K}_2} a_{\mathbf{K}_3} a_{\mathbf{K}_4} \delta_{\mathbf{K}_1 + \mathbf{K}_2,\, \mathbf{K}_3 + \mathbf{K}_4} \int_{-\tau/2}^{\tau/2} e^{i(W_{K_1} + W_{K_2} - W_{K_3} - W_{K_4})t}\, dt,$$

$$P'_{\mathbf{K}_1 \mathbf{K}_2 \mathbf{K}_3 \mathbf{K}_4} = 4a^\dagger_{\mathbf{K}_1} a^\dagger_{\mathbf{K}_3} a_{\mathbf{K}_2} a_{\mathbf{K}_4} \delta_{\mathbf{K}_1 + \mathbf{K}_3,\, \mathbf{K}_2 + \mathbf{K}_4} \int_{-\tau/2}^{\tau/2} e^{i(W_{K_1} + W_{K_3} - W_{K_2} - W_{K_4})t}\, dt.$$

$\Longrightarrow$

$$[S^{(1)}_{fi}]_{v\tau} = -\frac{(2\pi)^3 i}{\hbar v} T_{fi} \delta_{\underline{k}+\underline{k}', \underline{K}+\underline{K}'} \int_{-\tau/2}^{\tau/2} e^{i(E_f-E_i)t/\hbar} dt,$$

where $E_f = E_K + E_{K'}, \quad E_i = E_k + E_{k'},$

$$T_{fi} = 8[\Omega_{\underline{k}\underline{k}'\underline{K}\underline{K}'} + \Omega_{\underline{k}\underline{K}\underline{k}'\underline{K}'} + \Omega_{\underline{k}\underline{K}'\underline{k}'\underline{K}}].$$

$\Longrightarrow$

$$[W_{fi}]_{v\tau} = \frac{(2\pi)^6 |T_{fi}|^2}{\hbar^2 v^2} \frac{|\delta_{\underline{k}+\underline{k}', \underline{K}+\underline{K}'}|^2}{v} \frac{\left|\int_{-\tau/2}^{\tau/2} e^{i(E_f-E_i)t/\hbar} dt\right|^2}{\tau}$$

$$= \frac{(2\pi)^6 |T_{fi}|^2}{v^4 \hbar^2} \frac{\left|\int_v e^{i(\underline{K}_f-\underline{K}_i)\cdot\underline{x}} d^3x\right|^2}{v} \frac{\left|\int_{-\tau/2}^{\tau/2} e^{i(E_f-E_i)t/\hbar} dt\right|^2}{\tau}$$

$$\approx \frac{|T_{fi}|^2}{\hbar v^4} (2\pi)^{10} \delta(\underline{K}_f-\underline{K}_i) \delta(E_f-E_i),$$

where $\underline{K}_f = \underline{K}+\underline{K}', \quad \underline{K}_i = \underline{k}+\underline{k}'.$

The error introduced tends to zero as $v, \tau \to \infty$. The reason for retaining a finite v will be seen presently.

Consider the scattering of $|i\rangle$ into a group of final states $|\underline{K}\underline{K}'\rangle$ with $\underline{K}$ in the range $(\underline{K}, \underline{K}+d\underline{K})$ and $\underline{K}'$ in the range $(\underline{K}', \underline{K}'+d\underline{K}')$. The number of these states is

$$\frac{v^2}{(2\pi)^6} d^3K\, d^3K'.$$

The transition probability per unit time per unit volume into this group of final states is

$$[W_{fi}]_{v\tau}\,\frac{v^2}{(2\pi)^6}\,d^3K\,d^3K' = \frac{(2\pi)^4\,|T_{fi}|^2}{\hbar v^2}\,\delta(\underline{K}_f-\underline{K}_i)\,\delta(E_f-E_i)\,d^3K\,d^3K'.$$

Define the differential scattering cross section $d\sigma$ to be the ratio between the transition probability over the space v per unit time and the incident flux J, then we have

$$d\sigma = \frac{(2\pi)^4\,|T_{fi}|^2}{\hbar vJ}\,\delta(\underline{K}_f-\underline{K}_i)\,\delta(E_f-E_i)\,d^3K\,d^3K'.$$

In the special case in which the initial state consists of two quanta head-on colliding, i.e.,

$$\underline{K}_i = \underline{k}+\underline{k}' = 0,$$

the expression for J is

$$J = \frac{2c}{v}\,, \quad \text{c being the speed of light.}$$

$$\Longrightarrow \qquad d\sigma = \frac{(2\pi)^4\,|T_{fi}|^2}{2\hbar c}\,\delta(\underline{K}+\underline{K}')\,\delta(E_f-E_i)\,d^3K\,d^3K'.$$

It is seen that this expression is independent of v and τ which may now go to infinity. The total scattering cross section is

$$\sigma = \int \frac{(2\pi)^4\,|T_{fi}|^2}{2\hbar c}\,\delta(\underline{K}+\underline{K}')\,\delta(E_f-E_i)\,d^3K\,d^3K'.$$

Note that in evaluating σ, use must be made of the special condition $\underline{k}+\underline{k}' = 0$ in the expression for T_{fi}.

$$\sigma = \frac{32(2\pi)^4}{(\hbar c)^2}\int [\Omega_{\underline{k}\underline{k}'\underline{K}\underline{K}'} + \Omega_{\underline{k}\underline{K}\underline{k}'\underline{K}'} + \Omega_{\underline{k}\underline{K}'\underline{k}'\underline{K}}]^2\,\delta(\underline{K}+\underline{K}')\,\delta(K+K'-2k)\,d^3K\;d^3K'$$

$$= \frac{16(2\pi)^4}{(\hbar c)^2}\int [\Omega_{\underline{k}\underline{k}'\underline{K}\underline{K}'} + \Omega_{\underline{k}\underline{K}\underline{k}'\underline{K}'} + \Omega_{\underline{k}\underline{K}'\underline{k}'\underline{K}}]^2\,\delta(K-k)\,d^3K,$$

where $\underline{k}' = -\underline{k}$, $\underline{K}' = -\underline{K}$.

$\Longrightarrow$

$$\sigma = \frac{1}{64(2\pi)^2}\left(\frac{\hbar c}{Q}\right)^2\int d^3K\;k^2K^2\;\delta(K-k)\;\left[6 + 2(\underline{k}^o\cdot\underline{K}^o)^2\right]^2$$

$$= \frac{7}{10\pi}\left(\frac{\hbar c}{Q}\right)^2 k^6$$

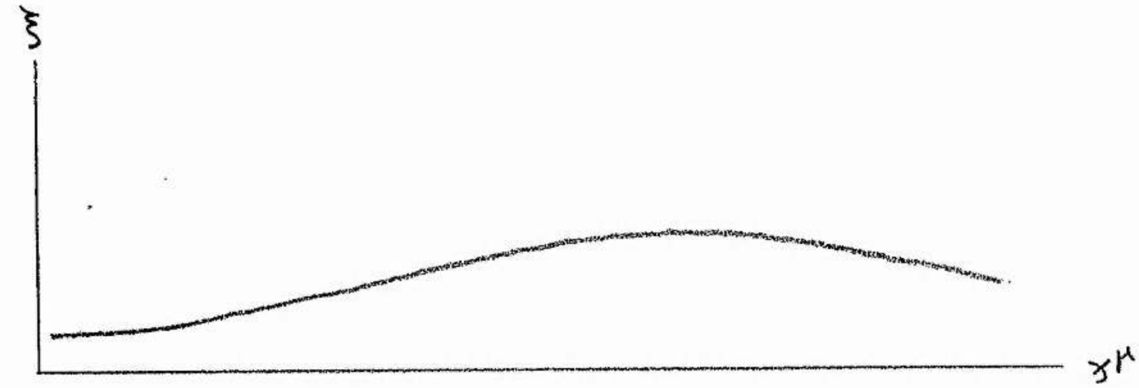
$$= \frac{7\,E_k^6}{10\pi(\hbar c)^4 Q^2}\;.$$

CHAPTER 7

MODEL FIELD II TREATMENT III

7.1 Introduction

In the previous chapter a weak field approximation, which is common in the treatment of Einstein's equations of General Relativity, was used. This approximation is a very restrictive one. A surface of the shape shown schematically below is then outside the scope of the approximation because $\xi_{,\mu}$ may not be small everywhere.



However such surface may still have an important property which we shall call the "everywhere-slowly-varying property". By this we mean that the deviation of the surface from the tangent plane at a nearby point on the surface is always small compared with unity. It is this property which corresponds to the local flatness in the Einstein's theory of General Relativity. Efforts were made to devise an approximation scheme to accommodate this more realistic situation for our Model Field II. In this chapter, the same coordinate conditions as those employed in the previous chapter will be used, and we shall assume that our field $\xi(x)$ has the everywhere-slowly-varying property. In terms of canonical variables the latter is equivalent to the assumption that $\nabla\xi, \pi$ be slowly varying functions of x^μ.

7.2 The Idea

The central problem is to obtain an approximation expansion of the Hamiltonian. Now divide the spatial space into cubic boxes. The $\underline{n}$th box is taken as the domain $V_{\underline{n}}$ which is defined by

$$(\underline{n}-\tfrac{1}{2})L \leq \underline{x} \leq (\underline{n}+\tfrac{1}{2})L, \quad \text{where } L \text{ is a numerical constant and}$$
$$\underline{n} = (n^1, n^2, n^3);\ n^i = \text{zero or integers.}$$

The size of the box specified by L is to be such that $\nabla\xi, \pi$ vary very little over it. The assumption that $\nabla\xi$ and π be slowly varying functions of the spatial coordinates means that L may be large (for a more precise analysis, see Appendix 7.4.) It is apparent that certain approximations can be made within each box. To explore this possibility, let us consider a general situation. Suppose $\theta_a(\underline{x})$ with $a = 1, 2, \ldots, M$ are a set of N real functions which vary slowly with $\underline{x}$ over a region of dimension L. The problem posed is to obtain an approximation expression for the integral

$$I = \int_{-\infty}^{\infty} d^3x\, F(\theta_a(\underline{x})) \ .$$

Let $I = \sum_{\underline{n}} I_{\underline{n}}$, where $I_{\underline{n}} = \int_{V_{\underline{n}}} d^3x\, F(\theta_a(\underline{x}))$.

We can make a Fourier expansion of $\theta_a(\underline{x})$ within each $V_{\underline{n}}$.

$$\theta_a = \sum_{\underline{K}} \theta_{a\underline{K}\underline{n}} \frac{e^{i\underline{K}\cdot\underline{x}}}{L^{3/2}} \qquad \text{for } (\underline{n}-\tfrac{1}{2})L < \underline{x} < (\underline{n}+\tfrac{1}{2})L \ ,$$

where $\underline{K} = \frac{2\pi}{L}(K^1, K^2, K^3)$; K^j = zero or integers.

Since θ_a changes little over the box, we may consider

$$<\theta_a>_{\underline{n}} = \frac{1}{V_{\underline{n}}} \int_{V_{\underline{n}}} d^3x\ \theta_a(\underline{x}) = \frac{1}{L^{3/2}} \theta_{ao\underline{n}} \ ,$$

as the average value of θ_a inside the box and

$$\theta'_{a\underline{n}} = \theta_a(\underline{x}) - <\theta_a>_{\underline{n}} = \sum_{K\neq 0} \theta_{a\underline{K}\underline{n}} \frac{e^{i\underline{K}\cdot\underline{x}}}{L^{3/2}} = \text{small in } V_{\underline{n}}.$$

Therefore we may develop a Taylor series expansion of $F(\theta_a)$ in $V_{\underline{n}}$ by

$$\begin{aligned} F(\theta_a) &= F(<\theta_a>_{\underline{n}} + \theta'_{a\underline{n}}) \\ &= F(<\theta_a>_{\underline{n}}) + \sum_a \left(\frac{\partial F(\theta_d)}{\partial \theta_a}\right)_{\theta_a = <\theta_a>_{\underline{n}}} \theta'_{a\underline{n}} + \frac{1}{2!} \sum_{ab} \left(\frac{\partial^2 F(\theta_d)}{\partial\theta_a \partial\theta_b}\right)_{\substack{\theta_a=<\theta_a>_{\underline{n}} \\ \theta_b=<\theta_b>_{\underline{n}}}} \\ &\quad \theta'_{a\underline{n}}\theta'_{b\underline{n}} + \dots \ . \end{aligned}$$

Hence

$$I_{\underline{n}} = \int_{V_{\underline{n}}} d^3x\ F(\theta_a) \approx F(<\theta_a>_n)\ V_{\underline{n}} + \tfrac{1}{2} \sum_{ab} \sum_{\underline{K}\neq 0} \left(\frac{\partial^2 F}{\partial\theta_a \partial\theta_b}\right)_{\substack{\theta_a=<\theta_a>_{\underline{n}} \\ \theta_b=<\theta_b>_{\underline{n}}}} \theta^*_{a\underline{K}\underline{n}}\theta_{b\underline{K}\underline{n}} \ ,$$

and

$$I = \sum_{\underline{n}} I_{\underline{n}} \approx \sum_{\underline{n}} \left(L^3 F(\langle\theta_a\rangle_{\underline{n}}) + \tfrac{1}{2} \sum_{ab} \sum_{K \neq 0} \left\{ \frac{\partial^2 F}{\partial\theta_a \partial\theta_b} \right\}_{\substack{\theta_a = \langle\theta_a\rangle_{\underline{n}} \\ \theta_b = \langle\theta_b\rangle_{\underline{n}}}} \theta^*_{\underline{aKn}} \theta_{\underline{bKn}} \right).$$

We can apply this approximate expression to our Hamiltonian

$$H = \int d^3x\, \mathcal{H}, \quad \text{where } \mathcal{H} = Q\sqrt{(1+(\nabla\xi)^2)(1+\frac{c^2}{G^2}\pi^2)} \;,$$

$$\approx \sum_{\underline{n}} L^3 \mathcal{H}_{\underline{n}} + \tfrac{1}{2} \sum_{\substack{\underline{n} \\ \underline{K} \neq 0}} [(\frac{\partial^2 \mathcal{H}}{\partial\pi^2})_{\underline{n}} P^\dagger_{\underline{Kn}} P_{\underline{Kn}} + (\frac{\partial^2 \mathcal{H}}{\partial\pi\partial\xi_{,j}})_{\underline{n}} P^\dagger_{\underline{Kn}} q_{\underline{Kn}j} + (\frac{\partial^2 \mathcal{H}}{\partial\xi_i \partial\xi_j})_{\underline{n}} q^\dagger_{\underline{Kn}i} q_{\underline{Kn}j}],$$

where ① the subscript $\underline{n}$ means that functions of $\nabla\xi, \pi$ are evaluated at $\nabla\xi = \langle\nabla\xi\rangle_{\underline{n}}$,

$\pi = \langle\pi\rangle_n$.

② $P_{\underline{Kn}}$, $q_{\underline{Kn}}$, $q_{\underline{Kn}j}$ are the coefficients of the Fourier Expansions of π, ξ, $\xi_{,j}$ respectively.

$$P_{\underline{Kn}} = \int_{V_{\underline{n}}} \pi \frac{e^{-i\underline{K}.\underline{x}}}{L^{3/2}} d^3x \;;\; q_{\underline{Kn}} = \int_{V_{\underline{n}}} \xi \frac{e^{-i\underline{K}.\underline{x}}}{L^{3/2}} d^3x \;;$$

$$q_{\underline{Kn}j} = \int_{V_{\underline{n}}} \xi_{,j} \frac{e^{-i\underline{K}.\underline{x}}}{L^{3/2}} d^3x = iK^j q_{\underline{Kn}} + \int_{S_{\underline{n}}} \xi \frac{e^{-i\underline{K}.\underline{x}}}{L^{3/2}} dS_j \;,$$

where $S_{\underline{n}}$ is the surface of $V_{\underline{n}}$

$dS_1 = dx^2\, dx^3$; $dS_2 = dx^3\, dx^1$; $dS_3 = dx^1\, dx^2$.

It is tempting now to treat $q_{\underline{Kn}}$, $p^\dagger_{\underline{Kn}}$ as canonical coordinates and momenta to effect a quantization.

There are difficulties. Awkward continuity conditions on the boundary surfaces between the boxes will inevitably appear and one has the difficulty of expressing $q_{\underline{Kn}j}$ purely in terms of $q_{\underline{Kn}}$ because of the term involving a surface integral.

Another treatment using the same basic idea while avoiding the above difficulties will be discussed in the remainder of this chapter. In this treatment a new complete orthonormal set of functions is used for the decomposition of ξ, $\nabla\xi$, π to preserve continuity.

7.3 A Complete Orthonormal set of Functions $U_{Kn}^{(x)}$

Define a set of functions $U_{Kn}^{(x)}$ of a single variable x by

$$U_{Kn}^{(x)} = \frac{1}{\sqrt{L}} e^{iKx} \frac{\sin\pi\left(\frac{x}{L} - n\right)}{\pi\left(\frac{x}{L} - n\right)} ,$$

where: (1) L is a numerical constant,

(2) $K = \frac{2\pi}{L}$ *(integers or zero),

(3) n = integers or zero.

Note that U_{Kn} is defined for all x from $-\infty$ to $+\infty$.

The relevant properties of this set of functions are listed in Appendix 7.1 at the end of this chapter. In particular, we note that $U_{Kn}^{(x)}$ are a complete orthonormal set of functions and $|U_{Kn}|$ has the absolute maximum at $x_n = nL$. $|U_{Kn}|$ decreases as x moves away from x_n and becomes small compared with $|U_{Kn}^{(x_n)}|$ as $|x-x_n| >> L$.

7.4 An Approximation Procedure

We want to consider the same problem as studied in section 7.2 again, that is, to obtain an approximate expression for the integral

$$I = \int_{-\infty}^{\infty} dx\ F(\theta_a(x)) \ ,$$

where $\theta_a(x)$ are a set of slowly-varying real functions of x. For the moment we consider only a single independent variable x, but shall later extend the results to the case of three spatial variables $\underline{x}$. To make use of the set of functions $U_{Kn}^{(x)}$, we firstly adjust the value of the L in $U_{Kn}^{(x)}$ to be such that $\theta_a(x)$ changes only slightly over a dimension L. Again L may be large on account of the slowly-varying nature of $\theta_a(x)$. We can write

$$\theta_a = \sum_{Kn} \theta_{aKn}\, U_{Kn}^{(x)} \ , \qquad \text{where } \theta_{aKn} = \int_{-\infty}^{\infty} \theta_a\, U_{Kn}^{*}\, dx \ .$$

Define

$$<\theta_a> = \frac{\int U_{on}\, \theta_a\, dx}{\int U_{on}\, dx} = \frac{\theta_{aon}}{L^{\frac{1}{2}}} \ , \text{ and}$$

$$\theta'_{an} = \theta_a(x) - <\theta_a>_n \ .$$

By our assumption, we must have

$$\theta'_{an} = \text{small}.$$

A more precise analysis of this statement may be found in Appendix 7.4.

we have parameters τ and V, . $x = V$

$$t = \tau - \sqrt{b^2 + V^2} \qquad b \text{ a constant.}$$

in other words. looking at the equations of Wan page 28. to see that we have $y^0 = V$

$$y^1 = \tau - \sqrt{b^2+V^2}$$

This not b very clear on decide c.f. page 67

V is the coordinate on the surface.

The tangent vector to the surface. will have components

$$Y^0_{,1} \quad \frac{\partial y^0}{\partial V^1} = 1 \quad \frac{\partial y^1}{\partial V^1} = -\frac{1}{2}(b^2+V^2)^{-\frac{1}{2}} 2V = -\frac{V}{\sqrt{b^2+V^2}} = Y^1_{,1}$$

now require surface metric. Only only 1 component from the formula at bottom of page 28

$$Y_{11} = g_{\mu\nu} Y^\mu_1 Y^\nu_1 = g_{00}(Y^0)^2 + g_{11}(Y^1)^2$$

$$= V^2 - (\tau - \sqrt{b^2+V^2})^2 \text{ - diff of two squares,}$$

∴ c.f. page 72. So we have.

$$g_{\pi/2} \quad g = \sqrt{-V^2 + (\tau - \sqrt{b^2+V^2})^2} \text{ — this is wrong altogether probably not.}$$

page 31 tells us how to go from a lagrangian on 4 spacetime $\mathcal{L}$ to one over the restricted spacetime $\bar{\mathcal{L}}$ In our case is the very scalar field $\sqrt{-g}$.

pg 67 the x^μ and on the V's. and the Σ^A are the x's.

$$Y_{11} = \cancel{g}\, Y^0_{,1} Y^0_{,1} - Y^1_{,1} Y^1_{,1}$$

$$Y_{11} = g_{\mu\nu} \quad 1 - \frac{V^2}{(b^2+V^2)} = \frac{b^2}{b^2+V^2}$$

For use. $\Sigma^A_{,\mu} = Y$ $A = (0,1)$ $\rho = r = 1$

Σ^A = x and t. = y^0 and y^1

$$\mathcal{L} = \sqrt{-g} = \sqrt{\frac{U^2}{U^2+V^2}} = \frac{U}{\sqrt{U^2+V^2}}$$

Where has Σ^A_0 ... [illegible]

I think ? should be zero.

The only is [illegible] to the other end. [illegible] that

$$\frac{\partial \mathcal{L}}{\Sigma^A_{10}} = \frac{\partial \mathcal{L}}{\partial \Sigma^A_{11}} = \frac{\partial \mathcal{L}}{\partial Y^0_{11}}$$

0 1 2 3 4
0 1 2 3

0 1 2 3
1 2 3

$$\mathcal{L} = \sqrt{(Y^0_{11})^2 - (Y^1_{11})^2}$$

Π_A = Π_0 Π_1 Π_2 Π_3

0 1 2 3 4
0 1 2 3

$$\Pi_0 = \frac{\partial \mathcal{L}}{\partial Y^0_{11}} = \frac{1}{2}\left((Y^0_{11})^2 - (Y^1_{11})^2\right)^{-\frac{1}{2}} 2\, Y^0_{11} = \left((Y^0_{11})^2 - (Y^1_{11})^2\right)^{-\frac{1}{2}} Y^0_{11}$$

$$\Pi_1 = \frac{\partial \mathcal{L}}{\partial Y^1_{11}} = -\frac{1}{2}\left((Y^0_{11})^2 - (Y^1_{11})^2\right)^{-\frac{1}{2}} 2\, Y^1_{11} = -\left((Y^0_{11})^2 - (Y^1_{11})^2\right)^{-\frac{1}{2}} Y^1_{11}$$

[illegible]. $\Sigma^A_{10} = t_{12} \quad X_{12} = 0$

$\parallel$

1

$$\Pi^A = \frac{\partial \mathcal{L}}{\partial Y_{A,0}}.$$

$\langle\theta_a\rangle_n$ may be regarded as the average value of θ_a in a neighbourhood of $x = nL$ of dimensions of the order of L.

Now we are in a position to state a

Theorem:

If $\theta_a(x)$ is a set of slowly-varying real functions of x, then

$$I = \int_{-\infty}^{\infty} dx\, F(\theta_a(x))$$

$$\approx \sum_n \left(L\, F(\langle\theta_a\rangle_n) + \tfrac{1}{2}\left(\frac{\partial^2 F}{\partial\theta_a \partial\theta_b}\right)_n \sum_{K\neq 0} \theta^*_{aKn}\, \theta_{bKn} \right) ,$$

where $\left(\frac{\partial^2 F}{\partial\theta_a\partial\theta_b}\right)_n \equiv \left(\frac{\partial^2 F}{\partial\theta_a\partial\theta_b}\right)_{\substack{\theta_a=\langle\theta_a\rangle_n \\ \theta_b=\langle\theta_b\rangle_n}}$, $\theta_{aKn} = \int \theta_a\, U^*_{Kn}\, dx$, and

summations over a,b are implied.

The proof together with an examination of the approximation involved is given in Appendix 7.2. Extension to 3-dimensional case is easily obtained by defining the complete orthonormal set of functions

$$U_{\underline{Kn}}(\underline{x}) = \frac{1}{L^{3/2}}\, e^{i\underline{K}\cdot\underline{x}} \prod_{j=1,2,3} \frac{\sin\pi(\frac{x^j}{L} - n^j)}{\pi(\frac{x^j}{L} - n^j)} .$$

The Theorem takes exactly the same form with L in the first sum replaced by $L^{3/2}$.

In the following section our Model Field II will be examined using this approximation scheme.

7.5 Quantization

Expand

$$\xi(x) = \sum_{\underline{Kn}} q_{\underline{Kn}} U_{\underline{Kn}}(\underline{x}), \quad \pi(x) = \sum_{\underline{Kn}} P_{\underline{Kn}} U_{\underline{Kn}}(\underline{x}) ,$$

where

$$q_{\underline{Kn}} = \int d^3\underline{x}\ \xi(x)\ U^*_{\underline{Kn}}(\underline{x}) = q^\dagger_{-\underline{Kn}}$$

$$P_{\underline{Kn}} = \int d^3\underline{x}\ \Pi(x)\ U^*_{\underline{Kn}}(\underline{x}) = \pi^\dagger_{-\underline{Kn}} .$$

We can now use the Theorem to evaluate the Hamiltonian of our Model Field II. As given in Appendix 7.3 the result is

$$H = QL^3 \sum_{\underline{n}} \sqrt{(1+(\nabla\xi)_{\underline{n}})(1+\frac{c^2}{Q^2}\langle\pi\rangle^2_{\underline{n}})} + \tfrac{1}{2} \sum_{\substack{\underline{n},\underline{K}\neq 0\\ j}} \frac{g_{\underline{n}}^{oj}c}{g_{\underline{n}}^{oo}} (q^\dagger_{\underline{Kn}}\ iK^j\ P_{\underline{Kn}} - P^\dagger_{\underline{Kn}}\ iK^j\ q_{\underline{Kn}})$$

$$+ \sum_{\underline{n}} \tfrac{1}{2} \frac{c^2}{Q} \frac{\sqrt{-g_{\underline{n}}}}{g_{\underline{n}}^{oo}} \sum_{\underline{K}\neq 0} P^\dagger_{\underline{Kn}} P_{\underline{Kn}} + \tfrac{1}{2} \sum_{\substack{\underline{n},\underline{K}\neq 0\\ j,k}} \frac{-Q}{\sqrt{-g_{\underline{n}}}} \left(\frac{g_{\underline{n}}^{oj} g_{\underline{n}}^{ok}}{g_{\underline{n}}^{oo}} - g_{\underline{n}}^{jk}\right) \sum_{\underline{K}\neq 0} K^j K^k q^\dagger_{\underline{Kn}} q_{\underline{Kn}}$$

+ higher terms, where $g^{\tau\kappa}$ is the metric tensor for the 4-surface. $g^{\tau\kappa}$ and $\sqrt{-g}$ may be expressed in terms of $\xi_{,j}$ and π as seen in Appendix 7.3 and $g_{\underline{n}}^{\tau\kappa} = [g^{\tau\kappa}]_{\substack{\xi_{,j} = \langle\xi_j\rangle_{\underline{n}} \\ \pi = \langle\pi\rangle_{\underline{n}}}}$. Note that $\langle\xi_j\rangle_{\underline{n}}$, $\langle\pi\rangle_{\underline{n}}$, $g_{\underline{n}}^{\tau\kappa}$ depend solely on $q_{o\underline{n}}$ and $p_{o\underline{n}}$.

Now define

$$a_{\underline{\underline{Kn}}} = \left\{\frac{(g_{\underline{n}}^{oj} g_{\underline{n}}^{ok} - g_{\underline{n}}^{jk} g_{\underline{n}}^{oo}) K^j K^k \; Q^2}{4\hbar^2 (-g_{\underline{n}})}\right\}^{\frac{1}{4}} q_{\underline{\underline{Kn}}} + ic\left\{\frac{(-g_{\underline{n}})}{Q^2 4\hbar^2 (g_{\underline{n}}^{oj} g_{\underline{n}}^{ok} - g_{\underline{n}}^{ik} g_{\underline{n}}^{oo}) K^j K^k}\right\}^{\frac{1}{4}} p_{\underline{\underline{Kn}}},$$

$$w_{\underline{\underline{Kn}}} = \frac{\left\{(g_{\underline{n}}^{oj} g_{\underline{n}}^{ok} - g_{\underline{n}}^{jk} g_{\underline{n}}^{oo}) K^j K^k\right\}^{\frac{1}{2}} g_{\underline{n}}^{oj} K^j}{g_{\underline{n}}^{oo}} \quad ,$$

where repeated indices imply a summation.

Then

$$H = QL^3 \sum_{\underline{n}} \sqrt{(1+(\nabla\xi)_{\underline{n}})(1+ \frac{c^2}{Q^2} <\pi>_{\underline{n}}^2)}$$

$$+ \sum_{\substack{\underline{K}\neq 0 \\ \underline{n}}} \hbar w_{\underline{\underline{Kn}}} a_{\underline{\underline{Kn}}}^{\dagger} a_{\underline{\underline{Kn}}} + \text{higher terms.} \qquad (7.5\text{-}1)$$

To quantize it, the canonical variables ξ, π are to be regarded as operators in a linear vector space with the standard equal-time commutation rules

$$[\xi(\underline{x}), \pi(\underline{x}')] = i\hbar \; \delta(\underline{x}-\underline{x}') \; ; \; [\xi(\underline{x}), \xi(\underline{x}')] = [\pi(\underline{x}), \pi(\underline{x}')] = 0$$

These imply the commutation rules for $q_{\underline{\underline{Kn}}}$, $p_{\underline{\underline{Kn}}}$

$$[q_{\underline{\underline{Kn}}}, p_{\underline{\underline{K'n'}}}^{\dagger}] = i\hbar \; \delta_{\underline{\underline{KK'}}} \delta_{\underline{\underline{nn'}}} \; ; \; [q_{\underline{\underline{Kn}}}, q_{\underline{\underline{K'n'}}}] = [p_{\underline{\underline{Kn}}}, p_{\underline{\underline{K'n'}}}] = 0$$

The everywhere-slowly-varying nature of the field ξ means that $<\xi_j>_{\underline{n}}$, $<\frac{c\pi}{Q}>_{\underline{n}}$ may be taken as unquantized c-numbers (see Appendix 7.4 for details). As a result, $g_{\underline{n}}^{\tau\kappa}$ may also be similarly treated as c-numbers.

To this approximation, we see that $a_{\underline{Kn}}$, $a^{\dagger}_{\underline{Kn}}$ defined above obey the standard commutation rules for creation and annihilation operators, i.e.

$$[a_{\underline{Kn}}, a^{\dagger}_{\underline{K'n'}}] = \delta_{\underline{KK'}}\delta_{\underline{nn'}} \; ; \; [a_{\underline{Kn}}, a_{\underline{K'n'}}] = [a^{\dagger}_{\underline{Kn}}, a^{\dagger}_{\underline{K'n'}}] = 0$$

Indeed with the above expression for the Hamiltonian, we can regard $a_{\underline{Kn}}, a^{\dagger}_{\underline{Kn}}$ respectively as the annihilation and creation operators for the quantum of energy $\hbar w_{\underline{Kn}}$.

Note that if we write

$$(k_{\underline{Kn}})_o = w_{\underline{Kn}}/c \quad \text{and} \quad (k_{\underline{Kn}})_j = -K^j ,$$

where K^i are the three components of $\underline{K}$, $(i = 1,2,3)$, then

$$g^{\lambda\mu}_{\underline{n}} \left(k_{\underline{Kn}}\right)_\lambda \left(k_{\underline{Kn}}\right)_\mu = 0 \; .$$

This means that

$$(w_{\underline{Kn}}/c \; , \; -\underline{K}) \; ,$$

are the covariant components of a null vector with respect to the metric $g^{\tau\kappa}_{\underline{n}}$.

Now the situation is that we have on the one hand the slowly-varying classical field $g^{\tau\kappa}_{\underline{n}}$ and on the other hand the quantized field with the Hamiltonian (7.5-1). The background metric $g^{\tau\kappa}_{\underline{n}}$ depends on the classical variables $\langle\xi_{,j}\rangle_{\underline{n}}$ and $\langle\frac{c\pi}{Q}\rangle_{\underline{n}}$ so that it will vary slowly with time as these variables develop in time according to the classical canonical equations generated by the Hamiltonian

$$QL^3 \sum_{\underline{n}} \sqrt{[1+<\nabla\xi>^2_{\underline{n}}][1+<\frac{c\pi}{Q}>^2_{\underline{n}}]} \; .$$

The quanta created by $a^{\dagger}_{\underline{K}\underline{n}}$ from the vacuum state are mainly in the region $n^i - L/2 < x^i < n^i + L/2$, but there is some overlap with neighbouring regions. The total number operator is

$$N = \sum_{\underline{n}} N_{\underline{n}} \; ,$$

where $N_{\underline{n}} = \sum_{\underline{K}} a^{\dagger}_{\underline{K}\underline{n}} a_{\underline{K}\underline{n}}$ and $[N_{\underline{n}}, N_{\underline{n}'}] = 0$.

It is seen that we have essentially a free field theory. To bring in interaction, one may proceed to higher order terms in the Taylor series. However the calculation becomes very lengthy and the usual divergence problem remains.

7.6 Appendix 7.1 Properties of $U_{kn}(x)$

(A) Fourier Transform

$$U_{Kn}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, U_{Kn}^{(x)}\, e^{-ikx} = \sqrt{\frac{L}{2\pi}}\, e^{-iknL} \quad \text{if } K - \frac{\pi}{L} < k < K + \frac{\pi}{L}$$

$$= \frac{1}{2}\sqrt{\frac{L}{2\pi}}\, e^{-iknL} \quad \text{if } k = K \pm \frac{\pi}{L}$$

$$= 0 \quad \text{otherwise}$$

(B) Orthogonality

$$\int_{-\infty}^{\infty} d\chi\, U_{Kn}^{*(x)}\, U_{K'n'}^{(x)} = \delta_{KK'}\, \delta_{nn'} .$$

(C) Completeness

$$\sum_{K,n} U_{Kn}^{*(x)}\, U_{Kn}^{(x')} = \delta(x-x') .$$

(D) $$\frac{\partial U_{Kn}}{\partial x} = iK\, U_{Kn} + \sum_{m \neq n} \frac{(-1)^{m-n}}{L(m-n)}\, U_{Km} .$$

(E) $$I_{pnn'}^{K-K'} = \int_{-\infty}^{\infty} d\chi\, U_{op}^{*}\, U_{Kn}^{*}\, U_{K'n'}$$

Case I: $K' - K = 0$

$$I_{pnn'}^{o} = \frac{3}{4\sqrt{L}} \quad \text{if } p = n = n'$$

$$= \frac{1}{2\pi^2\sqrt{L}\,(p-n')^2}\,(1-(-1)^{p-n'}) \quad \text{if } p = n \neq n'$$

$$= \frac{(-1)^{p+n+n'}}{2\pi^2\sqrt{L}} \left(\frac{(-1)^p}{(p-n)(p-n')} + \frac{(-1)^n}{(n-p)(n-n')} + \frac{(-1)^{n'}}{(n'-p)(n'-n)}\right) .$$

$$\text{if } p \neq n \neq n'$$

Case II: $K' - K = \frac{2\pi}{L}$

$$I^{2\pi/L}_{pnn'} = \frac{1}{8\sqrt{L}} \qquad \text{if } p = n = n'$$

$$= \frac{(-1)^{p-n'} - 1}{4\pi^2\sqrt{L}(p-n')^2} + \frac{i\ (-1)^{p-n}}{4\pi\sqrt{L}(p-n')} \qquad \text{if } p = n \neq n'$$

$$= -\frac{(-1)^{p+n+n'}}{4\pi^2\sqrt{L}}\left(\frac{(-1)^p}{(p-n)(p-n')} + \frac{(-1)^n}{(n-p)(n-n')} + \frac{(-1)^{n'}}{(n'-p)(n'-n)}\right).$$

$$\text{if } p \neq n \neq n'$$

Case III:

$I^{K'-K}_{pnn'} = 0$ in all other cases apart from those obtainable by the symmetry properties the expression $I^{K'-K}_{pnn'}$.

Symmetry Properties of $I^{K'-K}_{pnn'}$

(a) $I^{K-K'}_{pnn'} = (I^{K'-K}_{pnn'})^*$

(b) The order of the indices pnn' is irrelevant to the value of

$I^{K'-K}_{pnn'}$, e.g.,

$I^{K'-K}_{pnn} = I^{K'-K}_{nn'p} = I_{n\ pn}$ and so on,

$I^{K'-K}_{ppn'} = I^{K'-K}_{n'pp}$ and so on.

Our present set of functions may have certain similarities with the Block functions expressed in terms of the Wannier functions in solid state physics.[G1] But they are in fact quite different.

7.7 Appendix 7.2 The Theorem

$$I = \int_{-\infty}^{\infty} dx\ F(\theta_a(x)) \approx \sum_n \left(L\ F(<\theta_a>_n) + \tfrac{1}{2}\left(\frac{\partial^2 F}{\partial\theta_a \partial\theta_b}\right)_n \sum_{K \neq 0} \theta^{\dagger}_{aKn}\ \theta_{bKn} \right)$$

Proof:

Expanding the number 1 in terms of the complete orthonormal set U_{Kn} we obtain

$$1 = \sqrt{L}\ \sum_n U_{on}^{(x)} ,$$

so

$$I = \int dx\ F(\theta_a(x)) = \sqrt{L}\ \sum_n \int d^3x\ U_{on}\ F(\theta_a) = \sum I_n$$

$$I_m = \sqrt{L}\ \int dx\ U_{on}\ F(\theta_a)$$

We assume θ_a to be slowly varying functions of x so that the main contribution to I_n comes from the values of θ_a near $x_n = nL$ in which region $\theta'_{an} = \theta_a(x) - <\theta_a>_n << 1.$

Let $F(\theta_a) = F(<\theta_a>_n + \theta'_{an})$, then a Taylor expansion gives

$$F(\theta_a) = F(<\theta_a>_n) + \left(\frac{\partial F}{\partial\theta_a}\right)_n \theta'_{an} + \tfrac{1}{2}\left(\frac{\partial^2 F}{\partial\theta_a \partial\theta_b}\right)_n \theta'_{an}\ \theta'_{bn} + \ldots\ldots.$$

$$I_n = L\ F(<\theta_a>_n) + \sqrt{L}\ \left(\frac{\partial F}{\partial\theta_a}\right)_n \int U_{on}\ \theta'_{an}\ dx + \tfrac{1}{2}\sqrt{L}\ \left(\frac{\partial^2 F}{\partial\theta_a \partial\theta_b}\right)_n \int dx\ U_{on}\ \theta'_{an}\ \theta'_{bn} + \ldots\ldots$$

Now our assumption that I_m depends mainly on the values of θ_a near $x_n = nL$ shall mean that the same applies to each integral over a Taylor series term. Let $\theta'_{an}(x)$ be of the order of, say $\lambda << 1$, for χ near $x_n = nL$, then we can roughly estimate that $\int dx\, U_{on}\, \theta'_{an}\, \theta'_{bn}\, \theta'_{cn}$ is smaller than $\int dx\, U_{on}\, \theta'_{an}\, \theta'_{bn}$ by a factor λ. In this way we are able to establish a series approximation to I_n.

Now the proof of the Theorem rests on a

Lemma: If f_n is a slowly varying real function of n, then

$$\sqrt{L} \sum_n f_n \int U_{on}\, \theta'_{an}\, \theta'_{bn}\, dx \approx \sum_{n,K\neq 0} f_n\, \theta^*_{aKn}\, \theta_{bKn}$$

Proof:

$$\sqrt{L} \sum_n f_n \int U_{on}\, \theta_a\, \theta_b\, dx = \sum_{\substack{p,n,n',\\ K,K'}} \sqrt{L}\, f_p\, \theta^*_{aKn}\, \theta_{bK'n'} \int dx\, U_{op}\, U^*_{Kn}\, U_{K'n'}$$

The property (E) of U_{Kn} as given in Appendix 7.1 is used to evaluate the right-hand expression which gives

$$\text{R.H.S.} = \sum_K \sqrt{L} \left\{ ① + ② + ③ + ④ + ⑤ \right\},$$

where

$$① = \sum_{p=n=n'} f_n \left[\frac{3}{4}\, \theta^*_{aKn}\, \theta_{bKn} + \frac{1}{8}\, \theta^*_{aKn}\, \theta_{bK+\frac{2\pi}{L}n} + \frac{1}{8}\, \theta^*_{aKn}\, \theta_{bK-\frac{2\pi}{L}n}\right],$$

$$② = \sum_{p=n\neq n'} f_n \left(\frac{1-(-1)^{n-n'}}{2\pi^2 (n-n')^2}\, \theta^{\dagger}_{aKn}\, \theta_{bKn'} + \left[-\frac{1-(-1)^{n-n'}}{4\pi^2 (n-n')^2} + \frac{i(-1)^{n-n'}}{4\pi(n-n')}\right] \theta^*_{aKn}\, \theta_{bK+\frac{2\pi}{L}n'} + \left[-\frac{1-(-1)^{n-n'}}{4\pi^2 (n-n')^2} - \frac{i(-1)^{n-n'}}{4\pi(n-n')}\right] \theta^*_{aKn}\, \theta_{bK-\frac{2\pi}{L}n'} \right),$$

$$③ = \sum_{p=n'\neq n} f_{n'}\left(\frac{1-(-1)^{n-n'}}{2\pi^2 (n-n')^2}\,\theta^*_{aKn}\,\theta_{bKn'} + \left[-\frac{1-(-1)^{n-n'}}{4\pi^2 (n-n')^2} - \frac{i(-1)^{n-n'}}{4\pi(n-n')}\right]\theta^*_{aKn}\times\right.$$

$$\left.\theta_{bK+\frac{2\pi}{L}n'} + \left[-\frac{1-(-1)^{n-n'}}{4\pi^2 (n-n')^2} + \frac{i(-1)^{n-n'}}{4\pi(n-n')}\right]\theta^*_{aKn}\,\theta_{bK-\frac{2\pi}{L}n'}\right),$$

$$④ = \sum_{n=n'\neq p} f_p\left(\frac{1-(-1)^{p-n}}{2\pi^2 (p-n)^2}\,\theta^*_{aKn}\,\theta_{bKn} + \left[-\frac{1-(-1)^{p-n}}{4\pi^2 (p-n)^2} - \frac{i(-1)^{p-n}}{4\pi(p-n)}\right]\theta^*_{aKn}\times\right.$$

$$\left.\theta_{bK+\frac{2\pi}{L}n} + \left[-\frac{1-(-1)^{p-n}}{4\pi^2 (p-n)^2} + \frac{i(-1)^{p-n}}{4\pi(p-n)}\right]\theta^*_{aKn}\,\theta_{bK-\frac{2\pi}{L}n}\right).$$

Since f_p is slowly-varying with p we may make the assumption

$$\sum_{p\neq n} f_p \frac{1-(-1)^{p-n}}{(p-n)^2} \approx f_n \sum_{p\neq n} \frac{1-(-1)^{p-n}}{(p-n)^2} = \frac{\pi^2}{2} f_n\ ;$$

$$\sum_{p\neq n} f_p \frac{(-1)^{p-n}}{p-n} \approx f_n \sum_{p\neq n} \frac{(-1)^{p-n}}{p-n} = 0\ .$$

Hence the $f_{n'}$ in ③ may be replaced by f_n and

$$④ \simeq \sum_n f_n\left[\frac{1}{4}\theta^*_{aKn}\,\theta_{bKn} - \frac{1}{8}\theta^*_{aKn}\,\theta_{bK+\frac{2\pi}{L}n} - \frac{1}{8}\theta^*_{aKn}\,\theta_{bK-\frac{2\pi}{L}n}\right].$$

Similarly

$$⑤ \simeq \sum_{n\neq n'} \frac{-f_n}{\pi^2 (n-n')^2}\left(1-(-1)^{n-n'}\right)\left(\theta^*_{aKn}\,\theta_{bKn'} - \frac{1}{2}\theta^*_{aKn}\,\theta_{bK+\frac{2\pi}{L}n'} - \frac{1}{2}\theta^*_{aKn}\times\right.$$

$$\left.\theta_{bK-\frac{2\pi}{L}n'}\right).$$

Adding up we obtain finally

$$\sqrt{L} \sum_n f_n \int dx\, U_{on}\, \theta_a \theta_b \simeq \sum_{Kn} f_n\, \theta^*_{aKn}\, \theta_{bKn} \,.$$

Some remarks on the above approximation are worth mentioning. Let $y(x)$ be a function of x, then we can readily show

$$\text{(A)} \quad L < \frac{\partial y(x)}{\partial x} >_n = \sum_{n'} <y>_{n'} \frac{(-1)^{n-n'}}{(n-n')} \,,$$

$$\text{(B)} \quad L^2 < \frac{\partial^2 y(x)}{\partial x^2} >_n = - \Big[\sum_{n' \neq n} \frac{2(-1)^{n-n'}}{(n-n')^2} + \frac{\pi^2}{6} <y>_n \Big] \,,$$

where

$$<y>_n = \frac{\int y(x)\, U_{on}\, dx}{\sqrt{L}} \,.$$

Therefore our approximation used in proving the Lemma is equivalent to the approximation

$$< \frac{\partial}{\partial x} F(x) >_n \simeq 0 \text{ and } < \frac{\partial^2}{\partial x^2} F(x) >_n \simeq 0 \,,$$

where

$$F(x) = \sum_n f_n\, U_{on}^{(x)} \,.$$

Now to apply the Lemma to prove the Theorem we have

$$\sqrt{L} \sum_n f_n \int dx\, U_{on}\, \theta'_{an}\, \theta'_{bn} = \sqrt{L} \sum_n f_n \Big(\int U_{on}\, \theta_a \theta_b\, dx - \sqrt{L} <\theta_a>_n{}^* <\theta_b>_n \Big) \approx \sum_{\substack{K \neq 0 \\ n}} f_n\, \theta^*_{aKn}\, \theta_{bKn} \,, \text{ by the Lemma.}$$

Applying this result to

$$I = \int_{-\infty}^{\infty} dx\, F(\theta_a^{(x)}) = \sum_n I_n \,,$$

we immediately obtain the Theorem.

7.8 Appendix 7.3 Evaluation of the Hamiltonian

$$H = \int d^3x\, \mathcal{H}, \quad \text{where} \quad \mathcal{H} = Q\sqrt{(1+(\nabla\xi)^2)(1+\frac{C^2}{G^2}\pi^2)} \ .$$

Firstly approximation may be made for the derivatives of ξ.

$$\xi_{,1} = \sum_{\underline{Km}} q_{\underline{Km}}[i\, K^1 U^{(x)}_{\underline{Km}} + \sum_{m'^1 \neq m^1} \frac{(-1)^{m'^1 - m^1}}{L(m'^1 - m^1)} U_{\underline{K}m'^1 m^2 m^3}]$$

$$= \sum_{\underline{Km}} q_{\underline{Km}1} U^{(x)}_{\underline{Km}} \ ,$$

where $q_{\underline{Km}1} = iK^1 q_{Km} + \sum_{m'^1 \neq m^1} \frac{1}{L} \frac{(-1)^{m'^1 - m^1}}{(m'^1 - m^1)} q_{Km'^1 m^2 m^3}$.

Since $q_{Km'^1 m^2 m^3}$ is assumed to be slowly-varying with m'^1, the second sum in the above expression is very small and may be neglected in order to be consistent with the approximations made in Appendix 7.2. Now apply the Theorem to evaluate H.

$$H \simeq \sum_{\underline{n}} L^3 \sqrt{(1+\langle\nabla\xi\rangle^2_{\underline{n}})(1+\frac{C^2}{Q^2}\langle\pi\rangle^2_{\underline{n}})} + \tfrac{1}{2}\sum_{\underline{n}}(\frac{\partial^2 \mathcal{H}}{\partial\theta_a \partial\theta_b})_{\underline{n}} \sum_{K\neq 0} \theta^*_{aK\underline{n}}\, \theta_{bK\underline{n}} \ , \tag{7.8-1}$$

where $\theta_a \equiv (\xi_{,1}, \xi_{,2}, \xi_{,3}, c\pi/Q)$.

$\frac{\partial^2 \mathcal{H}}{\partial\theta_a \partial\theta_b}$ may be expressed in terms of the metric tensors $g_{\tau\kappa}$ and $g^{\tau\kappa}$.

$$\frac{\partial^2 \mathcal{H}}{\partial\pi^2} = \frac{C^2}{Q} \frac{(1+(\nabla\xi)^2)^{\frac{1}{2}}}{(1+\frac{C^2}{G^2}\pi^2)^{3/2}} = \frac{C^2}{Q}\frac{\sqrt{-g}}{g^{oo}} \ ; \quad \frac{\partial^2 \mathcal{H}}{\partial\pi\partial\xi_j} = \frac{C^2}{Q}\frac{\pi\xi_{,j}}{\sqrt{(1+(\nabla\xi)^2)(1+\frac{C^2}{G^2}\pi^2)}}$$

$$= -C\frac{g^{oj}}{g^{oo}} \ ;$$

$$\frac{\partial^2 \mathcal{H}}{\partial\xi_{,j}\partial\xi_{,k}} = Q\sqrt{\frac{1+\frac{C^2}{G^2}\pi^2}{1+(\nabla\xi)^2}}\left(\delta_{jk} - \frac{\xi_{,j}\xi_{,k}}{1+(\nabla\xi)^2}\right) = -\frac{Q}{\sqrt{-g}}\left(g^{jk} - \frac{g^{oj}g^{ok}}{g^{oo}}\right) .$$

$$H \simeq \sum_{\underline{n}} QL^3 \sqrt{(1+\langle\nabla\xi\rangle^2_{\underline{n}})(1+\frac{C^2}{Q^2}\langle\pi\rangle^2_{\underline{n}})} + \frac{1}{2}\sum_{\substack{\underline{n}\\ \underline{K}\neq 0\\ j}} \frac{g^{oj}_{\underline{n}}C}{g^{oo}_{\underline{n}}}\left(q^{\dagger}_{\underline{K}\underline{n}}\, iK^j\, p_{\underline{K}\underline{n}} - p^{\dagger}_{\underline{K}\underline{n}}\, iK^j q_{\underline{K}\underline{n}}\right)$$

$$+ \sum_{\underline{n}} \frac{1}{2}\frac{C^2}{Q}\frac{\sqrt{-g_{\underline{n}}}}{g^{oo}_{\underline{n}}}\sum_{\underline{K}\neq 0} p^{\dagger}_{\underline{K}\underline{n}}\, p_{\underline{K}\underline{n}} + \frac{1}{2}\sum_{\substack{\underline{n}\\ \underline{K}\neq 0\\ j,k}} \frac{Q(-1)}{\sqrt{-g_{\underline{n}}}}\left(\frac{g^{oj}_{\underline{n}}\, g^{ok}_{\underline{n}}}{g^{oo}_{\underline{n}}} - g^{jk}_{\underline{n}}\right)\sum_{\underline{K}\neq 0} K^j K^k\, q^{\dagger}_{\underline{K}\underline{n}} q_{\underline{K}\underline{n}},$$

where $g^{\tau\kappa}_{\underline{n}} = [g^{\tau\kappa}]_{\pi = \langle\pi\rangle_{\underline{n}}}$

$$\xi_j = \langle\xi_j\rangle_{\underline{n}} .$$

7.9. Appendix 7.4 The Approximately Classical Nature of $g_{\underline{n}}^{\tau\kappa}$

Two assumptions are made in relation to the everywhere-slowly-varying nature of the field.

(1) $\theta_{an}^{\prime(x)} = \theta_a^{(x)} - \langle\theta_a\rangle_n \ll \theta_a^{(x)}$ and $\theta_{an}^{\prime(x)} \ll 1$ in a neighbourhood of $x = nL$ of dimensions of the order L. Note that θ_a stand for the dimensionless quantities $\xi_{,j}$, $\frac{c\pi}{Q}$ so that the above inequalities are independent of the units employed.

(2) L can be large, or more precisely, we require that $\frac{\hbar c}{QL^4} \ll 1$, where $\frac{\hbar c}{QL^4}$ is a dimensionless quantity.

In C.G.S. units with $Q = 10^{-7}$ erg $cm^{-3} \approx$ the mean energy density of the universe, we get $L \gg (\frac{\hbar c}{Q})^{\frac{1}{4}} \approx 10^{-2.5}$cm.

Firstly these assumptions enable us to count the orders of smallness of a quantity (see Appendix 7.2.) Secondly they lead to the result that $g_{\underline{n}}^{\tau\kappa}$ may be regarded as c-numbers. This result is seen in the following analysis based on a finite difference method.

$$\xi_{,1}^{(x)} \approx \frac{1}{2L}(\xi(x^1+L, x^2, x^3) - \xi(x^1-L, x^2, x^3)) .$$

$\Longrightarrow$ $\langle\xi_{,1}\rangle_{\underline{n}} \approx \frac{1}{2L^{5/2}}(q_{o\underline{n}+} - q_{o\underline{n}-})$, where $\underline{n}+ = (n^1+1, n^2, n^3)$; $\underline{n}- = (n^1-1, n^2, n^3)$.

$$\langle\frac{c\pi}{Q}\rangle_{\underline{n}'} = \frac{c}{QL^{3/2}} P_{o\underline{n}'} .$$

The commutator

$$[\langle\xi_{,1}\rangle_{\underline{n}}, \langle\frac{c\pi}{Q}\rangle_{\underline{n}'}] \approx \tfrac{1}{2}\frac{\hbar c}{QL^4}(i\ \delta_{\underline{n}+\underline{n}'} - i\ \delta_{\underline{n}-\underline{n}'})$$

$$\rightarrow 0 \text{ as } \frac{\hbar c}{QL^4} \rightarrow 0 \ .$$

Since $\langle\xi_{,j}\rangle_{\underline{n}}$, $\langle\frac{c\pi}{Q}\rangle_{\underline{n}'}$ also commute with all the other operators appearing in the theory, we conclude that $\langle\xi_{,j}\rangle_{\underline{n}}$, $\langle\frac{c\pi}{Q}\rangle_{\underline{n}'}$ can be treated as c-numbers. The set of quantities $g_{\underline{n}}^{\tau\kappa}$ depends solely on $\langle\xi_{,j}\rangle_{\underline{n}}$, $\langle\frac{c\pi}{Q}\rangle_{\underline{n}}$. Hence they may be similarly taken as c-numbers.

CHAPTER 8

MODEL FIELD II A VARIATIONAL TREATMENT

8.1 Introduction

The treatments given in Chapters 6 and 7 lead to divergences in the calculation for the energy eigenvalues of various states. However it may be possible that the divergences obtained are a spurious result of the perturbation method used. It is therefore desirable to devise a non-perturbative approach for the treatment of the field. In this chapter a variational approach is adopted for the calculation of energy eigenvalues.

8.2 The Idea

The essential idea lies in the approximation of the continuum field by a system of countable degrees of freedom. Once this is done many of the usual techniques of quantum mechanics, in particular the variational method for the calculation of eigenvalues, may readily be employed to study the system. There are a number of ways to achieve this. We shall adopt a method of approximation by finite differences.[H1]

To begin with, confine the field in a large yet finite cubic box of side L with the usual periodic boundary conditions. Divide this spatial box into $M = (2N+1)^3$ small cubic cells, each of side $d = L/(2N+1)$, N being a positive integer. The centre of the $\underline{n}$th cell is specified by $\underline{x} = \underline{x}_{\underline{n}} = (n_1, n_2, n_3)d$, where $-N \leqslant n_i \leqslant N$. Then we may make the following approximations for the quantities in the $\underline{n}$th cell:

(1) the field variable $\xi(\underline{x}) \approx \xi_{\underline{n}} = \xi(\underline{x}_{\underline{n}})$, where $\underline{x}_{\underline{n}} = (n_1, n_2, n_3)d$,

(2) $\nabla\xi(x) \approx \Delta\xi_{\underline{n}}$

$$= \frac{1}{2d}\Big(\xi_{n_1+1,\, n_2,\, n_3} - \xi_{n_1-1,\, n_2,\, n_3},\ \xi_{n_1,\, n_2+1,\, n_3} - \xi_{n_1,\, n_2-1,\, n_3},\ \xi_{n_1,\, n_2,\, n_3+1} - \xi_{n_1,\, n_2,\, n_3-1}\Big),$$

(3) the canonical momentum density $\pi(x) \approx p_{\underline{n}}/d^3$, where $p_{\underline{n}}$ is the momentum conjugate to $\xi_{\underline{n}}$,

(4) the total Hamiltonian $H \approx \sum_{\underline{n}} H_{\underline{n}}$, where $H_{\underline{n}} \approx$

$$Qd^3\sqrt{[1+(\Delta\xi_{\underline{n}})^2][1+\left(\frac{cp_{\underline{n}}}{Qd^3}\right)^2]},$$

(5) the total linear momentum $\underline{P} \approx \sum_{\underline{n}} \underline{P}_{\underline{n}}$, where $\underline{P}_{\underline{n}} \approx -p_{\underline{n}} \Delta \xi_{\underline{n}}$.

The quantization may be effected by the usual procedure with the explicit representation

$$\xi_{\underline{n}} \to \xi_{\underline{n}} \;;\; p_{\underline{n}} \to -i\hbar \frac{\partial}{\partial \xi_{\underline{n}}} \;.$$

Note that $[1+(\Delta\xi_{\underline{n}})^2]$ commutes with $[1+\left(\frac{cp_{\underline{n}}}{Qd^3}\right)^2]$. Hence there is no ambiguity in the expression for $H_{\underline{n}}$.

To illustrate the variational method to be used for the study of the Hamiltonian $H = \sum_{\underline{n}} Qd^3 \sqrt{[1+(\Delta\xi_{\underline{n}})^2][1+\left(\frac{cp_{\underline{n}}}{Qd^3}\right)^2]}$, let us consider the much simpler and well-defined case of the massless real Klein-Gordon field. In the finite difference approximation, the Hamiltonian is

$$H = \frac{Qd^3}{2} \sum_{\underline{n}} [(\Delta\xi_{\underline{n}})^2 + \left(\frac{cp_{\underline{n}}}{Qd^3}\right)^2]$$

$$= \frac{Qd^3}{2} \sum_{\underline{n}} [(\Delta\xi_{\underline{n}})^2 - \left(\frac{\hbar c}{Qd^3}\right)^2 \frac{\partial^2}{\partial \xi_{\underline{n}}^2}] .$$

The eigenfunctionals of H in the functional representation in the continuum case are explicitly known.[H2] The exact eigenfunctions of H in our present discrete case may also be similarly established. The vacuum state is

$$\Psi_o = A \exp[-\sigma^2 \sum_{\underline{k},\underline{m},\underline{m}'} w_k \, e^{i\underline{k}\cdot(\underline{m}-\underline{m}')d} \, \xi_{\underline{m}} \xi_{\underline{m}'}] ,$$

where (1) $\sigma^2 = \frac{1}{2cM}\left(\frac{Qd^3}{\hbar c}\right)$;

(2) A = normalization constant;

(3) $\underline{k} = \frac{2\pi}{L}(n_1, n_2, n_3)$ and $-N \leqslant n_i \leqslant N$;

(4) $w_{\underline{k}} = \frac{c}{d}\sqrt{(\sin k_x d)^2 + (\sin k_y d)^2 + (\sin k_z d)^2}$.

One can verify that

$$H\Psi_o = E_o\Psi_o \ ; \ \underline{P}\Psi_o = 0,$$

where $E_o = \frac{1}{2}\sum_{\underline{k}} \hbar w_{\underline{k}}$ which is clearly seen to approach the well-known infinite zero point energy $\frac{1}{2}\sum_{\underline{k}} \hbar kc$ in the continuum case as $d \to o$ and $V = Md^3 \to \infty$. For $d \to o$ keeping the volume $V = Md^3$ large but finite we have

$$E_o = \frac{1}{2}\sum_{\underline{k}} \hbar w_{\underline{k}} \to \frac{\hbar cV}{d^4}\ \frac{1}{16\pi^3}\iiint_{-\pi}^{\pi} dx\,dy\,dz\sqrt{\sin^2 x + \sin^2 y + \sin^2 z} + O\left(\frac{\hbar cV^{\frac{1}{3}}}{d^2}\right). \qquad (8.2\text{-}1)$$

The one-particle eigenstates are of the form

$$\Psi_{\underline{k}} = B\sum_{\underline{m}} e^{i\underline{k}\cdot\underline{m}d}\,\xi_{\underline{m}}\Psi_o,$$

noting that

$$\frac{1}{M}\sum_{\underline{k}} e^{i\underline{k}\cdot(\underline{n}-\underline{n}')d} = \delta_{\underline{n}\underline{n}'} \ ; \ \frac{1}{M}\sum_{\underline{n}} e^{i\underline{n}\cdot(\underline{k}-\underline{k}')} = \delta_{\underline{k},\underline{k}'}.$$

We have

$$H\Psi_{\underline{k}} = E_{\underline{k}} \Psi_{\underline{k}} \; ; \; \underline{P}\Psi_{\underline{k}} = \underline{p}\Psi_{\underline{k}},$$

where $E_{\underline{k}} = \frac{1}{2} \sum_{\underline{k}'} \hbar w_{\underline{k}'} + \hbar w_{\underline{k}} \; ; \; \underline{p} = \hbar\left(\frac{\sin dk_x}{d}, \frac{\sin dk_y}{d}, \frac{\sin dk_z}{d}\right).$

These eigenvalues tend to the original continuum values as d approaches zero. All other many-particle eigenstates may be similarly obtained.

Now suppose we did not know the exact eigenfunctions. We can use the variational technique to estimate the eigenvalues now that we have a discrete system. Take the normalized trial wave function for the vacuum to be

$\Phi_o = \prod_{\underline{n}} \Phi_{o\underline{n}}$, where $\Phi_{o\underline{n}} = \frac{e^{-\xi_{\underline{n}}^2/2\sigma^2}}{(\pi\sigma^2)^{\frac{1}{4}}}$ and σ^2 is to be the variational parameter. The vacuum expectation value is then

$$(\Phi_o, H\Phi_o) = \frac{Qd^3 M}{2} \left(\frac{3\sigma^2}{4d^2} + \frac{d^2}{2\rho^2\sigma^2}\right).$$

where $\rho = \frac{Qd^4}{c\hbar}$ = dimensionless. Hence the estimated zero point energy is given by the minimum of $(\Phi_o, H\Phi_o)$.

$$(\Phi_o, H\Phi_o)_{min} = \sqrt{\frac{3}{8}} \frac{M\hbar c}{d} = \sqrt{\frac{3}{8}} \frac{\hbar cV}{d^4}. \qquad (8.2\text{-}2)$$

Thus our variational method gives a factor $\sqrt{\frac{3}{8}} \approx 0.612$ compared with the true factor

$$\frac{1}{16\pi^3} \iiint_{-\pi}^{\pi} dx\,dy\,dz \sqrt{\sin^2 x + \sin^2 y + \sin^2 z} \approx 0.593 \pm 0.004.$$

The above integration was performed numerically. Φ_o is seen to be a very good trial wave function as far as the most singular contribution to E_o is concerned, i.e. the term proportional to V/d^4. The natural trial wave function for a one-particle state would be

$$\Phi_{\underline{k}} = \sqrt{\frac{2}{M\sigma^2}} \sum_{\underline{m}} e^{i\underline{k}\cdot\underline{m}d} \xi_{\underline{m}} \Phi_o,$$

where $\sqrt{\frac{2}{M\sigma^2}}$ is the normalization constant and σ^2 is the variational parameter. Some properties of $\Phi_{\underline{k}}$ are

$$(\Phi_{\underline{k}}, \Phi_o) = 0 \; ; \; (\Phi_{\underline{k}}, \Phi_{\underline{k}'}) = \delta_{\underline{k},\underline{k}'} \; ;$$

$$(\Phi_o, \xi_{\underline{n}} \Phi_{\underline{k}}) = \frac{\sigma}{\sqrt{2M}} e^{i\underline{k}\cdot\underline{x}_n} \; ; \; \underline{P}\Phi_{\underline{k}} = \underline{p}\Phi_{\underline{k}} \; .$$

$$(\Phi_{\underline{k}}, H\Phi_{\underline{k}}) = \frac{d^3 QM}{2}\left(\frac{3\sigma^2}{4d^2} + \tfrac{1}{2}\frac{d^2}{\rho^2\sigma^2}\right) + \tfrac{1}{2}d^3 Q\left(\frac{d^2}{\rho^2\sigma^2} + \frac{3}{2}(1-Z_{\underline{k}})\frac{\sigma^2}{d^2}\right),$$

where $Z_{\underline{k}} = \frac{1}{3}(\cos 2dk_x + \cos 2dk_y + \cos 2dk_z)$ and

$\underline{k} = \frac{2\pi}{L}(n_1, n_2, n_3), \; -N \leqslant n_i \leqslant N.$

Observe that

$$(\Phi_{\underline{k}}, H\Phi_{\underline{k}}) = (\Phi_o, H\Phi_o) + \tfrac{1}{2}d^3 Q\left(\frac{d^2}{\rho^2\sigma^2} + \frac{3}{2}(1-Z_{\underline{k}})\frac{\sigma^2}{d^2}\right) \text{ and}$$

that

$$(\Phi_o, H\Phi_o) >> \tfrac{1}{2}d^3 Q\left(\frac{d^2}{\rho^2\sigma^2} + \frac{3}{2}(1-Z_{\underline{k}})\frac{\sigma^2}{d^2}\right) \text{ since } M \to \infty.$$

Furthermore one finds that

$$[(\Phi_{\underline{k}}, H\Phi_{\underline{k}}) - (\Phi_o, H\Phi_o)]_{min} = [\tfrac{1}{2}d^3 Q\left(\frac{d^2}{\rho^2\sigma^2} + \frac{3}{2}(1-Z_{\underline{k}})\frac{\sigma^2}{d^2}\right)]_{min}$$

$$= d^3 Q \frac{1}{\rho}\sqrt{\frac{3}{2}(1-Z_{\underline{k}})}$$

$$\rightarrow \hbar ck \quad \text{as} \quad d \rightarrow o.$$

Hence an exact eigenvalue spacing is obtained. The conventional variational procedure would be to minimize $(\Phi_{\underline{k}}, H\Phi_{\underline{k}})$ giving the eigenvalue spacing as

$$(\Phi_{\underline{k}}, H\Phi_{\underline{k}})_{min} - (\Phi_o, H\Phi_o)_{min} \rightarrow \sqrt{\frac{3}{8}}\,\frac{\hbar c}{d} ,$$

which has quite the wrong form, and indeed gives an infinite value in the small d limit instead of the correct spacing $\hbar ck$. The reason is that for a fixed V, $(\Phi_{\underline{k}}, H\Phi_{\underline{k}})_{min}$ contains terms in $\frac{1}{d^4}$, $\frac{1}{d}$, ... while $(\Phi_o, H\Phi_o)_{min}$ contains $\frac{1}{d^4}$ only. On subtraction the $\frac{1}{d^4}$ terms cancel but we are left with $\frac{1}{d}$ which diverges as $d \rightarrow o$. Since we know that the correct one-particle energy relative to the vacuum is finite, viz., $\hbar ck$, we must conclude that more sophisticated trial wave functions are needed to effect cancellation of the singular terms. Such functions may readily be constructed for the massless real Klein-Gordon field, but unfortunately these functions prove impracticable in the nonlinear case. We shall therefore retain the same simple forms for Φ_o and $\Phi_{\underline{k}}$ when we turn to the nonlinear field and shall adopt the following rather questionable procedure. Instead of estimating the eigenvalue spacing from $(\Phi_{\underline{k}}, H\Phi_{\underline{k}})_{min} - (\Phi_o, H\Phi_o)_{min}$

we shall take $[(\Phi_{\underline{k}}, H\Phi_{\underline{k}}) - (\Phi_o, H\Phi_o)]_{min}$. Since we are trying to estimate the finite differences between two indefinitely large energies it is not unreasonable to use the same parameter σ in both $\Phi_{\underline{k}}$ and Φ_o. On minimizing this difference with respect to σ we find complete cancellation of all the singular terms, both for the linear massless Klein-Gordon field above and also for the nonlinear case. As we have already shown that this rather dubious procedure yields the correct answer $\hbar ck$ for the energy of a massless Klein-Gordon particle of momentum $\hbar\underline{k}$, we hope it will also give reasonable results when applied to our nonlinear field in the following sections.

8.3 The Hamiltonian in the Finite Difference Approximation

The Hamiltonian of our Model Field II is given in the previous section as

$$H = \sum_{\underline{n}} H_{\underline{n}} \; ; \; H_{\underline{n}} = Qd^3 \sqrt{[1+(\Delta\xi_{\underline{n}})^2][1+\left(\frac{cp_{\underline{n}}}{Qd^3}\right)^2]},$$

where $p_{\underline{n}}{}^2 = -\hbar^2 \frac{\partial^2}{\partial \xi_{\underline{n}}{}^2}$. The exact meaning of the square root needs to be ascertained. We shall define it through the Fourier transform of the wave functions. Observe the following expressions:

$$\frac{\partial^2}{\partial x^2} e^{ikx} = -k^2 e^{ikx},$$

$$f\left(\frac{\partial^2}{\partial x^2}\right) e^{ikx} = f(-k^2)\, e^{ikx},$$ where f is a polynomial in $\frac{\partial^2}{\partial x^2}$.

Hence we may define

$$\sqrt{1 - \gamma \frac{\partial^2}{\partial x^2}}\; e^{ikx} \equiv \sqrt{1 + \gamma k^2}\; e^{ikx},$$ γ being a numerical constant.

Since any wave function $\psi(x,t)$ may be written as

$$\psi(x,t) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} dk\; \hat{\psi}(k,t)\, e^{ikx},$$

we can define

$$\sqrt{1 - \gamma \frac{\partial^2}{\partial x^2}}\ \psi(x,t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \sqrt{1+\gamma k^2}\ \hat{\psi}(k,t)\ e^{ikx}.$$

An unambiguous meaning for H is therefore established for the subset of wave functions for which all relevant integrals of the above type exist. We are in a position to attempt a variational treatment for this nonlinear H.

8.4 The Vacuum State

In view of the similarity of our Model Field II and the massless Klein-Gordon field as seen in the weak field treatment of Chapter 6, we again take

$$\Phi_o = \prod_{\underline{n}} \Phi_{o\underline{n}}, \text{ where } \Phi_{o\underline{n}} = \frac{e^{-\xi_{\underline{n}}^2/2\sigma^2}}{(\pi\sigma^2)^{\frac{1}{4}}} ,$$

as the trial wave function for the vacuum state. Since

$$\Phi_{o\underline{n}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\eta_{\underline{n}} \left(\sigma e^{-\frac{1}{2}\sigma^2 \eta_{\underline{n}}^2}\right) \frac{e^{i\eta_{\underline{n}}\xi_{\underline{n}}}}{(\pi\sigma^2)^{\frac{1}{4}}} ,$$

we have

$$H_{\underline{n}} \Phi_{o\underline{n}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\eta_{\underline{n}} (\sigma e^{-\frac{1}{2}\sigma^2 \eta_{\underline{n}}^2}) \; Qd^3 \; \sqrt{1+(\Delta\xi_{\underline{n}})^2} \sqrt{1+\left(\frac{c\hbar\eta_{\underline{n}}}{Qd^3}\right)^2} \; \frac{e^{i\eta_{\underline{n}}\xi_{\underline{n}}}}{(\pi\sigma^2)^{\frac{1}{4}}} .$$

The vacuum expectation value may be calculated exactly in terms of modified Bessel functions of the second kind $K_\nu(z)$.[H3] The result is

$$(\Phi_o, H\Phi_o) = \frac{Q^2 d^7 M}{\sqrt{2}\pi c\hbar} \; e^{\lambda+\beta} \; K_1(\beta)[K_o(\lambda)+K_1(\lambda)], \tag{8.4-1}$$

where $\beta = \dfrac{d^2}{\sigma^2}$; $\lambda = \dfrac{\sigma^2 Q^2 d^6}{2c^2\hbar^2} = \dfrac{\rho}{2\beta}$; $\rho = \dfrac{Qd^4}{c\hbar}$.

The minimization of (8.4-1) gives (Appendix 8.1)

$$(\Phi_o, H\Phi_o)_{min} \approx \frac{\sqrt{2}M}{\pi} \frac{\hbar c}{d} = \frac{\sqrt{2}\hbar cV}{\pi d^4} \tag{8.4-2}$$

This is a significant result. Comparing this with (8.2-1), (8.2-2) we see that the present zero point energy and the vacuum energy of the Klein-Gordon field diverge in exactly the same manner. In contrast the corresponding expression for the perturbation Hamiltonian of Chapter 6 has a much higher order of divergence. One might then expect a similar behaviour of the one-particle states.

8.5 One-Particle States

In general we talk about a one-particle state in a nonlinear theory only in the context of a perturbation approach. The situation is however different in the present case. Our nonlinear field equation admits individual plane wave solutions. Moreover the Hamiltonian and the linear momentum take on a linear form if we confine ourselves to a plane wave solution. Hence we may be able to formulate one-particle states in an exact manner. Indeed this can be done as will be seen in section 10.2, Chapter 10. Therefore in our variational treatment it is reasonable to use the trial wave function

$$\Phi_{\underline{k}} = \sqrt{\frac{2}{M\sigma^2}} \sum_{\underline{m}} e^{i\underline{k}.\underline{m}d} \xi_{\underline{m}} \Phi_o,$$

hoping that it would at least give a qualitatively correct result. After some calculation, whose details are available in Appendix 8.2, the energy expectation value is found to be

$$(\Phi_{\underline{k}}, H\Phi_{\underline{k}}) = (\Phi_o, H\Phi_o) + \frac{Q^2 d^7}{\sqrt{2}\pi c\hbar} e^{\lambda+\beta} (A+B),$$

where
$$A = K_1(\beta)[2K_1(\lambda) - (K_o(\lambda)+K_1(\lambda))Z_k] > 0,$$

$$B = 2\beta(1-Z_k)[K_o(\beta) - K_1(\beta)][K_o(\lambda)+K_1(\lambda)] < 0,$$

$$Z_k = \frac{1}{3}[\cos 2k_x d + \cos 2k_y d + \cos 2k_z d].$$

Let

$$\langle H\rangle^{\underline{k}} = (\Phi_{\underline{k}}, H\Phi_{\underline{k}}) - (\Phi_o, H\Phi_o)$$

For reasons specified in section 8.2 we shall take $\langle H\rangle^{\underline{k}}_{min}$ as our estimate for the energy of the one-particle state concerned. Appendix 8.2 gives

$$\langle H\rangle^{\underline{k}}_{min} \approx \sqrt{\frac{8}{3\pi}}\ \hbar ck.$$

Observe that $\sqrt{\frac{8}{3\pi}} \approx 0.9$ which, under the circumstances, may be regarded as a good approximation to unity. This means that $\langle H\rangle^{\underline{k}}_{min}$ is approximately the same as the corresponding value of the one-particle state in the massless Klein-Gordon field case. This result appears very reasonable. Since the theory ought to be Lorentz invariant one would expect that the energy of something which resembles a free and massless particle should be $\hbar ck$ in order to give a Lorentz energy-momentum 4-vector with momentum $\hbar\underline{k}$.

In conclusion we note that if we attempt to estimate the one-particle energy by

$$(\Phi_{\underline{k}}, H\Phi_{\underline{k}})_{min} - (\Phi_o, H\Phi_o)_{min},$$

we again are faced with a divergent result. The same applies to the expression

$$[(\Phi_{\underline{k}}, H\Phi_{\underline{k}}) - (\Phi_o, H\Phi_o)]_{\sigma_o}$$

where both expectation values are evaluated at the parameter value $\sigma = \sigma_o$ which optimizes $(\Phi_o, H\Phi_o)$.

8.6 Two-Particle States

Considerable difficulties begin to emerge as we try to construct states resembling those two-particle states of linear theories. The inevitably approximate or even precarious nature of the idea of two-particle states in a nonlinear theory which allows strong interaction is obvious. In our present case however we expect, from the knowledge of plane wave solutions, that well-defined two-particle states may be formulated at least for two particles moving in the same direction. There should be no interaction between these particles. One formulation which leads to these results is given in section 10.2, Chapter 10. Interference effects begin to appear when we try to bring together particles travelling at an angle with one another. Our task is to estimate the interaction energy with the present variational method.

8.6.1 Two Particles Moving in Opposite Directions

When two particles are travelling in the same direction, there is no interaction (see section 10.2, Chapter 10). Hence we only have to consider two particles moving at angle with each other, i.e. $\underline{k}_1 \neq$ positive constant $* \underline{k}_2$ where $\underline{k}_1$, $\underline{k}_2$ are the two $\underline{k}$-vectors specifying the states of the two particles concerned. With such $\underline{k}_1$, $\underline{k}_2$ we can always effect a Lorentz transformation to the centre-of-momentum frame of reference where the two vectors will be seen to be equal in magnitude but exactly opposite in direction. Therefore it is sufficient to investigate two-particle states with $\underset{=}{k}_1+\underset{=}{k}_2 = 0$ without loss of generality. A trial wave function which readily comes into one's mind is

$$\Phi_{\underset{=}{k},-\underset{=}{k}} = \frac{2}{\sqrt{M(M-2)\sigma^2}} \sum_{\underset{=}{n}_1 \neq \underset{=}{n}_2} e^{i(\underset{=}{n}_1-\underset{=}{n}_2).\underset{=}{k}d} \, \xi_{\underset{=}{n}_1} \xi'_{\underset{=}{n}_2} \Phi_o .$$

This function is orthogonal to Φ_o, $\Phi_{\underset{=}{k}}$ and is a null eigenvector of the linear momentum operator and it also gives the correct two-particle energy for the massless Klein-Gordon field. However this trial wave function is not satisfactory because it does not lead to any interaction between the two particles (see Appendix 8.3 for details). We have to attempt some other trial wave functions. Since the state involved is of momentum zero it is not unreasonable to combine the two null momentum eigenvectors Φ_o and $\Phi_{\underset{=}{k},-\underset{=}{k}}$ to form a new trial wave function

$$\phi = a\Phi_o + b\Phi_{\underset{=}{k},-\underset{=}{k}} , \qquad (8.6.1\text{-}1)$$

where a, b are constants to be regarded as two independent variational parameters in addition to the original σ in Φ_o and $\Phi_{\underset{=}{k},-\underset{=}{k}}$. The optimization of

$$E = \frac{(\phi, H\phi)}{(\phi, \phi)} ,$$

with respect to a, b leads to

$$\begin{pmatrix} (\Phi_o, H\Phi_o) & (\Phi_o, H\Phi_{\underset{=}{k},-\underset{=}{k}}) \\ (\Phi_o, H\Phi_{\underset{=}{k},-\underset{=}{k}})^* & (\Phi_{\underset{=}{k},-\underset{=}{k}} H\Phi_{\underset{=}{k},-\underset{=}{k}}) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = E \begin{pmatrix} a \\ b \end{pmatrix} ,$$

where use has been made of the orthonormality property of Φ_o and $\Phi_{\underline{k},-\underline{k}}$.

$$\Rightarrow E^{\pm} = \frac{[(\Phi_o,H\Phi_o)+(\Phi_{\underline{k},-\underline{k}},H\Phi_{\underline{k},-\underline{k}})]\pm\sqrt{[(\Phi_{\underline{k},-\underline{k}},H\Phi_{\underline{k},-\underline{k}})-(\Phi_o,H\Phi_o)]^2+4|(\Phi_o,H\Phi_{\underline{k},-\underline{k}})|^2}}{2};$$

$$E^+ - E^- = \sqrt{[(\Phi_{\underline{k},-\underline{k}},H\Phi_{\underline{k},-\underline{k}})-(\Phi_o,H\Phi_o)]^2 + 4|(\Phi_o,H\Phi_{\underline{k},-\underline{k}})|^2}.$$

The corresponding trial wave functions ϕ^+, ϕ^- are orthogonal to each other. We then regard E^- as an estimate for the vacuum energy and E^+ as an estimate of the energy of the two-particle state with linear momentum zero. Then

$$\Delta E = E^+ - E^- > (\Phi_{\underline{k},-\underline{k}},H\Phi_{\underline{k},-\underline{k}})-(\Phi_o,H\Phi_o),$$

will serve as an estimate of the two-particle energy relative to the vacuum which may well bring in nonvanishing interaction energy. There is still an unspecified parameter σ to be determined. The obvious choice which is in harmony with the procedure adopted in section 8.2 is to employ the σ which optimizes ΔE. Appendix 8.4 gives

$$(\Delta E)_{min} \simeq \frac{\sqrt{2}\hbar c}{\pi d},$$

which diverges as $d \to \infty$. Some other choices of σ are tried without avail in Appendix 8.4. However there is reason to believe that again it is the trial wave function which is at fault. To see this we can apply the trial wave function (8.6.1-1) to the massless Klein-Gordon field. We find that the corresponding expression is (see Appendix 8.4 for details)

$$\Delta E_{KG} = E^{+}_{KG} - E^{-}_{KG} > (\Phi_{\underline{k},-\underline{k}}, H_{KG}\Phi_{\underline{k},-\underline{k}}) - (\Phi_o, H\Phi_o),$$

leading to a spurious interaction energy which also diverges like $\frac{1}{d}$. Under these circumstances we cannot reach any definite conclusion about the interaction between two particles.

8.7 Some Remarks

The variational approach we studied so far has been able to lead to some positive results for one-particle states. The exact reason for its failure in the two-particle case is an open question. Some variants of the two-particle trial wave functions and of the variational procedures have been studied without much success. It is quite possible that we have just not hit upon a sufficiently good trial wave function. The appearance of a divergent spurious interaction energy of order $\frac{1}{d}$ between two massless Klein-Gordon particles lends support to this view — the trial wave functions are just not good enough to give complete cancellation of all the divergent terms. In the linear case it is certainly true that a more sophisticated trial wave function will effect such cancellation and yield the correct result. However the situation may not be so simple in the nonlinear case. It may well be that there are no such things as two-particle states and we are quite wrong in attempting to simulate such a state by our choice of trial wave function. Further work is needed to resolve this problem. Although the variational method in its present form has only limited success for our Model Field II, there is no reason why the method cannot be further developed into a general theory applicable for other field theories, especially those of more conventional types. Obviously there is much scope for further development.

8.8 Appendix 8.1 The Vacuum Expectation Value

(1) $$R_o^{(\underline{n})} = (\Phi_{o\underline{n}}, \sqrt{1+\left(\frac{cP_{\underline{n}}}{Qd^3}\right)^2}\, \Phi_{o\underline{n}}) = \frac{\sigma Q d^3}{2\sqrt{\pi}c\hbar} e^{\lambda}[K_o(\lambda)+K_1(\lambda)].$$

$$\Phi_{o\underline{n}} = \frac{1}{(\pi\sigma^2)^{\frac{1}{4}}} e^{-\xi_{\underline{n}}^2/2\sigma^2}.$$

(2) $$\Delta_o^{(\underline{n})} = (\Phi_o, \sqrt{1+(\Delta\xi_{\underline{n}})^2}\, \Phi_o) = (\prod_{\underline{m}\neq\underline{n}} \Phi_{o\underline{m}}, \sqrt{1+(\Delta\xi_{\underline{n}})^2} \prod_{\underline{m}\neq\underline{n}} \Phi_{o\underline{m}})$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{d}{\sigma}\, e^{\beta} K_1(\beta).$$

(3) $$(\Phi_o, H_n \Phi_o) = Qd^3\, \Delta_o^{(\underline{n})}\, R_o^{(\underline{n})}$$

$$= \frac{Q^2 d^7}{\sqrt{2}\pi c\hbar} e^{\lambda+\beta} K_1(\beta)\, [K_o(\lambda)+K_1(\lambda)].$$

(4) Minimization of $(\Phi_o, H_n \Phi_o)$ in the limit of vanishing d.

There are only 5 possibilities for the behaviour of the optimum λ and β as $d \to o$:

$\lambda \to \infty$, $\beta \to o$; $\lambda \to o$, $\beta \to o$; $\lambda \to o$, $\beta \to \infty$;

$\lambda \to$ finite and non-zero, $\beta \to o$; $\lambda \to o$, $\beta \to$ finite and non-zero.

Using the known asymptotic behaviour of the relevant Bessel functions[H3] one can verify that a consistent minimization is possible only for the case $\lambda_{opt} \to o$, $\beta_{opt} \to o$. From numerical computation we also know that $(\Phi_o, H_{\underline{n}} \Phi_o)$ possesses a minimum for small d. Hence we conclude that as $d \to o$, the minimum occurs with $\lambda_{opt} \to o$, $\beta_{opt} \to o$ and we have

$$(\Phi_o, H\, \Phi_o)_{min} \approx \frac{Q^2 d^7 M}{\sqrt{2}\pi\hbar c} \left(\frac{2}{\rho^2} + \frac{2\sqrt{2}\,\ell n(1/\rho)}{\rho} + \ldots\right).$$

8.9 Appendix 8.2 One-Particle States

(1) $$R_1^{(\underline{n})} = (\xi_{\underline{n}}\,\Phi_{o\underline{n}},\ \sqrt{1+\left(\frac{cP_{\underline{n}}}{Qd^3}\right)^2}\ \xi_{\underline{n}}\,\Phi_{o\underline{n}})$$

$$= \frac{\sigma^3}{2\sqrt{\pi}}\left(\frac{Qd^3}{c\hbar}\right) e^{\lambda} K_1(\lambda).$$

(2) $$\Delta_{\underline{m}\underline{m}'}^{(\underline{n})} = (\xi_{\underline{m}}\,\Phi_o,\ \sqrt{1+(\Delta\xi_{\underline{n}})^2}\ \xi_{\underline{m}'}\Phi_o)$$

$$= \begin{cases} \frac{d\sigma}{6\sqrt{2\pi}}\ e^{\beta}\ [2\beta(K_o(\beta)-K_1(\beta))+7K_1(\beta)] & \text{if } \underline{m}=\underline{m}' \in S_{\pm}^{(\underline{n})} \\ -\frac{d\sigma}{6\sqrt{2\pi}}\ e^{\beta}\ [2\beta(K_o(\beta)-K_1(\beta))+K_1(\beta)] & \text{if } \underline{m},\underline{m}' \in P_{\pm}^{(\underline{n})} \\ \frac{d\sigma}{\sqrt{2\pi}}\ e^{\beta}\ K_1(\beta) & \text{if } \underline{m}=\underline{m}' \notin S_{\pm}^{(\underline{n})} \\ 0 & \text{otherwise,} \end{cases}$$

where $S_{\pm}^{(\underline{n})}$ is the following set

$$(n_1\pm1,\ n_2,\ n_3);\ (n_1,\ n_2\pm1,\ n_3);\ (n_1,\ n_2,\ n_3\pm1),$$

and $P_{\pm}^{(\underline{n})}$ is the following set of pairs

$$(n_1+1,n_2,n_3),\ (n_1-1,n_2,n_3);\ (n_1-1,n_2,n_3),\ (n_1+1,n_2,n_3);\ (n_1,n_2+1,n_3),\ (n_1,n_2-1,n_3);$$

$$(n_1,n_2+1,n_3),\ (n_1,n_2-1,n_3);\ (n_1,n_2,n_3+1),\ (n_1,n_2,n_3-1);\ (n_1,n_2,n_3-1),\ (n_1,n_2,n_3+1).$$

Let $\Delta_1^{(n)} = \Delta_{m,m}^{(n)}$ with $m \in S_{\pm}^{(n)}$,

$\Delta_{1\pm}^{(n)} = \Delta_{m,m'}^{(n)}$ with $m,m' \in P_{\pm}^{(n)}$.

Then

$$\Delta_1^{(n)} + \Delta_{1\pm}^{(n)} = \frac{d\sigma}{\sqrt{2\pi}} e^{\beta} K_1(\beta) = \frac{\sigma^2}{2} \Delta_o$$

(3) $$(\Phi_k, H_n \Phi_k) = \frac{2}{M\sigma^2} \sum_{m,\neq m'} e^{i(m'-m).kd} (\xi_m \Phi_o, H_n \xi_{m'} \Phi_o)$$

$$= \frac{2Qd^3}{M\sigma^2} [6R_o\Delta_1 + 6R_o\Delta_{1\pm}Z_k + R_1\Delta_o - \frac{7}{2}\sigma^2\Delta_o R_o]$$

$$= (\Phi_o, H_n \Phi_o) + \frac{Q^3 d^7}{\sqrt{2\pi} c\hbar M} e^{\lambda+\beta} (A+B),$$

where

$$A = K_1(\beta)[2K_1(\lambda) - (K_o(\lambda) + K_1(\lambda)) Z_k] > 0,$$

$$B = 2\beta(1-Z_k)[K_o(\beta) - K_1(\beta)][K_o(\lambda) + K_1(\lambda)] < 0.$$

(4) The minimization of

$$\langle H\rangle^k = (\Phi_k, H\Phi_k) - (\Phi_o, H\Phi_o) ,$$

in the limit of vanishing d may be carried out in the same way as for the case of the vacuum expectation value. A consistent minimization is possible only if $\lambda_{opt} \to \infty$, $\beta_{opt} \to o$. Using the known asymptotic properties of the relevant Bessel functions we obtain

$$\langle H\rangle^k \approx \frac{Qd^4}{\sqrt{2\pi}} (\frac{1}{\rho^2\sigma} + \frac{4}{3} k^2\sigma) \qquad \text{(A8.3-1)}$$

$$\Longrightarrow \qquad \langle H\rangle^k_{min} \approx \sqrt{\frac{8}{3\pi}} \hbar ck.$$

8.10 Appendix 8.3 Two-Particle States I

$(\Phi_{\underline{k},-\underline{k}}, H_{\underline{n}}\ \Phi_{\underline{k},-\underline{k}})$

$$= \frac{2}{M(M-1)\sigma^4} \sum_{\substack{\underline{n}_1 \neq \underline{n}_2 \\ \underline{n}_1' \neq \underline{n}_2'}} e^{i[(\underline{n}_1'-\underline{n}_2')-(\underline{n}_1-\underline{n}_2)].\underline{k}d} (\xi_{\underline{n}_1}\xi_{\underline{n}_2}\Phi_o, H_{\underline{n}}\xi_{\underline{n}_1'}\xi_{\underline{n}_2'}\Phi_o).$$

After lengthy calculation we get

$$(\Phi_{\underline{k},-\underline{k}}, H\Phi_{\underline{k},-\underline{k}}) = (\Phi_{\underline{k}}, H\Phi_{\underline{k}}) + (\Phi_{-\underline{k}}, H\Phi_{-\underline{k}}) - (\Phi_o, H\Phi_o) + \frac{Qd^3}{(M-1)\sigma^4} f$$

$$= 2(\Phi_{\underline{k}}, H\Phi_{\underline{k}}) - (\Phi_o, H\Phi_o) + \frac{Qd^3}{(M-1)\sigma^4} f,$$

where f stands for the rest of the terms and f does not contain M. As we enlarge the box V within which the field is confined, M will tend to infinity irrespective of whether or not d approaches zero. It is therefore apparent that

$$\frac{Qd^3}{(M-1)\sigma^4} f$$

may be igonored. We can pursue this a bit further. Let

$$\langle H\rangle^{\underline{k},-\underline{k}} = (\Phi_{\underline{k},-\underline{k}}, H\Phi_{\underline{k},-\underline{k}}) - (\Phi_o, H\Phi_o)$$

$$= 2[(\Phi_{\underline{k}}, H\Phi_{\underline{k}}) - (\Phi_o, H\Phi_o)] + \frac{Qd^3}{(M-1)\sigma^4} f.$$

We know that at $\langle H\rangle^{\underline{k}}_{min}$, $\beta_{opt} \to o$ and $\lambda_{opt} \to \infty$. We can now show explicitly that $\frac{Qd^3}{(M-1)\sigma^4}$ f may indeed be neglected.

f consists of terms of the following form:

$\sigma^4 \Delta_o R_o$; $\sigma^2 R_o \Delta_{1\pm}$; $\sigma^2 R_1 \Delta_o$; $\sigma^2 R_{o2} \Delta_o$; $\sigma^2 R_o \Delta_1$; $R_o \Delta_{2\pm}$;

$R_o \Delta_{3\pm}$; $R_o \Delta_2$; $\Delta_o R_2$; $\Delta_1 R_1$; $R_1 \Delta_{1\pm}$; $R_{o2} \Delta_{1\pm}$; $R_{o2} \Delta_1$;

$R_o \Delta_{\underline{mmmm}''}$ (where $\underline{m} \in S_{\pm}^{(\underline{n})}$; $\underline{m}' \in S_{\pm}^{(\underline{n})}$); $R_o \Delta_{\underline{m}_1 \underline{m}_2 \underline{m}_3 \underline{m}_4}$ (where $\underline{m}_1 \neq \underline{m}_2 \neq \underline{m}_3 \neq \underline{m}_4$),

where

(1) $R_2 = \left(\xi_{\underline{n}}^2 \Phi_o, \sqrt{1+\left(\frac{cP_{\underline{n}}}{Qd^3}\right)^2} \;\; \xi_{\underline{n}}^2 \Phi_o\right)$, $R_{o2} = \left(\Phi_o, \sqrt{1+\left(\frac{cP_{\underline{n}}}{Qd^3}\right)^2} \;\; \xi_{\underline{n}}^2 \Phi_o\right)$;

(2) $\Delta_{\underline{m}_1 \underline{m}_2 \underline{m}_3 \underline{m}_4} = \left(\xi_{\underline{m}_1} \xi_{\underline{m}_2} \Phi_o, \sqrt{1+(\Delta\xi_{\underline{m}})^2} \;\; \xi_{\underline{m}_3} \xi_{\underline{m}_4} \Phi_o\right)$:

$\Delta_2 \quad = \Delta_{\underline{mmmm}}$ with $\underline{m} \in S_{\pm}^{(\underline{n})}$;

$\Delta_{2\pm} = \Delta_{\underline{mmmm}'''}$ with $\underline{m} \in S_{\pm}^{(\underline{n})}$, $(\underline{m}', \underline{m}'') \in P_{\pm}^{(\underline{n})}$, $\underline{m} \neq \underline{m}' \neq \underline{m}''$;

$\Delta_{3\pm} = \Delta_{\underline{mmmm}'}$ with $(\underline{m}, \underline{m}') \in P_{\pm}^{(\underline{n})}$.

Now all these terms may be expressed explicitly in terms of the modified Bessel functions of the second kind. Then using the asymptotic expansions of these relevant Bessel functions for

$$\beta \to o \; ; \; \lambda \to \infty$$

we find that all those terms in f approach

$$\frac{\sigma^5}{d} .$$

$$\Longrightarrow \qquad \frac{Qd^3}{(M-1)\sigma^4} \; \frac{\sigma^5}{d} \approx \frac{Qd^3}{Md}\,\sigma = \frac{Q}{V} \; d^5 \sigma \qquad \text{(A8.4-1)}$$

where $V = Md^3$. Comparing this with equation (A8.3-1) we see that indeed (A8.4-1) is smaller than $\langle H \rangle^{\underline{k}}$ by a vanishing factor $\frac{d}{V}$.

8.11 Appendix 8.4 Two-Particle States II

The trial wave function

$$a\ \Phi_o + b\ \Phi_{\underline{k},-\underline{k}},$$

$$\Rightarrow E^{\pm} = \tfrac{1}{2}\{[(\Phi_{\underline{k},-\underline{k}},H\Phi_{\underline{k},-\underline{k}})+(\Phi_o,H\Phi_o)] \pm$$

$$\sqrt{[(\Phi_{\underline{k},-\underline{k}},H\Phi_{\underline{k},-\underline{k}})-(\Phi_o,H\Phi_o)]^2 + 4|(\Phi_o,H\Phi_{\underline{k},-\underline{k}})|^2}\ \};$$

$$\Delta E = E^+ - E^- = \sqrt{[(\Phi_{\underline{k},-\underline{k}},H\Phi_{\underline{k},-\underline{k}})-(\Phi_o,H\Phi_o)]^2 + 4|(\Phi_o,H\Phi_{\underline{k},-\underline{k}})|^2}.$$

(1) The Nonlinear Field

$$(\Phi_{\underline{k},-\underline{k}},H\Phi_{\underline{k},-\underline{k}}) \approx 2(\Phi_{\underline{k}},H\Phi_{\underline{k}})-(\Phi_o,H\Phi_o);$$

$$(\Phi_o,H\Phi_{\underline{k},-\underline{k}}) = \frac{M}{\sqrt{M(M-1)}} \quad \frac{12Qd^3 R_o\Delta_{1\pm}Z_{\underline{k}}}{\sigma^2} \simeq \frac{12Qd^3 R_o\Delta_{1\pm}Z_{\underline{k}}}{\sigma^2}.$$

clearly

$$\Delta E \approx 2\sqrt{<H>^{\underline{k}} + |(\Phi_o,H\Phi_{\underline{k},-\underline{k}})|^2} \geqslant 2\ <H>^{\underline{k}}$$

$$\Longrightarrow \quad (\Delta E)_{min} \geqslant 2\sqrt{\frac{8}{3\pi}}\ \hbar ck.$$

To actually estimate $(\Delta E)_{min}$ let

$$f = <H>^{\underline{k}} + |(\Phi_o,H\Phi_{\underline{k},-\underline{k}})|^2;$$

$$Y_{\underline{k}} = 1-Z_{\underline{k}} \to \frac{2}{3}d^2k^2 + O(d^4) \quad \text{as } d \to o.$$

Now

$$\langle H\rangle^{\underline{k}} = \frac{2Qd^3}{\sigma^2}\,[6R_o\Delta_1 + 6R_o\Delta_{1\pm}\,Z_{\underline{k}} + R_1\Delta_o - \frac{7}{2}\sigma^2\Delta_o R_o]$$

$$= \frac{2Qd^3}{\sigma^2}\,[(R_1 - \frac{\sigma^2}{2}R_o)\Delta_o - 6Y_{\underline{k}}R_o\Delta_{1\pm}].$$

The rather tedious procedure of minimization of ΔE may be carried out as before. The only consistent minimization occurs when

$$\lambda_{opt} \to o \; ; \; \beta_{opt} \to o \quad \text{as } d \to o.$$

The result is

$$(\Delta E)_{min} \simeq \frac{\sqrt{2}\hbar c}{\pi d}, \text{ which diverges as } d \to o.$$

Two other assignments of the value of ΔE are also considered. The first alternative is to take ΔE calculated with the value of σ which minimizes E^-. Obviously this also leads to a divergent energy since $\Delta E > (\Delta E)_{min}$. The other alternative is to consider

$$\Delta E = E^+_{min} - E^-_{min}.$$

This again may readily be shown to be bigger than $(\Delta E)_{min}$, and hence divergent.

(2) The Massless Klein-Gordon Field

Let us apply the above trial wave function to the real massless Klein-Gordon field to estimate the corresponding two-particle energy. Then

$$\Delta E_{KG} = \sqrt{[(\Phi_{\underline{k},-\underline{k}}, H_{KG}\Phi_{\underline{k},-\underline{k}}) - (\Phi_o, H_{KG}\Phi_o)]^2 + 4|(\Phi_o, H_{KG}\Phi_{\underline{k},-\underline{k}})|^2}\,.$$

Calculation gives

$$(\Phi_o, H_{KG}\Phi_{\underline{k},-\underline{k}}) = -\frac{3}{4}\mathbf{Q}d\sigma^2 Z_{\underline{k}},$$

and

$$(\Delta E_{KG})_{min} = \sqrt{3}\ \frac{\hbar c}{d}\ \sqrt{(1-Z_{\underline{k}}) + [Z_{\underline{k}}^2 + (1-Z_{\underline{k}})^2]^{\frac{1}{2}}}$$

$$\simeq \frac{\sqrt{3}\hbar c}{d},$$

which gives a spurious and divergent interaction energy. The other two choices of the value of σ in ΔE only lead to values bigger than this one.

CHAPTER 9

SOME REMARKS ON THE THREE QUANTIZATION SCHEMES AND A DISCUSSION OF THE TREATMENTS OF MODEL FIELD II

9.1 Introduction

In Chapter 3, three different quantization schemes were reviewed. These schemes were later applied to the two model field theories, in particular to the Model Field II. In addition, we also studied various special treatments of the Model Field II. In this chapter, some problems related to the three quantization schemes will be examined. A detailed analysis will also be made which enables us to unify the treatments of the Model Field II given in Chapters 6 and 7. We shall also compare the variational theory and the perturbative theories.

9.2 Some General Problems Related to the Three Quantization Schemes.

The Quantization Scheme I given in Chapter 3 appears to be quite different from the schemes of Quantization II and III discussed in the same chapters. The apparent difference gives rise to some perplexing problems. Let us confine our attention to the Scheme I and Scheme II. In Quantization I the Hamiltonian which is formed by a linear combination of the constraints vanishes. In quantum theory, the constraints are taken as subsidiary conditions imposed on physical vectors. As a result, the state vectors in the Schrödinger picture are "time-independent" in the sense that

$$\frac{\partial}{\partial x^o} |\Psi\rangle = 0, \qquad (9.2\text{-}1)$$

$$\because \quad H|\Psi\rangle = 0. \qquad (9.2\text{-}2)$$

So things appear to be "dead" and changeless.[I1] However, working in a special coordinate frame in Quantization II, we surely have a non-vanishing Hamiltonian and time-dependent state vectors. All this is clearly seen in the treatments in Chapter 5 and 6 for the Model Field II. A dilemma therefore appears. The way out of this dilemma may be found with the help of an analysis similar to that given in sections 4.2 and 4.3 in Chapter 4. In the generally covariant formulation, the coordinate variables x^μ may not be identified with the actual (metre-stick and clock) coordinates of an individual observer. This means that as far as an individual observer is concerned results obtained in the covariant theory cannot be interpreted literally.

To illustrate the situation more vividly, let us consider the parameterized or parametric formulation[I2] of classical mechanics. In such theory the time t is promoted to the status of an additional canonical variable q_o while another variable τ is introduced to act as the independent variable. The original variational principle is

$$\delta \int L(q_1, \ldots q_n, t, \dot{q}_1, \ldots \dot{q}_n) dt = 0, \quad \text{where } \dot{q}_i = \frac{dq_i}{dt} .$$

In the parametric formulation, the variational principle becomes

$$\delta \int L_1 d\tau = 0$$

where

$$L_1 = L(q_1, \ldots q_n, q_o, \frac{q_1'}{q_o'}, \ldots \frac{q_n'}{q_o'}) q_o' ,$$

$$q_\mu' = \frac{dq_\mu}{d\tau} \text{ and } \mu = 0,1,2,..n.$$

One can define the canonical momentum conjugate to q_o and establish the Hamiltonian formulation in the usual way. L_1 being homogeneous in the velocity variables q_μ' of degree one implies a vanishing Hamiltonian for the system. There will certainly be one primary constraint due to the introduction of an additional variable.[I3] Let it be

$$\phi(q_1, \ldots q_o, p_1, \ldots p_o) \approx 0. \tag{9.2-3}$$

Let us further suppose that there are no more constraints nor consistency equations. Then the total Hamiltonian is

$$H_T = u\phi \approx 0$$

where u is an arbitrary coefficient. With this Hamiltonian, equations of motion of various canonical variables may be obtained. This parametric formulation is considered to be of the most advanced form.[14] Among other things, this formulation is highly valued for its consistency with the spirit of special relativity in that space coordinates and the time are treated on an equal footing. Instead of the variable t which is the actual time coordinate of an individual observer (measured by his clock), an unspecified variable τ is introduced to act as the independent variable of the theory. The τ, being unspecified, cannot be identified with the actual time coordinate of an observer. Now one may go one step further to quantize the parametric theory using the method of Quantization I. The constraint (9.2-3) will become the subsidiary condition imposed on physical vectors. As a result, physical vectors in the Schrödinger picture are "time-independent" because

$$\frac{\partial}{\partial\tau}\,|\Psi\rangle = 0 \qquad \because \quad H_T|\Psi\rangle = 0. \tag{9.2-4}$$

To illustrate the situation consider a concrete one-dimensional example of a particle in an external potential V. The Lagrangian is

$$L = \frac{m}{2}\,\dot{q}_1^{\,2} - V(q_1).$$

In the parametric formulation we introduce 2 new variables $q_o = t$ and τ.

$$L_1 = [\frac{m}{2}\left(\frac{q_1'}{q_0'}\right)^2 - V(q_1)]q_0' \ ;$$

$$p_1 = \frac{\partial L_1}{\partial q_1'} = m\left(\frac{q_1'}{q_0'}\right) \ ; \quad p_o = \frac{\partial L_1}{\partial q_o'} = -\frac{m}{2}\left(\frac{q_1'}{q_o'}\right)^2 - V(q_1).$$

$\Longrightarrow$ the primary constraint in the classical theory

$$\phi = p_o + \frac{p_1^2}{2m} + V(q_1) = 0.$$

The Hamiltonian is proportional to ϕ and therefore vanishes. The equations of motion are obtained from the total Hamiltonian

$$H_T = u\phi,$$

where u is an arbitrary function of q_o, q_1.

To quantize we have

$$p_o \to \hat{p}_o = -i\hbar\frac{\partial}{\partial q_o} \ ; \quad p_1 \to \hat{p}_1 = -i\hbar\frac{\partial}{\partial q_1}.$$

The constraint equation now takes the form of

$$(\hat{p}_o + \frac{\hat{p}_1^2}{2m} + V(q_1))\ |\Psi\rangle = 0,$$

or $$i\hbar\frac{\partial}{\partial q_o}\Psi(q_o,q_1,\tau) = [-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q_1^2} + V(q_1)]\Psi(q_o,q_1,\tau).$$

The Schrödinger equation of motion is

$$i\hbar\frac{\partial}{\partial\tau}|\Psi\rangle = H_T|\Psi\rangle = 0.$$

Thus the wave function $\Psi(q_o,q_1,\tau)$ is actually independent of the parameter τ while its q_o-development is given by the standard

Schrödinger equation.

We have the apparent dilemma showing up clearly now. However in the present case we know with certainty that the physical system as viewed by an individual observer is not changeless — the wave function does indeed depend on this time coordinate q_o. This example shows that expressions like (9.2-1), (9.2-4) cannot be interpreted literally and that they do not imply a changeless situation. In general x^o and τ in these expressions cannot be identified with clock readings of a physical observer, nor can H or H_T be interpreted as a physical energy. Perhaps it should be stressed that while the generally covariant formalism and the Quantization I have great theoretical advantages, we actually have to go into the theory formulated in a special frame in order to be able to see what is going on from the point of view of a particular observer.

9.3 A Unified Treatment of The Model Field II

We shall show that the treatments of the Model Field II given in Chapters 6 and 7 may be unified as a perturbation theory about an exact solution of the field equation.

9.3.1 A General Formulation of the Perturbation Theory

Let $\xi^{(o)}$ be an exact solution of the field equation (6.2.1-1). The corresponding Langrangian density and the Hamiltonian density are

$$\mathcal{L}^{(o)} = -Q\sqrt{1-\eta^{\rho\sigma}\xi^{(o)}_{,\rho}\xi^{(o)}_{,\sigma}} \;;\; \mathcal{H}^{(o)} = Q\sqrt{[1+(\nabla\xi^{(o)})^2][1+(\frac{c\pi^{(o)}}{Q})^2]} \;,$$

where $\pi^{(o)} = \dfrac{\partial\mathcal{L}^{(o)}}{\partial\dot{\xi}^{(o)}}$. Now try a perturbation solution of the form

$$\xi = \xi^{(o)} + \xi^{(1)} + \xi^{(2)} + \ldots\ldots,$$

where $\xi^{(1)}$ is regarded as being a small quantity of first order, $\xi^{(2)}$ as small of second order and so on. Let

$$\phi = \xi^{(1)} + \xi^{(2)} + \ldots\ldots$$

Then

$$\xi = \xi^{(o)} + \phi.$$

The Lagrangian density and the Hamiltonian density are

$$\mathcal{L} = -Q\sqrt{1-\eta^{\rho\sigma}\xi_{,\rho}\xi_{,\sigma}} \;;\; \tilde{\mathcal{H}} = Q\sqrt{[1+(\nabla\xi)^2][1+(\frac{c\Pi}{Q})^2]} \;,$$

where $\Pi = \dfrac{\partial\mathcal{L}}{\partial\dot{\xi}}$.

9.3.1.1 General Theory

Consider a transformation

$$\begin{array}{lcl} \xi & & \phi = (\xi - \xi^{(o)}) \\ \Pi & \rightarrow & \pi = (\Pi - \pi^{(o)}) \\ \bar{\mathcal{H}} & & \mathcal{H} = \bar{\mathcal{H}} - \dot{\xi}^{(o)}\pi + \dot{\pi}^{(o)}\phi, \end{array}$$

where $\xi^{(o)}$, $\pi^{(o)}$ are taken as known functions. This is a canonical transformation in the sense that the form of canonical equations of motion are preserved, that is,

$$\begin{array}{lcl} \dot{\xi} = \dfrac{\partial \bar{H}}{\partial \Pi} & & \dot{\phi} = \dfrac{\partial H}{\partial \pi} \\ & \Longleftrightarrow & \\ \dot{\Pi} = -\dfrac{\partial \bar{H}}{\partial \xi} & & \dot{\pi} = -\dfrac{\partial H}{\partial \phi}\,. \end{array}$$

Poisson brackets between canonical variables are unchanged by the transformation. Hence we may consider ϕ, π as a pair of canonical variables for the description of the field in place of ξ, Π. More explicitly we have now for the characterization of the field a pair of canonical variables ϕ, π with the Hamiltonian density

$$\mathcal{H} = Q\sqrt{[1+(\nabla\xi^{(o)} + \nabla\phi)^2][1+(\frac{c\pi^{o}}{Q} + \frac{c\pi}{Q})^2]} - \dot{\xi}^{(o)}\pi + \dot{\pi}^{(o)}\phi. \quad (9.3.1\text{-}1)$$

The Lagrangian density is

$$\mathcal{L} = -Q\sqrt{1-\eta^{\rho\sigma}(\xi^{(o)} + \phi)_{,\rho}(\xi^{(o)} + \phi)_{,\sigma}} - \frac{\partial}{\partial t}(\pi^{(o)}\phi) - 3\pi^{(o)}\dot{\xi}^{(o)}.$$

In the following two sections we shall see that the perturbation theories we had before may now be obtained by particular choices for $\xi^{(o)}$.

9.3.2 Weak-Field Theory

The weak field approximation discussed in Chapter 6 is based on the binomial expansion (6.3-1) of the Hamiltonian density

$$Q\sqrt{[1+(\nabla\phi)^2][1+(\frac{c\pi}{Q})^2]},\quad \nabla\phi,\ \frac{c\pi}{Q}\text{ being small.}$$

Comparing with (9.3.1-1) we can see that this weak field theory is really a perturbation theory about an exact solution to the field equation

$$\xi^{(o)} = \text{constant}.$$

The weak field theory may be extended slightly by considering a perturbation about

$$\xi^{(o)} = a_\mu x^\mu \ ;\ \pi^{(o)} = b,$$

where a_μ, b are constants. On substitution into (9.3.1.1-1) this leads to a Hamiltonian density

$$Q\sqrt{[1+\sum_j(a_j+\phi_{,j})^2][1+(b+\pi)^2]}.$$

9.3.3 Theory of Everywhere-Slowly-Varying Field

Suppose $\xi^{(o)}$ is a solution which is everywhere-slowly-varying. With $\nabla\phi$, $\frac{c\pi}{Q}$ being small we can effect an essentially binomial expansion of the Hamiltonian again. But in order to achieve greater similarity with the treatment in Chapter 7 we proceed in the following way. Let

$$\bar{\xi} = \xi^{(o)}+\phi \ ; \ \bar{\Pi} = \pi^{(o)}+\pi \ ; \ \bar{\xi}' = \bar{\xi}-\xi^{(o)} \ ; \ \bar{\Pi}' = \bar{\Pi}-\pi^{(o)} .$$

Note that we are not performing another transformation and $\bar{\xi}$, $\bar{\Pi}$ are purely short-hand symbols. Rewrite the Hamiltonian in (9.3.1.1-1) in terms of $\bar{\xi}$, $\bar{\Pi}$. Then carry out a Taylor expansion of it about $\xi^{(o)}$ in the same way as in Appendix 7.2 in Chapter 7 we obtain

$$H = \int d^3x\, \mathcal{H}^{(o)} + \tfrac{1}{2}\int d^3x \Big[\left(\frac{\partial^2 \mathcal{H}}{\partial \bar{\Pi}^2}\right)^{(o)} \bar{\Pi}'^2 + \left(\frac{\partial^2 \mathcal{H}}{\partial \bar{\xi}_{,i}\partial \bar{\xi}_{,j}}\right)^{(o)} \bar{\xi}'_{,i}\bar{\xi}'_{,j} + \left(\frac{\partial^2 \mathcal{H}}{\partial \bar{\Pi}\partial \bar{\xi}_{,j}}\right)^{(o)} \times \bar{\Pi}'\bar{\xi}'_{,j}\Big] + \ldots\ldots, \qquad (9.3.3\text{-}1)$$

where the superscript (o) refers to values at $\bar{\xi} = \xi^{(o)}$, $\bar{\Pi} = \pi^{(o)}$. We shall now make use of the $U_{\underline{K}\underline{n}}$ functions introduced in Chapter 7 and the slowly varying nature of $\nabla\xi^{(o)}$, $\pi^{(o)}$ which also implies the same property of any function of $\nabla\xi^{(o)}$, $\pi^{(o)}$ appearing in (9.3.3-1).

$L^{3/2} \sum_{\underline{m}} U_{o\underline{m}}(\underline{x}) = 1$, using the same notation as before.

$$\Longrightarrow \qquad H = \int d^3x\, \mathcal{H} = \sum_{\underline{m}} L^{3/2} \int d^3x \ U_{o\underline{m}}(\underline{x})\, \mathcal{H}$$

The following approximation may be made:

$$U_{om}(\underline{x})\ f(\xi^{(o)}_{,j}, \pi^{(o)}) \approx f^{(o)}_{\underline{m}}\ U_{om}(\underline{x}), \quad \text{(no summation over } \underline{m})$$

where f stands for any function of $\xi^{(o)}_{,j}$, $\pi^{(o)}$ in (9.3.3-1) and $f^{(o)}_{\underline{m}}$ is its average value around $\underline{x} = \underline{m}L$. $f_{\underline{m}}$ may be taken to be

$$f^{(o)}_{\underline{m}} = f^{(o)}(\xi^{(o)}_{,j},\pi^{(o)}) \Big|_{\substack{\xi^{(o)}_{,j} = <\xi^{(o)}_{,j}>_{\underline{m}} \\ \pi^{(o)} = <\pi^{(o)}>_{\underline{m}}}} ,$$

where $<\xi^{(o)}_{,j}>_{\underline{m}} = \frac{1}{L^{3/2}} \int d^3x \; U_{o\underline{m}} \; \xi^{(o)}_{,j}$; $<\pi^{(o)}>_{\underline{m}} = \frac{1}{L^{3/2}} \int d^3x \; U_{o\underline{m}} \; \pi^{(o)}$.

This is justified since the slowly varying nature of $\xi^{(o)}_{,j}$, $\pi^{(o)}$ makes it immaterial how such average is precisely defined.

Then (9.3.3-1) becomes

$$H \approx L^3 \sum_{\underline{m}} \mathcal{H}^{(o)}_{\underline{m}} +$$

$$\frac{L^{3/2}}{2} \sum_{\underline{m}} \Big[\left(\frac{\partial^2 \mathcal{H}}{\partial \bar{\Pi}^2}\right)^{(o)}_{\underline{m}} \int d^3x \; U_{o\underline{m}} \; \bar{\Pi}'^2_{\underline{m}} + \left(\frac{\partial^2 \mathcal{H}}{\partial \bar{\xi}_{,i} \partial \bar{\xi}_{,j}}\right)^{(o)}_{\underline{m}} \int d^3x \; U_{o\underline{m}} \; (\bar{\xi}'_{,i})_{\underline{m}} \; (\bar{\xi}'_{,j})_{\underline{m}} +$$

$$\left(\frac{\partial^2 \mathcal{H}}{\partial \bar{\Pi} \partial \bar{\xi}_{,j}}\right)^{(o)}_{\underline{m}} \int d^3x \; U_{o\underline{m}} \; \bar{\Pi}'_{\underline{m}} (\bar{\xi}'_{,j})_{\underline{m}} +$$

$$\cdots\cdots\cdots \qquad (9.3.3\text{-}2)$$

Comparing this with equation (7.8-1) and identifying $\left(\frac{\partial^2 \mathcal{H}}{\partial \bar{\Pi}^2}\right)^{(o)}_{\underline{m}}$ with $\left(\frac{\partial^2 \mathcal{H}}{\partial \pi^2}\right)_{\underline{m}}$ in (7.8-1) we can conclude that our present theory does indeed reduce to the treatment in Chapter 7. The form of (9.3.3-1), (9.3.3-2) is specially written for an easy comparison with equations (7.5-1), (7.5-8). Actually the present theory may be made neater by expressing (9.3.3-1), (9.3.3-2) in terms of the canonical variables ϕ, π. Then the Hamiltonian becomes

$$H \approx L^3 \sum_{\underline{m}} \mathcal{H}^{(o)}_{\underline{m}} +$$

$$\frac{L^{3/2}}{2} \sum_{\underline{m}} \left[\left(\frac{\partial^2 \mathcal{H}}{\partial \bar{\Pi}^2}\right)^{(o)}_{\underline{m}} \int d^3x \, U_{o\underline{m}} \, \pi^2 + \left(\frac{\partial^2 \mathcal{H}}{\partial \bar{\xi}_{,i} \partial \bar{\xi}_{,j}}\right)^{(o)}_{\underline{m}} \int d^3x \, U_{o\underline{m}} \, \phi_{,i} \phi_{,j} + \right.$$

$$\left. \left(\frac{\partial^2 \mathcal{H}}{\partial \bar{\Pi} \partial \bar{\xi}_{,j}}\right)^{(o)}_{\underline{m}} \int d^3x \, U_{o\underline{m}} \, \pi \phi_{,j} \right] + \ldots .$$

The quantization is effected by imposing the usual commutation rules on ϕ, π, while keeping $\xi^{(o)}$, $\pi^{(o)}$ classical.

9.3.4 Some Qualitative Features

The possibility of a unified treatment for the weak field and the slowly-varying field cases should hint at some common features in these two cases. They are

(1) In the lowest approximation, both treatments lead to a Klein-Gordon type field with the corresponding set of non-interacting particles.

(2) Higher order interaction terms are obtained essentially from a binomial expansion of the square root expression for the Hamiltonian density. As one goes to a higher order, one has a higher order product of $\phi_{,j}$, π leading to higher order divergence in the quantum theory. The type of divergence in both treatments is of the same nature.

There are also substantial differences in the two cases. They are

(1) Even in the lowest approximation, there are fundamental differences.

While the weak field treatment leads exactly to a massless Klein-Gordon field, the slowly-varying field theory gives a quantized field which is superimposed on a classical background field $\xi_{\underline{m}}^{(o)}$, $\pi_{\underline{m}}^{(o)}$. This background field also contributes to the energy eigenvalues of the quanta.

(2) The use of $U_{\underline{Kn}}$ functions leads to quanta which are localized in domains of volume of the order of L^3 as shown in section 7.5, Chapter 7.

The appearance of the classical background field is one of the most striking features in an intrinsically nonlinear field. We also have noted that the background field may vary slowly with time. Hence the energy associated with each quantum will depend on time as well as the spatial position of the cell in which it is created. These properties appear to be in accord with the very concept of curved space-time in which not all world points are equivalent as in the case of a flat space-time.

9.4 Variational Treatment versus Perturbative Theories

The variational method used in Chapter 8 is certainly very different from those perturbative theories discussed in some previous chapters. The fact that some finite and reasonable results are obtained in the variational calculation could be significant. It might mean that in Chapters 6 and 7 the divergences, at least some of them like the one-particle self-energy, are spurious effects due to the particular perturbation theories employed rather than the inevitable consequence of the nonlinearity of the field concerned. Therefore it would be well worthwhile to effect further work along these lines. Some practical schemes are outlined in the next chapter.

CHAPTER 10

SUGGESTIONS FOR FURTHER WORK

10.1 Introduction

In this thesis, a study has been made on the quantization of some generally covariant model field theories. Certain results are obtained on the generally covariant aspects of the problem. However a tremendous amount of work is still needed in order to obtain a totally satisfactory quantum theory of such nonlinear fields in a curved space-time. We shall not enter into discussion of any such general problems. In this chapter we shall only suggest for further research some definite problems directly related to our present work on the Model Field II. There is of course room for more work to be done in order to make the various present treatments of the Model Field II more complete and satisfactory. However in this chapter we shall confine ourselves to suggestions for further work along completely new lines.

10.2 Attempt for a New Treatment of the Model Field II

10.2.1 Classical Theory

As shown in section 6.2.1, Chapter 6 our Model Field II possesses a set of plane wave solutions. Each such solution has the form of a free wave propagating along a straight line which happens to be a null geodesic in the space-time defined by the solution itself. No interaction effect exists as long as the field is built up by waves travelling in the same direction. Let us confine the field in a box V with the usual periodic boundary conditions. Let

$$\phi_{n\underline{k}} = \frac{1}{\sqrt{V}} \exp(in\, k_\rho x^\rho),$$

where $k_j = \frac{2\pi}{L} \times$ integer; $k_o = |\underline{k}| = k$; n = integer; $k_\rho k^\rho = \eta^{\mu\nu} k_\mu k_\nu = 0$. Then

$$\xi_{n\underline{k}} = A_n \phi^*_{n\underline{k}} + A^*_n \phi_{n\underline{k}} \qquad \text{(no summation over n)}$$

$$= A_n \phi_{-n\underline{k}} + A^*_n \phi_{n\underline{k}} = A_n \phi_{-n\underline{k}} + A^*_n \phi^*_{-n\underline{k}}, \qquad (10.2\text{-}1)$$

is a real solution to our nonlinear equation (6.2.1-1). The general solution representing a wave travelling in the direction specified by a $\underline{k}$ is

$$\xi_{\underline{k}} = \sum_{n>o} \xi_{n\underline{k}}. \qquad (10.2\text{-}2)$$

$$\Longrightarrow \qquad \pi_{\underline{k}} = \frac{Q\,\dot{\xi}_{\underline{k}}/c^2}{\sqrt{1-\eta^{\rho\sigma}\xi_{\underline{k},\rho}\xi_{\underline{k},\sigma}}} = \frac{Q\,\dot{\xi}_{\underline{k}}}{c^2} \quad \text{and} \quad \left(\frac{c\pi_{\underline{k}}}{Q}\right)^2 = (\nabla\xi_{\underline{k}})^2.$$

$$\Longrightarrow \qquad \mathcal{H}_{\underline{k}} = Q\sqrt{[1+(\nabla\xi_{\underline{k}})^2][1+(\frac{c\pi_{\underline{k}}}{Q})^2]}$$

$$= Q[1+(\nabla\xi_{\underline{k}})^2] = Q + \tfrac{1}{2}[(\nabla\xi_{\underline{k}})^2 + \left(\frac{c\pi_{\underline{k}}}{Q}\right)^2]. \qquad (10.2\text{-}3)$$

We end up with a Hamiltonian which is linear in the sense that the energy contributions from different n-values are additive. Rewrite (10.2-2) as

$$\xi_{\underline{k}} = \sum_{n>o} \sqrt{\frac{\hbar c}{2Qnk}}\,(a_{n\underline{k}}\phi^*_{n\underline{k}} + a^*_{n\underline{k}}\phi_{n\underline{k}}).$$

Expressing the Hamiltonian and the momentum associated with the field in terms of $a_{n\underline{k}}$, $a^*_{n\underline{k}}$ we obtain

$$H_{\underline{k}} = \int d^3x\,\mathcal{H}_{\underline{k}} \qquad (10.2\text{-}4)$$

$$= E_o + \sum_{n>o} \hbar c\, nk\, a^\dagger_{n\underline{k}} a_{n\underline{k}}, \quad \text{where } E_o = QV + \tfrac{1}{2}\sum_{n>o} \hbar c\, nk.$$

$$P_{\underline{k}} = \int d^3x(-\nabla\xi_{\underline{k}})\pi_{\underline{k}}$$

$$= \sum_{n>o} \hbar\, n\, \underline{k}\, a^\dagger_{n\underline{k}} a_{n\underline{k}} + \tfrac{1}{2}\sum_{n>o} \hbar\, n\, \underline{k} \qquad (10.2\text{-}5)$$

10.2.2 Quantization

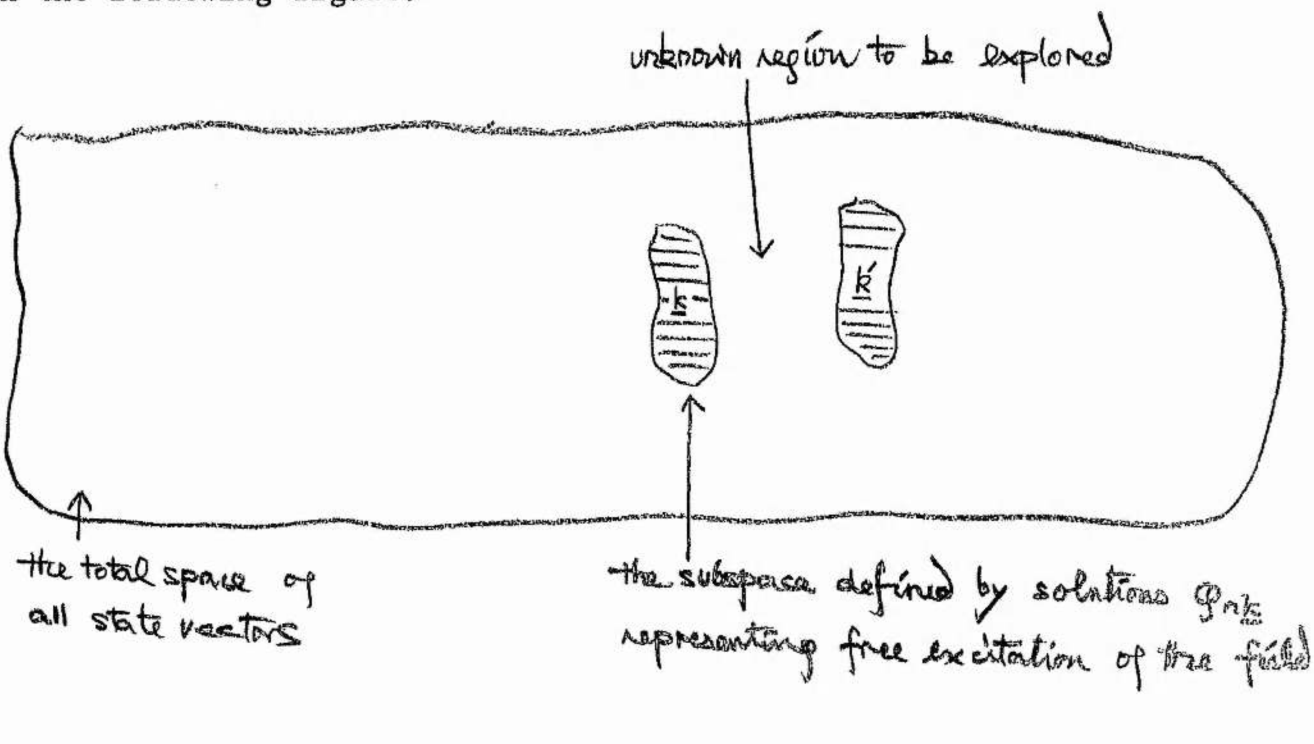
We now postulate that each plane wave solution becomes in the quantum theory a quantized harmonic oscillator in the same way as in linear field theories, that is, $a_{n\underline{k}}$, $a^*_{n\underline{k}}$ are regarded as creation and annihilation operators with the usual boson commutation rules

$$[a_{n\underline{k}}, a_{n'\underline{k}}] = [a^\dagger_{n\underline{k}}, a^\dagger_{n'\underline{k}}] = 0; \quad [a_{n\underline{k}}, a^\dagger_{n'\underline{k}}] = \delta_{nn'}.$$

$H_{\underline{k}}$, $P_{\underline{k}}$ are now operators. We see that if we confine ourselves to the field excitation which is formed by plane wave travelling in the same direction, we can perform the quantization which leads to the exact solution of various problems about the particular field excitation. The procedure may be applied to any specific $\underline{k}$. Observe that the present results agree with the corresponding ones obtained by the variational method of Chapter 8. Indeed the two theories reinforce each other.

10.2.3 Some Ideas for Future Research

A general picture in the quantum theory begins to emerge after we carry out the above quantization procedure for all $\underline{k}$ values. What one has now is that some special subspaces exist in the linear vector space of quantum states for the nonlinear field as depicted in the following figure.



In such subspaces the field behaves like free field consisting of non-interacting particles moving along the same direction. The next step is to explore the "unknown" region outside those "known" subspaces and this presents great difficulties. Firstly one would try to extend the above exact treatment. It is possible to obtain exact solutions other than plane wave ones to the field equations for our Model Field II. The problem is whether such knowledge could help to bring about the quantization of the corresponding field excitations. Secondly we should seek now approximation methods in the light of the above exact quantum theory of plane wave states. The most obvious thing to do is to examine two sets of plane waves travelling along almost parallel paths. It would be reasonable to expect that if the two sets of waves differ only slightly in their direction of propagation then their interaction may be treated by a perturbation theory. Further efforts are needed to develop the above ideas.

10.3 Interaction with other Fields

So far our discussion has been confined to the ξ-field itself without any reference to its possible interaction with other things. One could study this problem in the usual way. Consider the simplest type of coupling determined by the total Hamiltonian

$$H_T = H_\xi + H',$$

where

$$H_\xi = \int d^3x \;\; Q\sqrt{[1+(\nabla\xi)^2][1+(\frac{c\pi}{Q})^2]};$$

$$H' = -\int d^3x \;\; Q\, f(\underline{x},t)\,\xi,$$ and $f(\underline{x},t)$ is a prescribed function.

The field equation becomes

$$\left(\frac{\eta^{\mu\nu}\,\xi_{,\mu}}{\sqrt{1+\eta^{\rho\sigma}\xi_{,\rho}\xi_{,\sigma}}}\right)_{,\nu} = f(\underline{x},t).$$

$f(\underline{x},t)$ may now be interpreted as the source for the ξ-field. The usual static point source may be incorporated here if we take

$$f \propto \sum_{\underline{n}} \delta(\underline{x} - \underline{x}_n).$$

This source then consists of point particles fixed spatially. These particles may be considered to possess a sort of "charge" which generates the ξ-field.

For a more general theory of coupling ξ-field to other fields defined on curved space-time one may follow the treatment adopted in the theory of General Relativity.[J1] In General Relativity the field variables are the metric $g_{\mu\nu}$. The coupling with other fields may be formulated by taking a Lagrangian density of the form

$$\mathcal{L}_T = \mathcal{L}_G + \mathcal{L}_\phi,$$

where $\mathcal{L}_G$ is the original gravitational Lagrangian density. $\mathcal{L}_\phi$ will in general contain the metric $g_{\mu\nu}$ in addition to the new field variable ϕ and its derivatives. The derivatives of $g_{\mu\nu}$ should not appear in $\mathcal{L}_\phi$. Then the total Hamiltonian density takes the form

$$\mathcal{H}_T = \mathcal{H}_G + \mathcal{H}_\phi, \tag{10.3-1}$$

where $\mathcal{H}_G$ is the original gravitational Hamiltonian density. The situation becomes more involved when this scheme is applied to our Model Field II whose field variables are ξ^A. The additional Lagrangian $\mathcal{L}_\phi$ would then contain the derivatives of ξ^A. This derivative type of interaction is not unknown and the scheme could be carried through.

10.4 The de-Sitter Universe

Consider the generally covariant field theory derived from the following Lagrangian density

$$\mathcal{L} = -Q\sqrt{-g} + \frac{Q\,\varepsilon_{ABCDE}\,\xi^A}{\sqrt{d}\quad 3!}\quad \frac{\partial(\xi^B \xi^C \xi^D \xi^E)}{\partial(x^0 x^1 x^2 x^3)} \;, \qquad (10.4\text{-}1)$$

where d is a numerical constant of dimension $[(\text{length})^2]$;

$$A = 0,\ 1,\ 2,\ 3,\ 4\ ;\ g_{\tau\kappa} = \eta_{AB}\ \xi^A_{\ ,\tau}\xi^B_{\ ,\kappa};$$

$$\frac{\partial(\xi^B \xi^C \xi^D \xi^E)}{\partial(x^0 x^1 x^2 x^3)} = \text{the Jacobian determinant;}$$

ε_{ABCDE} = the 5-dimensional permutation symbol.

It is seen that

$$\int \mathcal{L}\ d^3 x$$

is an invariant under coordinate transformations on the 4-dimensional space. In the special coordinate frame $x^\mu = \xi^\mu$, the Lagrangian density becomes

$$\mathcal{L} = -Q\sqrt{1-\eta^{\rho\sigma}\xi_{,\rho}\xi_{,\sigma}} + \frac{4Q\xi}{\sqrt{d}} \;. \qquad (10.4\text{-}2)$$

The field equation is

$$\frac{1}{\sqrt{-g}}\left(\frac{\eta^{\mu\nu}\ \xi_{,\mu}}{\sqrt{1-\eta^{\rho\sigma}\xi_{,\rho}\xi_{,\sigma}}}\right)_{,\nu} + \frac{4}{\sqrt{d}} = 0 \qquad (10.4\text{-}3)$$

To relate all this to the de-Sitter universe of constant curvature we recall that the de-Sitter space can be embedded in a 5-dimensional Pseudo-Euclidean space and that a 4-surface defined by $\eta_{AB}\ \xi^A \xi^B = r$, where r = a numerical constant, is a de-Sitter space of constant curvature $\frac{1}{r}$.[J2] In the special coordinate frame $\xi^\mu = x^\mu$, the de-Sitter space is given by

$$\xi^2 = \eta_{\mu\nu} x^\mu x^\nu - r. \tag{10.4-4}$$

This expression satisfies equation (10.4-3). Therefore we conclude that indeed the de-Sitter universe is derivable from the Lagrangian density (10.4-1) if we identify d in (10.4-1) with the r in (10.4-4). Comparing all this with our Model Field II, we appear to have the surprising result that the Model Field II, which is a 4-surface of stationary volume, seems to be a de-Sitter world of a constant curvature $\frac{1}{d} = 0$. This is no cause for alarm however. While the flat space is a solution to our Model Field II, it has other non-trivial solutions as well. Now one can set up the Hamiltonian theory for the new field and quantize it. It will be most interesting to investigate the final results to see how they are related to the work of other people[J3] using mostly group theoretic methods. In passing we note that the original flat metric η_{AB} has to be modified to achieve a constant positive curvature while maintaining a negative value of g. A choice is[J4]

$$\eta'_{AB} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & +1 \end{pmatrix}.$$

The treatment of curved space-time using such an embedding technique may be extended to other Riemannian spaces. It is known[J5] that various Riemannian spaces commonly occurring in general relativity are immersible in pseudo-Euclidean spaces of appropriate dimension. As an example consider those vacuum solutions in general relativity, that is, solutions which give a vanishing Ricci tensor $R_{\mu\nu}$. We know that all vacuum solutions are immersible in a ten-dimensional flat space, while on the other hand, many vacuum solutions of physical significance are immersible in a six-dimensional flat space. All vacuum solutions immersible in a five-dimensional flat space are trivial, leading to Minkowskian space-times only [e.g. see L.P. Eisenhart, Riemannian Geometry (Princeton, 1949) p.200]. Therefore one can formulate theories for the above non-trivial space-times in a way similar to our treatment of the Model Field II. The results obtained in section 5.1, Chapter 5 may be easily extended to higher-dimensional embedding spaces. The field equations will be

$$R_{\mu\nu} = 0, \qquad (10.4\text{-}5)$$

where the Ricci tensor $R_{\mu\nu}$ is expressed in terms of the coordinates ξ^A ($A = 0,1,2,\ldots,N$; $6 \leqslant N \leqslant 10$) and their derivatives of the particular embedding flat space concerned. One may then proceed to set up a Lagrangian formulation for the theory, bearing in mind that in the variational principle the variation is effected by $\delta\xi^A$. After this the Hamiltonian formulation may be established and various quantization schemes can then be attempted. In view of the complexity of the field equations (10.4-5), various technical difficulties may arise in actually carrying out the above programme. There is obviously room for much further work on this.

10.5 New Field Theories

Some interesting new quantum field theories may be derived from the Model Field II by new interpretations of the Hamiltonian density in a similar way to that in which the Dirac equation was obtained. The classical Hamiltonian density for the Model Field II is

$$\mathcal{H} = \sqrt{[1+(\nabla\xi)^2][1+\pi^2]},$$

where we have taken Q = c = 1 for brevity. Now an example of a new interpretation of this expression would be to take

$$\sqrt{[1+(\nabla\xi)^2][1+\pi^2]} = \Omega_o + \underline{\Omega}.\nabla\xi + \omega_o\pi + \underline{\omega}.\nabla\xi\pi, \qquad (10.5\text{-}1)$$

in the sense that both sides should be identical on squaring. The quantities $\Omega_\mu = (\Omega_o,\underline{\Omega})$; $\omega_\mu = (\omega_o,\underline{\omega})$ are constants to be determined by the squaring procedure. It may be readily shown that squaring leads to the following expressions for the determination of Ω_μ and ω_μ:

$$\{\Omega_\mu,\Omega_\nu\} = 2\delta_{\mu\nu} \ ; \ \{\omega_\mu,\omega_\nu\} = 2\delta_{\mu\nu}; \qquad (10.5\text{-}2)$$

$$\{\Omega_\mu,\omega_\nu\} + \{\Omega_\nu,\omega_\mu\} = 0, \qquad (10.5\text{-}3)$$

where the brackets mean anticommutators, that is, Ω_μ, ω_μ separately satisfy the Dirac algebra (10.5-2) while at the same time they are related by (10.5-3). Obviously Ω_μ, ω_μ are matrices. Lengthy calculations show that at least an 8-dimensional matrix representation is needed. An explicit 8-dimensional representation for Ω_μ, ω_μ is

$$\Omega_o = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}; \qquad \underline{\Omega} = \begin{pmatrix} \underline{\alpha} & 0 \\ 0 & -\underline{\alpha} \end{pmatrix},$$

$$\omega_o = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \qquad \underline{\omega} = \begin{pmatrix} 0 & \underline{\alpha} \\ \underline{\alpha} & 0 \end{pmatrix},$$

where I = the 4x4 identity matrix and

$$\underline{\alpha} = \begin{pmatrix} 0 & \underline{\sigma} \\ \underline{\sigma} & 0 \end{pmatrix}$$, $\underline{\sigma}$ being the Pauli Matrices.

Hence we obtain a field theory defined by the Hamiltonian (10.5-1). The field may be quantized by imposing the usual Boson commutation rules on ξ and π. Since it is a linear theory, we can readily calculate the exact energy eigenvalues and the corresponding multi-component eigenvectors. Many similar, yet different, interpretations of $\mathcal{H}$ are possibly leading to similar theories. However this type of procedure leads only to linear quantum theories. It is hard to see how such linear quantum theories can possibly be related to the original nonlinear classical theory. Perhaps further investigation should be made to see if there is any other interpretation of the Hamiltonian density in quantum theory which will bring in nonlinear terms, hence bearing a closer relationship to the original classical theory.

10.6 Further Remarks

A lot of work has been done on the quantization of the gravitational field. However in most cases people are unable to carry their schemes right through because of the sheer mathematical complexity of the gravitational field equations. The Model Field II studied in this thesis is much simpler in comparison while retaining many essential features of the gravitational theory. It would be well worthwhile to test out these schemes on the Model Field II instead of the frequently used electromagnetic field which is physically very different from the gravitational theory of curved space-time. In particular many of the conceptual problems arising from a curved space-time may be discussed through the Model Field II. There are also many people working on the quantization of nonlinear Lorentz covariant field theories. As far as the problem of non-linearity is concerned one may try to use their methods to treat the nonlinearity of the Model Field II. Of course many of these methods may not be applicable in our case, e.g. those valid only for massive fields.[J6]

One hopes that further investigation along these lines may help towards a better understanding of the quantum theory of fields obeying nonlinear differential equations in a curved space-time.

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