HIGHER SPIN RELATIVISTIC WAVE-EQUATIONS

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"Higher spin relativistic wave-equations".

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Abstract:

In this work we have undertaken the study of some aspects of higher spin relativistic wave equations of the first order. These aspects are their definitness of charge, their second quantization and their causal propagation.

The work is divided into three parts. In part one we have studied the Bhabha equations based on the sixteen and twenty dimensional representations and showed that the charge associated with them is indefinite. In part two we have studied in detail the Gel'fand Yaglom equations and showed that certain equations can have definite charge. Finally in part three we have studied the propagation in the external electromagnetic field of certain higher spin relativistic wave equations.

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CERTIFICATE.

I certify that the conditions of the Ordinance and Regulations have been fulfilled.

....
Research Supervisor.

I was admitted as a research student for the Ph.D. degree under Ordinance General No. 12 and resolution of the University Court 1967 No. 1.

DECLARATION.

The accompanying thesis is my own composition. It is based on work carried out by me and no part of it has previously been presented in application for a Higher Degree.

PRELIMINARIES.

1: Introduction.

The field of relativistic wave-equations is fifty four years old and aims at the description of particles in terms of wave-functions and equations of motion.

The wave equation first proposed for this purpose was the equation of Schrödinger [1-4] (1926) which is not relativistic and describes particles without spin. Later Schrödinger and separately Klein and Gordon [5-6] tried to incorporate relativity into the field of wave equations by proposing a wave equation which is relativistic and describes scalar particles. This equation is known as the Klein-Gordon equation.

Dirac [7-8] in (1928) proposed a relativistic wave-equation which describes particles of spin 1/2 and mass m. Since then many other relativistic wave-equations have been discovered to describe particles of higher spin. Dirac [9] in (1936) proposed a set of relativistic wave equations for the description of particles of higher spin, but as Pauli and Fierz [10-11] showed (1939), these equations become inconsistent if minimal coupling is introduced. Thus they proposed a Lagrangian derivation of a new set of equations suitable for the description of higher spin particles.

Other equations for the description of higher spin particles are:

The Kemmer [12-13] equation used for the description of particles of maximum spin 1 and mass m.

The Rarita-Schwinger [14] equation used for the description of particles of higher spin.

The Bhabha equation[15-19] which describes multimass and multispin fields and others.

2: The Bhabha wave-equation:

Bhabha in his effort to free the higher spin theories from the presence of the subsidiary conditions, proposed an equation which is similar in appearance to the Dirac wave-equation and which in the absence of interactions reads

$$\mathbb{L}_0 \frac{\partial \psi}{\partial x_0} + \mathbb{L}_1 \frac{\partial \psi}{\partial x_1} + \mathbb{L}_2 \frac{\partial \psi}{\partial x_2} + \mathbb{L}_3 \frac{\partial \psi}{\partial x_3} + i\chi\psi = 0 , \qquad (1)$$

where $\mathbb{L}_k$, $k = 0, 1, 2, 3$ are four matrices of appropriate dimension depending on the representation according to which the wave function ψ transforms and χ a constant related to the mass of the particle.

The Bhabha field is a multimass and multispin field. For example a field of maximum spin 3/2 appears with two possible rest masses one three times the other. A field of maximum spin 2 has also two rest masses one twice the other. In a field of maximum spin s all the lower values of spin appear as well.

Bhabha studied equation (1); a) in the case when the underlying representation belongs to the group $SO(4,1)$ and b) in the case when the underlying representation is a general representation.

In the first case the matrices $\mathbb{L}_k$ satisfy a relation of the form

$$\left[\mathbb{L}_m , \mathbb{L}_n \right]_- = \mathbb{I}_{mn} \qquad (2)$$

where $\mathbb{I}_{mn}$ are the infinitesimal generators of the Lorentz group. In the second case, (2) does not hold.

Some aspects of both of these theories we study in parts I and II of this work.

PART I.

CHAPTER 1.

A method of determining the 16 and 20-dimensional realizations of the matrices $\mathbb{L}_k, (k=0,1,2,3)$ of the Bhabha wave-equation.

Making use of the methods of Lie algebras[20-21] in this chapter, we shall find matrix interpretations of the 16 and 20-dimensional representations of the group $SO(4,1)$ in terms of which the matrices $\mathbb{L}_k\ (k=0,1,2,3)$ appearing in the wave-equation of Bhabha can be expressed.

1: Preliminaries.

According to Bhabha there are two possible representations of the group $SO(4,1)$ which can be used to describe a particle of spin 3/2 namely the 16-dimensional representation and the 20-dimensional.

The Lie algebra corresponding to $SO(4,1)$ is the complex Lie algebra B_2 . Its generators are $\vec{h}_{\alpha_1}$, $\vec{h}_{\alpha_2}$, $\vec{e}_{\pm\alpha_1}$, $\vec{e}_{\pm\alpha_2}$, $\vec{e}_{\pm(\alpha_1+\alpha_2)}$, $\vec{e}_{\pm(\alpha_1+2\alpha_2)}$. We recall that for any Lie algebra $\mathcal{L}$:

i) any element $\vec{h} \in \mathcal{H}$ is given by

$$\vec{h} = \mu_1 \vec{h}_{\alpha_1} + \mu_2 \vec{h}_{\alpha_2} ,$$

where $\mathcal{H}$ the Cartan subalgebra of $\mathcal{L}$, $\vec{h}_{\alpha_1}$, $\vec{h}_{\alpha_2}$, the basis elements of the Cartan subalgebra and μ_1 , μ_2 coefficients,

ii) $[\vec{e}_\alpha , \vec{h}]_- = \alpha(\vec{h})\vec{e}_\alpha$,

iii) $[\vec{h} , \vec{h}']_- = 0$, $\forall\ \vec{h}, \vec{h}' \in \mathcal{H}$,

iv) $[\vec{e}_\alpha , \vec{e}_{-\alpha}]_- = \vec{h}_\alpha$,

v) $[\vec{e}_\alpha , \vec{e}_\beta]_- = N_{\alpha,\beta}\vec{e}_{\alpha+\beta}$,

where $N_{\alpha,\beta}=0$ if $\alpha+\beta$ is not a root of $\mathcal{L}$. $N_{\beta,\alpha}=-N_{\alpha,\beta}$ and by convention $N_{-\alpha,-\beta}=N_{\alpha,\beta}$. Moreover if the α-string of roots containing β is $\beta-r\alpha$, $\beta-(r-1)\alpha, \cdots \beta \cdots \beta+q\alpha$, then the magnitude of $N_{\alpha,\beta}$ is given by $(N_{\alpha,\beta})^2 = \frac{1}{2}q(r+1)(\alpha,\alpha)$ with the signs of $N_{\alpha,\beta}$ to some extent being arbitrary.

Five dimensional matrix realizations of the complex Lie algebra of $SO(4,1)$ i.e. B_2 and hence of the canonical form of B_2 are given by J.F. Cornwell.[22] These formulae for any algebra B_ℓ are:

$$\vec{h}_{\alpha_j} = \begin{cases} -\frac{1}{2(2\ell-1)}\left\{\vec{e}_{j+1,j+1} - \vec{e}_{j+\ell+1,j+\ell+1} - \vec{e}_{j+2,j+2} + \vec{e}_{j+\ell+2,j+\ell+2}\right\}, & j = 1,\cdots,\ell-1, \\ -\frac{1}{2(2\ell-1)}\left\{\vec{e}_{\ell+1,\ell+1} - \vec{e}_{2\ell+1,2\ell+1}\right\}, & j = \ell, \end{cases}$$

for the basis elements of the Cartan subalgebra, and

$$\vec{e}_{\alpha_j} = \begin{cases} \left[2(2\ell-1)\right]^{-1/2}\left\{\vec{e}_{j+1,j+2} - \vec{e}_{j+\ell+2,j+\ell+1}\right\}, & j = 1,\cdots,\ell-1, \\ \left[2(2\ell-1)\right]^{-1/2}\left\{\vec{e}_{1,2\ell+1} - \vec{e}_{\ell+1,1}\right\}, & j = \ell. \end{cases}$$

$$\vec{e}_{-\alpha_j} = \begin{cases} -\left[2(2\ell-1)\right]^{-1/2}\left\{\vec{e}_{j+2,j+1} - \vec{e}_{j+\ell+1,j+\ell+2}\right\}, & j = 1,\cdots,\ell-1, \\ -\left[2(2\ell-1)\right]^{-1/2}\left\{\vec{e}_{2\ell+1,1} - \vec{e}_{1,\ell+1}\right\}, & j = \ell, \end{cases}$$

for the other elements corresponding to the simple roots.

In the above $\vec{e}_{m,n}$ are square matrices of appropriate dimension in which the (m,n) element is unit and all other elements are zero.

The inner product of a simple root α_j by itself, for the algebra B_ℓ is given by the formula

$$(\alpha_j, \alpha_j) = \begin{cases} \frac{1}{2\ell-1}, & j = 1,\cdots,\ell-1, \\ \frac{1}{2(2\ell-1)}, & j = \ell. \end{cases}$$

The above formulae in the case of the algebra B_2, $(\ell=2)$, become:

i) <u>Basis elements of the Cartan subalgebra.</u>

a) For $j=1$, $\ell=2$,

$$\vec{h}_{\alpha_1} = -\frac{1}{6}\left\{\vec{e}_{2,2} - \vec{e}_{4,4} - \vec{e}_{3,3} + \vec{e}_{5,5}\right\}.$$

b) For $j = 2$, $\ell = 2$,

$$\vec{h}_{\alpha_2} = -\frac{1}{6}\left(\vec{e}_{3,3} - \vec{e}_{5,5}\right).$$

ii) Basis elements corresponding to the simple roots.

a) For $j = 1$, $\ell = 2$,

$$\vec{e}_{\alpha_1} = \frac{1}{\sqrt{6}}\left\{\vec{e}_{2,3} - \vec{e}_{5,4}\right\},$$

and

$$\vec{e}_{-\alpha_1} = -\frac{1}{\sqrt{6}}\left\{\vec{e}_{3,2} - \vec{e}_{4,5}\right\}.$$

b) For $j = 2$, $\ell = 2$,

$$\vec{e}_{\alpha_2} = \frac{1}{\sqrt{6}}\left\{\vec{e}_{1,5} - \vec{e}_{3,1}\right\},$$

and

$$\vec{e}_{-\alpha_2} = -\frac{1}{\sqrt{6}}\left\{\vec{e}_{5,1} - \vec{e}_{1,3}\right\}.$$

iii) Inner products (α_j, α_j).

a) For $j = 1$, $\ell = 2$,

$$(\alpha_1, \alpha_1) = \frac{1}{3}.$$

b) For $j = 2$, $\ell = 2$,

$$(\alpha_2, \alpha_2) = \frac{1}{6}.$$

Some authors define as basis elements of the Cartan subalgebra instead of $\vec{h}_{\alpha_j}$ the elements $\vec{h}_j$ related to $\vec{h}_{\alpha_j}$ by

$$\vec{h}_{\alpha_j} = \frac{(\alpha_j, \alpha_j)}{2}\vec{h}_j .$$

Thus for B_2, $\vec{h}_{\alpha_1} = \frac{1}{6}\vec{h}_1$ and $\vec{h}_{\alpha_2} = \frac{1}{12}\vec{h}_2$. For $\vec{e}_{\alpha_j}$ they use $\vec{e}_j$ related as follows

$$\vec{e}_{\alpha_j} \equiv \vec{e}_j ,$$

and for $\vec{e}_{-\alpha_j}$ they use $\vec{f}_j$ connected by

$$\vec{f}_j = \frac{2\,\vec{e}_{-\alpha_j}}{(\alpha_j, \alpha_j)} .$$

In the following we shall distinguish the matrices associated with the elements $\vec{h}_{\alpha_1}, \vec{h}_{\alpha_2}, \vec{e}_{\pm\alpha_1}, \vec{e}_{\pm\alpha_2}$ by writing $\Gamma(\vec{h}_{\alpha_1})$,

$\Gamma(\vec{h}_{\alpha_2})$, $\Gamma(\vec{e}_{\pm\alpha_1})$, $\Gamma(\vec{e}_{\pm\alpha_2})$.

To determine the basis elements $\vec{e}_{(\alpha_1+\alpha_2)}$, $\vec{e}_{-(\alpha_1+\alpha_2)}$, $\vec{e}_{(\alpha_1+2\alpha_2)}$, $\vec{e}_{-(\alpha_1+2\alpha_2)}$ of B_2 in the five dimensional representation we make use of the following formulae

$$\vec{e}_{(\alpha_1+\alpha_2)} = \frac{1}{N_{\alpha_1,\alpha_2}}[\vec{e}_{\alpha_1}, \vec{e}_{\alpha_2}]_- , \quad \vec{e}_{-(\alpha_1+\alpha_2)} = \frac{1}{N_{-\alpha_1,-\alpha_2}}[\vec{e}_{-\alpha_1}, \vec{e}_{-\alpha_2}]_- ,$$

$$\vec{e}_{(\alpha_1+2\alpha_2)} = \frac{1}{N_{\alpha_2,\alpha_1+\alpha_2}}[\vec{e}_{\alpha_2}, \vec{e}_{\alpha_1+\alpha_2}]_- , \quad \vec{e}_{-(\alpha_1+2\alpha_2)} = \frac{1}{N_{-\alpha_2,-\alpha_1-\alpha_2}}[\vec{e}_{-\alpha_2}, \vec{e}_{-\alpha_1-\alpha_2}]_- ,$$

or in matrix notation

$$\Gamma(\vec{e}_{\alpha_1+\alpha_2}) = \frac{1}{N_{\alpha_1,\alpha_2}}[\Gamma(\vec{e}_{\alpha_1}), \Gamma(\vec{e}_{\alpha_2})]_- ,$$

$$\Gamma(\vec{e}_{-\alpha_1-\alpha_2}) = \frac{1}{N_{-\alpha_1-\alpha_2}}[\Gamma(\vec{e}_{-\alpha_1}), \Gamma(\vec{e}_{-\alpha_2})]_- ,$$

$$\Gamma(\vec{e}_{\alpha_1+2\alpha_2}) = \frac{1}{N_{\alpha_2,\alpha_1+\alpha_2}}[\Gamma(\vec{e}_{\alpha_2}), \Gamma(\vec{e}_{\alpha_1+\alpha_2})]_- ,$$

$$\Gamma(\vec{e}_{-\alpha_1-2\alpha_2}) = \frac{1}{N_{-\alpha_2,-\alpha_1-\alpha_2}}[\Gamma(\vec{e}_{-\alpha_2}), \Gamma(e_{-\alpha_1-\alpha_2})]_- .$$

Thus, using $\Gamma(\vec{e}_{\alpha_1})$, $\Gamma(\vec{e}_{\alpha_2})$, $\Gamma(\vec{e}_{-\alpha_1})$, $\Gamma(e_{-\alpha_2})$ and $N_{\alpha_1,\alpha_2} = N_{-\alpha_1,-\alpha_2} = \pm\frac{1}{\sqrt{6}}$ (to be determined later), we find choosing the positive sign,

$$\vec{e}_{(\alpha_1+\alpha_2)} = \frac{1}{\sqrt{6}}\{-\vec{e}_{2,1} + \vec{e}_{1,4}\} ,$$

$$\vec{e}_{(-\alpha_1-\alpha_2)} = \frac{1}{\sqrt{6}}\{-\vec{e}_{4,1} + \vec{e}_{1,2}\} .$$

Using $\Gamma(\vec{e}_{\alpha_2})$, $\Gamma(\vec{e}_{\alpha_1+\alpha_2})$, $\Gamma(\vec{e}_{-\alpha_2})$, $\Gamma(e_{-\alpha_1-\alpha_2})$ and $N_{\alpha_2,\alpha_1+\alpha_2} = N_{-\alpha_2,-\alpha_1-\alpha_2} = \pm\frac{1}{\sqrt{6}}$ we find choosing the positive sign

$$\vec{e}_{(\alpha_1+2\alpha_2)} = \frac{1}{\sqrt{6}}\{-\vec{e}_{3,4} + \vec{e}_{2,5}\} ,$$

$$\vec{e}_{(\alpha_1+2\alpha_2)} = \frac{1}{\sqrt{6}}\{-\vec{e}_{5,2} + \vec{e}_{4,3}\} .$$

To determine N_{α_1,α_2} we use the formula

$$N^2_{\alpha_1,\alpha_2} = \frac{1}{2} q(r+1)(\alpha_1, \alpha_1) ,$$

where q, r are determined from the α_1-series of roots containing α_2 which is

$$\{\alpha_2 , \alpha_2 + 1\alpha_1\} ,$$

and hence $r=0$, $q=1$. Using $(\alpha_1, \alpha_1) = \frac{1}{3}$ we find

$$(N_{\alpha_1,\alpha_2})^2 = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6} \quad \text{or} \quad N_{\alpha_1,\alpha_2} = \pm\frac{1}{\sqrt{6}} .$$

Similarly to determine $N_{\alpha_2,\alpha_1+\alpha_2}$, we use the formula $N^2_{\alpha_2,\alpha_1+\alpha_2} = \frac{1}{2} q(r+1)(\alpha_2,\alpha_2)$ where q, r are now determined from the α_2-series of roots containing $\alpha_1+\alpha_2$ which is

$$\{(\alpha_1+\alpha_2)-1\alpha_2 , (\alpha_1+\alpha_2) , (\alpha_1+\alpha_2)+1\alpha_2\}$$

and hence $r=1, q=1$. Using $(\alpha_2,\alpha_2)=\frac{1}{6}$ we get

$$N^2_{\alpha_2,\alpha_1+\alpha_2} = \frac{1}{2}(1+1)\left(\frac{1}{6}\right) = \frac{1}{6} \quad \text{or} \quad N_{\alpha_1,\alpha_1+\alpha_2} = \pm\frac{1}{\sqrt{6}} .$$

Bhabha, in defining the five dimensional realizations of his matrices $\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$, extended the group $SO(3,1)$ to the group $SO(4,1)$ by identifying

$$\mathbb{L}_0 = I_{0,4} , \quad \mathbb{L}_1 = I_{1,4} , \quad \mathbb{L}_2 = I_{2,4} , \quad \mathbb{L}_3 = I_{3,4} ,$$

where $I_{0,4}, I_{1,4}, I_{2,4}, I_{3,4}$ are generators of the five dimensional Lorentz group.

To derive the generators I_{ij} we consider a rotation $R_{ij}(\varphi)$ through an angle φ in the (i,j) plane and identify the generator I_{ij} with the derivative of $R_{ij}(\varphi)$ with respect to φ at $\varphi=0$, i.e.

$$I_{ij} = \frac{d}{d\varphi} R_{ij}(\varphi)\bigg|_{\varphi=0} .$$

J.F. Cornwell[22] gives a similarity transformation $\mathbb{S}$ which maps the canonical form of B_ℓ to $so(2\ell+1-2r, 2r)$, $r=0,1,\dots,\ell$ and is defined in the following theorem.

<u>Theorem</u>: Let $\mathbb{b}$ be an element of the matrix realization of the canonical form $\mathcal{L}$. Then the similarity transformation to the $so(2\ell+1-2r, 2r)$ Lie algebra (for $r=0,1,\dots,\ell$) is given by

$$\alpha = \mathbb{S}\,\mathbb{b}\,\mathbb{S}^{-1} ,$$

where $\mathbb{S} = \sqrt{g}\,\mathbb{T}$,

$\mathbb{T}$ being given by,

$$T_{jk} = \begin{cases} 1 \quad , & j = 2k-2 \, , \; k = 2, \cdots, \ell+1, \\ & \text{and } j = 2k-2\ell-2 \, , \; k = \ell+2, \cdots, 2\ell+1 \\ i \quad , & j = 2k-3 \, , \; k = 2, \cdots, \ell+1 , \\ -i \quad , & j = 2k-2\ell-3 \, , \; k = \ell+2, \cdots, 2\ell+1 , \\ \sqrt{2} & j = 2\ell+1 \, , \; k = 1 \, , \\ 0 & \text{all other } j, k \, , \end{cases}$$

provided that the diagonal elements of g of

$$\alpha^{tr} g + g \alpha = 0 ,$$

(tr means transpose), are arranged so that

$$g_{2j,2j} = g_{2j-1,\,2j-1} \quad , \; j = 1, 2, \cdots, \ell \, ,$$

and

$$g_{2j+1,\,2j+1} = g_{2j-1,\,2j-1} \exp\left\{ \alpha_j(\vec{h}) \right\} , \quad j = 1, 2, \cdots \ell ,$$

where $\exp\left\{\alpha_j(\vec{h})\right\}$ are given by

$$\exp\{\alpha_j(\vec{h})\} = \begin{cases} +1 \; , & j = 1, \cdots, \ell-1 \; , \; j \neq \ell - r \\ -1 \; , & j = \ell - r, \ell \, , \end{cases}$$

where $\alpha_1, \alpha_2, \cdots, \alpha_\ell$, are the simple roots of B_ℓ. It will be noted that the only dependence of S on the elements of g lies in the factor $\sqrt{g}$ which is defined to be the diagonal matrix such that

$$\left(\sqrt{g}\right)_{jj} = \begin{cases} 1 \text{ if } g_{jj} = 1 \, , \\ i \text{ if } g_{jj} = -1 \, . \end{cases}$$

Taking $\ell = 2$, $r = 2$ in the above theorem, we have a mapping of the canonical form of the Lie algebra B_2 onto $SO(4,1)$. In this case

$$T = \begin{bmatrix} 0 & i & 0 & -i & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & i & 0 & -i \\ 0 & 0 & 1 & 0 & 1 \\ \sqrt{2} & 0 & 0 & 0 & 0 \end{bmatrix} ,$$

$g = \mathrm{dig}(1,1,1,1,-1)$ and $\sqrt{g} = \mathrm{dig}(1,1,1,1,i)$.

Hence the similarity transformation S is

$$S = \sqrt{g}\,\mathbb{T} = \begin{bmatrix} 0 & i & 0 & -i & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & i & 0 & -i \\ 0 & 0 & 1 & 0 & 1 \\ i\sqrt{2} & 0 & 0 & 0 & 0 \end{bmatrix},$$

with inverse

$$S^{-1} = \frac{1}{\det S}\,\mathrm{Adj}\,S = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{i}{\sqrt{2}} \\ \frac{-i}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{-i}{2} & \frac{1}{2} & 0 \\ \frac{i}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

Using the similarity transformation S and constructing $S^{-1}\mathbb{L}_k S$, $k=0,1,2,3$ we find

$$S^{-1}\mathbb{L}_0 S = \begin{bmatrix} 0 & 0 & \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{-i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{-i}{\sqrt{2}} & 0 & 0 & 0 & 0 \end{bmatrix}, \quad S^{-1}\mathbb{L}_1 S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 & \frac{i}{2} \\ 0 & \frac{i}{2} & 0 & \frac{-i}{2} & 0 \\ 0 & 0 & \frac{-i}{2} & 0 & \frac{-i}{2} \\ 0 & \frac{i}{2} & 0 & \frac{-i}{2} & 0 \end{bmatrix},$$

$$S^{-1}\mathbb{L}_2 S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{2} & 0 & \frac{-1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{-1}{2} & 0 & \frac{-1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}, \quad S^{-1}\mathbb{L}_3 S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \end{bmatrix}.$$

The matrices $S^{-1}\mathbb{L}_k S$ form a linear combination of the basis elements $\Gamma_{5\text{-dim}}(\vec{h}_{\alpha_1})$, $\Gamma_{5\text{-dim}}(\vec{h}_{\alpha_2})$, $\Gamma_{5\text{-dim}}(\vec{e}_{\alpha_1})$, $\Gamma_{5\text{-dim}}(\vec{e}_{-\alpha_1})$, $\Gamma_{5\text{-dim}}(\vec{e}_{\alpha_2})$, $\Gamma_{5\text{-dim}}(\vec{e}_{-\alpha_2})$, $\Gamma_{5\text{-dim}}(\vec{e}_{\alpha_1+\alpha_2})$, $\Gamma_{5\text{-dim}}(\vec{e}_{-\alpha_1-\alpha_2})$, $\Gamma_{5\text{-dim}}(\vec{e}_{\alpha_1+2\alpha_2})$, $\Gamma_{5\text{-dim}}(\vec{e}_{-\alpha_1-2\alpha_2})$ of the Lie algebra B_2.

Thus

$$S^{-1}\mathbb{L}_k S = \alpha\,\Gamma_{5\text{-dim}}(\vec{h}_{\alpha_1}) + \beta\,\Gamma_{5\text{-dim}}(\vec{h}_{\alpha_2}) + \gamma\,\Gamma_{5\text{-dim}}(\vec{e}_{\alpha_1}) + \delta\,\Gamma_{5\text{-dim}}(\vec{e}_{-\alpha_1}) + \epsilon\,\Gamma_{5\text{-dim}}(\vec{e}_{\alpha_2}) + \zeta\,\Gamma_{5\text{-dim}}(\vec{e}_{-\alpha_2}) + \eta\,\Gamma_{5\text{-dim}}(\vec{e}_{\alpha_1+\alpha_2}) + \theta\,\Gamma_{5\text{-dim}}(\vec{e}_{-\alpha_1-\alpha_2}) + \kappa\,\Gamma_{5\text{-dim}}(\vec{e}_{\alpha_1+2\alpha_2}) + \lambda\,\Gamma_{5\text{-dim}}(\vec{e}_{-\alpha_1-2\alpha_2}),$$

where $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \kappa, \lambda$ coefficients. As an example we find the linear combination $S^{-1}\mathbb{L}_0 S$. We have

$$S^{-1}\mathbb{L}_0 S = \begin{bmatrix} 0, & \frac{\theta}{\sqrt{6}}, & \frac{\zeta}{\sqrt{6}}, & \frac{\eta}{\sqrt{6}}, & \frac{\epsilon}{\sqrt{6}} \\ \frac{-\eta}{\sqrt{6}}, & \frac{-\alpha}{6}, & \frac{\gamma}{\sqrt{6}}, & 0, & \frac{\kappa}{\sqrt{6}} \\ \frac{-\epsilon}{\sqrt{6}}, & \frac{-\delta}{\sqrt{6}}, & \frac{\alpha}{6}-\frac{\beta}{6}, & \frac{-\kappa}{\sqrt{6}}, & 0 \\ \frac{-\theta}{\sqrt{6}}, & 0, & \frac{\lambda}{\sqrt{6}}, & \frac{\alpha}{6}, & \frac{\delta}{\sqrt{6}} \\ \frac{-\zeta}{\sqrt{6}}, & \frac{-\lambda}{\sqrt{6}}, & 0, & \frac{-\gamma}{\sqrt{6}}, & -\frac{\alpha}{6}+\frac{\beta}{6} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{\iota}{\sqrt{2}} & 0 & \frac{\iota}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{-\iota}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{-\iota}{\sqrt{2}} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Equating corresponding elements we get

$$\zeta = \iota\sqrt{3}, \quad \epsilon = \iota\sqrt{3},$$

with all the other coefficients zero. Thus

$$\begin{aligned} S^{-1}\mathbb{L}_0 S &= \epsilon\,\Gamma_{5\text{-dim}}(\vec{e}_{\alpha_2}) + \zeta\,\Gamma_{5\text{-dim}}(\vec{e}_{-\alpha_2}) \\ &= \iota\sqrt{3}\,\Gamma_{5\text{-dim}}(\vec{e}_{\alpha_2}) + \iota\sqrt{3}\,\Gamma_{5\text{-dim}}(\vec{e}_{-\alpha_2}). \end{aligned}$$

Similarly we find:

$$\begin{aligned} S^{-1}\mathbb{L}_1 S &= \gamma\,\Gamma_{5\text{-dim}}(\vec{e}_{\alpha_1}) + \delta\,\Gamma_{5\text{-dim}}(\vec{e}_{-\alpha_1}) + \kappa\,\Gamma_{5\text{-dim}}(\vec{e}_{\alpha_1+2\alpha_2}) + \lambda\,\Gamma_{5\text{-dim}}(\vec{e}_{-\alpha_1-2\alpha_2}) \\ &= \frac{\iota\sqrt{3}}{\sqrt{2}}\Gamma_{5\text{-dim}}(\vec{e}_{\alpha_1}) - \frac{\iota\sqrt{3}}{\sqrt{2}}\Gamma_{5\text{-dim}}(\vec{e}_{-\alpha_1}) + \frac{\iota\sqrt{3}}{\sqrt{2}}\Gamma_{5\text{-dim}}(\vec{e}_{\alpha_1+2\alpha_2}) - \frac{\iota\sqrt{3}}{\sqrt{2}}\Gamma_{5\text{-dim}}(e_{-\alpha_1-2\alpha_2}), \end{aligned}$$

$$S^{-1}\mathbb{L}_2 S = \gamma\Gamma_{5\text{-dim}}(\vec{e}_{\alpha_1}) + \delta\Gamma_{5\text{-dim}}(\vec{e}_{-\alpha_1}) + \kappa\Gamma_{5\text{-dim}}(\vec{e}_{\alpha_1+2\alpha_2}) + \lambda\Gamma_{5\text{-dim}}(\vec{e}_{-\alpha_1-2\alpha_2})$$

$$= -\frac{\sqrt{3}}{\sqrt{2}}\Gamma_{5\text{-dim}}(\vec{e}_{\alpha_1}) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma_{5\text{-dim}}(\vec{e}_{-\alpha_1}) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma_{5\text{-dim}}(\vec{e}_{\alpha_1+2\alpha_2}) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma_{5\text{-dim}}(\vec{e}_{-\alpha_1-2\alpha_2}),$$

$$S^{-1}\mathbb{L}_3 S = \beta\Gamma_{5\text{-dim}}(\vec{h}_{\alpha_2})$$

$$= -6\iota\Gamma_{5\text{-dim}}(\vec{h}_{\alpha_2}).$$

The 20-dimensional realizations of the matrices $\mathbb{L}_k$ are given by the same linear combinations of the basis elements of B_2 as for the 5-dimensional representation, except that $\Gamma_{5\text{-dim}}(\vec{h}_{\alpha_1})$, $\Gamma_{5\text{-dim}}(\vec{h}_{\alpha_2}) \cdots \Gamma_{5\text{-dim}}(\vec{e}_{-\alpha_1-2\alpha_2})$, have to be replaced by the 20-dimensional matrices $\Gamma_{20\text{-dim}}(\vec{h}_{\alpha_1})$, $\Gamma_{20\text{-dim}}(\vec{h}_{\alpha_2}) \cdots \Gamma_{20\text{-dim}}(\vec{e}_{-\alpha_1-2\alpha_2})$ forming the basis of B_2 in the 20-dimensional representation.

Thus we have:

$$\mathbb{L}_0^{20\text{-dim}} = \epsilon\Gamma_{20\text{-dim}}(\vec{e}_{\alpha_2}) + \zeta\Gamma_{20\text{-dim}}(\vec{e}_{-\alpha_2}) = \iota\sqrt{3}\Gamma_{20\text{-dim}}(\vec{e}_{\alpha_2}) + \iota\sqrt{3}\Gamma_{20\text{-dim}}(\vec{e}_{-\alpha_2}),$$

$$\mathbb{L}_1^{20\text{-dim}} = \gamma\Gamma_{20\text{-dim}}(\vec{e}_{\alpha_1}) + \delta\Gamma_{20\text{-dim}}(\vec{e}_{-\alpha_1}) + \kappa\Gamma_{20\text{-dim}}(\vec{e}_{\alpha_1+2\alpha_2}) + \lambda\Gamma_{20\text{-dim}}(\vec{e}_{-\alpha_1-2\alpha_2})$$

$$= \iota\frac{\sqrt{3}}{\sqrt{2}}\Gamma_{20\text{-dim}}(\vec{e}_{\alpha_1}) - \iota\frac{\sqrt{3}}{\sqrt{2}}\Gamma_{20\text{-dim}}(\vec{e}_{-\alpha_1}) + \iota\frac{\sqrt{3}}{\sqrt{2}}\Gamma_{20\text{-dim}}(\vec{e}_{\alpha_1+2\alpha_2}) - \iota\frac{\sqrt{3}}{\sqrt{2}}\Gamma_{20\text{-dim}}(\vec{e}_{-\alpha_1-2\alpha_2}),$$

$$\mathbb{L}_2^{20\text{-dim}} = \gamma\Gamma_{20\text{-dim}}(\vec{e}_{\alpha_1}) + \delta\Gamma_{20\text{-dim}}(\vec{e}_{-\alpha_1}) + \kappa\Gamma_{20\text{-dim}}(\vec{e}_{\alpha_1+2\alpha_2}) + \lambda\Gamma_{20\text{-dim}}(\vec{e}_{-\alpha_1-2\alpha_2})$$

$$= -\frac{\sqrt{3}}{\sqrt{2}}\Gamma_{20\text{-dim}}(\vec{e}_{\alpha_1}) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma_{20\text{-dim}}(\vec{e}_{-\alpha_1}) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma_{20\text{-dim}}(\vec{e}_{\alpha_1+2\alpha_2}) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma_{20\text{-dim}}(\vec{e}_{-\alpha_1-2\alpha_2}),$$

$$\mathbb{L}_3^{20\text{-dim}} = \beta\Gamma_{20\text{-dim}}(\vec{h}_{\alpha_2}) = -6\iota\Gamma_{20\text{-dim}}(\vec{h}_{\alpha_2}).$$

Hence for the determination of the matrices $\mathbb{L}_k$, $(k = 0, 1, 2, 3)$ in the 20-dimensional representation the basis elements $\Gamma_{20\text{-dim}}(\vec{h}_{\alpha_1})$, $\Gamma_{20\text{-dim}}(\vec{h}_{\alpha_2}) \cdots \Gamma_{20\text{-dim}}(\vec{e}_{-\alpha_1-2\alpha_2})$, of B_2 are necessary. These we find in the next paragraph.

2: 20-dimensional basis of B_2 .

We determine in this paragraph the 20-dimensional basis elements of B_2 .

The fundamental weights of B_2 are

$$\lambda_1 = \alpha_1 + \alpha_2 \quad , \quad \lambda_2 = \frac{1}{2}\alpha_1 + \alpha_2 .$$

The highest weight is given by

$$\Lambda = q_1\lambda_1 + q_2\lambda_2 \quad ,$$

where q_1 and q_2 integers satisfying the relation

$$N = \frac{1}{6}(q_1+1)(q_2+1)(q_1+q_2+2)(2q_1+q_2+3).$$

N gives the degree of the representation with highest weight Λ . For $N=20$ the formula is satisfied when $q_1 = 0$, $q_2 = 3$. The highest weight then is

$$\Lambda = 0\lambda_1 + 3\lambda_2 = 3\left(\frac{1}{2}\alpha_1 + \alpha_2\right) = \frac{3}{2}\alpha_1 + 3\alpha_2 .$$

To find the other weights we proceed as follows. We start with the highest weight $\Lambda = \frac{3}{2}\alpha_1 + 3\alpha_2$ and evaluate the ratios

$$\frac{2(\Lambda, \alpha_1)}{(\alpha_1, \alpha_1)} \quad , \quad \frac{2(\Lambda, \alpha_2)}{(\alpha_2, \alpha_2)} .$$

Evaluation of $\frac{2(\Lambda, \alpha_1)}{(\alpha_1, \alpha_1)}$:

We have

$$\frac{2(\Lambda, \alpha_1)}{(\alpha_1, \alpha_1)} = \frac{2\left(\frac{3}{2}\alpha_1 + 3\alpha_2 , \alpha_1\right)}{(\alpha_1, \alpha_1)} = \frac{2\left(\left(\frac{3}{2}\alpha_1, \alpha_1\right) + (3\alpha_2, \alpha_1)\right)}{(\alpha_1, \alpha_1)}$$

$$= \frac{2\left(\frac{3}{2}(\alpha_1, \alpha_1) + 3(\alpha_2, \alpha_1)\right)}{(\alpha_1, \alpha_1)} .$$

For the evaluation of this ratio the values of the inner products (α_1,α_1), (α_2,α_1), are necessary. (α_1,α_1) was found earlier to be $(\alpha_1,\alpha_1)=\frac{1}{3}$. To find (α_2,α_1) we make use of the Cartan matrix of B_2 defined by

$$A_{jk}=\frac{2(\alpha_j,\alpha_k)}{(\alpha_j,\alpha_j)}\equiv\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}=\begin{bmatrix}2 & -1\\ -2 & 2\end{bmatrix}.$$

Setting

$$A_{21}=\frac{2(\alpha_2,\alpha_1)}{(\alpha_2,\alpha_2)}=-2,$$

we find after using $(\alpha_2,\alpha_2)=\frac{1}{6}$ given earlier,

$$(\alpha_2,\alpha_1)=-1\left(\frac{1}{6}\right)=-\frac{1}{6}.$$

Thus $\frac{2(\Lambda,\alpha_1)}{(\alpha_1,\alpha_1)}=0$ and the α_1-series of weights containing Λ is just $\{\Lambda=\frac{3}{2}\alpha_1+3\alpha_2\}$.

Evaluation of $\frac{2(\Lambda,\alpha_2)}{(\alpha_2,\alpha_2)}$:

We have

$$\frac{2(\Lambda,\alpha_2)}{(\alpha_2,\alpha_2)}=\frac{2(\frac{3}{2}\alpha_1+3\alpha_2,\ \alpha_2)}{(\alpha_2,\alpha_2)}=\frac{2\left(\frac{3}{2}(\alpha_1,\alpha_2)+3(\alpha_2,\alpha_2)\right)}{(\alpha_2,\alpha_2)}.$$

Using $(\alpha_2,\alpha_2)=\frac{1}{6}$ and $(\alpha_1,\alpha_2)=-\frac{1}{6}$ we find

$$\frac{2(\Lambda,\alpha_2)}{(\alpha_2,\alpha_2)}=3\equiv j-k.$$

Where j is the lower integer and k the upper integer, in the α_2-series of weights containing Λ i.e.

$$\{\Lambda-j\alpha_2,\ \Lambda-(j-1)\alpha_2\ \cdots\ \Lambda+(k-1)\alpha_2,\ \Lambda+k\alpha_2\}.$$

The equation $j-k=3$ accepts the solution $k=0$, $j=3$ and hence the α_2-series of weights containing Λ is

$$\{\Lambda-3\alpha_2,\ \Lambda-2\alpha_2,\ \Lambda-\alpha_2,\ \Lambda\}$$

or

$$\{\tfrac{3}{2}\alpha_1+3\alpha_2,\ \tfrac{3}{2}\alpha_1+2\alpha_2,\ \tfrac{3}{2}\alpha_1+1\alpha_2,\ \tfrac{3}{2}\alpha_1\}.$$

Choosing any weight of the above set for example $\frac{3}{2}\alpha_1$ and proceeding in the same way as for the highest weight we find some new weights of the representation. Continuing in the same way with every new weight we find finally the following sixteen weights

$$\Lambda \equiv M_1 = \frac{3}{2}\alpha_1 + 3\alpha_2 , \; M_2 = \frac{3}{2}\alpha_1 + 2\alpha_2 , \; M_3 = \frac{3}{2}\alpha_1 + 1\alpha_2 , \; M_4 = \frac{3}{2}\alpha_1 , \; M_5 = \frac{1}{2}\alpha_1 + 2\alpha_2 ,$$
$$M_6 = \frac{1}{2}\alpha_1 + \alpha_2 , \; M_7 = \frac{1}{2}\alpha_1 , \; M_8 = \frac{1}{2}\alpha_1 - \alpha_2 , \; M_9 = -\frac{1}{2}\alpha_1 + \alpha_2 , M_{10} = -\frac{1}{2}\alpha_1 , M_{11} = -\frac{1}{2}\alpha_1 - \alpha_2 ,$$
$$M_{12} = -\frac{3}{2}\alpha_1 , \; M_{13} = -\frac{1}{2}\alpha_1 - 2\alpha_2 , \; M_{14} = -\frac{3}{2}\alpha_1 - \alpha_2 , \quad M_{15} = -\frac{3}{2}\alpha_1 - 2\alpha_2 ,$$
$$M_{16} = -\frac{3}{2}\alpha_1 - 3\alpha_2 .$$

Because the number of the weights found is less than the degree of the representation, some weights are multiple. In general the method of calculating the multiplicity n_M of a weight M is given by Freudentals[21] recursion formula

$$\{(\Lambda+\delta,\Lambda+\delta)-(M+\delta,M+\delta)\} n_M = 2\sum_{k=1}^{\infty}\sum_{\alpha>0} n_{M+k\alpha}(M+k\alpha,\alpha) ,$$

where $\delta = \frac{1}{2}\sum_{\alpha>0}\alpha$, and the sum is taken over all the positive roots. In applying Freudentals formula we have to bear in mind that the multiplicity of the highest weight is always one i.e.

$$n_\Lambda = 1 .$$

In the case of B_2 the positive roots are

$$\alpha_1 \quad , \quad \alpha_2 \quad , \quad (\alpha_1 + \alpha_2) \quad , \quad (\alpha_1 + 2\alpha_2)$$

and δ is

$$\delta = \frac{1}{2}\{\alpha_1 + \alpha_2 + (\alpha_1 + \alpha_2) + (\alpha_1 + 2\alpha_2)\} = \frac{3}{2}\alpha_1 + 2\alpha_2 .$$

As an example let us find the multiplicity of the weight $M_2 = \frac{3}{2}\alpha_1 + 2\alpha_2$. Substituting in Freudental's formula

$$\Lambda = \frac{3}{2}\alpha_1 + 3\alpha_2 , \quad M = M_2 = \frac{3}{2}\alpha_1 + 2\alpha_2 ,$$

$$n_M \equiv n_{M_2} , \quad M + k\alpha \equiv M_2 + k\alpha , \quad \delta = \frac{3}{2}\alpha_1 + 2\alpha_2 ,$$

the left hand side gives,

$$\{(\Lambda+\delta,\Lambda+\delta)-(M_2+\delta,M_2+\delta)\} n_{M_2} = \{3(\alpha_1,\alpha_2) + 3(\alpha_2,\alpha_1) + 9(\alpha_2,\alpha_2)\} n_{M_2} =$$

$= \frac{1}{2} n_{M_2}$.

The right hand side using $n_\Lambda = 1$ gives,

$$2\sum_{\alpha>0}\sum_{k=1}^{1} n_{M_2+k\alpha}\left(M_2+k\alpha,\alpha\right) = 2\sum_{\alpha>0} n_{M_2+1\alpha_2}\left(M_2+1\alpha_2,\alpha_2\right)$$

$$= 2n_\Lambda\left(\tfrac{3}{2}\alpha_1+3\alpha_2,\alpha_2\right) = 2\cdot1\cdot\left(\tfrac{3}{2}(\alpha_1,\alpha_2)+3(\alpha_2,\alpha_2)\right) = \tfrac{1}{2} \cdot$$

Equating the two results we find $n_{M_2} = 1$. Similarly we find for the other multiplicities,

$n_\Lambda = 1$, $n_{M_3} = 1$, $n_{M_4} = 1$, $n_{M_5} = 1$, $n_{M_6} = 2$, $n_{M_7} = 2$,
$n_{M_8} = 1$, $n_{M_9} = 1$, $n_{M_{10}} = 2$, $n_{M_{11}} = 2$, $n_{M_{12}} = 1$, $n_{M_{13}} = 1$,
$n_{M_{14}} = 1$, $n_{M_{15}} = 1$, $n_{M_{16}} = 1$.

Taking into consideration the multiplicities of the weights and renaming them as Λ_j , $j = 1, 2, \cdots 20$, we have the following twenty weights:

$\Lambda \equiv \Lambda_1 = \frac{3}{2}\alpha_1+3\alpha_2$, $\Lambda_2 = \frac{3}{2}\alpha_1+2\alpha_2$, $\Lambda_3 = \frac{3}{2}\alpha_1+\alpha_2$, $\Lambda_4 = \frac{3}{2}\alpha_1$, $\Lambda_5 = \frac{1}{2}\alpha_1+2\alpha_2$,
$\Lambda_6 = \frac{1}{2}\alpha_1+\alpha_2$, $\Lambda_7 = \frac{1}{2}\alpha_1+\alpha_2$, $\Lambda_8 = \frac{1}{2}\alpha_1$, $\Lambda_9 = \frac{1}{2}\alpha_1$, $\Lambda_{10} = \frac{1}{2}\alpha_1-\alpha_2$,
$\Lambda_{11} = -\frac{1}{2}\alpha_1+\alpha_2$, $\Lambda_{12} = -\frac{1}{2}\alpha_1$, $\Lambda_{13} = -\frac{1}{2}\alpha_1$, $\Lambda_{14} = -\frac{1}{2}\alpha_1-\alpha_2$, $\Lambda_{15} = -\frac{1}{2}\alpha_1-\alpha_2$,
$\Lambda_{16} = -\frac{3}{2}\alpha_1$, $\Lambda_{17} = -\frac{1}{2}\alpha_1-2\alpha_2$, $\Lambda_{18} = -\frac{3}{2}\alpha_1-\alpha_2$, $\Lambda_{19} = -\frac{3}{2}\alpha_1-2\alpha_2$,
$\Lambda_{20} = -\frac{3}{2}\alpha_1-3\alpha_2$.

These weights can be arranged in a weight diagram which we omit.

Using the test of reflections we can check that the above weights are all the weights of the representation. We use as lines of reflection the lines perpendicular to the roots α_1, α_2 , $\alpha_1+\alpha_2$, $\alpha_1+2\alpha_2$. The test of reflections states that for every non-zero root α relative to the space of roots, a linear transformation S_α in the linear space of weights is defined, such that

$$S_\alpha(\Lambda_i) \equiv \Lambda_i - \frac{2(\Lambda_i,\alpha)}{(\alpha,\alpha)}\alpha ,$$

for any weight Λ_i. As an example let us find the reflection of the highest weight $\Lambda = \frac{3}{2}\alpha_1 + 3\alpha_2$ through the line perpendicular to the root $\alpha = \alpha_1 + 2\alpha_2$. We have

$$S_\alpha(\Lambda) = \left(\frac{3}{2}\alpha_1 + 3\alpha_2\right) - \frac{2\left(\frac{3}{2}\alpha_1 + 3\alpha_2 \, , \, \alpha_1 + 2\alpha_2\right)}{(\alpha_1 + 2\alpha_2 \, , \, \alpha_1 + 2\alpha_2)} (\alpha_1 + 2\alpha_2)$$

$$= -\frac{3}{2}\alpha_1 - 3\alpha_2 \equiv \Lambda_{20} \; .$$

<u>Basis of the Cartan subalgebra:</u>

We find now the matrices $\Gamma(\vec{h}_{\alpha_1})$, $\Gamma(\vec{h}_{\alpha_2})$, forming the basis of the Cartan subalgebra. In doing so we find first the matrices $\Gamma(\vec{h}_1)$, $\Gamma(\vec{h}_2)$.

We require:

a) $\Gamma(\vec{h}_j)_{pp} = \Lambda_p(\vec{h}_j)$, $j = 1, 2$ $\quad p = 1, 2, \cdots 20$,

where Λ_p is the p^{th} weight of the representation.

b) $\alpha_k(\vec{h}_j) = A_{jk}$, where A_{jk} are the elements of the Cartan matrix of B_2 .

i) $\underline{\Gamma(\vec{h}_{\alpha_1})}$:

The elements of $\Gamma(\vec{h}_1)$ are:

$\Gamma(\vec{h}_1)_{1,1} = \Lambda_1(\vec{h}_1) = \frac{3}{2}\alpha_1(\vec{h}_1) + 3\alpha_2(\vec{h}_1) = \frac{3}{2}A_{11} + 3A_{12} = \frac{3}{2}(2) + 3(-1) = 0$,

$\Gamma(\vec{h}_1)_{2,2} = 1$, $\Gamma(\vec{h}_1)_{3,3} = 2$, $\Gamma(\vec{h}_1)_{44} = 3$, $\Gamma(\vec{h}_1)_{5,5} = -1$, $\Gamma(\vec{h}_1)_{6,6} = 0$,

$\Gamma(\vec{h}_1)_{7,7} = 0$, $\Gamma(\vec{h}_1)_{8,8} = 1$, $\Gamma(\vec{h}_1)_{9,9} = 1$, $\Gamma(\vec{h}_1)_{10,10} = 2$, $\Gamma(\vec{h}_1)_{11,11} = -2$, $\Gamma(\vec{h}_1)_{12,12} = -1$

$\Gamma(\vec{h}_1)_{13,13} = -1$, $\Gamma(\vec{h}_1)_{14,14} = 0$, $\Gamma(\vec{h}_1)_{15,15} = 0$, $\Gamma(\vec{h}_1)_{16,16} = -3$, $\Gamma(\vec{h}_1)_{17,17} = 1$

$\Gamma(\vec{h}_1)_{18,18} = -2$, $\Gamma(\vec{h}_1)_{19,19} = -1$, $\Gamma(\vec{h}_1)_{20,20} = 0$,

with all the off diagonal elements zero.

Thus

$$\Gamma(\vec{h}_1) = \text{dig}\,(0, 1, 2, 3, -1, 0, 0; 1, 1, 2, -2, -1, -1, 0, 0, -3, 1, -2, -1, 0)$$

and

$$\Gamma(\vec{h}_{\alpha_1}) = \frac{1}{6}\Gamma(\vec{h}_1).$$

ii) $\underline{\Gamma(\vec{h}_{\alpha_2})}$:

The elements of $\Gamma(\vec{h}_2)$ are:

$$\Gamma(\vec{h}_2)_{1,1} = \Lambda_1(\vec{h}_2) = \tfrac{3}{2}\alpha_1(\vec{h}_2) + 3\alpha_2(\vec{h}_2) = \tfrac{3}{2}A_{2,1} + 3A_{2,2} = \tfrac{3}{2}(-2) + 3\cdot 2 = -3 + 6 = 3,$$

$$\Gamma(\vec{h}_2)_{2,2} = 1, \Gamma(\vec{h}_2)_{3,3} = -1, \Gamma(\vec{h}_2)_{44} = -3, \Gamma(\vec{h}_2)_{5,5} = 3, \Gamma(\vec{h}_2)_{6,6} = 1,$$

$$\Gamma(\vec{h}_2)_{77} = 1, \Gamma(\vec{h}_2)_{8,8} = -1, \Gamma(\vec{h}_2)_{9,9} = -1, \Gamma(\vec{h}_2)_{10,10} = -3, \Gamma(\vec{h}_2)_{11,11} = 3,$$

$$\Gamma(\vec{h}_2)_{12,12} = 1, \Gamma(\vec{h}_2)_{13,13} = 1, \Gamma(\vec{h}_2)_{14,14} = -1, \Gamma(\vec{h}_2)_{15,15} = -1, \Gamma(\vec{h}_2)_{16,16} = 3,$$

$$\Gamma(\vec{h}_2)_{17,17} = -3, \Gamma(\vec{h}_2)_{18,18} = 1, \Gamma(\vec{h}_2)_{19,19} = -1, \Gamma(\vec{h}_2)_{20,20} = -3,$$

with all the off diagonal elements zero.

Thus

$$\Gamma(\vec{h}_2) = \mathrm{dig}(3,1,-1,-3,3,1,1,-1,-1,-3,3,1,1,-1,-1,3,-3,1,-1,-3)$$

and

$$\Gamma(\vec{h}_{\alpha_2}) = \tfrac{1}{12}\Gamma(\vec{h}_2).$$

Notice that $\Gamma(\vec{h}_{\alpha_1}), \Gamma(\vec{h}_{\alpha_2})$ have vanishing trace.

<u>The other basis elements of B_2 :</u>

By definition

$$[\Gamma(\vec{h}), \Gamma(\vec{e}_\alpha)] = -\alpha(\vec{h})\Gamma(\vec{e}_\alpha).$$

From this we have

$$\{\Gamma(\vec{h})_{pp} - \Gamma(\vec{h})_{qq} + \alpha(\vec{h})\}\Gamma(\vec{e}_\alpha)_{pq} = 0.$$

The matrix element $\Gamma(\vec{e}_\alpha)_{pq} \neq 0$ only if

$$\Gamma(\vec{h})_{pp} - \Gamma(\vec{h})_{qq} = -\alpha(\vec{h}),$$

or if the difference between the p^{th} weight and the q^{th} weight is equal to $-\alpha(\vec{h})$ i.e.

$$\Lambda_p(\vec{h}) - \Lambda_q(\vec{h}) = -\alpha(\vec{h}).$$

These differences give the positions (p,q) at which the matrix $\Gamma(\vec{e}_\alpha)$ has elements different from zero.

i) $\underline{\Gamma(\vec{e}_{\alpha_1})}$:

Using the weights $\Lambda_1, \Lambda_2, \cdots \Lambda_{20}$, found earlier and constructing the differences $\Lambda_p(\vec{h})-\Lambda_q(\vec{h})$ and selecting out of them, those for which the relation $\Lambda_p(\vec{h})-\Lambda_q(\vec{h})=-\alpha_1(\vec{h})$ is satisfied, we get:

$\Lambda_5(\vec{h})-\Lambda_2(\vec{h})=-\alpha_1(\vec{h})$, $\Lambda_6(\vec{h})-\Lambda_3(\vec{h})=-\alpha_1(\vec{h})$,

$\Lambda_7(\vec{h})-\Lambda_3(\vec{h})=-\alpha_1(\vec{h})$, $\Lambda_8(\vec{h})-\Lambda_4(\vec{h})=-\alpha_1(\vec{h})$,

$\Lambda_9(\vec{h})-\Lambda_4(\vec{h})=-\alpha_1(\vec{h})$, $\Lambda_{11}(\vec{h})-\Lambda_6(\vec{h})=-\alpha_1(\vec{h})$,

$\Lambda_{11}(\vec{h})-\Lambda_7(\vec{h})=-\alpha_1(\vec{h})$, $\Lambda_{12}(\vec{h})-\Lambda_8(\vec{h})=-\alpha_1(\vec{h})$,

$\Lambda_{12}(\vec{h})-\Lambda_9(\vec{h})=-\alpha_1(\vec{h})$, $\Lambda_{13}(\vec{h})-\Lambda_8(\vec{h})=-\alpha_1(\vec{h})$,

$\Lambda_{13}(\vec{h})-\Lambda_9(\vec{h})=-\alpha_1(\vec{h})$, $\Lambda_{14}(\vec{h})-\Lambda_{10}(\vec{h})=-\alpha_1(\vec{h})$,

$\Lambda_{15}(\vec{h})-\Lambda_{10}(\vec{h})=-\alpha_1(\vec{h})$, $\Lambda_{16}(\vec{h})-\Lambda_{12}(\vec{h})=-\alpha_1(\vec{h})$,

$\Lambda_{16}(\vec{h})-\Lambda_{13}(\vec{h})=-\alpha_1(\vec{h})$, $\Lambda_{18}(\vec{h})-\Lambda_{14}(\vec{h})=-\alpha_1(\vec{h})$,

$\Lambda_{18}(\vec{h})-\Lambda_{15}(\vec{h})=-\alpha_1(\vec{h})$, $\Lambda_{19}(\vec{h})-\Lambda_{17}(\vec{h})=-\alpha_1(\vec{h})$.

Thus the non-zero elements of $\Gamma(\vec{e}_{\alpha_1})$ appear at the positions:

$(p, q)=(5,2), (6,3), (7,3), (8,4), (9,4), (11,6), (11,7), (12,8), (12,9), (13,8),$

$(13,9), (14,10), (15,10), (16,12), (16,13), (18,14), (18,15), (19,17).$

We call the non-zero elements of $\Gamma(\vec{e}_{\alpha_1})$ respectively

$e_{5,2}, e_{6,3}, e_{7,3}, e_{8,4}, e_{9,4}, e_{11,6}, e_{11,7}, e_{12,8}, e_{12,9}, e_{13,8}, e_{13,9},$

$e_{14,10}, e_{15,10}, e_{16,12}, e_{16,13}, e_{18,14}, e_{18,15}, e_{19,17}.$

ii) $\underline{\Gamma(\vec{e}_{-\alpha_1})}$:

We choose $\Gamma(\vec{e}_{-\alpha_1})=-\Gamma^{tr}(\vec{e}_{\alpha_1})$. Thus the non-zero elements of $\Gamma(\vec{e}_{-\alpha_1})$ appear at the positions:

$(p, q)=(2,5), (3,6), (3,7), (4,8), (4,9), (6,11), (7,11), (8,12), (9,12), (8,13),$

$(9,13), (10,14), (10,15), (12,16), (13,16), (14,18), (15,18), (17,19).$

We call these elements respectively,

$e_{2,5} = -e_{5,2}$, $e_{3,6} = -e_{6,3}$, $e_{3,7} = -e_{7,3}$, $e_{4,8} = -e_{8,4}$, $e_{4,9} = -e_{9,4}$,
$e_{6,11} = -e_{11,6}$, $e_{7,11} = -e_{11,7}$, $e_{8,12} = -e_{12,8}$, $e_{9,12} = -e_{12,9}$, $e_{8,13} = -e_{13,8}$, $e_{9,13} = -e_{13,9}$
$e_{10,14} = -e_{14,10}$, $e_{10,15} = -e_{15,10}$, $e_{12,16} = -e_{16,12}$, $e_{13,16} = -e_{16,13}$, $e_{14,18} = -e_{18,14}$, $e_{15,18} = -e_{18,15}$
$e_{17,19} = -e_{19,17}$.

iii) $\underline{\Gamma(\vec{e}_{\alpha_2}):}$

Constructing again the differences $\Lambda_p(\vec{h}) - \Lambda_q(\vec{h})$, and selecting those for which $\Lambda_p(\vec{h}) - \Lambda_q(\vec{h}) = -\alpha_2(\vec{h})$ is satisfied, we get:

$$\Lambda_2(\vec{h}) - \Lambda_1(\vec{h}) = -\alpha_2(\vec{h}) \quad , \quad \Lambda_3(\vec{h}) - \Lambda_2(\vec{h}) = -\alpha_2(\vec{h})$$

$$\Lambda_4(\vec{h}) - \Lambda_3(\vec{h}) = -\alpha_2(\vec{h}) \quad , \quad \Lambda_6(\vec{h}) - \Lambda_5(\vec{h}) = -\alpha_2(\vec{h})$$

$$\Lambda_7(\vec{h}) - \Lambda_5(\vec{h}) = -\alpha_2(\vec{h}) \quad , \quad \Lambda_8(\vec{h}) - \Lambda_6(\vec{h}) = -\alpha_2(\vec{h})$$

$$\Lambda_8(\vec{h}) - \Lambda_7(\vec{h}) = -\alpha_2(\vec{h}) \quad , \quad \Lambda_9(\vec{h}) - \Lambda_6(\vec{h}) = -\alpha_2(\vec{h})$$

$$\Lambda_9(\vec{h}) - \Lambda_7(\vec{h}) = -\alpha_2(\vec{h}) \quad , \quad \Lambda_{10}(\vec{h}) - \Lambda_8(\vec{h}) = -\alpha_2(\vec{h})$$

$$\Lambda_{10}(\vec{h}) - \Lambda_9(\vec{h}) = -\alpha_2(\vec{h}) \quad , \quad \Lambda_{12}(\vec{h}) - \Lambda_{11}(\vec{h}) = -\alpha_2(\vec{h})$$

$$\Lambda_{13}(\vec{h}) - \Lambda_{11}(\vec{h}) = -\alpha_2(\vec{h}) \quad , \quad \Lambda_{14}(\vec{h}) - \Lambda_{12}(\vec{h}) = -\alpha_2(\vec{h})$$

$$\Lambda_{14}(\vec{h}) - \Lambda_{13}(\vec{h}) = -\alpha_2(\vec{h}) \quad , \quad \Lambda_{15}(\vec{h}) - \Lambda_{12}(\vec{h}) = -\alpha_2(\vec{h})$$

$$\Lambda_{15}(\vec{h}) - \Lambda_{13}(\vec{h}) = -\alpha_2(\vec{h}) \quad , \quad \Lambda_{17}(\vec{h}) - \Lambda_{14}(\vec{h}) = -\alpha_2(\vec{h})$$

$$\Lambda_{17}(\vec{h}) - \Lambda_{15}(\vec{h}) = -\alpha_2(\vec{h}) \quad , \quad \Lambda_{18}(\vec{h}) - \Lambda_{16}(\vec{h}) = -\alpha_2(\vec{h})$$

$$\Lambda_{19}(\vec{h}) - \Lambda_{18}(\vec{h}) = -\alpha_2(\vec{h}) \quad , \quad \Lambda_{20}(\vec{h}) - \Lambda_{19}(\vec{h}) = -\alpha_2(\vec{h}).$$

Thus $\Gamma(\vec{e}_{\alpha_2})$ has non-zero elements at the positions:

$(p,q) =$ (2,1), (3,2), (4,3), (6,5), (7,5), (8,6), (8,7), (9,6), (9,7), (10,8), (10,9)
(12,11), (13,11), (14,12), (14,13), (15,12), (15,13), (17,14), (17,15), (18,16)
(19,18), (20,19).

We call the non-zero elements of $\Gamma(\vec{e}_{\alpha_2})$ respectively

$\epsilon_{2,1}$, $\epsilon_{3,2}$, $\epsilon_{4,3}$, $\epsilon_{6,5}$, $\epsilon_{7,5}$, $\epsilon_{8,6}$, $\epsilon_{8,7}$, $\epsilon_{9,6}$, $\epsilon_{9,7}$, $\epsilon_{10,8}$, $\epsilon_{10,9}$, $\epsilon_{12,11}$

$\epsilon_{13,11}$, $\epsilon_{14,12}$, $\epsilon_{14,13}$, $\epsilon_{15,12}$, $\epsilon_{15,13}$, $\epsilon_{17,14}$, $\epsilon_{17,15}$, $\epsilon_{18,16}$, $\epsilon_{19,18}$, $\epsilon_{20,19}$.

iv) $\underline{\Gamma(\vec{e}_{-\alpha_2}):}$

We choose $\Gamma(\vec{e}_{-\alpha_2}) = -\Gamma^{tr}(\vec{e}_{\alpha_2})$. Thus $\Gamma(\vec{e}_{-\alpha_2})$ has non-zero elements at the positions:

$(p,q) = (1,2)$, $(2,3)$, $(3,4)$, $(5,6)$, $(5,7)$, $(6,8)$, $(7,8)$, $(6,9)$, $(7,9)$, $(8,10)$

$(9,10)$, $(11,12)$, $(11,13)$, $(12,14)$, $(13,14)$, $(12,15)$, $(13,15)$, $(14,17)$, $(15,17)$

$(16,18)$, $(18,19)$, $(19,20)$.

These elements we call,

$\epsilon_{1,2} = -\epsilon_{2,1}$, $\epsilon_{2,3} = -\epsilon_{3,2}$, $\epsilon_{3,4} = -\epsilon_{4,3}$, $\epsilon_{5,6} = -\epsilon_{6,5}$, $\epsilon_{5,7} = -\epsilon_{7,5}$, $\epsilon_{6,8} = -\epsilon_{8,6}$

$\epsilon_{7,8} = -\epsilon_{8,7}$, $\epsilon_{6,9} = -\epsilon_{9,6}$, $\epsilon_{7,9} = -\epsilon_{9,7}$, $\epsilon_{8,10} = -\epsilon_{10,8}$, $\epsilon_{9,10} = -\epsilon_{10,9}$, $\epsilon_{11,12} = -\epsilon_{12,11}$

$\epsilon_{11,13} = -\epsilon_{13,11}$, $\epsilon_{12,14} = -\epsilon_{14,12}$, $\epsilon_{13,14} = -\epsilon_{14,13}$, $\epsilon_{12,15} = -\epsilon_{15,12}$, $\epsilon_{13,15} = -\epsilon_{15,13}$

$\epsilon_{14,17} = -\epsilon_{17,14}$, $\epsilon_{15,17} = -\epsilon_{17,15}$, $\epsilon_{16,18} = -\epsilon_{18,16}$, $\epsilon_{18,19} = -\epsilon_{19,18}$, $\epsilon_{19,20} = -\epsilon_{20,19}$.

v) $\underline{\Gamma(\vec{e}_{\alpha_1+\alpha_2}):}$

To find $\Gamma(\vec{e}_{\alpha_1+\alpha_2})$ we use the formula,

$$\Gamma(\vec{e}_{\alpha_1+\alpha_2}) = \frac{1}{N_{\alpha_1,\alpha_2}} \left\{ \Gamma(\vec{e}_{\alpha_1}) \cdot \Gamma(\vec{e}_{\alpha_2}) - \Gamma(\vec{e}_{\alpha_2}) \Gamma(\vec{e}_{\alpha_1}) \right\} ,$$

where $N_{\alpha_1,\alpha_2} = \pm \frac{1}{\sqrt{6}}$. In the sequel we use N_{α_1,α_2} with the positive sign. The elements of $\Gamma(\vec{e}_{\alpha_1+\alpha_2})$ are functions of the elements e_{ij} and ϵ_{ij} .

vi) $\underline{\Gamma(\vec{e}_{-\alpha_1-\alpha_2}):}$

The matrix $\Gamma(\vec{e}_{-\alpha_1-\alpha_2})$ is determined either by using

$$\Gamma(\vec{e}_{-\alpha_1-\alpha_2}) = \frac{1}{N_{-\alpha_1,-\alpha_2}} \left\{ \Gamma(\vec{e}_{-\alpha_1}) \Gamma(\vec{e}_{-\alpha_2}) - \Gamma(\vec{e}_{-\alpha_2}) \Gamma(\vec{e}_{-\alpha_1}) \right\}$$

or by taking $\Gamma(\vec{e}_{-\alpha_1-\alpha_2}) = -\Gamma^{tr}(\vec{e}_{\alpha_1+\alpha_2})$. The elements of $\Gamma(\vec{e}_{-\alpha_1-\alpha_2})$ are again functions of the elements e_{ij} , ϵ_{ij} .

vii) $\Gamma(\vec{e}_{\alpha_1+2\alpha_2})$:

To find $\Gamma(\vec{e}_{\alpha_1+2\alpha_2})$ we use the formula

$$\Gamma(\vec{e}_{\alpha_1+2\alpha_2}) = \frac{1}{N_{\alpha_2,\alpha_1+\alpha_2}} \left\{ \Gamma(\vec{e}_{\alpha_2})\Gamma(\vec{e}_{\alpha_1+\alpha_2}) - \Gamma(\vec{e}_{\alpha_1+\alpha_2})\Gamma(\vec{e}_{\alpha_2}) \right\},$$

where $N_{\alpha_2,\alpha_1+\alpha_2} = \pm \frac{1}{\sqrt{6}}$. In the sequel we use the positive sign. The elements of $\Gamma(\vec{e}_{\alpha_1+2\alpha_2})$ are again functions of the elements e_{ij}, ϵ_{ij}.

viii) $\Gamma(\vec{e}_{-\alpha_1-2\alpha_2})$:

The matrix $\Gamma(e_{-\alpha_1-2\alpha_2})$ is determined either by using

$$\Gamma(\vec{e}_{-\alpha_1-2\alpha_2}) = \frac{1}{N_{-\alpha_2,-\alpha_1-\alpha_2}} \left\{ \Gamma(\vec{e}_{-\alpha_2})\Gamma(\vec{e}_{-\alpha_1-\alpha_2}) - \Gamma(\vec{e}_{-\alpha_1-\alpha_2})\Gamma(\vec{e}_{-\alpha_2}) \right\}$$

or by taking $\Gamma(\vec{e}_{-\alpha_1-2\alpha_2}) = -\Gamma^{tr}(\vec{e}_{\alpha_1+2\alpha_2})$. Its elements are functions of the elements e_{ij}, ϵ_{ij}.

To find the values of e_{ij}, ϵ_{ij} we make use of the commutation relations of the Lie algebra; namely,

$$[\Gamma(\vec{e}_{\alpha_1}), \Gamma(\vec{e}_{-\alpha_1})]_- = \Gamma(\vec{h}_{\alpha_1}),$$

$$[\Gamma(\vec{e}_{\alpha_2}), \Gamma(\vec{e}_{-\alpha_2})]_- = \Gamma(\vec{h}_{\alpha_2}),$$

$$[\Gamma(\vec{e}_{\alpha_1}), \Gamma(\vec{e}_{-\alpha_2})]_- = 0 ,$$

(This vanishes because $N_{\alpha_1,-\alpha_2} = 0$ since $\alpha_1-\alpha_2$ is not a root of B_2.)

$$[\Gamma(\vec{e}_{\alpha_1+\alpha_2}), \Gamma(\vec{e}_{-\alpha_1-\alpha_2})]_- = \Gamma(\vec{h}_{\alpha_1}) + \Gamma(\vec{h}_{\alpha_2}),$$

$$[\Gamma(\vec{e}_{\alpha_1}), \Gamma(\vec{e}_{\alpha_1+\alpha_2})]_- = 0, \quad [\Gamma(\vec{e}_{-\alpha_1}), \Gamma(\vec{e}_{\alpha_1+2\alpha_2})]_- = 0 ,$$

$$[\Gamma(\vec{e}_{\alpha_2}), \Gamma(\vec{e}_{\alpha_1+2\alpha_2})]_- = 0 , \quad [\Gamma(\vec{e}_{\alpha_1}), \Gamma(\vec{e}_{\alpha_1+2\alpha_2})]_- = 0 ,$$

$$[\Gamma(\vec{e}_{\alpha_1+\alpha_2}), \Gamma(\vec{e}_{\alpha_1+2\alpha_2})]_- = 0 ,$$

$$[\Gamma(\vec{e}_{\alpha_1+2\alpha_2}), \Gamma(\vec{e}_{-\alpha_1-2\alpha_2})]_- = \Gamma(\vec{h}_{\alpha_1}) + 2\Gamma(\vec{h}_{\alpha_2}).$$

From each one of these commutation relations we derive a set of equations satisfied by the elements e_{ij}, ϵ_{ij}. Solving these sets of equations simultaneously we find,

$\epsilon_{2,1} = \pm \frac{1}{2}$, $\epsilon_{4,3} = \pm \frac{1}{2}$, $\epsilon_{18,16} = \pm \frac{1}{2}$, $\epsilon_{20,19} = \pm \frac{1}{2}$, $\epsilon_{3,2} = \pm \frac{1}{\sqrt{3}}$, $\epsilon_{19,18} = \pm \frac{1}{\sqrt{3}}$,

$\epsilon_{17,15} = 0$, $\epsilon_{17,14} = \pm \frac{1}{2}$, $\epsilon_{12,11} = 0$, $\epsilon_{13,11} = \pm \frac{1}{2}$, $\epsilon_{14,12} = 0$, $\epsilon_{15,12} = \pm \frac{1}{2\sqrt{3}}$, $\epsilon_{15,13} = 0$,

$\epsilon_{14,11} = \pm \frac{1}{\sqrt{3}}$, $\epsilon_{6,5} = 0$, $\epsilon_{7,5} = \pm \frac{1}{2}$, $\epsilon_{8,6} = 0$, $\epsilon_{9,6} = \pm \frac{1}{2\sqrt{3}}$, $\epsilon_{9,7} = 0$, $\epsilon_{8,7} = \pm \frac{1}{\sqrt{3}}$,

$\epsilon_{10,8} = \pm \frac{1}{2}$, $\epsilon_{10,9} = 0$,

$e_{5,2} = \pm \frac{1}{\sqrt{6}}$, $e_{19,17} = \pm \frac{1}{\sqrt{6}}$, $e_{18,14} = \pm \frac{\sqrt{2}}{3}$, $e_{18,15} = \pm \frac{1}{3}$, $e_{15,10} = \pm \frac{1}{3}$, $e_{11,17} = \pm \frac{\sqrt{2}}{3}$,

$e_{11,6} = \pm \frac{1}{3}$, $e_{7,3} = \pm \frac{\sqrt{2}}{3}$, $e_{6,3} = \pm \frac{1}{3}$, $e_{16,13} = \pm \frac{1}{\sqrt{6}}$, $e_{16,12} = \pm \frac{1}{\sqrt{3}}$, $e_{8,4} = \pm \frac{1}{\sqrt{6}}$,

$e_{9,4} = \pm \frac{1}{\sqrt{3}}$, $e_{14,10} = \pm \frac{\sqrt{2}}{3}$, $e_{13,9} = \pm \frac{1}{3\sqrt{3}}$, $e_{12,9} = \pm \frac{5}{3\sqrt{6}}$, $e_{12,8} = \pm \frac{1}{3\sqrt{3}}$,

$e_{13,8} = \pm \frac{2\sqrt{2}}{3\sqrt{3}}$.

Choosing for the ϵ_{ij} the positive sign this fixes the sign of e_{ij} to positive.

Thus the 20-dimensional basis matrices $\Gamma(\vec{e}_{\alpha_1})$, $\Gamma(e_{-\alpha_1}), \Gamma(e_{\alpha_2}), \ldots, \Gamma(\vec{e}_{-\alpha_1-2\alpha_2})$ of B_2 are:

$\underline{\Gamma(\vec{e}_{\alpha_1}) :}$

The non-zero elements of $\Gamma(\vec{e}_{\alpha_1})$ are:

$\Gamma_{5,2}(\vec{e}_{\alpha_1}) = \frac{1}{\sqrt{6}}$, $\Gamma_{6,3}(\vec{e}_{\alpha_1}) = \frac{1}{3}$, $\Gamma_{7,3}(\vec{e}_{\alpha_1}) = \frac{\sqrt{2}}{3}$, $\Gamma_{8,4}(\vec{e}_{\alpha_1}) = \frac{1}{\sqrt{6}}$, $\Gamma_{9,4}(\vec{e}_{\alpha_1}) = \frac{1}{\sqrt{3}}$,

$\Gamma_{11,6}(\vec{e}_{\alpha_1}) = \frac{1}{3}$, $\Gamma_{11,7}(\vec{e}_{\alpha_1}) = \frac{\sqrt{2}}{3}$, $\Gamma_{12,8}(\vec{e}_{\alpha_1}) = \frac{1}{3\sqrt{3}}$, $\Gamma_{12,9}(\vec{e}_{\alpha_1}) = \frac{5}{3\sqrt{6}}$,

$\Gamma_{13,8}(\vec{e}_{\alpha_1}) = \frac{2\sqrt{2}}{3\sqrt{3}}$, $\Gamma_{13,9}(\vec{e}_{\alpha_1}) = \frac{1}{3\sqrt{3}}$, $\Gamma_{14,10}(\vec{e}_{\alpha_1}) = \frac{\sqrt{2}}{3}$, $\Gamma_{15,10}(\vec{e}_{\alpha_1}) = \frac{1}{3}$,

$\Gamma_{16,12}(\vec{e}_{\alpha_1}) = \frac{1}{\sqrt{3}}$, $\Gamma_{16,13}(\vec{e}_{\alpha_1}) = \frac{1}{\sqrt{6}}$, $\Gamma_{18,14}(\vec{e}_{\alpha_1}) = \frac{\sqrt{2}}{3}$, $\Gamma_{18,15}(\vec{e}_{\alpha_1}) = \frac{1}{3}$,

$\Gamma_{19,17}(\vec{e}_{\alpha_1}) = \frac{1}{\sqrt{6}}$.

$\Gamma(\vec{e}_{-\alpha_1})$:

The non-zero elements of $\Gamma(\vec{e}_{-\alpha_1})$ **are:**

$\Gamma_{2,5}(\vec{e}_{-\alpha_1}) = -\frac{1}{\sqrt{6}}$, $\Gamma_{3,6}(\vec{e}_{-\alpha_1}) = -\frac{1}{3}$, $\Gamma_{3,7}(\vec{e}_{-\alpha_1}) = -\frac{\sqrt{2}}{3}$, $\Gamma_{4,8}(\vec{e}_{-\alpha_1}) = \frac{-1}{\sqrt{6}}$, $\Gamma_{4,9}(\vec{e}_{-\alpha_1}) = -\frac{1}{\sqrt{3}}$, $\Gamma_{6,11}(\vec{e}_{-\alpha_1}) = -\frac{1}{3}$, $\Gamma_{7,11}(\vec{e}_{-\alpha_1}) = -\frac{\sqrt{2}}{3}$, $\Gamma_{8,12}(\vec{e}_{-\alpha_1}) = -\frac{1}{3\sqrt{3}}$, $\Gamma_{8,13}(\vec{e}_{-\alpha_1}) = -\frac{2\sqrt{2}}{3\sqrt{3}}$, $\Gamma_{9,12}(\vec{e}_{-\alpha_1}) = -\frac{5}{3\sqrt{6}}$, $\Gamma_{9,13}(\vec{e}_{-\alpha_1}) = -\frac{1}{3\sqrt{3}}$, $\Gamma_{10,14}(\vec{e}_{-\alpha_1}) = -\frac{\sqrt{2}}{3}$, $\Gamma_{10,15}(\vec{e}_{-\alpha_1}) = -\frac{1}{3}$, $\Gamma_{12,16}(\vec{e}_{-\alpha_1}) = -\frac{1}{\sqrt{3}}$, $\Gamma_{13,16}(\vec{e}_{-\alpha_1}) = -\frac{1}{\sqrt{6}}$, $\Gamma_{14,18}(\vec{e}_{-\alpha_1}) = -\frac{\sqrt{2}}{3}$, $\Gamma_{15,18}(\vec{e}_{-\alpha_1}) = -\frac{1}{3}$, $\Gamma_{17,19}(\vec{e}_{-\alpha_1}) = -\frac{1}{\sqrt{6}}$.

$\Gamma(\vec{e}_{\alpha_2})$:

The non-zero elements of $\Gamma(\vec{e}_{\alpha_2})$ **are:**

$\Gamma_{2,1}(\vec{e}_{\alpha_2}) = \frac{1}{2}$, $\Gamma_{3,2}(\vec{e}_{\alpha_2}) = \frac{1}{\sqrt{3}}$, $\Gamma_{4,3}(\vec{e}_{\alpha_2}) = \frac{1}{2}$, $\Gamma_{7,5}(\vec{e}_{\alpha_2}) = \frac{1}{2}$, $\Gamma_{9,6}(\vec{e}_{\alpha_2}) = \frac{1}{2\sqrt{3}}$, $\Gamma_{8,7}(\vec{e}_{\alpha_2}) = \frac{1}{\sqrt{3}}$, $\Gamma_{10,8}(\vec{e}_{\alpha_2}) = \frac{1}{2}$, $\Gamma_{13,11}(\vec{e}_{\alpha_2}) = \frac{1}{2}$, $\Gamma_{15,12}(\vec{e}_{\alpha_2}) = \frac{1}{2\sqrt{3}}$, $\Gamma_{14,13}(\vec{e}_{\alpha_2}) = \frac{1}{\sqrt{3}}$, $\Gamma_{17,14}(\vec{e}_{\alpha_2}) = \frac{1}{2}$, $\Gamma_{18,16}(\vec{e}_{\alpha_2}) = \frac{1}{2}$, $\Gamma_{19,18}(\vec{e}_{\alpha_2}) = \frac{1}{\sqrt{3}}$, $\Gamma_{20,19}(\vec{e}_{\alpha_2}) = \frac{1}{2}$.

$\Gamma(\vec{e}_{-\alpha_2})$:

The non-zero elements of $\Gamma(\vec{e}_{-\alpha_2})$ **are:**

$\Gamma_{1,2}(\vec{e}_{-\alpha_2}) = -\frac{1}{2}$, $\Gamma_{2,3}(\vec{e}_{-\alpha_2}) = -\frac{1}{\sqrt{3}}$, $\Gamma_{3,4}(\vec{e}_{-\alpha_2}) = -\frac{1}{2}$, $\Gamma_{5,7}(\vec{e}_{-\alpha_2}) = -\frac{1}{2}$,

$\Gamma_{7,8}(\vec{e}_{-\alpha_2}) = -\frac{1}{\sqrt{3}}$, $\Gamma_{6,9}(\vec{e}_{-\alpha_2}) = -\frac{1}{2\sqrt{3}}$, $\Gamma_{8,10}(\vec{e}_{-\alpha_2}) = -\frac{1}{2}$, $\Gamma_{11,13}(\vec{e}_{-\alpha_2}) = -\frac{1}{2}$,

$\Gamma_{13,14}(\vec{e}_{-\alpha_2}) = -\frac{1}{\sqrt{3}}$, $\Gamma_{12,15}(\vec{e}_{-\alpha_2}) = -\frac{1}{2\sqrt{3}}$, $\Gamma_{14,17}(\vec{e}_{-\alpha_2}) = -\frac{1}{2}$, $\Gamma_{16,18}(\vec{e}_{-\alpha_2}) = -\frac{1}{2}$,

$\Gamma_{18,19}(\vec{e}_{-\alpha_2}) = -\frac{1}{\sqrt{3}}$, $\Gamma_{19,20}(e_{-\alpha_2}) = -\frac{1}{2}$.

$\Gamma(\vec{e}_{\alpha_1+\alpha_2})$:

The non-zero elements of $\Gamma(\vec{e}_{\alpha_1+\alpha_2})$ apart from the multiplying factor $\sqrt{6}$ are:

$\Gamma_{5,1}(\vec{e}_{\alpha_1+\alpha_2})=\frac{1}{2\sqrt{6}}$, $\Gamma_{6,2}(e_{\alpha_1+\alpha_2})=\frac{1}{3\sqrt{3}}$, $\Gamma_{7,2}(\vec{e}_{\alpha_1+\alpha_2})=\frac{1}{6\sqrt{6}}$,

$\Gamma_{8,3}(\vec{e}_{\alpha_1+\alpha_2})=-\frac{1}{6\sqrt{6}}$, $\Gamma_{9,3}(\vec{e}_{\alpha_1+\alpha_2})=\frac{1}{3\sqrt{3}}$, $\Gamma_{10,4}(\vec{e}_{\alpha_1+\alpha_2})=-\frac{1}{2\sqrt{6}}$,

$\Gamma_{11,5}(\vec{e}_{\alpha_1+\alpha_2})=\frac{\sqrt{2}}{6}$, $\Gamma_{12,6}(\vec{e}_{\alpha_1+\alpha_2})=\frac{5}{6\sqrt{18}}$, $\Gamma_{12,7}(\vec{e}_{\alpha_1+\alpha_2})=\frac{1}{9}$,

$\Gamma_{13,6}(\vec{e}_{\alpha_1+\alpha_2})=-\frac{1}{9}$, $\Gamma_{13,7}(\vec{e}_{\alpha_1+\alpha_2})=\frac{\sqrt{2}}{18}$, $\Gamma_{14,8}(\vec{e}_{\alpha_1+\alpha_2})=-\frac{\sqrt{2}}{18}$,

$\Gamma_{14,9}(\vec{e}_{\alpha_1+\alpha_2})=-\frac{1}{9}$, $\Gamma_{15,8}(\vec{e}_{\alpha_1+\alpha_2})=\frac{1}{9}$, $\Gamma_{15,9}(\vec{e}_{\alpha_1+\alpha_2})=-\frac{5}{6\sqrt{18}}$,

$\Gamma_{17,10}(\vec{e}_{\alpha_1+\alpha_2})=-\frac{\sqrt{2}}{6}$, $\Gamma_{16,11}(\vec{e}_{\alpha_1+\alpha_2})=\frac{1}{2\sqrt{6}}$, $\Gamma_{18,12}(\vec{e}_{\alpha_1+\alpha_2})=-\frac{1}{3\sqrt{3}}$,

$\Gamma_{18,13}(\vec{e}_{\alpha_1+\alpha_2})=\frac{1}{6\sqrt{6}}$, $\Gamma_{19,14}(\vec{e}_{\alpha_1+\alpha_2})=-\frac{1}{6\sqrt{6}}$, $\Gamma_{19,15}(\vec{e}_{\alpha_1+\alpha_2})=-\frac{1}{3\sqrt{3}}$,

$\Gamma_{20,17}(e_{\alpha_1+\alpha_2})=-\frac{1}{2\sqrt{6}}$.

$\Gamma(\vec{e}_{-\alpha_1-\alpha_2})$:

The non-zero elements of $\Gamma(\vec{e}_{-\alpha_1-\alpha_2})$ apart from the multiplying factor $\sqrt{6}$ are:

$\Gamma_{1,5}(\vec{e}_{-\alpha_1-\alpha_2})=-\frac{1}{2\sqrt{6}}$, $\Gamma_{2,6}(\vec{e}_{-\alpha_1-\alpha_2})=-\frac{1}{3\sqrt{3}}$, $\Gamma_{2,7}(\vec{e}_{-\alpha_1-\alpha_2})=-\frac{1}{6\sqrt{6}}$,

$\Gamma_{3,8}(\vec{e}_{-\alpha_1-\alpha_2})=\frac{1}{6\sqrt{6}}$, $\Gamma_{3,9}(\vec{e}_{-\alpha_1-\alpha_2})=-\frac{1}{3\sqrt{3}}$, $\Gamma_{4,10}(\vec{e}_{-\alpha_1-\alpha_2})=\frac{1}{2\sqrt{6}}$,

$\Gamma_{5,11}(\vec{e}_{-\alpha_1-\alpha_2})=-\frac{\sqrt{2}}{6}$, $\Gamma_{6,12}(\vec{e}_{-\alpha_1-\alpha_2})=-\frac{5}{6\sqrt{18}}$, $\Gamma_{6,13}(\vec{e}_{-\alpha_1-\alpha_2})=\frac{1}{9}$,

$\Gamma_{7,12}(\vec{e}_{-\alpha_1-\alpha_2})=-\frac{1}{9}$, $\Gamma_{7,13}(\vec{e}_{-\alpha_1-\alpha_2})=-\frac{\sqrt{2}}{18}$, $\Gamma_{8,14}(\vec{e}_{-\alpha_1-\alpha_2})=\frac{\sqrt{2}}{18}$,

$\Gamma_{8,15}(\vec{e}_{-\alpha_1-\alpha_2}) = -\frac{1}{9}$, $\Gamma_{9,14}(\vec{e}_{-\alpha_1-\alpha_2}) = \frac{1}{9}$, $\Gamma_{9,15}(\vec{e}_{-\alpha_1-\alpha_2}) = \frac{5}{6\sqrt{18}}$,

$\Gamma_{11,16}(\vec{e}_{-\alpha_1-\alpha_2}) = -\frac{1}{2\sqrt{6}}$, $\Gamma_{10,17}(\vec{e}_{-\alpha_1-\alpha_2}) = \frac{\sqrt{2}}{6}$, $\Gamma_{12,18}(\vec{e}_{-\alpha_1-\alpha_2}) = \frac{1}{3\sqrt{3}}$,

$\Gamma_{13,18}(\vec{e}_{-\alpha_1-\alpha_2}) = -\frac{1}{6\sqrt{6}}$, $\Gamma_{14,19}(\vec{e}_{-\alpha_1-\alpha_2}) = \frac{1}{6\sqrt{6}}$, $\Gamma_{15,19}(\vec{e}_{-\alpha_1-\alpha_2}) = \frac{1}{3\sqrt{3}}$,

$\Gamma_{17,20}(\vec{e}_{-\alpha_1-\alpha_2}) = \frac{1}{2\sqrt{6}}$.

$\Gamma(\vec{e}_{\alpha_1+2\alpha_2})$:

The non-zero elements of $\Gamma(\vec{e}_{\alpha_1+2\alpha_2})$ apart from the multiplying factor 6 are:

$\Gamma_{6,1}(\vec{e}_{\alpha_1+2\alpha_2}) = -\frac{1}{6\sqrt{3}}$, $\Gamma_{7,1}(\vec{e}_{\alpha_1+2\alpha_2}) = \frac{1}{6\sqrt{6}}$, $\Gamma_{8,2}(\vec{e}_{\alpha_1+2\alpha_2}) = \frac{1}{3\sqrt{18}}$,

$\Gamma_{9,2}(\vec{e}_{\alpha_1+2\alpha_2}) = -\frac{1}{18}$, $\Gamma_{10,3}(\vec{e}_{\alpha_1+2\alpha_2}) = \frac{1}{6\sqrt{6}}$, $\Gamma_{12,5}(\vec{e}_{\alpha_1+2\alpha_2}) = -\frac{1}{18}$,

$\Gamma_{13,5}(\vec{e}_{\alpha_1+2\alpha_2}) = \frac{1}{3\sqrt{18}}$, $\Gamma_{14,6}(\vec{e}_{\alpha_1+2\alpha_2}) = -\frac{1}{18\sqrt{3}}$, $\Gamma_{14,7}(\vec{e}_{\alpha_1+2\alpha_2}) = \frac{2}{9\sqrt{6}}$,

$\Gamma_{15,6}(\vec{e}_{\alpha_1+2\alpha_2}) = \frac{5}{18\sqrt{6}}$, $\Gamma_{15,7}(\vec{e}_{\alpha_1+2\alpha_2}) = -\frac{1}{18\sqrt{3}}$, $\Gamma_{17,8}(\vec{e}_{\alpha_1+2\alpha_2}) = \frac{1}{9\sqrt{2}}$,

$\Gamma_{17,9}(\vec{e}_{\alpha_1+2\alpha_2}) = -\frac{1}{18}$, $\Gamma_{18,11}(\vec{e}_{\alpha_1+2\alpha_2}) = \frac{1}{6\sqrt{6}}$, $\Gamma_{19,12}(\vec{e}_{\alpha_1+2\alpha_2}) = -\frac{1}{18}$,

$\Gamma_{19,13}(\vec{e}_{\alpha_1+2\alpha_2}) = \frac{1}{3\sqrt{18}}$, $\Gamma_{20,14}(\vec{e}_{\alpha_1+2\alpha_2}) = \frac{1}{6\sqrt{6}}$, $\Gamma_{20,15}(\vec{e}_{\alpha_1+2\alpha_2}) = -\frac{1}{6\sqrt{3}}$.

$\Gamma(\vec{e}_{-\alpha_1-2\alpha_2})$:

The non-zero elements of $\Gamma(\vec{e}_{-\alpha_1-2\alpha_2})$ apart from the multiplying factor 6 are:

$\Gamma_{1,6}(\vec{e}_{-\alpha_1-2\alpha_2}) = \frac{1}{6\sqrt{3}}$, $\Gamma_{1,7}(\vec{e}_{-\alpha_1-2\alpha_2}) = -\frac{1}{6\sqrt{6}}$, $\Gamma_{2,8}(\vec{e}_{-\alpha_1-2\alpha_2}) = -\frac{1}{3\sqrt{18}}$,

$\Gamma_{2,9}(\vec{e}_{-\alpha_1-2\alpha_2}) = \frac{1}{18}$, $\Gamma_{3,10}(\vec{e}_{-\alpha_1-2\alpha_2}) = -\frac{1}{6\sqrt{6}}$, $\Gamma_{5,12}(\vec{e}_{-\alpha_1-2\alpha_2}) = \frac{1}{18}$,

$\Gamma_{5,13}(\vec{e}_{-\alpha_1-2\alpha_2}) = -\frac{1}{3\sqrt{18}}$, $\Gamma_{6,14}(\vec{e}_{-\alpha_1-2\alpha_2}) = \frac{1}{18\sqrt{3}}$, $\Gamma_{6,15}(\vec{e}_{-\alpha_1-2\alpha_2}) = -\frac{5}{18\sqrt{6}}$,

$$\Gamma_{7,14}(\vec{e}_{-\alpha_1-2\alpha_2}) = -\frac{2}{9\sqrt{6}} \;,\; \Gamma_{7,15}(\vec{e}_{-\alpha_1-2\alpha_2}) = \frac{1}{18\sqrt{3}} \;,\; \Gamma_{8,17}(\vec{e}_{-\alpha_1-2\alpha_2}) = -\frac{1}{9\sqrt{2}} \;,$$

$$\Gamma_{9,17}(\vec{e}_{-\alpha_1-2\alpha_2}) = \frac{1}{18} \;,\; \Gamma_{11,18}(\vec{e}_{-\alpha_1-2\alpha_2}) = -\frac{1}{6\sqrt{6}} \;,\; \Gamma_{12,19}(\vec{e}_{-\alpha_1-2\alpha_2}) = \frac{1}{18} \;,$$

$$\Gamma_{13,19}(\vec{e}_{-\alpha_1-2\alpha_2}) = -\frac{1}{3\sqrt{18}} \;,\; \Gamma_{14,20}(\vec{e}_{-\alpha_1-2\alpha_2}) = -\frac{1}{6\sqrt{6}} \;,\; \Gamma_{15,20}(\vec{e}_{-\alpha_1-2\alpha_2}) = \frac{1}{6\sqrt{3}} .$$

These matrices satisfy all the commutation relations of the Lie algebra.

3: <u>The matrices $\mathbb{L}_k^{20\text{-dim}}$, $k = 0,1,2,3$.</u>

We are now in the position to construct the matrices $\mathbb{L}_k^{20\text{-dim}}$ for which we have;

1) $\mathbb{L}_0^{20\text{-dim}} = i\sqrt{3}\,\Gamma(\vec{e}_{\alpha_2}) + i\sqrt{3}\,\Gamma(\vec{e}_{-\alpha_2}) =$

$$i\sqrt{3}\left|\begin{array}{cccccccccccccccccccc}
\cdot & -\frac{1}{2} & \cdot & \cdot & \cdot & & & & & & & & & & & & & & & \\
\frac{1}{2} & \cdot & -\frac{1}{\sqrt{3}} & \cdot & \cdot & & & & & & & & & & & & & & & \\
\cdot & \frac{1}{\sqrt{3}} & \cdot & -\frac{1}{2} & \cdot & & & & & & & & & & & & & & & \\
\cdot & \cdot & \frac{1}{2} & \cdot & \cdot & & & & & & & & & & & & & & & \\
 & & & & \cdot & \cdot & -\frac{1}{2} & \cdot & \cdot & \cdot & & & & & & & & & & \\
 & & & & \cdot & \cdot & \cdot & \cdot & -\frac{1}{2\sqrt{3}} & \cdot & & & & & & & & & & \\
 & & & & \frac{1}{2} & \cdot & \cdot & -\frac{1}{\sqrt{3}} & \cdot & \cdot & & & & & & & & & & \\
 & & & & \cdot & \cdot & \frac{1}{\sqrt{3}} & \cdot & \cdot & -\frac{1}{2} & & & & & & & & & & \\
 & & & & \cdot & \frac{1}{2\sqrt{3}} & \cdot & \cdot & \cdot & \cdot & & & & & & & & & & \\
 & & & & \cdot & \cdot & \cdot & \frac{1}{2} & \cdot & \cdot & \cdot & & & & & & & & & \\
 & & & & & & & & & & \cdot & \cdot & -\frac{1}{2} & \cdot & \cdot & \cdot & & & & \\
 & & & & & & & & & & \cdot & \cdot & \cdot & \cdot & -\frac{1}{2\sqrt{3}} & \cdot & & & & \\
 & & & & & & & & & & \frac{1}{2} & \cdot & \cdot & -\frac{1}{\sqrt{3}} & \cdot & \cdot & & & & \\
 & & & & & & & & & & \cdot & \cdot & \frac{1}{\sqrt{3}} & \cdot & \cdot & \cdot & -\frac{1}{2} & \cdot & \cdot & \cdot \\
 & & & & & & & & & & \cdot & \frac{1}{2\sqrt{3}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\frac{1}{2} & \cdot & \cdot \\
 & & & & & & & & & & \cdot & \cdot & \cdot & \frac{1}{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & & & & & & & & & \frac{1}{2} & \cdot & \cdot & -\frac{1}{\sqrt{3}} & \cdot \\
 & & & & & & & & & & & & & & & \cdot & \cdot & \frac{1}{\sqrt{3}} & \cdot & -\frac{1}{2} \\
 & & & & & & & & & & & & & & & \cdot & \cdot & \cdot & \frac{1}{2} & \cdot
\end{array}\right| .$$

The matrix $\mathbb{L}_0^{20\text{-dim}}$ is hermitian.

2) $\mathbb{L}_1^{20\text{-dim}} = \iota\frac{\sqrt{3}}{\sqrt{2}}\Gamma\left(\vec{e}_{\alpha_1}\right) - \iota\frac{\sqrt{3}}{\sqrt{2}}\Gamma\left(\vec{e}_{-\alpha_1}\right) + \iota\frac{\sqrt{3}}{\sqrt{2}}\Gamma\left(\vec{e}_{\alpha_1+2\alpha_2}\right) - \iota\frac{\sqrt{3}}{\sqrt{2}}\Gamma\left(\vec{e}_{-\alpha_1-2\alpha_2}\right) =$

$$\iota\frac{\sqrt{3}}{\sqrt{2}}\left|\begin{array}{cccccccccccccccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & & & & & & & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{6}} & \cdot & \cdot & \frac{2}{\sqrt{18}} & -\frac{1}{3} & & & & & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \frac{\sqrt{2}}{3} & \cdot & \cdot & \frac{1}{\sqrt{6}} & & & & & & & & & & \\
\cdot & \cdot & \cdot & & & & & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \cdot & \cdot & \cdot & & & & & & & & \\
\cdot & \frac{1}{\sqrt{6}} & \cdot & & & & & \cdot & \cdot & \cdot & \cdot & -\frac{1}{3} & \frac{2}{\sqrt{18}} & & & & & & & \\
-\frac{1}{\sqrt{3}} & \cdot & \frac{1}{3} & & & & & & & & \frac{1}{3} & \cdot & \cdot & -\frac{1}{3\sqrt{3}} & \frac{5}{3\sqrt{6}} & & & & & \\
\frac{1}{\sqrt{6}} & \cdot & \frac{\sqrt{2}}{3} & & & & & & & & \frac{\sqrt{2}}{3} & \cdot & \cdot & \frac{4}{3\sqrt{6}} & -\frac{1}{3\sqrt{3}} & & & & & \\
\cdot & \frac{2}{\sqrt{18}} & \cdot & \frac{1}{\sqrt{6}} & & & & & & & \cdot & \frac{1}{3\sqrt{3}} & \frac{2\sqrt{2}}{3\sqrt{3}} & \cdot & \cdot & \cdot & \frac{2}{3\sqrt{2}} & & & \\
\cdot & -\frac{1}{3} & \cdot & \frac{1}{\sqrt{3}} & \cdot & & & & & & & \frac{5}{3\sqrt{6}} & \frac{1}{3\sqrt{3}} & \cdot & \cdot & \cdot & -\frac{1}{3} & & & \\
 & & \frac{1}{\sqrt{6}} & \cdot & \cdot & & & & & & & \cdot & \cdot & \frac{\sqrt{2}}{3} & \frac{1}{3} & \cdot & \cdot & \cdot & & \\
 & & & & \cdot & \frac{1}{3} & \frac{\sqrt{2}}{3} & \cdot & & & & & & & & \cdot & \cdot & \frac{1}{\sqrt{6}} & & \\
 & & & & -\frac{1}{3} & \cdot & \cdot & \frac{1}{3\sqrt{3}} & \frac{5}{3\sqrt{6}} & & & & & & & \frac{1}{\sqrt{3}} & \cdot & \cdot & -\frac{1}{3} & \\
 & & & & \frac{2}{\sqrt{18}} & \cdot & \cdot & \frac{2\sqrt{2}}{3\sqrt{3}} & \frac{1}{3\sqrt{3}} & & & & & & & \frac{1}{\sqrt{6}} & \cdot & \cdot & \frac{2}{\sqrt{18}} & \\
 & & & & & -\frac{1}{3\sqrt{3}} & \frac{4}{3\sqrt{6}} & \cdot & \cdot & \frac{\sqrt{2}}{3} & & & & & & \cdot & \cdot & \frac{\sqrt{2}}{3} & \cdot & \frac{1}{\sqrt{6}} \\
 & & & & & \frac{5}{3\sqrt{6}} & -\frac{1}{3\sqrt{3}} & \cdot & \cdot & \frac{1}{3} & \cdot & & & & & & & \frac{1}{3} & \cdot & -\frac{1}{\sqrt{3}} \\
 & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & & & & & \cdot & \cdot & \cdot \\
 & & & \cdot & & & & \frac{2}{3\sqrt{2}} & -\frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \cdot & \frac{1}{\sqrt{6}} & \cdot \\
 & & & & & & & & & & \frac{1}{\sqrt{6}} & \cdot & \cdot & \frac{\sqrt{2}}{3} & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & & & & & -\frac{1}{3} & \frac{2}{\sqrt{18}} & \cdot & \cdot & \cdot & \frac{1}{\sqrt{6}} & \cdot & \cdot & \cdot \\
 & & & & & & & & & & & & & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right|$$

3) $\mathbb{L}_2^{20\text{-dim}} = -\frac{\sqrt{3}}{\sqrt{2}}\Gamma\left(\vec{e}_{\alpha_1}\right) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma\left(\vec{e}_{-\alpha_1}\right) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma\left(\vec{e}_{\alpha_1+2\alpha_2}\right) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma\left(\vec{e}_{-\alpha_1-2\alpha_2}\right) =$

$$-\frac{\sqrt{3}}{\sqrt{2}}\left|\begin{array}{cccccccccccccccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & & & & & & & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & -\frac{1}{\sqrt{6}} & \cdot & \cdot & -\frac{2}{\sqrt{18}} & \frac{1}{3} & & & & & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & -\frac{1}{3} & -\frac{\sqrt{2}}{3} & \cdot & \cdot & -\frac{1}{\sqrt{6}} & & & & & & & & & & \\
\cdot & \cdot & \cdot & & & & & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \cdot & \cdot & \cdot & & & & & & & & \\
\cdot & \frac{1}{\sqrt{6}} & \cdot & & & & & & & & \cdot & \frac{1}{3} & -\frac{2}{\sqrt{18}} & & & & & & & \\
-\frac{1}{\sqrt{3}} & \cdot & -\frac{1}{3} & & & & & & & & -\frac{1}{3} & \cdot & \cdot & \frac{1}{3\sqrt{3}} & -\frac{5}{3\sqrt{6}} & & & & & \\
\frac{1}{\sqrt{6}} & \cdot & \frac{\sqrt{2}}{3} & & & & & & & & -\frac{\sqrt{2}}{3} & \cdot & \cdot & -\frac{4}{3\sqrt{6}} & \frac{1}{3\sqrt{3}} & & & & & \\
\cdot & \frac{2}{\sqrt{18}} & \cdot & \frac{1}{\sqrt{6}} & & & & & & & & -\frac{1}{3\sqrt{3}} & -\frac{2\sqrt{2}}{3\sqrt{3}} & \cdot & \cdot & \cdot & -\frac{2}{3\sqrt{2}} & & & \\
 & -\frac{1}{3} & \cdot & \frac{1}{\sqrt{3}} & & & & & & & & -\frac{5}{3\sqrt{6}} & -\frac{1}{3\sqrt{3}} & \cdot & \cdot & \cdot & \frac{1}{3} & & & \\
 & & \frac{1}{\sqrt{6}} & \cdot & \cdot & & & & & & & & & -\frac{\sqrt{2}}{3} & -\frac{1}{3} & \cdot & \cdot & \cdot & & \\
 & & & \cdot & \cdot & \frac{1}{3} & \frac{\sqrt{2}}{3} & & & & & & & & & \cdot & \cdot & -\frac{1}{\sqrt{6}} & & \\
 & & & & -\frac{1}{3} & \cdot & \cdot & \frac{1}{3\sqrt{3}} & \frac{5}{3\sqrt{6}} & & & & & & & -\frac{1}{\sqrt{3}} & \cdot & \cdot & \frac{1}{3} & \\
 & & & & \frac{2}{\sqrt{18}} & \cdot & \cdot & \frac{2\sqrt{2}}{3\sqrt{3}} & \frac{1}{3\sqrt{3}} & & & & & & & -\frac{1}{\sqrt{6}} & \cdot & \cdot & -\frac{2}{\sqrt{18}} & \\
 & & & & & -\frac{1}{3\sqrt{3}} & \frac{4}{3\sqrt{6}} & \cdot & \cdot & \frac{\sqrt{2}}{3} & & & & & & & & -\frac{\sqrt{2}}{3} & \cdot & -\frac{1}{\sqrt{6}} \\
 & & & & & \frac{5}{3\sqrt{6}} & -\frac{1}{3\sqrt{3}} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & & & & & & -\frac{1}{3} & \cdot & \frac{1}{\sqrt{3}} \\
 & & & & & & & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & & & & & \cdot & \cdot & \cdot \\
 & & & & & & & \frac{2}{3\sqrt{2}} & -\frac{1}{3} & \cdot & \cdot & \cdot & \cdot & & & & & \cdot & -\frac{1}{\sqrt{6}} & \cdot \\
 & & & & & & & & & & \frac{1}{\sqrt{6}} & \cdot & \cdot & \frac{\sqrt{2}}{3} & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & & & & & -\frac{1}{3} & \frac{2}{\sqrt{18}} & \cdot & \cdot & \cdot & \frac{1}{\sqrt{6}} & \cdot & \cdot & \cdot \\
 & & & & & & & & & & & & & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right|$$

$$\mathbb{L}_3^{20-\dim} = -6i\, \Gamma(\vec{h}_{\alpha_2})$$

$$= -\tfrac{1}{2}\, \mathrm{dig}\left\{3,1,-1,-3,3,1,1,-1,-1,-3,3,1,1,-1,-1,3,-3,1,-1,-3\right\}.$$

The matrices $\mathbb{L}_1^{20-\dim}$, $\mathbb{L}_2^{20-\dim}$, $\mathbb{L}_3^{20-\dim}$, are anti-hermitian.

4: Hermitianizing matrix $\mathbb{A}$:

Using the hermiticity of the matrix $\mathbb{L}_0$ and the anti-hermiticity of the matrices $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$ a matrix $\mathbb{A}$ called the hermitianizing matrix can be found satisfying the following properties

$$(\mathbb{A})^2 = 1 \;, \quad [\mathbb{L}_0, \mathbb{A}]_- = 0 \;, \quad \{\mathbb{L}_k, \mathbb{A}\}_+ = 0 \;, \quad k = 1,2,3 .$$

The matrix $\mathbb{A}$ can be expressed in terms of the matrix $\mathbb{L}_0$. A general formula for the matrix $\mathbb{A}$ and for any value of the spin s is given by Madhavarao, Thiruvenkatachar and Venkatachalienger[23-24] and is

$$\mathbb{A}(s) = f(\mathbb{L}_0, s)$$

with

$$f(x,s) = \frac{(x-s)(x-s+1)\cdots(x+s-1)(x+s)}{2s!} \sum_{n=0}^{2s} \binom{2s}{n} \frac{1}{(x-s+n)} .$$

If s is half integer this formula reduces to

$$f(x, s=\text{half integer}) = \frac{2x\,(x^2-\frac{1}{4})(x^2-\frac{9}{4})\cdots(x^2-s^2)}{2s!} \sum_{n=1}^{s+1/2} \binom{2s}{s+1/2-n} \frac{1}{[x^2-(n-1/2)^2]} ,$$

and if s is integer the formula reduces to

$$f(x, s=\text{integer}) = \frac{(x^2-1^2)(x^2-2^2)\cdots(x^2-s^2)}{(s!)^2} + \frac{2x^2(x^2-1^2)\cdots(x^2-s^2)}{(2s)!} \sum_{n=1}^{s} \binom{2s}{s-n} \frac{1}{[x^2-n^2]} .$$

Specific examples are:

for $S=\frac{1}{2}$, $\mathbb{A}=\mathbb{L}_0$, (Dirac case) (i)

for $S=1$, $\mathbb{A}=2(\mathbb{L}_0)^2-1$, (Kemmer case) (ii)

for $S=\frac{3}{2}$, $\mathbb{A}=\frac{1}{3}\mathbb{L}_0\{4(\mathbb{L}_0)^2-7\}$, (iii)

for $S=2$, $\mathbb{A}=\frac{2}{3}(\mathbb{L}_0)^4-\frac{8}{3}(\mathbb{L}_0)^2+1$. (iv)

Thus in the case $S=\frac{3}{2}$ using the formula (iii) and $\mathbb{L}_0^{20\text{-dim}}$, we find for the hermitianizing matrix:

$$\mathbb{A}_{20\text{-dim}}=\frac{1}{3}\mathbb{L}_0^{20\text{-dim}}\left\{4\left(\mathbb{L}_0^{20\text{-dim}}\right)^2-7\right\}=$$

$$\left|\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c}
\cdot & \cdot & \cdot & 1 & & & & & & & & & & & & & & & & \\ \hline
\cdot & \cdot & -1 & \cdot & & & & & & & & & & & & & & & & \\ \hline
\cdot & 1 & \cdot & \cdot & & & & & & & & & & & & & & & & \\ \hline
-1 & \cdot & \cdot & \cdot & & & & & & & & & & & & & & & & \\ \hline
 & & & & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & & & & & & & & & & \\ \hline
 & & & & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & & & & & & & & & & \\ \hline
 & & & & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & & & & & & & & & & \\ \hline
 & & & & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & & & & & & & & & & \\ \hline
 & & & & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & & & & & & & & & & \\ \hline
 & & & & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & & & & \\ \hline
 & & & & & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \hline
 & & & & & & & & & & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
 & & & & & & & & & & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
 & & & & & & & & & & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
 & & & & & & & & & & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
 & & & & & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \hline
 & & & & & & & & & & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
 & & & & & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \hline
 & & & & & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \hline
 & & & & & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot
\end{array}\right| .$$

$\mathbb{A}_{20\text{-dim}}$ satisfies the relations,

$$(\mathbb{A}_{20\text{-dim}})^2 = 1 \, , \quad [\mathbb{L}_0^{20\text{-dim}}, \mathbb{A}]_- = 0 \, , \quad \{\mathbb{L}_k^{20\text{-dim}}, \mathbb{A}\}_+ = 0 \, , \quad k = 1,2,3 \, .$$

5: Eigenvalues of $\mathbb{L}_0^{20\text{-dim}}$.

The eigenvalues of $\mathbb{L}_0^{20\text{-dim}}$ give the masses with which the particles of spin $\frac{3}{2}$ may appear. These are the roots of the polynomial

$$\det\left(\mathbb{L}_0^{20\text{-dim}} - \lambda \mathbb{1}\right) = 0 \, .$$

Evaluating this determinant by standard methods we find for the eigenvalues:

$$\lambda_1 = \frac{3}{2} \, , \; \lambda_2 = -\frac{3}{2} \, , \; \lambda_3 = \frac{1}{2} \, , \; \lambda_4 = -\frac{1}{2} \, , \; \lambda_5 = \frac{1}{2} \, , \; \lambda_6 = -\frac{1}{2} \, ,$$
$$\lambda_7 = -\frac{1}{2} \, , \; \lambda_8 = \frac{1}{2} \, , \; \lambda_9 = \frac{3}{2} \, , \; \lambda_{10} = -\frac{3}{2} \, , \; \lambda_{11} = \frac{1}{2} \, , \; \lambda_{12} = -\frac{1}{2} \, ,$$
$$\lambda_{13} = \frac{3}{2} \, , \; \lambda_{14} = -\frac{3}{2} \, , \; \lambda_{15} = \frac{1}{2} \, , \; \lambda_{16} = -\frac{1}{2} \, , \; \lambda_{17} = \frac{1}{2} \, , \; \lambda_{18} = -\frac{1}{2} \, ,$$
$$\lambda_{19} = \frac{3}{2} \, , \; \lambda_{20} = -\frac{3}{2} \, .$$

The masses are given by the formula $m_i = \frac{\chi}{\lambda_i}$ and) thus for a spin $\frac{3}{2}$ field the possible masses are,

$$m_{1,2} = \frac{\chi}{\pm 3/2} = \pm \frac{2\chi}{3} \, , \quad m_{3,4} = \frac{\chi}{\pm 1/2} = \pm 2\chi \, .$$

6: 16-dimensional representation of B_2.

For the 16-dimensional representation we find $q_1 = 1$ $q_2 = 1$ and the highest weight of the representation is

$$\Lambda_1 = q_1 \lambda_1 + q_2 \lambda_2 = \frac{3}{2}\alpha_1 + 2\alpha_2 \, .$$

The other weights are

$\Lambda_2 = \frac{3}{2}\alpha_1 + \alpha_2$, $\Lambda_3 = \frac{1}{2}\alpha_1 + 2\alpha_2$, $\Lambda_4 = \Lambda_5 = \frac{1}{2}\alpha_1 + \alpha_2$ (of multiplicity 2), $\Lambda_6 = \Lambda_7 = \frac{1}{2}\alpha_1$ (of multiplicity 2), $\Lambda_8 = \frac{1}{2}\alpha_1 - \alpha_2$, $\Lambda_9 = -\frac{1}{2}\alpha_1 + \alpha_2$, $\Lambda_{10} = \Lambda_{11} = -\frac{1}{2}\alpha_1$ (of multiplicity 2), $\Lambda_{12} = \Lambda_{13} = -\frac{1}{2}\alpha_1 - \alpha_2$ (of multiplicity 2)

$\Lambda_{14} = -\frac{1}{2}\alpha_1 - 2\alpha_2$, $\Lambda_{15} = -\frac{3}{2}\alpha_1 - \alpha_2$, $\Lambda_{16} = -\frac{3}{2}\alpha_1 - 2\alpha_2$.

The weights can be arranged in a weight diagram which we omit.

The basis elements of the Cartan subalgebra are

$$\Gamma(\vec{h}_{\alpha_1}) = \frac{1}{6}\Gamma(\vec{h}_1) = \frac{1}{6}\,\mathrm{dig}\left\{1,2,-1,0,0,1,1,2,-2,-1,-1,0,0,1,-2,-1\right\} ,$$

and

$$\Gamma(\vec{h}_{\alpha_2}) = \frac{1}{12}\Gamma(\vec{h}_2) = \frac{1}{12}\,\mathrm{dig}\left\{1,-1,3,1,1,-1,-1,-3,3,1,1,-1,-1,-3,1,-1\right\}.$$

Constructing the differences $\Lambda_i - \Lambda_j$, $i,j = 1,2,\cdots,16$, we find that the matrix $\Gamma(\vec{e}_{\alpha_1})$ has elements different from zero at the positions

$$(p,q) = (3,1),\ (4,2),\ (5,2),\ (9,4),\ (9,5),\ (10,6),\ (10,7),\ (11,6),$$
$$(11,7),\ (12,8),\ (13,8),\ (15,12),\ (15,13),\ (16,14).$$

The matrix $\Gamma(\vec{e}_{-\alpha_1})$ is taken to be equal to $-\Gamma^{tr}(\vec{e}_{\alpha_1})$. Similarly the matrix $\Gamma(\vec{e}_{\alpha_2})$ has elements different from zero at the positions

$$(p,q) = (2,1),\ (4,3),\ (5,3),\ (6,4),\ (6,5),\ (7,4),\ (7,5),\ (8,6),\ (8,7),$$
$$(10,9),\ (11,9),\ (12,10),\ (12,11),\ (13,10),\ (13,11),\ (14,12),\ (14,13),$$
$$(16,15).$$

The matrix $\Gamma(\vec{e}_{-\alpha_2})$ is taken to be equal to $-\Gamma^{tr}(\vec{e}_{\alpha_2})$. The matrix $\Gamma(\vec{e}_{\alpha_1+\alpha_2})$ is given by

$$\Gamma(\vec{e}_{\alpha_1+\alpha_2}) = \frac{1}{N_{\alpha_1,\alpha_2}}\left[\Gamma(\vec{e}_{\alpha_1}),\Gamma(\vec{e}_{\alpha_2})\right]_- .$$

The matrix $\Gamma(\vec{e}_{-\alpha_1-\alpha_2})$ is taken equal to $-\Gamma^{tr}(\vec{e}_{\alpha_1+\alpha_2})$. The matrix $\Gamma(\vec{e}_{\alpha_1+2\alpha_2})$ is given by

$$\Gamma(\vec{e}_{\alpha_1+2\alpha_2}) = \frac{1}{N_{\alpha_2,\alpha_1+\alpha_2}}\left[\Gamma(\vec{e}_{\alpha_2}),\Gamma(\vec{e}_{\alpha_1+\alpha_2})\right]_- .$$

The matrix $\Gamma(\vec{e}_{-\alpha_1-2\alpha_2})$ is taken equal to $-\Gamma^{tr}(\vec{e}_{\alpha_1+2\alpha_2})$. Working as for the 20-dimensional representation we find by solving the set of simultaneous equations in the quantities e_{ij}, ϵ_{ij} that a possible solution for the ϵ_{ij} is

$$\epsilon_{2,1} = \pm \frac{1}{2\sqrt{3}} \;,\; \epsilon_{5,3} = 0 \;,\; \epsilon_{4,3} = \pm \frac{1}{2} \;,\; \epsilon_{6,4} = 0 \;,\; \epsilon_{7,4} = \pm \frac{1}{\sqrt{3}} \;,$$

$$\epsilon_{7,5} = 0 \;,\; \epsilon_{6,5} = \pm \frac{1}{2\sqrt{3}} \;,\; \epsilon_{8,6} = 0 \;,\; \epsilon_{8,7} = \pm \frac{1}{2} \;,\; \epsilon_{11,9} = 0 \;,$$

$$\epsilon_{10,9} = \pm \frac{1}{2} \;,\; \epsilon_{12,10} = 0 \;,\; \epsilon_{14,13} = \pm \frac{1}{2} \;,\; \epsilon_{18,10} = \pm \frac{1}{\sqrt{3}} \;,\; \epsilon_{13,11} = 0 \;,$$

$$\epsilon_{12,11} = \pm \frac{1}{2\sqrt{3}} \;,\; \epsilon_{14,12} = 0 \;.$$

Choosing the positive sign for the ϵ_{ij} and using

$$\mathbb{L}_0^{16\text{-dim}} = i\sqrt{3}\,\Gamma\!\left(\vec{e}_{\alpha_2}\right) + i\sqrt{3}\,\Gamma\!\left(\vec{e}_{-\alpha_2}\right)$$

we find,

$$\mathbb{L}_0^{16\text{-dim}} = i\sqrt{3} \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\cdot & -1/2\sqrt{3} & \cdot & \cdot & & & & & & & & & & & & \\ \hline
1/2\sqrt{3} & \cdot & \cdot & \cdot & & & & & & & & & & & & \\ \hline
\cdot & \cdot & \cdot & -1/2 & & & & & & & & & & & & \\ \hline
\cdot & \cdot & 1/2 & \cdot & \cdot & \cdot & -1/\sqrt{3} & & & & & & & & & \\ \hline
 & & & \cdot & \cdot & -1/2\sqrt{3} & \cdot & & & & & & & & & \\ \hline
 & & \cdot & \cdot & 1/2\sqrt{3} & \cdot & \cdot & & & & & & & & & \\ \hline
 & & & 1/\sqrt{3} & \cdot & \cdot & \cdot & -1/2 & & & & & & & & \\ \hline
 & & & & & & 1/2 & \cdot & \cdot & \cdot & & & & & & \\ \hline
 & & & & & & \cdot & \cdot & \cdot & -1/2 & & & & & & \\ \hline
 & & & & & & & & 1/2 & \cdot & \cdot & \cdot & -1/\sqrt{3} & & & \\ \hline
 & & & & & & & & & \cdot & \cdot & -1/2\sqrt{3} & \cdot & & & \\ \hline
 & & & & & & & & & \cdot & 1/2\sqrt{3} & \cdot & \cdot & & & \\ \hline
 & & & & & & & & & 1/\sqrt{3} & \cdot & \cdot & \cdot & -1/2 & & \\ \hline
 & & & & & & & & & & & & 1/2 & \cdot & \cdot & \cdot \\ \hline
 & & & & & & & & & & & & & & \cdot & -1/2\sqrt{3} \\ \hline
 & & & & & & & & & & & & & & 1/2\sqrt{3} & \cdot
\end{array} .$$

This matrix has eigenvalues,

$$\lambda_1 = \lambda_3 = \lambda_7 = \lambda_9 = \lambda_{13} = \lambda_{15} = \frac{1}{2} \;,$$

$$\lambda_2 = \lambda_4 = \lambda_8 = \lambda_{10} = \lambda_{14} = \lambda_{16} = -\frac{1}{2} \;,$$

$$\lambda_5 = \lambda_{11} = \frac{3}{2} \qquad , \quad \lambda_6 = \lambda_{12} = -\frac{3}{2} \;,$$

and the masses associated with the field are

$$m_{1,2} = \frac{\chi}{\pm 3/2} = \pm \frac{2\chi}{3} \quad , \quad m_{3,4} = \frac{\chi}{\pm 1/2} = \pm 2\chi \,.$$

7: Transformation properties of the hermitianizing matrix $\mathbb{A}$.

We show in this paragraph that the hermitianizing matrix $\mathbb{A}$ satisfies the relation

$$\mathbb{T}^{+}\mathbb{A}\mathbb{T} = \mathbb{A}, \quad (+ = \text{Hermitian conjugate}), \quad (1)$$

for every transformation of $SO(4,1)$ which also belongs to $SO(3,1)$. We demonstrate the validity of this relation in the case of the spin $\frac{3}{2}$ equation of Bhabha based on the 20-dimensional representation. It is sufficient to prove it for the infinitesimal transformations. Let $T = 1 + \epsilon I_{ij}$ be an infinitesimal transformation which can be either a rotation on the plane (i,j) or a boost in the direction perpendicular to the plane (i,j). I_{ij} stands for the infinitesimal generators and ϵ is the infinitesimal parameter. Expressing the generators I_{ij} by their 20-dimensional representation matrices $\Gamma_{20\text{-dim}}(I_{ij})$ we have

$$\mathbb{T} = \mathbb{1} + \epsilon\,\Gamma_{20\text{-dim}}(I_{ij}).$$

Substituting for $\mathbb{T}$ in (1) we get

$$\begin{aligned}\mathbb{T}^{+}\mathbb{A}\mathbb{T} &= \{\mathbb{1} + \epsilon\,\Gamma_{20\text{-dim}}(I_{ij})\}^{+}\mathbb{A}\{\mathbb{1} + \epsilon\,\Gamma_{20\text{-dim}}(I_{ij})\} \\ &= \{\mathbb{A} + \epsilon\,\Gamma^{+}_{20\text{-dim}}(I_{ij})\mathbb{A}\}\{\mathbb{1} + \epsilon\,\Gamma_{20\text{-dim}}(I_{ij})\} \\ &= \mathbb{A} + \epsilon\,\{\Gamma^{+}_{20\text{-dim}}(I_{ij})\mathbb{A} + \mathbb{A}\,\Gamma_{20\text{-dim}}(I_{ij})\} + O(\epsilon^{2}).\end{aligned}$$

Thus in order that (1) holds it is necessary that

$$\{\Gamma^{+}_{20\text{-dim}}(I_{ij})\mathbb{A} + \mathbb{A}\,\Gamma_{20\text{-dim}}(I_{ij})\} = 0.$$

The matrices $\Gamma_{20\text{-dim}}(I_{ij})$ are connected to the matrices

$\Gamma_{20\text{-dim}}(\vec{h}_{\alpha_1}), \Gamma_{20\text{-dim}}(\vec{h}_{\alpha_2}) \cdots \Gamma_{20\text{-dim}}(\vec{e}_{-\alpha_1-2\alpha_2})$ which form the basis elements of the Lie algebra, by the relations:

(i) $\Gamma_{20\text{-dim}}\left(I_{1,2}\right) = i\,\Gamma_{20\text{-dim}}\left(\vec{h}_{\alpha_1}\right) + \frac{1}{2}\,i\,\Gamma_{20\text{-dim}}\left(\vec{h}_{\alpha_2}\right)$,

(ii) $\Gamma_{20\text{-dim}}\left(I_{1,3}\right) = -\frac{\sqrt{3}}{\sqrt{2}}\,\Gamma_{20\text{-dim}}\left(\vec{e}_{\alpha_1}\right) - \frac{\sqrt{3}}{\sqrt{2}}\,\Gamma_{20\text{-dim}}\left(\vec{e}_{-\alpha_1}\right) + \frac{\sqrt{3}}{\sqrt{2}}\,\Gamma_{20\text{-dim}}\left(\vec{e}_{\alpha_1+2\alpha_2}\right) + \frac{\sqrt{3}}{\sqrt{2}}\,\Gamma_{20\text{-dim}}\left(\vec{e}_{-\alpha_1-2\alpha_2}\right)$,

(iii) $\Gamma_{20\text{-dim}}\left(I_{1,0}\right) = -\sqrt{3}\,\Gamma_{20\text{-dim}}\left(\vec{e}_{\alpha_1+\alpha_2}\right) + \sqrt{3}\,\Gamma_{20\text{-dim}}\left(\vec{e}_{-\alpha_1-\alpha_2}\right)$,

(iv) $\Gamma_{20\text{-dim}}\left(I_{2,3}\right) = -i\frac{\sqrt{3}}{\sqrt{2}}\,\Gamma_{20\text{-dim}}\left(\vec{e}_{\alpha_1}\right) + i\frac{\sqrt{3}}{\sqrt{2}}\,\Gamma_{20\text{-dim}}\left(\vec{e}_{-\alpha_1}\right) + i\frac{\sqrt{3}}{\sqrt{2}}\,\Gamma_{20\text{-dim}}\left(\vec{e}_{\alpha_1+2\alpha_2}\right) - i\frac{\sqrt{3}}{\sqrt{2}}\,\Gamma_{20\text{-dim}}\left(\vec{e}_{-\alpha_1-2\alpha_2}\right)$,

(v) $\Gamma_{20\text{-dim}}\left(I_{2,0}\right) = -i\sqrt{3}\,\Gamma_{20\text{-dim}}\left(\vec{e}_{\alpha_1+\alpha_2}\right) + i\sqrt{3}\,\Gamma_{20\text{-dim}}\left(\vec{e}_{-\alpha_1-\alpha_2}\right)$,

(vi) $\Gamma_{20\text{-dim}}\left(I_{3,0}\right) = -\sqrt{3}\,\Gamma_{20\text{-dim}}\left(\vec{e}_{\alpha_2}\right) + \sqrt{3}\,\Gamma_{20\text{-dim}}\left(\vec{e}_{-\alpha_2}\right)$.

As an example, let us show how one proves the validity of $\mathbb{T}^{+}\mathbb{A}\mathbb{T} = \mathbb{A}$ when $\mathbb{T}$ is chosen to be

$$\mathbb{T} = \mathbb{1} + \epsilon\,\Gamma_{20\text{-dim}}\left(I_{2,0}\right). \tag{2}$$

Using the matrix realizations of the matrices $\Gamma_{20\text{-dim}}\left(\vec{e}_{\alpha_1+\alpha_2}\right)$ $\Gamma_{20\text{-dim}}\left(\vec{e}_{-\alpha_1-\alpha_2}\right)$ we find applying (v) the matrix $\Gamma_{20\text{-dim}}\left(I_{2,0}\right)$ and then by taking its complex conjugate transpose we find $\Gamma^{+}_{20\text{-dim}}\left(I_{2,0}\right)$. Then using the hermitianizing matrix $\mathbb{A}_{20\text{-dim}}$ given in §4 we construct $\Gamma^{+}_{20\text{-dim}}\left(I_{2,0}\right)\mathbb{A}$ and also $\mathbb{A}\,\Gamma_{20\text{-dim}}\left(I_{2,0}\right)$. Adding these two matrices together we find that

$$\left\{\Gamma^{+}_{20\text{-dim}}\left(I_{2,0}\right)\mathbb{A} + \mathbb{A}\,\Gamma_{20\text{-dim}}\left(I_{2,0}\right)\right\} = 0,$$

and hence $\mathbb{T}^{+}\mathbb{A}\mathbb{T} = \mathbb{A}$ when $\mathbb{T}$ is chosen as in (2). Similarly we find that the same is true for all the other generators $\Gamma_{20\text{-dim}}\left(I_{ij}\right)$.

8: Summary

In this chapter we have concentrated on the Bhabha equation based on the 16 and 20-dimensional representations of the group $SO(4,1)$ describing particles of maximum spin $\frac{3}{2}$. By using the methods of Lie algebras we gave matrix representations of the basis elements $\Gamma(\vec{h}_{\alpha_1})$, $\Gamma(\vec{h}_{\alpha_2})$, $\Gamma(\vec{e}_{\pm\alpha_1})$, $\Gamma(\vec{e}_{\pm\alpha_2})$, $\Gamma(\vec{e}_{\pm(\alpha_1+\alpha_2)})$, $\Gamma(\vec{e}_{\pm(\alpha_1+2\alpha_2)})$ of the complex Lie algebra B_2 associated with the 16 and 20-dimensional representations of $SO(4,1)$. It was shown then that the matrices $\mathbb{L}_k$, $k=0,1,2,3$, appearing in the Bhabha equation can be expressed in terms of the basis elements of B_2 by the linear combinations

$$\mathbb{L}_0 = \iota\sqrt{3}\,\Gamma(\vec{e}_{\alpha_2}) + \iota\sqrt{3}\,\Gamma(e_{-\alpha_2}),$$

$$\mathbb{L}_1 = \iota\frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{\alpha_1}) - \iota\frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{-\alpha_1}) + \iota\frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{\alpha_1+2\alpha_2}) - \iota\frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{-\alpha_1-2\alpha_2}),$$

$$\mathbb{L}_2 = -\frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{\alpha_1}) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{-\alpha_1}) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{\alpha_1+2\alpha_2}) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{-\alpha_1-2\alpha_2}),$$

$$\mathbb{L}_3 = -6\iota\,\Gamma(\vec{h}_{\alpha_2}).$$

These matrices were given explicitly. Then by means of the matrix $\mathbb{L}_0$ the hermitianizing matrix $\mathbb{A}$ was constructed employing the formula

$$\mathbb{A} = \frac{1}{3}\mathbb{L}_0\left\{4(\mathbb{L}_0)^2 - 7\right\}.$$

Finding the eigenvalues of the matrix $\mathbb{L}_0$ the masses $m_\iota = \frac{\chi}{\lambda_\iota}$ of the particles were determined.

Finally it was shown that the hermitianizing matrix $\mathbb{A}$ satisfies the property $\mathbb{T}^{+}\mathbb{A}\mathbb{T} = \mathbb{A}$ for every transformation of the group $SO(4,1)$ which belongs also to the group $SO(3,1)$.

CHAPTER 2.

Charge of the Bhabha field.

In this chapter we shall use some of our previous results to study the charge associated with the Bhabha field.

1: Charge of a particle of spin 3/2.

Case a) 20-dim. representation:

With every Bhabha wave-equation one can associate the quantity S_0 known as the charge density and given by the formula

$$S_0 = \psi^{+} \mathbb{A} \mathbb{L}_0 \psi ,$$

where ψ is a vector with the same number of components as the dimension of the representation, ψ^{+} its hermitian conjugate, $\mathbb{A}$ the hermitianizing matrix and $\mathbb{L}_0$ the Bhabha matrix. The total charge is given by

$$S = \int S_0 \, dv \qquad , \quad (dv = \text{volume element}).$$

Since the charge is independent of the frame of reference, it is better to consider that frame in which the matrix $\mathbb{A}\mathbb{L}_0$ is diagonal. If Λ_n, $n = 1,2\ldots 20$, are the eigenvalues of $\mathbb{A}\mathbb{L}_0$ and ψ_n the components of ψ then

$$S_0 = \psi^{+} (\mathbb{A}\mathbb{L}_0)_{diag} \psi = \sum_{n=1}^{20} \Lambda_n \psi_n^{*} \psi_n .$$

This is positive definite if and only if all the eigenvalues $\Lambda_n > 0$.

To find the eigenvalues of $\mathbb{A}\mathbb{L}_0$ we use the following theorem: If $\mathbb{A} = f(\mathbb{L}_0)$ and λ is an eigenvalue of $\mathbb{L}_0$ then $f(\lambda)$ is an eigenvalue of $\mathbb{A}$ and $\lambda f(\lambda) = \Lambda$ is an eigenvalue of $\mathbb{A}\mathbb{L}_0$.

In the case of the spin 3/2 field associated with the 20-dim. representation $\mathbb{A}_{20\text{-dim}}$ has the functional form

$$\mathbb{A}_{20\text{-dim}} = f\left(\mathbb{L}_0^{20\text{-dim}}\right) = \frac{1}{3} \mathbb{L}_0^{20\text{-dim}} \left\{ 4 \left(\mathbb{L}_0^{20\text{-dim}} \right)^2 - 7 \right\},$$

and for the eigenvalues $\lambda_1 = \lambda_9 = \lambda_{13} = \lambda_{19} = \frac{3}{2}$, of $\mathbb{L}_0^{20\text{-dim}}$ we find

$$f(\lambda_1 = 3/2) = \frac{1}{3}\lambda_1\left(4\lambda_1^2 - 7\right) = 1 = f(\lambda_9) = f(\lambda_{13}) = f(\lambda_{19}).$$

Thus the corresponding eigenvalues of $\mathbb{A}_{20\text{-dim}}\mathbb{L}_0^{20\text{-dim}}$, are

$$\Lambda_1 = \lambda_1 f(\lambda_1) = \frac{3}{2} = \Lambda_9 = \Lambda_{13} = \Lambda_{19}.$$

Similarly for the other eigenvalues of $\mathbb{L}_0^{20\text{-dim}}$ we find that the corresponding eigenvalues of $\mathbb{A}_{20\text{-dim}}\mathbb{L}_0^{20\text{-dim}}$ are:

$$\Lambda_2 = \lambda_2 f\left(\lambda_2 = \frac{1}{2}\right) = \frac{1}{2}(-1) = -\frac{1}{2} = \Lambda_5 = \Lambda_8 = \Lambda_{11} = \Lambda_{15} = \Lambda_{17},$$

$$\Lambda_3 = \lambda_3 f\left(\lambda_3 = -\frac{3}{2}\right) = -\frac{3}{2}(-1) = \frac{3}{2} = \Lambda_{10} = \Lambda_{14} = \Lambda_{20},$$

$$\Lambda_4 = \lambda_4 f\left(\lambda_4 = -\frac{1}{2}\right) = -\frac{1}{2}(1) = -\frac{1}{2} = \Lambda_6 = \Lambda_7 = \Lambda_{12} = \Lambda_{16} = \Lambda_{18}.$$

Substituting in $S_0 = \psi^{\dagger}\left(\mathbb{A}_{20\text{-dim}}\mathbb{L}_0^{20\text{-dim}}\right)_{diag}\psi$ we find for the charge density

$$\begin{aligned} S_0 = {} & \frac{3}{2}\psi_1^*\psi_1 - \frac{1}{2}\psi_2^*\psi_2 + \frac{3}{2}\psi_3^*\psi_3 - \frac{1}{2}\psi_4^*\psi_4 - \frac{1}{2}\psi_5^*\psi_5 - \\ & - \frac{1}{2}\psi_6^*\psi_6 - \frac{1}{2}\psi_7^*\psi_7 - \frac{1}{2}\psi_8^*\psi_8 + \frac{3}{2}\psi_9^*\psi_9 + \frac{3}{2}\psi_{10}^*\psi_{10} - \\ & - \frac{1}{2}\psi_{11}^*\psi_{11} - \frac{1}{2}\psi_{12}^*\psi_{12} + \frac{3}{2}\psi_{13}^*\psi_{13} + \frac{3}{2}\psi_{14}^*\psi_{14} - \frac{1}{2}\psi_{15}^*\psi_{15} - \\ & - \frac{1}{2}\psi_{16}^*\psi_{16} - \frac{1}{2}\psi_{17}^*\psi_{17} - \frac{1}{2}\psi_{18}^*\psi_{18} + \frac{3}{2}\psi_{19}^*\psi_{19} + \frac{3}{2}\psi_{20}^*\psi_{20}. \end{aligned}$$

This is not definite, and hence the charge is not definite.

Case b) 16-dimensional representation:

In this case again the functional form of $\mathbb{A}$ is

$$\mathbb{A}_{16\text{-dim}} = \frac{1}{3}\mathbb{L}_0^{16\text{-dim}}\left\{4\left(\mathbb{L}_0^{16\text{-dim}}\right)^2 - 7\right\}$$

and both $\mathbb{A}_{16\text{-dim}}$ and $\mathbb{L}_0^{16\text{-dim}}$ were given in chapter 1. For the eigenvalues of $\mathbb{A}_{16\text{-dim}}\mathbb{L}_0^{16\text{-dim}}$ we find

$$\Lambda_1=\Lambda_2=\Lambda_3=\Lambda_4=\Lambda_7=\Lambda_8=\Lambda_9=\Lambda_{10}=\Lambda_{13}=\Lambda_{14}=\Lambda_{15}=\Lambda_{16}=-\frac{1}{2},$$

$$\Lambda_5=\Lambda_6=\Lambda_{11}=\Lambda_{12}=\frac{3}{2}.$$

The charge density is

$$S_0=\psi^+\left(\mathbb{A}_{16\text{-dim}}\mathbb{L}_0^{16\text{-dim}}\right)_{\text{diag}}\psi=-\frac{1}{2}\psi_1^*\psi_1-\frac{1}{2}\psi_2^*\psi_2-\frac{1}{2}\psi_3^*\psi_3-\frac{1}{2}\psi_4^*\psi_4+$$

$$+\frac{3}{2}\psi_5^*\psi_5+\frac{3}{2}\psi_6^*\psi_6-\frac{1}{2}\psi_7^*\psi_7-\frac{1}{2}\psi_8^*\psi_8-\frac{1}{2}\psi_9^*\psi_9-\frac{1}{2}\psi_{10}^*\psi_{10}+$$

$$+\frac{3}{2}\psi_{11}^*\psi_{11}+\frac{3}{2}\psi_{12}^*\psi_{12}-\frac{1}{2}\psi_{13}^*\psi_{13}-\frac{1}{2}\psi_{14}^*\psi_{14}-\frac{1}{2}\psi_{15}^*\psi_{15}-\frac{1}{2}\psi_{16}^*\psi_{16}.$$

This is not definite.

2: Charge associated with the Bhabha-Gupta field.

The Bhabha-Gupta field describes particles with two different mass states $m_{1,2}=\pm\frac{\chi}{3/2}$, $m_{3,4}=\pm\frac{\chi}{1/2}$ and spins 3/2 and 1/2.

The space of the 20-dimensional representation of the field is reduced to the direct sum of a 16-dimensional and a 4-dimensional (Dirac) subspaces, and the matrix $\mathbb{A}_{20\text{-dim}}$ of the Bhabha-Gupta field breaks into the blocks $\mathbb{A}_{16\text{-dim}}$ and $\mathbb{A}_{4\text{-dim}}=\gamma_0$, where γ_0 is the Dirac matrix $\gamma_0=\text{diag}\{1,1,-1,-1\}$. Similarly the matrix $\mathbb{L}_0^{20\text{-dim}}$ of the Bhabha-Gupta field breaks into the blocks $\mathbb{L}_0^{16\text{-dim}}$ and $\mathbb{L}_0^{4\text{-dim}}=\gamma_0$ and hence the matrix $\mathbb{A}_{20\text{-dim}}\mathbb{L}_0^{20\text{-dim}}$ of the Bhabha-Gupta field admits the blocks $\mathbb{A}_{16\text{-dim}}\mathbb{L}_0^{16\text{-dim}}$ and $\mathbb{A}_{4\text{-dim}}\mathbb{L}_0^{4\text{-dim}}=\gamma_0\cdot\gamma_0$. The eigenvalues Λ_i corresponding to these blocks are respectively

$$\Lambda_1=\Lambda_2=\Lambda_3=\Lambda_4=\Lambda_7=\Lambda_8=\Lambda_9=\Lambda_{10}=\Lambda_{13}=\Lambda_{14}=\Lambda_{15}=\Lambda_{16}=-\frac{1}{2},$$

$$\Lambda_5=\Lambda_6=\Lambda_{11}=\Lambda_{12}=\frac{3}{2}$$

and

$$\Lambda_{17} = \Lambda_{18} = \Lambda_{19} = \Lambda_{20} = 1 .$$

The charge density is

$$S_0 = \psi^+ \left(\mathbb{A}_{20\text{-dim}} \mathbb{L}_0^{20\text{-dim}} \right) \psi = -\tfrac{1}{2} \psi_1^* \psi_1 - \tfrac{1}{2} \psi_2^* \psi_2 - \tfrac{1}{2} \psi_3^* \psi_3 - \\ -\tfrac{1}{2} \psi_4^* \psi_4 + \tfrac{3}{2} \psi_5^* \psi_5 + \tfrac{3}{2} \psi_6^* \psi_6 - \tfrac{1}{2} \psi_7^* \psi_7 - \tfrac{1}{2} \psi_8^* \psi_8 - \tfrac{1}{2} \psi_9^* \psi_9 - \\ -\tfrac{1}{2} \psi_{10}^* \psi_{10} + \tfrac{3}{2} \psi_{11}^* \psi_{11} + \tfrac{3}{2} \psi_{12}^* \psi_{12} - \tfrac{1}{2} \psi_{13}^* \psi_{13} - \tfrac{1}{2} \psi_{14}^* \psi_{14} - \tfrac{1}{2} \psi_{15}^* \psi_{15} - \\ -\tfrac{1}{2} \psi_{16}^* \psi_{16} + \left(1 \psi_{17}^* \psi_{17} + 1 \psi_{18}^* \psi_{18} + 1 \psi_{19}^* \psi_{19} + 1 \psi_{20}^* \psi_{20} \right).$$

This is not definite.

3: Charge associated with the Bhabha field for any half integer spin.

We have seen in detail above that the charge associated with the spin 3/2 Bhabha field is indefinite. The question which arises next is: what is the charge associated with the Bhabha field for spins higher than 3/2. The answer is given in this paragraph.

As we have said already the charge density is given by the formula $S_0 = \psi^+ \mathbb{A} \mathbb{L}_0 \psi$. To be able to find this for any spin we need to know the eigenvalues of the product $\mathbb{A} \mathbb{L}_0$. An explicit expression of $\mathbb{L}_0$ is not necessary because in the case of the Bhabha field for any spin S the matrix $\mathbb{L}_0$ is diagonalizable and has eigenvalues $\pm s , \pm (s-1) , \pm (s-2) \cdots$ with multiplicities $\geqslant 1$. An expression for $\mathbb{A}$ as a function of $\mathbb{L}_0$ for any spin S which also gives the eigenvalues of $\mathbb{A}$, corresponding to the eigenvalues of $\mathbb{L}_0$ is given by Madava-Rao <u>et al</u>.[23] and is

$$\mathbb{A}(s) = f(\mathbb{L}_0 , s) \equiv f(x, s) =$$

$$= \frac{(x-s)(x-s+1)\cdots(x+s-1)(x+s)}{2s\,!} \sum_{n=0}^{2s} \binom{2s}{n} \frac{1}{(x-s+n)} ,$$

where S = maximum value of spin associated with the considered field and x any of the eigenvalues of $\mathbb{L}_0$. For S half integer which is what we shall be concerned with in this paragraph the above formula reduces to

$$f(x,s)=\frac{2x\left(x^2-\frac{1}{4}\right)\left(x^2-\frac{9}{4}\right)\cdots\left(x^2-s^2\right)}{(2s)!}\sum_{n=1}^{(s+1/2)}\binom{2s}{s+\frac{1}{2}-n}\frac{1}{\left[x^2-\left(n-\frac{1}{2}\right)^2\right]}\,. \quad (1)$$

As it has been said for a knowledge of the charge associated with a particular Bhabha field corresponding to maximum spin S, we need to know the eigenvalues $\Lambda = x f(x,s)$ of $\mathbb{A}\mathbb{L}_0$. Thus let us consider (1) and use it to study the charge associated with the Bhabha fields of higher spin. Before dealing with the general case we shall study a particular case which will let us see what kind of regularities exist in calculating the eigenvalues $f(x,s)$ of $\mathbb{A}$ corresponding to the different eigenvalues of $\mathbb{L}_0$. These regularities will help us to find a simpler formula expressing $f(x,S)$ which in turn will be used to study the charge in the general case.

Bhabha field for spin 5/2.

Let us consider (1) and write it for the case $S=\frac{5}{2}$. We have

$$f\left(x,s=\tfrac{5}{2}\right)=\frac{2x\left(x^2-1/4\right)\left(x^2-9/4\right)\left(x^2-25/4\right)}{5!}\sum_{n=1}^{3}\binom{5}{3-n}\frac{1}{\left[x^2-\left(n-\frac{1}{2}\right)^2\right]}=$$

$$=\frac{2x\left(x^2-1/4\right)\left(x^2-9/4\right)\left(x^2-25/4\right)}{5!}\left\{\frac{10}{\left[x^2-\frac{1}{4}\right]}+\frac{5}{\left[x^2-\frac{9}{4}\right]}+\frac{1}{\left[x^2-\frac{25}{4}\right]}\right\}=$$

$$=\left\{\frac{2x\left(x^2-9/4\right)\left(x^2-25/4\right)}{5!}10+\frac{2x\left(x^2-1/4\right)\left(x^2-25/4\right)}{5!}5+\frac{2x\left(x^2-1/4\right)\left(x^2-9/4\right)}{5!}\right\}. \quad (2)$$

Let us now evaluate (2) for the different eigenvalues of $\mathbb{L}_0$ namely $x=\pm\frac{1}{2}$, $x=\pm\frac{3}{2}$, $x=\pm\frac{5}{2}$.

$x=\pm\frac{1}{2}$:

For the eigenvalue $x = 1/2$ only the first term in the summation (2) survives. The other two terms vanish because the factor $(x^2 - 1/4)$ for $x = 1/2$ vanishes.

Thus

$$f(x = \tfrac{1}{2}, s = \tfrac{5}{2}) = \frac{2(\frac{1}{2})(\frac{1}{4} - \frac{9}{4})(\frac{1}{4} - \frac{25}{4})}{5!} 10 = 1.$$

Likewise $f(x = -1/2, s = 5/2) = -1.$

$x = \pm \frac{3}{2}$:

For the eigenvalue $x = 3/2$ only the second term in the summation (2) survives. The other two terms vanish because the factor $(x^2 - 9/4)$ vanishes.

Thus $f(x = 3/2, s = 5/2) = \dfrac{2(\frac{3}{2})(\frac{9}{4} - \frac{1}{4})(\frac{9}{4} - \frac{25}{4})}{5!} 5 = -1.$

Likewise $f(x = -3/2, s = 5/2) = 1.$

$x = \pm \frac{5}{2}$:

For the eigenvalue $x = 5/2$ only the third term in (2) survives. The other two terms vanish because $(x^2 - 25/4)$ vanishes.

Thus $f(x = 5/2, s = 5/2) = \dfrac{2(5/2)(25/4 - 1/4)(25/4 - 9/4)}{5!} = 1.$

Likewise $f(x = -5/2, s = 5/2) = -1.$

Having found the eigenvalues of $\mathbb{A}$ then the different eigenvalues $\Lambda = x f(x, s)$ of $\mathbb{A}\mathbb{L}_0$ are:

$$\Lambda_1 = \frac{1}{2} f(x = 1/2, s = 5/2) = \frac{1}{2}, \quad \Lambda_2 = -\frac{1}{2} f(x = -1/2, s = 5/2) = \frac{1}{2},$$

$$\Lambda_3 = \frac{3}{2} f(x = 3/2, s = 5/2) = -3/2, \quad \Lambda_4 = -\frac{3}{2} f(x = -3/2, s = 5/2) = -\frac{3}{2},$$

$$\Lambda_5 = \frac{5}{2} f(x = 5/2, s = 5/2) = 5/2, \quad \Lambda_6 = -\frac{5}{2} f(x = -5/2, s = 5/2) = 5/2.$$

Thus the charge density is

$$s_0 = \psi^+ \mathbb{A}\mathbb{L}_0 \psi = \frac{1}{2}\psi_1^*\psi_1 + \frac{1}{2}\psi_2^*\psi_2 - \frac{3}{2}\psi_3^*\psi_3 - \frac{3}{2}\psi_4^*\psi_4 + \frac{5}{2}\psi_5^*\psi_5 + \cdots$$

which is indefinite. Hence the charge for $S = 5/2$ is indefinite.

Following the method outlined above we have studied also the charge in the cases $S = 7/2$, $S = 9/2$, $S = 11/2$ and found that this is again indefinite for all of them. The calculations for higher values of spin become progressively harder and thus for the study of the charge in the general case the method must be simplified.

From the example studied above in detail we observe that the eigenvalues $f(x,s)$ of $\mathbb{A}$ have either the value 1 or -1. Also we observe that in a given summation giving $f(x,s)$ only one term survives each time. These observations make it clear that formula (1) can be simplified further.

The simpler formula which can be used to calculate the eigenvalues of $\mathbb{A}$ corresponding to the eigenvalues of $\mathbb{L}_0$ is

$$f(x,s) = (-1)^{s-p-\frac{1}{2}} \qquad (3)$$

where p is a number related to x by the formula $x=p+\frac{1}{2}$ and $x, s,$ have the same meaning as before.

<u>Proof</u>: Let us consider the formula

$$\frac{2x(x^2-1/4)(x^2-9/4)\cdots(x^2-s^2)}{\lfloor\!\underline{2s}} \sum_{n=1}^{(s+1/2)} \binom{2s}{s+\frac{1}{2}-n} \frac{1}{[x^2-(n-\frac{1}{2})^2]} ,$$

where we are using $\lfloor\!\underline{\quad}$ for the factorial, in order to avoid confusion which might arise from the use of $(!)$. In the above formula for fixed x the number n takes the values $1, 2, 3, \cdots (s-\frac{1}{2}), (s+\frac{1}{2})$. The variable x takes the values $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots \pm(s-1), \pm s$. In each summation (obtained for fixed x and n running from 1 to $s+\frac{1}{2}$ only one term survives. Finally all the non-zero contributions of the above formula when x takes values from $\pm\frac{1}{2}$ to $\pm s$ correspond to the following combinations of x and n :

$$\left(x=\pm\frac{1}{2}, n=1\right), \left(x=\pm\frac{3}{2}, n=2\right), \left(x=\pm\frac{5}{2}, n=3\right)\cdots$$

$$\cdots\left(x=\pm\frac{q}{2}, n=\frac{q}{2}+\frac{1}{2}\right)\cdots \left(x=\pm(s-1), n=\left(s-\frac{1}{2}\right)\right), \left(x=\pm s, n=s+\frac{1}{2}\right).$$

Let us now calculate the contribution corresponding to $\left(x=\pm\frac{q}{2},\ n=\frac{q}{2}+\frac{1}{2}\right)$ where $\frac{q}{2}$ indicates an arbitrary choice of x in the interval $[-s,\ s]$.

For $x=\frac{q}{2}$, $n=\frac{q}{2}+\frac{1}{2}$, the surviving term is:

$$\frac{2x\left(x^2-\frac{1}{4}\right)\left(x^2-\frac{9}{4}\right)\cdots\left(x^2-\left(\frac{q}{2}-2\right)^2\right)\left(x^2-\left(\frac{q}{2}-1\right)^2\right)\left(x^2-\left(\frac{q}{2}+1\right)^2\right)\left(x^2-\left(\frac{q}{2}+2\right)^2\right)\cdots\left(x^2-(s-1)^2\right)\left(x^2-s^2\right)\underline{\lfloor 2s}}{\underline{\lfloor 2s}\;\;\underline{\lfloor s-q/2}\;\;\underline{\lfloor s+q/2}}=$$

$$=\frac{2x(x-s)(x-s+1)\cdots\left(x-\left(\frac{q}{2}+2\right)\right)\left(x-\left(\frac{q}{2}+1\right)\right)\left(x-\left(\frac{q}{2}-1\right)\right)\left(x-\left(\frac{q}{2}-2\right)\right)\cdots\left(x-\frac{3}{2}\right)\left(x-\frac{1}{2}\right)\cdot}{1}$$

$$\cdot\frac{\left(x+\frac{1}{2}\right)\left(x+\frac{3}{2}\right)\cdots\left(x+\left(\frac{q}{2}-2\right)\right)\left(x+\left(\frac{q}{2}-1\right)\right)\left(x+\left(\frac{q}{2}+1\right)\right)\left(x+\left(\frac{q}{2}+2\right)\right)\cdots(x+s-1)(x+s)}{\underline{\lfloor s-q/2}\;\;\underline{\lfloor s+q/2}}=$$

$$=\frac{2\,\frac{q}{2}\left(\frac{q}{2}-s\right)\left(\frac{q}{2}-s+1\right)\cdots\left(\frac{q}{2}-\frac{q}{2}-2\right)\left(\frac{q}{2}-\frac{q}{2}-1\right)\left(\frac{q}{2}-\frac{q}{2}+1\right)\left(\frac{q}{2}-\frac{q}{2}+2\right)\cdots\left(\frac{q}{2}-\frac{3}{2}\right)\left(\frac{q}{2}-\frac{1}{2}\right)\cdot}{1}$$

$$\cdot\frac{\left(\frac{q}{2}+\frac{1}{2}\right)\left(\frac{q}{2}+\frac{3}{2}\right)\cdots\left(\frac{q}{2}+\frac{q}{2}-2\right)\left(\frac{q}{2}+\frac{q}{2}-1\right)\left(\frac{q}{2}+\frac{q}{2}+1\right)\left(\frac{q}{2}+\frac{q}{2}+2\right)\cdots\left(\frac{q}{2}+s-1\right)\left(\frac{q}{2}+s\right)}{\underline{\lfloor s-q/2}\;\;\underline{\lfloor s+q/2}}=$$

$$=\frac{q\left(\frac{q}{2}-s\right)\left(\frac{q}{2}-s+1\right)\cdots(-2)(-1)(+1)(+2)\cdots\left(\frac{q}{2}-\frac{3}{2}\right)\left(\frac{q}{2}-\frac{1}{2}\right)\left(\frac{q}{2}+\frac{1}{2}\right)\left(\frac{q}{2}+\frac{3}{2}\right)\cdots}{1}$$

$$\cdot\frac{\cdots(q-2)(q-1)(q+1)(q+2)\cdots\left(\frac{q}{2}+s-1\right)\left(s+\frac{q}{2}\right)}{\underline{\lfloor s-q/2}\;\;\underline{\lfloor s+q/2}}=$$

$$=\frac{q\left\{-\left(s-\frac{q}{2}\right)\right\}\left\{-\left(s-\frac{q}{2}-1\right)\right\}\cdots\{-2\}\{-1\}(+1)(+2)\cdots\left(\frac{q}{2}-\frac{3}{2}\right)\left(\frac{q}{2}-\frac{1}{2}\right)\left(\frac{q}{2}+\frac{1}{2}\right)\left(\frac{q}{2}+\frac{3}{2}\right)\cdots}{1}$$

$$\cdot\frac{\cdots(q-2)(q-1)(q+1)(q+2)\cdots\left(\frac{q}{2}+s-1\right)\left(s+\frac{q}{2}\right)}{\underline{\lfloor s-q/2}\;\;\underline{\lfloor s+q/2}}\,.$$

Shifting q from the front to the position between $(q-1)$ and $(q+1)$ and observing that the number of negative signs is $(s-q/2)$ (and hence a sign $(-1)^{s-q/2}$ comes out at the front) and also observing that the descending and ascending terms make factorials we find

$$f(x=\frac{q}{2}, s) = (-1)^{s-q/2} \frac{|\underline{s-q/2}\;|\underline{s+q/2}}{|\underline{s-q/2}\;|\underline{s+q/2}} = (-1)^{s-q/2}.$$

Similarly working we find for $x=-q/2$, $n=q/2+1/2$ that the surviving term has the value

$$f(x=-q/2, s) = (-1)^{s+q/2}.$$

Hence for any x with $x=\pm\frac{1}{2}, \pm\frac{3}{2}, \dots \pm s$ we find

$$f(x,s) = (-1)^{s-x}.$$

Putting $x=p+\frac{1}{2}$ where p a number we have

$$f(x,s) = (-1)^{s-p-\frac{1}{2}},$$

which is what we wanted to prove.

As applications of this formula we use it to study the charge of the Bhabha fields for $s=1/2$, $s=3/2$ and the general case $s=$ half integer.

1) Dirac field for spin 1/2.

The eigenvalues of $\mathbb{L}_0$ are $x=\pm\frac{1}{2}$ (both double).

For $x=\frac{1}{2} \Rightarrow p=0$ and $f(x=\frac{1}{2}, s=\frac{1}{2}) = (-1)^{1/2-1/2} = (-1)^0 = 1.$

For $x=-\frac{1}{2} \Rightarrow p=-1$ and $f(x=-\frac{1}{2}, s=\frac{1}{2}) = (-1)^{1/2+1-1/2} = (-1)^1 = -1.$

Hence $\Lambda_l = x\,f(x,s)$ are respectively:

$$\Lambda_1=\Lambda_2=\frac{1}{2}, \quad \Lambda_3=\Lambda_4=\frac{1}{2},$$

and the charge density is then

$$s_0 = \frac{1}{2}\left(\psi_1^*\psi_1+\psi_2^*\psi_2+\psi_3^*\psi_3+\psi_4^*\psi_4\right),$$

which gives definite charge for the Dirac equation.

2) Bhabha field for spin 3/2.

The eigenvalues of $\mathbb{L}_0$ are $x = \pm\frac{1}{2}$, $\pm\frac{3}{2}$,

For $x = \frac{1}{2} \Rightarrow p = 0$ and $f(x=\frac{1}{2}, s=\frac{3}{2}) = (-1)^{3/2-1/2} = (-1)^1 = -1.$

For $x = -\frac{1}{2} \Rightarrow p = -1$ and $f(x=-\frac{1}{2}, s=\frac{3}{2}) = (-1)^{3/2+1-1/2} = (-1)^2 = 1.$

For $x = \frac{3}{2} \Rightarrow p = 1$ and $f(x=\frac{3}{2}, s=\frac{3}{2}) = (-1)^{3/2-1-1/2} = (-1)^0 = 1.$

For $x = -\frac{3}{2} \Rightarrow p = -2$ and $f(x=-\frac{3}{2}, s=\frac{3}{2}) = (-1)^{3/2+2-1/2} = (-1)^3 = -1.$

Hence $\Lambda_l = x f(x,s)$ are respectively:

$$\Lambda_1 = \tfrac{1}{2}(-1) = -\tfrac{1}{2} \quad , \Lambda_2 = -\tfrac{1}{2}(1) = -\tfrac{1}{2} \ ,$$

$$\Lambda_3 = \tfrac{3}{2}(1) = \tfrac{3}{2} \quad , \Lambda_4 = -\tfrac{3}{2}(-1) = \tfrac{3}{2} \ ,$$

and the charge density is

$$S_0 = -\frac{1}{2}\psi_1^*\psi_1 - \frac{1}{2}\psi_2^*\psi_2 + \frac{3}{2}\psi_3^*\psi_3 + \frac{3}{2}\psi_4^*\psi_4 + \cdots$$

which gives indefinite charge for the Bhabha field of spin 3/2. The cases $S = 5/2$, $s = 7/2$, $s = 9/2$, $s = 11/2$ were also studied and the charge was found to be indefinite.

General case S = half integer $> 1/2$.

$\mathbb{L}_0$ has eigenvalues $\pm s$, $\pm(s-1)$, $\pm(s-2)\cdots$ with multiplicities $\geqslant 1$. Using formula (3) we find that the hermitianizing matrix $\mathbb{A}$ has eigenvalues:

for $x = s \Rightarrow p = s - \frac{1}{2}$ and $f(x=s,s) = (-1)^{s-s+1/2-1/2} = (-1)^0 = 1$;

for $x = -s \Rightarrow p = -s - \frac{1}{2}$ and $f(x=-s,s) = (-1)^{s+s+1/2-1/2} = (-1)^{2s} = -1$,

(since S is half integer and $2S$ is then an odd number);

for $x = s-1 \Rightarrow p = s - 1 - \frac{1}{2}$ and $f(x=s-1,s) = (-1)^{s-s+1+1/2-1/2} = (-1)^1 = -1$;

for $x = -s+1 \Rightarrow p = -s+1-\frac{1}{2}$ and $f(x=-s+1,s) = (-1)^{s+s-1+1/2-1/2} = (-1)^{2s-1} = 1$,

(since $2S$ is odd and hence $2S-1$ is even);

for $x = s-2 \Rightarrow p = s-2-\frac{1}{2}$ and $f(x=s-2,s) = (-1)^{s-s+2+1/2-1/2} = (-1)^2 = 1$;

for $x = -s+2 \Rightarrow p = -s+2-\frac{1}{2}$ and $f(x=-s+2,s) = (-1)^{s+s-2+1/2-1/2} = (-1)^{2s-2} = -1$,

(since $2S-2$ is odd)

and so on. The corresponding eigenvalues of $\mathbb{A}\mathbb{L}_0$ are $\Lambda_1 = s$, $\Lambda_2 = s$, $\Lambda_3 = -(s-1)$, $\Lambda_4 = -(s-1)$, $\Lambda_5 = (s-2)$, $\Lambda_6 = (s-2)\cdots$. Hence the charge density is

$$s_0 = \psi^+ \mathbb{A}\mathbb{L}_0 \psi = s\psi_1^*\psi_1 + s\psi_2^*\psi_2 - (s-1)\psi_3^*\psi_3 - (s-1)\psi_4^*\psi_4 + $$
$$+ (s-2)\psi_5^*\psi_5 + (s-2)\psi_6^*\psi_6 + \cdots$$

which gives an indefinite charge.

4: Summary

In this chapter we were concerned with the charge of the Bhabha fields. It was verified that the charge of the Bhabha field for spin 3/2 based either on the 16 or 20-dimensional representations is indefinite. It was shown also that the charge of the Bhabha-Gupta field is indefinite. Then using the fact that for any spin the matrix $\mathbb{L}_0$ of the Bhabha wave-equation can be written in diagonal form with the elements on the main diagonal being the values of the spin and using also the fact that the hermitianizing matrix $\mathbb{A}$ likewise can be written in diagonal form with elements given by the formula

$$f(x,s) = (-1)^{s-p-\frac{1}{2}} ,$$

we were able to prove that the charge of any Bhabha wave-equation for higher spin with half integer value is indefinite. Examples illustrating this were studied, covering values of spin from 3/2 to $11/2$. In the case of the Dirac wave-equation ($s = \frac{1}{2}$) the charge is definite.

PART II.

CHAPTER 3.

1: Preliminaries.

In this part we study some aspects of the Bhabha equation in the case when the representation according to which ψ transforms is a general representation. This equation is better known in the scientific literature as the Gel'fand-Yaglom equation[25-28] and is described by the system of first order differential equations

$$\mathbb{L}_0 \frac{\partial \psi}{\partial x_0} + \mathbb{L}_1 \frac{\partial \psi}{\partial x_1} + \mathbb{L}_2 \frac{\partial \psi}{\partial x_2} + \mathbb{L}_3 \frac{\partial \psi}{\partial x_3} + i\chi\psi = 0 \;,$$

where $\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$ are four matrices whose dimension depends on the representation chosen to describe the invariance of the equation. ψ is the wavefunction assuming values in the same space in which the matrices $\mathbb{L}_k$, $k=0,1,2,3$ act. Our notation is the same as that of Gel'fand, Minlos and Shapiro[28].

This and the following chapters are devoted to the study of the Gel'fand-Yaglom equation in the case when the maximum spin described by the equation is $\frac{3}{2}$. We shall concentrate on the following three representations.

1: The 20-dimensional representation with chain diagram between its irreducibles of the form

$$\begin{tikzcd}[column sep=large] \tau_1 \sim \left(\frac{3}{2}, \frac{5}{2}\right) \arrow[r, shift left, "c^{\tau_1\tau_2}"] & \tau_2 \sim \left(\frac{1}{2}, \frac{5}{2}\right) \arrow[l, shift left, "c^{\tau_2\tau_1}"] \arrow[r, shift left, "c^{\tau_2\dot{\tau}_2}"] & \dot{\tau}_2 \sim \left(-\frac{1}{2}, \frac{5}{2}\right) \arrow[l, shift left, "c^{\dot{\tau}_2\tau_2}"] \arrow[r, shift left, "c^{\dot{\tau}_2\dot{\tau}_1}"] & \dot{\tau}_1 \sim \left(-\frac{3}{2}, \frac{5}{2}\right) \arrow[l, shift left, "c^{\dot{\tau}_1\dot{\tau}_2}"] \end{tikzcd} \tag{I}$$

2: The 16-dimensional representation with chain diagram between its irreducibles of the form

$$\begin{tikzcd}[column sep=huge, row sep=huge] \left(\frac{1}{2}, \frac{3}{2}\right) \sim \tau_1 \arrow[r, shift left, "c^{\tau_1\dot{\tau}_1}"] \arrow[d, shift left, "c^{\tau_1\tau_2}"] & \dot{\tau}_1 \sim \left(-\frac{1}{2}, \frac{3}{2}\right) \arrow[l, shift left, "c^{\dot{\tau}_1\tau_1}"] \arrow[d, shift left, "c^{\dot{\tau}_1\dot{\tau}_2}"] \\ \left(\frac{1}{2}, \frac{5}{2}\right) \sim \tau_2 \arrow[u, shift left, "c^{\tau_2\tau_1}"] \arrow[r, shift left, "c^{\tau_2\dot{\tau}_2}"] & \dot{\tau}_2 \sim \left(-\frac{1}{2}, \frac{5}{2}\right) \arrow[l, shift left, "c^{\dot{\tau}_2\tau_2}"] \arrow[u, shift left, "c^{\dot{\tau}_2\dot{\tau}_1}"] \end{tikzcd} \tag{II}$$

.

3: The 20-dimensional representation with chain diagram between its irreducibles of the form

$$\begin{tikzcd}
\left(\frac{1}{2},\frac{5}{2}\right)\sim\tau_2 \arrow[rr, shift left, leftarrow, "c^{\tau_1\tau_2}"] \arrow[rr, shift right, "c^{\tau_2\tau_1}"'] \arrow[dr, shift left, "c^{\tau_2\tau_1'}"] \arrow[dr, shift right, leftarrow, "c^{\tau_1'\tau_2}"'] \arrow[ddd, shift right, leftarrow, "c^{\dot\tau_2\tau_2}"'] \arrow[ddd, shift left, "c^{\tau_2\dot\tau_2}"] & & \tau_1\sim\left(\frac{1}{2},\frac{3}{2}\right) \arrow[ddd, shift right, leftarrow, "c^{\dot\tau_1\tau_1}"'] \arrow[ddd, shift left, "c^{\tau_1\dot\tau_1}"] \arrow[ddl, shift right, leftarrow, "c^{\dot\tau_1'\tau_1}"'] \arrow[ddl, shift left, "c^{\tau_1\dot\tau_1'}"] \\
& \tau_1'\sim\left(\frac{1}{2},\frac{3}{2}\right) \arrow[d, shift right, leftarrow, "c^{\dot\tau_1'\tau_1'}"'] \arrow[d, shift left, "c^{\tau_1'\dot\tau_1'}"] \arrow[ddr, shift left, "c^{\tau_1'\dot\tau_1}"] \arrow[ddr, shift right, leftarrow, "c^{\dot\tau_1\tau_1'}"'] & \\
& \dot\tau_1'\sim\left(-\frac{1}{2},\frac{3}{2}\right) & \\
\left(-\frac{1}{2},\frac{5}{2}\right)\sim\dot\tau_2 \arrow[uur, shift left, "c^{\dot\tau_2\dot\tau_1'}"] \arrow[uur, shift right, leftarrow, "c^{\dot\tau_1'\dot\tau_2}"'] \arrow[rr, shift left, "c^{\dot\tau_2\dot\tau_1}"] \arrow[rr, shift right, leftarrow, "c^{\dot\tau_1\dot\tau_2}"'] & & \dot\tau_1\sim\left(-\frac{1}{2},\frac{5}{2}\right)
\end{tikzcd} \qquad (\mathrm{III})$$

Our primary concern is to find examples of the Gel'fand-Yaglom equation based on these representations having definite charge.

In part (I) we saw that none of the Bhabha equations whose underlying representation belongs to the group $SO(4,1)$ has definite charge. This class of equations is characterized by a diagonalizable matrix $\mathbb{L}_0$ and as far as the charge is concerned they form a particular case of a general theorem stating that all Gel'fand-Yaglom equations with diagonalizable matrix $\mathbb{L}_0$ have indefinite charge except the Dirac equation. Thus we shall be interested in finding examples with non-diagonalizable matrix $\mathbb{L}_0$ having definite charge. In the present chapter we shall be concerned with those equations for which the wavefunction Ψ transforms according to the 20-dimensional representation with components

$$\tau_1\sim\left(\frac{3}{2},\frac{5}{2}\right) \ , \ \dot\tau_1\sim\left(-\frac{3}{2},\frac{5}{2}\right) , \ \tau_2\sim\left(\frac{1}{2},\frac{5}{2}\right) , \ \dot\tau_2\sim\left(-\frac{1}{2},\frac{5}{2}\right)$$

interlocking according to the scheme (I).

To start with, we need to find $\mathbb{L}_0$. We use as basis for expressing $\mathbb{L}_0$ the canonical basis $\{\xi_{\ell m}\}$, namely that basis in which the operator of rotations in the z-direction is diagonal. The elements of the matrix $\mathbb{L}_0$ with respect to the canonical basis[29] have the form

$$c^{\tau\tau'}_{\ell m,\ell' m'} = c^{\tau\tau'}_{\ell}\,\delta_{\ell\ell'}\,\delta_{mm'} ,$$

where the numbers $c^{\tau\tau'}_{\ell}$ are different from zero, only in the case when the components $\tau=(\ell_0,\ell_1)$ and $\tau'=(\ell_0',\ell_1')$ of the representation under which Ψ transforms are interlocking.

The numbers $c^{\tau\tau'}_{\ell}$ are given by the formulae:

1) For $(\ell_0',\ell_1')=(\ell_0+1,\ell_1)$:

$$c^{\tau\tau'}_{\ell} = c^{\tau\tau'}\sqrt{(\ell+\ell_0+1)(\ell-\ell_0)} , \qquad (1)$$

$$c^{\tau'\tau}_{\ell} = c^{\tau'\tau}\sqrt{(\ell+\ell_0+1)(\ell-\ell_0)} . \qquad (2)$$

2) For $(\ell_0',\ell_1')=(\ell_0,\ell_1+1)$:

$$c^{\tau\tau'}_{\ell} = c^{\tau\tau'}\sqrt{(\ell+\ell_1+1)(\ell-\ell_1)} , \qquad (3)$$

$$c^{\tau'\tau}_{\ell} = c^{\tau'\tau}\sqrt{(\ell+\ell_1+1)(\ell-\ell_1)} , \qquad (4)$$

where $c^{\tau\tau'}$ and $c^{\tau'\tau}$ are arbitrary complex numbers. In all the other cases, $c^{\tau\tau'}=0$, $c^{\tau'\tau}=0$.

By interlocking representations τ,τ' we mean that the pairs (ℓ_0,ℓ_1) and (ℓ_0',ℓ_1') characterizing these representations are either connected according to the scheme

$$(\ell_0',\ell_1')=(\ell_0\pm1,\ell_1)$$

or according to the scheme

$$(\ell_0',\ell_1')=(\ell_0,\ell_1\pm1).$$

Similar formulae exist for the matrices $\mathbb{L}_1,\mathbb{L}_2,\mathbb{L}_3$.

Using the formulae (1), (2), (3), (4) and the canonical basis

$$\{\xi_{\ell m}\} \equiv \{ \xi^{\tau_1}_{3/2,3/2}, \xi^{\tau_1}_{3/2,1/2}, \xi^{\tau_1}_{3/2,-1/2}, \xi^{\tau_1}_{3/2,-3/2}, \xi^{\tau_2}_{1/2,1/2},$$

$$\xi^{\tau_2}_{1/2,-1/2}, \xi^{\tau_2}_{3/2,3/2}, \xi^{\tau_2}_{3/2,1/2}, \xi^{\tau_2}_{3/2,-1/2}, \xi^{\tau_2}_{3/2,-3/2},$$

$$\xi^{\dot{\tau}_2}_{1/2,1/2}, \xi^{\dot{\tau}_2}_{1/2,-1/2}, \xi^{\dot{\tau}_2}_{3/2,3/2}, \xi^{\dot{\tau}_2}_{3/2,1/2}, \xi^{\dot{\tau}_2}_{3/2,-1/2},$$

$$\xi^{\dot{\tau}_2}_{3/2,-3/2}, \xi^{\dot{\tau}_1}_{3/2,3/2}, \xi^{\dot{\tau}_1}_{3/2,1/2}, \xi^{\dot{\tau}_1}_{3/2,-1/2}, \xi^{\dot{\tau}_1}_{3/2,-3/2} \} ,$$

associated with the 20-dimensional representation under consideration, we find for the elements of $\mathbb{L}_0$,

a) $\underline{\text{for} \quad \tau_1 \sim (3/2, 5/2) \longleftrightarrow \tau_2 \sim (1/2, 3/2)}$

$$c^{\tau_1\tau_2}_{3/2} = c^{\tau_1\tau_2}\sqrt{3} \quad , \quad c^{\tau_2\tau_1}_{3/2} = c^{\tau_2\tau_1}\sqrt{3} ,$$

b) $\underline{\text{for} \quad \tau_2 \sim (1/2, 5/2) \longleftrightarrow \dot{\tau}_2 \sim (-1/2, 5/2)}$

$$c^{\tau_2\dot{\tau}_2}_{1/2} = c^{\tau_2\dot{\tau}_2} \quad , \quad c^{\dot{\tau}_2\tau_2}_{1/2} = c^{\dot{\tau}_2\tau_2} , \quad c^{\tau_2\dot{\tau}_2}_{3/2} = 2c^{\tau_2\dot{\tau}_2} , \quad c^{\dot{\tau}_2\tau_2}_{3/2} = 2c^{\dot{\tau}_2\tau_2} ,$$

c) $\underline{\text{for} \quad \dot{\tau}_2 \sim (-1/2, 5/2) \longleftrightarrow \dot{\tau}_1 \sim (-3/2, 5/2)}$

$$c^{\dot{\tau}_1\dot{\tau}_2}_{3/2} = \sqrt{3}\, c^{\dot{\tau}_1\dot{\tau}_2} \quad , \quad c^{\dot{\tau}_2\dot{\tau}_1}_{3/2} = \sqrt{3}\, c^{\dot{\tau}_2\dot{\tau}_1} .$$

These elements can be arranged in a 20-dimensional matrix $\mathbb{L}_0$. In rearranging the rows and columns of $\mathbb{L}_0$ by grouping together those rows and columns corresponding to the same value of the spin ℓ we can cast it into block form with two blocks corresponding to $\ell = \frac{3}{2}$ and $\ell = \frac{1}{2}$ respectively. Suppressing the index m of the canonical basis $\{\xi_{\ell m}\}$ we can associate with $\mathbb{L}_0$ the matrix

$$\begin{array}{c|cccc|cc|}
 & \xi^{\tau_1}_{3/2} & \xi^{\tau_2}_{3/2} & \xi^{\dot{\tau}_2}_{3/2} & \xi^{\dot{\tau}_1}_{3/2} & \xi^{\tau_2}_{1/2} & \xi^{\dot{\tau}_2}_{1/2} \\
\xi^{\tau_1}_{3/2} & 0 & \sqrt{3}c^{\tau_1\tau_2} & 0 & 0 & & \\
\xi^{\tau_2}_{3/2} & \sqrt{3}c^{\tau_2\tau_1} & 0 & 2c^{\tau_2\dot{\tau}_2} & 0 & & \\
\xi^{\dot{\tau}_2}_{3/2} & 0 & 2c^{\dot{\tau}_2\tau_2} & 0 & \sqrt{3}c^{\dot{\tau}_2\dot{\tau}_1} & & \\
\xi^{\dot{\tau}_1}_{3/2} & 0 & 0 & \sqrt{3}c^{\dot{\tau}_1\dot{\tau}_2} & 0 & & \\ \hline
\xi^{\tau_2}_{1/2} & & & & & 0 & c^{\tau_2\dot{\tau}_2} \\
\xi^{\dot{\tau}_2}_{1/2} & & & & & c^{\dot{\tau}_2\tau_2} & 0
\end{array}
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right\} \ell=\frac{3}{2}\ \text{block} \\
\left.\begin{array}{c} \\ \\ \end{array}\right\} \ell=\frac{1}{2}\ \text{block}
\end{array}
.$$

Separating off the blocks corresponding to each value of the spin we have

for $\ell = 3/2$

$$\mathbb{L}_0^{3/2} = \begin{array}{c}
 \\ \tau_1 \\ \tau_2 \\ \dot{\tau}_2 \\ \dot{\tau}_1
\end{array}
\begin{array}{c}
\begin{array}{cccc} \tau_1 & \tau_2 & \dot{\tau}_2 & \dot{\tau}_1 \end{array} \\
\left[\begin{array}{cccc}
0 & \sqrt{3}c^{\tau_1\tau_2} & 0 & 0 \\
\sqrt{3}c^{\tau_2\tau_1} & 0 & 2c^{\tau_2\dot{\tau}_2} & 0 \\
0 & 2c^{\dot{\tau}_2\tau_2} & 0 & \sqrt{3}c^{\dot{\tau}_2\dot{\tau}_1} \\
0 & 0 & \sqrt{3}c^{\dot{\tau}_1\dot{\tau}_2} & 0
\end{array}\right]
\end{array},$$

and for $\ell = 1/2$

$$\mathbb{L}_0^{1/2} = \begin{array}{c}
 \\ \tau_2 \\ \dot{\tau}_2
\end{array}
\begin{array}{c}
\begin{array}{cc} \tau_2 & \dot{\tau}_2 \end{array} \\
\left[\begin{array}{cc}
0 & c^{\tau_2\dot{\tau}_2} \\
c^{\dot{\tau}_2\tau_2} & 0
\end{array}\right]
\end{array}.$$

Furthermore if invariance of the wave-equation with respect to the complete group is required the condition which has to be satisfied among the elements of $\mathbb{L}_0$ is

$$c^{\tau\tau'} = c^{\dot{\tau}\dot{\tau}'},$$

for $\tau \neq \dot{\tau}$ and $\tau' \neq \dot{\tau}'$.

Thus in our case the following conditions must be satisfied

$$c^{\tau_1\tau_2} = c^{\dot{\tau}_1\dot{\tau}_2} ,$$
$$c^{\tau_2\tau_1} = c^{\dot{\tau}_2\dot{\tau}_1} ,$$
$$c^{\dot{\tau}_2\tau_2} = c^{\tau_2\dot{\tau}_2} .$$

Hence the matrix $\mathbb{L}_0$ which is invariant under the complete group has the blocks

$$\mathbb{L}_0^{3/2} = \begin{array}{c|cccc} & \tau_1 & \tau_2 & \dot{\tau}_2 & \dot{\tau}_1 \\ \hline \tau_1 & 0 & \sqrt{3}\,c^{\tau_1\tau_2} & 0 & 0 \\ \tau_2 & \sqrt{3}\,c^{\tau_2\tau_1} & 0 & 2c^{\tau_2\dot{\tau}_2} & 0 \\ \dot{\tau}_2 & 0 & 2c^{\tau_2\dot{\tau}_2} & 0 & \sqrt{3}\,c^{\tau_2\tau_1} \\ \dot{\tau}_1 & 0 & 0 & \sqrt{3}\,c^{\tau_1\tau_2} & 0 \end{array} , \qquad (5)$$

$$\mathbb{L}_0^{1/2} = \begin{array}{c|cc} & \tau_2 & \dot{\tau}_2 \\ \hline \tau_2 & 0 & c^{\tau_2\dot{\tau}_2} \\ \dot{\tau}_2 & c^{\tau_2\dot{\tau}_2} & 0 \end{array} .$$

Let us now construct bilinear forms consistent with the representation (I). The condition for the existence of a bilinear form for a given finite representation is that in the analysis of the representation into its irreducibles each component τ appears together with its conjugate $\dot{\tau}$. This condition is satisfied for the representation (I) and hence we have the following general expression for it,

$$(\Psi_1, \Psi_2) = \sum_{\substack{\tau=(\tau_1,\dot{\tau}_1,\dot{\tau}_2,\tau_2) \\ \ell = 1/2,\ 3/2, \\ m = 3/2,\ 1/2,\ -1/2,\ -3/2,}} a^{\tau\dot{\tau}}\, s_\ell^{\tau\dot{\tau}}\, x^{\tau}_{\ell m}\, \bar{y}^{\dot{\tau}}_{\ell m} ,$$

where $s_\ell^{\tau\dot{\tau}} = s_\ell^{\dot{\tau}\tau} = (-1)^{[\ell]}$, and $[\ell]$ = integer part of ℓ. An equivalent way of describing a bilinear form is by means of the matrix[30]

$$\mathbb{A} = (-1)^{[\ell]} \alpha^{\tau\dot{\tau}} \delta_{\ell\ell'} \delta_{mm'} ,$$

corresponding to it, known as the hermitianizing matrix. The matrix $\mathbb{A}$ can be written out in the canonical basis and by arguments similar to those made about $\mathbb{L}_0$, can be cast into block form as follows

$$\mathbb{A}_{3/2} = \begin{array}{c|c|c|c|c|} & \tau_1 & \tau_2 & \dot{\tau}_2 & \dot{\tau}_1 \\ \tau_1 & & & & (-1)^{[3/2]} \alpha^{\tau_1\dot{\tau}_1} \\ \hline \tau_2 & & & (-1)^{[3/2]} \alpha^{\tau_2\dot{\tau}_2} & \\ \hline \dot{\tau}_2 & & (-1)^{[3/2]} \alpha^{\dot{\tau}_2\tau_2} & & \\ \hline \dot{\tau}_1 & (-1)^{[3/2]} \alpha^{\dot{\tau}_1\tau_1} & & & \end{array} ,$$

$$\mathbb{A}_{1/2} = \begin{array}{c} \\ \tau_2 \\ \\ \dot{\tau}_2 \end{array} \begin{array}{cc} \tau_2 & \dot{\tau}_2 \\ \left[\begin{matrix} 0 \\ \\ (-1)^{[1/2]} \alpha^{\dot{\tau}_2\tau_2} \end{matrix} \right. & \left. \begin{matrix} (-1)^{[1/2]} \alpha^{\tau_2\dot{\tau}_2} \\ \\ 0 \end{matrix} \right] \end{array} .$$

Depending on how we choose the numbers $\alpha^{\tau_1\dot{\tau}_1}$, $\alpha^{\tau_2\dot{\tau}_2}$ and how we relate $\alpha^{\tau_1\dot{\tau}_1}$ to $\alpha^{\dot{\tau}_1\tau_1}$ and $\alpha^{\tau_2\dot{\tau}_2}$ to $\alpha^{\dot{\tau}_2\tau_2}$ we get different bilinear forms. With a suitable choice of the basis $\{\xi_{\ell m}\}$ the numbers $\alpha^{\tau\dot{\tau}}$, in a form invariant under a representation of the complete group, are given by $\alpha^{\tau\dot{\tau}} = \pm 1$ for ℓ_1 real (case of finite representations).

Thus the 20-dimensional representation (I) admits the following four different bilinear forms, given below in terms of their corresponding hermitianizing matrices cast in block form.

1: **For** $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1\ ,\ \alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = 1\ ,$

$$A_{3/2} = \begin{array}{c|c|c|c|c|} & \tau_1 & \tau_2 & \dot{\tau}_2 & \dot{\tau}_1 \\ \tau_1 & & & & -1 \\ \hline \tau_2 & & & -1 & \\ \hline \dot{\tau}_2 & & -1 & & \\ \hline \dot{\tau}_1 & -1 & & & \end{array}\ ,\quad A_{1/2} = \begin{array}{c|cc|} & \tau_2 & \dot{\tau}_2 \\ \tau_2 & 0 & 1 \\ \dot{\tau}_2 & 1 & 0 \end{array}\ . \tag{6}$$

2: **For** $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1\ ,\ \alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1\ ,$

$$A_{3/2} = \begin{array}{c|c|c|c|c|} & \tau_1 & \tau_2 & \dot{\tau}_2 & \dot{\tau}_1 \\ \tau_1 & & & & -1 \\ \hline \tau_2 & & & 1 & \\ \hline \dot{\tau}_2 & & 1 & & \\ \hline \dot{\tau}_1 & -1 & & & \end{array}\ ,\quad A_{1/2} = \begin{array}{c|cc|} & \tau_2 & \dot{\tau}_2 \\ \tau_2 & 0 & -1 \\ \dot{\tau}_2 & -1 & 0 \end{array}\ . \tag{7}$$

3: **For** $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = -1\ ,\ \alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = 1,$

$$A_{3/2} = \begin{array}{c|c|c|c|c|} & \tau_1 & \tau_2 & \dot{\tau}_2 & \dot{\tau}_1 \\ \tau_1 & & & & 1 \\ \hline \tau_2 & & & -1 & \\ \hline \dot{\tau}_2 & & -1 & & \\ \hline \dot{\tau}_1 & 1 & & & \end{array}\ ,\quad A_{1/2} = \begin{array}{c|cc|} & \tau_2 & \dot{\tau}_2 \\ \tau_2 & 0 & 1 \\ \dot{\tau}_2 & 1 & 0 \end{array}\ . \tag{8}$$

imposed on the elements of the matrix $\mathbb{L}_0$.

<u>First case: $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = 1$</u> ,

$c^{\tau_1\tau_2} = \bar{c}^{\dot{\tau}_2\dot{\tau}_1}$, $c^{\dot{\tau}_1\dot{\tau}_2} = \bar{c}^{\tau_2\tau_1}$, $c^{\tau_2\dot{\tau}_2} = \bar{c}^{\tau_2\dot{\tau}_2}$ (real).

Combining these relations with those of invariance of the equation under the complete group we find for $\mathbb{L}_0$ the sub-blocks

$$\mathbb{L}_0^{3/2} = \begin{array}{c|cccc} & \tau_1 & \tau_2 & \dot{\tau}_2 & \dot{\tau}_1 \\ \hline \tau_1 & 0 & \sqrt{3}\,c^{\tau_1\tau_2} & 0 & 0 \\ \tau_2 & \sqrt{3}\,\bar{c}^{\tau_1\tau_2} & 0 & 2c^{\tau_2\dot{\tau}_2} & 0 \\ \dot{\tau}_2 & 0 & 2c^{\tau_2\dot{\tau}_2} & 0 & \sqrt{3}\,\bar{c}^{\tau_1\tau_2} \\ \dot{\tau}_1 & 0 & 0 & \sqrt{3}\,c^{\tau_1\tau_2} & 0 \end{array} , \quad \mathbb{L}_0^{1/2} = \begin{array}{c|cc} & \tau_2 & \dot{\tau}_2 \\ \hline \tau_2 & 0 & c^{\tau_2\dot{\tau}_2} \\ \dot{\tau}_2 & c^{\tau_2\dot{\tau}_2} & 0 \end{array} .$$

Renaming $c^{\tau_1\tau_2} = \alpha$, $c^{\tau_2\dot{\tau}_2} = \beta$, we have

$$\mathbb{L}_0^{3/2} = \begin{bmatrix} 0 & \sqrt{3}\alpha & 0 & 0 \\ \sqrt{3}\,\bar{\alpha} & 0 & 2\beta & 0 \\ 0 & 2\beta & 0 & \sqrt{3}\,\bar{\alpha} \\ 0 & 0 & \sqrt{3}\alpha & 0 \end{bmatrix} , \quad \mathbb{L}_0^{1/2} = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix} , \qquad (10)$$

where α = complex number, β = real.

<u>Second case: $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1$</u> ,

$c^{\tau_1\tau_2} = -\bar{c}^{\dot{\tau}_2\dot{\tau}_1}$, $c^{\tau_1\dot{\tau}_2} = -\bar{c}^{\tau_2\tau_1}$, $c^{\tau_2\dot{\tau}_2} = \bar{c}^{\tau_2\dot{\tau}_2}$ (real),

and the matrix $\mathbb{L}_0$ which is invariant under the complete group and derivable from an invariant Lagrangian has sub-blocks

$$\mathbb{L}_0^{3/2} = \begin{bmatrix} 0 & \sqrt{3}\alpha & 0 & 0 \\ -\sqrt{3}\bar{\alpha} & 0 & 2\beta & 0 \\ 0 & 2\beta & 0 & -\sqrt{3}\bar{\alpha} \\ 0 & 0 & \sqrt{3}\alpha & 0 \end{bmatrix}, \quad \mathbb{L}_0^{1/2} = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}. \tag{11}$$

<u>**Third case:** $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = -1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = 1$</u>,
$c^{\tau_1\tau_2} = -\bar{c}^{\dot{\tau}_2\dot{\tau}_1}$, $c^{\dot{\tau}_1\dot{\tau}_2} = -\bar{c}^{\tau_2\tau_1}$, $c^{\tau_2\dot{\tau}_2} = \bar{c}^{\tau_2\dot{\tau}_2}$ (real),
and the matrix $\mathbb{L}_0$ which is invariant under the complete group and derivable from an invariant Lagrangian has sub-blocks

$$\mathbb{L}_0^{3/2} = \begin{bmatrix} 0 & \sqrt{3}\alpha & 0 & 0 \\ -\sqrt{3}\bar{\alpha} & 0 & 2\beta & 0 \\ 0 & 2\beta & 0 & -\sqrt{3}\bar{\alpha} \\ 0 & 0 & \sqrt{3}\alpha & 0 \end{bmatrix}, \quad \mathbb{L}_0^{1/2} = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}. \tag{12}$$

<u>**Fourth case:** $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = -1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1$</u>,

$c^{\tau_1\tau_2} = \bar{c}^{\dot{\tau}_2\dot{\tau}_1}$, $c^{\dot{\tau}_1\dot{\tau}_2} = \bar{c}^{\tau_2\tau_1}$, $c^{\tau_2\dot{\tau}_2} = \bar{c}^{\tau_2\dot{\tau}_2}$ (real),
and the matrix $\mathbb{L}_0$ which is invariant under the complete group and derivable from an invariant Lagrangian has sub-blocks

$$\mathbb{L}_0^{3/2} = \begin{bmatrix} 0 & \sqrt{3}\alpha & 0 & 0 \\ \sqrt{3}\bar{\alpha} & 0 & 2\beta & 0 \\ 0 & 2\beta & 0 & \sqrt{3}\bar{\alpha} \\ 0 & 0 & \sqrt{3}\alpha & 0 \end{bmatrix}, \quad \mathbb{L}_0^{1/2} = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}. \tag{13}$$

Because in the sequel we shall be dealing with non-diagonalizable matrices we give here a necessary and sufficient condition for a matrix to be diagonalizable.

A necessary and sufficient condition, that a mrtrix $\mathbb{M}$ be similar to a diagonal matrix, is that its minimal polynomial $m(x)$ factorizes into distinct linear terms.[33-39]

We give now a method by which the minimal polynomial of a matrix $\mathbb{M} \neq 0$ can be determined.

i) If $\mathbb{M} = \alpha_0 \mathbb{I}$ then $m(x) = x - \alpha_0$, (where $\mathbb{I}$ = unit matrix.)

ii) If $\mathbb{M} \neq \alpha_0 \mathbb{I}$ for all α_0 but $\mathbb{M}^2 = \alpha_1 \mathbb{M} + \alpha_0 \mathbb{I}$ then $m(x) = x^2 - \alpha_1 x - \alpha_0$.

iii) If $\mathbb{M}^2 \neq \alpha_0 \mathbb{M} + b\mathbb{I}$ for all α and b but $\mathbb{M}^3 = \alpha_2 \mathbb{M}^2 + \alpha_1 \mathbb{M} + \alpha_0 \mathbb{I}$ then $m(x) = x^3 - \alpha_2 x^2 - \alpha_1 x - \alpha_0$ and so on.

We notice that a real symmetric matrix or a Hermitian matrix is always diagonalizable.

2: Diagonalizability or not of $\mathbb{L}_0$.

Case 1 $\left(\alpha^{\tau_1 \dot{\tau}_1} = \alpha^{\dot{\tau}_1 \tau_1} = 1 \ , \ \alpha^{\tau_2 \dot{\tau}_2} = \alpha^{\dot{\tau}_2 \tau_2} = 1 \right)$.

The general form of $\mathbb{L}_0$ in this case is given by (10). The block $\mathbb{L}_0^{3/2}$ is hermitian and the block $\mathbb{L}_0^{1/2}$ is symmetric. Thus $\mathbb{L}_0$ is hermitian and is always diagonalizable. The same is true about $\mathbb{L}_0$ in the case $\left(\alpha^{\tau_1 \dot{\tau}_1} = \alpha^{\dot{\tau}_1 \tau_1} = -1 \ , \ \alpha^{\tau_2 \dot{\tau}_2} = \alpha^{\dot{\tau}_2 \tau_2} = -1 \right)$.

Case 2 $\left(\alpha^{\tau_1 \dot{\tau}_1} = \alpha^{\dot{\tau}_1 \tau_1} = 1 \ , \ \alpha^{\tau_2 \dot{\tau}_2} = \alpha^{\dot{\tau}_2 \tau_2} = -1 \right)$.

The general form of $\mathbb{L}_0$ in this case is given by (11). The block $\mathbb{L}_0^{3/2}$ is not hermitian and is likely that for some values of the constants α, β, we get a matrix $\mathbb{L}_0$ which is not diagonalizable. This will be so if the minimal equation of $\mathbb{L}_0$ does not factorize into linear terms, which implies that $\mathbb{L}_0$ must have eigenvalues appearing with multiplicities higher than one. This leads to the problem of finding the characteristic polynomial of $\mathbb{L}_0^{3/2}$ and by requiring that it should have multiple roots we find the relations which the constants α and β must satisfy, for this to be so. Then by substituting the values of α and β calculated from these relations, into $\mathbb{L}_0$ we check if the minimal polynomial factorizes into linear terms or not.

The characteristic polynonial of $\mathbb{L}_0^{3/2}$ is

$$f(\lambda) = \det\left\{ \mathbb{L}_0^{3/2} - \lambda\mathbb{I} \right\} = \lambda^4 + \left(6\alpha\bar{\alpha} - 4\beta^2\right)\lambda^2 + 9\left(\alpha\bar{\alpha}\right)^2 = 0.$$

For equal roots it is necessary that

$$\beta^2\left(16\beta^2 - 48\,\alpha\bar{\alpha}\right) = 0\,. \tag{14}$$

(We notice that the eigenvalues of $\mathbb{L}_0$ must appear in pairs of equal and opposite eigenvalues so that each particle appears together with its antiparticle. They have to be real too).

From (14) we get either $\beta=0$ (double) or $\beta = +\sqrt{3\alpha\bar{\alpha}}$ or $\beta=-\sqrt{3\alpha\bar{\alpha}}$. Thus β can be chosen in three different ways for any α, leading thus to three possible forms for the matrix $\mathbb{L}_0$.

i) <u>Choice $\beta=0$, α = any complex number.</u>

This choice leads to $\mathbb{L}_0$ with sub-blocks

$$\mathbb{L}_0^{3/2} = \begin{bmatrix} 0 & \sqrt{3}\,\alpha & 0 & 0 \\ -\sqrt{3}\,\bar{\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3}\,\bar{\alpha} \\ 0 & 0 & \sqrt{3}\,\alpha & 0 \end{bmatrix}, \quad \mathbb{L}_0^{1/2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{15}$$

The minimal equation of $\mathbb{L}_0^{3/2}$ is

$$m\left(\mathbb{L}_0^{3/2}\right) = \left(\mathbb{L}_0^{3/2}\right)^2 - 3\alpha\bar{\alpha}\,\mathbb{I} = \left(\mathbb{L}_0^{3/2} - \sqrt{3\alpha\bar{\alpha}}\right)\left(\mathbb{L}_0^{3/2} + \sqrt{3\alpha\bar{\alpha}}\right) = 0\,,$$

and so $\mathbb{L}_0^{3/2}$ is diagonalizable. Thus $\mathbb{L}_0$ is diagonalizable too.

ii) <u>Choice $\beta=+\sqrt{3\alpha\bar{\alpha}}$, α = any complex number.</u>

This choice leads to $\mathbb{L}_0$ with sub-blocks

$$\mathbb{L}_0^{3/2} = \begin{bmatrix} 0 & \sqrt{3}\,\alpha & 0 & 0 \\ -\sqrt{3}\,\bar{\alpha} & 0 & 2\sqrt{3\alpha\bar{\alpha}} & 0 \\ 0 & 2\sqrt{3\alpha\bar{\alpha}} & 0 & -\sqrt{3}\,\bar{\alpha} \\ 0 & 0 & \sqrt{3}\,\alpha & 0 \end{bmatrix}, \quad \mathbb{L}_0^{1/2} = \begin{bmatrix} 0 & \sqrt{3\alpha\bar{\alpha}} \\ \sqrt{3\alpha\bar{\alpha}} & 0 \end{bmatrix}. \tag{16}$$

The minimal equations of the two sub-blocks are respectively

$$m\left(\mathbb{L}_0^{3/2}\right) = \left\{\mathbb{L}_0^{3/2} - \sqrt{3\alpha\bar{\alpha}}\right\}^2\left\{\mathbb{L}_0^{3/2} + \sqrt{3\alpha\bar{\alpha}}\right\}^2 = 0\,,$$

$$m\left(\mathbb{L}_0^{1/2}\right) = \left\{\mathbb{L}_0^{1/2} - \sqrt{3\alpha\bar{\alpha}}\right\}\left\{\mathbb{L}_0^{1/2} + \sqrt{3\alpha\bar{\alpha}}\right\} = 0\,,$$

and the minimal equation of $\mathbb{L}_0$ is

$$m\left(\mathbb{L}_0\right) = \left\{\mathbb{L}_0 - \sqrt{3\alpha\bar{\alpha}}\right\}^2\left\{\mathbb{L}_0 + \sqrt{3\alpha\bar{\alpha}}\right\}^2 = 0.$$

Hence $\mathbb{L}_0$ is not diagonalizable.

We notice that the minimal polynomial of $\mathbb{L}_0$ is the same as the least common multiple of the minimal polynomials of the blocks corresponding to it with the matrices of the blocks being replaced by $\mathbb{L}_0$ i.e.

$m\left(\mathbb{L}_0\right)$ = least common multiple of $m\left(\mathbb{L}_0^{3/2}\right)$ and $m\left(\mathbb{L}_0^{1/2}\right)$ with $\mathbb{L}_0^{3/2}$ and $\mathbb{L}_0^{1/2}$ being replaced by $\mathbb{L}_0$.

iii) <u>Choice $\beta = -\sqrt{3\alpha\bar{\alpha}}$, α = any complex number.</u>

This choice leads to a matrix $\mathbb{L}_0$ with sub-blocks

$$\mathbb{L}_0^{3/2} = \begin{bmatrix} 0 & \sqrt{3}\,\alpha & 0 & 0 \\ -\sqrt{3}\,\bar{\alpha} & 0 & -2\sqrt{3\alpha\bar{\alpha}} & 0 \\ 0 & -2\sqrt{3\alpha\bar{\alpha}} & 0 & -\sqrt{3}\,\bar{\alpha} \\ 0 & 0 & \sqrt{3}\,\alpha & 0 \end{bmatrix}, \quad \mathbb{L}_0^{1/2} = \begin{bmatrix} 0 & -\sqrt{3\alpha\bar{\alpha}} \\ -\sqrt{3\alpha\bar{\alpha}} & 0 \end{bmatrix}, \qquad (17)$$

and minimal equation the same as for the previous case. Thus $\mathbb{L}_0$ is not diagonalizable.

3: <u>Charge.</u>

We study in this paragraph the charge associated with the matrix $\mathbb{L}_0$ corresponding to the bilinear form $\left(\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1,\ \alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1\right)$ and for the choices of the constants β and α mentioned above. We make first some general comments concerning the charge.

To each equation derivable from an invariant Lagrangian function corresponds a vector with components

$$s_k = \left(\mathbb{L}_k \psi, \psi \right) , \qquad k = 0, 1, 2, 3 ,$$

known as the four current vector. Its time component

$$s_0 = \left(\mathbb{L}_0 \psi, \psi \right) = \psi^{+} \mathbb{A} \mathbb{L}_0 \psi$$

is known as the charge density. $\mathbb{A}$ is the hermitianizing matrix and ψ^{+} the complex conjugate transpose of ψ .

Gel'fand, Minlos and Shapiro give the following definition for a wave-equation to have positive charge density.

The condition for a Gel'fand-Yaglom equation to acquire positive charge density is that for all the eigenvectors ψ_λ of the matrix $\mathbb{L}_0$ with non-zero eigenvalue λ the corresponding charge density ρ_λ is positive i.e.

$$\rho_\lambda = \left(\mathbb{L}_0 \psi_\lambda , \psi_\lambda \right) > 0 .$$

Likewise one can define negative charge density. Thus we shall be using below the term definite charge density to mean that s_0 is either positive or negative.

<u>Examples:</u> 1. We give one example in some detail and distribute the rest in a table.

Let the constants have the values $\beta = \sqrt{3\alpha\bar{\alpha}}$, α= any complex number and the bilinear form be $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1$. The matrix $\mathbb{L}_0$ has sublocks given by (16), and satisfies the minimal equation

$$m\left(\mathbb{L}_0\right) = \left\{ \mathbb{L}_0 + \sqrt{3\alpha\bar{\alpha}} \right\}^2 \left\{ \mathbb{L}_0 - \sqrt{3\alpha\bar{\alpha}} \right\}^2 = 0 .$$

Thus $\mathbb{L}_0$ is non-diagonalizable. The eigenvalues of the block $\mathbb{L}_0^{3/2}$ are $\overset{1}{\lambda}_{3/2} = \overset{2}{\lambda}_{3/2} = +\sqrt{3\alpha\bar{\alpha}}$, $\overset{3}{\lambda}_{3/2} = \overset{4}{\lambda}_{3/2} = -\sqrt{3\alpha\bar{\alpha}}$, and the corresponding eigenvectors are

$$\phi_1 \left(\lambda = \sqrt{3\alpha\bar{\alpha}} , \ell = 3/2 \right) = \left\{ 1 , \frac{\bar{\alpha}}{\sqrt{\alpha\bar{\alpha}}} , \frac{\bar{\alpha}}{\sqrt{\alpha\bar{\alpha}}} , 1 \right\}^{tr} ,$$

$$\phi_2 \left(\lambda = -\sqrt{3\alpha\bar{\alpha}} , \ell = 3/2 \right) = \left\{ -1 , \frac{\bar{\alpha}}{\sqrt{\alpha\bar{\alpha}}} , \frac{-\bar{\alpha}}{\sqrt{\alpha\bar{\alpha}}} , 1 \right\}^{tr} .$$

Notice that there is only one eigenvector corresponding to the double eigenvalue $\sqrt{3\alpha\bar{\alpha}}$, because the nullity of the matrix $\mathbb{L}_0^{3/2}$ for this eigenvalue is one. For the same reason there is one eigenvector corresponding to the double eigenvalue $-\sqrt{3\alpha\bar{\alpha}}$. The eigenvalues of the block $\mathbb{L}_0^{1/2}$ are $\lambda^1_{1/2} = \sqrt{3\alpha\bar{\alpha}}$, and $\lambda^2_{1/2} = -\sqrt{3\alpha\bar{\alpha}}$. The corresponding eigenvectors are

$$\phi_3\left(\lambda = \sqrt{3\alpha\bar{\alpha}}, \ell = 1/2\right) = \{1, 1\}^{tr}, \quad \phi_4\left(\lambda = -\sqrt{3\alpha\bar{\alpha}}, \ell = 1/2\right) = \{-1, 1\}^{tr}.$$

We notice that an eigenvector belonging to the block $\mathbb{L}_0^{\ell}$ is a $(2\ell + 1)$-fold degenerate eigenvector of $\mathbb{L}_0$.

Using $\rho^i = \phi_i^+ \mathbb{A} \mathbb{L}_0 \phi_i$ and the eigenvectors given above we find for the charge densities

$$\rho^1_{3/2} = \phi_1^+ \mathbb{A}_{3/2} \mathbb{L}_0^{3/2} \phi_1 = 0 , \quad \rho^2_{3/2} = \phi_2^+ \mathbb{A}_{3/2} \mathbb{L}_0^{3/2} \phi_2 = 0 ,$$

$$\rho^3_{1/2} = \phi_3^+ \mathbb{A}_{1/2} \mathbb{L}_0^{1/2} \phi_3 = -2\sqrt{3\alpha\bar{\alpha}} < 0 , \quad \rho^4_{1/2} = \phi_4^+ \mathbb{A}_{1/2} \mathbb{L}_0^{1/2} \phi_4 = -2\sqrt{3\alpha\bar{\alpha}} < 0.$$

All the off diagonal charge densities of the form $\phi_i \mathbb{A}_\ell \mathbb{L}_0^\ell \phi_j$ are zero $\forall\ \ell = 3/2, 1/2$. Thus the charge is zero in the state $\frac{3}{2}$ and negative in the state $\frac{1}{2}$. This example does not satisfy the definition of Gel'fand, Minlos and Shapiro in that for some eigenvectors corresponding to non-zero eigenvalues the charge densities are zero.

More examples are given in the table below.

Overall charge	indefinite	It does not satisfy the definition of definite charge.
Charge in the state 3/2		zero
Charge in the state ½		positive
Charge densities		$\rho^1_{3/2} = 0$, $\rho^2_{3/2} = 0$, $\rho^3_{1/2} = 2\sqrt{3\alpha\bar{\alpha}} > 0$, $\rho^4_{1/2} = 2\sqrt{3\alpha\bar{\alpha}} > 0$.
Minimal equation	diagonalizable matrix.	$m(\mathbb{L}_0) = \{\mathbb{L}_0 - \sqrt{3\alpha\bar{\alpha}}\}^2 \{\mathbb{L}_0 + \sqrt{3\alpha\bar{\alpha}}\}^2 = 0$, non-diagonalizable matrix.
Bilinear form	$\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1$.	$\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1$.
Constants of $\mathbb{L}_0$	$\beta = 0$, α = any complex number.	$\beta = -\sqrt{3\alpha\bar{\alpha}}$, α = any complex number.
Form of $\mathbb{L}_0$	$\mathbb{L}_0$ has blocks given by (15).	$\mathbb{L}_0$ has blocks given by (17).
	Example 2	**Example 3**

Overall Charge	It does not satisfy the definition of definite charge	It does not satisfy the definition of definite charge
Charge in the state 3/2	zero (eigenvalues of $\mathbb{L}_0^{3/2}$ not zero)	zero (eigenvalues of $\mathbb{L}_0^{3/2}$ not zero)
Charge in the state $\frac{1}{2}$	positive	negative
Charge densities	$\rho^1_{3/2} = 0$, $\rho^2_{3/2} = 0$, $\rho^3_{1/2} = 2\sqrt{3\alpha\bar{\alpha}} > 0$, $\rho^4_{1/2} = 2\sqrt{3\alpha\bar{\alpha}} > 0.$	$\rho^1_{3/2} = 0$, $\rho^2_{3/2} = 0$, $\rho^3_{1/2} = -2\sqrt{3\alpha\bar{\alpha}} < 0$, $\rho^4_{1/2} = -2\sqrt{3\alpha\bar{\alpha}} < 0$.
Minimal equation	$m(\mathbb{L}_0) = \{\mathbb{L}_0 + \sqrt{3\alpha\bar{\alpha}}\}^2 \{\mathbb{L}_0 - \sqrt{3\alpha\bar{\alpha}}\}^2 = 0,$ non-diagonalizable matrix.	$m(\mathbb{L}_0) = \{\mathbb{L}_0 - \sqrt{3\alpha\bar{\alpha}}\}^2 \{\mathbb{L}_0 + \sqrt{3\alpha\bar{\alpha}}\}^2 = 0$, non-diagonalizable matrix.
Bilinear form	$\alpha^{\tau_1 \dot{\tau}_1} = \alpha^{\dot{\tau}_1 \tau_1} = -1$, $\alpha^{\tau_2 \dot{\tau}_2} = \alpha^{\dot{\tau}_2 \tau_2} = 1$.	$\alpha^{\tau_1 \dot{\tau}_1} = \alpha^{\dot{\tau}_1 \tau_1} = -1$, $\alpha^{\dot{\tau}_2 \tau_2} = \alpha^{\tau_2 \dot{\tau}_2} = 1$.
Constants of $\mathbb{L}_0$	$\beta = \sqrt{3\alpha\bar{\alpha}}$, $\alpha =$ any complex number.	$\beta = -\sqrt{3\alpha\bar{\alpha}}$, $\alpha =$ any complex number.
Form of $\mathbb{L}_0$	$\mathbb{L}_0$ has blocks given by (16).	$\mathbb{L}_0$ has blocks given by (17).
	Example 4	Example 5

Examples (1), (3), (4), (5) are the only ones based on the 20-dimensional representation $\tau_1 \oplus \dot{\tau}_1 \oplus \tau_2 \oplus \dot{\tau}_2$, with non-diagonalizable matrix $\mathbb{L}_0$ and as was demonstrated they do not satisfy the definition of definite charge.

As we shall see in the third part of this work equations whose matrix $\mathbb{L}_0$ satisfies a minimal equation of the form

$$m(\mathbb{L}_0) = \mathbb{L}_0 \prod_{l=1}^{k} \left\{ (\mathbb{L}_0)^2 - \lambda_l^2 \right\}^{r_l} = 0 \ ,$$

where $r_l \geqslant 1$ and $\lambda_l \neq 0$ the eigenvalues of $\mathbb{L}_0$, are very interesting because they propagate causally in the presence of an external electromagnetic field. This is the criterion of Amar and Dozzio[40] of causal propagation. Also from Cox[41] we have that if the matrix $\mathbb{L}_0$ of a wave equation satisfies a minimal equation of the form

$$(\mathbb{L}_0)^m \prod_{l=1}^{k} \left\{ (\mathbb{L}_0)^2 - \lambda_l^2 \right\} = 0 \quad ,$$

where $m=0$ or 1 , then the field can be quantized.

We give now an example satisfying both criteria.

Example 6. Let the constants have the values $\alpha=0, \beta\neq 0$ and the bilinear form be $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1$.

The matrix $\mathbb{L}_0$ has sublocks given by (11) and satisfies the minimal equation

$$m(\mathbb{L}_0) = \mathbb{L}_0 \left\{ (\mathbb{L}_0)^2 - 4\beta^2 \right\} \left\{ (\mathbb{L}_0)^2 - \beta^2 \right\} = 0.$$

Thus $\mathbb{L}_0$ is diagonalizable and hence the charge is indefinite. Despite this the field propagates causally in an external electromagnetic field and can be quantized too.

Example 7. Another example we have if in (13) we choose $\beta=0$ α = any complex number and bilinear form $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = -1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1$. The resulting matrix $\mathbb{L}_0$ satisfies the minimal equation

$$m(\mathbb{L}_0) = \mathbb{L}_0 \left\{ (\mathbb{L}_0)^2 - \left(\sqrt{3\alpha\bar{\alpha}}\right)^2 \right\} = 0 .$$

This example represents a field propagating causally and quantizable but with indefinite charge.

Let us relax the condition on $\mathbb{L}_0$ being derivable from an invariant Lagrangian (i.e. no hermitianizing matrix $\mathbb{A}$ can be defined). There are some interesting examples satisfying the conditions of causal propagation and second quantization but no charge can be defined for them.

Example 8. The general form which $\mathbb{L}_0$ has when only the condition of invariance under the complete group is imposed is given by (5).

Let us give to the constants the values $c^{\tau_2\dot{\tau}_2}=1$, $c^{\tau_1\tau_2}=1$, $c^{\tau_2\tau_1}=-\frac{1}{3}$. The resulting matrix $\mathbb{L}_0$ satisfies the minimal equation

$$m(\mathbb{L}_0)=\left\{(\mathbb{L}_0)^2-1\right\}^2=0.$$

This is an example of equation with unique mass and two spin states $\ell=\frac{3}{2}$, $\ell=\frac{1}{2}$ and matrix $\mathbb{L}_0$ non-diagonalizable. No charge can be defined for it.

Example 9. Let us give to the constants the values $c^{\tau_2\tau_1}=0$, $c^{\tau_1\tau_2}\neq 0$, $c^{\tau_2\dot{\tau}_2}\neq 0$. The resulting matrix $\mathbb{L}_0$ satisfies the minimal equation

$$m(\mathbb{L}_0)=\mathbb{L}_0\left\{(\mathbb{L}_0)^2-4\left(c^{\tau_2\dot{\tau}_2}\right)^2\right\}\left\{(\mathbb{L}_0)^2-\left(c^{\tau_2\dot{\tau}_2}\right)^2\right\}=0.$$

Thus the field propagates causally in an external electromagnetic field and can be quantized but no charge can be attributed to it.

Another example we have if we choose $c^{\tau_2\tau_1}\neq 0$, $c^{\tau_2\dot{\tau}_2}\neq 0$, $c^{\tau_1\tau_2}=0$, with the same minimal equation and conclusions as above.

4: Conclusions.

In this chapter we have studied the charge associated with the Gel'fand-Yaglom equation based on the 20-dimensional representation $\tau_1\oplus\dot{\tau}_1\oplus\tau_2\oplus\dot{\tau}_2$. Our conclusions are:

1. There is no example based on this representation with non-diagonalizable matrix $\mathbb{L}_0$ and definite charge either positive or negative.

2. There are examples which satisfy the Amar and Dozzio criterion of causal propagation and the criterion of Cox of second quantization but they have either indefinite

charge ($\mathbb{L}_0$ diagonalizable) or the charge cannot be defined because $\mathbb{L}_0$ is not derivable from an invariant Lagrangian.

CHAPTER 4.

1. Preliminaries.

In this chapter we shall be concerned with equations of the Gel'fand-Yaglom type for which the wave-function ψ transforms according to the 16-dimensional representation with components $\tau_1 \sim (\frac{1}{2}, \frac{3}{2})$, $\dot{\tau}_1 \sim (-\frac{1}{2}, \frac{3}{2})$, $\tau_2 \sim (\frac{1}{2}, \frac{5}{2})$, $\dot{\tau}_2 \sim (-\frac{1}{2}, \frac{5}{2})$ interlocking according to the scheme

$$\begin{tikzcd}
(\frac{1}{2}, \frac{3}{2}) \sim \tau_1 \arrow[r, shift left, "c^{\tau_1\dot{\tau}_1}"] \arrow[d, shift left, "c^{\tau_1\tau_2}"] & \dot{\tau}_1 \sim (-\frac{1}{2}, \frac{3}{2}) \arrow[l, shift left, "c^{\dot{\tau}_1\tau_1}"] \arrow[d, shift left, "c^{\dot{\tau}_1\dot{\tau}_2}"] \\
(\frac{1}{2}, \frac{5}{2}) \sim \tau_2 \arrow[r, shift left, "c^{\tau_2\dot{\tau}_2}"] \arrow[u, shift left, "c^{\tau_2\tau_1}"] & \dot{\tau}_2 \sim (-\frac{1}{2}, \frac{5}{2}) \arrow[l, shift left, "c^{\dot{\tau}_2\tau_2}"] \arrow[u, shift left, "c^{\dot{\tau}_2\dot{\tau}_1}"]
\end{tikzcd} \qquad (II)$$

This representation admits the cononical basis

$$\{\xi_{\ell m}\} \equiv \{ \xi^{\tau_1}_{1/2,1/2}, \xi^{\tau_1}_{1/2,-1/2}, \xi^{\dot{\tau}_1}_{1/2,1/2}, \xi^{\tau_1}_{1/2,-1/2}, \xi^{\tau_2}_{1/2,1/2}, \xi^{\tau_2}_{1/2,-1/2},$$
$$\xi^{\tau_2}_{3/2,3/2}, \xi^{\tau_2}_{3/2,1/2}, \xi^{\tau_2}_{3/2,-1/2}, \xi^{\tau_2}_{3/2,-3/2}, \xi^{\dot{\tau}_2}_{1/2,1/2};$$
$$\xi^{\dot{\tau}_2}_{1/2,-1/2}, \xi^{\dot{\tau}_2}_{3/2,+3/2}, \xi^{\dot{\tau}_2}_{3/2,1/2}, \xi^{\dot{\tau}_2}_{3/2,-1/2}, \xi^{\dot{\tau}_2}_{3/2,-3/2} \}. \qquad (1)$$

The matrix $\mathbb{L}_0$ with respect to this basis has blocks

$$\mathbb{L}^{1/2}_0 = \begin{array}{c|cccc|} & \tau_1 & \dot{\tau}_1 & \tau_2 & \dot{\tau}_2 \\ \tau_1 & 0 & c^{\tau_1\dot{\tau}_1} & c^{\tau_1\tau_2} & 0 \\ \dot{\tau}_1 & c^{\dot{\tau}_1\tau_1} & 0 & 0 & c^{\dot{\tau}_1\dot{\tau}_2} \\ \tau_2 & c^{\tau_2\tau_1} & 0 & 0 & c^{\tau_2\dot{\tau}_2} \\ \dot{\tau}_2 & 0 & c^{\dot{\tau}_2\dot{\tau}_1} & c^{\dot{\tau}_2\tau_2} & 0 \end{array}, \qquad \mathbb{L}^{3/2}_0 = \begin{array}{c|cc|} & \tau_2 & \dot{\tau}_2 \\ \tau_2 & 0 & 2c^{\tau_2\dot{\tau}_2} \\ \dot{\tau}_2 & 2c^{\dot{\tau}_2\tau_2} & 0 \end{array}. \qquad (2)$$

(c.f. Gel'fand, Minlos and Shapiro p. 315).

For invariance of the equation under the complete group the following relations must be satisfied among the elements of $\mathbb{L}_0$.

$$c^{\tau_1\dot{\tau}_1}=c^{\dot{\tau}_1\tau_1}\,,\quad c^{\tau_2\tau_1}=c^{\dot{\tau}_2\dot{\tau}_1}\,,\quad c^{\tau_1\tau_2}=c^{\dot{\tau}_1\dot{\tau}_2}\,,\quad c^{\dot{\tau}_2\tau_2}=c^{\tau_2\dot{\tau}_2}\,. \tag{3}$$

Since in the reduction of the representation (II) each irreducible appears together with its conjugate this guarantees the existence of the following bilinear forms.

1. For $\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1\,,\quad \alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=1\,,$

$$A_{1/2}=\begin{array}{c|cccc|} & \tau_1 & \dot{\tau}_1 & \tau_2 & \dot{\tau}_2 \\ \tau_1 & 0 & 1 & 0 & 0 \\ \dot{\tau}_1 & 1 & 0 & 0 & 0 \\ \tau_2 & 0 & 0 & 0 & 1 \\ \dot{\tau}_2 & 0 & 0 & 1 & 0 \end{array}\,,\quad A_{3/2}=\begin{array}{c|cc|} & \tau_2 & \dot{\tau}_2 \\ \tau_2 & 0 & -1 \\ \dot{\tau}_2 & -1 & 0 \end{array}\,. \tag{4}$$

2. For $\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1\,,\quad \alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1\,,$

$$A_{1/2}=\begin{array}{c|cccc|} & \tau_1 & \dot{\tau}_1 & \tau_2 & \dot{\tau}_2 \\ \tau_1 & 0 & 1 & 0 & 0 \\ \dot{\tau}_1 & 1 & 0 & 0 & 0 \\ \tau_2 & 0 & 0 & 0 & -1 \\ \dot{\tau}_2 & 0 & 0 & -1 & 0 \end{array}\,,\quad A_{3/2}=\begin{array}{c|cc|} & \tau_2 & \dot{\tau}_2 \\ \tau_2 & 0 & 1 \\ \dot{\tau}_2 & 1 & 0 \end{array}\,. \tag{5}$$

3. For $\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=-1\,,\quad \alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=1\,,$

$$A_{1/2}=\begin{array}{c|cccc|} & \tau_1 & \dot{\tau}_1 & \tau_2 & \dot{\tau}_2 \\ \tau_1 & 0 & -1 & 0 & 0 \\ \dot{\tau}_1 & -1 & 0 & 0 & 0 \\ \tau_2 & 0 & 0 & 0 & 1 \\ \dot{\tau}_2 & 0 & 0 & 1 & 0 \end{array}\,,\quad A_{3/2}=\begin{array}{c|cc|} & \tau_2 & \dot{\tau}_2 \\ \tau_2 & 0 & -1 \\ \dot{\tau}_2 & -1 & 0 \end{array}\,. \tag{6}$$

4. For $\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=-1\,,\quad \alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1\,,$

$$\mathbb{A}_{1/2}=\begin{array}{c|cccc} & \tau_1 & \dot{\tau}_1 & \tau_2 & \dot{\tau}_2 \\ \hline \tau_1 & 0 & -1 & 0 & 0 \\ \dot{\tau}_1 & -1 & 0 & 0 & 0 \\ \tau_2 & 0 & 0 & 0 & -1 \\ \dot{\tau}_2 & 0 & 0 & -1 & 0 \end{array}\,,\quad \mathbb{A}_{3/2}=\begin{array}{c|cc} & \tau_2 & \dot{\tau}_2 \\ \hline \tau_2 & 0 & 1 \\ \dot{\tau}_2 & 1 & 0 \end{array}\,. \qquad (7)$$

The above bilinear forms are not degenerate. In order that the equation be derivable from an invariant Lagranigian the following relations must be satisfied among the elements of the matrix $\mathbb{L}_0$.

First case: $\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1\,,\ \alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=1\,,$

$$c^{\tau_1\dot{\tau}_1}=\bar{c}^{\tau_1\dot{\tau}_1}\,,\quad c^{\tau_1\tau_2}=\bar{c}^{\dot{\tau}_2\dot{\tau}_1}\,,\quad c^{\dot{\tau}_1\dot{\tau}_2}=c^{\tau_2\tau_1}\,,$$
$$c^{\tau_2\dot{\tau}_2}=\bar{c}^{\tau_2\dot{\tau}_2}\,,\quad c^{\dot{\tau}_2\tau_2}=\bar{c}^{\dot{\tau}_2\tau_2}\,,\quad c^{\dot{\tau}_2\dot{\tau}_1}=\bar{c}^{\tau_1\tau_2}\,,$$
$$c^{\tau_2\tau_1}=\bar{c}^{\dot{\tau}_1\dot{\tau}_2}\,,\quad c^{\dot{\tau}_1\tau_1}=\bar{c}^{\dot{\tau}_1\tau_1}\,. \qquad (8)$$

Combining these relations with those of invariance of the equation under the complete group, we find for $\mathbb{L}_0$ the blocks

$$\mathbb{L}_0^{1/2}=\begin{array}{c|cccc} & \tau_1 & \dot{\tau}_1 & \tau_2 & \dot{\tau}_2 \\ \hline \tau_1 & 0 & c^{\tau_1\dot{\tau}_1} & c^{\tau_1\tau_2} & 0 \\ \dot{\tau}_1 & c^{\tau_1\dot{\tau}_1} & 0 & 0 & c^{\tau_1\tau_2} \\ \tau_2 & \bar{c}^{\tau_1\tau_2} & 0 & 0 & c^{\tau_2\dot{\tau}_2} \\ \dot{\tau}_2 & 0 & \bar{c}^{\tau_1\tau_2} & c^{\tau_2\dot{\tau}_2} & 0 \end{array}\,,\quad \mathbb{L}_0^{3/2}=\begin{array}{c|cc} & \tau_2 & \dot{\tau}_2 \\ \hline \tau_2 & 0 & 2c^{\tau_2\dot{\tau}_2} \\ \dot{\tau}_2 & 2c^{\tau_2\dot{\tau}_2} & 0 \end{array}\,, \qquad (9)$$

or after renaming the constants $c^{\tau_1\dot{\tau}_1}=\alpha$ (real), $c^{\tau_1\tau_2}=\beta$, $c^{\tau_2\dot{\tau}_2}=\gamma$ (real)

$$\mathbb{L}_0^{1/2} = \begin{array}{c|cccc} & \tau_1 & \dot{\tau}_1 & \tau_2 & \dot{\tau}_2 \\ \hline \tau_1 & 0 & \alpha & \beta & 0 \\ \dot{\tau}_1 & \alpha & 0 & 0 & \beta \\ \tau_2 & \bar{\beta} & 0 & 0 & \gamma \\ \dot{\tau}_2 & 0 & \bar{\beta} & \gamma & 0 \end{array} , \qquad \mathbb{L}_0^{3/2} = \begin{array}{c|cc} & \tau_2 & \dot{\tau}_2 \\ \hline \tau_2 & 0 & 2\gamma \\ \dot{\tau}_2 & 2\gamma & 0 \end{array} . \qquad (10)$$

Second case: $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1,\ \alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1,$

$$c^{\tau_1\dot{\tau}_1} = \bar{c}^{\,\tau_1\dot{\tau}_1}\ (\text{real}),\quad -c^{\tau_1\tau_2} = \bar{c}^{\,\dot{\tau}_2\dot{\tau}_1},\quad -c^{\dot{\tau}_1\dot{\tau}_2} = \bar{c}^{\,\tau_2\tau_1},$$
$$\bar{c}^{\,\dot{\tau}_2\tau_2} = c^{\dot{\tau}_2\tau_2},\quad c^{\tau_2\dot{\tau}_2} = \bar{c}^{\,\tau_2\dot{\tau}_2}\ (\text{real}),\quad c^{\dot{\tau}_2\dot{\tau}_1} = -\bar{c}^{\,\tau_1\tau_2},$$
$$c^{\tau_2\tau_1} = -\bar{c}^{\,\dot{\tau}_1\dot{\tau}_2},\quad c^{\dot{\tau}_1\tau_1} = \bar{c}^{\,\dot{\tau}_1\tau_1} . \qquad (11)$$

The matrix $\mathbb{L}_0$ which is invariant under the complete group and derivable from an invariant Lagrangian has blocks

$$\mathbb{L}_0^{1/2} = \begin{array}{c|cccc} & \tau_1 & \dot{\tau}_1 & \tau_2 & \dot{\tau}_2 \\ \hline \tau_1 & 0 & \alpha & \beta & 0 \\ \dot{\tau}_1 & \alpha & 0 & 0 & \beta \\ \tau_2 & -\bar{\beta} & 0 & 0 & \gamma \\ \dot{\tau}_2 & 0 & -\bar{\beta} & \gamma & 0 \end{array} , \qquad \mathbb{L}_0^{3/2} = \begin{array}{c|cc} & \tau_2 & \dot{\tau}_2 \\ \hline \tau_2 & 0 & 2\gamma \\ \dot{\tau}_2 & 2\gamma & 0 \end{array} . \qquad (12)$$

Third case: $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = -1,\ \alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = 1,$

$$c^{\tau_1\dot{\tau}_1} = \bar{c}^{\,\tau_1\dot{\tau}_1},\quad -c^{\tau_1\tau_2} = \bar{c}^{\,\dot{\tau}_2\dot{\tau}_1},\quad -c^{\dot{\tau}_1\dot{\tau}_2} = c^{\tau_2\tau_1},$$
$$c^{\dot{\tau}_2\tau_2} = \bar{c}^{\,\dot{\tau}_2\tau_2},\quad c^{\tau_2\dot{\tau}_2} = \bar{c}^{\,\tau_2\dot{\tau}_2},\quad c^{\dot{\tau}_2\dot{\tau}_1} = -c^{\tau_1\tau_2},$$
$$c^{\tau_2\tau_1} = -\bar{c}^{\,\dot{\tau}_1\dot{\tau}_2},\quad c^{\dot{\tau}_1\tau_1} = \bar{c}^{\,\dot{\tau}_1\tau_1} .$$

The matrix $\mathbb{L}_0$ which is invariant under the complete group and derivable from an invariant Lagrangian is the same as (12).

<u>Fourth case: $\alpha^{\tau_1 \dot{\tau}_1} = \alpha^{\dot{\tau}_1 \tau_1} = -1$, $\alpha^{\dot{\tau}_2 \tau_2} = \alpha^{\tau_2 \dot{\tau}_2}$</u>$= -1$

The matrix $\mathbb{L}_0$ in this case is the same as (10).

2: <u>Charge</u>

Let us now study the charge corresponding to the above cases.

We observe that for the first and fourth cases the matrix $\mathbb{L}_0$ is hermitian and so is always diagonalizable for any values of the constants.

The charge then is indefinite. Hence any cases of equations with positive charge will belong to the second and third cases.

Let us then engage ourselves in the study of the charge of particular examples based on these two cases. These examples are enlisted in the table below.

Overall charge	indefinite for $\forall\ \beta \neq 1/2$	indefinite for $\forall\ \beta$.
Charge in the state 3/2		
Charge in the state 1/2		
Charge densities		
Minimal equation	$m(\mathbb{L}_0) = \{(\mathbb{L}_0)^2 - 1\}\{(\mathbb{L}_0)^2 - (\frac{1}{4} - \beta^2)\} = 0,$ diagonalizable matrix	$m(\mathbb{L}_0) = \{(\mathbb{L}_0)^2 - (1/2+\beta)^2\}\{(\mathbb{L}_0)^2 - (1/2-\beta)^2\}\{(\mathbb{L}_0)^2 - 1\} = 0,$ diagonalizable matrix.
Bilinear form	$\alpha^{\tau_1 \dot{\tau}_1} = \alpha^{\dot{\tau}_1 \tau_1} = -1$, or 1, $\alpha^{\tau_2 \dot{\tau}_2} = \alpha^{\dot{\tau}_2 \tau_2} = 1$, or -1,	$\alpha^{\tau_1 \dot{\tau}_1} = \alpha^{\dot{\tau}_1 \tau_1} = -1$, or 1, $\alpha^{\tau_2 \dot{\tau}_2} = \alpha^{\dot{\tau}_2 \tau_2} = 1$, or -1,
Constants of $\mathbb{L}_0$	$\alpha = -\frac{1}{2}$, $\gamma = \frac{1}{2}$, β = real	$\alpha = \frac{1}{2} \pm 2\beta$, $\gamma = \frac{1}{2}$, β = real
Form of $\mathbb{L}_0$	$\mathbb{L}_0$ has blocks given by (12).	$\mathbb{L}_0$ has blocks given by (12).
	Example 1	Example 2

Overall Charge	indefinite for $\beta^2 \neq \frac{1}{4}$.	positive. (This is an example of a spin 3/2 particle with definite charge.)	negative. (This is another example of a spin 3/2 particle with definite charge .)
Charge in the state 3/2		positive.	negative.
Charge in the state 1/2		zero, (because the eigen-values of $\mathbb{L}_0^{1/2}$ are zero.)	zero. (eigenvalues of $\mathbb{L}_0^{1/2}$ are zero.)
Charge densities		$\rho^1_{1/2} = 0$, $\rho^2_{1/2} = 0$ $\rho^3_{3/2} = 2 > 0$, $\rho^4_{3/2} = 2 > 0$.	$\rho^1_{1/2} = 0$, $\rho^2_{1/2} = 0$, $\rho^3_{3/2} = -2 < 0$, $\rho^4_{3/2} = -2 < 0$.
Minimal equation	$m(\mathbb{L}_0) = \mathbb{L}_0\{(\mathbb{L}_0)^2 - (\beta^2 - \frac{1}{4})^2\}\{(\mathbb{L}_0)^2 - 1\} = 0,$ diagonalizable matrix if $\beta^2 \neq \frac{1}{4}$.	$m(\mathbb{L}_0) = (\mathbb{L}_0)^2\{(\mathbb{L}_0)^2 - 1\} = 0,$ non-diagonalizable matrix.	$m(\mathbb{L}_0) = (\mathbb{L}_0)^2\{(\mathbb{L}_0)^2 - 1\} = 0,$ non-diagonalizable matrix.
Bilinear form	$\alpha^{\tau\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = -1$ or 1 , $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = 1$ or -1 . .	$\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1,$ $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1.$	$\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = -1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = 1$.
Constants of $\mathbb{L}_0$	$\alpha = -2\beta^2$, $\gamma = \frac{1}{2}$, β = real .	$\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{2}$.	$\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{2}$.
Form of $\mathbb{L}_0$	$\mathbb{L}_0$ has blocks given by (12).	$\mathbb{L}_0$ has blocks given by (12).	$\mathbb{L}_0$ has blocks given by (12).
	Example 3	Example 4	Example 5

Overall charge	positive. (This is a spin 3/2 particle with definite charge.)	negative. (This is a spin 3/2 particle with definite charge .)	It does not satisfy the definition of definite charge.	indefinite	It does not satisfy the definition of definite charge.
charge in the state 3/2	positive.	negative.	zero, (eigenvalues of $\mathbb{L}_0^{3/2}$ are zero.)		positive.
charge in the state 1/2	zero , (eigenvalues of $\mathbb{L}_0^{1/2}$ are zero.)	zero , (eigenvalues of $\mathbb{L}_0^{1/2}$ are zero .)	zero (eigenvalues of $\mathbb{L}_0^{1/2}$ not zero.)		zero, (eigenvalues of $\mathbb{L}_0^{1/2}$ not zero.)
charge densities	$\rho^1_{1/2}=0$, $\rho^2_{1/2}=0$, $\rho^3_{3/2}=2>0$, $\rho^4_{3/2}=2>0$.	$\rho^1_{1/2}=0$, $\rho^2_{1/2}=0$, $\rho^3_{3/2}=-2<0$, $\rho^4_{3/2}=-2<0$.	$\rho^1_{1/2}=0$, $\rho^2_{1/2}=0$, $\rho^3_{3/2}=0$, $\rho^3_{3/2}=0$.		$\rho^1_{1/2}=0$, $\rho^2_{1/2}=0$, $\rho^3_{3/2}=2>0$, $\rho^4_{3/2}=2>0$.
minimal equation	$m(\mathbb{L}_0)=(\mathbb{L}_0)^2\{(\mathbb{L}_0)^2-1\}=0$, non-diagonalizable matrix.	$m(\mathbb{L}_0)=(\mathbb{L}_0)^2\{(\mathbb{L}_0)^2-1\}=0$, non-diagonalizable matrix	$m(\mathbb{L}_0)=\mathbb{L}\{(\mathbb{L}_0)^2-(\frac{\alpha}{2})^2\}^2=0$, non-diagonalizable matrix.	diagonalizable matrix.	$m(\mathbb{L}_0)=\{(\mathbb{L}_0)^2-1\}\{(\mathbb{L}_0)^2-(\frac{1}{4})^2\}^2=0$ non-diagonalizable matrix.
Bilinear form	$\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1$, $\alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1$.	$\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=-1$, $\alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=1$.	$\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=-1$ or 1, $\alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=1$ or -1.	$\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=-1$, or 1, $\alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=1$, or -1.	$\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1$, $\alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1$.
constants of $\mathbb{L}_0$	$\alpha=-\frac{1}{2}$, $\beta=-\frac{1}{2}$, $\gamma=\frac{1}{2}$.	$\alpha=-\frac{1}{2}$, $\beta=-\frac{1}{2}$, $\gamma=\frac{1}{2}$.	$\beta=\pm\frac{\alpha}{2}$, $\alpha\neq0$, $\gamma=0$.	$\frac{1}{4}>\beta^2$, $\alpha=0$, $\gamma=1/2$.	$\beta=\pm\frac{1}{4}$, $\alpha=0$, $\gamma=1/2$.
Form of $\mathbb{L}_0$	$\mathbb{L}_0$ has blocks given by (12).	$\mathbb{L}_0$ has blocks given by (12).	$\mathbb{L}_0$ has blocks given by (12).	$\mathbb{L}_0$ has blocks given by (12).	$\mathbb{L}_0$ has blocks given by (12).
	Example 6 (Pauli-Fierz eq.)	Example 7 (Pauli-Fierz eq.)	Example 7	Example 8	Example 9

Overall charge	positive if $\alpha>0$.	indefinite	indefinite
charge in the state 3/2	zero		
charge in the state 1/2	positive if $\alpha>0$.		
charge densities	$\rho^1_{1/2}=2\alpha>0$, $\rho^2_{1/2}=2\alpha>0$, $\rho^3_{3/2}=0$, $\rho^4_{3/2}=0$.		
Minimal equation	$m(\mathbb{L}_0)=\mathbb{L}_0\{(\mathbb{L}_0)^2-\alpha^2\}=0$ diagonalizable matrix.	$m(\mathbb{L}_0)=\{(\mathbb{L}_0)^2-1\}\{(\mathbb{L}_0)^2-\frac{1}{4}\}=0$ diagonalizable matrix.	$m(\mathbb{L}_0)=\{(\mathbb{L}_0)^2-1\}\{(\mathbb{L}_0)^2-2^2\}=0$, diagonalizable matrix.
Bilinear form	$\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1$, $\alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1$.	$\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1$ or -1, $\alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1$ or 1.	$\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1$ or -1, $\alpha^{\dot{\tau}_2\tau_2}=\alpha^{\dot{\tau}_2\tau_2}=-1$ or 1.
Constants of $\mathbb{L}_0$	$\alpha\neq 0$, $\beta=0$, $\gamma=0$.	$\alpha=1$, $\beta=0$, $\gamma=-\frac{1}{2}$.	$\alpha=1$, $\beta=0$, $\gamma=1$.
Form of $\mathbb{L}_0$	$\mathbb{L}_0$ has blocks given by (12).	$\mathbb{L}_0$ has blocks given by (12).	$\mathbb{L}_0$ has blocks given by (12).
	Example 10 (Dirac equation.)	Example 11	Example 12

Let us now consider the matrix $\mathbb{L}_0$ with sublocks given by (10) and corresponding bilinear form defined by $\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1$, $\alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=1$. As was said at the beginning of the paragraph the matrix $\mathbb{L}_0$ is hermitian and so is diagonalizable. Thus the charge associated with it is indefinite. We verify this statement proving that for all possible values of the constants α, β, γ , except the case $\alpha\neq 0$, $\beta=0$, $\gamma=0$ the eigenvalues of $\mathbb{A}\,\mathbb{L}_0$ appear with alternating signs.

It is sufficient to prove this for the sub-blocks $\mathbb{A}_\ell \mathbb{L}^\ell_0$, $\ell = 1/2 , 3/2$. The block $\mathbb{A}_{1/2}\mathbb{L}^{1/2}_0$ has eigenvalues

$$\Lambda^1_{1/2}=\frac{(\alpha+\gamma)+\sqrt{(\alpha-\gamma)^2+4\bar{\beta}\beta}}{2} , \quad \Lambda^2_{1/2}=\frac{(\alpha+\gamma)-\sqrt{(\alpha-\gamma)^2+4\bar{\beta}\beta}}{2} ,$$

$$\Lambda^3_{1/2}=\frac{(\alpha+\gamma)+\sqrt{(\alpha-\gamma)^2+4\bar{\beta}\beta}}{2} , \quad \Lambda^4_{1/2}=\frac{(\alpha+\gamma)-\sqrt{(\alpha-\gamma)^2+4\bar{\beta}\beta}}{2} ,$$

and the block $\mathbb{A}_{3/2}\mathbb{L}^{3/2}_0$ has eigenvalues

$$\Lambda^1_{3/2}=-2\gamma \quad , \quad \Lambda^2_{3/2}=-2\gamma .$$

The following cases are possible.

Case 1. $\alpha = \gamma$.

In this case the eigenvalues of the two blocks become respectively

$$\Lambda^1_{1/2} = \Lambda^3_{1/2} = \frac{2\alpha + \sqrt{4\beta\bar{\beta}}}{2}, \quad \Lambda^2_{1/2} = \Lambda^4_{1/2} = \frac{2\alpha - \sqrt{4\beta\bar{\beta}}}{2},$$

and

$$\Lambda^1_{3/2} = \Lambda^2_{3/2} = -2\alpha.$$

We have the following subcases.

i) If $\alpha > 0, -\alpha < \beta < \alpha$, then the eigenvalues are

$$\Lambda^1_{1/2} = \Lambda^3_{1/2} > 0, \quad \Lambda^2_{1/2} = \Lambda^4_{1/2} > 0, \quad \Lambda^1_{3/2} = \Lambda^2_{3/2} < 0,$$

and the charge associated with $\mathbb{L}_0$ is indefinite.

ii) If $\alpha > 0$ and $\beta > \alpha$ or $\beta < -\alpha$, then the eigenvalues are $\Lambda^1_{1/2} = \Lambda^3_{1/2} > 0$, $\Lambda^2_{1/2} = \Lambda^4_{1/2} > 0$, $\Lambda^1_{3/2} = \Lambda^2_{3/2} < 0$, and the charge is indefinite.

iii) If $\alpha = 0, \beta \neq 0$, then the eigenvalues are

$$\Lambda^1_{1/2} = \Lambda^3_{1/2} > 0, \quad \Lambda^2_{1/2} = \Lambda^4_{1/2} < 0, \quad \Lambda^1_{3/2} = \Lambda^2_{3/2} = 0,$$

and the charge is indefinite.

iv) If $\alpha < 0, -|\alpha| < |\beta| < |\alpha|$ then the eigenvalues are

$$\Lambda^1_{1/2} = \Lambda^3_{1/2} < 0, \quad \Lambda^2_{1/2} = \Lambda^4_{1/2} < 0, \quad \Lambda^1_{3/2} = \Lambda^2_{3/2} > 0,$$

and the charge is indefinite.

v) If $\alpha < 0, |\beta| > |\alpha|$ or $|\beta| < -|\alpha|$ then the eigenvalues are

$$\Lambda^1_{1/2} = \Lambda^3_{1/2} > 0, \quad \Lambda^2_{1/2} = \Lambda^4_{1/2} < 0, \quad \Lambda^1_{3/2} = \Lambda^2_{3/2} > 0,$$

and the charge is indefinite.

Case 2. $\alpha > 0, \gamma = 0$.

In this case the eigenvalues of the two blocks become respectively

$$\Lambda^1_{1/2} = \Lambda^3_{1/2} = \frac{\alpha + \sqrt{\alpha^2 + 4\bar{\beta}\beta}}{2}, \quad \Lambda^2_{1/2} = \Lambda^4_{1/2} = \frac{\alpha - \sqrt{\alpha^2 + 4\bar{\beta}\beta}}{2},$$

$$\Lambda^1_{3/2} = \Lambda^2_{3/2} = 0.$$

We have the following subcases.

i) If $\beta\neq 0$ then the eigenvalues are

$$\Lambda^1_{1/2}=\Lambda^3_{1/2}=\frac{\alpha+\sqrt{\alpha^2+4\bar{\beta}\beta}}{2}>0,\ \Lambda^2_{1/2}=\Lambda^4_{1/2}=\frac{\alpha-\sqrt{\alpha^2+4\bar{\beta}\beta}}{2}<0$$

and the charge is indefinite.

ii) If $\beta=0$ then the eigenvalues are

$$\Lambda^1_{1/2}=\Lambda^3_{1/2}=\alpha>0,\ \Lambda^2_{1/2}=\Lambda^4_{1/2}=0,\ \Lambda^1_{3/2}=\Lambda^2_{3/2}=0.$$

The charge in this case is definite. This corresponds to the Dirac equation.

Case 3. $\alpha>0,\ \gamma>0.$

The general formula for the eigenvalues is

$$\Lambda^1_{1/2}=\Lambda^3_{1/2}=\frac{(\alpha+\gamma)+\sqrt{(\alpha-\gamma)^2+4\bar{\beta}\beta}}{2},\ \Lambda^2_{1/2}=\Lambda^4_{1/2}=\frac{(\alpha+\gamma)-\sqrt{(\alpha-\gamma)^2+4\bar{\beta}\beta}}{2}$$

$$\Lambda^1_{3/2}=\Lambda^2_{3/2}=-2\gamma.$$

We have the following subcases.

i) If $\alpha>\gamma$ and $(\alpha+\gamma)>\sqrt{(\alpha-\gamma)^2+4\bar{\beta}\beta}$ then

$$\Lambda^1_{1/2}=\Lambda^3_{1/2}>0,\ \Lambda^2_{1/2}=\Lambda^4_{1/2}>0,\ \Lambda^1_{3/2}=\Lambda^2_{3/2}<0,$$

and the charge is indefinite.

ii) If $\alpha>\gamma$ and $\sqrt{(\alpha-\gamma)^2+4\bar{\beta}\beta}>(\alpha+\gamma)$ then

$$\Lambda^1_{1/2}=\Lambda^3_{1/2}>0,\ \Lambda^2_{1/2}=\Lambda^4_{1/2}<0,\ \Lambda^1_{3/2}=\Lambda^2_{3/2}<0,$$

and the charge is indefinite.

iii) If $\gamma>\alpha$ and $(\alpha+\gamma)>\sqrt{(\alpha-\gamma)^2+4\bar{\beta}\beta}$ then

$$\Lambda^1_{1/2}=\Lambda^3_{1/2}>0,\ \Lambda^2_{1/2}=\Lambda^4_{1/2}>0,\ \Lambda^1_{3/2}=\Lambda^2_{3/2}<0,$$

and the charge is indefinite.

iv) If $\gamma>\alpha$ and $\sqrt{(\alpha-\gamma)^2+4\bar{\beta}\beta}>(\alpha+\gamma)$ then

$$\Lambda^1_{1/2}=\Lambda^3_{1/2}>0,\ \Lambda^2_{1/2}=\Lambda^4_{1/2}<0,\ \Lambda^1_{3/2}=\Lambda^2_{3/2}<0,$$

and the charge is indefinite.

Case 4. $\alpha>0,\ \gamma<0.$

We have the following subcases.

i) If $|\alpha|>|\gamma|$ then $\sqrt{(\alpha-\gamma)^2+4\bar{\beta}\beta}>(\alpha+\gamma)$ for $\forall\,\beta$, and $\Lambda^1_{1/2}=\Lambda^3_{1/2}>0,\ \Lambda^2_{1/2}=\Lambda^4_{1/2}<0,\ \Lambda^1_{3/2}=\Lambda^2_{3/2}>0.$ Hence the charge is indefinite.

?

ii) If $|\gamma|>|\alpha|$ then $\sqrt{(\alpha-\gamma)^2+4\bar{\beta}\beta} > |(\alpha+\gamma)|$ for $\forall\ \beta$ and $\Lambda^1_{1/2}=\Lambda^3_{1/2}>0$, $\Lambda^2_{1/2}=\Lambda^4_{1/2}<0$, $\Lambda^1_{3/2}=\Lambda^2_{3/2}>0$. Hence the charge is indefinite.

<u>Case 5.</u> $\alpha=0$, $\gamma>0$.

i) If $\beta\neq 0$ then $\Lambda^1_{1/2}=\Lambda^3_{1/2}>0$, $\Lambda^2_{1/2}=\Lambda^4_{1/2}<0$, $\Lambda^1_{3/2}=\Lambda^2_{3/2}<0$, and the charge is indefinite.

ii) If $\beta=0$, then $\Lambda^1_{1/2}=\Lambda^3_{1/2}>0$, $\Lambda^2_{1/2}=\Lambda^4_{1/2}=0$, $\Lambda^1_{3/2}=\Lambda^2_{3/2}<0$ and the charge is indefinite.

<u>Case 6.</u> $\alpha=0$, $\gamma<0$.

i) If $\beta\neq 0$ then $\Lambda^1_{1/2}=\Lambda^3_{1/2}>0$, $\Lambda^2_{1/2}=\Lambda^4_{1/2}<0$, $\Lambda^1_{3/2}=\Lambda^2_{3/2}>0$, and the charge is indefinite.

ii) If $\beta=0$ then $\Lambda^1_{1/2}=\Lambda^3_{1/2}=0$, $\Lambda^2_{1/2}=\Lambda^4_{1/2}<0$, $\Lambda^1_{3/2}=\Lambda^2_{3/2}>0$, and the charge is indefinite.

<u>Case 7.</u> $\alpha<0$, $\gamma>0$.

i) If $|\alpha|>|\gamma|$ then $\sqrt{(\alpha-\gamma)^2+4\bar{\beta}\beta} > |\alpha-\gamma|$ for $\forall\beta$ and $\Lambda^1_{1/2}=\Lambda^3_{1/2}>0$, $\Lambda^2_{1/2}=\Lambda^4_{1/2}<0$, $\Lambda^1_{3/2}=\Lambda^2_{3/2}<0$. Hence the charge is indefinite.

ii) If $|\gamma|>|\alpha|$ then $\sqrt{(\alpha-\gamma)^2+4\bar{\beta}\beta} > (\alpha+\gamma)$ for $\forall\ \beta$ and $\Lambda^1_{1/2}=\Lambda^3_{1/2}>0$, $\Lambda^2_{1/2}=\Lambda^4_{1/2}<0$, $\Lambda^1_{3/2}=\Lambda^2_{3/2}<0$. Hence the charge is indefinite.

<u>Case 8.</u> $\alpha<0$, $\gamma=0$.

i) If $\beta\neq 0$ then $\Lambda^1_{1/2}=\Lambda^3_{1/2}>0$, $\Lambda^2_{1/2}=\Lambda^4_{1/2}<0$, $\Lambda^1_{3/2}=\Lambda^2_{3/2}=0$. and the charge is indefinite.

ii) If $\beta=0$ then $\Lambda^1_{1/2}=\Lambda^3_{1/2}=0$, $\Lambda^2_{1/2}=\Lambda^4_{1/2}<0$, $\Lambda^1_{3/2}=\Lambda^2_{3/2}=0$. The charge is definite in this case. This again corresponds to the Dirac equation.

<u>Case 9.</u> $\alpha<0$, $\gamma<0$.

i) If $\sqrt{(\alpha-\gamma)^2+4\bar{\beta}\beta} > |\alpha+\gamma|$ then $\Lambda^1_{1/2}=\Lambda^3_{1/2}>0$, $\Lambda^2_{1/2}=\Lambda^4_{1/2}<0$, $\Lambda^1_{3/2}=\Lambda^2_{3/2}>0$, and the charge is indefinite.

ii) If $\sqrt{(\alpha-\gamma)^2+4\bar{\beta}\beta} < |\alpha+\gamma|$ then $\Lambda^{1}_{1/2}=\Lambda^{3}_{1/2} < 0$, $\Lambda^{2}_{1/2}=\Lambda^{4}_{1/2}<0$, $\Lambda^{1}_{3/2}=\Lambda^{2}_{3/2}>0$, and the charge is indefinite.

iii) If $\beta=0$, then $\Lambda^{1}_{1/2}=\Lambda^{3}_{1/2}<0$, $\Lambda^{2}_{1/2}=\Lambda^{4}_{1/2}<0$, $\Lambda^{1}_{3/2}=\Lambda^{2}_{3/2}>0$, and the charge is again indefinite.

Thus we conclude that for all the cases of the constants α,β,γ except the case $\alpha\neq 0$, $\beta=0$, $\gamma=0$ the charge of the matrix $\mathbb{L}_0$ given by (10) is indefinite. Similar results are obtained if the bilinear form $\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=-1$, $\alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1$, is used instead.

3. Conclusions and discussion.

In this chapter we have studied equations for which the wavefunction ψ transforms according to the 16-dimensional representation with components

$$\tau_1\sim(\tfrac{1}{2},\tfrac{3}{2}) \;,\; \dot{\tau}_1\sim(-\tfrac{1}{2},\tfrac{3}{2}),\; \tau_2\sim(\tfrac{1}{2},\tfrac{5}{2}) \;,\; \dot{\tau}_2\sim(-\tfrac{1}{2},\tfrac{5}{2}),$$

interlocking according to the scheme (II). We have reached the following conclusions.

1. There is only one equation of higher spin $\frac{3}{2}$ having definite charge and matrix $\mathbb{L}_0$ non-diagonalizable. This is the Pauli-Fierz equation (c.f. examples 4, 5, 6, 7).

2. There are equations which satisfy the criterion of Amar and Dozzio of causal propagation and the criterion of Cox of second quantization but the corresponding charge is indefinite, (c.f. examples 1, 2, 3, 11, 12).

3. There are equations which satisfy the criterion of Amar and Dozzio but they present difficulties as far as the charge is concerned because there are certain states corresponding to non-zero eigenvalues and vanishing charge densities respectively, (c.f. examples 7, 9).

4. There is an equation with definite charge which satisfies the criteria of causal propagation and of second quantization but it describes particles of spin 1/2. This is another form of the equation of Dirac, (c.f. example 10).

5. All the cases with diagonalizable matrix $\mathbb{L}_0$ except the case $\alpha \neq 0, \beta = 0, \gamma = 0$, give indefinite charge.

CHAPTER 5.

1. Preliminaries.

In this chapter we shall be concerned with equations of the Gel'fand-Yaglom type for which the wave-function ψ transforms according to the 20-dimensional representation with components

$$\tau_1 \sim \left(\frac{1}{2}, \frac{3}{2}\right), \quad \dot{\tau}_1 \sim \left(-\frac{1}{2}, \frac{3}{2}\right), \quad \tau_2 \sim \left(\frac{1}{2}, \frac{5}{2}\right), \quad \dot{\tau}_2 \sim \left(-\frac{1}{2}, \frac{5}{2}\right),$$
$$\tau_1' \sim \left(\frac{1}{2}, \frac{3}{2}\right), \quad \dot{\tau}_1' \sim \left(-\frac{1}{2}, \frac{3}{2}\right),$$

interlocking according to the scheme

$$\begin{tikzcd}
\left(\frac{1}{2}, \frac{5}{2}\right) \sim \tau_2 \arrow[rr, shift left, "c^{\tau_2\tau_1}"] \arrow[rr, leftarrow, shift right, "c^{\tau_1\tau_2}"'] \arrow[dr, shift left, "c^{\tau_2\tau_1'}"] \arrow[dr, leftarrow, shift right, "c^{\tau_1'\tau_2}"'] \arrow[ddd, shift left, "c^{\tau_2\dot{\tau}_2}"] \arrow[ddd, leftarrow, shift right, "c^{\dot{\tau}_2\tau_2}"'] & & \tau_1 \sim \left(\frac{1}{2}, \frac{3}{2}\right) \arrow[ddl, shift left, "c^{\tau_1\dot{\tau}_1'}"] \arrow[ddl, leftarrow, shift right, "c^{\dot{\tau}_1'\tau_1}"'] \arrow[ddd, shift left, "c^{\tau_1\dot{\tau}_1}"] \arrow[ddd, leftarrow, shift right, "c^{\dot{\tau}_1\tau_1}"'] \\
& \tau_1' \sim \left(\frac{1}{2}, \frac{3}{2}\right) \arrow[d, shift left, "c^{\tau_1'\dot{\tau}_1'}"] \arrow[d, leftarrow, shift right, "c^{\dot{\tau}_1'\tau_1'}"'] \arrow[ddr, shift left, "c^{\tau_1'\dot{\tau}_1}"] \arrow[ddr, leftarrow, shift right, "c^{\dot{\tau}_1\tau_1'}"'] & \\
& \dot{\tau}_1' \sim \left(-\frac{1}{2}, \frac{3}{2}\right) \arrow[dl, shift left, "c^{\dot{\tau}_1'\dot{\tau}_2}"] \arrow[dl, leftarrow, shift right, "c^{\dot{\tau}_2\dot{\tau}_1'}"'] & \\
\left(-\frac{1}{2}, \frac{5}{2}\right) \sim \dot{\tau}_2 \arrow[rr, shift left, "c^{\dot{\tau}_2\dot{\tau}_1}"] \arrow[rr, leftarrow, shift right, "c^{\dot{\tau}_1\dot{\tau}_2}"'] & & \dot{\tau}_1 \sim \left(-\frac{1}{2}, \frac{3}{2}\right)
\end{tikzcd} \qquad \text{(III)}$$

.

This representation admits the canonical basis

$$\{\xi_{\ell m}\} = \{ \xi^{\tau_1}_{1/2, 1/2}, \xi^{\tau_1}_{1/2, -1/2}, \xi^{\dot{\tau}_1}_{1/2, 1/2}, \xi^{\dot{\tau}_1}_{1/2, -1/2}, \xi^{\dot{\tau}_2}_{1/2, 1/2},$$
$$\xi^{\dot{\tau}_2}_{1/2, -1/2}, \xi^{\dot{\tau}_2}_{3/2, 3/2}, \xi^{\dot{\tau}_2}_{3/2, 1/2}, \xi^{\dot{\tau}_2}_{3/2, -1/2}, \xi^{\dot{\tau}_2}_{3/2, -3/2},$$
$$\xi^{\tau_2}_{1/2, 1/2}, \xi^{\tau_2}_{1/2, -1/2}, \xi^{\tau_2}_{3/2, 3/2}, \xi^{\tau_2}_{3/2, 1/2}, \xi^{\tau_2}_{3/2, -1/2},$$
$$\xi^{\tau_2}_{3/2, -3/2}, \xi^{\tau_1'}_{1/2, 1/2}, \xi^{\tau_1'}_{1/2, -1/2}, \xi^{\dot{\tau}_1'}_{1/2, 1/2}, \xi^{\dot{\tau}_1'}_{1/2, -1/2} \}.$$

(1)

The matrix $\mathbb{L}_0$ with respect to this basis has blocks

$$\mathbb{L}_0^{1/2} = \begin{array}{c|c|c|c|c|c|c|}
 & \tau_1 & \dot\tau_1 & \dot\tau_2 & \tau_2 & \tau_1' & \dot\tau_1' \\
\tau_1 & 0 & c^{\tau_1\dot\tau_1} & 0 & i\sqrt{3}\,c^{\tau_1\tau_2} & 0 & c^{\tau_1\dot\tau_1'} \\ \hline
\dot\tau_1 & c^{\dot\tau_1\tau_1} & 0 & i\sqrt{3}\,c^{\dot\tau_1\dot\tau_2} & 0 & c^{\dot\tau_1\tau_1'} & 0 \\ \hline
\dot\tau_2 & 0 & i\sqrt{3}\,c^{\dot\tau_2\dot\tau_1} & 0 & c^{\dot\tau_2\tau_2} & 0 & i\sqrt{3}\,c^{\dot\tau_2\dot\tau_1'} \\ \hline
\tau_2 & i\sqrt{3}\,c^{\tau_2\tau_1} & 0 & c^{\tau_2\dot\tau_2} & 0 & i\sqrt{3}\,c^{\tau_2\tau_1'} & 0 \\ \hline
\tau_1' & 0 & c^{\tau_1'\dot\tau_1} & 0 & i\sqrt{3}\,c^{\tau_1'\tau_2} & 0 & c^{\tau_1'\dot\tau_1'} \\ \hline
\dot\tau_1' & c^{\dot\tau_1'\tau_1} & 0 & i\sqrt{3}\,c^{\dot\tau_1'\dot\tau_2} & 0 & c^{\dot\tau_1'\tau_1'} & 0
\end{array} , \quad \mathbb{L}_0^{3/2} = \begin{array}{c}
 \\ \dot\tau_2 \\ \tau_2
\end{array}\!\!\begin{array}{cc}
\dot\tau_2 & \tau_2 \\
\multicolumn{2}{c}{\left[\begin{array}{cc} 0 & 2c^{\dot\tau_2\tau_2} \\ 2c^{\tau_2\dot\tau_2} & 0 \end{array}\right]}
\end{array} .$$

(2)

For invariance of the equation under the complete group the following relations must be satisfied among the elements of $\mathbb{L}_0$,

$$\begin{array}{lll}
c^{\tau_1\dot\tau_1} = c^{\dot\tau_1\tau_1} , & c^{\tau_1\tau_2} = c^{\dot\tau_1\dot\tau_2} , & c^{\tau_1\dot\tau_1'} = c^{\dot\tau_1\tau_1'} , \\
c^{\dot\tau_2\dot\tau_1} = c^{\tau_2\tau_1} , & c^{\dot\tau_2\tau_2} = c^{\tau_2\dot\tau_2} , & c^{\tau_2\tau_1'} = c^{\dot\tau_2\dot\tau_1'} , \\
c^{\tau_1'\tau_2} = c^{\dot\tau_1'\dot\tau_2} , & c^{\tau_1'\dot\tau_1'} = c^{\dot\tau_1'\tau_1'} , & c^{\tau_1'\dot\tau_1} \pm{,}c^{\dot\tau_1'\tau_1} .
\end{array} \qquad (3)$$

Since in the reduction of the 20-dimensional representation each irreducible τ_i appears together with its conjugate $\dot\tau_i$, this guarantees the existence of the following bilinear forms $(\psi_{\tau_1} , \psi_{\tau_2})$ which we give in terms of the constants $\alpha^{\tau_i\tau_j}$ defining them:

(i) $\alpha^{\tau_1\dot\tau_1} = \alpha^{\dot\tau_1\tau_1} = 1$, $\alpha^{\tau_2\dot\tau_2} = \alpha^{\dot\tau_2\tau_2} = -1$, $\alpha^{\tau_1'\dot\tau_1'} = \alpha^{\dot\tau_1'\tau_1'} = 1$,

(ii) $\alpha^{\tau_1\dot\tau_1} = \alpha^{\dot\tau_1\tau_1} = -1$, $\alpha^{\tau_2\dot\tau_2} = \alpha^{\dot\tau_2\tau_2} = 1$, $\alpha^{\tau_1'\dot\tau_1'} = \alpha^{\dot\tau_1'\tau_1'} = 1$,

(iii) $\alpha^{\tau_1\dot\tau_1} = \alpha^{\dot\tau_1\tau_1} = 1$, $\alpha^{\tau_2\dot\tau_2} = \alpha^{\dot\tau_2\tau_2} = 1$, $\alpha^{\tau_1'\dot\tau_1'} = \alpha^{\dot\tau_1'\tau_1'} = 1$,

(iv) $\alpha^{\tau_1\dot\tau_1} = \alpha^{\dot\tau_1\tau_1} = 1$, $\alpha^{\tau_2\dot\tau_2} = \alpha^{\dot\tau_2\tau_2} = 1$, $\alpha^{\tau_1'\dot\tau_1'} = \alpha^{\dot\tau_1'\tau_1'} = -1$,

(v) $\alpha^{\tau_1\dot\tau_1} = \alpha^{\dot\tau_1\tau_1} = -1$, $\alpha^{\tau_2\dot\tau_2} = \alpha^{\dot\tau_2\tau_2} = -1$, $\alpha^{\tau_1'\dot\tau_1'} = \alpha^{\dot\tau_1'\tau_1'} = 1$,

(vi) $\alpha^{\tau_1\dot\tau_1} = \alpha^{\dot\tau_1\tau_1} = 1$, $\alpha^{\tau_2\dot\tau_2} = \alpha^{\dot\tau_2\tau_2} = -1$, $\alpha^{\tau_1'\dot\tau_1'} = \alpha^{\dot\tau_1'\tau_1'} = -1$,

(vii) $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = -1, \quad \alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = 1, \quad \alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1'} = -1,$

(viii) $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = -1, \quad \alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1, \quad \alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1'} = -1.$

Finally if we require that the Gel'fand-Yaglom wave-equation be derivable from an invariant Lagrangian with respect to the bilinear form (ψ_1, ψ_2) defined by the constants

$$\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1, \quad \alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1, \quad \alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1'} = 1,$$

and having hermitianizing matrix

$$\mathbb{A}_{1/2} = \begin{array}{c|cc|c|c|c|c|} & \tau_1 & \dot{\tau}_1 & \dot{\tau}_2 & \tau_2 & \tau_1' & \dot{\tau}_1' \\ \tau_1 & 0 & 1 & & & & \\ \dot{\tau}_1 & 1 & 0 & & & & \\ \dot{\tau}_2 & & & 0 & -1 & & \\ \tau_2 & & & -1 & 0 & & \\ \tau_1' & & & & & 0 & 1 \\ \dot{\tau}_1' & & & & & 1 & 0 \end{array}, \quad \mathbb{A}_{3/2} = \begin{array}{c|cc|} & \dot{\tau}_2 & \tau_2 \\ \dot{\tau}_2 & 0 & 1 \\ \tau_2 & 1 & 0 \end{array}, \qquad (4)$$

we find the following relations satisfied among the elements of the matrix $\mathbb{L}_0$,

$$\begin{array}{lll} c^{\tau_1\dot{\tau}_1} = \bar{c}^{\tau_1\dot{\tau}_1}\ ; & c^{\tau_1\tau_2} = c^{\dot{\tau}_2\dot{\tau}_1}, & c^{\tau_1\dot{\tau}_1'} = \bar{c}^{\tau_1'\dot{\tau}_1}, \\ c^{\dot{\tau}_1\tau_1} = \bar{c}^{\dot{\tau}_1\tau_1}, & c^{\dot{\tau}_1\dot{\tau}_2} = \bar{c}^{\tau_2\tau_1}, & c^{\dot{\tau}_1\tau_1'} = \bar{c}^{\dot{\tau}_1'\tau_1}, \\ c^{\dot{\tau}_2\dot{\tau}_1} = \bar{c}^{\tau_1\tau_2}, & c^{\dot{\tau}_2\tau_2} = \bar{c}^{\dot{\tau}_2\tau_2}, & c^{\dot{\tau}_2\dot{\tau}_1'} = c^{\tau_1'\tau_2}, \\ c^{\tau_2\tau_1} = \bar{c}^{\dot{\tau}_1\dot{\tau}_2}, & c^{\tau_2\dot{\tau}_2} = c^{\tau_2\dot{\tau}_2}, & c^{\tau_2\tau_1'} = \bar{c}^{\dot{\tau}_1'\dot{\tau}_2}, \\ c^{\tau_1'\dot{\tau}_1} = \bar{c}^{\tau_1\dot{\tau}_1'}, & c^{\tau_1'\tau_2} = \bar{c}^{\dot{\tau}_2\dot{\tau}_1'}, & c^{\tau_1'\dot{\tau}_1'} = \bar{c}^{\tau_1'\dot{\tau}_1'}, \\ c^{\dot{\tau}_1'\tau_1} = \bar{c}^{\dot{\tau}_1\tau_1'}, & c^{\dot{\tau}_1'\dot{\tau}_2} = \bar{c}^{\tau_2\tau_1'}, & c^{\tau_1'\tau_1'} = \bar{c}^{\dot{\tau}_1'\tau_1'}. \end{array} \qquad (5)$$

Combining these relations with the relations (3) of invariance of the equation under the complete Lorentz group, we find for the matrix $\mathbb{L}_0$ which is invariant under the complete group and derivable from an invariant Lagrangian the blocks

$$\mathbb{L}_0^{1/2} = \begin{array}{c|c|c|c|c|c|c|} & \tau_1 & \dot{\tau}_1 & \dot{\tau}_2 & \tau_2 & \tau_1' & \dot{\tau}_1' \\ \tau_1 & 0 & c^{\tau_1\dot{\tau}_1} & 0 & i\sqrt{3}\, c^{\tau_1\tau_2} & 0 & c^{\tau_1\dot{\tau}_1'} \\ \hline \dot{\tau}_1 & c^{\tau_1\dot{\tau}_1} & 0 & i\sqrt{3}\, c^{\tau_1\tau_2} & 0 & c^{\tau_1\dot{\tau}_1'} & 0 \\ \hline \dot{\tau}_2 & 0 & i\sqrt{3}\, \bar{c}^{\tau_1\tau_2} & 0 & c^{\dot{\tau}_2\tau_2} & 0 & i\sqrt{3}\, c^{\dot{\tau}_2\dot{\tau}_1'} \\ \hline \tau_2 & i\sqrt{3}\, \bar{c}^{\tau_1\tau_2} & 0 & c^{\dot{\tau}_2\tau_2} & 0 & i\sqrt{3}\, c^{\dot{\tau}_2\dot{\tau}_1'} & 0 \\ \hline \tau_1' & 0 & \bar{c}^{\tau_1\dot{\tau}_1'} & 0 & i\sqrt{3}\, \bar{c}^{\dot{\tau}_2\dot{\tau}_1'} & 0 & c^{\tau_1'\dot{\tau}_1'} \\ \hline \dot{\tau}_1' & \bar{c}^{\tau_1\dot{\tau}_1'} & 0 & i\sqrt{3}\, \bar{c}^{\dot{\tau}_2\dot{\tau}_1'} & 0 & c^{\tau_1'\dot{\tau}_1'} & 0 \end{array}\ , \quad \mathbb{L}_0^{3/2} = \begin{array}{c|cc|} & \dot{\tau}_2 & \tau_2 \\ \dot{\tau}_2 & 0 & 2c^{\dot{\tau}_2\tau_2} \\ \tau_2 & 2c^{\dot{\tau}_2\tau_2} & 0 \end{array}\ , \qquad (6)$$

where $c^{\tau_1\dot{\tau}_1}$ = real, $c^{\dot{\tau}_2\tau_2}$ = real, $c^{\tau_1'\dot{\tau}_1'}$ = real. Renaming the constants $c^{\tau_i\tau_j}$ of $\mathbb{L}_0$ as follows, $c^{\tau_1\dot{\tau}_1} \equiv \alpha$ (real), $c^{\tau_1\tau_2} \equiv \beta$, $c^{\tau_1\dot{\tau}_1'} \equiv \gamma$, $c^{\dot{\tau}_2\tau_2} \equiv \varepsilon$ (real) $c^{\dot{\tau}_2\dot{\tau}_1'} \equiv \zeta$, $c^{\tau_1'\dot{\tau}_1'} \equiv \theta$ (real)

we have

$$\mathbb{L}_0^{1/2} = \begin{array}{c|c|c|c|c|c|c|} & \tau_1 & \dot{\tau}_1 & \dot{\tau}_2 & \tau_2 & \tau_1' & \dot{\tau}_1' \\ \tau_1 & 0 & \alpha & 0 & i\sqrt{3}\,\beta & 0 & \gamma \\ \hline \dot{\tau}_1 & \alpha & 0 & i\sqrt{3}\,\beta & 0 & \gamma & 0 \\ \hline \dot{\tau}_2 & 0 & i\sqrt{3}\,\bar{\beta} & 0 & \varepsilon & 0 & i\sqrt{3}\,\zeta \\ \hline \tau_2 & i\sqrt{3}\,\bar{\beta} & 0 & \varepsilon & 0 & i\sqrt{3}\,\zeta & 0 \\ \hline \tau_1' & 0 & \bar{\gamma} & 0 & i\sqrt{3}\,\bar{\zeta} & 0 & \theta \\ \hline \dot{\tau}_1' & \bar{\gamma} & 0 & i\sqrt{3}\,\bar{\zeta} & 0 & \theta & 0 \end{array}\ , \quad \mathbb{L}_0^{1/2} = \begin{array}{c|cc|} & \dot{\tau}_2 & \tau_2 \\ \dot{\tau}_2 & 0 & 2\varepsilon \\ \tau_2 & 2\varepsilon & 0 \end{array}\ . \qquad (7)$$

The matrix $\mathbb{L}_0$ for the other possible choices of bilinear form is:

ii) for $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = -1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = 1$, $\alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1'} = 1$,

$$\mathbb{L}_0^{1/2} = \left|\begin{array}{c|c|c|c|c|c} 0 & \alpha=\text{real} & 0 & \iota\sqrt{3}\,\beta & 0 & \gamma \\ \hline \alpha=\text{real} & 0 & \iota\sqrt{3}\,\beta & 0 & \gamma & 0 \\ \hline 0 & \iota\sqrt{3}\,\bar{\beta} & 0 & \varepsilon=\text{real} & 0 & \iota\sqrt{3}\,\zeta \\ \hline \iota\sqrt{3}\,\bar{\beta} & 0 & \varepsilon=\text{real} & 0 & \iota\sqrt{3}\,\zeta & 0 \\ \hline 0 & -\bar{\gamma} & 0 & -\iota\sqrt{3}\,\bar{\zeta} & 0 & \theta=\text{real} \\ \hline -\bar{\gamma} & 0 & -\iota\sqrt{3}\,\bar{\zeta} & 0 & \theta=\text{real} & 0 \end{array}\right| , \quad \mathbb{L}_0^{3/2} = \begin{bmatrix} 0 & 2\varepsilon \\ 2\varepsilon & 0 \end{bmatrix} ;$$

iii) for $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = 1$, $\alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1'} = 1$,

$$\mathbb{L}_0^{1/2} = \left(\begin{array}{c|c|c|c|c|c} 0 & \alpha=\text{real} & 0 & \iota\sqrt{3}\,\beta & 0 & \gamma \\ \hline \alpha=\text{real} & 0 & \iota\sqrt{3}\,\beta & 0 & \gamma & 0 \\ \hline 0 & -\iota\sqrt{3}\,\bar{\beta} & 0 & \varepsilon=\text{real} & 0 & \iota\sqrt{3}\,\zeta \\ \hline -\iota\sqrt{3}\,\bar{\beta} & 0 & \varepsilon=\text{real} & 0 & \iota\sqrt{3}\,\zeta & 0 \\ \hline 0 & \bar{\gamma} & 0 & -\iota\sqrt{3}\,\bar{\zeta} & 0 & \theta=\text{real} \\ \hline \bar{\gamma} & 0 & -\iota\sqrt{3}\,\bar{\zeta} & 0 & \theta=\text{real} & 0 \end{array}\right) , \quad \mathbb{L}_0^{3/2} = \begin{bmatrix} 0 & 2\varepsilon \\ 2\varepsilon & 0 \end{bmatrix} ;$$

iv) for $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = 1$, $\alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1'} = -1$,

$$\mathbb{L}_0^{1/2} = \left|\begin{array}{c|c|c|c|c|c} 0 & \alpha=\text{real} & 0 & \iota\sqrt{3}\,\beta & 0 & \gamma \\ \hline \alpha=\text{real} & 0 & \iota\sqrt{3}\,\beta & 0 & \gamma & 0 \\ \hline 0 & -\iota\sqrt{3}\,\bar{\beta} & 0 & \varepsilon=\text{real} & 0 & \iota\sqrt{3}\,\zeta \\ \hline -\iota\sqrt{3}\,\bar{\beta} & 0 & \varepsilon=\text{real} & 0 & \iota\sqrt{3}\,\zeta & 0 \\ \hline 0 & -\bar{\gamma} & 0 & \iota\sqrt{3}\,\bar{\zeta} & 0 & \theta=\text{real} \\ \hline -\bar{\gamma} & 0 & \iota\sqrt{3}\,\bar{\zeta} & 0 & \theta=\text{real} & 0 \end{array}\right| , \quad \mathbb{L}_0^{3/2} = \begin{bmatrix} 0 & 2\varepsilon \\ 2\varepsilon & 0 \end{bmatrix} ;$$

v) for $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = -1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1$, $\alpha^{\tau'_1\dot{\tau}'_1} = \alpha^{\dot{\tau}'_1\tau'_1} = 1$,

$$\mathbb{L}^{1/2}_{0} = \left|\begin{array}{c|c|c|c|c|c} 0 & \alpha=\text{real} & 0 & \iota\sqrt{3}\beta & 0 & \gamma \\ \hline \alpha=\text{real} & 0 & \iota\sqrt{3}\beta & 0 & \gamma & 0 \\ \hline 0 & -\iota\sqrt{3}\bar{\beta} & 0 & \varepsilon=\text{real} & 0 & \iota\sqrt{3}\zeta \\ \hline -\iota\sqrt{3}\bar{\beta} & 0 & \varepsilon=\text{real} & 0 & \iota\sqrt{3}\zeta & 0 \\ \hline 0 & -\bar{\gamma} & 0 & \iota\sqrt{3}\bar{\zeta} & 0 & \theta=\text{real} \\ \hline -\bar{\gamma} & 0 & \iota\sqrt{3}\bar{\zeta} & 0 & \theta=\text{real} & 0 \end{array}\right| , \quad \mathbb{L}^{3/2}_{0} = \begin{bmatrix} 0 & 2\varepsilon \\ 2\varepsilon & 0 \end{bmatrix};$$

vi) for $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1$, $\alpha^{\tau'_1\dot{\tau}'_1} = \alpha^{\dot{\tau}'_1\tau'_1} = -1$,

$$\mathbb{L}^{1/2}_{0} = \left|\begin{array}{c|c|c|c|c|c} 0 & \alpha=\text{real} & 0 & \iota\sqrt{3}\beta & 0 & \gamma \\ \hline \alpha=\text{real} & 0 & \iota\sqrt{3}\beta & 0 & \gamma & 0 \\ \hline 0 & \iota\sqrt{3}\bar{\beta} & 0 & \varepsilon=\text{real} & 0 & \iota\sqrt{3}\zeta \\ \hline \iota\sqrt{3}\bar{\beta} & 0 & \varepsilon=\text{real} & 0 & \iota\sqrt{3}\zeta & 0 \\ \hline 0 & -\bar{\gamma} & 0 & -\iota\sqrt{3}\bar{\zeta} & 0 & \theta=\text{real} \\ \hline -\bar{\gamma} & 0 & -\iota\sqrt{3}\bar{\zeta} & 0 & \theta=\text{real} & 0 \end{array}\right| , \quad \mathbb{L}^{3/2}_{0} = \begin{bmatrix} 0 & 2\varepsilon \\ 2\varepsilon & 0 \end{bmatrix};$$

vii) for $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = -1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = 1$, $\alpha^{\tau'_1\dot{\tau}'_1} = \alpha^{\dot{\tau}'_1\tau'_1} = -1$,

$$\mathbb{L}^{1/2}_{0} = \left|\begin{array}{c|c|c|c|c|c} 0 & \alpha=\text{real} & 0 & \iota\sqrt{3}\beta & 0 & \gamma \\ \hline \alpha=\text{real} & 0 & \iota\sqrt{3}\beta & 0 & \gamma & 0 \\ \hline 0 & \iota\sqrt{3}\bar{\beta} & 0 & \varepsilon=\text{real} & 0 & \iota\sqrt{3}\zeta \\ \hline \iota\sqrt{3}\bar{\beta} & 0 & \varepsilon=\text{real} & 0 & \iota\sqrt{3}\zeta & 0 \\ \hline 0 & \bar{\gamma} & 0 & \iota\sqrt{3}\bar{\zeta} & 0 & \theta=\text{real} \\ \hline \bar{\gamma} & 0 & \iota\sqrt{3}\bar{\zeta} & 0 & \theta=\text{real} & 0 \end{array}\right| , \quad \mathbb{L}^{3/2}_{0} = \begin{bmatrix} 0 & 2\varepsilon \\ 2\varepsilon & 0 \end{bmatrix};$$

viii) for $\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=-1$, $\alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1$, $\alpha^{\tau_1'\dot{\tau}_1'}=\alpha^{\dot{\tau}_1'\tau_1'}=-1$,

$$\mathbb{L}_0^{1/2}=\left[\begin{array}{c|c|c|c|c|c} 0 & a=\text{real} & 0 & i\sqrt{3}\beta & 0 & \gamma \\ \hline a=\text{real} & 0 & i\sqrt{3}\beta & 0 & \gamma & 0 \\ \hline 0 & -i\sqrt{3}\bar{\beta} & 0 & \epsilon=\text{real} & 0 & i\sqrt{3}\zeta \\ -i\sqrt{3}\bar{\beta} & 0 & \epsilon=\text{real} & 0 & i\sqrt{3}\zeta & 0 \\ \hline 0 & \bar{\gamma} & 0 & -i\sqrt{3}\bar{\zeta} & 0 & \theta=\text{real} \\ \hline \bar{\gamma} & 0 & -i\sqrt{3}\bar{\zeta} & 0 & \theta=\text{real} & 0 \end{array}\right], \quad \mathbb{L}_0^{3/2}=\begin{bmatrix} 0 & 2\epsilon \\ 2\epsilon & 0 \end{bmatrix}.$$

2: <u>Charge.</u>

Let us now study the charge of particular examples of wave equations based on the 20-dimensional representation $\tau_1\oplus\dot{\tau}_1\oplus\dot{\tau}_2\oplus\tau_2\oplus\tau_1'\oplus\dot{\tau}_1'$. For the sake of brevity we enlist these examples in the following table.

Overall charge and comments.	This example does not satisfy the definition of definite charge because there are eigenvectors corresponding to $\mathbb{L}_0^{1/2}$ having charge densities which vanish without the eigenvalues being zero.	<u>Not definable</u> according to the adopted definition. For the same reason as for example 1.
Charge in the state 3/2	positive	negative
Charge in the state 1/2	not definable.	Could be considered positive if we ignore the vanishing charge densities corresponding to $\lambda_1\neq 0$
Charge densities	$\rho^1_{1/2}=0$, with eigenvalue being $1\neq 0$, $\rho^2_{1/2}=0$, with eigenvalue being $-1\neq 0$, $\rho^3_{1/2}=2>0$, $\rho^4_{1/2}=2>0$; $\rho^1_{3/2}=8>0$, $\rho^2_{3/2}=8>0$.	$\rho^1_{1/2}=0$, with eigenvalue being $1\neq 0$. $\rho^2_{1/2}=0$, with eigevalue being $-1\neq 0$. $\rho^3_{1/2}=\rho^4_{1/2}=2>0$; $\rho^1_{3/2}=\rho^2_{3/2}=-8<0$.
Minimal equation	$m(\mathbb{L}_0)=\{\mathbb{L}_0^2-4^2\}\{\mathbb{L}_0^2-1\}^2=0$, matrix $\mathbb{L}_0$ non-diagonalizable.	$m(\mathbb{L}_0)=\{\mathbb{L}_0^2-4^2\}\{\mathbb{L}_0^2-1\}^2=0$, matrix $\mathbb{L}_0$ non-diagonalizable.
Bilinear form	$\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1$, $\alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1$, $\alpha^{\tau_1'\dot{\tau}_1'}=\alpha^{\dot{\tau}_1'\tau_1'}=1$.	$\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1$, $\alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1$, $\alpha^{\tau_1'\dot{\tau}_1'}=\alpha^{\dot{\tau}_1'\tau_1'}=1$.
Constants of $\mathbb{L}_0$	$\alpha=0$, $\beta=1/\sqrt{3}$, $\gamma=0$, $\epsilon=2$, $\zeta=0$, $\theta=1$.	$\alpha=0$, $\beta=1/\sqrt{3}$, $\gamma=0$, $\epsilon=-2$, $\zeta=0$, $\theta=1$.
	Example 1	Example 2

Overall Charge and Comments		<u>Definite</u>. This is an example of spin 3/2 wave equation based on the 20-dim representation $\tau_1 \oplus \dot{\tau}_1 \oplus \tau_2 \oplus \dot{\tau}_2 \oplus \tau_1' \oplus \dot{\tau}_1'$, with definite charge.	<u>Positive</u> if $\alpha<0$. This is an example of a spin 3/2 wave equation with definite charge. It is a 16-dim equation. It is like the Pauli-Fierz equation. The representations which survive are $\tau_1, \dot{\tau}_1, \tau_2, \dot{\tau}_2$.
Charge in the state 3/2		negative.	positive if $\alpha<0$.
Charge in the state 1/2		zero: because the eigenvalues of $\mathbb{L}_0^{1/2}$ are all zero.	zero: because the eigenvalues of $\mathbb{L}_0^{1/2}$, are zero.
Charge densities		$\rho^{\iota}_{1/2}=0$. (The charge densities in the state 1/2 are zero because all the eigenvalues vanish.) $\rho^{1}_{3/2}=-4<0$, $\rho^{2}_{3/2}=-4<0$.	$\rho^{\iota}_{1/2}=0$, (The charge densities in the state 1/2 vanish because $\lambda_1=0$.) $\rho^{1}_{3/2}=-4\alpha$, $\rho^{2}_{3/2}=-4\alpha$.
Minimal equation	matrix $\mathbb{L}_0$ non-diagonalizable	$m(\mathbb{L}_0)=\mathbb{L}_0^3\{\mathbb{L}_0^2-2^2\}=0$, matrix $\mathbb{L}_0$ non-diagonalizable	$m(\mathbb{L}_0)=\mathbb{L}_0^2\{\mathbb{L}_0^2-(2\alpha)^2\}=0$, matrix $\mathbb{L}_0$ non-diagonalizable.
Bilinear form	$\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1$, $\alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1$, $\alpha^{\tau_1'\dot{\tau}_1'}=\alpha^{\dot{\tau}_1'\tau_1'}=1$	$\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1$, $\alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1$, . $\alpha^{\tau_1'\dot{\tau}_1'}=\alpha^{\dot{\tau}_1'\tau_1'}=1$.	$\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1$, $\alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1$, $\alpha^{\tau_1'\dot{\tau}_1'}=\alpha^{\dot{\tau}_1'\tau_1'}=1$.
Constants of $\mathbb{L}_0$	Let us require that $\alpha=\iota\sqrt{3}\,\bar{\beta}$, $\epsilon=\iota\sqrt{3}\,\beta$, $\gamma=\iota\sqrt{3}\,\zeta$. Hence $\alpha=-\varepsilon$. Taking $\theta=0$ we find an $\mathbb{L}_0^{1/2}$ whose eigenvalues are all zero.	In the previous general case take for the constants $\alpha=1$ $\varepsilon=-1, \gamma=1, \theta=\underline{0}$, $\iota\sqrt{3}\,\bar{\beta}=1$, $\iota\sqrt{3}\,\beta=-1$, $\iota\sqrt{3}\,\zeta=1$, $\iota\sqrt{3}\,\bar{\zeta}=-1$.	Let $\alpha=\iota\sqrt{3}\,\bar{\beta}\neq 0$, $\gamma=\iota\sqrt{3}\,\zeta=0$, $\varepsilon=\iota\sqrt{3}\,\beta$, $\Rightarrow \alpha=-\varepsilon$, $\theta=0$.
	general case. Example 3	special case.	 Example 4

Overall charge and comments	**Definite.** This is another example of a spin 3/2 wave equation with definite charge. It is based on the representations τ_1 , $\dot{\tau}_1$, τ_2, $\dot{\tau}_2$, like the Pauli-Fierz eq.	**Positive.** This is another example of a spin 3/2 wave equation based on the 20-dim representation $\tau_1 \oplus \dot{\tau}_1 \oplus \dot{\tau}_2 \oplus \tau_2 \oplus \tau_1' \oplus \dot{\tau}_1'$ with definite charge. This example is very interesting because the subsidiary conditions of the second kind can be derived for it.	The interest of this example lies with the fact that the subsidiary conditions of the second kind can be found for it.
Charge in the state 3/2	negative.	positive.	
Charge in the state 1/2	zero because the eigenvalues of $\mathbb{L}_0^{1/2}$ are all zero.	zero: because the eigenvalues of $\mathbb{L}_0^{1/2}$ are all zero.	
Charge densities	$\rho^1_{1/2}=0$. (The charge densities in the state 1/2 vanish because the eigenvalues $\lambda_l=0$.) $\rho^1_{3/2}=-4<0$, $\rho^2_{3/2}=-4<0$.	$\rho^1_{1/2}=0$. (The charge densities in the state 1/2 vanish because all the eigenvalues $\lambda_l=0$.) $\rho^1_{3/2}=2>0$, $\rho^2_{3/2}=2>0$.	
Minimal equation	$m(\mathbb{L}_0)=\mathbb{L}_0^2\{\mathbb{L}_0^2-2^2\}=0$, matrix $\mathbb{L}_0$ non-diagonalizable.	$m(\mathbb{L}_0)=\mathbb{L}_0^2\{\mathbb{L}_0^2-1\}=0$, matrix $\mathbb{L}_0$ non-diagonalizable.	$m(\mathbb{L}_0)=\mathbb{L}_0^2\{\mathbb{L}_0^2-(1/2)^2\}\{\mathbb{L}_0^2-1\}=0$, matrix $\mathbb{L}_0$ non-diagonalizable.
Bilinear form	$\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1$, $\alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1$, $\alpha^{\tau_1'\dot{\tau}_1'}=\alpha^{\dot{\tau}_1'\tau_1'}=1$.	$\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1$, $\alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1$, $\alpha^{\tau_1'\dot{\tau}_1'}=\alpha^{\dot{\tau}_1'\tau_1'}=1$.	$\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1$, $\alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1$, $\alpha^{\tau_1'\dot{\tau}_1'}=\alpha^{\dot{\tau}_1'\tau_1'}=1$.
Constants of $\mathbb{L}_0$	$\alpha=1$, $\beta=\frac{1}{\sqrt{3}}$, $\gamma=0$, $\varepsilon=-1$, $\zeta=0$, $\theta=0$.	$\alpha=-\frac{1}{4}$, $\iota\sqrt{3}\,\beta=\frac{1}{2\sqrt{2}}$, $\iota\sqrt{3}\,\bar{\beta}=-\frac{1}{2\sqrt{2}}$, $\varepsilon=\frac{1}{2}$, $\gamma=-\frac{1}{4}$, $\bar{\gamma}=-\frac{1}{4}$, $\iota\sqrt{3}\,\zeta=-\frac{1}{2\sqrt{2}}$, $\iota\sqrt{3}\,\bar{\zeta}=\frac{1}{2\sqrt{2}}$, $\theta=-\frac{1}{4}$	$\alpha=0$, $\iota\sqrt{3}\,\beta=\frac{1}{2\sqrt{2}}$, $\iota\sqrt{3}\,\bar{\beta}=-\frac{1}{2\sqrt{2}}$, $\varepsilon=\frac{1}{2}$, $\iota\sqrt{3}\,\zeta=-\frac{1}{2\sqrt{2}}$, $\iota\sqrt{3}\,\bar{\zeta}=\frac{1}{2\sqrt{2}}$, $\gamma=-\frac{1}{2}$, $\bar{\gamma}=-\frac{1}{2}$, $\theta=0$.
	Example 5	Example 6	Example 7

Overall Charge and Comments	This is another example of a 20-dim eq. for which the subsidiary conditions of the second kind can be found.	<u>Definite</u> This example describes a spin 1/2 Dirac particle based on the representations $\tau_1 , \dot{\tau}_1$.	<u>Definite</u> if α, θ have the same sign. This example describes two spin 1/2 Dirac particles based on the representations $\tau_1 , \dot{\tau}_1 , \tau_1' , \dot{\tau}_1'$.
Charge in the state 3/2		zero.	zero.
Charge in the state 1/2		positive if $\alpha > 0$.	positive if $\alpha > 0 , \theta > 0$.
Charge densities		$\rho^1_{1/2} = 2\alpha$, $\rho^2_{1/2} = 2\alpha$, all the other charge densities vanish.	$\rho^1_{1/2} = 2\alpha$, $\rho^2_{1/2} = 2\alpha$, $\rho^5_{1/2} = 2\theta$, $\rho^6_{1/2} = 2\theta$, all other charge densities vanish.
Minimal equation	$m(\mathbb{L}_0) = \mathbb{L}_0^2\{\mathbb{L}_0^2-(1/2)^2\}\{\mathbb{L}_0^2-1\}=0$, matrix $\mathbb{L}_0$ non-diagonalizable.	$m(\mathbb{L}_0) = \mathbb{L}_0\{\mathbb{L}_0^2-\alpha^2\}=0$, matrix $\mathbb{L}_0$ diagonalizable.	$m(\mathbb{L}_0) = \mathbb{L}_0\{\mathbb{L}_0^2-\alpha^2\}\{\mathbb{L}_0^2-\theta^2\}=0$, matrix $\mathbb{L}_0$ diagonalizable.
Bilinear form	$\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1$, $\alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1'} = 1$.	$\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1$ or 1 or 1 or 1 , $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1$ or 1 or 1 or -1 , $\alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1'} = 1$ or 1 or -1 or -1.	$\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1$ or 1, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1$ or 1, $\alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1'} = 1$ or 1.
Constants of $\mathbb{L}_0$	$\alpha = -1/2$, $\iota\sqrt{3}\beta = \frac{1}{2\sqrt{2}}$, $\iota\sqrt{3}\bar{\beta} = -\frac{1}{2\sqrt{2}}$, $\varepsilon = 1/2$, $\gamma = 0$, $\bar{\gamma} = 0$, $\iota\sqrt{3}\zeta = -\frac{1}{2\sqrt{2}}$, $\iota\sqrt{3}\bar{\zeta} = \frac{1}{2\sqrt{2}}$, $\theta = -1/2$.	$\alpha \neq 0$, $\beta = 0$, $\gamma = 0$, $\varepsilon = 0$, $\zeta = 0$, $\theta = 0$.	$\alpha \neq 0$, $\beta = 0$, $\gamma = 0$, $\varepsilon = 0$, $\zeta = 0$, $\theta \neq 0$.
	Example 8	Example 9	Example 10.

Overall Charge and Comments	Positive. This is another example of a spin 1/2 wave-equation with definite charge.	Positive. This is another example of a spin 1/2 wave-equation with definite charge.
Charge in the state 3/2	zero: because the eigenvalues vanish.	zero: because the eigenvalues vanish.
Charge in the state 1/2	positive.	positive.
Charge densities	$\rho^1_{1/2} = \rho^2_{1/2} = 0$, $\rho^3_{1/2} = \rho^4_{1/2} = 4 > 0$ $\rho^5_{1/2} = \rho^6_{1/2} = 4 > 0$; $\rho^1_{3/2} = \rho^2_{3/2} = 0$	$\rho^1_{1/2} = \rho^2_{1/2} = 0$, $\rho^3_{1/2} = \rho^4_{1/2} = 4 > 0$ $\rho^5_{1/2} = \rho^6_{1/2} = 4 > 0$; $\rho^1_{3/2} = \rho^2_{3/2} = 0$
Minimal equation	$m(\mathbb{L}_0) = \mathbb{L}_0\{\mathbb{L}_0^2 - 1\}^2 = 0$, matrix $\mathbb{L}_0$ non-diagonalizable.	$m(\mathbb{L}_0) = \mathbb{L}_0\{\mathbb{L}_0^2 - 1\}^2 = 0$, matrix $\mathbb{L}_0$ non-diagonalizable.
Bilinear form	$\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = 1$, $\alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1'} = 1$.	$\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = 1$, $\alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1'} = 1$.
Constants of $\mathbb{L}_0$	$\alpha = 0$, $\beta = 0$, $\varepsilon = 0$, $\gamma = 0$, $\zeta = \frac{1}{\sqrt{3}}$, $\theta = 2$.	$\alpha = 2$, $\beta = \frac{1}{\sqrt{3}}$, $\varepsilon = 0$ $\gamma = 0$, $\zeta = 0$, $\theta = 0$.
	Example 11	Example 12.

Let us now give examples which satisfy the criterion of Cox of second quantization and also the criterion of Amar and Dozzio of causal propagation in an external electromagnetic field.

Overall Charge and Comments	Indefinite This example satisfies the criteria of Cox and Amar and Dozzio but the charge is indefinite because $\mathbb{L}_0$ is diagonalizable.	Indefinite This example satisfies the criteria of Cox and Amar and Dozzio but the charge is indefinite because $\mathbb{L}_0$ is diagonalizable.	Indefinite This example satisfies the criteria of Cox and Amar and Dozzio but the charge is indefinite because $\mathbb{L}_0$ is diagonalizable.
Charge in the state 3/2			
Charge in the state 1/2			
Charge densities	Charge densities appearing with alternating signs.	Charge densities appearing with alternating signs.	Charge densities appearing with alternating signs.
Minimal equation	$m(\mathbb{L}_0)=\mathbb{L}_0(\mathbb{L}_0^2-1)(\mathbb{L}_0^2-(2\sqrt{2})^2)=0$, matrix $\mathbb{L}_0$ diagonalizable.	$m(\mathbb{L}_0)=\mathbb{L}_0\{\mathbb{L}_0^2-\varepsilon^2\}\{\mathbb{L}_0^2-(2\varepsilon)^2\}=0$, matrix $\mathbb{L}_0$ diagonalizable.	$m(\mathbb{L}_0)=\mathbb{L}_0(\mathbb{L}_0^2-1)(\mathbb{L}_0^2-2^2)=0$, matrix $\mathbb{L}_0$ diagonalizable
Bilinear form	$\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1, \alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1$ $\alpha^{\tau_1'\dot{\tau}_1'}=\alpha^{\dot{\tau}_1'\tau_1}=1.$	$\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1, \alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1$ $\alpha^{\tau_1'\dot{\tau}_1'}=\alpha^{\dot{\tau}_1'\tau_1'}=1.$	$\alpha^{\tau_1\dot{\tau}_1}=\alpha^{\dot{\tau}_1\tau_1}=1, \quad \alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1,$ $\alpha^{\tau_1'\dot{\tau}_1'}=\alpha^{\dot{\tau}_1'\tau_1'}=1.$
Constants of $\mathbb{L}_0$	$\alpha=\sqrt{2}$, $\beta=\frac{1}{\sqrt{3}}$, $\gamma=0$, $\varepsilon=-\sqrt{2}$, $\zeta=0$, $\theta=0$.	$\alpha=0$, $\beta=0$, $\gamma=0$, $\varepsilon\neq 0$, $\zeta=0$, $\theta=0$.	$\alpha=1$, $\beta=0$, $\gamma=0$, $\varepsilon=-1$, $\zeta=0$, $\theta=0$.
	Example 13	Example 14	Example 15

Overall Charge and Comments	Not definable. This example does not satisfy the definition of definite charge because there are eigenvectors corresponding to $\mathbb{L}_0^{1/2}$ having charge densities which vanish without the eigenvalues being zero. It satisfies the criterion of Amar and Dozzio.
Charge in the state 3/2	negative or positive depending on the sign of ε.
Charge in the state 1/2	The charge in this state vanishes without the eigenvalues being all zero.
Charge densities	$\rho^1_{1/2} = \rho^2_{1/2} = 0$, $\rho^3_{1/2} = 0$ without the eigenvalue being zero. $\rho^4_{1/2} = 0$, without the eigenvalue being zero. $\rho^1_{3/2} = \mp 8$, $\rho^2_{3/2} = \mp 8$.
Minimal equation	$m(\mathbb{L}_0) = \mathbb{L}_0(\mathbb{L}_0^2 - 1)^2(\mathbb{L}_0^2 - (4)^2) = 0$, matrix $\mathbb{L}_0$ non-diagonalizable.
Bilinear form	$\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1$, $\alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1'} = 1$.
Constants of $\mathbb{L}_0$	$\alpha = 0$, $\beta = \frac{1}{\sqrt{3}}$, $\gamma = 0$, $\varepsilon = \mp 2$, $\zeta = 0$, $\theta = 0$.
	Example 16.

Ex. 17:

We have given to date examples of wave equations describing spin 3/2 particles with definite charge and based on a 20-dim representation. But we have not yet given any examples of wave-equations describing particles of spin 1/2 and 3/2 existing in the field simultaneously and having the same charge. Thus we are faced with the problem if there are wave-equations describing particles of spin 1/2 and 3/2 having the same charge. The answer is in the affirmative and we give examples below.

Let us consider the matrix $\mathbb{L}_0$ which is invariant under the complete group and derivable from an invariant Lagrangian with associated bilinear form defined by the constants

$$\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1 , \quad \alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1 , \quad \alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1'} = 1 .$$

We first make the observation that the constant ε featuring in both blocks $\mathbb{L}_0^{1/2}$ and $\mathbb{L}_0^{3/2}$ should be different from zero in order that particles of spin 1/2 and 3/2 appear simultaneously. One way of constructing an example having the same charge in both states is to assume that the matrix block $\mathbb{L}_0^{1/2}$ brakes into two sub-blocks one including the representations $\tau_1, \dot{\tau}_1, \dot{\tau}_2, \tau_2$ and the other, the representations $\tau_1', \dot{\tau}_1'$ This is possible if we choose $\gamma=0, \zeta=0$, in which case $\mathbb{L}_0$, reads

$$\mathbb{L}_0^{1/2} = \begin{array}{c|cccccc|} & \tau_1 & \dot{\tau}_1 & \dot{\tau}_2 & \tau_2 & \tau_1' & \dot{\tau}_1' \\ \hline \tau_1 & 0 & \alpha & 0 & i\sqrt{3}\beta & & \\ \dot{\tau}_1 & \alpha & 0 & i\sqrt{3}\beta & 0 & & \\ \dot{\tau}_2 & 0 & i\sqrt{3}\bar{\beta} & 0 & \varepsilon & & \\ \tau_2 & i\sqrt{3}\bar{\beta} & 0 & \varepsilon & 0 & & \\ \tau_1' & & & & & 0 & \theta \\ \dot{\tau}_1' & & & & & \theta & 0 \end{array} , \quad \mathbb{L}_0^{3/2} = \begin{array}{c|cc|} & \dot{\tau}_2 & \tau_2 \\ \hline \dot{\tau}_2 & 0 & 2\varepsilon \\ \tau_2 & 2\varepsilon & 0 \end{array} .$$

Then we require that the sub-block of $\mathbb{L}_0^{1/2}$ corresponding to the representations $\tau_1, \dot{\tau}_1, \dot{\tau}_2, \tau_2$ should have eigenvalues equal to zero while the block corresponding to the representations $\tau_1', \dot{\tau}_1'$ should have eigenvalues different from zero. One such case for which this is so is the following where the constants in $\mathbb{L}_0$ have the values

$$\alpha=1 , \ \beta=\frac{1}{\sqrt{3}} , \ \varepsilon=-1 , \ \gamma=0 , \ \zeta=0 , \ \theta=-1 .$$

The matrix $\mathbb{L}_0$ satisfies the minimal equation

$$m(\mathbb{L}_0) = \mathbb{L}_0^2\{\mathbb{L}_0^2-1\}\{\mathbb{L}_0^2-2^2\} = 0 .$$

The block $\mathbb{L}_0^{1/2}$ has eigenvalues

$$\lambda_1=\lambda_2=\lambda_3=\lambda_4=0 , \qquad \lambda_5=1 , \ \lambda_6=-1 ,$$

and the eigenvectors corresponding to the non-zero eigenvalues are respectively

$$\phi_5(\lambda_5=1, \ell=1/2) = \{0,0,0,0,-1,1\}^{tr}$$

$$\phi_6(\lambda_6=-1, \ell=1/2) = \{0,0,0,0,1,1\}^{tr}.$$

The charge densities in the state $\ell=\frac{1}{2}$ corresponding to the non-zero eigenvalues are

$$\rho^5_{1/2} = \phi^+_5(\lambda_5=1, \ell=1/2)\mathbb{A}_{1/2}\mathbb{L}^{1/2}_0\phi_5(\lambda_5=1, \ell=1/2) = -2 < 0,$$

$$\rho^6_{1/2} = \phi^+_6(\lambda_6=-1, \ell=1/2)\mathbb{A}_{1/2}\mathbb{L}^{1/2}_0\phi_6(\lambda_6=-1, \ell=1/2) = -2 < 0.$$

The block $\mathbb{L}^{3/2}_0$ has eigenvalues $\lambda_1=2$, $\lambda_2=-2$, and the corresponding eigenvectors are respectively,

$$\phi_1(\lambda_1=2, \ell=3/2) = \{-1, 1\}^{tr}$$

$$\phi_2(\lambda_2=-2, \ell=3/2) = \{1, 1\}^{tr}.$$

The charge densities in the state $\ell=3/2$ are

$$\rho^1_{3/2} = \phi^+_1(\lambda_1=2, \ell=3/2)\mathbb{A}_{3/2}\mathbb{L}^{3/2}_0\phi_1(\lambda_1=2, \ell=3/2) = -4 < 0,$$

$$\rho^2_{3/2} = \phi^+_2(\lambda_2=-2, \ell=3/2)\mathbb{A}_{3/2}\mathbb{L}^{3/2}_0\phi_2(\lambda_2=-2, \ell=3/2) = -4 < 0.$$

Thus the sign of the charge in both spin states is the same. The example studied here describes two particles of spin 1/2 and 3/2 respectively with different masses and the same charge. In the previous example if we make the choice of the bilinear form defined by the constants

$$\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = -1, \quad \alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = 1, \quad \alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1'} = -1,$$

we get positive charge for both spin states.

More examples are given in the table below.

Overall Charge and Comments	**Positive.** This example describes particles of spin 1/2 and 3/2 respectively with positive charge.
Charge in the state 3/2	positive.
Charge in the state $1/2$	positive.
Charge densities	$\rho^{1}_{1/2} = \rho^{2}_{1/2} = \rho^{3}_{1/2} = \rho^{4}_{1/2} = 0$, $\rho^{5}_{1/2} = \rho^{6}_{1/2} = 2 > 0$; $\rho^{1}_{3/2} = \rho^{2}_{3/2} = 2 > 0$.
Minimal equation	$m(\mathbb{L}_0) = \mathbb{L}_0^2\{\mathbb{L}_0 - 1\} = 0$, matrix $\mathbb{L}_0$ non-diagonalizable.
Bilinear form	$\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1$ $\alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1'} = 1$.
Constants of $\mathbb{L}_0$.	$\alpha = -\frac{1}{2}$, $\iota\sqrt{3}\beta = \frac{1}{2}$, $\iota\sqrt{3}\bar{\beta} = -\frac{1}{2}$, $\varepsilon = \frac{1}{2}$, $\gamma = \bar{\gamma} = 0$, $\zeta = \bar{\zeta} = 0$, $\theta = 1$.
	Example 18

Positive.
This example describes particles of spin 1/2 and 3/2 respectively with positive charge.

positive .

positive.

$$\rho^{1}_{1/2} = \rho^{2}_{1/2} = \rho^{3}_{1/2} = \rho^{4}_{1/2} = 0,$$

$$\rho^{5}_{1/2} = \rho^{6}_{1/2} = 2 ; \; \rho^{1}_{3/2} = \rho^{2}_{3/2} = 4 > 0.$$

$$m(\mathbb{L}_0) = \mathbb{L}_0^2 \{\mathbb{L}_0^2 - 1\}\{\mathbb{L}_0^2 - 2^2\} = 0,$$

matrix $\mathbb{L}_0$ non-diagonalizable.

$$\alpha^{\tau_1 \dot{\tau}_1} = \alpha^{\dot{\tau}_1 \tau_1} = 1 , \quad \alpha^{\tau_2 \dot{\tau}_2} = \alpha^{\dot{\tau}_2 \tau_2} = -1 ,$$

$$\alpha^{\tau_1' \dot{\tau}_1'} = \alpha^{\dot{\tau}_1' \tau_1'} = 1 .$$

$$\alpha = -1 , \; \beta = \frac{1}{\sqrt{3}} , \; \varepsilon = 1 , \; \gamma = 0 , \; \zeta = 0 , \; \theta = 1$$

Example 19.

Overall Charge and Comments	<u>Negative.</u> This example describes particles of spin 1/2 and 3/2 respectively with negative charge.	<u>Negative.</u> This is another example describing particles of spin 1/2 and 3/2 respectively with the same sign of charge.	This example describes particles of two different spin states. The overall sign of the charge depends on the signs of α and θ. See the investigation following.
Charge in the state 3/2	Negative.	Negative.	sign depending on the sign of α.
Charge in the state 1/2	Negative.	Negative.	sign depending on the sign of θ.
Charge densities	$\rho^1_{1/2} = \rho^2_{1/2} = \rho^3_{1/2} = \rho^4_{1/2} = 0,$ $\rho^5_{1/2} = \rho^6_{1/2} = -4 < 0;\ \rho^1_{3/2} = \rho^2_{3/2} = -4 < 0$	$\rho^1_{1/2} = \rho^2_{1/2} = \rho^3_{1/2} = \rho^4_{1/2} = 0,$ $\rho^5_{1/2} = \rho^6_{1/2} = -4 < 0;\ \rho^1_{3/2} = \rho^2_{3/2} = -4 < 0.$	$\rho^1_{1/2} = \rho^2_{1/2} = \rho^3_{1/2} = \rho^4_{1/2} = 0,$ $\rho^5_{1/2} = \rho^6_{1/2} = 2\theta,\ \rho^1_{3/2} = \rho^2_{3/2} = -4\alpha.$
Minimal equation	$m(\mathbb{L}_0) = \mathbb{L}_0^2\{\mathbb{L}_0^2 - 2^2\} = 0,$ matrix $\mathbb{L}_0$ non-diagonalizable.	$m(\mathbb{L}_0) = \mathbb{L}_0^2\{\mathbb{L}_0^2 - 2^2\} = 0$ matrix $\mathbb{L}_0$ non-diagonalizable.	$m(\mathbb{L}_0) = \mathbb{L}_0^2\{\mathbb{L}_0^2 - \theta^2\}\{\mathbb{L}_0^2 - (2\alpha)^2\} = 0,$ matrix $\mathbb{L}_0$ non-diagonalizable.
Bilinear form	$\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1,$ $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1,$ $\alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1'} = 1.$	$\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1,$ $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1,$ $\alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1'} = -1.$	$\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1,$ $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1,$ $\alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1'} = 1.$
Constants of $\mathbb{L}_0$	$\alpha = 1,\ \beta = \frac{1}{\sqrt{3}},\ \varepsilon = -1$ $\gamma = 0,\ \zeta = 0,\ \theta = -2.$	$\alpha = 1,\ \beta = \frac{1}{\sqrt{3}},\ \varepsilon = -1,$ $\gamma = 0,\ \zeta = 0,\ \theta = 2.$	$\alpha \neq 0,\ \beta = \frac{\alpha}{\sqrt{3}},\ \varepsilon = -\alpha,$ $\gamma = 0,\ \zeta = 0,\ \theta \neq 0.$
	Example 20	Example 21	Example 22

Investigation of Ex. 22.

Case (i). If $\alpha>0$, $\theta<0$ then we have negative charge in both states of spin and the overall charge is negative.

Case (ii). If $\alpha>0$ and $\theta=-2\alpha$ then we have negative charge in both spin states with the overall charge negative and minimal equation

$$m(\mathbb{L}_0)=\mathbb{L}_0^2\left\{\mathbb{L}_0^2-(2\alpha)^2\right\}=0.$$

Case (iii). If $\alpha<0$, $\theta>0$ then we have positive charge in both states of spin with the overall charge positive.

Case (iv). If $\alpha<0$ and $\theta=-2\alpha$ then we have positive charge in both spin states with the overall charge positive and minimal equation

$$m(\mathbb{L}_0)=\mathbb{L}_0^2\left\{\mathbb{L}_0^2-(2\alpha)^2\right\}=0.$$

Case (v). If α and θ have the same sign then the charge densities corresponding to the two different spin states appear with opposite signs and according to the adopted definition of definite charge, the overall charge is indefinite.

Overall Charge and Comments	This example describes particles of two different spin states. The overall charge is: 1) Negative if $\alpha > 0$, $\theta < 0$. 2) Positive if $\alpha < 0$, $\theta > 0$, and indefinite if α and θ have the same sign. See for more details the investigation of Ex. 22.
Charge in the state 3/2	Negative if $\alpha > 0$. Positive if $\alpha < 0$.
Charge in the state 1/2	Negative if $\theta < 0$. Positive if $\theta > 0$.
Charge densities	$\rho^1_{1/2} = \rho^2_{1/2} = \rho^3_{1/2} = \rho^4_{1/2} = 0,$ $\rho^5_{1/2} = \rho^6_{1/2} = 2\theta;\ \rho^1_{3/2} = \rho^2_{3/2} = -4\alpha.$
Minimal equation	$m(\mathbb{L}_0) = \mathbb{L}^2_0\{\mathbb{L}^2_0 - \theta^2\}\{\mathbb{L}^2_0 - (2\alpha)^2\} = 0,$ matrix $\mathbb{L}_0$ non-diagonalizable.
Bilinear form	$a^{\tau_1 \dot{\tau}_1} = a^{\dot{\tau}_1 \tau_1} = 1,$ $a^{\tau_2 \dot{\tau}_2} = a^{\dot{\tau} \tau_2} = -1,$ $a^{\tau'_1 \dot{\tau}'_1} = a^{\dot{\tau}'_1 \tau'_1} = 1.$
Constants of $\mathbb{L}_0$	$\alpha \neq 0,\ \beta = \frac{\alpha}{i\sqrt{3}},\ \varepsilon = -\alpha,$ $\gamma = 0,\ \zeta = 0,\ \theta \neq 0.$
	Example 23

This example describes particles of two different spin states. The overall charge is: 1) Positive if $\alpha > 0$, $\theta < 0$. 2) Negative if $\alpha < 0$, $\theta > 0$, and indefinite if α and θ, have the same sign.	This example describes particles of two different spin states. The overall charge is: 1) Positive if $\alpha > 0$, $\theta < 0$. 2) Negative if $\alpha < 0$, $\vartheta > 0$. and indefinite if α and θ have the same sign.
Positive if $\theta < 0$. Negative if $\theta > 0$.	Positive if $\theta < 0$. Negative if $\theta > 0$.
Positive if $\alpha > 0$. Negative if $\alpha < 0$.	Positive if $\alpha > 0$. Negative if $\alpha < 0$.
$\rho^1_{1/2} = \rho^2_{1/2} = 2\alpha$, $\rho^3_{1/2} = \rho^4_{1/2} = \rho^5_{1/2} = \rho^6_{1/2} = 0$; $\rho^1_{3/2} = \rho^2_{3/2} = -4\theta$.	$\rho^1_{1/2} = \rho^2_{1/2} = 2\alpha$, $\rho^3_{1/2} = \rho^4_{1/2} = \rho^5_{1/2} = \rho^6_{1/2} = 0$; $\rho^1_{3/2} = \rho^2_{3/2} = -4\theta$.
$m(\mathbb{L}_0) = \mathbb{L}_0^2\{\mathbb{L}_0^2 - \alpha^2\}\{\mathbb{L}_0^2 - (2\theta)^2\} = 0$, matrix $\mathbb{L}_0$ non-diagonalizable	$m(\mathbb{L}_0) = \mathbb{L}_0^2\{\mathbb{L}_0^2 - \alpha^2\}\{\mathbb{L}_0^2 - (2\theta)^2\} = 0$, matrix $\mathbb{L}_0$ non-diagonalizable.
$\alpha^{\tau_1 \dot{\tau}_1} = \alpha^{\dot{\tau}_1 \tau_1} = 1$, $\alpha^{\tau_2 \dot{\tau}_2} = \alpha^{\dot{\tau}_2 \tau_2} = -1$, $\alpha^{\tau'_1 \dot{\tau}'_1} = \alpha^{\dot{\tau}'_1 \tau'_1} = 1$.	$\alpha^{\tau_1 \dot{\tau}_1} = \alpha^{\dot{\tau}_1 \tau_1} = 1$, $\alpha^{\tau_2 \dot{\tau}_2} = \alpha^{\dot{\tau}_2 \tau_2} = -1$, $\alpha^{\tau'_1 \dot{\tau}'_1} = \alpha^{\dot{\tau}'_1 \tau'_1} = 1$.
$\alpha \neq 0$, $\beta = 0$, $\gamma = 0$, $\varepsilon = -\theta$, $\zeta = \frac{\theta}{\sqrt{3}}$, $\theta \neq 0$.	$\alpha \neq 0$, $\beta = 0$, $\gamma = 0$, $\varepsilon = -\theta$, $\zeta = \frac{\theta}{\sqrt{3}}$, $\theta \neq 0$.
Example 24	Example 25

In this paragraph we have given some representative examples of 20-dim. wave-equations based on the representation $\tau_1 \oplus \dot{\tau}_1 \oplus \tau_2 \oplus \dot{\tau}_2 \oplus \tau_1' \oplus \dot{\tau}_1'$, describing spin 3/2 particles with definite charge and also examples describing spin 1/2 and 3/2 particles existing simultaneously and having definite charge (i.e. the same charge). The above table of examples does not exhaust all possible cases of wave equations with defined charge based on the 20-dim. representation. An exhaustive study will require studying the charge for the eight different possible cases of bilinear forms allowed by the representation.

3.

We prove in this paragraph that there are no examples of wave-equations based on the 20-dim. representation $\tau_1 \sim (1/2, 3/2)$, $\dot{\tau}_1 \sim (-1/2, 3/2)$, $\tau_2 \sim (1/2, 5/2)$, $\dot{\tau}_2 \sim (-1/2, 5/2)$, $\tau_1' \sim (1/2, 3/2)$, $\dot{\tau}_1' \sim (-1/2, 3/2)$, describing a spin 3/2 particle with matrix $\mathbb{L}_0$ nondiagonalizable and satisfying a minimal equation of the form

$$m(\mathbb{L}_0) = \mathbb{L}_0 \{ \mathbb{L}_0^2 - \lambda^2 \}^{\mu} = 0 ,$$

with $\mu \geqslant 2$, $\lambda \neq 0$.

(The Amar-Dozzio condition then implies causal propagation in an external electromagnetic field).

<u>Proof.</u> Let us consider the matrix $\mathbb{L}_0$ which is invariant under the complete group and derivable from an invariant Lagrangian. This matrix as we have seen can be cast into the blocks $\mathbb{L}_0^{1/2}$ and $\mathbb{L}_0^{3/2}$. The block $\mathbb{L}_0^{3/2}$ is always diagonalizable. Since we want particles of spin 3/2 to be present in the field the block $\mathbb{L}_0^{3/2}$ must not vanish. Then the only possible minimal equation that this block can satisfy is $m(\mathbb{L}_0^{3/2}) = \{ (\mathbb{L}_0^{3/2})^2 - \lambda^2 \} = 0$ where λ are the eigenvalues of the block $\mathbb{L}_0^{3/2}$. In the meantime the block $\mathbb{L}_0^{1/2}$ must either vanish or have all its eigenvalues zero in order that particles of spin 1/2 will not be present in the field. The case of the block $\mathbb{L}_0^{1/2}$ being identically zero

is not allowed because this would also cause the block $\mathbb{L}^{3/2}_0$ to disappear. Thus the only possible case is the case in which the block $\mathbb{L}^{1/2}_0$ has all its eigenvalues equal to zero. The minimal equation satisfied by the block $\mathbb{L}^{1/2}_0$ in this case is of the form $m(\mathbb{L}^{1/2}_0) = \{\mathbb{L}^{1/2}_0\}^n = 0$, with $n \geqslant 2$. Since the minimal equation for the matrix $\mathbb{L}_0$ will be the common multiple of the minimal equations for the two blocks $\mathbb{L}^{1/2}_0$ and $\mathbb{L}^{3/2}_0$ we find that

$$m(\mathbb{L}_0) = \mathbb{L}^n_0 \{\mathbb{L}^2_0 - \lambda^2\} = 0 , \quad n \geqslant 2 .$$

is the only possible form of minimal equation that the matrix $\mathbb{L}_0$ can satisfy for the description of spin 3/2 particles only.

Notice that the minimal equation does not satisfy the criterion of Amar and Dozzio of causal propagation in an external electromagnetic field nor the criterion of Cox of second quantization.

As a general conclusion we have that there are no examples of wave equations based on the representations

(I) $\tau_1 \sim (3/2, 5/2) \oplus \dot{\tau}_1 \sim (-3/2, 5/2) \oplus \tau_2 \sim (1/2, 5/2) \oplus \dot{\tau}_2 \sim (-1/2, 5/2)$,

(II) $\tau_1 \sim (1/2, 3/2) \oplus \dot{\tau}_1 \sim (-1/2, 3/2) \oplus \tau_2 \sim (1/2, 5/2) \oplus \dot{\tau}_2 \sim (-1/2, 5/2)$,

(III) $\tau_1 \sim (1/2, 3/2) \oplus \dot{\tau}_1 \sim (-1/2, 3/2) \oplus \tau_2 \sim (1/2, 5/2) \oplus \dot{\tau}_2 \sim (-1/2, 5/2) \oplus$
$\oplus \tau_1' \sim (1/2, 3/2) \oplus \dot{\tau}_1' \sim (-1/2, 3/2)$,

studied in chapters (3), (4), (5) respectively describing particles of spin 3/2 only with non-diagonalizable matrix $\mathbb{L}_0$ satisfying a minimal equation of the form

$$m(\mathbb{L}_0) = \mathbb{L}_0 \prod \{\mathbb{L}^2_0 - \lambda^2_i\}^{\mu_i} = 0 ,$$

with $\mu_i \geqslant 2$ and having definite charge.

4: Summary.

In this chapter we have studied examples of the Gel'fand-Yaglom wave-equation based on the 20-dim. representation with components $\tau_1 \sim (1/2, 3/2)$, $\dot{\tau}_1 \sim (-1/2, 3/2)$, $\tau_2 \sim (1/2, 5/2)$, $\dot{\tau}_2 \sim (-1/2, 5/2)$, $\tau_1' \sim (1/2, 3/2)$,

$\dot{\tau}_1' \sim (-1/2, 3/2)$, interlocking according to the scheme (III). We have reached the following conclusions.

1)a) There are examples of wave equations describing particles of spin 3/2 with matrix $\mathbb{L}_0$ non-diagonalizable and definite charge (c.f. examples 3, 6). These examples describe 20-dim. wave-equations.

b) There are other examples of wave equations describing particles of spin 3/2 with matrix $\mathbb{L}_0$ non-diagonalizable and definite charge but they are 16-dimensional like the Pauli-Fierz equation, (c.f. examples 4, 5).

2) There are examples satisfying the Amar-Dozzio condition of causal propagation in the presence of an external electromagnetic field and the condition of Cox of second quantization but the charge is indefinite.

3) There are examples which satisfy the Amar-Dozzio condition as well as the condition of Cox with definite charge but they are based on the same irreducible representations as the Dirac equation.

4) There are two examples which satisfy the Amar-Dozzio condition of causal propagation with matrix $\mathbb{L}_0$ non-diagonalizable and definite charge describing spin 1/2 particles (c.f. examples 11, 12). These examples are interesting because they show that the theorem of Udgaonkar[42] concerning the charge of relativisfic wave-equations is partly correct. As it will be shown in a later chapter these examples can be second quantized also.

5) There are examples with non-diagonalizable matrix $\mathbb{L}_0$, describing spin 1/2 and spin 3/2 particles existing simultaneously in the field and having the same sign of charge i.e. definite, (c.f. examples 18, 19, 20, 21, 22, 23, 24, 25).

6) As a general conclusion we have that there are no

examples of wave-equations based on the representations (I), (II), (III) describing particles of spin 3/2 only with matrix $\mathbb{L}_0$ non-diagonalizable, satisfying a minimal equation of the form

$$m(\mathbb{L}_0) = \mathbb{L}_0 \prod_i \{ \mathbb{L}_0^2 - \lambda_i^2 \}^{\mu_i} = 0,$$

$\mu_i \geqslant 2$, $\lambda_i \neq 0$ and having definite charge.

CHAPTER 6.

The Bhabha wave-equations studied from the Gel'fand-Yaglom viewpoint.

In this chapter we study the Bhabha wave-equations for which the wave-function ψ transforms according to some representation of the group $SO(4,1)$. These wave-equations were studied in Part I, but here we look upon them as Gel'fand-Yaglom wave-equations and investigate their charge.

1. Study of the Bhabha wave-equation based on the 16-dim. rep. of the group $SO(4,1)$.

Let us consider the Bhabha wave-equation for which the wave-function ψ transforms according to the 16-dim. representation $R_5(3/2, 1/2)$ of the group $SO(4,1)$ (Bhabha notation).

The wave-function ψ consists of the spinors α_μ, $\alpha^{\dot\mu}$, $\alpha^{\dot\nu}_{\lambda\mu}, \alpha^{\dot\lambda\dot\mu}_{\nu}$, $(\mu, \lambda, \nu = 1,2)$. These spinors are realized in the spaces of 1-rank undotted spinors (2-dimensional), 1-rank dotted spinors (2-dimensional), 3-rank spinors with one dotted and two undotted indeces (6-dimensional) and 3-rank spinors with two dotted and one undotted index (6-dimensional) and are described by the irreducibles $R_4(1/2, 0)$, $R_4(0, 1/2)$, $R_4(1, 1/2)$, $R_4(1/2, 1)$ of $R_5(3/2, 1/2)$ with respect to the proper group. The above irreducibles interlock according to the scheme

$$\begin{tikzcd}
R_4(1, 1/2) \arrow[r, shift left] \arrow[r, <-, shift right] \arrow[d, shift left] \arrow[d, <-, shift right] & R_4(1/2, 0) \arrow[d, shift left] \arrow[d, <-, shift right] \\
R_4(1/2, 1) \arrow[r, shift left] \arrow[r, <-, shift right] & R_4(0, 1/2)
\end{tikzcd}.$$

According to Bhabha the linkage between two irreducibles is:

i) of the first kind if $(k, \ell) \longrightarrow (k+1/2, \ell-1/2)$ or

$$(k, \ell) \longrightarrow (k-1/2, \ell+1/2),$$

ii) of the second kind if $(k,\ell) \longrightarrow (k+1/2,\ell+1/2)$ or

$(k,\ell) \longrightarrow (k-1/2,\ell-1/2)$.

Changing the above notation into that of Gel'fand and Yaglom, the irreducibles $R_4(1/2,0)$, $R_4(0,1/2)$, $R_4(1,1/2)$, $R_4(1/2,1)$ get replaced by the irreducibles

$$\tau_1 \sim (1/2, 3/2), \quad \dot{\tau}_1 \sim (-1/2, 3/2), \quad \tau_2 \sim (1/2, 3/2), \quad \dot{\tau}_2 \sim (-1/2, 3/2)$$

and the interlocking scheme by the scheme

$$\begin{tikzcd}
(1/2, 3/2) \sim \tau_1 \arrow[rr, shift left, "c^{\tau_1\dot{\tau}_1}"] \arrow[d, shift left, "c^{\tau_1\tau_2}"] & & \dot{\tau}_1 \sim (-1/2, 3/2) \arrow[ll, shift left, "c^{\dot{\tau}_1\tau_1}"] \arrow[d, shift left, "c^{\dot{\tau}_1\dot{\tau}_2}"] \\
(1/2, 5/2) \sim \tau_2 \arrow[u, shift left, "c^{\tau_2\tau_1}"] \arrow[rr, shift left, "c^{\tau_2\dot{\tau}_2}"] & & \dot{\tau}_2 \sim (-1/2, 5/2) \arrow[u, shift left, "c^{\dot{\tau}_2\dot{\tau}_1}"] \arrow[ll, shift left, "c^{\dot{\tau}_2\tau_2}"]
\end{tikzcd}$$

The 16-dim. rep. under consideration admits the canonical basis

$$\{\xi_{\ell m}\} = \{\xi^{\tau_1}_{1/2,1/2}, \xi^{\tau_1}_{1/2,-1/2}\} \cup \{\xi^{\dot{\tau}_1}_{1/2,1/2}, \xi^{\dot{\tau}_1}_{1/2,-1/2}\} \cup$$

$$\{\xi^{\tau_2}_{1/2,1/2}, \xi^{\tau_2}_{1/2,-1/2}, \xi^{\tau_2}_{3/2,3/2}, \xi^{\tau_2}_{3/2,1/2}, \xi^{\tau_2}_{3/2,-1/2}, \xi^{\tau_2}_{3/2,-3/2}\} \cup$$

$$\{\xi^{\dot{\tau}_2}_{1/2,1/2}, \xi^{\dot{\tau}_2}_{1/2,-1/2}, \xi^{\dot{\tau}_2}_{3/2,3/2}, \xi^{\dot{\tau}_2}_{3/2,1/2}, \xi^{\dot{\tau}_2}_{3/2,-1/2}, \xi^{\dot{\tau}_2}_{3/2,-3/2}\},$$

where the individual sub-bases are the sub-bases of the irreducibles into which it decomposes. The matrix $\mathbb{L}_0$ with respect to this basis has the blocks

$$\mathbb{L}_0^{1/2} = \begin{array}{c|c|c|c|c|}
 & \tau_1 & \dot{\tau}_1 & \tau_2 & \dot{\tau}_2 \\
\tau_1 & 0 & c^{\tau_1\dot{\tau}_1} & c^{\tau_1\tau_2} & 0 \\
\hline
\dot{\tau}_1 & c^{\dot{\tau}_1\tau_1} & 0 & 0 & c^{\dot{\tau}_1\dot{\tau}_2} \\
\hline
\tau_2 & c^{\tau_2\tau_1} & 0 & 0 & c^{\tau_2\dot{\tau}_2} \\
\hline
\dot{\tau}_2 & 0 & c^{\dot{\tau}_2\dot{\tau}_1} & c^{\dot{\tau}_2\tau_2} & 0
\end{array}, \qquad
\mathbb{L}_0^{3/2} = \begin{array}{c|cc|}
 & \tau_2 & \dot{\tau}_2 \\
\tau_2 & 0 & 2c^{\tau_2\dot{\tau}_2} \\
\dot{\tau}_2 & 2c^{\dot{\tau}_2\tau_2} & 0
\end{array}.$$

From the requirement of invariance of the equation under the complete group we obtain the following relations among the elements $c^{\tau\tau'}$ of $\mathbb{L}_0$,

$$c^{\tau_1\dot{\tau}_1} = c^{\dot{\tau}_1\tau_1}, \quad c^{\tau_2\tau_1} = c^{\dot{\tau}_2\dot{\tau}_1}, \quad c^{\tau_1\tau_2} = c^{\dot{\tau}_1\dot{\tau}_2}, \quad c^{\dot{\tau}_2\tau_2} = c^{\tau_2\dot{\tau}_2}.$$

Since the matrix $\mathbb{L}_0$ for every Bhabha equation is diagonalizable, this implies that the minimal equation for $\mathbb{L}_0$ includes each eigenvalue only once with multiplicity one. For the 16-dim. representation $\mathbb{L}_0$ has the eigenvalues $\lambda = 1/2$ (6-fold), $\lambda = -1/2$ (6-fold), $\lambda = 3/2$ (2-fold) $\lambda = -3/2$ (2-fold) as was shown in Part I.

The minimal equation of $\mathbb{L}_0$ must then be of the form

$$m(\mathbb{L}_0) = \{\mathbb{L}_0^2 - (1/2)^2\}\{\mathbb{L}_0^2 - (3/2)^2\} = 0.$$

This translated into the minimal equations of the two blocks of $\mathbb{L}_0$, gives rise to the following possible cases:

(1) $m(\mathbb{L}_0^{1/2}) = (\mathbb{L}_0^{1/2})^2 - (1/2)^2$ and $m(\mathbb{L}_0^{3/2}) = (\mathbb{L}_0^{3/2})^2 - (3/2)^2 = 0$,

(2) $m(\mathbb{L}_0^{1/2}) = \{(\mathbb{L}_0^{1/2})^2 - (3/2)^2\}\{(\mathbb{L}_0^{1/2})^2 - (1/2)^2\} = 0$, and $m(\mathbb{L}_0^{3/2}) = (\mathbb{L}_0^{3/2})^2 - (3/2)^2 = 0$,

(3) $m(\mathbb{L}_0^{1/2}) = (\mathbb{L}_0^{1/2})^2 - (3/2)^2 = 0$ and $m(\mathbb{L}_0^{3/2}) = (\mathbb{L}_0^{3/2})^2 - (1/2)^2 = 0$,

(4) $m(\mathbb{L}_0^{1/2}) = \{(\mathbb{L}_0^{1/2})^2 - (3/2)^2\}\{(\mathbb{L}_0^{1/2})^2 - (1/2)^2\} = 0$, and $m(\mathbb{L}_0^{3/2}) = (\mathbb{L}_0^{3/2})^2 - (1/2)^2 = 0$.

To find which of the above possibilities is acceptable we use the fact that each eigenvalue λ corresponding to the block $\mathbb{L}_0^{\ell}$ is a $(2\ell+1)$-fold eigenvalue of $\mathbb{L}_0$. Thus for the above four cases we have the following analysis.

<u>Case 1.</u> In this case we have associated the eigenvalue 1/2 with the block $\mathbb{L}_0^{1/2}$ and the eigenvalue 3/2 with the block $\mathbb{L}_0^{3/2}$. This means that the eigenvalue 1/2 is of multiplicity $(2\ell+1) = (2(\frac{1}{2})+1) = 2$ considered as an eigenvalue of $\mathbb{L}_0$ while 3/2 is of multiplicity 4. But this is a false conclusion since we know already that the eigenvalue 1/2 is of multiplicity 6 and the eigenvalue 3/2 is of multiplicity 2.

Thus case 1 has to be discarded.

Case 2. This case also has to be discarded because it requires that the eigenvalue 3/2 has to be a 6-fold eigenvalue of $\mathbb{L}_0$ which is incorrect.

Case 3. This case also has to be discarded because it requires that the eigenvalues 1/2 and 3/2 have to be 4-fold eigenvalues of $\mathbb{L}_0$ each, which is not true.

Case 4. This case is the only possible case of distribution of the eigenvalues of $\mathbb{L}_0$ (or masses of the particles) among the spin states. Thus the eigenvalues 3/2 and 1/2 appear simultaneously as eigenvalues of the block $\mathbb{L}_0^{1/2}$ and 1/2 as the only kind of eigenvalue of the block $\mathbb{L}_0^{3/2}$. This distribution of the eigenvalues is compatible with their multiplicities.

Since $\mathbb{L}_0$ is a diagonalizable matrix, then there is a frame not the same as $\{\xi_{\ell m}\}$ in which it acquires a diagonal form and hence $\mathbb{L}_0^{1/2} = \operatorname{diag}\{3/2, -3/2, 1/2, -1/2\}$, $\mathbb{L}_0^{3/2} = \operatorname{diag}\{1/2, -1/2\}$.

From Part I we have that the hermitianizing matrix $\mathbb{A}$ for the 16-dim. representation is given by the formula $\mathbb{A} = \frac{1}{3}\mathbb{L}_0\{4\mathbb{L}_0^2 - 7\}$ and thus

$$\mathbb{A}_{1/2} = \operatorname{diag}\{1, -1, -1, 1\} \quad , \quad \mathbb{A}_{3/2} = \operatorname{diag}\{-1, 1\}.$$

The eigenvectors of $\mathbb{L}_0^{1/2}$ are,

$$\Phi_1(\lambda_1 = 3/2, \ell = 1/2) = \{1, 0, 0, 0\}^{tr}, \quad \Phi_2(\lambda_2 = -3/2, \ell = 1/2) = \{0, 1, 0, 0\}^{tr},$$
$$\Phi_3(\lambda = 1/2, \ell = 1/2) = \{0, 0, 1, 0\}^{tr}, \quad \Phi_4(\lambda_4 = -1/2, \ell = 1/2) = \{0, 0, 0, 1\}^{tr},$$

and those of $\mathbb{L}_0^{3/2}$ are,

$$\Phi_1(\lambda_1 = 1/2, \ell = 3/2) = \{1, 0\}^{tr}, \quad \Phi_2(\lambda_2 = -1/2, \ell = 3/2) = \{0, 1\}^{tr}.$$

The charge densities for $\ell = 1/2$ are:

$$\rho^1_{1/2} = \Phi_1^+(3/2, 1/2)\mathbb{A}_{1/2}\mathbb{L}_0^{1/2}\Phi_1(3/2, 1/2) = 3/2 > 0, \quad \rho^2_{1/2} = 3/2 > 0,$$
$$\rho^3_{1/2} = -1/2 < 0, \quad \rho^4_{1/2} = -1/2 < 0.$$

The charge densities for $\ell = 3/2$ are

$$\rho^1_{3/2} = \Phi_1^+(1/2, 3/2)\mathbb{A}_{3/2}\mathbb{L}_0^{3/2}\Phi_1(1/2, 3/2) = -1/2 < 0, \quad \rho^2_{3/2} = -1/2 < 0.$$

Thus the total charge corresponding to $\mathbb{L}_0$ is indefinite since some charge densities are positive and some negative.

2. Study of the Bhabha wave-equation based on the 20-dim. rep. of the group SO(4,1).

Let us consider the Bhabha wave-equation for which the wave-function ψ transforms according to the 20-dim. rep. $R_5(3/2, 3/2)$ of the group $SO(4,1)$ (Bhabha notation). The wave-function consists of the spinors $\alpha_{\lambda\mu}, \alpha^{\dot\lambda\dot\mu}, \alpha^{\dot\nu}_{\lambda\mu}, \alpha^{\dot\lambda\dot\mu}_{\nu}$. These spinors are realized in the spaces of 2-rank undotted spinors (4-dimensional) 2-rank dotted spinors (4-dimensional), 3-rank spinors with one dotted and two undotted indices (6-dimensional) and 3-rank spinors with two dotted and one undotted index and are described by the irreducibles $R_4(3/2,0), R_4(0,3/2), R_4(1,1/2)$, $R_4(1/2,1)$ with respect to the proper group. These irreducibles interlock according to the scheme

$$R_4(3/2,0) \rightleftarrows R_4(1,1/2) \rightleftarrows R_4(1/2,1) \rightleftarrows R_4(0,3/2).$$

In our notation the irreducibles $R_4(3/2,0)$, $R_4(0,3/2)$, $R_4(1,1/2)$, $R_4(1/2,1)$ are replaced by

$$\tau_1 \sim (3/2, 5/2), \quad \dot\tau_1 \sim (-3/2, 5/2), \quad \tau_2 \sim (1/2, 5/2), \quad \dot\tau_2 \sim (-1/2, 5/2),$$

respectively and the interlocking scheme by

$$\tau_1 \sim (3/2, 5/2) \xrightleftharpoons[c^{\tau_2\tau_1}]{c^{\tau_1\tau_2}} \tau_2 \sim (1/2, 5/2) \xrightleftharpoons[c^{\dot\tau_2\tau_2}]{c^{\tau_2\dot\tau_2}} \dot\tau_2 \sim (-1/2, 5/2) \xrightleftharpoons[c^{\dot\tau_1\dot\tau_2}]{c^{\dot\tau_2\dot\tau_1}} \dot\tau_1 \sim (-3/2, 5/2).$$

The 20-dim. representation under consideration admits the canonical basis

$$\{\xi_{\ell m}\} = \{\xi^{\tau_1}_{3/2,3/2}, \xi^{\tau_1}_{3/2,1/2}, \xi^{\tau_1}_{3/2,-1/2}, \xi^{\tau_1}_{3/2,-3/2}\ \xi^{\dot\tau_1}_{3/2,3/2}, \xi^{\dot\tau_1}_{3/2,1/2},$$
$$\xi^{\dot\tau_1}_{3/2,-1/2}, \xi^{\dot\tau_1}_{3/2,-3/2}, \xi^{\tau_2}_{1/2,1/2}, \xi^{\tau_2}_{1/2,-1/2}, \xi^{\tau_2}_{3/2,3/2}, \xi^{\tau_2}_{3/2,1/2},$$
$$\xi^{\tau_2}_{3/2,-1/2}, \xi^{\tau_2}_{3/2,-3/2}, \xi^{\dot\tau_2}_{1/2,1/2}, \xi^{\dot\tau_2}_{1/2,-1/2}, \xi^{\dot\tau_2}_{3/2,3/2}, \xi^{\dot\tau_2}_{3/2,1/2},$$
$$\xi^{\dot\tau_2}_{3/2,-1/2}, \xi^{\dot\tau_2}_{3/2,-3/2}\ \},$$

and $\mathbb{L}_0$ with respect to this basis has the blocks

$$\mathbb{L}^{3/2}_0 = \begin{array}{c|cccc|} & \tau_1 & \tau_2 & \dot{\tau}_2 & \dot{\tau}_1 \\ \hline \tau_1 & 0 & \sqrt{3}\,c^{\tau_1\tau_2} & 0 & 0 \\ \tau_2 & \sqrt{3}\,c^{\tau_2\tau_1} & 0 & 2c^{\tau_2\dot{\tau}_2} & 0 \\ \dot{\tau}_2 & 0 & 2c^{\dot{\tau}_2\tau_2} & 0 & \sqrt{3}\,c^{\dot{\tau}_2\dot{\tau}_1} \\ \dot{\tau}_1 & 0 & 0 & \sqrt{3}\,c^{\dot{\tau}_1\dot{\tau}_2} & 0 \end{array}\ , \qquad \mathbb{L}^{1/2}_0 = \begin{array}{c|cc|} & \tau_2 & \dot{\tau}_2 \\ \hline \tau_2 & 0 & c^{\tau_2\dot{\tau}_2} \\ \dot{\tau}_2 & c^{\dot{\tau}_2\tau_2} & 0 \end{array}.$$

The eigenvalues of $\mathbb{L}_0$ are $\lambda = 3/2$ (4-fold), $\lambda = 1/2$ (6-fold), $\lambda = -1/2$ (6-fold) $\lambda = -3/2$ (4-fold) (c.f. Part I). Since $\mathbb{L}_0$ is diagonalizable its minimal equation is of the form

$$m(\mathbb{L}_0) = \{\mathbb{L}_0^2 - (1/2)^2\}\{\mathbb{L}_0^2 - (3/2)^2\} = 0 .$$

This expressed in terms of the minimal equations of the two sub-blocks gives rise to the following possible cases.

1) $m(\mathbb{L}^{3/2}_0) = \{(\mathbb{L}^{3/2}_0)^2 - (1/2)^2\} = 0$ and $m(\mathbb{L}^{1/2}_0) = \{(\mathbb{L}^{1/2}_0)^2 - (3/2)^2\} = 0$,

2) $m(\mathbb{L}^{3/2}_0) = \{(\mathbb{L}^{3/2}_0)^2 - (3/2)^2\}\{(\mathbb{L}^{3/2}_0)^2 - (1/2)^2\} = 0$, and $m(\mathbb{L}^{1/2}_0) = \{(\mathbb{L}^{1/2}_0)^2 - (3/2)^2\} = 0$,

3) $m(\mathbb{L}^{3/2}_0) = \{(\mathbb{L}^{3/2}_0)^2 - (3/2)^2\} = 0$, and $m(\mathbb{L}^{1/2}_0) = \{(\mathbb{L}^{1/2}_0)^2 - (1/2)^2\} = 0$,

4) $m(\mathbb{L}^{3/2}_0) = \{(\mathbb{L}^{3/2}_0)^2 - (3/2)^2\}\{(\mathbb{L}^{3/2}_0)^2 - (1/2)^2\} = 0$, and $m(\mathbb{L}^{1/2}_0) = \{(\mathbb{L}^{1/2}_0)^2 - (1/2)^2\} = 0$.

Cases (1), (2) and (3) have to be discarded because they require that the eigenvalues have multiplicities not the same as the ones given above. Case (4) is the only acceptable case and thus the eigenvalues ±3/2 and ±1/2 appear simultaneously as eigenvalues of the block $\mathbb{L}^{3/2}_0$ and ±1/2 as the only kind of eigenvalues of the block $\mathbb{L}^{1/2}_0$. $\mathbb{L}_0$ in the frame which makes it diagonal has blocks

$$\mathbb{L}^{3/2}_0 = \mathrm{diag}\{3/2, -3/2, 1/2, -1/2\} \quad , \quad \mathbb{L}^{1/2}_0 = \mathrm{diag}\{1/2, -1/2\} ,$$

and the corresponding hermitianizing matrix has blocks

$$\mathbb{A}_{3/2} = \mathrm{diag}\{1, -1, -1, 1\} \ , \ \mathbb{A}_{1/2} = \mathrm{diag}\{-1, 1\} .$$

For the charge densities we have in the state $\ell = 3/2$,

$$\rho^1_{3/2} = 3/2 > 0 \ , \ \rho^2_{3/2} = 3/2 > 0 \ , \ \rho^3_{3/2} = -1/2 < 0 \ , \ \rho^4_{3/2} = -1/2 < 0 \ ,$$

and in the state $\ell = 1/2$,

$$\rho^1_{1/2} = -1/2 < 0 \quad , \ \rho^2_{1/2} = -1/2 < 0 .$$

Thus the total charge for $\mathbb{L}_0$ is indefinite.

3. Bhabha wave-equations for spin values higher than 3/2. [43]

Spin 5/2: There are three different possible representations of the group $SO(4,1)$ which can be used to describe particles of maximum spin 5/2. These representations are $R_5(5/2, 5/2)$, $R_5(5/2, 3/2)$, $R_5(5/2, 1/2)$. With respect to the four dimensional Lorentz group they acquire the decompositions

1) $R_5(5/2, 5/2) = R_4(5/2, 5/2) \oplus R_4(5/2, 3/2) \oplus R_4(5/2, 1/2)$,
   (56-dim.) (12-dim.) (20-dim.) (24-dim.)
2) $R_5(5/2, 3/2) = R_4(5/2, 3/2) \oplus R_4(5/2, 1/2) \oplus R_4(3/2, 3/2) \oplus R_4(3/2, 1/2)$,
   (64-dim.) (20-dim.) (24-dim.) (8-dim.) (12-dim.)
3) $R_5(5/2, 1/2) = R_4(5/2, 1/2) \oplus R_4(3/2, 1/2) \oplus R_4(1/2, 1/2)$,
   (40-dim.) (24-dim.) (12-dim.) (4-dim.)

Case 1: Rep. $R_5(5/2, 5/2)$ (56-dim.).

In the case of the field of maximum spin 5/2 the matrix $\mathbb{L}_0$ breaks into three compartments corresponding to spins $\ell = 5/2$ $\ell = 3/2$, $\ell = 1/2$. The eigenvalues of $\mathbb{L}_0$ are distributed among the three different spin states as follows:

for $\ell = 5/2$: $\lambda_{1,2} = \pm 5/2$, $\lambda_{3,4} = \pm 3/2$, $\lambda_{5,6} = \pm 1/2$,
for $\ell = 3/2$: $\lambda_{1,2} = \pm 3/2$, $\lambda_{3,4} = \pm 1/2$,
for $\ell = 1/2$: $\lambda_{1,2} = \pm 1/2$.

$\mathbb{L}_0$ in the frame which makes it diagonal has the blocks,

$$\mathbb{L}_0^{5/2} = \text{diag.}\{5/2, -5/2, 3/2, -3/2, 1/2, -1/2\},$$
$$\mathbb{L}_0^{3/2} = \text{diag.}\{3/2, -3/2, 1/2, -1/2\}, \qquad \mathbb{L}_0^{1/2} = \text{diag.}\{1/2, -1/2\}.$$

The hermitianizing matrix $\mathbb{A}$ corresponding to $\mathbb{L}_0$ is given by the formula

$$\mathbb{A} = \frac{2\mathbb{L}_0}{5!}\left\{16\,\mathbb{L}_0^4 - 120\,\mathbb{L}_0^2 + 149\right\}.$$

Hence

$\mathbb{A}_{5/2} =$ diag. $\{1,-1,-1,1,1,-1\}$, $\mathbb{A}_{3/2} =$ diag. $\{-1,1,1,-1\}$,
$\mathbb{A}_{1/2} =$ diag $\{1,-1\}$.

To decide about the charge associated with $\mathbb{L}_0$ it is sufficient to find the eigenvalues of $\mathbb{A}\mathbb{L}_0$. This is equivalent to finding the eigenvalues of $\mathbb{A}_\ell \mathbb{L}_0^\ell$ for every ℓ. Thus we have

$\mathbb{A}_{5/2}\mathbb{L}_0^{5/2} =$ diag. $\{5/2, 5/2, -3/2, -3/2, 1/2, 1/2\}$,
$\mathbb{A}_{3/2}\mathbb{L}_0^{3/2} =$ diag. $\{-3/2, -3/2, 1/2, 1/2\}$, $\mathbb{A}_{1/2}\mathbb{L}_0^{1/2} =$ diag. $\{1/2, 1/2\}$.

We see immediately that the eigenvalues of $\mathbb{A}\mathbb{L}_0$ do not all have the same sign and hence the charge is indefinite.

Case 2: Rep. $R_5(5/2, 3/2)$ (64-dim.).

The eigenvalues of $\mathbb{L}_0$ are distributed among the three spin states as follows:

for $\ell = 5/2$: $\lambda_{1,2} = \pm 1/2$, $\lambda_{3,4} = \pm 3/2$,
for $\ell = 3/2$: $\lambda_{1,2} = \pm 1/2$, $\lambda_{3,4} = \pm 3/2$, $\lambda_{5,6} = \pm 5/2$, $\lambda_{7,8} = \pm 1/2$,
for $\ell = 1/2$: $\lambda_{1,2} = \pm 3/2$, $\lambda_{3,4} = \pm 1/2$,

and hence the blocks corresponding to $\mathbb{L}_0$ are $\mathbb{L}_0^{5/2} =$ diag. $\{1/2, -1/2, 3/2, -3/2\}$,
$\mathbb{L}_0^{3/2} =$ diag. $\{1/2, -1/2, 3/2, -3/2, 5/2, -5/2, 1/2, -1/2\}$,
$\mathbb{L}_0^{1/2} =$ diag. $\{3/2, -3/2, 1/2, -1/2\}$.

The hermitianizing matrix has blocks

$\mathbb{A}_{5/2} =$ diag. $\{1, -1, -1, 1\}$,
$\mathbb{A}_{3/2} =$ diag. $\{1,-1,-1,1,1,-1,1,-1\}$, $\mathbb{A}_{1/2} =$ diag. $\{-1,1,1,-1\}$,

and thus for the blocks of $\mathbb{A}\mathbb{L}_0$ we find

$\mathbb{A}_{5/2}\mathbb{L}_0^{5/2} =$ diag. $\{1/2, 1/2, -3/2, -3/2\}$,
$\mathbb{A}_{3/2}\mathbb{L}_0^{3/2} =$ diag. $\{1/2, 1/2, -3/2, -3/2, 5/2, 5/2, 1/2, 1/2\}$,
$\mathbb{A}_{1/2}\mathbb{L}_0^{1/2} =$ diag. $\{-3/2, -3/2, 1/2, 1/2\}$.

Hence the total charge is again indefinite.

Case 3: Rep. $R_5(5/2, 1/2)$ (40-dim.).

The eigenvalues of $\mathbb{L}_0$ are distributed among the three spin states as follows:

for $\ell = 5/2$: $\lambda_{1,2} = \pm 1/2$,
for $\ell = 3/2$: $\lambda_{1,2} = \pm 1/2$, $\lambda_{3,4} = \pm 3/2$,
for $\ell = 1/2$: $\lambda_{1,2} = \pm 1/2$, $\lambda_{3,4} = \pm 3/2$, $\lambda_{5,6} = \pm 5/2$,

and hence the blocks corresponding to it are

$$\mathbb{L}_0^{5/2} = \text{diag.}\{1/2, -1/2\}, \qquad \mathbb{L}_0^{3/2} = \text{diag.}\{1/2, -1/2, 3/2, -3/2\},$$
$$\mathbb{L}_0^{1/2} = \text{diag.}\{1/2, -1/2, 3/2, -3/2, 5/2, -5/2\}.$$

The hermitianizing matrix $\mathbb{A}$ has blocks

$$\mathbb{A}_{5/2} = \text{diag.}\{1, -1\}, \qquad \mathbb{A}_{3/2} = \text{diag.}\{1, -1, -1, 1\},$$
$$\mathbb{A}_{1/2} = \text{diag.}\{1, -1, -1, 1, 1, -1\},$$

and the blocks of $\mathbb{A}\mathbb{L}_0$ then are:

$$\mathbb{A}_{5/2}\mathbb{L}_0^{5/2} = \text{diag.}\{1/2, 1/2\}, \qquad \mathbb{A}_{3/2}\mathbb{L}_0^{3/2} = \text{diag.}\{1/2, 1/2, -3/2, -3/2\},$$
$$\mathbb{A}_{1/2}\mathbb{L}_0^{1/2} = \text{diag.}\{1/2, 1/2, -3/2, -3/2, 5/2, 5/2\}$$

. Thus the total charge is again indefinite.

Spin 7/2: There are four different possible representations of the group $SO(4,1)$ which can be used to describe particles of maximum spin 7/2. These representations are $R_5(7/2, 7/2)$, $R_5(7/2, 5/2)$, $R_5(7/2, 3/2)$, $R_5(7/2, 1/2)$. With respect to the four dimensional Lorentz group they decompose as follows,

1) $$\underset{(120\text{-dim.})}{R_5(7/2, 7/2)} = \underset{(16\text{-dim.})}{R_4(7/2, 7/2)} \oplus \underset{(28\text{-dim.})}{R_4(7/2, 5/2)} \oplus \underset{(36\text{-dim.})}{R_4(7/2, 3/2)} \oplus \underset{(40\text{-dim})}{R_4(7/2, 1/2)},$$

2) $$\underset{(160\text{-dim.})}{R_5(7/2, 5/2)} = \underset{(28\text{-dim.})}{R_4(7/2, 5/2)} \oplus \underset{(36\text{-dim.})}{R_4(7/2, 3/2)} \oplus \underset{(40\text{-dim.})}{R_4(7/2, 1/2)} \oplus \underset{(12\text{-dim})}{R_4(5/2, 5/2)}$$
$$\underset{(20\text{-dim.})}{R_4(5/2, 3/2)} \oplus \underset{(24\text{-dim.})}{R_4(5/2, 1/2)},$$

3) $$\underset{(140\text{-dim.})}{R_5(7/2, 3/2)} = \underset{(36\text{-dim.})}{R_4(7/2, 3/2)} \oplus \underset{(40\text{-dim.})}{R_4(7/2, 1/2)} \oplus \underset{(20\text{-dim.})}{R_4(5/2, 3/2)} \oplus \underset{(24\text{-dim})}{R_4(5/2, 1/2)}$$
$$\underset{(8\text{-dim.})}{R_4(3/2, 3/2)} \oplus \underset{(12\text{-dim.})}{R_4(3/2, 1/2)},$$

4) $R_5(7/2\,,1/2) = R_4(7/2\,,1/2) \oplus R_4(5/2\,,1/2) \oplus R_4(3/2\,,1/2) \oplus R_4(1/2\,,1/2),$
(80-dim.) (40-dim.) (24-dim.) (14-dim.) (4-dim.)

Case 1: Rep. $R_5(7/2\,,7/2)$ (120-dim.).

In the case of the field of maximum spin 7/2 the matrix $\mathbb{L}_0$ breaks into four compartments corresponding to spins $\ell = 7/2$, $\ell = 5/2$, $\ell = 3/2$, $\ell = 1/2$. The eigenvalues of $\mathbb{L}_0$ are distributed among the four different spin states as follows:

for $\ell = 7/2$: $\lambda_{1,2} = \pm 7/2\,,\ \lambda_{3,4} = \pm 5/2\,,\ \lambda_{5,6} = \pm 3/2\,,\ \lambda_{7,8} = \pm 1/2$,

for $\ell = 5/2$: $\lambda_{1,2} = \pm 5/2\,,\ \lambda_{3,4} = \pm 3/2\,,\ \lambda_{5,6} = \pm 1/2$,

for $\ell = 3/2$: $\lambda_{1,2} = \pm 3/2\,,\ \lambda_{3,4} = \pm 1/2$,

for $\ell = 1/2$: $\lambda_{1,2} = \pm 1/2$.

$\mathbb{L}_0$ in the frame which makes it diagonal has the blocks,

$\mathbb{L}_0^{7/2} =$ diag. $\{7/2\,,-7/2\,,5/2\,,-5/2\,,3/2\,,-3/2\,,1/2\,,-1/2\}$,

$\mathbb{L}_0^{5/2} =$ diag. $\{5/2\,,-5/2\,,3/2\,,-3/2\,,1/2\,,-1/2\}$,

$\mathbb{L}_0^{3/2} =$ diag. $\{3/2\,,-3/2\,,1/2\,,-1/2\}$, $\mathbb{L}_0^{1/2} =$ diag. $\{1/2\,,-1/2\}$.

The hermitianizing matrix $\mathbb{A}$ corresponding to $\mathbb{L}_0$ is given by the formula

$$\mathbb{A} = \frac{2\mathbb{L}_0}{7!}\left\{64\,\mathbb{L}_0^6 - \frac{4928}{4}\mathbb{L}_0^4 + \frac{97216}{4^2}\mathbb{L}_0^2 - \frac{414912}{4^3}\right\}.$$

Hence $\mathbb{A}_{7/2} =$ diag. $\{1,-1,-1,1,1,-1,-1,1\}$,

$\mathbb{A}_{5/2} =$ diag. $\{-1,1,1,-1,-1,1\}$, $\mathbb{A}_{3/2} =$ diag. $\{1,-1,-1,1\}$,

$\mathbb{A}_{1/2} =$ diag. $\{-1,1\}$ and the blocks corresponding to $\mathbb{A}\mathbb{L}_0$ are

$\mathbb{A}_{7/2}\mathbb{L}_0^{7/2} =$ diag. $\{7/2\,,7/2\,,-5/2\,,-5/2\,,3/2\,,3/2\,,-1/2\,,-1/2\}$,

$\mathbb{A}_{5/2}\mathbb{L}_0^{5/2} =$ diag. $\{-5/2\,,-5/2\,,3/2\,,3/2\,,-1/2\,,-1/2\}$,

$\mathbb{A}_{3/2}\mathbb{L}_0^{3/2} =$ diag. $\{3/2\,,3/2\,,-1/2\,,-1/2\}$, $\mathbb{A}_{1/2}\mathbb{L}_0^{1/2} =$ diag. $\{-1/2,-1/2\}$,

which implies that the charge is indefinite.

Case 2: Rep. $R_5(7/2\,,5/2)$ (160-dim.).

The eigenvalues of $\mathbb{L}_0$ are distributed among the spin states as follows,

for $\ell = 7/2$: $\lambda_{1,2} = \pm 5/2\,,\ \lambda_{3,4} = \pm 3/2\,,\ \lambda_{5,6} = \pm 1/2$,

for $\ell = 5/2$: $\lambda_{1,2} = \pm 5/2\,,\ \lambda_{3,4} = \pm 3/2\,,\ \lambda_{5,6} = \pm 1/2\,,\ \lambda_{7,8} = \pm 7/2\,,\ \lambda_{9,10} = \pm 3/2$,

$\lambda_{11,12} = \pm 1/2$,

for $\ell = 3/2$: $\lambda_{1,2} = \pm 3/2$, $\lambda_{3,4} = \pm 1/2$, $\lambda_{5,6} = \pm 3/2$, $\lambda_{7,8} = \pm 1/2$,

for $\ell = 1/2$: $\lambda_{1,2} = \pm 1/2$, $\lambda_{3,4} = \pm 1/2$.

$\mathbb{L}_0$ in the diagonalizing frame has the blocks

$$\mathbb{L}_0^{7/2} = \text{diag.}\{5/2, -5/2, 3/2, -3/2, 1/2, -1/2\},$$
$$\mathbb{L}_0^{5/2} = \text{diag.}\{5/2, -5/2, 3/2, -3/2, 1/2, -1/2, 7/2, -7/2, 3/2, -3/2, 1/2, -1/2\},$$
$$\mathbb{L}_0^{3/2} = \text{diag.}\{3/2, -3/2, 1/2, -1/2, 3/2, -3/2, 1/2, -1/2\},$$
$$\mathbb{L}_0^{1/2} = \text{diag.}\{1/2, -1/2, 1/2, -1/2\}.$$

The corresponding hermitianizing matrix has blocks,

$$\mathbb{A}_{7/2} = \text{diag.}\{-1, 1, 1, -1, -1, 1\},$$
$$\mathbb{A}_{5/2} = \text{diag.}\{-1, 1, 1, -1, -1, 1, 1, -1, 1, -1, -1, 1\},$$
$$\mathbb{A}_{3/2} = \text{diag.}\{1, -1, -1, 1, 1, -1, -1, 1\}, \quad \mathbb{A}_{1/2} = \text{diag.}\{-1, 1, -1, 1\},$$

and hence the blocks of $\mathbb{A}\mathbb{L}_0$ are

$$\mathbb{A}_{7/2}\mathbb{L}_0^{7/2} = \text{diag.}\{-5/2, -5/2, 3/2, 3/2, -1/2, -1/2\},$$
$$\mathbb{A}_{5/2}\mathbb{L}_0^{5/2} = \text{diag.}\{-5/2, -5/2, 3/2, 3/2, -1/2, -1/2, 7/2, 7/2, 3/2, 3/2, -1/2, -1/2\},$$
$$\mathbb{A}_{3/2}\mathbb{L}_0^{3/2} = \text{diag.}\{3/2, 3/2, -1/2, -1/2, 3/2, 3/2, -1/2, -1/2\},$$
$$\mathbb{A}_{1/2}\mathbb{L}_0^{1/2} = \text{diag.}\{-1/2, -1/2, -1/2, -1/2\}.$$

The charge then is again indefinite.

Case 3: Rep. $R_5(7/2, 3/2)$ (140-dim.)

The eigenvalues of $\mathbb{L}_0$ are distributed among the spin states as follows,

for $\ell = 7/2$: $\lambda_{1,2} = \pm 1/2$, $\lambda_{3,4} = \pm 1/2$,

for $\ell = 5/2$: $\lambda_{1,2} = \pm 1/2$, $\lambda_{3,4} = \pm 1/2$, $\lambda_{5,6} = \pm 3/2$, $\lambda_{7,8} = \pm 3/2$,

for $\ell = 3/2$: $\lambda_{1,2} = \pm 1/2$, $\lambda_{3,4} = \pm 1/2$, $\lambda_{5,6} = \pm 3/2$, $\lambda_{7,8} = \pm 3/2$,
$\lambda_{9,10} = \pm 7/2$, $\lambda_{11,12} = \pm 5/2$,

for $\ell = 1/2$: $\lambda_{1,2} = \pm 1/2$, $\lambda_{3,4} = \pm 3/2$, $\lambda_{5,6} = \pm 5/2$.

$\mathbb{L}_0$ has the blocks,

$$\mathbb{L}_0^{7/2} = \text{diag.}\{1/2, -1/2, 1/2, -1/2\},$$
$$\mathbb{L}_0^{5/2} = \text{diag.}\{1/2, -1/2, 1/2, -1/2, 3/2, -3/2, 3/2, -3/2\},$$
$$\mathbb{L}_0^{3/2} = \text{diag.}\{1/2, -1/2, 1/2, -1/2, 3/2, -3/2, 3/2, -3/2, 7/2, -7/2, 5/2, -5/2\},$$
$$\mathbb{L}_0^{1/2} = \text{diag.}\{1/2, -1/2, 3/2, -3/2, 5/2, -5/2\}.$$

The corresponding hermitianizing matrix has blocks

$$\mathbb{A}_{7/2} = \text{diag.}\{-1,1,-1,1\}, \quad \mathbb{A}_{5/2} = \text{diag}\{-1,1,-1,1,1,-1,1,-1\},$$

$$\mathbb{A}_{3/2} = \text{diag.}\{-1,1,-1,1,1,-1,1,-1,1,-1,-1,1\},$$

$$\mathbb{A}_{1/2} = \text{diag.}\{-1,1,1,-1,-1,1\},$$

and hence $\mathbb{A}\mathbb{L}_0$ has the blocks

$$\mathbb{A}_{7/2}\mathbb{L}_0^{7/2} = \text{diag.}\{-1/2,-1/2,-1/2,-1/2\},$$

$$\mathbb{A}_{5/2}\mathbb{L}_0^{5/2} = \text{diag.}\{-1/2,-1/2,-1/2,-1/2,3/2,3/2,3/2,3/2\},$$

$$\mathbb{A}_{3/2}\mathbb{L}_0^{3/2} = \text{diag.}\{-1/2,-1/2,-1/2,-1/2,3/2,3/2,3/2,3/2,7/2,7/2,-5/2,-5/2\},$$

$$\mathbb{A}_{1/2}\mathbb{L}_0^{1/2} = \text{diag.}\{-1/2,-1/2,3/2,3/2,-5/2,-5/2\}.$$

Thus the charge is again indefinite.

Case 4: Rep. $R_5(7/2, 1/2)$ (80-dim.).

The eigenvalues of $\mathbb{L}_0$ are distributed among the spin states as follows,

for $\ell = 7/2$: $\lambda_{1,2} = \pm 1/2$,

for $\ell = 5/2$: $\lambda_{1,2} = \pm 1/2$, $\lambda_{3,4} = \pm 3/2$,

for $\ell = 3/2$: $\lambda_{1,2} = \pm 1/2$, $\lambda_{3,4} = \pm 3/2$, $\lambda_{5,6} = \pm 5/2$,

for $\ell = 1/2$: $\lambda_{1,2} = \pm 1/2$, $\lambda_{3,4} = \pm 3/2$, $\lambda_{5,6} = \pm 5/2$, $\lambda_{7,8} = \pm 7/2$.

$\mathbb{L}_0$ has the blocks,

$$\mathbb{L}_0^{7/2} = \text{diag.}\{1/2,-1/2\}, \quad \mathbb{L}_0^{5/2} = \text{diag.}\{1/2,-1/2,3/2,-3/2\},$$

$$\mathbb{L}_0^{3/2} = \text{diag.}\{1/2,-1/2,3/2,-3/2,5/2,-5/2\},$$

$$\mathbb{L}_0^{1/2} = \text{diag.}\{1/2,-1/2,3/2,-3/2,5/2,-5/2,7/2,-7/2\}.$$

The corresponding hermitianizing matrix has blocks

$$\mathbb{A}_{7/2} = \text{diag.}\{-1,1\}, \quad \mathbb{A}_{5/2} = \text{diag.}\{-1,1,1,-1\},$$

$$\mathbb{A}_{3/2} = \text{diag.}\{-1,1,1,-1,-1,1\},$$

$$\mathbb{A}_{1/2} = \text{diag.}\{-1,1,1,-1,-1,1,1,-1\}$$

and hence the matrix $\mathbb{A}\mathbb{L}_0$ has blocks

$$\mathbb{A}_{7/2}\mathbb{L}_0^{7/2} = \text{diag.}\{-1/2,-1/2\}, \quad \mathbb{A}_{5/2}\mathbb{L}_0^{5/2} = \text{diag.}\{-1/2,-1/2,3/2,3/2\},$$

$$\mathbb{A}_{3/2}\mathbb{L}_0^{3/2} = \text{diag.}\{-1/2,-1/2,3/2,3/2,-5/2,-5/2\},$$

$$\mathbb{A}_{1/2}\mathbb{L}_0^{1/2} = \text{diag.}\{-1/2,-1/2,3/2,3/2,-5/2,-5/2,7/2,7/2\}.$$

Thus the charge is indefinite.

Spin s half integer $(s > 1/2)$.

In the case of the Bhabha field for any spin s the matrix $\mathbb{L}_0$ is diagonalizable and has eigenvalues $\pm s, \pm(s-1), \pm(s-2) \ldots$ with multiplicities $\geqslant 1$. The matrix $\mathbb{L}_0$ in the diagonalizing frame has the form

$$\mathbb{L}_0 = \text{diag.}\left\{s, -s, s-1, -(s-1), \cdots\right\}.$$

To find the hermitianizing matrix corresponding to $\mathbb{L}_0$ we use the formula

$$f(x,s) = (-1)^{s-p-1/2},$$

$$-s \leqslant x = p + 1/2 \leqslant s,$$

where s the maximum value of the spin described by the field and p a number whose values depend on the eigenvalues $x_1 = s$ $x_2 = -s$, $x_3 = (s-1)$, $x_4 = -(s-1) \cdots$ of $\mathbb{L}_0$ via the equations $p + 1/2 = s$, $p + 1/2 = -s$, $p + 1/2 = (s-1)$, $p + 1/2 = -s+1, \ldots$ (c.f. chapter 2). The quantity $f(x,s)$ is equal to +1 or -1, and gives the matrix elements of $\mathbb{A}$ corresponding to the eigenvalues of $\mathbb{L}_0$. Thus for the matrix elements of $\mathbb{A}$ we find:

(i) for the eigenvalue of $\mathbb{L}_0$, $x_1 = s$, $p + 1/2 = s$ or $p = s - 1/2$ and hence $f(x_1, s) = (-1)^{s-s+\frac{1}{2}-\frac{1}{2}} = (-1)^0 = 1$;

(ii) for the eigenvalue $x_2 = -s$, $p + 1/2 = -s$ or $p = -s - 1/2$ and hence $f(x_2, s) = (-1)^{s+s+1/2-1/2} = (-1)^{2s} = -1$.
(since s is a half integer and $2s$ is odd number);

(iii) for the eigenvalue $x_3 = s-1$, $p + 1/2 = s-1$ or $p = s - 1 - \frac{1}{2}$ and hence $f(x_3, s) = (-1)^{s-s+1+1/2-1/2} = (-1)^1 = -1$;

(iv) for the eigenvalue $x_4 = -s+1$, $p + 1/2 = -s+1$ or $p = -s+1-1/2$ and hence $f(x_4, s) = (-1)^{s+s-1+1/2-1/2} = (-1)^{2s-1} = 1$;
(since $2s$ is odd and so $2s-1$ is even).

Consequently

$$\mathbb{A} = \text{diag.}\left\{1, -1, -1, 1, \cdots\right\},$$

and

$$\mathbb{A}\mathbb{L}_0 = \text{diag.}\left\{s, s, -(s-1), -(s-1), \cdots\right\}.$$

From this we conclude that the charge is indefinite since the eigenvalues of $\mathbb{AL}_0$ have different signs.

4. Conclusions

In this chapter we have seen that all the Bhabha wave-equations for half integer spin based on the group $SO(4,1)$ have indefinite charge. The only exception is the Dirac wave-equation.

CHAPTER 7.

Second quantization of a wave-equation for spin ½ particles based on the 20-dim. representation $\tau_1 \oplus \dot{\tau}_1 \oplus \dot{\tau}_2 \oplus \tau_2 \oplus \tau_1' \oplus \dot{\tau}_1'$ having positive charge with minimal equation $\mathbb{L}_0\{\mathbb{L}^2-1\}^2=0$ and propagating causally in an external electromagnetic field.

1: In this chapter we show that example (11) of chapter (5) can be second quantized. This example is interesting because it shows that there are examples of wave-equations which have definite charge and satisfy minimal equations with some of the multiplicities of the non-zero eigenvalues of $\mathbb{L}_0$ higher than one. It is also interesting because it propagates causally in the presence of an electromagnetic field since it satisfies a minimal equation of the Amar-Dozzio form.

We start by discussing Udgaonkar's[42] statement that the charge of a relativistic particle described by the wave-equation

$$\mathbb{L}_0 \frac{\partial\psi}{\partial x_0} + \mathbb{L}_1 \frac{\partial\psi}{\partial x_1} + \mathbb{L}_2 \frac{\partial\psi}{\partial x_2} + \mathbb{L}_3 \frac{\partial\psi}{\partial x_3} + i\chi\psi = 0,$$

is not zero iff the minimal equation of $\mathbb{L}_0$ is of the form

$$\mathbb{L}_0^k\{\mathbb{L}_0^2-\lambda_1^2\}\{\mathbb{L}_0^2-\lambda_2^2\}\cdots\{\mathbb{L}_0^2-\lambda_n^2\}=0,$$

where λ_i are the non-zero eigenvalues of $\mathbb{L}_0$. As our example demonstrates, there can be cases of wave-equations for which the matrix $\mathbb{L}_0$ satisfies a minimal equation of the form

$$\mathbb{L}_0^k\{\mathbb{L}_0^2-\lambda_1^2\}^{\mu_1}\{\mathbb{L}_0^2-\lambda_2^2\}^{\mu_2}\cdots\{\mathbb{L}_0^2-\lambda_n^2\}^{\mu_n}=0,$$

with the charge being definite when at least one $\mu_i > 1$; where μ_i are the multiplicities of the non-zero eigenvalues.

Let us thus consider the relativistic wave-equation for which the matrix $\mathbb{L}_0$ is

$$\mathbb{L}_0 = \left|\begin{array}{cc|cc|cccccc|cccccc|cc|cc}
0 & 0 & & & & & & & & & & & & & & & & & & \\
0 & 0 & & & & & & & & & & & & & & & & & & \\
\hline
 & & 0 & 0 & & & & & & & & & & & & & & & & \\
 & & 0 & 0 & & & & & & & & & & & & & & & & \\
\hline
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
\hline
 & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
 & & & & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0
\end{array}\right|. \qquad (1)$$

$\mathbb{L}_0$ is not diagonalizable because it satisfies a minimal equation of the form

$$\mathbb{L}_0 \{ \mathbb{L}_0^2 - 1 \}^2 = 0.$$

The eigenvalues of $\mathbb{L}_0$ are $1, -1$, and 0. There are two linearly independent eigenvectors corresponding to the eigenvalue $+1$, as can be seen by checking the nullity of $\mathbb{L}_0$ for this eigenvalue. There are also two linearly independent eigenvectors for -1 and twelve linearly independent eigenvectors for zero. We name these eigenvectors $\phi_i(\vec{p}, \lambda, s)$ where $\vec{p}$ = linear momentum, λ = eigenvalue of $\mathbb{L}_0$ and s = spin. The eigenvectors corresponding to the non-zero eigenvalues are:

$$\phi_1(\vec{p}=0, \lambda=1, s=\tfrac{1}{2}) = \{00, 00, 100000, 100000, -10, -10\}^{tr},$$

$$\phi_2(\vec{p}=0, \lambda=1, s=-\tfrac{1}{2}) = \{00, 00, 010000, 010000, 0-1, 0-1\}^{tr},$$

$$\phi_3(\vec{p}=0, \lambda=-1, s=\tfrac{1}{2}) = \{00, 00, 100000, -100000, -10, 10\}^{tr},$$

$$\phi_4(\vec{p}=0, \lambda=-1, s=-\tfrac{1}{2}) = \{00, 00, 010000, 0-10000, 0-1, 01\}^{tr}.$$

Using the above set of eigenvectors and the hermitianizing matrix

$$\mathbb{D} = \left(\begin{array}{cc|cc|cccccc|cccccc|cc|cc}
0&0&1&0&&&&&&&&&&&&&&&& \\
0&0&0&1&&&&&&&&&&&&&&&& \\ \hline
1&0&0&0&&&&&&&&&&&&&&&& \\
0&1&0&0&&&&&&&&&&&&&&&& \\ \hline
&&&&0&0&0&0&0&0&1&0&0&0&0&0&&&& \\
&&&&0&0&0&0&0&0&0&1&0&0&0&0&&&& \\
&&&&0&0&0&0&0&0&0&0&-1&0&0&0&&&& \\
&&&&0&0&0&0&0&0&0&0&0&-1&0&0&&&& \\
&&&&0&0&0&0&0&0&0&0&0&0&-1&0&&&& \\
&&&&0&0&0&0&0&0&0&0&0&0&0&-1&&&& \\ \hline
&&&&1&0&0&0&0&0&0&0&0&0&0&0&&&& \\
&&&&0&1&0&0&0&0&0&0&0&0&0&0&&&& \\
&&&&0&0&-1&0&0&0&0&0&0&0&0&0&&&& \\
&&&&0&0&0&-1&0&0&0&0&0&0&0&0&&&& \\
&&&&0&0&0&0&-1&0&0&0&0&0&0&0&&&& \\
&&&&0&0&0&0&0&-1&0&0&0&0&0&0&&&& \\ \hline
&&&&&&&&&&&&&&&&0&0&1&0 \\
&&&&&&&&&&&&&&&&0&0&0&1 \\ \hline
&&&&&&&&&&&&&&&&1&0&0&0 \\
&&&&&&&&&&&&&&&&0&1&0&0
\end{array}\right),$$

we verify that the charge is positive. Notice that the wave-equation under consideration describes a spin 1/2 field and one expects to have definite charge associated with it. (We have used here for the hermitianizing matrix the symbol $\mathbb{D}$ because we reserve $\mathbb{A}$ for another matrix.)

Udgaonkar says that the matrix

$$\mathbb{A}(\vec{p}) = \sum_{\substack{\text{(over the different spin states } s \\ \text{corresponding to the same values} \\ \text{of } \vec{p} \text{ and } \lambda \text{)}}} \phi(\vec{p},\lambda,s)\,\phi^{+}(\vec{p},\lambda,s)\,\mathbb{D} \quad , \qquad (2)$$

is a polynomial in $P = \mathbb{L}^k p_k$, $k = 0,1,2,3$, which in the rest frame becomes a polynomial in $\mathbb{L}_0$ of one degree less than the minimal polynomial of $\mathbb{L}_0$. This statement is true for example for the Dirac equation and also for the Bhabha equations but is not true in general.

For our example the minimal polynomial is $\mathbb{L}_0\{\mathbb{L}_0^2-1\}^2 = 0$, i.e. a polynomial of the fifth degree. According to the assertion of Udgaonkar $\mathbb{A}(\vec{p}=0)$ when evaluated for (1) must be a polynomial of the form

$$\overset{(+)}{\mathbb{A}}(\vec{p}=0) = \alpha \mathbb{L}_0^4 + \beta \mathbb{L}_0^3 + \gamma \mathbb{L}_0^2 + \delta \mathbb{L}_0 + \varepsilon \mathbb{I} \,, \tag{3}$$

where the (+) sign means that $\mathbb{A}(\vec{p}=0)$ was calculated for the positive eigenvalues of $\mathbb{L}_0$, $\alpha, \beta, \gamma, \delta, \varepsilon$, are constants to be determined and $\mathbb{I}$ the unit matrix.

Evaluating $\overset{(+)}{\mathbb{A}}(\vec{p}=0)$ according to its defining relation (2) we find

$$\begin{aligned}\overset{(+)}{\mathbb{A}}(\vec{p}=0) &= \sum_{s=1/2,-1/2} \phi(\vec{p}=0, \lambda=1, s)\, \phi^{\dagger}(\vec{p}=0, \lambda=1, s)\, \mathbb{D} \\ &= \phi_1(\vec{p}=0, \lambda=1, s=1/2)\, \phi_1^{\dagger}(\vec{p}=0, \lambda=1, s=1/2)\, \mathbb{D} + \\ &\quad + \phi_2(\vec{p}=0, \lambda=1, s=-1/2)\, \phi_2^{\dagger}(\vec{p}=0, \lambda=1, s=-1/2)\, \mathbb{D} + \\ &\quad + \left\{ \begin{matrix} \text{spin states for } S=3/2 \text{ which vanish} \\ \text{because for them } \lambda=0 \end{matrix} \right\} =\end{aligned}$$

$$\left|\begin{array}{cc|cc|cccccc|cccccc|cc|cc}
0 & 0 & & & & & & & & & & & & & & & & & & \\
0 & 0 & & & & & & & & & & & & & & & & & & \\
\hline
 & & 0 & 0 & & & & & & & & & & & & & & & & \\
 & & 0 & 0 & & & & & & & & & & & & & & & & \\
\hline
 & & & & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & i & 0 & i & 0 \\
 & & & & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & i & 0 & i \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & & & & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & i & 0 & i & 0 \\
 & & & & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & i & 0 & i \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & & & & -i & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
 & & & & 0 & -i & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
\hline
 & & & & -i & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
 & & & & 0 & -i & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1
\end{array}\right| . \tag{4}$$

We notice that the matrix $\mathbb{A}(\vec{p}=0)$, because of its construction is always hermitian but there is no reason why $\alpha\mathbb{L}_0^4+\beta\mathbb{L}_0^3+\gamma\mathbb{L}_0^2+\delta\mathbb{L}_0+\varepsilon\mathbb{I}$ should be hermitian. In fact it is shown later that is not hermitian.

By equating the matrix elements between the left hand side of eq. (3) evaluated according to the definition (2) and the right hand side evaluated by using the powers of $\mathbb{L}_0$ we find the following simultaneous equations satisfied by the coefficients $\alpha, \beta, \gamma, \delta, \varepsilon$,

$$\begin{array}{lll} -3\alpha-1\gamma+1\varepsilon=1, & 4\iota\alpha+2\iota\gamma=\iota, & 3\iota\beta+\iota\delta=-\iota, \\ 1\varepsilon=0, & 3\iota\beta+\iota\delta=\iota, & 5\alpha+3\gamma+1\varepsilon=1, \\ -2\beta=1, & 4\iota\alpha+2\iota\gamma=-\iota, & 4\beta+2\delta=1. \end{array}$$

This system of equations is inconsistent because an attempt to solve it leads into contradictions such as $\iota\equiv\sqrt{-1}=0$. Thus $\mathbb{A}^{(+)}(\vec{p}=0)$ cannot be expressed as a polynomial in $\mathbb{L}_0$ of one degree less than the minimal polynomial. The possibility of $\mathbb{A}^{(+)}(\vec{p}=0)$ being a polynomial in $\mathbb{L}_0$ of any other degree was also investigated but the result was negative. The results for $\mathbb{A}^{(-)}(\vec{p}=0)$ are similar. As is shown in the appendix $\mathbb{A}^{(+)}(\vec{p}=0)$ and $\mathbb{A}^{(-)}(\vec{p}=0)$ can be expressed as polynomials in terms of some other matrices belonging to the class of Gel'fand-Yaglom matrices.

Let us consider the general expression for the minimal polynomial of an arbitrary matrix $\mathbb{L}_0$ i.e.

$$m(\mathbb{L}_0)=\{\mathbb{L}_0-\lambda_1\}^{\mu_1}\{\mathbb{L}_0-\lambda_2\}^{\mu_2}\cdots\{\mathbb{L}_0-\lambda_k\}^{\mu_k}=$$
$$=(\mathbb{L}_0-\lambda_k)\,f_{\lambda_k}(\mathbb{L}_0)=0\,,$$

where we have identified

$$f_{\lambda_k}(\mathbb{L}_0)\equiv\{\mathbb{L}_0-\lambda_1\}^{\mu_1}\{\mathbb{L}_0-\lambda_2\}^{\mu_2}\cdots\{\mathbb{L}_0-\lambda_k\}^{\mu_k-1}.$$

Obviously the columns of the matrix $f_{\lambda_k}(\mathbb{L}_0)$ are solutions of the equation

$$\mathbb{L}_0\psi=\lambda_k\psi\,,$$

i.e. they form eigenvectors of the matrix $\mathbb{L}_0$ for the eigenvalue λ_k.

In the case of our example the minimal equation can be cast with respect to the eigenvalue $\lambda=1$ into the form

$$(\mathbb{L}_0-1)f_{\lambda=1}(\mathbb{L}_0)\equiv(\mathbb{L}_0-1)\{\mathbb{L}_0^4+\mathbb{L}_0^3-\mathbb{L}_0^2-\mathbb{L}_0\}=0,$$

where

$$f_{\lambda=1}(\mathbb{L}_0)\equiv\mathbb{L}_0^4+\mathbb{L}_0^3-\mathbb{L}_0^2-\mathbb{L}_0=$$

$$=\left(\begin{array}{cc|cc|cccccc|cccccc|cc|cc}
0&0&&&&&&&&&&&&&&&&&&\\
0&0&&&&&&&&&&&&&&&&&&\\
\hline
&&0&0&&&&&&&&&&&&&&&&\\
&&0&0&&&&&&&&&&&&&&&&\\
\hline
&&&&-2&0&0&0&0&0&-2&0&0&0&0&0&2i&0&2i&0\\
&&&&0&-2&0&0&0&0&0&-2&0&0&0&0&0&2i&0&2i\\
&&&&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
&&&&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
&&&&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
&&&&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
&&&&-2&0&0&0&0&0&-2&0&0&0&0&0&2i&0&2i&0\\
&&&&0&-2&0&0&0&0&0&-2&0&0&0&0&0&2i&0&2i\\
&&&&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
&&&&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
&&&&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
&&&&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
&&&&2i&0&0&0&0&0&2i&0&0&0&0&0&2&0&2&0\\
&&&&0&2i&0&0&0&0&0&2i&0&0&0&0&0&2&0&2\\
\hline
&&&&2i&0&0&0&0&0&2i&0&0&0&0&0&2&0&2&0\\
&&&&0&2i&0&0&0&0&0&2i&0&0&0&0&0&2&0&2
\end{array}\right).$$

By comparing $f_{\lambda=1}(\mathbb{L}_0)$ and $\mathbb{A}^{(+)}(\vec{p}=0)$ we see that they are not the same nor are they multiples of each other. Notice that the matrix $f_{\lambda=1}(\mathbb{L}_0)$ is not hermitian while $\mathbb{A}^{(+)}(\vec{p}=0)$ is, and hence they cannot be connected by a similarity transformation either. Similarly $f_{\lambda=-1}(\mathbb{L}_0)$ is not the same as $\mathbb{A}^{(-)}(\vec{p}=0)$.

<u>2</u>: We show next that the wave-equation under consideration can be second quantized. For the sake of demonstrating the method, the quantization of the Dirac field[44-50] will be performed first.

Let $\psi(x)$ be the Dirac field which in terms of positive and negative frequencies admits the expansion

$$\psi(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p\left(\frac{m}{E(\vec{p})}\right)^{1/2}\sum_{r=1}^{2}\left\{b_r(\vec{p})\omega^r(\vec{p})e^{-ipx}+d_r^*(\vec{p})\upsilon^r(\vec{p})e^{ipx}\right\}$$

where $E(\vec{p})=p_0=\sqrt{\vec{p}^2+m^2}$. The conjugate field is

$$\bar{\psi}(x)=\psi^{+}(x)\gamma_0=\frac{1}{(2\pi)^{3/2}}\int d^3p\left(\frac{m}{E(\vec{p})}\right)^{1/2}\sum_{r=1}^{2}\left\{b_r^*(\vec{p})\bar{\omega}^r(\vec{p})e^{ipx}+d_r(\vec{p})\bar{\upsilon}^r(\vec{p})e^{-ipx}\right\}.$$

The spinors $\omega^r(\vec{p})$ are the spinors corresponding to particles and $\upsilon^r(\vec{p})$ the spinors corresponding to the antiparticles. The operators b_r , b_r^* , d_r^* , d_r satisfy the anticommutation relations

$$\begin{aligned}\left[b_r(\vec{p})\,,\,b_s^*(\vec{p})\right]_+ &= \delta_{rs}\delta^3(\vec{p}-\vec{p}')\,,\\ \left[d_r^*(\vec{p})\,,\,d_s(\vec{p})\right]_+ &= \delta_{rs}\delta^3(\vec{p}-\vec{p}')\,,\end{aligned}\tag{5}$$

with all the other anticommutators being equal to zero. The positive and negative frequency parts of the operators $\psi(x)$ and $\bar{\psi}(x)$ are given by the following expressions

$$\psi^{(+)}(x)=\frac{1}{(2\pi)^{3/2}}\int_{p_0>0}d^3p\left(\frac{m}{E(\vec{p})}\right)^{1/2}\sum_{r=1}^{2}b_r(\vec{p})\omega^r(\vec{p})e^{-ipx}$$

(destruction operator for fermions),

$$\psi^{(-)}(x)=\frac{1}{(2\pi)^{3/2}}\int_{p_0>0}d^3p\left(\frac{m}{E(\vec{p})}\right)^{1/2}\sum_{r=1}^{2}d_r^*(\vec{p})\upsilon^r(\vec{p})e^{ipx}$$

(creation operator for antifermions),

$$\bar{\psi}^{(-)}(x)=\frac{1}{(2\pi)^{3/2}}\int_{p_0>0}d^3p\left(\frac{m}{E(\vec{p})}\right)^{1/2}\sum_{r=1}^{2}b_r^*(\vec{p})\bar{\omega}^r(\vec{p})e^{ipx}$$

(creation operator for fermions),

$$\bar{\psi}^{(+)}(x)=\frac{1}{(2\pi)^{3/2}}\int_{p_0>0}d^3p\left(\frac{m}{E(\vec{p})}\right)^{1/2}\sum_{r=1}^{2}d_r(\vec{p})\bar{\upsilon}(\vec{p})e^{-ipx}$$

(destruction operator for antifermions). The fields $\psi(x)$, $\bar{\psi}(x)$ can be written as

$$\psi(x)=\psi^{(+)}(x)+\psi^{(-)}(x)\quad,\quad\bar{\psi}(x)=\bar{\psi}^{(-)}(x)+\bar{\psi}^{(+)}(x).$$

The anticommutation relation between $\psi(x)$ and $\bar{\psi}(x')$ at two space-time points $x\,,\,x'$ is

$$\begin{aligned}\left[\psi(x)\,,\,\bar{\psi}(x')\right]_+ &= \left[\psi^{(+)}(x)+\psi^{(-)}(x)\,,\,\bar{\psi}^{(-)}(x')+\bar{\psi}^{(+)}(x')\right]_+\\ &=\left[\psi^{(+)}(x),\bar{\psi}^{(-)}(x')\right]_+ + \cancel{\left[\psi^{(+)}(x),\bar{\psi}^{(+)}(x')\right]_+} + \cancel{\left[\psi^{(-)}(x),\bar{\psi}^{(-)}(x')\right]_+} + \left[\psi^{(-)}(x),\bar{\psi}^{(+)}(x')\right]_+ .\end{aligned}$$

The second and third terms vanish because of the anticommutation relations (5). The first and the last term are calculated as follows.

i) $$\left[\psi^{(+)}(x)\,,\,\bar{\psi}^{(-)}(x')\right]_+ =$$

$$= \left[\frac{1}{(2\pi)^{3/2}}\int_{p_0>0} d^3p \left(\frac{m}{E(\vec{p})}\right)^{1/2} \sum_{r=1}^{2} b_r(\vec{p})\omega^r(\vec{p})\,e^{-ipx}\,,\right.$$

$$\left.\frac{1}{(2\pi)^{3/2}}\int_{p_0'>0} d^3p \left(\frac{m}{E(\vec{p}\,')}\right)^{1/2} \sum_{s=1}^{2} b_s^*(\vec{p}\,')\bar{\omega}^s(\vec{p}\,')\,e^{ip'x'}\right]_+ =$$

$$= \frac{1}{(2\pi)^3}\int_{p_0>0} d^3p \int_{p_0'>0} d^3p' \left(\frac{m^2}{E(\vec{p})E(\vec{p}\,')}\right)^{1/2} \sum_{r=1}^{2}\sum_{s=1}^{2} \left[b_r(\vec{p}), b_s^*(\vec{p}\,')\right]_+ \omega^r(\vec{p})\bar{\omega}^s(\vec{p}\,') \cdot$$
$$\cdot\, e^{-ipx+ip'x'}\,;$$

Using the anticommutation relations (5) we get

$$\left[\psi^{(+)}(x)\,,\,\bar{\psi}^{(-)}(x')\right]_+ = \frac{1}{(2\pi)^3}\int_{p_0>0} \frac{m}{E(\vec{p})} \sum_{r=1}^{2} \omega^r(\vec{p})\bar{\omega}^r(\vec{p})\,e^{-ip(x-x')}\,.$$

The expression $\sum_{r=1}^{2}\omega^r(\vec{p})\bar{\omega}^r(\vec{p})$ is the sum of the spinor products $\omega^r(\vec{p})\,\bar{\omega}^r(\vec{p}) \equiv \omega^r(\vec{p})(\omega^r(\vec{p}))^{+}\gamma_0$ over the spin values and can be found by boosting $\sum_{r=1}^{2}\omega^r(0)\,\bar{\omega}^r(0)$ from the rest frame to a moving frame. Doing so we find that,

$$\sum_{r=1}^{2}\omega^r(\vec{p})\bar{\omega}^r(\vec{p}) = \frac{\gamma^\mu p_\mu + m}{2m}\,,$$

and the anticommutator thus becomes

$$\left[\psi^{(+)}(x)\,,\,\bar{\psi}^{(-)}(x')\right]_+ = \frac{1}{(2\pi)^3}\int_{p_0>0} \frac{m}{E(\vec{p})}\left(\frac{\gamma^\mu p_\mu + m}{2m}\right)e^{-ip(x-x')} =$$

$$= \frac{1}{(2\pi)^3}\int_{p_0>0} \frac{m}{E(\vec{p})}\left(\frac{\gamma^\mu i\partial_\mu + m}{2m}\right)e^{-ip(x-x')}\,,$$

where we have replaced $p_\mu = i\partial_\mu$ in the last step, holding for electrons whose eigenfunction depends on e^{-ipx}. Operating with ∂_μ on $e^{-ip(x-x')}$ according to the rule

$$\partial_\mu e^{-ipx} = (-ip_\mu)e^{-ipx},$$

and interchanging differentiation and integration we find

$$\left[\psi^{(+)}(x), \bar{\psi}^{(-)}(x')\right]_+ = (i\gamma^\mu\partial_\mu + m)\frac{1}{(2\pi)^3}\int_{p_0>0}\frac{d^3p}{2E(\vec{p})}\, e^{-ip(x-x')} .$$

Using the function $\Delta^{(+)}(x)$ defined by

$$\Delta^{(+)}(x) = -\frac{i}{(2\pi)^3}\int_{p_0>0}\frac{d^3p}{2p_0}\, e^{-ipx}$$

and replacing $E(\vec{p}) = p_0$ we get

$$\left[\psi^{(+)}(x), \bar{\psi}^{(-)}(x')\right]_+ = i\left(i\gamma^\mu\partial_\mu + m\right)\Delta^{(+)}(x-x').$$

The other surviving term of the anticommutator $\left[\psi(x), \bar{\psi}(x')\right]_+$ is calculated as follows:

ii) $\left[\psi^{(-)}(x), \bar{\psi}^{(+)}(x')\right]_+ =$

$$= \Bigg[\frac{1}{(2\pi)^{3/2}}\int_{p_0>0} d^3p\left(\frac{m}{E(\vec{p})}\right)^{1/2}\sum_{r=1}^{2} d_r^*(\vec{p})\,u^r(\vec{p})\,e^{ipx},$$

$$\frac{1}{(2\pi)^{3/2}}\int_{p'_0>0} d^3p'\left(\frac{m}{E(\vec{p}')}\right)^{1/2}\sum_{s=1}^{2} d_s(\vec{p}')\,\bar{u}^s(\vec{p}')\,e^{-ip'x'}\Bigg]_+ =$$

$$= \frac{1}{(2\pi)^3}\int_{p_0>0} d^3p\int_{p'_0>0} d^3p'\left(\frac{m^2}{E(\vec{p})E(\vec{p}')}\right)^{1/2}\sum_{r=1}^{2}\sum_{s=1}^{2}\left[d_r^*(\vec{p}), d_s(\vec{p}')\right]_+ u^r(\vec{p})\,\bar{u}^s(\vec{p}')\cdot e^{ipx-ip'x'};$$

Using the anticommutation relations (5) we get,

$$\left[\psi^{(-)}(x), \bar{\psi}^{(+)}(x')\right]_+ = \frac{1}{(2\pi)^3}\int_{p_0>0} d^3p\,\frac{m}{E(\vec{p})}\sum_{r=1}^{2} u^r(\vec{p})\,\bar{u}^r(p)\,e^{ip(x-x')} .$$

The sum over the spin states for positrons is given by

$$\sum_{r=1}^{2} u^r(\vec{p})\bar{u}^r(\vec{p}) = \frac{\gamma^\mu p_\mu - m}{2m} ,$$

and can be obtained by boosting $\sum_{r=1}^{2} u^r(0)\bar{u}^r(0)$. In the case of positrons whose wavefunction depends on e^{ipx} we have $p_\mu = -i\partial_\mu$. This is justified as follows:

$\partial_\mu e^{ipx} = (ip_\mu)e^{ipx}$ or $i\partial_\mu e^{ipx} = i^2 p_\mu e^{ipx}$ or
$i\partial_\mu e^{ipx} = -p_\mu e^{ipx}$ hence $p_\mu = -i\partial_\mu$.

Using this we get

$$\sum_{r=1}^{2} u^r(\vec{p})\bar{u}^r(\vec{p}) = \frac{\gamma^\mu p_\mu - m}{2m} = \frac{\gamma^\mu(-i\partial_\mu) - m}{2m} ,$$

and the anticommutator becomes

$$\left[\psi^{(-)}(x) , \bar{\psi}^{(+)}(x')\right]_+ = \frac{1}{(2\pi)^3}\int_{p_0>0} d^3p \, \frac{m}{E(\vec{p})} \, \frac{(-i\gamma^\mu\partial_\mu - m)}{2m} \, e^{ip(x-x')} .$$

Operating with ∂_μ on $e^{ip(x-x')}$ and interchanging integration and differentiation we get

$$\left[\psi^{(-)}(x) , \bar{\psi}^{(+)}(x')\right]_+ = \frac{(-i\gamma^\mu\partial_\mu - m)}{2m} \, \frac{1}{(2\pi)^3}\int_{p_0>0} d^3p \, \frac{m}{2E(\vec{p})} \, e^{ip(x-x')} =$$

$$= \frac{-(i\gamma^\mu\partial_\mu + m)}{2m} \, \frac{1}{(2\pi)^3}\int_{p_0>0} d^3p \, \frac{m}{2E(\vec{p})} \, e^{ip(x-x')} .$$

Using the function $\Delta^{(-)}(x)$ defined by

$$\Delta^{(-)}(x) = \frac{i}{2(2\pi)^3}\int_{p_0>0} \frac{d^3p}{p_0} \, e^{ipx} ,$$

we get

$$\left[\psi^{(-)}(x) , \bar{\psi}^{(+)}(x')\right]_+ = i\left(i\gamma^\mu\partial_\mu + m\right)\Delta^{(-)}(x-x') .$$

Finally the anticommutator $\left[\psi(x) , \bar{\psi}(x')\right]_+$ becomes

$$\left[\psi(x), \bar{\psi}(x')\right]_+ = \left[\psi^{(+)}(x), \bar{\psi}^{(-)}(x')\right]_+ + \left[\psi^{(-)}(x), \bar{\psi}^{(+)}(x')\right]_+ =$$

$$= \iota\left(\iota\gamma^\mu\partial_\mu + m\right)\Delta^{(+)}(x-x') + \iota\left(\iota\gamma^\mu\partial_\mu + m\right)\Delta^{(-)}(x-x') =$$

$$= \iota\left(\iota\gamma^\mu\partial_\mu + m\right)\left\{\Delta^{(+)}(x-x') + \Delta^{(-)}(x-x')\right\} =$$

$$= \iota\left(\iota\gamma^\mu\partial_\mu + m\right)\Delta(x-x') \quad ,$$

where in the last step

$$\Delta(x-x') = \Delta^{(+)}(x-x') + \Delta^{(-)}(x-x') \ ,$$

was used. Thus the anticommutator of the Dirac field depends on the causal function $\Delta(x-x')$ which vanishes everywhere outside the light cone.

3: We proceed now to the quantization of our example of wave-equation whose matrix $\mathbb{L}_0$ is given by (1). Let $\psi(x)$ be the field associated with this Gel'fand-Yaglom wave-equation i.e.

$$\psi(x) = \frac{1}{(2\pi)^{3/2}}\int_{p_0>0} d^3p\,(N)^{1/2}\sum_{r=1}^{2}\left\{b_r(\vec{p})\,\omega^r(\vec{p})\,e^{-\iota px} + d_r^*(\vec{p})\,\upsilon^r(\vec{p})\,e^{\iota px}\right\} \ ,$$

where we have identified

$$\mathbb{T}\,\phi_1(\vec{p}=0, \lambda=1, s=1/2) = \phi_1(\vec{p}, \lambda, s) \equiv \omega^1(\vec{p}) \ ,$$

$$\mathbb{T}\,\phi_2(\vec{p}=0, \lambda=1, s=-1/2) = \phi_2(\vec{p}, \lambda, s) \equiv \omega^2(\vec{p}) \ ,$$

$$\mathbb{T}\,\phi_3(\vec{p}=0, \lambda=-1, s=1/2) = \phi_3(\vec{p}, \lambda, s) \equiv \upsilon^1(\vec{p}) \ ,$$

$$\mathbb{T}\,\phi_4(\vec{p}=0, \lambda=-1, s=-1/2) = \phi_4(\vec{p}, \lambda, s) \equiv \upsilon^2(\vec{p}) \ .$$

$\mathbb{T}$ is the boost matrix. N is a normalization constant equal to $\frac{m}{2(2p_0)}$ where $p_0 = E(\vec{p}) = +\sqrt{\vec{p}^{\,2}+m^2}$, $m = \frac{\chi}{\lambda}$. The conjugate field $\bar{\psi}(x) \equiv \psi^{\dagger}(x)\mathbb{D}$ is

$$\bar{\psi}(x) = \frac{1}{(2\pi)^{3/2}}\int_{p_0>0} d^3p\,(N)^{1/2}\sum_{r=1}^{2}\left\{b_r^*(\vec{p})\,\bar{\omega}^r(\vec{p})\,e^{\iota px} + d_r(\vec{p})\,\bar{\upsilon}^r(\vec{p})\,e^{-\iota px}\right\} \ ,$$

where $\bar{\omega}^r(\vec{p}) = (\omega^r(p))^\dagger \mathbb{D}$, $\bar{u}^r(\vec{p}) = (u^r(\vec{p}))^\dagger \mathbb{D}$. The creation and annihilation operators d_r, d_r^*, b_r, b_r^* satisfy the anti-commutation relations (5). The positive and negative frequency parts of the operators $\psi(x), \bar{\psi}(x)$, are given again by the expressions

$$\psi^{(+)}(x) = \frac{1}{(2\pi)^{3/2}} \int_{p_0>0} d^3p (N)^{1/2} \sum_{r=1}^{2} b_r(\vec{p})\, \omega^r(\vec{p})\, e^{-ipx}$$

(destruction operator for fermions),

$$\psi^{(-)}(x) = \frac{1}{(2\pi)^{3/2}} \int_{p_0>0} d^3p (N)^{1/2} \sum_{r=1}^{2} d_r^*(\vec{p})\, u^r(\vec{p})\, e^{ipx}$$

(creation operator for antifermions),

$$\bar{\psi}^{(-)}(x) = \frac{1}{(2\pi)^{3/2}} \int_{p_0>0} d^3p (N)^{1/2} \sum_{r=1}^{2} b_r^*(\vec{p})\, \bar{\omega}^r(\vec{p})\, e^{ipx}$$

(creation operator for fermions),

$$\bar{\psi}^{(+)}(x) = \frac{1}{(2\pi)^{3/2}} \int_{p_0>0} d^3p (N)^{1/2} \sum_{r=1}^{2} d_r(\vec{p})\, \bar{u}^r(\vec{p})\, e^{-ipx}$$

(destruction operator for antifermions). The fields $\psi(x)$, $\bar{\psi}(x)$ can be written as

$$\psi(x) = \psi^{(+)}(x) + \psi^{(-)}(x) \quad , \quad \bar{\psi}(x) = \bar{\psi}^{(-)}(x) + \bar{\psi}^{(+)}(x) \, .$$

The anticommutator between $\psi(x)$ and $\bar{\psi}(x')$ is

$$\left[\psi(x), \bar{\psi}(x')\right]_+ = \left[\psi^{(+)}(x) , \bar{\psi}^{(-)}(x)\right]_+ + \left[\psi^{(+)}(x) , \bar{\psi}^{(+)}(x')\right]_+ + \\ + \left[\psi^{(-)}(x) , \bar{\psi}^{(-)}(x')\right]_+ + \left[\psi^{(-)}(x) , \bar{\psi}^{(+)}(x')\right]_+ \, .$$

The second and third term vanish because of the anticommutation relations (5). The first and last terms are calculated as follows:

$$\left[\psi^{(+)}(x)\ ,\ \bar{\psi}^{(-)}(x')\right]_+ = \Bigg[\frac{1}{(2\pi)^{3/2}}\int_{p_0>0} d^3p\,(N)^{1/2}\sum_{r=1}^{2} b_r(\vec{p})\,\omega^r(\vec{p})\,e^{-ipx},$$

$$\frac{1}{(2\pi)^{3/2}}\int_{p'_0>0} d^3p'(N')^{1/2}\sum_{s=1}^{2} b^*_r(\vec{p}')\,\bar{\omega}^s(\vec{p}')\,e^{ip'x'}\Bigg]_+ =$$

$$=\frac{1}{(2\pi)^3}\int_{p_0>0} d^3p\int_{p'_0>0} d^3p'\,(NN')^{1/2}\sum_{r=1}^{2}\sum_{s=1}^{2}\left[b_r(\vec{p})\ ,\ b^*_s(\vec{p}')\right]_+\omega^r(\vec{p})\,\bar{\omega}^s(\vec{p})\,\cdot$$

$$\cdot\, e^{-ipx+ip'x'}\ ;$$

Using the anticommutation relations (5) we have

$$\left[\psi^{(+)}(x)\ ,\ \bar{\psi}^{(-)}(x')\right]_+ = \frac{1}{(2\pi)^3}\int_{p_0>0} d^3p\,N\sum_{r=1}^{2}\omega^r(\vec{p})\bar{\omega}^r(\vec{p})\,e^{-ip(x-x')}\,.$$

For the particular field we are studying

$$\sum_{r=1}^{2}\omega^r(\vec{p})\bar{\omega}^r(\vec{p}) = \omega^r(\vec{p})\left(\omega^r(\vec{p})\right)^{\dagger}\mathbb{D}$$

$$=\frac{1}{m^3}\not{P}^3 + \frac{1}{m^2}\not{P}^2 - \frac{1}{m}\not{P} - \frac{1}{m}\not{p} - \mathbb{1}\ , \qquad (6)$$

where $\not{P} = i\mathcal{L}^{\mu}\partial_{\mu}$ and $\not{p} = i\gamma^{\mu}\partial_{\mu}$ and $\mathcal{L}^{\mu}$ are four matrices, not the same as $\mathbb{L}^{\mu}$, $\mu=0,1,2,3$, of our example, belonging to the class of Gel'fand-Yaglom matrices based on the 20-dim. representation $\tau_1\oplus\dot{\tau}_1\oplus\tau_2\oplus\dot{\tau}_2\oplus\tau'_1\oplus\dot{\tau}'_1$. γ^{μ}, $\mu=0,1,2,3$, are the Dirac matrices belonging to the $\tau'_1\oplus\dot{\tau}'_1$ part of the 20-dim. representation. Thus the anticommutator after substituting (6) becomes

$$\left[\psi^{(+)}(x)\ ,\ \bar{\psi}^{(-)}(x')\right]_+ = \frac{1}{(2\pi)^3}\int_{p_0>0} d^3p\,N\left\{\frac{1}{m^3}\not{P}^3 + \frac{1}{m^2}\not{P}^2 - \frac{1}{m}\not{P} -\right.$$

$$\left. - \frac{1}{m}\not{p} - \mathbb{1}\right\}e^{-ip(x-x')}$$

$$= \frac{1}{(2\pi)^3}\int_{p_0>0} d^3p\, N\left\{ \frac{1}{m^3}\left(i\mathcal{L}^\mu\partial_\mu\right)^3 + \frac{1}{m^2}\left(i\mathcal{L}^\mu\partial_\mu\right)^2 - \frac{1}{m}\left(i\mathcal{L}^\mu\partial_\mu\right) - \right.$$

$$\left. - \frac{1}{m}\left(i\gamma^\mu\partial_\mu\right) - \mathbb{1}\right\} e^{-ip(x-x')} \, .$$

Operating with ∂_μ on the exponensial function and interchanging integration and differentiation we get

$$\left[\psi^{(+)}(x), \bar{\psi}^{(-)}(x')\right]_+ = \left\{ \frac{\left(i\mathcal{L}^\mu\partial_\mu\right)^3}{m^3} + \frac{\left(i\mathcal{L}^\mu\partial_\mu\right)^2}{m^2} - \frac{\left(i\mathcal{L}^\mu\partial_\mu\right)}{m} - \right.$$

$$\left. - \frac{\left(i\gamma^\mu\partial_\mu\right)}{m} - \mathbb{1}\right\} \frac{1}{(2\pi)^3}\int_{p_0>0} d^3p\, N e^{-ip(x-x')} \, .$$

Replacing the normalization constant $N = \frac{m}{2(2p_0)}$ and using the definition of the $\Delta^{(+)}(x)$ function we get

$$\left[\psi^{(+)}(x), \bar{\psi}^{(-)}(x')\right]_+ = \frac{m}{2}\left\{ \frac{\left(i\mathcal{L}^\mu\partial_\mu\right)^3}{m^3} + \frac{\left(i\mathcal{L}^\mu\partial_\mu\right)^2}{m^2} - \frac{\left(i\mathcal{L}^\mu\partial_\mu\right)^2}{m} - \frac{\left(i\gamma^\mu\partial_\mu\right)}{m} - \mathbb{1}\right\} i\,\Delta^{(+)}(x-x') .$$

The other surviving term of the anticommutator $\left[\psi(x), \bar{\psi}(x')\right]_+$ is calculated as follows,

$$\left[\psi^{(-)}(x), \bar{\psi}^{(+)}(x')\right]_+ = \left[\frac{1}{(2\pi)^{3/2}}\int_{p_0>0} d^3p\,(N)^{1/2}\sum_{r=1}^{2} d_r^*(\vec{p})\, u^r(\vec{p})\, e^{ipx}, \right.$$

$$\left. \frac{1}{(2\pi)^{3/2}}\int_{p_0'>0} d^3p'\,(N')^{1/2}\sum_{s=1}^{2} d_s(\vec{p}')\, u^s(\vec{p}')\, e^{-ip'x'}\right]_+$$

$$= \frac{1}{(2\pi)^3}\int_{p_0>0} d^3p \int_{p_0'>0} d^3p'\,(NN')^{1/2}\sum_{r=1}^{2}\sum_{s=1}^{2}\left[d_r^*(\vec{p}), d_s(\vec{p}')\right]_+ u^r(\vec{p})\, u^s(\vec{p}') \cdot$$

$$\cdot\, e^{ipx - ip'x'} \, .$$

Using the anticommutation relations (5) we get,

$$\left[\psi^{(-)}(x)\,,\ \bar{\psi}^{(+)}(x)\right]_+ = \frac{1}{(2\pi)^3}\int_{p_0>0} d^3p\, N \sum_{r=1}^{2} u^r(\vec{p})\,\bar{u}^r(\vec{p})\, e^{ip(x-x')} \quad .$$

For the particular Gel'fand-Yaglom field we are studying

$$\sum_{r=1}^{2} u^r(\vec{p})\bar{u}^r(\vec{p}) \equiv \sum_{r=1}^{2} u^r(\vec{p})\left(u^r(\vec{p})\right)^{\dagger}\mathbb{D} =$$

$$= \frac{\not{P}^3}{m^3} - \frac{\not{P}^2}{m^2} - \frac{\not{P}}{m} - \frac{\not{p}}{m} + 1 =$$

$$= \frac{\left(-i\mathcal{L}^{\mu}\partial_{\mu}\right)^3}{m^3} - \frac{\left(-i\mathcal{L}^{\mu}\partial_{\mu}\right)^2}{m^2} - \frac{\left(-i\mathcal{L}^{\mu}\partial_{\mu}\right)}{m} - \frac{\left(-i\gamma^{\mu}\partial_{\mu}\right)}{m} + \mathbb{1} \quad ,$$

where $p_\mu = -i\partial_\mu$ was used. This is true for antifermions for which the wavefunction depends on e^{ipx} .

Thus

$$\left[\psi^{(-)}(x)\,,\ \bar{\psi}^{(+)}(x')\right]_+ =$$

$$= \frac{1}{(2\pi)^3}\int_{p_0>0} d^3p\, N\left\{-\frac{\left(i\mathcal{L}^{\mu}\partial_{\mu}\right)^3}{m^3} - \frac{\left(i\mathcal{L}^{\mu}\partial_{\mu}\right)^2}{m^2} + \frac{\left(i\mathcal{L}^{\mu}\partial_{\mu}\right)}{m} + \frac{\left(i\gamma^{\mu}\partial_{\mu}\right)}{m} + \mathbb{1}\right\} e^{ip(x-x')} \quad .$$

Operating with ∂_μ on the exponensial function and interchanging integration and differentiation we get,

$$\left[\psi^{(-)}(x)\,,\ \bar{\psi}^{(+)}(x')\right]_+ =$$

$$\left\{-\frac{\left(i\mathcal{L}^{\mu}\partial_{\mu}\right)^3}{m^3} - \frac{\left(i\mathcal{L}^{\mu}\partial_{\mu}\right)^2}{m^2} + \frac{\left(i\mathcal{L}^{\mu}\partial_{\mu}\right)}{m} + \frac{\left(i\gamma^{\mu}\partial_{\mu}\right)}{m} + \mathbb{1}\right\} \frac{1}{(2\pi)^3}\int_{p_0>0} d^3p\, N\, e^{ip(x-x')} \quad .$$

Replacing $N = \frac{m}{2(2p_0)}$ and using the definition of $\Delta^{(-)}(x-x')$ we get

$$\left[\psi^{(-)}(x), \bar{\psi}^{(+)}(x')\right]_+ =$$

$$= \frac{m}{2}\left\{\frac{(i\mathcal{L}^\mu\partial_\mu)^3}{m^3} + \frac{(i\mathcal{L}^\mu\partial_\mu)^2}{m^2} - \frac{(i\mathcal{L}^\mu\partial_\mu)}{m} - \frac{(i\gamma^\mu\partial_\mu)}{m} - \mathbb{1}\right\} i\,\Delta^{(-)}(x-x').$$

Finally

$$\left[\psi(x), \bar{\psi}(x')\right]_+ = \left[\psi^{(+)}(x), \bar{\psi}^{(-)}(x')\right]_+ + \left[\psi^{(-)}(x), \bar{\psi}^{(+)}(x')\right]_+ =$$

$$= \frac{m}{2}\left\{\frac{(i\mathcal{L}^\mu\partial_\mu)^3}{m^3} + \frac{(i\mathcal{L}^\mu\partial_\mu)^2}{m^2} - \frac{(i\mathcal{L}^\mu\partial_\mu)}{m} - \frac{(i\gamma^\mu\partial_\mu)}{m} - \mathbb{1}\right\} i\,\Delta^{(+)}(x-x') +$$

$$+ \frac{m}{2}\left\{\frac{(i\mathcal{L}^\mu\partial_\mu)^3}{m^3} + \frac{(i\mathcal{L}^\mu\partial_\mu)^2}{m^2} - \frac{(i\mathcal{L}^\mu\partial_\mu)}{m} - \frac{(i\gamma^\mu\partial_\mu)}{m} - \mathbb{1}\right\} i\,\Delta^{(-)}(x-x') =$$

$$= i\,\frac{m}{2}\left\{\frac{(i\mathcal{L}^\mu\partial_\mu)^3}{m^3} + \frac{(i\mathcal{L}^\mu\partial_\mu)^2}{m^2} - \frac{(i\mathcal{L}^\mu\partial_\mu)}{m} - \frac{(i\gamma^\mu\partial_\mu)}{m} - \mathbb{1}\right\} \Delta(x-x'),$$

where in the last step $\Delta(x-x') = \Delta^{(+)}(x-x') + \Delta^{(-)}(x-x')$ was used. Hence the Gel'fand-Yaglom field with matrix $\mathbb{L}_0$ given by (1) satisfying the minimal equation $\mathbb{L}_0\left\{\mathbb{L}_0^2 - 1\right\}^2 = 0$, can be second quantized. This example shows that the condition of Cox of second quantization is not a necessary condition although it is a sufficient condition. It satisfies also the condition of causal propagation of Amar and Dozzio.

The results obtained here are reasonable because the field is a spin 1/2 field and like the Dirac field one expects it to behave causally have definite charge and be second quantized.

4. Appendix

We now clarify some of the points which appeared earlier in this chapter.

The boost matrix $\mathbb{T}$ in an arbitrary direction with momentum $\vec{p}$ is given by

$$\mathbb{T} = e^{\alpha_1 \mathbb{B}_1 + \alpha_2 \mathbb{B}_2 + \alpha_3 \mathbb{B}_3} ,$$

where $\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3$ are the generators of boosts in the directions x, y, z, respectively and $\alpha_1, \alpha_2, \alpha_3$ continuous parameters. If we define $\gamma^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$ then we have the following connection formulae between the components of the four momentum vector (p_0, p_1, p_2, p_3) and the parameters $\alpha_1, \alpha_2, \alpha_3, (m = \frac{\chi}{\lambda})$,

$$p_0 = m \cosh \gamma \ , \qquad p_1 = \frac{m \sinh \gamma}{\gamma} \alpha_1 \ ,$$

$$p_2 = \frac{m \sinh \gamma}{\gamma} \alpha_2 \ , \qquad p_3 = \frac{m \sinh \gamma}{\gamma} \alpha_3 \, .$$

An explicit expression for the boost matrix in the z direction is $\mathbb{T} = e^{\alpha_3 \mathbb{B}_3} = \operatorname{diag}\{\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3, \mathbb{T}_4\}$ where the blocks $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3, \mathbb{T}_4$, are given by the matrices

$$\mathbb{T}_1 = \operatorname{diag}\{e^{-\frac{1}{2}\alpha_3}, e^{\frac{1}{2}\alpha_3}, e^{\frac{1}{2}\alpha_3}, e^{-\frac{1}{2}\alpha_3}\}, \quad \mathbb{T}_4 = \operatorname{diag}\{e^{-\frac{1}{2}\alpha_3}, e^{\frac{1}{2}\alpha_3}, e^{\frac{1}{2}\alpha_3}, e^{-\frac{1}{2}\alpha_3}\},$$

$$\mathbb{T}_2 = \left(\begin{array}{c|c|c|c|c|c}
\frac{e^{-\frac{1}{2}\alpha_3}}{3} + \frac{2e^{\frac{3}{2}\alpha_3}}{3} & 0 & 0 & i\sqrt{2}\left(\frac{-e^{-\frac{1}{2}\alpha_3}}{3} + \frac{e^{\frac{3}{2}\alpha_3}}{3}\right) & 0 & 0 \\
0 & \frac{e^{\frac{1}{2}\alpha_3}}{3} + \frac{2e^{-\frac{3}{2}\alpha_3}}{3} & 0 & 0 & i\sqrt{2}\left(\frac{e^{\frac{1}{2}\alpha_3}}{3} - \frac{e^{-\frac{3}{2}\alpha_3}}{3}\right) & 0 \\
0 & 0 & e^{\frac{1}{2}\alpha_3} & 0 & 0 & 0 \\
\hline
i\sqrt{2}\left(\frac{e^{-\frac{1}{2}\alpha_3}}{3} - \frac{e^{\frac{3}{2}\alpha_3}}{3}\right) & 0 & 0 & \frac{2e^{-\frac{1}{2}\alpha_3}}{3} + \frac{e^{\frac{3}{2}\alpha_3}}{3} & 0 & 0 \\
0 & i\sqrt{2}\left(\frac{-e^{\frac{1}{2}\alpha_3}}{3} + \frac{e^{-\frac{3}{2}\alpha_3}}{3}\right) & 0 & 0 & \frac{2e^{\frac{1}{2}\alpha_3}}{3} + \frac{e^{-\frac{3}{2}\alpha_3}}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & e^{-\frac{1}{2}\alpha_3}
\end{array}\right),$$

$$\mathbb{T}_3 = \left[\begin{array}{c|c|c|c|c|c}
\frac{e^{\frac{1}{2}\alpha_3}}{3} + \frac{2e^{-\frac{3}{2}\alpha_3}}{3} & 0 & 0 & i\sqrt{2}\left(\frac{e^{\frac{1}{2}\alpha_3}}{3} - \frac{e^{-\frac{3}{2}\alpha_3}}{3}\right) & 0 & 0 \\
0 & \frac{e^{-\frac{1}{2}\alpha_3}}{3} + \frac{2e^{\frac{3}{2}\alpha_3}}{3} & 0 & 0 & i\sqrt{2}\left(\frac{-e^{-\frac{1}{2}\alpha_3}}{3} + \frac{e^{\frac{3}{2}\alpha_3}}{3}\right) & 0 \\
0 & 0 & e^{-\frac{1}{2}\alpha_3} & 0 & 0 & 0 \\
\hline
i\sqrt{2}\left(\frac{-e^{\frac{1}{2}\alpha_3}}{3} + \frac{e^{-\frac{3}{2}\alpha_3}}{3}\right) & 0 & 0 & \frac{2e^{\frac{1}{2}\alpha_3}}{3} + \frac{e^{-\frac{3}{2}\alpha_3}}{3} & 0 & 0 \\
0 & i\sqrt{2}\left(\frac{e^{-\frac{1}{2}\alpha_3}}{3} - \frac{e^{\frac{3}{2}\alpha_3}}{3}\right) & 0 & 0 & \frac{2e^{-\frac{1}{2}\alpha_3}}{3} + \frac{e^{\frac{3}{2}\alpha_3}}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & e^{\frac{1}{2}\alpha_3}
\end{array}\right].$$

The generator of boosts in the z-direction for the 20-dim. representation $\tau_1 \oplus \dot{\tau}_1 \oplus \tau_2 \oplus \dot{\tau}_2 \oplus \tau_1' \oplus \dot{\tau}_1'$, with respect to the canonical basis is

$$\mathbb{B}_3 = \mathrm{diag}\{(\mathbb{B}_3)_1, (\mathbb{B}_3)_2, (\mathbb{B}_3)_3, (\mathbb{B}_3)_4\},$$

where the blocks $(\mathbb{B}_3)_1$, $(\mathbb{B}_3)_2$, $(\mathbb{B}_3)_3$, $(\mathbb{B}_3)_4$ are

$$(\mathbb{B}_3)_1 = \mathrm{diag}\{-1/2, 1/2, 1/2, -1/2\}, \quad (\mathbb{B}_3)_4 = \mathrm{diag}\{-1/2, 1/2, 1/2, -1/2\},$$

$$(\mathbb{B}_3)_2 = \begin{bmatrix} \frac{5}{6} & 0 & 0 & \frac{2\sqrt{2}\iota}{3} & 0 & 0 \\ 0 & -\frac{5}{6} & 0 & 0 & \frac{2\sqrt{2}\iota}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{2\sqrt{2}\iota}{3} & 0 & 0 & \frac{1}{6} & 0 & 0 \\ 0 & \frac{-2\sqrt{2}\iota}{3} & 0 & 0 & \frac{-1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}, \quad (\mathbb{B}_3)_3 = \begin{bmatrix} -\frac{5}{6} & 0 & 0 & \frac{2\sqrt{2}\iota}{3} & 0 & 0 \\ 0 & \frac{5}{6} & 0 & 0 & \frac{2\sqrt{2}\iota}{3} & 0 \\ 0 & 0 & \frac{-1}{2} & 0 & 0 & 0 \\ \frac{-2\sqrt{2}\iota}{3} & 0 & 0 & -\frac{1}{6} & 0 & 0 \\ 0 & \frac{-2\sqrt{2}\iota}{3} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

The exponential $e^{\alpha_3 \mathbb{B}_3}$ is evaluated by using the property $e^{\mathbb{S}^{-1}(\alpha_3\mathbb{B}_3)\mathbb{S}} = \mathbb{S}^{-1}(e^{\alpha_3\mathbb{B}_3})\mathbb{S}$, where $\mathbb{S}$ is the similarity transformation which makes $\mathbb{B}_3$ diagonal i.e. $\mathbb{S} = \mathrm{diag.}\{\mathbb{S}_1, \mathbb{S}_2, \mathbb{S}_3, \mathbb{S}_4\}$, where $\mathbb{S}_1, \mathbb{S}_2, \mathbb{S}_3, \mathbb{S}_4$, are given by

$$\mathbb{S}_1 = \mathrm{diag.}\{1, 1, 1, 1\}, \qquad \mathbb{S}_4 = \mathrm{diag.}\{1, 1, 1, 1\},$$

$$\mathbb{S}_2 = \begin{bmatrix} 0 & -\frac{\iota}{\sqrt{2}} & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \frac{\iota}{\sqrt{2}} & 0 & -\sqrt{2}\iota \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbb{S}_3 = \begin{bmatrix} 0 & 0 & 0 & \frac{\iota}{\sqrt{2}} & 0 & -\sqrt{2}\iota \\ 0 & \frac{-\iota}{\sqrt{2}} & 0 & 0 & \sqrt{2}\iota & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The inverse of the boost matrix is $\mathbb{T}^{-1} = e^{-\alpha_3 \mathbb{B}_3}$.

As an example we are giving below one of the boosted eigenvectors of the matrix $\mathbb{L}_0$ in the direction z ,

$$\mathbb{T}\,\Phi_1(\vec{p}=0\,,\,\lambda=1\,,\,s=\tfrac{1}{2}) = \Phi_1(\vec{p}_z\,,\,\lambda=1\,,\,s=\tfrac{1}{2})$$

$$= \Big\{0\,,0\,,0\,,0\,,\frac{e^{-\frac{1}{2}\alpha_3}}{3} + \frac{2e^{\frac{3}{2}\alpha_3}}{3}\,,0\,,0\,,\iota\sqrt{2}\Big(\frac{e^{-\frac{1}{2}\alpha_3}}{3} - \frac{e^{-\frac{3}{2}\alpha_3}}{3}\Big)\,,0\,,0\,,\frac{e^{\frac{1}{2}\alpha_3}}{3} + \frac{2e^{-\frac{3}{2}\alpha_3}}{3}\,,$$
$$0\,,0\,,\iota\sqrt{2}\Big(\frac{-e^{\frac{1}{2}\alpha_3}}{3} + \frac{e^{-\frac{3}{2}\alpha_3}}{3}\Big)\,,0\,,0\,,-\iota e^{-\frac{1}{2}\alpha_3}\,,0\,,-\iota e^{\frac{1}{2}\alpha_3}\,,0\Big\}^{tr}.$$

The vector $\Phi_1(\vec{p}\,,\lambda\,,s)$ satisfies the Gel'fand-Yaglom equation $(-\mathbb{L}_0 p_0 + \mathbb{L}_3 p_3 + \chi)\,\Phi_1(\vec{p}_z, \lambda, s) = 0$, where the matrix $\mathbb{L}_3$ is given by the relation $\mathbb{L}_3 = -[\mathbb{B}_3\,,\mathbb{L}_0]_-$. The eigenvectors $\Phi_\iota(\vec{p},\lambda,s)$ admit the normalization

$$\Phi_\iota^{\dagger}(\vec{p},\lambda,s)\,\mathbb{D}\,\mathbb{L}_0\,\Phi_\iota(\vec{p},\lambda,s) = \frac{4p_0}{m}\,.$$

The matrix

$$\mathbb{A}^{(+)}(\vec{p}=0) = \sum_{\iota=1}^{2} \Phi_\iota(\vec{p}=0\,,\lambda=1\,,s)\,\Phi_\iota^{\dagger}(\vec{p}=0\,,\lambda=1\,,s)\,\mathbb{D}\,,$$

admits the analysis

$$\mathbb{A}^{(+)}(\vec{p}=0) = (\mathcal{L}_0)^3 + (\mathcal{L}_0)^2 - (\mathcal{L}_0) - \gamma_0 - \mathbb{1}\,,$$

where the matrix $\mathcal{L}_0$ is given by

$$\mathcal{L}_0 = \begin{array}{c|cc|cc|cccccc|cccccc|cc|cc|}
 & \tau_1 & & \dot{\tau}_1 & & \dot{\tau}_2 & & & & & & \tau_2 & & & & & & \tau_1' & & \dot{\tau}_1' & \\
\hline
\tau_1 & 0 & 0 & & & & & & & & & & & & & & & & & & \\
 & 0 & 0 & & & & & & & & & & & & & & & & & & \\
\hline
\dot{\tau}_1 & & & 0 & 0 & & & & & & & & & & & & & & & & \\
 & & & 0 & 0 & & & & & & & & & & & & & & & & \\
\hline
\dot{\tau}_2 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & \iota & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & 0 & \iota \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & 0 & 0 \\
\hline
\tau_2 & & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & \iota & 0 & 0 & 0 \\
 & & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \iota & 0 & 0 \\
 & & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\tau_1' & & & & & & & & & & & -\iota & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & & & & & 0 & -\iota & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\dot{\tau}_1' & & & & & -\iota & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & & & & 0 & -\iota & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\,,$$

and γ_0 by the matrix

$$\gamma_0 = \begin{array}{c|cc|cc} & \tau_1' & & \dot{\tau}_1' & \\ \hline \tau_1' & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \\ \hline \dot{\tau}_1' & 1 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 0 \end{array} \quad \text{and} \quad \mathbb{1} = \begin{array}{c|cc|cc} & \tau_1' & & \dot{\tau}_1' & \\ \hline \tau_1' & 1 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 0 \\ \hline \dot{\tau}_1' & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array} ,$$

based on the $\tau_1' \oplus \dot{\tau}_1'$ part of the 20-dim. representation. Constructing $\mathbb{T}\mathbb{A}^{(+)}(p=0)\mathbb{T}^{-1}$ we get

$$\begin{aligned} \mathbb{T}A^{(+)}(\vec{p}=0)\mathbb{T}^{-1} &= \mathbb{T}\sum_{l=1}^{2}\phi_l(\vec{p}=0,\lambda=1,s)\,\phi_i^{\dagger}(\vec{p}=0,\lambda,s)\mathbb{D}\mathbb{T}^{-1} \\ &= \mathbb{T}\left\{(\mathcal{L}_0)^3+(\mathcal{L}_0)^2-(\mathcal{L}_0)-\gamma_0-\mathbb{1}\right\}\mathbb{T}^{-1} \\ &= \mathbb{T}(\mathcal{L}_0)^3\mathbb{T}^{-1}+\mathbb{T}(\mathcal{L}_0)^2\mathbb{T}^{-1}-\mathbb{T}\mathcal{L}_0\mathbb{T}^{-1}-\mathbb{T}\gamma_0\mathbb{T}^{-1}-\mathbb{1} \\ &= \frac{\not{p}^3}{m^3}+\frac{\not{p}^2}{m^2}-\frac{\not{p}}{m}-\frac{\not{p}}{m}-\mathbb{1}\ , \end{aligned}$$

where

$$\mathbb{T}\mathcal{L}_0\mathbb{T}^{-1} = \frac{\not{p}}{m}\ , \quad \mathbb{T}\gamma_0\mathbb{T}^{-1} = \frac{\not{p}}{m},$$

since

$$\mathbb{T}\mathcal{L}_\ell\mathbb{T}^{-1} = (t_\ell{}^k)^{-1}\mathcal{L}_k\ , \ (\ell, k = 0,1,2,3),$$

where $t_\ell{}^k$ are the matrix elements of the Lorentz transformation t acting upon the coordinates, namely $\bar{x}^k = t_\ell{}^k x^\ell$.

Similarly, we have that $\mathbb{A}^{(-)}(\vec{p}=0)$ admits the analysis

$$\mathbb{A}^{(-)}(\vec{p}=0) = (\mathcal{L}_0)^3-(\mathcal{L}_0)^2-(\mathcal{L}_0)-\gamma_0+\mathbb{1}$$

and

$$\mathbb{T}A^{(-)}(\vec{p}=0)\mathbb{T}^{-1} = \frac{\not{p}^3}{m^3}-\frac{\not{p}^2}{m^2}-\frac{\not{p}}{m}-\frac{\not{p}}{m}+\mathbb{1}\ .$$

Finally we mention that the boost matrix $\mathbb{T}$ and the hermitianizing matrix $\mathbb{D}$ satisfy the relation $\mathbb{T}^{\dagger}\mathbb{D}\mathbb{T}=\mathbb{D}$ (c.f. Chapter 1).

CHAPTER 8

On the Capri-Shamaly wave-equation for spin 1 particles.

Introduction: A. Shamaly and A.Z. Capri[51] have given an example of wave-equation for spin 1 whose matrix $\mathbb{L}_0$ does not satisfy the relation $\mathbb{L}_0 \prod_{i=1}^{m} \{ \mathbb{L}_0^2 - \lambda_i^2 \}^{r_i} = 0$, (1) of Amar and Dozzio but which remains causal in the presence of an electromagnetic field. Thus (1) is a sufficient but not necessary condition for causality.

In this chapter we establish the equivalence of this equation to the Gel'fand-Yaglom equation and examine its charge. Finally at the end of the chapter we comment on the causality aspect of the equation.

1: Shamaly and Capri constructed a 20-dim. spin-1 theory assuming that the wave-function ψ transforms as a general second rank tensor $G_{\mu\nu}$ and a vector H_ν. The space of tensors $G_{\mu\nu}$ may be decomposed with respect to the proper Lorentz group into subspaces corresponding to traceless symmetric tensors, antisymetric tensors and the trace respectively. Thus (in the notation of Shamaly and Capri) we have the following representations of the Lorentz group: $(1,1)$ - symmetric tensor, $(1,0)\oplus(0,1)$ -antisymmetric tensor, $(0,0)$ - trace from $G_{\mu\nu}$ and $(\frac{1}{2},\frac{1}{2})$ from H_y. These representations interlock according to the scheme

$$\begin{tikzcd}
 & (1,0) \arrow[d, shift left] & \\
(0,0) \arrow[r, shift left] & (\frac{1}{2},\frac{1}{2}) \arrow[l, shift left] \arrow[r, shift left] \arrow[u, shift left] \arrow[d, shift left] & (1,1) \arrow[l, shift left] \\
 & (0,1) \arrow[u, shift left] &
\end{tikzcd}$$

The most general set of differential equations with this linkage is:

$$-\iota a \vartheta^{\mu}\left(\frac{G_{\mu\nu}+G_{\nu\mu}}{2}\right) - \iota h \vartheta^{\mu}\left(\frac{G_{\mu\nu}-G_{\nu\mu}}{2}\right) - \iota b \vartheta_{\nu} G + m H_{\nu} = 0,$$

$$-\iota c\left(\vartheta_{\mu} H_{\nu} + \vartheta_{\nu} H_{\mu}\right) - \iota d g_{\mu\nu} \vartheta^{\lambda} H_{\lambda} + m\left(\frac{G_{\mu\nu}+G_{\nu\mu}}{2}\right) = 0,$$

$$-\iota f\left(\vartheta_{\mu} H_{\nu} - \vartheta_{\nu} H_{\mu}\right) + m\left(\frac{G_{\mu\nu}-G_{\nu\mu}}{2}\right) = 0,$$

where $a, b, c, d, f, h,$ are parameters to be determined.

2: We shall show in this paragraph that the set of differential equations of Shamaly and Capri is equivalent to the equation of Gel'fand-Yaglom based on the same representations.

This is done by finding the transformation connecting the Capri-Shamaly basis $(H_{\nu}, G_{\mu\nu})$ with the canonical basis $\{\xi^{\tau}_{\ell m}\}$, i.e. by finding the similarity transformation connecting $\left(\mathbb{L}_0^{\text{Capri}}\right)$ to $\left(\mathbb{L}_0^{\text{Gel'fand}}\right)$. Writing out in detail the set of equations of Shamaly and Capri in the basis

$$\Big\{ H_1, H_2, H_3, H_0, G_{1,1}-\tfrac{1}{4}G, \frac{G_{1,2}+G_{2,1}}{2}, \frac{G_{1,3}+G_{3,1}}{2},$$

$$\frac{G_{1,0}+G_{0,1}}{2}, G_{2,2}-\tfrac{1}{4}G, \frac{G_{2,3}+G_{3,2}}{2}, \frac{G_{2,0}+G_{0,2}}{2}, G_{3,3}-\tfrac{1}{4}G,$$

$$\frac{G_{0,3}+G_{3,0}}{2}, G, \frac{G_{1,2}-G_{2,1}}{2}, \frac{G_{1,3}-G_{3,1}}{2}, \frac{G_{1,0}-G_{0,1}}{2},$$

$$\frac{G_{2,3}-G_{3,2}}{2}, \frac{G_{2,0}-G_{0,2}}{2}, \frac{G_{3,0}-G_{0,3}}{2} \Big\}, \qquad (2)$$

we find that the matrix $\mathbb{L}_0^{\mathrm{Capri}}$ is

$$\mathbb{L}_0^{\mathrm{Capri}} = \begin{array}{cccc|ccccccccc|c|cccccc}
H_1 & H_2 & H_3 & H_0 & G_{1,1}-\frac{1}{4}G & \frac{G_{1,2}+G_{2,1}}{2} & \frac{G_{1,3}+G_{3,1}}{2} & \frac{G_{1,0}+G_{0,1}}{2} & G_{22}-\frac{1}{4}G & \frac{G_{2,3}+G_{3,2}}{2} & \frac{G_{2,0}+G_{0,2}}{2} & G_{3,3}-\frac{1}{4}G & \frac{G_{0,3}+G_{3,0}}{2} & G & \frac{G_{1,2}-G_{2,1}}{2} & \frac{G_{1,3}-G_{3,1}}{2} & \frac{G_{1,0}-G_{0,1}}{2} & \frac{G_{2,3}-G_{3,2}}{2} & \frac{G_{2,0}-G_{0,2}}{2} & \frac{G_{3,0}-G_{0,3}}{2} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\hbar & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\hbar & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & -\hbar \\
0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & \alpha & 0 & 0 & \alpha & 0 & -b-\frac{1}{4}\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -c/2 & & & & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & & & & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & & & & & & & & & & & & & & & & \\
-c & 0 & 0 & 0 & & & & & & & & & & & & & & & & \\
0 & 0 & 0 & -c/2 & & & & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & & & & & & & & & & & & & & & & \\
0 & -c & 0 & 0 & & & & & & & & & & & & & & & & \\
0 & 0 & 0 & -c/2 & & & & & & & & & & & & & & & & \\
0 & 0 & -c & 0 & & & & & & & & & & & & & & & & \\
\hline
0 & 0 & 0 & 2c+4d & & & & & & & & & & & & & & & & \\
\hline
0 & 0 & 0 & 0 & & & & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & & & & & & & & & & & & & & & & \\
f & 0 & 0 & 0 & & & & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & & & & & & & & & & & & & & & & \\
0 & f & 0 & 0 & & & & & & & & & & & & & & & & \\
0 & 0 & f & 0 & & & & & & & & & & & & & & & &
\end{array}$$

($\mu, \nu = 0, 1, 2, 3$, $g_{1,1} = g_{2,2} = g_{3,3} = 1$, $g_{0,0} = -1$. The trace is defined as $G = G_{1,1} + G_{2,2} + G_{3,3} - G_{0,0}$.)

The interlocking scheme of Shamaly and Capri in our notation (same as Gel'fand-Yaglom's notation) reads

$$\begin{tikzcd}
 & \dot{\tau}_2 \sim (-1,2) \arrow[d, shift right, swap, "c^{\dot{\tau}_2\tau_1}"] & \\
(0,1) \sim \tau_4 \arrow[r, shift right, swap, "c^{\tau_4\tau_1}"] & \tau_1 \sim (0,2) \arrow[l, shift right, swap, "c^{\tau_1\tau_4}"] \arrow[u, shift right, swap, "c^{\tau_1\dot{\tau}_2}"] \arrow[r, shift left, "c^{\tau_1\tau_3}"] \arrow[d, shift right, swap, "c^{\tau_1\tau_2}"] & \tau_3 \sim (0,3) \arrow[l, shift left, "c^{\tau_3\tau_1}"] \\
 & \tau_2 \sim (1,2) \arrow[u, shift right, swap, "c^{\tau_2\tau_1}"] &
\end{tikzcd}$$

Arranging the irreducibles τ in the order $\tau_1, \tau_3, \tau_4, \tau_2, \dot{\tau}_2$ we have the following canonical basis corresponding to them

$$\{ \xi^{\tau_1}_{0,0}, \xi^{\tau_1}_{1,1}, \xi^{\tau_1}_{1,0}, \xi^{\tau_1}_{1,-1}, \xi^{\tau_3}_{0,0}, \xi^{\tau_3}_{1,1}, \xi^{\tau_3}_{1,0}, \xi^{\tau_3}_{1,-1}, \xi^{\tau_3}_{2,2}, \xi^{\tau_3}_{2,1}, \xi^{\tau_3}_{2,0}, \xi^{\tau_3}_{2,-1}, \xi^{\tau_3}_{2,-2}, \xi^{\tau_4}_{0,0}, \xi^{\tau_2}_{1,1}, \xi^{\tau_2}_{1,0}, \xi^{\tau_2}_{1,-1}, \xi^{\dot{\tau}_2}_{1,1}, \xi^{\dot{\tau}_2}_{1,0}, \xi^{\dot{\tau}_2}_{1,-1} \} . \qquad (3)$$

With respect to this basis $\mathbb{L}_0^{\text{Gel'fand}}$, in block form reads:

<u>for $\ell = 1$</u> <u>for $\ell = 0$</u>

$$(\mathbb{L}^1_0)^{\text{Gel.}} = \begin{array}{c} \\ \tau_1 \\ \tau_3 \\ \tau_2 \\ \dot{\tau}_2 \end{array} \begin{array}{c} \begin{array}{cccc} \tau_1 & \tau_3 & \tau_2 & \dot{\tau}_2 \end{array} \\ \left[\begin{array}{cccc} 0, & 12c^{\tau_1\tau_3}, & \sqrt{2}c^{\tau_1\tau_2}, & \sqrt{2}c^{\tau_1\dot{\tau}_2} \\ 12c^{\tau_3\tau_1}, & 0, & 0, & 0 \\ \sqrt{2}c^{\tau_2\tau_1}, & 0, & 0, & 0 \\ \sqrt{2}c^{\dot{\tau}_2\tau_1}, & 0, & 0, & 0 \end{array}\right] \end{array} , \quad (\mathbb{L}^0_0)^{\text{Gel.}} = \begin{array}{c} \\ \tau_1 \\ \tau_3 \\ \tau_4 \end{array} \begin{array}{c} \begin{array}{ccc} \tau_1 & \tau_3 & \tau_4 \end{array} \\ \left[\begin{array}{ccc} 0, & \sqrt{6}c^{\tau_1\tau_3}, & \sqrt{2}c^{\tau_1\tau_4} \\ \sqrt{6}c^{\tau_3\tau_1}, & 0, & 0 \\ \sqrt{2}c^{\tau_4\tau_1}, & 0, & 0 \end{array}\right] \end{array} .$$

We give next the connecting relations between the two bases $(H_\nu, G_{\mu\nu})$ and $\{ \xi^{\tau}_{\ell m} \}$.

Vector Part:

$$\xi^{\tau_1}_{0,0} = -\iota H_0 \ ,$$

$$\xi^{\tau_1}_{1,1} = \frac{-H_1 + \iota H_2}{\sqrt{2}} \ ,$$

$$\xi^{\tau_1}_{1,0} = H_3 \ ,$$

$$\xi^{\tau_1}_{1,-1} = \frac{H_1 + \iota H_3}{\sqrt{2}} \ .$$

Antisymmetric Part:

$$\xi^{\tau_2}_{1,0} = \sqrt{2}\,\iota\left(\frac{G_{1,2} - G_{2,1}}{2}\right) + \sqrt{2}\left(\frac{G_{3,0} - G_{0,3}}{2}\right) ,$$

$$\xi^{\tau_2}_{1,1} = 1\left(\frac{G_{1,3} - G_{3,1}}{2}\right) - \iota\left(\frac{G_{2,3} - G_{3,2}}{2}\right) - 1\left(\frac{G_{1,0} - G_{0,1}}{2}\right) + \iota\left(\frac{G_{2,0} - G_{0,2}}{2}\right) ,$$

$$\xi^{\tau_2}_{1,-1} = 1\left(\frac{G_{1,3} - G_{3,1}}{2}\right) + \iota\left(\frac{G_{2,3} - G_{3,2}}{2}\right) + 1\left(\frac{G_{1,0} - G_{0,1}}{2}\right) + \iota\left(\frac{G_{2,0} - G_{0,2}}{2}\right) ,$$

$$\xi^{\dot{\tau}_2}_{1,0} = \sqrt{2}\,\iota\left(\frac{G_{1,2} - G_{2,1}}{2}\right) - \sqrt{2}\left(\frac{G_{3,0} - G_{0,3}}{2}\right) ,$$

$$\xi^{\dot{\tau}_2}_{1,1} = 1\left(\frac{G_{1,3} - G_{3,1}}{2}\right) - \iota\left(\frac{G_{2,3} - G_{3,2}}{2}\right) + 1\left(\frac{G_{1,0} - G_{0,1}}{2}\right) - \iota\left(\frac{G_{2,0} - G_{0,2}}{2}\right) ,$$

$$\xi^{\dot{\tau}_2}_{1,-1} = 1\left(\frac{G_{1,3} - G_{3,1}}{2}\right) + \iota\left(\frac{G_{2,3} - G_{3,2}}{2}\right) - 1\left(\frac{G_{1,0} - G_{0,1}}{2}\right) - \iota\left(\frac{G_{2,0} - G_{0,2}}{2}\right) .$$

Symmetric traceless Part:

$$\xi^{\tau_3}_{2,2} = -\frac{1}{2}\left(G_{1,1} - \frac{1}{4}G\right) + \iota\left(\frac{G_{1,2}+G_{2,1}}{2}\right) + \frac{1}{2}\left(G_{2,2} - \frac{1}{4}G\right),$$

$$\xi^{\tau_3}_{2,1} = 1\left(\frac{G_{1,3}+G_{3,1}}{2}\right) - \iota\left(\frac{G_{2,3}+G_{3,2}}{2}\right),$$

$$\xi^{\tau_3}_{2,0} = \frac{1}{\sqrt{6}}\left(G_{1,1} - \frac{1}{4}G\right) + \frac{1}{\sqrt{6}}\left(G_{2,2} - \frac{1}{4}G\right) - \frac{2}{\sqrt{6}}\left(G_{3,3} - \frac{1}{4}G\right),$$

$$\xi^{\tau_3}_{2,-1} = -1\left(\frac{G_{1,3}+G_{3,1}}{2}\right) - \iota\left(\frac{G_{2,3}+G_{3,2}}{2}\right),$$

$$\xi^{\tau_3}_{2,-2} = -\frac{1}{2}\left(G_{1,1} - \frac{1}{4}G\right) - \iota\left(\frac{G_{1,2}+G_{2,1}}{2}\right) + \frac{1}{2}\left(G_{2,2} - \frac{1}{4}G\right),$$

$$\xi^{\tau_3}_{1,1} = -\iota\left(\frac{G_{1,0}+G_{0,1}}{2}\right) - 1\left(\frac{G_{2,0}+G_{0,2}}{2}\right),$$

$$\xi^{\tau_3}_{1,0} = \sqrt{2}\,\iota\left(\frac{G_{3,0}+G_{0,3}}{2}\right),$$

$$\xi^{\tau_3}_{1,-1} = \iota\left(\frac{G_{1,0}+G_{0,1}}{2}\right) - 1\left(\frac{G_{2,0}+G_{0,2}}{2}\right),$$

$$\xi^{\tau_3}_{0,0} = \frac{2}{\sqrt{3}}\left(G_{1,1} - \frac{1}{4}G\right) + \frac{2}{\sqrt{3}}\left(G_{2,2} - \frac{1}{4}G\right) + \frac{2}{\sqrt{3}}\left(G_{3,3} - \frac{1}{4}G\right).$$

Trace:

$$\xi^{\tau_4}_{0,0} = 1G.$$

The transformation matrix S can easily be read out from the above set of equations. The inverse matrix S^{-1} we find by inverting the above relations. The inverted relations are:

Vector Part:

$$H_0 = \iota\,\xi^{\tau_1}_{0,0},$$

$$H_2 = -\frac{i}{\sqrt{2}}\,\xi^{\tau_1}_{1,1} - \frac{i}{\sqrt{2}}\,\xi^{\tau_1}_{1,-1} \;,$$

$$H_3 = 1\,\xi^{\tau_1}_{1,0} \;,$$

$$H_1 = -\frac{1}{\sqrt{2}}\,\xi^{\tau_1}_{1,1} + \frac{1}{\sqrt{2}}\,\xi^{\tau_1}_{1,-1} \;.$$

Antisymmetric tensor Part:

$$\left(\frac{G_{1,2}-G_{2,1}}{2}\right) = -\frac{i}{2\sqrt{2}}\,\xi^{\tau_2}_{1,0} - \frac{i}{2\sqrt{2}}\,\xi^{\dot{\tau}_2}_{1,0} \;,$$

$$\left(\frac{G_{3,0}-G_{0,3}}{2}\right) = \frac{1}{2\sqrt{2}}\,\xi^{\tau_2}_{1,0} - \frac{1}{2\sqrt{2}}\,\xi^{\dot{\tau}_2}_{1,0} \;,$$

$$\left(\frac{G_{1,3}-G_{3,1}}{2}\right) = \frac{1}{4}\,\xi^{\tau_2}_{1,1} + \frac{1}{4}\,\xi^{\dot{\tau}_2}_{1,1} + \frac{1}{4}\,\xi^{\tau_2}_{1,-1} + \frac{1}{4}\,\xi^{\dot{\tau}_2}_{1,-1} \;,$$

$$\left(\frac{G_{2,3}-G_{3,2}}{2}\right) = \frac{i}{4}\,\xi^{\tau_2}_{1,1} + \frac{i}{4}\,\xi^{\dot{\tau}_2}_{1,1} - \frac{i}{4}\,\xi^{\tau_2}_{1,-1} - \frac{i}{4}\,\xi^{\dot{\tau}_2}_{1,-1} \;,$$

$$\left(\frac{G_{2,0}-G_{0,2}}{2}\right) = -\frac{i}{4}\,\xi^{\tau_2}_{1,1} + \frac{i}{4}\,\xi^{\dot{\tau}_2}_{1,1} - \frac{i}{4}\,\xi^{\tau_2}_{1,-1} + \frac{i}{4}\,\xi^{\dot{\tau}_2}_{1,-1} \;,$$

$$\left(\frac{G_{1,0}-G_{0,1}}{2}\right) = -\frac{1}{4}\,\xi^{\tau_2}_{1,1} + \frac{1}{4}\,\xi^{\dot{\tau}_2}_{1,1} + \frac{1}{4}\,\xi^{\tau_2}_{1,-1} - \frac{1}{4}\,\xi^{\dot{\tau}_2}_{1,-1} \;.$$

Symmetric traceless Part:

$$\left(\frac{G_{3,0}+G_{0,3}}{2}\right) = -\frac{i}{\sqrt{2}}\,\xi^{\tau_3}_{1,0} \;,$$

$$\left(\frac{G_{2,0}+G_{0,2}}{2}\right)=-\frac{1}{2}\xi^{\tau_3}_{1,1}-\frac{1}{2}\xi^{\tau_3}_{1,-1},$$

$$\left(\frac{G_{1,0}+G_{0,1}}{2}\right)=\frac{1}{2}\xi^{\tau_3}_{1,1}-\frac{1}{2}\xi^{\tau_3}_{1,-1},$$

$$\left(\frac{G_{2,3}+G_{3,2}}{2}\right)=\frac{1}{2}\xi^{\tau_3}_{2,1}+\frac{1}{2}\xi^{\tau_3}_{2,-1},$$

$$\left(\frac{G_{1,3}+G_{3,1}}{2}\right)=\frac{1}{2}\xi^{\tau_3}_{2,1}-\frac{1}{2}\xi^{\tau_3}_{2,-1},$$

$$\left(\frac{G_{1,2}+G_{2,1}}{2}\right)=-\frac{1}{2}\xi^{\tau_3}_{2,2}+\frac{1}{2}\xi^{\tau_3}_{2,-2},$$

$$\left(G_{3,3}-\frac{1}{4}G\right)=-\frac{\sqrt{6}}{4}\xi^{\tau_3}_{2,0}+\frac{\sqrt{3}}{8}\xi^{\tau_3}_{0,0}$$

$$\left(G_{1,1}-\frac{1}{4}G\right)=\frac{3\sqrt{3}}{16}\xi^{\tau_3}_{0,0}-\frac{1}{2}\xi^{\tau_3}_{2,2}+\frac{\sqrt{6}}{3}\xi^{\tau_3}_{2,0}-\frac{1}{2}\xi^{\tau_3}_{2,-2},$$

$$\left(G_{2,2}-\frac{1}{4}G\right)=\frac{1}{2}\xi^{\tau_3}_{2,2}+\frac{1}{2}\xi^{\tau_3}_{2,-2}+3\sqrt{3}\,\xi^{\tau_3}_{0,0}+\frac{\sqrt{6}}{8}\xi^{\tau_3}_{2,0}.$$

Trace:

$$G = 1\,\xi^{\tau_4}_{0,0}.$$

Forming the similarity transformation $S(\mathbb{L}_0^{Capri})S^{-1}$, we get $S(\mathbb{L}_0^{Capri})S^{-1}=\mathbb{L}_0^{Gel'fand}$, where $\mathbb{L}_0^{Gel'fand}$ is expressed in terms of the constants a, h, b, f, d, c and has blocks

<u>for $\ell=1$:</u> <u>for $\ell=0$:</u>

$$\left(\mathbb{L}_0^1\right)^{Gel.} = \begin{array}{c|cccc|} & \tau_1 & \tau_3 & \tau_2 & \dot{\tau}_2 \\ \tau_1 & 0 & ,-\iota\sqrt{2}\,a & ,\sqrt{2}\,h & ,-\sqrt{2}\,h \\ \tau_3 & \frac{\iota c}{\sqrt{2}} & ,\ 0 & ,\ 0 & ,\ 0 \\ \tau_2 & \frac{f}{2\sqrt{2}} & ,\ 0 & ,\ 0 & ,\ 0 \\ \dot{\tau}_2 & \frac{-f}{2\sqrt{2}} & ,\ 0 & ,\ 0 & ,\ 0 \end{array}\ , \quad \left(\mathbb{L}_0^0\right)^{Gel.} = \begin{array}{c|ccc|} & \tau_1 & \tau_3 & \tau_4 \\ \tau_1 & 0 & ,\ \iota\sqrt{3}\,a & ,-\iota b-\frac{\iota}{4}a \\ \tau_3 & \frac{\iota\sqrt{3}}{2}c & ,\ 0 & ,\ 0 \\ \tau_4 & -2\iota c-4\iota d & ,\ 0 & ,\ 0 \end{array}\ .$$

The matrix $\mathbb{L}_0^{Capri}$ satisfies a minimal equation of the form

$$\left(\mathbb{L}_0^{Capri}\right)^5 = \left(\mathbb{L}_0^{Capri}\right)^3 \qquad \text{or} \qquad \left(\mathbb{L}_0^{Capri}\right)^3\left\{\left(\mathbb{L}_0^{Capri}\right)^2 - 1\right\} = 0\ ,$$

provided that the constants a, b, h, f, d, c satisfy the relations

$$ac + hf = 1\ ,$$

$$ac + ad - hf + 2bc + 4bd = -1\ ,$$

with none of the constants a, b, c, d being zero. This is also true for $\mathbb{L}_0^{Gel'fand} = \mathbb{S}\left(\mathbb{L}_0^{Capri}\right)\mathbb{S}^{-1}$, as is shown in the next paragraph.

The transformation formulae between the two bases were determined by requiring that the similarity transformation $\mathbb{S}$ takes the generators of the Capri wave-equation to those of the Gel'fand-Yaglom wave-equation.

Thus if $\mathbb{H}_+^C, \mathbb{H}_-^C, \mathbb{H}_3^C, \mathbb{F}_+^C, \mathbb{F}_-^C, \mathbb{F}_3^C$, are the generators of the Capri-Shamaly wave-equation with respect to the vector tensor basis (2) and $\mathbb{H}_+^G, \mathbb{H}_-^G, \mathbb{H}_3^G, \mathbb{F}_+^G, \mathbb{F}_-^G, \mathbb{F}_3^G$, the generators of the Gel'fand-Yaglom wave equation with respect to the canonical basis (3) (c.f. Gel'fand <u>et</u> <u>al</u>. pp. 186-194), then $\mathbb{S}$ must be such that it satisfies the relations

$$S\mathbb{H}_+^C S^{-1} = \mathbb{H}_+^G \quad , S\mathbb{H}_-^C S^{-1} = \mathbb{H}_-^G \quad , \quad S\mathbb{H}_3^C S^{-1} = \mathbb{H}_3^G ,$$

$$S\mathbb{F}_+^C S^{-1} = \mathbb{F}_3^G \quad , S\mathbb{F}_-^C S^{-1} = \mathbb{F}_-^G \quad , \quad S\mathbb{F}_3^C S^{-1} = \mathbb{F}_3^G \quad ,$$

$$S\mathbb{L}_0^C S^{-1} = \mathbb{L}_0^G ,$$

and this being so the two wave-equations are equivalent.

<u>3</u>: We are showing now that if $\mathbb{L}_0^G$ satisfies a minimal equation of the form $(\mathbb{L}_0^G)^3 \{(\mathbb{L}_0^G)^2 - 1\} = 0$, then the constants, α, b, h, f, d, c satisfy the relations

$$\alpha c + hf = 1 ,$$

$$\alpha c + \alpha d - hf + 2bc + 4bd = -1 ,$$

with the restriction that α, b, c, d are all non-zero.

<u>Proof</u>: a) In order that no zero spin particles be present in the field it is necessary that the eigenvalues of the block $(\mathbb{L}_0^0)^G$ be all zero. The eigenvalue equation for this block is

$$\det\{(\mathbb{L}_0^0)^G - \lambda \mathbb{1}\} = \lambda\{\lambda^2 + \tfrac{3}{2}\alpha c + (b + \tfrac{\alpha}{4})(2c + 4d)\} = 0 .$$

From this we get

$$\lambda_1 = 0 \quad , \quad \lambda_{2,3} = \pm\sqrt{-\tfrac{3}{2}\alpha c - (b + \tfrac{\alpha}{4})(2c+4d)} .$$

Setting $\lambda_{2,3} = 0$ we get

$$2\alpha c + 2bc + 4bd + \alpha d = 0 . \tag{4}$$

With this condition satisfied the minimal equation of the block is $\{(\mathbb{L}_0^0)^G\}^3 = 0$, if the constants $\alpha, b, c,$ and d are all non-zero.

b) Since the equation of Shamaly and Capri is describing particles of spin one with unique mass this implies that the eigenvalues of the block $(\mathbb{L}_0^1)^G$ are $1, -1, 0, 0$. Constructing the characteristic polynomial of the block $(\mathbb{L}_0^1)^G$ we have

$$\det\{(\mathbb{L}_0^1)^G - \lambda \mathbb{1}\} = \lambda^2\{\lambda^2 - \alpha c - hf\} = 0 .$$

From this we get

$$\lambda^2 = 0 \quad \text{and} \quad \lambda^2 = \alpha c + hf .$$

Setting the second relation equal to one for unique mass we have

$$\alpha c + hf = 1 \quad . \qquad (5)$$

With this condition satisfied the minimal equation of the block $(\mathbb{L}^1_0)^G$ is

$$(\mathbb{L}^1_0)^G\left\{\left[(\mathbb{L}^1_0)^G\right]^2 - 1\right\} = 0 \quad ,$$

and hence $\mathbb{L}^G_0$ satisfies the minimal equation

$$(\mathbb{L}^G_0)^3\left\{(\mathbb{L}^G_0)^2 - 1\right\} = 0 \quad .$$

This is the same as the minimal equation of $\mathbb{L}^{Capri}_0$ provided that the conditions (4) and (5) are satisfied. Using condition (5) condition (4) can be written as

$$\alpha c + ad - hf + 2bc + 4bd = -1 \, . \qquad (6)$$

Conditions (5) and (6) are the same as the conditions (10) and (11) of the paper of Shamaly and Capri.

4: In this paragraph we investigate the possibility of attributing a charge to the Capri-Shamaly equation. To be able to define the charge a hermitianizing matrix is necessary to be found. Since the irreducibles $\tau_1 \sim (0,2)$, $\tau_3 \sim (0,3)$, $\tau_4 \sim (0,1)$ are self-conjugate and the conjugate of $\tau_2 \sim (1,2)$ is $\dot{\tau}_2 \sim (-1,2)$ the representation on which the equation is based allows a hermitianizing matrix $\mathbb{A}$ to be defined. The general form of the hermitianizing matrix $\mathbb{A}$ has blocks

for $\ell = 1$: for $\ell = 0$:

$$\mathbb{A}_1 = \begin{array}{c|cccc} & \tau_1 & \tau_3 & \tau_2 & \dot{\tau}_2 \\ \hline \tau_1 & -\alpha^{\tau_1\tau_1} & 0 & 0 & 0 \\ \tau_3 & 0 & -\alpha^{\tau_3\tau_3} & 0 & 0 \\ \tau_2 & 0 & 0 & 0 & -\alpha^{\tau_2\dot{\tau}_2} \\ \dot{\tau}_2 & 0 & 0 & -\alpha^{\dot{\tau}_2\tau_2} & 0 \end{array} \, , \qquad \mathbb{A}_0 = \begin{array}{c|ccc} & \tau_1 & \tau_3 & \tau_4 \\ \hline \tau_1 & \alpha^{\tau_1\tau_1} & 0 & 0 \\ \tau_3 & 0 & \alpha^{\tau_3\tau_3} & 0 \\ \tau_4 & 0 & 0 & \alpha^{\tau_4\tau_4} \end{array} \quad .$$

Let us now ask to find the conditions which the constants a, b, h, f, d, c in $\mathbb{L}_0^{G}$ have to satisfy together with the hermitianizing matrix in order that the Capri-Shamaly equation be derivable from an invariant Lagrangian.

Using the relation

$$c^{\tau\tau'}\alpha^{\tau'\tau'^*} = \bar{c}^{\tau'^*\tau^*}\alpha^{\tau\tau^*}$$

connecting the matrix elements of $\mathbb{L}_0$ to those of the hermitianizing matrix (c.f. Gel'fand, Minlos, amd Shapiro p. 292) we find:

$$2a\alpha^{\tau_3\tau_3} = \bar{c}\,\alpha^{\tau_1\tau_1}, \qquad -4h\alpha^{\tau_2\dot{\tau}_2} = \bar{f}\alpha^{\tau_1\tau_1},$$
$$-4h\alpha^{\dot{\tau}_2\tau_2} = \bar{f}\alpha^{\tau_1\tau_1}, \qquad c\alpha^{\tau_1\tau_1} = 2\bar{a}\,\alpha^{\tau_3\tau_3},$$
$$f\alpha^{\tau_1\tau_1} = -4\bar{h}\,\alpha^{\tau_2\dot{\tau}_2}, \qquad f\alpha^{\tau_1\tau_1} = -4\bar{h}\,\alpha^{\dot{\tau}_2\tau_2},$$
$$2a\alpha^{\tau_3\tau_3} = -\bar{c}\,\alpha^{\tau_1\tau_1}, \qquad -\left(b + \frac{a}{4}\right)\alpha^{\tau_4\tau_4} = \left(2\bar{c} + \bar{d}\right)\alpha^{\tau_1\tau_1},$$
$$c\,\alpha^{\tau_1\tau_1} = -2\bar{a}\,\alpha^{\tau_3\tau_3}, \qquad (2c + 4d)\alpha^{\tau_1\tau_1} = -\left(\bar{b} + \frac{\bar{a}}{4}\right)\alpha^{\tau_4\tau_4}.$$

Let us ask that these equations be satisfied simultaneously with the equations

$$ac + hf = 1 \quad , \quad ac + ad - hf + 2bc + 4bd = -1 ,$$

with the restriction that none of the constants a, b, c, d, be zero which is the case for which $\mathbb{L}_0$ satisfies a minimal equation of the form $\mathbb{L}_0^3\{\mathbb{L}_0^2 - 1\} = 0$. In this case no solution to the above set of simultaneous equations exists and hence no charge can be attributed to the equation because no hermitianizing matrix can be found.

Let us relax the restriction that a, b, c, d, be non-zero. In this case the above set of equations accepts the following solution:

i) for the constants entering $\mathbb{L}_0$,

$$a = 0 \ , \ c = 0 \ , \ b = 0 \ , \ d = 0 \ , \ f = \pm 2 \ , \ h = \pm \frac{1}{2} \ ,$$

ii) for the constants entering $\mathbb{A}$,

$$\alpha^{\tau_1\tau_1}=1\ ,\ \alpha^{\tau_2\dot{\tau}_2}=\alpha^{\dot{\tau}_2\tau_2}=-1\ ,\ \alpha^{\tau_3\tau_3}=1\ ,\ \alpha^{\tau_4\tau_4}=1.$$

The resulting wave-equation has matrix $\mathbb{L}_0$ diagonalizable and the charge is not definite.

5: Finally in this paragraph we comment on the causality aspect of the equation of Shamaly and Capri. As they have shown and we verified as well their equation in the presence of the electromagnetic field acquires the form

$$\pi^2 H_\nu - \pi_\nu \pi^\lambda H_\lambda + ie(\alpha c-\hbar f)F_\nu{}^\lambda H_\lambda + m^2 H_\nu = 0\ , \qquad (7)$$

where $\pi_\mu = \partial_\mu - ieA_\mu$, A_μ the four vector electromagnetic potential and $F_\nu{}^\lambda$ the electromagnetic field tensor. Equation (7) has the same principal part (i.e. terms in the highest order derivatives) as the Proca equation and so the propagation is causal.

We remark here that in eq. (22) of the paper of Shamaly and Capri the sign in front of the term $ie(\alpha c-\hbar f)\cdot$ $\cdot F_\nu{}^\lambda H_\lambda$ should be positive, but this does not change their result concerning causality because it is a lower order term with respect to the derivatives.

An intermediate step in the calculation of (7) is the equation

$$\pi^2 H_\nu - \pi_\nu \pi^\lambda H_\lambda + ie(\alpha c-\hbar f)F_\nu{}^\lambda H_\lambda +$$
$$\alpha c\, \pi^\mu \pi_\nu H_\mu - \hbar f \pi^\mu \pi_\nu H_\mu - \alpha c\, \pi^\lambda \pi_\nu H_\lambda + \hbar f \pi^\lambda \pi_\nu H_\lambda +$$
$$\alpha d\, \pi_\nu \pi^\lambda H_\lambda - \alpha d\, \pi_\nu \pi^\lambda H_\lambda + m^2 H_\nu = 0\ ,$$

which shows which terms cancel to give (7). These are the terms which in other cases of higher spin equations may lead to acausal propagation if they do not cancel like it happens here.

6: Summary.

In this chapter it was shown that the 20-dim. Capri-Shamaly equation for the description of spin-1 particles with unique mass is equivalent to the Gel'fand-Yaglom equation based on the same representation.

It was also shown that in the case that the constants $a, b, c, d, h, f,$ are such that the wave-equation has a matrix L_0 satisfying a minimal equation of the form $L_0^3\{L_0^2-1\}=0$, no hermitianizing matrix can be found and hence no charge can be attributed to it.

In the case the constants have the values $a=0$, $b=0$, $d=0$, $f=\pm 2$, $h=\pm\frac{1}{2}$, a hermitianizing matrix can be found but the resulting wave-equation has indefinite charge.

PART III.

CHAPTER 9.

Propagation and Causality.

1. Introduction

In this part we study in detail the problem of propagation of the solutions of wave-equations in the presence of an external electromagnetic field. We shall say that the solutions of a wave-equation propagate causally if the velocity U_p of propagation is not greater than the velocity of light i.e. if $U_p \leqslant c$. Otherwise we shall say that they propagate acausally.

The first time that it was realized that the solutions of wave-equations may not propagate causally in an external field was in 1961 when K. Johnson and E.C.G. Sudarshan [52] working on the quantum level discovered that for the spin 3/2 Rarita-Schwinger field the equal-time commutation relations do not vanish at space-like points, when the equation is minimally coupled to an external electromagnetic field. As is mentioned by the same authors before them Weinberg and Kusaka in an unpublished manuscript claimed to have shown that in the presence of interactions with an external electromagnetic field all higher spin fields ($S>1$) both integral and half-integral acquire non-local (anti-) commutators which do not vanish for space-like separations.

In 1968 Wightman proposed the investigation of stability of relativistic wave-equations and a year later in 1969 Velo and Zwanziger [53] working on the classical level and using the method of characteristics, showed that when minimal electromagnetic coupling is introduced into the Rarita-Schwinger spin - 3/2 field certain modes propagate non-causally even for very weak fields, while in the strong field case the equation ceases to be hyperbolic. In the case of the Proca equation describing spin one particles, they found that when minimally coupled to an external electromagnetic field via a magnetic dipole moment the propagation is causal, but if the vector particle is coupled to an external tensor field the propagation is not causal. With an arbitrary electric quadrupole moment the spin one particle propagates non-causally in an electrostatic field. They also studied minimally coupled spin two particles and found that in this case there is loss of constraints.[54]

Soon after these results were obtained there was a big flood of papers studying various kinds of fields in the presence of various combinations of external fields. It is worth mentioning here a few representative works like that of Shamaly and Capri (1972)[55-56] who studied the spin one theory of Takahashi and Palmer and also the Rarita-Schwinger spin 3/2 theory in interaction with various kinds of external fields. Their study is on the classical level and employs the method of characteristics.

On the quantum level Baisya (1970)[57], employing the theory of Takahashi of the Klein-Gordon devisor,[58] showed that the minimally coupled spin-3/2 Bhabha field propagates causally. Later in (1973) Nagpal[59] extended these calculations by an iteration method to show that the Bhabha equation for any integer or half integer spin in the presence of minimally coupled electromagnetic field is causal. Similar are the results of R.A. Krajick and M.M. Nieto[60] on the propagation of the Bhabha fields.

In (1972) V. Amar and U. Dozzio[61] discovered that a sufficient condition for a Gel'fand-Yaglom equation with unique mass minimally coupled to an external electromagnetic field to propagate causally is that $\mathbb{L}_0$ be diagonalizable. In (1975)[40] they generalized their results to include fields with several masses, some of which may appear with multiplicities higher than one. These conclusions are stated in the following theorem.

<u>Theorem</u>: A sufficient condition for causal propagation of a Gel'fand-Yaglom equation in interaction with an external electromagnetic field is that $\mathbb{L}_0$ satisfies the minimal equation

$$\mathbb{L}_0 \left\{ \prod_{l=1}^{m} \left(\mathbb{L}_0^2 - \lambda_l^2 \right)^{r_l} \right\} = 0,$$

where λ_l the non-zero eigenvalues of $\mathbb{L}_0$ and $r_l = 1, 2 \ldots$ are their multiplicities. Thus the sufficient condition is that the factor $\mathbb{L}_0$ only appears linearly in the minimal equation.

2: Method of characteristics.[62-63]

This is a method by which one proceeds to find solutions to partial differential equations. Since in the rest of this work we shall be using this method, we give here a few concepts concerning first order systems of partial differential equations.

Let us consider the first order quasilinear system of partial differential equations

$$\mathbb{A}_0(u,\vec{x},t)\, u_t + \sum_{l=1}^{n} \mathbb{A}_l(u,\vec{x},t)\, u_{x_l} + \mathbb{B}(u,\vec{x},t) = 0 \quad , \qquad (1)$$

defined in the space $R^n \times t$ and where $u(\vec{x},t)$ is a column vector with m elements $u_1(\vec{x},t)$, $u_2(\vec{x},t)\cdots$, $u_m(\vec{x},t)$, $\vec{x} = (x_1, x_2 \cdots x_n)$ is a vector in R^n, the $\mathbb{A}_l(u,\vec{x},t)$ are $m\times m$ matrices, with elements dependent on $u(\vec{x},t)$ and $\mathbb{B}(u,\vec{x},t)$ a column vector with elements $b_1(u,\vec{x},t)$, $b_2(u,\vec{x},t),\ldots b_m(u,\vec{x},t)$, while the suffixes t and x_l denote partial differentiation with respect to these variables.

Closely connected with the system (1) is the expression

$$Q(\lambda) = \det\left(\sum_{l=1}^{n} \nu_l \mathbb{A}_l - \lambda \mathbb{A}_0\right).$$

This is a homogeneous polynomial of degree m in the quantities $\{-\lambda, \nu_1, \nu_2, \cdots \nu_n\}$ and is known as the characteristic polynomial. In terms of the roots of the above polynomial we have the following definition of hyperbolicity.

Definition (Hyperbolicity):[64]

The first order quasilinear system (1) will be said to be strictly hyperbolic in the t-direction at P if the zeros $\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(m)}$, of the characteristic polynomial $Q(\lambda)$ are all real and distinct for all choices of the unit vector $\vec{\nu}$ and if the right eigenvectors $\vec{r}^{(1)}, \vec{r}^{(2)}, \ldots \vec{r}^{(m)}$, satisfying

$$\sum_{l=1}^{n} \{\nu_l \mathbb{A}_l - \lambda^{(j)} \mathbb{A}_0\} \vec{r}^{(j)} = 0,$$

span the space E^m occupied by the m element eigenvectors; that is to say if they comprise a set of m linearly independent vectors in the space E^m.

The system (1) will be said to be merely hyperbolic in the t-direction if the eigenvectors span the space E^m but the eigenvalues, although all real are not all distinct.

With each eigenvalue $\lambda^{(j)}$ of the characteristic polynomial, a characteristic manifold $\mathcal{J}^{(l)}$ is associated. In connection with the characteristic manifolds the following very important theorem can be proved: characteristic manifolds are carriers of discontinuities of derivatives, or stated differently, discontinuities propagate along characteristic manifolds.

The velocity of propagation of these discontinuities is connected with the roots λ of the characteristic polynomial by the relation

$$\lambda = \vec{\nu} \cdot \frac{d\vec{x}}{dt} \quad ,$$

where $\vec{\nu}$ the unit vector normal to the individual characteristic manifold.

CHAPTER 10.

On the Rarita-Schwinger wave-equation for spin 3/2 particles.

In this chapter we give a detail analysis of the Rarita-Schwinger wave-equation for spin 3/2 particles and then by using the method of characteristics and the criterion of hyperbolicity for quasilinear systems of partial differential equations stated in chapter 9, we investigate its propagation behaviour in an external electromagnetic field. In a later chapter we show that the charge associated with it is definite.

1: The Rarita-Schwinger wave-equation.[14]

The Rarita-Schwinger wave-equation for spin 3/2 particles can be written in several equivalent forms. Among these we choose the form used by Velo and Zwanziger (1969)[53] with their notations and conventions.

Thus let us consider the Rarita-Schwinger Lagrangian density

$$\mathcal{L} = \bar{\psi}(\Gamma \cdot \pi - B)\psi ,$$

where ψ is the Rarita-Schwinger vector-spinor $\psi_\mu{}^\kappa$, μ being the vector index running from 0 to 3 and κ the Dirac index running likewise from 0 to 3 . It is customary to suppress the Dirac index and simply write ψ_μ. $\bar{\psi}$ or $\bar{\psi}^\mu$ is given by $\bar{\psi}^\mu = (\psi^\mu)^\dagger \gamma_0$ and $\pi_\mu = i\frac{\partial}{\partial x^\mu} + eA_\mu$, where A_μ is the four-vector electromagnetic potential. The matrices Γ^μ and B are given by

$$(\Gamma \cdot \pi)_\kappa{}^\lambda = g_\kappa{}^\lambda \gamma\cdot\pi - (\gamma_\kappa \pi^\lambda + \pi_\kappa \gamma^\lambda) + \gamma_\kappa \gamma\cdot\pi \gamma^\lambda ,$$

$$B_\kappa{}^\lambda = m(g_\kappa{}^\lambda - \gamma_\kappa \gamma^\lambda) ,$$

where γ^μ , $\mu = 0,1,2,3$, are the Dirac matrices, satisfying the anticommutation relations $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}$ and $g^{\mu\nu}$ the metric tensor taken to be $(1,-1,-1,-1)$. m is the mass constant. The other conventions are $\hbar = c = 1$.

Variation of the Lagrangian with respect to the sixteen components of ψ and $\bar{\psi}$ independently yields the equations of motion

$$\left(\Gamma \cdot \pi - B \right)_\kappa{}^\lambda \psi_\lambda = 0 ,$$

$$\bar{\psi}^\kappa \left(\Gamma \cdot \pi - B \right)_\kappa{}^\lambda = 0 .$$

We have chosen here the Rarita-Schwinger equation in the presence of the electromagnetic field because this form is of more interest to us in what will follow. The free field equation is obtained from this by setting $A_\mu = 0$.

The field ψ of the Rarita-Schwinger wave-equation for spin 3/2 particles transforms according to the 16-dim. representation of the Lorentz group with components

$$\tau_1 \sim (1/2 , 3/2) , \quad \dot{\tau}_1 \sim (-1/2 , 3/2) , \quad \tau_2 \sim (1/2 , 5/2) , \quad \dot{\tau}_2 \sim (-1/2 , 5/2) .$$

<u>Proof</u>: The field quantity

$$\psi \equiv \psi_\mu{}^\kappa = \{ \psi_0^0 , \psi_0^1 , \psi_0^2 , \psi_0^3 , \psi_1^0 , \psi_1^1 , \psi_1^2 , \psi_1^3 , \psi_2^0 , \psi_2^1 , \psi_2^2 , \psi_2^3 , \psi_3^0 , \psi_3^1 , \psi_3^2 , \psi_3^3 \}^{tr} ,$$

has the transformation properties of a four-vector and a Dirac spinor at the same time. Thus it transforms according to the representation

$$\underbrace{T^{(1,1)}}_{\text{four vector representation}} \otimes \underbrace{\{ T^{(1,0)} \oplus T^{(0,1)} \}}_{\text{Dirac spinor representation}} = \{ T^{(1,1)} \otimes T^{(1,0)} \} \oplus [T^{(1,1)} \otimes T^{(0,1)}] .$$

Each of these terms can be analysed into a direct sum as follows.

i) $T^{(1,1)} \otimes T^{(1,0)} = \sum T^{(k',n')}$,

where k' takes the values $|1-1|, |1+1|$ or $0, 2$ and n' takes the values $|1-0|$, $|1+0|$ or 1.

Thus we have the following table of representations $T^{(k',n')}$ into which the direct product $T^{(1,1)} \otimes T^{(1,0)}$ can be analysed

$n' \diagdown k'$	0	2
1	$T^{(0,1)}$	$T^{(2,1)}$

i.e.

$$T^{(1,1)} \otimes T^{(1,0)} = T^{(0,1)} \oplus T^{(2,1)}.$$

Translating this into the usual notation $\tau = (\ell_0, \ell_1)$ we find for $T^{(0,1)}$ that $\ell_0 = \frac{0-1}{2} = -\frac{1}{2}$, $\ell_1 = \frac{0+1}{2} + 1 = \frac{3}{2}$ and hence $T^{(0,1)}$ corresponds to $\dot{\tau}_1 \sim (-\frac{1}{2}, \frac{3}{2})$. Likewise $T^{(2,1)}$ corresponds to $\tau_2 \sim (\frac{1}{2}, \frac{5}{2})$. Thus we have that

$$T^{(1,1)} \otimes T^{(1,0)} = T^{(0,1)} \oplus T^{(2,1)} \Rightarrow \dot{\tau}_1 \sim (-1/2, 3/2) \oplus \tau_2 \sim (1/2, 5/2).$$

ii) $T^{(1,1)} \otimes T^{(0,1)} = \sum T^{(k',n')}$,

where k' takes the values $|1-0|$ i.e. 1 and n' takes the values $|1-1|$, $|1+1|$ or $0, 2$. Thus we have the following table of representations $T^{(k',n')}$ into which the direct product $T^{(1,1)} \otimes T^{(0,1)}$ can be analysed

$n' \diagdown k'$	1
0	$T^{(1,0)}$
2	$T^{(1,2)}$

i.e.

$$T^{(1,1)} \otimes T^{(0,1)} = T^{(1,0)} \oplus T^{(1,2)}.$$

Tanslating this into the usual notation we find for $T^{(1,0)}$ that $\ell_0 = \frac{1-0}{2} = \frac{1}{2}$, $\ell_1 = \frac{1+0}{2} + 1 = \frac{3}{2}$ and hence $T^{(1,0)}$ corresponds to $\tau_1 \sim (\frac{1}{2}, \frac{3}{2})$. Likewise $T^{(1,2)}$ corresponds to $\dot{\tau}_2 \sim (-\frac{1}{2}, \frac{5}{2})$. Thus

$$T^{(1,1)} \otimes T^{(1,0)} = T^{(1,1)} \oplus T^{(1,2)} \Longrightarrow \tau_1 \sim (1/2, 3/2) \oplus \dot{\tau}_2 \sim (-1/2, 5/2).$$

Combining results i) and ii) we find that the quantity ψ transforms according to the representation with components

$$\tau_1 \sim (1/2, 3/2) \ , \ \dot{\tau}_1 \sim (-1/2, 3/2) , \ \tau_2 \sim (1/2, 5/2) , \ \dot{\tau}_2 \sim (-1/2, 5/2).$$

These components interlock according to the scheme

$$\begin{tikzcd}
(1/2, 3/2) \sim \tau_1 \arrow[r, shift left] \arrow[d, shift left] & \dot{\tau}_1 \sim (-1/2, 3/2) \arrow[l, shift left] \arrow[d, shift left] \\
(1/2, 5/2) \sim \tau_2 \arrow[r, shift left] \arrow[u, shift left] & \dot{\tau}_2 \sim (-1/2, 5/2) \arrow[l, shift left] \arrow[u, shift left]
\end{tikzcd}$$

.

Let us now determine the matrices $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, B$, appearing in the Rarita-Schwinger equation. The Rarita-Schwinger equation after some algebraic manipulations involving giving to the indices κ, λ their values $0, 1, 2, 3$, summing over repeated indices, lowering indices by means of the metric tensor $g_{\mu\nu}$ and employing the properties of the Dirac matrices namely:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} \ , \ (\gamma_0)^2 = 1 \ , \ (\gamma_i)^2 = -1 \ , \ i = 1, 2, 3,$$

can be written as follows:

$$(\Gamma \cdot \pi - B)_\kappa{}^\lambda \psi_\lambda = (\Gamma_0 \pi_0 - \Gamma_1 \pi_1 - \Gamma_2 \pi_2 - \Gamma_3 \pi_3 - m B') \psi =$$

$$
= \Bigg[\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma_1\gamma_0\gamma_2 & -\gamma_1\gamma_0\gamma_3 \\ 0 & -\gamma_2\gamma_0\gamma_1 & 0 & -\gamma_2\gamma_0\gamma_3 \\ 0 & -\gamma_3\gamma_0\gamma_1 & -\gamma_3\gamma_0\gamma_2 & 0 \end{bmatrix} \pi_0 - \begin{bmatrix} 0 & 0 & -\gamma_0\gamma_1\gamma_2 & -\gamma_0\gamma_1\gamma_3 \\ 0 & 0 & 0 & 0 \\ \gamma_2\gamma_1\gamma_0 & 0 & 0 & -\gamma_2\gamma_1\gamma_3 \\ \gamma_3\gamma_1\gamma_0 & 0 & -\gamma_3\gamma_1\gamma_2 & 0 \end{bmatrix} \pi_1 -
$$

$$
- \begin{bmatrix} 0 & -\gamma_0\gamma_2\gamma_1 & 0 & -\gamma_0\gamma_2\gamma_3 \\ \gamma_1\gamma_2\gamma_0 & 0 & 0 & -\gamma_1\gamma_2\gamma_3 \\ 0 & 0 & 0 & 0 \\ \gamma_3\gamma_2\gamma_0 & -\gamma_3\gamma_2\gamma_1 & 0 & 0 \end{bmatrix} \pi_2 - \begin{bmatrix} 0 & -\gamma_0\gamma_3\gamma_1 & -\gamma_0\gamma_3\gamma_2 & 0 \\ \gamma_1\gamma_3\gamma_0 & 0 & -\gamma_1\gamma_3\gamma_2 & 0 \\ \gamma_2\gamma_3\gamma_0 & -\gamma_2\gamma_3\gamma_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \pi_3 -
$$

$$
- m \begin{bmatrix} 0 & \gamma_0\gamma_1 & \gamma_0\gamma_2 & \gamma_0\gamma_3 \\ -\gamma_1\gamma_0 & 0 & \gamma_1\gamma_2 & \gamma_1\gamma_3 \\ -\gamma_2\gamma_0 & \gamma_2\gamma_1 & 0 & \gamma_2\gamma_3 \\ -\gamma_3\gamma_0 & \gamma_3\gamma_1 & \gamma_3\gamma_2 & 0 \end{bmatrix} \Bigg\} \begin{bmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = 0 \quad ,
$$

where each block is a four by four matrix and

$$
\psi_0 = \{ \psi_0^0 , \psi_0^1 , \psi_0^2 , \psi_0^3 \}^{tr} \quad , \quad \psi_1 = \{ \psi_1^0 , \psi_1^1 , \psi_1^2 , \psi_1^3 \}^{tr}
$$

$$
\psi_2 = \{ \psi_2^0 , \psi_2^1 , \psi_2^2 , \psi_2^3 \}^{tr} \quad , \quad \psi_3 = \{ \psi_3^0 , \psi_3^1 , \psi_3^2 , \psi_3^3 \}^{tr} .
$$

Thus from the above form of the equation the following matrices can be separated out:

$$\Gamma_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma_1\gamma_0\gamma_2 & -\gamma_1\gamma_0\gamma_3 \\ 0 & -\gamma_2\gamma_0\gamma_1 & 0 & -\gamma_2\gamma_0\gamma_3 \\ 0 & -\gamma_3\gamma_0\gamma_1 & -\gamma_3\gamma_0\gamma_2 & 0 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0 & 0 & -\gamma_0\gamma_1\gamma_2 & -\gamma_0\gamma_1\gamma_3 \\ 0 & 0 & 0 & 0 \\ \gamma_2\gamma_1\gamma_0 & 0 & 0 & -\gamma_2\gamma_1\gamma_3 \\ \gamma_3\gamma_1\gamma_0 & 0 & -\gamma_3\gamma_1\gamma_2 & 0 \end{bmatrix},$$

$$\Gamma_2 = \begin{bmatrix} 0 & -\gamma_0\gamma_2\gamma_1 & 0 & -\gamma_0\gamma_2\gamma_3 \\ \gamma_1\gamma_2\gamma_0 & 0 & 0 & -\gamma_1\gamma_2\gamma_3 \\ 0 & 0 & 0 & 0 \\ \gamma_3\gamma_2\gamma_0 & -\gamma_3\gamma_2\gamma_1 & 0 & 0 \end{bmatrix}, \quad \Gamma_3 = \begin{bmatrix} 0 & -\gamma_0\gamma_3\gamma_1 & -\gamma_0\gamma_3\gamma_2 & 0 \\ \gamma_1\gamma_3\gamma_0 & 0 & -\gamma_1\gamma_3\gamma_2 & 0 \\ \gamma_2\gamma_3\gamma_0 & -\gamma_2\gamma_3\gamma_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbb{B}' = \begin{bmatrix} 0 & \gamma_0\gamma_1 & \gamma_0\gamma_2 & \gamma_0\gamma_3 \\ -\gamma_1\gamma_0 & 0 & \gamma_1\gamma_2 & \gamma_1\gamma_3 \\ -\gamma_2\gamma_0 & \gamma_2\gamma_1 & 0 & \gamma_2\gamma_3 \\ -\gamma_3\gamma_0 & \gamma_3\gamma_1 & \gamma_3\gamma_2 & 0 \end{bmatrix}.$$

Using the following representation of the Dirac matrices

$$\gamma_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \gamma_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix},$$

$$\gamma_3 = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] ,$$

we find for the matrices Γ_0, Γ_1, Γ_2, Γ_3, $\mathbb{B}'$, the following 16 x 16 matrices:

$$\Gamma_0 = \begin{array}{cccc|cccc|cccc|cccc} \psi_0^0 & \psi_0^1 & \psi_0^2 & \psi_0^3 & \psi_1^0 & \psi_1^1 & \psi_1^2 & \psi_1^3 & \psi_2^0 & \psi_2^1 & \psi_2^2 & \psi_2^3 & \psi_3^0 & \psi_3^1 & \psi_3^2 & \psi_3^3 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} ,$$

$$\Gamma_1 = \left[\begin{array}{cccc|cccc|cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] ,$$

$$\Gamma_2 = \left(\begin{array}{cccc|cccc|cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\
0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\
\hline
0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\
0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\
-i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\
0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & i & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & i & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & i & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right),$$

$$\Gamma_3 = \left(\begin{array}{cccc|cccc|cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -i & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -i & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-i & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right),$$

$$\mathbb{B}' = \left(\begin{array}{cccc|cccc|cccc|cccc}
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & i & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -i & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & -1 \\
\hline
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & -1 & 0 \\
\hline
0 & i & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\
-i & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\
0 & 0 & 0 & -i & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\
0 & 0 & i & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\
\hline
-1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

The matrices $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3$, are singular i.e. they have vanishing determinant. The matrix $\mathbb{B}'$ has $\det \mathbb{B}' \neq 0$ and hence it has an inverse. The inverse of $\mathbb{B}'$ is

$$(\mathbb{B}')^{-1} = \left[\begin{array}{cccc|cccc|cccc|cccc}
\frac{2}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 \\
0 & \frac{2}{3} & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} \\
\hline
0 & -\frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\
-\frac{1}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} \\
0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\
\hline
0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 \\
-\frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} \\
0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & -\frac{1}{3} & 0 \\
\hline
-\frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\
0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{2}{3}
\end{array}\right].$$

This was found by applying the following method.[36] Let $\mathbb{M}$ be an $n \times n$ invertible matrix. To find its inverse we "augment" $\mathbb{M}$ with the $n \times n$ identity matrix $\mathbb{I}$ on the left to obtain the $n \times 2n$ matrix $\{\mathbb{I} \mid \mathbb{M}\}$ and then apply to it those row operations that transform $\mathbb{M}$ to $\mathbb{I}$. At the same time the augmented $\mathbb{I}$ is transformed into the inverse matrix $\mathbb{M}^{-1}$. To illustrate the method we compute the inverse of the invertible matrix

$$\mathbb{M} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 2 \\ -1 & -3 & -2 \end{bmatrix}.$$

Augmenting $\mathbb{M}$ with the unit matrix $\mathbb{I}$ and applying row operations we have:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & -3 & 2 \\ 0 & 0 & 1 & -1 & -3 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ -2 & 1 & 0 & 0 & -5 & 0 \\ 1 & 0 & 1 & 0 & -2 & -1 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ -5 & 1 & -3 & 0 & 1 & 3 \\ 1 & 0 & 1 & 0 & -2 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 6 & -1 & 3 & 1 & 0 & -2 \\ -5 & 1 & -3 & 0 & 1 & 3 \\ -9 & 2 & -5 & 0 & 0 & 5 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{ccc|ccc} 6 & -1 & 3 & 1 & 0 & -2 \\ -5 & 1 & -3 & 0 & 1 & 3 \\ \frac{-9}{5} & \frac{2}{5} & -1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} \frac{12}{5} & -\frac{1}{5} & 1 & 1 & 0 & 0 \\ \frac{2}{5} & -\frac{1}{5} & 0 & 0 & 1 & 0 \\ -\frac{9}{5} & \frac{2}{5} & -1 & 0 & 0 & 1 \end{array}\right].$$

Hence

$$\mathbb{M}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 2 \\ -1 & -3 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{12}{5} & -\frac{1}{5} & 1 \\ \frac{2}{5} & -\frac{1}{5} & 0 \\ -\frac{9}{5} & \frac{2}{5} & -1 \end{bmatrix},$$

which can be checked by forming $\mathbb{M}\,\mathbb{M}^{-1} = \mathbb{I}$.

Let us consider the Rarita-Schwinger equation

$$(\Gamma_0 \pi_0 - \Gamma_1 \pi_1 - \Gamma_2 \pi_2 - \Gamma_3 \pi_3 - m\mathbb{B}')\psi = 0 .$$

Multiplying through by the inverse $(\mathbb{B}')^{-1}$ we have

$$[(\mathbb{B}')^{-1}\Gamma_0 \pi_0 - (\mathbb{B}')^{-1}\Gamma_1 \pi_1 - (\mathbb{B}')^{-1}\Gamma_2 \pi_2 - (\mathbb{B}')^{-1}\Gamma_3 \pi_3 - m\,]\psi = 0.$$

Identifying

$$(\mathbb{B}')^{-1}\Gamma_0 \equiv \mathbb{L}_0 \;, (\mathbb{B}')^{-1}\Gamma_1 \equiv \mathbb{L}_1 \;, (\mathbb{B}')^{-1}\Gamma_2 \equiv \mathbb{L}_2 \;, \; (\mathbb{B}')^{-1}\Gamma_3 \equiv \mathbb{L}_3 ,$$

we have

$$\{ \mathbb{L}_0 \pi_0 - \mathbb{L}_1 \pi_1 - \mathbb{L}_2 \pi_2 - \mathbb{L}_3 \pi_3 - m \}\psi = 0 \;,$$

where $\mathbb{L}_0$, $\mathbb{L}_1$, $\mathbb{L}_2$, $\mathbb{L}_3$ are the following 16 x 16 matrices,

$$\mathbb{L}_0=\left[\begin{array}{cccc|cccc|cccc|cccc}
0&0&0&0&0&0&0&\frac{2}{3}&0&0&0&-\frac{2i}{3}&0&0&\frac{2}{3}&0\\
0&0&0&0&0&0&\frac{2}{3}&0&0&0&\frac{2i}{3}&0&0&0&0&-\frac{2}{3}\\
0&0&0&0&0&-\frac{2}{3}&0&0&0&\frac{2i}{3}&0&0&-\frac{2}{3}&0&0&0\\
0&0&0&0&-\frac{2}{3}&0&0&0&-\frac{2i}{3}&0&0&0&0&\frac{2}{3}&0&0\\
\hline
0&0&0&0&0&0&\frac{2}{3}&0&0&0&-\frac{i}{3}&0&0&0&0&\frac{1}{3}\\
0&0&0&0&0&0&0&\frac{2}{3}&0&0&0&\frac{i}{3}&0&0&-\frac{1}{3}&0\\
0&0&0&0&\frac{2}{3}&0&0&0&-\frac{i}{3}&0&0&0&0&\frac{1}{3}&0&0\\
0&0&0&0&0&\frac{2}{3}&0&0&0&\frac{i}{3}&0&0&-\frac{1}{3}&0&0&0\\
\hline
0&0&0&0&0&0&\frac{i}{3}&0&0&0&\frac{2}{3}&0&0&0&0&-\frac{i}{3}\\
0&0&0&0&0&0&0&-\frac{i}{3}&0&0&0&\frac{2}{3}&0&0&-\frac{i}{3}&0\\
0&0&0&0&\frac{i}{3}&0&0&0&\frac{2}{3}&0&0&0&0&-\frac{i}{3}&0&0\\
0&0&0&0&0&-\frac{i}{3}&0&0&0&\frac{2}{3}&0&0&-\frac{i}{3}&0&0&0\\
\hline
0&0&0&0&0&0&0&-\frac{1}{3}&0&0&0&\frac{i}{3}&0&0&\frac{2}{3}&0\\
0&0&0&0&0&0&\frac{1}{3}&0&0&0&\frac{i}{3}&0&0&0&0&\frac{2}{3}\\
0&0&0&0&0&-\frac{1}{3}&0&0&0&\frac{i}{3}&0&0&\frac{2}{3}&0&0&0\\
0&0&0&0&\frac{1}{3}&0&0&0&\frac{i}{3}&0&0&0&0&\frac{2}{3}&0&0
\end{array}\right],$$

$$\mathbb{L}_1=\left(\begin{array}{cccc|cccc|cccc|cccc}
0&0&0&\frac{2}{3}&0&0&0&0&0&0&\frac{i}{3}&0&0&0&0&-\frac{1}{3}\\
0&0&\frac{2}{3}&0&0&0&0&0&0&0&0&-\frac{i}{3}&0&0&\frac{1}{3}&0\\
0&-\frac{2}{3}&0&0&0&0&0&0&\frac{i}{3}&0&0&0&0&-\frac{1}{3}&0&0\\
-\frac{2}{3}&0&0&0&0&0&0&0&0&-\frac{i}{3}&0&0&\frac{1}{3}&0&0&0\\
\hline
0&0&\frac{2}{3}&0&0&0&0&0&0&0&0&\frac{2i}{3}&0&0&-\frac{2}{3}&0\\
0&0&0&\frac{2}{3}&0&0&0&0&0&0&-\frac{2i}{3}&0&0&0&0&\frac{2}{3}\\
\frac{2}{3}&0&0&0&0&0&0&0&0&-\frac{2i}{3}&0&0&\frac{2}{3}&0&0&0\\
0&\frac{2}{3}&0&0&0&0&0&0&\frac{2i}{3}&0&0&0&0&-\frac{2}{3}&0&0\\
\hline
0&0&\frac{i}{3}&0&0&0&0&0&0&0&0&\frac{2}{3}&0&0&-\frac{i}{3}&0\\
0&0&0&-\frac{i}{3}&0&0&0&0&0&0&\frac{2}{3}&0&0&0&0&-\frac{i}{3}\\
\frac{i}{3}&0&0&0&0&0&0&0&0&-\frac{2}{3}&0&0&\frac{i}{3}&0&0&0\\
0&-\frac{i}{3}&0&0&0&0&0&0&-\frac{2}{3}&0&0&0&0&\frac{i}{3}&0&0\\
\hline
0&0&0&-\frac{1}{3}&0&0&0&0&0&0&\frac{i}{3}&0&0&0&0&\frac{2}{3}\\
0&0&\frac{1}{3}&0&0&0&0&0&0&0&0&\frac{i}{3}&0&0&\frac{2}{3}&0\\
0&-\frac{1}{3}&0&0&0&0&0&0&-\frac{i}{3}&0&0&0&0&-\frac{2}{3}&0&0\\
\frac{1}{3}&0&0&0&0&0&0&0&0&-\frac{i}{3}&0&0&-\frac{2}{3}&0&0&0
\end{array}\right),$$

$$\mathbb{L}_2=\left(\begin{array}{cccc|cccc|cccc|cccc}
0 & 0 & 0 & -\frac{2i}{3} & 0 & 0 & -\frac{i}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{3} \\
0 & 0 & \frac{2i}{3} & 0 & 0 & 0 & 0 & \frac{i}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{3} & 0 \\
0 & \frac{2i}{3} & 0 & 0 & -\frac{i}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{3} & 0 & 0 \\
-\frac{2i}{3} & 0 & 0 & 0 & 0 & \frac{i}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{3} & 0 & 0 & 0 \\
\hline
0 & 0 & -\frac{i}{3} & 0 & 0 & 0 & 0 & -\frac{2i}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{3} & 0 \\
0 & 0 & 0 & \frac{i}{3} & 0 & 0 & \frac{2i}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{3} \\
\frac{-i}{3} & 0 & 0 & 0 & 0 & \frac{2i}{3} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{3} & 0 & 0 & 0 \\
0 & \frac{i}{3} & 0 & 0 & -\frac{2i}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{3} & 0 & 0 \\
\hline
0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{3} & 0 \\
0 & 0 & 0 & \frac{2}{3} & 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \\
\frac{2}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 \\
0 & \frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{3} & 0 & 0 \\
\hline
0 & 0 & 0 & \frac{i}{3} & 0 & 0 & -\frac{i}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2i}{3} \\
0 & 0 & \frac{i}{3} & 0 & 0 & 0 & 0 & -\frac{i}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2i}{3} & 0 \\
0 & \frac{i}{3} & 0 & 0 & \frac{i}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2i}{3} & 0 & 0 \\
\frac{i}{3} & 0 & 0 & 0 & 0 & \frac{i}{3} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2i}{3} & 0 & 0 & 0
\end{array}\right),$$

$$\mathbb{L}_3=\left(\begin{array}{cccc|cccc|cccc|cccc}
0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & -\frac{i}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{2}{3} & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{i}{3} & 0 & 0 & 0 & 0 & 0 \\
-\frac{2}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & -\frac{i}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2}{3} & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{i}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & -\frac{i}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & -\frac{i}{3} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & \frac{i}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{i}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -\frac{i}{3} & 0 & 0 & \frac{i}{3} & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{i}{3} & 0 & 0 & 0 & 0 & \frac{i}{3} & 0 & 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & 0 \\
0 & -\frac{i}{3} & 0 & 0 & -\frac{i}{3} & 0 & 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{i}{3} & 0 & 0 & 0 & 0 & -\frac{i}{3} & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & \frac{2i}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2}{3} & 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & -\frac{2i}{3} & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & -\frac{2i}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{2i}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

Thus by multiplying through the original form of the Rarita-Schwinger equation with $(\mathbb{B}')^{-1}$ we get a new form in which the mass matrix is a multiple of the unit matrix. From this form we see immediately that $\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$, are singular. $\mathbb{L}_0$ satisfies the minimal equation

$$\mathbb{L}_0^2 \{ \mathbb{L}_0^2 - 1 \} = 0 .$$

2: Generators of the Rarita-Schwinger equation.

We give now in this paragraph the generators of the Rarita-Schwinger equation in the vector-spinor basis $\{\psi_\mu^\kappa\}$, $\mu = 0,1,2,3$, $\kappa = 0,1,2,3$, used to express the matrices $\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$, given in the previous paragraph. For the generators of rotations we use the symbols $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$, to represent rotations about the axes $0x_1, 0x_2, 0x_3$, and for the generators of boosts we use the symbols $\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3$, to represent boosts along the axis $0x_1, 0x_2, 0x_3$, respectively. For convenience we introduce instead of the generators $\mathbb{A}_\iota$, $\mathbb{B}_\iota$, $\iota = 1,2,3$, the generators $\mathbb{H}_+, \mathbb{H}_-, \mathbb{H}_3, \mathbb{F}_+, \mathbb{F}_-, \mathbb{F}_3$, defined by the following combinations:

$$\mathbb{H}_+ = \iota\mathbb{A}_1 - \mathbb{A}_2 , \quad \mathbb{H}_- = \iota\mathbb{A}_1 + \mathbb{A}_2 , \quad \mathbb{H}_3 = \iota\mathbb{A}_3 ,$$

$$\mathbb{F}_+ = \iota\mathbb{B}_1 - \mathbb{B}_2 , \quad \mathbb{F}_- = \iota\mathbb{B}_1 + \mathbb{B}_2 , \quad \mathbb{F}_3 = \iota\mathbb{B}_3 .$$

The generators then $\mathbb{H}_+, \mathbb{H}_-, \mathbb{H}_3, \mathbb{F}_+, \mathbb{F}_-, \mathbb{F}_3$, of the Rarita-Schwinger equation in the vector spinor basis are:

$$\mathbb{H}_+ = \left(\begin{array}{cccc|cccc|cccc|cccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\iota & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\iota & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\iota & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\iota \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \iota & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \iota & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \iota & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \iota & 0 & 0 & 0 & 0
\end{array}\right) ,$$

$$\mathbb{H}_{-}=\left[\begin{array}{cccc|cccc|cccc|cccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&-i&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&-i&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-i&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&-i\\
\hline
0&0&0&0&-1&0&0&0&i&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&0&0&0&i&0&0&1&0&0&0\\
0&0&0&0&0&0&-1&0&0&0&i&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&i&0&0&1&0
\end{array}\right],$$

$$\mathbb{H}_{3}=\left[\begin{array}{cccc|cccc|cccc|cccc}
\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&\frac{1}{2}&0&0&0&-i&0&0&0&0&0&0&0\\
0&0&0&0&0&-\frac{1}{2}&0&0&0&-i&0&0&0&0&0&0\\
0&0&0&0&0&0&\frac{1}{2}&0&0&0&-i&0&0&0&0&0\\
0&0&0&0&0&0&0&-\frac{1}{2}&0&0&0&-i&0&0&0&0\\
\hline
0&0&0&0&i&0&0&0&\frac{1}{2}&0&0&0&0&0&0&0\\
0&0&0&0&0&i&0&0&0&-\frac{1}{2}&0&0&0&0&0&0\\
0&0&0&0&0&0&i&0&0&0&\frac{1}{2}&0&0&0&0&0\\
0&0&0&0&0&0&0&i&0&0&0&-\frac{1}{2}&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-\frac{1}{2}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-\frac{1}{2}
\end{array}\right],$$

$$\mathbb{F}_{+}=\left[\begin{array}{cccc|cccc|cccc|cccc}
0&-i&0&0&i&0&0&0&-1&0&0&0&0&0&0&0\\
0&0&0&0&0&i&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&i&0&0&i&0&0&0&-1&0&0&0&0&0\\
0&0&0&0&0&0&0&i&0&0&0&-1&0&0&0&0\\
\hline
i&0&0&0&0&-i&0&0&0&0&0&0&0&0&0&0\\
0&i&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&i&0&0&0&0&i&0&0&0&0&0&0&0&0\\
0&0&0&i&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
-1&0&0&0&0&0&0&0&0&-i&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&0&0&0&0&i&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&-i&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&i\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right],$$

$$\mathbb{F}_{-} = \left[\begin{array}{cccc|cccc|cccc|cccc}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right],$$

$$\mathbb{F}_{3} = \left[\begin{array}{cccc|cccc|cccc|cccc}
-\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}
\end{array}\right].$$

These generators satisfy the following commutation relations:

$$[H_+ , H_3]_- = -H_+ , \; [H_-, H_3]_- = H_- , \; [H_+ , H_-]_- = 2H_3 ,$$

$$[H_+ , F_+]_- = 0 , \; [H_-, F_-]_- = 0 , \; [H_3 , F_3]_- = 0 ,$$

$$[F_+ , F_3]_- = H_+ , \; [F_- , F_3]_- = -H_- , \; [F_+ , F_-]_- = -2H_3 ,$$

$$[H_+ , F_3]_- = -F_+ , \; [H_- , F_3]_- = F_- , \; [H_+ , F_-]_- = 2F_3 ,$$

$$[H_- , F_+]_- = -2F_3 , \; [F_+ , H_3]_- = -F_+ , \; [F_- , H_3] = F_- .$$

The generators of the Rarita-Schwinger equation given above were constructed from the generators of the four vector representation,

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} , \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} , \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} ,$$

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \quad B_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \quad B_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} ,$$

or the equivalent ones,

$$H_+ = iA_1 - A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -i \\ 0 & 1 & i & 0 \end{bmatrix} , \; H_- = iA_1 + A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i \\ 0 & -1 & i & 0 \end{bmatrix} , \; H_3 = iA_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} ,$$

$$\mathbb{F}_+ = \iota\mathbb{B}_1 - \mathbb{B}_2 = \begin{bmatrix} 0 & \iota & -1 & 0 \\ \iota & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbb{F}_- = \iota\mathbb{B}_1 + \mathbb{B}_2 = \begin{bmatrix} 0 & \iota & 1 & 0 \\ \iota & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbb{F}_3 = \iota\mathbb{B}_3 = \begin{bmatrix} 0 & 0 & 0 & \iota \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \iota & 0 & 0 & 0 \end{bmatrix},$$

and the generators of the Dirac spinor representation

$$\mathbb{H}_+ = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbb{H}_- = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbb{H}_3 = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix},$$

$$\mathbb{F}_+ = \begin{bmatrix} 0 & -\iota & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \iota \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbb{F}_- = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\iota & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \iota & 0 \end{bmatrix}, \quad \mathbb{F}_3 = \begin{bmatrix} -\frac{\iota}{2} & 0 & 0 & 0 \\ 0 & \frac{\iota}{2} & 0 & 0 \\ 0 & 0 & \frac{\iota}{2} & 0 \\ 0 & 0 & 0 & -\frac{\iota}{2} \end{bmatrix},$$

by constructing the following combinations of direct products

$$\mathbb{H}_+^{R.S.} = \mathbb{I}^V \otimes \mathbb{H}_+^S + \mathbb{H}_+^V \otimes \mathbb{I}^S \quad , \quad \mathbb{H}_-^{R.S.} = \mathbb{I}^V \otimes \mathbb{H}_-^S + \mathbb{H}_-^V \otimes \mathbb{I}^S ,$$

$$\mathbb{H}_3^{R.S.} = \mathbb{I}^V \otimes \mathbb{H}_3^S + \mathbb{H}_3^V \otimes \mathbb{I}^S , \quad \mathbb{F}_+^{R.S.} = \mathbb{I}^V \otimes \mathbb{F}_+^S + \mathbb{F}_+^V \otimes \mathbb{I}^S ,$$

$$\mathbb{F}_-^{R.S.} = \mathbb{I}^V \otimes \mathbb{F}_-^S + \mathbb{F}_-^V \otimes \mathbb{I}^S \quad , \quad \mathbb{F}_3^{R.S.} = \mathbb{I}^V \otimes \mathbb{F}_3^S + \mathbb{F}_3^V \otimes \mathbb{I}^S ,$$

where R.S. stands for Rarita-Schwinger, V for four-vector, S for Dirac spinor and $\mathbb{I}$ for the 4 x 4 unit matrix.

The generators $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3$, like the generators $\mathbb{H}_+, \mathbb{H}_-, \mathbb{H}_3, \mathbb{F}_+, \mathbb{F}_-, \mathbb{F}_3$, satisfy the following commutation relations:

$$[\mathbb{A}_1, \mathbb{A}_2]_- = \mathbb{A}_3 \quad , \quad [\mathbb{A}_2, \mathbb{A}_3]_- = \mathbb{A}_1 \quad , \quad [\mathbb{A}_3, \mathbb{A}_1]_- = \mathbb{A}_2 ,$$

$$[\mathbb{B}_1, \mathbb{B}_2]_- = -\mathbb{A}_3, \quad [\mathbb{B}_2, \mathbb{B}_3]_- = -\mathbb{A}_1, \quad [\mathbb{B}_3, \mathbb{B}_1]_- = -\mathbb{A}_2,$$

$$[\mathbb{A}_1, \mathbb{B}_1]_- = 0, \quad [\mathbb{A}_2, \mathbb{B}_2]_- = 0, \quad [\mathbb{A}_3, \mathbb{B}_3]_- = 0,$$

$$[\mathbb{A}_1, \mathbb{B}_2]_- = \mathbb{B}_3, [\mathbb{A}_1, \mathbb{B}_3]_- = -\mathbb{B}_2, [\mathbb{A}_2, \mathbb{B}_3]_- = \mathbb{B}_1,$$

$$[\mathbb{A}_2, \mathbb{B}_1]_- = -\mathbb{B}_3, [\mathbb{A}_3, \mathbb{B}_1]_- = \mathbb{B}_2, \quad [\mathbb{A}_3, \mathbb{B}_2]_- = -\mathbb{B}_1 .$$

The matrix $\mathbb{L}_0$ (of the Rarita-Schwinger equation) and the generators $\mathbb{H}_+, \mathbb{H}_-, \mathbb{H}_3, \mathbb{F}_3$, satisfy the following commutation relations

$$[\mathbb{L}_0, \mathbb{H}_+]_- = 0, \quad [\mathbb{L}_0, \mathbb{H}_-]_- = 0, \quad [\mathbb{L}_0, \mathbb{H}_3]_- = 0,$$

$$[[\mathbb{F}_3, \mathbb{L}_0]_-, \mathbb{F}_3]_- = \mathbb{L}_0 .$$

Also the matrices $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$ (of the Rarita-Schwinger equation) with the generators of boosts $\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3$ (of the Rarita-Schwinger equation) together with the matrix $\mathbb{L}_0$ (of the Rarita-Schwinger equation) satisfy the relations

$$\mathbb{L}_1 = -[\mathbb{B}_1, \mathbb{L}_0]_-, \quad \mathbb{L}_2 = -[\mathbb{B}_2, \mathbb{L}_0]_-, \quad \mathbb{L}_3 = -[\mathbb{B}_3, \mathbb{L}_0]_-,$$

where the generators of boosts are:

$$\mathbb{B}_1=\frac{\mathbb{F}_+ + \mathbb{F}_-}{2i}=\left[\begin{array}{cccc|cccc|cccc|cccc}
0 & -1/2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1/2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1/2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
1 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0
\end{array}\right],$$

$$\mathbb{B}_2=\frac{-\mathbb{F}_+ + \mathbb{F}_-}{2}=\left[\begin{array}{cccc|cccc|cccc|cccc}
0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0
\end{array}\right],$$

$$\mathbb{B}_3 = \frac{\mathbb{F}_3}{l} = \left[\begin{array}{cccc|cccc|cccc|cccc}
-1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2
\end{array}\right].$$

3: Subsidiary conditions of the Rarita-Schwinger wave-equation. Modified Rarita-Schwinger equation.

Let us consider the Rarita-Schwinger equation in the presence of the electromagnetic field

$$\left(\Gamma \cdot \pi - \mathbb{B}\right)_{\kappa}{}^{\lambda} \psi_{\lambda} = 0. \qquad (1)$$

Because ψ has more components than needed to describe a spin 3/2 particle (only eight components are necessary for a spin 3/2 particle), some of the 16-equations are constraint equations, that is, equations which do not involve the time derivatives. (These assertions we clarify later in chapter (11)).

Setting $\kappa = 0$ in (1) we get the primary constraint equations

$$\left(\vec{\pi} - \vec{h}\vec{\alpha} \right) \cdot \vec{\psi} = 0 , \qquad (2)$$

where $\vec{\pi} = (\pi^{i})$, $\bar{\psi} = (\psi^{i})$, $i = 1, 2, 3$, and

$$\vec{h} = \vec{\alpha} \cdot \vec{\pi} + \beta m ,$$

with $\vec{\alpha} = (\gamma^{0}\gamma^{i})$ for $i = 1, 2, 3$, $\beta = \gamma^{0}$.

The time derivatives of the four field components $\psi^{0} \equiv \{ \psi^{0}_{0} , \psi^{0}_{1} , \psi^{0}_{2}, \psi^{0}_{3} \}$ do not appear in the differential equations (1) and they cannot be determined by the primary constraints (2) either. To be able to determine the time evolution of the four components ψ^{0} we multiply equation (1) successively by γ^{κ} and π^{κ} which yields respectively,

$$2 \left(\gamma \cdot \pi \gamma - \pi \right) \psi + 3 m \gamma \cdot \psi = 0 , \qquad (3)$$

and

$$m \left(\gamma \cdot \pi \gamma - \pi \right) \cdot \psi - i e \gamma^{5} \gamma \cdot F^{d} \psi = 0 , \qquad (4)$$

where $F = (F)_{\mu}{}^{\nu} = \partial_{\mu} A^{\nu} - \partial^{\nu} A_{\mu}$, and

$$F^{d} = (F^{d})_{\mu}{}^{\nu} = \frac{1}{2} \epsilon_{\mu}{}^{\nu\kappa}{}_{\lambda} F_{\kappa}{}^{\lambda} =$$

$$\begin{bmatrix} 0 & \frac{F^{2,3}-F^{3,2}}{2} & \frac{F^{1,3}-F^{3,1}}{2} & \frac{F^{1,2}-F^{2,1}}{2} \\ \frac{F^{2,3}-F^{3,2}}{2} & 0 & \frac{F^{0,3}-F^{3,0}}{2} & \frac{F^{2,0}-F^{0,2}}{2} \\ \frac{F^{3,1}-F^{1,3}}{2} & \frac{F^{3,0}-F^{0,3}}{2} & 0 & \frac{F^{0,1}-F^{1,0}}{2} \\ \frac{F^{1,2}-F^{2,1}}{2} & \frac{F^{0,2}-F^{2,0}}{2} & \frac{F^{1,0}-F^{0,1}}{2} & 0 \end{bmatrix},$$

$\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$, and $\epsilon_\mu{}^{\nu\kappa}{}_\lambda$ the permutation symbol defined as follows:

$\epsilon^{\iota jk\ell} = 1$, for an even permutation of ι, j, k, ℓ,

$\epsilon^{\iota jk\ell} = -1$, for an odd permutation of ι, j, k, ℓ,

$\epsilon^{\iota jk\ell} = 0$, if two or more indices are the same.

For instance $\epsilon^{0123} = 1$, $\epsilon^{0132} = -1$. $F^{\mu\nu}$ is the electromagnetic field tensor.

The derivation of equation (3) involves taking into account the properties of the Dirac matrices and the operations of raising and lowering tensor indices. The derivation of equation (4) involves taking into account the commutation relations of the Dirac matrices, the operations of raising and lowering tensor indices and the commutation relations among the components of the electromagnetic four-momentum,

$$[\pi^\mu, \pi^\nu]_- = \iota e F^{\mu\nu}.$$

From (3) and (4) by comparison we get the secondary constraints

$$\gamma\cdot\psi = -\frac{2}{3} m^{-2} \iota e \gamma^5 \gamma\cdot F^d \cdot \psi \qquad (5)$$

which determine ψ^0 . Inserting (5) into (4) we get also the relation

$$\pi\cdot\psi = -(\gamma\cdot\pi + \frac{3}{2} m)\frac{2}{3} \iota e m^{-2} \gamma^5 \gamma\cdot F^d \cdot \psi . \qquad (6)$$

By means of (5) and (6) one can convert the original Rarita-Schwinger equation into a new equation involving the time derivatives of all the field components. Thus by substituting

(5) and (6) into (1) we get as new equation of motion, the equation

$$(\gamma\cdot\pi - m)\psi_\mu + (\pi_\mu + \tfrac{1}{2}m\gamma_\mu)\tfrac{2}{3}\iota e m^{-2}\gamma^5\gamma\cdot F^d\cdot\psi = 0 , \qquad (7)$$

which together with the primary and secondary constraints determines completely the time evolution of all the field components. We shall call equation (7) the "modified" Rarita-Schwinger equation in the presence of an external electromagnetic field.

The need of converting the original wave-equation into a new equation of motion using the subsidiary conditions of the second kind, arises from the fact that the existing criteria of hyperbolicity for quasilinear systems of partial differential equations do not apply to systems with singular matrices. Thus by converting the original wave-equation into a new equation of motion with non-singular matrices we make it possible for the existing criteria to apply.

In the following paragraphs we consider the propagation behaviour of the Rarita-Schwinger equation in an external electromagnetic field. For this purpose the matrices $\mathcal{T}_0 , \mathcal{T}_1 , \mathcal{T}_2 , \mathcal{T}_3$ of the 'modified' equation are necessary.

4: The matrices $\mathcal{T}_0 , \mathcal{T}_1 , \mathcal{T}_2 , \mathcal{T}_3$ of the modified equation.

We give now the matrices $\mathcal{T}_0 , \mathcal{T}_1 , \mathcal{T}_2 , \mathcal{T}_3$ of the modified equation (7) corresponding to the Rarita-Schwinger equation in an external electromagnetic field. Equation (7) which corresponds to equation (2.12) of Velo and Zwanziger can be rearranged as follows,

$$(\gamma\cdot\pi)\psi_\mu - m\psi_\mu + \pi_\mu\frac{2}{3}\frac{\iota e}{m^2}\gamma^5\gamma\cdot F^d\cdot\psi + \frac{1}{2}m\gamma_\mu\frac{2}{3}\frac{\iota e}{m^2}\gamma^5\gamma\cdot F^d\cdot\psi = 0 .$$

The principle part of this i.e. the part involving the first order derivatives

$$(\gamma\cdot\pi)\psi_\mu + \pi_\mu\omega\gamma^5\gamma\cdot F^d\cdot\psi ,$$

after some manipulations gives the following four matrices in block form, each block being a 4 x 4 matrix:

$$\mathcal{T}_0 = \left[\begin{array}{c|c|c|c} i\gamma_0 - i\omega\gamma_5\gamma^\mu F_\mu^{d_0} & -i\omega\gamma_5\gamma^\mu F_\mu^{d_1} & -i\omega\gamma_5\gamma^\mu F_\mu^{d_2} & -i\omega\gamma_5\gamma^\mu F_\mu^{d_3} \\ \hline 0 & i\gamma_0 & 0 & 0 \\ \hline 0 & 0 & i\gamma_0 & 0 \\ \hline 0 & 0 & 0 & i\gamma_0 \end{array}\right],$$

$$\mathcal{T}_1 = \left[\begin{array}{c|c|c|c} i\gamma_1 & 0 & 0 & 0 \\ \hline i\omega\gamma_5\gamma^\mu F_\mu^{d_0} & i\gamma_1 + i\omega\gamma_5\gamma^\mu F_\mu^{d_1} & i\omega\gamma_5\gamma^\mu F_\mu^{d_2} & i\omega\gamma_5\gamma^\mu F_\mu^{d_3} \\ \hline 0 & 0 & i\gamma_1 & 0 \\ \hline 0 & 0 & 0 & i\gamma_1 \end{array}\right],$$

$$\mathcal{T}_2 = \left[\begin{array}{c|c|c|c} i\gamma_2 & 0 & 0 & 0 \\ \hline 0 & i\gamma_2 & 0 & 0 \\ \hline i\omega\gamma_5\gamma^\mu F_\mu^{d_0} & i\omega\gamma_5\gamma^\mu F_\mu^{d_1} & i\gamma_2 + i\omega\gamma_5\gamma^\mu F_\mu^{d_2} & i\omega\gamma_5\gamma^\mu F_\mu^{d_3} \\ \hline 0 & 0 & 0 & i\gamma_2 \end{array}\right],$$

$$\mathcal{T}_3 = \left[\begin{array}{c|c|c|c} i\gamma_3 & 0 & 0 & 0 \\ \hline 0 & i\gamma_3 & 0 & 0 \\ \hline 0 & 0 & i\gamma_3 & 0 \\ \hline i\omega\gamma_5\gamma^\mu F_\mu^{d_0} & i\omega\gamma_5\gamma^\mu F_\mu^{d_1} & i\omega\gamma_5\gamma^\mu F_\mu^{d_2} & i\gamma_3 + i\omega\gamma_5\gamma^\mu F_\mu^{d_3} \end{array}\right],$$

where $\omega = \frac{2ie}{3m^2}$, and the basis with respect to which these matrices are considered is

$$\{\psi_0^0, \psi_0^1, \psi_0^2, \psi_0^3, \psi_1^0, \psi_1^1, \psi_1^2, \psi_1^3, \psi_2^0, \psi_2^1, \psi_2^2, \psi_2^3, \psi_3^0, \psi_3^1, \psi_3^2, \psi_3^3\}.$$

5: Study of propagation according to Velo and Zwanziger.

Let us now study the propagation of the modified equation. We do this first following the method of Velo and Zwanziger which consists in finding the determinant of the matrix $\mathcal{T}_0$ after multiplying it through by the time normal $n \equiv (n_0, 0, 0, 0)$ and subsequently writing the result in a covariant form, taking thus into account the fact that the normal (n_0, n_1, n_2, n_3) on the characteristic surfaces may have any other direction different than that of the time normal. Thus considering $\mathcal{T}_0$ given in the previous paragraph, multiplying it through by the time normal $n = (n_0, 0, 0, 0)$ and evaluating the determinant $D(n) \equiv \det(\mathcal{T}_0 n)$ we find

$$D(n) \equiv \det(\mathcal{T}_0 n) = n^{16} \det \mathcal{T}_0 =$$

$$= n^{16} \det \left| \begin{array}{c|c|c|c} i\gamma_0 - i\omega\gamma_5\gamma^\mu \cdot F_\mu^{d_0} & -i\omega\gamma_5\gamma^\mu \cdot F_\mu^{d_1} & -i\omega\gamma_5\gamma^\mu \cdot F_\mu^{d_2} & -i\omega\gamma_5\gamma^\mu \cdot F_\mu^{d_3} \\ \hline 0 & i\gamma_0 & 0 & 0 \\ \hline 0 & 0 & i\gamma_0 & 0 \\ \hline 0 & 0 & 0 & i\gamma_0 \end{array} \right|$$

$$= n^{16}\det\left(\gamma_0 - \omega\gamma_5\gamma^{\mu}\cdot F_{\mu}^{\ d_0}\right)\det\gamma_0\,\det\gamma_0\,\det\gamma_0 =$$

$$= \det\left|\begin{array}{c|c|c|c} 0 & 0 & 1+\omega\iota F^{1,2} & \omega\iota F^{2,3} - \omega F^{1,3} \\ \hline 0 & 0 & \omega\iota F^{2,3} + \omega F^{1,3} & 1 - \omega\iota F^{1,2} \\ \hline 1+\omega\iota F^{1,2} & \omega\iota F^{2,3} - \omega F^{1,3} & 0 & 0 \\ \hline \omega\iota F^{2,3} + \omega F^{1,3} & 1 - \omega\iota F^{1,2} & 0 & 0 \end{array}\right| \cdot$$

$$\cdot\det\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\cdot\det\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\cdot\det\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} =$$

$$= \det\begin{bmatrix} 1+\omega\iota F^{1,2} & , \ \omega\iota F^{2,3} - \omega F^{1,3} \\ \omega\iota F^{2,3} + \omega F^{1,3}, & 1 - \omega\iota F^{1,2} \end{bmatrix}\cdot\det\begin{bmatrix} 1+\omega\iota F^{1,2} & , \ \omega\iota F^{2,3} - \omega F^{1,3} \\ \omega\iota F^{2,3} + \omega F^{1,3}, & 1 - \omega\iota F^{1,2} \end{bmatrix}$$

$$= \left[(1+\omega\iota F^{1,2})(1-\omega\iota F^{1,2}) - (\omega\iota F^{2,3} - \omega F^{1,3})(\omega\iota F^{2,3} + \omega F^{1,3})\right]\cdot$$

$$\cdot\left[(1+\omega\iota F^{1,2})(1-\omega\iota F^{1,2}) - (\omega\iota F^{2,3} - \omega F^{1,3})(\omega\iota F^{2,3} + \omega F^{1,3})\right] =$$

$$= \left[1 + \omega^2\left((F^{1,2})^2 + (F^{2,3})^2 + (F^{1,3})^2\right)\right]\cdot\left[1 + \omega^2\left((F^{1,2})^2 + (F^{2,3})^2 + (F^{1,3})^2\right)\right].$$

Replacing $F^{1,2} = B_3$, $F^{1,3} = -B_2$, $F^{2,3} = B_1$, $\omega = \frac{\iota 2e}{3m^2}$ we get

$$D(n) = \det(\mathcal{S}_0 n) = n^{16}\left[1 - \left(\tfrac{2}{3}\, e\, m^{-2}\right)\vec{B}^2\right]^2.$$

(Here B_1, B_2, B_3, were used instead of H_1, H_2, H_3, used in the sequel for compliance of the result with that of Velo and Zwanziger.)

Writing this in covariant form and setting it equal to zero we have

$$D(n) = (n^2)^6 \left[n^2 + (\tfrac{2}{3} e m^{-2})^2 (F^d \cdot n)^2 \right]^2 = 0 \quad . \qquad (8)$$

This equation determines the normals to the characteristic surfaces. Velo and Zwanziger, before analysing equation (8), introduce the term "weak-field case", to refer to the case for which the inequality

$$(\tfrac{2}{3} e m^{-2})^2 \vec{B}^2 < 1 ,$$

is satisfied. Otherwise they refer to the "strong-field case".

In the weak-field case equation (8) has eight positive and eight negative roots n_0 for any given $\vec{n} = (n_1, n_2, n_3)$. This establishes hyperbolicity of the modified equation and also of the Rarita-Schwinger equation. In this case the characteristic surfaces determined by equation (8) are not all tengent to the light cone and some signals are propagated at velocities greater than the speed of light, violating causality.

In the strong-field case the modified equation and also the Rarita-Schwinger equation cease to be hyperbolic.

Finally in the case the external field $F_{\mu\nu}$ goes to zero the equation remains hyperbolic and propagates causally.

6: Another approach to the problem of propagation.

Let us consider again the modified equation (7) and study its propagation behaviour using this time the criterion of hyperbolicity for quasilinear first order systems of partial differential equations given in chapter 9 paragraph 2.

As it was said there the first order quasilinear system

$$\mathcal{J}_0(\psi,\vec{x},x_0)\psi_{x_0} + \sum_{l=1}^{3} \mathcal{J}_l(\psi,\vec{x},x_0)\psi_{x_l} + \mathbb{B}(\psi,\vec{x},x_0) = 0 ,$$

is strictly hyperbolic in the x_0-direction at the point P if the zeros $\lambda^{(1)}, \lambda^{(2)} \dots \lambda^{(m)}$, of the characteristic polynomial

$$Q(\lambda) = \det\left(\sum_{l=1}^{3} \nu_l \mathcal{J}_l - \lambda \mathcal{J}_0 \right) = 0 ,$$

are all real and distinct for all choices of the unit vector $\vec{\nu}$ and if the right eigenvectors $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_m$ satisfying

$$\left\{ \sum_{l=1}^{3} \nu_l \mathcal{J}_l - \lambda_j \mathcal{J}_0 \right\} \vec{r}_j = 0 ,$$

span the space E^m. The system will be said to be merely hyperbolic in the X_0-direction if the eigenvectors span the space E^m but the eigenvalues although real are not all distinct.

For convenience we shall restrict ourselves to one spacial dimension, i.e. X_3 in which case the unit normal vector involved in the definition of the characteristic polynomial simplifies to the unit vector in the X_3-direction so that $\nu_1 = 1$. Thus using the matrices $\mathcal{J}_0$, $\mathcal{J}_3$ of the modified equation given in §5 the characteristic polynomial $Q(\lambda)$ reads: $Q(\lambda) = \det\{ 1 \cdot \mathcal{J}_3 - \lambda \mathcal{J}_0 \} =$

$$= \det\left\{ \begin{bmatrix} \iota\gamma_3 & 0 & 0 & 0 \\ 0 & \iota\gamma_3 & 0 & 0 \\ 0 & 0 & \iota\gamma_3 & 0 \\ \iota\omega\gamma_5\gamma^\mu F_\mu^{d_0}, & \iota\omega\gamma_5\gamma^\mu F_\mu^{d_1}, & \iota\omega\gamma_5\gamma^\mu F_\mu^{d_2}, & \iota\gamma_3 + \iota\omega\gamma_5\gamma^\mu F_\mu^{d_3} \end{bmatrix} - \right.$$

$$\left. - \lambda \begin{bmatrix} \iota\gamma_0 - \iota\omega\gamma_5\gamma^\mu F_\mu^{d_0}, & -\iota\omega\gamma_5\gamma^\mu F_\mu^{d_1}, & -\iota\omega\gamma_5\gamma^\mu F_\mu^{d_2}, & -\iota\omega\gamma_5\gamma^\mu F_\mu^{d_3} \\ 0 & \iota\gamma_0 & 0 & 0 \\ 0 & 0 & \iota\gamma_0 & 0 \\ 0 & 0 & 0 & \iota\gamma_0 \end{bmatrix} \right\} =$$

$$= \det\begin{bmatrix} \iota\gamma_3 - \lambda\iota\gamma_0 + \lambda\iota\omega\gamma_5\gamma^\mu F_\mu^{d_0} & , \lambda\iota\omega\gamma_5\gamma^\mu F_\mu^{d_3} \\ \iota\omega\gamma_5\gamma^\mu F_\mu^{d_0} & , \iota\gamma_3 + \iota\omega\gamma_5\gamma^\mu F_\mu^{d_3} - \lambda\iota\gamma_0 \end{bmatrix} \cdot$$

$$\cdot \det(\iota\gamma_3 - \lambda\iota\gamma_0) \cdot \det(\iota\gamma_3 - \lambda\iota\gamma_0) =$$

$$= \det\left(i\gamma_3 + i\omega\gamma_5\gamma^\mu F_\mu^{d_3} - \lambda i\gamma_0 + \lambda i\omega\gamma_5\gamma^\mu F_\mu^{d_0} \right)\cdot$$

$$\cdot \det\left(i\gamma_3 - \lambda i\gamma_0 \right)\cdot \det\left(i\gamma_3 - \lambda i\gamma_0 \right)\cdot \det\left(i\gamma_3 - \lambda i\gamma_0 \right) = 0.$$

The roots of this polynomial are:

$$\lambda_i = +1 \quad \text{for} \quad i = 1, 2, 3, 4, 5, 6,$$
$$\lambda_j = -1 \quad \text{for} \quad j = 7, 8, 9, 10, 11, 12,$$

$$\lambda_{13,14} = \frac{-\left(\frac{2e}{3m^2}\right)^2\left[F^{0,2}F^{2,3}+F^{0,1}F^{1,3}\right] \pm \sqrt{\begin{aligned}&\left(\frac{2e}{3m^2}\right)^4\left[F^{0,2}F^{2,3}+F^{0,1}F^{1,3}\right]^2 + \\ &+\left[1-\left(\frac{2e}{3m^2}\right)^2\left((F^{1,2})^2+(F^{2,3})^2+(F^{1,3})^2\right)\right]\cdot \\ &\cdot\left[1-\left(\frac{2e}{3m^2}\right)^2(F^{1,2})^2+\left(\frac{2e}{3m^2}\right)^2\left((F^{0,1})^2+(F^{0,2})^2\right)\right]\end{aligned}}}{\left[1-\left(\frac{2e}{3m^2}\right)^2\left((F^{1,2})^2+(F^{2,3})^2+(F^{1,3})^2\right)\right]},$$

$$\lambda_{15,16} = \frac{-\left(\frac{2e}{3m^2}\right)^2\left[F^{0,2}F^{2,3}+F^{0,1}F^{1,3}\right] \pm \sqrt{\begin{aligned}&\left(\frac{2e}{3m^2}\right)^4\left[F^{0,2}F^{2,3}+F^{0,1}F^{1,3}\right]^2 + \\ &+\left[1-\left(\frac{2e}{3m^2}\right)^2\left((F^{1,2})^2+(F^{2,3})^2+(F^{1,3})^2\right)\right]\cdot \\ &\cdot\left[1-\left(\frac{2e}{3m^2}\right)^2(F^{1,2})^2+\left(\frac{2e}{3m^2}\right)^2\left((F^{0,1})^2+(F^{0,2})^2\right]\right.\end{aligned}}}{\left[1-\left(\frac{2e}{3m^2}\right)^2\left((F^{1,2})^2+(F^{2,3})^2+(F^{1,3})^2\right)\right]}.$$

Using

$$F^{\mu\nu} = \begin{bmatrix} 0 & H_3 & -H_2 & -E_1 \\ -H_3 & 0 & H_1 & -E_2 \\ H_2 & -H_1 & 0 & -E_3 \\ E_1 & E_2 & E_3 & 0 \end{bmatrix}$$

and replacing $F^{1,2} = H_3$, $F^{1,3} = -H_2$, $F^{2,3} = H_1$, $F^{0,2} = E_2$, $F^{0,1} = E_1$, the field dependent roots become

$$\lambda_{13} = \frac{-\left(\frac{2e}{3m^2}\right)^2\left[E_2H_1 - E_1H_2\right] + \sqrt{\begin{array}{l}\left(\frac{2e}{3m^2}\right)^4\left[E_2H_1 - E_1H_2\right]^2 + \\ +\left[1 - \left(\frac{2e}{3m^2}\right)^2\left(H_1^2 + H_2^2 + H_3^2\right)\right]\cdot \\ \cdot\left[1 - \left(\frac{2e}{3m^2}\right)^2 H_3^2 + \left(\frac{2e}{3m^2}\right)^2 E_1^2 + \left(\frac{2e}{3m^2}\right)^2 E_2^2\right]\end{array}}}{\left[1 - \left(\frac{2e}{3m^2}\right)^2\left(H_1^2 + H_2^2 + H_3^2\right)\right]} ,$$

$$\lambda_{14} = \frac{-\left(\frac{2e}{3m^2}\right)^2\left[E_2H_1 - E_1H_2\right] - \sqrt{\begin{array}{l}\left(\frac{2e}{3m^2}\right)^4\left[E_2H_1 - E_1H_2\right]^2 + \\ +\left[1 - \left(\frac{2e}{3m^2}\right)^2\left(H_1^2 + H_2^2 + H_3^2\right)\right]\cdot \\ \cdot\left[1 - \left(\frac{2e}{3m^2}\right)^2 H_3^2 + \left(\frac{2e}{3m^2}\right)^2 E_1^2 + \left(\frac{2e}{3m^2}\right)^2 E_2^2\right]\end{array}}}{\left[1 - \left(\frac{2e}{3m^2}\right)^2\left(H_1^2 + H_2^2 + H_3^2\right)\right]} ,$$

$$\lambda_{15} = \frac{-\left(\frac{2e}{3m^2}\right)^2\left[E_2H_1 - E_1H_2\right] + \sqrt{\begin{array}{l}\left(\frac{2e}{3m^2}\right)^4\left[E_2H_1 - E_1H_2\right]^2 + \\ +\left[1 - \left(\frac{2e}{3m^2}\right)^2\left(H_1^2 + H_2^2 + H_3^2\right)\right]\cdot \\ \cdot\left[1 - \left(\frac{2e}{3m^2}\right)^2 H_3^2 + \left(\frac{2e}{3m^2}\right)^2 E_1^2 + \left(\frac{2e}{3m^2}\right)^2 E_2^2\right]\end{array}}}{\left[1 - \left(\frac{2e}{3m^2}\right)^2\left(H_1^2 + H_2^2 + H_3^2\right)\right]} ,$$

$$\lambda_{16}=\frac{-\left(\frac{2e}{3m^2}\right)^2\left[E_2H_1-E_1H_2\right]-\sqrt{\begin{array}{l}\left(\frac{2e}{3m^2}\right)^4\left[E_2H_1-E_1H_2\right]^2+\\+\left[1-\left(\frac{2e}{3m^2}\right)^2\left(H_1^2+H_2^2+H_3^2\right)\right]\cdot\\\cdot\left[1-\left(\frac{2e}{3m^2}\right)^2H_3^2+\left(\frac{2e}{3m^2}\right)^2E_1^2+\left(\frac{2e}{3m^2}\right)^2E_2^2\right]\end{array}}}{\left[1-\left(\frac{2e}{3m^2}\right)^2\left(H_1^2+H_2^2+H_3^2\right)\right]}.$$

The number of linearly independent eigenvectors satisfying $\{\mathcal{T}_3-\lambda_j\mathcal{T}_0\}\vec{r}_j=0$, is sixteen, spanning thus the entire space E^{16}. That there are sixteen linearly independent eigenvectors can be verified either by straightforward calculation or as follows.

The matrix $(\mathcal{T}_3-\lambda\mathcal{T}_0)$ for the eigenvalues $\lambda_l=1$, $l=1,2,3,4,5,6$, has nullity six and since the number of linearly independent eigenvectors is equal to the nullity of the matrix there are thus six linearly independent eigenvectors. Similarly the nullity of $\{\mathcal{T}_3-\lambda\mathcal{T}_0\}$ for the eigenvalues $\lambda_j=-1$, $j=7,8,9,10,11,12$, is six and thus there are six linearly independent eigenvectors corresponding to them. The nullity of $\{\mathcal{T}_3-\lambda\mathcal{T}_0\}$ for the eigenvalues $\lambda_{13}(F^{\mu\nu})=\lambda_{15}(F^{\mu\nu})$ is two and thus there are two linearly independent eigenvectors corresponding to them and finally the nullity for the eigenvalues $\lambda_{14}(F^{\mu\nu})=\lambda_{16}(F^{\mu\nu})$ is two and so we find two more linearly independent eigenvectors completing thus a total of sixteen linearly independent eigenvectors.

We investigate now the circumstances under which the modified equation of motion and in consequence the Rarita-Schwinger equation is hyperbolic in an external electromagnetic field.

Case 1: (Zero field case).

In the case of zero field i.e. $F^{\mu\nu} \longrightarrow 0$, the eigenvalues are all real, namely:

$$\lambda_i = 1 \ , \ \text{for} \ \ i = 1,2,3,4,5,6,$$

$$\lambda_j = -1, \ \text{for} \ \ j = 7,8,9,10,11,12,$$

$$\lambda_{13} = 1 \ , \quad \lambda_{14} = -1 \ , \quad \lambda_{15} = 1 \ , \quad \lambda_{16} = -1,$$

and the equation is then hyperbolic. Furthermore the propagation along each characteristic is causal since by the formula

$$\lambda = 1 \frac{dx}{d(ct)} = \frac{1}{c} \frac{dx}{dt} ,$$

if $\lambda = \pm 1$, then the velocity of propagation

$$u_p \equiv \frac{dx}{dt} = \pm c ,$$

where c is the velocity of light.

Case 2: (Weak field case).

Let us consider

$$\lambda_{13} = \frac{-\left(\frac{2e}{3m^2}\right)^2 \left[E_2H_1 - E_1H_2\right] + \sqrt{\begin{array}{l}\left(\frac{2e}{3m^2}\right)^4 \left[E_2H_1 - E_1H_2\right]^2 + \\ + \left[1 - \left(\frac{2e}{3m^2}\right)^2 \left(H_1^2 + H_2^2 + H_3^2\right)\right] \cdot \\ \cdot \left[1 + \left(\frac{2e}{3m^2}\right)^2 \left(-H_3^2 + E_1^2 + E_2^2\right)\right]\end{array}}}{\left[1 - \left(\frac{2e}{3m^2}\right)^2 \left(H_1^2 + H_2^2 + H_3^2\right)\right]} .$$

Using the expansions

$$\left[1 - \left(\frac{2e}{3m^2}\right)^2 \left(H_1^2 + H_2^2 + H_3^2\right)\right]^{1/2} \cong 1 - \frac{1}{2}\left(\frac{2e}{3m^2}\right)^2 \left(H_1^2 + H_2^2 + H_3^2\right) + \cdots$$

holding for $-1 < (2e/3m^2)^2(H_1^2 + H_2^2 + H_3^2) < 1$,

$$\left[1+\left(\frac{2e}{3m^2}\right)^2\left(-H_3^2+E_1^2+E_2^2\right)\right]^{1/2} \cong 1+\frac{1}{2}\left(\frac{2e}{3m^2}\right)^2\left(-H_3^2+E_1^2+E_2^2\right)+\cdots$$

holding for $-1<(2e/3m^2)^2(-H_3^2+E_1^2+E_2^2)<1$,

$$\left[1-\left(\frac{2e}{3m^2}\right)^2\left(H_1^2+H_2^2+H_3^2\right)\right]^{-1} \cong 1+\left(\frac{2e}{3m^2}\right)^2\left(H_1^2+H_2^2+H_3^2\right)+\cdots$$

holding for $-1<(2e/3m^2)^2(H_1^2+H_2^2+H_3^2)<1$,

we find keeping terms up to the second order,

$$\lambda_{13}=\left\{\left(\frac{2e}{3m^2}\right)^2\left[E_1H_2-E_2H_1\right]+\left[1-\frac{1}{2}\left(\frac{2e}{3m^2}\right)^2\left(H_1^2+H_2^2+H_3^2\right)\right]\left[1+\frac{1}{2}\left(\frac{2e}{3m^2}\right)^2\left(-H_3^2+E_1^2+E_2^2\right)\right]\right\}\cdot$$
$$\cdot\left\{1+\left(\frac{2e}{3m^2}\right)^2\left(H_1^2+H_2^2+H_3^2\right)\right\}$$

$$\cong\left\{\left(\frac{2e}{3m^2}\right)^2\left[E_1H_2-E_2H_1\right]+1+\frac{1}{2}\left(\frac{2e}{3m^2}\right)^2\left(-H_3^2+E_1^2+E_2^2\right)-\frac{1}{2}\left(\frac{2e}{3m^2}\right)^2\left(H_1^2+H_2^2+H_3^2\right)\right\}\cdot$$
$$\cdot\left\{1+\left(\frac{2e}{3m^2}\right)^2\left(H_1^2+H_2^2+H_3^2\right)\right\}$$

$$\cong\left(\frac{2e}{3m^2}\right)^2\left[E_1H_2-E_2H_1\right]+1+\frac{1}{2}\left(\frac{2e}{3m^2}\right)^2\left(-H_3^2+E_1^2+E_2^2\right)-$$
$$-\frac{1}{2}\left(\frac{2e}{3m^2}\right)^2\left(H_1^2+H_2^2+H_3^2\right)+\left(\frac{2e}{3m^2}\right)^2\left(H_1^2+H_2^2+H_3^2\right)$$

$$\cong 1+\frac{1}{2}\left(\frac{2e}{3m^2}\right)^2\left(E_1+H_2\right)^2+\frac{1}{2}\left(\frac{2e}{3m^2}\right)^2\left(E_2-H_1\right)^2 .$$

Similarly

$$\lambda_{14}\cong -1-\frac{1}{2}\left(\frac{2e}{3m^2}\right)^2\left(E_1-H_2\right)^2-\frac{1}{2}\left(\frac{2e}{3m^2}\right)^2\left(E_2+H_1\right)^2 ,$$

$$\lambda_{15} \cong 1 + \frac{1}{2}\left(\frac{2e}{3m^2}\right)^2 \left(E_1 + H_2\right)^2 + \frac{1}{2}\left(\frac{2e}{3m^2}\right)^2 \left(E_2 - H_1\right)^2 ,$$

$$\lambda_{16} \cong -1 - \frac{1}{2}\left(\frac{2e}{3m^2}\right)^2 \left(E_1 - H_2\right)^2 - \frac{1}{2}\left(\frac{2e}{3m^2}\right)^2 \left(E_2 + H_1\right)^2 .$$

The other roots are

$$\lambda_i = 1 \quad \text{for} \quad i = 1,2,3,4,5,6 \quad , \quad \lambda_j = -1 \quad \text{for} \quad j = 7,8,9,10,11,12 .$$

Thus in the weak field case all the eigenvalues are real and the equation is hyperbolic. If the field components H_1, H_2, E_1, E_2, are chosen to be different from zero and within the radius of convergence then the field dependent roots are

$$\lambda_{13}(F^{\mu\nu}) = \lambda_{15}(F^{\mu\nu}) > 1 , \quad \lambda_{14}(F^{\mu\nu}) = \lambda_{16}(F^{\mu\nu}) < -1 ,$$

and the equation is not causal. We notice though that the field dependent roots for weak fields are independent of H_3, E_3 and hence for $H_1 = H_2 = E_1 = E_2 = 0$, the equation is hyperbolic and causal.

Case 3: (Strong field case)

Case 3.1: (Field along x_3).

Let us consider the case $H_3 \neq 0, E_3 \neq 0, H_1 = H_2 = E_1 = E_2 = 0$. In this case the field dependent roots become

$$\left.\begin{matrix} \lambda_{13} = \lambda_{15} \\ \\ \lambda_{14} = \lambda_{16} \end{matrix}\right\} = \frac{\pm\sqrt{\left[1 - \left(\frac{2e}{3m^2}\right)^2 H_3^2\right]\left[1 - \left(\frac{2e}{3m^2}\right)^2 H_3^2\right]}}{\left[1 - \left(\frac{2e}{3m^2}\right)^2 H_3^2\right]} = \pm 1 .$$

The other roots are

$$\lambda_i = 1 \quad \text{for} \quad i = 1,2,3,4,5,6, \quad \text{and} \quad \lambda_j = -1 \quad \text{for} \quad j = 7,8,9,10,11,12 .$$

Thus the equation in this case is hyperbolic and causal. The case of vanishing denominator must be excluded.

Case 3.ii: $(H_1 \neq 0 \,,\, H_2 \neq 0 \,,\, H_3 = 0 \,,\, E_1 = 0 \,,\, E_2 = 0 .)$

In this case we have

$$\lambda_i = 1 \text{ for } i = 1, 2, 3, 4, 5, 6, \qquad \lambda_j = -1 \text{ for } j = 7, 8, 9, 10, 11, 12,$$

and

$$\left.\begin{matrix} \lambda_{13} = \lambda_{15} \\ \\ \lambda_{14} = \lambda_{16} \end{matrix}\right\} = \frac{\pm \sqrt{\left[1 - \left(\frac{2e}{3m^2}\right)^2 \left(H_1^2 + H_2^2\right)\right.}}{\left[1 - \left(\frac{2e}{3m^2}\right)^2 \left(H_1^2 + H_2^2\right)\right]} =$$

$$= \frac{\pm 1}{\sqrt{1 - \left(\frac{2e}{3m^2}\right)^2 \left(H_1^2 + H_2^2\right)}} \,.$$

This is real with $\lambda_{13} = \lambda_{15} > 1$ and $\lambda_{14} = \lambda_{16} < -1$ if $1 > \left(\frac{2e}{3m^2}\right)^2 \left(H_1^2 + H_2^2\right)$. Thus the equation in this case is hyperbolic but not causal. If $\left(\frac{2e}{3m^2}\right)^2 \left(H_1^2 + H_2^2\right) > 1$, the roots λ_{13}, λ_{15}, λ_{14}, λ_{16}, become imaginary and hyperbolicity is lost.

Case 3.iii: $(H_1 \neq 0 \,,\, H_2 \neq 0 \,,\, H_3 \neq 0 \,,\, E_1 = 0 \,,\, E_2 = 0 .)$

In this case we have

$$\lambda_i = 1 \text{ for } i = 1, 2, 3, 4, 5, 6, \qquad \lambda_j = -1 \text{ for } j = 7, 8, 9, 10, 11, 12,$$

$$\left.\begin{matrix} \lambda_{13} = \lambda_{15} \\ \\ \lambda_{14} = \lambda_{16} \end{matrix}\right\} = \frac{\pm \sqrt{\left[1 - \left(\frac{2e}{3m^2}\right)^2 \left(H_1^2 + H_2^2 + H_3^2\right)\right]\left[1 - \left(\frac{2e}{3m^2}\right)^2 H_3^2\right]}}{\left[1 - \left(\frac{2e}{3m^2}\right)^2 \left(H_1^2 + H_2^2 + H_3^2\right)\right]}$$

$$= \frac{\pm \sqrt{1 - \left(\frac{2e}{3m^2}\right)^2 H_3^2}}{\sqrt{1 - \left(\frac{2e}{3m^2}\right)^2 \left(H_1^2 + H_2^2 + H_3^2\right)}} \,.$$

If $1 > \left(\frac{2e}{3m^2}\right)^2 H_3^2$ and $1 > \left(\frac{2e}{3m^2}\right)^2 \left(H_1^2 + H_2^2 + H_3^2\right)$ then the field dependent roots are real with $\lambda_{13} = \lambda_{15} > 1$ and $\lambda_{14} = \lambda_{16} < -1$. Thus in this case the equation is hyperbolic but non-causal. If $1 > \left(\frac{2e}{3m^2}\right)^2 H_3^2$ and $\left(\frac{2e}{3m^2}\right)^2 \left(H_1^2 + H_2^2 + H_3^2\right) > 1$ the field dependent roots are imaginary and so hyperbolicity is lost.

If $\left(\frac{2e}{3m^2}\right)^2 H_3^2 > 1$, both the numerator and denominator are imaginary but the roots are real and less than $|1|$. So the equation is hyperbolic and propagates causally in this case.

Case 3.iv: $(H_1 = 0 , H_2 = 0 , H_3 = 0 , E_1 \neq 0 , E_2 \neq 0 .)$

In this case we have

$\lambda_l = 1$ for $l = 1, 2, 3, 4, 5, 6,$ $\lambda_j = -1$ for $j = 7, 8, 9, 10, 11, 12,$

and

$$\left.\begin{array}{l} \lambda_{13} = \lambda_{15} \\ \\ \lambda_{14} = \lambda_{16} \end{array}\right\} = \pm\sqrt{\left[1 + \left(\frac{2e}{3m^2}\right)^2 \left(E_1^2 + E_2^2\right)\right]} .$$

This is always real and greater than $|1|$. Thus in this case the equation is hyperbolic but non-causal.

Case 3.v: $(H_1 \neq 0 , H_2 \neq 0 , H_3 \neq 0 , E_1 \neq 0 , E_2 \neq 0 .)$

In this case we have

$\lambda_l = 1$ for $l = 1, 2, 3, 4, 5, 6,$ $\lambda_j = -1$ for $j = 7, 8, 9, 10, 11, 12,$

and

$$\left.\begin{array}{l} \lambda_{13} = \lambda_{15} \\ \\ \lambda_{14} = \lambda_{16} \end{array}\right\} = \frac{\left(\frac{2e}{3m^2}\right)^2 \left[E_1 H_2 - E_2 H_1\right] \pm \sqrt{\begin{array}{l} \left(\frac{2e}{3m^2}\right)^4 \left[E_1 H_2 - E_2 H_1\right] + \\ + \left[1 - \left(\frac{2e}{3m^2}\right)^2 \left(H_1^2 + H_2^2 + H_3^2\right)\right] \cdot \\ \cdot \left[1 + \left(\frac{2e}{3m^2}\right)^2 \left(-H_3^2 + E_1^2 + E_2^2\right)\right] \end{array}}}{\left[1 - \left(\frac{2e}{3m^2}\right)^2 \left(H_1^2 + H_2^2 + H_3^2\right)\right]} .$$

If $1+\left(\frac{2e}{3m^2}\right)^2\left(E_1^2+E_2^2\right)=\left(\frac{2e}{3m^2}\right)^2 H_3^2$, then we have

$$\left.\begin{matrix}\lambda_{13}=\lambda_{15}\\ \\ \lambda_{14}=\lambda_{16}\end{matrix}\right\}=\left[\begin{matrix}\to=\dfrac{2\left(2e/3m^2\right)^2\left[E_1H_2-E_2H_1\right]}{\left[1-\left(2e/3m^2\right)^2\left(H_1^2+H_2^2+H_3^2\right)\right]}.\\ \\ \to=0\end{matrix}\right.$$

i) These roots if $E_1H_2=E_2H_1$ become zero and the equation is then hyperbolic and causal.

ii) If $H_2=H_1$ and $E_1\neq E_2$ then $\lambda_{14}=\lambda_{16}=0$ and

$$\lambda_{13}=\lambda_{15}=\frac{2\left(2e/3m^2\right)^2\left[E_1-E_2\right]H_1}{\left[1-\left(2e/3m^2\right)^2\left(H_1^2+H_1^2+H_3^2\right)\right]}\longrightarrow 0 \quad \text{for } H_1\to\infty.$$

Thus there is an optimum value of H_1 above which $\lambda_{13}=\lambda_{15}$, become less than $|1|$ and the equation then is hyperbolic and causal.

iii) If $E_1H_2\neq E_2H_1$ and $H_2\neq H_1$, there are again optimum values of H_1 , H_2 , so that when H_1 , H_2, are chosen greater than these values , $\lambda_{13}=\lambda_{15}$, become less than $|1|$ and the equation remains hyperbolic and causal.

In connection with the strong field case we had investigated particular cases since facing the general case is very difficult.

7: Summary:

In this chapter we have given a detailed analysis of the Rarita-Schwinger equation for spin 3/2 particles. Its matrices $\mathbb{L}_0$, $\mathbb{L}_1$, $\mathbb{L}_2$, $\mathbb{L}_3$, and its generators $\mathbb{H}_+$, $\mathbb{H}_-$, $\mathbb{H}_3$, $\mathbb{F}_+$, $\mathbb{F}_-$, $\mathbb{F}_3$, were given explicitly. The commutation relations satisfied by the generators and also among the generators and matrices $\mathbb{L}_0 , \mathbb{L}_1 , \mathbb{L}_2 , \mathbb{L}_3$, were verified. $\mathbb{L}_0$ satisfies a minimal equation of the form $\mathbb{L}_0^2 \{ \mathbb{L}_0^2 - 1 \} = 0$. Then by suitably modifying the Rarita-Schwinger equation in the presence of the electromagnetic field by means of the subsidiary conditions of the second kind we were able to study its propagation behaviour by following two separate methods. Both methods showed that for certain values of the electromagnetic field the propagation is not causal.

CHAPTER 11

On the Pauli - Fierz - Gupta wave-equation for spin 3/2 particles.

In this chapter we give a detailed analysis of the Pauli-Fierz[11] wave-equation for spin 3/2 particles. The Pauli-Fierz equation can be written in several equivalent forms.

One of the aims of this chapter is to establish these equivalences, but our ultimate aim is to study its propagation behaviour which we perform by employing the method of characteristics.

1: The Pauli-Fierz wave-equation expressed in the canonical frame.

One of the possible forms in which the Pauli-Fierz equation can be written is that of Gel'fand and Yaglom

$$\mathbb{L}_0 \frac{\partial\psi}{\partial x_0} + \mathbb{L}_1 \frac{\partial\psi}{\partial x_1} + \mathbb{L}_2 \frac{\partial\psi}{\partial x_2} + \mathbb{L}_3 \frac{\partial\psi}{\partial x_3} + i\kappa\psi = 0,$$

where the wave-function ψ transforms under the 16-dim. representation with components $\tau_1 \sim (1/2, 3/2)$, $\dot{\tau}_1 \sim (-1/2, 3/2)$, $\tau_2 \sim (1/2, 5/2)$, $\dot{\tau}_2 \sim (-1/2, 5/2)$ interlocking according to the scheme

$$\begin{tikzcd} \tau_1 \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \dot{\tau}_1 \arrow[d, leftrightarrow] \\ \tau_2 \arrow[r, leftrightarrow] & \dot{\tau}_2 \end{tikzcd} .$$

The canonical basis associated with the representation is

$$\{\xi_{\ell m}\} \equiv \{ \xi^{\tau_1}_{1/2\, 1/2},\ \xi^{\tau_1}_{1/2, -1/2},\ \xi^{\dot{\tau}_1}_{1/2, 1/2},\ \xi^{\dot{\tau}_1}_{1/2, -1/2},\ \xi^{\tau_2}_{1/2, 1/2},\ \xi^{\tau_2}_{1/2, -1/2},$$

$$\xi^{\tau_2}_{3/2, 3/2},\ \xi^{\tau_2}_{3/2, 1/2},\ \xi^{\tau_2}_{3/2 -1/2},\ \xi^{\tau_2}_{3/2, -3/2},\ \xi^{\dot{\tau}_2}_{1/2, 1/2},\ \xi^{\dot{\tau}_2}_{1/2, -1/2},$$

$$\xi^{\dot{\tau}_2}_{3/2, 3/2},\ \xi^{\dot{\tau}_2}_{3/2, 1/2},\ \xi^{\dot{\tau}_2}_{3/2, -1/2},\ \xi^{\dot{\tau}_2}_{3/2, -3/2} \}.$$

With respect to this basis the matrix $\mathbb{L}_0$ of the Pauli-Fierz equation is

$$\begin{array}{cc|cc|cccccc|cccccc}
\xi^{\tau_1}_{1/2,1/2} & \xi^{\tau_1}_{1/2,-1/2} & \dot{\xi}^{\tau_1}_{1/2,1/2} & \dot{\xi}^{\tau_1}_{1/2,-1/2} & \xi^{\tau_2}_{1/2,1/2} & \xi^{\tau_2}_{1/2,-1/2} & \xi^{\tau_2}_{3/2,3/2} & \xi^{\tau_2}_{3/2,1/2} & \xi^{\tau_2}_{3/2,-1/2} & \xi^{\tau_2}_{3/2,-3/2} & \dot{\xi}^{\tau_2}_{1/2,1/2} & \dot{\xi}^{\tau_2}_{1/2,-1/2} & \dot{\xi}^{\tau_2}_{3/2,3/2} & \dot{\xi}^{\tau_2}_{3/2,1/2} & \dot{\xi}^{\tau_2}_{3/2,-1/2} & \dot{\xi}^{\tau_2}_{3/2,-3/2} \\
\hline
0 & 0 & -1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\
0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\
\hline
-1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\
0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & -1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} .$$

The matrix $\mathbb{L}_0$ has zero determinant and satisfies the minimal equation $\mathbb{L}_0^2 \{ \mathbb{L}_0^2 - 1 \} = 0$.

The other three matrices $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$, are determined from $\mathbb{L}_0$ and the generators of boosts $\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3$ by the formulae

$$\mathbb{L}_1 = -[\mathbb{B}_1, \mathbb{L}_0]_- , \quad \mathbb{L}_2 = -[\mathbb{B}_2, \mathbb{L}_0]_- , \quad \mathbb{L}_3 = -[\mathbb{B}_3, \mathbb{L}_0]_- .$$

The generators $\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3$, are expressed in terms of the generators $\mathbb{F}_+, \mathbb{F}_-, \mathbb{F}_3$ by the formulae

$$\mathbb{B}_1 = \frac{\mathbb{F}_+ + \mathbb{F}_-}{2i} , \quad \mathbb{B}_2 = \frac{-\mathbb{F}_+ + \mathbb{F}_-}{2} , \quad \mathbb{B}_3 = \frac{\mathbb{F}_3}{i} ,$$

where the generators $\mathbb{F}_+, \mathbb{F}_-, \mathbb{F}_3$ are respectively:

$$\mathbb{F}_+ = \begin{array}{c|cc|cc|cccccc|cccccc}
 & \xi^{\tau_1}_{1/2,1/2} & \xi^{\tau_1}_{1/2,-1/2} & \dot\xi^{\tau_1}_{1/2,1/2} & \dot\xi^{\tau_1}_{1/2,-1/2} & \xi^{\tau_2}_{1/2,1/2} & \xi^{\tau_2}_{1/2,-1/2} & \xi^{\tau_2}_{3/2,3/2} & \xi^{\tau_2}_{3/2,1/2} & \xi^{\tau_2}_{3/2,-1/2} & \xi^{\tau_2}_{3/2,-3/2} & \dot\xi^{\tau_2}_{1/2,1/2} & \dot\xi^{\tau_2}_{1/2,-1/2} & \dot\xi^{\tau_2}_{3/2,3/2} & \dot\xi^{\tau_2}_{3/2,1/2} & \dot\xi^{\tau_2}_{3/2,-1/2} & \dot\xi^{\tau_2}_{3/2,-3/2} \\
\hline
\tau_1 & 0 & -1 & & & & & & & & & & & & & & \\
 & 0 & 0 & & & & & & & & & & & & & & \\
\hline
\dot\tau_1 & & & 0 & i & & & & & & & & & & & & \\
 & & & 0 & 0 & & & & & & & & & & & & \\
\hline
\tau_2 & & & & & 0 & \frac{-i5}{3} & 0 & 0 & \frac{-2\sqrt{2}}{3} & 0 & & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & \frac{-2\sqrt{6}}{3} & & & & & & \\
 & & & & & \frac{-2\sqrt{6}}{3} & 0 & 0 & \frac{-i\sqrt{3}}{3} & 0 & 0 & & & & & & \\
 & & & & & 0 & \frac{-2\sqrt{2}}{3} & 0 & 0 & \frac{-2i}{3} & 0 & & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & \frac{-i\sqrt{3}}{3} & & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & \\
\hline
\dot\tau_2 & & & & & & & & & & & 0 & \frac{i5}{3} & 0 & 0 & \frac{-2\sqrt{2}}{3} & 0 \\
 & & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & \frac{-2\sqrt{6}}{3} \\
 & & & & & & & & & & & \frac{-2\sqrt{6}}{3} & 0 & 0 & \frac{i\sqrt{3}}{3} & 0 & 0 \\
 & & & & & & & & & & & 0 & \frac{-2\sqrt{2}}{3} & 0 & 0 & \frac{i2}{3} & 0 \\
 & & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & \frac{i\sqrt{3}}{3} \\
 & & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0
\end{array},$$

$$\mathbb{F}_- = \begin{array}{c|cc|cc|cccccc|cccccc}
 & \tau_1 & & \dot\tau_1 & & & & \tau_2 & & & & & & & \dot\tau_2 & & \\
\hline
\tau_1 & 0 & 0 & & & & & & & & & & & & & & \\
 & -1 & 0 & & & & & & & & & & & & & & \\
\hline
\dot\tau_1 & & & 0 & 0 & & & & & & & & & & & & \\
 & & & i & 0 & & & & & & & & & & & & \\
\hline
\tau_2 & & & & & 0 & 0 & \frac{2\sqrt{6}}{3} & 0 & 0 & 0 & & & & & & \\
 & & & & & \frac{-i5}{3} & 0 & 0 & \frac{2\sqrt{2}}{3} & 0 & 0 & & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & \\
 & & & & & 0 & 0 & \frac{-i\sqrt{3}}{3} & 0 & 0 & 0 & & & & & & \\
 & & & & & \frac{2\sqrt{2}}{3} & 0 & 0 & \frac{-2i}{3} & 0 & 0 & & & & & & \\
 & & & & & 0 & \frac{2\sqrt{6}}{3} & 0 & 0 & \frac{-i\sqrt{3}}{3} & 0 & & & & & & \\
\hline
\dot\tau_2 & & & & & & & & & & & 0 & 0 & \frac{2\sqrt{6}}{3} & 0 & 0 & 0 \\
 & & & & & & & & & & & \frac{i5}{3} & 0 & 0 & \frac{2\sqrt{2}}{3} & 0 & 0 \\
 & & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & 0 & 0 & \frac{i\sqrt{3}}{3} & 0 & 0 & 0 \\
 & & & & & & & & & & & \frac{2\sqrt{2}}{3} & 0 & 0 & \frac{i2}{3} & 0 & 0 \\
 & & & & & & & & & & & 0 & \frac{2\sqrt{6}}{3} & 0 & 0 & \frac{i\sqrt{3}}{3} & 0
\end{array},$$

$$\mathbb{F}_3 = \begin{array}{c|cc|cc|cccccc|cccccc|}
 & \multicolumn{2}{c}{\tau_1} & \multicolumn{2}{c}{\dot{\tau}_1} & \multicolumn{6}{c}{\tau_2} & \multicolumn{6}{c}{\dot{\tau}_2} \\
\tau_1 & -\frac{1}{2} & 0 \\
 & 0 & \frac{1}{2} \\ \hline
\dot{\tau}_1 & & & \frac{1}{2} & 0 \\
 & & & 0 & -\frac{1}{2} \\ \hline
 & & & & & -\frac{15}{6} & 0 & 0 & -\frac{2\sqrt{2}}{3} & 0 & 0 \\
 & & & & & 0 & \frac{15}{6} & 0 & 0 & -\frac{2\sqrt{2}}{3} & 0 \\
\tau_2 & & & & & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
 & & & & & \frac{2\sqrt{2}}{3} & 0 & 0 & -\frac{1}{6} & 0 & 0 \\
 & & & & & 0 & \frac{2\sqrt{2}}{3} & 0 & 0 & \frac{1}{6} & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ \hline
 & & & & & & & & & & & \frac{15}{6} & 0 & 0 & -\frac{2\sqrt{2}}{3} & 0 & 0 \\
 & & & & & & & & & & & 0 & -\frac{15}{6} & 0 & 0 & -\frac{2\sqrt{2}}{3} & 0 \\
\dot{\tau}_2 & & & & & & & & & & & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
 & & & & & & & & & & & \frac{2\sqrt{2}}{3} & 0 & 0 & \frac{1}{6} & 0 & 0 \\
 & & & & & & & & & & & 0 & \frac{2\sqrt{2}}{3} & 0 & 0 & -\frac{1}{6} & 0 \\
 & & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}
\end{array} .$$

The other three generators $\mathbb{H}_+, \mathbb{H}_-, \mathbb{H}_3$, (related to the generators of rotations) are respectively:

$$\mathbb{H}_+ = \begin{array}{c|cc|cc|cccccc|cccccc|}
 & \multicolumn{2}{c}{\tau_1} & \multicolumn{2}{c}{\dot{\tau}_1} & \multicolumn{6}{c}{\tau_2} & \multicolumn{6}{c}{\dot{\tau}_2} \\
\tau_1 & 0 & 1 \\
 & 0 & 0 \\ \hline
\dot{\tau}_1 & & & 0 & 1 \\
 & & & 0 & 0 \\ \hline
 & & & & & 0 & 1 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\
\tau_2 & & & & & 0 & 0 & 0 & \sqrt{3} & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 2 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & \sqrt{3} \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
 & & & & & & & & & & & 0 & 1 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\
\dot{\tau}_2 & & & & & & & & & & & 0 & 0 & 0 & \sqrt{3} & 0 & 0 \\
 & & & & & & & & & & & 0 & 0 & 0 & 0 & 2 & 0 \\
 & & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & \sqrt{3} \\
 & & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0
\end{array} ,$$

$$\mathbb{H}_- = \begin{array}{c|c|c|c|c} & \tau_1 & \dot{\tau}_1 & \tau_2 & \dot{\tau}_2 \\ \hline \tau_1 & \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} & & & \\ \hline \dot{\tau}_1 & & \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} & & \\ \hline \tau_2 & & & \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} & 0 \end{matrix} & \\ \hline \dot{\tau}_2 & & & & \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} & 0 \end{matrix} \end{array},$$

$$\mathbb{H}_3 = \mathrm{diag}\left[\tfrac{1}{2}, -\tfrac{1}{2} \,\middle|\, \tfrac{1}{2}, -\tfrac{1}{2} \,\middle|\, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2} \,\middle|\, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{-1}{2}, \tfrac{-3}{2}\right].$$

The generators $\mathbb{H}_+, \mathbb{H}_-, \mathbb{H}_3, \mathbb{F}_+, \mathbb{F}_-, \mathbb{F}_3$, were found by using the following formulae,

$$H_3 \xi^\tau_{\ell m} = m \, \xi^\tau_{\ell m},$$

$$H_- \xi^\tau_{\ell m} = \sqrt{(\ell+m)(\ell-m+1)} \; \xi^\tau_{\ell, m-1},$$

$$H_+ \xi^\tau_{\ell m} = \sqrt{(\ell+m+1)(\ell-m)} \; \xi^\tau_{\ell, m+1},$$

$$m = -\ell, -\ell+1, \cdots, \ell-1, \ell;$$

$$F_3 \xi^\tau_{\ell m} = c_\ell \sqrt{\ell^2 - m^2} \; \xi^\tau_{\ell-1, m} - A_\ell m \xi^\tau_{\ell m} - c_{\ell+1} \sqrt{(\ell+1)^2 - m^2} \; \xi^\tau_{\ell+1, m},$$

$$F_+ \xi^\tau_{\ell m} = c_\ell \sqrt{(\ell-m)(\ell-m-1)} \; \xi^\tau_{\ell-1, m-1} - A_\ell \sqrt{(\ell-m)(\ell+m+1)} \; \xi^\tau_{\ell, m+1} + \\ + c_{\ell+1} \sqrt{(\ell+m+1)(\ell+m+2)} \; \xi^\tau_{\ell+1, m+1},$$

$$F_- \xi^\tau_{\ell m} = -c_\ell \sqrt{(\ell+m)(\ell+m-1)} \; \xi^\tau_{\ell-1, m-1} - A_\ell \sqrt{(\ell+m)(\ell-m+1)} \; \xi^\tau_{\ell, m-1} - \\ - c_{\ell+1} \sqrt{(\ell-m+1)(\ell-m+2)} \; \xi^\tau_{\ell+1, m-1},$$

$$A_\ell = \frac{i \ell_0 \ell_1}{\ell(\ell+1)}, \qquad c_\ell = \frac{i}{\ell} \sqrt{\frac{(\ell^2 - \ell_0^2)(\ell^2 - \ell_1^2)}{4\ell^2 - 1}},$$

$$m = -\ell, -\ell+1, \ldots, \ell-1, \ell; \quad \ell = \ell_0, \ell_0+1, \ldots$$

$$\tau \sim (\ell_0, \ell_1) \equiv \tau_1, \dot{\tau}_1, \tau_2, \dot{\tau}_2,$$

and the fact that the generators of the 16-dim. representation can be written in block form; for instance

$$\mathbb{H}_3^{16\text{-dim}} = \left[\begin{array}{c|c|c|c} \mathbb{H}_3^{\tau_1} & & & \\ \hline & \mathbb{H}_3^{\dot{\tau}_1} & & \\ \hline & & \mathbb{H}_3^{\tau_2} & \\ \hline & & & \mathbb{H}_3^{\dot{\tau}_2} \end{array}\right].$$

The generators $\mathbb{H}_+, \mathbb{H}_-, \mathbb{H}_3, \mathbb{F}_+, \mathbb{F}_-, \mathbb{F}_3$, satisfy the commutation relations

$$[\mathbb{H}_+, \mathbb{H}_3]_- = -\mathbb{H}_+, \quad [\mathbb{H}_-, \mathbb{H}_3]_- = \mathbb{H}_-, \quad [\mathbb{H}_+, \mathbb{H}_-]_- = 2\mathbb{H}_3,$$

$$[\mathbb{H}_+, \mathbb{F}_+]_- = 0, \quad [\mathbb{H}_3, \mathbb{F}_3]_- = 0, \quad [\mathbb{F}_+, \mathbb{F}_3]_- = \mathbb{H}_+, \quad [\mathbb{F}_-, \mathbb{F}_3]_- = -\mathbb{H}_-,$$

$$[\mathbb{F}_+, \mathbb{F}_-]_- = -2\mathbb{H}_3, \quad [\mathbb{H}_+, \mathbb{F}_3]_- = -\mathbb{F}_+, \quad [\mathbb{H}_-, \mathbb{F}_3]_- = \mathbb{F}_-,$$

$$[\mathbb{H}_+, \mathbb{F}_-]_- = 2\mathbb{F}_3, \quad [\mathbb{H}_-, \mathbb{F}_+]_- = -2\mathbb{F}_3, \quad [\mathbb{F}_+, \mathbb{H}_3]_- = -\mathbb{F}_+,$$

$$[\mathbb{F}_-, \mathbb{H}_3]_- = \mathbb{F}_-, \quad [\mathbb{H}_-, \mathbb{F}_-]_- = 0.$$

Also the matrix $\mathbb{L}_0$ and the generators $\mathbb{H}_+, \mathbb{H}_-, \mathbb{H}_3, \mathbb{F}_3$, satisfy the commutation relations

$$[\mathbb{L}_0, \mathbb{H}_+]_- = 0, \quad [\mathbb{L}_0, \mathbb{H}_-]_- = 0, \quad [\mathbb{L}_0, \mathbb{H}_3]_- = 0,$$

$$[[\mathbb{F}_3, \mathbb{L}_0]_-, \mathbb{F}_3]_- = \mathbb{L}_0.$$

The reason of giving $\mathbb{L}_0$ and the generators of the Pauli-Fierz equation explicitly is because they are essential in establishing the equivalence of the present form (canonical form) of the equation to its other forms.

Another form used to express the Pauli-Fierz equation is the spinor form. This is actually the form in which the equation was given by Pauli and Fierz. Since spinor calculus is important in understanding the structure of the equation we give in the next paragraph a brief summary of the most important concepts relevant to our work.

2. Elements of spinor calculus.[65-66]

Definition: Any two quantities $a_\kappa\ (\kappa=1,2)$ transforming according to the rule

$$\xi'_1 = a_{11}\xi_1 + a_{12}\xi_2 ,$$
$$\xi'_2 = a_{21}\xi_1 + a_{22}\xi_2 ,$$

with determinant

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = 1 ,$$

we call a covariant undotted spinor of the first rank.

Definition: Any two quantities $b_{\dot\rho}$, $(\dot\rho = \dot1, \dot2)$ transforming according to the rule

$$\bar\xi'_1 = \bar a_{11}\bar\xi_1 + \bar a_{12}\bar\xi_2 \quad ,$$
$$\bar\xi'_2 = \bar a_{21}\bar\xi_1 + \bar a_{22}\bar\xi_2 \quad ,$$

(complex conjugate of the previous) we call a covariant dotted spinor of the first rank. The bar indicates complex conjugate.

Definition: Any four quantities $a_{\kappa\lambda}$, $(\kappa, \lambda = 1, 2)$ transforming like the products $\xi_1\xi_1$, $\xi_1\xi_2$, $\xi_2\xi_1$, $\xi_2\xi_2$, we call an undotted spinor of the second rank.

Analogously any four quantities $b_{\dot\rho\dot\sigma}$, $(\dot\rho = \dot1, \dot2 , \dot\sigma = \dot1 , \dot2)$ transforming like the products $\bar\xi_1\bar\xi_1$, $\bar\xi_1\bar\xi_2$, $\bar\xi_2\bar\xi_1$, $\bar\xi_2\bar\xi_2$, we call a dotted spinor of the second rank.

Definition: Any four quantities $c_{\dot\rho\kappa}$, $(\dot\rho = \dot1 , \dot2 , \kappa = 1, 2)$ transforming like the products $\bar\xi_1\xi_1$, $\bar\xi_1\xi_2$, $\bar\xi_2\xi_1$, $\bar\xi_2\xi_2$, we call a mixed spinor of the second rank.

Similarly one defines spinors of higher rank like $a_{\kappa\lambda\mu}$ or $a_{\dot{\kappa}\lambda\mu}$, etc.

One can define also contravariant spinors by raising indices according to the rule

$$a^1 = a_2 \,, \qquad b^{\dot{1}} = b_{\dot{2}} \,,$$
$$a^2 = -a_1 \,, \qquad b^{\dot{2}} = -b_{\dot{1}} \,.$$

Dotted and undotted indices can be interchanged i.e.

$$a_{\dot{\rho}\lambda} = a_{\lambda\dot{\rho}} \,.$$

The complex conjugate of any spinor equation is obtained by replacing all undotted indices by dotted ones and vice versa.

Vectors a_k , $(k=0,1,2,3)$ of the four dimensional space-time $(t = x_0 \,, x_1 \,, x_2 \,, x_3)$ with the metric $g_{11} = g_{22} = g_{33} = 1$, $g_{00} = -1$, are put into correspondence with the spinors $a_{\dot{\alpha}\beta}$ according to the formula

$$a_{\dot{\alpha}\beta} = \sum_k a_k \, \sigma^k_{\dot{\alpha}\beta} \qquad , \; (k = 0,1,2,3) \,,$$

where

$$\sigma^1_{\dot{\alpha}\beta} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \; \sigma^2_{\dot{\alpha}\beta} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \; \sigma^3_{\dot{\alpha}\beta} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \; \sigma^0_{\dot{\alpha}\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

In particular the four momentum vector $\left(p_j = -i \frac{\partial}{\partial x_j} \,, (j = 1,2,3), \; p_0 = i \frac{\partial}{\partial x_0} \,, \hbar = 1\right)$ is connected with the spinor $p_{\dot{\alpha}\beta}$ according to the formulae

$$p_{\dot{1}1} = -i \frac{\partial}{\partial x_3} - i \frac{\partial}{\partial x_0} = p_3 - p_0 \,, \quad p_{\dot{2}1} = -i \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} = p_1 + i p_2 \,,$$
$$p_{\dot{1}2} = -i \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} = p_1 - i p_2 \,, \quad p_{\dot{2}2} = i \frac{\partial}{\partial x_3} - i \frac{\partial}{\partial x_0} = -p_3 - p_0 \,.$$

Using the formulae of raising and lowering spinor indices we have the following relations as well:

$$p^{\dot{1}1} = p_{\dot{2}2} = -p_0 - p_3 \quad , \; p^{\dot{1}2} = -p_{\dot{2}1} = -p_1 - i p_2 \,, \quad p^{\dot{1}}{}_2 = p_{\dot{2}2} = -p_0 - p_3 \,,$$

$$p^{\dot{2}1} = -p_{\dot{1}2} = -p_1 + i p_2 \,, \; p^{\dot{2}2} = p_{\dot{1}1} = -p_0 + p_3 \quad , \quad p^{\dot{1}}{}_1 = p_{\dot{2}1} = p_1 + i p_2 \,,$$

$$p^{\dot{2}}{}_1 = -p_{\dot{1}1} = p_0 - p_3 \qquad , \; p^{\dot{2}}{}_2 = -p_{\dot{1}2} = -p_1 + i p_2 \,, \quad p^1{}_{\dot{1}} = p_{2\dot{1}} = p_1 - i p_2 \,,$$

$$p^2{}_{\dot{1}} = -p_{1\dot{1}} = p_0 - p_3 \qquad , \; p^1{}_{\dot{2}} = p_{2\dot{2}} = -p_0 - p_3 \,, \quad p^2{}_{\dot{2}} = -p_{1\dot{2}} = -p_1 - i p_2 \,.$$

Similarly the electromagnetic four momentum vector $(\pi_0, \pi_1, \pi_2, \pi_3)$ is connected with the spinor $\pi_{\dot{\alpha}\beta}$ as follows:

$$\pi_{\dot{1}1} = -\pi_0 + \pi_3 \quad , \quad \pi_{\dot{2}1} = \pi_1 + i\pi_2 \quad ,$$

$$\pi_{\dot{1}2} = \pi_1 - i\pi_3 \quad , \quad \pi_{\dot{2}2} = -\pi_0 - \pi_3 \; .$$

By raising and lowering spinor indices we have also the formulae:

$$\pi^{\dot{1}1} = \pi_{\dot{2}2} = -\pi_0 - \pi_3 \; , \quad \pi^{\dot{1}2} = -\pi_{\dot{2}1} = -\pi_1 - i\pi_2 \; , \quad \pi^{\dot{1}}_{2} = \pi_{\dot{2}2} = -\pi_0 - \pi_3 \; ,$$

$$\pi^{\dot{2}1} = -\pi_{\dot{1}2} = -\pi_1 + i\pi_2 \; , \quad \pi^{\dot{2}2} = \pi_{\dot{1}1} = -\pi_0 + \pi_3 \; , \quad \pi^{\dot{1}}_{1} = \pi_{\dot{2}1} = \pi_1 + i\pi_2 \; ,$$

$$\pi^{\dot{2}}_{1} = -\pi_{\dot{1}1} = \pi_0 - \pi_3 \; , \quad \pi^{\dot{2}}_{2} = -\pi_{\dot{1}2} = -\pi_1 + i\pi_2 \; , \quad \pi^{1}_{\dot{1}} = \pi_{2\dot{1}} = \pi_1 - i\pi_2 \; ,$$

$$\pi^{2}_{\dot{1}} = -\pi_{1\dot{1}} = \pi_0 - \pi_3 \; , \quad \pi^{1}_{\dot{2}} = \pi_{2\dot{2}} = -\pi_0 - \pi_3 \; , \quad \pi^{2}_{\dot{2}} = -\pi_{1\dot{2}} = -\pi_1 - i\pi_2 \; .$$

3: Spinorial form of the Pauli-Fierz equation.

In this paragraph we shall be concerned with the free field Pauli-Fierz equation in its spinor form for spin 3/2 particles.

Dirac (1936) gave the following set of spinor equations

$$\kappa \, b^{\dot{\alpha}\dot{\beta}}_{\gamma} = p^{\dot{\alpha}\rho} a^{\dot{\beta}}_{\gamma\rho} \; ,$$

$$\kappa \, a^{\dot{\gamma}}_{\alpha\beta} = p_{\alpha\dot{\rho}} b^{\dot{\rho}\dot{\gamma}}_{\beta} \; ,$$

suitable for the description of spin 3/2 particles where $b^{\dot{\alpha}\dot{\beta}}_{\gamma}$, $a^{\dot{\beta}}_{\gamma\rho}$, symmetrical spinors, $p_{\alpha\dot{\rho}}$ the spinor associated with the four momentum p_k and κ constant associated with the mass of the particle; but it was shown by Fierz and Pauli that inconsistencies arise as soon as the electromagnetic field is introduced into them by minimal substitution. In avoiding this difficulty Fierz and Pauli assumed that a field corresponding to particles of spin 3/2 is described by the symmetrical spinors

$$a^{\dot{\alpha}}_{\beta\gamma} = a^{\dot{\alpha}}_{\gamma\beta} \quad , \quad b^{\dot{\alpha}\dot{\beta}}_{\gamma} = b^{\dot{\beta}\dot{\alpha}}_{\gamma} \; ,$$

and the auxiliary spinors c_α , $d^{\dot\alpha}$, and that the field equations are obtained from a Lagrangian density by variation. The Lagrangian density for the field is

$$\mathcal{L} = -\left(a^{*\gamma}_{\dot\alpha\dot\beta}\, p^{\dot\beta\rho} a^{\dot\alpha}_{\gamma\rho} + b^{*\alpha\beta}_{\dot\gamma}\, p_{\alpha\dot\rho}\, b^{\dot\gamma\dot\rho}_{\beta} \right) +$$

$$+ \kappa\left(a^{*\gamma}_{\dot\alpha\dot\beta}\, b^{\dot\alpha\dot\beta}_{\gamma} + b^{*\alpha\beta}_{\dot\gamma}\, a^{\dot\gamma}_{\alpha\beta} \right) + \left(p^{\beta}_{\dot\gamma}\, d^{*\alpha} a^{\dot\gamma}_{\alpha\beta} + p^{\gamma}_{\dot\beta}\, c^{*}_{\dot\alpha}\, b^{\dot\alpha\dot\beta}_{\gamma} -\right.$$

$$\left. - a^{*\gamma}_{\dot\alpha\dot\beta}\, p^{\dot\beta}_{\gamma}\, d^{\dot\alpha} - b^{*\alpha\beta}_{\dot\gamma}\, p^{\dot\gamma}_{\beta}\, c_\alpha \right) + 3\left(d^{*\alpha} p_{\alpha\dot\beta}\, d^{\dot\beta} + c^{*}_{\dot\alpha}\, p^{\dot\alpha\beta} c_\beta \right) +$$

$$+ 6\kappa\left(d^{*\alpha} c_\alpha + c^{*}_{\dot\alpha}\, d^{\dot\alpha} \right).$$

By variation one obtaines the field equations

$$- p^{\dot\beta\rho} a^{\dot\alpha}_{\gamma\rho} - p^{\dot\alpha\rho} a^{\dot\beta}_{\gamma\rho} - p^{\dot\beta}_{\gamma} d^{\dot\alpha} - p^{\dot\alpha}_{\gamma} d^{\dot\beta} + 2\kappa\, b^{\dot\alpha\dot\beta}_{\gamma} = 0 ,$$

$$- p_{\alpha\dot\rho}\, b^{\dot\gamma\dot\rho}_{\beta} - p_{\beta\dot\rho}\, b^{\dot\gamma\dot\rho}_{\alpha} - p^{\dot\gamma}_{\beta} c_\alpha - p^{\dot\gamma}_{\alpha} c_\beta + 2\kappa\, a^{\dot\gamma}_{\alpha\beta} = 0 ,$$

$$- p^{\beta}_{\dot\gamma}\, a^{\dot\gamma}_{\alpha\beta} + 3 p_{\alpha\dot\beta}\, d^{\dot\beta} + 6\kappa c_\alpha = 0 ,$$

$$- p^{\gamma}_{\dot\beta}\, b^{\dot\alpha\dot\beta}_{\gamma} + 3 p^{\dot\alpha\beta} c_\beta + 6\kappa d^{\dot\alpha} = 0 .$$

(We write the Pauli-Fierz wave-equation according to S.N. Gupta[67] and not according to Pauli and Fierz because there are misprints in their equations leading to a wave-equation with matrix $\mathbb{L}_0$ non-singular.).

Giving to the indices α, γ, β, the values $1, 2$, and to the indices $\dot\alpha, \dot\gamma, \dot\beta$, the values $\dot 1, \dot 2$ and summing over $\rho = 1, 2$, the above set of equations reads in detail:

$$-p^{\dot{1}1}a^{\dot{1}}_{11}-p^{\dot{1}2}a^{\dot{1}}_{12}-p^{\dot{1}}_{1}d^{\dot{1}}+\kappa b^{\dot{1}\dot{1}}_{1}=0 \quad ,$$

$$-p^{\dot{1}1}a^{\dot{1}}_{12}-p^{\dot{1}2}a^{\dot{1}}_{22}-p^{\dot{1}}_{2}d^{\dot{1}}+\kappa b^{\dot{1}\dot{1}}_{2}=0 \quad ,$$

$$-\frac{1}{2}p^{\dot{1}1}a^{\dot{2}}_{11}-\frac{1}{2}p^{\dot{1}2}a^{\dot{2}}_{12}-\frac{1}{2}p^{\dot{2}1}a^{\dot{1}}_{11}-\frac{1}{2}p^{\dot{2}2}a^{\dot{1}}_{12}-\frac{1}{2}p^{\dot{1}}_{1}d^{\dot{2}}-\frac{1}{2}p^{\dot{2}}_{1}d^{\dot{1}}+\kappa b^{\dot{1}\dot{2}}_{1}=0 \quad ,$$

$$-\frac{1}{2}p^{\dot{1}1}a^{\dot{2}}_{12}-\frac{1}{2}p^{\dot{1}2}a^{\dot{2}}_{22}-\frac{1}{2}p^{\dot{2}1}a^{\dot{1}}_{12}-\frac{1}{2}p^{\dot{2}2}a^{\dot{1}}_{22}-\frac{1}{2}p^{\dot{1}}_{2}d^{\dot{2}}-\frac{1}{2}p^{\dot{2}}_{2}d^{\dot{1}}+\kappa b^{\dot{1}\dot{2}}_{2}=0 \quad ,$$

$$-p^{\dot{2}1}a^{\dot{2}}_{11}-p^{\dot{2}2}a^{\dot{2}}_{12}-p^{\dot{2}}_{1}d^{\dot{2}}+\kappa b^{\dot{2}\dot{2}}_{1}=0 \quad ,$$

$$-p^{\dot{2}1}a^{\dot{2}}_{12}-p^{\dot{2}2}a^{\dot{2}}_{22}-p^{\dot{2}}_{2}d^{\dot{2}}+\kappa b^{\dot{2}\dot{2}}_{2}=0 \quad ,$$

$$-p_{1\dot{1}}b^{\dot{1}\dot{1}}_{1}-p_{1\dot{2}}b^{\dot{1}\dot{2}}_{1}-p^{\dot{1}}_{1}c_{1}+\kappa a^{\dot{1}}_{11}=0 \quad ,$$

$$-p_{1\dot{1}}b^{\dot{1}\dot{2}}_{1}-p_{1\dot{2}}b^{\dot{2}\dot{2}}_{1}-p^{\dot{2}}_{1}c_{1}+\kappa a^{\dot{2}}_{11}=0 \quad ,$$

$$-\frac{1}{2}p_{2\dot{1}}b^{\dot{1}\dot{1}}_{1}-\frac{1}{2}p_{2\dot{2}}b^{\dot{1}\dot{2}}_{1}-\frac{1}{2}p_{1\dot{1}}b^{\dot{1}\dot{1}}_{2}-\frac{1}{2}p_{1\dot{2}}b^{\dot{1}\dot{2}}_{2}-\frac{1}{2}p^{\dot{1}}_{1}c_{2}-\frac{1}{2}p^{\dot{1}}_{2}c_{1}+\kappa a^{\dot{1}}_{12}=0 \quad ,$$

$$-\frac{1}{2}p_{2\dot{1}}b^{\dot{1}\dot{2}}_{1}-\frac{1}{2}p_{2\dot{2}}b^{\dot{2}\dot{2}}_{1}-\frac{1}{2}p_{1\dot{1}}b^{\dot{1}\dot{2}}_{2}-\frac{1}{2}p_{1\dot{2}}b^{\dot{2}\dot{2}}_{2}-\frac{1}{2}p^{\dot{2}}_{1}c_{2}-\frac{1}{2}p^{\dot{2}}_{2}c_{1}+\kappa a^{\dot{2}}_{12}=0 \quad ,$$

$$-p_{2\dot{1}}b^{\dot{1}\dot{1}}_{2}-p_{2\dot{2}}b^{\dot{1}\dot{2}}_{2}-p^{\dot{1}}_{2}c_{2}+\kappa a^{\dot{1}}_{22}=0 \quad ,$$

$$-p_{2\dot{1}}b^{\dot{1}\dot{2}}_{2}-p_{2\dot{2}}b^{\dot{2}\dot{2}}_{2}-p^{\dot{2}}_{2}c_{2}+\kappa a^{\dot{2}}_{22}=0 \quad ,$$

$$-\frac{1}{6}p^{1}_{\dot{1}}a^{\dot{1}}_{11}-\frac{1}{6}p^{1}_{\dot{2}}a^{\dot{2}}_{11}-\frac{1}{6}p^{2}_{\dot{1}}a^{\dot{1}}_{12}-\frac{1}{6}p^{2}_{\dot{2}}a^{\dot{2}}_{12}+\frac{3}{6}p_{1\dot{1}}d^{\dot{1}}+\frac{3}{6}p_{1\dot{2}}d^{\dot{2}}+\kappa c_{1}=0 \quad ,$$

$$-\frac{1}{6}p^{1}_{\dot{1}}a^{\dot{1}}_{12}-\frac{1}{6}p^{1}_{\dot{2}}a^{\dot{2}}_{12}-\frac{1}{6}p^{2}_{\dot{1}}a^{\dot{1}}_{22}-\frac{1}{6}p^{2}_{\dot{2}}a^{\dot{2}}_{22}+\frac{3}{6}p_{2\dot{1}}d^{\dot{1}}+\frac{3}{6}p_{2\dot{2}}d^{\dot{2}}+\kappa c_{2}=0 \quad ,$$

$$-\frac{1}{6}p^{1}_{\dot{1}}b^{\dot{1}\dot{1}}_{1}-\frac{1}{6}p^{2}_{\dot{1}}b^{\dot{1}\dot{1}}_{2}-\frac{1}{6}p^{1}_{\dot{2}}b^{\dot{1}\dot{2}}_{1}-\frac{1}{6}p^{2}_{\dot{2}}b^{\dot{1}\dot{2}}_{2}+\frac{3}{6}p^{\dot{1}1}c_{1}+\frac{3}{6}p^{\dot{1}2}c_{2}+\kappa d^{\dot{1}}=0 \quad ,$$

$$-\frac{1}{6}p^{1}_{\dot{1}}b^{\dot{1}\dot{2}}_{1}-\frac{1}{6}p^{2}_{\dot{1}}b^{\dot{1}\dot{2}}_{2}-\frac{1}{6}p^{1}_{\dot{2}}b^{\dot{2}\dot{2}}_{1}-\frac{1}{6}p^{2}_{\dot{2}}b^{\dot{2}\dot{2}}_{2}+\frac{3}{6}p^{\dot{2}1}c_{1}+\frac{3}{6}p^{\dot{2}2}c_{2}+\kappa d^{\dot{2}}=0 \,.$$

Making use of the formulae connecting the spinor components $p_{\dot{\alpha}\rho}$ with the four momentum vector components p_{k} ,

$(k=0,1,2,3)$, the above equations can be put in four vector form. Replacing we have:

$$-(-P_0-P_3)\alpha^{\dot{1}}_{11}-(-P_1-\iota P_2)\alpha^{\dot{1}}_{12}-(P_1+\iota P_2)d^{\dot{1}}+\kappa b_1^{\dot{1}\dot{1}}=0,$$

$$-(-P_0-P_3)\alpha^{\dot{1}}_{12}-(-P_1-\iota P_2)\alpha^{\dot{1}}_{22}-(-P_0-P_3)d^{\dot{1}}+\kappa b_2^{\dot{1}\dot{1}}=0,$$

$$-\frac{1}{2}(-P_0-P_3)\alpha^{\dot{2}}_{11}-\frac{1}{2}(-P_1-\iota P_2)\alpha^{\dot{2}}_{12}-\frac{1}{2}(-P_1+\iota P_2)\alpha^{\dot{1}}_{11}-\frac{1}{2}(-P_0+P_3)\alpha^{\dot{1}}_{12}-$$
$$-\frac{1}{2}(P_1+\iota P_2)d^{\dot{2}}-\frac{1}{2}(P_0-P_3)d^{1}+\kappa b_1^{\dot{1}\dot{2}}=0,$$

$$-\frac{1}{2}(-P_0-P_3)\alpha^{\dot{2}}_{12}-\frac{1}{2}(-P_1-\iota P_2)\alpha^{\dot{2}}_{22}-\frac{1}{2}(-P_1+\iota P_2)\alpha^{\dot{1}}_{12}-\frac{1}{2}(-P_0+P_3)\alpha^{\dot{1}}_{22}-$$
$$-\frac{1}{2}(-P_0-P_3)d^{\dot{2}}-\frac{1}{2}(-P_1+\iota P_2)d^{\dot{1}}+\kappa b_2^{\dot{1}\dot{2}}=0,$$

$$-(-P_1+\iota P_2)\alpha^{\dot{2}}_{11}-(-P_0+P_3)\alpha^{\dot{2}}_{12}-(P_0-P_3)d^{\dot{2}}+\kappa b_1^{\dot{2}\dot{2}}=0,$$

$$-(-P_1+\iota P_2)\alpha^{\dot{2}}_{12}-(-P_0+P_3)\alpha^{\dot{2}}_{22}-(-P_1+\iota P_2)d^{\dot{2}}+\kappa b_2^{\dot{2}\dot{2}}=0,$$

$$-(-P_0+P_3)b_1^{\dot{1}\dot{1}}-(P_1+\iota P_2)b_1^{\dot{1}\dot{2}}-(P_1+\iota P_2)c_1+\kappa\alpha^{\dot{1}}_{11}=0,$$

$$-(-P_0+P_3)b_1^{\dot{1}\dot{2}}-(P_1+\iota P_2)b_1^{\dot{2}\dot{2}}-(P_0-P_3)c_1+\kappa\alpha^{\dot{2}}_{11}=0,$$

$$-\frac{1}{2}(P_1-\iota P_2)b_1^{\dot{1}\dot{1}}-\frac{1}{2}(-P_0-P_3)b_1^{\dot{1}\dot{2}}-\frac{1}{2}(-P_0-P_3)b_2^{\dot{1}\dot{1}}-\frac{1}{2}(P_1+\iota P_2)b_2^{\dot{1}\dot{2}}-$$
$$-\frac{1}{2}(P_1+\iota P_2)c_2-\frac{1}{2}(-P_0-P_3)c_1-\kappa\alpha^{\dot{1}}_{12}=0,$$

$$-\frac{1}{2}(P_1-\iota P_2)b_1^{\dot{1}\dot{2}}-\frac{1}{2}(-P_0-P_3)b_1^{\dot{2}\dot{2}}-\frac{1}{2}(-P_0+P_3)b_2^{\dot{1}\dot{2}}-\frac{1}{2}(P_1+\iota P_2)b_2^{\dot{2}\dot{2}}-$$
$$-\frac{1}{2}(P_0-P_3)c_2-\frac{1}{2}(-P_1+\iota P_2)c_1+\kappa\alpha^{2}_{12}=0,$$

$$-(P_1-\iota P_2)b_2^{\dot{1}\dot{1}}-(-P_0-P_3)b_2^{\dot{1}\dot{2}}-(-P_0-P_3)c_2+\kappa\alpha^{\dot{1}}_{22}=0,$$

$$-(P_1-\iota P_2)b_2^{\dot{1}\dot{2}}-(-P_0-P_3)b_2^{\dot{2}\dot{2}}-(-P_1+\iota P_2)c_2+\kappa\alpha^{\dot{2}}_{22}=0,$$

$$-\frac{1}{6}\left(p_1-\iota p_2\right)\alpha^{\dot{1}}_{11}-\frac{1}{6}\left(-p_0-p_3\right)\alpha^{\dot{2}}_{11}-\frac{1}{6}\left(p_0-p_3\right)\alpha^{\dot{1}}_{12}-\frac{1}{6}\left(-p_1-\iota p_2\right)\alpha^{\dot{2}}_{12}+$$

$$+\frac{3}{6}\left(-p_0+p_3\right)d^{\dot{1}}+\frac{3}{6}\left(p_1+\iota p_2\right)d^{\dot{2}}+\kappa c_1=0,$$

$$-\frac{1}{6}\left(p_1-\iota p_2\right)\alpha^{\dot{1}}_{12}-\frac{1}{6}\left(-p_0-p_3\right)\alpha^{\dot{2}}_{12}-\frac{1}{6}\left(p_0-p_3\right)\alpha^{\dot{1}}_{22}-\frac{1}{6}\left(-p_1-\iota p_2\right)\alpha^{\dot{2}}_{22}+$$

$$+\frac{3}{6}\left(p_1-\iota p_2\right)d^{\dot{1}}+\frac{3}{6}\left(-p_0-p_3\right)d^{\dot{2}}+\kappa c_2=0,$$

$$-\frac{1}{6}\left(p_1-\iota p_2\right)b^{\dot{1}\dot{1}}_1-\frac{1}{6}\left(p_0-p_3\right)b^{\dot{1}\dot{1}}_2-\frac{1}{6}\left(-p_0-p_3\right)b^{\dot{1}\dot{2}}_1-\frac{1}{6}\left(-p_1-\iota p_2\right)b^{\dot{1}\dot{2}}_2+$$

$$+\frac{3}{6}\left(-p_0-p_3\right)c_1+\frac{3}{6}\left(-p_1-\iota p_2\right)c_2+\kappa d^{\dot{1}}=0,$$

$$-\frac{1}{6}\left(p_1-\iota p_2\right)b^{\dot{1}\dot{2}}_1-\frac{1}{6}\left(p_0-p_3\right)b^{\dot{1}\dot{2}}_2-\frac{1}{6}\left(-p_0-p_3\right)b^{\dot{2}\dot{2}}_1-\frac{1}{6}\left(-p_1-\iota p_2\right)b^{\dot{2}\dot{2}}_2+$$

$$+\frac{3}{6}\left(-p_1+\iota p_2\right)c_1+\frac{3}{6}\left(-p_0+p_3\right)c_2+\kappa d^{\dot{2}}=0.$$

After some rearrangements these equations can be brought to the form

$$\left(-\mathbb{L}_0 p_0+\mathbb{L}_1 p_1+\mathbb{L}_2 p_2+\mathbb{L}_3 p_3+\kappa\right)\psi=0,$$

where

$$\mathbb{L}_0=\begin{array}{cccccccc|cccccccc}
\alpha^{\dot{1}}_{11} & \alpha^{\dot{1}}_{12} & \alpha^{\dot{1}}_{22} & \alpha^{\dot{2}}_{11} & \alpha^{\dot{2}}_{12} & \alpha^{\dot{2}}_{22} & d^{\dot{1}} & d^{\dot{2}} & b^{\dot{1}\dot{1}}_1 & b^{\dot{1}\dot{2}}_1 & b^{\dot{2}\dot{2}}_1 & b^{\dot{1}\dot{1}}_2 & b^{\dot{1}\dot{2}}_2 & b^{\dot{2}\dot{2}}_2 & c_1 & c_2 \\
\hline
 & & & & & & & & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & 0 & -1/2 & 0 & -1/2 & 0 & 0 & -1/2 & 0 \\
 & & & & & & & & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\
 & & & & & & & & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & & 0 & 0 & -1/2 & 0 & -1/2 & 0 & 0 & 1/2 \\
 & & & & & & & & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
 & & & & & & & & 0 & -1/6 & 0 & 1/6 & 0 & 0 & 1/2 & 0 \\
 & & & & & & & & 0 & 0 & -1/6 & 0 & 1/6 & 0 & 0 & 1/2 \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & -1/2 & 0 & -1/2 & 0 & 0 & 1/2 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & & & & & & & & \\
0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & & & & & & & & \\
0 & 0 & -1/2 & 0 & -1/2 & 0 & 0 & -1/2 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & & & & & & & & \\
0 & 1/6 & 0 & -1/6 & 0 & 0 & 1/2 & 0 & & & & & & & & \\
0 & 0 & 1/6 & 0 & -1/6 & 0 & 0 & 1/2 & & & & & & & &
\end{array},$$

$$\mathbb{L}_{\cdot 1} = \left[\begin{array}{|c|c|c|c|c|c|c|c||c|c|c|c|c|c|c|c|}
 & & & & & & & & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ \hline
 & & & & & & & & -1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 & -1/2 \\ \hline
 & & & & & & & & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline
 & & & & & & & & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline
 & & & & & & & & 0 & -1/2 & 0 & 0 & 0 & -1/2 & 1/2 & 0 \\ \hline
 & & & & & & & & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ \hline\hline
 & & & & & & & & -1/6 & 0 & 0 & 0 & 1/6 & 0 & 0 & -1/2 \\ \hline
 & & & & & & & & 0 & -1/6 & 0 & 0 & 0 & 1/6 & -1/2 & 0 \\ \hline\hline
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & & & & & & & & \\ \hline
1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & -1/2 & & & & & & & & \\ \hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & & & & & & & & \\ \hline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\ \hline
0 & 1/2 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & & & & & & & & \\ \hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & & & & & & & & \\ \hline\hline
-1/6 & 0 & 0 & 0 & 1/6 & 0 & 0 & 1/2 & & & & & & & & \\ \hline
0 & -1/6 & 0 & 0 & 0 & 1/6 & 1/2 & 0 & & & & & & & &
\end{array}\right],$$

$$\mathbb{L}_2 = \begin{array}{|cccccccc|cccccccc|}
 & & & & & & & & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\
 & & & & & & & & 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 & -1/2 \\
 & & & & & & & & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & & & & & & & & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & 0 & 1/2 & 0 & 0 & 0 & -1/2 & -1/2 & 0 \\
 & & & & & & & & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
 & & & & & & & & 1/6 & 0 & 0 & 0 & 1/6 & 0 & 0 & -1/2 \\
 & & & & & & & & 0 & 1/6 & 0 & 0 & 0 & 1/6 & 1/2 & 0 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & & & & & & & & \\
-1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & -1/2 & & & & & & & & \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & -1/2 & 0 & 0 & 0 & 1/2 & -1/2 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & & & & & & & & \\
1/6 & 0 & 0 & 0 & 1/6 & 0 & 0 & 1/2 & & & & & & & & \\
0 & 1/6 & 0 & 0 & 0 & 1/6 & -1/2 & 0 & & & & & & & &
\end{array} ,$$

$$\mathbb{L}_3=\left(\begin{array}{cccccccc|cccccccc}
 & & & & & & & & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & 0 & 1/2 & 0 & -1/2 & 0 & 0 & 1/2 & 0 \\
 & & & & & & & & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
 & & & & & & & & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & & 0 & 0 & 1/2 & 0 & -1/2 & 0 & 0 & 1/2 \\
 & & & & & & & & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & & & & & & & 0 & 1/6 & 0 & 1/6 & 0 & 0 & -1/2 & 0 \\
 & & & & & & & & 0 & 0 & 1/6 & 0 & 1/6 & 0 & 0 & 1/2 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & -1/2 & 0 & 1/2 & 0 & 0 & 1/2 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & & & & & & & & \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & & & & & & & & \\
0 & 0 & -1/2 & 0 & 1/2 & 0 & 0 & 1/2 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & & & & & & & & \\
0 & 1/6 & 0 & 1/6 & 0 & 0 & 1/2 & 0 & & & & & & & & \\
0 & 0 & 1/6 & 0 & 1/6 & 0 & 0 & -1/2 & & & & & & & &
\end{array}\right),$$

and $\psi = \{ \alpha^{\dot1}_{11}, \alpha^{\dot1}_{12}, \alpha^{\dot1}_{22}, \alpha^{\dot2}_{11}, \alpha^{\dot2}_{12}, \alpha^{\dot2}_{22}, d^{\dot1}, d^{\dot2}, b^{\dot1\dot1}_{1}, b^{\dot1\dot2}_{1}, b^{\dot2\dot2}_{1}, b^{\dot1\dot1}_{2}, b^{\dot1\dot2}_{2}, b^{\dot2\dot2}_{2}, c_1, c_2 \}^{tr}.$

The matrix $\mathbb{L}_0$ has $\det \mathbb{L}_0 = 0$ and satisfies the minimal equation

$$\mathbb{L}_0^2 \{ \mathbb{L}_0^2 - 1 \} = 0 .$$

The electromagnetic field is introduced into the Pauli-Fierz equations by implementing the minimal substitution on the Lagrangian density i.e. by replacing p_0, p_1, p_2, p_3 respectively by $\pi_0, \pi_1, \pi_2, \pi_3$ or equivalently by replacing the spinor $p_{\alpha\dot\beta}$ by the spinor $\pi_{\alpha\dot\beta}$. Again variation of the Lagrangian density gives the equations

$$-\pi^{\dot\beta\rho}\alpha^{\dot\alpha}_{\gamma\rho} - \pi^{\dot\alpha\rho}\alpha^{\dot\beta}_{\gamma\rho} - \pi^{\dot\beta}_{\gamma} d^{\dot\alpha} - \pi^{\dot\alpha}_{\gamma} d^{\dot\beta} + 2K b^{\dot\alpha\dot\beta}_{\gamma} = 0,$$

$$-\pi_{\alpha\dot\rho} b^{\dot\gamma\dot\rho}_{\beta} - \pi_{\beta\dot\rho} b^{\dot\gamma\dot\rho}_{\alpha} - \pi^{\dot\gamma}_{\beta} c_{\alpha} - \pi^{\dot\gamma}_{\alpha} c_{\beta} + 2K\alpha^{\dot\gamma}_{\alpha\beta} = 0,$$

$$-\pi^{\beta}_{\dot\gamma}\alpha^{\dot\gamma}_{\alpha\beta} + 3\pi_{\alpha\dot\beta} d^{\dot\beta} + 6K c_{\alpha} = 0,$$

$$-\pi^{\gamma}_{\dot\beta} b^{\dot\alpha\dot\beta}_{\gamma} + 3\pi^{\dot\alpha\beta} c_{\beta} + 6K d^{\dot\alpha} = 0.$$

These equations, after giving to the indices their respective values and replacing the spinors $\pi_{\alpha\dot\beta}$ by their four vector equivalents, acquire the form

$$\left(-\mathbb{L}_0\pi_0+\mathbb{L}_1\pi_1+\mathbb{L}_2\pi_2+\mathbb{L}_3\pi_3+\kappa\right)\psi=0,$$

where the matrices $\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$, are the same as for the free field case.

4: Generators of the Pauli-Fierz equation in the spinor basis.

We give now the generators of the Pauli-Fierz equation in the spinor basis $\{a^{\dot\alpha}_{\beta\gamma}, d^{\dot\alpha}, b^{\dot\alpha\dot\beta}_{\gamma}, c_\alpha\}$:

$$\mathbb{H}_+=\imath A_1-A_2=\begin{array}{cccccc|cc|cccccc|cc}
a^{\dot1}_{11} & a^{\dot1}_{12} & a^{\dot1}_{22} & a^{\dot2}_{11} & a^{\dot2}_{12} & a^{\dot2}_{22} & d^{\dot1} & d^{\dot2} & b^{\dot1\dot1}_{1} & b^{\dot1\dot2}_{1} & b^{\dot2\dot2}_{1} & b^{\dot1\dot1}_{2} & b^{\dot1\dot2}_{2} & b^{\dot2\dot2}_{2} & c_1 & c_2 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & & \\
-1 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & & \\
0 & -2 & 0 & 0 & 0 & 0 & & & & & & & & & & \\
-1 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & & \\
0 & -1 & 0 & -1 & 0 & 0 & & & & & & & & & & \\
0 & 0 & -1 & 0 & -2 & 0 & & & & & & & & & & \\
\hline
 & & & & & & 0 & 0 & & & & & & & & \\
 & & & & & & -1 & 0 & & & & & & & & \\
\hline
 & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & & \\
 & & & & & & & & -1 & 0 & 0 & 0 & 0 & 0 & & \\
 & & & & & & & & 0 & -2 & 0 & 0 & 0 & 0 & & \\
 & & & & & & & & -1 & 0 & 0 & 0 & 0 & 0 & & \\
 & & & & & & & & 0 & -1 & 0 & -1 & 0 & 0 & & \\
 & & & & & & & & 0 & 0 & -1 & 0 & -2 & 0 & & \\
\hline
 & & & & & & & & & & & & & & 0 & 0 \\
 & & & & & & & & & & & & & & -1 & 0
\end{array},$$

$$\mathbb{H}_3=\imath A_3=\operatorname{diag}\left\{-3/2, -1/2, 1/2, -1/2, 1/2, 3/2 \middle| -1/2, 1/2 \middle| -3/2, -1/2, 1/2, -1/2, 1/2, 3/2 \middle| -1/2, 1/2\right\},$$

$$\mathbb{H}_-=\iota\mathbb{A}_1+\mathbb{A}_2=\left[\begin{array}{cccccccccccccccc}
0&-2&0&-1&0&0&&&&&&&&&&\\
0&0&-1&0&-1&0&&&&&&&&&&\\
0&0&0&0&0&-1&&&&&&&&&&\\
0&0&0&0&-2&0&&&&&&&&&&\\
0&0&0&0&0&-1&&&&&&&&&&\\
0&0&0&0&0&0&&&&&&&&&&\\
&&&&&&0&-1&&&&&&&&\\
&&&&&&0&0&&&&&&&&\\
&&&&&&&&0&-2&0&-1&0&0&&\\
&&&&&&&&0&0&-1&0&-1&0&&\\
&&&&&&&&0&0&0&0&0&-1&&\\
&&&&&&&&0&0&0&0&-2&0&&\\
&&&&&&&&0&0&0&0&0&-1&&\\
&&&&&&&&0&0&0&0&0&0&&\\
&&&&&&&&&&&&&&0&-1\\
&&&&&&&&&&&&&&0&0
\end{array}\right],$$

$$\mathbb{F}_+=\iota\mathbb{B}_1-\mathbb{B}_2=\left[\begin{array}{cccccccccccccccc}
0&0&0&0&0&0&&&&&&&&&&\\
-\iota&0&0&0&0&0&&&&&&&&&&\\
0&-2\iota&0&0&0&0&&&&&&&&&&\\
\iota&0&0&0&0&0&&&&&&&&&&\\
0&\iota&0&-\iota&0&0&&&&&&&&&&\\
0&0&\iota&0&-2\iota&0&&&&&&&&&&\\
&&&&&&0&0&&&&&&&&\\
&&&&&&\iota&0&&&&&&&&\\
&&&&&&&&0&0&0&0&0&0&&\\
&&&&&&&&\iota&0&0&0&0&0&&\\
&&&&&&&&0&2\iota&0&0&0&0&&\\
&&&&&&&&-\iota&0&0&0&0&0&&\\
&&&&&&&&0&-\iota&0&\iota&0&0&&\\
&&&&&&&&0&0&-\iota&0&2\iota&0&&\\
&&&&&&&&&&&&&&0&0\\
&&&&&&&&&&&&&&-\iota&0
\end{array}\right],$$

$$\mathbb{F}_3 = \iota\,\mathbb{B}_3 = \left\{ -\iota/2, \iota/2, 3\iota/2, -3\iota/2, -\iota/2, \iota/2 \,\middle|\, \iota/2, -\iota/2 \,\middle|\, \iota/2, -\iota/2, -3\iota/2, 3\iota/2, \iota/2, -\iota/2 \,\middle|\, -\iota/2, \iota/2 \right\},$$

$$\mathbb{F}_- = \iota\,\mathbb{B}_1 + \mathbb{B}_2 = \left[\begin{array}{cccccc|cc|cccccc|cc}
0 & -2\iota & 0 & \iota & 0 & 0 & & & & & & & & & & \\
0 & 0 & -\iota & 0 & \iota & 0 & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \iota & & & & & & & & & & \\
0 & 0 & 0 & 0 & -2\iota & 0 & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & -\iota & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & & \\
\hline
 & & & & & & 0 & \iota & & & & & & & & \\
 & & & & & & 0 & 0 & & & & & & & & \\
\hline
 & & & & & & & & 0 & 2\iota & 0 & -\iota & 0 & 0 & & \\
 & & & & & & & & 0 & 0 & \iota & 0 & -\iota & 0 & & \\
 & & & & & & & & 0 & 0 & 0 & 0 & 0 & -\iota & & \\
 & & & & & & & & 0 & 0 & 0 & 0 & 2\iota & 0 & & \\
 & & & & & & & & 0 & 0 & 0 & 0 & 0 & \iota & & \\
 & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & & \\
\hline
 & & & & & & & & & & & & & & 0 & -\iota \\
 & & & & & & & & & & & & & & 0 & 0
\end{array}\right].$$

The above generators satisfy the commutation relations:

$$[\mathbb{H}_+,\mathbb{H}_3]_- = -\mathbb{H}_+,\ [\mathbb{H}_-,\mathbb{H}_3]_- = \mathbb{H}_-,\ [\mathbb{H}_+,\mathbb{H}_-]_- = 2\mathbb{H}_3,\ [\mathbb{H}_+,\mathbb{F}_+] = 0,\ [\mathbb{H}_-,\mathbb{F}_-]_- = 0,$$

$$[\mathbb{H}_3,\mathbb{F}_3]_- = 0,\ [\mathbb{F}_+,\mathbb{F}_3]_- = \mathbb{H}_+,\ [\mathbb{F}_-,\mathbb{F}_3]_- = -\mathbb{H}_-,\ [\mathbb{F}_+,\mathbb{F}_-]_- = -2\mathbb{H}_3,\ [\mathbb{H}_+,\mathbb{F}_3]_- = -\mathbb{F}_+,$$

$$[\mathbb{H}_-,\mathbb{F}_3]_- = \mathbb{F}_-,\ [\mathbb{H}_+,\mathbb{F}_-]_- = 2\mathbb{F}_3,\ [\mathbb{H}_-,\mathbb{F}_+]_- = -2\mathbb{F}_3,\ [\mathbb{F}_+,\mathbb{H}_3]_- = -\mathbb{F}_+,\ [\mathbb{F}_-,\mathbb{H}_3]_- = \mathbb{F}_-.$$

The matrix $\mathbb{L}_0$ and the generators $\mathbb{F}_3, \mathbb{H}_\pm, \mathbb{H}_3$, satisfy the relations

$$[\mathbb{L}_0,\mathbb{H}_+]_- = [\mathbb{L}_0,\mathbb{H}_-]_- = [\mathbb{L}_0,\mathbb{H}_3]_- = 0,\quad [[\mathbb{F}_3,\mathbb{L}_0]_-,\mathbb{F}_3]_- = \mathbb{L}_0,$$

while the matrices $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$, are connected with $\mathbb{L}_0$ via the relations

$$\mathbb{L}_1 = -[\mathbb{B}_1, \mathbb{L}_0]_- \,, \quad \mathbb{L}_2 = -[\mathbb{B}_2, \mathbb{L}_0]_- \,, \quad \mathbb{L}_3 = -[\mathbb{B}_3, \mathbb{L}_0]_- \,.$$

If instead of the generators, $\mathbb{H}_\pm, \mathbb{H}_3, \mathbb{F}_\pm, \mathbb{F}_3$, the generators $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3$, are considered, then with the matrices $\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$, satisfy the following commutation relations necessary for relativistic invariance of the equation:

$$\mathbb{L}_1 = [\mathbb{L}_0, \mathbb{B}_1]_- \,, \quad \mathbb{L}_2 = [\mathbb{L}_0, \mathbb{B}_2]_- \,, \quad \mathbb{L}_3 = [\mathbb{L}_0, \mathbb{B}_3]_- \,, \quad [\mathbb{L}_1, \mathbb{B}_2]_- = 0 \,,$$

$$[\mathbb{L}_1, \mathbb{B}_3]_- = 0 \,, \quad [\mathbb{L}_1, \mathbb{B}_1]_- = \mathbb{L}_0 \,, \quad [\mathbb{L}_2, \mathbb{B}_1]_- = 0, \quad [\mathbb{L}_2, \mathbb{B}_3]_- = 0, \quad [\mathbb{L}_2, \mathbb{B}_2]_- = \mathbb{L}_0 \,,$$

$$[\mathbb{L}_3, \mathbb{B}_1]_- = 0 \,, \quad [\mathbb{L}_3, \mathbb{B}_2]_- = 0 \,, \quad [\mathbb{L}_3, \mathbb{B}_3]_- = \mathbb{L}_0 \,, \quad [\mathbb{L}_0, \mathbb{A}_1]_- = 0 \,, \quad [\mathbb{L}_0, \mathbb{A}_2]_- = 0 \,,$$

$$[\mathbb{L}_0, \mathbb{A}_3]_- = 0 \,, \quad [\mathbb{L}_1, \mathbb{A}_1]_- = 0 \,, \quad [\mathbb{L}_2, \mathbb{A}_2]_- = 0 \,, \quad [\mathbb{L}_3, \mathbb{A}_3]_- = 0 \,, \quad [\mathbb{L}_1, \mathbb{A}_2]_- = \mathbb{L}_3 \,,$$

$$[\mathbb{L}_1, \mathbb{A}_3]_- = -\mathbb{L}_2 \,, \quad [\mathbb{L}_2, \mathbb{A}_1]_- = -\mathbb{L}_3 \,, \quad [\mathbb{L}_2, \mathbb{A}_3]_- = \mathbb{L}_1 \,, \quad [\mathbb{L}_3, \mathbb{A}_1]_- = \mathbb{L}_2 \,,$$

$$[\mathbb{L}_3, \mathbb{A}_2]_- = -\mathbb{L}_1 \,, \quad [[\mathbb{L}_0, \mathbb{B}_3]_-, \mathbb{B}_3]_- = \mathbb{L}_0 \,.$$

The generators $\mathbb{A}_i, \mathbb{B}_i, (i=1,2,3)$ satisfy the following commutation relations:

$$[\mathbb{A}_1, \mathbb{A}_2]_- = \mathbb{A}_3 \,, \quad [\mathbb{A}_2, \mathbb{A}_3]_- = \mathbb{A}_1 \,, \quad [\mathbb{A}_3, \mathbb{A}_1]_- = \mathbb{A}_2 \,, \quad [\mathbb{B}_1, \mathbb{B}_2]_- = -\mathbb{A}_3 \,,$$

$$[\mathbb{B}_2, \mathbb{B}_3]_- = -\mathbb{A}_1 \,, \quad [\mathbb{B}_3, \mathbb{B}_1]_- = -\mathbb{A}_2 \,, \quad [\mathbb{A}_1, \mathbb{B}_1]_- = 0 \,, \quad [\mathbb{A}_2, \mathbb{B}_3]_- = 0 \,,$$

$$[\mathbb{A}_3, \mathbb{B}_3]_- = 0 \,, \quad [\mathbb{A}_1, \mathbb{B}_2]_- = \mathbb{B}_3 \,, \quad [\mathbb{A}_1, \mathbb{B}_3]_- = -\mathbb{B}_2 \,, \quad [\mathbb{A}_2, \mathbb{B}_3]_- = \mathbb{B}_1 \,,$$

$$[\mathbb{A}_2, \mathbb{B}_1]_- = -\mathbb{B}_3 \,, \quad [\mathbb{A}_3, \mathbb{B}_1]_- = \mathbb{B}_2 \,, \quad [\mathbb{A}_3, \mathbb{B}_2]_- = -\mathbb{B}_1 \,.$$

5: The equivalence between the canonical and the spinorial form of the Pauli-Fierz equation.

We establish now the similarity transformation $\mathbb{S}$ transforming the Pauli-Fierz equation from the canonical basis to the Pauli-Fierz equation expressed in the spinor basis. This transformation must be such that it transforms the generators $\mathbb{H}_{\pm}^{C}, \mathbb{H}_{3}^{C}, \mathbb{F}_{\pm}^{C}, \mathbb{F}_{3}^{C}$, and the matrices $\mathbb{L}_{0}^{C}, \mathbb{L}_{1}^{C}, \mathbb{L}_{2}^{C}, \mathbb{L}_{3}^{C}$, of the equation expressed in the canonical frame, to the generators $\mathbb{H}_{\pm}^{S}, \mathbb{H}_{3}^{S}, \mathbb{F}_{\pm}^{S}, \mathbb{F}_{3}^{S}$, and the matrices $\mathbb{L}_{0}^{S}, \mathbb{L}_{1}^{S}, \mathbb{L}_{2}^{S}, \mathbb{L}_{3}^{S}$, of the equation expressed in the spinor basis, i.e.

$$S\mathbb{H}_{3}^{C}S^{-1} = \mathbb{H}_{3}^{S} \quad , \quad S\mathbb{H}_{+}^{C}S^{-1} = \mathbb{H}_{+}^{S} \quad , \quad S\mathbb{H}_{-}^{C}S^{-1} = \mathbb{H}_{-}^{S} \, ,$$

$$S\mathbb{F}_{3}^{C}S^{-1} = \mathbb{F}_{3}^{S} \quad , \quad S\mathbb{F}_{+}^{C}S^{-1} = \mathbb{F}_{+}^{S} \quad , \quad S\mathbb{F}_{-}^{C}S^{-1} = \mathbb{F}_{-}^{S} \, ,$$

$$S\mathbb{L}_{0}^{C}S^{-1} = \mathbb{L}_{0}^{S} \, , \; S\mathbb{L}_{1}S^{-1} = \mathbb{L}_{1}^{S} \, , \; S\mathbb{L}_{2}^{C}S^{-1} = \mathbb{L}_{2}^{S} \, , \; S\mathbb{L}_{3}^{C}S^{-1} = \mathbb{L}_{3}^{S} \, ,$$

where the superscripts C and S indicate the canonical and the spinor bases respectively. From these relations only the first seven are absolutely necessary. The last three follow from them as follows,

$$S\mathbb{L}_{1}^{C}S^{-1} = S[\mathbb{L}_{0}^{C}, \mathbb{B}_{1}^{C}]_{-}S^{-1} = S\left[\mathbb{L}_{0}^{C}, \frac{\mathbb{F}_{+}^{C}+\mathbb{F}_{-}^{C}}{2\iota}\right]_{-}S^{-1} =$$

$$= \left[S\mathbb{L}_{0}^{C}S^{-1}, \frac{S\mathbb{F}_{+}^{C}S^{-1}+S\mathbb{F}_{-}^{C}S^{-1}}{2\iota}\right]_{-} = \left[\mathbb{L}_{0}^{S}, \frac{\mathbb{F}_{+}^{S}+\mathbb{F}_{-}^{S}}{2\iota}\right]_{-} = [\mathbb{L}_{0}^{S}, \mathbb{B}_{1}^{S}] = \mathbb{L}_{1}^{S} \, ,$$

$$S\mathbb{L}_{2}^{C}S^{-1} = S[\mathbb{L}_{0}^{C}, \mathbb{B}_{2}^{C}]_{-}S^{-1} = S\left[\mathbb{L}_{0}^{C}, \frac{-\mathbb{F}_{+}^{C}+\mathbb{F}_{-}^{C}}{2}\right]_{-}S^{-1} =$$

$$= \left[S\mathbb{L}_{0}^{C}S^{-1}, \frac{-S\mathbb{F}_{+}^{C}S^{-1}+S\mathbb{F}_{-}^{C}S^{-1}}{2}\right]_{-} = \left[\mathbb{L}_{0}^{S}, \frac{-\mathbb{F}_{+}^{S}+\mathbb{F}_{-}^{S}}{2}\right]_{-} = [\mathbb{L}_{0}^{S}, \mathbb{B}_{2}^{S}]_{-} = \mathbb{L}_{2}^{S} \, ,$$

$$S\mathbb{L}_{3}^{C}S^{-1} = S[\mathbb{L}_{0}^{C}, \mathbb{B}_{3}^{C}]_{-}S^{-1} = S\left[\mathbb{L}_{0}^{C}, \frac{\mathbb{F}_{3}^{C}}{\iota}\right]_{-}S^{-1} = \left[S\mathbb{L}_{0}^{C}S^{-1}, \frac{S\mathbb{F}_{3}^{C}S^{-1}}{\iota}\right]_{-} =$$

$$= \left[\mathbb{L}_{0}^{S}, \frac{\mathbb{F}_{3}^{S}}{\iota}\right]_{-} = [\mathbb{L}_{0}^{S}, \mathbb{B}_{3}^{S}]_{-} = \mathbb{L}_{3}^{S} \; . \; .$$

Thus assuming that S is 16 x 16 matrix (c_{ij}) and requring that it satisfies the relations

$$S\mathbb{H}_{3}^{C} = \mathbb{H}_{3}^{S}S \; , \; S\mathbb{H}_{+}^{C} = \mathbb{H}_{+}^{S}S \, , \; S\mathbb{H}_{-}^{C} = \mathbb{H}_{-}^{S}S \, , \; S\mathbb{F}_{3}^{C} = \mathbb{F}_{3}^{S}S \; , S\mathbb{F}_{+}^{C} = \mathbb{F}_{+}^{S}S \, ,$$

$$S\mathbb{F}_{-}^{C} = \mathbb{F}_{-}^{S}S \; , \; S\mathbb{L}_{0}^{C} = \mathbb{L}_{0}^{S}S \, ,$$

we find a number of relations which the matrix elements c_{ij} must satisfy among themselves. After solving these equations simultaneously we find that the matrix S of the similarity transformation is,

$$S = \left[\begin{array}{cccccccccccccccc}
 & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 1 \\
 & & & & & & & & & & 0 & i/\sqrt{6} & 0 & 0 & -1/\sqrt{3} & 0 \\
 & & & & & & & & & & -2i/\sqrt{6} & 0 & 0 & 1/\sqrt{3} & 0 & 0 \\
 & & & & & & & & & & 0 & -2i/\sqrt{6} & 0 & 0 & -1/\sqrt{3} & 0 \\
 & & & & & & & & & & i/\sqrt{6} & 0 & 0 & 1/\sqrt{3} & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -i/\sqrt{6} & & & & & & & & & & & & & & \\
i/\sqrt{6} & 0 & & & & & & & & & & & & & & \\
 & & & & 0 & 0 & 0 & 0 & 0 & -1 & & & & & & \\
 & & & & 0 & i/\sqrt{6} & 0 & 0 & 1/\sqrt{3} & 0 & & & & & & \\
 & & & & -2i/\sqrt{6} & 0 & 0 & -1/\sqrt{3} & 0 & 0 & & & & & & \\
 & & & & 0 & -2i/\sqrt{6} & 0 & 0 & 1/\sqrt{3} & 0 & & & & & & \\
 & & & & i/\sqrt{6} & 0 & 0 & -1/\sqrt{3} & 0 & 0 & & & & & & \\
 & & & & 0 & 0 & 1 & 0 & 0 & 0 & & & & & & \\
 & & 0 & i/\sqrt{6} & & & & & & & & & & & & \\
 & & -i/\sqrt{6} & 0 & & & & & & & & & & & &
\end{array}\right].$$

The inverse of S is,

$$S^{-1} = \left[\begin{array}{cccccccccccccccc}
 & & & & & & 0 & -i\sqrt{6} & & & & & & & & \\
 & & & & & & i\sqrt{6} & 0 & & & & & & & & \\
 & & & & & & & & & & & & & & 0 & i\sqrt{6} \\
 & & & & & & & & & & & & & & -i\sqrt{6} & 0 \\
 & & & & & & & & 0 & 0 & i\sqrt{6}/3 & 0 & -i\sqrt{6}/3 & 0 & & \\
 & & & & & & & & 0 & -i\sqrt{6}/3 & 0 & i\sqrt{6}/3 & 0 & 0 & & \\
 & & & & & & & & 0 & 0 & 0 & 0 & 0 & 1 & & \\
 & & & & & & & & 0 & 0 & -1/\sqrt{3} & 0 & -2/\sqrt{3} & 0 & & \\
 & & & & & & & & 0 & 2/\sqrt{3} & 0 & 1/\sqrt{3} & 0 & 0 & & \\
 & & & & & & & & -1 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & i\sqrt{6}/3 & 0 & -i\sqrt{6}/3 & 0 & & & & & & & & & & \\
0 & -i\sqrt{6}/3 & 0 & i\sqrt{6}/3 & 0 & 0 & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & -1 & & & & & & & & & & \\
0 & 0 & 1/\sqrt{3} & 0 & 2/\sqrt{3} & 0 & & & & & & & & & & \\
0 & -2/\sqrt{3} & 0 & -1/\sqrt{3} & 0 & 0 & & & & & & & & & & \\
1 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & &
\end{array}\right].$$

In terms of transformation formulae connecting the two bases we have:

$$\alpha^{\dot{1}}_{11} = 1\,\xi^{\dot{\tau}_2}_{3/2,-3/2} ,$$

$$\alpha^{\dot{1}}_{12} = \frac{i}{\sqrt{6}}\,\xi^{\dot{\tau}_2}_{1/2,-1/2} - \frac{1}{\sqrt{3}}\,\xi^{\dot{\tau}_2}_{3/2,-1/2} ,$$

$$\alpha^{\dot{1}}_{22} = -\frac{2i}{\sqrt{6}}\,\xi^{\dot{\tau}_2}_{1/2,1/2} + \frac{1}{\sqrt{3}}\,\xi^{\dot{\tau}_2}_{3/2,1/2} ,$$

$$\alpha^{\dot{2}}_{11} = -\frac{2i}{\sqrt{6}}\,\xi^{\dot{\tau}_2}_{1/2,-1/2} - \frac{1}{\sqrt{3}}\,\xi^{\dot{\tau}_2}_{3/2,-1/2} ,$$

$$\alpha^{\dot{2}}_{12} = \frac{i}{\sqrt{6}}\,\xi^{\dot{\tau}_2}_{1/2,1/2} + \frac{1}{\sqrt{3}}\,\xi^{\dot{\tau}_2}_{3/2,1/2} ,$$

$$\alpha^{\dot{2}}_{22} = -1\,\xi^{\dot{\tau}_2}_{3/2,3/2} ,$$

$$d^{\dot{1}} = -\frac{i}{\sqrt{6}}\,\xi^{\tau_1}_{1/2,-1/2} ,$$

$$d^{\dot{2}} = \frac{i}{\sqrt{6}}\,\xi^{\tau_1}_{1/2,1/2} ,$$

$$b^{\dot{1}\dot{1}}_{1} = -1\,\xi^{\tau_2}_{3/2,-3/2} ,$$

$$b^{\dot{1}\dot{2}}_{1} = \frac{i}{\sqrt{6}}\,\xi^{\tau_2}_{1/2,-1/2} + \frac{1}{\sqrt{3}}\,\xi^{\tau_2}_{3/2,-1/2} ,$$

$$b^{\dot{2}\dot{2}}_{1} = -\frac{2i}{\sqrt{6}}\,\xi^{\tau_2}_{1/2,1/2} - \frac{1}{\sqrt{3}}\,\xi^{\tau_2}_{3/2,1/2} ,$$

$$b^{\dot{1}\dot{1}}_{2} = -\frac{2i}{\sqrt{6}}\,\xi^{\tau_2}_{1/2,-1/2} + \frac{1}{\sqrt{3}}\,\xi^{\tau_2}_{3/2,-1/2} ,$$

$$b^{\dot{1}\dot{2}}_{2} = \frac{i}{\sqrt{6}}\,\xi^{\tau_2}_{1/2,1/2} - \frac{1}{\sqrt{3}}\,\xi^{\tau_2}_{3/2,1/2} ,$$

$$b^{\dot{2}\dot{2}}_{2} = 1\,\xi^{\tau_2}_{3/2,3/2} .$$

$$c_1 = \frac{i}{\sqrt{6}}\,\xi^{\dot{\tau}_1}_{1/2,-1/2} ,$$

$$c_2 = -\frac{i}{\sqrt{6}}\,\xi^{\dot{\tau}_1}_{1/2,1/2} .$$

The inverted formulae are:

$$\xi^{\tau_1}_{1/2,\,1/2} = -i\sqrt{6}\, d^{\dot{2}} \ ,$$

$$\xi^{\tau_1}_{1/2,\,-1/2} = i\sqrt{6}\, d^{\dot{1}} \ ,$$

$$\xi^{\dot{\tau}_1}_{1/2,\,1/2} = i\sqrt{6}\, c_2 \ ,$$

$$\xi^{\dot{\tau}_1}_{1/2,\,-1/2} = -i\sqrt{6}\, c_1 \ ,$$

$$\xi^{\tau_2}_{1/2,\,1/2} = \frac{i\sqrt{6}}{3}\, b^{\dot{2}\dot{2}}_{1} - \frac{i\sqrt{6}}{3}\, b^{\dot{1}\dot{2}}_{2} \ ,$$

$$\xi^{\tau_2}_{1/2,\,-1/2} = -\frac{i\sqrt{6}}{3}\, b^{\dot{1}\dot{2}}_{1} + \frac{i\sqrt{6}}{3}\, b^{\dot{1}\dot{1}}_{2} \ ,$$

$$\xi^{\tau_2}_{3/2,\,3/2} = i\, b^{\dot{2}\dot{2}}_{2} \ ,$$

$$\xi^{\tau_2}_{3/2,\,1/2} = -\frac{1}{\sqrt{3}}\, b^{\dot{2}\dot{2}}_{1} - \frac{2}{\sqrt{3}}\, b^{\dot{1}\dot{2}}_{2} \ ,$$

$$\xi^{\tau_2}_{3/2,\,-1/2} = \frac{2}{\sqrt{3}}\, b^{\dot{1}\dot{2}}_{1} + \frac{1}{\sqrt{3}}\, b^{\dot{1}\dot{1}}_{2} \ ,$$

$$\xi^{\tau_2}_{3/2,\,-3/2} = -i\, b^{\dot{1}\dot{1}}_{1} \ ,$$

$$\xi^{\dot{\tau}_2}_{1/2,\,1/2} = \frac{i\sqrt{6}}{3}\, \alpha^{\dot{1}}_{22} - \frac{i\sqrt{6}}{3}\, \alpha^{\dot{2}}_{12} \ ,$$

$$\xi^{\dot{\tau}_2}_{1/2,\,-1/2} = -\frac{i\sqrt{6}}{3}\, \alpha^{\dot{1}}_{12} + \frac{i\sqrt{6}}{3}\, \alpha^{\dot{2}}_{11} \ ,$$

$$\xi^{\dot{\tau}_2}_{3/2,\,3/2} = -i\, \alpha^{\dot{2}}_{22} \ ,$$

$$\xi^{\dot{\tau}_2}_{3/2,\,1/2} = \frac{1}{\sqrt{3}}\, \alpha^{\dot{1}}_{22} + \frac{2}{\sqrt{3}}\, \alpha^{\dot{2}}_{12} \ ,$$

$$\xi^{\dot{\tau}_2}_{3/2,\,-1/2} = -\frac{2}{\sqrt{3}}\, \alpha^{\dot{1}}_{12} - \frac{1}{\sqrt{3}}\, \alpha^{\dot{2}}_{11} \ ,$$

$$\xi^{\dot{\tau}_2}_{3/2,\,-3/2} = i\, \alpha^{\dot{1}}_{11} \ .$$

By implementing the similarity transformation $\mathcal{S}$ on the Pauli-Fierz equation expressed in the canonical basis (c.f. § 1) we find the Pauli-Fierz equation expressed in the spinor basis (c.f. § 3). Thus the equivalence of the two forms of the equation is established, i.e.

$$\left(\begin{array}{l}\text{Pauli-Fierz equation}\\ \text{in the cononical frame}\end{array}\right) \xleftrightarrow[\text{equivalent}]{\text{is}} \left(\begin{array}{l}\text{Pauli-Fierz equation}\\ \text{in the spinorial frame}\end{array}\right).$$

6: The equivalence between the Rarita-Schwinger equation expressed in the vector spinor basis and the Pauli-Fierz equation expressed in the canonical basis.

We are now establishing in this paragraph the equivalence between the Rarita-Schwinger equation for spin 3/2 expressed in the vector spinor basis, and the Pauli-Fierz equation for spin 3/2 expressed in the canonical basis by giving the similarity transformation connecting them. Before doing so we give those common features of the two equations which point in the direction of the existence of a similarity transformation between them. These common features are:

Rarita-Schwinger eq.	Pauli-Fierz eq.
1) Is based on the 16-dim. representation $(1/2,3/2)\oplus(-1/2,3/2)\oplus(1/2,5/2)\oplus(-1/2,5/2)$.	1) Is based on the 16-dim. representation $(1/2,3/2)\oplus(-1/2,3/2)\oplus(1/2,5/2)\oplus(-1/2,5/2)$.
2) $\mathbb{L}_0$ is singular, has rank 12, and satisfies the minimal equation $\mathbb{L}_0^2\{\mathbb{L}_0^2-1\}=0$.	2) $\mathbb{L}_0$ is singular, has rank 12, and satisfies the minimal equation $\mathbb{L}_0^2\{\mathbb{L}_0^2-1\}=0$.
3) The equation has four subsidiary conditions of the first kind (i.e. equations involving only the space derivatives)	3) The equation has four subsidiary conditions of the first kind and four of the second kind which we shall give in due course.

and four subsidiary conditions of the second kind (i.e. conditions not involving any derivatives).

4) $\mathbb{L}_0$ accepts the Jordan canonical form:

$$\left(\begin{array}{c|c|c|c|c} \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & & & & \\ \hline & \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & & & \\ \hline & & \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & & \\ \hline & & & \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & \\ \hline & & & & \begin{smallmatrix}1&&&&&&&\\&-1&&&&&&\\&&1&&&&&\\&&&-1&&&&\\&&&&1&&&\\&&&&&-1&&\\&&&&&&1&\\&&&&&&&-1\end{smallmatrix} \end{array}\right).$$

4) $\mathbb{L}_0$ accepts the Jordan canonical form:

$$\left(\begin{array}{c|c|c|c|c} \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & & & & \\ \hline & \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & & & \\ \hline & & \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & & \\ \hline & & & \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & \\ \hline & & & & \begin{smallmatrix}1&&&&&&&\\&-1&&&&&&\\&&1&&&&&\\&&&-1&&&&\\&&&&1&&&\\&&&&&-1&&\\&&&&&&1&\\&&&&&&&-1\end{smallmatrix} \end{array}\right).$$

We establish now the similarity transformation connecting the two equations.

In chapter (10) the generators $\mathbb{H}_+^R$, $\mathbb{H}_-^R$, $\mathbb{H}_3^R$, $\mathbb{F}_+^R$, $\mathbb{F}_-^R$, $\mathbb{F}_3^R$, and the matrices $\mathbb{L}_0^R, \mathbb{L}_1^R, \mathbb{L}_2^R, \mathbb{L}_3^R$, of the Rarita-Schwinger equation in the vector spinor basis $\{\psi_i{}^j\}$, $(i, j = 0, 1, 2, 3)$ were given. Likewise in the present chapter the generators $\mathbb{H}_+^C, \mathbb{H}_-^C, \mathbb{H}_3^C, \mathbb{F}_+^C$, $\mathbb{F}_-^C$, $\mathbb{F}_3^C$, and the matrices $\mathbb{L}_0^C$, $\mathbb{L}_1^C$, $\mathbb{L}_2^C$, $\mathbb{L}_3^C$, of the Pauli-Fierz equation in the canonical basis $\{\xi^T_{\ell m}\}$ were given.

If a similarity transformation $\mathbb{S}$ exists connecting the two equations, must be such that it satisfies the following relations:

$$\mathbb{S}\,\mathbb{H}_+^R\mathbb{S}^{-1} = \mathbb{H}_+^C \quad , \quad \mathbb{S}\,\mathbb{H}_-^R\mathbb{S}^{-1} = \mathbb{H}_-^C \quad , \quad \mathbb{S}\,\mathbb{H}_3^R\mathbb{S}^{-1} = \mathbb{H}_3^C ,$$

$$\mathbb{S}\,\mathbb{F}_+^R\mathbb{S}^{-1} = \mathbb{F}_+^C \quad , \quad \mathbb{S}\,\mathbb{F}_-^R\mathbb{S}^{-1} = \mathbb{F}_-^C \quad , \quad \mathbb{S}\,\mathbb{F}_3^R\mathbb{S}^{-1} = \mathbb{F}_3^C ,$$

$$\mathbb{S}\,\mathbb{L}_0^R\mathbb{S}^{-1} = \mathbb{L}_0^C \ , \ \mathbb{S}\,\mathbb{L}_1^R\mathbb{S}^{-1} = \mathbb{L}_1^C \ , \ \mathbb{S}\,\mathbb{L}_2^R\mathbb{S}^{-1} = \mathbb{L}_2^C \ , \ \mathbb{S}\,\mathbb{L}_3^R\mathbb{S}^{-1} = \mathbb{L}_3^C .$$

Indeed such a transformation can be found by the same procedure outlined in the previous paragraph and is:

$$\mathcal{S} = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & i & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -i & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1/3 & 0 & 0 & 0 & i/3 & 0 & 0 & -1/3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1/3 & 0 & 0 & 0 & -i/3 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \\
0 & 0 & 0 & 0 & i\sqrt{2}/\sqrt{3} & 0 & 0 & 0 & \sqrt{2}/\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & i\sqrt{2}/3 & 0 & 0 & 0 & \sqrt{2}/3 & 0 & 0 & 2i\sqrt{2}/3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i\sqrt{2}/3 & 0 & 0 & 0 & \sqrt{2}/3 & 0 & 0 & 0 & 0 & -2i\sqrt{2}/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -i\sqrt{2}/\sqrt{3} & 0 & 0 & 0 & \sqrt{2}/\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 & 0 & -i/3 & 0 & 0 & 1/3 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1/3 & 0 & 0 & 0 & i/3 & 0 & 0 & 0 & 0 & -1/3 \\
0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{2}/\sqrt{3} & 0 & 0 & 0 & \sqrt{2}/\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{2}/3 & 0 & 0 & 0 & \sqrt{2}/3 & 0 & 0 & -2i\sqrt{2}/3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{2}/3 & 0 & 0 & 0 & \sqrt{2}/3 & 0 & 0 & 0 & 0 & -2i\sqrt{2}/3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{2}/\sqrt{3} & 0 & 0 & 0 & \sqrt{2}/\sqrt{3} & 0 & 0 & 0 & 0
\end{bmatrix}.$$

The inverse is:

$$\mathcal{S}^{-1} = \begin{bmatrix}
0 & 0 & 1/4 & 0 & 3/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/4 & 0 & 3/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/4 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/4 & 0 & -1/4 & \frac{-i\sqrt{3}}{2\sqrt{2}} & 0 & i/2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/4 & 0 & -1/4 & 0 & 0 & -i/2\sqrt{2} & 0 & \frac{i\sqrt{3}}{2\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & \frac{-i\sqrt{3}}{2\sqrt{2}} & 0 & i/2\sqrt{2} & 0 \\
-1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 & -i/2\sqrt{2} & 0 & \frac{i\sqrt{3}}{2\sqrt{2}} \\
0 & 0 & 0 & -i/4 & 0 & i/4 & \sqrt{3}/2\sqrt{2} & 0 & 1/2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & i/4 & 0 & -i/4 & 0 & 0 & i/2\sqrt{2} & 0 & \sqrt{3}/2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & i/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i/4 & \sqrt{3}/2\sqrt{2} & 0 & 1/2\sqrt{2} & 0 \\
-i/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i/4 & 0 & 0 & 1/2\sqrt{2} & 0 & \sqrt{3}/2\sqrt{2} \\
0 & 0 & 1/4 & 0 & -1/4 & 0 & 0 & i/\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/4 & 0 & 1/4 & 0 & 0 & i/\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 & i/\sqrt{2} & 0 & 0 \\
0 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/4 & 0 & 0 & i/\sqrt{2} & 0
\end{bmatrix}.$$

In terms of transformation equations connecting the two bases we have:

$$\xi^{\tau_1}_{1/2,1/2} = 1\psi^2_0 - 1\psi^3_1 + \iota\psi^3_2 - 1\psi^2_3 \ ,$$

$$\xi^{\tau_1}_{1/2,-1/2} = 1\psi^3_0 - 1\psi^2_1 - \iota\psi^2_2 + 1\psi^3_3 \ ,$$

$$\xi^{\dot{\tau}_1}_{1/2,1/2} = 1\psi^0_0 + 1\psi^1_1 - \iota\psi^1_2 + 1\psi^0_3 \ ,$$

$$\xi^{\dot{\tau}_1}_{1/2,-1/2} = 1\psi^1_0 + 1\psi^0_1 + \iota\psi^0_2 - 1\psi^1_3 \ ,$$

$$\xi^{\tau_2}_{1/2,1/2} = 1\psi^0_0 - \frac{1}{3}\psi^1_1 + \frac{\iota}{3}\psi^1_2 - \frac{1}{3}\psi^0_3 \ ,$$

$$\xi^{\tau_2}_{1/2,-1/2} = 1\psi^1_0 - \frac{1}{3}\psi^0_1 - \frac{\iota}{3}\psi^0_2 + \frac{1}{3}\psi^1_3 \ ,$$

$$\xi^{\tau_2}_{3/2,3/2} = \frac{\iota\sqrt{2}}{\sqrt{3}}\psi^0_1 + \frac{\sqrt{2}}{\sqrt{3}}\psi^0_2 \ ,$$

$$\xi^{\tau_2}_{3/2,1/2} = \frac{\iota\sqrt{2}}{3}\psi^1_1 + \frac{\sqrt{2}}{3}\psi^1_2 - \frac{2\iota\sqrt{2}}{3}\psi^0_3 \ ,$$

$$\xi^{\tau_2}_{3/2,-1/2} = -\frac{\iota\sqrt{2}}{3}\psi^0_1 + \frac{\sqrt{2}}{3}\psi^0_2 - \frac{2\iota\sqrt{2}}{3}\psi^1_3 \ ,$$

$$\xi^{\tau_2}_{3/2,-3/2} = -\frac{\iota\sqrt{2}}{\sqrt{3}}\psi^1_1 + \frac{\sqrt{2}}{\sqrt{3}}\psi^1_2 \ ,$$

$$\xi^{\dot{\tau}_2}_{1/2,1/2} = 1\psi^2_0 + \frac{1}{3}\psi^3_1 - \frac{\iota}{3}\psi^3_2 + \frac{1}{3}\psi^2_3 \ ,$$

$$\xi^{\dot{\tau}_2}_{1/2,-1/2} = 1\psi^3_0 + \frac{1}{3}\psi^2_1 + \frac{\iota}{3}\psi^2_2 - \frac{1}{3}\psi^3_3 \ ,$$

$$\xi^{\dot{\tau}_2}_{3/2,3/2} = \frac{\iota\sqrt{2}}{\sqrt{3}}\psi^2_1 + \frac{\sqrt{2}}{\sqrt{3}}\psi^2_2 \ ,$$

$$\xi^{\dot{\tau}_2}_{3/2,1/2} = \frac{\iota\sqrt{2}}{3}\psi^3_1 + \frac{\sqrt{2}}{3}\psi^3_2 - \frac{2\iota\sqrt{2}}{3}\psi^2_3 \ ,$$

$$\xi^{\dot{\tau}_2}_{3/2,-1/2} = -\frac{\iota\sqrt{2}}{3}\psi^2_1 + \frac{\sqrt{2}}{3}\psi^2_2 - \frac{2\iota\sqrt{2}}{3}\psi^3_3 \ ,$$

$$\xi^{\dot{\tau}_2}_{3/2,-3/2} = -\frac{\iota\sqrt{2}}{\sqrt{3}}\psi^3_1 + \frac{\sqrt{2}}{\sqrt{3}}\psi^3_2 \ .$$

The inverted equations are:

$$\psi_0^0 = \frac{1}{4}\xi^{\dot\tau_1}_{1/2,1/2} + \frac{3}{4}\xi^{\tau_2}_{1/2,1/2},$$

$$\psi_0^1 = \frac{1}{4}\xi^{\dot\tau_1}_{1/2,-1/2} + \frac{3}{4}\xi^{\tau_2}_{1/2,-1/2},$$

$$\psi_0^2 = \frac{1}{4}\xi^{\tau_1}_{1/2,1/2} + \frac{3}{4}\xi^{\dot\tau_2}_{1/2,1/2},$$

$$\psi_0^3 = \frac{1}{4}\xi^{\tau_1}_{1/2,-1/2} + \frac{3}{4}\xi^{\dot\tau_2}_{1/2,-1/2},$$

$$\psi_1^0 = \frac{1}{4}\xi^{\dot\tau_1}_{1/2,-1/2} - \frac{1}{4}\xi^{\tau_2}_{1/2,-1/2} - \frac{i\sqrt{3}}{2\sqrt{2}}\xi^{\tau_2}_{3/2,3/2} + \frac{i}{2\sqrt{2}}\xi^{\tau_2}_{3/2,-1/2},$$

$$\psi_1^1 = \frac{1}{4}\xi^{\dot\tau_1}_{1/2,1/2} - \frac{1}{4}\xi^{\tau_2}_{1/2,1/2} - \frac{i}{2\sqrt{2}}\xi^{\tau_2}_{3/2,1/2} + \frac{i\sqrt{3}}{2\sqrt{2}}\xi^{\tau_2}_{3/2,-3/2},$$

$$\psi_1^2 = -\frac{1}{4}\xi^{\tau_1}_{1/2,-1/2} + \frac{1}{4}\xi^{\dot\tau_2}_{1/2,-1/2} - \frac{i\sqrt{3}}{2\sqrt{2}}\xi^{\dot\tau_2}_{3/2,3/2} + \frac{i}{3\sqrt{2}}\xi^{\dot\tau_2}_{3/2,-1/2},$$

$$\psi_1^3 = -\frac{1}{4}\xi^{\tau_1}_{1/2,1/2} + \frac{1}{4}\xi^{\dot\tau_2}_{1/2,1/2} - \frac{i}{2\sqrt{2}}\xi^{\dot\tau_2}_{3/2,1/2} + \frac{i\sqrt{3}}{2\sqrt{2}}\xi^{\dot\tau_2}_{3/2,-3/2},$$

$$\psi_2^0 = -\frac{i}{4}\xi^{\dot\tau_1}_{1/2,-1/2} + \frac{i}{4}\xi^{\tau_2}_{1/2,-1/2} + \frac{\sqrt{3}}{2\sqrt{2}}\xi^{\tau_2}_{3/2,3/2} + \frac{1}{2\sqrt{2}}\xi^{\tau_2}_{3/2,-1/2},$$

$$\psi_2^1 = \frac{i}{4}\xi^{\dot\tau_1}_{1/2,1/2} - \frac{i}{4}\xi^{\tau_2}_{1/2,1/2} + \frac{1}{2\sqrt{2}}\xi^{\tau_2}_{3/2,1/2} + \frac{\sqrt{3}}{2\sqrt{2}}\xi^{\tau_2}_{3/2,-3/2},$$

$$\psi_2^2 = \frac{i}{4}\xi^{\tau_1}_{1/2,-1/2} - \frac{i}{4}\xi^{\dot\tau_2}_{1/2,-1/2} + \frac{\sqrt{2}}{2\sqrt{2}}\xi^{\dot\tau_2}_{3/2,3/2} + \frac{1}{2\sqrt{2}}\xi^{\dot\tau_2}_{3/2,-1/2},$$

$$\psi_2^3 = -\frac{i}{4}\xi^{\tau_1}_{1/2,1/2} + \frac{i}{4}\xi^{\dot\tau_2}_{1/2,1/2} + \frac{1}{2\sqrt{2}}\xi^{\dot\tau_2}_{3/2,1/2} + \frac{\sqrt{3}}{2\sqrt{2}}\xi^{\dot\tau_2}_{3/2,-3/2},$$

$$\psi_3^0 = \frac{1}{4}\xi^{\dot\tau_1}_{1/2,1/2} - \frac{1}{4}\xi^{\tau_2}_{1/2,1/2} + \frac{i}{\sqrt{2}}\xi^{\tau_2}_{3/2,1/2},$$

$$\psi_3^1 = -\frac{1}{4}\xi^{\dot\tau_1}_{1/2,-1/2} + \frac{1}{4}\xi^{\tau_2}_{1/2,-1/2} + \frac{i}{\sqrt{2}}\xi^{\tau_2}_{3/2,-1/2},$$

$$\psi_3^2 = -\frac{1}{4}\xi^{\tau_1}_{1/2,1/2} + \frac{1}{4}\xi^{\dot\tau_2}_{1/2,1/2} + \frac{i}{\sqrt{2}}\xi^{\dot\tau_2}_{3/2,1/2},$$

$$\psi_3^3 = \frac{1}{4}\xi^{\tau_1}_{1/2,-1/2} - \frac{1}{4}\xi^{\dot\tau_2}_{1/2,-1/2} + \frac{i}{\sqrt{2}}\xi^{\dot\tau_2}_{3/2,-1/2}.$$

Thus by giving the similarity transformation, the equivalence of the two equations follows: i.e.

$$\left\{\begin{array}{l}\text{Rarita-Schwinger equation}\\ \text{expressed in the vector}\\ \text{spinor basis } \psi_l^{\delta}\,.\end{array}\right\} \xleftrightarrow[\text{equivalent}]{\text{is}} \left\{\begin{array}{l}\text{Pauli-Fierz equation}\\ \text{expressed in the canonical}\\ \text{basis } \xi^{\tau}_{\ell m}\,.\end{array}\right\}$$

As an immediate conclusion of this equivalence we have that the charge of the Rarita-Schwinger equation is definite.

7: The equivalence between the Rarita-Schwinger equation expressed in the vector spinor basis ψ_l^{δ} and the Pauli-Fierz equation expressed in the spinor basis $\{ a^{\dot\alpha}_{\beta\gamma}$, $b^{\dot\alpha\beta}_{\gamma}$, c_α , $d^{\dot\alpha}$ $\}$.

Having found the similarity transformation $\mathbb{S}_1$ (which we have renamed for convenience in what follows) taking the Rarita-Schwinger equation from the vector spinor basis $\{\psi_l^{\delta}\}$ to the canonical basis $\{\xi^{\tau}_{\ell m}\}$, and the similarity transformation $\mathbb{S}_2$ (renamed again) taking the Pauli-Fierz equation from the canonical basis to the spinor basis $\{a^{\dot\alpha}_{\beta\gamma}, b^{\dot\alpha\beta}_{\gamma}, c_\alpha, d^{\dot\alpha}\}$, we construct the similarity transformation $\mathbb{T}$ which takes the Rarita-Schwinger equation from the vector spinor basis to the Pauli-Fierz equation in the spinor basis by multiplying the two transformations as indicated below

$$\mathbb{T} = \mathbb{S}_2\, \mathbb{S}_1 \,.$$

This similarity transformation could also be calculated directly by requiring that $\mathbb{T}$ be such that it satisfies:

$$\mathbb{T}\mathbb{F}^R_\pm\mathbb{T}^{-1} = \mathbb{F}^S_\pm \;,\quad \mathbb{T}\mathbb{F}^R_3\mathbb{T}^{-1} = \mathbb{F}^S_3 \;,\quad \mathbb{T}\mathbb{H}^R_\pm\mathbb{T}^{-1} = \mathbb{H}^S_\pm \;,$$

$$\mathbb{T}\mathbb{H}^R_3\mathbb{T}^{-1} = \mathbb{H}^S_3 \,,\quad \mathbb{T}\mathbb{L}^R_0\mathbb{T}^{-1} = \mathbb{L}^S_0 \;,\quad \mathbb{T}\mathbb{L}^R_k\mathbb{T}^{-1} = \mathbb{L}^S_k \;,\; (k=1,2,3).$$

No matter which method is followed we find for $\mathbb{T}$:

$$T = \left|\begin{array}{cccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{2}/\sqrt{3} & 0 & 0 & 0 & \sqrt{2}/\sqrt{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & i/\sqrt{6} & 0 & 0 & i/\sqrt{6} & 0 & 0 & 0 & -1/\sqrt{6} & 0 & 0 & 0 & 0 & i/\sqrt{6} \\
0 & 0 & -i\sqrt{2}/\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{2}/\sqrt{3} & 0 \\
0 & 0 & 0 & -i\sqrt{2}/\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{2}/\sqrt{3} \\
0 & 0 & i/\sqrt{6} & 0 & 0 & 0 & 0 & i/\sqrt{6} & 0 & 0 & 0 & 1/\sqrt{6} & 0 & 0 & -i/\sqrt{6} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{2}/\sqrt{3} & 0 & 0 & 0 & -\sqrt{2}/\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i/\sqrt{6} & 0 & 0 & i/\sqrt{6} & 0 & 0 & 0 & -1/\sqrt{6} & 0 & 0 & 0 & 0 & -i/\sqrt{6} \\
0 & 0 & i/\sqrt{6} & 0 & 0 & 0 & 0 & -i/\sqrt{6} & 0 & 0 & 0 & -1/\sqrt{6} & 0 & 0 & -i/\sqrt{6} & 0 \\
0 & 0 & 0 & 0 & 0 & i\sqrt{2}/\sqrt{3} & 0 & 0 & 0 & -\sqrt{2}/\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & i/\sqrt{6} & 0 & 0 & -i/\sqrt{6} & 0 & 0 & 0 & 1/\sqrt{6} & 0 & 0 & 0 & 0 & -i/\sqrt{6} & 0 & 0 \\
-i\sqrt{2}/\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{2}/\sqrt{3} & 0 & 0 & 0 \\
0 & -i\sqrt{2}/\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{2}/\sqrt{3} & 0 & 0 \\
i/\sqrt{6} & 0 & 0 & 0 & 0 & -i/\sqrt{6} & 0 & 0 & 0 & -1/\sqrt{6} & 0 & 0 & i/\sqrt{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i\sqrt{2}/\sqrt{3} & 0 & 0 & 0 & \sqrt{2}/\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & i/\sqrt{6} & 0 & 0 & i/\sqrt{6} & 0 & 0 & 0 & -1/\sqrt{6} & 0 & 0 & 0 & 0 & -i/\sqrt{6} & 0 & 0 \\
-i/\sqrt{6} & 0 & 0 & 0 & 0 & -i/\sqrt{6} & 0 & 0 & 0 & -1/\sqrt{6} & 0 & 0 & -i/\sqrt{6} & 0 & 0 & 0
\end{array}\right|.$$

The inverse is

$$T^{-1} = \left|\begin{array}{cccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{6}/4 & 0 & -i\sqrt{6}/4 & 0 & 0 & i\sqrt{6}/4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{6}/4 & 0 & i\sqrt{6}/4 & 0 & 0 & -i\sqrt{6}/4 & 0 \\
0 & 0 & i\sqrt{6}/4 & 0 & -i\sqrt{6}/4 & 0 & 0 & -i\sqrt{6}/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -i\sqrt{6}/4 & 0 & i\sqrt{6}/4 & 0 & 0 & i\sqrt{6}/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{6}/4 & 0 & 0 & 0 & -i\sqrt{6}/4 & -i\sqrt{6}/4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{6}/4 & 0 & 0 & 0 & i\sqrt{6}/4 & 0 & 0 & i\sqrt{6}/4 \\
0 & -i\sqrt{6}/4 & 0 & 0 & 0 & i\sqrt{6}/4 & -i\sqrt{6}/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{i\sqrt{3}}{2\sqrt{2}} & 0 & 0 & 0 & -i\sqrt{6}/4 & 0 & 0 & i\sqrt{6}/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6}/4 & 0 & 0 & 0 & \sqrt{6}/4 & -\sqrt{6}/4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{6}/4 & 0 & 0 & 0 & -\sqrt{6}/4 & 0 & 0 & -\sqrt{6}/4 \\
0 & -\sqrt{6}/4 & 0 & 0 & 0 & -\sqrt{6}/4 & -\sqrt{6}/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{6}/4 & 0 & 0 & 0 & \sqrt{6}/4 & 0 & 0 & -\sqrt{6}/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{6}/4 & 0 & -i\sqrt{6}/4 & 0 & 0 & i\sqrt{6}/4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{6}/4 & 0 & i\sqrt{6}/4 & 0 & 0 & i\sqrt{6}/4 & 0 \\
0 & 0 & i\sqrt{6}/4 & 0 & i\sqrt{6}/4 & 0 & 0 & i\sqrt{6}/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -i\sqrt{6}/4 & 0 & -i\sqrt{6}/4 & 0 & 0 & i\sqrt{6}/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right|.$$

In terms of transformation formulae connecting the two bases

we have:

$$a^{\dot{1}}_{11} = -\frac{\iota\sqrt{2}}{\sqrt{3}}\psi^3_1 + \frac{\sqrt{2}}{\sqrt{3}}\psi^3_2 ,$$

$$a^{\dot{1}}_{12} = \frac{\iota}{\sqrt{6}}\psi^3_0 + \frac{\iota}{\sqrt{6}}\psi^2_1 - \frac{1}{\sqrt{6}}\psi^2_2 + \frac{\iota}{\sqrt{6}}\psi^3_3 ,$$

$$a^{\dot{1}}_{22} = -\frac{\iota\sqrt{2}}{\sqrt{3}}\psi^2_0 - \frac{\iota\sqrt{2}}{\sqrt{3}}\psi^2_3 ,$$

$$a^{\dot{2}}_{11} = -\frac{\iota\sqrt{2}}{\sqrt{3}}\psi^3_0 + \frac{\iota\sqrt{2}}{\sqrt{3}}\psi^3_3 ,$$

$$a^{\dot{2}}_{12} = \frac{\iota}{\sqrt{6}}\psi^2_0 + \frac{\iota}{\sqrt{6}}\psi^3_1 + \frac{1}{\sqrt{6}}\psi^3_2 - \frac{\iota}{\sqrt{6}}\psi^2_3 ,$$

$$a^{\dot{2}}_{22} = -\frac{\iota\sqrt{2}}{\sqrt{3}}\psi^2_1 - \frac{\sqrt{2}}{\sqrt{3}}\psi^2_2 ,$$

$$d^{\dot{1}} = -\frac{\iota}{\sqrt{6}}\psi^3_0 + \frac{\iota}{\sqrt{6}}\psi^2_1 - \frac{1}{\sqrt{6}}\psi^2_2 - \frac{\iota}{\sqrt{6}}\psi^3_3 ,$$

$$d^{\dot{2}} = \frac{\iota}{\sqrt{6}}\psi^2_0 - \frac{\iota}{\sqrt{6}}\psi^3_1 - \frac{1}{\sqrt{6}}\psi^3_2 - \frac{\iota}{\sqrt{6}}\psi^2_3 ,$$

$$b^{\dot{1}\dot{1}}_1 = \frac{\iota\sqrt{2}}{\sqrt{3}}\psi^1_1 - \frac{\sqrt{2}}{\sqrt{3}}\psi^1_2 ,$$

$$b^{\dot{1}\dot{2}}_1 = \frac{\iota}{\sqrt{6}}\psi^1_0 - \frac{\iota}{\sqrt{6}}\psi^0_1 + \frac{1}{\sqrt{6}}\psi^0_2 - \frac{\iota}{\sqrt{6}}\psi^1_3 ,$$

$$b^{\dot{2}\dot{2}}_1 = -\frac{\iota\sqrt{2}}{\sqrt{3}}\psi^0_0 + \frac{\iota\sqrt{2}}{\sqrt{3}}\psi^0_3 ,$$

$$b^{\dot{1}\dot{1}}_2 = -\frac{\iota\sqrt{2}}{\sqrt{3}}\psi^1_0 - \frac{\iota\sqrt{2}}{\sqrt{3}}\psi^1_3 ,$$

$$b^{\dot{1}\dot{2}}_2 = \frac{\iota}{\sqrt{6}}\psi^0_0 - \frac{\iota}{\sqrt{6}}\psi^1_1 - \frac{1}{\sqrt{6}}\psi^1_2 + \frac{\iota}{\sqrt{6}}\psi^0_3 ,$$

$$b^{\dot{2}\dot{2}}_2 = \frac{\iota\sqrt{2}}{\sqrt{3}}\psi^0_1 + \frac{\sqrt{2}}{\sqrt{3}}\psi^0_2 ,$$

$$c_1 = \frac{\iota}{\sqrt{6}}\psi^1_0 + \frac{\iota}{\sqrt{6}}\psi^0_1 - \frac{1}{\sqrt{6}}\psi^0_2 - \frac{\iota}{\sqrt{6}}\psi^1_3 ,$$

$$c_2 = -\frac{\iota}{\sqrt{6}}\psi^0_0 - \frac{\iota}{\sqrt{6}}\psi^1_1 - \frac{1}{\sqrt{6}}\psi^1_2 - \frac{\iota}{\sqrt{6}}\psi^0_3 .$$

The inverted formulae are:

$$\psi_0^0=\frac{\iota\sqrt{6}}{4}\,b_1^{\dot{2}\dot{2}}-\frac{\iota\sqrt{6}}{4}\,b_2^{\dot{1}\dot{2}}+\frac{\iota\sqrt{6}}{4}\,c_2\,,$$

$$\psi_0^1=-\frac{\iota\sqrt{6}}{4}\,b_1^{\dot{1}\dot{2}}+\frac{\iota\sqrt{6}}{4}\,b_2^{\dot{1}\dot{1}}-\frac{\iota\sqrt{6}}{4}\,c_1\,,$$

$$\psi_0^2=\frac{\iota\sqrt{6}}{4}\,\alpha_{22}^{\dot{1}}-\frac{\iota\sqrt{6}}{4}\,\alpha_{12}^{\dot{2}}-\frac{\iota\sqrt{6}}{4}\,d^{\dot{2}}\,,$$

$$\psi_0^3=-\frac{\iota\sqrt{6}}{4}\,\alpha_{12}^{\dot{1}}+\frac{\iota\sqrt{6}}{4}\,\alpha_{11}^{\dot{2}}+\frac{\iota\sqrt{6}}{4}\,d^{\dot{1}}\,,$$

$$\psi_1^0=\frac{\iota\sqrt{6}}{4}\,b_1^{\dot{1}\dot{2}}-\frac{\iota\sqrt{6}}{4}\,b_2^{\dot{2}\dot{2}}-\frac{\iota\sqrt{6}}{4}\,c_1\,,$$

$$\psi_1^1=-\frac{\iota\sqrt{6}}{4}\,b_1^{\dot{1}\dot{1}}+\frac{\iota\sqrt{6}}{4}\,b_2^{\dot{1}\dot{2}}+\frac{\iota\sqrt{6}}{4}\,c_2\,,$$

$$\psi_1^2=-\frac{\iota\sqrt{6}}{4}\,\alpha_{12}^{\dot{1}}+\frac{\iota\sqrt{6}}{4}\,\alpha_{22}^{\dot{2}}-\frac{\iota\sqrt{6}}{4}\,d^{\dot{1}}\,,$$

$$\psi_1^3=\frac{\iota\sqrt{6}}{4}\,\alpha_{11}^{\dot{1}}-\frac{\iota\sqrt{6}}{4}\,\alpha_{12}^{\dot{2}}+\frac{\iota\sqrt{6}}{4}\,d^{\dot{2}}\,,$$

$$\psi_2^0=\frac{\sqrt{6}}{4}\,b_1^{\dot{1}\dot{2}}+\frac{\sqrt{6}}{4}\,b_2^{\dot{2}\dot{2}}-\frac{\sqrt{6}}{4}\,c_1\,,$$

$$\psi_2^1=-\frac{\sqrt{6}}{4}\,b_1^{\dot{1}\dot{1}}-\frac{\sqrt{6}}{4}\,b_2^{\dot{1}\dot{2}}-\frac{\sqrt{6}}{4}\,c_2\,,$$

$$\psi_2^2=-\frac{\sqrt{6}}{4}\,\alpha_{12}^{\dot{1}}-\frac{\sqrt{6}}{4}\,\alpha_{22}^{\dot{2}}-\frac{\sqrt{6}}{4}\,d^{\dot{1}}\,,$$

$$\psi_2^3=\frac{\sqrt{6}}{4}\,\alpha_{11}^{\dot{1}}+\frac{\sqrt{6}}{4}\,\alpha_{12}^{\dot{2}}-\frac{\sqrt{6}}{4}\,d^{\dot{2}}\,,$$

$$\psi_3^0=-\frac{\iota\sqrt{6}}{4}\,b_1^{\dot{2}\dot{2}}-\frac{\iota\sqrt{6}}{4}\,b_2^{\dot{1}\dot{2}}+\frac{\iota\sqrt{6}}{4}\,c_2\,,$$

$$\psi_3^1=\frac{\iota\sqrt{6}}{4}\,b_1^{\dot{1}\dot{2}}+\frac{\iota\sqrt{6}}{4}\,b_2^{\dot{1}\dot{1}}+\frac{\iota\sqrt{6}}{4}\,c_1\,,$$

$$\psi_3^2=\frac{\iota\sqrt{6}}{4}\,\alpha_{22}^{\dot{1}}+\frac{\iota\sqrt{6}}{4}\,\alpha_{12}^{\dot{2}}+\frac{\iota\sqrt{6}}{4}\,d^{\dot{2}}\,,$$

$$\psi_3^3=-\frac{\iota\sqrt{6}}{4}\,\alpha_{12}^{\dot{1}}-\frac{\iota\sqrt{6}}{4}\,\alpha_{11}^{\dot{2}}+\frac{\iota\sqrt{6}}{4}\,d^{\dot{1}}\,.$$

Thus we have established the following equivalence

$$\begin{tikzcd} \left(\begin{array}{l}\text{Rarita-Schwinger equation}\\ \text{expressed in the vector-}\\ \text{spinor basis } \{\psi_\iota\}.\end{array}\right) \arrow[r, leftrightarrow, "\text{is}", "\text{equivalent}"'] & \left\{\begin{array}{l}\text{Pauli-Fierz equation}\\ \text{expressed in the spinor}\\ \text{basis } \{a^{\dot\alpha}_{\beta\gamma}, b^{\dot\alpha\beta}_{\gamma}, c_\alpha, d^{\dot\alpha}\}.\end{array}\right\} \end{tikzcd}$$

Finally we summarize all the equivalence relations among the different forms of the equation in the following diagram:

$$\begin{tikzcd} \left\{\begin{array}{l}\text{Rarita-Schwinger}\\ \text{equation, } \{\psi_\iota\}.\end{array}\right\} \arrow[rr, leftrightarrow, "S_1"] & & \left\{\begin{array}{c}\text{Pauli-Fierz equation,}\\ \{\xi^{\tau}_{\ell m}\}.\end{array}\right\} \arrow[d, leftrightarrow, "S_2"] \\ & & \left\{\begin{array}{c}\text{Pauli-Fierz equation,}\\ \{a^{\dot\alpha}_{\beta\gamma}, b^{\dot\alpha\beta}_{\gamma}, c_\alpha, d^{\dot\alpha}\}.\end{array}\right\} \arrow[ull, leftrightarrow, "T = S_2 S_1"] \end{tikzcd}$$

8: Subsidiary conditions of the first kind.

Before proceeding to the main object of this chapter which is the study of propagation of the Pauli-Fierz equation in an external electromagnetic field we engage ourselves with the problem of subsidiary conditions.

Statement 1: If the matrix $\mathbb{L}_0$ of a Gel'fand-Yaglom equation is singular then the first order system of differential equations

$$\mathbb{L}_0 \frac{\partial\psi}{\partial x_0} + \mathbb{L}_1 \frac{\partial\psi}{\partial x_1} + \mathbb{L}_2 \frac{\partial\psi}{\partial x_2} + \mathbb{L}_3 \frac{\partial\psi}{\partial x_3} + \iota\kappa\psi = 0,$$

acquires constraints of the first kind i.e. differential equations not involving the time derivative but only the space derivatives.

Proof: **Case 1:** ($\mathbb{L}_0$ singular and diagonalizable.)

If $\mathbb{L}_0$ is diagonalizable there exists a similarity transformation S which puts it into diagonal form with the eigenvalues on the main diagonal and elsewhere zeros.

Acting on the set of equations

$$\mathbb{L}_0 \frac{\partial\psi}{\partial x_0} + \mathbb{L}_1 \frac{\partial\psi}{\partial x_1} + \mathbb{L}_2 \frac{\partial\psi}{\partial x_2} + \mathbb{L}_3 \frac{\partial\psi}{\partial x_3} + \iota K\psi = 0,$$

with the similarity transformation S we have

$$S\mathbb{L}_0 S^{-1} \frac{\partial}{\partial x_0}(S\psi) + S\mathbb{L}_1 S^{-1} \frac{\partial}{\partial x_1}(S\psi) + S\mathbb{L}_2 S^{-1} \frac{\partial}{\partial x_2}(S\psi) + S\mathbb{L}_3 S^{-1} \frac{\partial}{\partial x_3}(S\psi) + \iota K S\psi = 0,$$

or

$$\mathbb{L}'_0 \frac{\partial\psi'}{\partial x_0} + \mathbb{L}'_1 \frac{\partial\psi'}{\partial x_1} + \mathbb{L}'_2 \frac{\partial\psi'}{\partial x_2} + \mathbb{L}'_3 \frac{\partial\psi'}{\partial x_3} + \iota K\psi' = 0, \qquad (1)$$

where $\mathbb{L}'_k = S\mathbb{L}_k S^{-1}$, $(k = 0,1,2,3)$, $\psi' = S\psi$, with $\mathbb{L}_0 = S\mathbb{L}_0 S^{-1}$ a diagonal matrix. Since $\mathbb{L}_0$ is singular then $\mathbb{L}'_0 = S\mathbb{L}_0 S^{-1}$ is of the form

$$\mathbb{L}'_0 = S\mathbb{L}_0 S^{-1} = \begin{bmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & & & & \\ & & & & \lambda_1 & & & \\ & & & & & -\lambda_1 & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_\nu \\ & & & & & & & & -\lambda_\nu \end{bmatrix}.$$

This form implies that the time derivatives of some field components ψ'_i will be missing from some differential equations of the set of differential equations (1).

For the sake of demonstration let us consider (4 x 4) matrices and confine ourselves to a two dimensional space-time $(t \equiv x_0\ , x_1)$. Taking

$$\mathbb{L}'_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & -\lambda_1 \end{bmatrix}, \quad \mathbb{L}'_1 = \begin{bmatrix} \alpha_{11}, & \alpha_{12}, & \alpha_{13}, & \alpha_{14} \\ \alpha_{21}, & \alpha_{22}, & \alpha_{23}, & \alpha_{24} \\ \alpha_{31}, & \alpha_{32}, & \alpha_{33}, & \alpha_{34} \\ \alpha_{41}, & \alpha_{42}, & \alpha_{43}, & \alpha_{44} \end{bmatrix}, \quad \psi' = \begin{bmatrix} \psi'_1 \\ \psi'_2 \\ \psi'_3 \\ \psi'_4 \end{bmatrix},$$

we have the set of equations:

i) $\alpha_{11} \frac{\partial}{\partial x_1}\psi'_1 + \alpha_{12}\frac{\partial}{\partial x_1}\psi'_2 + \alpha_{13}\frac{\partial}{\partial x_1}\psi'_3 + \alpha_{14}\frac{\partial}{\partial x_1}\psi'_4 + \iota K\psi'_1 = 0$,

ii) $\alpha_{21} \frac{\partial}{\partial x_1}\psi'_1 + \alpha_{22}\frac{\partial}{\partial x_1}\psi'_2 + \alpha_{23}\frac{\partial}{\partial x_1}\psi'_3 + \alpha_{24}\frac{\partial}{\partial x_1}\psi'_4 + \iota K\psi'_2 = 0$,

iii) $\lambda_1 \frac{\partial}{\partial t}\psi'_3 + \alpha_{31}\frac{\partial}{\partial x_1}\psi'_1 + \alpha_{32}\frac{\partial}{\partial x_1}\psi'_2 + \alpha_{33}\frac{\partial}{\partial x_1}\psi'_3 + \alpha_{34}\frac{\partial}{\partial x_1}\psi'_4 + \iota K\psi_3 = 0$,

iv) $-\lambda_1 \frac{\partial}{\partial t}\psi'_4 + \alpha_{41}\frac{\partial}{\partial x_1}\psi'_1 + \alpha_{42}\frac{\partial}{\partial x_1}\psi'_2 + \alpha_{43}\frac{\partial}{\partial x_1}\psi'_3 + \alpha_{44}\frac{\partial}{\partial x_1}\psi'_4 + \iota K\psi_4 = 0$.

Equations (i) and (ii) do not involve the time derivatives and form the constraints of the first kind.

<u>Case 2</u>: ($\mathbb{L}_0$ singular and non-diagonalizable.)

If $\mathbb{L}_0$ is non-diagonalizable still there is a similarity transformation S which brings it into the Jordan canonical form $\mathbb{L}^J_0 = S\mathbb{L}_0S^{-1}$, and since is singular some of its eigenvalues are zero. The most general minimal equation satisfied by a matrix of this kind is

$$\mathbb{L}_0^{\mu}\{\mathbb{L}_0^2 - \lambda_1^2\}^{\nu}\cdots\{\mathbb{L}_0^2 - \lambda_n^2\}^{\xi} = 0,$$

where $\lambda_1, \cdots, \lambda_n$ the non-zero eigenvalues of $\mathbb{L}_0$ with at least one of the exponents μ, ν, ξ, greater than one. A possible form of $\mathbb{L}^J_0 = S\mathbb{L}_0S^{-1}$ is

$$\mathbb{L}^J_0 = S\mathbb{L}_0S^{-1} = \left[\begin{array}{c|c|c|c} \begin{matrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{matrix} & & & \\ \hline & \begin{matrix}\lambda_1 & 1\\ 0 & -\lambda_1\end{matrix} & & \\ \hline & & \ddots & \\ \hline & & & \begin{matrix}\lambda_\nu & 0\\ 0 & -\lambda_\nu\end{matrix}\end{array}\right].$$

The form of $\mathbb{L}^J_0$ implies that the time derivatives of some field components ψ_i do not appear in the differential equations. It also implies that there are certain equations not involving the time derivatives but having only space derivatives. These are the subsidiary conditions of the first kind.

For the sake of demonstration let us consider 4 x 4 matrices and confine ourselves to a two dimensional space-time $(t \equiv x_0, x_1)$. Taking

$$S\mathbb{L}_0S^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & -\lambda_1 \end{bmatrix}, \quad S\mathbb{L}_1S^{-1} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{bmatrix}, \quad \psi' = \begin{bmatrix} \psi'_1 \\ \psi'_2 \\ \psi'_3 \\ \psi'_4 \end{bmatrix}$$

we have the set of equations:

$$\text{i) } 1\,\frac{\partial \psi_2'}{\partial t} + \alpha_{11}\frac{\partial \psi_1'}{\partial x_1} + \alpha_{12}\frac{\partial \psi_2'}{\partial x_1} + \alpha_{13}\frac{\partial \psi_3'}{\partial x_1} + \alpha_{14}\frac{\partial \psi_4'}{\partial x_1} + i\kappa\psi_1' = 0\,,$$

$$\text{ii) } \alpha_{21}\frac{\partial \psi_1'}{\partial x_1} + \alpha_{22}\frac{\partial \psi_2'}{\partial x_1} + \alpha_{23}\frac{\partial \psi_3'}{\partial x_1} + \alpha_{24}\frac{\partial \psi_4'}{\partial x_1} + i\kappa\psi_2' = 0\,,$$

$$\text{iii) } \lambda_1\frac{\partial \psi_3'}{\partial t} + \alpha_{31}\frac{\partial \psi_1'}{\partial x_1} + \alpha_{32}\frac{\partial \psi_2'}{\partial x_1} + \alpha_{33}\frac{\partial \psi_3'}{\partial x_1} + \alpha_{34}\frac{\partial \psi_4'}{\partial x_1} + i\kappa\psi_3' = 0\,,$$

$$\text{iv) } -\lambda_1\frac{\partial \psi_4'}{\partial t} + \alpha_{41}\frac{\partial \psi_1'}{\partial x_1} + \alpha_{42}\frac{\partial \psi_2'}{\partial x_1} + \alpha_{43}\frac{\partial \psi_3'}{\partial x_1} + \alpha_{44}\frac{\partial \psi_4'}{\partial x_1} + i\kappa\psi_4' = 0\,.$$

Equation (ii) does not involve the time derivatives and acts as a subsidiary condition of the first kind. Also from the above equation we have that the time derivative of ψ_1' does not appear in any of the equations nor can be determined from the subsidiary conditions of the first kind.

<u>Statement 2:</u> The Pauli-Fierz equation has subsidiary conditions of the first kind build in it.

<u>Proof</u>: This follows immediately from case 2 of the previous statement since the matrix $\mathbb{L}_0$ of the Pauli-Fierz equation is singular and non-diagonalizable satisfying a minimal equation of the form

$$\mathbb{L}_0^2\left\{\mathbb{L}_0^2 - 1\right\} = 0\,.$$

Let us consider the Pauli-Fierz equation in the canonical frame $\{\xi^{\tau}_{\ell m}\}$ and in a two dimensional space-time $(x_0\,, x_3)$ i.e.

$$\mathbb{L}_0\frac{\partial \xi}{\partial x_0} + \mathbb{L}_3\frac{\partial \xi}{\partial x_3} + i\kappa\xi = 0\,, \qquad (2)$$

with $\mathbb{L}_0$ given explicitly in paragraph (1) of the present chapter and $\mathbb{L}_3$ given by

$$\mathbb{L}_3 = -[\mathbb{B}_3, \mathbb{L}_0] = \left[\begin{array}{cc|cc|cccccc|cccccc}
\multicolumn{2}{c}{\tau_1} & \multicolumn{2}{c}{\dot{\tau}_1} & \multicolumn{6}{c}{\tau_2} & \multicolumn{6}{c}{\dot{\tau}_2} \\
0 & 0 & -1/2 & 0 & -1/6 & 0 & 0 & \iota\sqrt{2}/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 0 & 1/6 & 0 & 0 & \iota\sqrt{2}/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 0 & 0 & \iota\sqrt{2}/3 & 0 & 0 \\
0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/6 & 0 & 0 & \iota\sqrt{2}/3 & 0 \\
-1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5/6 & 0 & 0 & -\iota\sqrt{2}/3 & 0 & 0 \\
0 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5/6 & 0 & 0 & -\iota\sqrt{2}/3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-\iota\sqrt{2}/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\iota\sqrt{2}/3 & 0 & 0 & 1/3 & 0 & 0 \\
0 & -\iota\sqrt{2}/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\iota\sqrt{2}/3 & 0 & 0 & -1/3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\hline
0 & 0 & 1/6 & 0 & -5/6 & 0 & 0 & -\iota\sqrt{2}/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/6 & 0 & 5/6 & 0 & 0 & -\iota\sqrt{2}/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\iota\sqrt{2}/3 & 0 & -\iota\sqrt{2}/3 & 0 & 0 & -1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\iota\sqrt{2}/3 & 0 & -\iota\sqrt{2}/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],$$

where $\mathbb{B}_3$ is the generator of boosts in the x_3 direction. Operating on (2) with the similarity transformation

$$\mathbb{S} = \left[\begin{array}{cc|cc|cccccc|cccccc}
0 & 0 & -1/2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
\hline
0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right],$$

whose inverse is:

$$S^{-1} = \left[\begin{array}{cc|cc|cccccc|cccccc}
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 \\
\hline
0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 1/2
\end{array}\right],$$

$\mathbb{L}_0$ is put into the following canonical form

$$S\,\mathbb{L}_0 S^{-1} = \left[\begin{array}{cc|cc|cccccc|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right].$$

This form is not the Jordan canonical form but is sufficient for our purpose. $\mathbb{L}_3$ under the similarity transformation goes to

$$
S\mathbb{L}_3S^{-1}=\left(\begin{array}{cc|cc|cccccc|cccccc}
0 & 0 & 0 & 0 & 2/3 & 0 & 0 & \frac{2i\sqrt{2}}{3} & 0 & 0 & 0 & 0 & 0 & \frac{2i\sqrt{2}}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2/3 & 0 & 0 & \frac{2i\sqrt{2}}{3} & 0 & 0 & 0 & 0 & 0 & \frac{2i\sqrt{2}}{3} & 0 \\
\hline
-1/3 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2i\sqrt{2}}{3} & 0 & 0 & -2/3 & 0 & 0 & -\frac{2i\sqrt{2}}{3} & 0 & 0 \\
0 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2i\sqrt{2}}{3} & 0 & 0 & 2/3 & 0 & 0 & -\frac{2i\sqrt{2}}{3} & 0 \\
\hline
-2/3 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2i\sqrt{2}}{3} & 0 & 0 & 0 & 0 & 0 & -\frac{2i\sqrt{2}}{3} & 0 & 0 \\
0 & 2/3 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2i\sqrt{2}}{3} & 0 & 0 & 0 & 0 & 0 & -\frac{2i\sqrt{2}}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-\frac{i\sqrt{2}}{6} & 0 & \frac{i\sqrt{2}}{6} & 0 & -\frac{i\sqrt{2}}{6} & 0 & 0 & 0 & 0 & 0 & \frac{i\sqrt{2}}{6} & 0 & 0 & 1/3 & 0 & 0 \\
0 & -\frac{i\sqrt{2}}{6} & 0 & \frac{i\sqrt{2}}{6} & 0 & -\frac{i\sqrt{2}}{6} & 0 & 0 & 0 & 0 & 0 & \frac{i\sqrt{2}}{6} & 0 & 0 & -1/3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\hline
0 & 0 & 2/3 & 0 & 1/3 & 0 & 0 & \frac{2i\sqrt{2}}{3} & 0 & 0 & 0 & 0 & 0 & \frac{2i\sqrt{2}}{3} & 0 & 0 \\
0 & 0 & 0 & -2/3 & 0 & -1/3 & 0 & 0 & \frac{2i\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & \frac{2i\sqrt{2}}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{i\sqrt{2}}{6} & 0 & -\frac{i\sqrt{2}}{6} & 0 & \frac{i\sqrt{2}}{6} & 0 & 0 & -1/3 & 0 & 0 & \frac{i\sqrt{2}}{6} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{i\sqrt{2}}{6} & 0 & -\frac{i\sqrt{2}}{6} & 0 & \frac{i\sqrt{2}}{6} & 0 & 0 & 1/3 & 0 & 0 & \frac{i\sqrt{2}}{6} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right),
$$

and

$$S\xi=\{\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6,\psi_7,\psi_8,\psi_9,\psi_{10},\psi_{11},\psi_{12},\psi_{13},\psi_{14},\psi_{15},\psi_{16}\}^{tr.}.$$

So the Pauli-Fierz equation under the similarity transformation S goes to

$$\left(S\mathbb{L}_0S^{-1}\frac{\partial}{\partial x_0}+S\mathbb{L}_3S^{-1}\frac{\partial}{\partial x_3}+iK\right)S\xi=0,$$

which is equivalent to the following set of equations:

$$1)\ \frac{2}{3}\frac{\partial}{\partial x_3}\psi_5+\frac{2i\sqrt{2}}{3}\frac{\partial}{\partial x_3}\psi_8+\frac{2i\sqrt{2}}{3}\frac{\partial}{\partial x_3}\psi_{14}+iK\psi_1=0,$$

$$2)\ -\frac{2}{3}\frac{\partial}{\partial x_3}\psi_6+\frac{2i\sqrt{2}}{3}\frac{\partial}{\partial x_3}\psi_9+\frac{2i\sqrt{2}}{3}\frac{\partial}{\partial x_3}\psi_{15}+iK\psi_2=0,$$

$$3)\ 1\frac{\partial}{\partial x_0}\psi_1-\frac{1}{3}\frac{\partial}{\partial x_3}\psi_1+\frac{2i\sqrt{2}}{3}\frac{\partial}{\partial x_3}\psi_8-\frac{2}{3}\frac{\partial}{\partial x_3}\psi_{11}-\frac{2i\sqrt{2}}{3}\frac{\partial}{\partial x_3}\psi_{14}+iK\psi_3=0,$$

$$4)\ 1\frac{\partial}{\partial x_0}\psi_2+\frac{1}{3}\frac{\partial}{\partial x_3}\psi_2+\frac{2i\sqrt{2}}{3}\frac{\partial}{\partial x_3}\psi_9+\frac{2}{3}\frac{\partial}{\partial x_3}\psi_{12}-\frac{2i\sqrt{2}}{3}\frac{\partial}{\partial x_3}\psi_{15}+iK\psi_4=0,$$

$$5)\ -\frac{2}{3}\frac{\partial}{\partial x_3}\psi_1+\frac{2i\sqrt{2}}{3}\frac{\partial}{\partial x_3}\psi_8-\frac{2i\sqrt{2}}{3}\frac{\partial}{\partial x_3}\psi_{14}+iK\psi_5=0,$$

$$6)\ \frac{2}{3}\frac{\partial}{\partial x_3}\psi_2+\frac{2i\sqrt{2}}{3}\frac{\partial}{\partial x_3}\psi_9-\frac{2i\sqrt{2}}{3}\frac{\partial}{\partial x_3}\psi_{15}+iK\psi_6=0,$$

$$7)\ 1\frac{\partial}{\partial x_0}\psi_7+1\frac{\partial}{\partial x_3}\psi_{13}+iK\psi_7=0,$$

$$8)\ 1\frac{\partial}{\partial x_0}\psi_8-\frac{i\sqrt{2}}{6}\frac{\partial}{\partial x_3}\psi_1+\frac{i\sqrt{2}}{6}\frac{\partial}{\partial x_3}\psi_3-\frac{i\sqrt{2}}{6}\frac{\partial}{\partial x_3}\psi_5+\frac{i\sqrt{2}}{6}\frac{\partial}{\partial x_3}\psi_{11}+\frac{1}{3}\frac{\partial}{\partial x_3}\psi_{14}+iK\psi_8=0,$$

$$9)\ 1\frac{\partial}{\partial x_0}\psi_9-\frac{\iota\sqrt{2}}{6}\frac{\partial}{\partial x_3}\psi_2+\frac{\iota\sqrt{2}}{6}\frac{\partial}{\partial x_3}\psi_4-\frac{\iota\sqrt{2}}{6}\frac{\partial}{\partial x_3}\psi_6+\frac{\iota\sqrt{2}}{6}\frac{\partial}{\partial x_3}\psi_{12}-\frac{1}{3}\frac{\partial}{\partial x_3}\psi_{15}+\iota\kappa\psi_9=0,$$

$$10)\ 1\frac{\partial}{\partial x_0}\psi_{10}-1\frac{\partial}{\partial x_3}\psi_{16}+\iota\kappa\psi_{10}=0,$$

$$11)\ 1\frac{\partial}{\partial x_0}\psi_5+\frac{2}{3}\frac{\partial}{\partial x_3}\psi_3+\frac{1}{3}\frac{\partial}{\partial x_3}\psi_5+\frac{2\iota\sqrt{2}}{3}\frac{\partial}{\partial x_3}\psi_8+\frac{2\iota\sqrt{2}}{3}\frac{\partial}{\partial x_3}\psi_{14}+\iota\kappa\psi_{11}=0,$$

$$12)\ 1\frac{\partial}{\partial x_0}\psi_6-\frac{2}{3}\frac{\partial}{\partial x_3}\psi_4-\frac{1}{3}\frac{\partial}{\partial x_3}\psi_6+\frac{2\iota\sqrt{2}}{3}\frac{\partial}{\partial x_3}\psi_9+\frac{2\iota\sqrt{2}}{3}\frac{\partial}{\partial x_3}\psi_{15}+\iota\kappa\psi_{12}=0,$$

$$13)\ -1\frac{\partial}{\partial x_0}\psi_{13}-1\frac{\partial}{\partial x_3}\psi_7+\iota\kappa\psi_{13}=0,$$

$$14)\ -1\frac{\partial}{\partial x_0}\psi_{14}-\frac{\iota\sqrt{2}}{6}\frac{\partial}{\partial x_3}\psi_1-\frac{\iota\sqrt{2}}{6}\frac{\partial}{\partial x_3}\psi_3+\frac{\iota\sqrt{2}}{6}\frac{\partial}{\partial x_3}\psi_5-\frac{1}{3}\frac{\partial}{\partial x_3}\psi_3+\frac{\iota\sqrt{2}}{6}\frac{\partial}{\partial x_3}\psi_{11}+\iota\kappa\psi_{14}=0,$$

$$15)\ -1\frac{\partial}{\partial x_0}\psi_{15}-\frac{\iota\sqrt{2}}{6}\frac{\partial}{\partial x_3}\psi_2-\frac{\iota\sqrt{2}}{6}\frac{\partial}{\partial x_3}\psi_4+\frac{\iota\sqrt{2}}{6}\frac{\partial}{\partial x_3}\psi_6+\frac{1}{3}\frac{\partial}{\partial x_3}\psi_9+\frac{\iota\sqrt{2}}{6}\frac{\partial}{\partial x_3}\psi_{15}+\iota\kappa\psi_{15}=0,$$

$$16)\ -1\frac{\partial}{\partial x_0}\psi_{16}+1\frac{\partial}{\partial x_3}\psi_{10}+\iota\kappa\psi_{16}=0.$$

Equations (1), (2), (5), (6) do not involve the time derivative and constitute the subsidiary conditions of the first kind or primary constraints. Notice also that the time derivative of the components ψ_3, ψ_4, ψ_{11}, ψ_{12}, do not appear in the above differential equations nor can be determined from the primary constraints.

We are giving now another, much more straightforward method of obtaining the subsidiary conditions of the first kind, for the Pauli-Fierz equation. For this purpose let us consider the Pauli-Fierz equation in its spinor form and in an external electromagnetic field after having replaced the spinors $\pi_{\dot{\alpha}\rho}$ by their four momentum components π_k i.e.

$$1)\ -(-\pi_0-\pi_3)\alpha^{\dot{1}}_{11}-(-\pi_1-\iota\pi_2)\alpha^{\dot{1}}_{12}-(\pi_1+\iota\pi_2)d^{\dot{1}}+\kappa b^{\dot{1}\dot{1}}_1=0,$$

$$2)\ -(-\pi_0-\pi_3)\alpha^{\dot{1}}_{12}-(-\pi_1-\iota\pi_2)\alpha^{\dot{1}}_{22}-(-\pi_0-\pi_3)d^{\dot{1}}+\kappa b^{\dot{1}\dot{1}}_2=0,$$

$$3)\ -\frac{1}{2}(-\pi_0-\pi_3)\alpha^{\dot{2}}_{11}-\frac{1}{2}(-\pi_1-\iota\pi_2)\alpha^{\dot{2}}_{12}-\frac{1}{2}(-\pi_1+\iota\pi_2)\alpha^{\dot{1}}_{11}-\frac{1}{2}(-\pi_0+\pi_3)\alpha^{\dot{1}}_{12}-$$
$$-\frac{1}{2}(\pi_1+\iota\pi_2)d^{\dot{2}}-\frac{1}{2}(\pi_0-\pi_3)d^{\dot{1}}+\kappa b^{\dot{1}\dot{2}}_{1}=0,$$

$$4)\ -\frac{1}{2}(-\pi_0-\pi_3)\alpha^{\dot{2}}_{12}-\frac{1}{2}(-\pi_1-\iota\pi_2)\alpha^{\dot{2}}_{22}-\frac{1}{2}(-\pi_1+\iota\pi_2)\alpha^{\dot{1}}_{12}-\frac{1}{2}(-\pi_0+\pi_3)\alpha^{\dot{1}}_{22}-$$
$$-\frac{1}{2}(-\pi_0-\pi_3)d^{\dot{2}}-\frac{1}{2}(-\pi_1+\iota\pi_2)d^{\dot{1}}+\kappa b^{\dot{1}\dot{2}}_{2}=0,$$

$$5)\ -(-\pi_1-\iota\pi_2)\alpha^{\dot{2}}_{11}-(-\pi_0+\pi_3)\alpha^{\dot{2}}_{12}-(\pi_0-\pi_3)d^{\dot{2}}+\kappa b^{\dot{2}\dot{2}}_{1}=0,$$

$$6)\ -(-\pi_1+\iota\pi_2)\alpha^{\dot{2}}_{12}-(-\pi_0+\pi_3)\alpha^{\dot{2}}_{22}-(-\pi_1+\iota\pi_2)d^{\dot{2}}+\kappa b^{\dot{2}\dot{2}}_{2}=0,$$

$$7)\ -(-\pi_0+\pi_3)b^{\dot{1}\dot{1}}_{1}-(\pi_1+\iota\pi_2)b^{\dot{1}\dot{2}}_{1}-(\pi_1+\iota\pi_2)c_1+\kappa\alpha^{\dot{1}}_{11}=0,$$

$$8)\ -(-\pi_0+\pi_3)b^{\dot{1}\dot{2}}_{1}-(\pi_1+\iota\pi_2)b^{\dot{2}\dot{2}}_{1}-(\pi_0-\pi_3)c_1+\kappa\alpha^{\dot{2}}_{11}=0,$$

$$9)\ -\frac{1}{2}(\pi_1-\iota\pi_2)b^{\dot{1}\dot{1}}_{1}-\frac{1}{2}(-\pi_0-\pi_3)b^{\dot{1}\dot{2}}_{1}-\frac{1}{2}(-\pi_0-\pi_3)b^{\dot{1}\dot{1}}_{2}-\frac{1}{2}(\pi_1+\iota\pi_2)b^{\dot{1}\dot{2}}_{2}-$$
$$-\frac{1}{2}(\pi_1+\iota\pi_2)c_2-\frac{1}{2}(-\pi_0-\pi_3)c_1+\kappa\alpha^{\dot{1}}_{22}=0,$$

$$10)\ -\frac{1}{2}(\pi_1-\iota\pi_2)b^{\dot{1}\dot{2}}_{1}-\frac{1}{2}(-\pi_0-\pi_3)b^{\dot{2}\dot{2}}_{1}-\frac{1}{2}(-\pi_0+\pi_3)b^{\dot{1}\dot{2}}_{2}-\frac{1}{2}(\pi_1+\iota\pi_2)b^{\dot{2}\dot{2}}_{2}-$$
$$-\frac{1}{2}(\pi_0-\pi_3)c_2-\frac{1}{2}(-\pi_1+\iota\pi_2)c_1+\kappa\alpha^{\dot{2}}_{12}=0,$$

$$11)\ -(\pi_1-\iota\pi_2)b^{\dot{1}\dot{1}}_{2}-(-\pi_0-\pi_3)b^{\dot{1}\dot{2}}_{2}-(-\pi_0-\pi_3)c_2+\kappa\alpha^{\dot{1}}_{22}=0,$$

$$12)\ -(\pi_1-\iota\pi_2)b^{\dot{1}\dot{2}}_{2}-(-\pi_0-\pi_3)b^{\dot{2}\dot{2}}_{2}-(-\pi_1+\iota\pi_2)c_2+\kappa\alpha^{\dot{2}}_{22}=0,$$

$$13)\ -\frac{1}{6}(\pi_1-\iota\pi_2)\alpha^{\dot{1}}_{11}-\frac{1}{6}(-\pi_0-\pi_3)\alpha^{\dot{2}}_{11}-\frac{1}{6}(\pi_0-\pi_3)\alpha^{\dot{1}}_{12}-\frac{1}{6}(-\pi_1-\iota\pi_2)\alpha^{\dot{2}}_{12}+$$
$$+\frac{1}{2}(-\pi_0+\pi_3)d^{\dot{1}}+\frac{1}{2}(\pi_1+\iota\pi_2)d^{\dot{2}}+\kappa c_1=0,$$

$$14)\ -\frac{1}{6}(\pi_1-\iota\pi_2)\alpha^{\dot{1}}_{12}-\frac{1}{6}(-\pi_0-\pi_3)\alpha^{\dot{2}}_{12}-\frac{1}{6}(\pi_0-\pi_3)\alpha^{\dot{1}}_{22}-\frac{1}{6}(-\pi_1-\iota\pi_2)\alpha^{\dot{2}}_{22}+$$
$$+\frac{1}{2}(\pi_1-\iota\pi_2)d^{\dot{1}}+\frac{1}{2}(-\pi_0-\pi_3)d^{\dot{2}}+\kappa c_2=0,$$

$$15)\ -\frac{1}{6}(\pi_1-\iota\pi_2)b^{\dot{1}\dot{1}}_{1}-\frac{1}{6}(\pi_0-\pi_3)b^{\dot{1}\dot{1}}_{2}-\frac{1}{6}(-\pi_0-\pi_3)b^{\dot{1}\dot{2}}_{1}-\frac{1}{6}(-\pi_1-\iota\pi_2)b^{\dot{1}\dot{2}}_{2}+$$
$$+\frac{1}{2}(-\pi_0-\pi_3)c_1+\frac{1}{2}(-\pi_1-\iota\pi_2)c_2+\kappa d^{\dot{1}}=0,$$

$$16)\ -\frac{1}{6}(\pi_1-\iota\pi_2)b^{\dot{1}\dot{2}}_{1}-\frac{1}{6}(\pi_0-\pi_3)b^{\dot{1}\dot{2}}_{2}-\frac{1}{6}(-\pi_0-\pi_3)b^{\dot{2}\dot{2}}_{1}-\frac{1}{6}(-\pi_1-\iota\pi_2)b^{\dot{2}\dot{2}}_{2}+$$
$$+\frac{1}{2}(-\pi_1+\iota\pi_2)c_1+\frac{1}{2}(-\pi_0+\pi_3)c_2+\kappa d^{\dot{2}}=0.$$

The subsidiary conditions of the first kind can then be obtained as follows:

i) Consider equation (15) and multiply it through by the scalar 3. Then add to it equation (9) and from the resulting equation subtract equation (8) to find the following differential equation involving no time derivatives:

$$\{3\cdot(eq_{15})+(eq_{9})\}-(eq_{8})\Longrightarrow -\iota\pi_1 b_1^{\dot1\dot1}+\iota\pi_1 b_1^{\dot2\dot2}-2\pi_1 c_2+\iota\pi_2 b_1^{\dot1\dot1}+$$
$$+\iota\pi_2 b_1^{\dot2\dot2}-2\iota\pi_2 c_2+2\pi_3 b_1^{\dot1\dot2}-2\pi_3 c_1+\chi\alpha_{12}^{\dot1}-\chi\alpha_{11}^{\dot2}+3\chi d^{1}=0.$$

This differential equation forms a primary constraint. Similarly the other three primary constraints are obtained by performing analogous operations on the differential equations as indicated below:

ii) $$\{3\cdot(eq_{16})-(eq_{10})\}+(eq_{11})\Longrightarrow -\iota\pi_1 b_2^{\dot1\dot1}+\iota\pi_1 b_2^{\dot2\dot2}-2\pi_1 c_1+$$
$$+\iota\pi_2 b_2^{\dot1\dot1}+\iota\pi_2 b_2^{\dot2\dot2}+2\iota\pi_2 c_1+2\pi_3 b_2^{\dot1\dot2}+2\pi_3 c_2+\chi\alpha_{22}^{\dot1}-\chi\alpha_{12}^{\dot2}+3\chi d^{\dot2}=0,$$

iii) $$\{3\cdot(eq_{13})-(eq_{3})\}+(eq_{2})\Longrightarrow -\iota\pi_1\alpha_{11}^{\dot1}+\iota\pi_1\alpha_{22}^{\dot1}+2\pi_1 d^{\dot2}+$$
$$+\iota\pi_2\alpha_{11}^{\dot1}+\iota\pi_2\alpha_{22}^{\dot1}+2\iota\pi_2 d^{\dot2}+2\pi_3\alpha_{12}^{\dot1}+2\pi_3 d^{\dot1}-\chi b_1^{\dot1\dot2}+\chi b_2^{\dot1\dot1}+3\chi c_1=0,$$

iv) $$\{3\cdot(eq_{14})+(eq_{4})\}-(eq_{5})\Longrightarrow -\iota\pi_1\alpha_{11}^{\dot2}+\iota\pi_1\alpha_{22}^{\dot2}+2\pi_1 d^{\dot1}+$$
$$+\iota\pi_2\alpha_{11}^{\dot2}+\iota\pi_2\alpha_{22}^{\dot2}-2\iota\pi_2 d^{\dot1}+2\pi_3\alpha_{12}^{\dot2}-2\pi_3 d^{\dot2}-\chi b_1^{\dot2\dot2}+\chi b_2^{\dot1\dot2}+3\chi c_2=0.$$

Besides the subsidiary conditions of the first kind the Pauli-Fierz equation acquires another set of subsidiary conditions involving no derivatives at all. These are the subsidiary conditions of the second kind or secondary constraints and are derived in the following paragraph.

9: Subsidiary conditions of the second kind.

Let us consider again the spinor form of the Pauli-Fierz equation in an external electromagnetic field i.e.

1A) $-\pi^{\dot\beta\rho}\alpha^{\dot\alpha}_{\gamma\rho} - \pi^{\dot\alpha\rho}\alpha^{\dot\beta}_{\gamma\rho} - \pi^{\dot\beta}_{\gamma} d^{\dot\alpha} - \pi^{\dot\alpha}_{\gamma} d^{\dot\beta} + 2\kappa b^{\dot\alpha\dot\beta}_{\gamma} = 0,$

2A) $-\pi_{\alpha\dot\rho} b^{\dot\gamma\dot\rho}_{\beta} - \pi_{\beta\dot\rho} b^{\dot\gamma\dot\rho}_{\alpha} - \pi^{\dot\gamma}_{\beta} c_\alpha - \pi^{\dot\gamma}_{\alpha} c_\beta + 2\kappa \alpha^{\dot\gamma}_{\alpha\beta} = 0,$

3A) $-\pi^{\beta}_{\dot\gamma}\alpha^{\dot\gamma}_{\alpha\beta} + 3\pi_{\alpha\dot\beta} d^{\dot\beta} + 6\kappa c_\alpha = 0,$

4A) $-\pi^{\gamma}_{\dot\beta} b^{\dot\alpha\dot\beta}_{\gamma} + 3\pi^{\dot\alpha\beta} c_\beta + 6\kappa d^{\dot\alpha} = 0,$

or in detail:

1B) $-\pi^{\dot 1 1}\alpha^{\dot 1}_{11} - \pi^{\dot 1 2}\alpha^{\dot 1}_{12} - \pi^{\dot 1}_{1} d^{\dot 1} + \kappa b^{\dot 1\dot 1}_{1} = 0,$

2B) $-\pi^{\dot 1 1}\alpha^{\dot 1}_{12} - \pi^{\dot 1 2}\alpha^{\dot 1}_{22} - \pi^{\dot 1}_{2} d^{\dot 1} + \kappa b^{\dot 1\dot 1}_{2} = 0,$

3B) $-\frac{1}{2}\pi^{\dot 1 1}\alpha^{\dot 2}_{11} - \frac{1}{2}\pi^{\dot 1 2}\alpha^{\dot 2}_{12} - \frac{1}{2}\pi^{\dot 2 1}\alpha^{\dot 1}_{11} - \frac{1}{2}\pi^{\dot 2 2}\alpha^{\dot 1}_{12} - \frac{1}{2}\pi^{\dot 1}_{1} d^{\dot 2} - \frac{1}{2}\pi^{\dot 2}_{1} d^{\dot 1} + \kappa b^{\dot 1\dot 2}_{1} = 0,$

4B) $-\frac{1}{2}\pi^{\dot 1 1}\alpha^{\dot 2}_{12} - \frac{1}{2}\pi^{\dot 1 2}\alpha^{\dot 2}_{22} - \frac{1}{2}\pi^{\dot 2 1}\alpha^{\dot 1}_{12} - \frac{1}{2}\pi^{\dot 2 2}\alpha^{\dot 1}_{22} - \frac{1}{2}\pi^{\dot 1}_{2} d^{\dot 2} - \frac{1}{2}\pi^{\dot 2}_{2} d^{\dot 1} + \kappa b^{\dot 1\dot 2}_{2} = 0,$

5B) $-\pi^{\dot 2 1}\alpha^{\dot 2}_{11} - \pi^{\dot 2 2}\alpha^{\dot 2}_{12} - \pi^{\dot 2}_{1} d^{\dot 2} + \kappa b^{\dot 2\dot 2}_{1} = 0,$

6B) $-\pi^{\dot 2 1}\alpha^{\dot 2}_{12} - \pi^{\dot 2 2}\alpha^{\dot 2}_{22} - \pi^{\dot 2}_{2} d^{\dot 2} + \kappa b^{\dot 2\dot 2}_{2} = 0,$

7B) $-\pi_{1\dot 1} b^{\dot 1\dot 1}_{1} - \pi_{1\dot 2} b^{\dot 1\dot 2}_{1} - \pi^{\dot 1}_{1} c_1 + \kappa \alpha^{\dot 1}_{11} = 0,$

8B) $-\pi_{1\dot 1} b^{\dot 1\dot 2}_{1} - \pi_{1\dot 2} b^{\dot 2\dot 2}_{1} - \pi^{\dot 2}_{1} c_1 + \kappa \alpha^{\dot 2}_{11} = 0,$

9B) $-\frac{1}{2}\pi_{2\dot 1} b^{\dot 1\dot 1}_{1} - \frac{1}{2}\pi_{2\dot 2} b^{\dot 1\dot 2}_{1} - \frac{1}{2}\pi_{1\dot 1} b^{\dot 1\dot 1}_{2} - \frac{1}{2}\pi_{1\dot 2} b^{\dot 1\dot 2}_{2} - \frac{1}{2}\pi^{\dot 1}_{1} c_1 - \frac{1}{2}\pi^{\dot 1}_{2} c_1 + \kappa \alpha^{\dot 1}_{12} = 0,$

10B) $-\frac{1}{2}\pi_{2\dot 1} b^{\dot 1\dot 2}_{1} - \frac{1}{2}\pi_{2\dot 2} b^{\dot 2\dot 2}_{1} - \frac{1}{2}\pi_{1\dot 1} b^{\dot 1\dot 2}_{2} - \frac{1}{2}\pi_{1\dot 2} b^{\dot 2\dot 2}_{2} - \frac{1}{2}\pi^{\dot 2}_{1} c_2 - \frac{1}{2}\pi^{\dot 2}_{2} c_1 + \kappa \alpha^{\dot 2}_{12} = 0,$

11B) $-\pi_{2\dot 1} b^{\dot 1\dot 1}_{2} - \pi_{2\dot 2} b^{\dot 1\dot 2}_{2} - \pi^{\dot 1}_{2} c_2 + \kappa \alpha^{\dot 1}_{22} = 0,$

12B) $-\pi_{2\dot 1} b^{\dot 1\dot 2}_{2} - \pi_{2\dot 2} b^{\dot 2\dot 2}_{2} - \pi^{\dot 2}_{2} c_2 + \kappa \alpha^{\dot 2}_{22} = 0,$

$$13B)\ -\frac{1}{6}\pi^{1}_{\dot 1}\alpha^{\dot 1}_{11}-\frac{1}{6}\pi^{1}_{\dot 2}\alpha^{\dot 2}_{11}-\frac{1}{6}\pi^{2}_{\dot 1}\alpha^{\dot 1}_{12}-\frac{1}{6}\pi^{2}_{\dot 2}\alpha^{\dot 2}_{12}+\frac{3}{6}\pi_{1\dot 1}d^{\dot 1}+\frac{3}{6}\pi_{1\dot 2}d^{\dot 2}+\kappa c_{1}=0,$$

$$14B)\ -\frac{1}{6}\pi^{1}_{\dot 1}\alpha^{\dot 1}_{12}-\frac{1}{6}\pi^{1}_{\dot 2}\alpha^{\dot 2}_{12}-\frac{1}{6}\pi^{2}_{\dot 1}\alpha^{\dot 1}_{22}-\frac{1}{6}\pi^{2}_{\dot 2}\alpha^{\dot 2}_{22}+\frac{3}{6}\pi_{2\dot 1}d^{\dot 1}+\frac{3}{6}\pi_{2\dot 2}d^{\dot 2}+\kappa c_{2}=0,$$

$$15B)\ -\frac{1}{6}\pi^{1}_{\dot 1}b_{1}^{\dot 1\dot 1}-\frac{1}{6}\pi^{2}_{\dot 1}b_{2}^{\dot 1\dot 1}-\frac{1}{6}\pi^{1}_{\dot 2}b_{1}^{\dot 1\dot 2}-\frac{1}{6}\pi^{2}_{\dot 2}b_{2}^{\dot 1\dot 2}+\frac{3}{6}\pi^{\dot 1 1}c_{1}+\frac{3}{6}\pi^{\dot 1 2}c_{2}+\kappa d^{\dot 1}=0,$$

$$16B)\ -\frac{1}{6}\pi^{1}_{\dot 1}b_{1}^{\dot 1\dot 2}-\frac{1}{6}\pi^{2}_{\dot 1}b_{2}^{\dot 1\dot 2}-\frac{1}{6}\pi^{1}_{\dot 2}b_{1}^{\dot 2\dot 2}-\frac{1}{6}\pi^{2}_{\dot 2}b_{2}^{\dot 2\dot 2}+\frac{3}{6}\pi^{\dot 2 1}c_{1}+\frac{3}{6}\pi^{\dot 2 2}c_{2}+\kappa d^{\dot 2}=0.$$

Multiplying (1A) from the left by $\pi^{\gamma}_{\dot\beta}$ we get (after giving to the spinor indices their values) the pair of equations:

$$\begin{aligned}
&2\kappa\pi^{1}_{\dot 1}b_{1}^{\dot 1\dot 1}+2\kappa\pi^{1}_{\dot 2}b_{1}^{\dot 1\dot 2}+2\kappa\pi^{2}_{\dot 1}b_{2}^{\dot 1\dot 1}+2\kappa\pi^{2}_{\dot 2}b_{2}^{\dot 1\dot 2}-\pi^{1}_{\dot 1}\pi^{\dot 1 1}\alpha^{\dot 1}_{11}-\pi^{1}_{\dot 1}\pi^{\dot 1 2}\alpha^{\dot 1}_{12}-\\
&-\pi^{1}_{\dot 2}\pi^{\dot 2 1}\alpha^{\dot 1}_{11}-\pi^{1}_{\dot 2}\pi^{\dot 2 2}\alpha^{\dot 1}_{12}-\pi^{2}_{\dot 1}\pi^{\dot 1 1}\alpha^{\dot 1}_{21}-\pi^{2}_{\dot 1}\pi^{\dot 1 2}\alpha^{\dot 1}_{22}-\pi^{2}_{\dot 2}\pi^{\dot 2 1}\alpha^{\dot 1}_{21}-\pi^{2}_{\dot 2}\pi^{\dot 2 2}\alpha^{\dot 1}_{22}-\\
&-\pi^{1}_{\dot 1}\pi^{\dot 1 1}\alpha^{\dot 1}_{11}-\pi^{1}_{\dot 1}\pi^{\dot 1 2}\alpha^{\dot 1}_{12}-\pi^{1}_{\dot 2}\pi^{\dot 1 1}\alpha^{\dot 2}_{11}-\pi^{1}_{\dot 2}\pi^{\dot 1 2}\alpha^{\dot 2}_{12}-\pi^{2}_{\dot 1}\pi^{\dot 1 1}\alpha^{\dot 1}_{21}-\pi^{2}_{\dot 1}\pi^{\dot 1 2}\alpha^{\dot 1}_{22}-\\
&-\pi^{2}_{\dot 2}\pi^{\dot 1 1}\alpha^{\dot 2}_{21}-\pi^{2}_{\dot 2}\pi^{\dot 1 2}\alpha^{\dot 2}_{22}-\pi^{1}_{\dot 1}\pi^{\dot 1}_{1}d^{\dot 1}-\pi^{1}_{\dot 2}\pi^{\dot 2}_{1}d^{\dot 1}-\pi^{2}_{\dot 1}\pi^{\dot 1}_{2}d^{\dot 1}-\pi^{2}_{\dot 2}\pi^{\dot 2}_{2}d^{\dot 1}-\\
&-\pi^{1}_{\dot 1}\pi^{\dot 1}_{1}d^{\dot 1}-\pi^{1}_{\dot 2}\pi^{\dot 1}_{1}d^{\dot 2}-\pi^{2}_{\dot 1}\pi^{\dot 1}_{2}d^{\dot 1}-\pi^{2}_{\dot 2}\pi^{\dot 1}_{2}d^{\dot 2}=0,
\end{aligned}\tag{1}$$

$$\begin{aligned}
&2\kappa\pi^{1}_{\dot 1}b_{1}^{\dot 2\dot 1}+2\kappa\pi^{1}_{\dot 2}b_{1}^{\dot 2\dot 2}+2\kappa\pi^{2}_{\dot 1}b_{2}^{\dot 2\dot 1}+2\kappa\pi^{2}_{\dot 2}b_{2}^{\dot 2\dot 2}-\pi^{1}_{\dot 1}\pi^{\dot 1 1}\alpha^{\dot 2}_{11}-\pi^{1}_{\dot 1}\pi^{\dot 1 2}\alpha^{\dot 2}_{12}-\pi^{1}_{\dot 2}\pi^{\dot 2 1}\alpha^{\dot 2}_{11}-\\
&-\pi^{1}_{\dot 2}\pi^{\dot 2 2}\alpha^{\dot 2}_{12}-\pi^{2}_{\dot 1}\pi^{\dot 1 1}\alpha^{\dot 2}_{21}-\pi^{2}_{\dot 1}\pi^{\dot 1 2}\alpha^{\dot 2}_{22}-\pi^{2}_{\dot 2}\pi^{\dot 2 1}\alpha^{\dot 2}_{21}-\pi^{2}_{\dot 2}\pi^{\dot 2 2}\alpha^{\dot 2}_{22}-\pi^{1}_{\dot 1}\pi^{\dot 2 1}\alpha^{\dot 1}_{11}-\pi^{1}_{\dot 1}\pi^{\dot 2 2}\alpha^{\dot 1}_{12}-\\
&-\pi^{1}_{\dot 2}\pi^{\dot 2 1}\alpha^{\dot 2}_{11}-\pi^{1}_{\dot 2}\pi^{\dot 2 2}\alpha^{\dot 2}_{12}-\pi^{2}_{\dot 1}\pi^{\dot 2 1}\alpha^{\dot 1}_{21}-\pi^{2}_{\dot 1}\pi^{\dot 2 2}\alpha^{\dot 1}_{22}-\pi^{2}_{\dot 2}\pi^{\dot 2 1}\alpha^{\dot 2}_{21}-\pi^{2}_{\dot 2}\pi^{\dot 2 2}\alpha^{\dot 2}_{22}-\pi^{1}_{\dot 1}\pi^{\dot 1}_{1}d^{\dot 2}-\\
&-\pi^{1}_{\dot 2}\pi^{\dot 2}_{1}d^{\dot 2}-\pi^{2}_{\dot 1}\pi^{\dot 1}_{2}d^{\dot 2}-\pi^{2}_{\dot 2}\pi^{\dot 2}_{2}d^{\dot 2}-\pi^{1}_{\dot 1}\pi^{\dot 2}_{1}d^{\dot 1}-\pi^{1}_{\dot 2}\pi^{\dot 2}_{1}d^{\dot 2}-\pi^{2}_{\dot 1}\pi^{\dot 2}_{2}d^{\dot 1}-\\
&-\pi^{2}_{\dot 2}\pi^{\dot 2}_{2}d^{\dot 2}=0\ .
\end{aligned}\tag{2}$$

Multiplying (3A) from the left by $\pi^{\alpha\dot{\alpha}}$ we get the pair of equations:

$$-\pi^{1\dot{1}}\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{11} - \pi^{11}\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{11} - \pi^{1\dot{1}}\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{12} - \pi^{1\dot{1}}\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{12} - \pi^{2\dot{1}}\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{21} -$$
$$-\pi^{2\dot{1}}\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{21} - \pi^{2\dot{1}}\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{22} - \pi^{2\dot{1}}\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{22} + 3\pi^{1\dot{1}}\pi_{1\dot{1}}d^{\dot{1}} + 3\pi^{1\dot{1}}\pi_{1\dot{2}}d^{\dot{2}} +$$
$$+ 3\pi^{2\dot{1}}\pi_{2\dot{1}}d^{\dot{1}} + 3\pi^{2\dot{1}}\pi_{2\dot{2}}d^{\dot{2}} + 6K\pi^{1\dot{1}}c_1 + 6K\pi^{2\dot{1}}c_2 = 0 , \qquad (3)$$

$$-\pi^{1\dot{2}}\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{11} - \pi^{1\dot{2}}\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{11} - \pi^{1\dot{2}}\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{12} - \pi^{1\dot{2}}\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{12} - \pi^{2\dot{2}}\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{21} -$$
$$-\pi^{2\dot{2}}\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{21} - \pi^{2\dot{2}}\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{22} - \pi^{2\dot{2}}\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{22} + 3\pi^{1\dot{2}}\pi_{1\dot{1}}d^{\dot{1}} + 3\pi^{1\dot{2}}\pi_{1\dot{2}}d^{\dot{2}} +$$
$$+ 3\pi^{2\dot{2}}\pi_{2\dot{1}}d^{\dot{1}} + 3\pi^{2\dot{2}}\pi_{2\dot{2}}d^{\dot{2}} + 6K\pi^{1\dot{2}}c_1 + 6K\pi^{2\dot{2}}c_2 = 0 . \qquad (4)$$

Subtracting the second pair of equations from the first namely subtracting (3) from (1) and (4) from (2) using the formulae connecting the spinor components $\pi_{\alpha\dot{\beta}}$ with the four vector components π_k and using also the commutation relations

$$[\pi_k , \pi_\ell]_- = \pi_k\pi_\ell - \pi_\ell\pi_k = \iota e F_{k\ell} \equiv f_{k\ell} \ , \quad k,\ell = 0,1,2,3 ,$$

where $F_{k\ell} = \left(\frac{\partial A_\ell}{\partial x_k}\right) - \left(\frac{\partial A_k}{\partial x_\ell}\right)$ the electromagnetic field tensor and A_ℓ the four vector potential, we get the pair of equations:

$$2K\pi^{1}_{\dot{1}}b_1^{\dot{1}\dot{1}} + 2K\pi^{1}_{\dot{2}}b_1^{\dot{1}\dot{2}} + 2K\pi^{2}_{\dot{1}}b_2^{\dot{1}\dot{1}} + 2K\pi^{2}_{\dot{2}}b_2^{\dot{1}\dot{2}} + 2(-f_{01}+f_{13}-\iota f_{20}+\iota f_{32})\alpha^{\dot{1}}_{11} +$$
$$+ 2(\iota f_{12} - f_{30} - f_{30} + \iota f_{12})\alpha^{\dot{1}}_{12} + 2(-f_{10} - \iota f_{20} + f_{13} + \iota f_{23})\alpha^{\dot{1}}_{22} + 4f_{03}d^{\dot{1}} - 4\iota f_{12}d^{\dot{1}} + 4f_{01}d^{\dot{2}} +$$
$$+ 4\iota f_{02}d^{\dot{2}} - 4f_{13}d^{\dot{2}} - 4\iota f_{23}d^{\dot{2}} - 6K\pi^{1\dot{1}}c_1 - 6K\pi^{\dot{2}1}c_2 = 0 , \qquad (5)$$

$$2K\pi^{1}_{\dot{1}}b_1^{\dot{2}\dot{1}} + 2K\pi^{1}_{\dot{2}}b_1^{\dot{2}\dot{2}} + 2K\pi^{2}_{\dot{1}}b_2^{\dot{2}\dot{1}} + 2K\pi^{2}_{\dot{2}}b_2^{\dot{2}\dot{2}} + 2\left(-f_{01}-\iota f_{20}+f_{13}+\iota f_{32}\right)\alpha_{11}^{\dot{2}} +$$

$$+2\left(-f_{30}+\iota f_{12}+\iota f_{12}-f_{30}\right)\alpha_{12}^{\dot{2}} + 2\left(-f_{10}+f_{13}-\iota f_{20}+\iota f_{23}\right)\alpha_{22}^{\dot{2}} - 4\iota f_{21}d^{2} +$$

$$+4f_{30}d^{\dot{2}} + 4f_{01}d^{\dot{1}} - 4f_{31}d^{\dot{1}} + 4\iota f_{20}d^{\dot{1}} - 4\iota f_{23}d^{\dot{1}} - 6K\pi^{1\dot{2}}c_1 - 6K\pi^{2\dot{2}}c_2 = 0 \,. \qquad (6)$$

Using equations (15B) and (16B) of the original Pauli-Fierz set of equations, and substituting into (5) and (6) respectively we get the equations:

$$6K^2d^{\dot{1}} + \left(f_{10}+f_{13}+\iota f_{02}+\iota f_{32}\right)\alpha_{11}^{\dot{1}} + 2\left(f_{03}+\iota f_{12}\right)\alpha_{12}^{\dot{1}} + \left(f_{01}+\iota f_{02}+f_{13}+\iota f_{23}\right)\alpha_{22}^{\dot{1}} +$$

$$+2\left(f_{03}-\iota f_{12}\right)d^{\dot{1}} + 2\left(f_{01}+\iota f_{02}-f_{13}-\iota f_{23}\right)d^{\dot{2}} = 0 \,, \qquad (7)$$

$$6K^2d^{\dot{2}} + \left(-f_{01}-\iota f_{20}+f_{13}+\iota f_{32}\right)\alpha_{11}^{\dot{2}} + 2\left(-f_{30}+\iota f_{12}\right)\alpha_{12}^{\dot{2}} + \left(-f_{10}+f_{13}-\iota f_{20}+\iota f_{23}\right)\alpha_{22}^{\dot{2}} +$$

$$+2\left(f_{30}-\iota f_{21}\right)d^{\dot{2}} + 2\left(f_{01}-f_{31}+\iota f_{20}-\iota f_{23}\right)d^{\dot{1}} = 0 \,. \qquad (8)$$

These equations do not involve any time or space derivatives and constitute two of the subsidiary conditions of the second kind.

Similarly multiplying (2A) by $\pi_{\dot{\gamma}}^{\beta}$ and (4A) by $\pi_{\alpha\dot{\alpha}}$ both from the left and substracting we get the pair of equations:

$$2K\pi^{1}{}_{\dot{1}}\alpha^{\dot{1}}_{11}+2K\pi^{2}{}_{\dot{1}}\alpha^{\dot{1}}_{12}+2K\pi^{1}{}_{\dot{2}}\alpha^{\dot{2}}_{11}+2K\pi^{2}{}_{\dot{2}}\alpha^{\dot{2}}_{12}+2(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23})b_1^{\dot{1}\dot{1}}+$$
$$+2(\iota f_{21}-f_{30}-f_{30}+\iota f_{21})b_1^{\dot{1}\dot{2}}+2(-f_{10}-\iota f_{20}+f_{31}+\iota f_{32})b_1^{\dot{2}\dot{2}}+4f_{30}c_1-4\iota f_{12}c_1+$$
$$+4f_{10}c_2+4\iota f_{20}c_2-4f_{13}c_2-4\iota f_{23}c_2-6K\pi_{\dot{1}1}d^{\dot{1}}-6K\pi_{\dot{2}1}d^{\dot{2}}=0\,, \qquad (9)$$

$$2K\pi^{1}{}_{\dot{1}}\alpha^{\dot{1}}_{21}+2K\pi^{2}{}_{\dot{1}}\alpha^{\dot{1}}_{22}+2K\pi^{1}{}_{\dot{2}}\alpha^{\dot{2}}_{21}+2K\pi^{2}{}_{\dot{2}}\alpha^{\dot{2}}_{22}+2(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23})b_2^{\dot{1}\dot{1}}+$$
$$+2(-f_{30}+\iota f_{21}+\iota f_{21}-f_{30})b_2^{\dot{1}\dot{2}}+2(-f_{10}+f_{31}-\iota f_{20}+\iota f_{32})b_2^{\dot{2}\dot{2}}+4f_{10}c_1-4f_{31}c_1+$$
$$+4\iota f_{02}c_1-4\iota f_{23}c_1-4\iota f_{21}c_2+4f_{03}c_2-6K\pi_{\dot{1}2}d^{\dot{1}}-6K\pi_{\dot{2}2}d^{\dot{2}}=0\,. \qquad (10)$$

Using equations (13B) and (14B) and substituting into (9) and (10) respectively we get the equations:

$$6K^2c_1+(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23})b_1^{\dot{1}\dot{1}}+2(\iota f_{21}-f_{30})b_1^{\dot{1}\dot{2}}+(-f_{10}-\iota f_{20}+f_{31}+\iota f_{32})b_1^{\dot{2}\dot{2}}+$$
$$+2(f_{30}-\iota f_{12})c_1+2(f_{10}+\iota f_{20}-f_{13}-\iota f_{23})c_2=0\,, \qquad (11)$$

$$6K^2c_2+(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23})b_2^{\dot{1}\dot{1}}+2(-f_{30}+\iota f_{21})b_2^{\dot{1}\dot{2}}+(-f_{10}+f_{31}-\iota f_{20}+\iota f_{32})b_2^{\dot{2}\dot{2}}+$$
$$+2(f_{10}-f_{31}+\iota f_{02}-\iota f_{23})c_1+2(-\iota f_{21}+f_{03})c_2=0\,. \qquad (12)$$

These equations constitute the extra two subsidiary conditions of the second kind.

10. Equivalence between the subsidiary conditions of the Rarita-Schwinger and the Pauli-Fierz equations.

Since we have proved that the Rarita-Schwinger equation and the Pauli-Fierz equation are equivalent then one expects that their subsidiary conditions (c.f. chapter 10) are the same. Indeed this is the case as we show in this paragraph. As far as the subsidiary conditions of the first kind are concerned no extra proof is needed because they are derived directly from the equation itself and since the two equations are equivalent then their primary constraints are the same. Thus we shall show that the secondary constraints of the Rarita-Schwinger equation expressed in the vector spinor basis $\{\psi_\iota^{\ \alpha}\}$ go into the secondary constraints of the Pauli-Fierz equation expressed in the spinor basis $\{\alpha^{\dot\alpha}_{\ \beta\gamma}, b^{\dot\alpha\dot\beta}_{\ \ \gamma}, c_\alpha, d^{\dot\alpha}\}$. This we do, not only for the sake of demonstrating that the two sets of subsidiary conditions are the same, but also for showing that the spinorial method followed in §(9) to derive the secondary constraints leads to the same constraints as the method of the Dirac γ matrices used by Velo and Zwanziger. The advantage of the spinorial method is that it can be generalized with slight modifications to apply to other equations based on higher dimension representations, as we demonstrate in a later chapter, while the method of the Dirac matrices becomes exceedingly difficult to apply.

Thus let us consider the subsidiary conditions of the second kind of the Rarita-Schwinger equation, i.e.

$$\gamma\cdot\psi = -\tfrac{2}{3}m^{-2}\iota e\gamma^5\gamma\cdot F^{d}\cdot\psi$$

(c.f. chapter 10), which written out in detail are:

(I) $\iota\psi_0^2-\iota\psi_1^3+\iota\psi_2^3-\iota\psi_3^2=\omega\iota F^{12}\psi_0^2+\omega(\iota F^{23}-\iota F^{13})\psi_0^3+\omega(-\iota F^{23}+\iota F^{02})\psi_1^2-$
$-\omega F^{03}\psi_1^3+\omega(\iota F^{13}-\iota F^{01})\psi_2^2+\omega\iota F^{03}\psi_2^3-\omega\iota F^{12}\psi_3^2+\omega(-\iota F^{02}+F^{01})\psi_3^3$,

(II) $\iota\psi_0^2-\iota\psi_1^2-\iota\psi_2^2+\iota\psi_3^3=\omega(\iota F^{23}+\iota F^{13})\psi_0^2-\omega\iota F^{12}\psi_0^3+\omega F^{03}\psi_1^2+$
$+\omega(-\iota F^{23}-\iota F^{02})\psi_1^3+\omega\iota F^{03}\psi_2^2+\omega(\iota F^{13}+\iota F^{01})\psi_2^3+\omega(-\iota F^{02}-\iota F^{01})\psi_3^2-$
$-\omega\iota F^{12}\psi_3^3$,

(III) $\iota\psi_0^0+\iota\psi_1^1-\iota\psi_2^1+\psi_3^0=\omega\iota F^{12}\psi_0^0+\omega(\iota F^{23}-\iota F^{13})\psi_0^1+\omega(\iota F^{23}+\iota F^{02})\psi_1^0-$
$-\omega F^{03}\psi_1^1+\omega(-\iota F^{13}-\iota F^{01})\psi_2^0+\omega\iota F^{03}\psi_2^1+\omega\iota F^{12}\psi_3^0+\omega(-\iota F^{02}+\iota F^{01})\psi_3^1$,

(IV): $1\psi_0^1 + 1\psi_1^0 + \iota\psi_2^0 - 1\psi_3^1 = \omega\left(\iota F^{23} + 1F^{13}\right)\psi_0^0 - \omega\iota F^{12}\psi_0^1 + \omega F^{03}\psi_1^0 +$
$+ \omega\left(\iota F^{23} - \iota F^{02}\right)\psi_1^1 + \omega\iota F^{03}\psi_2^0 + \omega\left(-\iota F^{13} + \iota F^{01}\right)\psi_2^1 + \omega\left(-\iota F^{02} - 1F^{01}\right)\psi_3^0 +$
$+ \omega\iota F^{12}\psi_3^1$,

where $\omega = -\frac{2}{3}m^{-2}\iota e$. After replacing the vector spinor components $\psi_\iota^{\,j}$ by their spinor equivalent given by the transformation formulae of § 7, (I) goes to

$$1\left(\frac{\iota\sqrt{6}}{4}\alpha_{22}^{\dot 1} - \frac{\iota\sqrt{6}}{4}\alpha_{12}^{\dot 2} - \frac{\iota\sqrt{6}}{4}d^{\dot 2}\right) - 1\left(\frac{\iota\sqrt{3}}{2\sqrt{2}}\alpha_{11}^{\dot 1} - \frac{\iota\sqrt{6}}{4}\alpha_{12}^{\dot 2} + \frac{\iota\sqrt{6}}{4}d^{\dot 2}\right) +$$
$$+\iota\left(\frac{\sqrt{6}}{4}\alpha_{11}^{\dot 1} + \frac{\sqrt{6}}{4}\alpha_{12}^{\dot 2} - \frac{\sqrt{6}}{4}d^{\dot 2}\right) - \left(\frac{\iota\sqrt{6}}{4}\alpha_{22}^{\dot 1} + \frac{\iota\sqrt{6}}{4}\alpha_{12}^{\dot 2} + \frac{\iota\sqrt{6}}{4}d^{\dot 2}\right) =$$
$$= \omega\iota F^{12}\left(\frac{\iota\sqrt{6}}{4}\alpha_{22}^{\dot 1} - \frac{\iota\sqrt{6}}{4}\alpha_{12}^{\dot 2} - \frac{\iota\sqrt{6}}{4}d^{\dot 2}\right) + \omega\left(\iota F^{23} - 1F^{13}\right)\left(-\frac{\iota\sqrt{6}}{4}\alpha_{12}^{\dot 1} + \frac{\iota\sqrt{6}}{4}\alpha_{11}^{\dot 2} + \frac{\iota\sqrt{6}}{4}d^{\dot 1}\right) +$$
$$+ \omega\left(-\iota F^{23} + \iota F^{02}\right)\left(-\frac{\iota\sqrt{6}}{4}\alpha_{12}^{\dot 1} + \frac{\iota\sqrt{6}}{4}\alpha_{22}^{\dot 2} - \frac{\iota\sqrt{6}}{4}d^{\dot 1}\right) - \omega F^{03}\left(\frac{\iota\sqrt{6}}{4}\alpha_{11}^{\dot 1} - \frac{\iota\sqrt{6}}{4}\alpha_{12}^{\dot 2} + \frac{\iota\sqrt{6}}{4}d^{\dot 2}\right) +$$
$$+ \omega\left(\iota F^{13} - \iota F^{01}\right)\left(-\frac{\sqrt{6}}{4}\alpha_{12}^{\dot 1} - \frac{\sqrt{6}}{4}\alpha_{22}^{\dot 2} - \frac{\sqrt{6}}{4}d^{\dot 1}\right) + \omega\iota F^{03}\left(\frac{\sqrt{6}}{4}\alpha_{11}^{\dot 1} + \frac{\sqrt{6}}{4}\alpha_{12}^{\dot 2} - \frac{\sqrt{6}}{4}d^{\dot 2}\right) -$$
$$- \omega\iota F^{12}\left(\frac{\iota\sqrt{6}}{4}\alpha_{22}^{\dot 1} + \frac{\iota\sqrt{6}}{4}\alpha_{12}^{\dot 2} + \frac{\iota\sqrt{6}}{4}d^{\dot 2}\right) + \omega\left(-\iota F^{02} + F^{01}\right)\left(-\frac{\iota\sqrt{6}}{4}\alpha_{12}^{\dot 1} - \frac{\iota\sqrt{6}}{4}\alpha_{11}^{\dot 2} + \frac{\iota\sqrt{6}}{4}d^{\dot 1}\right),$$

which after some algebraic manipulations gives:

$$6K^2 d^{\dot 2} + \left(-f_{01} - \iota f_{20} + f_{13} + \iota f_{32}\right)\alpha_{11}^{\dot 2} + 2\left(-f_{30} + \iota f_{12}\right)\alpha_{12}^{\dot 2} + \left(-f_{10} + f_{13} - \iota f_{20} + \iota f_{23}\right)\alpha_{22}^{\dot 2} +$$
$$+ 2\left(f_{30} - \iota f_{21}\right)d^{\dot 2} + 2\left(f_{01} - f_{31} + \iota f_{20} - \iota f_{23}\right)d^{\dot 1} = 0,$$

where the mass constant m is renamed as K. Comparing this with equation (8) of § 9 we see that they are the same. Hence we have proved that the secondary constraint (I) of the Rarita-Schwinger equation is the same as the secondary constraint (eq.8 § 9) of the Pauli-Fierz equation.

Similarly we find that the other three secondary constraints of the Rarita-Schwinger equation are the same as those of the Pauli-Fierz equation i.e.

$$(\mathrm{II}) \longrightarrow 6\kappa^2 d^{\dot{1}} + (f_{10} + f_{13} + \iota f_{02} + \iota f_{32})\alpha^{\dot{1}}_{11} +$$

$$+ 2(f_{03} + \iota f_{12})\alpha^{\dot{1}}_{12} + (f_{01} + \iota f_{02} + f_{13} + \iota f_{23})\alpha^{\dot{1}}_{22} +$$

$$+ 2(f_{03} - \iota f_{12})d^{\dot{1}} + 2(f_{01} + \iota f_{02} - f_{13} - \iota f_{23})d^{\dot{2}} = 0 ,$$

$$(\mathrm{III}) \longrightarrow 6K^2 c_2 + (-f_{01} + f_{31} - \iota f_{20} + \iota f_{23})b_2^{\dot{1}\dot{1}} + 2(-f_{30} + \iota f_{21})b_2^{\dot{1}\dot{2}} +$$

$$+ (-f_{10} + f_{31} - \iota f_{20} + \iota f_{32})b_2^{\dot{2}\dot{2}} + 2(f_{10} - f_{31} + \iota f_{02} - \iota f_{23})c_1 +$$

$$+ 2(-\iota f_{21} + f_{03})c_2 = 0 ,$$

$$(\mathrm{IV}) \longrightarrow 6\kappa^2 c_1 + (-f_{01} + f_{31} - \iota f_{20} + \iota f_{23})b_1^{\dot{1}\dot{1}} +$$

$$+ 2(\iota f_{21} - f_{30})b_1^{\dot{1}\dot{2}} + (-f_{10} - \iota f_{20} + f_{31} + \iota f_{32})b_1^{\dot{2}\dot{2}} +$$

$$+ 2(f_{30} - \iota f_{21})c_1 + 2(f_{10} + \iota f_{20} - f_{13} - \iota f_{23})c_2 = 0 .$$

11: Derivation of the modified Pauli-Fierz equation. Propagation.

(A): Let us consider the spinor form of the Pauli-Fierz equation in an external electromagnetic field. If we wish to study the propagation behaviour of this equation by using the method of characteristics complications arise because of the singular nature of $\mathbb{L}_0$. Thus it becomes imperative that the Pauli-Fierz equation is modified by means of the subsidiary conditions into a new equation of motion with non-singular matrix $\mathbb{L}_0$ so that the method of characteristics can be implemented.

In paragraph 8 we saw that because of the singular nature of $\mathbb{L}_0$ the time derivatives of certain field components do not appear in the original equations and that the system of partial differential equations is separated into two sets of equations composed of twelve equations involving the time and space derivatives of the field components and the other composed of four equations involving only their space derivatives. A way of getting round the difficulty set up by the singular nature of $\mathbb{L}_0$ was proposed by Velo and Zwanziger who by modifying the original equation managed to introduce the time dependence of the four field components missing from the original set of differential equations, and hence derive a new set of equations of motion involving the time derivatives of all the sixteen field components (with non-singular matrix $\mathbb{L}_0(16\times16)$) which together with the primary and secondary constraints describe the motion of the system as the Pauli-Fierz equation does. That every solution of the modified equation satisfies the subsidiary conditions and also the original equation of motion is something which always has to be proved. Velo and Zwanziger prove this in the case of the Rarita-Schwinger equation, but if the method is to be applied to other cases then this point has to be secured first, before the method of characteristics is to be applied to the modified equation which we show below how it is derived by using spinor formulation. At this point we thought that we could avoid the method of Velo and Zwanziger by solving the subsidiary conditions of the second kind for $c_1, c_2, d^{\dot 1}, d^{\dot 2}$, i.e. expressing them as functions of the other field components and introducing them into the original set of equations deriving thus a set of twelve differential equations which will have only twelve components to be determined, but we faced the difficulty of how to handle the subsidiary conditions of the first kind.

Let us now derive the modified Pauli-Fierz equation. For this purpose consider the subsidiary conditions of the second kind of the Pauli-Fierz equation, eqs. (11) and (12) of § 9. Solving these constraint equations for κc_1 and κc_2 we have respectively

$$\begin{aligned} \kappa c_1 = &-\frac{1}{6\kappa}\left(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23}\right)b_1^{\dot 1\dot 1}-\frac{1}{6\kappa}\left(\iota f_{21}-f_{30}-f_{30}+\iota f_{21}\right)b_1^{\dot 1\dot 2}- \\ &-\frac{1}{6\kappa}\left(-f_{10}-\iota f_{20}+f_{31}+\iota f_{32}\right)b_1^{\dot 2\dot 2}-\frac{1}{6\kappa}\left(2f_{30}-2\iota f_{12}\right)c_1-\frac{1}{6\kappa}\left(2f_{10}+2\iota f_{20}-2f_{13}-2\iota f_{23}\right)c_2, \\ \kappa c_2 = &-\frac{1}{6\kappa}\left(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23}\right)b_2^{\dot 1\dot 1}-\frac{1}{6\kappa}\left(-f_{30}+\iota f_{21}+\iota f_{21}-f_{30}\right)b_2^{\dot 1\dot 2}- \\ &-\frac{1}{6\kappa}\left(-f_{10}+f_{31}-\iota f_{20}+\iota f_{32}\right)b_2^{\dot 2\dot 2}-\frac{1}{6\kappa}\left(2f_{10}-2f_{31}+2\iota f_{02}-2\iota f_{23}\right)c_1-\frac{1}{6\kappa}\left(-2\iota f_{21}+2\iota f_{03}\right)c_2 . \end{aligned}$$

Let us also consider eqs. (5) and (6) which we rewrite as follows:

$$\begin{aligned}
&\kappa\pi^{1}_{\dot{1}}b_{1}^{\dot{1}\dot{1}}+\kappa\pi^{1}_{\dot{2}}b_{1}^{\dot{1}\dot{2}}+\kappa\pi^{2}_{\dot{1}}b_{2}^{\dot{1}\dot{1}}+\kappa\pi^{2}_{\dot{2}}b_{2}^{\dot{1}\dot{2}}+\kappa\pi^{1\dot{1}}c_{1}+\kappa\pi^{2\dot{1}}c_{2}+\\
&+(-f_{01}+f_{13}-\iota f_{20}+\iota f_{32})\alpha^{\dot{1}}_{11}+(\iota f_{12}-f_{30}-f_{30}+\iota f_{12})\alpha^{\dot{1}}_{12}+(-f_{10}-\iota f_{20}+f_{13}+\iota f_{23})\alpha^{\dot{1}}_{22}+(2f_{08}-2\iota f_{12})d^{\dot{1}}+\\
&+(2f_{01}+2\iota f_{02}-2f_{13}-2\iota f_{23})d^{\dot{2}}-4\kappa\pi^{1\dot{1}}c_{1}-4\kappa\pi^{2\dot{1}}c_{2}=0\,,
\end{aligned}$$

$$\begin{aligned}
&\kappa\pi^{1}_{\dot{1}}b_{1}^{\dot{2}\dot{1}}+\kappa\pi^{1}_{\dot{2}}b_{1}^{\dot{2}\dot{2}}+\kappa\pi^{2}_{\dot{1}}b_{2}^{\dot{2}\dot{1}}+\kappa\pi^{2}_{\dot{2}}b_{2}^{\dot{2}\dot{2}}+\kappa\pi^{1\dot{2}}c_{1}+\kappa\pi^{2\dot{2}}c_{2}+(-f_{01}-\iota f_{20}+f_{13}+\iota f_{32})\alpha^{\dot{2}}_{11}+\\
&+(-f_{30}+\iota f_{12}+\iota f_{12}-f_{30})\alpha^{\dot{2}}_{12}+(-f_{10}+f_{13}-\iota f_{20}+\iota f_{23})\alpha^{\dot{2}}_{22}+(-2\iota f_{21}+2f_{30})d^{\dot{2}}+\\
&+(2f_{01}-2f_{31}+2\iota f_{20}-2\iota f_{23})d^{\dot{1}}-4\kappa\pi^{1\dot{2}}c_{1}-4\kappa\pi^{2\dot{2}}c_{2}=0\,.
\end{aligned}$$

After partly substituting KC_1 , KC_2 , into them we find the equations:

$$\begin{aligned}
&\kappa\pi^{1}_{\dot{1}}b_{1}^{\dot{1}\dot{1}}+\kappa\pi^{1}_{\dot{2}}b_{1}^{\dot{1}\dot{2}}+\kappa\pi^{2}_{\dot{1}}b_{2}^{\dot{1}\dot{1}}+\kappa\pi^{2}_{\dot{2}}b_{2}^{\dot{1}\dot{2}}+\kappa\pi^{1\dot{1}}c_{1}+\kappa\pi^{2\dot{1}}c_{2}+\\
&+(-f_{01}+f_{13}-\iota f_{20}+\iota f_{32})\alpha^{\dot{1}}_{11}+(\iota f_{12}-f_{30}-f_{30}+\iota f_{12})\alpha^{\dot{1}}_{12}+(-f_{10}-\iota f_{20}+f_{13}+\iota f_{23})\alpha^{\dot{1}}_{22}+(2f_{03}-2\iota f_{12})d^{\dot{1}}+\\
&+(2f_{01}+2\iota f_{02}-2f_{13}-2\iota f_{23})d^{\dot{2}}+\frac{2}{3K}\pi^{1\dot{1}}\Big\{(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23})b_{1}^{\dot{1}\dot{1}}+(\iota f_{21}-f_{30}-f_{30}+\\
&+\iota f_{21})b_{1}^{\dot{1}\dot{2}}+(-f_{10}-\iota f_{20}+f_{31}+\iota f_{32})b_{1}^{\dot{2}\dot{2}}+(2f_{30}-2\iota f_{12})c_{1}+(2f_{10}+2\iota f_{20}-2f_{13}-2\iota f_{23})c_{2}\Big\}+\\
&+\frac{2}{3K}\pi^{2\dot{1}}\Big\{(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23})b_{2}^{\dot{1}\dot{1}}+(-f_{30}+\iota f_{21}+\iota f_{21}-f_{30})b_{2}^{\dot{1}\dot{2}}+(-f_{10}+f_{31}-\iota f_{20}+\\
&+\iota f_{32})b_{2}^{\dot{2}\dot{2}}+(2f_{10}-2f_{31}+2\iota f_{02}-2\iota f_{23})c_{1}+(-2\iota f_{21}+2f_{03})c_{2}\Big\}=0, \qquad (13)
\end{aligned}$$

$$\begin{aligned}
&\kappa\pi^{1}_{\dot{1}}b_{1}^{\dot{2}\dot{1}}+\kappa\pi^{1}_{\dot{2}}b_{1}^{\dot{2}\dot{2}}+\kappa\pi^{2}_{\dot{1}}b_{2}^{\dot{2}\dot{1}}+\kappa\pi^{2}_{\dot{2}}b_{2}^{\dot{2}\dot{2}}+\kappa\pi^{1\dot{2}}c_{1}+\kappa\pi^{2\dot{2}}c_{2}+(-f_{01}-\iota f_{20}+f_{13}+\iota f_{32})\alpha^{\dot{2}}_{11}+\\
&+(-f_{30}+\iota f_{12}+\iota f_{12}-f_{30})\alpha^{\dot{2}}_{12}+(-f_{10}+f_{13}-\iota f_{20}+\iota f_{23})\alpha^{\dot{2}}_{22}+(-2\iota f_{21}+2f_{30})d^{\dot{2}}+(2f_{01}-\\
&-2f_{31}+2\iota f_{20}-2\iota f_{23})d^{\dot{1}}+\frac{2}{3K}\pi^{1\dot{2}}\Big\{(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23})b_{1}^{\dot{1}\dot{1}}+(\iota f_{21}-f_{30}-f_{30}+\iota f_{21})b_{1}^{\dot{1}\dot{2}}+\\
&+(-f_{10}-\iota f_{20}+f_{31}+\iota f_{32})b_{1}^{\dot{2}\dot{2}}+(2f_{30}-2\iota f_{12})c_{1}+(2f_{10}+2\iota f_{20}-2f_{13}-2\iota f_{23})c_{2}\Big\}+\frac{2}{3K}\pi^{2\dot{2}}\Big\{(-f_{01}+\\
&+f_{31}-\iota f_{20}+\iota f_{23})b_{2}^{\dot{1}\dot{1}}+(-f_{30}+\iota f_{21}+\iota f_{21}-f_{30})b_{2}^{\dot{1}\dot{2}}+(-f_{10}+f_{31}-\iota f_{20}+\iota f_{32})b_{2}^{\dot{2}\dot{2}}+(2f_{10}-\\
&-2f_{31}+2\iota f_{02}-2\iota f_{23})c_{1}+(-2\iota f_{21}+2f_{03})c_{2}\Big\}=0\,. \qquad (14)
\end{aligned}$$

Finally after completing the substitution of κc_1 , κc_2, in the above equations we find:

$$\begin{aligned}
&2\kappa\pi^{1}_{\dot{1}}b_1^{\dot{1}\dot{1}}+2\kappa\pi^{1}_{\dot{2}}b_1^{\dot{1}\dot{2}}+2\kappa\pi^{2}_{\dot{1}}b_2^{\dot{1}\dot{1}}+2\kappa\pi^{2}_{\dot{2}}b_2^{\dot{1}\dot{2}}+2(-f_{01}+f_{13}-\iota f_{20}+\iota f_{32})\alpha^{\dot{1}}_{11}+\\
&+2(\iota f_{12}-f_{30}-f_{30}+\iota f_{12})\alpha^{\dot{1}}_{12}+2(-f_{10}-\iota f_{20}+f_{13}+\iota f_{23})\alpha^{\dot{1}}_{22}+4(f_{03}-\iota f_{12})d^{\dot{1}}+4(f_{01}+\\
&+\iota f_{02}-f_{13}-\iota f_{23})d^{\dot{2}}-\pi^{1\dot{1}}\{-\tfrac{1}{\kappa}(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23})b_1^{\dot{1}\dot{1}}-\tfrac{1}{\kappa}(\iota f_{21}-f_{30}-\\
&-f_{30}+\iota f_{21})b_1^{\dot{1}\dot{2}}-\tfrac{1}{\kappa}(-f_{10}-\iota f_{20}+f_{31}+\iota f_{32})b_1^{\dot{2}\dot{2}}-\tfrac{1}{\kappa}(2f_{30}-2\iota f_{12})c_1-\\
&-\tfrac{1}{\kappa}(2f_{10}+2\iota f_{20}-2f_{13}-2\iota f_{23})c_2\}-\pi^{2\dot{1}}\{-\tfrac{1}{\kappa}(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23})b_2^{\dot{1}\dot{1}}-\\
&-\tfrac{1}{\kappa}(-f_{30}+\iota f_{21}+\iota f_{21}-f_{30})b_2^{\dot{1}\dot{2}}-\tfrac{1}{\kappa}(-f_{10}+f_{31}-\iota f_{20}+\iota f_{32})b_2^{\dot{2}\dot{2}}-\tfrac{1}{\kappa}(2f_{10}-\\
&-2f_{31}+2\iota f_{02}-2\iota f_{23})c_1-\tfrac{1}{\kappa}(-2\iota f_{21}+2\iota f_{03})c_2\}=0,
\end{aligned}\tag{13'}$$

$$\begin{aligned}
&2\kappa\pi^{1}_{\dot{1}}b_1^{\dot{2}\dot{1}}+2\kappa\pi^{1}_{\dot{2}}b_1^{\dot{2}\dot{2}}+2\kappa\pi^{2}_{\dot{1}}b_2^{\dot{2}\dot{1}}+2\kappa\pi^{2}_{\dot{2}}b_2^{\dot{2}\dot{2}}+2(-f_{01}-\iota f_{20}+f_{13}+\iota f_{32})\alpha^{\dot{2}}_{11}+\\
&+2(-f_{30}+\iota f_{12}+\iota f_{12}-f_{30})\alpha^{\dot{2}}_{12}+2(-f_{10}+f_{13}-\iota f_{20}+\iota f_{23})\alpha^{\dot{2}}_{22}+4(-\iota f_{21}+\\
&+f_{30})d^{\dot{2}}+4(f_{01}-f_{31}+\iota f_{20}-\iota f_{23})d^{\dot{1}}-\pi^{1\dot{2}}\{-\tfrac{1}{\kappa}(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23})b_1^{\dot{1}\dot{1}}-\\
&-\tfrac{1}{\kappa}(\iota f_{21}-f_{30}-f_{30}+\iota f_{21})b_1^{\dot{1}\dot{2}}-\tfrac{1}{\kappa}(-f_{10}-\iota f_{20}+f_{31}+\iota f_{32})b_1^{\dot{2}\dot{2}}-\tfrac{1}{\kappa}(2f_{30}-2\iota f_{12})c_1-\\
&-\tfrac{1}{\kappa}(2f_{10}+2\iota f_{20}-2f_{13}-2\iota f_{23})c_2\}-\pi^{2\dot{2}}\{-\tfrac{1}{\kappa}(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23})b_2^{\dot{1}\dot{1}}-\\
&-\tfrac{1}{\kappa}(-f_{30}+\iota f_{21}+\iota f_{21}-f_{30})b_2^{\dot{1}\dot{2}}-\tfrac{1}{\kappa}(-f_{10}+f_{31}-\iota f_{20}+\iota f_{32})b_2^{\dot{2}\dot{2}}-\tfrac{1}{\kappa}(2f_{10}-2f_{31}+\\
&+2\iota f_{02}-2\iota f_{23})c_1-\tfrac{1}{\kappa}(-2\iota f_{21}+2\iota f_{03})c_2\}=0\ .
\end{aligned}\tag{14'}$$

Equations (13') and (14') can be obtained easier by solving eqs. (11) and (12) for $6KC_1$, $6KC_2$, and replacing into eqs. (5) and (6). The reason of deriving the intermediate equations (13) and (14) is because we need them in the next paragraph.

Consider now the other pair of secondary constraints, namely equations (7) and (8) of § 9. Solving them for $Kd^{\dot{1}}$, and $Kd^{\dot{2}}$ we get respectively:

$$Kd^{\dot{1}} = -\frac{1}{6K}\left(-f_{01}+f_{13}-\iota f_{20}+\iota f_{32}\right)\alpha^{\dot{1}}_{11} - \frac{1}{6K}\left(\iota f_{12}-f_{30}-f_{30}+\iota f_{12}\right)\alpha^{\dot{1}}_{12} - \frac{1}{6K}\left(-f_{10}- -\iota f_{20}+f_{13}+\iota f_{23}\right)\alpha^{\dot{1}}_{22} - \frac{1}{6K}\left(2f_{03}-2\iota f_{12}\right)d^{\dot{1}} - \frac{1}{6K}\left(2f_{01}+2\iota f_{02}-2f_{13}-2\iota f_{23}\right)d^{\dot{2}},$$

$$Kd^{\dot{2}} = -\frac{1}{6K}\left(-f_{01}-\iota f_{20}+f_{13}+\iota f_{32}\right)\alpha^{\dot{2}}_{11} - \frac{1}{6K}\left(-f_{30}+\iota f_{12}+\iota f_{12}-f_{30}\right)\alpha^{\dot{2}}_{12} - \frac{1}{6K}\left(-f_{10}+ +f_{13}-\iota f_{20}+\iota f_{23}\right)\alpha^{\dot{2}}_{22} - \frac{1}{6K}\left(-2\iota f_{21}+2f_{30}\right)d^{\dot{2}} - \frac{1}{6K}\left(2f_{01}-2f_{31}+2\iota f_{20}-2\iota f_{23}\right)d^{\dot{1}}.$$

Let us also consider equations (9) and (10) of § 9 which we rewrite as follows:

$$K\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{11} + K\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{12} + K\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{11} + K\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{12} + K\pi_{\dot{1}1}d^{\dot{1}} + K\pi_{\dot{2}1}d^{\dot{2}} + +\left(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23}\right)b^{\dot{1}\dot{1}}_{1} + \left(\iota f_{21}-f_{30}-f_{30}+\iota f_{21}\right)b^{\dot{1}\dot{2}}_{1} + \left(-f_{10}-\iota f_{20}+f_{31}+\iota f_{32}\right)b^{\dot{2}\dot{2}}_{1} + +\left(2f_{30}-2\iota f_{12}\right)c_1 + \left(2f_{10}+2\iota f_{20}-2f_{13}-2\iota f_{23}\right)c_2 - 4K\pi_{\dot{1}1}d^{\dot{1}} - 4K\pi_{\dot{2}1}d^{\dot{2}} = 0,$$

$$K\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{21} + K\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{22} + K\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{21} + K\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{22} + K\pi_{\dot{1}2}d^{\dot{1}} + K\pi_{\dot{2}2}d^{\dot{2}} + +\left(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23}\right)b^{\dot{1}\dot{1}}_{2} + \left(-f_{30}+\iota f_{21}+\iota f_{21}-f_{30}\right)b^{\dot{1}\dot{2}}_{2} + \left(-f_{10}+f_{31}-\iota f_{20}+\iota f_{32}\right)b^{\dot{2}\dot{2}}_{2} + +\left(2f_{10}-2f_{31}+2\iota f_{02}-2\iota f_{23}\right)c_1 + \left(-2\iota f_{21}+2f_{03}\right)c_2 - 4K\pi_{\dot{1}2}d^{\dot{1}} - 4K\pi_{\dot{2}2}d^{\dot{2}} = 0.$$

After partly substituting $Kd^{\dot{1}}$, $Kd^{\dot{2}}$ into them we find the equations:

$$\begin{aligned}
&\kappa\pi^{1}_{\dot1}\alpha^{\dot1}_{11}+\kappa\pi^{2}_{\dot1}\alpha^{\dot1}_{12}+\kappa\pi^{1}_{\dot2}\alpha^{\dot2}_{11}+\kappa\pi^{2}_{\dot2}\alpha^{\dot2}_{12}+\kappa\pi_{\dot1 1}d^{1}+\kappa\pi_{\dot2 1}d^{\dot2}+(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23})b_1^{\dot1\dot1}+\\
&+(\iota f_{21}-f_{30}-f_{30}+\iota f_{21})b_1^{\dot1\dot2}+(-f_{10}-\iota f_{20}+f_{31}+\iota f_{32})b_1^{\dot2\dot2}+(2f_{30}-2\iota f_{12})c_1+(2f_{10}+2\iota f_{20}-\\
&-2f_{13}-2\iota f_{23})c_2+\frac{2}{3K}\pi_{\dot1 1}\{(-f_{01}+f_{13}-\iota f_{20}+\iota f_{32})\alpha^{\dot1}_{11}+(\iota f_{12}-f_{30}-f_{30}+\iota f_{12})\alpha^{\dot1}_{12}+(-f_{10}-\iota f_{20}+f_{13}+\\
&+\iota f_{23})\alpha^{\dot1}_{22}+(2f_{03}-2\iota f_{12})d^{\dot1}+(2f_{01}+2\iota f_{02}-2f_{13}-2\iota f_{23})d^{\dot2}\}+\frac{2}{3K}\pi_{\dot2 1}\{(-f_{01}-\iota f_{20}+f_{13}+\iota f_{32})\alpha^{\dot2}_{11}+\\
&+(-f_{30}+\iota f_{12}+\iota f_{12}-f_{30})\alpha^{\dot2}_{12}+(-f_{10}+f_{13}-\iota f_{20}+\iota f_{23})\alpha^{\dot2}_{22}+(-2\iota f_{21}+2f_{30})d^{\dot2}+\\
&+(2f_{01}-2f_{31}+2\iota f_{20}-2\iota f_{23})d^{1}\}=0\,,
\end{aligned}\tag{15}$$

$$\begin{aligned}
&\kappa\pi^{1}_{\dot1}\alpha^{\dot1}_{21}+\kappa\pi^{2}_{\dot1}\alpha^{\dot1}_{22}+\kappa\pi^{1}_{\dot2}\alpha^{\dot2}_{21}+\kappa\pi^{2}_{\dot2}\alpha^{\dot2}_{22}+\kappa_{\dot1 2}d^{\dot1}+\kappa\pi_{\dot2 2}d^{\dot2}+(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23})b_2^{\dot1\dot1}+\\
&+(-f_{30}+\iota f_{21}+\iota f_{21}-f_{30})b_2^{\dot1\dot2}+(-f_{10}+f_{31}-\iota f_{20}+\iota f_{32})b_2^{\dot2\dot2}+(2f_{10}-2f_{31}+2\iota f_{02}-2\iota f_{23})c_1+\\
&+(-2\iota f_{21}+2f_{03})c_2+\frac{2}{3K}\pi_{\dot1 2}\{(-f_{01}+f_{13}-\iota f_{20}+\iota f_{32})\alpha^{\dot1}_{11}+(\iota f_{12}-f_{30}-f_{30}+\iota f_{12})\alpha^{\dot1}_{12}+\\
&+(-f_{10}-\iota f_{20}+f_{13}+\iota f_{23})\alpha^{\dot1}_{22}+(2f_{03}-2\iota f_{12})d^{\dot1}+(2f_{01}+2\iota f_{02}-2f_{13}-2\iota f_{23})d^{\dot2}\}+\\
&+\frac{2}{3K}\pi_{\dot2 2}\{(-f_{01}-\iota f_{20}+f_{13}+\iota f_{32})\alpha^{\dot2}_{11}+(-f_{30}+\iota f_{12}+\iota f_{12}-f_{30})\alpha^{\dot2}_{12}+(-f_{10}+f_{13}-\\
&-\iota f_{20}+\iota f_{23})\alpha^{\dot2}_{22}+(-2\iota f_{21}+2f_{30})d^{\dot2}+(2f_{01}-2f_{31}+\\
&+2\iota f_{20}-2\iota f_{23})d^{\dot1}\}=0\,.
\end{aligned}\tag{16}$$

Finally after completing the substitution of $\kappa d^{\dot1}, \kappa d^{\dot2}$, in the above equations we find:

$$\begin{aligned}&2K\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{11}+2K\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{12}+2K\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{11}+2K\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{12}+2(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23})b_1^{\dot{1}\dot{1}}+\\&+2(\iota f_{21}-f_{30}-f_{30}+\iota f_{21})b_1^{\dot{1}\dot{2}}+2(-f_{10}-\iota f_{20}+f_{31}+\iota f_{32})b_1^{\dot{2}\dot{2}}+4(f_{30}-\iota f_{21})c_1+\\&+4(f_{10}+\iota f_{20}-f_{13}-\iota f_{23})c_2-\pi_{\dot{1}1}\{-\frac{1}{K}(-f_{01}+f_{13}-\iota f_{20}+\iota f_{32})\alpha^{\dot{1}}_{11}-\\&-\frac{1}{K}(\iota f_{12}-f_{30}-f_{30}+\iota f_{12})\alpha^{\dot{1}}_{12}-\frac{1}{K}(-f_{10}-\iota f_{20}+f_{13}+\iota f_{23})\alpha^{\dot{1}}_{22}-\frac{1}{K}(2f_{03}-\\&-2\iota f_{12})d^{\dot{1}}-\frac{1}{K}(2f_{01}+2\iota f_{02}-2f_{13}-2\iota f_{23})d^{\dot{2}}\}-\pi_{\dot{2}1}\{-\frac{1}{K}(-f_{01}-\iota f_{20}+f_{13}+\\&+\iota f_{32})\alpha^{\dot{2}}_{11}-\frac{1}{K}(-f_{30}+\iota f_{12}+\iota f_{12}-f_{30})\alpha^{\dot{2}}_{12}-\frac{1}{K}(-f_{10}+f_{13}-\iota f_{20}+\iota f_{23})\alpha^{\dot{2}}_{22}-\\&-\frac{1}{K}(-2\iota f_{21}+2f_{30})d^{\dot{2}}-\frac{1}{K}(2f_{01}-2f_{31}+2\iota f_{20}-2\iota f_{23})d^{\dot{1}}\}=0,\end{aligned}\tag{15'}$$

$$\begin{aligned}&2K\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{21}+2K\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{22}+2K\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{21}+2K\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{22}+2(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23})b_2^{\dot{1}\dot{1}}+\\&+2(-f_{30}+\iota f_{21}+\iota f_{21}-f_{30})b_2^{\dot{1}\dot{2}}+2(-f_{10}+f_{31}-\iota f_{20}+\iota f_{32})b_2^{\dot{2}\dot{2}}+4(f_{10}-f_{31}+\iota f_{02}-\iota f_{23})c_1+\\&+4(-\iota f_{21}+f_{03})c_2-\pi_{\dot{1}2}\{-\frac{1}{K}(-f_{01}+f_{13}-\iota f_{20}+\iota f_{32})\alpha^{\dot{1}}_{11}-\frac{1}{K}(\iota f_{12}-f_{30}-\\&-f_{30}+\iota f_{12})\alpha^{\dot{1}}_{12}-\frac{1}{K}(-f_{10}-\iota f_{20}+f_{13}+\iota f_{23})\alpha^{\dot{1}}_{22}-\frac{1}{K}(2f_{03}-2\iota f_{12})d^{\dot{1}}-\\&-\frac{1}{K}(2f_{01}+2\iota f_{02}-2f_{13}-2\iota f_{23})d^{\dot{2}}\}-\pi_{\dot{2}2}\{-\frac{1}{K}(-f_{01}-\iota f_{20}+f_{13}+\iota f_{32})\alpha^{\dot{2}}_{11}-\\&-\frac{1}{K}(-f_{30}+\iota f_{12}+\iota f_{12}-f_{30})\alpha^{\dot{2}}_{12}-\frac{1}{K}(-f_{10}+f_{13}-\iota f_{20}+\iota f_{23})\alpha^{\dot{2}}_{22}-\\&-\frac{1}{K}(-2\iota f_{21}+2f_{30})d^{\dot{2}}-\frac{1}{K}(2f_{01}-2f_{31}+2\iota f_{20}-2\iota f_{23})d^{\dot{1}}\}=0.\end{aligned}\tag{16'}$$

Equations (15') and (16') can be obtained easier by solving eqs. (7) and (8) for $6Kd^{\dot{1}}$, $6Kd^{\dot{2}}$, and substituting into eqs. (9) and (10). The reason of deriving equations (15) and (16) is because we need them in the next paragraph.

Solving eq. (13') for $\pi_{2\dot{1}} b_1^{\dot{1}\dot{1}} \equiv \pi^{1}_{\dot{1}} b_1^{\dot{1}\dot{1}}$, eq. (14') for $\pi_{2\dot{1}} b_1^{\dot{1}\dot{2}} \equiv \pi^{1}_{\dot{1}} b_1^{\dot{1}\dot{2}}$, eq. (15') for $\pi^{\dot{2}1}\alpha_{11}^{\dot{1}} \equiv -\pi^{1}_{\dot{1}}\alpha_{11}^{\dot{1}}$, eq. (16') for $\pi^{\dot{2}1}\alpha_{21}^{\dot{1}} \equiv -\pi^{1}_{\dot{1}}\alpha_{21}^{\dot{1}}$ and substituting in (9B), (10B) (3B), (4B) respectively and converting the spinors $\pi_{\alpha\dot{\beta}}$ into their four vector components π_k we get for the modified Pauli-Fierz equation the following set:

$$-(-\pi_0-\pi_3)\alpha_{11}^{\dot{1}}-(-\pi_1-i\pi_2)\alpha_{12}^{\dot{1}}-(\pi_1+i\pi_2)d^{\dot{1}}+\kappa b_1^{\dot{1}\dot{1}}=0, \tag{1}$$

$$-(-\pi_0-\pi_3)\alpha_{12}^{\dot{1}}-(-\pi_1-i\pi_2)\alpha_{22}^{\dot{1}}-(-\pi_0-\pi_3)d^{\dot{1}}+\kappa b_2^{\dot{1}\dot{1}}=0, \tag{2}$$

$$\begin{aligned}
&-(-\pi_0-\pi_3)\alpha_{11}^{\dot{2}}-(-\pi_1-i\pi_2)\alpha_{12}^{\dot{2}}-\frac{1}{2K}(-f_{01}+f_{31}-if_{20}+if_{23})b_1^{\dot{1}\dot{1}}-\\
&-\frac{1}{2K}(if_{21}-f_{30}-f_{30}+if_{21})b_1^{\dot{1}\dot{2}}-\frac{1}{2K}(-f_{10}-if_{20}+f_{31}+if_{32})b_1^{\dot{2}\dot{2}}-\frac{1}{K}(f_{30}-if_{12})c_1-\\
&-\frac{1}{K}(f_{10}+if_{20}-f_{13}-if_{23})c_2+\frac{(-\pi_0-\pi_3)}{4K}\{-\frac{1}{K}(-f_{01}+f_{13}-if_{20}+if_{32})\alpha_{11}^{\dot{1}}-\\
&-\frac{1}{K}(if_{12}-f_{30}-f_{30}+if_{12})\alpha_{12}^{\dot{1}}-\frac{1}{K}(-f_{10}-if_{20}+f_{13}+if_{23})\alpha_{22}^{\dot{1}}-\frac{1}{K}(2f_{03}-2if_{12})d^{\dot{1}}-\\
&-\frac{1}{K}(2f_{01}+2if_{02}-2f_{13}-2if_{23})d^{\dot{2}}\}+\frac{(\pi_1+i\pi_2}{4K}\{-\frac{1}{K}(-f_{01}-if_{20}+f_{13}+if_{32})\alpha_{11}^{\dot{2}}-\\
&-\frac{1}{K}(-f_{30}+if_{12}+if_{12}-f_{30})\alpha_{12}^{\dot{2}}-\frac{1}{K}(-f_{10}+f_{13}-if_{20}+if_{23})\alpha_{22}^{\dot{2}}-\frac{1}{K}(-2if_{21}+2f_{30})d^{\dot{2}}-\\
&-\frac{1}{K}(2f_{01}-2f_{31}+2if_{20}-2if_{23})d^{\dot{1}}\}-\frac{1}{2}(\pi_1+i\pi_2)d^{\dot{2}}-\frac{1}{2}(-\pi_0-\pi_3)d^{\dot{1}}+\kappa b_1^{\dot{1}\dot{2}}=0,
\end{aligned} \tag{3}$$

$$\begin{aligned}
&-(-\pi_0-\pi_3)\alpha_{12}^{\dot{2}}-(-\pi_1-i\pi_2)\alpha_{22}^{\dot{2}}-\frac{1}{2K}(-f_{01}+f_{31}-if_{20}+if_{23})b_2^{\dot{1}\dot{1}}-\\
&-\frac{1}{2K}(-f_{30}+if_{21}+if_{21}-f_{30})b_2^{\dot{1}\dot{2}}-\frac{1}{2K}(-f_{10}+f_{31}-if_{20}+if_{32})b_2^{\dot{2}\dot{2}}-\frac{1}{K}(f_{10}-f_{31}+\\
&+if_{02}-if_{23})c_1-\frac{1}{K}(-if_{21}+f_{03})c_2+\frac{(\pi_1-i\pi_2)}{4K}\{-\frac{1}{K}(-f_{01}+f_{13}-if_{20}+if_{32})\alpha_{11}^{\dot{1}}-\\
&-\frac{1}{K}(if_{12}-f_{30}-f_{30}+if_{12})\alpha_{12}^{\dot{1}}-\frac{1}{K}(-f_{10}-if_{20}+f_{13}+if_{23})\alpha_{22}^{\dot{1}}-\frac{1}{K}(2f_{03}-2if_{12})d^{\dot{1}}-\\
&-\frac{1}{K}(2f_{01}+2if_{02}-2f_{13}-2if_{23})d^{\dot{2}}\}+\frac{(-\pi_0-\pi_3)}{4K}\{-\frac{1}{K}(-f_{01}-if_{20}+f_{13}+if_{32})\alpha_{11}^{\dot{2}}-\\
&-\frac{1}{K}(-f_{30}+if_{12}+if_{12}-f_{30})\alpha_{12}^{\dot{2}}-\frac{1}{K}(-f_{10}+f_{13}-if_{20}+if_{23})\alpha_{22}^{\dot{2}}-\frac{1}{K}(-2if_{21}+\\
&+2f_{30})d^{\dot{2}}-\frac{1}{K}(2f_{01}-2f_{31}+2if_{20}-2if_{23})d^{\dot{1}}\}-\frac{1}{2}(-\pi_0-\pi_3)d^{\dot{2}}-\\
&-\frac{1}{2}(-\pi_1+i\pi_2)d^{\dot{1}}+\kappa b_2^{\dot{1}\dot{2}}=0,
\end{aligned} \tag{4}$$

(5) $$-(-\pi_1+\iota\pi_2)\alpha^{\dot{2}}_{11}-(-\pi_0+\pi_3)\alpha^{\dot{2}}_{12}-(\pi_0-\pi_3)d^{\dot{2}}+\kappa b^{\dot{2}\dot{2}}_1=0,$$

(6) $$-(-\pi_1+\iota\pi_2)\alpha^{\dot{2}}_{12}-(-\pi_0+\pi_3)\alpha^{\dot{2}}_{22}-(-\pi_1+\iota\pi_2)d^{\dot{2}}+\kappa b^{\dot{2}\dot{2}}_2=0,$$

(7) $$-(-\pi_0+\pi_3)b^{11}_1-(\pi_1+\iota\pi_2)b^{1\dot{2}}_1-(\pi_1+\iota\pi_2)c_1+\kappa\alpha^{\dot{1}}_{11}=0,$$

(8) $$-(-\pi_0+\pi_3)b^{\dot{1}\dot{2}}_1-(\pi_1+\iota\pi_2)b^{\dot{2}\dot{2}}_1-(\pi_0-\pi_3)c_1+\kappa\alpha^{\dot{2}}_{11}=0,$$

(9) $$\begin{aligned}&-(-\pi_0+\pi_3)b^{\dot{1}\dot{1}}_2-(\pi_1+\iota\pi_2)b^{\dot{1}\dot{2}}_2+\frac{1}{2\kappa}(-f_{01}+f_{13}-\iota f_{20}+\iota f_{32})\alpha^{\dot{1}}_{11}+\\&+\frac{1}{2\kappa}(\iota f_{12}-f_{30}-f_{30}+\iota f_{12})\alpha^{\dot{1}}_{12}+\frac{1}{2\kappa}(-f_{10}-\iota f_{20}+f_{13}+\iota f_{23})\alpha^{\dot{1}}_{22}+\frac{1}{\kappa}(f_{03}-\iota f_{12})d^{\dot{1}}+\\&+\frac{1}{\kappa}(f_{01}+\iota f_{02}-f_{13}-\iota f_{23})d^{\dot{2}}-\frac{(-\pi_0-\pi_3)}{4\kappa}\Big\{-\frac{1}{\kappa}(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23})b^{\dot{1}\dot{1}}_1-\\&-\frac{1}{\kappa}(\iota f_{21}-f_{30}-f_{30}+\iota f_{21})b^{\dot{1}\dot{2}}_1-\frac{1}{\kappa}(-f_{10}-\iota f_{20}+f_{31}+\iota f_{32})b^{\dot{2}\dot{2}}_1-\frac{1}{\kappa}(2f_{30}-2\iota f_{12})c_1-\\&-\frac{1}{\kappa}(2f_{10}+2\iota f_{20}-2f_{13}-2\iota f_{23})c_2\Big\}-\frac{(-\pi_1-\iota\pi_2)}{4\kappa}\Big\{-\frac{1}{\kappa}(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23})b^{\dot{1}\dot{1}}_2-\\&-\frac{1}{\kappa}(-f_{30}+\iota f_{21}+\iota f_{21}-f_{30})b^{\dot{1}\dot{2}}_2-\frac{1}{\kappa}(-f_{10}+f_{31}-\iota f_{20}+\iota f_{32})b^{\dot{2}\dot{2}}_2-\frac{1}{\kappa}(2f_{10}-2f_{31}+2\iota f_{02}-\\&-2\iota f_{23})c_1-\frac{1}{\kappa}(-2\iota f_{21}+2\iota f_{03})c_2\Big\}-\frac{1}{2}(\pi_1+\iota\pi_2)c_2-\frac{1}{2}(-\pi_0-\pi_3)c_1+\kappa\alpha^{\dot{1}}_{12}=0,\end{aligned}$$

(10) $$\begin{aligned}&-(-\pi_0+\pi_3)b^{\dot{1}\dot{2}}_2-(\pi_1+\iota\pi_2)b^{\dot{2}\dot{2}}_2+\frac{1}{2\kappa}(-f_{01}-\iota f_{20}+f_{13}+\iota f_{32})\alpha^{\dot{2}}_{11}+\frac{1}{2\kappa}(-f_{30}+\iota f_{12}+\\&+\iota f_{12}-f_{30})\alpha^{\dot{2}}_{12}+\frac{1}{2\kappa}(-f_{10}+f_{13}-\iota f_{20}+\iota f_{23})\alpha^{\dot{2}}_{22}+\frac{1}{\kappa}(-\iota f_{21}+f_{30})d^{\dot{2}}+\frac{1}{\kappa}(f_{01}-\\&-f_{31}+\iota f_{20}-\iota f_{23})d^{\dot{1}}-\frac{(-\pi_1+\iota\pi_2)}{4\kappa}\Big\{-\frac{1}{\kappa}(-f_{01}+f_{31}-\iota f_{20}+\iota f_{23})b^{\dot{1}\dot{1}}_1-\frac{1}{\kappa}(\iota f_{21}-\\&-f_{30}-f_{30}+\iota f_{21})b^{\dot{1}\dot{2}}_1-\frac{1}{\kappa}(-f_{10}-\iota f_{20}+f_{31}+\iota f_{32})b^{\dot{2}\dot{2}}_1-\frac{1}{\kappa}(2f_{30}-2\iota f_{12})c_1-\\&-\frac{1}{\kappa}(2f_{10}+2\iota f_{20}-2f_{13}-2\iota f_{23})c_2\Big\}-\frac{(\mp\pi_0+\pi_3)}{4\kappa}\Big\{-\frac{1}{\kappa}(-f_{01}+f_{31}-\iota f_{20}+\\&+\iota f_{23})b^{\dot{1}\dot{1}}_2-\frac{1}{\kappa}(-f_{30}+\iota f_{21}+\iota f_{21}-f_{30})b^{\dot{1}\dot{2}}_2-\frac{1}{\kappa}(-f_{10}+f_{31}-\iota f_{20}+\iota f_{32})b^{\dot{2}\dot{2}}_2-\\&-\frac{1}{\kappa}(2f_{10}-2f_{31}+2\iota f_{02}-2\iota f_{23})c_1-\frac{1}{\kappa}(-2\iota f_{21}+2\iota f_{03})c_2\Big\}-\frac{1}{2}(\pi_0-\pi_3)c_2-\\&-\frac{1}{2}(-\pi_1+\iota\pi_2)c_1+\kappa\alpha^{\dot{2}}_{12}=0,\end{aligned}$$

$$(11) \qquad -(\pi_1 - \iota\pi_2) b_2^{\dot{1}\dot{1}} - (-\pi_0 - \pi_3) b_2^{\dot{1}\dot{2}} - (-\pi_0 - \pi_3) c_2 + \kappa\alpha_{22}^{\dot{1}} = 0 \,,$$

$$(12) \qquad -(\pi_1 - \iota\pi_2) b_2^{\dot{1}\dot{2}} - (-\pi_0 - \pi_3) b_2^{\dot{2}\dot{2}} - (-\pi_1 + \iota\pi_2) c_2 + \kappa\alpha_{22}^{\dot{2}} = 0 \,,$$

$$(13) \qquad -(\pi_1 - \iota\pi_2)\alpha_{11}^{\dot{1}} - (-\pi_0 - \pi_3)\alpha_{11}^{\dot{2}} - (\pi_0 - \pi_3)\alpha_{12}^{\dot{1}} - (-\pi_1 - \iota\pi_2)\alpha_{12}^{\dot{2}} + \\ + 3(-\pi_0 + \pi_3) d^{\dot{1}} + 3(\pi_1 + \iota\pi_2) d^{\dot{2}} + 6\kappa c_1 = 0 \,,$$

$$(14) \qquad -(\pi_1 - \iota\pi_2)\alpha_{12}^{\dot{1}} - (-\pi_0 - \pi_3)\alpha_{12}^{\dot{2}} - (\pi_0 - \pi_3)\alpha_{22}^{\dot{1}} - (-\pi_1 - \iota\pi_2)\alpha_{22}^{\dot{2}} + \\ + 3(\pi_1 - \iota\pi_2) d^{\dot{1}} + 3(-\pi_0 - \pi_3) d^{\dot{2}} + 6\kappa c_2 = 0 \,,$$

$$(15) \qquad -(\pi_1 - \iota\pi_2) b_1^{\dot{1}\dot{1}} - (\pi_0 - \pi_3) b_2^{\dot{1}\dot{1}} - (-\pi_0 - \pi_3) b_1^{\dot{1}\dot{2}} - (-\pi_1 - \iota\pi_2) b_2^{\dot{1}\dot{2}} + \\ + 3(-\pi_0 - \pi_3) c_1 + 3(-\pi_1 - \iota\pi_2) c_2 + 6\kappa d^{\dot{1}} = 0 \,,$$

$$(16) \qquad -(\pi_1 - \iota\pi_2) b_1^{\dot{1}\dot{2}} - (\pi_0 - \pi_3) b_2^{\dot{1}\dot{2}} - (-\pi_0 - \pi_3) b_1^{\dot{2}\dot{2}} - (-\pi_1 - \iota\pi_2) b_2^{\dot{2}\dot{2}} + \\ + 3(-\pi_1 + \iota\pi_2) c_1 + 3(-\pi_0 + \pi_3) c_2 + 6\kappa d^{\dot{2}} = 0 \,.$$

These equations under spacial reflections go into each other. That this is so can be verified using the following formulae showing how spinors and other quantities behave under space relfections;

$$d^{\dot{1}} \longrightarrow -\iota c_1 \,, \quad d^{\dot{2}} \longrightarrow -\iota c_2 \,, \quad c_1 \longrightarrow -\iota d^{\dot{1}} \,, \quad c_2 \longrightarrow -\iota d^{\dot{2}} \,,$$

$$b_1^{\dot{1}\dot{1}} \longrightarrow \iota\alpha_{11}^{\dot{1}} \,, \quad b_1^{\dot{1}\dot{2}} \longrightarrow \iota\alpha_{12}^{\dot{1}} \,, \quad b_1^{\dot{2}\dot{2}} \longrightarrow \iota\alpha_{22}^{\dot{1}} \,, \quad b_2^{\dot{1}\dot{1}} \longrightarrow \iota\alpha_{11}^{\dot{2}} \,,$$

$$b_2^{\dot{1}\dot{2}} \longrightarrow \iota\alpha_{12}^{\dot{2}} \,, \quad b_2^{\dot{2}\dot{2}} \longrightarrow \iota\alpha_{22}^{\dot{2}} \,, \quad \alpha_{11}^{\dot{1}} \longrightarrow \iota b_1^{\dot{1}\dot{1}} \,, \quad \alpha_{12}^{\dot{1}} \longrightarrow \iota b_1^{\dot{1}\dot{2}} \,,$$

$$\alpha_{22}^{\dot{1}} \longrightarrow \iota b^{\dot{2}\dot{2}} \,, \quad \alpha_{11}^{\dot{2}} \longrightarrow \iota b_2^{\dot{1}\dot{1}} \,, \quad \alpha_{12}^{\dot{2}} \longrightarrow \iota b_2^{\dot{1}\dot{2}} \,, \quad \alpha_{22}^{\dot{2}} \longrightarrow \iota b_2^{\dot{2}\dot{2}} \,,$$

$$\pi_1 \longrightarrow -\pi_1 \,, \quad \pi_2 \longrightarrow -\pi_2 \,, \quad \pi_3 \longrightarrow -\pi_3 \,, \quad \pi_0 \longrightarrow \pi_0 \,,$$

$$f_{12} \longrightarrow f_{12} \,, \quad f_{13} \longrightarrow f_{13} \,, \quad f_{23} \longrightarrow f_{23} \,, \quad f_{31} \longrightarrow f_{31} \,,$$

$$f_{32} \longrightarrow f_{32} \,, \quad f_{21} \longrightarrow f_{21} \,, \quad f_{10} \longrightarrow -f_{10} \,, \quad f_{20} \longrightarrow -f_{20} \,,$$

$$f_{30} \longrightarrow -f_{30} \,, \quad f_{01} \longrightarrow -f_{01} \,, \quad f_{02} \longrightarrow -f_{02} \,, \quad f_{03} \longrightarrow -f_{03} \,.$$

(B): Propagation.

Having derived the modified Pauli-Fierz equation, we now study its propagation. The modified equation after some rearrangements can be written in the form:

$$\{\mathbb{L}_0(\kappa,f_{k\ell})\pi_0+\mathbb{L}_1(\kappa,f_{k\ell})\pi_1+\mathbb{L}_2(\kappa,f_{k\ell})\pi_2+\mathbb{L}_3(\kappa,f_{k\ell})\pi_3+\mathbb{B}(\kappa,f_{k\ell})\}\psi=0$$

(where the negative sign of the metric is incorporated in $\mathbb{L}_0(\kappa,f_{k\ell})$) or finally in the form:

$$\{(-\mathbb{L}_0(\kappa,f_{k\ell}))\frac{\partial}{\partial x_0}+\mathbb{L}_1(\kappa,f_{k\ell})\frac{\partial}{\partial x_1}+\mathbb{L}_2(\kappa,f_{k\ell})\frac{\partial}{\partial x_2}+\mathbb{L}_3(\kappa,f_{k\ell})\frac{\partial}{\partial x_3}+\mathbb{G}(f_{k\ell},eA_k)\}\psi=0,$$

where the matrices $\mathbb{L}_0(\kappa,f_{k\ell})$, $\mathbb{L}_1(\kappa,f_{k\ell})$, $\mathbb{L}_2(\kappa,f_{k\ell})$, $\mathbb{L}_3(\kappa,f_{k\ell})$, $\mathbb{B}(\kappa,f_{k\ell})$, $\mathbb{G}(f_{k\ell},eA_k)$ depend on the electromagnetic field. Restricting ourselves to a two dimensional space-time (x_0,x_3) and constructing the characteristic determinant

$$Q(\lambda)=\det\{1\mathbb{L}_3(\kappa,f_{k\ell})-\lambda(-\mathbb{L}_0(\kappa,f_{k\ell}))\}=\det\{\mathbb{L}_3+\lambda\mathbb{L}_0\}=0,$$

(where the matrices $\mathbb{L}_0$, $\mathbb{L}_3$ are the coefficients of π_0, π_3, of the modified equation), we find that the roots of the polynomial $Q(\lambda)=0$ are:

$$\lambda_i=1 \quad \text{for} \quad i=1,2,3,4,5,6,$$

$$\lambda_j=-1 \quad \text{for} \quad j=7,8,9,10,11,12,$$

$$\lambda_{13,14}=\frac{-\left(\frac{2e}{3K^2}\right)^2[E_2H_1-E_1H_2]\pm\sqrt{\begin{aligned}&\left(\tfrac{2e}{3K^2}\right)^4[E_2H_1-E_1H_2]^2+\\&\left[1-\left(\tfrac{2e}{3K^2}\right)^2(H_1^2+H_2^2+H_3^2)\right]\cdot\\&\cdot\left[1-\left(\tfrac{2e}{3K^2}\right)^2H_3^2+\left(\tfrac{2e}{3K^2}\right)^2(E_1^2+E_2^2)\right]\end{aligned}}}{\left[1-\left(\frac{2e}{3K^2}\right)^2(H_1^2+H_2^2+H_3^2)\right]},$$

$$\lambda_{15,16} = \frac{-\left(\frac{2e}{3K^2}\right)^2\left[E_2H_1 - E_1H_2\right] \pm \sqrt{\begin{array}{l}\left(\frac{2e}{3K^2}\right)^4\left[E_2H_1 - E_1H_2\right]^2 + \\ + \left[1 - \left(\frac{2e}{3K^2}\right)^2\left(H_1^2 + H_2^2 + H_3^2\right)\right] \cdot \\ \cdot \left[1 - \left(\frac{2e}{3K^2}\right)^2 H_3^2 + \left(\frac{2e}{3K^2}\right)^2\left(E_1^2 + E_2^2\right)\right]\end{array}}}{\left[1 - \left(\frac{2e}{3K^2}\right)^2\left(H_1^2 + H_2^2 + H_3^2\right)\right]} .$$

These roots are exactly the same as the ones obtained for the Rarita-Schwinger equation. Thus the propagation behaviour of the Pauli-Fierz equation is the same as for the Rarita-Schwinger equation. For a detailed analysis of this behaviour we refer you to chapter 10 paragraph 6.

12: Intermediate steps involved in the derivation of the modified Rarita-Schwinger equation, the same as those involved in the derivation of the modified Pauli-Fierz equation.

We show now in this paragraph that all the intermediate steps involved in the derivation of the modified Rarita-Schwinger equation are the same as those involved in the derivation of the modified Pauli-Fierz equation. It is shown already that the unmodified Rarita-Schwinger equation is equivalent to the unmodified Pauli-Fierz equation. It was also shown that the secondary constraints of the Rarita-Schwinger equation are the same as those of the Pauli-Fierz equation.

We show now that equation

$$m(\gamma\cdot\pi\gamma - \pi)\cdot\psi - \iota e\gamma^5\gamma\cdot F^d\cdot\psi = 0 , \qquad (2.9 , V.Z.)$$

(i.e. eq. 2.9 of Velo and Zwanziger) goes to equations (5), (6), (9), (10) of § 9. Equation (2.9, V.Z.) above, written out in detail is the same as the following four equations:

$$
\begin{aligned}
\text{I)}\quad & m(\pi_0-\pi_3)(\psi_0^0+\psi_1^1-\iota\psi_2^1+\psi_3^0)+m(-\pi_1+\iota\pi_2)(\psi_0^1+\psi_1^0+\iota\psi_2^0-\psi_3^1)-\\
&-m(\pi_0\psi_0^0-\pi_1\psi_1^0-\pi_2\psi_2^0-\pi_3\psi_3^0)+\lambda\iota F^{12}\psi_0^2+\lambda(\iota F^{23}-\iota F^{13})\psi_0^3+\\
&+\lambda(-\iota F^{23}+\iota F^{02})\psi_1^2-\lambda F^{03}\psi_1^3+\lambda(\iota F^{13}-\iota F^{01})\psi_2^2+\lambda\iota F^{03}\psi_2^3-\\
&-\lambda\iota F^{12}\psi_3^2+\lambda(-\iota F^{02}+F^{01})\psi_3^3=0,
\end{aligned}
$$

$$
\begin{aligned}
\text{II)}\quad & m(-\pi_1-\iota\pi_2)(\psi_0^0+\psi_1^1-\iota\psi_2^1+\psi_3^0)+m(\pi_0+\pi_3)(\psi_0^1+\psi_1^0+\psi_2^0-\psi_3^1)-\\
&-m(\pi_0\psi_0^1-\pi_1\psi_1^1-\pi_2\psi_2^1-\pi_3\psi_3^1)+\lambda(\iota F^{23}+F^{13})\psi_0^2-\lambda\iota F^{12}\psi_0^3+\\
&+\lambda F^{03}\psi_1^2+\lambda(-\iota F^{23}-\iota F^{02})\psi_1^3+\lambda\iota F^{03}\psi_2^2+\lambda(\iota F^{13}+\iota F^{01})\psi_2^3+\\
&+\lambda(-\iota F^{02}-\iota F^{01})\psi_3^2-\lambda\iota F^{12}\psi_3^3=0,
\end{aligned}
$$

$$
\begin{aligned}
\text{III)}\quad & m(\pi_0+\pi_3)(\psi_0^2-\psi_1^3+\iota\psi_2^3-\psi_3^2)+m(\pi_1-\iota\pi_2)(\psi_0^3-\psi_1^2-\iota\psi_2^2+\psi_3^3)-\\
&-m(\pi_0\psi_0^2-\pi_1\psi_1^2-\pi_2\psi_2^2-\pi_3\psi_3^2)+\lambda\iota F^{12}\psi_0^0+\lambda(\iota F^{23}-\iota F^{13})\psi_0^1+\\
&+\lambda(\iota F^{23}+\iota F^{02})\psi_1^0-\lambda F^{03}\psi_1^1+\lambda(-\iota F^{13}-\iota F^{01})\psi_2^0+\lambda\iota F^{03}\psi_2^1+\\
&+\lambda\iota F^{12}\psi_3^0+\lambda(-\iota F^{02}+\iota F^{01})\psi_3^1=0,
\end{aligned}
$$

$$
\begin{aligned}
\text{IV)}\quad & m(\pi_1+\iota\pi_2)(\psi_0^2-\psi_1^3+\iota\psi_2^3-\psi_3^2)+m(\pi_0-\pi_3)(\psi_0^3-\psi_1^2-\iota\psi_2^2+\psi_3^3)-\\
&-m(\pi_0\psi_0^3-\pi_1\psi_1^3-\pi_2\psi_2^3-\pi_3\psi_3^3)+\lambda(\iota F^{23}+F^{13})\psi_0^0-\lambda\iota F^{12}\psi_0^1+\\
&+\lambda F^{03}\psi_1^0+\lambda(\iota F^{23}-\iota F^{02})\psi_1^1+\lambda\iota F^{03}\psi_2^0+\lambda(-\iota F^{13}+\iota F^{01})\psi_2^1+\\
&+\lambda(-\iota F^{02}-F^{01})\psi_3^0+\lambda\iota F^{12}\psi_3^1=0,
\end{aligned}
$$

where $\lambda=-\iota e$. Let us consider equation (I). Replacing the vector spinor components $\psi_\iota{}^{l}$ by their spinor equivalent given by the transformation formulae of § 7 we find for the individual terms in (I) the following results:

a)
$$\begin{aligned}&\psi^0_0 + 1\psi^1_1 - \iota\psi^1_2 + 1\psi^0_3 = 1\left(\frac{\iota\sqrt6}{4} b_1^{\dot2\dot2} - \frac{\iota\sqrt6}{4} b_2^{\dot1\dot2} + \frac{\iota\sqrt6}{4} c_2\right) + \left(-\frac{\iota\sqrt6}{4} b_1^{\dot1\dot1} +\right.\\ &\left.+ \frac{\iota\sqrt6}{4} b_2^{\dot1\dot2} + \frac{\iota\sqrt6}{4} c_2\right) - \iota\left(-\frac{\sqrt6}{4} b_1^{\dot1\dot1} - \frac{\sqrt6}{4} b_2^{\dot1\dot2} - \frac{\sqrt6}{4} c_2\right) + 1\left(-\frac{\iota\sqrt6}{4} b_1^{\dot2\dot2} -\right.\\ &\left.- \frac{\iota\sqrt6}{4} b_2^{\dot1\dot2} + \frac{\iota\sqrt6}{4} c_2\right) = \frac{\iota\sqrt6}{4}(4c_2),\end{aligned}$$

b)
$$\begin{aligned}&\psi^1_0 + \psi^0_1 + \iota\psi^0_2 - \psi^1_3 = 1\left(-\frac{\iota\sqrt6}{4} b_1^{\dot1\dot2} - \frac{\iota\sqrt6}{4} b_2^{\dot1\dot1} - \frac{\iota\sqrt6}{4} c_1\right) + 1\left(\frac{\iota\sqrt6}{4} b_1^{\dot1\dot2} - \iota\frac{\sqrt6}{4} b_2^{\dot2\dot2} -\right.\\ &\left.- \frac{\iota\sqrt6}{4} c_1\right) + \iota\left(\frac{\sqrt6}{4} b_1^{\dot1\dot2} + \frac{\sqrt6}{4} b_2^{\dot2\dot2} - \frac{\sqrt6}{4} c_1\right) - 1\left(\frac{\iota\sqrt6}{4} b_1^{\dot1\dot2} + \frac{\iota\sqrt6}{4} b_2^{\dot1\dot1} + \frac{\iota\sqrt6}{4} c_1\right) =\\ &= -\frac{\iota\sqrt6}{4}(4c_1),\end{aligned}$$

c)
$$\begin{aligned}&\pi_0\psi^0_0 - \pi_1\psi^0_1 - \pi_2\psi^0_2 - \pi_3\psi^0_3 = \pi_0\left(\frac{\iota\sqrt6}{4} b_1^{\dot2\dot2} - \frac{\iota\sqrt6}{4} b_2^{\dot1\dot2} + \frac{\iota\sqrt6}{4} c_2\right) -\\ &- \pi_1\left(\frac{\iota\sqrt6}{4} b_1^{\dot1\dot2} - \frac{\iota\sqrt6}{4} b_2^{\dot2\dot2} - \frac{\iota\sqrt6}{4} c_2\right) - \pi_2\left(\frac{\sqrt6}{4} b_1^{\dot1\dot2} + \frac{\sqrt6}{4} b_2^{\dot2\dot2} - \frac{\sqrt6}{4} c_1\right) -\\ &- \pi_3\left(-\frac{\iota\sqrt6}{4} b_1^{\dot2\dot2} - \frac{\iota\sqrt6}{4} b_2^{\dot1\dot2} + \frac{\iota\sqrt6}{4} c_2\right) = \frac{\iota\sqrt6}{4}(\pi_0 + \pi_3) b_1^{\dot2\dot2} + \frac{\iota\sqrt6}{4}(-\pi_0 + \pi_3) b_2^{\dot1\dot2} +\\ &+ \frac{\iota\sqrt6}{4}(\pi_0 - \pi_3) c_2 + \frac{\iota\sqrt6}{4}(-\pi_1 + \iota\pi_2) b_1^{\dot1\dot2} + \frac{\iota\sqrt6}{4}(\pi_1 + \iota\pi_2) b_2^{\dot2\dot2} + \frac{\iota\sqrt6}{4}(\pi_1 - \iota\pi_2) c_1,\end{aligned}$$

d)
$$\begin{aligned}&\{\text{field depended terms}\} = \frac{\iota\sqrt6}{4}\{\iota f_{12} - f_{30} - f_{30} + \iota f_{12}\} a^{\dot2}_{12} +\\ &+ \frac{\iota\sqrt6}{4}\{-\iota f_{21} + f_{30} + f_{30} - \iota f_{21}\} d^{\dot2} + \frac{\iota\sqrt6}{4}\{\iota f_{32} + f_{13} - \iota f_{20} - f_{01}\} a^{\dot2}_{11} + \frac{\iota\sqrt6}{4}\{\iota f_{23} -\\ &- \iota f_{20} + f_{13} - f_{10}\} a^{\dot2}_{22} + \frac{\iota\sqrt6}{4}\{-2\iota f_{23} + 2\iota f_{20} + 2f_{01} - 2f_{31}\} d^{\dot1}.\end{aligned}$$

Substituting all these results back into (I) we find:

$$\begin{aligned}&2\kappa\pi^1_{\dot2} b_1^{\dot2\dot2} + 2\kappa\pi^2_{\dot1} b_2^{\dot1\dot2} + 2\kappa\pi^1_{\dot1} b_1^{\dot1\dot2} + 2\kappa\pi^2_{\dot2} b_2^{\dot2\dot2} + 2\{\iota f_{12} - f_{30} - f_{30} + \iota f_{12}\} a^{\dot2}_{12} +\\ &+ 2\{-\iota f_{21} + f_{30} + f_{30} - \iota f_{21}\} d^{\dot2} + 2\{\iota f_{32} + f_{13} - \iota f_{20} - f_{01}\} a^{\dot2}_{11} + 2\{\iota f_{23} - \iota f_{20} +\\ &+ f_{13} - f_{10}\} a^{\dot2}_{22} + 2\{-2\iota f_{23} + 2\iota f_{20} + 2f_{01} - 2f_{31}\} d^{\dot1} - 6\kappa\pi^{\dot2 2} c_2 - 6\kappa\pi^{\dot2 1} c_1 = 0,\end{aligned}$$

where we have replaced $\iota e F_{k\ell}$ by $f_{k\ell}$ and the mass constant m by κ. The above equation is the same as equation (6) of § 9, which is one of the intermediate steps in the derivation of the modified Pauli-Fierz equation. It is actually one of the equations which will be used to generate the time derivatives of the components c_1, c_2, $\dot{d}^1$, $\dot{d}^2$, which are not appearing in the original Pauli-Fierz equation. Similarly working as for equation (I) we find:

(II) $\underrightarrow{\text{is equivalent to}}$ eq. (5) of § 9,
(III) $\underrightarrow{\text{is equivalent to}}$ eq. (10) of § 9,
(IV) $\underrightarrow{\text{is equivalent to}}$ eq. (9) of § 9.

Thus we have proved that eq. (2.9, V.Z.) of Velo and Zwanziger is equivalent to equations (5), (6), (9) and (10) of § 9.

We show next that equation

$$\pi\cdot\psi = -\left(\gamma\cdot\pi - \frac{3}{2}m\right)\frac{2}{3}\iota e m^{-2}\gamma^5\gamma\cdot F^{d}\cdot\psi \quad , \qquad (2.11,\ \text{V.Z.})$$

(i.e. equation 2.11 of Velo and Zwanziger) goes to equations (13), (14), (15), (16) of § 11. Equation (2.11, V.Z.) written out in detail is the same as the following four equations:

$$\iota)\ \underbrace{m\left(\pi_0\psi^0_0 - \pi_1\psi^0_1 - \pi_2\psi^0_2 - \pi_3\psi^0_3\right)}_{\text{part one}} + \underbrace{\left\{\lambda'\iota F^{12}\psi^2_0 + \lambda'(\iota F^{23} - \iota F^{13})\psi^3_0 +\right.}_{\text{part two} \ \ldots}$$

$$+ \lambda'(-\iota F^{23} + \iota F^{02})\psi^2_1 - \lambda' F^{03}\psi^3_1 + \lambda'(\iota F^{13} - \iota F^{01})\psi^2_2 + \lambda'\iota F^{03}\psi^3_2 -$$

$$\underbrace{\left.- \lambda'\iota F^{12}\psi^2_3 + \lambda'(-\iota F^{02} + F^{01})\psi^3_3\right\}}_{\ldots} + \underbrace{\mu(\pi_0 - \pi_3)\left\{\lambda'\iota F^{12}\psi^0_0 +\right.}_{\text{part three} \ \ldots}$$

$$+ \lambda'(\iota F^{23} - \iota F^{13})\psi^1_0 + \lambda'(\iota F^{23} + \iota F^{02})\psi^0_1 - \lambda' F^{03}\psi^1_1 + \lambda'(-\iota F^{13} - \iota F^{01})\psi^0_2 +$$

$$\underbrace{\left.+ \lambda'\iota F^{03}\psi^1_2 + \lambda'\iota F^{12}\psi^0_3 + \lambda'(-\iota F^{02} + F^{01})\psi^1_3\right\}}_{\ldots} + \underbrace{\mu(-\pi_1 + \iota\pi_2)\left\{\lambda'(\iota F^{23} + F^{13})\psi^0_0 -\right.}_{\text{part four} \ \ldots}$$

$$- \lambda'\iota F^{12}\psi^1_0 + \lambda' F^{03}\psi^0_1 + \lambda'(\iota F^{23} - \iota F^{02})\psi^1_1 + \lambda'\iota F^{03}\psi^0_2 + \lambda'(-\iota F^{13} + \iota F^{01})\psi^1_2 +$$

$$\underbrace{\left.+ \lambda'(-\iota F^{02} - F^{01})\psi^0_3 + \lambda'\iota F^{12}\psi^1_3\right\} = 0}_{\ldots} \quad ,$$

ii)
$$\begin{aligned}
&m(\pi_0\psi_0^1-\pi_1\psi_1^1-\pi_2\psi_2^1-\pi_3\psi_3^1)+\lambda'(\iota F^{23}+F^{13})\psi_0^2-\lambda'\iota F^{12}\psi_0^3+\\
&+\lambda'F^{03}\psi_1^2+\lambda'(-\iota F^{23}-\iota F^{02})\psi_1^3+\lambda'\iota F^{03}\psi_2^2+\lambda'(\iota F^{13}+\iota F^{01})\psi_2^3+\\
&+\lambda'(-\iota F^{02}-F^{01})\psi_3^2-\lambda'\iota F^{12}\psi_3^3+\mu(-\pi_1-\iota\pi_2)\{\lambda'\iota F^{12}\psi_0^0+\\
&+\lambda'(\iota F^{23}-F^{13})\psi_0^1+\lambda'(\iota F^{23}+\iota F^{02})\psi_1^0-\lambda'F^{03}\psi_1^1+\lambda'(-\iota F^{13}-\iota F^{01})\psi_2^0+\\
&+\lambda'\iota F^{03}\psi_2^1+\lambda'\iota F^{12}\psi_3^0+\lambda'(-\iota F^{02}+F^{01})\psi_3^1\}+\mu(\pi_0+\pi_3)\{\lambda'(\iota F^{23}+\\
&+F^{13})\psi_0^0-\lambda'\iota F^{12}\psi_0^1+\lambda'F^{03}\psi_1^0+\lambda'(\iota F^{23}-\iota F^{02})\psi_1^1+\lambda'\iota F^{03}\psi_2^0+\\
&+\lambda'(-\iota F^{12}+\iota F^{01})\psi_2^1+\lambda'(-\iota F^{02}-F^{01})\psi_3^0+\lambda'\iota F^{12}\psi_3^1\}=0\,,
\end{aligned}$$

iii)
$$\begin{aligned}
&m(\pi_0\psi_0^2-\pi_1\psi_1^2-\pi_2\psi_2^2-\pi_3\psi_3^2)+\lambda'\iota F^{12}\psi_0^0+\lambda'(\iota F^{23}-F^{13})\psi_0^1+\\
&+\lambda'(\iota F^{23}+\iota F^{02})\psi_1^0-\lambda'F^{03}\psi_1^1+\lambda'(-\iota F^{13}-\iota F^{01})\psi_2^0+\lambda'\iota F^{03}\psi_2^1+\\
&+\lambda'\iota F^{12}\psi_3^0+\lambda'(-\iota F^{02}+F^{01})\psi_3^1+\mu(\pi_0+\pi_3)\{\lambda'\iota F^{12}\psi_0^2+\\
&+\lambda'(\iota F^{23}-F^{13})\psi_0^3+\lambda'(-\iota F^{23}+\iota F^{02})\psi_1^2-\lambda'F^{03}\psi_1^3+\lambda'(\iota F^{13}-\iota F^{01})\psi_2^2+\\
&+\lambda'\iota F^{03}\psi_2^3-\lambda'\iota F^{12}\psi_3^2+\lambda'(-\iota F^{02}+F^{01})\psi_3^3\}+\mu(\pi_1-\iota\pi_2)\{\lambda'(\iota F^{23}+\\
&+F^{13})\psi_0^2-\lambda'\iota F^{12}\psi_0^3+\lambda'F^{03}\psi_1^2+\lambda'(-\iota F^{23}-\iota F^{02})\psi_1^3+\\
&+\lambda'\iota F^{03}\psi_2^2+\lambda'(\iota F^{13}+\iota F^{01})\psi_2^3+\lambda'(-\iota F^{02}-F^{01})\psi_3^2-\lambda'\iota F^{12}\psi_3^3\}=0\,,
\end{aligned}$$

iv)
$$\begin{aligned}
&m(\pi_0\psi_0^3-\pi_1\psi_1^3-\pi_2\psi_2^3-\pi_3\psi_3^3)+\lambda'(\iota F^{23}+F^{13})\psi_0^0-\lambda'\iota F^{12}\psi_0^1+\\
&+\lambda'F^{03}\psi_1^0+\lambda'(\iota F^{23}-\iota F^{02})\psi_1^1+\lambda'\iota F^{03}\psi_2^0+\lambda'(-\iota F^{13}+\iota F^{01})\psi_2^1+\\
&+\lambda'(-\iota F^{02}-F^{01})\psi_3^0+\lambda'\iota F^{12}\psi_3^1+\mu(\pi_1+\iota\pi_2)\{\lambda'\iota F^{12}\psi_0^2+\lambda'(\iota F^{23}-\\
&-\iota F^{13})\psi_0^3+\lambda'(-\iota F^{23}+\iota F^{02})\psi_1^2-\lambda'F^{03}\psi_1^3+\lambda'(\iota F^{13}-\iota F^{01})\psi_2^2+\\
&+\lambda'\iota F^{03}\psi_2^3-\lambda'\iota F^{12}\psi_3^2+\lambda'(-\iota F^{02}+F^{01})\psi_3^3\}+\mu(\pi-\pi_3)\{\lambda'(\iota F^{23}+\\
&+F^{13})\psi_0^2-\lambda'\iota F^{12}\psi_0^3+\lambda'F^{03}\psi_1^2+\lambda'(-\iota F^{23}-\iota F^{02})\psi_1^3+\lambda'\iota F^{03}\psi_2^2+\\
&+\lambda'(\iota F^{13}+\iota F^{01})\psi_2^3+\lambda'(-\iota F^{02}-\iota F^{01})\psi_3^2-\lambda'\iota F^{12}\psi_3^3\}=0\,,
\end{aligned}$$

where $\lambda' = \iota e$, $\mu = \frac{2}{3m}$. Let us consider equation (ι). Replacing the vector spinor components ψ^{j}_{ι} by their spinor equivalent given by the transformation formulae of § 7 we find for the individual terms in (ι) the following results:

$$a)\ m\left(\pi_0\psi^0_0-\pi_1\psi^0_1-\pi_2\psi^0_2-\pi_3\psi^0_3\right)=-\frac{\iota\sqrt{6}}{4}m\pi^{1}_{\dot{2}}b_1^{\dot{2}\dot{2}}-\frac{\iota\sqrt{6}}{4}m\pi^{2}_{\dot{1}}b_2^{\dot{1}\dot{2}}-$$

$$-\frac{\iota\sqrt{6}}{4}m\pi^{2\dot{2}}c_2-\frac{\iota\sqrt{6}}{4}m\pi^{1}_{\dot{1}}b_1^{\dot{1}\dot{2}}-\frac{\iota\sqrt{6}}{4}m\pi^{2}_{\dot{2}}b_2^{\dot{2}\dot{2}}-\frac{\iota\sqrt{6}}{4}m\pi^{\dot{2}1}c_1 ,$$

$$b)\ \{\text{Part two}\}=\frac{\iota\sqrt{6}}{4}\left\{-\iota f_{12}-f_{03}-f_{03}-\iota f_{12}\right\}\alpha^{\dot{2}}_{12}+\frac{\iota\sqrt{6}}{4}\left\{-\iota f_{12}+f_{03}+f_{03}-\right.$$

$$\left.-\iota f_{12}\right\}d^{\dot{2}}+\frac{\iota\sqrt{6}}{4}\left\{\iota f_{23}-f_{13}-\iota f_{02}+f_{01}\right\}\alpha^{\dot{2}}_{11}+\frac{\iota\sqrt{6}}{4}\left\{-\iota f_{23}-\iota f_{02}-f_{13}-f_{01}\right\}\alpha^{\dot{2}}_{22}+$$

$$+\frac{\iota\sqrt{6}}{4}\left\{2\iota f_{23}+2\iota f_{02}-2f_{01}-2f_{13}\right\}d^{\dot{1}} ,$$

$$c)\ \{\text{Part three}\}=-\frac{2}{3m}\pi^{\dot{2}2}\left\{\frac{\iota\sqrt{6}}{4}\left(-\iota f_{12}+f_{03}+f_{03}-\iota f_{12}\right)b_2^{\dot{1}\dot{2}}+\right.$$

$$+\frac{\iota\sqrt{6}}{4}\left(\iota f_{12}+f_{03}+f_{03}+\iota f_{12}\right)c_2+\frac{\iota\sqrt{6}}{4}\left(\iota f_{23}-f_{13}+\iota f_{02}-f_{01}\right)b_2^{\dot{1}\dot{1}}+\frac{\iota\sqrt{6}}{4}\left(-2\iota f_{23}+\right.$$

$$\left.2f_{13}+2\iota f_{02}-2f_{01}\right)c_1+\frac{\iota\sqrt{6}}{4}\left(-\iota f_{23}+\iota f_{02}-f_{13}+f_{01}\right)b_2^{\dot{2}\dot{2}} ,$$

$$d)\ \{\text{Part four}\}=\frac{2}{3m}\pi^{1\dot{2}}\left\{\frac{\iota\sqrt{6}}{4}\left(\iota f_{23}+f_{13}-\iota f_{02}-f_{01}\right)b_1^{\dot{2}\dot{2}}+\frac{\iota\sqrt{6}}{4}\left(2\iota f_{23}+2f_{13}+\right.\right.$$

$$\left.+2\iota f_{02}+2f_{01}\right)c_2+\frac{\iota\sqrt{6}}{4}\left(\iota f_{12}-f_{03}-f_{03}+\iota f_{12}\right)b_1^{\dot{1}\dot{2}}+\frac{\iota\sqrt{6}}{4}\left(\iota f_{12}+f_{03}+f_{03}+\right.$$

$$\left.+\iota f_{12}\right)c_1+\frac{\iota\sqrt{6}}{4}\left(-\iota f_{23}-\iota f_{02}+f_{13}+f_{01}\right)b_1^{\dot{1}\dot{1}} .$$

Substituting all these results back into (ι) we find:

$$K\pi^{1}_{\dot{2}}b_1^{\dot{2}\dot{2}}+K\pi^{2}_{\dot{1}}b_2^{\dot{1}\dot{2}}+K\pi^{2\dot{2}}c_2+K\pi^{1}_{\dot{1}}b_1^{\dot{1}\dot{2}}+K\pi^{2}_{\dot{2}}b_2^{\dot{2}\dot{2}}+K\pi^{\dot{2}1}c_1+$$

$$+\left\{\iota f_{12}-f_{30}-f_{30}+\iota f_{12}\right\}\alpha^{\dot{2}}_{12}+\left\{-\iota f_{21}+f_{30}+f_{30}-\iota f_{21}\right\}d^{\dot{2}}+\left\{\iota f_{32}+f_{13}-\iota f_{20}-f_{01}\right\}\alpha^{\dot{2}}_{11}+$$

$$+\left\{\iota f_{23}-\iota f_{20}+f_{13}-f_{10}\right\}\alpha^{\dot{2}}_{22}+\left\{-2\iota f_{23}+2\iota f_{20}+2f_{01}-2f_{31}\right\}d^{\dot{1}}+\frac{2}{3m}\pi^{\dot{2}2}\left\{\left(\iota f_{21}-f_{30}-\right.\right.$$

$$\left.-f_{30}+\iota f_{21}\right)b_2^{\dot{1}\dot{2}}+\left(-\iota f_{21}+f_{03}+f_{03}-\iota f_{21}\right)c_2+\left(\iota f_{23}+f_{31}-\iota f_{20}-f_{01}\right)b_2^{\dot{1}\dot{1}}+\left(-2\iota f_{23}-\right.$$

$$\left.\left.-2f_{31}+2\iota f_{02}+2f_{10}\right)c_1+\left(\iota f_{32}-\iota f_{20}+f_{31}-f_{10}\right)b_2^{\dot{2}\dot{2}}\right\}+\frac{2}{3m}\pi^{1\dot{2}}\left\{\left(\iota f_{32}+f_{31}-\iota f_{20}-\right.\right.$$

$$-f_{10})\, b_1^{\dot{2}\dot{2}} + (-2\iota f_{23} - 2f_{13} + 2\iota f_{20} + 2f_{10})\, c_2 + (\iota f_{21} - f_{30} - f_{30} - \iota f_{21})\, b_1^{\dot{1}\dot{2}} + (-\iota f_{12} + f_{30} +$$

$$+ f_{30} - \iota f_{12})\, c_1 + (\iota f_{23} - \iota f_{20} + f_{31} - f_{01})\, b_1^{\dot{1}\dot{1}} = 0 ,$$

where we have used $\iota e F_{k\ell} = f_{k\ell}$ and $m = \kappa$. The above equation is the same as equation (14) of § 11, which is another intermediate step in the derivation of the modified Pauli-Fierz equation. Similarly working as for equation (ι) we find:

(ii) is equivalent to eq. (13) of § 11,

(iii) is equivalent to eq. (16) of § 11,

(iv) is equivalent to eq. (15) of § 11.

Thus we have proved that eq. (2.11, V.Z.) is equivalent to equations (13), (14), (15), (16) of § 11. Substituting equation (2.11, V.Z.) and equation

$$\gamma \cdot \psi = -\frac{2}{3}\, m^{-2} \iota e\, \gamma^5 \gamma \cdot \overset{d}{F} \cdot \psi ,$$

(i.e. the subsidiary conditions of the second kind) back into the original Rarita-Schwinger equation (c.f. Chapter 10) we find the modified Rarita-Schwinger equation

$$(\gamma \cdot \pi - m)\, \psi_\mu + (\pi_\mu + \tfrac{1}{2} m \gamma_\mu)\, \tfrac{2}{3} \iota e m^{-2} \gamma^5 \gamma \cdot \overset{d}{F} \cdot \psi = 0 \text{ , (2.12, V.Z.)}$$

(i.e. eq. 2.12, of Velo and Zwanziger). The propagation behaviour of this equation was studied in chapter (10) and is the same as that of the Pauli-Fierz equation studied in the previous paragraph.

CHAPTER 12.

Spinor formulation of the general 16-dimensional Gel'fand-Yaglom equation.

In this chapter we shall be concerned with the general 16-dimensional Gel'fand-Yaglom equation based on the representation with components

$$\tau_1 \sim (1/2\,, 3/2)\,,\quad \dot{\tau}_1 \sim (-1/2\,, 3/2)\,,\quad \tau_2 \sim (1/2\,, 5/2)\,,\quad \dot{\tau}_2 \sim (-1/2\,, 5/2).$$

The canonical form of this equation was studied in chapter 4 but here the equation is reformulated in spinor language.

1: Spinorial form of the 16-dimensional equation.

From chapter (4) we have that the general 16-dimensional Gel'fand-Yaglom equation based on the representation with components $\tau_1 \sim (1/2\,, 3/2)$, $\dot{\tau}_1 \sim (-1/2\,, 3/2)$, $\tau_2 \sim (1/2\,, 5/2)$, $\dot{\tau}_2 \sim (-1/2\,, 5/2)$, invariant under the complete group, derivable from an invariant Lagrangian and associated with the bilinear form $\alpha^{\tau_1 \dot{\tau}_1} = \alpha^{\dot{\tau}_1 \tau_1} = 1$, $\alpha^{\tau_2 \dot{\tau}_2} = \alpha^{\dot{\tau}_2 \tau_2} = -1$, has matrix $\mathbb{L}_0$ which in the canonical basis $\{\xi^{\tau}_{\ell m}\}$ acquires the block form:

$$\mathbb{L}_0^{1/2} = \begin{array}{c} \begin{array}{cccc} \tau_1 & \dot{\tau}_1 & \tau_2 & \dot{\tau}_2 \end{array} \\ \begin{bmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & \beta \\ -\bar{\beta} & 0 & 0 & \gamma \\ 0 & -\bar{\beta} & \gamma & 0 \end{bmatrix} \end{array}, \qquad \mathbb{L}_0^{3/2} = \begin{array}{c} \begin{array}{cc} \tau_2 & \dot{\tau}_2 \end{array} \\ \begin{bmatrix} 0 & 2\gamma \\ 2\gamma & 0 \end{bmatrix} \end{array},$$

(γ = real, α = real) or after dividing the equation through by $2\gamma \neq 0$, and renaming the constants we have for $\mathbb{L}_0$ the blocks:

$$\mathbb{L}_0^{1/2} = \begin{array}{c} \begin{array}{cccc} \tau_1 & \dot{\tau}_1 & \tau_2 & \dot{\tau}_2 \end{array} \\ \begin{bmatrix} 0 & a & b & 0 \\ a & 0 & 0 & b \\ -\bar{b} & 0 & 0 & 1/2 \\ 0 & -\bar{b} & 1/2 & 0 \end{bmatrix} \end{array} \quad , \qquad \mathbb{L}_0^{3/2} = \begin{array}{c} \begin{array}{cc} \tau_2 & \dot{\tau}_2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} ,$$

(a = real). Notice that dividing through by $2\gamma \neq 0$ has as a result restricting ourselves to fields which will have always present spin 3/2 particles since the spin 3/2 block has non-zero eigenvalues. Since our main interest is in spin 3/2 equations, thus the generality is not affected by this division.

From chapter (11) we know also that there exists a similarity transformation which connects the canonical basis $\{\xi^{\tau}_{\ell m}\}$ with the spinor basis $\{a^{\dot{\alpha}}_{\beta\gamma}, b^{\dot{\alpha}\dot{\beta}}_{\gamma}, c_{\alpha}, d^{\dot{\alpha}}\}$. Thus it is possible to express the general 16-dimensional Gel'fand-Yaglom equation in spinor form. This we do presently. Acting on the Gel'fand-Yaglom equation (considered minimally coupled to the electromagnetic field) with the similarity transformation S connecting the canonical basis with the spinor basis we find:

$$S\{-\mathbb{L}_0\pi_0 + \mathbb{L}_1\pi_1 + \mathbb{L}_2\pi_2 + \mathbb{L}_3\pi_3 + \kappa\}\psi =$$

$$= \{S\mathbb{L}_0 S^{-1}\pi_0 + S\mathbb{L}_1 S^{-1}\pi_1 + S\mathbb{L}_2 S^{-1}\pi_2 + S\mathbb{L}_3 S^{-1}\pi_3 + S\kappa S^{-1}\}S\psi =$$

$$= \{-\mathbb{L}'_0\pi_0 + \mathbb{L}'_1\pi_1 + \mathbb{L}'_2\pi_2 + \mathbb{L}'_3\pi_3 + \kappa\}\psi' = 0.$$

where the matrices $\mathbb{L}'_0$, $\mathbb{L}'_1$, $\mathbb{L}'_2$, $\mathbb{L}'_3$, ψ', in the new basis are:

$$\mathbb{L}'_0=\mathbb{S}\mathbb{L}_0\mathbb{S}^{-1}=\left(\begin{array}{cccccccc|cccccccc}
a^{\dot1}_{11} & a^{\dot1}_{12} & a^{\dot1}_{22} & a^{\dot2}_{11} & a^{\dot2}_{12} & a^{\dot2}_{22} & d^{\dot1} & d^{\dot2} & b_1^{\dot1\dot1} & b_1^{\dot1\dot2} & b_1^{\dot2\dot2} & b_2^{\dot1\dot1} & b_2^{\dot1\dot2} & b_2^{\dot2\dot2} & c_1 & c_2 \\
 & & & & & & & & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & 0 & -1/2 & 0 & -1/2 & 0 & 0 & -\bar{b} & 0 \\
 & & & & & & & & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -2\bar{b} \\
 & & & & & & & & 0 & -1 & 0 & 0 & 0 & 0 & 2\bar{b} & 0 \\
 & & & & & & & & 0 & 0 & -1/2 & 0 & -1/2 & 0 & 0 & \bar{b} \\
 & & & & & & & & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
 & & & & & & & & 0 & -b/3 & 0 & b/3 & 0 & 0 & -a & 0 \\
 & & & & & & & & 0 & 0 & -b/3 & 0 & b/3 & 0 & 0 & -a \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & -1/2 & 0 & -1/2 & 0 & 0 & \bar{b} & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 2\bar{b} & & & & & & & & \\
0 & -1 & 0 & 0 & 0 & 0 & -2\bar{b} & 0 & & & & & & & & \\
0 & 0 & -1/2 & 0 & -1/2 & 0 & 0 & -\bar{b} & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & & & & & & & & \\
0 & b/3 & 0 & -b/3 & 0 & 0 & -a & 0 & & & & & & & & \\
0 & 0 & b/3 & 0 & -b/3 & 0 & 0 & -a & & & & & & & &
\end{array}\right),$$

$$\mathbb{L}'_1=\mathbb{S}\mathbb{L}_1\mathbb{S}^{-1}=\left(\begin{array}{cccccccc|cccccccc}
 & & & & & & & & 0 & -1 & 0 & 0 & 0 & 0 & -2\bar{b} & 0 \\
 & & & & & & & & -1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 & -\bar{b} \\
 & & & & & & & & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
 & & & & & & & & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & 0 & -1/2 & 0 & 0 & 0 & -1/2 & \bar{b} & 0 \\
 & & & & & & & & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2\bar{b} \\
 & & & & & & & & -b/3 & 0 & 0 & 0 & b/3 & 0 & 0 & a \\
 & & & & & & & & 0 & -b/3 & 0 & 0 & 0 & b/3 & a & 0 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & -2\bar{b} & 0 & & & & & & & & \\
1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & -\bar{b} & & & & & & & & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & 1/2 & 0 & 0 & 0 & 1/2 & \bar{b} & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 2\bar{b} & & & & & & & & \\
-b/3 & 0 & 0 & 0 & b/3 & 0 & 0 & -a & & & & & & & & \\
0 & -b/3 & 0 & 0 & 0 & b/3 & -a & 0 & & & & & & & &
\end{array}\right),$$

$$\mathbb{L}'_2 = S\mathbb{L}_2 S^{-1} = \left[\begin{array}{cccccccc|cccccccc}
 & & & & & & & & 0 & -i & 0 & 0 & 0 & 0 & -2ib & 0 \\
 & & & & & & & & 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 & -i\bar{b} \\
 & & & & & & & & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\
 & & & & & & & & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & 0 & 1/2 & 0 & 0 & 0 & -1/2 & -i\bar{b} & 0 \\
 & & & & & & & & 0 & 0 & 0 & 0 & i & 0 & 0 & -2i\bar{b} \\
 & & & & & & & & ib/3 & 0 & 0 & 0 & ib/3 & 0 & 0 & ia \\
 & & & & & & & & 0 & ib/3 & 0 & 0 & 0 & ib/3 & -ia & 0 \\
\hline
0 & i & 0 & 0 & 0 & 0 & -2i\bar{b} & 0 & & & & & & & & \\
-1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & -i\bar{b} & & & & & & & & \\
0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & i & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & -1/2 & 0 & 0 & 0 & 1/2 & -i\bar{b} & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & -i & 0 & 0 & -2i\bar{b} & & & & & & & & \\
ib/3 & 0 & 0 & 0 & ib/3 & 0 & 0 & -ia & & & & & & & & \\
0 & ib/3 & 0 & 0 & 0 & ib/3 & ia & 0 & & & & & & & &
\end{array}\right],$$

$$\mathbb{L}'_3 = S\mathbb{L}_3 S^{-1} = \left[\begin{array}{cccccccc|cccccccc}
 & & & & & & & & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & 0 & 1/2 & 0 & -1/2 & 0 & 0 & \bar{b} & 0 \\
 & & & & & & & & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2\bar{b} \\
 & & & & & & & & 0 & -1 & 0 & 0 & 0 & 0 & 2\bar{b} & 0 \\
 & & & & & & & & 0 & 0 & 1/2 & 0 & -1/2 & 0 & 0 & \bar{b} \\
 & & & & & & & & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & & & & & & & 0 & b/3 & 0 & b/3 & 0 & 0 & a & 0 \\
 & & & & & & & & 0 & 0 & b/3 & 0 & b/3 & 0 & 0 & -a \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & -1/2 & 0 & 1/2 & 0 & 0 & \bar{b} & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 2\bar{b} & & & & & & & & \\
0 & 1 & 0 & 0 & 0 & 0 & 2\bar{b} & 0 & & & & & & & & \\
0 & 0 & -1/2 & 0 & 1/2 & 0 & 0 & \bar{b} & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & & & & & & & & \\
0 & b/3 & 0 & b/3 & 0 & 0 & -a & 0 & & & & & & & & \\
0 & 0 & b/3 & 0 & b/3 & 0 & 0 & a & & & & & & & &
\end{array}\right],$$

$$\psi' = \left\{ \alpha^{\dot{1}}_{11}, \alpha^{\dot{1}}_{12}, \alpha^{\dot{1}}_{22}, \alpha^{\dot{2}}_{11}, \alpha^{\dot{2}}_{12}, \alpha^{\dot{2}}_{22}, d^{\dot{1}}, d^{\dot{2}}, b_1^{\dot{1}\dot{1}}, b_1^{\dot{1}\dot{2}}, b_1^{\dot{2}\dot{2}}, b_2^{\dot{1}\dot{1}}, b_2^{\dot{1}\dot{2}}, b_2^{\dot{2}\dot{2}}, c_1, c_2 \right\}.$$

Finally the wave equation

$$\left\{ \mathbb{L}'_0\pi_0 + \mathbb{L}'_1\pi_1 + \mathbb{L}'_2\pi_2 + \mathbb{L}'_3\pi_3 + \kappa \right\}\psi' = 0,$$

is equivalent to the following set of differential equations

$$-(-\pi_0+\pi_3)b_1^{\dot{1}\dot{1}}-(\pi_1+\iota\pi_2)b_1^{\dot{1}\dot{2}}-2\bar{b}(\pi_1+\iota\pi_2)c_1+\kappa\alpha^{\dot{1}}_{11}=0,$$

$$\begin{aligned}-\tfrac{1}{2}(\pi_1-\iota\pi_2)b_1^{\dot{1}\dot{1}}-\tfrac{1}{2}(-\pi_0-\pi_3)b_1^{\dot{1}\dot{2}}-\tfrac{1}{2}(-\pi_0+\pi_3)b_2^{\dot{1}\dot{1}}-\tfrac{1}{2}(\pi_1+\iota\pi_2)b_2^{\dot{1}\dot{2}}-&\\ -\bar{b}(\pi_1+\iota\pi_2)c_2-\bar{b}(-\pi_0-\pi_3)c_1+\kappa\alpha^{\dot{1}}_{12}=0,&\end{aligned}$$

$$-(\pi_1-\iota\pi_2)b_2^{\dot{1}\dot{1}}-(-\pi_0-\pi_3)b_2^{\dot{1}\dot{2}}-2\bar{b}(-\pi_0-\pi_3)c_2+\kappa\alpha^{\dot{1}}_{22}=0,$$

$$-(-\pi_0+\pi_3)b_1^{\dot{1}\dot{2}}-(\pi_1+\iota\pi_2)b_1^{\dot{2}\dot{2}}-2\bar{b}(\pi_0-\pi_3)c_1+\kappa\alpha^{\dot{2}}_{11}=0,$$

$$\begin{aligned}-\tfrac{1}{2}(\pi_1-\iota\pi_2)b_1^{\dot{1}\dot{2}}-\tfrac{1}{2}(-\pi_0-\pi_3)b_1^{\dot{2}\dot{2}}-\tfrac{1}{2}(-\pi_0+\pi_3)b_2^{\dot{1}\dot{2}}-\tfrac{1}{2}(\pi_1+\iota\pi_2)b_2^{\dot{2}\dot{2}}-&\\ -\bar{b}(\pi_0-\pi_3)c_2-\bar{b}(-\pi_1+\iota\pi_2)c_1+\kappa\alpha^{\dot{2}}_{12}=0,&\end{aligned}$$

$$-(\pi_1-\iota\pi_2)b_1^{\dot{1}\dot{2}}-(-\pi_0-\pi_3)b_2^{\dot{2}\dot{2}}-2\bar{b}(-\pi_1+\iota\pi_2)c_2+\kappa\alpha^{\dot{2}}_{22}=0,$$

$$\begin{aligned}-\tfrac{b}{3}(\pi_1-\iota\pi_2)b_1^{\dot{1}\dot{1}}-\tfrac{b}{3}(\pi_0-\pi_3)b_2^{\dot{1}\dot{1}}-\tfrac{b}{3}(-\pi_0-\pi_3)b_1^{\dot{1}\dot{2}}-\tfrac{b}{3}(-\pi_1-\iota\pi_2)b_2^{\dot{1}\dot{2}}-&\\ -a(-\pi_0-\pi_3)c_1-a(-\pi_1-\iota\pi_2)c_2+\kappa d^{\dot{1}}=0,&\end{aligned}$$

$$\begin{aligned}-\tfrac{b}{3}(\pi_1-\iota\pi_2)b_1^{\dot{1}\dot{2}}-\tfrac{b}{3}(\pi_0-\pi_3)b_2^{\dot{1}\dot{2}}-\tfrac{b}{3}(-\pi_0-\pi_3)b_1^{\dot{2}\dot{2}}-\tfrac{b}{3}(-\pi_1-\iota\pi_2)b_2^{\dot{2}\dot{2}}-&\\ -a(-\pi_1+\iota\pi_2)c_1-a(-\pi_0+\pi_3)c_2+\kappa d^{\dot{2}}=0,&\end{aligned}$$

$$-(-\pi_0-\pi_3)\alpha^{\dot{1}}_{11}-(-\pi_1-\iota\pi_2)\alpha^{\dot{1}}_{12}-2\bar{b}(\pi_1+\iota\pi_2)d^{\dot{1}}+\kappa b_1^{\dot{1}\dot{1}}=0,$$

$$\begin{aligned}-\tfrac{1}{2}(-\pi_0-\pi_3)\alpha^{\dot{2}}_{11}-\tfrac{1}{2}(-\pi_1-\iota\pi_2)\alpha^{\dot{2}}_{12}-\tfrac{1}{2}(-\pi_1+\iota\pi_2)\alpha^{\dot{1}}_{11}-\tfrac{1}{2}(-\pi_0+\pi_3)\alpha^{\dot{1}}_{12}-&\\ -\bar{b}(\pi_1+\iota\pi_2)d^{\dot{2}}-\bar{b}(\pi_0-\pi_3)d^{\dot{1}}+\kappa b_1^{\dot{1}\dot{2}}=0,&\end{aligned}$$

$$-(-\pi_1+\iota\pi_2)\alpha^{\dot{2}}_{11}-(-\pi_0+\pi_3)\alpha^{\dot{2}}_{12}-2\bar{b}(\pi_0-\pi_3)d^{\dot{2}}+\kappa b_1^{\dot{2}\dot{2}}=0,$$

$$-(-\pi_0-\pi_3)\alpha^{\dot{1}}_{12}-(-\pi_1-\iota\pi_2)\alpha^{\dot{1}}_{22}-2\bar{b}(-\pi_0-\pi_3)d^{\dot{1}}+\kappa b_2^{\dot{1}\dot{1}}=0,$$

$$\begin{aligned}-\tfrac{1}{2}(-\pi_0-\pi_3)\alpha^{\dot{2}}_{12}-\tfrac{1}{2}(-\pi_1-\iota\pi_2)\alpha^{\dot{2}}_{22}-\tfrac{1}{2}(-\pi_1+\iota\pi_2)\alpha^{\dot{1}}_{12}-\tfrac{1}{2}(-\pi_0+\pi_3)\alpha^{\dot{1}}_{22}-&\\ -\bar{b}(-\pi_0-\pi_3)d^{\dot{2}}-\bar{b}(-\pi_1+\iota\pi_2)d^{\dot{1}}+\kappa b_2^{\dot{1}\dot{2}}=0,&\end{aligned}$$

$$-(-\pi_1+\iota\pi_2)\alpha^{\dot{2}}_{12}-(-\pi_0+\pi_3)\alpha^{\dot{2}}_{22}-2\bar{b}(-\pi_1+\iota\pi_2)d^{\dot{2}}+\kappa b_2^{\dot{2}\dot{2}}=0,$$

$$\begin{aligned}-\tfrac{b}{3}(\pi_1-\iota\pi_2)\alpha^{\dot{1}}_{11}-\tfrac{b}{3}(-\pi_0-\pi_3)\alpha^{\dot{2}}_{11}-\tfrac{b}{3}(\pi_0-\pi_3)\alpha^{\dot{1}}_{12}-\tfrac{b}{3}(-\pi_1-\iota\pi_2)\alpha^{\dot{2}}_{12}-&\\ -a(-\pi_0+\pi_3)d^{\dot{1}}-a(\pi_1+\iota\pi_2)d^{\dot{2}}+\kappa c_1=0,&\end{aligned}$$

$$\begin{aligned}-\tfrac{b}{3}(\pi_1-\iota\pi_2)\alpha^{\dot{1}}_{12}-\tfrac{b}{3}(-\pi_0-\pi_3)\alpha^{\dot{2}}_{12}-\tfrac{b}{3}(\pi_0-\pi_3)\alpha^{\dot{1}}_{22}-\tfrac{b}{3}(-\pi_1-\iota\pi_2)\alpha^{\dot{2}}_{22}-&\\ -a(\pi_1-\iota\pi_2)d^{\dot{1}}-a(-\pi_0-\pi_3)d^{\dot{2}}+\kappa c_2=0.&\end{aligned}$$

Making use of the formulae connecting the four vector components π_k, $(k=0,1,2,3)$ with the spinor components $\pi_{\alpha\dot{\rho}}$, $(\alpha=1,2, \dot{\rho}=\dot{1},\dot{2})$ the above equations can be put into the following spinor form:

$$1)\ -\pi_{\dot{1}1}b_1^{\dot{1}\dot{1}}-\pi_{\dot{2}1}b_1^{\dot{1}\dot{2}}-2\bar{b}\,\pi_{\dot{2}1}c_1+\kappa a_{11}^{\dot{1}}=0,$$

$$2)\ -\tfrac{1}{2}\pi_{\dot{1}2}b_1^{\dot{1}\dot{1}}-\tfrac{1}{2}\pi_{\dot{2}2}b_1^{\dot{1}\dot{2}}-\tfrac{1}{2}\pi_{\dot{1}1}b_2^{\dot{1}\dot{1}}-\tfrac{1}{2}\pi_{\dot{2}1}b_2^{\dot{1}\dot{2}}-\bar{b}\,\pi_{\dot{2}1}c_2-\bar{b}\,\pi_{\dot{2}2}c_1+\kappa a_{12}^{\dot{1}}=0,$$

$$3)\ -\pi_{\dot{1}2}b_2^{\dot{1}\dot{1}}-\pi_{\dot{2}2}b_2^{\dot{1}\dot{2}}-2\bar{b}\,\pi_{\dot{2}2}c_2+\kappa a_{22}^{\dot{1}}=0,$$

$$4)\ -\pi_{\dot{1}1}b_1^{\dot{1}\dot{2}}-\pi_{\dot{2}1}b_1^{\dot{2}\dot{2}}+2\bar{b}\,\pi_{\dot{1}1}c_1+\kappa a_{11}^{\dot{2}}=0,$$

$$5)\ -\tfrac{1}{2}\pi_{\dot{1}2}b_1^{\dot{1}\dot{2}}-\tfrac{1}{2}\pi_{\dot{2}2}b_1^{\dot{2}\dot{2}}-\tfrac{1}{2}\pi_{\dot{1}1}b_2^{\dot{1}\dot{2}}-\tfrac{1}{2}\pi_{\dot{2}1}b_2^{\dot{2}\dot{2}}+\bar{b}\,\pi_{\dot{1}1}c_2+\bar{b}\,\pi_{\dot{1}2}c_1+\kappa a_{12}^{\dot{2}}=0,$$

$$6)\ -\pi_{\dot{1}2}b_1^{\dot{1}\dot{2}}-\pi_{\dot{2}2}b_2^{\dot{2}\dot{2}}+2\bar{b}\,\pi_{\dot{1}2}c_2+\kappa a_{22}^{\dot{2}}=0,$$

$$7)\ -\tfrac{b}{3}\pi_{\dot{1}2}b_1^{\dot{1}\dot{1}}+\tfrac{b}{3}\pi_{\dot{1}1}b_2^{\dot{1}\dot{1}}-\tfrac{b}{3}\pi_{\dot{2}2}b_1^{\dot{1}\dot{2}}+\tfrac{b}{3}\pi_{\dot{2}1}b_2^{\dot{1}\dot{2}}-a\pi_{\dot{2}2}c_1+a\pi_{\dot{2}1}c_2+\kappa d^{\dot{1}}=0,$$

$$8)\ -\tfrac{b}{3}\pi_{\dot{1}2}b_1^{\dot{1}\dot{2}}+\tfrac{b}{3}\pi_{\dot{1}1}b_2^{\dot{1}\dot{2}}-\tfrac{b}{3}\pi_{\dot{2}2}b_1^{\dot{2}\dot{2}}+\tfrac{b}{3}\pi_{\dot{2}1}b_2^{\dot{2}\dot{2}}+a\pi_{\dot{1}2}c_1-a\pi_{\dot{1}1}c_2+\kappa d^{\dot{2}}=0,$$

$$9)\ -\pi_{\dot{2}2}a_{11}^{\dot{1}}+\pi_{\dot{2}1}a_{12}^{\dot{1}}-2\bar{b}\pi_{\dot{2}1}d^{\dot{1}}+\kappa b_1^{\dot{1}\dot{1}}=0,$$

$$10)\ -\tfrac{1}{2}\pi_{\dot{2}2}a_{11}^{\dot{2}}+\tfrac{1}{2}\pi_{\dot{2}1}a_{12}^{\dot{2}}+\tfrac{1}{2}\pi_{\dot{1}2}a_{11}^{\dot{1}}-\tfrac{1}{2}\pi_{\dot{1}1}a_{12}^{\dot{1}}-\bar{b}\pi_{\dot{2}1}d^{\dot{2}}+\bar{b}\pi_{\dot{1}1}d^{\dot{1}}+\kappa b_1^{\dot{1}\dot{2}}=0,$$

$$11)\ \pi_{\dot{1}2}a_{11}^{\dot{2}}-\pi_{\dot{1}1}a_{12}^{\dot{2}}+2\bar{b}\,\pi_{\dot{1}1}d^{\dot{2}}+\kappa b_1^{\dot{2}\dot{2}}=0,$$

$$12)\ -\pi_{\dot{2}2}a_{12}^{\dot{1}}+\pi_{\dot{2}1}a_{22}^{\dot{1}}-2\bar{b}\pi_{\dot{2}2}d^{\dot{1}}+\kappa b_2^{\dot{1}\dot{1}}=0,$$

$$13)\ -\tfrac{1}{2}\pi_{\dot{2}2}a_{12}^{\dot{2}}+\tfrac{1}{2}\pi_{\dot{2}1}a_{22}^{\dot{2}}+\tfrac{1}{2}\pi_{\dot{1}2}a_{12}^{\dot{1}}-\tfrac{1}{2}\pi_{\dot{1}1}a_{22}^{\dot{1}}-\bar{b}\pi_{\dot{2}2}d^{\dot{2}}+\bar{b}\pi_{\dot{1}2}d^{\dot{1}}+\kappa b_2^{\dot{1}\dot{2}}=0,$$

$$14)\ \pi_{\dot{1}2}a_{12}^{\dot{2}}-\pi_{\dot{1}1}a_{22}^{\dot{2}}+2\bar{b}\,\pi_{\dot{1}2}d^{\dot{2}}+\kappa b_2^{\dot{2}\dot{2}}=0,$$

$$15)\ -\tfrac{b}{3}\pi_{\dot{1}2}a_{11}^{\dot{1}}-\tfrac{b}{3}\pi_{\dot{2}2}a_{11}^{\dot{2}}+\tfrac{b}{3}\pi_{\dot{1}1}a_{12}^{\dot{1}}+\tfrac{b}{3}\pi_{\dot{2}1}a_{12}^{\dot{2}}-a\pi_{\dot{1}1}d^{\dot{1}}-a\pi_{\dot{2}1}d^{\dot{2}}+\kappa c_1=0,$$

$$16)\ -\tfrac{b}{3}\pi_{\dot{1}2}a_{12}^{\dot{1}}-\tfrac{b}{3}\pi_{\dot{2}2}a_{12}^{\dot{2}}+\tfrac{b}{3}\pi_{\dot{1}1}a_{22}^{\dot{1}}+\tfrac{b}{3}\pi_{\dot{2}1}a_{22}^{\dot{2}}-a\pi_{\dot{1}2}d^{\dot{1}}-a\pi_{\dot{2}2}d^{\dot{2}}+\kappa c_2=0.$$

This we shall call the spinorial form of the 16-dimensional Gel'fand-Yaglom equation. Notice that if the constants are given the values $b=\bar{b}=\frac{1}{2}$, $a=-\frac{1}{2}$, then we find the spinor form of the Pauli-Fierz equation (c.f. Ch. 11).

2: The class of 16-dimensional spin 3/2 equations accepting subsidiary conditions of the second kind.

We shall find now in this paragraph all those spin 3/2 equations which accept subsidiary conditions of the second kind. For this let us consider equations (9), (10), (12), (13) of the previous paragraph and let us multiply (9) by $\pi^{1}{}_{\dot{1}}$, (10) by $\pi^{1}{}_{\dot{2}}$, (12) by $\pi^{2}{}_{\dot{1}}$, (13) by $\pi^{2}{}_{\dot{2}}$, and add them all up to find:

$$\pi^{1}{}_{\dot{1}}(eq_{9}) + \pi^{1}{}_{\dot{2}}(eq_{10}) + \pi^{2}{}_{\dot{1}}(eq_{12}) + \pi^{2}{}_{\dot{2}}(eq_{13}) =$$

$$= \kappa\,\pi^{1}{}_{\dot{1}} b_{1}^{\dot{1}\dot{1}} + \kappa\,\pi^{1}{}_{\dot{2}} b_{1}^{\dot{1}\dot{2}} + \kappa\,\pi^{2}{}_{\dot{1}} b_{2}^{\dot{1}\dot{1}} + \kappa\,\pi^{2}{}_{\dot{2}} b_{2}^{\dot{1}\dot{2}} - \pi^{1}{}_{\dot{1}}\pi_{\dot{2}2}\alpha^{\dot{1}}_{11} + \pi^{1}{}_{\dot{1}}\pi_{\dot{2}1}\alpha^{\dot{1}}_{12} -$$

$$-2\bar{b}\,\pi^{1}{}_{\dot{1}}\pi_{\dot{2}1}d^{\dot{1}} - \frac{1}{2}\pi^{1}{}_{\dot{2}}\pi_{\dot{2}2}\alpha^{\dot{2}}_{11} + \frac{1}{2}\pi^{1}{}_{\dot{2}}\pi_{\dot{2}1}\alpha^{\dot{2}}_{12} + \frac{1}{2}\pi^{1}{}_{\dot{2}}\pi_{\dot{1}2}\alpha^{\dot{1}}_{11} -$$

$$-\frac{1}{2}\pi^{1}{}_{\dot{2}}\pi_{\dot{1}1}\alpha^{\dot{1}}_{12} - \bar{b}\,\pi^{1}{}_{\dot{2}}\pi_{\dot{2}1}d^{\dot{2}} + \bar{b}\,\pi^{1}{}_{\dot{2}}\pi_{\dot{1}1}d^{\dot{1}} - \pi^{2}{}_{\dot{1}}\pi_{\dot{2}2}\alpha^{\dot{1}}_{12} + \pi^{2}{}_{\dot{1}}\pi_{\dot{2}1}\alpha^{\dot{1}}_{22} -$$

$$-2\bar{b}\,\pi^{2}{}_{\dot{1}}\pi_{\dot{2}2}d^{\dot{1}} - \frac{1}{2}\pi^{2}{}_{\dot{2}}\pi_{\dot{2}2}\alpha^{\dot{2}}_{12} + \frac{1}{2}\pi^{2}{}_{\dot{2}}\pi_{\dot{2}1}\alpha^{\dot{2}}_{22} + \frac{1}{2}\pi^{2}{}_{\dot{2}}\pi_{\dot{1}2}\alpha^{\dot{1}}_{12} -$$

$$-\frac{1}{2}\pi^{2}{}_{\dot{2}}\pi_{\dot{1}1}\alpha^{\dot{1}}_{22} - \bar{b}\,\pi^{2}{}_{\dot{2}}\pi_{\dot{2}2}d^{\dot{2}} + \bar{b}\,\pi^{2}{}_{\dot{2}}\pi_{\dot{1}2}d^{\dot{1}} = 0 .$$

Let us consider also equations (15) and (16) and multiply (15) by $3\pi^{1\dot{1}}$, and (16) by $3\pi^{2\dot{1}}$, and then add them up to obtain:

$$3\pi^{1\dot{1}}(eq_{15}) + 3\pi^{2\dot{1}}(eq_{16}) = -b\,\pi^{1\dot{1}}\pi_{\dot{1}2}\alpha^{\dot{1}}_{11} - b\,\pi^{1\dot{1}}\pi_{\dot{2}2}\alpha^{\dot{2}}_{11} +$$

$$+ b\,\pi^{1\dot{1}}\pi_{\dot{1}1}\alpha^{\dot{1}}_{12} + b\,\pi^{1\dot{1}}\pi_{\dot{2}1}\alpha^{\dot{2}}_{12} - 3a\,\pi^{1\dot{1}}\pi_{\dot{1}1}d^{\dot{1}} - 3a\,\pi^{1\dot{1}}\pi_{\dot{2}1}d^{\dot{2}} + 3\kappa\,\pi^{1\dot{1}}c_{1} -$$

$$- b\,\pi^{2\dot{1}}\pi_{\dot{1}2}\alpha^{\dot{1}}_{12} - b\,\pi^{2\dot{1}}\pi_{\dot{2}2}\alpha^{\dot{2}}_{12} + b\,\pi^{2\dot{1}}\pi_{\dot{1}1}\alpha^{\dot{1}}_{22} + b\,\pi^{2\dot{1}}\pi_{\dot{2}1}\alpha^{\dot{2}}_{22} - 3a\,\pi^{2\dot{1}}\pi_{\dot{1}2}d^{\dot{1}} -$$

$$- 3a\,\pi^{2\dot{1}}\pi_{\dot{2}2}d^{\dot{2}} + 3\kappa\,\pi^{2\dot{1}}c_{2} = 0 .$$

Subtracting the second equation from the first and rearranging we find:

$$\text{I)}\quad \kappa\,\pi^{1}{}_{\dot{1}} b_{1}^{\dot{1}\dot{1}} + \kappa\,\pi^{1}{}_{\dot{2}} b_{1}^{\dot{1}\dot{2}} + \kappa^{2}{}_{\dot{1}} b_{2}^{\dot{1}\dot{1}} + \kappa\,\pi^{2}{}_{\dot{2}} b_{2}^{\dot{1}\dot{2}} - 3\kappa\,\pi^{1\dot{1}}c_{1} - 3\kappa\,\pi^{2\dot{1}}c_{2} +$$

$$\left\{-1\pi^{1}{}_{\dot{1}}\pi_{\dot{2}2} + \frac{1}{2}\pi^{1}{}_{\dot{2}}\pi_{\dot{1}2} + b\,\pi^{1\dot{1}}\pi_{\dot{1}2}\right\}\alpha^{\dot{1}}_{11} + \left\{\pi^{1}{}_{\dot{1}}\pi_{\dot{2}1} - \frac{1}{2}\pi^{1}{}_{\dot{2}}\pi_{\dot{1}1} - \pi^{2}{}_{\dot{1}}\pi_{\dot{2}2} +\right.$$

$$+\frac{1}{2}\pi^{2}_{\dot{2}}\pi_{\dot{1}2}-b\pi^{1\dot{1}}\pi_{\dot{1}1}+b\pi^{2\dot{1}}\pi_{\dot{1}2}\Big\}\alpha^{\dot{1}}_{12}+\Big\{-2\bar{b}\,\pi^{1}_{\dot{1}}\pi_{\dot{2}1}+\bar{b}\,\pi^{1}_{\dot{2}}\pi_{\dot{1}1}-$$

$$-2\bar{b}\,\pi^{2}_{\dot{1}}\pi_{\dot{2}2}+\bar{b}\,\pi^{2}_{\dot{2}}\pi_{\dot{1}2}+3a\pi^{1\dot{1}}\pi_{\dot{1}1}+3a\pi^{2\dot{1}}\pi_{\dot{1}2}\Big\}d^{\dot{1}}+\Big\{-\bar{b}\,\pi^{1}_{\dot{2}}\pi_{\dot{2}1}-$$

$$-\bar{b}\,\pi^{2}_{\dot{2}}\pi_{\dot{2}2}+3a\pi^{1\dot{1}}\pi_{\dot{2}1}+3a\pi^{2\dot{1}}\pi_{\dot{2}2}\Big\}d^{\dot{2}}+\Big\{\pi^{2}_{\dot{1}}\pi_{\dot{2}1}-\frac{1}{2}\pi^{2}_{\dot{2}}\pi_{\dot{1}1}-$$

$$-b\pi^{2\dot{1}}\pi_{\dot{1}1}\Big\}\alpha^{\dot{1}}_{22}+\Big\{-\frac{1}{2}\pi^{1}_{\dot{2}}\pi_{\dot{2}2}+b\pi^{1\dot{1}}\pi_{\dot{2}2}\Big\}\alpha^{\dot{2}}_{11}+\Big\{\frac{1}{2}\pi^{1}_{\dot{2}}\pi_{\dot{2}1}\quad-$$

$$-b\pi^{1\dot{1}}\pi_{\dot{2}1}+b\pi^{2\dot{1}}\pi_{\dot{2}2}-\frac{1}{2}\pi^{2}_{\dot{2}}\pi_{\dot{2}2}\Big\}\alpha^{\dot{2}}_{12}+\Big\{\frac{1}{2}\pi^{2}_{\dot{2}}\pi_{\dot{2}1}-b\pi^{2\dot{1}}\pi_{\dot{2}1}\Big\}\alpha^{\dot{2}}_{22}=0\,.$$

Let us consider the bracketed terms and replace the spinor components $\pi_{\alpha\dot{\rho}}$ by their four vector equivalents. Doing so we find for the individual terms the following:

a): $\Big\{-1\pi^{1}_{\dot{1}}\pi_{\dot{2}2}+\frac{1}{2}\pi^{1}_{\dot{2}}\pi_{\dot{1}2}+b\pi^{1\dot{1}}\pi_{\dot{1}2}\Big\}\alpha^{\dot{1}}_{11}=\Big\{(\pi_1\pi_0-(b+\frac{1}{2})\pi_0\pi_1)+$

$+(\pi_1\pi_3-(b+\frac{1}{2})\pi_3\pi_1)+\iota((b+\frac{1}{2})\pi_0\pi_2-\pi_2\pi_0)+\iota((b+\frac{1}{2})\pi_3\pi_2-\pi_2\pi_3)\Big\}\alpha^{\dot{1}}_{11}.$

If $b+\frac{1}{2}=1$, i.e. $b=\frac{1}{2}$, the commutation relations $[\pi_k\,,\,\pi_\ell]_-=$ $=\iota eF_{k\ell}=f_{k\ell}$, can be used and the term becomes

$$=\{f_{10}+f_{13}+\iota f_{02}+\iota f_{32}\}\alpha^{\dot{1}}_{11}\ .$$

b): $\Big\{\pi^{1}_{\dot{1}}\pi_{\dot{2}1}-\frac{1}{2}\pi^{1}_{\dot{2}}\pi_{\dot{1}1}-\pi^{2}_{\dot{1}}\pi_{\dot{2}2}+\frac{1}{2}\pi^{2}_{\dot{2}}\pi_{\dot{1}2}-b\pi^{1\dot{1}}\pi_{\dot{1}1}+b\pi^{2\dot{1}}\pi_{\dot{1}2}\Big\}\alpha^{\dot{1}}_{12}=$

$=\{2\iota f_{12}+2f_{03}\}\alpha^{\dot{1}}_{12}$, if $b=\frac{1}{2}$.

c): $\Big\{-2\bar{b}\pi^{1}_{\dot{1}}\pi_{\dot{2}1}+\bar{b}\pi^{1}_{\dot{2}}\pi_{\dot{1}1}-2\bar{b}\pi^{2}_{\dot{1}}\pi_{\dot{2}2}+\bar{b}\pi^{2}_{\dot{2}}\pi_{\dot{1}2}+3a\pi^{1\dot{1}}\pi_{\dot{1}1}+3a\pi^{2\dot{1}}\pi_{\dot{1}2}\Big\}d^{\dot{1}}=$

$=(3\bar{b}+3a)\{-(\pi_1)^2-(\pi_2)^2-(\pi_3)^2+(\pi_0)^2\}d^{\dot{1}}+\{\bar{b}\iota f_{21}+\bar{b}f_{03}+3a\iota f_{12}+3af_{30}\}d^{\dot{1}}.$

Imposing the condition $3\bar{b}+3a=0$, we make the terms involving the second order derivatives π^2_k , $k=0,1,2,3$, to vanish since such terms must not appear in the subsidiary conditions of the second kind.

d): $\Big\{-\bar{b}\,\pi^{1}_{\dot{2}}\pi_{\dot{2}1}-\bar{b}\,\pi^{2}_{\dot{2}}\pi_{\dot{2}2}+3a\pi^{1\dot{1}}\pi_{\dot{2}1}+3a\pi^{2\dot{1}}\pi_{\dot{2}2}\Big\}d^{\dot{2}}=$

$=(\bar{b}-3a)\{f_{01}+f_{31}+\iota f_{02}+\iota f_{32}\}d^{\dot{2}}\ .$

$$\text{e)}: \left\{ \pi^{2}_{\dot{1}}\pi_{\dot{2}1} - \frac{1}{2}\pi^{2}_{\dot{2}}\pi_{\dot{1}1} - b\pi^{2\dot{1}}\pi_{\dot{1}1} \right\}\alpha^{\dot{1}}_{22} =$$

$$= \left\{ f_{01} + \iota f_{02} + f_{13} + \iota f_{23} \right\}\alpha^{\dot{1}}_{22}, \qquad \text{if} \quad b = \frac{1}{2} .$$

$$\text{f)}: \left\{ -\frac{1}{2}\pi^{1}_{\dot{2}}\pi_{\dot{2}2} + b\pi^{1\dot{1}}\pi_{\dot{2}2} \right\}\alpha^{\dot{2}}_{11} = 0 \quad , \text{ if } \quad b = \frac{1}{2} .$$

$$\text{g)}: \left\{ \frac{1}{2}\pi^{1}_{\dot{2}}\pi_{\dot{2}1} - b\pi^{1\dot{1}}\pi_{\dot{2}1} + b\pi^{2\dot{1}}\pi_{\dot{2}2} - \frac{1}{2}\pi^{2}_{\dot{2}}\pi_{\dot{2}2} \right\}\alpha^{\dot{2}}_{12} = 0 \text{ , if } b = \frac{1}{2} .$$

$$\text{h)}: \left\{ \frac{1}{2}\pi^{2}_{\dot{2}}\pi_{\dot{2}1} - b\pi^{2\dot{1}}\pi_{\dot{2}1} \right\}\alpha^{\dot{2}}_{22} = 0 \quad \text{if} \quad b = \frac{1}{2} .$$

Let us consider equation (7) (of the set of Pauli-Fierz equations) which after multiplying through by $\frac{3K}{b}$, $(b \neq 0)$, raising some spinor indices and setting $\frac{a}{b} = -1$, can be written finally in the form:

$$K\pi^{1}_{\dot{1}}b_{1}^{\dot{1}\dot{1}} + K\pi^{2}_{\dot{1}}b_{2}^{\dot{1}\dot{1}} + K\pi^{1}_{\dot{2}}b_{1}^{\dot{1}\dot{2}} + K\pi^{2}_{\dot{2}}b_{2}^{\dot{1}\dot{2}} - 3K\pi^{1\dot{1}}c_{1} - 3K\pi^{2\dot{1}}c_{2} = \frac{3}{b}K^{2}d^{\dot{1}} .$$

Substituting this and all the above calculated terms into (I) we find the subsidiary condition of the second kind:

$$\frac{3}{b}K^{2}d^{\dot{1}} + \left\{ f_{10} + f_{13} + \iota f_{02} + \iota f_{32} \right\}\alpha^{\dot{1}}_{11} + 2\left\{ \iota f_{12} + f_{03} \right\}\alpha^{\dot{1}}_{12} + \left\{ \bar{b}\iota f_{21} + \bar{b}f_{03} + 3a\iota f_{12} + \right.$$

$$\left. + 3af_{30} \right\}d^{\dot{1}} + \left\{ \bar{b} - 3a \right\}\left\{ f_{01} + f_{31} + \iota f_{02} + \iota f_{32} \right\}d^{\dot{2}} + \left\{ f_{01} + \iota f_{02} + f_{13} + \iota f_{23} \right\}\alpha^{\dot{1}}_{22} = 0 ,$$

which holds if the following conditions are satisfied

$$b = \frac{1}{2} \quad , \quad 3\bar{b} + 3a = 0 \quad , \quad \frac{a}{b} = -1 . \qquad \text{(A)}$$

Three more subsidiary conditions of the second kind can be obtained by similar operations. The operations involved in deriving these other subsidiary conditions are respectively:

(II): Construct

$$\left\{ \pi^{1}_{\dot{1}}(eq_{10}) + \pi^{1}_{\dot{2}}(eq_{11}) + \pi^{2}_{\dot{1}}(eq_{13}) + \pi^{2}_{\dot{2}}(eq_{14}) \right\} - \left\{ 3\pi^{1\dot{2}}(eq_{15}) + 3\pi^{2\dot{2}}(eq_{16}) \right\},$$

substitute into it equation (8) (of the P.F. equations) multiplied by $\frac{3K}{b}$ and replace the spinor components $\pi_{\alpha\dot{\beta}}$ by their four vector equivalent to obtain terms involving $[\pi_{k}, \pi_{\ell}]_{-}$.

(III): Construct

$$\left\{ \pi^{1}_{\dot{1}}(eq_{1}) + \pi^{2}_{\dot{1}}(eq_{2}) + \pi^{1}_{\dot{2}}(eq_{4}) + \pi^{2}_{\dot{2}}(eq_{5}) \right\} - \left\{ 3\,\pi_{\dot{1}1}(eq_{7}) + 3\,\pi_{\dot{2}1}(eq_{8}) \right\},$$

substitute into it $\frac{3K}{b}(eq_{15})$, and replace $\pi_{\alpha\dot{\rho}}$ to create terms involving $[\pi_k, \pi_\ell]_-$.

(IV): Construct

$$\left\{ \pi^{1}_{\dot{1}}(eq_{2}) + \pi^{2}_{\dot{1}}(eq_{3}) + \pi^{1}_{\dot{2}}(eq_{5}) + \pi^{2}_{\dot{2}}(eq_{6}) \right\} - \left\{ 3\pi_{2\dot{1}}(eq_{7}) + 3\pi_{2\dot{2}}(eq_{8}) \right\},$$

substitute into it $\frac{3K}{b}(eq_{16})$ and replace $\pi_{\alpha\dot{\rho}}$ to produce terms involving $[\pi_k, \pi_\ell]_-$.

The conditions which have to be satisfied by the constants $a, b, \bar{b}$, in order that the above operations lead to subsidiary conditions of the second kind are the same as relations (A). Relations (A) define a class of 16-dim. wave equations for spin 3/2 particles for which the subsidiary conditions of the second kind can be found. The set of equations (A) accepts only one solution namely:

$$b = \bar{b} = \frac{1}{2} \quad , \quad a = -\frac{1}{2} \ ,$$

which corresponds to the Pauli-Fierz equation. Hence we conclude that there is only one spin 3/2 equation which accepts subsidiary conditions of the second kind based on the representation $(1/2, 3/2) \oplus (-1/2, 3/2) \oplus (1/2, 5/2) \oplus (-1/2, 5/2)$.

3: Summary.

In this chapter the spinorial form of the general 16-dim. Gel'fand-Yaglom equation was given and shown that there is only one spin 3/2 eq. namely the Pauli-Fierz equation which accepts subsidiary conditions of the second kind.

CHAPTER 13.

Spinor formulation of the 20-dimensional general Gel'fand-Yaglom wave-equation.

We are formulating now in this chapter the general 20-dimensional Gel'fand-Yaglom equation based on the representation $(1/2\,,3/2)\oplus(-1/2\,,3/2)\oplus(1/2\,,5/2)\oplus(-1/2\,,5/2)\oplus\oplus(1/2\,,3/2)\oplus(-1/2\,,3/2)$ in spinor form and determine those classes of equations accepting subsidiary conditions of the second kind. Finally we give an example of a 20-dimensional wave-equation for spin 3/2 and study its propagation.

1: Spinorial form of the 20-dimensional equation.

To be able to express the 20-dimensional wave-equation in spinor form the (similarity) transformation connecting the canonical basis with the spinor basis is necessary. The canonical basis associated with the representation is:

$$\{\xi_{\ell m}\}=\{\xi^{\tau_1}_{1/2,1/2},\xi^{\tau_1}_{1/2,-1/2},\xi^{\dot\tau_1}_{1/2,1/2},\xi^{\dot\tau_1}_{1/2,-1/2},\xi^{\dot\tau_2}_{1/2,1/2},\xi^{\dot\tau_2}_{1/2,-1/2}$$
$$\xi^{\dot\tau_2}_{3/2,3/2},\xi^{\dot\tau_2}_{3/2,1/2},\xi^{\dot\tau_2}_{3/2,-1/2},\xi^{\dot\tau_2}_{3/2,-3/2},\xi^{\tau_2}_{1/2,1/2},\xi^{\tau_2}_{1/2,-1/2},\xi^{\tau_2}_{3/2,3/2},$$
$$\xi^{\tau_2}_{3/2,1/2},\xi^{\tau_2}_{3/2,-1/2},\xi^{\tau_2}_{3/2,-3/2},\xi^{\tau_1'}_{1/2,1/2},\xi^{\tau_1'}_{1/2,-1/2},\xi^{\dot\tau_1'}_{1/2,1/2},\xi^{\dot\tau_1'}_{1/2,-1/2}\}.$$

For the spinor basis we take:

$$\{\alpha^{\dot1}_{11},\alpha^{\dot1}_{12},\alpha^{\dot1}_{22},\alpha^{\dot2}_{11},\alpha^{\dot2}_{12},\alpha^{\dot2}_{22},d^{\dot1},d^{\dot2},\delta^{\dot1},\delta^{\dot2},b_1^{\dot1\dot1},b_1^{\dot1\dot2},b_1^{\dot2\dot2},$$
$$b_2^{\dot1\dot1},b_2^{\dot1\dot2},b_2^{\dot2\dot2},c_1,c_2,\gamma_1,\gamma_2\},$$

which differs from the sixteen dimensional spinor basis by the introduction of the extra spin 1/2 spinors $\gamma_1,\gamma_2,\delta^{\dot1},\delta^{\dot2}$.

The (similarity) transformation connecting the canonical basis with the spinor basis is:

$$S = \begin{array}{cccccccccccccccccccc}
&&&& 0 & 0 & 0 & 0 & 0 & 1 &&&&&&&&&& \\
&&&& 0 & i/\sqrt{6} & 0 & 0 & -1/\sqrt{3} & 0 &&&&&&&&&& \\
&&&& \frac{-2i}{\sqrt{6}} & 0 & 0 & 1/\sqrt{3} & 0 & 0 &&&&&&&&&& \\
&&&& 0 & \frac{-2i}{\sqrt{6}} & 0 & 0 & -1/\sqrt{3} & 0 &&&&&&&&&& \\
&&&& i/\sqrt{6} & 0 & 0 & 1/\sqrt{3} & 0 & 0 &&&&&&&&&& \\
&&&& 0 & 0 & -1 & 0 & 0 & 0 &&&&&&&&&& \\
0 & -i/\sqrt{6} &&&&&&&&&&&&&&&&&& \\
i/\sqrt{6} & 0 &&&&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&&& 0 & -i/\sqrt{6} && \\
&&&&&&&&&&&&&&&& i/\sqrt{6} & 0 && \\
&&&&&&&&&& 0 & 0 & 0 & 0 & 0 & -1 &&&& \\
&&&&&&&&&& 0 & i/\sqrt{6} & 0 & 0 & 1/\sqrt{3} & 0 &&&& \\
&&&&&&&&&& \frac{-2i}{\sqrt{6}} & 0 & 0 & -1/\sqrt{3} & 0 & 0 &&&& \\
&&&&&&&&&& 0 & \frac{-2i}{\sqrt{6}} & 0 & 0 & 1/\sqrt{3} & 0 &&&& \\
&&&&&&&&&& i/\sqrt{6} & 0 & 0 & -1/\sqrt{3} & 0 & 0 &&&& \\
&&&&&&&&&& 0 & 0 & 1 & 0 & 0 & 0 &&&& \\
&& 0 & i/\sqrt{6} &&&&&&&&&&&&&&&& \\
&& -i/\sqrt{6} & 0 &&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&&&&& 0 & i/\sqrt{6} \\
&&&&&&&&&&&&&&&&&& -i/\sqrt{6} & 0
\end{array} ,$$

with inverse,

$$S^{-1} = \begin{array}{cccccccccccccccccccc}
&&&&&& 0 & -i\sqrt{6} &&&&&&&&&&&& \\
&&&&&& i\sqrt{6} & 0 &&&&&&&&&&&& \\
&&&&&&&&&&&&&&&& 0 & i\sqrt{6} && \\
&&&&&&&&&&&&&&&& -i\sqrt{6} & 0 && \\
0 & 0 & \frac{i\sqrt{6}}{3} & 0 & \frac{-i\sqrt{6}}{3} & 0 &&&&&&&&&&&&&& \\
0 & \frac{-i\sqrt{6}}{3} & 0 & \frac{i\sqrt{6}}{3} & 0 & 0 &&&&&&&&&&&&&& \\
0 & 0 & 0 & 0 & 0 & -1 &&&&&&&&&&&&&& \\
0 & 0 & 1/\sqrt{3} & 0 & 2/\sqrt{3} & 0 &&&&&&&&&&&&&& \\
0 & -2/\sqrt{3} & 0 & -1/\sqrt{3} & 0 & 0 &&&&&&&&&&&&&& \\
1 & 0 & 0 & 0 & 0 & 0 &&&&&&&&&&&&&& \\
&&&&&&&&&& 0 & 0 & \frac{i\sqrt{6}}{3} & 0 & \frac{-i\sqrt{6}}{3} & 0 &&&& \\
&&&&&&&&&& 0 & \frac{-i\sqrt{6}}{3} & 0 & \frac{i\sqrt{6}}{3} & 0 & 0 &&&& \\
&&&&&&&&&& 0 & 0 & 0 & 0 & 0 & 1 &&&& \\
&&&&&&&&&& 0 & 0 & -1/\sqrt{3} & 0 & -2/\sqrt{3} & 0 &&&& \\
&&&&&&&&&& 0 & 2/\sqrt{3} & 0 & 1/\sqrt{3} & 0 & 0 &&&& \\
&&&&&&&&&& -1 & 0 & 0 & 0 & 0 & 0 &&&& \\
&&&&&&&& 0 & -i\sqrt{6} &&&&&&&&&& \\
&&&&&&&& i\sqrt{6} & 0 &&&&&&&&&& \\
&&&&&&&&&&&&&&&&&& 0 & i\sqrt{6} \\
&&&&&&&&&&&&&&&&&& -i\sqrt{6} & 0
\end{array} .$$

The similarity transformation $\mathbb{S}$ is such that it transforms the generators $\mathbb{H}_3^c, \mathbb{H}_+^c, \mathbb{H}_-^c, \mathbb{F}_3^c, \mathbb{F}_+^c, \mathbb{F}_-^c$, of the equation expressed in the canonical basis to the generators $\mathbb{H}_3^s, \mathbb{H}_+^s, \mathbb{H}_-^s, \mathbb{F}_3^s, \mathbb{F}_+^s, \mathbb{F}_-^s$, of the equation expressed in the spinor basis i.e.

$$\mathbb{S}\mathbb{H}_3^c\mathbb{S}^{-1} = \mathbb{H}_3^s \,, \quad \mathbb{S}\mathbb{H}_\pm^c\mathbb{S}^{-1} = \mathbb{H}_\pm^s \,, \quad \mathbb{S}\mathbb{F}_3^c\mathbb{S}^{-1} = \mathbb{F}_3^s \,, \quad \mathbb{S}\mathbb{F}_\pm^c\mathbb{S}^{-1} = \mathbb{F}_\pm^s \,.$$

The generators either in the canonical basis or in the spinor basis can be written in block form and can be constructed from those of the 16-dim. equation (i.e. Pauli-Fierz eq.) by adding onto them two extra blocks corresponding to the spin 1/2 representation together with its conjugate. For the sake of brevity we omit them.

In terms of transformation formulae connecting the two bases we have:

$$a_{11}^{\dot{1}} = 1\,\xi^{\dot{\tau}_2}_{3/2,3/2} \,,$$

$$a_{12}^{\dot{1}} = \frac{i}{\sqrt{6}}\,\xi^{\dot{\tau}_2}_{1/2,-1/2} - \frac{1}{\sqrt{3}}\,\xi^{\dot{\tau}_2}_{3/2,-1/2} \,,$$

$$a_{22}^{\dot{1}} = -\frac{2i}{\sqrt{6}}\,\xi^{\dot{\tau}_2}_{1/2,1/2} + \frac{1}{\sqrt{3}}\,\xi^{\dot{\tau}_2}_{3/2,1/2} \,,$$

$$a_{11}^{\dot{2}} = -\frac{2i}{\sqrt{6}}\,\xi^{\dot{\tau}_2}_{1/2,-1/2} - \frac{1}{\sqrt{3}}\,\xi^{\dot{\tau}_2}_{3/2,-1/2} \,,$$

$$a_{12}^{\dot{2}} = \frac{i}{\sqrt{6}}\,\xi^{\dot{\tau}_2}_{1/2,1/2} + \frac{1}{\sqrt{3}}\,\xi^{\dot{\tau}_2}_{3/2,1/2} \,,$$

$$a_{22}^{\dot{2}} = -1\,\xi^{\dot{\tau}_2}_{3/2,3/2} \,,$$

$$d^{\dot{1}} = -\frac{i}{\sqrt{6}}\,\xi^{\tau_1}_{1/2,-1/2} \,,$$

$$d^{\dot{2}} = \frac{i}{\sqrt{6}}\,\xi^{\tau_1}_{1/2,1/2} \,,$$

$$\delta^{\dot{1}} = -\frac{i}{\sqrt{6}}\,\xi^{\tau_1'}_{1/2,-1/2} \,,$$

$$\delta^{\dot{2}} = \frac{i}{\sqrt{6}}\,\xi^{\tau_1'}_{1/2,1/2} \,,$$

$$b_1^{\dot{1}\dot{1}} = -1\,\xi^{\tau_2}_{3/2,-3/2} \,,$$

$$b_1^{\dot{1}\dot{2}} = \frac{i}{\sqrt{6}}\,\xi^{\tau_2}_{1/2,-1/2} + \frac{1}{\sqrt{3}}\,\xi^{\tau_2}_{3/2,-1/2} \,,$$

$$b_1^{\dot{2}\dot{2}} = -\frac{2i}{\sqrt{6}}\,\xi^{\tau_2}_{1/2,1/2} - \frac{1}{\sqrt{3}}\,\xi^{\tau_2}_{3/2,1/2} \,,$$

$$b_2^{\dot{1}\dot{1}} = -\frac{2i}{\sqrt{6}}\,\xi^{\tau_2}_{1/2,-1/2} + \frac{1}{\sqrt{3}}\,\xi^{\tau_2}_{3/2,-1/2} \,,$$

$$b_2^{\dot{1}\dot{2}} = \frac{i}{\sqrt{6}}\,\xi^{\tau_2}_{1/2,1/2} - \frac{1}{\sqrt{3}}\,\xi^{\tau_2}_{3/2,1/2} \,,$$

$$b_2^{\dot{2}\dot{2}} = 1\,\xi^{\tau_2}_{3/2,3/2} \,,$$

$$c_1 = \frac{i}{\sqrt{6}}\,\xi^{\dot{\tau}_1}_{1/2,-1/2} \,,$$

$$c_2 = -\frac{i}{\sqrt{6}}\,\xi^{\dot{\tau}_1}_{1/2,1/2} \,,$$

$$\gamma_1 = \frac{i}{\sqrt{6}}\,\xi^{\dot{\tau}_1'}_{1/2,-1/2} \,,$$

$$\gamma_2 = -\frac{i}{\sqrt{6}}\,\xi^{\dot{\tau}_1'}_{1/2,1/2} \,.$$

The inverted formulae are:

$$\xi^{\tau_1}_{1/2,1/2} = -\iota\sqrt{6}\, d^{\dot{2}} \quad ,$$

$$\xi^{\tau_1}_{1/2,-1/2} = \iota\sqrt{6}\, d^{\dot{1}} \quad ,$$

$$\xi^{\dot{\tau}_1}_{1/2,1/2} = \iota\sqrt{6}\, c_2 \quad ,$$

$$\xi^{\dot{\tau}_1}_{1/2,-1/2} = -\iota\sqrt{6}\, c_1 \, ,$$

$$\xi^{\dot{\tau}_2}_{1/2,1/2} = \frac{\iota\sqrt{6}}{3}\alpha^{\dot{1}}_{22} - \frac{\iota\sqrt{6}}{3}\alpha^{\dot{2}}_{12} \, ,$$

$$\xi^{\dot{\tau}_2}_{1/2,-1/2} = -\frac{\iota\sqrt{6}}{3}\alpha^{\dot{1}}_{12} + \frac{\iota\sqrt{6}}{3}\alpha^{\dot{2}}_{11} \, ,$$

$$\xi^{\dot{\tau}_2}_{3/2,3/2} = -\iota\,\alpha^{\dot{2}}_{22} \, .$$

$$\xi^{\dot{\tau}_2}_{3/2,1/2} = \frac{1}{\sqrt{3}}\alpha^{\dot{1}}_{22} + \frac{2}{\sqrt{3}}\alpha^{\dot{2}}_{12} \quad ,$$

$$\xi^{\dot{\tau}_2}_{3/2,-1/2} = -\frac{2}{\sqrt{3}}\alpha^{\dot{1}}_{12} - \frac{1}{\sqrt{3}}\alpha^{\dot{2}}_{11} \, ,$$

$$\xi^{\dot{\tau}_2}_{3/2,-3/2} = \iota\,\alpha^{\dot{1}}_{11} \quad ,$$

$$\xi^{\tau_2}_{1/2,1/2} = \frac{\iota\sqrt{6}}{3} b^{\dot{2}\dot{2}}_1 - \frac{\iota\sqrt{6}}{3} b^{\dot{1}\dot{2}}_2 \, ,$$

$$\xi^{\tau_2}_{1/2,-1/2} = -\frac{\iota\sqrt{6}}{3} b^{\dot{1}\dot{2}}_1 + \frac{\iota\sqrt{6}}{3} b^{\dot{1}\dot{1}}_2 \, ,$$

$$\xi^{\tau_2}_{3/2,3/2} = \iota\, b^{\dot{2}\dot{2}}_2 \, ,$$

$$\xi^{\tau_2}_{3/2,1/2} = -\frac{1}{\sqrt{3}} b^{\dot{2}\dot{2}}_1 - \frac{2}{\sqrt{3}} b^{\dot{1}\dot{2}}_2 \, ,$$

$$\xi^{\tau_2}_{3/2,-1/2} = \frac{2}{\sqrt{3}} b^{\dot{1}\dot{2}}_1 + \frac{1}{\sqrt{3}} b^{\dot{1}\dot{1}}_2 \, ,$$

$$\xi^{\tau_2}_{3/2,-3/2} = -\iota\, b^{\dot{1}\dot{1}}_1 \, ,$$

$$\xi^{\tau_1'}_{1/2,1/2} = -\iota\sqrt{6}\,\delta^{\dot{2}} \quad ,$$

$$\xi^{\tau_1'}_{1/2,-1/2} = \iota\sqrt{6}\,\delta^{1} \, ,$$

$$\xi^{\dot{\tau}_1'}_{1/2,1/2} = \iota\sqrt{6}\,\gamma_2 \, ,$$

$$\xi^{\dot{\tau}_1'}_{1/2,-1/2} = -\iota\sqrt{6}\,\gamma_1 \, .$$

From chapter (5) we have that the general 20-dimensional Gel'fand-Yaglom equation based on the representation with components $\tau_1 \sim (1/2\,,3/2)$, $\dot{\tau}_1 \sim (-1/2\,,3/2)$, $\tau_2 \sim (1/2\,,5/2)$, $\dot{\tau}_2 \sim (-1/2\,,5/2)$, $\tau_1' \sim (1/2\,,3/2)$, $\dot{\tau}_1' \sim (-1/2\,,3/2)$ invariant under the complete group, derivable from an invariant Lagrangian and associated with the bilinear form $\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1$, $\alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1$, $\alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1} = 1$, has matrix $\mathbb{L}_0$, which in the canonical basis $\{\xi^{\tau}_{\ell m}\}$ reads:

$$\mathbb{L}_0^{1/2}=\begin{array}{c|cccccc|}
 & \tau_1 & \dot{\tau}_1 & \dot{\tau}_2 & \tau_2 & \tau_1' & \dot{\tau}_1' \\
\tau_1 & 0 & \alpha & 0 & \iota\sqrt{3}\,\beta & 0 & \gamma \\
\dot{\tau}_1 & \alpha & 0 & \iota\sqrt{3}\,\beta & 0 & \gamma & 0 \\
\dot{\tau}_2 & 0 & \iota\sqrt{3}\,\bar{\beta} & 0 & \varepsilon & 0 & \iota\sqrt{3}\,\zeta \\
\tau_2 & \iota\sqrt{3}\,\bar{\beta} & 0 & \varepsilon & 0 & \iota\sqrt{3}\,\zeta & 0 \\
\tau_1' & 0 & \bar{\gamma} & 0 & \iota\sqrt{3}\,\bar{\zeta} & 0 & \theta \\
\dot{\tau}_1' & \bar{\gamma} & 0 & \iota\sqrt{3}\,\zeta & 0 & \theta & 0
\end{array}\,, \qquad \mathbb{L}_0^{3/2}=\begin{array}{c|cc|}
 & \dot{\tau}_2 & \tau_2 \\
\dot{\tau}_2 & 0 & 2\varepsilon \\
\tau_2 & 2\varepsilon & 0
\end{array}\,,$$

or after replacing $b=\iota\sqrt{3}\,\beta$, $c=\iota\sqrt{3}\,\bar{\beta}$, $z=\iota\sqrt{3}\,\zeta$, $k=\iota\sqrt{3}\,\bar{\zeta}$, reads:

$$\mathbb{L}_0^{1/2}=\begin{bmatrix}
0 & \alpha & 0 & b & 0 & \gamma \\
\alpha & 0 & b & 0 & \gamma & 0 \\
0 & c & 0 & \varepsilon & 0 & z \\
c & 0 & \varepsilon & 0 & z & 0 \\
0 & \bar{\gamma} & 0 & k & 0 & \theta \\
\bar{\gamma} & 0 & k & 0 & \theta & 0
\end{bmatrix}, \quad \mathbb{L}_0^{3/2}=\begin{bmatrix} 0 & 2\varepsilon \\ 2\varepsilon & 0 \end{bmatrix},$$

where $\alpha =$ real, $\varepsilon =$ real, $\theta =$ real. The other three matrices $\mathbb{L}_k$, $k = 1, 2, 3$, are given in terms of $\mathbb{L}_0$ and the boosts $\mathbb{B}_k$, $k = 1, 2, 3$, by the formula $\mathbb{L}_k = -[\mathbb{B}_k\,, \mathbb{L}_0]_-$.

Acting on the Gel'fand-Yaglom equation (considered minimally coupled to the electromagnetic field) with the similarity transformation $\mathbb{S}$ we find:

$$\mathbb{S}\{-\mathbb{L}_0\pi_0+\mathbb{L}_1\pi_1+\mathbb{L}_2\pi_2+\mathbb{L}_3\pi_3+\kappa\}\psi=$$

$$=\{-\mathbb{S}\mathbb{L}_0\mathbb{S}^{-1}\pi_0+\mathbb{S}\mathbb{L}_1\mathbb{S}^{-1}\pi_1+\mathbb{S}\mathbb{L}_2\mathbb{S}^{-1}\pi_2+\mathbb{S}\mathbb{L}_3\mathbb{S}^{-1}\pi_3+\mathbb{S}\kappa\mathbb{S}^{-1}\}\mathbb{S}\psi=$$

$$=\{-\mathbb{L}_0'\pi_0+\mathbb{L}_1'\pi_1+\mathbb{L}_2'\pi_2+\mathbb{L}_3\pi_3+\kappa\}\psi'=0\,,$$

where the matrices $\mathbb{L}'_0$, $\mathbb{L}'_1$, $\mathbb{L}'_2$, $\mathbb{L}'_3$, ψ', in the spinor basis are:

$$\psi' = \{\alpha^{\dot1}_{11}, \alpha^{\dot1}_{12}, \alpha^{\dot1}_{22}, \alpha^{\dot2}_{11}, \alpha^{\dot2}_{12}, \alpha^{\dot2}_{22}, d^{\dot1}, d^{\dot2}, \delta^{\dot1}, \delta^{\dot2}, b_1^{\dot1\dot1}, b_1^{\dot1\dot2}, b_1^{\dot2\dot2}, b_2^{\dot1\dot1}, b_2^{\dot1\dot2}, b_2^{\dot2\dot2}, c_1, c_2, \gamma_1, \gamma_2\},$$

$$\mathbb{L}'_0 = \mathbb{S}\mathbb{L}_0\mathbb{S}^{-1} = \begin{array}{|cccccccccc|cccccccccc|}
\alpha^{\dot1}_{11} & \alpha^{\dot1}_{12} & \alpha^{\dot1}_{22} & \alpha^{\dot2}_{11} & \alpha^{\dot2}_{12} & \alpha^{\dot2}_{22} & d^{\dot1} & d^{\dot2} & \delta^{\dot1} & \delta^{\dot2} & b_1^{\dot1\dot1} & b_1^{\dot1\dot2} & b_1^{\dot2\dot2} & b_2^{\dot1\dot1} & b_2^{\dot1\dot2} & b_2^{\dot2\dot2} & c_1 & c_2 & \gamma_1 & \gamma_2 \\
\hline
 & & & & & & & & & & -2\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & -\varepsilon & 0 & -\varepsilon & 0 & 0 & c & 0 & z & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & -2\varepsilon & 0 & 0 & 2c & 0 & 2z \\
 & & & & & & & & & & 0 & -2\varepsilon & 0 & 0 & 0 & 0 & -2c & 0 & -2z & 0 \\
 & & & & & & & & & & 0 & 0 & -\varepsilon & 0 & -\varepsilon & 0 & 0 & -c & 0 & -z \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & -2\varepsilon & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & -b/3 & 0 & b/3 & 0 & 0 & -\alpha & 0 & -\gamma & 0 \\
 & & & & & & & & & & 0 & 0 & -b/3 & 0 & b/3 & 0 & 0 & -\alpha & 0 & -\gamma \\
 & & & & & & & & & & 0 & -k/3 & 0 & k/3 & 0 & 0 & -\bar{\gamma} & 0 & -\theta & 0 \\
 & & & & & & & & & & 0 & 0 & -k/3 & 0 & k/3 & 0 & 0 & -\bar{\gamma} & 0 & -\theta \\
\hline
-2\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & & \\
0 & -\varepsilon & 0 & -\varepsilon & 0 & 0 & -c & 0 & -z & 0 & & & & & & & & & & \\
0 & 0 & 0 & 0 & -2\varepsilon & 0 & 0 & -2c & 0 & -2z & & & & & & & & & & \\
0 & -2\varepsilon & 0 & 0 & 0 & 0 & 2c & 0 & 2z & 0 & & & & & & & & & & \\
0 & 0 & -\varepsilon & 0 & -\varepsilon & 0 & 0 & c & 0 & z & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & -2\varepsilon & 0 & 0 & 0 & 0 & & & & & & & & & & \\
0 & b/3 & 0 & -b/3 & 0 & 0 & -\alpha & 0 & -\gamma & 0 & & & & & & & & & & \\
0 & 0 & b/3 & 0 & -b/3 & 0 & 0 & -\alpha & 0 & -\gamma & & & & & & & & & & \\
0 & k/3 & 0 & -k/3 & 0 & 0 & -\bar{\gamma} & 0 & -\theta & 0 & & & & & & & & & & \\
0 & 0 & k/3 & 0 & -k/3 & 0 & 0 & -\bar{\gamma} & 0 & -\theta & & & & & & & & & & \\
\end{array},$$

$$\mathbb{L}'_1 = \mathbb{S}\,\mathbb{L}_1\mathbb{S}^{-1} = \left(\begin{array}{cccccccccc|cccccccccc}
&&&&&&&&&& 0 & -2\varepsilon & 0 & 0 & 0 & 0 & 2c & 0 & 2z & 0 \\
&&&&&&&&&& -\varepsilon & 0 & 0 & 0 & -\varepsilon & 0 & 0 & c & 0 & z \\
&&&&&&&&&& 0 & 0 & 0 & -2\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 \\
&&&&&&&&&& 0 & 0 & -2\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
&&&&&&&&&& 0 & -\varepsilon & 0 & 0 & 0 & -\varepsilon & -c & 0 & -z & 0 \\
&&&&&&&&&& 0 & 0 & 0 & 0 & -2\varepsilon & 0 & 0 & -2c & 0 & -2z \\
&&&&&&&&&& -b/3 & 0 & 0 & 0 & b/3 & 0 & 0 & \alpha & 0 & \gamma \\
&&&&&&&&&& 0 & -b/3 & 0 & 0 & 0 & b/3 & \alpha & 0 & \gamma & 0 \\
&&&&&&&&&& -k/3 & 0 & 0 & 0 & k/3 & 0 & 0 & \bar{\gamma} & 0 & \theta \\
&&&&&&&&&& 0 & -k/3 & 0 & 0 & 0 & k/3 & \bar{\gamma} & 0 & \theta & 0 \\
\hline
0 & 2\varepsilon & 0 & 0 & 0 & 0 & 2c & 0 & 2z & 0 &&&&&&&&&& \\
\varepsilon & 0 & 0 & 0 & \varepsilon & 0 & 0 & c & 0 & z &&&&&&&&&& \\
0 & 0 & 0 & 2\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 &&&&&&&&&& \\
0 & 0 & 2\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 &&&&&&&&&& \\
0 & \varepsilon & 0 & 0 & 0 & \varepsilon & -c & 0 & -z & 0 &&&&&&&&&& \\
0 & 0 & 0 & 0 & 2\varepsilon & 0 & 0 & -2c & 0 & -2z &&&&&&&&&& \\
-b/3 & 0 & 0 & 0 & b/3 & 0 & 0 & -\alpha & 0 & -\gamma &&&&&&&&&& \\
0 & -b/3 & 0 & 0 & 0 & b/3 & -\alpha & 0 & -\gamma & 0 &&&&&&&&&& \\
-k/3 & 0 & 0 & 0 & k/3 & 0 & 0 & -\bar{\gamma} & 0 & -\theta &&&&&&&&&& \\
0 & -k/3 & 0 & 0 & 0 & k/3 & -\bar{\gamma} & 0 & -\theta & 0 &&&&&&&&&&
\end{array}\right),$$

$$\mathbb{L}'_2 = \mathbb{S}\,\mathbb{L}_2\mathbb{S}^{-1} = \left(\begin{array}{cccccccccc|cccccccccc}
&&&&&&&&&& 0 & -\iota 2\varepsilon & 0 & 0 & 0 & 0 & \iota 2c & 0 & \iota 2z & 0 \\
&&&&&&&&&& \iota\varepsilon & 0 & 0 & 0 & -\iota\varepsilon & 0 & 0 & \iota c & 0 & \iota z \\
&&&&&&&&&& 0 & 0 & 0 & 2\iota\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 \\
&&&&&&&&&& 0 & 0 & -2\iota\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
&&&&&&&&&& 0 & \iota\varepsilon & 0 & 0 & 0 & -\iota\varepsilon & \iota c & 0 & \iota z & 0 \\
&&&&&&&&&& 0 & 0 & 0 & 0 & \iota 2\varepsilon & 0 & 0 & \iota 2c & 0 & \iota 2z \\
&&&&&&&&&& \iota b/3 & 0 & 0 & 0 & \iota b/3 & 0 & 0 & \iota\alpha & 0 & \iota\gamma \\
&&&&&&&&&& 0 & \iota b/3 & 0 & 0 & 0 & \iota b/3 & -\iota\alpha & 0 & -\iota\gamma & 0 \\
&&&&&&&&&& \iota k/3 & 0 & 0 & 0 & \iota k/3 & 0 & 0 & \iota\bar{\gamma} & 0 & \iota\theta \\
&&&&&&&&&& 0 & \iota k/3 & 0 & 0 & 0 & \iota k/3 & -\iota\bar{\gamma} & 0 & -\iota\theta & 0 \\
\hline
0 & \iota 2\varepsilon & 0 & 0 & 0 & 0 & \iota 2c & 0 & \iota 2z & 0 &&&&&&&&&& \\
-\iota\varepsilon & 0 & 0 & 0 & \iota\varepsilon & 0 & 0 & \iota c & 0 & \iota z &&&&&&&&&& \\
0 & 0 & 0 & -2\iota\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 &&&&&&&&&& \\
0 & 0 & 2\iota\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 &&&&&&&&&& \\
0 & -\iota\varepsilon & 0 & 0 & 0 & \iota\varepsilon & \iota c & 0 & \iota z & 0 &&&&&&&&&& \\
0 & 0 & 0 & 0 & -2\iota\varepsilon & 0 & 0 & \iota 2c & 0 & \iota 2z &&&&&&&&&& \\
\iota b/3 & 0 & 0 & 0 & \iota b/3 & 0 & 0 & -\iota\alpha & 0 & -\iota\gamma &&&&&&&&&& \\
0 & \iota b/3 & 0 & 0 & 0 & \iota b/3 & \iota\alpha & 0 & \iota\gamma & 0 &&&&&&&&&& \\
\iota k/3 & 0 & 0 & 0 & \iota k/3 & 0 & 0 & -\iota\bar{\gamma} & 0 & -\iota\theta &&&&&&&&&& \\
0 & \iota k/3 & 0 & 0 & 0 & \iota k/3 & \iota\bar{\gamma} & 0 & \iota\theta & 0 &&&&&&&&&&
\end{array}\right),$$

$$\mathbb{L}_3' = S\mathbb{L}_3S^{-1} = \left(\begin{array}{cccccccccc|cccccccccc}
 & & & & & & & & & & -2\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & \varepsilon & 0 & -\varepsilon & 0 & 0 & -c & 0 & -z & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & 2\varepsilon & 0 & 0 & -2c & 0 & -2z \\
 & & & & & & & & & & 0 & -2\varepsilon & 0 & 0 & 0 & 0 & -2c & 0 & -2z & 0 \\
 & & & & & & & & & & 0 & 0 & \varepsilon & 0 & -\varepsilon & 0 & 0 & -c & 0 & -z \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 2\varepsilon & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & b/3 & 0 & b/3 & 0 & 0 & \alpha & 0 & \gamma & 0 \\
 & & & & & & & & & & 0 & 0 & b/3 & 0 & b/3 & 0 & 0 & -\alpha & 0 & -\gamma \\
 & & & & & & & & & & 0 & k/3 & 0 & k/3 & 0 & 0 & \bar{\gamma} & 0 & \theta & 0 \\
 & & & & & & & & & & 0 & 0 & k/3 & 0 & k/3 & 0 & 0 & -\bar{\gamma} & 0 & -\theta \\
\hline
2\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & & \\
0 & -\varepsilon & 0 & \varepsilon & 0 & 0 & -c & 0 & -z & 0 & & & & & & & & & & \\
0 & 0 & 0 & 0 & -2\varepsilon & 0 & 0 & -2c & 0 & -2z & & & & & & & & & & \\
0 & 2\varepsilon & 0 & 0 & 0 & 0 & -2c & 0 & -2z & 0 & & & & & & & & & & \\
0 & 0 & -\varepsilon & 0 & \varepsilon & 0 & 0 & -c & 0 & -z & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & -2\varepsilon & 0 & 0 & 0 & 0 & & & & & & & & & & \\
0 & b/3 & 0 & b/3 & 0 & 0 & -\alpha & 0 & -\gamma & 0 & & & & & & & & & & \\
0 & 0 & b/3 & 0 & b/3 & 0 & 0 & \alpha & 0 & \gamma & & & & & & & & & & \\
0 & k/3 & 0 & k/3 & 0 & 0 & -\bar{\gamma} & 0 & -\theta & 0 & & & & & & & & & & \\
0 & 0 & k/3 & 0 & k/3 & 0 & 0 & \bar{\gamma} & 0 & \theta & & & & & & & & & &
\end{array}\right).$$

Finally the wave equation

$$\left\{-\mathbb{L}_0'\pi_0 + \mathbb{L}_1'\pi_1 + \mathbb{L}_2'\pi_2 + \mathbb{L}_3'\pi_3 + \kappa\right\}\psi' = 0,$$

is equivalent to the following set of differential equations:

$$2\varepsilon(\pi_0-\pi_3)b_1^{\dot{1}\dot{1}} + 2\varepsilon(-\pi_1-\iota\pi_2)b_1^{\dot{1}\dot{2}} + 2c(\pi_1+\iota\pi_2)c_1 + 2z(\pi_1+\iota\pi_2)\gamma_1 + \kappa\alpha_{11}^{\dot{1}} = 0,$$

$$\varepsilon(\pi_0+\pi_3)b_1^{\dot{1}\dot{2}} + \varepsilon(\pi_0-\pi_3)b_2^{\dot{1}\dot{1}} + c(-\pi_0-\pi_3)c_1 + z(-\pi_0-\pi_3)\gamma_1 + \varepsilon(-\pi_1+\iota\pi_2)b_1^{\dot{1}\dot{1}} +$$

$$+\varepsilon(-\pi_1-\iota\pi_2)b_2^{\dot{1}\dot{2}} + c(\pi_1+\iota\pi_2)c_2 + z(\pi_1+\iota\pi_2)\gamma_2 + \kappa\alpha_{12}^{\dot{1}} = 0,$$

$$2\varepsilon(\pi_0+\pi_3)b_2^{\dot{1}\dot{2}} + 2c(-\pi_0-\pi_3)c_2 + 2z(-\pi_0-\pi_3)\gamma_2 + 2\varepsilon(-\pi_1+\iota\pi_2)b_2^{\dot{1}\dot{1}} + \kappa\alpha_{22}^{\dot{1}} = 0,$$

$$2\varepsilon(\pi_0-\pi_3)b_1^{\dot{1}\dot{2}} + 2c(\pi_0-\pi_3)c_1 + 2z(\pi_0-\pi_3)\gamma_1 + 2\varepsilon(-\pi_1-\iota\pi_2)b_1^{\dot{2}\dot{2}} + \kappa\alpha_{11}^{\dot{2}} = 0,$$

$$\varepsilon(\pi_0-\pi_3)b_1^{\dot{2}\dot{2}} + \varepsilon(\pi_0-\pi_3)b_2^{\dot{1}\dot{2}} + c(\pi_0-\pi_3)c_2 + z(\pi_0-\pi_3)\gamma_2 + \varepsilon(-\pi_1+\iota\pi_2)b_1^{\dot{1}\dot{2}} +$$

$$+\varepsilon(-\pi_1-\iota\pi_2)b_2^{\dot{2}\dot{2}} + c(-\pi_1+\iota\pi_2)c_1 + z(-\pi_1+\iota\pi_2)\gamma_1 + \kappa\alpha_{12}^{\dot{2}} = 0,$$

$$2\varepsilon(\pi_0+\pi_3)b_2^{\dot{2}\dot{2}} + 2\varepsilon(-\pi_1+\iota\pi_2)b_2^{\dot{1}\dot{2}} + 2c(-\pi_1+\iota\pi_2)c_2 + 2z(-\pi_1+\iota\pi_2)\gamma_2 + \kappa\alpha_{22}^{\dot{2}} = 0,$$

$$\frac{b}{3}(\pi_0+\pi_3)b_1^{\dot{1}\dot{2}} + \frac{b}{3}(-\pi_0+\pi_3)b_2^{\dot{1}\dot{1}} + \alpha(\pi_0+\pi_3)c_1 + \gamma(\pi_0+\pi_3)\gamma_1 + \frac{b}{3}(-\pi_1+\iota\pi_2)b_1^{\dot{1}\dot{1}} +$$

$$+\frac{b}{3}(\pi_1+\iota\pi_2)b_2^{\dot{1}\dot{2}} + \alpha(\pi_1+\iota\pi_2)c_2 + \gamma(\pi_1+\iota\pi_2)\gamma_2 + \kappa d^{\dot{1}} = 0,$$

$$\frac{b}{3}(\pi_0+\pi_3)b_1^{\dot{2}\dot{2}} + \frac{b}{3}(-\pi_0+\pi_3)b_2^{\dot{1}\dot{2}} + \alpha(\pi_0-\pi_3)c_2 + \gamma(\pi_0-\pi_3)\gamma_2 + \frac{b}{3}(-\pi_1+\iota\pi_2)b_1^{\dot{1}\dot{2}} +$$

$$+\frac{b}{3}(\pi_1+\iota\pi_2)b_2^{\dot{2}\dot{2}} + \alpha(\pi_1-\iota\pi_2)c_1 + \gamma(\pi_1-\iota\pi_2)\gamma_1 + \kappa d^{\dot{2}} = 0,$$

$$\frac{k}{3}(\pi_0+\pi_3)b_1^{\dot{1}\dot{2}}+\frac{k}{3}(-\pi_0+\pi_3)b_2^{\dot{1}\dot{1}}+\bar{\gamma}(\pi_0+\pi_3)c_1+\theta(\pi_0+\pi_3)\gamma_1+\frac{k}{3}(-\pi_1+\iota\pi_2)b_1^{\dot{1}\dot{1}}+$$

$$+\frac{k}{3}(\pi_1+\iota\pi_2)b_2^{\dot{1}\dot{2}}+\bar{\gamma}(\pi_1+\iota\pi_2)c_2+\theta(\pi_1+\iota\pi_2)\gamma_2+\kappa\delta^{\dot{1}}=0,$$

$$\frac{k}{3}(\pi_0+\pi_3)b_1^{\dot{2}\dot{2}}+\frac{k}{3}(-\pi_0+\pi_3)b_2^{\dot{1}\dot{2}}+\bar{\gamma}(\pi_0-\pi_3)c_2+\theta(\pi_0-\pi_3)\gamma_2+\frac{k}{3}(-\pi_1+\iota\pi_2)b_1^{\dot{1}\dot{2}}+$$

$$+\frac{k}{3}(\pi_1+\iota\pi_2)b_2^{\dot{2}\dot{2}}+\bar{\gamma}(\pi_1-\iota\pi_2)c_1+\theta(\pi_1-\iota\pi_2)\gamma_1+\kappa\delta^{\dot{2}}=0,$$

$$2\varepsilon(\pi_0+\pi_3)\alpha_{11}^{\dot{1}}+2\varepsilon(\pi_1+\iota\pi_2)\alpha_{12}^{\dot{1}}+2c(\pi_1+\iota\pi_2)d^{\dot{1}}+2z(\pi_1+\iota\pi_2)\delta^{\dot{1}}+\kappa b_1^{\dot{1}\dot{1}}=0,$$

$$\varepsilon(\pi_0-\pi_3)\alpha_{12}^{\dot{1}}+\varepsilon(\pi_0+\pi_3)\alpha_{11}^{\dot{2}}+c(\pi_0-\pi_3)d^{\dot{1}}+z(\pi_0-\pi_3)\delta^{\dot{1}}+\varepsilon(\pi_1-\iota\pi_2)\alpha_{11}^{\dot{1}}+$$

$$+\varepsilon(\pi_1+\iota\pi_2)\alpha_{12}^{\dot{2}}+c(\pi_1+\iota\pi_2)d^{\dot{2}}+z(\pi_1+\iota\pi_2)\delta^{\dot{2}}+\kappa b_1^{\dot{1}\dot{2}}=0,$$

$$2\varepsilon(\pi_0-\pi_3)\alpha_{12}^{\dot{2}}+2c(\pi_0-\pi_3)d^{\dot{2}}+2z(\pi_0-\pi_3)\delta^{\dot{2}}+2\varepsilon(\pi_1-\iota\pi_2)\alpha_{11}^{\dot{2}}+\kappa b_1^{\dot{2}\dot{2}}=0,$$

$$2\varepsilon(\pi_0+\pi_3)\alpha_{12}^{\dot{1}}+2c(-\pi_0-\pi_3)d^{\dot{1}}+2z(-\pi_0-\pi_3)\delta^{\dot{1}}+2\varepsilon(\pi_1+\iota\pi_2)\alpha_{22}^{\dot{1}}+\kappa b_2^{\dot{1}\dot{1}}=0,$$

$$\varepsilon(\pi_0-\pi_3)\alpha_{22}^{\dot{1}}+\varepsilon(\pi_0+\pi_3)\alpha_{12}^{\dot{2}}+c(-\pi_0-\pi_3)d^{\dot{2}}+z(-\pi_0-\pi_3)\delta^{\dot{2}}+\varepsilon(\pi_1-\iota\pi_2)\alpha_{22}^{\dot{1}}+$$

$$+\varepsilon(\pi_1+\iota\pi_2)\alpha_{22}^{\dot{2}}+c(-\pi_1+\iota\pi_2)d^{\dot{1}}+z(-\pi_1+\iota\pi_2)\delta^{\dot{1}}+\kappa b_2^{\dot{1}\dot{2}}=0,$$

$$2\varepsilon(\pi_0-\pi_3)\alpha_{22}^{\dot{2}}+2\varepsilon(\pi_1-\iota\pi_2)\alpha_{12}^{\dot{2}}+2c(-\pi_1+\iota\pi_2)d^{\dot{2}}+2z(-\pi_1+\iota\pi_2)\delta^{\dot{2}}+\kappa b_2^{\dot{2}\dot{2}}=0,$$

$$\frac{b}{3}(-\pi_0+\pi_3)\alpha_{12}^{\dot{1}}+\frac{b}{3}(\pi_0+\pi_3)\alpha_{11}^{\dot{2}}+\alpha(\pi_0-\pi_3)d^{\dot{1}}+\gamma(\pi_0-\pi_3)\delta^{\dot{1}}+\frac{b}{3}(-\pi_1+\iota\pi_2)\alpha_{11}^{\dot{1}}+$$

$$+\frac{b}{3}(\pi_1+\iota\pi_2)\alpha_{12}^{\dot{2}}+\alpha(-\pi_1-\iota\pi_2)d^{\dot{2}}+\gamma(-\pi_1-\iota\pi_2)\delta^{\dot{2}}+\kappa c_1=0,$$

$$\frac{b}{3}(-\pi_0+\pi_3)\alpha_{22}^{\dot{1}}+\frac{b}{3}(\pi_0+\pi_3)\alpha_{12}^{\dot{2}}+\alpha(\pi_0+\pi_3)d^{\dot{2}}+\gamma(\pi_0+\pi_3)\delta^{\dot{2}}+\frac{b}{3}(-\pi_1+\iota\pi_2)\alpha_{12}^{\dot{1}}+$$

$$+\frac{b}{3}(\pi_1+\iota\pi_2)\alpha_{22}^{\dot{2}}+\alpha(-\pi_1+\iota\pi_2)d^{\dot{1}}+\gamma(-\pi_1+\iota\pi_2)\delta^{\dot{1}}+\kappa c_2=0,$$

$$\frac{k}{3}(-\pi_0+\pi_3)\alpha_{12}^{\dot{1}}+\frac{k}{3}(\pi_0+\pi_3)\alpha_{11}^{\dot{2}}+\bar{\gamma}(\pi_0-\pi_3)d^{\dot{1}}+\theta(\pi_0-\pi_3)\delta^{\dot{1}}+\frac{k}{3}(-\pi_1+\iota\pi_2)\alpha_{11}^{\dot{1}}+$$

$$+\frac{k}{3}(\pi_1+\iota\pi_2)\alpha_{12}^{\dot{2}}+\bar{\gamma}(-\pi_1-\iota\pi_2)d^{\dot{2}}+\theta(-\pi_1-\iota\pi_2)\delta^{\dot{2}}+\kappa\gamma_1=0,$$

$$\frac{k}{3}(-\pi_0-\pi_3)\alpha_{22}^{\dot{1}}+\frac{k}{3}(\pi_0+\pi_3)\alpha_{12}^{\dot{2}}+\bar{\gamma}(\pi_0+\pi_3)d^{\dot{2}}+\theta(\pi_0+\pi_3)\delta^{\dot{2}}+\frac{k}{3}(-\pi_1+\iota\pi_2)\alpha_{12}^{\dot{1}}+$$

$$+\frac{k}{3}(\pi_1+\iota\pi_2)\alpha_{22}^{\dot{2}}+\bar{\gamma}(-\pi_1+\iota\pi_2)d^{\dot{1}}+\theta(-\pi_1+\iota\pi_2)\delta^{\dot{1}}+\kappa\gamma_2=0.$$

Making use of the formulae connecting the four vector components π_k , $k = 0, 1, 2, 3$, with the spinor components $\pi_{\alpha\dot{\rho}}$ $(\alpha = 1, 2 , \dot{\rho} = \dot{1}, \dot{2})$ the above equations can be put into the following spinor form:

$$-2\varepsilon\pi_{1\dot1}b_1^{\dot1\dot1} - 2\varepsilon\pi_{1\dot2}b_1^{\dot1\dot2} + 2c\pi^{\dot1}_1c_1 + 2z\pi^{\dot1}_1\gamma_1 + \kappa\alpha^{\dot1}_{11} = 0,$$

$$-\varepsilon\pi_{2\dot2}b_1^{\dot1\dot2} - \varepsilon\pi_{1\dot1}b_2^{\dot1\dot1} + c\pi^{\dot1}_2c_1 + z\pi^{\dot1}_2\gamma_1 - \varepsilon\pi_{12}b_1^{\dot1\dot1} - \varepsilon\pi_{1\dot2}b_2^{\dot1\dot2} +$$

$$+ c\pi^{\dot1}_1c_2 + z\pi^{\dot1}_1\gamma_2 + \kappa\alpha^{\dot1}_{12} = 0,$$

$$-2\varepsilon\pi_{2\dot2}b_2^{\dot1\dot2} + 2c\pi^{\dot1}_2c_2 + 2z\pi^{\dot1}_2\gamma_2 - 2\varepsilon\pi_{2\dot1}b_2^{\dot1\dot1} + \kappa\alpha^{\dot1}_{22} = 0,$$

$$-2\varepsilon\pi_{1\dot1}b_1^{\dot1\dot2} + 2c\pi^{\dot2}_1c_1 + 2z\pi^{\dot2}_1\gamma_1 - 2\varepsilon\pi_{1\dot2}b_1^{\dot2\dot2} + \kappa\alpha^{\dot2}_{11} = 0,$$

$$-\varepsilon\pi_{\dot22}b_1^{\dot2\dot2} - \varepsilon\pi_{1\dot1}b_2^{\dot1\dot2} + c\pi^{\dot2}_1c_2 + z\pi^{\dot2}_1\gamma_2 - \varepsilon\pi_{2\dot1}b_1^{\dot1\dot2} - \varepsilon\pi_{1\dot2}b_2^{\dot2\dot2} +$$

$$+ c\pi^{\dot2}_2c_1 + z\pi^{\dot2}_2\gamma_1 + \kappa\alpha^{\dot2}_{12} = 0,$$

$$-2\varepsilon\pi_{2\dot2}b_2^{\dot2\dot2} - 2\varepsilon\pi_{2\dot1}b_2^{\dot1\dot2} + 2c\pi^{\dot2}_2c_2 + 2z\pi^{\dot2}_2\gamma_2 + \kappa\alpha^{\dot2}_{22} = 0,$$

$$-\frac{b}{3}\pi^1_{\dot2}b_2^{\dot1\dot2} - \frac{b}{3}\pi^2_{\dot1}b_2^{\dot1\dot1} - \alpha\pi^{1\dot1}c_1 - \gamma\pi^{1\dot1}\gamma_1 - \frac{b}{3}\pi^1_{\dot1}b_1^{\dot1\dot1} - \frac{b}{3}\pi^2_{\dot2}b_2^{\dot1\dot2} -$$

$$- \alpha\pi^{\dot12}c_2 - \gamma\pi^{\dot12}\gamma_2 + \kappa d^{\dot1} = 0,$$

$$-\frac{b}{3}\pi^1_{\dot2}b_1^{\dot2\dot2} - \frac{b}{3}\pi^2_{\dot1}b_2^{\dot1\dot2} - \alpha\pi^{\dot22}c_2 - \gamma\pi^{\dot22}\gamma_2 - \frac{b}{3}\pi^1_{\dot1}b_1^{\dot1\dot2} - \frac{b}{3}\pi^2_{\dot2}b_2^{\dot2\dot2} -$$

$$- \alpha\pi^{\dot21}c_1 - \gamma\pi^{\dot21}\gamma_1 + \kappa d^{\dot2} = 0,$$

$$-\frac{k}{3}\pi^1_{\dot2}b_1^{\dot1\dot2} - \frac{k}{3}\pi^2_{\dot1}b_2^{\dot1\dot1} - \bar{\gamma}\pi^{1\dot1}c_1 - \theta\pi^{1\dot1}\gamma_1 - \frac{k}{3}\pi^1_{\dot1}b_1^{\dot1\dot1} - \frac{k}{3}\pi^2_{\dot2}b_2^{\dot1\dot2} -$$

$$- \bar{\gamma}\pi^{\dot12}c_2 - \theta\pi^{\dot12}\gamma_2 + \kappa\delta^{\dot1} = 0,$$

$$-\frac{k}{3}\pi^1_{\dot2}b_1^{\dot2\dot2} - \frac{k}{3}\pi^2_{\dot1}b_2^{\dot1\dot2} - \bar{\gamma}\pi^{\dot22}c_2 - \theta\pi^{\dot22}\gamma_2 - \frac{k}{3}\pi^1_{\dot1}b_1^{\dot1\dot2} - \frac{k}{3}\pi^2_{\dot2}b_2^{\dot2\dot2} -$$

$$- \bar{\gamma}\pi^{\dot21}c_1 - \theta\pi^{\dot21}\gamma_1 + \kappa\delta^{\dot2} = 0,$$

$$-2\varepsilon\pi^{\dot{1}1}\alpha^{\dot{1}}_{11} - 2\varepsilon\pi^{\dot{1}2}\alpha^{\dot{1}}_{12} + 2c\pi^{\dot{1}}_{1}d^{\dot{1}} + 2z\pi^{\dot{1}}_{1}\delta^{\dot{1}} + \kappa b^{\dot{1}\dot{1}}_{1} = 0,$$

$$-\varepsilon\pi^{\dot{2}2}\alpha^{\dot{1}}_{12} - \varepsilon\pi^{\dot{1}1}\alpha^{\dot{2}}_{11} + c\pi^{\dot{2}}_{1}d^{\dot{1}} + z\pi^{\dot{2}}_{1}\delta^{\dot{1}} - \varepsilon\pi^{\dot{2}1}\alpha^{\dot{1}}_{11} - \varepsilon\pi^{\dot{1}2}\alpha^{\dot{2}}_{12} +$$
$$+ c\pi^{\dot{1}}_{1}d^{\dot{2}} + z\pi^{\dot{1}}_{1}\delta^{\dot{2}} + \kappa b^{\dot{1}\dot{2}}_{1} = 0,$$

$$-2\varepsilon\pi^{\dot{2}2}\alpha^{\dot{2}}_{12} + 2c\pi^{\dot{2}}_{1}d^{\dot{2}} + 2z\pi^{\dot{2}}_{1}\delta^{\dot{2}} - 2\varepsilon\pi^{\dot{2}1}\alpha^{\dot{2}}_{11} + \kappa b^{\dot{2}\dot{2}}_{1} = 0,$$

$$-2\varepsilon\pi^{\dot{1}1}\alpha^{\dot{1}}_{12} + 2c\pi^{\dot{1}}_{2}d^{\dot{1}} + 2z\pi^{\dot{1}}_{2}\delta^{\dot{1}} - 2\varepsilon\pi^{\dot{1}2}\alpha^{\dot{1}}_{22} + \kappa b^{\dot{1}\dot{1}}_{2} = 0,$$

$$-\varepsilon\pi^{\dot{2}2}\alpha^{\dot{1}}_{22} - \varepsilon\pi^{\dot{1}1}\alpha^{\dot{2}}_{12} + c\pi^{\dot{1}}_{2}d^{\dot{2}} + z\pi^{\dot{1}}_{2}\delta^{\dot{2}} - \varepsilon\pi^{\dot{2}1}\alpha^{\dot{1}}_{12} - \varepsilon\pi^{\dot{1}2}\alpha^{\dot{2}}_{22} +$$
$$+ c\pi^{\dot{2}}_{2}d^{\dot{1}} + z\pi^{\dot{2}}_{2}\delta^{\dot{1}} + \kappa b^{\dot{1}\dot{2}}_{2} = 0,$$

$$-2\varepsilon\pi^{\dot{2}2}\alpha^{\dot{2}}_{22} - 2\varepsilon\pi^{\dot{2}1}\alpha^{\dot{2}}_{12} + 2c\pi^{\dot{2}}_{2}d^{\dot{2}} + 2z\pi^{\dot{2}}_{2}\delta^{\dot{2}} + \kappa b^{\dot{2}\dot{2}}_{2} = 0,$$

$$-\frac{b}{3}\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{12} - \frac{b}{3}\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{11} - \alpha\pi_{\dot{1}1}d^{\dot{1}} - \gamma\pi_{\dot{1}1}\delta^{\dot{1}} - \frac{b}{3}\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{11} - \frac{b}{3}\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{12} -$$
$$-\alpha\pi_{1\dot{2}}d^{\dot{2}} - \gamma\pi_{1\dot{2}}\delta^{\dot{2}} + \kappa c_{1} = 0,$$

$$-\frac{b}{3}\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{22} - \frac{b}{3}\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{12} - \alpha\pi_{\dot{2}2}d^{\dot{2}} - \gamma\pi_{\dot{2}2}\delta^{\dot{2}} - \frac{b}{3}\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{12} - \frac{b}{3}\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{22} -$$
$$-\alpha\pi_{2\dot{1}}d^{\dot{1}} - \gamma\pi_{2\dot{1}}\delta^{\dot{1}} + \kappa c_{2} = 0,$$

$$-\frac{k}{3}\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{12} - \frac{k}{3}\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{11} - \bar{\gamma}\pi_{\dot{1}1}d^{\dot{1}} - \theta\pi_{\dot{1}1}\delta^{\dot{1}} - \frac{k}{3}\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{11} - \frac{k}{3}\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{12} -$$
$$-\bar{\gamma}\pi_{1\dot{2}}d^{\dot{2}} - \theta\pi_{1\dot{2}}\delta^{\dot{2}} + \kappa\gamma_{1} = 0,$$

$$-\frac{k}{3}\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{22} - \frac{k}{3}\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{12} - \bar{\gamma}\pi_{\dot{2}2}d^{\dot{2}} - \theta\pi_{\dot{2}2}\delta^{\dot{2}} - \frac{k}{3}\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{12} - \frac{k}{3}\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{22} -$$
$$-\bar{\gamma}\pi_{\dot{2}1}d^{\dot{1}} - \theta\pi_{2\dot{1}}\delta^{\dot{1}} + \kappa\gamma_{2} = 0.$$

Dividing through each of the above equations by $2\varepsilon \neq 0$ and renaming the constants by capital letters i.e.

$$\frac{c}{2\varepsilon} = C \ , \ \frac{z}{2\varepsilon} = Z \ , \ \frac{b}{2\varepsilon} = B \ , \ \frac{\alpha}{2\varepsilon} = A \ , \ \frac{\gamma}{2\varepsilon} = \Gamma \ , \ \frac{k}{2\varepsilon} = K \ ,$$

$$\frac{\theta}{2\varepsilon} = \Theta \ , \ \frac{\kappa}{2\varepsilon} = \chi \ ,$$

we have :

$$-1\,\pi_{1\dot{1}}b_1^{\dot{1}\dot{1}}-1\,\pi_{1\dot{2}}b_1^{\dot{1}\dot{2}}+2C\,\pi^{\dot{1}}_1c_1+2Z\,\pi^{\dot{1}}_1\gamma_1+\chi\,\alpha^{\dot{1}}_{11}=0, \tag{1}$$

$$-\frac{1}{2}\pi_{2\dot{2}}b_1^{\dot{1}\dot{2}}-\frac{1}{2}\pi_{1\dot{1}}b_2^{\dot{1}\dot{1}}+C\,\pi^{\dot{1}}_2c_1+Z\,\pi^{\dot{1}}_2\gamma_1-\frac{1}{2}\pi_{12}b_1^{\dot{1}\dot{1}}-\frac{1}{2}\pi_{1\dot{2}}b_2^{\dot{1}\dot{2}}+$$

$$+C\,\pi^{\dot{1}}_1c_2+Z\,\pi^{\dot{1}}_1\gamma_2+\chi\,\alpha^{\dot{1}}_{12}=0, \tag{2}$$

$$-1\,\pi_{2\dot{2}}b_2^{\dot{1}\dot{2}}+2C\,\pi^{\dot{1}}_2c_2+2Z\,\pi^{\dot{1}}_2\gamma_2-1\,\pi_{2\dot{1}}b_2^{\dot{1}\dot{1}}+\chi\,\alpha^{\dot{1}}_{22}=0, \tag{3}$$

$$-1\,\pi_{1\dot{1}}b_1^{\dot{1}\dot{2}}+2C\,\pi^{\dot{2}}_1c_1+2Z\,\pi^{\dot{2}}_1\gamma_1-1\,\pi_{1\dot{2}}b_1^{\dot{2}\dot{2}}+\chi\,\alpha^{\dot{2}}_{11}=0, \tag{4}$$

$$-\frac{1}{2}\pi_{\dot{2}2}b_1^{\dot{2}\dot{2}}-\frac{1}{2}\pi_{1\dot{1}}b_2^{\dot{1}\dot{2}}+C\,\pi^{\dot{2}}_1c_2+Z\,\pi^{\dot{2}}_1\gamma_2-\frac{1}{2}\pi_{2\dot{1}}b_1^{\dot{1}\dot{2}}-\frac{1}{2}\pi_{1\dot{2}}b_2^{\dot{2}\dot{2}}+$$

$$+C\,\pi^{\dot{2}}_2c_1+Z\,\pi^{\dot{2}}_2\gamma_1+\chi\,\alpha^{\dot{2}}_{12}=0, \tag{5}$$

$$-1\,\pi_{2\dot{2}}b_2^{\dot{2}\dot{2}}-1\,\pi_{2\dot{1}}b_2^{\dot{1}\dot{2}}+2C\,\pi^{\dot{2}}_2c_2+2Z\,\pi^{\dot{2}}_2\gamma_2+\chi\,\alpha^{\dot{2}}_{22}=0, \tag{6}$$

$$-\frac{1}{3}B\,\pi^1_{\dot{2}}b_1^{\dot{1}\dot{2}}-\frac{1}{3}B\,\pi^2_{\dot{1}}b_2^{\dot{1}\dot{1}}-A\,\pi^{\dot{1}1}c_1-\Gamma\,\pi^{\dot{1}1}\gamma_1-\frac{1}{3}B\,\pi^1_{\dot{1}}b_1^{\dot{1}\dot{1}}-\frac{1}{3}B\,\pi^2_{\dot{2}}b_2^{\dot{1}\dot{2}}-$$

$$-A\,\pi^{\dot{1}2}c_2-\Gamma\,\pi^{\dot{1}2}\gamma_2+\chi\,d^{\dot{1}}=0, \tag{7}$$

$$-\frac{1}{3}B\,\pi^1_{\dot{2}}b_1^{\dot{2}\dot{2}}-\frac{1}{3}B\,\pi^2_{\dot{1}}b_2^{\dot{1}\dot{2}}-A\,\pi^{\dot{2}2}c_2-\Gamma\,\pi^{\dot{2}2}\gamma_2-\frac{1}{3}B\,\pi^1_{\dot{1}}b_1^{\dot{1}\dot{2}}-\frac{1}{3}B\,\pi^2_{\dot{2}}b_2^{\dot{2}\dot{2}}-$$

$$-A\,\pi^{\dot{2}1}c_1-\Gamma\,\pi^{\dot{2}1}\gamma_1+\chi\,d^{\dot{2}}=0, \tag{8}$$

$$-\frac{1}{3}K\,\pi^1_{\dot{2}}b_1^{\dot{1}\dot{2}}-\frac{1}{3}K\,\pi^2_{\dot{1}}b_2^{\dot{1}\dot{1}}-\bar{\Gamma}\,\pi^{\dot{1}1}c_1-\Theta\,\pi^{\dot{1}1}\gamma_1-\frac{1}{3}K\,\pi^1_{\dot{1}}b_1^{\dot{1}\dot{1}}-\frac{1}{3}K\,\pi^2_{\dot{2}}b_2^{\dot{1}\dot{2}}-$$

$$-\bar{\Gamma}\,\pi^{\dot{1}2}c_2-\Theta\,\pi^{\dot{1}2}\gamma_2+\chi\,\delta^{\dot{1}}=0, \tag{9}$$

$$-\frac{1}{3}K\,\pi^1_{\dot{2}}b_1^{\dot{2}\dot{2}}-\frac{1}{3}K\,\pi^2_{\dot{1}}b_2^{\dot{1}\dot{2}}-\bar{\Gamma}\,\pi^{\dot{2}2}c_2-\Theta\,\pi^{\dot{2}2}\gamma_2-\frac{1}{3}K\,\pi^1_{\dot{1}}b_1^{\dot{1}\dot{2}}-\frac{1}{3}K\,\pi^2_{\dot{2}}b_2^{\dot{2}\dot{2}}-$$

$$-\bar{\Gamma}\,\pi^{\dot{2}1}c_1-\Theta\,\pi^{\dot{2}1}\gamma_1+\chi\,\delta^{\dot{2}}=0, \tag{10}$$

$$-1\pi^{\dot{1}1}\alpha^{\dot{1}}_{11} - 1\pi^{\dot{1}2}\alpha^{\dot{1}}_{12} + 2C\pi^{\dot{1}}_{1}d^{\dot{1}} + 2Z\pi^{\dot{1}}_{1}\delta^{\dot{1}} + \chi b^{\dot{1}\dot{1}}_{1} = 0, \tag{11}$$

$$\begin{aligned}-\tfrac{1}{2}\pi^{\dot{2}2}\alpha^{\dot{1}}_{12} - \tfrac{1}{2}\pi^{\dot{1}1}\alpha^{\dot{2}}_{11} + C\pi^{\dot{2}}_{1}d^{\dot{1}} + Z\pi^{\dot{2}}_{1}\delta^{\dot{1}} - \tfrac{1}{2}\pi^{\dot{2}1}\alpha^{\dot{1}}_{11} - \tfrac{1}{2}\pi^{\dot{1}2}\alpha^{\dot{2}}_{12} + {} \\ + C\pi^{\dot{1}}_{1}d^{\dot{2}} + Z\pi^{\dot{1}}_{1}\delta^{\dot{2}} + \chi b^{\dot{1}\dot{2}}_{1} = 0,\end{aligned} \tag{12}$$

$$-1\pi^{\dot{2}2}\alpha^{\dot{2}}_{12} + 2C\pi^{\dot{2}}_{1}d^{\dot{2}} + 2Z\pi^{\dot{2}}_{1}\delta^{\dot{2}} - 1\pi^{\dot{2}1}\alpha^{\dot{2}}_{11} + \chi b^{\dot{2}\dot{2}}_{1} = 0, \tag{13}$$

$$-1\pi^{\dot{1}1}\alpha^{\dot{1}}_{12} + 2C\pi^{\dot{1}}_{2}d^{\dot{1}} + 2Z\pi^{\dot{1}}_{2}\delta^{\dot{1}} - 1\pi^{\dot{1}2}\alpha^{\dot{1}}_{22} + \chi b^{\dot{1}\dot{1}}_{2} = 0, \tag{14}$$

$$\begin{aligned}-\tfrac{1}{2}\pi^{\dot{2}2}\alpha^{\dot{1}}_{22} - \tfrac{1}{2}\pi^{\dot{1}1}\alpha^{\dot{2}}_{12} + C\pi^{\dot{1}}_{2}d^{\dot{2}} + Z\pi^{\dot{1}}_{2}\delta^{\dot{2}} - \tfrac{1}{2}\pi^{\dot{2}1}\alpha^{\dot{1}}_{12} - \tfrac{1}{2}\pi^{\dot{1}2}\alpha^{\dot{2}}_{22} + {} \\ + C\pi^{\dot{2}}_{2}d^{\dot{1}} + Z\pi^{\dot{2}}_{2}\delta^{\dot{1}} + \chi b^{\dot{1}\dot{2}}_{2} = 0,\end{aligned} \tag{15}$$

$$-1\pi^{\dot{2}2}\alpha^{\dot{2}}_{22} - 1\pi^{\dot{2}1}\alpha^{\dot{2}}_{12} + 2C\pi^{\dot{2}}_{2}d^{\dot{2}} + 2Z\pi^{\dot{2}}_{2}\delta^{\dot{2}} + \chi b^{\dot{2}\dot{2}}_{2} = 0, \tag{16}$$

$$\begin{aligned}-\tfrac{1}{3}B\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{12} - \tfrac{1}{3}B\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{11} - A\pi_{\dot{1}1}d^{\dot{1}} - \Gamma\pi_{\dot{1}1}\delta^{\dot{1}} - \tfrac{1}{3}B\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{11} - \tfrac{1}{3}B\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{12} - {} \\ - A\pi_{1\dot{2}}d^{\dot{2}} - \Gamma\pi_{1\dot{2}}\delta^{\dot{2}} + \chi c_{1} = 0,\end{aligned} \tag{17}$$

$$\begin{aligned}-\tfrac{1}{3}B\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{22} - \tfrac{1}{3}B\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{12} - A\pi_{\dot{2}2}d^{\dot{2}} - \Gamma\pi_{\dot{2}2}\delta^{\dot{2}} - \tfrac{1}{3}B\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{12} - {} \\ - \tfrac{1}{3}B\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{22} - A\pi_{2\dot{1}}d^{\dot{1}} - \Gamma\pi_{2\dot{1}}\delta^{\dot{1}} + \chi c_{2} = 0,\end{aligned} \tag{18}$$

$$\begin{aligned}-\tfrac{1}{3}K\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{12} - \tfrac{1}{3}K\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{11} - \bar{\Gamma}\pi_{\dot{1}1}d^{\dot{1}} - \Theta\pi_{\dot{1}1}\delta^{\dot{1}} - \tfrac{1}{3}K\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{11} - {} \\ - \tfrac{1}{3}K\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{12} - \bar{\Gamma}\pi_{1\dot{2}}d^{\dot{2}} - \Theta\pi_{1\dot{2}}\delta^{\dot{2}} + \chi\gamma_{1} = 0,\end{aligned} \tag{19}$$

$$\begin{aligned}-\tfrac{1}{3}K\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{22} - \tfrac{1}{3}K\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{12} - \bar{\Gamma}\pi_{\dot{2}2}d^{\dot{2}} - \Theta\pi_{\dot{2}2}\delta^{\dot{2}} - \tfrac{1}{3}K\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{12} - {} \\ - \tfrac{1}{3}K\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{22} - \bar{\Gamma}\pi_{2\dot{1}}d^{\dot{1}} - \Theta\pi_{2\dot{1}}\delta^{\dot{1}} + \chi\gamma_{2} = 0.\end{aligned} \tag{20}$$

This set of equations carries less information than the previous one since by dividing through by 2ε we are restricting the block $\mathbb{L}^{3/2}_{0}$ to having non vanishing eigenvalues and hence the wave-equation will describe spin 3/2 particles with or without spin 1/2 particles present, depending on the eigenvalues of the block $\mathbb{L}^{1/2}_{0}$.

We show now how from the above 20-dim. wave-equation in spinor form the 16-dim. spinor form of the Pauli-Fierz equation can be obtained. Giving to the constants the values $\frac{b}{2\varepsilon} = B = \frac{1}{2}$, $\frac{c}{2\varepsilon} = C = -\frac{1}{2}$, $\frac{a}{2\varepsilon} = A = -\frac{1}{2}$, $\frac{z}{2\varepsilon} = Z = 0$, $\frac{k}{2\varepsilon} = K = 0$, $\frac{\theta}{2\varepsilon} = \Theta = 0$, $\frac{\gamma}{2\varepsilon} = \Gamma = 0$, $\frac{\bar{\gamma}}{2\varepsilon} = \bar{\Gamma} = 0$, and substituting into the wave-equation we find that equations (9), (10), (19) and (20) become respectively

$$\chi\delta^{\dot{1}} = 0\ , \quad \chi\delta^{\dot{2}} = 0\ , \quad \chi\gamma_1 = 0\ , \quad \chi\gamma_2 = 0\ ,$$

and since $\chi \neq 0$, it follows that the spinors $\delta^{\dot{1}}$, $\delta^{\dot{2}}$, γ_1, γ_2, vanish. The remaining equations are:

$$-1\pi_{\dot{1}1} b_1^{\dot{1}\dot{1}} - 1\pi_{1\dot{2}} b_1^{\dot{1}\dot{2}} - 1\pi^{\dot{1}}_{1} c_1 + \chi\alpha^{\dot{1}}_{11} = 0,$$

$$-\frac{1}{2}\pi_{2\dot{2}} b_1^{\dot{1}\dot{2}} - \frac{1}{2}\pi_{1\dot{1}} b_2^{\dot{1}\dot{1}} - \frac{1}{2}\pi^{\dot{1}}_{2} c_1 - \frac{1}{2}\pi_{\dot{1}2} b_1^{\dot{1}\dot{1}} - \frac{1}{2}\pi_{1\dot{2}} b_2^{\dot{1}\dot{2}} - \frac{1}{2}\pi^{\dot{1}}_{1} c_2 + \chi\alpha^{\dot{1}}_{12} = 0,$$

$$-1\pi_{2\dot{2}} b_2^{\dot{1}\dot{2}} - 1\pi^{\dot{1}}_{2} c_2 - 1\pi_{2\dot{1}} b_2^{\dot{1}\dot{1}} + \chi\alpha^{\dot{1}}_{22} = 0,$$

$$-1\pi_{1\dot{1}} b_1^{\dot{1}\dot{2}} - \frac{1}{2}\pi^{\dot{2}}_{1} c_1 - 1\pi_{1\dot{2}} b_1^{\dot{2}\dot{2}} + \chi\alpha^{\dot{2}}_{11} = 0,$$

$$-\frac{1}{2}\pi_{\dot{2}2} b_1^{\dot{2}\dot{2}} - \frac{1}{2}\pi_{1\dot{1}} b_2^{\dot{1}\dot{2}} - \frac{1}{2}\pi^{\dot{2}}_{1} c_2 - \frac{1}{2}\pi_{2\dot{1}} b_1^{\dot{1}\dot{2}} - \frac{1}{2}\pi_{1\dot{2}} b_2^{\dot{2}\dot{2}} - \frac{1}{2}\pi^{\dot{2}}_{2} c_1 + \chi\alpha^{\dot{2}}_{12} = 0,$$

$$-1\pi_{2\dot{2}} b_2^{\dot{2}\dot{2}} - 1\pi_{2\dot{1}} b_2^{\dot{1}\dot{2}} - 1\pi^{\dot{2}}_{2} c_2 + \chi\alpha^{\dot{2}}_{22} = 0,$$

$$-\frac{1}{6}\pi^{1}_{\dot{2}} b_1^{\dot{1}\dot{2}} - \frac{1}{6}\pi^{2}_{\dot{1}} b_2^{\dot{1}\dot{1}} + \frac{3}{6}\pi^{1\dot{1}} c_1 - \frac{1}{6}\pi^{1}_{\dot{1}} b_1^{\dot{1}\dot{1}} - \frac{1}{6}\pi^{2}_{\dot{2}} b_2^{\dot{1}\dot{2}} + \frac{3}{6}\pi^{\dot{1}2} c_2 + \chi d^{\dot{1}} = 0,$$

$$-\frac{1}{6}\pi^{1}_{\dot{2}} b_1^{\dot{2}\dot{2}} - \frac{1}{6}\pi^{2}_{\dot{1}} b_2^{\dot{1}\dot{2}} + \frac{3}{6}\pi^{\dot{2}2} c_2 - \frac{1}{6}\pi^{1}_{\dot{1}} b_1^{\dot{1}\dot{2}} - \frac{1}{6}\pi^{2}_{\dot{2}} b_2^{\dot{2}2} + \frac{3}{6}\pi^{\dot{2}1} c_1 + \chi d^{\dot{2}} = 0,$$

$$-1\pi^{\dot{1}1}\alpha^{\dot{1}}_{11} - 1\pi^{\dot{1}2}\alpha^{\dot{1}}_{12} - 1\pi^{\dot{1}}_{1}d^{\dot{1}} + \chi b^{\dot{1}\dot{1}}_{1} = 0,$$

$$-\tfrac{1}{2}\pi^{\dot{2}2}\alpha^{\dot{1}}_{22} - \tfrac{1}{2}\pi^{\dot{1}1}\alpha^{\dot{2}}_{11} - \tfrac{1}{2}\pi^{\dot{2}}_{1}d^{\dot{1}} - \tfrac{1}{2}\pi^{\dot{2}1}\alpha^{\dot{1}}_{11} - \tfrac{1}{2}\pi^{\dot{1}2}\alpha^{\dot{2}}_{12} - \tfrac{1}{2}\pi^{\dot{1}}_{1}d^{\dot{2}} + \chi b^{\dot{1}\dot{2}}_{1} = 0,$$

$$-1\pi^{\dot{2}2}\alpha^{\dot{2}}_{12} - 1\pi^{\dot{2}}_{1}d^{\dot{2}} - 1\pi^{\dot{2}1}\alpha^{\dot{2}}_{11} + \chi b^{\dot{2}\dot{2}}_{1} = 0,$$

$$-1\pi^{\dot{1}1}\alpha^{\dot{1}}_{12} - 1\pi^{\dot{1}}_{2}d^{\dot{1}} - 1\pi^{\dot{1}2}\alpha^{\dot{1}}_{22} + \chi b^{\dot{1}\dot{1}}_{2} = 0,$$

$$-\tfrac{1}{2}\pi^{\dot{2}2}\alpha^{\dot{1}}_{22} - \tfrac{1}{2}\pi^{\dot{1}1}\alpha^{\dot{2}}_{12} - \tfrac{1}{2}\pi^{\dot{1}}_{2}d^{\dot{2}} - \tfrac{1}{2}\pi^{\dot{2}1}\alpha^{\dot{1}}_{12} - \tfrac{1}{2}\pi^{\dot{1}2}\alpha^{\dot{2}}_{22} - \tfrac{1}{2}\pi^{\dot{2}}_{2}d^{\dot{1}} + \chi b^{\dot{1}\dot{2}}_{2} = 0,$$

$$-1\pi^{\dot{2}2}\alpha^{\dot{2}}_{22} - \pi^{\dot{2}1}\alpha^{\dot{2}}_{12} - 1\pi^{\dot{2}}_{2}d^{\dot{2}} + \chi b^{\dot{2}\dot{2}}_{2} = 0,$$

$$-\tfrac{1}{6}\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{12} - \tfrac{1}{6}\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{11} + \tfrac{3}{6}\pi_{\dot{1}1}d^{\dot{1}} - \tfrac{1}{6}\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{11} - \tfrac{1}{6}\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{12} + \tfrac{3}{6}\pi_{1\dot{2}}d^{\dot{2}} + \chi c_{1} = 0,$$

$$-\tfrac{1}{6}\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{22} - \tfrac{1}{6}\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{12} + \tfrac{3}{6}\pi_{\dot{2}2}d^{\dot{2}} - \tfrac{1}{6}\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{12} - \tfrac{1}{6}\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{22} + \tfrac{3}{6}\pi_{2\dot{1}}d^{\dot{1}} + \chi c_{2} = 0,$$

which comprise the 16-dim. Pauli-Fierz equation. A simpler way of showing that the 20-dim. equation for the values of the constants given above goes to the 16-dim. Pauli-Fierz equation is by considering $\mathbb{L}_0$ in the canonical basis and in block form in which case after substituting we have the blocks:

$$\mathbb{L}^{1/2}_{0} = \begin{array}{c|c|c|c|c|c|c|} & \tau_1 & \dot{\tau}_1 & \dot{\tau}_2 & \tau_2 & \tau_1' & \dot{\tau}_1' \\ \tau_1 & 0 & -1/2 & 0 & 1/2 & 0 & 0 \\ \dot{\tau}_1 & -1/2 & 0 & 1/2 & 0 & 0 & 0 \\ \dot{\tau}_2 & 0 & -1/2 & 0 & 1/2 & 0 & 0 \\ \tau_2 & -1/2 & 0 & 1/2 & 0 & 0 & 0 \\ \tau_1' & 0 & 0 & 0 & 0 & 0 & 0 \\ \dot{\tau}_1' & 0 & 0 & 0 & 0 & 0 & 0 \end{array}, \qquad \mathbb{L}^{3/2}_{0} = \begin{array}{c|cc|} & \dot{\tau}_2 & \tau_2 \\ \dot{\tau}_2 & 0 & 1 \\ \tau_2 & 1 & 0 \end{array},$$

or after rearranging the components τ_l in the order τ_1, $\dot{\tau}_1$, τ_2, $\dot{\tau}_2$, τ_1', $\dot{\tau}_1'$ and removing the zero rows and columns we have:

$$\mathbb{L}^{1/2}_0 = \begin{array}{c|cccc|} & \tau_1 & \dot\tau_1 & \tau_2 & \dot\tau_2 \\ \tau_1 & 0 & -1/2 & 1/2 & 0 \\ \dot\tau_1 & -1/2 & 0 & 0 & 1/2 \\ \tau_2 & -1/2 & 0 & 0 & 1/2 \\ \dot\tau_2 & 0 & -1/2 & 1/2 & 0 \end{array} , \qquad \mathbb{L}^{3/2}_0 = \begin{array}{c|cc|} & \tau_2 & \dot\tau_2 \\ \tau_2 & 0 & 1 \\ \dot\tau_2 & 1 & 0 \end{array} ,$$

which are the blocks of $\mathbb{L}_0$ of the Pauli-Fierz equation.

2: The classes of equations accepting subsidiary conditions of the second kind.

We find now which wave-equations based on the representation we are studying accept subsidiary conditions of the second kind by finding what relations the constants entering the general form of the equation must satisfy among themselves. Thus let us consider equations (11), (12), (14) and (15) of the previous paragraph and let us multiply (11) by $\lambda\pi^1_{\dot 1}$, (12) by $\lambda\pi^1_{\dot 2}$, (14) by $\lambda\pi^2_{\dot 1}$, (15) by $\lambda\pi^2_{\dot 2}$ and add them up to obtain:

$$\begin{aligned}
&\lambda\pi^1_{\dot 1}(eq_{11}) + \lambda\pi^1_{\dot 2}(eq_{12}) + \lambda\pi^2_{\dot 1}(eq_{14}) + \lambda\pi^2_{\dot 2}(eq_{15}) = \\
&\chi\lambda\pi^1_{\dot 1}b_1^{\dot 1\dot 1} + \chi\lambda\pi^1_{\dot 2}b_1^{\dot 1\dot 2} + \chi\lambda\pi^2_{\dot 1}b_2^{\dot 1\dot 1} + \chi\lambda\pi^2_{\dot 2}b_2^{\dot 1\dot 2} - \lambda\pi^1_{\dot 1}\pi^{\dot 1 1}\alpha^{\dot 1}_{11} - \\
&- \lambda\pi^1_{\dot 1}\pi^{\dot 1 2}\alpha^{\dot 1}_{12} + \lambda 2C\pi^1_{\dot 1}\pi^{\dot 1}_1 d^{\dot 1} + \lambda 2Z\pi^1_{\dot 1}\pi^{\dot 1}_1\delta^{\dot 1} - \lambda\tfrac{1}{2}\pi^1_{\dot 2}\pi^{\dot 2 2}\alpha^{\dot 1}_{12} - \\
&- \lambda\tfrac{1}{2}\pi^1_{\dot 2}\pi^{\dot 1 1}\alpha^{\dot 2}_{11} + \lambda C\pi^1_{\dot 2}\pi^{\dot 2}_1 d^{\dot 1} + \lambda Z\pi^1_{\dot 2}\pi^{\dot 2}_1\delta^{\dot 1} - \lambda\tfrac{1}{2}\pi^1_{\dot 2}\pi^{\dot 2 1}\alpha^{\dot 1}_{11} - \\
&- \lambda\tfrac{1}{2}\pi^1_{\dot 2}\pi^{\dot 1 2}\alpha^{\dot 2}_{12} + \lambda C\pi^1_{\dot 2}\pi^{\dot 1}_1 d^{\dot 2} + \lambda Z\pi^1_{\dot 2}\pi^{\dot 1}_1\delta^{\dot 2} - \lambda\pi^2_{\dot 1}\pi^{\dot 1 1}\alpha^{\dot 1}_{12} + \\
&+ \lambda 2C\pi^2_{\dot 1}\pi^{\dot 1}_2 d^{\dot 1} + \lambda 2Z\pi^2_{\dot 1}\pi^{\dot 1}_2\delta^{\dot 1} - \lambda\pi^2_{\dot 1}\pi^{\dot 1 2}\alpha^{\dot 1}_{22} - \lambda\tfrac{1}{2}\pi^2_{\dot 2}\pi^{\dot 2 2}\alpha^{\dot 1}_{22} - \\
&- \lambda\tfrac{1}{2}\pi^2_{\dot 2}\pi^{\dot 1 1}\alpha^{\dot 2}_{12} + \lambda C\pi^2_{\dot 2}\pi^{\dot 1}_2 d^{\dot 2} + \lambda Z\pi^2_{\dot 2}\pi^{\dot 1}_2\delta^{\dot 2} - \lambda\tfrac{1}{2}\pi^2_{\dot 2}\pi^{\dot 2 1}\alpha^{\dot 1}_{12} - \\
&- \lambda\tfrac{1}{2}\pi^2_{\dot 2}\pi^{\dot 1 2}\alpha^{\dot 2}_{22} + \lambda C\pi^2_{\dot 2}\pi^{\dot 2}_2 d^{\dot 1} + \lambda Z\pi^2_{\dot 2}\pi^{\dot 2}_2\delta^{\dot 1} = 0 ,
\end{aligned}$$

where λ a constant to be determined.

Let us consider also equations (17), (18), (19) and (20) and multiply (17) by $\xi 3\pi^{1\dot{1}}$, (18) by $\xi 3\pi^{2\dot{1}}$ (19) by $\xi 3\pi^{1\dot{1}}$, (20) by $\xi 3\pi^{2\dot{1}}$ and add them up to obtain:

$$\xi 3\pi^{1\dot{1}}(eq_{17}) + \xi 3\pi^{2\dot{1}}(eq_{18}) + \xi 3\pi^{1\dot{1}}(eq_{19}) + \xi 3\pi^{2\dot{1}}(eq_{20}) =$$

$$- \xi(B+K)\pi^{1\dot{1}}\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{12} - \xi(B+K)\pi^{1\dot{1}}\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{11} - 3\xi(A+\bar{\Gamma})\pi^{1\dot{1}}\pi_{\dot{1}1}d^{\dot{1}} -$$

$$-3\xi(\Gamma+\Theta)\pi^{1\dot{1}}\pi_{\dot{1}1}\delta^{\dot{1}} - \xi(B+K)\pi^{1\dot{1}}\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{11} - \xi(B+K)\pi^{1\dot{1}}\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{12} -$$

$$-3\xi(A+\bar{\Gamma})\pi^{1\dot{1}}\pi_{1\dot{2}}d^{\dot{2}} - 3\xi(\Gamma+\Theta)\pi^{1\dot{1}}\pi_{1\dot{2}}\delta^{\dot{2}} + 3\xi\chi\pi^{1\dot{1}}c_{1} + 3\xi\chi\pi^{1\dot{1}}\gamma_{1} -$$

$$-\xi(B+K)\pi^{2\dot{1}}\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{22} - \xi(B+K)\pi^{2\dot{1}}\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{12} - 3\xi(A+\bar{\Gamma})\pi^{2\dot{1}}\pi_{\dot{2}2}d^{\dot{2}} -$$

$$-3\xi(\Gamma+\Theta)\pi^{2\dot{1}}\pi_{\dot{2}2}\delta^{\dot{2}} - \xi(B+K)\pi^{2\dot{1}}\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{12} - \xi(B+K)\pi^{2\dot{1}}\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{22} -$$

$$-3\xi(A+\bar{\Gamma})\pi^{2\dot{1}}\pi_{2\dot{1}}d^{\dot{1}} - 3\xi(\Gamma+\Theta)\pi^{2\dot{1}}\pi_{2\dot{1}}\delta^{\dot{1}} + 3\xi\chi\pi^{2\dot{1}}c_{2} + 3\xi\chi\pi^{2\dot{1}}\gamma_{2} = 0,$$

where ξ a constant to be determined.

Subtracting the second equation from the first after raising and lowering some spinor indices, rearranging and grouping terms, and aiming in creating terms involving $[\pi_k, \pi_\ell]_- = \iota e F_{k\ell} = f_{k\ell}$, which is possible if we impose on the constants the condition

$$-\frac{\lambda}{2} + (B+K)\xi = 0, \qquad (\iota)$$

after having replaced the spinor components $\pi_{\alpha\dot{\rho}}$ by their four vector equivalents we find:

$$(\mathrm{I}): \quad \lambda\chi\pi^{1}_{\dot{1}}b^{\dot{1}\dot{1}}_{1} + \lambda\chi\pi^{1}_{\dot{2}}b^{\dot{1}\dot{2}}_{1} + \lambda\chi\pi^{2}_{\dot{1}}b^{\dot{1}\dot{1}}_{2} + \lambda\chi\pi^{2}_{\dot{2}}b^{\dot{1}\dot{2}}_{2} - \xi 3\chi\pi^{1\dot{1}}c_{1} -$$

$$-\xi 3\chi\pi^{1\dot{1}}\gamma_{1} - \xi 3\chi\pi^{2\dot{1}}c_{2} - \xi 3\chi\pi^{2\dot{1}}\gamma_{2} + \lambda(f_{10}+f_{13}+\iota f_{32}+\iota f_{02})\alpha^{\dot{1}}_{11} +$$

$$+\lambda 2(f_{03}+\iota f_{12})\alpha^{\dot{1}}_{12} + \{\lambda 3C - \xi 3(A+\bar{\Gamma})\}\{-(\pi_0)^2+(\pi_1)^2+(\pi_2)^2+(\pi_3)^2\}d^{\dot{1}} +$$

$$+(\lambda C+\xi 3(A+\bar{\Gamma}))(\iota f_{12}+f_{30})d^{\dot{1}}+\{\lambda 3Z-\xi 3(\Gamma+\Theta)\}\{-(\pi_0)^2+(\pi_1)^2+(\pi_2)^2+(\pi_3)^2\}\delta^{\dot{1}}+$$

$$+(\lambda Z+\xi 3(\Gamma+\Theta))(\iota f_{12}+f_{30})\delta^{\dot{1}}+(\lambda C+\xi 3(A+\bar{\Gamma}))(f_{10}+\iota f_{20}+f_{13}+\iota f_{23})d^{\dot{2}}+$$

$$+(\lambda Z+\xi 3(\Gamma+\Theta))(f_{10}+f_{13}+\iota f_{20}+\iota f_{23})\delta^{\dot{2}}+\lambda(f_{01}+\iota f_{02}+\iota f_{23}+f_{13})\alpha^{\dot{1}}_{22}=0\ ,$$

where the terms involving $\alpha^{\dot{2}}_{11}$, $\alpha^{\dot{2}}_{12}$, $\alpha^{\dot{2}}_{22}$ have vanished by virtue of (I).

Let us now consider equations (7) and (9) of the previous paragraph and multiply (7) by -6χ and (9) by $-\mu 6\chi$ (where μ a constant to be determined) and add them up to obtain:

$$-6\chi(eq_7)-\mu 6\chi(eq_9)=(2B+\mu 2K)\chi\pi^{1}_{\dot{2}}b_1^{\dot{1}\dot{2}}+(2B+\mu 2K)\chi\pi^{2}_{\dot{1}}b_2^{\dot{1}\dot{1}}+$$

$$+(2A+\mu 2\bar{\Gamma})\chi 3\pi^{1\dot{1}}c_1+(2\Gamma+\mu 2\Theta)3\chi\pi^{1\dot{1}}\gamma_1+(2B+\mu 2K)\chi\pi^{1}_{\dot{1}}b_1^{\dot{1}\dot{1}}+$$

$$+(2B+\mu 2K)\chi\pi^{2}_{\dot{2}}b_2^{\dot{1}\dot{2}}+(2A+\mu 2\bar{\Gamma})3\chi\pi^{\dot{1}2}c_2+(2\Gamma+\mu 2\Theta)3\chi\pi^{\dot{1}2}\gamma_2-$$

$$-6\chi^2 d^{\dot{1}}-\mu 6\chi^2\delta^{\dot{1}}=0.$$

If the constants in this expression satisfy the relations

$$2B+2K\mu=\lambda,\quad 2A+2\bar{\Gamma}\mu=-\xi,\quad 2\Gamma+2\Theta\mu=-\xi \qquad (II)$$

we have after some rearrangements:

$$\lambda\chi\pi^{1}_{\dot{1}}b_1^{\dot{1}\dot{1}}+\lambda\chi\pi^{1}_{\dot{2}}b_1^{\dot{1}\dot{2}}+\lambda\chi\pi^{2}_{\dot{1}}b_2^{\dot{1}\dot{1}}+\lambda\chi\pi^{2}_{\dot{2}}b_2^{\dot{1}\dot{2}}-\xi 3\chi\pi^{1\dot{1}}c_1-\xi 3\chi\pi^{1\dot{1}}\gamma_1-$$
$$-\xi 3\chi\pi^{2\dot{1}}c_2-\xi 3\chi\pi^{2\dot{1}}\gamma_2=6\chi^2 d^{\dot{1}}+\mu 6\chi^2\delta^{\dot{1}}\ .$$

Substituting this into (I) and making the terms involving the second order derivatives $(\pi_\iota)^2$, $\iota=0,1,2,3$ vanish (since such terms must not appear in the subsidiary conditions) which is possible if we set

$$\lambda 3C-\xi 3(A+\bar{\Gamma})=0\ ,\qquad \lambda 3Z-\xi 3(\Gamma+\Theta)=0\ , \qquad (III)$$

we find the secondary constraint:

$$6\chi^2 d^{\dot{1}} + 6\mu\chi^2\delta^{\dot{1}} + \lambda(f_{10}+f_{13}+\iota f_{32}+\iota f_{02})\alpha^{\dot{1}}_{11} + \lambda 2(f_{03}+\iota f_{12})\alpha^{\dot{1}}_{12} +$$
$$+(\lambda C+\xi 3(A+\bar{\Gamma}))(\iota f_{12}+f_{30})d^{\dot{1}} + (\lambda Z+\xi 3(\Gamma+\Theta))(\iota f_{12}+f_{30})\delta^{\dot{1}} +$$
$$+(\lambda C+\xi 3(A+\bar{\Gamma}))(f_{10}+\iota f_{20}+f_{13}+\iota f_{23})d^{\dot{2}} + (\lambda Z+\xi 3(\Gamma+\Theta))(f_{10}+f_{13}+$$
$$+\iota f_{20}+\iota f_{23})\delta^{\dot{2}} + \lambda(f_{01}+\iota f_{02}+\iota f_{23}+f_{13})\alpha^{\dot{1}}_{22} = 0.$$

Three more subsidiary conditions can be obtained by similar operations. The operations involved in deriving these extra subsidiary conditions are respectively:

(II): Construct

$$\lambda\pi^{1}_{\dot{1}}(eq_{12}) + \lambda\pi^{1}_{\dot{2}}(eq_{13}) + \lambda\pi^{2}_{\dot{1}}(eq_{15}) + \lambda\pi^{2}_{\dot{2}}(eq_{16}) -$$
$$-\{\xi 3\pi^{1\dot{2}}(eq_{17}) + \xi 3\pi^{2\dot{2}}(eq_{18}) + \xi 3\pi^{1\dot{2}}(eq_{19}) + \xi 3\pi^{2\dot{2}}(eq_{20})\},$$

and substitute into it $-6\chi(eq_8) - \mu 6\chi(eq_{10})$ having imposed conditions $(\iota\iota)$. Replace the spinor components $\pi_{\alpha\dot{\rho}}$ by their four vector equivalent and impose condition (ι) to create terms involving $[\pi_k, \pi_\ell]_-$. Finally impose conditions $(\iota\iota\iota)$ to make the second order derivatives $(\pi_\iota)^2$, $\iota = 0,1,2,3$, vanish and obtain the secondary constraint:

$$6\chi^2 d^{\dot{2}} + \mu 6\chi^2\delta^{\dot{2}} + \lambda(f_{10}+f_{13}+\iota f_{02}+\iota f_{32})\alpha^{\dot{2}}_{11} + (\lambda C+\xi 3(A+\bar{\Gamma}))(f_{10}+\iota f_{02}+f_{31}+$$
$$+\iota f_{23})d^{\dot{1}} + (\lambda Z+\xi 3(\Gamma+\Theta))(f_{10}+\iota f_{02}+f_{31}+\iota f_{23})\delta^{\dot{1}} + 2\lambda(\iota f_{12}+f_{03})\alpha^{\dot{2}}_{12} +$$
$$+(\lambda C+\xi 3(A+\bar{\Gamma}))(\iota f_{21}+f_{03})d^{\dot{2}} + (\lambda Z+\xi 3(\Gamma+\Theta))(\iota f_{21}+f_{03})\delta^{\dot{2}} +$$
$$+\lambda(f_{01}+\iota f_{02}+f_{13}+\iota f_{23})\alpha^{\dot{2}}_{12} = 0.$$

(III): Construct

$$\lambda\pi^{1}_{\dot{1}}(eq_1) + \lambda\pi^{2}_{\dot{1}}(eq_2) + \lambda\pi^{1}_{\dot{2}}(eq_4) + \lambda\pi^{2}_{\dot{2}}(eq_5) -$$
$$-\{\xi 3\pi_{\dot{1}1}(eq_7) + \xi 3\pi_{\dot{2}1}(eq_8) + \xi 3\pi_{\dot{1}1}(eq_9) + \xi 3\pi_{\dot{2}1}(eq_{10})\},$$

and substitute into it $-6\chi(eq_{17}) - \mu 6\chi(eq_{19})$ having imposed $(\iota\iota)$. Replace $\pi_{\alpha\dot{\rho}}$ and impose (ι) to create terms $[\pi_k, \pi_\ell]_-$.

Finally impose $(\iota\iota\iota)$ to make $(\pi_\iota)^2$ vanish and obtain the secondary constraint:

$$6\chi^2 c_1 + \mu 6\chi^2 \gamma_1 + \lambda(f_{10}+f_{31}+\iota f_{02}+\iota f_{23})b_1^{\dot{1}\dot{1}} + 2\lambda(\iota f_{21}+f_{03})b_1^{\dot{1}\dot{2}} +$$
$$+(\lambda C+\xi 3(A+\bar{\Gamma}))(\iota f_{12}+f_{03})c_1 + (\lambda Z+\xi 3(\Gamma+\Theta))(\iota f_{12}+f_{03})\gamma_1 +$$
$$+(\lambda C+\xi 3(A+\bar{\Gamma}))(f_{01}+\iota f_{02}+\iota f_{23}+f_{13})c_2 + (\lambda Z+\xi(\Gamma+\Theta))(f_{01}+\iota f_{02}+$$
$$+f_{13}+\iota f_{23})\gamma_2 + \lambda(f_{01}+\iota f_{02}+f_{31}+\iota f_{32})b_1^{\dot{2}\dot{2}} = 0 .$$

(IV): Construct

$$\lambda\pi_{\dot{1}}^{1}(eq_2) + \lambda\pi_{\dot{1}}^{2}(eq_3) + \lambda\pi_{\dot{2}}^{1}(eq_5) + \lambda\pi_{\dot{2}}^{2}(eq_6) -$$
$$-\{\xi 3\pi_{2\dot{1}}(eq_7) + \xi 3\pi_{\dot{2}2}(eq_8) + \xi 3\pi_{2\dot{1}}(eq_9) + \xi 3\pi^{2\dot{2}}(eq_{10})\},$$

and substitute into it $-6\chi(eq_{18}) - \mu 6\chi(eq_{20})$ having imposed $(\iota\iota)$. Replace $\pi_{\alpha\dot{p}}$ and impose (ι) to create $[\pi_k,\pi_\ell]_-$. Finally impose $(\iota\iota\iota)$ to make $(\pi_\iota)^2$ vanish and obtain the secondary constraint:

$$6\chi^2 c_2 + 6\mu\chi^2\gamma_2 + \lambda(f_{10}+f_{31}+\iota f_{02}+\iota f_{23})b_2^{\dot{1}\dot{1}} + (\lambda C+\xi 3(A+\bar{\Gamma}))(f_{01}+\iota f_{02}+f_{31}+$$
$$+\iota f_{23})c_1 + (\lambda Z+\xi 3(\Gamma+\Theta))(f_{01}+f_{31}+\iota f_{20}+\iota f_{23})\gamma_1 + \lambda 2(f_{03}+\iota f_{21})b_2^{\dot{1}\dot{2}} +$$
$$+(\lambda C+\xi 3(A+\bar{\Gamma}))(\iota f_{21}+f_{30})c_2 + (\lambda Z+\xi 3(\Gamma+\Theta))(\iota f_{21}+f_{30})\gamma_2 +$$
$$+\lambda(f_{01}+\iota f_{02}+f_{31}+\iota f_{32})b_2^{\dot{2}\dot{2}} = 0 .$$

Thus if the constants entering the general 20-dim. wave equation satisfy the following set of simultaneous equations

$$-\frac{\lambda}{2} + (B+K)\xi = 0 \ , \ (2B+2K\mu) = \lambda \ , \ (2A+2\bar{\Gamma}\mu) = -\xi \ ,$$
$$(2\Gamma+2\Theta\mu) = -\xi \ , \ \lambda 3C - \xi 3(A+\bar{\Gamma}) = 0, \ \lambda 3Z - \xi 3(\Gamma+\Theta) = 0,$$

then the equation accepts subsidiary conditions of the kind given above, i.e. involving the field $f_{k\ell}$. As we shall see in an example later the above method does not give all the subsidiary conditions of the second kind. Still for every particular example of wave equation more subsidiary conditions of the second kind not involving the field $f_{k\ell}$ may be found by adding or subtracting certain equations of the original wave equation.

If we introduce new constants:

$$\frac{\beta}{2\varepsilon}=\mathcal{B}\ ,\quad \frac{\bar{\beta}}{2\varepsilon}=\bar{\mathcal{B}}\ ,\quad \frac{\zeta}{2\varepsilon}=\mathcal{Z}\ ,\quad \frac{\bar{\xi}}{2\varepsilon}=\bar{\mathcal{Z}}$$

and remember all the connecting relations between the ones used earlier which we summarize below:

$$C=\frac{c}{2\varepsilon}=i\sqrt{3}\,\frac{\bar{\beta}}{2\varepsilon}=i\sqrt{3}\,\bar{\mathcal{B}}\ ,$$

$$B=\frac{b}{2\varepsilon}=i\sqrt{3}\,\frac{\beta}{2\varepsilon}=i\sqrt{3}\,\mathcal{B}\ ,$$

$$Z=\frac{z}{2\varepsilon}=i\sqrt{3}\,\frac{\zeta}{2\varepsilon}=i\sqrt{3}\,\mathcal{Z}\ ,$$

$$K=\frac{k}{2\varepsilon}=i\sqrt{3}\,\frac{\bar{\zeta}}{2\varepsilon}=i\sqrt{3}\,\bar{\mathcal{Z}}\ ,$$

$$A=\frac{\alpha}{2\varepsilon}\ ,\quad \Theta=\frac{\theta}{2\varepsilon}\ ,\quad \bar{\Gamma}=\frac{\bar{\gamma}}{2\varepsilon}\ ,\quad \Gamma=\frac{\gamma}{2\varepsilon}\ ,$$

$$\alpha=\text{real},\quad \theta=\text{real},\quad \varepsilon=\text{real},$$

then the conditions for the derivation of the secondary constraints can equivalently be written

$$-\frac{\lambda}{2}+\left(i\sqrt{3}\,\mathcal{B}+i\sqrt{3}\,\bar{\mathcal{Z}}\right)\xi=0\ ,$$

$$i\sqrt{3}\,\mathcal{B}+i\sqrt{3}\,\bar{\mathcal{Z}}\,\mu=\frac{\lambda}{2}\ ,$$

$$A+\bar{\Gamma}\mu=-\frac{\xi}{2}\ ,$$

$$\Gamma+\Theta\mu=-\frac{\xi}{2}\ ,$$

$$\lambda\, i\sqrt{3}\,\bar{\mathcal{B}}-(A+\bar{\Gamma})\xi=0,$$

$$\lambda\, i\sqrt{3}\,\mathcal{Z}-(\Gamma+\Theta)\xi=0\,.$$

One choice of the constants ξ, μ, λ, which makes this set of equations consistent is:

$$\xi=1\ ,\quad \mu=1\ ,\quad \lambda=\pm\sqrt{2}\,.$$

Choosing $\lambda = +\sqrt{2}$ the above set becomes:

$$\iota\sqrt{3}\,\mathcal{B} + \iota\sqrt{3}\,\bar{\mathcal{Z}} = \frac{\sqrt{2}}{2} \;,\; A + \bar{\Gamma} = -\frac{1}{2} \;,\; \Gamma + \Theta = -\frac{1}{2} \;,$$

$$\sqrt{2}\,\iota\sqrt{3}\,\mathcal{B} - (A + \bar{\Gamma}) = 0 \;,\; \sqrt{2}\,\iota\sqrt{3}\,\mathcal{Z} - (\Gamma + \Theta) = 0 \;.$$

This defines a class of wave equations which accept subsidiary conditions of the second kind and is not empty. We are giving below three representative examples belonging to it.

<u>Ex. 1</u>: If the constants have the values:

$$\iota\sqrt{3}\,\mathcal{B} = \frac{1}{2\sqrt{2}} \;,\; \iota\sqrt{3}\,\bar{\mathcal{B}} = -\frac{1}{2\sqrt{2}} \;,\; \iota\sqrt{3}\,\mathcal{Z} = -\frac{1}{2\sqrt{2}} \;,\; \iota\sqrt{3}\,\bar{\mathcal{Z}} = \frac{1}{2\sqrt{2}} \;,$$

$$A = -\frac{1}{4} \;,\; \Gamma = -\frac{1}{4} \;,\; \bar{\Gamma} = -\frac{1}{4} \;,\; \Theta = -\frac{1}{4} \;,$$

the above set of simultaneous equations is satisfied and a spin 3/2 wave equation is defined whose matrix $\mathbb{L}_0$ has in the canonical basis the blocks

$$, \overset{1/2}{\mathbb{L}_0} = \begin{array}{c|c|c|c|c|c|c|} & \tau_1 & \dot{\tau}_1 & \dot{\tau}_2 & \tau_2 & \tau_1' & \dot{\tau}_1' \\ \tau_1 & 0 & -1/4 & 0 & \frac{1}{2\sqrt{2}} & 0 & -1/4 \\ \hline \dot{\tau}_1 & -1/4 & 0 & \frac{1}{2\sqrt{2}} & 0 & -1/4 & 0 \\ \hline \dot{\tau}_2 & 0 & -\frac{1}{2\sqrt{2}} & 0 & 1/2 & 0 & -\frac{1}{2\sqrt{2}} \\ \hline \tau_2 & -\frac{1}{2\sqrt{2}} & 0 & 1/2 & 0 & -\frac{1}{2\sqrt{2}} & 0 \\ \hline \tau_1' & 0 & -1/4 & 0 & \frac{1}{2\sqrt{2}} & 0 & -1/4 \\ \dot{\tau}_1' & -1/4 & 0 & \frac{1}{2\sqrt{2}} & 0 & -1/4 & 0 \end{array} \;,\; \overset{3/2}{\mathbb{L}_0} = \begin{array}{c|cc|} & \dot{\tau}_2 & \tau_2 \\ \dot{\tau}_2 & 0 & 1 \\ \tau_2 & 1 & 0 \end{array} \;.$$

Notice that the eigenvalues of the block $\overset{1/2}{\mathbb{L}_0}$ are all zero and hence the equation describes only spin 3/2 particles. The matrix $\mathbb{L}_0$ satisfies the minimal equation $\mathbb{L}_0^2\{\mathbb{L}_0^2 - 1\} = 0$, and the charge associated with it is positive. Thus we have here an example of a spin 3/2, 20-dim. wave equation with definite charge and which accepts subsidiary conditions of the second kind.

<u>Ex. 2</u>: A second example of an equation accepting subsidiary conditions of the second kind we have if the constants have the values:

$$i\sqrt{3}\,\mathcal{B}=\frac{1}{2\sqrt{2}}\ ,\ i\sqrt{3}\,\bar{\mathcal{B}}=-\frac{1}{2\sqrt{2}}\quad ,\ i\sqrt{3}\,\mathcal{Z}=-\frac{1}{2\sqrt{2}}\ ,\ i\sqrt{3}\,\bar{\mathcal{Z}}=\frac{1}{2\sqrt{2}}\ ,$$

$$A=0\quad ,\quad \Theta=0\quad ,\quad \bar{\Gamma}=-\frac{1}{2}\ ,\ \Gamma=-\frac{1}{2}\ ,$$

in which case the resulting equation describes spin 3/2 particles together with spin 1/2 particles since the eigenvalues of the block $\mathbb{L}_0^{1/2}$ are not all zero. $\mathbb{L}_0$ satisfies the minimal equation $\mathbb{L}_0^2\{\mathbb{L}_0^2-(1/2)^2\}\{\mathbb{L}_0^2-1\}=0$.

<u>Ex. 3</u>: A third example we have if the constants take the values: $i\sqrt{3}\,\mathcal{B}=\frac{1}{2\sqrt{2}}$, $i\sqrt{3}\,\bar{\mathcal{B}}=-\frac{1}{2\sqrt{2}}$, $i\sqrt{3}\,\mathcal{Z}=-\frac{1}{2\sqrt{2}}$, $i\sqrt{3}\,\bar{\mathcal{Z}}=\frac{1}{2\sqrt{2}}$, $A=-\frac{1}{2}$, $\Theta=\frac{1}{2}$, $\Gamma=\bar{\Gamma}=0$. This example describes again two kinds of particles namely spin 3/2 and 1/2. $\mathbb{L}_0$ satisfies the minimal equation $\mathbb{L}_0^2\{\mathbb{L}_0^2-(\frac{1}{2})^2\}\{\mathbb{L}_0^2-1\}=0$.

Let us now give the most general operations which will lead to the most general set of equations to be satisfied among the constants such that the subsidiary conditions of the second kind be obtainable. Thus let us construct:

$$(I'):\ p\,\pi^{1}_{\dot 1}(eq_{11})+q\,\pi^{1}_{\dot 2}(eq_{12})+r\,\pi^{2}_{\dot 1}(eq_{14})+s\,\pi^{2}_{\dot 2}(eq_{15})-$$

$$-\{3\ell\pi^{1\dot 1}(eq_{17})+3m\pi^{2\dot 1}(eq_{18})+3n\pi^{1\dot 1}(eq_{19})+3t\pi^{2\dot 1}(eq_{20})\}=$$

$$p\chi\,\pi^{1}_{\dot 1}b_1^{\dot 1\dot 1}+q\chi\,\pi^{1}_{\dot 2}b_1^{\dot 1\dot 2}+r\chi\,\pi^{2}_{\dot 1}b_2^{\dot 1\dot 1}+s\chi\,\pi^{2}_{\dot 2}b_2^{\dot 1\dot 2}-3\ell\chi\,\pi^{1\dot 1}c_1-3n\chi\,\pi^{1\dot 1}\gamma_1-$$

$$-3m\chi\,\pi^{2\dot 1}c_2-3t\chi\,\pi^{2\dot 1}\gamma_2+\underbrace{\{-p\pi^{1}_{\dot 1}\pi^{1\dot 1}-\frac{q}{2}\pi^{1}_{\dot 2}\pi^{2\dot 1}-(\ell B+nK)\pi^{1}_{\dot 2}\pi^{1\dot 2}\}a^{\dot 1}_{11}}_{\text{Term 1}}+$$

$$+\underbrace{\{-p\pi^{1}_{\dot 1}\pi^{\dot 1 2}-\frac{q}{2}\pi^{1}_{\dot 2}\pi^{\dot 2 2}-r\,\pi^{2}_{\dot 1}\pi^{\dot 1 1}-\frac{s}{2}\pi^{2}_{\dot 2}\pi^{\dot 2 1}-(\ell B+nK)\pi^{1}_{\dot 2}\pi^{2\dot 2}-}_{\text{Term 2}\ \cdots}$$

$$\underbrace{-(mB+tK)\pi^{2}_{\dot 2}\pi^{1\dot 2}\}a^{\dot 1}_{12}}_{\cdots}+\underbrace{\{q\,C\pi^{1}_{\dot 2}\pi^{\dot 2}_{1}+p\,2C\pi^{1}_{\dot 1}\pi^{\dot 1}_{1}+}_{\text{Term 3}\ \cdots}$$

$$+r\,2C\pi^{2}_{\dot 1}\pi^{\dot 1}_{2}+sC\pi^{2}_{\dot 2}\pi^{\dot 2}_{2}-3(\ell A+n\bar{\Gamma})\pi^{1}_{\dot 2}\pi^{\dot 2}_{1}-3(mA+$$

$$\underbrace{+t\bar{\Gamma})\pi^{2}_{\dot 2}\pi^{\dot 2}_{2}\}d^{\dot 1}}_{\cdots}+\underbrace{\{q\,Z\pi^{1}_{\dot 2}\pi^{\dot 2}_{1}+r\,2Z\pi^{2}_{\dot 1}\pi^{\dot 1}_{2}+s\,Z\pi^{2}_{\dot 2}\pi^{\dot 2}_{2}-}_{\text{Term 4}\ \cdots}$$

$$\underbrace{-3(\ell\Gamma+n\Theta)\pi^{1}_{\dot 2}\pi^{\dot 2}_{1}-3(m\Gamma+t\Theta)\pi^{2}_{\dot 2}\pi^{\dot 2}_{2}-p\,2Z\pi^{1}_{\dot 1}\pi^{\dot 1}_{1}\}\delta^{\dot 1}}_{\cdots}+$$

$$+\underbrace{\{qC\,\pi^{1}_{\dot 2}\pi^{\dot 1}_{1}+sC\,\pi^{2}_{\dot 2}\pi^{\dot 1}_{2}+3(\ell A+n\bar\Gamma)\pi^{1}_{\dot 2}\pi^{\dot 1}_{1}+3(mA+t\bar\Gamma)\pi^{2}_{\dot 2}\pi^{\dot 1}_{2}\}d^{\dot 2}}_{\text{Term 5}}+$$

$$+\underbrace{\{qZ\,\pi^{1}_{\dot 2}\pi^{\dot 1}_{1}+sZ\pi^{2}_{\dot 2}\pi^{\dot 1}_{2}+3(\ell\Gamma+n\Theta)\pi^{1}_{\dot 2}\pi^{\dot 1}_{1}+3(m\Gamma+t\Theta)\pi^{2}_{\dot 2}\pi^{\dot 1}_{2}\}\delta^{\dot 2}}_{\text{Term 6}}+$$

$$+\underbrace{\{-r\,\pi^{\dot 2}_{1}\pi^{\dot 1 2}-\tfrac{s}{2}\pi^{2}_{\dot 2}\pi^{\dot 2 2}-(mB+tK)\pi^{2}_{\dot 2}\pi^{2\dot 2}\}\alpha^{\dot 1}_{22}}_{\text{Term 7}}+\underbrace{\{-\tfrac{q}{2}\pi^{1}_{\dot 2}\pi^{\dot 1 1}+}_{\text{Term 8}\cdots}$$

$$\underbrace{+(\ell B+nK)\pi^{1}_{\dot 2}\pi^{1\dot 1}\}\alpha^{\dot 2}_{11}}_{\cdots}+\underbrace{\{-\tfrac{q}{2}\pi^{1}_{\dot 2}\pi^{\dot 1 2}+(\ell B+nK)\pi^{1}_{\dot 2}\pi^{2\dot 1}-}_{\text{Term 9}\cdots}$$

$$\underbrace{-\tfrac{s}{2}\pi^{2}_{\dot 2}\pi^{\dot 1 1}+(mB+tK)\pi^{2}_{\dot 2}\pi^{1\dot 1}\}\alpha^{\dot 2}_{12}}_{\cdots}+\underbrace{\{-\tfrac{s}{2}\pi^{2}_{\dot 2}\pi^{\dot 1 2}+(mB+tK)\pi^{2}_{\dot 2}\pi^{2\dot 1}\}\alpha^{\dot 2}_{22}}_{\text{Term 10}}=0,$$

where p, q, r, s, t, ℓ, m, n, **constants to be determined. Replacing the spinor components** $\pi_{\alpha\dot\beta}$ **by their four vector equivalent the individual terms in the above expression become respectively:**

$\{\text{Term 1}\}=p(f_{10}+f_{13}+\iota f_{02}+\iota f_{32})\alpha^{\dot 1}_{11}$,
if $p=q$ and $\ell B+nK=\frac{q}{2}=\frac{p}{2}$.

$\{\text{Term 2}\}=2p(\iota f_{12}+f_{03})\alpha^{\dot 1}_{12}$,
if $p=r=s=q$ and $mB+tK=\frac{s}{2}=\frac{r}{2}$.

$\{\text{Term 3}\}=\{pC+3(mA+t\bar\Gamma)\}\iota f_{12}d^{\dot 1}+\{pC+3(\ell A+t\bar\Gamma)\}f_{30}d^{\dot 1}$,
if $3pC-3(mA+t\bar\Gamma)=0$, $3pC-3(\ell A+n\bar\Gamma)=0$.

$\{\text{Term 4}\}=\{pZ+3(m\Gamma+t\Theta)\}\iota f_{12}\delta^{\dot 1}+\{pZ+3(\ell\Gamma+n\Theta)\}f_{30}\delta^{\dot 1}$,
if $3pZ-3(m\Gamma+t\Theta)=0$, $3pZ-3(\ell\Gamma+n\Theta)=0$.

$\{\text{Term 5}\}=\{pC+3(mA+t\bar\Gamma)\}(f_{10}+\iota f_{20}+f_{13}+\iota f_{23})d^{\dot 2}$,
if $3(mA+t\bar\Gamma)=3(\ell A+n\bar\Gamma)$.

$\{\text{Term 6}\}=\{pZ+3(m\Gamma+t\Theta)\}(f_{10}+f_{13}+\iota f_{20}+\iota f_{23})\delta^{\dot 2}$,
if $3(\ell\Gamma+n\Theta)=3(m\Gamma+t\Theta)$.

$\{\text{Term 7}\}=r(f_{01}+f_{13}+\iota f_{02}+\iota f_{23})\alpha^{\dot 1}_{22}$,
if $r=s$ and $mB+tK=\frac{s}{2}=\frac{r}{2}$.

$\{\text{Term}8\} = \{\text{Term}\,9\} = \{\text{Term}\,10\} = 0$, by virtue of the imposed already conditions on the previous terms. Let us consider also equations (7) and (9) and multiply (7) by $-6\nu\chi$ and (9) by $-6\mu\chi$ and add them up (ν, μ, constants) i.e.

$$-6\nu\chi(eq_7) - 6\mu\chi(eq_9) = (\nu 2B + \mu 2K)\chi\pi^{1}_{;2}b_1^{\dot{1}\dot{2}} + (\nu 2B + \mu 2K)\chi\pi^{2}_{;1}b_2^{\dot{1}\dot{1}} + (\nu 2A + \mu 2\bar{\Gamma})\chi 3\pi^{1\dot{1}}c_1 + (\nu 2\Gamma + \mu 2\Theta)\chi 3\pi^{1\dot{1}}\gamma_1 + (\nu 2B + \mu 2K)\chi\pi^{1}_{;1}b_1^{\dot{1}\dot{1}} + (\nu 2B + \mu 2K)\chi\pi^{2}_{;2}b_2^{\dot{1}\dot{2}} + (\nu 2A + \mu 2\bar{\Gamma})\chi 3\pi^{\dot{1}2}c_2 + (\nu 2\Gamma + \mu 2\Theta)\chi 3\pi^{\dot{1}2}\gamma_2 - \nu 6\chi^2 d^{\dot{1}} - \mu 6\chi^2\delta^{\dot{1}} = 0 .$$

If the constants in this expression satisfy the relations:

$$(\nu 2B + \mu 2K) = q\ , \quad (\nu 2B + \mu 2K) = p\ , \quad (\nu 2B + \mu 2K) = r,$$

$$(\nu 2B + \mu 2K) = s\ , \quad (\nu 2A + \mu 2\bar{\Gamma}) = -\ell, \quad (\nu 2\Gamma + \mu 2\Theta) = -n,$$

$$(\nu 2A + \mu 2\bar{\Gamma}) = -m, \quad (\nu 2\Gamma + \mu 2\Theta) = -t\ ,$$

then we have:

$$p\chi\pi^{1}_{;1}b_1^{\dot{1}\dot{1}} + q\chi\pi^{1}_{;2}b_1^{\dot{1}\dot{2}} + r\chi\pi^{2}_{;1}b_2^{\dot{1}\dot{1}} + s\chi\pi^{2}_{;2}b_2^{\dot{1}\dot{2}} - 3\ell\chi\pi^{1\dot{1}}c_1 - 3n\chi\pi^{1\dot{1}}\gamma_1 - 3m\chi\pi^{2\dot{1}}c_2 - 3t\chi\pi^{2\dot{1}}\gamma_2 = \nu 6\chi^2 d^{\dot{1}} + \mu 6\chi^2\delta^{\dot{1}} .$$

An even further generalization will be to consider

$$p\chi\pi^{1}_{;1}b_1^{\dot{1}\dot{1}} + q\chi\pi^{1}_{;2}b_1^{\dot{1}\dot{2}} + r\chi\pi^{2}_{;1}b_2^{\dot{1}\dot{1}} + s\chi\pi^{2}_{;2}b_2^{\dot{1}\dot{2}} - 3\ell\chi\pi^{1\dot{1}}c_1 - 3n\chi\pi^{1\dot{1}}\gamma_1 - 3m\chi\pi^{2\dot{1}}c_2 - 3t\chi\pi^{2\dot{1}}\gamma_2 = \frac{\nu 6\chi^2 d^{\dot{1}} + \mu 6\chi^2\delta^{\dot{1}}}{\Phi}$$

in which case the following relations should hold instead:

$$(\nu 2B + \mu 2K) = q\Phi\ , \quad (\nu 2B + \mu 2K) = p\Phi\ , \quad (\nu 2B + \mu 2K) = r\Phi,$$

$$(\nu 2B + \mu 2K) = s\Phi\ , \quad (\nu 2A + \mu 2\bar{\Gamma}) = -\ell\Phi, \quad (\nu 2\Gamma + \mu 2\Theta) = -n\Phi,$$

$$(\nu 2A + \mu 2\bar{\Gamma}) = -m\Phi\ , \quad (\nu 2\Gamma + \mu 2\Theta) = -t\,\Phi\ ,$$

where Φ constant.

Substituting back into (I') we find the subsidiary condition:

$$\begin{aligned}
&\nu 6\chi^2 d^{i} + \mu 6\chi^2\delta^{i} + p(f_{10}+f_{13}+\iota f_{02}+\iota f_{32})\alpha^{i}_{11} + r(f_{01}+f_{13}+ \\
&+\iota f_{02}+\iota f_{23})\alpha^{i}_{22} + 2p(\iota f_{12}+f_{03})\alpha^{i}_{12} + \{pC+3(mA+t\bar{\Gamma})\}\iota f_{12}d^{i}+ \\
&+\{pC+3(\ell A+n\bar{\Gamma})\}f_{30}d^{i} + \{pZ+3(m\Gamma+t\Theta)\}\iota f_{12}\delta^{i}+ \\
&+\{pZ+3(\ell\Gamma+n\Theta)\}f_{30}\delta^{i} + \{pC+3(mA+t\bar{\Gamma})\}(f_{10}+\iota f_{20}+ \\
&+f_{13}+\iota f_{23})d^{\dot{2}} + \{pZ+3(m\Gamma+t\Theta)\}(f_{10}+f_{13}+\iota f_{20}+\iota f_{23})\delta^{\dot{2}} = 0,
\end{aligned}$$

or the more general one:

$$\frac{\nu 6\chi^2 d^{i} + \mu 6\chi^2\delta^{i}}{\Phi} + \text{(all the remaining terms of the above subsidiary condition)} = 0.$$

Three more subsidiary conditions of the type obtained already can be found by generalizing the operations used in (II), (III) and (IV) using multipliers $p, q, r, s, t, \ell, m, n$ and without having to impose extra conditions among the constants. Finally we summarize the conditions which have to be satisfied for the derivation of these secondary constraints.

$$\ell B + nK = \frac{q}{2} = \frac{p}{2}, \qquad p = q, \qquad p = q = r = s,$$

$$mB + tK = \frac{s}{2} = \frac{r}{2}, . \quad 3pC - 3(mA + t\bar{\Gamma}) = 0,$$

$$3pC - 3(\ell A + n\bar{\Gamma}) = 0, \quad 3pZ - 3(m\Gamma + t\Theta) = 0,$$

$$3pZ - 3(\ell\Gamma + n\Theta) = 0, \quad 3(mA + t\bar{\Gamma}) = 3(\ell A + n\bar{\Gamma}),$$

$$3(\ell\Gamma + n\Theta) = 3(m\Gamma + t\Theta), \quad (\nu 2B + \mu 2K) = q,$$

$$(\nu 2B + \mu 2K) = p, \ (\nu 2B + \mu 2K) = r, \ (\nu 2B + \mu 2K) = s,$$

$$(\nu 2A + \mu 2\bar{\Gamma}) = -\ell, \quad (\nu 2\Gamma + \mu 2\Theta) = -n, \quad (\nu 2A + \mu 2\bar{\Gamma}) = -m$$

$$(\nu 2\Gamma + \mu 2\Theta) = -t \quad ,$$ (or the ones involving Φ).

The class defined by these relations is not empty because the class given at the beginning of the paragraph is a subclass of this class.

3: Propagation of a particular spin 3/2 20-dim. wave-equation.

We study in this paragraph the propagation of the spin 3/2, 20-dim. wave-equation with definite charge defined by the constants:

$$B = i\sqrt{3}\,\mathcal{B} = \frac{1}{2\sqrt{2}}\,;\; C = i\sqrt{3}\,\bar{\mathcal{B}} = -\frac{1}{2\sqrt{2}}\,,\; Z = i\sqrt{3}\,\mathcal{Z} = -\frac{1}{2\sqrt{2}}\,,$$

$$K = i\sqrt{3}\,\bar{\mathcal{Z}} = \frac{1}{2\sqrt{2}}\,,\; A = -\frac{1}{4}\,,\; \Gamma = -\frac{1}{4}\,,\; \bar{\Gamma} = -\frac{1}{4}\,,\; \Theta = -\frac{1}{4}\,,$$

(Ex. 1 of the previous paragraph). Substituting the values of these constants back into the spinor form of the general 20-dim. wave equation we find for this particular example the form:

$$-\pi_{\dot{1}1} b_1^{\dot{1}\dot{1}} - i\pi_{1\dot{2}} b_1^{\dot{1}\dot{2}} - \frac{1}{\sqrt{2}}\pi^{\dot{1}}_1 c_1 - \frac{1}{\sqrt{2}}\pi^{\dot{1}}_1 \gamma_1 + \chi\alpha^{\dot{1}}_{11} = 0, \tag{1'}$$

$$-\frac{1}{2}\pi_{2\dot{2}} b_1^{\dot{1}\dot{2}} - \frac{1}{2}\pi_{1\dot{1}} b_2^{\dot{1}\dot{1}} - \frac{1}{2\sqrt{2}}\pi^{\dot{1}}_2 c_1 - \frac{1}{2\sqrt{2}}\pi^{\dot{1}}_2 \gamma_1 - \frac{1}{2}\pi_{\dot{1}2} b_1^{\dot{1}\dot{1}} - \frac{1}{2}\pi_{1\dot{2}} b_2^{\dot{1}\dot{2}} - \\ - \frac{1}{2\sqrt{2}}\pi^{\dot{1}}_1 c_2 - \frac{1}{2\sqrt{2}}\pi^{\dot{1}}_1 \gamma_2 + \chi\alpha^{\dot{1}}_{12} = 0, \tag{2'}$$

$$-i\pi_{2\dot{2}} b_2^{\dot{1}\dot{2}} - \frac{1}{\sqrt{2}}\pi^{\dot{1}}_2 c_2 - \frac{1}{\sqrt{2}}\pi^{\dot{1}}_2 \gamma_2 - i\pi_{2\dot{1}} b_2^{\dot{1}\dot{1}} + \chi\alpha^{\dot{1}}_{22} = 0, \tag{3'}$$

$$-i\pi_{1\dot{1}} b_1^{\dot{1}\dot{2}} - \frac{1}{\sqrt{2}}\pi^{\dot{2}}_1 c_1 - \frac{1}{\sqrt{2}}\pi^{\dot{2}}_1 \gamma_1 - i\pi_{1\dot{2}} b_1^{\dot{2}\dot{2}} + \chi\alpha^{\dot{2}}_{11} = 0, \tag{4'}$$

$$-\frac{1}{2}\pi_{\dot{2}2} b_1^{\dot{2}\dot{2}} - \frac{1}{2}\pi_{1\dot{1}} b_2^{\dot{1}\dot{2}} - \frac{1}{2\sqrt{2}}\pi^{\dot{2}}_1 c_2 - \frac{1}{2\sqrt{2}}\pi^{\dot{2}}_1 \gamma_2 - \frac{1}{2}\pi_{2\dot{1}} b_1^{\dot{1}\dot{2}} - \frac{1}{2}\pi_{1\dot{2}} b_2^{\dot{2}\dot{2}} - \\ - \frac{1}{2\sqrt{2}}\pi^{\dot{2}}_2 c_1 - \frac{1}{2\sqrt{2}}\pi^{\dot{2}}_2 \gamma_1 + \chi\alpha^{\dot{2}}_{12} = 0, \tag{5'}$$

$$-i\pi_{2\dot{2}} b_2^{\dot{2}\dot{2}} - i\pi_{2\dot{1}} b_2^{\dot{1}\dot{2}} - \frac{1}{\sqrt{2}}\pi^{\dot{2}}_2 c_2 - \frac{1}{\sqrt{2}}\pi^{\dot{2}}_2 \gamma_2 + \chi\alpha^{\dot{2}}_{22} = 0, \tag{6'}$$

$$\begin{aligned} &-\frac{1}{6\sqrt{2}}\pi^{1}_{\dot{2}}b^{\dot{1}\dot{2}}_{1}-\frac{1}{6\sqrt{2}}\pi^{2}_{\dot{1}}b^{\dot{1}\dot{1}}_{2}+\frac{1}{4}\pi^{1\dot{1}}c_{1}+\frac{1}{4}\pi^{1\dot{1}}\gamma_{1}-\frac{1}{6\sqrt{2}}\pi^{1}_{\dot{1}}b^{\dot{1}\dot{1}}_{1}-\frac{1}{6\sqrt{2}}\pi^{2}_{\dot{2}}b^{\dot{1}\dot{2}}_{2}+ \\ &\qquad +\frac{1}{4}\pi^{\dot{1}2}c_{2}+\frac{1}{4}\pi^{\dot{1}2}\gamma_{2}+\chi d^{\dot{1}}=0, \end{aligned} \qquad (7')$$

$$\begin{aligned} &-\frac{1}{6\sqrt{2}}\pi^{1}_{\dot{2}}b^{\dot{2}\dot{2}}_{1}-\frac{1}{6\sqrt{2}}\pi^{2}_{\dot{1}}b^{\dot{1}\dot{2}}_{2}+\frac{1}{4}\pi^{\dot{2}2}c_{2}+\frac{1}{4}\pi^{\dot{2}2}\gamma_{2}-\frac{1}{6\sqrt{2}}\pi^{1}_{\dot{1}}b^{\dot{1}\dot{2}}_{1}-\frac{1}{6\sqrt{2}}\pi^{2}_{\dot{2}}b^{\dot{2}\dot{2}}_{2}+ \\ &\qquad +\frac{1}{4}\pi^{\dot{2}1}c_{1}+\frac{1}{4}\pi^{\dot{2}1}\gamma_{1}+\chi d^{\dot{2}}=0, \end{aligned} \qquad (8')$$

$$\begin{aligned} &-\frac{1}{6\sqrt{2}}\pi^{1}_{\dot{2}}b^{\dot{1}\dot{2}}_{1}-\frac{1}{6\sqrt{2}}\pi^{2}_{\dot{1}}b^{\dot{1}\dot{1}}_{2}+\frac{1}{4}\pi^{1\dot{1}}c_{1}+\frac{1}{4}\pi^{1\dot{1}}\gamma_{1}-\frac{1}{6\sqrt{2}}\pi^{1}_{\dot{1}}b^{\dot{1}\dot{1}}_{2}-\frac{1}{6\sqrt{2}}\pi^{2}_{\dot{2}}b^{\dot{1}\dot{2}}_{2}+ \\ &\qquad +\frac{1}{4}\pi^{\dot{1}2}c_{2}+\frac{1}{4}\pi^{\dot{1}2}\gamma_{2}+\chi \delta^{\dot{1}}=0, \end{aligned} \qquad (9')$$

$$\begin{aligned} &-\frac{1}{6\sqrt{2}}\pi^{1}_{\dot{2}}b^{\dot{2}\dot{2}}_{1}-\frac{1}{6\sqrt{2}}\pi^{2}_{\dot{1}}b^{\dot{1}\dot{2}}_{2}+\frac{1}{4}\pi^{\dot{2}2}c_{2}+\frac{1}{4}\pi^{\dot{2}2}\gamma_{2}-\frac{1}{6\sqrt{2}}\pi^{1}_{\dot{1}}b^{\dot{1}\dot{2}}_{1}-\frac{1}{6\sqrt{2}}\pi^{2}_{\dot{2}}b^{\dot{2}\dot{2}}_{2}+ \\ &\qquad +\frac{1}{4}\pi^{\dot{2}1}c_{1}+\frac{1}{4}\pi^{\dot{2}1}\gamma_{1}+\chi \delta^{\dot{2}}=0, \end{aligned} \qquad (10')$$

$$-1\pi^{\dot{1}1}\alpha^{\dot{1}}_{11}-1\pi^{\dot{1}2}\alpha^{\dot{1}}_{12}-\frac{1}{\sqrt{2}}\pi^{\dot{1}}_{1}d^{\dot{1}}-\frac{1}{\sqrt{2}}\pi^{\dot{1}}_{1}\delta^{\dot{1}}+\chi b^{\dot{1}\dot{1}}_{1}=0, \qquad (11')$$

$$\begin{aligned} &-\frac{1}{2}\pi^{\dot{2}2}\alpha^{\dot{1}}_{12}-\frac{1}{2}\pi^{\dot{1}1}\alpha^{\dot{2}}_{11}-\frac{1}{2\sqrt{2}}\pi^{\dot{2}}_{1}d^{\dot{1}}-\frac{1}{2\sqrt{2}}\pi^{\dot{2}}_{1}\delta^{\dot{1}}-\frac{1}{2}\pi^{\dot{2}1}\alpha^{\dot{1}}_{11}-\frac{1}{2}\pi^{\dot{1}2}\alpha^{\dot{2}}_{12}- \\ &\qquad -\frac{1}{2\sqrt{2}}\pi^{\dot{1}}_{1}d^{\dot{2}}-\frac{1}{2\sqrt{2}}\pi^{\dot{1}}_{1}\delta^{\dot{2}}+\chi b^{\dot{1}\dot{2}}_{1}=0, \end{aligned} \qquad (12')$$

$$-1\pi^{\dot{2}2}\alpha^{\dot{2}}_{12}-\frac{1}{\sqrt{2}}\pi^{\dot{2}}_{1}d^{\dot{2}}-\frac{1}{\sqrt{2}}\pi^{\dot{2}}_{1}\delta^{\dot{2}}-1\pi^{\dot{2}1}\alpha^{\dot{2}}_{11}+\chi b^{\dot{2}\dot{2}}_{1}=0, \qquad (13')$$

$$-1\pi^{\dot{1}1}\alpha^{\dot{1}}_{12}-\frac{1}{\sqrt{2}}\pi^{\dot{1}}_{2}d^{\dot{1}}-\frac{1}{\sqrt{2}}\pi^{\dot{1}}_{2}\delta^{\dot{1}}-1\pi^{\dot{1}2}\alpha^{\dot{1}}_{22}+\chi b^{\dot{1}\dot{1}}_{2}=0, \qquad (14')$$

$$\begin{aligned} &-\frac{1}{2}\pi^{\dot{2}2}\alpha^{\dot{1}}_{22}-\frac{1}{2}\pi^{\dot{1}1}\alpha^{\dot{2}}_{12}-\frac{1}{2\sqrt{2}}\pi^{\dot{1}}_{2}d^{\dot{2}}-\frac{1}{2\sqrt{2}}\pi^{\dot{1}}_{2}\delta^{\dot{2}}-\frac{1}{2}\pi^{\dot{2}1}\alpha^{\dot{1}}_{12}-\frac{1}{2}\pi^{\dot{1}2}\alpha^{\dot{2}}_{22}- \\ &\qquad -\frac{1}{2\sqrt{2}}\pi^{\dot{2}}_{2}d^{\dot{1}}-\frac{1}{2\sqrt{2}}\pi^{\dot{2}}_{2}\delta^{\dot{1}}+\chi b^{\dot{1}\dot{2}}_{2}=0, \end{aligned} \qquad (15')$$

$$-1\pi^{\dot{2}2}\alpha^{\dot{2}}_{22}-1\pi^{\dot{2}1}\alpha^{\dot{2}}_{12}-\frac{1}{\sqrt{2}}\pi^{\dot{2}}_{2}d^{\dot{2}}-\frac{1}{\sqrt{2}}\pi^{\dot{2}}_{2}\delta^{\dot{2}}+\chi b^{\dot{2}\dot{2}}_{2}=0, \qquad (16')$$

$$\begin{aligned} &-\frac{1}{6\sqrt{2}}\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{12}-\frac{1}{6\sqrt{2}}\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{11}+\frac{1}{4}\pi_{\dot{1}1}d^{\dot{1}}+\frac{1}{4}\pi_{\dot{1}1}\delta^{\dot{1}}-\frac{1}{6\sqrt{2}}\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{11}-\frac{1}{6\sqrt{2}}\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{12}+ \\ &\qquad +\frac{1}{4}\pi_{1\dot{2}}d^{\dot{2}}+\frac{1}{4}\pi_{1\dot{2}}\delta^{\dot{2}}+\chi c_{1}=0, \end{aligned} \qquad (17')$$

$$\begin{aligned} &-\frac{1}{6\sqrt{2}}\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{22}-\frac{1}{6\sqrt{2}}\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{12}+\frac{1}{4}\pi_{\dot{2}2}d^{\dot{2}}+\frac{1}{4}\pi_{\dot{2}2}\delta^{\dot{2}}-\frac{1}{6\sqrt{2}}\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{12}-\frac{1}{6\sqrt{2}}\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{22}+ \\ &\qquad +\frac{1}{4}\pi_{2\dot{1}}d^{\dot{1}}+\frac{1}{4}\pi_{2\dot{1}}\delta^{\dot{1}}+\chi c_{2}=0, \end{aligned} \qquad (18')$$

$$\begin{aligned} &-\frac{1}{6\sqrt{2}}\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{22}-\frac{1}{6\sqrt{2}}\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{11}+\frac{1}{4}\pi_{\dot{1}1}d^{\dot{1}}+\frac{1}{4}\pi_{\dot{1}1}\delta^{\dot{1}}-\frac{1}{6\sqrt{2}}\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{11}-\frac{1}{6\sqrt{2}}\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{12}+ \\ &\qquad +\frac{1}{4}\pi_{1\dot{2}}d^{\dot{2}}+\frac{1}{4}\pi_{1\dot{2}}\delta^{\dot{2}}+\chi \gamma_{1}=0, \end{aligned} \qquad (19')$$

$$\begin{aligned} &-\frac{1}{6\sqrt{2}}\pi^{2}_{\dot{1}}\alpha^{\dot{1}}_{22}-\frac{1}{6\sqrt{2}}\pi^{1}_{\dot{2}}\alpha^{\dot{2}}_{12}+\frac{1}{4}\pi_{\dot{2}2}d^{\dot{2}}+\frac{1}{4}\pi_{\dot{2}2}\delta^{\dot{2}}-\frac{1}{6\sqrt{2}}\pi^{1}_{\dot{1}}\alpha^{\dot{1}}_{12}-\frac{1}{6\sqrt{2}}\pi^{2}_{\dot{2}}\alpha^{\dot{2}}_{22}+ \\ &\qquad +\frac{1}{4}\pi_{2\dot{1}}d^{\dot{1}}+\frac{1}{4}\pi_{2\dot{1}}\delta^{\dot{1}}+\chi \gamma_{2}=0\ . \end{aligned} \qquad (20')$$

We show now that this wave-equation besides the four subsidiary conditions of the second kind involving the external field components $f_{k\ell}$ accepts also four more subsidiary conditions of the second kind involving no external field components. Thus let us consider equations (17') and (19'). Subtracting (19') from (17') we find

$$\chi c_1 - \chi \gamma_1 = 0 ,$$

which gives the secondary constraint

$$c_1 = \gamma_1 \quad . \tag{i'}$$

Similarly subtracting (20') from (18') we find

$$c_2 = \gamma_2 \quad . \tag{ii'}$$

Subtracting (9') from (7') we find

$$d^{\dot{1}} = \delta^{\dot{1}} \quad . \tag{iii'}$$

Finally subtracting (10') from (8') we find

$$d^{\dot{2}} = \delta^{\dot{2}} \quad . \tag{iv'}$$

The other four subsidiary conditions of the second kind are:

$$\begin{aligned} &6\chi^2 d^{\dot{1}} + 6\chi^2\delta^{\dot{1}} + \sqrt{2}\,(f_{10}+f_{13}+if_{32}+if_{02})\alpha^{\dot{1}}_{11} + 2\sqrt{2}\,(f_{03}+if_{12})\alpha^{\dot{1}}_{12} - \\ &-2(if_{12}+f_{30})d^{\dot{1}} - 2(f_{30}+if_{12})\delta^{\dot{1}} - 2(f_{10}+if_{20}+f_{13}+if_{23})d^{\dot{2}} - \\ &-2(f_{10}+f_{13}+if_{20}+if_{23})\delta^{\dot{2}} + \sqrt{2}\,(f_{01}+if_{02}+if_{23}+f_{13})\alpha^{\dot{1}}_{22} = 0 , \end{aligned} \tag{v'}$$

$$\begin{aligned} &6\chi^2 d^{\dot{2}} + 6\chi^2\delta^{\dot{2}} + \sqrt{2}\,(f_{10}+f_{13}+if_{02}+if_{32})\alpha^{\dot{2}}_{11} - 2(f_{10}+if_{02}+ \\ &+f_{31}+if_{23})d^{\dot{1}} - 2(f_{10}+if_{02}+f_{31}+if_{23})\delta^{\dot{1}} + 2\sqrt{2}\,(if_{12}+f_{03})\alpha^{\dot{2}}_{12} - \\ &-2(if_{21}+f_{03})d^{\dot{2}} - 2(if_{21}+f_{03})\delta^{\dot{2}} + \sqrt{2}\,(f_{01}+if_{02}+f_{13}+if_{23})\alpha^{\dot{2}}_{22} = 0, \end{aligned} \tag{vi'}$$

$$\begin{aligned} &6\chi^2 c_1 + 6\chi^2\gamma_1 + \sqrt{2}\,(f_{10}+f_{31}+if_{02}+if_{23})b^{\dot{1}\dot{1}}_1 + 2\sqrt{2}\,(if_{21}+f_{03})b^{\dot{1}\dot{2}}_1 - \\ &-2(if_{12}+f_{03})c_1 - 2(if_{12}+f_{03})\gamma_1 - 2(f_{01}+if_{02}+if_{23}+f_{13})c_2 - \\ &-2(f_{01}+if_{02}+f_{13}+if_{23})\gamma_2 + \sqrt{2}\,(f_{01}+if_{02}+f_{31}+if_{32})b^{\dot{2}\dot{2}}_1 = 0, \end{aligned} \tag{vii'}$$

$$\begin{aligned} &6\chi^2 c_2 + 6\chi^2\gamma_2 + \sqrt{2}\,(f_{10}+f_{31}+if_{02}+if_{23})b^{\dot{1}\dot{1}}_2 - 2(f_{01}+if_{20}+f_{31}+ \\ &+if_{23})c_1 - 2(f_{01}+f_{31}+if_{20}+if_{23})\gamma_1 + 2\sqrt{2}\,(f_{03}+if_{21})b^{\dot{1}\dot{2}}_2 - 2(if_{21}+ \\ &+f_{30})c_2 - 2(if_{21}+f_{30})\gamma_2 + \sqrt{2}\,(f_{01}+if_{02}+f_{31}+if_{32})b^{\dot{2}\dot{2}}_2 = 0 \quad . \end{aligned} \tag{viii'}$$

Substituting the constraints (i'), (ii'), (iii'), (iv'), into the wave-equation we find that (20') is the same as (18'), (19') is the same as (17'), (10') is the same as (8'), and (9') is the same as (7').

Thus we can dispense with equations (20'), (19') (10'), (9') and reduce the problem to one involving sixteen differential equations. Farther more changing the spinor basis $\{a_{11}^{\dot{1}}, a_{12}^{\dot{1}}, a_{22}^{\dot{1}}, a_{11}^{\dot{2}}, a_{12}^{\dot{2}}, a_{22}^{\dot{2}}, d^{\dot{1}}, d^{\dot{2}}, b_1^{\dot{1}\dot{1}}, b_1^{\dot{1}\dot{2}}, b_1^{\dot{2}\dot{2}}, b_2^{\dot{1}\dot{1}} ; b_2^{\dot{1}\dot{2}}, b_2^{\dot{2}\dot{2}}, c_1, c_2\}$ to the new basis $\{a_{11}^{\dot{1}}, a_{12}^{\dot{1}}, a_{22}^{\dot{1}}, a_{11}^{\dot{2}}, a_{12}^{\dot{2}}, a_{22}^{\dot{2}}, \sqrt{2}\, d^{\dot{1}}, \sqrt{2}\, d^{\dot{2}}, b_1^{\dot{1}\dot{1}}, b_1^{\dot{1}\dot{2}}, b_1^{\dot{2}\dot{2}}, b_2^{\dot{1}\dot{1}}, b_2^{\dot{1}\dot{2}}, b_2^{\dot{2}\dot{2}}, \sqrt{2}\, c_1, \sqrt{2}\, c_2\}$ obtained from the old one by scaling up the spin 1/2 components by the factor $\sqrt{2}$, the set of sixteen differential equations with respect to the new basis becomes the same as the Pauli-Fierz wave-equation, while the subsidiary conditions of the second kind involving the field $f_{k\ell}$ (with respect to the new basis) become the same as those of the Pauli-Fierz equation. Hence we conclude that the propagation of the spin 3/2, 20-dim. wave-equation with definite charge is the same as that of the Pauli-Fierz equation (c.f. Chapters 10, 11).

CHAPTER 14.

Matrix formulation of the problem of subsidiary conditions of the second kind.

In the previous chapters we saw how the subsidiary conditions of the second kind can be verified by employing spinor calculus. In this chapter we show how the same problem can be formulated in matrix language. We do this first for the Pauli-Fierz equation and then derive general conditions which have to be satisfied such that the subsidiary conditions of the second kind for any first order wave-equation be derivable.

1: Matrix formulation of the subsidiary conditions of the second kind for the Pauli-Fierz equation.

The subsidiary conditions of the second kind of the Pauli-Fierz equation can be formulated in matrix language. To demonstrate how this can be done, let us for instance consider the derivation of the subsidiary condition:

$$6\chi^2 d^{\dot{1}} + (f_{10}+f_{13}+\iota f_{02}+\iota f_{32})\alpha^{\dot{1}}_{11} + 2(f_{03}+\iota f_{12})\alpha^{\dot{1}}_{12} + (f_{01}+\iota f_{02}+f_{13}+$$
$$+\iota f_{23})\alpha^{\dot{1}}_{22} + 2(f_{03}-\iota f_{12})d^{\dot{1}} + 2(f_{01}+\iota f_{02}-f_{13}-\iota f_{23})d^{\dot{2}} = 0.$$

In deriving this subsidiary condition the following steps were involved: construct,

$$\{\pi^{1}_{\dot{1}}(eq_{9}) + \pi^{1}_{\dot{2}}(eq_{10}) + \pi^{2}_{\dot{1}}(eq_{12}) + \pi^{2}_{\dot{2}}(eq_{13})\} - \{3\pi^{1\dot{1}}(eq_{15}) - 3\pi^{2\dot{1}}(eq_{16})\} = 0,$$

and substitute into it $6\chi(eq_{7})$ having replaced $\pi_{\alpha\dot{\rho}}$ to produce terms involving $[\pi_k, \pi_\ell]_- = \iota e F_{k\ell} = f_{k\ell}$.

The above steps can be put into matrix language as follows:

$$\mathbb{S}^{(I)}(\mathbb{A}^{(I)}_0\pi_0 + \mathbb{A}^{(I)}_1\pi_1 + \mathbb{A}^{(I)}_2\pi_2 + \mathbb{A}^{(I)}_3\pi_3 + \mathbb{A}^{(I)}_4\chi)(-\mathbb{L}_0\pi_0 + \mathbb{L}_1\pi_1 + \mathbb{L}_2\pi_2 + \mathbb{L}_3\pi_3 + \mathbb{1}\chi)\psi = 0,$$

where $\mathbb{S}^{(I)}$ is 1 x 16 matrix and $\mathbb{A}_0^{(I)}, \mathbb{A}_1^{(I)}, \mathbb{A}_2^{(I)}, \mathbb{A}_3^{(I)}, \mathbb{A}_4^{(I)}$ are 16 x 16 matrices. The index (I) makes reference to the first subsidiary condition of the second kind. The matrices $\mathbb{S}^{(I)}$, $\mathbb{A}_j^{(I)}$, $j = 0,1,2,3,4,$ with respect to the spinor basis $\{a^{\dot 1}_{11}, a^{\dot 1}_{12}, a^{\dot 1}_{22}, a^{\dot 2}_{11}, a^{\dot 2}_{12}, a^{\dot 2}_{22}, d^{\dot 1}, d^{\dot 2}, b_1^{\dot 1\dot 1}, b_1^{\dot 1\dot 2}, b_1^{\dot 2\dot 2}, b_2^{\dot 1\dot 1}, b_2^{\dot 1\dot 2}, b_2^{\dot 2\dot 2}, c_1, c_2\}$ are:

$$\mathbb{S}^{(I)} = [\,0, 0, 0, 0, 0, 0, 6, 0, 1, 1, 0, 1, 1, 0, -3, -3\,] ,$$

$$\mathbb{A}_0^{(I)} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&-1&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&1&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&-1& \\
&&&&&&&&&&&&&&&
\end{array} ,$$

$$\mathbb{A}_1^{(I)} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&1&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&-1&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&&-1
\end{array} ,$$

$$\mathbb{A}_2^{(I)} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&-1&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&-1&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&&-1
\end{array} ,$$

$$\mathbb{A}_3^{(I)} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
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&&&&&&&&&&&&&&& \\
&&&&&&&&&-1&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&-1&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&-1& \\
&&&&&&&&&&&&&&&
\end{array} ,$$

$$\mathbb{A}_4^{(I)} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&1&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
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&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&&
\end{array}$$

The matrices for the other three subsidiary conditions are respectively:

$$\mathbb{S}^{(II)} = [0,0,0,0,0,0,0,6,0,1,1,0,1,1,-3,-3],$$

$$\mathbb{A}_0^{(II)} = \mathrm{diag}\{0,0,0,0,0,0,0,0,0,0,-1,0,1,0,0,-1\},$$

$$\mathbb{A}_1^{(II)} = \mathrm{diag}\{0,0,0,0,0,0,0,0,0,1,0,0,0,-1,-1,0\},$$

$$\mathbb{A}_2^{(II)} = \mathrm{diag}\{0,0,0,0,0,0,0,0,0,-1,0,0,0,-1,1,0\},$$

$$\mathbb{A}_3^{(II)} = \mathrm{diag}\{0,0,0,0,0,0,0,0,0,0,-1,0,-1,0,0,1\},$$

$$\mathbb{A}_4^{(II)} = \mathrm{diag}\{0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0),$$

$$\mathbb{S}^{(III)} = [1,1,0,1,1,0,-3,-3,0,0,0,0,0,0,6,0],$$

$$\mathbb{A}_0^{(III)} = \mathrm{diag}\{0,1,0,-1,0,0,-1,0,0,0,0,0,0,0,0,0\},$$

$$\mathbb{A}_1^{(III)} = \mathrm{diag}\{1,0,0,0,-1,0,0,1,0,0,0,0,0,0,0,0\},$$

$$\mathbb{A}_2^{(III)} = \mathrm{diag}\{-1,0,0,0,-1,0,0,1,0,0,0,0,0,0,0,0\},$$

$$\mathbb{A}_3^{(III)} = \mathrm{diag}\{0,-1,0,-1,0,0,1,0,0,0,0,0,0,0,0,0\},$$

$$\mathbb{A}_4^{(III)} = \mathrm{diag}\{0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0\},$$

$$\mathbb{S}^{(IV)} = [0,1,1,0,1,1,-3,-3,0,0,0,0,0,0,0,6],$$

$$\mathbb{A}_0^{(IV)} = \mathrm{diag}\{0,0,1,0,-1,0,0,-1,0,0,0,0,0,0,0,0\},$$

$$\mathbb{A}_1^{(IV)} = \mathrm{diag}\{1,0,0,0,-1,0,0,1,0,0,0,0,0,0,0,0\},$$

$$\mathbb{A}_2^{(IV)} = \mathrm{diag}\{-1,0,0,0,-1,0,0,1,0,0,0,0,0,0,0,0\},$$

$$\mathbb{A}_3^{(IV)} = \mathrm{diag}\{0,-1,0,-1,0,0,1,0,0,0,0,0,0,0,0,0\},$$

$$\mathbb{A}_4^{(IV)} = \mathrm{diag}\{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1\}.$$

If $\mathbb{S}$ is considered as a 4 x 20 matrix and $\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \mathbb{A}_4$, as 20 x 16 matrices then all four subsidiary conditions can be accommodated in one matrix formula:

$$\mathbb{S}(\mathbb{A}_0\pi_0+\mathbb{A}_1\pi_1+\mathbb{A}_2\pi_2+\mathbb{A}_3\pi_3+\mathbb{A}_4\chi)(-\mathbb{L}_0\pi_0+\mathbb{L}_1\pi_1+\mathbb{L}_2\pi_2+\mathbb{L}_3\pi_3+\mathbb{1}\chi)\psi=0.$$

2: Conditions to be satisfied for the derivation of the subsidiary conditions of the second kind for any first order relativistic wave-equation.

Let us consider the first order relativistic wave-equation

$$\{-\mathbb{L}_0\pi_0+\mathbb{L}_1\pi_1+\mathbb{L}_2\pi_2+\mathbb{L}_3\pi_3+\mathbb{1}\chi\}\psi=0,$$

realized in an n-dimensional space and let us multiply it from the left by the expression

$$\mathbb{S}(\mathbb{A}_0\pi_0+\mathbb{A}_1\pi_1+\mathbb{A}_2\pi_2+\mathbb{A}_3\pi_3+\mathbb{A}_4\chi),$$

where $\mathbb{S}$ is an $1\times n$ matrix and $\mathbb{A}_l$, $l=0,1,2,3,4,$ are $n\times n$ matrices. We shall show that the expression

$$\mathbb{S}(\mathbb{A}_0\pi_0+\mathbb{A}_1\pi_1+\mathbb{A}_2\pi_2+\mathbb{A}_3\pi_3+\mathbb{A}_4\chi)(-\mathbb{L}_0\pi_0+\mathbb{L}_1\pi_1+\mathbb{L}_2\pi_2+\mathbb{L}_3\pi_3+\mathbb{1}\chi)\psi=0,$$

can become a subsidiary condition of the second kind when certain conditions are satisfied. Thus, let us rewrite it as follows:

$$(\mathbb{S}\mathbb{A}_0\pi_0+\mathbb{S}\mathbb{A}_1\pi_1+\mathbb{S}\mathbb{A}_2\pi_2+\mathbb{S}\mathbb{A}_3\pi_3+\mathbb{S}\mathbb{A}_4\chi)(-\mathbb{L}_0\pi_0+\mathbb{L}_1\pi_1+\mathbb{L}_2\pi_2+\mathbb{L}_3\pi_3+\mathbb{1}\chi)\psi=0.$$

The products $\mathbb{S}\mathbb{A}_l$, $l=0,1,2,3,4,$ are $1\times n$ matrices and can be identified with the row vectors $\vec{a}_l$, $l=0,1,2,3,4$. Hence we can write:

$$\begin{aligned}
&(\vec{a}_0\pi_0+\vec{a}_1\pi_1+\vec{a}_2\pi_2+\vec{a}_3\pi_3+\vec{a}_4\chi)(-\mathbb{L}_0\pi_0+\mathbb{L}_1\pi_1+\mathbb{L}_2\pi_2+\mathbb{L}_3\pi_3+\mathbb{1}\chi)\psi=\\
&=-\vec{a}_0\mathbb{L}_0\pi_0^2\psi+\vec{a}_0\mathbb{L}_1\pi_0\pi_1\psi+\vec{a}_0\mathbb{L}_2\pi_0\pi_2\psi+\vec{a}_0\mathbb{L}_3\pi_0\pi_3\psi+\vec{a}_0\pi_0\mathbb{1}\chi\psi-\\
&\quad-\vec{a}_1\mathbb{L}_0\pi_1\pi_0\psi+\vec{a}_1\mathbb{L}_1\pi_1^2\psi+\vec{a}_1\mathbb{L}_2\pi_1\pi_2\psi+\vec{a}_1\mathbb{L}_3\pi_1\pi_3\psi+\vec{a}_1\pi_1\mathbb{1}\chi\psi-\\
&\quad-\vec{a}_2\mathbb{L}_0\pi_2\pi_0\psi+\vec{a}_2\mathbb{L}_1\pi_2\pi_1\psi+\vec{a}_2\mathbb{L}_2\pi_2^2\psi+\vec{a}_2\mathbb{L}_3\pi_2\pi_3\psi+\vec{a}_2\pi_2\mathbb{1}\chi\psi-\\
&\quad-\vec{a}_3\mathbb{L}_0\pi_3\pi_0\psi+\vec{a}_3\mathbb{L}_1\pi_3\pi_1\psi+\vec{a}_3\mathbb{L}_2\pi_3\pi_2\psi+\vec{a}_3\mathbb{L}_3\pi_3^2\psi+\vec{a}_3\pi_3\mathbb{1}\chi\psi-\\
&\quad-\vec{a}_4\chi\mathbb{L}_0\pi_0\psi+\vec{a}_4\chi\mathbb{L}_1\pi_1\psi+\vec{a}_4\chi\mathbb{L}_2\pi_2\psi+\vec{a}_4\chi\mathbb{L}_3\pi_3\psi+\vec{a}_4\chi\mathbb{1}\chi\psi=0.
\end{aligned}$$

To find what conditions have to be satisfied for this expression to become a subsidiary condition of the second kind let us consider the various kinds of terms separately.

First let us consider the terms linear in π_i, $i=0,1,2,3$, i.e.

$$(-\vec{\alpha}_4\mathbb{L}_0+\vec{\alpha}_0)\pi_0\chi\psi \;,\; (\vec{\alpha}_4\mathbb{L}_1+\vec{\alpha}_1)\pi_1\chi\psi \;,\; (\vec{\alpha}_4\mathbb{L}_2+\vec{\alpha}_2)\pi_2\chi\psi \;,$$
$$(\vec{\alpha}_4\mathbb{L}_3+\vec{\alpha}_3)\pi_3\chi\psi \;.$$

We do not want such terms to appear in the subsidiary conditions of the second kind and we can make them disappear by imposing the conditions:

$$-\vec{\alpha}_4\mathbb{L}_0+\vec{\alpha}_0=0 \;,\; \vec{\alpha}_4\mathbb{L}_1+\vec{\alpha}_1=0 \;,\; \vec{\alpha}_4\mathbb{L}_2+\vec{\alpha}_2=0 \;,\; \vec{\alpha}_4\mathbb{L}_3+\vec{\alpha}_3=0 \;,$$

or

$$\vec{\alpha}_0=\vec{\alpha}_4\mathbb{L}_0 \;,\; \vec{\alpha}_1=-\vec{\alpha}_4\mathbb{L}_1 \;,\; \vec{\alpha}_2=-\vec{\alpha}_4\mathbb{L}_2 \;,\; \vec{\alpha}_3=-\vec{\alpha}_4\mathbb{L}_3 \;. \tag{1}$$

Then let us consider the terms involving π_i^2, $i=0,1,2,3$, i.e.

$$-\vec{\alpha}_0\mathbb{L}_0\pi_0^2\psi \;,\; \vec{\alpha}_1\mathbb{L}_1\pi_1^2\psi \;,\; \vec{\alpha}_2\mathbb{L}_2\pi_2^2\psi \;,\; \vec{\alpha}_3\mathbb{L}_3\pi_3^2\psi \,.$$

Again such terms must disappear and we achieve this by imposing the conditions:

$$\vec{\alpha}_0\mathbb{L}_0=0 \;,\; \vec{\alpha}_1\mathbb{L}_1=0 \;,\; \vec{\alpha}_2\mathbb{L}_2=0 \;,\; \vec{\alpha}_3\mathbb{L}_3=0 \;. \tag{2}$$

Finally with the terms involving $\pi_i\pi_j$ we deal as follows:

$$\begin{aligned}
\vec{\alpha}_0\mathbb{L}_1\pi_0\pi_1\psi-\vec{\alpha}_1\mathbb{L}_0\pi_1\pi_0\psi &= (\vec{\alpha}_0\mathbb{L}_1-\alpha_1\mathbb{L}_0)\pi_0\pi_1\psi-\vec{\alpha}_1\mathbb{L}_0f_{10}\psi \,,\\
\vec{\alpha}_0\mathbb{L}_2\pi_0\pi_2\psi-\vec{\alpha}_2\mathbb{L}_0\pi_2\pi_0\psi &= (\vec{\alpha}_0\mathbb{L}_2-\vec{\alpha}_2\mathbb{L}_0)\pi_0\pi_2\psi-\vec{\alpha}_2\mathbb{L}_0f_{20}\psi \,,\\
\vec{\alpha}_0\mathbb{L}_3\pi_0\pi_3\psi-\vec{\alpha}_3\mathbb{L}_0\pi_3\pi_0\psi &= (\vec{\alpha}_0\mathbb{L}_3-\vec{\alpha}_3\mathbb{L}_0)\pi_0\pi_3\psi-\vec{\alpha}_3\mathbb{L}_0f_{30}\psi \,,\\
\vec{\alpha}_1\mathbb{L}_2\pi_1\pi_2\psi+\vec{\alpha}_2\mathbb{L}_1\pi_2\pi_1\psi &= (\vec{\alpha}_1\mathbb{L}_2+\vec{\alpha}_2\mathbb{L}_1)\pi_1\pi_2\psi+\vec{\alpha}_2\mathbb{L}_1f_{21}\psi \,,\\
\vec{\alpha}_1\mathbb{L}_3\pi_1\pi_3\psi+\vec{\alpha}_3\mathbb{L}_1\pi_3\pi_1\psi &= (\vec{\alpha}_1\mathbb{L}_3+\vec{\alpha}_3\mathbb{L}_1)\pi_1\pi_3\psi+\vec{\alpha}_3\mathbb{L}_1f_{31}\psi \,,\\
\vec{\alpha}_2\mathbb{L}_3\pi_2\pi_3\psi+\vec{\alpha}_3\mathbb{L}_2\pi_3\pi_2\psi &= (\vec{\alpha}_2\mathbb{L}_3+\vec{\alpha}_3\mathbb{L}_2)\pi_2\pi_3\psi+\vec{\alpha}_3\mathbb{L}_2f_{32}\psi \,,
\end{aligned}$$

where we have used $[\pi_k\,,\pi_\ell]_-=\iota eF_{k\ell}\equiv f_{k\ell}$.

Again the terms:

$$(\vec{\alpha}_0 \mathbb{L}_1 - \alpha_1 \mathbb{L}_0)\pi_0\pi_1\psi \quad , (\vec{\alpha}_0 \mathbb{L}_2 - \vec{\alpha}_2 \mathbb{L}_0)\pi_0\pi_2\psi , (\vec{\alpha}_0 \mathbb{L}_3 - \vec{\alpha}_3 \mathbb{L}_0)\pi_0\pi_3\psi ,$$

$$(\vec{\alpha}_1 \mathbb{L}_2 + \vec{\alpha}_2 \mathbb{L}_1)\pi_1\pi_2\psi \quad , (\vec{\alpha}_1 \mathbb{L}_3 + \vec{\alpha}_3 \mathbb{L}_1)\pi_1\pi_3\psi , (\vec{\alpha}_2 \mathbb{L}_3 + \vec{\alpha}_3 \mathbb{L}_0)\pi_2\pi_3\psi ,$$

should vanish, and we achieve this by imposing the conditions:

$$\begin{aligned} &\vec{\alpha}_0 \mathbb{L}_1 - \vec{\alpha}_1 \mathbb{L}_0 = 0 \ , \quad \vec{\alpha}_0 \mathbb{L}_2 - \vec{\alpha}_2 \mathbb{L}_0 = 0 \ , \quad \vec{\alpha}_0 \mathbb{L}_3 - \vec{\alpha}_3 \mathbb{L}_0 = 0 \ , \\ &\vec{\alpha}_1 \mathbb{L}_2 + \vec{\alpha}_2 \mathbb{L}_1 = 0 \ , \quad \vec{\alpha}_1 \mathbb{L}_3 + \vec{\alpha}_3 \mathbb{L}_1 = 0 \ , \quad \vec{\alpha}_2 \mathbb{L}_3 + \vec{\alpha}_3 \mathbb{L}_0 = 0 . \end{aligned} \qquad (3)$$

Hence we find that the general form of a subsidiary condition of the second kind for a first order relativistic wave-equation is:

$$\left(-\vec{\alpha}_1 \mathbb{L}_0 f_{10} - \vec{\alpha}_2 \mathbb{L}_0 f_{20} - \vec{\alpha}_3 \mathbb{L}_0 f_{30} + \vec{\alpha}_2 \mathbb{L}_1 f_{21} + \vec{\alpha}_3 \mathbb{L}_1 f_{31} + \vec{\alpha}_3 \mathbb{L}_2 f_{32} + \vec{\alpha}_4 \chi^2 \mathbb{1}\right)\psi = 0.$$

The above conditions (1), (2), (3) necessary for the derivation of the subsidiary conditions of the second kind can acquire other forms also. For instance taking the transpose of conditions (2) we find

$$\mathbb{L}_0^{tr}\vec{\alpha}_0^{tr} = 0 \ , \quad \mathbb{L}_1^{tr}\vec{\alpha}_1^{tr} = 0 \ , \quad \mathbb{L}_2^{tr}\vec{\alpha}_2^{tr} = 0 \ , \quad \mathbb{L}_3^{tr}\vec{\alpha}_3^{tr} = 0 \ , \qquad (2')$$

which say that $\vec{\alpha}_i^{tr}$, $i = 0, 1, 2, 3$, is an eigenvector of $\mathbb{L}_i^{tr}$ corresponding to zero eigenvalue. From these formulae we see immediately that the Dirac equation does not accept subsidiary conditions of the second kind because the matrices $\gamma_0^{tr}, \gamma_1^{tr}, \gamma_2^{tr}, \gamma_3^{tr}$, do not have zero eigenvalues. Also for the same reason the Bhabha equations for half integer spin do not accept subsidiary conditions of the second kind.

If we use conditions (1) conditions (2) acquire also the form:

$$(\mathbb{L}_0^{tr})^2\vec{\alpha}_4^{tr} = 0 \ , \quad (\mathbb{L}_1^{tr})^2\vec{\alpha}_4^{tr} = 0 \ , \quad (\mathbb{L}_2^{tr})^2\vec{\alpha}_4^{tr} = 0 \ , \quad (\mathbb{L}_3^{tr})^2\vec{\alpha}_4^{tr} = 0 \ , \qquad (2'')$$

and conditions (3) the form:

$$(\mathbb{L}_1^{tr}\mathbb{L}_0^{tr} + \mathbb{L}_0^{tr}\mathbb{L}_1^{tr})\vec{\alpha}_4^{tr} = 0 \ , \quad (\mathbb{L}_2^{tr}\mathbb{L}_0^{tr} + \mathbb{L}_0^{tr}\mathbb{L}_2^{tr})\vec{\alpha}_4^{tr} = 0 \ , \quad (\mathbb{L}_3^{tr}\mathbb{L}_0^{tr} + \mathbb{L}_0^{tr}\mathbb{L}_3^{tr})\vec{\alpha}_4^{tr} = 0 \ ,$$

$$(\mathbb{L}_2^{tr}\mathbb{L}_1^{tr} + \mathbb{L}_1^{tr}\mathbb{L}_2^{tr})\vec{\alpha}_4^{tr} = 0 \ , \quad (\mathbb{L}_3^{tr}\mathbb{L}_1^{tr} + \mathbb{L}_1^{tr}\mathbb{L}_3^{tr})\vec{\alpha}_4^{tr} = 0 \ , \quad (\mathbb{L}_3^{tr}\mathbb{L}_2^{tr} + \mathbb{L}_2^{tr}\mathbb{L}_3^{tr})\vec{\alpha}_4^{tr} = 0 \ .$$

$$(3')$$

Conditions (2") and (3') say that $\vec{\alpha}_4^{tr}$ must be an eigenvector corresponding to zero eigenvalue of the matrices $(\mathbb{L}_l^{tr})^2$, $l = 0, 1, 2, 3$, and also of the matrices $[\mathbb{L}_k^{tr}, \mathbb{L}_0^{tr}]_+$, $(k = 1, 2, 3)$, $[\mathbb{L}_m^{tr}, \mathbb{L}_1^{tr}]_+$, $(m = 2, 3)$, $[\mathbb{L}_3^{tr}, \mathbb{L}_2^{tr}]_+$ where $[\ ,\ ]_+$ indicates the anticommutator between two matrices. Conditions (3') acquire also the form:

$$(\mathbb{L}_0^{tr})^2\mathbb{B}_1^{tr}\vec{\alpha}_4^{tr} = 0\ ,\quad (\mathbb{L}_0^{tr})^2\mathbb{B}_2^{tr}\vec{\alpha}_4^{tr} = 0\ ,\quad (\mathbb{L}_0^{tr})^2\mathbb{B}_3^{tr}\vec{\alpha}_4^{tr} = 0\ ,$$

$$(\mathbb{L}_1^{tr})^2\mathbb{A}_3^{tr}\vec{\alpha}_4^{tr} = 0\ ,\quad (\mathbb{L}_1^{tr})^2\mathbb{A}_2^{tr}\vec{\alpha}_4^{tr} = 0\ ,\quad (\mathbb{L}_2^{tr})^2\mathbb{A}_1^{tr}\vec{\alpha}_4^{tr} = 0\ , \qquad (3")$$

where $\mathbb{A}_l$, $l = 1, 2, 3$, are the generators of rotations and $\mathbb{B}_l$, $l = 1, 2, 3$, the generators of boosts. Formulae (3") were obtained upon using $\mathbb{L}_k = -[\mathbb{B}_k, \mathbb{L}_0]_-$, $k = 1, 2, 3$, and also

$$\mathbb{L}_2 = -[\mathbb{L}_1, \mathbb{A}_3]_-\ ,\quad \mathbb{L}_3 = [\mathbb{L}_1, \mathbb{A}_2]_-\ ,\quad \mathbb{L}_3 = -[\mathbb{L}_2, \mathbb{A}_1]_-\ .$$

Finally conditions (2") and (3") say that $\vec{\alpha}_4^{tr}$ is an eigenvector of $(\mathbb{L}_l^{tr})^2$, $l = 0, 1, 2, 3$, corresponding to zero eigenvalue $\mathbb{B}_k^{tr}\vec{\alpha}_4^{tr}$, $k = 1, 2, 3$, are eigenvectors of $(\mathbb{L}_0^{tr})^2$ corresponding to zero eigenvalue, $\mathbb{A}_m^{tr}\vec{\alpha}_4^{tr}$, $m = 2, 3$, are eigenvectors of $(\mathbb{L}_1^{tr})^2$ corresponding to zero eigenvalue, and $\mathbb{A}_1^{tr}\vec{\alpha}_4^{tr}$ is an eigenvector of $(\mathbb{L}_2^{tr})^2$ corresponding to zero eigenvalue.

We give now examples of wave-equations which accept subsidiary conditions of the second kind and satisfy all the previous conditions.

Example 1. (Pauli-Fierz equation).

In the case of the Pauli-Fierz wave-equation conditions (1), (2"), (3") or the intermediate forms (2'), (3') are all satisfied as can be seen using the explicit form of the matrices $\mathbb{L}_l$, $l = 0, 1, 2, 3$, and the generators $\mathbb{A}_k, \mathbb{B}_k$, $k = 1, 2, 3$, expressed in the spinor basis $\{\alpha_{11}^{\dot 1}, \alpha_{12}^{\dot 1}, \alpha_{22}^{\dot 1}, \alpha_{11}^{\dot 2}, \alpha_{12}^{\dot 2}, \alpha_{22}^{\dot 2}, d^{\dot 1}, d^{\dot 2}, b_1^{\dot 1\dot 1}, b_1^{\dot 1\dot 2}, b_1^{\dot 2\dot 2}, b_2^{\dot 1\dot 1},$

$b_2^{\dot{1}\dot{2}}$, $b_2^{\dot{2}\dot{2}}, c_1, c_2 \}$ which turns out to be more convenient to work with.

The number of the subsidiary conditions of the second kind (i.e. four) is equal to the number of the spin 1/2 components i.e. c_1, c_2, $d^{\dot{1}}$, $d^{\dot{2}}$ which have to be constraint in order that the equation be a spin 3/2 wave-equation. This number is also equal to the number of the linearly independent eigenvectors $\vec{\alpha}_4^{tr}$ corresponding to the zero eigenvalues of the matrices $(\mathbb{L}_l^{tr})^2$, $l=0,1,2,3$, and such that they satisfy also conditions (3") i.e. $\mathbb{B}_k^{tr}\alpha_4^{tr}$, $k=1,2,3$ are eigenvectors of $(\mathbb{L}_0^{tr})^2$ corresponding to zero eigenvalue $\mathbb{A}_m^{tr}\vec{\alpha}_4^{tr}$, $m=2,3$, are eigenvectors of $(\mathbb{L}_1^{tr})^2$ corresponding to zero eigenvalue, and $\mathbb{A}_1^{tr}\vec{\alpha}_4^{tr}$ is an eigenvector of $(\mathbb{L}_2^{tr})^2$ corresponding to zero eigenvalue.

In the spinor basis the matrices $(\mathbb{L}_l^{tr})^2$, $l=0,1,2,3$, have zero rows and columns corresponding to the spin 1/2 components $d^{\dot{1}}, d^{\dot{2}}, c_1, c_2$, and then the eigenvectors $(\vec{\alpha}_4^{tr})_j$, $j=1,2,3,4$, of $(\mathbb{L}_l^{tr})^2$, $l=0,1,2,3$, corresponding to the zero eigenvalues and satisfying also the conditions (3") are:

$$
(\vec{\alpha}_4^{tr})_1 = \begin{array}{|c|l} \cline{1-1} 0 & \\ 0 & \\ 0 & \\ 0 & \\ 0 & \\ 0 & \\ \cline{1-1} 1 & \text{spin} \\ 0 & \tfrac{1}{2}\ \text{part} \\ \cline{1-1} 0 & \\ 0 & \\ 0 & \\ 0 & \\ 0 & \\ 0 & \\ \cline{1-1} 0 & \text{spin} \\ 0 & \tfrac{1}{2}\ \text{part} \\ \cline{1-1} \end{array}, \quad
(\vec{\alpha}_4^{tr})_2 = \begin{array}{|c|} \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline \end{array}, \quad
(\vec{\alpha}_4^{tr})_3 = \begin{array}{|c|} \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 1 \\ 0 \\ \hline \end{array}, \quad
(\vec{\alpha}_4^{tr})_4 = \begin{array}{|c|} \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 1 \\ \hline \end{array}.
$$

Notice that the blocks of the generators $\mathbb{A}_k^{tr}, \mathbb{B}_k^{tr}$, $k=1,2,3$ which have non-zero effect on the eigenvectors $\vec{\alpha}_4^{tr}$, when the products $\mathbb{A}_k^{tr}\vec{\alpha}_4^{tr}$, $\mathbb{B}_k^{tr}\vec{\alpha}_4^{tr}$ are constructed, are the ones corresponding to spin 1/2 and reproduce the eigenvectors $(\vec{\alpha}_4^{tr})_j$, $j=1,2,3,4$ with different order.

Example 2:

Let us now consider the 20-dim. spin 3/2 wave-equation with definite charge defined by the constants $B=\frac{1}{2\sqrt{2}}$, $C=-\frac{1}{2\sqrt{2}}$, $Z=-\frac{1}{2\sqrt{2}}$, $K=\frac{1}{2\sqrt{2}}$, $A=-\frac{1}{4}$, $\Gamma=\bar{\Gamma}=-\frac{1}{4}$, $\Theta=\frac{-1}{4}$, and based on the representation $(1/2,3/2)\oplus(-1/2,3/2)\oplus(1/2,5/2)\oplus(-1/2,5/2)\oplus(1/2,3/2)\oplus$ $\oplus(-1/2,3/2)$ (c.f. Chapter 13). For this example again all the conditions for the existence of the secondary constraints are satisfied as can be seen using the explicit form of the matrices $\mathbb{L}_l$, $l=0,1,2,3$, and the generators $\mathbb{A}_k, \mathbb{B}_k$, $(k=1,2,3)$ expressed for convenience in the spinor basis $\{\alpha_{11}^{\dot{1}}, \alpha_{12}^{\dot{1}}, \alpha_{22}^{\dot{1}}, \alpha_{11}^{\dot{2}}, \alpha_{12}^{\dot{2}}, \alpha_{22}^{\dot{2}}, d^{\dot{1}}, d^{\dot{2}}, \delta^{\dot{1}}, \delta^{\dot{2}}, b_1^{\dot{1}\dot{1}}, b_1^{\dot{1}\dot{2}}, b_1^{\dot{2}\dot{2}}, b_2^{\dot{1}\dot{1}}, b_2^{\dot{1}\dot{2}}, b_2^{\dot{2}\dot{2}}, c_1, c_2, \gamma_1, \gamma_2\}$.

The number of the subsidiary conditions of the second kind is eight and is equal to the number of the spin 1/2 components $d^{\dot{1}}, d^{\dot{2}}, \delta^{\dot{1}}, \delta^{\dot{2}}, c_1, c_2, \gamma_1, \gamma_2$, which need to be constraint in order that the equation be a spin 3/2 equation. This number is again equal to the number of the linearly independent eigenvectors $\vec{\alpha}_4^{tr}$ corresponding to the zero eigenvalues of $(\mathbb{L}_l^{tr})^2$, $l=0,1,2,3$, and associated with the spin 1/2 part of the matrices, and such that conditions (3") are satisfied also. The matrices $(\mathbb{L}_l^{tr})^2$, $l=0,1,2,3$, when considered in the spinor basis have zero rows and columns corresponding to the spin 1/2 components $d^{\dot{1}}, d^{\dot{2}}, \delta^{\dot{1}}, \delta^{\dot{2}}, c_1, c_2, \gamma_1, \gamma_2$, and their eigenvectors corresponding to zero eigenvalue and associated with the spin 1/2 part and satisfying (3") are:

$$(\vec{a}_4^{tr})_1 = \left[\begin{array}{c|l} 0 & \\ 0 & \\ 0 & \\ 0 & \\ 0 & \\ 0 & \\ \hline 1 & \text{spin} \\ 0 & 1/2 \\ \hline 0 & \text{spin} \\ 0 & 1/2 \\ \hline 0 & \\ 0 & \\ 0 & \\ 0 & \\ 0 & \\ 0 & \\ \hline 0 & \text{spin} \\ 0 & 1/2 \\ \hline 0 & \text{spin} \\ 0 & 1/2 \end{array}\right. , \quad (\vec{a}_4^{tr})_2 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 1 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] , \quad (\vec{a}_4^{tr})_3 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 1 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] , \quad (\vec{a}_4^{tr})_4 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] ,$$

$$(\vec{a}_4^{tr})_5 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 1 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] , \quad (\vec{a}_4^{tr})_6 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 1 \\ \hline 0 \\ 0 \end{array}\right] , \quad (\vec{a}_4^{tr})_7 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 1 \\ 0 \end{array}\right] , \quad (\vec{a}_4^{tr})_8 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 1 \end{array}\right] .$$

Notice again that the blocks of the generators $\mathbb{A}_k^{tr}$, $\mathbb{B}_k^{tr}$, $k=1,2,3$, which have non-zero effect on the eigenvectors $\vec{a}_4^{tr}$ when the products $\mathbb{A}_k^{tr}\vec{a}_4^{tr}$, $\mathbb{B}_k^{tr}\vec{a}_4^{tr}$ are constructed, are the ones corresponding to spin 1/2, and reproduce them with different order.

Another observation is that for both of the examples considered here $\mathbb{L}_0$ satisfies a minimal equation of the form

$$\mathbb{L}_0^2\left\{\mathbb{L}_0^2-1\right\}=0 .$$

3: On the number of subsidiary conditions of a first order relativistic wave equation.

The question arising now concerns the number of subsidiary conditions of a first order relativistic wave equation. Along these lines we state and prove below several relevant theorems.

Theorem 1: For the description of a spin S field associated with the first order relativistic wave equation $(\mathbb{L}^{\mu} p_{\mu} + \chi)\psi = 0$, $2(2S+1)$ field components are absolutely necessary.

Proof: For a wave equation to have physical meaning it is necessary that it describes charged particles. This is possible if a hermitianizing matrix A can be defined and this is the case if each irreducible τ appears together with its conjugate $\dot{\tau}$. Thus the minimum number of irreducibles necessary for a physically meaningful wave equation is two namely τ and $\dot{\tau}$. For a spin S particle the maximum weight of τ is S and the other weights are $S-1, S-2, S-3, \cdots, (-S+1), -S$. The corresponding field components are:

$$\underbrace{\psi^{\tau}_{S,S},\ \psi^{\tau}_{S,(S-1)},\ \psi^{\tau}_{S,(S-2)} \cdots \psi^{\tau}_{S,(-S+1)},\ \psi^{\tau}_{S,(-S)},}_{(2S+1)}$$

$$\underbrace{\psi^{\dot{\tau}}_{S,S},\ \psi^{\dot{\tau}}_{S,(S-1)},\ \psi^{\dot{\tau}}_{S,(S-2)} \cdots \psi^{\dot{\tau}}_{S,(-S+1)},\ \psi^{\dot{\tau}}_{S,-S},}_{(2S+1)}$$

and their total number is $2(2s+1)$. Any extra components used for the description of the field are constrained. For instance in the case of the Pauli-Fierz equation for spin 3/2, 16 components are used but only eight are absolutely necessary. The rest are constrained by eight relations.

Theorem 2: In a relativistic wave equation of the first order describing half integer spin particles and based on a representation of the form

$$\begin{tikzcd}
\tau_1 \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \tau_2 \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \tau_3 \arrow[d, leftrightarrow] & \cdots\cdots & \tau_l \arrow[d, leftrightarrow] \\
\dot{\tau}_1 \arrow[r, leftrightarrow] & \dot{\tau}_2 \arrow[r, leftrightarrow] & \dot{\tau}_3 & \cdots\cdots & \dot{\tau}_l
\end{tikzcd} \qquad (I)$$

without repeated components, the number of the primary constraints is equal to the number of the linearly independent eigenvectors of $\mathbb{L}_0$ corresponding to the zero eigenvalues.

<u>Proof</u>: The matrix $\mathbb{L}_0$ in the canonical frame breaks into the blocks

$$\mathbb{L}_0^{S} \ , \quad \mathbb{L}_0^{S-1} \ , \ \cdots \ , \ \mathbb{L}_0^{1/2} \ .$$

Since the equation must describe spin S particles the eigenvalues of $\mathbb{L}_0^{S}$ must not vanish (in here comes the assumption of the above considered interlocking scheme) while the eigenvalues of all the blocks corresponding to lower spin must vanish. Bringing $\mathbb{L}_0$ to the Jordan form the diagonalizable block $\mathbb{L}_0^{S}$ will have its eigenvalues arranged on the main diagonal while the other blocks will have zeros on the main diagonal and units above the main diagonal their number depending on the multiplicities of the zero eigenvalues. For instance when $m(\mathbb{L}_0)=\mathbb{L}_0^2\{\mathbb{L}_0^2-\lambda^2\}=0$, $\mathbb{L}_0^{\text{Jordan}}$ is:

$$\mathbb{L}_0^{\text{Jordan}} = \left[\begin{array}{cccccc|cccc}
0 & 1 & & & & & & & & \\
0 & 0 & & & & & & & & \\
 & & 0 & 1 & & & & & & \\
 & & 0 & 0 & & & & & & \\
 & & & & 0 & 1 & & & & \\
 & & & & 0 & 0 & & & & \\
\hline
 & & & & & & \lambda & & & \\
 & & & & & & & -\lambda & & \\
 & & & & & & & & \lambda & \\
 & & & & & & & & & -\lambda
\end{array}\right] .$$

The zero rows of $\mathbb{L}_0^{\text{Jordan}}$ give rise to the primary constraints. Thus the primary constraints are equal in number to the zero rows of $\mathbb{L}_0^{\text{Jordan}}$. The number of zero rows of $\mathbb{L}_0^{\text{Jordan}}$ is equal to the null eigenvectors of $\mathbb{L}_0^{\text{Jordan}}$ and hence the number of the primary constraints is equal to the number of the linearly independent eigenvectors of $\mathbb{L}_0$, corresponding to the zero eigenvalues.

Theorem 3: For the existence of the subsidiary conditions of the second kind of the first order relativistic wave equation i.e.

$$(-\vec{\alpha}_1 \mathbb{L}_0 f_{10} - \vec{\alpha}_2 \mathbb{L}_0 f_{20} - \vec{\alpha}_3 \mathbb{L}_0 f_{30} + \vec{\alpha}_2 \mathbb{L}_1 f_{21} + \vec{\alpha}_3 \mathbb{L}_1 f_{31} + \vec{\alpha}_3 \mathbb{L}_2 f_{32} + \vec{\alpha}_4 \chi)\psi = 0,$$

it is necessary that the matrices $\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$, be all singular.

Proof: We have seen that for the derivation of the subsidiary conditions of the second kind it is necessary that the second order derivatives π_0^2, π_1^2, π_2^2, π_3^2, appearing in

$$(\vec{\alpha}_0 \pi_0 + \vec{\alpha}_1 \pi_1 + \vec{\alpha}_2 \pi_2 + \vec{\alpha}_3 \pi_3 + \vec{\alpha}_4 \chi)(-\mathbb{L}_0 \pi_0 + \mathbb{L}_1 \pi_1 + \mathbb{L}_2 \pi_2 + \mathbb{L}_3 \pi_3 + \mathbb{1}\chi)\psi = 0,$$

should vanish. This is possible if

$$\vec{\alpha}_0 \mathbb{L}_0 = 0 \quad , \quad \vec{\alpha}_1 \mathbb{L}_1 = 0 \quad , \quad \vec{\alpha}_2 \mathbb{L}_2 = 0 \quad , \quad \vec{\alpha}_3 \mathbb{L}_3 = 0.$$

Taking the transpose we have

$$\mathbb{L}_0^{tr} \vec{\alpha}_0^{tr} = 0 \quad , \quad \mathbb{L}_1^{tr} \vec{\alpha}_1^{tr} = 0, \quad \mathbb{L}_2^{tr} \vec{\alpha}_2^{tr} = 0 \quad , \quad \mathbb{L}_3^{tr} \vec{\alpha}_3^{tr} = 0.$$

From these relations we have that $\vec{\alpha}_0^{tr}$ is an eigenvector of $\mathbb{L}_0^{tr}$ corresponding to zero eigenvalue, $\vec{\alpha}_1^{tr}$ is an eigenvector of $\mathbb{L}_1^{tr}$ corresponding to zero eigenvalue, $\vec{\alpha}_2^{tr}$ is an eigenvector of $\mathbb{L}_2^{tr}$ corresponding to zero eigenvalue, $\vec{\alpha}_3^{tr}$ is an eigenvector of $\mathbb{L}_3^{tr}$ corresponding to zero eigenvalue. Thus it follows that $\det(\mathbb{L}_0^{tr}) = 0$, $\det(\mathbb{L}_1^{tr}) = 0$, $\det(\mathbb{L}_2^{tr}) = 0$, $\det(\mathbb{L}_3^{tr}) = 0$.

Remembering that for any matrix M, $\det M = \det M^{tr}$, we have

$$\det \mathbb{L}_0 = 0 \quad , \quad \det \mathbb{L}_1 = 0 \quad , \quad \det \mathbb{L}_2 = 0 \quad , \quad \det \mathbb{L}_3 = 0,$$

i.e. $\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$, are singular.

Theorem 4: In order that a first order relativistic wave equation accepts subsidiary conditions of the second kind it is necessary that the eigenvectors $\vec{\alpha}_4^{tr}$ be simultaneous eigenvectors of $(\mathbb{L}_0^{tr})^2, (\mathbb{L}_1^{tr})^2, (\mathbb{L}_2^{tr})^2, (\mathbb{L}_3^{tr})^2$, corresponding to zero eigenvalue.

Proof: This follows from the relations

$$(\mathbb{L}_0^{tr})^2 \vec{a}_4^{tr} = 0 \quad , \quad (\mathbb{L}_1^{tr})^2 \vec{a}_4^{tr} = 0 \; , \quad (\mathbb{L}_2^{tr})^2 \vec{a}_4^{tr} = 0 \quad , \quad (\mathbb{L}_3^{tr})^2 \vec{a}_4^{tr} = 0 \, ,$$

derived in § 2.

Theorem 5: The matrix $\mathbb{L}_0$ of a first order relativistic wave equation based on a representation of the form (I) acquires the following block form

$$\mathbb{L}_0 = \begin{array}{c} \begin{array}{cc} \frac{N}{2} & \frac{N}{2} \end{array} \\ \left[\begin{array}{c|c} 0 & \text{Non-zero} \\ \hline \text{Non-zero} & 0 \end{array} \right] \end{array} ,$$

where N the dimension of the entire representation space.

Proof: We shall prove this theorem by induction.

a) Thus we prove it first for a wave equation based on a representation of the form

$$\begin{tikzcd} \overset{(1/2\,,3/2)}{\tau_1} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \overset{(1/2\,,5/2)}{\tau_2} \arrow[d, leftrightarrow] \\ \underset{(-1/2\,,3/2)}{\dot{\tau}_1} \arrow[r, leftrightarrow] & \underset{(-1/2\,,5/2)}{\dot{\tau}_2} \end{tikzcd} .$$

Let us arrange the components τ_1 , τ_2 , $\dot{\tau}_1$, $\dot{\tau}_2$ in the following order τ_2 , $\dot{\tau}_1$, $\dot{\tau}_2$, τ_1 . Then the matrix $\mathbb{L}_0$ will have elements different from zero for interlocking components τ_i and hence it will be of the form

$$\begin{array}{c|cc|cc|} & \tau_2 & \dot{\tau}_1 & \dot{\tau}_2 & \tau_1 \\ \hline \tau_2 & 0 & 0 & c^{\tau_2 \dot{\tau}_2} & c^{\tau_2 \tau_1} \\ \dot{\tau}_1 & 0 & 0 & c^{\dot{\tau}_1 \dot{\tau}_2} & c^{\dot{\tau}_1 \tau_1} \\ \hline \dot{\tau}_2 & c^{\dot{\tau}_2 \tau_2} & c^{\dot{\tau}_2 \dot{\tau}_1} & 0 & 0 \\ \tau_1 & c^{\tau_1 \tau_2} & c^{\tau_1 \dot{\tau}_1} & 0 & 0 \\ \hline & \multicolumn{2}{c}{\underbrace{\qquad\qquad}_{\frac{N}{2}}} & \multicolumn{2}{c}{\underbrace{\qquad\qquad}_{\frac{N}{2}}} \end{array} .$$

b) Next we prove the same for a wave equation based on a representation of the form

$$\begin{tikzcd}
(1/2\,,3/2) & (1/2\,,5/2) & (1/2\,,7/2) \\
\tau_1 \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \tau_2 \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \tau_3 \arrow[d, leftrightarrow] \\
\dot{\tau}_1 \arrow[r, leftrightarrow] & \dot{\tau}_2 \arrow[r, leftrightarrow] & \dot{\tau}_3 \\
(-1/2\,,3/2) & (-1/2\,,5/2) & (-1/2\,,7/2)
\end{tikzcd} .$$

Arranging the components τ_i in the order:

τ_3 , $\dot{\tau}_2$, τ_1 , . and $\dot{\tau}_3$, τ_2 , $\dot{\tau}_1$

i.e. diagonally we have for $\mathbb{L}_0$ the form:

$$\mathbb{L}_0 = \begin{array}{c|ccc|ccc|}
 & \tau_3 & \dot{\tau}_2 & \tau_1 & \dot{\tau}_3 & \tau_2 & \dot{\tau}_1 \\
\hline
\tau_3 & 0 & 0 & 0 & c^{\tau_3\dot{\tau}_3} & c^{\tau_3\tau_2} & 0 \\
\dot{\tau}_2 & 0 & 0 & 0 & c^{\dot{\tau}_2\dot{\tau}_3} & c^{\dot{\tau}_2\tau_2} & c^{\dot{\tau}_2\dot{\tau}_1} \\
\tau_1 & 0 & 0 & 0 & 0 & c^{\tau_1\tau_2} & c^{\tau_1\dot{\tau}_1} \\
\hline
\dot{\tau}_3 & c^{\dot{\tau}_3\tau_3} & c^{\tau_3\dot{\tau}_2} & 0 & 0 & 0 & 0 \\
\tau_2 & c^{\tau_2\tau_3} & c^{\tau_2\dot{\tau}_2} & c^{\tau_2\tau_1} & 0 & 0 & 0 \\
\dot{\tau}_1 & 0 & c^{\dot{\tau}_1\dot{\tau}_2} & c^{\dot{\tau}_1\tau_1} & 0 & 0 & 0 \\
\hline
 & \multicolumn{3}{c}{\underbrace{\hspace{8em}}_{\frac{N}{2}}} & \multicolumn{3}{c}{\underbrace{\hspace{8em}}_{\frac{N}{2}}}
\end{array} .$$

c) We now prove the same for a wave equation based on a representation of the form

$$\begin{tikzcd}
(1/2\,,3/2) & (1/2\,,5/2) & (1/2\,,7/2) & (1/2\,,9/2) \\
\tau_1 \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \tau_2 \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \tau_3 \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \tau_4 \arrow[d, leftrightarrow] \\
\dot{\tau}_1 \arrow[r, leftrightarrow] & \dot{\tau}_2 \arrow[r, leftrightarrow] & \dot{\tau}_3 \arrow[r, leftrightarrow] & \dot{\tau}_4 \\
(-1/2\,,3/2) & (-1/2\,,5/2) & (-1/2\,,7/2) & (-1/2\,,9/2)
\end{tikzcd} .$$

Arranging the components τ_ι in the order

$$\tau_4\ ,\ \dot{\tau}_3\ ,\ \tau_2\ ,\ \dot{\tau}_1 \quad \text{and} \quad \dot{\tau}_4\ ,\ \tau_3\ ,\ \dot{\tau}_2\ ,\ \tau_1\ ,$$

(i.e. diagonally along a zig-zag path /\/) we have for $\mathbb{L}_0$ the following form

$$\mathbb{L}_0 = \begin{array}{c|c|c|c|c|c|c|c|c|}
 & \tau_4 & \dot{\tau}_3 & \tau_2 & \dot{\tau}_1 & \dot{\tau}_4 & \tau_3 & \dot{\tau}_2 & \tau_1 \\
\tau_4 & 0 & 0 & 0 & 0 & c^{\tau_4\dot{\tau}_4} & c^{\tau_4\tau_3} & 0 & 0 \\
\dot{\tau}_3 & 0 & 0 & 0 & 0 & c^{\dot{\tau}_3\dot{\tau}_4} & c^{\dot{\tau}_3\tau_3} & c^{\dot{\tau}_3\dot{\tau}_2} & 0 \\
\tau_2 & 0 & 0 & 0 & 0 & 0 & c^{\tau_2\tau_3} & c^{\tau_2\dot{\tau}_2} & c^{\tau_2\tau_1} \\
\dot{\tau}_1 & 0 & 0 & 0 & 0 & 0 & 0 & c^{\dot{\tau}_1\dot{\tau}_2} & c^{\dot{\tau}_1\tau_1} \\
\dot{\tau}_4 & c^{\dot{\tau}_4\tau_4} & c^{\dot{\tau}_4\dot{\tau}_3} & 0 & 0 & 0 & 0 & 0 & 0 \\
\tau_3 & c^{\tau_3\tau_4} & c^{\tau_3\dot{\tau}_3} & c^{\tau_3\tau_2} & 0 & 0 & 0 & 0 & 0 \\
\dot{\tau}_2 & 0 & c^{\dot{\tau}_2\dot{\tau}_3} & c^{\dot{\tau}_2\tau_2} & c^{\dot{\tau}_2\dot{\tau}_1} & 0 & 0 & 0 & 0 \\
\tau_1 & 0 & 0 & c^{\tau_1\tau_2} & c^{\tau_1\dot{\tau}_1} & 0 & 0 & 0 & 0
\end{array} .$$

d) The general case follows from the observation that arranging the components τ_ι on two zig-zag paths /\/\/ , \/\/\ , of length $\frac{N}{2}$ each, has as a consequence that the two $\frac{N}{2} \times \frac{N}{2}$ blocks of $\mathbb{L}_0$ along the main diagonal vanish because the components τ_ι constituting them (i.e. those on the corners of each zig-zag path) do not interlock. The two other blocks of $\mathbb{L}_0$ are non-zero. Thus $\mathbb{L}_0$ has the form

$$\mathbb{L}_0 = \begin{array}{c|c}
N/2 & N/2 \\
0 & \text{Non-zero} \\
\hline
\text{Non-zero} & 0
\end{array} .$$

<u>Theorem 6</u>: The matrices $\mathbb{L}_1$, $\mathbb{L}_2$, $\mathbb{L}_3$, of a first order relativistic wave equation based on the representation (I) acquire the following block form:

$$\mathbb{L}_\iota = \begin{array}{c|c}
N/2 & N/2 \\
0 & \text{Non-zero} \\
\hline
\text{Non-zero} & 0
\end{array} , \qquad \iota = 1\ ,\ 2\ ,\ 3\ .$$

<u>**Proof:**</u> The matrices $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$, are given in terms of $\mathbb{L}_0$ and the boost generators $\mathbb{B}_1$, $\mathbb{B}_2$, $\mathbb{B}_3$, by the formulae:

$$\mathbb{L}_1 = [\mathbb{L}_0, \mathbb{B}_1]_-, \quad \mathbb{L}_2 = [\mathbb{L}_0, \mathbb{B}_2]_- \, . \quad \mathbb{L}_3 = [\mathbb{L}_0, \mathbb{B}_3]_- \, .$$

The generators $\mathbb{B}_1$, $\mathbb{B}_2$, $\mathbb{B}_3$, can be written in block form as follows

$$\mathbb{B}_i = \left[\begin{array}{c|c|c|c|c} \text{Non-zero} & & & & \\ \hline & \text{Non-zero} & & & \\ \hline & & \ddots & & \\ \hline & & & \text{Non-zero} & \\ \hline & & & & \text{Non-zero} \end{array}\right], \qquad i = 1, 2, 3 \, .$$

Then $\mathbb{L}_i = [\mathbb{L}_0, \mathbb{B}_i]_- = \mathbb{L}_0\mathbb{B}_i - \mathbb{B}_i\mathbb{L}_0$ is made up of:

$$\mathbb{L}_0\mathbb{B}_i = \begin{array}{c} \begin{array}{cc} N/2 & N/2 \end{array} \\ \left[\begin{array}{c|c} 0 & \text{Non-zero} \\ \hline \text{Non-zero} & 0 \end{array}\right] \end{array} \left[\begin{array}{c|c|c|c|c} \text{Non-zero} & & & & \\ \hline & \text{Non-zero} & & & \\ \hline & & \ddots & & \\ \hline & & & \text{Non-zero} & \\ \hline & & & & \text{Non-zero} \end{array}\right] = \begin{array}{c} \begin{array}{cc} N/2 & N/2 \end{array} \\ \left[\begin{array}{c|c} 0 & \text{Non-zero} \\ \hline \text{Non-zero} & 0 \end{array}\right] \end{array},$$

and also

$$\mathbb{B}_i\mathbb{L}_0 = \left[\begin{array}{c|c|c|c|c} \text{Non-zero} & & & & \\ \hline & \text{Non-zero} & & & \\ \hline & & \ddots & & \\ \hline & & & \text{Non-zero} & \\ \hline & & & & \text{Non-zero} \end{array}\right] \begin{array}{c} \begin{array}{cc} N/2 & N/2 \end{array} \\ \left[\begin{array}{c|c} 0 & \text{Non-zero} \\ \hline \text{Non-zero} & 0 \end{array}\right] \end{array} = \begin{array}{c} \begin{array}{cc} N/2 & N/2 \end{array} \\ \left[\begin{array}{c|c} 0 & \text{Non-zero} \\ \hline \text{Non-zero} & 0 \end{array}\right] \end{array}.$$

Hence $\mathbb{L}_l=[\mathbb{L}_0,\mathbb{B}_l]_-$ will be of the form

$$\mathbb{L}_l=\begin{array}{c|c} N/2 & N/2 \\ \left[\begin{array}{c|c} 0 & \text{Non-zero} \\ \hline \text{Non-zero} & 0 \end{array}\right] \end{array}, \quad l=1,\ 2,\ 3.$$

Theorem 7: Let us consider a first order relativistic wave equation based on a representation of the form (I), describing spin s particles, with $\mathbb{L}_0$ singular and satisfying a minimal equation of the form $\mathbb{L}_0^2\{\mathbb{L}_0^2-1\}=0$. Then $\mathbb{L}_0^2$ will be of the form

$$\mathbb{L}_0^2=\begin{array}{c|c|c|c|c|c|c|c|c|c|c} & \tau_l & \dot{\tau}_{l-1} & \tau_{l-2} & \dot{\tau}_{l-3} & \cdots & \dot{\tau}_l & \tau_{l-1} & \dot{\tau}_{l-2} & \tau_{l-3} & \cdots \\ \hline \tau_l & \text{Non-zero} & & & & & & & & & \\ \hline \dot{\tau}_{l-1} & & 0 & & & & & & & & \\ \hline \tau_{l-2} & & & 0 & & & & & & & \\ \hline \dot{\tau}_{l-3} & & & & 0 & & & & & & \\ \hline \vdots & & & & & \ddots & & & & & \\ \hline \dot{\tau}_l & & & & & & \text{Non-zero} & & & & \\ \hline \tau_{l-1} & & & & & & & 0 & & & \\ \hline \dot{\tau}_{l-2} & & & & & & & & 0 & & \\ \hline \tau_{l-3} & & & & & & & & & 0 & \\ \hline \vdots & & & & & & & & & & \ddots \end{array}.$$

Proof: The matrix $\mathbb{L}_0$ of a first order wave equation describing particles up to a maximum spin s acquires the blocks

$$\mathbb{L}_0^s\ ,\ \mathbb{L}_0^{s-1}\ ,\ \mathbb{L}_0^{s-2}\ ,\ \ldots,\ \mathbb{L}_0^{1/2}\ .$$

For spin s particles the eigenvalues of all the lower blocks should vanish while those of $\mathbb{L}_0^s$ be different from zero. In the case of the representation under consideration, $\mathbb{L}_0^{s}$ is always diagonalizable and the only kind of minimal equation that can satisfy is $\{(\mathbb{L}_0^{s})^2-1\}=0$. Since by assumption $\mathbb{L}_0$ satisfies $\mathbb{L}_0^{2}\{\mathbb{L}_0^{2}-1\}=0$, then the blocks $\mathbb{L}_0^{s-1}, \mathbb{L}_0^{s-2}, \ldots,$ $\mathbb{L}_0^{1/2}$, if they do not vanish identically they must satisfy minimal equations of the form

$$(\mathbb{L}_0^{s-1})^2=0\ ,\ (\mathbb{L}_0^{s-2})^2=0\ ,\ \ldots,\ (\mathbb{L}_0^{1/2})^2=0\ .$$

Thus $\mathbb{L}_0$ acquires the Jordan canonical form

$$\mathbb{L}_0^{\text{Jordan}} = \begin{array}{cccc} \overbrace{\qquad}^{1/2}\cdots & \overbrace{\qquad}^{s-2} & \overbrace{\qquad}^{s-1} & \overbrace{\qquad}^{s} \\ \left(\begin{array}{c|c|c|c} \begin{matrix}0&1&&&\\0&0&&&\\&&0&1&\\&&0&0&\\&&&&\ddots\end{matrix} & & & \\ \hline & \begin{matrix}0&1&&&\\0&0&&&\\&&0&1&\\&&0&0&\\&&&&\ddots\end{matrix} & & \\ \hline & & \begin{matrix}0&1&&&\\0&0&&&\\&&0&1&\\&&0&0&\\&&&&\ddots\end{matrix} & \\ \hline & & & \begin{matrix}1&&&&\\&-1&&&\\&&1&&\\&&&-1&\\&&&&\ddots\end{matrix} \end{array}\right) \end{array} .$$

Taking the square of this we have

$$\left(\mathbb{L}_0^{\text{Jordan}}\right)^2 = \begin{array}{cccc} \overbrace{\qquad}^{1/2}\cdots & \overbrace{\qquad}^{s-2} & \overbrace{\qquad}^{s-1} & \overbrace{\qquad}^{s} \\ \left(\begin{array}{c|c|c|c} \begin{matrix}0&&&&\\&0&&&\\&&0&&\\&&&0&\\&&&&\ddots\end{matrix} & & & \\ \hline & \begin{matrix}0&&&&\\&0&&&\\&&0&&\\&&&0&\\&&&&\ddots\end{matrix} & & \\ \hline & & \begin{matrix}0&&&&\\&0&&&\\&&0&&\\&&&0&\\&&&&\ddots\end{matrix} & \\ \hline & & & \begin{matrix}1&&&&\\&1&&&\\&&1&&\\&&&1&\\&&&&\ddots\end{matrix} \end{array}\right) \end{array} .$$

Thus the only block of $\left(\mathbb{L}_0^{\text{Jordan}}\right)^2$ which is different from zero is the one corresponding to the highest spin.

Let us now consider $\mathbb{L}_0$ with respect to the basis:

$$\underbrace{\tau_l\,,\ \dot{\tau}_{l-1}\,,\ \tau_{l-2}\,,\ \dot{\tau}_{l-3}\cdots}_{N/2}\ ,\ \underbrace{\dot{\tau}_l\,,\ \tau_{l-1}\,,\ \dot{\tau}_{l-2}\,,\ \tau_{l-3}\cdots}\ .$$

In this basis $\mathbb{L}_0$ has the form

$$\mathbb{L}_0 = \begin{array}{cc} N/2 & N/2 \\ \left[\begin{array}{c|c} 0 & \text{Non-zero} \\ \hline \text{Non-zero} & 0 \end{array}\right] \end{array},$$

and so $\mathbb{L}_0^2$ will be of the form

$$\mathbb{L}_0^2 = \begin{array}{cc} N/2 & N/2 \\ \left[\begin{array}{c|c} \text{Non-zero} & 0 \\ \hline 0 & \text{Non-zero} \end{array}\right] \end{array}.$$

Of the non-zero blocks of $\mathbb{L}_0^2$, only matrix elements corresponding to the spin s blocks $\tau_l(s)$, $\dot{\tau}_l(s)$, should not vanish, while all the other matrix elements must be zero because if any other elements were different from zero the matrix $\mathbb{L}_0$ would not satisfy a minimal equation of the form $\mathbb{L}_0^2\{\mathbb{L}_0^2 - 1\} = 0$, which in turn it means that in $\left(\mathbb{L}_0^{\text{Jordan}}\right)^2$ certain units would be appearing in the blocks corresponding to the lower spin blocks violating the assumed form of the minimal equation. Thus $\mathbb{L}_0^2$ has the form claimed by the theorem.

Corollary: $\left(\mathbb{L}_0^{tr}\right)^2$ has form

$$\begin{array}{c|c|c|c|c|} & \tau_l(s) & \cdots & \dot{\tau}_l(s) & \cdots \\ \hline \tau_l(s) & \text{Non-zero} & & & \\ \hline & & \ddots & & \\ \hline \dot{\tau}_l(s) & & & \text{Non-zero} & \\ \hline & & & & \ddots \end{array}$$

This follows from the previous theorem by taking the transpose of $\mathbb{L}^2_0$.

Theorem 8: Let us consider a first order relativistic wave equation based on a representation of the form (I) and describing spin S particles with matrix $\mathbb{L}_0$ singular and satisfying a minimal equation of the form $\mathbb{L}^2_0\{\mathbb{L}^2_0 - 1\} = 0$. Then $(\mathbb{L}^{tr}_0)^2$ has $\{N - 2(2S+1)\}$ linearly independent eigenvectors corresponding to zero eigenvalue.

Proof: Putting $\mathbb{L}_0$ into the Jordan canonical form, taking its transpose and squaring it we have

$$(\mathbb{L}^{tr}_0)^2 = \begin{array}{c|c} \overbrace{\qquad\qquad\qquad}^{dim = N-2(2S+1)} & \overbrace{\qquad\qquad}^{dim = 2(2S+1)} \\ \begin{matrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & \ddots \end{matrix} & \\ \hline & \begin{matrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{matrix} \end{array} .$$

From this we have immediately that the zero block acquires $N - 2(2S+1)$ linearly independent eigenvectors corresponding to zero eigenvalue.

Theorem 9: Let us consider a first order relativistic wave equation based on a representation of the form (I) describing spin S particles with matrix $\mathbb{L}_0$ singular and satisfying a minimal equation of the form $\mathbb{L}^2_0\{\mathbb{L}^2_0 - 1\} = 0$. Then $(\mathbb{L}^{tr}_0)^2$ accepts $N - \dim(\tau_1(S) \oplus \dot{\tau}_1(S))$ linearly independend eigenvectors $\vec{\alpha}^{tr}_4$ of the following form:

$$\begin{array}{l}
\tau_l(s) \\ \\ \\
\dot{\tau}_{l-1}(s-1) \\ \\ \\ \\
\tau_{l-2}(s-2) \\ \\ \\ \\
 \\
\dot{\tau}_l(s) \\ \\ \\
\tau_{l-1}(s-1) \\ \\ \\ \\
\dot{\tau}_{l-2}(s-2) \\ \\ \\
 \\
\end{array}
\underbrace{
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \vdots \end{bmatrix},
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \vdots \end{bmatrix},
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \vdots \end{bmatrix}
\cdots
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \vdots \end{bmatrix}
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \vdots \end{bmatrix}
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \vdots \end{bmatrix}
\cdots
}_{\left(N - \dim\left(\tau_l(s) \oplus \dot{\tau}_l(s)\right)\right)}
\qquad (A)$$

corresponding to zero eigenvalue.

Proof: From the corollary of theorem 7 we have that $(\mathbb{L}_0^{tr})^2$ accepts the form :

$$\left(\mathbb{L}_0^{tr}\right)^2 = \begin{array}{c|c|c|c|c|c|c|c|c|}
 & \tau_l(s) & \dot{\tau}_{l-1}(s-1) & \tau_{l-2}(s-2) & \cdots & \dot{\tau}_l(s) & \tau_{l-1}(s-1) & \dot{\tau}_{l-2}(s-2) & \cdots \\
\tau_l(s) & \text{Non-zero} & 0 & 0 & & & & & \\
\dot{\tau}_{l-1}(s-1) & 0 & 0 & 0 & & & & & \\
\tau_{l-2}(s-2) & 0 & 0 & 0 & & & & & \\
\vdots & & & & \ddots & & & & \\
\dot{\tau}_l(s) & & & & & \text{Non-zero} & 0 & 0 & \\
\tau_{l-1}(s-1) & & & & & 0 & 0 & 0 & \\
\dot{\tau}_{l-2}(s-2) & & & & & 0 & 0 & 0 & \\
\vdots & & & & & & & & \ddots \\
 & \multicolumn{4}{c}{\underbrace{\qquad\qquad\qquad\qquad}_{N/2\ -\dim\tau_l(s)}} & \multicolumn{4}{c}{\underbrace{\qquad\qquad\qquad\qquad}_{N/2\ -\dim\dot{\tau}_l(s)}}
\end{array} .$$

From this form we see that $\left(\mathbb{L}_0^{tr}\right)^2$ accepts $N-\dim(\tau_l(s)\oplus\dot{\tau}_l(s))$ linearly independent eigenvectors $\vec{\alpha}_4^{tr}$ of the form (A) corresponding to zero eigenvalue.

Theorem 10: Let us consider a first order relativistic wave equation based on a representation of the form (I) describing spin s particles with matrix $\mathbb{L}_0$ singular and satisfying a minimal equation of the form $\mathbb{L}_0^2\{\mathbb{L}_0^2-1\}=0$. If the matrices $\left(\mathbb{L}_1^{tr}\right)^2$, $\left(\mathbb{L}_2^{tr}\right)^2$, $\left(\mathbb{L}_3^{tr}\right)^2$ have the form

$$\begin{array}{c|c|c|c|c|c|c|c|c|}
 & \tau_l(s) & \dot{\tau}_{l-1}(s-1) & \tau_{l-2}(s-2) & \cdots & \dot{\tau}_l(s) & \tau_{l-1}(s-1) & \dot{\tau}_{l-2}(s-2) & \cdots \\
\tau_l(s) & \text{Non-zero} & 0 & 0 & & & & & \\
\dot{\tau}_{l-1}(s-1) & 0 & 0 & 0 & & & & & \\
\tau_{l-2}(s-2) & 0 & 0 & 0 & & & & & \\
\vdots & & & & \ddots & & & & \\
\dot{\tau}_l(s) & & & & & \text{Non-zero} & 0 & 0 & \\
\tau_{l-1}(s-1) & & & & & 0 & 0 & 0 & \\
\dot{\tau}_{l-2}(s-2) & & & & & 0 & 0 & 0 & \\
\vdots & & & & & & & & \ddots
\end{array} ,$$

with the non-zero blocks of one matrix at least non-singular or if singular with the eigenvectors associated with these blocks corresponding to zero eigenvalue being not simultaneous eigenvectors of $\{\mathbb{L}^{tr}_{\mu}\}^2$, $\mu = 0,1,2,3$, then the number of subsidiary conditions involving no derivatives is equal to the set of eigenvectors(A) i.e. $N - \dim(\tau_l(s) \oplus \dot{\tau}_l(s))$ (where $N =$ dimension of entire representation space).

<u>Proof</u>: What we have to prove is that the eigenvectors $\vec{\alpha}^{tr}_4$, of the set(A) which are eigenvectors of $(\mathbb{L}^{tr}_0)^2$ corresponding to zero eigenvalue satisfy the following relations

$$(\mathbb{L}^{tr}_0)^2\vec{\alpha}^{tr}_4 = 0\ ,\ (\mathbb{L}^{tr}_1)^2\vec{\alpha}^{tr}_4 = 0,\ (\mathbb{L}^{tr}_2)^2\vec{\alpha}^{tr}_4 = 0\ ,\ (\mathbb{L}^{tr}_3)^2\vec{\alpha}^{tr}_4 = 0,$$

$$(\mathbb{L}^{tr}_0)^2\mathbb{B}^{tr}_1\vec{\alpha}^{tr}_4 = 0\ ,\ (\mathbb{L}^{tr}_0)^2\mathbb{B}^{tr}_2\vec{\alpha}^{tr}_4 = 0\ ,\ (\mathbb{L}^{tr}_0)^2\mathbb{B}^{tr}_3\vec{\alpha}^{tr}_4 = 0,$$

$$(\mathbb{L}^{tr}_1)^2\mathbb{A}^{tr}_3\vec{\alpha}^{tr}_4 = 0\ ,\ (\mathbb{L}^{tr}_1)^2\mathbb{A}^{tr}_2\vec{\alpha}^{tr}_4 = 0\ ,\ (\mathbb{L}^{tr}_2)^2\mathbb{A}^{tr}_1\vec{\alpha}^{tr}_4 = 0,$$

necessary for the existence of the subsidiary conditions of the second kind (i.e. involving no derivatives).

These relations were derived in § 2.

From theorem (9) we have that $(\mathbb{L}^{tr}_0)^2$ accepts $N - \dim(\tau_l(s) \oplus \dot{\tau}_l(s))$ linearly independent eigenvectors $\vec{\alpha}^{tr}_4$ of the form given by(A), corresponding to zero eigenvalue. From the assumed form of $(\mathbb{L}^{tr}_l)^2$, $l = 1,2,3$, we see that these eigenvectors form also eigenvectors of $(\mathbb{L}^{tr}_l)^2$, $l = 1,2,3$, corresponding to zero eigenvalue. Thus the relations

$$(\mathbb{L}^{tr}_0)^2\vec{\alpha}^{tr}_4 = 0,\ (\mathbb{L}^{tr}_1)^2\vec{\alpha}^{tr}_4 = 0\ ,\ (\mathbb{L}^{tr}_2)^2\vec{\alpha}^{tr}_4 = 0\ ,\ (\mathbb{L}^{tr}_3)^2\vec{\alpha}^{tr}_4 = 0,$$

are satisfied. Next we prove that the eigenvectors(A) satisfy also the relations:

$$(\mathbb{L}^{tr}_0)^2\mathbb{B}^{tr}_1\vec{\alpha}^{tr}_4 = 0\ ,\ (\mathbb{L}^{tr}_0)^2\mathbb{B}^{tr}_2\vec{\alpha}^{tr}_4 = 0,\ (\mathbb{L}^{tr}_0)^2\mathbb{B}^{tr}_3\vec{\alpha}^{tr}_4 = 0,$$

$$(\mathbb{L}^{tr}_1)^2\mathbb{A}^{tr}_3\vec{\alpha}^{tr}_4 = 0\ ,\ (\mathbb{L}^{tr}_1)^2\mathbb{A}^{tr}_2\vec{\alpha}^{tr}_4 = 0,\ (\mathbb{L}^{tr}_2)^2\mathbb{A}^{tr}_1\vec{\alpha}^{tr}_4 = 0.$$

The matrices $\mathbb{B}_l^{tr}, \mathbb{A}_l^{tr}$, $l=1,2,3$, i.e. the transpose of the generators $\mathbb{B}_l$, $\mathbb{A}_l$, can be written in block diagonal form. From the form of $(\mathbb{L}_\mu^{tr})^2$, $\mu=0,1,2,3$, and that of $\mathbb{B}_l^{tr}, \mathbb{A}_l^{tr}$, we have that the matrices:

$$\begin{array}{lll} (\mathbb{L}_0^{tr})^2\mathbb{B}_1^{tr}, & (\mathbb{L}_0^{tr})^2\mathbb{B}_2^{tr}, & (\mathbb{L}_0^{tr})^2\mathbb{B}_3^{tr}, \\ (\mathbb{L}_1^{tr})^2\mathbb{A}_3^{tr}, & (\mathbb{L}_1^{tr})^2\mathbb{A}_2^{tr}, & (\mathbb{L}_2^{tr})^2\mathbb{A}_1^{tr}, \end{array} \qquad (1)$$

would be of the form

$$\begin{array}{c|c|c|c|c|c|c|c|c} & \tau_l & \dot{\tau}_{l-1} & \tau_{l-2} & \cdots & \dot{\tau}_l & \tau_{l-1} & \dot{\tau}_{l-2} & \cdots \\ \hline \tau_l & \text{Non-zero} & 0 & 0 & & & & & \\ \dot{\tau}_{l-1} & 0 & 0 & 0 & & & & & \\ \tau_{l-2} & 0 & 0 & 0 & & & & & \\ \vdots & & & & \ddots & & & & \\ \dot{\tau}_l & & & & & \text{Non-zero} & 0 & 0 & \\ \tau_{l-1} & & & & & 0 & 0 & 0 & \\ \dot{\tau}_{l-2} & & & & & 0 & 0 & 0 & \\ \vdots & & & & & & & & \ddots \end{array} .$$

Hence the eigenvectors $\vec{\alpha}_4^{tr}$ of the set(A) are also eigenvectors of the matrices (1) corresponding to zero eigenvalue. Their number is $N - \dim(\tau_l(s) \oplus \dot{\tau}_l(s))$. From the assumptions of the theorem there are no other eigenvectors $\vec{\alpha}_4^{tr}$ which are simultaneous eigenvectors of $(\mathbb{L}_\mu^{tr})^2$, $\mu=0,1,2,3$, corresponding to zero eigenvalue.

——— . ———

In theorem (3) we saw that for the existence of the subsidiary conditions of the second kind it is necessary that the matrices $\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$, be singular. This leads to the following question. Given $\mathbb{L}_0$ singular does it follow that $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$, are also singular! This for instance is true: 1) for the spin 3/2 Pauli-Fierz equation, 2) for the Kemmer equation for spins 1 and 0 respectively, and 3) for the 20 dimensional spin 3/2 equation with definite charge given in chapter 13. We were able also to prove the same for every wave equation with matrix $\mathbb{L}_0$ singular based on the 16 dimensional representation. Namely we have proved the following theorem.

<u>Theorem 11</u>. If the matrix $\mathbb{L}_0$ of the first order relativistic wave equation based on the sixteen dimensional representation

$$\begin{tikzcd}
(\frac{1}{2}, \frac{3}{2}) \sim \tau_1 \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \tau_2 \sim (\frac{1}{2}, \frac{5}{2}) \arrow[d, leftrightarrow] \\
(-\frac{1}{2}, \frac{3}{2}) \sim \dot{\tau}_1 \arrow[r, leftrightarrow] & \dot{\tau}_2 \sim (-\frac{1}{2}, \frac{5}{2})
\end{tikzcd} ,$$

is singular it follows that the matrices $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$, are also singular.

<u>Proof</u>: In connection with the above representation there are three possible cases for which $\mathbb{L}_0$ can be singular. Of these one is the Pauli-Fierz equation for which we know that $\mathbb{L}_i$, $i = 1, 2, 3$, are also singular. The other two cases have matrices $\mathbb{L}_0$ given in block form below:

1)

$$\mathbb{L}_0^{1/2} = \begin{bmatrix} 0 & -2\beta^2 & \beta & 0 \\ -2\beta^2 & 0 & 0 & \beta \\ -\beta & 0 & 0 & 1/2 \\ 0 & -\beta & 1/2 & 0 \end{bmatrix}, \quad \mathbb{L}_0^{3/2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

2)

$$\mathbb{L}_0^{1/2} = \begin{bmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & \beta \\ -\beta & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{bmatrix}, \quad \mathbb{L}_0^{3/2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Constructing the other matrices $\mathbb{L}_i = [\mathbb{L}_0, \mathbb{B}_i]_-$ and evaluating their determinants we find that they are all singular for all the values of the constants β, α.

The general case is not easy to prove because for higher dimensions $\det \mathbb{L}_i$ cannot be calculated.

4: The modified equation for a first order wave equation.

As we saw in chapters 10 and 11 for being able to study the propagation of the solutions of the Pauli-Fierz equation in an external electromagnetic field by employing the method of characteristics it was necessary to modify the original wave equation by means of the subsidiary conditions into a new equation of motion with matrices $\mathbb{L}_\mu(F_{\kappa\lambda})$, $\mu,\kappa,\lambda=0,1,2,3$, non-singular. The question which now arises is the following; can the same method be employed and give information about the propagation of the solutions of the general first order relativistic wave equation in an external electromagnetic field? For this purpose let us consider

$$(-\mathbb{L}_0\pi_0 + \mathbb{L}_1\pi_1 + \mathbb{L}_2\pi_2 + \mathbb{L}_3\pi_3 + \chi)\psi = 0 \,. \tag{1}$$

Multiplying it from the left by the matrix operator

$$S^\rho(\mathbb{A}_0^\rho\pi_0 + \mathbb{A}_1^\rho\pi_1 + \mathbb{A}_2^\rho\pi_2 + \mathbb{A}_3^\rho\pi_3)$$

we find after using the conditions of § 2 for the existence of the subsidiary conditions of the second kind,

$$\vec{\alpha}_0^\rho\pi_0\chi\psi + \vec{\alpha}_1^\rho\pi_1\chi\psi + \vec{\alpha}_2^\rho\pi_2\chi\psi + \vec{\alpha}_3^\rho\pi_3\chi\psi - \vec{\alpha}_1^\rho\mathbb{L}_0 f_{10}\psi - \alpha_2^\rho\mathbb{L}_0 f_{20}\psi - \vec{\alpha}_3^\rho\mathbb{L}_0 f_{30}\psi +$$
$$+ \vec{\alpha}_2^\rho\mathbb{L}_1 f_{21}\psi + \vec{\alpha}_3^\rho\mathbb{L}_2 f_{32}\psi = 0$$

where $S^\rho, \mathbb{A}_0^\rho, \mathbb{A}_1^\rho, \mathbb{A}_2^\rho, \mathbb{A}_3^\rho, \vec{\alpha}_0^\rho, \vec{\alpha}_1^\rho, \vec{\alpha}_2^\rho, \vec{\alpha}_3^\rho$, have the same meaning as in § 2 and the index ρ introduced runs over the subsidiary conditions of the second kind. Using $\vec{\alpha}_0^\rho = \vec{\alpha}_4^\rho\mathbb{L}_0$, $\vec{\alpha}_1^\rho = -\vec{\alpha}_4^\rho\mathbb{L}_1$, $\vec{\alpha}_2^\rho = -\vec{\alpha}_4^\rho\mathbb{L}_2$, $\vec{\alpha}_3^\rho = -\vec{\alpha}_4^\rho\mathbb{L}_3$, after some rearrangements we find for $\chi \neq 0$

$$\vec{\alpha}_4^\rho(-\mathbb{L}_0\pi_0\psi + \mathbb{L}_1\pi_1\psi + \mathbb{L}_2\pi_2\psi + \mathbb{L}_3\pi_3\psi) - \frac{1}{\chi}\vec{\alpha}_4^\rho\left(\begin{smallmatrix}\text{non-derivative}\\ \text{part}\end{smallmatrix}\right) = 0 \,.$$

Multiplying by $(\alpha_4^\rho)^{tr}$ and summing over ρ we have:

$$\sum_\rho(\vec{\alpha}_4^\rho)^{tr}\vec{\alpha}_4^\rho(-\mathbb{L}_0\pi_0\psi + \mathbb{L}_1\pi_1\psi + \mathbb{L}_2\pi_2\psi + \mathbb{L}_3\pi_3\psi) - \frac{1}{\chi}\sum_\rho(\vec{\alpha}_4^\rho)^{tr}\vec{\alpha}_4^\rho\left(\begin{smallmatrix}\text{non-derivative}\\ \text{part}\end{smallmatrix}\right) = 0. \tag{2}$$

The expression $\sum_\rho(\vec{\alpha}_4^\rho)^{tr}\alpha_4^\rho$, in the case that the eigenvectors $\vec{\alpha}_4^p$ have the form (A) of theorem (9) § 3, is a projection operator. Let us now consider the first order wave equation (1) and project out of it the part which will be modified. Thus (1) breaks into the following two parts:

$$\Big(1-\sum_p (\vec{\alpha}_4^p)^{tr}\vec{\alpha}_4^p\Big)\big(-\mathbb{L}_0\pi_0\psi+\mathbb{L}_1\pi_1\psi+\mathbb{L}_2\pi_2\psi+\mathbb{L}_3\pi_3\psi+\chi\psi\big)=0 \qquad (3)$$

(part which remains unmodified)

$$\sum_p (\vec{\alpha}_4^p)^{tr}\vec{\alpha}_4^p\big(-\mathbb{L}_0\pi_0\psi+\mathbb{L}_1\pi_1\psi+\mathbb{L}_2\pi_2\psi+\mathbb{L}_3\pi_3\psi+\chi\psi\big)=0, \qquad (4)$$

(part which will be modified)

Adding (2) and (4) we find after dividing through by the factor 2 and renaming the non derivative part:

$$\sum_p (\vec{\alpha}_4^p)^{tr}\vec{\alpha}_4^p\big(-\mathbb{L}_0\pi_0\psi+\mathbb{L}_1\pi_1\psi+\mathbb{L}_2\pi_2\psi+\mathbb{L}_3\pi_3\psi\big)+\Big(\begin{matrix}\text{non-derivative}\\ \text{part}\end{matrix}\Big)=0. \qquad (5)$$

Considering a set of orthonormal basis vectors and expanding ψ as follows:

$$\psi=\sum_{j=1}^{k}(\vec{\beta}^j\psi)(\vec{\beta}^j)^{tr}+\sum_{p=k+1}^{N}(\vec{\alpha}_4^p\psi)(\vec{\alpha}_4^p)^{tr},$$

where $\vec{\alpha}_4^p$ are the eigenvectors of $(\mathbb{L}_0^{tr})^2$ corresponding to zero eigenvalue and belonging to the set (A) of eigenvectors and $\vec{\beta}^j$ are the remaining eigenvectors of $(\mathbb{L}_0^{tr})^2$ and N the dimension of the entire representation space, we have after substituting for ψ in (5)

$$\sum_p (\vec{\alpha}_4^p)^{tr}\vec{\alpha}_4^p(-\mathbb{L}_0\pi_0+\mathbb{L}_1\pi_1+\mathbb{L}_2\pi_2+\mathbb{L}_3\pi_3)\Big(\sum_{j=1}^{k}(\vec{\beta}^j\psi)(\vec{\beta}^j)^{tr}+\sum_{p=k+1}^{N}(\vec{\alpha}_4^p\psi)(\vec{\alpha}_4^p)^{tr}\Big)+$$

$$+\ (\text{non derivative part}) = 0. \qquad (6)$$

Replacing in (6) $\alpha_4^p\psi$ from the subsidiary conditions of the second kind i.e.

$$\alpha_4^p\psi=-\frac{1}{\chi^2}\big(-\vec{\alpha}_1^p\mathbb{L}_0f_{10}\psi-\vec{\alpha}_2^p\mathbb{L}_0f_{20}\psi-\vec{\alpha}_3^p\mathbb{L}_0f_{30}\psi+\vec{\alpha}_2^p\mathbb{L}_1f_{21}\psi+\vec{\alpha}_3^p\mathbb{L}_1f_{31}\psi+\vec{\alpha}_3^p\mathbb{L}_2f_{32}\psi\big),$$

we find for the modified part of the equation of motion:

$$\sum_p (\vec{\alpha}_4^p)^{tr}\vec{\alpha}_4^p(-\mathbb{L}_0\pi_0+\mathbb{L}_1\pi_1+\mathbb{L}_2\pi_2+\mathbb{L}_3\pi_3)\Big\{\sum_{j=1}^{k}(\vec{\beta}^j\psi)(\vec{\beta}^j)^{tr}+$$

$$+\sum_{p=k+1}^{N}\Big(-\frac{1}{\chi^2}\big(-\vec{\alpha}_1^p\mathbb{L}_0f_{10}\psi-\vec{\alpha}_2^p\mathbb{L}_0f_{20}\psi-\vec{\alpha}_3^p\mathbb{L}_0f_{30}\psi+\vec{\alpha}_2^p\mathbb{L}_1f_{21}\psi+\vec{\alpha}_3^p\mathbb{L}_1f_{31}\psi+\vec{\alpha}_3^p\mathbb{L}_2f_{32}\psi\big)(\vec{\alpha}_4^p)^{tr}\Big\}+$$

$$+\ (\text{non derivative part}) = 0. \qquad (7)$$

Thus (3) (the unmodified part) and (7) (the modified part) constitute the new equation of motion (modified equation).

Finally the only thing that can be said concerning the propagation of the solutions is that certain propagation velocities arising from the modified part of the equation depend on the external field, while those propagation velocities derived from the unmodified part are independent of the external field. No quantitative results can be obtained by considerations like the ones used above where the aim was to obtain the general modified equation and then apply the method of characteristics to obtain the propagation behaviour of the solutions.

Incidentally we mention that the modified equation obtained above is restricted to those cases for which $\left(\mathbb{L}_0^{tr}\right)^2$ is a diagonalizable matrix i.e. $\mathbb{L}_0$ satisfies a minimal equation of the form $\mathbb{L}_0^2\{\mathbb{L}_0^2-1\}=0$. We have investigated also the most general case and our conclusion again is that no quantitative results can be obtained (these calculations being very complicated we omit). Quantitative results can be obtained for individual cases of wave equations based on lower dimension representations because otherwise the calculations become very long and tedious.

EPILOGUE.

In this work we have undertaken the study of some aspects, such as are, the definitness of charge and the causal propagation in an external electromagnetic field, of higher spin relativistic wave-equations of the first order. As a result of our research we have reached the following conclusions.

In chapter one we concentrated on the Bhabha equation based on the 16 and 20-dimensional representations of the group $SO(4,1)$ describing particles of maximum spin $\frac{3}{2}$. By using the methods of Lie algebras we gave matrix representations of the basis elements $\Gamma(\vec{h}_{\alpha_1})$, $\Gamma(\vec{h}_{\alpha_2})$, $\Gamma(\vec{e}_{\pm\alpha_1})$, $\Gamma(\vec{e}_{\pm\alpha_2})$, $\Gamma(\vec{e}_{\pm(\alpha_1+\alpha_2)})$, $\Gamma(\vec{e}_{\pm(\alpha_1+2\alpha_2)})$, of the complex Lie algebra B_2 associated with the 16 and 20-dimensional representations of the group $SO(4,1)$. It was shown then that the matrices $\mathbb{L}_k$, $k=0,1,2,3$, appearing in the Bhabha equation can be expressed in terms of the basis elements of B_2 by the linear combinations

$$\mathbb{L}_0 = \iota\sqrt{3}\,\Gamma(\vec{e}_{\alpha_2}) + \iota\sqrt{3}\,\Gamma(\vec{e}_{-\alpha_2}) \,,$$

$$\mathbb{L}_1 = \iota\frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{\alpha_1}) - \iota\frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{-\alpha_1}) + \iota\frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{\alpha_1+2\alpha_2}) - \iota\frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{-\alpha_1-2\alpha_2}) \,,$$

$$\mathbb{L}_2 = -\frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{\alpha_1}) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{-\alpha_1}) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{\alpha_1+2\alpha_2}) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{-\alpha_1-2\alpha_2}) \,,$$

$$\mathbb{L}_3 = -6\iota\,\Gamma(\vec{h}_{\alpha_2}) \,.$$

These matrices were given explicitly. Then by means of the matrix $\mathbb{L}_0$ the harmitianizing matrix $\mathbb{A}$ was constructed using the formula

$$\mathbb{A} = \frac{1}{3}\mathbb{L}_0\{4(\mathbb{L}_0)^2 - 7\}.$$

Also, by finding the eigenvalues of the matrix $\mathbb{L}_0$, the masses $m_l = \frac{\chi}{\lambda_l}$ of the particles were determined. Finally it was shown that the hermitianizing matrix $\mathbb{A}$ satisfies the property $\mathbb{T}^+\mathbb{A}\mathbb{T}=\mathbb{A}$ for every transformation $\mathbb{T}$ of the group $SO(4,1)$ which belongs also to the group $SO(3,1)$.

In chapter two we were concerned with the charge of the Bhabha fields. It was verified that the charge of the Bhabha field for spin 3/2 based either on the 16 or 20-dimensional representations is indefinite. It was shown also that the charge of the Bhabha-Gupta field is indefinite. Then using the fact that for any spin the matrix $\mathbb{L}_0$ of the Bhabha wave-equation can be written in diagonal form with elements on the main diagonal being the values of the spin and using also the fact that the hermitianizing matrix $\mathbb{A}$ likewise can be written in diagonal form with elements given by the formula

$$f(x,s) = (-1)^{s-p-1/2},$$

we were able to prove that the charge of any Bhabha wave-equation for higher spin with half integer value is indefinite. Examples illustrating this were studied covering values of spin from $\frac{3}{2}$ to $\frac{11}{2}$. In the case of the Dirac wave-equation ($s=\frac{1}{2}$) the charge is definite.

In chapter three we studied the charge associated with the Gel'fand-Yaglom equation based on the 20-dimensional representation $(3/2, 5/2)\oplus(1/2, 5/2)\oplus(-1/2, 5/2)\oplus(-3/2, 5/2)$ and reached the following conclusions:

1. There is no example based on this representation with non-diagonalizable matrix $\mathbb{L}_0$ and definite charge.
2. There are examples which satisfy the Amar and Dozzio criterion of causal propagation and the criterion of Cox of second quantization but they have either indefinite charge ($\mathbb{L}_0$ diagonalizable) or the charge cannot be defined because $\mathbb{L}_0$ is not derivable from an invariant Lagrangian.

In chapter four we studied wave-equations for which the wave-function ψ transforms according to the 16 dimensional representation

$$(1/2\,,\,3/2) \oplus (-1/2\,,\,3/2) \oplus (1/2\,,\,5/2) \oplus (-1/2\,,\,5/2)\,,$$

and reached the following conclusions:

1. There is only one equation of higher spin 3/2 having definite charge and matrix $\mathbb{L}_0$ non-diagonalizable. This is the Pauli-Fierz equation (c.f. examples 4, 5, 6, 7).
2. There are equations which satisfy the criterion of Amar and Dozzio of causal propagation and the criterion of Cox of second quantization but the corresponding charge is indefinite, (c.f. examples 1, 2, 3, 11, 12).
3. There are equations which satisfy the criterion of Amar and Dozzio but they present difficulties as far as the charge is concerned because there are certain states corresponding to non-zero eigenvalues and vanishing charge densities respectively (c.f. examples 7, 9).
4. There is an equation with definite charge which satisfies the criteria of causal propagation and of

second quantization but it describes particles of spin 1/2. This is another form of the Dirac equation (c.f. example 10).

5. All the cases with diagonalizable matrix $\mathbb{L}_0$ except the case $\alpha \neq 0$, $\beta = 0$, $\gamma = 0$ (Dirac equation) give indefinite charge.

In chapter five we studied examples of the Gel'fand-Yaglom wave-equation based on the 20-dimensional representation

$$(1/2, 3/2) \oplus (-1/2, 3/2) \oplus (1/2, 5/2) \oplus (-1/2, 5/2) \oplus (1/2, 3/2) \oplus (-1/2, 3/2)$$

and reached the following conclusions:

1) a) There are examples of wave equations describing particles of spin 3/2 with matrix $\mathbb{L}_0$ non-diagonalizable and definite charge (c.f. examples 3, 6). These examples describe 20-dim. wave-equations.
   b) There are other examples of wave equations describing particles of spin 3/2 with matrix $\mathbb{L}_0$ non-diagonalizable and definite charge but they are 16-dimensional like the Pauli-Fierz equation, (c.f. examples 4, 5).

2) There are examples satisfying the Amar-Dozzio condition of causal propagation in the presence of an external electromagnetic field and the condition of Cox of second quantization but the charge is indefinite.

3) There are examples which satisfy the Amar-Dozzio condition as well as the condition of Cox with definite charge but they are based on the same irreducible representations as the Dirac equation.

4) There are two examples which satisfy the Amar-

Dozzio condition of causal propagation with matrix $\mathbb{L}_0$ non-diagonalizable and definite charge describing spin 1/2 particles (c.f. examples 11, 12). These examples are interesting because they show that the theorem of Udgaonkor concerning the charge of relativistic wave equations is partly correct.

5) There are examples with non-diagonalizable matrix $\mathbb{L}_0$ describing spin 1/2 and spin 3/2 particles existing simultaneously in the field and having the same sign of charge i.e. definite, (c.f. examples 18, 19, 20, 21, 22, 23, 24, 25).

6) As a general conclusion we have that there are no wave-equations based on the representations (I), (II), (III) studied in chapters (3, 4 and 5) describing particles of spin 3/2 only with matrix $\mathbb{L}_0$ non-diagonalizable satisfying a minimal equation of the form

$$m(\mathbb{L}_0) = \mathbb{L}_0 \prod_l \{\mathbb{L}^2 - \lambda_l^2\}^{\mu_l} = 0$$

$\mu_l \geqslant 2$, $\lambda_l \neq 0$, and having definite charge.

In chapter six we studied again the charge of the Bhabha wave-equations based on the group $SO(4,1)$ looking upon them as Gel'fand-Yaglom equations. Our conclusions are again that all the Bhabha wave equations for half integer spin based on the group $SO(4,1)$ have indefinite charge. The only exception is the Dirac wave-equation.

In chapter seven it was shown that examples (11) and (12) of chapter 5 describing spin 1/2 particles and satisfying minimal equations of the form $\mathbb{L}_0\{\mathbb{L}_0^2 - 1\}^2 = 0$ have definite charge and can be second quantized.

In chapter eight it was shown that the 20-dim. Capri-Shamaly equation for the description of spin-1 particles with unique mass is equivalent to the Gel'fand-Yaglom equation based on the same representation. This was done by finding the transformation formulae connecting the vector tensor basis $(H_\nu, G_{\mu\nu})$ of Capri and Shamaly to the canonical basis of Gel'fand and Yaglom.

It was also shown that in the case that the constants α, b, c, d, h, f, are such that the wave-equation has a matrix $\mathbb{L}_0$ satisfying a minimal equation of the form $\mathbb{L}_0^3\{\mathbb{L}_0^2-1\}=0$, no hermitianizing matrix can be found and hence no charge can be attributed to it. In the case the constants have the values $\alpha=0, b=0, d=0, f=\pm 2, h=\pm 1/2$, a hermitianizing matrix can be found but the resulting wave-equation has indefinite charge.

In chapter nine we gave a brief review of some of the most representative works associated with the propagation of the solutions of wave-equations in external fields. We gave also a criterion of hyperbolicity for a first order quasilinear system of partial differential equations which was used in chapters 10 and 11 to study the propagation of the Rarita-Schwinger and Pauli-Fierz equations in an external electromagnetic field.

In chapter ten a detailed analysis of the Rarita-Schwinger equation for spin 3/2 was given. Its matrices $\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$ and its generators $\mathbb{H}_\pm, \mathbb{H}_3, \mathbb{F}_\pm, \mathbb{F}_3$, were given explicitly. The commutation relations satisfied by the generators and also among the generators and matrices $\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$, were verified. $\mathbb{L}_0$ was shown to satisfy a minimal equation of the form $\mathbb{L}_0^2\{\mathbb{L}_0^2-1\}=0$. Then by suitably modifying the Rarita-Schwinger equation in the presence of the electromagnetic field by means of the subsidiary conditions of the second kind we were able to study its propagation behaviour by following two separate methods. Both methods showed that for certain values of the electromagnetic field the propagation is not causal.

In chapter eleven a detailed analysis of the Pauli-Fierz wave-equation for spin 3/2 was given and shown that the Rarita-Schwinger equation expressed in the vector spinor basis $\{\psi_\iota{}^\delta\}$, the Pauli-Fierz equation expressed in the canonical basis $\{\xi^{\tau}_{\ell m}\}$ and the Pauli-Fierz equation expressed in the spinor basis $\{\alpha^{\dot\alpha}_{\beta\gamma}, b^{\dot\alpha\dot\beta}_{\gamma}, c_\alpha, d^{\dot\alpha}\}$ are all equivalent. This was done by finding the similarity transformations connecting the different bases. Then by suitably modifying the Pauli-Fierz equation expressed in the spinor basis by means of the subsidiary conditions of the second kind and employing the method of characteristics its propagation behaviour in an external electromagnetic field was studied and found that is the same as for the Rarita-Schwinger equation i.e. for certain values of the electromagnetic field the propagation is not causal. Thus it seems that the original attempt of Pauli and Fierz to free the higher spin first order relativistic wave equations proposed by Dirac from the problems created by the electromagnetic field proposing their Lagrangian derivation was not successful since there are problems concerning the causal propagation of the solutions in an external electromagnetic field.

In chapter twelve the spinorial form of the general 16-dimensional Gel'fand-Yaglom equation (or generalized Pauli-Fierz equation) based on the representation $(1/2, 3/2) \oplus$ $\oplus (-1/2, 3/2) \oplus (1/2, 5/2) \oplus (-1/2, 5/2)$ was given and shown that there is only one spin 3/2 wave-equation namely the Pauli-Fierz equation which accepts subsidiary conditions of the second kind.

In chapter thirteen the spinorial form of the general 20-dimensional Gel'fand-Yaglom equation (or generalized Pauli-Fierz equation) based on the representation $(1/2, 3/2) \oplus (-1/2, 3/2) \oplus (1/2, 5/2) \oplus (-1/2, 5/2) \oplus (1/2, 3/2) \oplus (-1/2, 3/2)$, was given and those classes of equations accepting subsidiary conditions of the second kind were found. Finally an example of a 20-dimensional wave equation for spin 3/2 with definite charge was given for which the subsidiary conditions of the second kind can be determined by a method given in the chapter. The propagation of this example in external electromagnetic field was studied and found that it is the same as that of the Pauli-Fierz equation studied in chapter eleven.

In chapter fourteen we showed that the problem of finding the subsidiary conditions of the second kind which in chapters eleven and twelve was faced by employing spinor calculus can be formulated in matrix language. This was demonstrated for the Pauli-Fierz equation and for the 20-dimensional equation with definite charge mentioned in chapter thirteen. Finally, several theorems related to the number of the subsidiary conditions of a first order relativistic wave-equation were given.

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The present list of references is incomplete in that many other interesting books and papers we had to leave out for the sake of being brief.

CONTENTS.

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PART II