DYNAMICS AND KINEMATICS OF SYSTEMS CONSISTING OF SPHERICAL AND SPHEROIDAL BODIES

Dimitrios N. Papadakos

A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

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by

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A Thesis presented for the Degree of Doctor of Philosophy

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March 1981

To my wife and son

IOANNA and ΝΙΚΟΛΑΟΣ

'Εάν ταῖς γλώσσαις τῶν ἀνθρώπων λαλῶ καί τῶν ἀγγέλων, ἀγάπην δέ μή ἔχω, γέγονα χαλκός ἠχῶν ἤ κύμβαλον ἀλαλάζον.

Paul, Korinthians A, Ch. 13,V. 1

DECLARATION

Except where reference is made to the work of others, the research described in this thesis and the composition of the thesis are my own work. No part of this thesis has previously been submitted in application for a higher degree. I was admitted to the Faculty of Science of the University of St Andrews as a research student under Ordinance General No.12 on the 1st October, 1977. I was accepted as a candidate for the degree of Ph.D. on the 1st October, 1978, under Resolution of the University Court, 1967, No.1.

D.N. PAPADAKOS

CERTIFICATE

I certify that D.N. Papadakos has spent nine terms in research in the Department of Astronomy of the University of St Andrews, that he has fulfilled the conditions of Ordinance General No.12 and Senate Regulations under Resolution of the University Court, 1967, No.1, and that he is qualified to submit the accompanying thesis in application for the degree of Ph.D.

T.R. CARSON
Supervisor.

ABSTRACT

There are three principal aims of this work; firstly to derive the analytical expressions for the potential energy and the mutual gravitational attraction between two homogeneous or non-homogeneous oblate spheroids with coplanar equatorial planes; secondly, to construct and study the equations of motion of dynamical systems consisting of particles and rigid homogeneous spheroidal bodies whose equatorial planes are coplanar; thirdly, to investigate by numerical integration and compare the evolution of dynamical models of interacting galaxies. Two different types of dynamical models of galaxies were used in this work:

(a) galaxies consisting of gravitating particles,

(b) galaxies comprising of gravitationally interacting particles and heavy central rigid homogeneous oblate spheroids.

Chapter (2) and appendices (4),(5),(6) and (7) are an account of the method used to derive the expressions for the potential energy and the mutual gravitational attraction between two rigid bodies bounded by spheroidal surfaces with coplanar equatorial planes, when the densities of the bodies are either constant or inversely proportional to the square of the radial distance from the centres of the bodies. These expressions were at first obtained in the form of non-elementary integrals over the complete elliptic integrals of the first and second kind, a result due to the fact that the common volume between two ellipsoids cannot be expressed in finite terms. The evaluation of these integrals was achieved with the aid of MacLaurin's theorem by collapsing one of the spheroids to its confocal disc; their final form is that of rapidly convergent series in terms of the parameters which determine the shape and orientation of the spheroids. In the course of

obtaining the expressions mentioned above we derived some other useful formulae, for example, the formulae giving the gravitational attraction between two homoeoids and the potential and force law of a non-homogeneous oblate spheroid.

Having found the expressions for the gravitational potential and attraction between two spheroids we proceeded by constructing and solving the equations of motion of dynamical systems consisting of either particles or particles and rigid homogeneous oblate spheroids with coplanar equatorial planes [chapters (1) and (3), appendices (2) and (3)]. The solutions of the equations of motion - being non-algebraic - were obtained in the form of power series. In particular, for the first type of dynamical systems (particles only) they were found in the form of three different types of series. The first type is based on the development and implementation of generalized f and g series for the N-body problem, the second type is based on recurrent formulae used for the evaluation of the terms of the series and the last type of series is a power series in terms of $\ln\left(\frac{r'_{ij}\tau}{r_{ij}}\right)$ and polynomial of $\frac{r'_{ij}\tau}{r_{ij}}$. All the series mentioned above lend themselves easily to numerical calculations since their convergence, which was analytically proved, is a rapid one.

We concluded the present work by numerically integrating the equations of motion of dynamical models of pairs of interacting galaxies. The galaxies were of the types (a) and (b) mentioned previously. The numerical integration was performed with variable time steps in order to reduce computing time. A general method was developed for the precise evaluation of the length of the time step determined by the desired integration accuracy. The calculations were carried out to an extremely high degree of accuracy.

The most significant of the results of the numerical experiments on the evolution of dynamical models of interacting galaxies were:

(i) the development of long-lived spiral structure when galaxies of type (b) were used even for intrinsic velocities and impact parameters so high that galaxies of type (a) were unable to produce such structure,

(ii) the roughly periodic appearance and disappearance of the spiral structure,

(iii) the fact that in the early stages of their evolution the dynamical models take up rather long-lived triaxial ellipsoidal shapes.

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C O N T E N T S

INTRODUCTION

There are three principal aims of this work; firstly to derive the analytical expressions for the potential energy and the mutual gravitational attraction between two homogeneous or non-homogeneous oblate spheroids with coplanar equatorial planes; secondly, to construct and study the equations of motion of dynamical systems consisting of particles and rigid homogeneous spheroidal bodies whose equatorial planes are coplanar; thirdly, to investigate by numerical integration and compare the evolution of dynamical models of interacting galaxies. Two different types of dynamical models of galaxies were used in this work:

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We concluded the present work by numerically integrating the equations of motion of dynamical models of pairs of interacting galaxies. The galaxies were of the types (a) and (b) mentioned previously. The numerical integration was performed with variable time steps in order to reduce computing time. A general method was developed for the precise evaluation of the length of the time step determined by the desired integration accuracy. The calculations were carried out to an extremely high degree of accuracy.

The most significant of the results of the numerical experiments on the evolution of dynamical models of interacting galaxies were:

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(iii) the fact that in the early stages of their evolution the dynamical models take up rather long-lived triaxial ellipsoidal shapes.

CHAPTER I

Power Series Solution of the n-body Problem

The n-body problem can briefly be described as follows: We consider the motion of n-bodies ($n>1$), with such an internal mass distribution that they may be considered as particles, interacting according to the Newtonian law of gravitation and moving freely in a three dimensional Euclidean space (initially in any given manner). The n-body problem consists of describing the complete solution of the equations of motion of the above system.

The used conventional notation is given below:

The position vectors of the particles and their co-ordinates in a Cartesian right handed inertial system of reference are:

$$\underset{\sim}{r_i} \equiv (x_i, y_i, z_i), \ (i = 1, \cdots n).$$

The vector connecting the i^{th} and j^{th} $(i, j = 1, \cdots n, \ i \neq j)$ particles are:

$$\underset{\sim}{r_{ij}} \equiv \underset{\sim}{r_i} - \underset{\sim}{r_j} \ .$$

The masses of the particles are:

$$m_i \ , (i = 1, \cdots n).$$

Assuming the masses of the particles remain constant in time, the equations of motion have the form

$$\frac{d^2 \underset{\sim}{r_i}}{dt^2} = -k^2 \sum_{j=1, j \neq i}^{n} m_j \frac{\underset{\sim}{r_{ij}}}{r_{ij}^{3}} \ , \ (i = 1, \cdots n), \tag{1.1}$$

where $r_{ij} \equiv |\underset{\sim}{r_{ij}}|$ and k is the gravitational constant. In the above equations the units for the time, distance and mass are the mean solar day, the astronomical unit and the mass of the Sun correspondingly.

We can achieve a useful modification of equations (1.1) as follows: Expressing the distances in a unit so that

$$1 \text{ unit of distance} = S \text{ Astronomical units}, \tag{1.2}$$

the foregoing equation becomes

$$S^3 k^{-2} \frac{d^2 \underset{\sim}{r_i}}{dt^2} = - \sum_{\substack{j=1 \\ j \neq i}}^{n} m_j \frac{\underset{\sim}{r_{ij}}}{r_{ij}^{3}} \ , (i = 1, \cdots n), \tag{1.3}$$

simplification of which is obtained by introduction of a new independent variable τ defined by the relation

$$\tau = S^{-3/2} \cdot K(t-t_0), \tag{1.4}$$

where t_0 is an arbitrary initial epoch and S the constant appearing in the relation (1.2). It is obvious from equation (1.4) that in the new system of units, the unit of time satisfies the relation

$$1 \text{ unit of time} = S^{3/2} K^{-1} \text{ mean solar days} \tag{1.5}$$

It can easily be proved that the vectors

$$\underset{\sim}{r}_i^{[\nu]} \equiv \frac{d^\nu \underset{\sim}{r}_i}{dt^\nu}, \quad \underset{\sim}{r}_i^{(\nu)} \equiv \frac{d^\nu \underset{\sim}{r}_i}{d\tau^\nu}, \quad \nu \geqslant 1, \ (i=1,\cdots n),$$

are connected through the relation

$$\underset{\sim}{r}_i^{(\nu)} = \underset{\sim}{r}_i^{[\nu]} \cdot S^{3\nu/2} \cdot K^{-\nu}, \quad \nu \geqslant 1, \ (i=1,\cdots n).$$

Introduction of this relation into equation (1.3) yields

$$\underset{\sim}{r}_i^{(2)} = -\sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \frac{\underset{\sim}{r}_{ij}}{r_{ij}^3}, \quad (i=1,\cdots n). \tag{1.6}$$

In this equation the masses are expressed in solar masses, while the units, in which the distances and the time are expressed, satisfy the relations (1.2) and (1.5).

For the sake of convenience the notation

$$\underset{\sim}{r}_i' \equiv \underset{\sim}{r}_i^{(1)}, \quad \underset{\sim}{r}_i'' \equiv \underset{\sim}{r}_i^{(2)}, \quad \underset{\sim}{r}_i''' \equiv \underset{\sim}{r}_i^{(3)} \ldots, \ (i=1,\cdots n).$$

will be used through out this work.

The system (1.6) consists of n second order differential equations for the vectors $\underset{\sim}{r}_i$ $(i=1,\cdots n)$. The existence of a unique solution for this system is secured from the known in the theory of differential equations as "existence theorem". It will be proved in the appendix (A2) that this unique solution, considered as function of the time τ, has derivatives of all orders and hence the Taylor's series for every vector $\underset{\sim}{r}_i(\tau)$ $(i=1,\cdots n)$ can be formed in powers of the independent variable τ about some epoch point, say $\tau=0$, that is

$$\underset{\sim}{r}_i = \sum_{\nu=0}^{\infty} \underset{\sim}{r}_{i0}^{(\nu)} \frac{\tau^\nu}{\nu!}, \quad (i=1, \cdots n), \quad \tau \in (-\tau^*, \tau^*) \tag{1.7}$$

In this equation we interpret $\underset{\sim}{r}_{i0}^{(0)}$ $(i=1, \cdots n)$ to mean $r_i \big|_{\tau=0}$ $(i=1, \cdots n)$ and

$$\underset{\sim}{r}_{i0}^{(\nu)} \equiv \frac{d^\nu \underset{\sim}{r}_i}{d\tau^\nu} \bigg|_{\tau=0} \quad \nu \geq 1, \ (i=1, \cdots n) .$$

The proof that the foregoing series converges in an open interval $(-\tau^*, \tau^*)$ is given in the appendix (A2), in the same appendix it will be shown that the derivatives of every order of the position vectors $\underset{\sim}{r}_i$ $(i=1, \cdots n)$ can be expressed as functions of the position and velocity vectors of the particles. Evidently, formulae giving the derivatives $\underset{\sim}{r}_i^{(\nu)}$, $(i=1, \cdots n)$, $\nu \geq 1$ as functions of the vectors $\underset{\sim}{r}_i$ and $\underset{\sim}{r}_i'$ $(i=1, \cdot n)$ permit a numerical integration of the system (1.6) with the help of the power series (1.7).

In the method described above a number of problems appear, particularly the laboriousness of the derivation of the formulae and the time consuming nature of the numerical calculations. These considerations, though not related to the theoretical aspect of the whole problem, make the numerical integration almost impossible for systems containing more than 20 particles.

An alternative approximate numerical solution of the system (1.6) can be obtained using a slightly different method based on recursive formulae. This method, though avoiding laborious algebraic manipulations, is found to be useless for dynamical system consisting of more than 15 particles because of the amount of time consumed by the arithmetical calculations.

The two methods described above have the advantage of the high accuracy for time steps rather big (0.5 of the unit in use).

The possibility of developing another method of numerical integration is provided by the form of the equations of motion. Successive differentiations of equation (1.6) give the derivatives, of order higher than one, of the position vectors $\mathbf{r}_i$ $(i=1,\cdots n)$ with respect to τ. These derivatives, at $\tau=0$, are the coefficients of the terms of the power series solution (1.7) and therefore a step by step numerical integration is possible.

The formulae for the third, fourth and fifth derivatives are given below

$$\mathbf{r}_i''' = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{m_j}{r_{ij}^3}\left\{\frac{3\mathbf{r}_{ij}(\mathbf{r}_{ij}\cdot\mathbf{r}_{ij}')}{r_{ij}^2} - \mathbf{r}_{ij}'\right\}, \quad (i=1,\cdots n),$$

$$\mathbf{r}_i^{(4)} = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{m_j}{r_{ij}^3}\left\{3\mathbf{r}_{ij}\left(-\frac{5(\mathbf{r}_{ij}\cdot\mathbf{r}_{ij}')^2}{r_{ij}^4} + \frac{(\mathbf{r}_{ij}'\cdot\mathbf{r}_{ij}')}{r_{ij}^2} + \frac{(\mathbf{r}_{ij}\cdot\mathbf{r}_{ij}'')}{r_{ij}^2}\right) + 6\mathbf{r}_{ij}'\frac{(\mathbf{r}_{ij}\cdot\mathbf{r}_{ij}')}{r_{ij}^2} - \mathbf{r}_{ij}''\right\} \quad (i=1,\cdots n),$$

$$\begin{aligned}\mathbf{r}_i^{(5)} = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{m_j}{r_{ij}^3}\Bigg\{&\mathbf{r}_{ij}\left(\frac{3(\mathbf{r}_{ij}\cdot\mathbf{r}_{ij}''')}{r_{ij}^2} + \frac{9(\mathbf{r}_{ij}'\cdot\mathbf{r}_{ij}'')}{r_{ij}^2} - \frac{45(\mathbf{r}_{ij}\cdot\mathbf{r}_{ij}')(\mathbf{r}_{ij}\cdot\mathbf{r}_{ij}'')}{r_{ij}^4} - \right.\\ &\left. - \frac{45(\mathbf{r}_{ij}\cdot\mathbf{r}_{ij}')(\mathbf{r}_{ij}'\cdot\mathbf{r}_{ij}')}{r_{ij}^4} + \frac{105(\mathbf{r}_{ij}\cdot\mathbf{r}_{ij}')^3}{r_{ij}^6}\right) + \mathbf{r}_{ij}'\left(\frac{9(\mathbf{r}_{ij}'\cdot\mathbf{r}_{ij}')}{r_{ij}^2} + \right.\\ &\left.\frac{9(\mathbf{r}_{ij}\cdot\mathbf{r}_{ij}'')}{r_{ij}^2} - \frac{45(\mathbf{r}_{ij}\cdot\mathbf{r}_{ij}')^2}{r_{ij}^4}\right) + \mathbf{r}_{ij}''\frac{9(\mathbf{r}_{ij}\cdot\mathbf{r}_{ij}')}{r_{ij}^2} - \mathbf{r}_{ij}'''\Bigg\}, \quad (i=1,\cdots n).\end{aligned} \tag{1.8}$$

Clearly the second and third order derivatives are determined once all the position and velocity vectors are known, while the determination of the fourth and fifth derivatives requires the additional knowledge of the second and third order derivatives.

In the numerical applications we begin by truncating the series (1.7) after its sixth term and accepting an approximate solution of the form

$$\underset{\sim}{r}_i^* = \underset{\sim}{r}_{i0} + \underset{\sim}{r}'_{i0}\tau + \sum_{\nu=2}^{5} \underset{\sim}{r}_{i0}^{(\nu)} \frac{\tau^\nu}{\nu!}, \quad (i=1,\cdots n), \quad \tau \in (-\tau^*, \tau^*) \tag{1.9}$$

which, by differentiation with respect to , gives an approximate formula for the velocity vector

$$\underset{\sim}{r}_i^{*\prime} = \underset{\sim}{r}'_{i0} + \sum_{\nu=2}^{5} \underset{\sim}{r}_{i0}^{(\nu)} \frac{\tau^{\nu-1}}{(\nu-1)!}, \quad (i=1,\cdots n), \quad \tau \in (-\tau^*, \tau^*) \tag{1.10}$$

From a set of initial values $\underset{\sim}{r}_{i0}$ and $\underset{\sim}{r}'_{i0}$ $(i=1,\cdots n)$ the coefficients of the polynomials (1.9) and (1.10) can be computed with the help of equations (1.6) and (1.8). By inserting in the foregoing polynomials a value for the time , the position and velocity vectors $\underset{\sim}{r}_i$ and $\underset{\sim}{r}'_i$ $(i=1,\cdots n)$ can be calculated at a subsequent moment, say $\Delta\tau$. Taking these new values as initial conditions we repeat the process, thereby detailing the paths of the bodies in space.

The method of numerical integration described above, although it reduces drastically the time needed for the arithmetical calculations, fails to reduce the consumption of the time to a level which would permit the treatment of dynamical systems with number of particles greater than 100. However, the accuracy of the integration is very high.

The formulae (1.9) and (1.10) give numerical values as closer to the values of the true solution the smaller the time step of the integration becomes. Hence the required accuracy determines the length of the time step. More details about the determination of the length of the time step are given in appendix (A1).

An attempt to develop a method of numerical integration of the system (1.6), which keeps the accuracy within the same limits as the foregoing methods and at the same time considerably reduces the consumption of the time required for the arithmetical calculations lead us to the relations (1.39) and (1.40).

Since the second order derivative $\underset{\sim}{r}_i''$ $(i=1,\dots n)$ is given by the relation

$$\underset{\sim}{r}_i'' = -\sum_{\substack{j=1\\ j\neq i}}^{n} m_j r_{ij}^{-3} \underset{\sim}{r}_{ij}\,, \quad (i=1,\dots n)\,, \tag{1.11}$$

and the derivative $(r_{ij}^{-3})'$ $(i,j=1,\dots n, i\neq j)$ is equal to

$$(r_{ij}^{-3})' = -3 r_{ij}^{-3} \frac{r_{ij}'}{r_{ij}}\,, \quad (i\neq j\,,\ i,j=1,\dots n)\,, \tag{1.12}$$

it can be proved with the help of mathematical induction that the following general relation holds for every derivative $\underset{\sim}{r}_i^{(k)}$, $(i=1,\dots n)$, $k\geqslant 2$

$$\underset{\sim}{r}_i^{(k)} = \sum_{\substack{j=1\\ j\neq i}}^{n} m_j r_{ij}^{-3} \sum_{\nu=0}^{k-2} \alpha_{ijk\nu}\, \underset{\sim}{r}_{ij}^{(\nu)}\,, \quad (i=1,\dots n)\,, \quad k\geqslant 2\,, \tag{1.13}$$

where $\alpha_{ijk\nu}$ $(i,j=1,\dots n,\ i\neq j)$, $k\geqslant 2$, $\nu=0,1,\dots k-2$ are functions of the variable T the form of which depends on $\underset{\sim}{r}_i, \underset{\sim}{r}_j$ and the order of the derivatives $\underset{\sim}{r}_i^{(k)}$ and $\underset{\sim}{r}_{ij}^{(\nu)}$. In the foregoing equation we interpret $\underset{\sim}{r}_{ij}^{(0)}$ to mean $\underset{\sim}{r}_{ij}$.

Introducing the above expression into relation

$$\underset{\sim}{r}_i^{(k+1)} \equiv \left(\underset{\sim}{r}_i^{(k)}\right)', \quad (i=1,\dots n)\,, \quad k\geqslant 2\,,$$

we obtain

$$\sum_{\substack{j=1\\ j\neq i}}^{n} m_j r_{ij}^{-3}\left\{\alpha_{ij\,k+1,0}\,\mathbf{r}_{ij} + \alpha_{ij\,k+1,k-1}\,\mathbf{r}_{ij}^{(k-1)} + \sum_{\nu=1}^{k-2}\alpha_{ij\,k+1,\nu}\,\mathbf{r}_{ij}^{(\nu)}\right\} \equiv$$

$$\equiv \sum_{\substack{j=1\\ j\neq i}}^{n} m_j r_{ij}^{-3}\left\{-3\frac{r'_{ij}}{r_{ij}}\sum_{\nu=1}^{k-2}\alpha_{ijk\nu}\,\mathbf{r}_{ij}^{(\nu)} + \sum_{\nu=1}^{k-2}\alpha'_{ijk\nu}\,\mathbf{r}_{ij}^{(\nu)} + \sum_{\nu=1}^{k-2}\alpha_{ijk,\nu-1}\,\mathbf{r}_{ij}^{(\nu)} - \right.$$

$$\left. -3\frac{r'_{ij}}{r_{ij}}\alpha_{ijk,0}\,\mathbf{r}_{ij} + \alpha'_{ijk,0}\,\mathbf{r}_{ij} + \alpha_{ijk,k-2}\,\mathbf{r}_{ij}^{(k-1)}\right\} \quad (i=1,\cdots n),\ k\geqslant 2 .$$

Since this relation is an identity we can equate the coefficients of the derivatives $\mathbf{r}_{ij}^{(\nu)}$ of the same order to obtain

$$\alpha_{ij\,k+1,k-1} = \alpha_{ijk,k-2}\ ,\quad k\geqslant 2,\ (i,j=1,\cdots n,\ i\neq j),$$

$$\alpha_{ij\,k+1,0} = -3\frac{r'_{ij}}{r_{ij}}\alpha_{ijk,0} + \alpha'_{ijk,0}\ ,\quad k\geqslant 2,\ (i,j=1,\cdots n,\ i\neq j), \tag{1.14}$$

$$\alpha_{ij\,k+1,\nu} = -3\frac{r'_{ij}}{r_{ij}}\alpha_{ijk,\nu} + \alpha'_{ijk,\nu} + \alpha_{ijk,\nu-1}\ ,\ k\geqslant 3,\ \nu=1,\cdots k-2,\ (i,j=1,\cdots n,\ i\neq j).$$

The first of these equations implies the relation

$$\alpha_{ij2,0} = \alpha_{ijk,k-2} \qquad k\geqslant 2 \quad (i,j=1,\cdots n,\ i\neq j). \tag{1.15}$$

The value of $\alpha_{ij2,0}$ can be found with the help of equation (1.13), which for $k=2$ becomes

$$\mathbf{r}''_i = \sum_{\substack{j=1\\ j\neq i}}^{n} m_j r_{ij}^{-3}\,\alpha_{ij2,0}\,\mathbf{r}_{ij}\ ,\quad (i=1,\cdots n). \tag{1.16}$$

Comparing equations (1.11), (1.15) and (1.16) we obtain the relation

$$\alpha_{ijk,k-2} = -1\ ,\quad k\geqslant 2\ ,\ (i,j=1,\cdots n\ \ i\neq j). \tag{1.17}$$

We consider now the second of equations (1.14), which may be rewritten in the form

$$\alpha_{ij\,k+1,0} = \frac{(\alpha_{ijk,0}\,r_{ij}^{-3})'}{r_{ij}^{-3}}\ ,\quad k\geqslant 2 \quad (i,j=1,\cdots n,\ i\neq j). \tag{1.18}$$

Let us assume that the equation

$$\alpha_{ijk,0} = -\frac{\left(r_{ij}^{-3}\right)^{(k-2)}}{r_{ij}^{-3}}, \quad (i,j=1,\cdots n, \; i\neq j), \tag{1.19}$$

is valid for an arbitrary k, then with the help of equation (1.18) it can easily be proved that the following relation also holds

$$\alpha_{ij\,k+1,0} = -\frac{\left(r_{ij}^{-3}\right)^{(k-1)}}{r_{ij}^{-3}}, \quad (i,j=1,\cdots n, \; i\neq j).$$

The validity of equation (1.19) for $k=2$ and $k=3$ can be seen at once from equations (1.17) and (1.18). Therefore, according to mathematical induction, this relation is true for every $k\geqslant 2$, that is

$$\alpha_{ijk,0} = -\frac{\left(r_{ij}^{-3}\right)^{(k-2)}}{r_{ij}^{-3}}, \quad k\geqslant 2, \; (i,j=1,\cdots n, \; i\neq j). \tag{1.20}$$

In the foregoing equation we interpret $\left(r_{ij}^{-3}\right)^{(0)}$ to mean r_{ij}^{-3} $(i,j=1,\cdots n, \; i\neq j)$.

Up to this point we found the expressions giving the functions $\alpha_{ijk,k-2}$ and $\alpha_{ijk,0}$, $(i,j=1,\cdots n, \; i\neq j)$, $k\geqslant 2$. We can find the rest of these functions as follows: The last of equations (1.14), in the particular case where $\nu=k-2$, becomes

$$\alpha_{ij\,k+1,k-2} - \alpha_{ij\,k\,k-3} = -3\frac{r_{ij}'}{r_{ij}}\alpha_{ijk,k-2} + \alpha_{ijk,k-2}', \quad k\geqslant 3 \; (i,j=1,\cdots n, \; i\neq j).$$

which, upon substitution of equation (1.17) and (1.20), takes the form

$$\alpha_{ij\,k+1,k-2} - \alpha_{ij\,k,k-3} = \alpha_{ij3,0}, \quad k\geqslant 3 \; (i,j=1,\cdots n, i\neq j).$$

Appling this relation in sequence for $k=3,4,\cdots(l-1)$ we obtain

$$\begin{aligned}
\alpha_{ij4,1} - \alpha_{ij3,0} &= \alpha_{ij3,0}, \\
\alpha_{ij5,2} - \alpha_{ij4,1} &= \alpha_{ij3,0}, \\
&\;\;\vdots \\
\alpha_{ij\,l-1,l-4} - \alpha_{ij\,l-2,l-5} &= \alpha_{ij3,0}, \\
\alpha_{ij\,l,l-3} - \alpha_{ij\,l-1,l-4} &= \alpha_{ij3,0}.
\end{aligned} \qquad (i,j=1,\cdots n, \; i\neq j)$$

Adding these equations we get

$$\alpha_{ij\,\ell,\ell-3} = (\ell-2)\,\alpha_{ij\,3,0}\;,\quad \ell \geq 3 \quad (i,j=1,\cdots n,\; i\neq j)\,.$$

In order to preserve the notation used up to this point, we introduce the index K instead of ℓ in the foregoing relation to obtain

$$\alpha_{ij\,K,K-3} = (K-2)\,\alpha_{ij\,3,0}\;,\quad K\geq 3 \quad (i,j=1,\cdots n,\; i\neq j). \tag{1.21}$$

With a process entirely analogous to the above described it can be proved that

$$\alpha_{ij\,K,K-4} = \frac{(K-2)(K-3)}{2}\,\alpha_{ij\,4,0}\;,\quad K\geq 4\;,\quad (i,j=1,\cdots n,\; i\neq j).$$

Let us now assume that the relation

$$\alpha_{ij\,K,K-\mu} = \frac{(K-2)!}{(K-\mu)!\,(\mu-2)!}\,\alpha_{ij\,\mu,0} \qquad K\geq\mu \quad (i,j=1,\cdots n,\; i\neq j)\,, \tag{1.22}$$

holds for an arbitrary μ. With this assumption it will be proved that the foregoing equation is also valid for $\mu+1$. From the last of equations (1.14) it is obvious that

$$\alpha_{ij\,K+1,K-\mu} - \alpha_{ij\,K,K-\mu-1} = -3\frac{r_{ij}'}{r_{ij}}\alpha_{ij\,K,K-\mu} + \alpha_{ij\,K,K-\mu}'\;,\quad K\geq\mu+1,\quad (i,j=1,\cdots n\;\; i\neq j)\,,$$

which, upon substitution of equation (1.22), becomes

$$\alpha_{ij\,K+1,K-\mu} - \alpha_{ij\,K,K-\mu-1} = \left(-3\frac{r_{ij}'}{r_{ij}}\alpha_{ij\,\mu,0} + \alpha_{ij\,\mu,0}'\right)\frac{(K-2)!}{(K-\mu)!\,(\mu-2)!}\,,\quad \begin{matrix} K\geq\mu+1, \\ (i,j=1,\cdots n\;\; i\neq j). \end{matrix}$$

Introduction of the second of equations (1.14) yields

$$\alpha_{ij\,K+1,K-\mu} - \alpha_{ij\,K,K-\mu-1} = \alpha_{ij\,\mu+1,0}\,\frac{(K-2)!}{(K-\mu)!\,(\mu-2)!}\,,\quad K\geq\mu+1,\quad (i,j=1,\cdots n,\; i\neq j)\,.$$

By applying this result in succession, for $K=(\mu+1),(\mu+2),\cdots\cdots(\ell-1)$, we find

$$\alpha_{ij\,\mu+2,1} - \alpha_{ij\,\mu+1,0} = \alpha_{ij\,\mu+1,0}\frac{(\mu-1)!}{1!\,(\mu-2)!},$$

$$\alpha_{ij\,\mu+3,2} - \alpha_{ij\,\mu+2,1} = \alpha_{ij\,\mu+1,0}\frac{\mu!}{2!\,(\mu-2)!},$$

$$\vdots \qquad \ell \geqslant \mu+2,\ (i,j=1,\cdots n,\ i\neq j)$$

$$\alpha_{ij\,\ell-1,\ell-\mu-2} - \alpha_{ij\,\ell-2,\ell-\mu-3} = \alpha_{ij\,\mu+1,0}\frac{(\ell-4)!}{(\ell-\mu-2)!\,(\mu-2)!},$$

$$\alpha_{ij\,\ell,\ell-\mu-1} - \alpha_{ij\,\ell-1,\ell-\mu-2} = \alpha_{ij\,\mu+1,0}\frac{(\ell-3)!}{(\ell-\mu-1)!\,(\mu-2)!}.$$

Adding the above equations and arranging the resulting expression we obtain

$$\alpha_{ij\,\ell,\ell-\mu-1} = \alpha_{ij\,\mu+1,0}\frac{1}{(\mu-2)!}\left(\frac{(\mu-2)!}{0!} + \frac{(\mu-1)!}{1!} + \cdots\cdots + \frac{(\ell-3)!}{(\ell-\mu-1)!}\right),$$

$$\ell \geqslant \mu+2\ ,\ (i,j=1,\cdots\cdots n,\ i\neq j),$$

which is equivalent to the relation

$$\alpha_{ij\,\ell,\ell-\mu-1} = \alpha_{ij\,\mu+1,0}\frac{(\ell-2)!}{(\ell-\mu-1)!\,(\mu-1)!},\quad \ell\geqslant\mu+1\ ,\ (i,j=1,\cdots\cdots n,\ i\neq j),$$

or changing the notation

$$\alpha_{ij\,\kappa,\kappa-\mu-1} = \alpha_{ij\,\mu+1,0}\frac{(\kappa-2)!}{(\kappa-\mu-1)!\,(\mu-1)!},\quad \kappa\geqslant\mu+1\ ,\ (i,j=1,\cdots n,\ i\neq j).$$

The foregoing result and equation (1.21) are the presuppositions with the help of which mathematical induction secures the validity of the following relation

$$\alpha_{ij\,\kappa,\kappa-\mu} = \alpha_{ij\,\mu,0}\frac{(\kappa-2)!}{(\kappa-\mu)!\,(\mu-2)!},\ \kappa\geqslant\mu,\ \mu\geqslant 3,\quad (i,j=1,\cdots n,\ i\neq j). \tag{1.23}$$

It is clear that equations (1.17), (1.20) and (1.23) determine the coefficients $\alpha_{ij\kappa,\nu}$, $\kappa\geqslant 2$, $\nu=0,1,\cdots\cdots\kappa-2$ $(i,j=1,\cdots\cdots n,\ i\neq j)$ as functions of the derivatives $(r_{ij}^{-3})^{(\ell)}$ $\ell\geqslant 0$, $(i,j=1,\cdots n, i\neq j)$.

It will be now proved, using the mathematical induction that an equation relating the derivatives $(r_{ij}^{-3})^{(k)}$, $k\geq 0$ and $r_{ij}^{(l)}$, $l\leq k$, $(i,j=1,\dots n,\ i\neq j)$ and having the form

$$-\frac{(r_{ij}^{-3})^{(k)}}{r_{ij}^{-3}}=(-1)^{k+1}\frac{(k+2)!}{2}\left(\frac{r_{ij}'}{r_{ij}}\right)^{k}+(-1)^{k}\frac{k(k-1)}{2!}\frac{(k+1)!}{2}\left(\frac{r_{ij}'}{r_{ij}}\right)^{k-2}\frac{r_{ij}''}{r_{ij}}+$$

$$+(-1)^{k+1}\frac{k(k-1)(k-2)}{3!}\frac{k!}{2}\left(\frac{r_{ij}'}{r_{ij}}\right)^{k-3}\frac{r_{ij}'''}{r_{ij}}+Q_k,\quad (i,j=1,\dots,n,\ i\neq j), \tag{1.24}$$

holds for every $k\geq 0$ where Q_k represents a summation of products different than these appearing in the right hand side of the above relation. The proof is not difficult; if we suppose that the foregoing relation is valid for an arbitrary k, then differentiation of both sides of the relation with respect to τ yields

$$-\frac{(r_{ij}^{-3})^{(k+1)}}{r_{ij}^{-3}}=3\frac{(r_{ij}^{-3})^{(k)}}{r_{ij}^{-3}}\cdot\frac{r_{ij}'}{r_{ij}}+(-1)^{k}\frac{(k+2)!}{2}k\left(\frac{r_{ij}'}{r_{ij}}\right)^{k+1}+(-1)^{k+1}\frac{(k+2)!}{2}k\left(\frac{r_{ij}'}{r_{ij}}\right)^{k-1}\frac{r_{ij}''}{r_{ij}}+$$

$$+(-1)^{k+1}\frac{k(k-1)^2}{2!}\frac{(k+1)!}{2}\left(\frac{r_{ij}'}{r_{ij}}\right)^{k-1}\frac{r_{ij}''}{r_{ij}}+(-1)^{k}\frac{k(k-1)(k-2)}{2!}\frac{(k+1)!}{2}\left(\frac{r_{ij}'}{r_{ij}}\right)^{k-3}\left(\frac{r_{ij}''}{r_{ij}}\right)^{2}+$$

$$+(-1)^{k}\frac{k(k-1)}{2!}\frac{(k+1)!}{2}\left(\frac{r_{ij}'}{r_{ij}}\right)^{k-2}\frac{r_{ij}'''}{r_{ij}}+(-1)^{k}\frac{k(k-1)(k-2)^2}{3!}\frac{k!}{2}\left(\frac{r_{ij}'}{r_{ij}}\right)^{k-2}\frac{r_{ij}'''}{r_{ij}}+ \tag{1.25}$$

$$+(-1)^{k+1}\frac{k(k-1)(k-2)(k-3)}{3!}\frac{k!}{2}\left(\frac{r_{ij}'}{r_{ij}}\right)^{k-4}\frac{r_{ij}''}{r_{ij}}\frac{r_{ij}'''}{r_{ij}}+(-1)^{k+1}\frac{k(k-1)(k-2)}{3!}\frac{k!}{2}\left(\frac{r_{ij}'}{r_{ij}}\right)^{k-3}\frac{r_{ij}^{(4)}}{r_{ij}}+$$

$$+Q_k',\quad (i,j=1,\dots,n,\ i\neq j).$$

Combining equations (1.24) and (1.25) and arranging the resulting expression we obtain

$$-\frac{\left(r_{ij}^{-3}\right)^{K+1}}{r_{ij}^{-3}}=(-1)^{K+2}\frac{(K+3)!}{2}\left(\frac{r_{ij}'}{r_{ij}}\right)^{K+1}+(-1)^{K+1}\frac{(K+1)K}{2!}\frac{(K+2)!}{2}\left(\frac{r_{ij}'}{r_{ij}}\right)^{K-1}\frac{r_{ij}''}{r_{ij}}+$$

$$+(-1)^{K+2}\frac{(K+1)K(K-1)}{3!}\frac{(K+1)!}{2}\left(\frac{r_{ij}'}{r_{ij}}\right)^{K-2}\frac{r_{ij}'''}{r_{ij}}+Q_{K+1},\ (i,j=1,\cdots n,\ i\neq j),$$

where Q_{K+1} is given by the relation

$$Q_{K+1}=(-1)^{K}\frac{K(K-1)(K-2)}{2!}\frac{(K+1)!}{2}\left(\frac{r_{ij}'}{r_{ij}}\right)^{K-3}\left(\frac{r_{ij}''}{r_{ij}}\right)^{2}+(-1)^{K+1}\frac{K(K-1)(K-2)}{3!}\frac{K!}{2}\left(\frac{r_{ij}'}{r_{ij}}\right)^{K-3}\frac{r_{ij}^{(4)}}{r_{ij}}+$$

$$+(-1)^{K+1}\frac{K(K-1)(K-2)(K-3)}{3!}\frac{K!}{2}\left(\frac{r_{ij}'}{r_{ij}}\right)^{K-4}\frac{r_{ij}''}{r_{ij}}\frac{r_{ij}'''}{r_{ij}}+Q_K'-3\frac{r_{ij}'}{r_{ij}}Q_K,$$

$$(i,j=1,\cdots\cdots n,\ i\neq j).$$

Comparing equations (1.20) and (1.24) we readily conclude

$$\alpha_{ij\,K+2,0}=(-1)^{K+1}\frac{(K+2)!}{2}\left(\frac{r_{ij}'}{r_{ij}}\right)^{K}+(-1)^{K}\frac{K(K-1)}{2!}\frac{(K+1)!}{2}\left(\frac{r_{ij}'}{r_{ij}}\right)^{K-2}\frac{r_{ij}''}{r_{ij}}+$$

$$+(-1)^{K+1}\frac{K(K-1)(K-2)}{3!}\frac{K!}{2}\left(\frac{r_{ij}'}{r_{ij}}\right)^{K-3}\frac{r_{ij}'''}{r_{ij}}+Q_K,\ K\geqslant 0,\ (i,j=1,\cdots n,\ i\neq j). \qquad (1.26)$$

We now return to equation (1.7), which can be written as follows:

$$\underset{\sim}{r}_i=\underset{\sim}{r}_{io}+\underset{\sim}{r}_{io}'\tau+\sum_{K=2}^{\infty}\underset{\sim}{r}_{io}^{(K)}\frac{\tau^K}{K!},\quad (i=1,\cdots n),\ \tau\in(-\tau^*,\tau^*).$$

In the interval of convergences of this series the last term of the above relation is equal to

$$\underset{\sim}{A}_i\equiv\sum_{K=2}^{\infty}\underset{\sim}{r}_{io}^{(K)}\frac{\tau^K}{K!}=\lim_{N\to\infty}\sum_{K=2}^{N}\underset{\sim}{r}_{io}^{(K)}\frac{\tau^K}{K!},\quad (i=1,\cdots n),\ \tau\in(-\tau^*,\tau^*). \qquad (1.27)$$

We recall equation (1.13), which at the point $\tau=0$ takes the form

$$\underset{\sim}{r}_{io}^{(K)} = \sum_{\substack{j=1\\ j\neq i}}^{n} \left(m_j r_{ijo}^{-3} \sum_{\nu=0}^{K-2} B_{ijK,\nu} \, \underset{\sim}{r}_{ijo}^{(\nu)} \right), \quad K \geqslant 2, \ (i=1,\cdots n), \tag{1.28}$$

where

$$B_{ijK,\nu} = \alpha_{ijK,\nu} \Big|_{\tau=0} \quad (i,j=1,\cdots n, i\neq j), \ K\geqslant 2, \ \nu=0,1,\cdots\cdots(K-2). \tag{1.29}$$

Combining equations (1.27) and (1.28) we find

$$\underset{\sim}{A}_i = \lim_{N\to\infty} \left(\sum_{K=2}^{N} \sum_{\substack{j=1\\ j\neq i}}^{n} \sum_{\nu=0}^{K-2} m_j r_{ijo}^{-3} B_{ijK\nu} \underset{\sim}{r}_{ijo}^{(\nu)} \frac{\tau^K}{K!} \right), \ (i=1,\cdots n), \ \tau\in(-\tau^*,\tau^*).$$

Changing the order of the sums in this relation we obtain

$$\underset{\sim}{A}_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \left[m_j r_{ijo}^{-3} \lim_{N\to\infty} \sum_{K=2}^{N} \sum_{\nu=0}^{K-2} B_{ijK\nu} \underset{\sim}{r}_{ijo}^{(\nu)} \frac{\tau^K}{K!} \right], \ (i=1,\cdots n), \ \tau\in(-\tau^*,\tau^*). \tag{1.30}$$

The next step is to rearrange the last double finite sum, appearing in the foregoing relation. We first observe that this is equal to the sum of the following terms

$$B_{ij2,0} \underset{\sim}{r}_{ijo} \frac{\tau^2}{2!},$$

$$B_{ij3,0} \underset{\sim}{r}_{ijo} \frac{\tau^3}{3!}, \quad B_{ij3,1} \underset{\sim}{r}'_{ijo} \frac{\tau^3}{3!}, \qquad (i,j=1,\cdots\cdots n, i\neq j), \tau\in(-\tau^*,\tau^*), N\geqslant 2$$

$$B_{ij4,0} \underset{\sim}{r}_{ijo} \frac{\tau^4}{4!}, \quad B_{ij4,1} \underset{\sim}{r}'_{ijo} \frac{\tau^4}{4!}, \quad B_{ij4,2} \underset{\sim}{r}''_{ijo} \frac{\tau^4}{4!},$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$B_{ijN,0} \underset{\sim}{r}_{ijo} \frac{\tau^N}{N!}, \quad B_{ijN,1} \underset{\sim}{r}'_{ijo} \frac{\tau^N}{N!}, \quad B_{ijN,2} \underset{\sim}{r}''_{ijo} \frac{\tau^N}{N!}, \cdots\cdots\cdots\cdots B_{ijN,N-2} \underset{\sim}{r}_{ijo}^{(N-2)} \frac{\tau^N}{N!},$$

which can be reordered as follows:

$$B_{ij2,0}\Gamma_{ijo}\frac{\tau^2}{2!},\ B_{ij3,1}\Gamma'_{ijo}\frac{\tau^3}{3!},\ B_{ij4,2}\Gamma''_{ijo}\frac{\tau^4}{4!},\cdots\cdots B_{ijN,N-2}\Gamma^{(N-2)}_{ijo}\frac{\tau^N}{N!},$$

$$B_{ij3,0}\Gamma_{ijo}\frac{\tau^3}{3!},\ B_{ij4,1}\Gamma'_{ijo}\frac{\tau^4}{4!},\cdots\cdots B_{ijN,N-3}\Gamma^{(N-3)}_{ijo}\frac{\tau^N}{N!},$$

$$\vdots \qquad \vdots$$

$$B_{ijN-1,0}\Gamma_{ijo}\frac{\tau^{N-1}}{(N-1)!},\ B_{ijN,1}\Gamma'_{ijo}\frac{\tau^N}{N!},$$

$$B_{ijN,0}\Gamma_{ijo}\frac{\tau^N}{N!}, \qquad N\geq 2,\ (i,j=1,\cdots n,\ i\neq j),\ \tau\in(-\tau^*,\tau^*).$$

We readily verify that, for every $N\geq 2$ the sum of the foregoing terms are given by the expression

$$\sum_{K=2}^{N}\sum_{V=K}^{N} B_{ijV,V-K}\Gamma^{(V-K)}_{ijo}\frac{\tau^V}{V!},\ N\geq 2,(i,j=1,\cdots\cdots n,i\neq j),\ \tau\in(-\tau^*,\tau^*).$$

It has been therefore proved (by writing down the terms of both sums) that

$$\sum_{K=2}^{N}\sum_{V=0}^{K-2} B_{ijKV}\Gamma^{(V)}_{ijo}\frac{\tau^K}{K!}=\sum_{K=2}^{N}\sum_{V=K}^{N} B_{ijV,V-K}\Gamma^{(V-K)}_{ijo}\frac{\tau^V}{V!},N\geq 2,(i,j=1,\cdots n,\ i\neq j),\tau\in(-\tau^*,\tau^*).$$

Introducing this equation into equation (1.30) we obtain

$$\underset{\sim}{A}_i=\sum_{\substack{j=1\\ j\neq i}}^{n}\left[m_j\Gamma^{-3}_{ijo}\lim_{N\to\infty}\left(\sum_{K=2}^{N}\sum_{V=K}^{N} B_{ijV,V-K}\Gamma^{(V-K)}_{ijo}\frac{\tau^V}{V!}\right)\right],(i=1,\cdots n),\ \tau\in(-\tau^*,\tau^*). \quad (1.31)$$

Comparison of equations (1.23) and (1.29) yields

$$B_{ijV,V-K}=\frac{(V-2)!}{(V-K)!\ (K-2)!}B_{ijK,0},\ V\geq K\quad K\geq 2\quad (i,j=1,\cdots\cdots n,\ i\neq j).$$

Substituting for $B_{ijV,V-K}$ according to this equation in equation (1.31), we find

$$\underset{\sim}{A}_i=\sum_{\substack{j=1\\j\neq i}}^{n}\left[m_j r_{ijo}^{-3}\lim_{N\to\infty}\left(\sum_{K=2}^{N}\sum_{V=K}^{N}B_{ijK,o}\,\underset{\sim}{r}_{ijo}^{(V-K)}\frac{\tau^{V}}{(V-1)V(K-2)!(V-K)!}\right)\right],\quad \begin{array}{l}(i=1,\cdots n),\\ \tau\in(-\tau^*,\tau^*).\end{array} \tag{1.32*}$$

The convergence of the power series (1.7) secures the convergence of the following series

$$\sum_{V=K}^{\infty}\underset{\sim}{r}_{ijo}^{(V-K)}\frac{\tau^{(V-K)}}{(V-1)V(V-K)!},\ (i,j=1,\cdots\cdots n,\ i\neq j),$$

for every $K\geq 2$ and $\tau\in(-\tau^*,\tau^*)$. Therefore, according to the theory of infinite series, the limit appearing in equation (1.32*) is equal to a double series. That is

$$\underset{\sim}{A}_i=\sum_{\substack{j=1\\j\neq i}}^{n}m_j r_{ijo}^{-3}\left[\sum_{K=2}^{\infty}\sum_{V=K}^{\infty}B_{ijK,o}\,\underset{\sim}{r}_{ijo}^{(V-K)}\frac{\tau^{V}}{(V-1)V(K-2)!(V-K)!}\right],(i=1,\cdots n),\tau\in(-\tau^*,\tau^*).$$

Shifting the origin of the second series of this equation to $V=o$ and interchanging the order of the series we obtain

$$\begin{aligned}\underset{\sim}{A}_i=\tau^2\sum_{\substack{j=1\\j\neq i}}^{n}m_j r_{ijo}^{-3}\Bigg[&\underset{\sim}{r}_{ijo}\sum_{K=2}^{\infty}B_{ijK,o}\frac{\tau^{K-2}}{K!}+\underset{\sim}{r}'_{ijo}\tau\sum_{K=2}^{\infty}B_{ijK,o}\frac{\tau^{K-2}}{K(K+1)(K-2)!}+\\ &+\underset{\sim}{r}''_{ijo}\frac{\tau^2}{2}\sum_{K=2}^{\infty}B_{ijK,o}\frac{\tau^{K-2}}{(K+1)(K+2)(K-2)!}+\underset{\sim}{r}'''_{ijo}\frac{\tau^3}{6}\sum_{K=2}^{\infty}B_{ijK,o}\frac{\tau^{K-2}}{(K+2)(K+3)(K-2)!}+\\ &+\sum_{V=4}^{\infty}\sum_{K=2}^{\infty}B_{ijK,o}\,\underset{\sim}{r}_{ijo}^{(V)}\frac{\tau^{V+K-2}}{(V+K-1)(V+K)(K-2)!\,V!}\Bigg],\ (i=1,\cdots\cdots n),\tau\in(-\tau^*,\tau^*).\end{aligned} \tag{1.32}$$

In the numerical applications we neglect the last term of the foregoing series (since the power of the variable τ is at least of the sixth order) accepting the approximate form

$$
\begin{aligned}
\underset{\sim}{A}{}_i^* \equiv \tau^2 \sum_{\substack{j=1\\ j\neq i}}^{n} m_j r_{ijo}^{-3} \Bigg\{ & \underset{\sim}{r}_{ijo} \sum_{k=2}^{\infty} B_{ijk,o} \frac{\tau^{k-2}}{k!} + \underset{\sim}{r}{}'_{ijo}\,\tau \sum_{k=2}^{\infty} B_{ijk,o} \frac{\tau^{k-2}}{k(k+1)(k-2)!} + \\
& + \underset{\sim}{r}{}''_{ijo} \frac{\tau^2}{2} \sum_{k=2}^{\infty} B_{ijk,o} \frac{\tau^{k-2}}{(k+1)(k+2)(k-2)!} + \underset{\sim}{r}{}'''_{ijo} \frac{\tau^3}{6} \sum_{k=2}^{\infty} B_{ijk,o} \frac{\tau^{k-2}}{(k+2)(k+3)(k-2)!} \Bigg\} ,
\end{aligned}
\tag{1.33}
$$

$$(i=1,\cdots n)\,,\ \tau \in (-\tau^*, \tau^*).$$

We now recall equation (A3-22), (A3-23), (A3-24) and (A3-25) of the appendix(A3). Introduction of these equations into relation (1.33) yields

$$
\begin{aligned}
\underset{\sim}{A}{}_i^* = \tau^2 \sum_{\substack{j=1\\ j\neq i}}^{n} m_j r_{ijo}^{-3} \Bigg\{ & \underset{\sim}{r}_{ijo} \Bigg[-\frac{1}{2(1+\varphi_{ij})} + \frac{\tau^2}{4}\left(\frac{r''_{ijo}}{r_{ijo}}\right)\left(\frac{2\varphi_{ij}^2+9\varphi_{ij}+6}{(1+\varphi_{ij})^2\varphi_{ij}^3} - \frac{6\ln(1+\varphi_{ij})}{\varphi_{ij}^4}\right) + \\
& + \frac{\tau^3}{12}\left(\frac{r'''_{ijo}}{r_{ijo}}\right)\left(\frac{7\varphi_{ij}+6}{(1+\varphi_{ij})^2\varphi_{ij}^3} + 6\,\frac{\ln(1+\varphi_{ij})-4}{\varphi_{ij}^4} + \frac{24\ln(1+\varphi_{ij})}{\varphi_{ij}^5}\right) + \sum_{k=0}^{\infty} Q_k^* \frac{\tau^k}{(k+2)!} \Bigg] + \\
& + \underset{\sim}{r}{}'_{ijo}\,\tau \Bigg[\frac{\ln(1+\varphi_{ij})}{\varphi_{ij}^3} - \frac{\varphi_{ij}+2}{2(1+\varphi_{ij})\varphi_{ij}^2} + \frac{\tau^2}{4}\left(\frac{r''_{ijo}}{r_{ijo}}\right)\left(\frac{7\varphi_{ij}+6}{(1+\varphi_{ij})^2\varphi_{ij}^3} + 6\,\frac{\ln(1+\varphi_{ij})-4}{\varphi_{ij}^4} + \right. \\
& \left. + \frac{24\ln(1+\varphi_{ij})}{\varphi_{ij}^5}\right) + \frac{\tau^3}{12}\left(\frac{r'''_{ijo}}{r_{ijo}}\right)\left(\frac{3}{\varphi_{ij}^4} - \frac{10\varphi_{ij}+9}{(1+\varphi_{ij})^2\varphi_{ij}^4} - 12\,\frac{2\ln(1+\varphi_{ij})-5}{\varphi_{ij}^5} - \right. \\
& \left. - \frac{60\ln(1+\varphi_{ij})}{\varphi_{ij}^6}\right) + \sum_{k=0}^{\infty} Q_k^* \frac{(k+1)\,\tau^k}{(k+3)!} \Bigg] + \underset{\sim}{r}{}''_{ijo}\frac{\tau^2}{2!}\Bigg[\frac{1}{2}\left(\frac{5\varphi_{ij}+6}{(1+\varphi_{ij})\varphi_{ij}^3} - \right. \\
& \left. - \frac{2\ln(1+\varphi_{ij})}{\varphi_{ij}^3} - \frac{6\ln(1+\varphi_{ij})}{\varphi_{ij}^4}\right) + \frac{\tau^2}{4}\left(\frac{r''_{ijo}}{r_{ijo}}\right)\left(\frac{3}{\varphi_{ij}^4} - \frac{10\varphi_{ij}+9}{(1+\varphi_{ij})^2\varphi_{ij}^4} - 12\,\frac{2\ln(1+\varphi_{ij})-5}{\varphi_{ij}^5} - \right.
\end{aligned}
\tag{1.34}
$$

$$-\frac{60\ln(1+\varphi_{ij})}{\varphi_{ij}^6}\Bigg)+\frac{\tau^3}{12}\left(\frac{\gamma_{ij0}'''}{\gamma_{ij0}}\right)\left(\frac{13\varphi_{ij}+12}{(1+\varphi_{ij})^2\varphi_{ij}^5}+60\frac{\ln(1+\varphi_{ij})-2}{\varphi_{ij}^6}+\frac{120\ln(1+\varphi_{ij})}{\varphi_{ij}^7}+\right.$$

$$\left.+\frac{\varphi_{ij}-12}{\varphi_{ij}^5}\right)+\sum_{K=0}^{\infty}Q_K^*\frac{\tau^K}{(K+3)(K+4)K!}\Bigg]+\underset{\sim}{\gamma}_{ij0}'''\frac{\tau^3}{3!}\Bigg[\frac{1}{2}\left(\frac{6\ln(1+\varphi_{ij})}{\varphi_{ij}^4}+\frac{12\ln(1+\varphi_{ij})}{\varphi_{ij}^5}-\right.$$

$$\left.-\frac{\varphi_{ij}^2+12\varphi_{ij}+12}{(1+\varphi_{ij})\varphi_{ij}^4}\right)+\frac{\tau^2}{4}\left(\frac{\gamma_{ij0}''}{\gamma_{ij0}}\right)\left(\frac{13\varphi_{ij}+12}{(1+\varphi_{ij})^2\varphi_{ij}^5}+60\frac{\ln(1+\varphi_{ij})-2}{\varphi_{ij}^6}+\frac{120\ln(1+\varphi_{ij})}{\varphi_{ij}^7}+\right.$$

$$\left.+\frac{\varphi_{ij}-12}{\varphi_{ij}^5}\right)+\sum_{K=0}^{\infty}(-1)^{K+1}\left(\frac{\gamma_{ij0}'}{\gamma_{ij0}}\right)^{K-3}\frac{\gamma_{ij0}'''}{\gamma_{ij0}}\frac{K(K-1)(K-2)\tau^K}{12(K+4)(K+5)}+$$

$$\left.\left.+\sum_{K=0}^{\infty}Q_K^*\frac{\tau^K}{(K+4)(K+5)K!}\right]\right\},$$

$$(i=1,\cdots n),\quad \tau\in(-\tau^*,\tau^*),\quad \varphi_{ij}\neq 0,\quad |\varphi_{ij}|<1 .$$

where according to equations, (A3-19) and (A3-21)

$$Q_K^*\equiv Q_K\Big|_{\tau=0},\quad K\geqslant 0,\quad \varphi_{ij}\equiv\frac{\tau\,\gamma_{ij0}'}{\gamma_{ij0}},\quad (i,j=1,\cdots n,\ i\neq j),\quad \tau\in(-\tau^*,\tau^*). \tag{1.35}$$

Differentiating both sides of equation (1.34) once with respect to τ we obtain

$$\underset{\sim}{A}_i^{*\prime}=\tau\sum_{\substack{j=1\\j\neq i}}^{n}m_j\gamma_{ij0}^{-3}\Bigg\{\underset{\sim}{\gamma}_{ij0}\Bigg[-\frac{\varphi_{ij}+2}{2(1+\varphi_{ij})^2}+\frac{\tau^2}{2}\left(\frac{\gamma_{ij0}''}{\gamma_{ij0}}\right)\frac{1}{(1+\varphi_{ij})^3}+\frac{\tau^3}{12}\left(\frac{\gamma_{ij0}'''}{\gamma_{ij0}}\right)\left(\frac{6\ln(1+\varphi_{ij})}{\varphi_{ij}^4}-\right.$$

$$\left.-\frac{11\varphi_{ij}^2+15\varphi_{ij}+6}{(1+\varphi_{ij})^3\varphi_{ij}^3}\right)+\sum_{K=0}^{\infty}\left(\frac{2Q_K^*}{(K+2)!}+\frac{Q_{K+1}^*(K+1)\tau}{(K+3)!}\right)\tau^K\Bigg]+\underset{\sim}{\gamma}_{ij0}'\frac{\tau}{2}\Bigg[-\frac{1}{(1+\varphi_{ij})^2}+ \tag{1.36}$$

$$+\frac{\tau^2}{2}\left(\frac{\Gamma''_{ij0}}{\Gamma_{ij0}}\right)\left(\frac{6\ln(1+\varphi_{ij})}{\varphi_{ij}^4}-\frac{11\varphi_{ij}^2+15\varphi_{ij}+6}{(1+\varphi_{ij})^3\,\varphi_{ij}^3}\right)+\frac{\tau^3}{6}\left(\frac{\Gamma'''_{ij0}}{\Gamma_{ij0}}\right)\left(\frac{6\varphi_{ij}^3+44\varphi_{ij}^2+60\varphi_{ij}+24}{(1+\varphi_{ij})^3\,\varphi_{ij}^4}-\right.$$

$$\left.-\frac{24\ln(1+\varphi_{ij})}{\varphi_{ij}^5}\right)+2\sum_{K=0}^{\infty}\left(\frac{3Q^*_K(K+1)}{(K+3)!}+\frac{Q^*_{K+1}(K+1)(K+2)\tau}{(K+4)!}\right)\tau^K\Bigg]+\underset{\sim}{\Gamma}''_{ij0}\frac{\tau^2}{4}\left[\frac{3\varphi_{ij}+2}{(1+\varphi_{ij})^2\varphi_{ij}^2}-\right.$$

$$-\frac{2\ln(1+\varphi_{ij})}{\varphi_{ij}^3}+\frac{\tau^2}{2}\left(\frac{\Gamma''_{ij0}}{\Gamma_{ij0}}\right)\left(\frac{6\varphi_{ij}^3+44\varphi_{ij}^2+60\varphi_{ij}+24}{(1+\varphi_{ij})^3\varphi_{ij}^4}-\frac{24\ln(1+\varphi_{ij})}{\varphi_{ij}^5}\right)+$$

$$+\frac{\tau^3}{6}\left(\frac{\Gamma'''_{ij0}}{\Gamma_{ij0}}\right)\left(\frac{60\ln(1+\varphi_{ij})}{\varphi_{ij}^6}+\frac{3\varphi_{ij}^4-15\varphi_{ij}^3-110\varphi_{ij}^2-150\varphi_{ij}-60}{(1+\varphi_{ij})^3\,\varphi_{ij}^5}\right)+$$

$$+2\sum_{K=0}^{\infty}\frac{4Q^*_K-\tau Q^*_{K+1}}{(K+3)(K+4)\,K!}\tau^K\Bigg]+\underset{\sim}{\Gamma}'''_{ij0}\frac{\tau^3}{12}\left[\frac{6\ln(1+\varphi_{ij})}{\varphi_{ij}^4}-\frac{2\varphi_{ij}^2+9\varphi_{ij}+6}{(1+\varphi_{ij})^2\,\varphi_{ij}^3}+\right.$$

$$+\frac{\tau^2}{2}\left(\frac{\Gamma''_{ij0}}{\Gamma_{ij0}}\right)\left(\frac{60\ln(1+\varphi_{ij})}{\varphi_{ij}^6}+\frac{3\varphi_{ij}^4-15\varphi_{ij}^3-110\varphi_{ij}^2-150\varphi_{ij}-60}{(1+\varphi_{ij})^3\varphi_{ij}^5}\right)+$$

$$+10\sum_{K=0}^{\infty}\left((-1)^{K+1}\left[\frac{\Gamma'_{ij0}}{\Gamma_{ij0}}\right]^{K-3}\frac{\Gamma'''_{ij0}}{\Gamma_{ij0}}\,\frac{K(K-1)(K-2)}{12}+\frac{Q^*_K}{K!}\right)\frac{\tau^K}{(K+4)(K+5)}+$$

$$+2\tau\left(\sum_{K=0}^{\infty}(-1)^{K+1}\left(\frac{\Gamma'_{ij0}}{\Gamma_{ij0}}\right)^{K-3}\frac{\Gamma'''_{ij0}}{\Gamma_{ij0}}\,\frac{K(K-1)(K-2)\,\tau^K}{12(K+4)(K+5)}+\sum_{K=0}^{\infty}\frac{Q^*_K\,\tau^K}{(K+4)(K+5)\,K!}\right)'\Bigg]\Bigg\},$$

$$(i=1,\cdots n),\quad \tau\in(-\tau^*,\tau^*),\quad \varphi_{ij}\neq 0,\quad |\varphi_{ij}|<1 .$$

Comparison of equations (1.7) and (1.27) yields

$$\underset{\sim}{\Gamma}_i=\underset{\sim}{\Gamma}_{i0}+\underset{\sim}{\Gamma}'_{i0}\tau+\underset{\sim}{A}_i,\quad (i=1,\cdots n),\quad \tau\in(-\tau^*,\tau^*). \tag{1.37}$$

Differentiating both sides of this equation once with respect to τ we obtain the velocity vectors $\underset{\sim}{r}_i'$, $(i=1,\cdots n)$,

$$\underset{\sim}{r}_i' = \underset{\sim}{r}_{i0}' + \underset{\sim}{A}_i', \quad (i=1,\cdots n), \quad \tau\in(-\tau^*,\tau^*). \tag{1.38}$$

In the numerical applications the evaluation of $\underset{\sim}{A}_i$ given by formula (1.32) and comprising infinite series, is impossible. Hence an approximate formula is needed. Although the formula (1.34) gives an approximation of $\underset{\sim}{A}_i$, it can not be used in the numerical applications for the same reason formula (1.32) can not be used. However an approximation of $\underset{\sim}{A}_i$ can be obtained by neglecting the series appearing in the right hand side of equation (1.34). Introduction of the resulting expression into equation (1.37) gives the following approximate solution for the system (1.6)

$$\begin{aligned}
\underset{\sim}{r}_i^* = {} & \underset{\sim}{r}_{i0} + \underset{\sim}{r}_{i0}'\tau + \frac{\tau^2}{2}\sum_{\substack{j=1\\ j\neq i}}^{n} m_j r_{ij0}^{-3}\Bigg\{\underset{\sim}{r}_{ij0}\Bigg[\frac{-1}{(1+\varphi_{ij})} + \left(\frac{r_{ij0}''}{r_{ij0}}\right)\frac{\tau^2}{2}\left(\frac{2\varphi_{ij}^2+9\varphi_{ij}+6}{(1+\varphi_{ij})^2\varphi_{ij}^3} - \frac{6\ln(1+\varphi_{ij})}{\varphi_{ij}^4}\right) + \\
& + \frac{\tau^3}{6}\left(\frac{r_{ij0}'''}{r_{ij0}}\right)\left(\frac{7\varphi_{ij}+6}{(1+\varphi_{ij})^2\varphi_{ij}^3} + 6\frac{\ln(1+\varphi_{ij})-4}{\varphi_{ij}^4} + 24\frac{\ln(1+\varphi_{ij})}{\varphi_{ij}^5}\right)\Bigg] + \underset{\sim}{r}_{ij0}'\tau\Bigg[\frac{2\ln(1+\varphi_{ij})}{\varphi_{ij}^3} - \\
& - \frac{\varphi_{ij}+2}{(1+\varphi_{ij})\varphi_{ij}^2} + \frac{\tau^2}{2}\left(\frac{r_{ij0}''}{r_{ij0}}\right)\left(\frac{7\varphi_{ij}+6}{(1+\varphi_{ij})^2\varphi_{ij}^3} + 6\frac{\ln(1+\varphi_{ij})-4}{\varphi_{ij}^4} + \frac{24\ln(1+\varphi_{ij})}{\varphi_{ij}^5}\right) + \\
& + \frac{\tau^3}{6}\left(\frac{r_{ij0}'''}{r_{ij0}}\right)\left(\frac{3}{\varphi_{ij}^4} - \frac{10\varphi_{ij}+9}{(1+\varphi_{ij})^2\varphi_{ij}^4} - 12\frac{2\ln(1+\varphi_{ij})-5}{\varphi_{ij}^5} - \frac{60\ln(1+\varphi_{ij})}{\varphi_{ij}^6}\right)\Bigg] + \\
& + \underset{\sim}{r}_{ij0}''\frac{\tau^2}{2}\Bigg[\frac{5\varphi_{ij}+6}{(1+\varphi_{ij})\varphi_{ij}^3} - \frac{2\ln(1+\varphi_{ij})}{\varphi_{ij}^3} - \frac{6\ln(1+\varphi_{ij})}{\varphi_{ij}^4} + \frac{\tau^2}{2}\left(\frac{r_{ij0}''}{r_{ij0}}\right)\left(\frac{3}{\varphi_{ij}^4} - \frac{10\varphi_{ij}+9}{(1+\varphi_{ij})^2\varphi_{ij}^4} - \right.
\end{aligned} \tag{1.39}$$

$$-12\frac{2\ln(1+\varphi_{ij})-5}{\varphi_{ij}^5}-\frac{60\ln(1+\varphi_{ij})}{\varphi_{ij}^6}\Bigg)+\frac{\tau^3}{6}\left(\frac{r_{ijo}'''}{r_{ijo}}\right)\left(\frac{13\varphi_{ij}+12}{(1+\varphi_{ij})^2\varphi_{ij}^5}+60\frac{\ln(1+\varphi_{ij})-2}{\varphi_{ij}^6}+\right.$$

$$\left.+\frac{120\ln(1+\varphi_{ij})}{\varphi_{ij}^7}+\frac{\varphi_{ij}-12}{\varphi_{ij}^5}\right)\Bigg]+\underset{\sim}{r}_{ijo}'''\frac{\tau^3}{6}\left[\frac{6\ln(1+\varphi_{ij})}{\varphi_{ij}^4}+\frac{\varphi_{ij}^2+12\varphi_{ij}+12}{(1+\varphi_{ij})\varphi_{ij}^4}+\frac{12\ln(1+\varphi_{ij})}{\varphi_{ij}^5}+\right.$$

$$\left.+\frac{\tau^2}{2}\left(\frac{r_{ijo}''}{r_{ijo}}\right)\left(\frac{13\varphi_{ij}+12}{(1+\varphi_{ij})^2\varphi_{ij}^5}+60\frac{\ln(1+\varphi_{ij})-2}{\varphi_{ij}^6}+\frac{120\ln(1+\varphi_{ij})}{\varphi_{ij}^7}+\frac{\varphi_{ij}-12}{\varphi_{ij}^5}\right)\right]\Bigg\},$$

$$(i=1,\cdots n),\quad \tau\in(-\tau^*,\tau^*),\quad \varphi_{ij}\neq 0,\quad |\varphi_{ij}|<1.$$

The velocity vectors $\underset{\sim}{r}_i^{*\prime}$ $(i=1,\cdots n)$ can be found with the help of the relation (1.36). Neglecting the series appearing in its right hand side and introducing the resulting expression in equation (1.38) we obtain

$$\underset{\sim}{r}_i^{*\prime}=\underset{\sim}{r}_{io}'+\frac{\tau}{2}\sum_{\substack{j=1\\ j\neq i}}^{n}m_j r_{ijo}^{-3}\left\{\underset{\sim}{r}_{ijo}\left[-\frac{\varphi_{ij}+2}{(1+\varphi_{ij})^2}+\tau^2\left(\frac{r_{ijo}''}{r_{ijo}}\right)\frac{1}{(1+\varphi_{ij})^3}+\frac{\tau^3}{6}\left(\frac{r_{ijo}'''}{r_{ijo}}\right)\left(\frac{6\ln(1+\varphi_{ij})}{\varphi_{ij}^4}-\right.\right.\right.$$

$$\left.\left.-\frac{11\varphi_{ij}^2+15\varphi_{ij}+6}{(1+\varphi_{ij})^3\varphi_{ij}^3}\right)\right]+\underset{\sim}{r}_{ijo}'\tau\left[-\frac{1}{(1+\varphi_{ij})^2}+\frac{\tau^2}{2}\left(\frac{r_{ijo}''}{r_{ijo}}\right)\left(\frac{6\ln(1+\varphi_{ij})}{\varphi_{ij}^4}-\frac{11\varphi_{ij}^2+15\varphi_{ij}+6}{(1+\varphi_{ij})^3\varphi_{ij}^3}\right)+\right.$$

$$\left.+\frac{\tau^3}{6}\left(\frac{r_{ijo}'''}{r_{ijo}}\right)\left(\frac{6\varphi_{ij}^3+44\varphi_{ij}^2+60\varphi_{ij}+24}{(1+\varphi_{ij})^3\varphi_{ij}^4}-\frac{24\ln(1+\varphi_{ij})}{\varphi_{ij}^5}\right)\right]+\underset{\sim}{r}_{ijo}''\frac{\tau^2}{2}\left[\frac{3\varphi_{ij}+2}{(1+\varphi_{ij})^2\varphi_{ij}^2}-\right.$$

$$-\frac{2\ln(1+\varphi_{ij})}{\varphi_{ij}^3}+\frac{\tau^2}{2}\left(\frac{r_{ijo}''}{r_{ijo}}\right)\left(\frac{6\varphi_{ij}^3+44\varphi_{ij}^2+60\varphi_{ij}+24}{(1+\varphi_{ij})^3\varphi_{ij}^4}-\frac{24\ln(1+\varphi_{ij})}{\varphi_{ij}^5}\right)+$$

$$+\frac{\tau^3}{6}\left(\frac{\Gamma'''_{ijo}}{\Gamma_{ijo}}\right)\left(\frac{60\ln(1+\varphi_{ij})}{\varphi_{ij}^6}+\frac{3\varphi_{ij}^4-15\varphi_{ij}^3-110\varphi_{ij}^2-150\varphi_{ij}-60}{(1+\varphi_{ij})^3\varphi_{ij}^5}\right)\Bigg]+\underset{\sim}{\Gamma}'''_{ijo}\frac{\tau^3}{6}\left(\frac{6\ln(1+\varphi_{ij})}{\varphi_{ij}^4}-\right.$$

$$\left.-\frac{2\varphi_{ij}^2+9\varphi_{ij}+6}{(1+\varphi_{ij})^2\varphi_{ij}^3}+\frac{\tau^2}{2}\left(\frac{\Gamma''_{ijo}}{\Gamma_{ijo}}\right)\left(\frac{60\ln(1+\varphi_{ij})}{\varphi_{ij}^6}-\frac{3\varphi_{ij}^4-15\varphi_{ij}^3-110\varphi_{ij}^2-150\varphi_{ij}-60}{(1+\varphi_{ij})^3\varphi_{ij}^5}\right)\right]\Bigg\}, \tag{1.40}$$

$$(i=1,\cdots n),\ \tau\in(-\tau^*,\tau^*),\ \varphi_{ij}\neq 0,\ |\varphi_{ij}|<1.$$

From a set of initial values $\underset{\sim}{\Gamma}_{io}$ and $\underset{\sim}{\Gamma}'_{io}$ $(i=1,\cdots n)$, and with the aid of equations (1.6), (1.8), (1.39), (1.40) and (A3-21) the position and velocity vectors $\underset{\sim}{\Gamma}_i$ and $\underset{\sim}{\Gamma}'_i$ $(i=1,\cdots n)$ can be calculated at a subsequent moment, say $\Delta\tau$, where $|\Delta\tau|<\tau^*$. Taking these new values as initial conditions we repeat the process, thereby detailing the paths of the bodies in space.

CHAPTER 2

Attraction between spheroids with co-planar equatorial planes

Homogeneous oblate spheroids

Having treated the problem of solution of the equations of motion of a dynamical system consisting of n particles, we now consider the equations describing the potential and the force law of a homogeneous oblate spheroid.

The solid with surface whose equation, in a Cartesian system of reference, is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

is called an ellipsoid. This solid is clearly symmetrical about each of the co-ordinate planes and its centre is located at the origin of the system of reference.

If two of the coefficients, appearing in the foregoing equation, are equal (a and b suppose) this equation describes the surface of a solid called a spheroid. Every spheroid is a solid of revolution of an ellipse about one of its axes. A spheroid formed by revolution about the minor axis is called an oblate spheroid and its surface is described by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{c^2}=1, \quad a^2>c^2. \qquad (2.1)$$

The general equation of a system of confocal oblate spheroids is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{a^2+\lambda}+\frac{z^2}{c^2+\lambda}=1, \quad a^2>c^2, \qquad (2.2)$$

where λ is either positive or negative but not less than $-c^2$. If λ is positive the principal axes of the solid will increase as λ increases and their ratio tends to one as $\lambda\to\infty$, so that the sphere of infinite radius is a limiting form of one of the confocals. If λ is negative the oblate spheroid becomes progressively flatter as λ approaches the value $-c^2$. Hence the disc whose periphery is given by

$$\frac{x^2}{\alpha^2 - c^2} + \frac{y^2}{\alpha^2 - c^2} = 1, \quad \alpha^2 > c^2, \tag{2.3}$$

is a limiting form of one of the confocal oblate spheroids.

For two confocal oblate spheroids with semi-axes of the internal α, c and the external α', c', we have

$$\alpha'^2 = \alpha^2 + \lambda, \quad c'^2 = c^2 + \lambda, \quad \lambda \geqslant -c^2,$$

that is,

$$\alpha'^2 - c'^2 = \alpha^2 - c^2 = \text{constant}.$$

For the sake of brevity we shall adopt in this chapter the terms homoeoid and focaloid. Homoeoid is a shell bounded by two similar and similarly situated concentric ellipsoids, and focaloid is a shell bounded by two confocal ellipsoids.

Infinitesimally thin homoeoids are said to be confocal when their inner boundaries are confocal.

Before proceeding to the discussion of the problem stated in the beginning of this chapter we shall give three theorems which we are going to use later on.

<u>Theorem 1</u> : Any two infinitesimally thin confocal homoeoids have the same external equipotential surfaces viz. the ellipsoids confocal with them and external to both, and therefore their attractions at any point, external to both, are proportional to their masses and in the same direction.

<u>Theorem 2</u>,:(MacLaurin's theorem) : The attraction on two homogeneous confocal ellipsoids by a particle external to both are proportional to their masses and in the same direction.

<u>Theorem 3</u> : The motion of the centre of gravity of a rigid body is the same as that of a particle of mass equal to the whole mass of the body, under the influence of forces equivalent to the total external forces acting to the

body, applied to the particle parallel to their actual directions.

We now consider an oblate spheroid with uniform density ρ, semi-major axis α, semi-minor axis c, eccentricity $e=(1-c^2/\alpha^2)^{\frac{1}{2}}$ and its centre at a point (x_ν, y_ν, z_ν), where the co-ordinates x_ν, y_ν and z_ν are referred to a Cartesian right-handed inertial system with its Z axis parallel to the minor axis of the spheroid, and its XY plane parallel to the equatorial plane of the spheroid. We also consider a test point located at (x_μ, y_μ, z_μ) and adopt such units of mass, distance and time that the gravitational constant K has numerical value equal to one, then the potential at the test point, due to the presence of the oblate spheroid, is

$$\phi=-\pi\rho\alpha^2 c\int_\lambda^\infty\left(1-\frac{x_{\mu\nu}^2}{\alpha^2+u}-\frac{y_{\mu\nu}^2}{\alpha^2+u}-\frac{z_{\mu\nu}^2}{c^2+u}\right)\frac{du}{(\alpha^2+u)(c^2+u)^{1/2}}, \tag{2.4}$$

where $x_{\mu\nu}\equiv x_\mu-x_\nu$, $y_{\mu\nu}\equiv y_\mu-y_\nu$ and $z_{\mu\nu}\equiv z_\mu-z_\nu$.

The lower limit of the integration for a point inside the spheroid has constant value

$$\lambda=0, \tag{2.5}$$

while for a test point located outside the oblate spheroid its value depends on the co-ordinates of both the centre of the oblate spheroid and the point and is given by the positive root of the equation

$$\frac{R_{\mu\nu}^2}{\alpha^2+\lambda}+\frac{z_{\mu\nu}^2}{c^2+\lambda}=1, \tag{2.6}$$

where

$$R_{\mu\nu}=(x_{\mu\nu}^2+y_{\mu\nu}^2)^{\frac{1}{2}}. \tag{2.7}$$

We may now deduce the components of the force acting on a particle of unit mass, and located at the point (x_μ, y_μ, z_μ). The force component F_x, parallel to the X axis, is

$$F_x = x''_\mu = -\frac{\partial \Phi}{\partial x_\mu} = -2\pi\rho a^2 c\, x_{\mu\nu} \int_\lambda^\infty \frac{du}{(a^2+u)^2(c^2+u)^{1/2}} -$$

$$-\pi\rho a^2 c \frac{\partial \lambda}{\partial x_\mu}\left(1 - \frac{x_{\mu\nu}^2}{a^2+\lambda} - \frac{y_{\mu\nu}^2}{a^2+\lambda} - \frac{z_{\mu\nu}^2}{c^2+\lambda}\right)\frac{1}{(a^2+\lambda)(c^2+\lambda)^{1/2}} .$$

The last term vanishes for all points, whether inside or outside of the spheroid, because of equations (2.5) and (2.6), hence

$$x''_\mu = -2\pi\rho a^2 c\, x_{\mu\nu} \int_\lambda^\infty \frac{du}{(a^2+u)^2(c^2+u)^{1/2}} , \tag{2.8}$$

and similarly

$$y''_\mu = -2\pi\rho a^2 c\, y_{\mu\nu} \int_\lambda^\infty \frac{du}{(a^2+u)^2(c^2+u)^{1/2}} ,$$

$$z''_\mu = -2\pi\rho a^2 c\, z_{\mu\nu} \int_\lambda^\infty \frac{du}{(a^2+u)(c^2+u)^{3/2}} . \tag{2.9}$$

Comparison of equations (2.4),(2.8) and (2.9) yields

$$\Phi = -\pi\rho a^2 c \int_\lambda^\infty \frac{du}{(a^2+u)(c^2+u)^{1/2}} - \frac{1}{2}\left(\underset{\sim}{r}_{\mu\nu} \cdot \underset{\sim}{r}''_\mu\right) , \tag{2.10}$$

where

$$\underset{\sim}{r}_{\mu\nu} \equiv (x_{\mu\nu}, y_{\mu\nu}, z_{\mu\nu}) .$$

The evaluation of the integrals occurring in the foregoing equations becomes easier with the help of a new independent variable defined by the following continuous transformation

$$\delta_{\mu\nu}=\arcsin\frac{\alpha e}{(\alpha^2+u)^{1/2}} \quad .$$

This variable satisfies the relations

$$\sin^2\delta_{\mu\nu}=(\alpha e)^2(\alpha^2+u)^{-1},$$
$$\tan^2\delta_{\mu\nu}=(\alpha e)^2(c^2+u)^{-1}.$$

Introducing these equations into equations (2.8), (2.9) and (2.10) and carrying out the integrations, we obtain

$$x''_{\mu}=-2\pi e^{-3}(1-e^2)^{1/2}\rho\, x_{\mu\nu}(\theta_{\mu\nu}-\sin\theta_{\mu\nu}\cos\theta_{\mu\nu}),$$

$$y''_{\mu}=-2\pi e^{-3}(1-e^2)^{1/2}\rho\, y_{\mu\nu}(\theta_{\mu\nu}-\sin\theta_{\mu\nu}\cos\theta_{\mu\nu}),$$

$$Z''_{\mu}=-4\pi e^{-3}(1-e^2)^{\frac{1}{2}}\rho\, Z_{\mu\nu}(\tan\theta_{\mu\nu}-\theta_{\mu\nu}), \qquad (2.11)$$

$$\phi=-2\pi e^{-1}(1-e^2)^{1/2}\alpha^2\rho\,\theta_{\mu\nu}-\frac{1}{2}(\underline{r}_{\mu\nu}\cdot\underline{r}''_{\mu}),$$

where

$$\theta_{\mu\nu}=\arcsin\frac{\alpha e}{\sqrt{\alpha^2+\lambda}} \quad . \qquad (2.12)$$

Obviously the parameter $\theta_{\mu\nu}$ satisfies the relations

$$\sin^2\theta_{\mu\nu}=\frac{(\alpha e)^2}{\alpha^2+\lambda},\quad \tan^2\theta_{\mu\nu}=\frac{(\alpha e)^2}{c^2+\lambda} \quad . \qquad (2.13)$$

Since the mass of a homogeneous oblate spheroid is given by equation

$$m=\frac{4}{3}\pi\rho\alpha^3(1-e^2)^{\frac{1}{2}}, \qquad (2.14)$$

equations (2.11) can take the form

$$x''_\mu = -\frac{3}{2} m (\alpha e)^{-3} x_{\mu\nu} (\theta_{\mu\nu} - \sin\theta_{\mu\nu}\cos\theta_{\mu\nu}),$$

$$y''_\mu = -\frac{3}{2} m (\alpha e)^{-3} y_{\mu\nu} (\theta_{\mu\nu} - \sin\theta_{\mu\nu}\cos\theta_{\mu\nu}),$$

$$z''_\mu = -3 m (\alpha e)^{-3} z_{\mu\nu} (\tan\theta_{\mu\nu} - \theta_{\mu\nu}), \tag{2.15}$$

$$\Phi = -\frac{1}{2}\left(3m(\alpha e)^{-1}\theta_{\mu\nu} + (\underset{\sim}{r}_{\mu\nu} \cdot \underset{\sim}{r}''_\mu)\right).$$

The parameter $\theta_{\mu\nu}$ can be evaluated as follows: for a test point inside the spheroid equation (2.5) holds and hence equation (2.12) becomes

$$\theta_{\mu\nu} = \arcsin(e), \tag{2.16}$$

while for a test point (x_μ, y_μ, z_μ) located outside the spheroid, λ is the positive root of equation (2.6), which upon substitution of equations (2.13) becomes

$$R^2_{\mu\nu}\sin^2\theta_{\mu\nu} + Z^2_{\mu\nu}\tan^2\theta_{\mu\nu} = (\alpha e)^2. \tag{2.17}$$

In this case $\theta_{\mu\nu}$ can be found in two ways; either by solving equation (2.6) with respect to $(\alpha^2+\lambda)$ and substituting the resulting expression into equation (2.12), or by solving equation (2.17) with respect to $\theta_{\mu\nu}$. For this purpose we rewrite equation (2.17) in the following form

$$R^2_{\mu\nu}\sin^2\theta_{\mu\nu} + Z^2_{\mu\nu}\frac{\sin^2\theta_{\mu\nu}}{1-\sin^2\theta_{\mu\nu}} = (\alpha e)^2.$$

Introducing the notation $r^2_{\mu\nu} = R^2_{\mu\nu} + Z^2_{\mu\nu}$ and rearranging the above equation, we obtain

$$R^2_{\mu\nu}\sin^4\theta_{\mu\nu} - (r^2_{\mu\nu} + \alpha^2 e^2)\sin^2\theta_{\mu\nu} + (\alpha e)^2 = 0,$$

The solution of this equation with respect to $\sin^2 \theta_{\mu\nu}$ gives the following two roots

$$\sin^2\theta_{\mu\nu}=\frac{1}{2R^2_{\mu\nu}}\left((r^2_{\mu\nu}+\alpha^2e^2)-\sqrt{(r^2_{\mu\nu}+\alpha^2e^2)^2-4R^2_{\mu\nu}\alpha^2e^2}\right),\ R_{\mu\nu}\neq 0,$$

$$\sin^2\theta_{\mu\nu}=\frac{1}{2R^2_{\mu\nu}}\left((r^2_{\mu\nu}+\alpha^2e^2)+\sqrt{(r^2_{\mu\nu}+\alpha^2e^2)^2-4R^2_{\mu\nu}\alpha^2e^2}\right),\ R_{\mu\nu}\neq 0. \tag{2.18}$$

From these roots we discard the second as not compatible with equations (2.6) and (2.12), at least for points located on the XY plane where $Z_{\mu\nu}=0$. For these points equation (2.12) in combination with equation (2.6) gives

$$\sin^2\theta_{\mu\nu}=\frac{\alpha^2e^2}{R^2_{\mu\nu}},\ R_{\mu\nu}\neq 0,\ Z_{\mu\nu}=0,$$

while the second of equations (2.18) gives $\sin^2\theta_{\mu\nu}=1$, $Z_{\mu\nu}=0$. On the other hand the first of these equations agrees completely with the relations (2.6) and (2.12) and implies that

$$\sin\theta_{\mu\nu}=\frac{1}{R_{\mu\nu}}\sqrt{\frac{(r^2_{\mu\nu}+\alpha^2e^2)-\sqrt{(r^2_{\mu\nu}+\alpha^2e^2)^2-4R^2_{\mu\nu}\alpha^2e^2}}{2}},\ R_{\mu\nu}\neq 0. \tag{2.19}$$

We choose the positive root, since the parameter $\theta_{\mu\nu}$, according to equation (2.12), obeys the relation

$$0<\theta_{\mu\nu}\leqslant\arcsin(e)<\frac{\pi}{2}, \tag{2.20}$$

for every value of η in the semi-open interval $[0,+\infty)$.

Solving equation (2.19) with respect to $\theta_{\mu\nu}$, we obtain

$$\theta_{\mu\nu}=\arcsin\left\{\frac{1}{R_{\mu\nu}}\sqrt{\frac{(r^2_{\mu\nu}+\alpha^2e^2)-\sqrt{(r^2_{\mu\nu}+\alpha^2e^2)^2-4R^2_{\mu\nu}\alpha^2e^2}}{2}}\right\},\ R_{\mu\nu}\neq 0. \tag{2.21}$$

It is obvious from equations (2.17) and (2.20) that if $R_{\mu\nu}=0$, $\theta_{\mu\nu}$ is given by the expression

$$\theta_{\mu\nu}=\arctan\left(\alpha e/|Z_{\mu\nu}|\right),\ R_{\mu\nu}=0. \tag{2.22}$$

In the limiting case, when the test point is on the surface of the spheroid, equation (2.21) reduces to the relation $\theta_{\mu\nu}=\arcsin(e)$. Thus equations (2.16) and (2.21) guarantee the continuity of $\theta_{\mu\nu}$ in passing across the boundary of the spheroid.

Equations (2.15), (2.16), (2.21) and (2.22) give the force and potential law of a homogeneous oblate spheroid.

We proceed with a very interesting problem, that is the evaluation of the potential energy and the force acting on two spheroids, due to their mutual gravitational attraction. The general problem, as it will become clear later on, is insoluble; we therefore confine ourselves to some particular cases.

Firstly we will evaluate the force acting on a homogeneous oblate spheroid and its potential energy, due to the presence of another homogeneous oblate spheroid, when the equatorial planes of both bodies coincide.

For the sake of convenience we assign to each spheroid an index, say i and j, and we shall refer to them by this index. Also, unless otherwise stated, hereafter the term "spheroid" will be taken to mean "oblate spheroid".

We choose as system of reference a Cartesian right-handed inertial system with its XY plane coinciding with the equatorial planes of the bodies. The used conventional notation is given below:

The position vectors of the centres of the spheroids are

$$\underset{\sim}{r}_i \equiv (x_i, y_i, z_i), \quad \underset{\sim}{r}_j \equiv (x_j, y_j, z_j),$$

while the vector $\underset{\sim}{r}_{ij}$ and its norm are defined as follows

$$\underset{\sim}{r}_{ij} \equiv \underset{\sim}{r}_i - \underset{\sim}{r}_j \equiv (x_{ij}, y_{ij}, z_{ij}), \quad r_{ij} \equiv |\underset{\sim}{r}_{ij}| \equiv \sqrt{x_{ij}^2 + y_{ij}^2 + z_{ij}^2}\,. \tag{2.23}$$

It is obvious from the position of the bodies that $z_{ij}=0$. Hence

$$\underset{\sim}{r}_{ij} = \underset{\sim}{R}_{ij} \equiv (x_{ij}, y_{ij}, 0), \quad r_{ij} = R_{ij} \equiv |\underset{\sim}{R}_{ij}| = \sqrt{x_{ij}^2 + y_{ij}^2}\,. \tag{2.24}$$

The masses of the bodies are m_i and m_j and their uniform densities ρ_i and ρ_j.

We symbolize by α_i and c_i the semi-major and semi-minor axes of the i^{th} spheroid, and by α_j and c_j the same axes of the j^{th} spheroid, while their eccentricities are e_i and e_j correspondingly.

As in the derivation of the potential and force law of a homogeneous oblate spheroid, we adopt units of mass, distance and time so that the gravitational constant K has numerical value equal to one.

Let us now consider a thin homoeoid of mass δm_j and semi-major and semi-minor axes θ_j and c_j^* correspondingly, where $\theta_j < \alpha_j$. Let this homoeoid have the density ρ_j of the jth spheroid and be similar, similarly situated and concentric with it. Evidently the eccentricity of this homoeoid is equal to e_j. Hereafter with the term "j^{th} homoeoid" we mean the above described shell.

According to the first of the theorems given at the beginning of this chapter the j^{th} homoeoid will produce around it a potential field with equipotential surfaces the confocal with it oblate spheroids. Suppose that one of these equipotential surfaces has semi-major and semi-minor axes α and c correspondingly. We shall call this surface "J^{th} spheroid". The semi-axes of this spheroid satisfy the relations

$$\alpha^2 = \theta_j^2 + \lambda, \quad c^2 = c_j^{*2} + \lambda, \tag{2.25}$$

where λ is a parameter obeying the relation $\lambda > -c_j^{*2}$. From the above two equations one easily concludes that

$$\alpha^2 - c^2 = \theta_j^2 e_j^2,$$

therefore

$$c = \sqrt{\alpha^2 - \theta_j^2 e_j^2}. \tag{2.26}$$

The potential of the j^{th} homoeoid at an external point located on its equipotential surface with semi-major axis α (J^{th} spheroid) is given by the formula

$$\varphi_{\delta m_j} = -\frac{\delta m_j}{2}\int_{\lambda}^{\infty}\frac{du}{(\theta_j^2+u)(c_j^{*2}+u)^{1/2}} = -\frac{\delta m_j}{2}\left(\frac{-2}{\sqrt{\theta_j^2-c_j^{*2}}}\arctan\sqrt{\frac{\theta_j^2-c_j^{*2}}{c_j^{*2}+u}}\right)_{\lambda}^{\infty}, \tag{2.27}$$

where λ is the parameter appearing in equations (2.25). Introducing these equations into equation (2.27) and arranging the resulting expression, we obtain

$$\varphi_{\delta m_j} = -\frac{\delta m_j}{\theta_j e_j} \alpha \arcsin\left(\frac{\theta_j e_j}{\alpha}\right). \tag{2.28}$$

The forces f_x, f_y and f_z acting along the X, Y and Z axes on a particle of mass equal to the unit of masses and located on the above described equipotential surface at a point, say $(x_\kappa, y_\kappa, z_\kappa)$, will be given by the partial derivatives of the expression (2.28) with respect to $x_{\kappa j} \equiv x_\kappa - x_j$ $y_{\kappa j} \equiv y_\kappa - y_j$ and $z_{\kappa j} \equiv z_\kappa - z_j$. That is

$$f_x = -\frac{\partial \varphi_{\delta m_j}}{\partial x_{\kappa j}} = \delta m_j (\theta_j e_j)^{-1} \frac{\partial\left(\alpha \arcsin\frac{\theta_j e_j}{\alpha}\right)}{\partial \alpha} \frac{\partial \alpha}{\partial x_{\kappa j}},$$

$$f_y = -\frac{\partial \varphi_{\delta m_j}}{\partial y_{\kappa j}} = \delta m_j (\theta_j e_j)^{-1} \frac{\partial\left(\alpha \arcsin\frac{\theta_j e_j}{\alpha}\right)}{\partial \alpha} \frac{\partial \alpha}{\partial y_{\kappa j}}, \tag{2.29}$$

$$f_z = -\frac{\partial \varphi_{\delta m_j}}{\partial z_{\kappa j}} = \delta m_j (\theta_j e_j)^{-1} \frac{\partial\left(\alpha \arcsin\frac{\theta_j e_j}{\alpha}\right)}{\partial \alpha} \frac{\partial \alpha}{\partial z_{\kappa j}}.$$

The partial derivatives $\partial\alpha/\partial x_{\kappa j}$, $\partial\alpha/\partial y_{\kappa j}$ and $\partial\alpha/\partial z_{\kappa j}$ can be found with the help of the equation describing the equipotential surface on which the particle is located.

$$\frac{x_{\kappa j}^2}{\alpha^2} + \frac{y_{\kappa j}^2}{\alpha^2} + \frac{z_{\kappa j}^2}{c^2} = 1.$$

Using equation (2.26) the above equation can be rewritten as follows

$$\alpha^4 - \alpha^2(r_{\kappa j}^2 + \theta_j^2 e_j^2) + (x_{\kappa j}^2 + y_{\kappa j}^2)\theta_j^2 e_j^2 = 0, \tag{2.30}$$

where $r_{\kappa j}^2 = (x_{\kappa j}^2 + y_{\kappa j}^2 + z_{\kappa j}^2)$. Combining equations (2.29) and (2.30), and after some calculations we obtain the relations

$$f_x = -\sum m_j x_{\kappa j} \frac{c}{a^2(a^2+c^2-r_{\kappa j}^2)} ,$$

$$f_y = -\sum m_j y_{\kappa j} \frac{c}{a^2(a^2+c^2-r_{\kappa j}^2)} ,$$

$$f_z = -\sum m_j z_{\kappa j} \frac{1}{c(a^2+c^2-r_{\kappa j}^2)} .$$

We note that, according to equations (2.26) and (2.30), a and c are given by the relations

$$a = \sqrt{\frac{r_{\kappa j}^2+\theta_j^2 e_j^2 + \sqrt{(r_{\kappa j}^2+\theta_j^2 e_j^2)^2 - 4(x_{\kappa j}^2+y_{\kappa j}^2)\theta_j^2 e_j^2}}{2}} ,$$

$$c = \sqrt{\frac{r_{\kappa j}^2-\theta_j^2 e_j^2 + \sqrt{(r_{\kappa j}^2+\theta_j^2 e_j^2)^2 - 4(x_{\kappa j}^2+y_{\kappa j}^2)\theta_j^2 e_j^2}}{2}} . \qquad (2.31)$$

Let us now assume that the semi-major axis of the J^{th} spheroid obeys the relation

$$R_{ij} - a_i \leqslant a \leqslant R_{ij} + a_i .$$

The internal and external boundaries of an infinitesimally thin focaloid, whose internal boundary is the surface of the J^{th} spheroid, will have semi-major axes a and $a+da$. By virtue of the foregoing relation this focaloid intersects the i^{th} spheroid. Figure (1) shows the intersections of i^{th}, j^{th}, J^{th} spheroids and the j^{th} homoeoid with the XY plane.

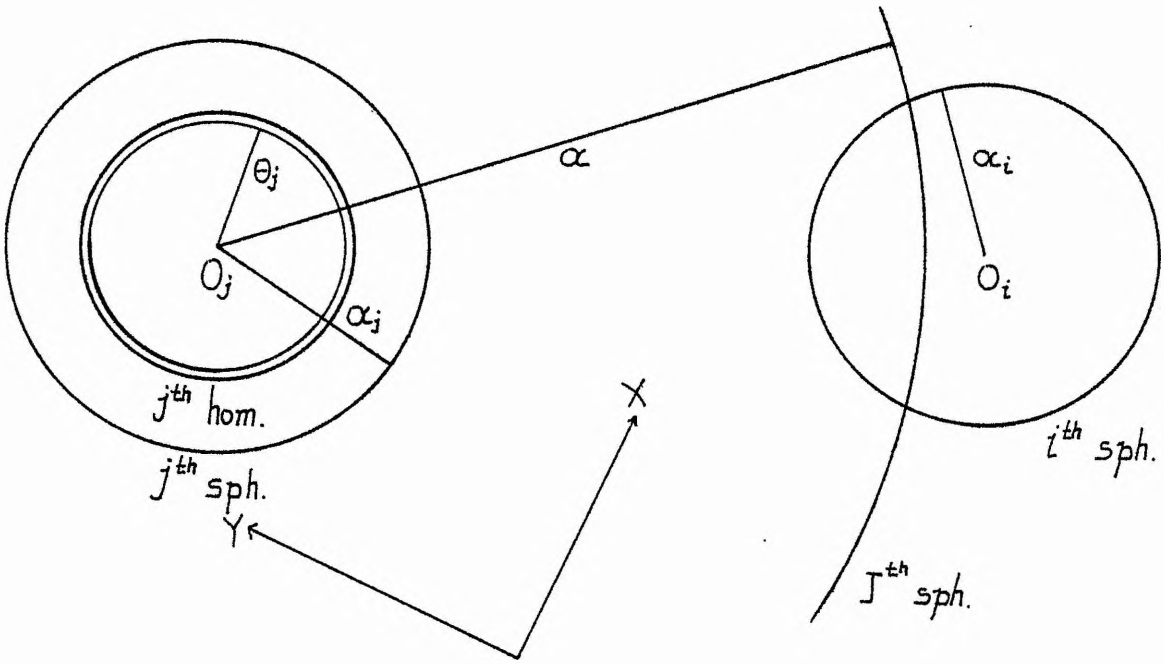


Figure 1

We define V to be the common volume between the i^{th} and J^{th} spheroids. According to equation (A4-41) of appendix A4 this volume is a function of the parameters α, c, α_i, c_i and R_{ij}, that is

$$V = V(\alpha, c, \alpha_i, c_i, R_{ij}) .$$

We also define dV^* to be the elementary volume of the i^{th} spheroid confined by the internal and external surfaces of the above described focaloid. From the above two definitions we readily conclude that dV^* is the infinitesimal change of the volume V when the semi-major axis of the J^{th} spheroid increases by $d\alpha$. Since V is a function of more than one parameter the above described relation between V and dV^* can mathematically be expressed by the relation

$$dV^* = \frac{\partial V}{\partial \alpha} d\alpha. \tag{2.32}$$

The infinitesimal mass dm_i included in the volume dV^* is given by the relation

$$dm_i = \rho_i \, dV^*.$$

Remembering the definitions of the focaloid the boundaries of which confine

dV^* and the J^{th} spheroid it is easy to show that the mass dm_i is distributed on one of the equipotential surfaces of the j^{th} homoeoid and that the constant potential along this surface is given by equation (2.28). Therefore, the elementary mass dm_i, being in the gravitational field of the j^{th} homoeoid, has potential energy

$$d\Phi_{i\delta m_j} = \varphi_{\delta m_j} dm_i = -\rho_i \delta m_j (\theta_j e_j)^{-1} \left(\arcsin \frac{\theta_j e_j}{\alpha}\right) dV^*,$$

which, upon substitution of equation (2.32) becomes

$$d\Phi_{i\delta m_j} = -\rho_i \delta m_j (\theta_j e_j)^{-1} \left(\arcsin \frac{\theta_j e_j}{\alpha}\right) \frac{\partial V}{\partial \alpha} d\alpha .$$

Integration of the foregoing equation over α with lower and upper limits $R_{ij}-\alpha_i$ and $R_{ij}+\alpha_i$ correspondingly gives the total potential energy of the i^{th} spheroid due to the presence of the j^{th} homoeoid

$$\Phi_{i\delta m_j} = -\rho_i \frac{\delta m_j}{\theta_j e_j} \int_{R_{ij}-\alpha_i}^{R_{ij}+\alpha_i} \frac{\partial V}{\partial \alpha} \arcsin \frac{\theta_j e_j}{\alpha} d\alpha . \tag{2.33}$$

Let us now consider the thickness of the j^{th} homoeoid infinitesimally small, $d\theta_j$ suppose. In this case its volume and its mass are given by the relations

$$dV_j = 4\pi(1-e_j^2)^{\frac{1}{2}} \theta_j^2 d\theta_j \;,\; dm_j = 4\pi(1-e_j^2)^{\frac{1}{2}} \rho_j \theta_j^2 d\theta_j . \tag{2.34}$$

Introducing the second of the equations (2.34) into equation (2.33) we obtain the potential energy $d\Phi_i$ of the i^{th} spheroid, being in the gravitational field of the infinitesimally thin homoeoid

$$d\Phi_i = -4\pi(1-e_j^2)^{1/2} \rho_i \rho_j \frac{\theta_j}{e_j} \left\{ \int_{R_{ij}-\alpha_i}^{R_{ij}+\alpha_i} \frac{\partial V}{\partial \alpha} \arcsin \frac{\theta_j e_j}{\alpha} d\alpha \right\} d\theta_j .$$

Integration of this equation with respect to θ_j, with lower and upper limits

0 and α_j, gives the potential energy of either of the oblate spheroids, i and j, due to the presence of the other

$$\phi_i = -4\pi(1-e_j^2)^{1/2} e_j^{-1} \rho_i \rho_j \int_0^{\alpha_j} \int_{R_{ij}-\alpha_i}^{R_{ij}+\alpha_i} \theta_j \frac{\partial V}{\partial \alpha} \arcsin\left(\frac{\theta_j e_j}{\alpha}\right) d\alpha\, d\theta_j \,. \tag{2.35}$$

The second of equations (2.24) implies the relations

$$\frac{\partial R_{ij}}{\partial x_{ij}} = \frac{x_{ij}}{R_{ij}}, \quad \frac{\partial R_{ij}}{\partial y_{ij}} = \frac{y_{ij}}{R_{ij}}, \quad \frac{\partial R_{ij}}{\partial z_{ij}} = 0 \,. \tag{2.36}$$

The force components F_{ix}, F_{iy} and F_{iz}, acting on the i^{th} spheroid along the X, Y and Z axes as a result of the attraction from the j^{th} spheroid, can be found with the help of equations (2.35), (2.36), (A6-10) and (A6-11); combining these equations we obtain

$$F_{ix} = 4\pi(1-e_j^2)^{1/2} e_j^{-1} \rho_i \rho_j \frac{x_{ij}}{R_{ij}} \frac{\partial}{\partial R_{ij}} \left\{ \int_0^{\alpha_j} \int_{R_{ij}-\alpha_i}^{R_{ij}+\alpha_i} \theta_j \frac{\partial V}{\partial \alpha} \arcsin\left(\frac{\theta_j e_j}{\alpha}\right) d\alpha\, d\theta_j \right\},$$

$$F_{iy} = 4\pi(1-e_j^2)^{1/2} e_j^{-1} \rho_i \rho_j \frac{y_{ij}}{R_{ij}} \frac{\partial}{\partial R_{ij}} \left\{ \int_0^{\alpha_j} \int_{R_{ij}-\alpha_i}^{R_{ij}+\alpha_i} \theta_j \frac{\partial V}{\partial \alpha} \arcsin\left(\frac{\theta_j e_j}{\alpha}\right) d\alpha\, d\theta_j \right\}, \tag{2.37}$$

$$F_{iz} = 0 \,.$$

The following general relation

$$\frac{d}{d\omega}\left\{ \int_{\psi^*(\omega)}^{\psi(\omega)} f(\omega,t)\, dt \right\} = f(\omega,\psi(\omega)) \frac{d\psi(\omega)}{d\omega} - f(\omega,\psi^*(\omega)) \frac{d\psi^*(\omega)}{d\omega} + \int_{\psi^*(\omega)}^{\psi(\omega)} \frac{\partial f(\omega,t)}{\partial \omega}\, dt \,, \tag{2.38}$$

is valid for every interval $(\psi(\omega), \psi^*(\omega))$ in which the function $f(\omega,t)$ has no singularity or discontinuity, properties which the function appearing under the integral signs in equations (2.37) holds in the interval $(R_{ij}-\alpha_i, R_{ij}+\alpha_i)$ as can be seen, at once, from equation (A4-43). We can therefore apply the above formula in the case of equations (2.37) to obtain

$$F_{ix}=4\pi\rho_i\rho_j(1-e_j^2)^{1/2}e_j^{-1}\frac{x_{ij}}{R_{ij}}\int_0^{\alpha_j}\int_{R_{ij}-\alpha_i}^{R_{ij}+\alpha_i}\theta_j\frac{\partial^2 V}{\partial R_{ij}\partial\alpha}\arcsin\left(\frac{\theta_j e_j}{\alpha}\right)d\alpha\, d\theta_j,$$

$$F_{iy}=4\pi\rho_i\rho_j(1-e_j^2)^{1/2}e_j^{-1}\frac{y_{ij}}{R_{ij}}\int_0^{\alpha_j}\int_{R_{ij}-\alpha_i}^{R_{ij}+\alpha_i}\theta_j\frac{\partial^2 V}{\partial R_{ij}\partial\alpha}\arcsin\left(\frac{\theta_j e_j}{\alpha}\right)d\alpha\, d\theta_j, \tag{2.39}$$

$$F_{iz}=0.$$

In deriving the above equations use has been made of relation (A4-48).

The nature of the function $V=V(\alpha, c, \alpha_i, c_i, R_{ij})$ and the relations (A4-41), (A4-42) and (A4-43) guarantee the continuity of this function with respect to the parameters α and R_{ij} in the box $\alpha\in(\alpha_j, +\infty)$ $R_{ij}\in(\alpha_i+\alpha_j, +\infty)$. They also guarantee the continuity and existence of the derivatives $\partial V/\partial\alpha$, $\partial V/\partial R_{ij}$ and $\partial^2 V/\partial R_{ij}\partial\alpha$ in the same box. Hence the existence of $\partial^2 V/\partial\alpha\partial R_{ij}$ is secured and

$$\frac{\partial^2 V}{\partial R_{ij}\partial\alpha}=\frac{\partial^2 V}{\partial\alpha\partial R_{ij}}.$$

By virtue of this equation the inside integrals of equations (2.39) become

$$I\equiv\int_{R_{ij}-\alpha_i}^{R_{ij}+\alpha_i}\theta_j\frac{\partial^2 V}{\partial R_{ij}\partial\alpha}\arcsin\left(\frac{\theta_j e_j}{\alpha}\right)d\alpha=\int_{R_{ij}-\alpha_i}^{R_{ij}+\alpha_i}\theta_j\frac{\partial^2 V}{\partial\alpha\partial R_{ij}}\arcsin\left(\frac{\theta_j e_j}{\alpha}\right)d\alpha. \tag{2.40}$$

Since the only variable in the above integral is the parameter α we can rewrite it in the form

$$I = \int_{R_{ij}-\alpha_i}^{R_{ij}+\alpha_i} \theta_j \arcsin\left(\frac{\theta_j e_j}{\alpha}\right) d\left(\frac{\partial V}{\partial R_{ij}}\right),$$

where $d(\partial V/\partial R_{ij})$ is the differential of the function $\partial V/\partial R_{ij}$ with respect to α. Integrating by parts and making use of equation (A4-49) we obtain

$$I = \int_{R_{ij}-\alpha_i}^{R_{ij}+\alpha_i} \frac{e_j\theta_j^2}{\alpha\sqrt{\alpha^2-\theta_j^2 e_j^2}} \frac{\partial V}{\partial R_{ij}} d\alpha .$$

Comparing this relation with equations (2.39) and (2.40) we readily conclude

$$F_{ix} = 4\pi\rho_j\rho_i (1-e_j^2)^{1/2} \frac{x_{ij}}{R_{ij}} \int_0^{\alpha_j} \int_{R_{ij}-\alpha_i}^{R_{ij}+\alpha_i} \frac{\theta_j^2}{\alpha\sqrt{\alpha^2-\theta_j^2 e_j^2}} \frac{\partial V}{\partial R_{ij}} d\alpha\, d\theta_j ,$$

$$F_{iy} = 4\pi\rho_i\rho_j (1-e_j^2)^{1/2} \frac{y_{ij}}{R_{ij}} \int_0^{\alpha_j} \int_{R_{ij}-\alpha_i}^{R_{ij}+\alpha_i} \frac{\theta_j^2}{\alpha\sqrt{\alpha^2-\theta_j^2 e_j^2}} \frac{\partial V}{\partial R_{ij}} d\alpha\, d\theta_j , \tag{2.41}$$

$$F_{iz} = 0 .$$

Due to the orientation and the symmetry of the bodies, the magnitude F_i of the total force $\underset{\sim}{F_i}$ acting on the i^{th} spheroid, being in the gravitational field of the j^{th} spheroid, must satisfy the relations

$$F_{ix} = -\frac{x_{ij}}{R_{ij}} F_i , \quad F_{iy} = -\frac{y_{ij}}{R_{ij}} F_i , \quad F_{iz} = -\frac{z_{ij}}{R_{ij}} F_i = 0 \tag{2.42}$$

Comparison of these equations with equations (2.41), yields

$$F_i = -4\pi \rho_i \rho_j (1-e_j^2)^{1/2} \int_0^{\alpha_j} \int_{R_{ij}-\alpha_i}^{R_{ij}+\alpha_i} \frac{\theta_j^2}{\alpha\sqrt{\alpha^2-\theta_j^2 e_j^2}} \frac{\partial V}{\partial R_{ij}} \, d\alpha \, d\theta_j . \tag{2.43}$$

We note here that the derivative $\partial V/\partial R_{ij}$ is obviously negative (the function $V=V(R_{ij})$ is a decreasing one), thus justifying the minus sign in the above expression.

In order to evaluate the double integral occurring in the right-hand side of the foregoing equation, the first problem is clearly one of expressing the derivative $\partial V/\partial R_{ij}$ analytically, in terms of the parameters α_j, θ_j, e_j α_i, c_i and R_{ij} . The answer to this problem is given by the relation (A4-42) of the appendix (A4). Substituting this relation in equation (2.43) and remembering also that

$$\rho_i = \frac{3m_i}{4\pi\alpha_i^2 c_i} , \quad \rho_j = \frac{3m_j}{4\pi\alpha_j^3 (1-e_j^2)^{1/2}} , \tag{2.44}$$

we have

$$F_i = \frac{3m_i m_j \sqrt{R_{ij}}}{\pi\alpha_j^3 \alpha_i^2} \int_0^{\alpha_j} \int_{R_{ij}-\alpha_i}^{R_{ij}+\alpha_i} \frac{\theta_j^2}{\alpha\sqrt{\eta+A}\sqrt{(\alpha^2-\theta_j^2 e_j^2)\alpha_i^2 - c_i^2\alpha^2}} \Big\{ \Big[2(\xi+A)(\eta+2A)+ \tag{2.45}$$

$$+(\Gamma+\alpha^2) \Big] K(\omega) - 4(\eta+A)(\xi+A)E(\omega) \Big\} d\alpha \, d\theta_j ,$$

where ξ, A, Γ, η and ω are given by equations (A4-44), (A4-45), (A4-46) and (A4-47). The functions $K(\omega)$ and $E(\omega)$ are the complete elliptic integrals of the first and second kinds respectively.

Further reduction of the above formula can be obtained by the following method. Consider a homogeneous oblate spheroid concentric and similarly situated with the i^{th} spheroid, and let its semi-major and semi-minor axis be

$(\alpha_i e_i)$ and C^*. Under the assumption that the mass of this spheroid (hereafter i^{th} spheroid) is equal to m_i we may deduce the force $\underset{\sim}{F}_i^*$ exerted on it by the j^{th} spheroid with the help of equation (2.45). According to this equation the magnitude F_i^* of force $\underset{\sim}{F}_i^*$ is given by

$$F_i^* = \frac{3 m_i m_j \sqrt{R_{ij}}}{\pi \alpha_j^3 (\alpha_i e_i)^2} \int_0^{\alpha_j} \int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i} \frac{\theta_j^2}{\alpha \sqrt{\mu+\beta} \sqrt{(\alpha^2-\theta_j^2 e_j^2)\alpha_i^2 e_i^2 - c^{*2}\alpha^2}} \Big\{ \Big[(\gamma+\alpha^2) + \tag{2.46}$$

$$+ 2(\beta+\zeta)(\mu+2\beta) \Big] K(\omega) - 4(\mu+\beta)(\zeta+\beta) E(\omega) \Big\} \, d\alpha \, d\theta_j \, ,$$

where, according to equations (A4-44), (A4-45), (A4-46) and (A4-47), $\zeta, \beta, \gamma, \mu$ and ω are given by the relations

$$\zeta \equiv \frac{R_{ij}^2 + \alpha^2 - \alpha_i^2 e_i^2}{2 R_{ij}} \, ,$$

$$\beta \equiv R_{ij} \frac{c^{*2}\alpha^2}{(c^2 \alpha_i^2 e_i^2 - c^{*2}\alpha^2)} \, , \quad \gamma \equiv \alpha^2 \frac{(c^2 - c^{*2})\alpha_i^2 e_i^2 + R_{ij}^2 c^{*2}}{(c^2 \alpha_i^2 e_i^2 - \alpha^2 c^{*2})} \, , \tag{2.47}$$

$$\mu \equiv -R_{ij} \frac{c^{*2}\alpha^2}{(c^2 \alpha_i^2 e_i^2 - \alpha^2 c^{*2})} + \alpha \alpha_i e_i \frac{\sqrt{(R_{ij}^2 - \alpha^2 - \alpha_i^2 e_i^2) c^2 c^{*2} + (c^4 \alpha_i^2 e_i^2 + c^{*4}\alpha^2)}}{(c^2 \alpha_i^2 e_i^2 - \alpha^2 c^{*2})} \, ,$$

$$\omega \equiv \sqrt{\frac{\mu - \zeta}{2(\mu+\beta)}} \, .$$

The limit of the function F_i^*, defined by equation (2.46), as $C^* \longrightarrow 0$ exists and has an important physical meaning. Its existence is guaranteed by the so called "theorem of dominated convergence", which (in a modified form) states that a function $f(x,t)$ of x, depending on a parameter t and continuous with respect to x in an open interval (α', β'), obeys the relation

$$\lim_{t\to\tau}\int_{\alpha'}^{\beta'} f(x,t)\,dx = \int_{\alpha'}^{\beta'} \lim_{t\to\tau} f(x,t)\,dx\,, \tag{2.48}$$

provided the limits and integrals appearing in this equation exist.

The integrand of the expression (2.46) satisfies the presuppositions of the foregoing integral and hence

$$\lim_{C^*\to 0} F_i^* = \frac{3m_i m_j R_{ij}^{1/2}}{\pi\alpha_j^3(\alpha_i e_i)^2}\int_0^{\alpha_j}\int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i}\lim_{C^*\to 0}\left\{\frac{\theta_j^2}{\alpha\sqrt{\mu+\beta}\,\sqrt{(\alpha^2-\theta_j^2 e_j^2)\alpha_i^2 e_i^2 - c^{*2}\alpha^2}}\left(\left[(\gamma+\alpha^2)+ +2(\beta+\zeta)(\mu+2\beta)\right]K(\omega)-4(\mu+\beta)(\zeta+\beta)E(\omega)\right)\right\}. \tag{2.49}$$

The relations

$$\lim_{C^*\to 0}\zeta = \zeta \qquad \lim_{C^*\to 0}\beta = 0 \qquad \lim_{C^*\to 0}\gamma = \alpha^2 \qquad \lim_{C^*\to 0}\mu = \alpha\,, \tag{2.50}$$

$$\lim_{C^*\to 0}\omega = u \equiv \sqrt{\frac{\alpha-\zeta}{2\alpha}}\,,$$

can easily be obtained from equations (2.47). On the other hand, application of the formula (2.48) in the case of the $K(\omega)$ and $E(\omega)$ complete elliptic integrals furnishes the following relations

$$\lim_{C^*\to 0} K(\omega) = K(u) \qquad \lim_{C^*\to 0} E(\omega) = E(u)\,. \tag{2.51}$$

By use of equations (2.50) and (2.51) the formula (2.49) reduces to

$$\lim_{C^*\to 0} F_i^* = \frac{6m_i m_j\sqrt{R_{ij}}}{\pi\alpha_j^3(\alpha_i e_i)^3}\int_0^{\alpha_j}\int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i}\frac{\theta_j^2}{\sqrt{\alpha(\alpha^2-\theta_j^2 e_j^2)}}\Big[(\zeta+\alpha)K(u)-2\zeta E(u)\Big]\,d\alpha\, d\theta_j\,.$$

Interchanging the order of integration in this formula we obtain

$$\lim_{C^*\to 0} F_i^* = \frac{6 m_i m_j \sqrt{R_{ij}}}{\pi \alpha_j^3 (\alpha_i e_i)^3} \int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i} \left\{ \frac{1}{\sqrt{\alpha}} \left[(\zeta+\alpha) K(u) - 2\zeta E(u) \right] \int_0^{\alpha_j} \frac{\theta_j^2}{\sqrt{\alpha^2-\theta_j^2 e_j^2}} d\theta_j \right\} d\alpha, \tag{2.52}$$

which upon substitution of the relation

$$\int_0^{\alpha_j} \frac{\theta_j^2}{\sqrt{\alpha^2-\theta_j^2 e_j^2}} d\theta_j = \frac{1}{2e_j^3} \left(\alpha^2 \arcsin \frac{\alpha_j e_j}{\alpha} - \alpha_j e_j \sqrt{\alpha^2-\alpha_j^2 e_j^2} \right),$$

yields

$$\lim_{C^*\to 0} F_i^* = \frac{3 m_i m_j \sqrt{R_{ij}}}{\pi (\alpha_i e_i)^3 (\alpha_j e_j)^3} \int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i} \frac{1}{\sqrt{\alpha}} \left(\alpha^2 \arcsin \frac{\alpha_j e_j}{\alpha} - \sqrt{\alpha^2-\alpha_j^2 e_j^2}\, \alpha_j e_j \right) \left[(\zeta+\alpha) K(u) - 2\zeta E(u) \right] d\alpha. \tag{2.53}$$

Now consider the physical meaning of the limit $\lim_{C^*\to 0} F_i^*$; since the i^{th} and ii^{th} oblate spheroids are concentric and similarly situated, as $C^*\to 0$ under the assumption that the total mass of the ii^{th} spheroid remains constant this spheroid tends to a disc with axis $(\alpha_i e_i)$, which according to the law of confocality is the innermost limiting form of the spheroids confocal with the i^{th} spheroid and whose mass is m_i. Therefore, according to MacLaurin's theorem given in the beginning of this chapter, the forces exerted by the j^{th} spheroid on the above described disc and the i^{th} spheroid are equal. That is

$$\lim_{C^*\to 0} F_i^* = F_i , \tag{2.54}$$

which upon substitution of equation (2.53), becomes

$$F_i = \frac{3m_i m_j \sqrt{R_{ij}}}{\pi(\alpha_i e_i)^3(\alpha_j e_j)^3} \int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i} \frac{1}{\sqrt{\alpha}} \left(\alpha^2 \arcsin\frac{\alpha_j e_j}{\alpha} - \alpha_j e_j \sqrt{\alpha^2 - \alpha_j^2 e_j^2} \right) \times \tag{2.55}$$

$$\times \left[(\zeta + \alpha) K(u) - 2\zeta E(u) \right] d\alpha .$$

The above integral not being elementary cannot be expressed in finite terms, it is therefore the simplest form the force between two homogeneous oblate spheroids with coplanar equatorial planes can take. However, a series expansion of the foregoing expression is possible. The use of this series expansion in numerical applications has two obvious advantages; (a) if it converges rapidly enough, is less time consuming than a direct numerical integration of the integral (2.55), and (b) relations of the form

$$|e| \leqslant \varepsilon, \quad |e^*| \leqslant \varepsilon^*, \tag{2.56}$$

for the overall and relative errors e and e^*, due to the truncation of the series, can be found, where ε and ε^* are parameters whose values depend on $\alpha_i, e_i, \alpha_j, e_j, R_{ij}$ and on the number of terms of the series used in the evaluation of F_i.

Numerical integration of the expression (2.55) for a range of different values of the parameters $\alpha_i, e_i, \alpha_j, e_j$ and R_{ij} and different methods of integration shows that, for the method giving the best results, at least an 800-point division of the interval of integration is required to obtain values for magnitudes F_i and F_j agreeing up to 10^{-13}. F_i and F_j are the magnitudes of the forces acting on the two oblate spheroids due to their mutual gravitational attraction. The cpu time needed for the calculations to be carried out on an IBM 360/44 computer is 2.99 minutes, while using the series expansion (2.70) the required time is 0.03 seconds and the overall error satisfies the relation

$$|e| \leqslant 10^{-16}.$$

Further details on these problems are given in chapter (4).

In order to obtain a series expansion of the expression (2.55) we firstly consider the non elementary integral

$$I \equiv \int_{\zeta}^{\alpha} \sqrt{\frac{\alpha^2-y^2}{y-\zeta}}\, dy, \tag{2.57}$$

which can be rewritten in the form

$$I \equiv \alpha^2 \int_{\zeta}^{\alpha} \frac{dy}{\sqrt{-(y-\alpha)(y-\zeta)(y+\alpha)}} - \int_{\zeta}^{\alpha} \frac{y^2 dy}{\sqrt{-(y-\alpha)(y-\zeta)(y+\alpha)}}.$$

This relation by virtue of equation (A4-24) yields

$$I \equiv \frac{2}{3}\alpha^2 \int_{\zeta}^{\alpha} \frac{dy}{\sqrt{-(y-\alpha)(y-\zeta)(y+\alpha)}} - \frac{2}{3}\zeta \int_{\zeta}^{\alpha} \frac{y\, dy}{\sqrt{-(y-\alpha)(y-\zeta)(y+\alpha)}}. \tag{2.58}$$

Reduction of the above two integrals to Legendre's standard form is accomplished by means of the continuous transformation

$$y = \alpha \frac{(\alpha-\zeta)\sin^2\varphi + 2\zeta}{2\alpha-(\alpha-\zeta)\sin^2\varphi}, \quad y \in [\alpha, \zeta]. \tag{2.59}$$

This transformation implies the relation

$$[\zeta, \alpha] \ni y \Rightarrow \varphi \in [0, \pi/2].$$

Introducing this transformation into equation (2.58), carrying out the integration and arranging the resulting expression we finally obtain

$$\frac{3\sqrt{2}}{4\alpha} \int_{\zeta}^{\alpha} \sqrt{\frac{\alpha^2-y^2}{y-\zeta}}\, dy = \frac{1}{\sqrt{\alpha}}\left[(\alpha+\zeta)K(u) - 2\zeta E(u)\right], \tag{2.60}$$

where $K(u)$ and $E(u)$ are, as in equation (2.55), the complete elliptic integrals of the first and second kinds. The modulus u is given by the last of equations (2.50).

Combining equations (2.52), (2.54) and (2.60) we obtain

$$F_i=\frac{9m_i m_j\sqrt{R_{ij}}}{\sqrt{2}\,\pi\alpha_j^3(\alpha_i e_i)^3}\int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i}\frac{1}{\alpha^2}\int_{\zeta}^{\alpha}\sqrt{\frac{\alpha^2-y^2}{y-\zeta}}\int_0^{\alpha_j}\frac{\theta_j^2}{\sqrt{1-\left(\frac{\theta_j e_j}{\alpha}\right)^2}}\,d\theta_j\,dy\,d\alpha\,. \tag{2.61}$$

In order to obtain the magnitude of the force $\underset{\sim}{F}_i$ in the form of a convergent series we start with the well-known expansion

$$\frac{1}{\sqrt{1-\left(\frac{\theta_j e_j}{\alpha}\right)^2}}=\sum_{m=0}^{\infty}\frac{(2m)!}{(m!)^2 2^{2m}}\left(\frac{\theta_j e_j}{\alpha}\right)^{2m}. \tag{2.62}$$

The foregoing series expansion is an absolutely convergent one for $\theta_j e_j < \alpha$. Since, in the numerical applications, $\theta_j e_j \ll \alpha$ it is obvious that this series converges very rapidly. Comparing equations (2.61) and (2.62) we readily conclude

$$F_i=\frac{9m_i m_j\sqrt{R_{ij}}}{\sqrt{2}\,\pi(\alpha_i e_i)^3(\alpha_j e_j)^3}\sum_{m=0}^{\infty}\frac{(2m)!\,e_j^{2m+3}}{2^{2m}(m!)^2}\left[\int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i}\frac{1}{\alpha^{2(m+1)}}\int_{\zeta}^{\alpha}\sqrt{\frac{\alpha^2-y^2}{y-\zeta}}\int_0^{\alpha_j}\theta_j^{2(m+1)}\,d\theta_j\,dy\,d\alpha\right]. \tag{2.63}$$

In deriving this equation we used term by term integration, since the presupposition for such integration, that is the convergence of the series, is satisfied. Carrying out the integration with respect to θ_j equation (2.63) takes the form

$$F_i=\frac{9m_i m_j\sqrt{R_{ij}}}{\sqrt{2}\,\pi(\alpha_i e_i)^3(\alpha_j e_j)^3}\sum_{m=0}^{\infty}\frac{(2m)!\,(\alpha_j e_j)^{2m+3}}{2^{2m}(m!)^2\,(2m+3)}\,W_m(R_{ij},\alpha_i,e_i)\,, \tag{2.64}$$

where

$$W_m(R_{ij},\alpha_i,e_i) \equiv \int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i} \frac{1}{\alpha^{2(m+1)}} \int_{\zeta}^{\alpha} \sqrt{\frac{\alpha^2-y^2}{y-\zeta}}\, dy\, d\alpha\ , \quad m=0,1,2,3,\cdots \tag{2.65}$$

We note here that the first term of the series occurring in equation (2.64) coincides with the magnitude of the force acting on the i^{th} spheroid being in the gravitational field of a particle of mass m_j, whose distance from the centre of the spheroid is R_{ij} and that from its equatorial plane is equal to zero. This can be seen at once from equations (2.61) and (2.63) if we choose the value zero for the eccentricity e_j.

Although the inner integral in equation (2.65) is elliptic, the double integral appearing in this equation is elementary and can be evaluated using the continuous transformation

$$\sqrt{\frac{\alpha^2-y^2}{y-\zeta}} = \sqrt{2R_{ij}}\,\tan\varphi\ , \quad y\in[\zeta,\alpha]\,. \tag{2.66}$$

This transformation implies the relations

$$[\zeta,\alpha]\ni y \Rightarrow \varphi\in[\pi/2,0]\ ,$$

$$y=-R_{ij}\tan^2\varphi \pm \sqrt{R_{ij}^2\tan^4\varphi+\alpha^2+2R_{ij}\zeta\tan^2\varphi}\ .$$

Introducing the first of equations (2.47) into the foregoing relation and remembering also that according to relation $0<\zeta\leqslant y\leqslant\alpha$, y must be positive, we have

$$y=-R_{ij}\tan^2\varphi+\frac{1}{\cos\varphi}\sqrt{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)+\alpha^2}\ , \quad \varphi\in[\pi/2,0]\ , \tag{2.67}$$

from which we easily obtain

$$dy=-2R_{ij}\frac{\tan\varphi}{\cos^2\varphi}\,d\varphi+\frac{\tan\varphi}{\cos\varphi}\,\frac{\alpha^2 d\varphi}{\sqrt{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)+\alpha^2}}+$$

$$+\frac{\tan\varphi}{\cos\varphi}\left(R_{ij}^2\tan^2\varphi+\frac{R_{ij}^2}{\cos^2\varphi}-\alpha_i^2e_i^2\right)\frac{d\varphi}{\sqrt{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)+\alpha^2}}\,, \tag{2.68}$$

$$\varphi\in[\pi/2,0].$$

Using equations (2.66) and (2.68) we can rewrite equation (2.65) in the form

$$W_m(R_{ij},\alpha_i,e_i)=\sqrt{2R_{ij}}\lim_{\substack{\varepsilon^*\to\frac{\pi}{2}-\\ \varepsilon\to 0+}}\left\{-2R_{ij}\int_{R_{ij}-\alpha_ie_i}^{R_{ij}+\alpha_ie_i}\frac{1}{\alpha^{2(m+1)}}\int_{\varepsilon^*}^{\varepsilon}\frac{\tan^2\varphi}{\cos^2\varphi}\,d\varphi\,d\alpha+\right.$$

$$+\int_{R_{ij}-\alpha_ie_i}^{R_{ij}+\alpha_ie_i}\frac{1}{\alpha^{2m}}\int_{\varepsilon^*}^{\varepsilon}\frac{\tan^2\varphi}{\cos\varphi\sqrt{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)+\alpha^2}}\,d\varphi\,d\alpha+$$

$$\left.\int_{R_{ij}-\alpha_ie_i}^{R_{ij}+\alpha_ie_i}\frac{1}{\alpha^{2(m+1)}}\int_{\varepsilon^*}^{\varepsilon}\frac{\tan^2\varphi}{\cos\varphi}\left(R_{ij}^2\tan^2\varphi+\frac{R_{ij}^2}{\cos^2\varphi}-\alpha_i^2e_i^2\right)\frac{d\varphi\,d\alpha}{\sqrt{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)+\alpha^2}}\right\},$$

$$m=0,1,2,\ldots$$

In the above formula the double integral $W_m(R_{ij},\alpha_i,e_i)$ has been expressed in terms of a limit to avoid the divergence of the inner integrals, occurring in the right-hand member of the formula, at the points 0 and $\pi/2$. Inverting the limits ε^* and ε of these integrals and interchanging the order of integration in the double integrals, we obtain

$$W_m(R_{ij},\alpha_i,e_i)=\sqrt{2R_{ij}}\lim_{\substack{\varepsilon\to 0+\\ \varepsilon^*\to\pi/2-}}\left\{2R_{ij}\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi}{\cos^2\varphi}d\varphi\int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i}\alpha^{-2(m+1)}d\alpha-\right.$$

$$-\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi}{\cos\varphi}\int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i}\frac{1}{\alpha^{2m}\sqrt{R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi+\alpha^2}}\,d\alpha\,d\varphi- \tag{2.69}$$

$$\left.-\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi}{\cos\varphi}\left(R_{ij}^2\tan^2\varphi+\frac{R_{ij}^2}{\cos^2\varphi}-\alpha_i^2e_i^2\right)\int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i}\frac{1}{\alpha^{2(m+1)}\sqrt{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)+\alpha^2}}\,d\alpha\,d\varphi\right\},$$

$$m=0,1,2,\cdots\cdots .$$

Evaluation of the first seven integrals $W_m(R_{ij},\alpha_i,e_i)$ for m = 0,1,2,3,4,5,6 is given in appendix (A5).

Introduction of equations (A5-14), (A5-17), (A5-20), (A5-23) and (A5-24) into equation (2.64), and after some algebra, yields

$$F_i=3m_im_jR_{ij}\left\{\frac{1}{2(\alpha_ie_i)^3}\left[\arcsin\left(\frac{\alpha_ie_i}{R_{ij}}\right)-\frac{\alpha_ie_i}{R_{ij}^2}\sqrt{R_{ij}^2-\alpha_i^2e_i^2}\right]+\right.$$

$$+\frac{(\alpha_je_j)^2}{(R_{ij}^2-\alpha_i^2e_i^2)\sqrt{R_{ij}^2-\alpha_i^2e_i^2}}\left[\frac{1}{10R_{ij}^2}+\frac{3(\alpha_je_j)^2}{56(R_{ij}^2-\alpha_i^2e_i^2)^2}+\frac{(\alpha_je_j)^4}{144(R_{ij}^2-\alpha_i^2e_i^2)^4}(5R_{ij}^2+4\alpha_i^2e_i^2)+\right.$$

$$+\frac{(\alpha_je_j)^6}{1{,}408\,(R_{ij}^2-\alpha_i^2e_i^2)^6}\left(35R_{ij}^4+84R_{ij}^2\alpha_i^2e_i^2+24\alpha_i^4e_i^4\right)+ \tag{2.70}$$

$$+\frac{3(\alpha_j e_j)^8}{16{,}640(R_{ij}^2-\alpha_i^2 e_i^2)^8}\left(105R_{ij}^6+504R_{ij}^4\alpha_i^2e_i^2+432R_{ij}^2\alpha_i^4e_i^4+64\alpha_i^6e_i^6\right)+$$

$$+\frac{(\alpha_j e_j)^{10}}{15{,}360(R_{ij}^2-\alpha_i^2 e_i^2)^{10}}\Big(231R_{ij}^8+1{,}848R_{ij}^6\alpha_i^2e_i^2+3{,}168R_{ij}^4\alpha_i^4e_i^4+$$

$$\left.\left.+1{,}408R_{ij}^2\alpha_i^6e_i^6+128\,\alpha_i^8e_i^8\Big)\right]\right\}+$$

$$+\frac{9m_i m_j\sqrt{R_{ij}}}{\sqrt{2}\,\pi\,(\alpha_i e_i)^3}\sum_{m=7}^{\infty}\frac{(2m)!\,(\alpha_j e_j)^{2m}}{2^{2m}(m!)^2(2m+3)}\,W_m(R_{ij},\alpha_i,e_i)\,.$$

Formulae (2.69) and (2.70) imply that the force between two homogeneous oblate spheroids with coplanar equatorial planes is completely specified when the distance of their centres and the quantities $(\alpha_i e_i)$ and $(\alpha_j e_j)$ are known. Further, since the semi-axes of confocal spheroids satisfy the relation

$$\alpha_1^2-c_1^2=\alpha_2^2-c_2^2=\cdots\cdots=\text{constant},$$

this force remains unchanged if we substitute one or both of the bodies by any one of their confocal spheroids. We note that formula (2.70) must be symmetrical with respect to $(\alpha_i e_i)$ and $(\alpha_j e_j)$ that is

$$F_i=F_j=3m_i m_j R_{ij}\left[\frac{1}{2(\alpha_j e_j)^3}\left[\alpha\arcsin\left(\frac{\alpha_j e_j}{R_{ij}}\right)-\frac{\alpha_j e_j}{R_{ij}^2}\sqrt{R_{ij}^2-\alpha_j^2e_j^2}\right]+\right.$$

$$+\frac{(\alpha_i e_i)^2}{(R_{ij}^2-\alpha_j^2 e_j^2)\sqrt{R_{ij}^2-\alpha_j^2 e_j^2}}\left[\frac{1}{10R_{ij}^2}+\frac{3(\alpha_i e_i)^2}{56(R_{ij}^2-\alpha_j^2 e_j^2)^2}+\frac{(\alpha_i e_i)^4}{144(R_{ij}^2-\alpha_j^2 e_j^2)^4}(5R_{ij}^2+4\alpha_j^2 e_j^2)+\right. \tag{2.71}$$

$$+\frac{(\alpha_i e_i)^6}{1,408\,(R_{ij}^2-\alpha_j^2 e_j^2)^6}(35R_{ij}^4+84R_{ij}^2\alpha_j^2 e_j^2+24\alpha_j^4 e_j^4)+$$

$$+\frac{3(\alpha_i e_i)^8}{16,640\,(R_{ij}^2-\alpha_j^2 e_j^2)^8}(105R_{ij}^6+504R_{ij}^4\alpha_j^2 e_j^2+432R_{ij}^2\alpha_j^4 e_j^4+64\alpha_j^6 e_j^6)+$$

$$+\frac{(\alpha_i e_i)^{10}}{15,360(R_{ij}^2-\alpha_j^2 e_j^2)^{10}}(231R_{ij}^8+1,848R_{ij}^6\alpha_j^2 e_j^2+3,168R_{ij}^4\alpha_j^4 e_j^4+$$

$$\left.\left.+1,408R_{ij}^2\alpha_j^6 e_j^6+128\alpha_j^8 e_j^8)\right]\right\}+$$

$$+\frac{9m_i m_j\sqrt{R_{ij}}}{\sqrt{2}\,\pi(\alpha_j e_j)^3}\sum_{m=7}^{\infty}\frac{(2m)!\,(\alpha_i e_i)^{2m}}{2^{2m}(m!)^2(2m+3)}W_m(R_{ij},\alpha_j,e_j).$$

The series expansions (2.70) and (2.71) are equivalent; however, the rates of convergence, depending on the values of the parameters α_i, e_i, α_j e_j and R_{ij}, are not the same; in fact, if $\alpha_i e_i > \alpha_j e_j$ then series (2.70) converges more rapidly than (2.71).

In the numerical applications we truncate the above series after its ν^{th} term, introducing in this way an error e, obeying the first of the relations (2.56). Proof of this relation is given in appendix (A7).

The discussions of this chapter provide a complete mathematical technique for describing the motion of two homogeneous oblate spheroids with coplanar equatorial planes under their mutual gravitational attraction.

According to theorem (3) given at the beginning of this chapter, we have

$$F_{ix}=m_i x_i'' , \quad F_{iy}=m_i y_i'' , \quad F_{iz}=m_i z_i'' . \tag{2.72}$$

Combining equations (2.42), (2.70), (2.72) and remembering that $Z_{ij}=0$, we obtain the relations

$$\begin{aligned}
\underset{\sim}{R}_i'' &\equiv (x_i'', y_i'') = -3\underset{\sim}{R}_{ij} m_j \left\{ \frac{1}{2(\alpha_i e_i)^3}\left[\arcsin\left(\frac{\alpha_i e_i}{R_{ij}}\right) - \frac{\alpha_i e_i}{R_{ij}^2}\sqrt{R_{ij}^2-\alpha_i^2 e_i^2}\right] + \right. \\
&+ \frac{(\alpha_j e_j)^2}{(R_{ij}^2-\alpha_i^2 e_i^2)\sqrt{R_{ij}^2-\alpha_i^2 e_i^2}}\left[\frac{1}{10R_{ij}^2} + \frac{3(\alpha_j e_j)^2}{56(R_{ij}^2-\alpha_i^2 e_i^2)^2} + \frac{(\alpha_j e_j)^4}{144(R_{ij}^2-\alpha_i^2 e_i^2)^4}\left(5R_{ij}^2+4\alpha_i^2 e_i^2\right) + \right. \\
&+ \frac{(\alpha_j e_j)^6}{1,408(R_{ij}^2-\alpha_i^2 e_i^2)^6}\left(35R_{ij}^4+84R_{ij}^2\alpha_i^2 e_i^2+24\alpha_i^4 e_i^4\right) + \\
&+ \frac{3(\alpha_j e_j)^8}{16,640(R_{ij}^2-\alpha_i^2 e_i^2)^8}\left(105R_{ij}^6+504R_{ij}^4\alpha_i^2 e_i^2+432R_{ij}^2\alpha_i^4 e_i^4+64\alpha_i^6 e_i^6\right) + \\
&+ \frac{(\alpha_j e_j)^{10}}{15,360(R_{ij}^2-\alpha_i^2 e_i^2)^{10}}\left(231R_{ij}^8+1,848R_{ij}^6\alpha_i^2 e_i^2+3,168R_{ij}^4\alpha_i^4 e_i^4 + \right. \\
&\left.\left.\left. +1,408R_{ij}^2\alpha_i^6 e_i^6+128\alpha_i^8 e_i^8\right)\right]\right\} - \\
&- \frac{9m_j R_{ij}}{\sqrt{2R_{ij}}\,\pi(\alpha_i e_i)^3}\sum_{m=7}^{\infty}\frac{(2m)!\,(\alpha_j e_j)^{2m}}{2^{2m}(m!)^2(2m+3)} W_m(R_{ij}, \alpha_i e_i) ,
\end{aligned} \tag{2.73}$$

$$Z_i''=0 .$$

Similar equations can be found for x_j, y_j, z_j .

$$\underset{\sim}{R}_j'' = 3\underset{\sim}{R}_{ij} m_i \left\{ \frac{1}{2(\alpha_i e_i)^3}\left[\arcsin\left(\frac{\alpha_i e_i}{R_{ij}}\right) - \frac{\alpha_i e_i}{R_{ij}^2}\sqrt{R_{ij}^2-\alpha_i^2 e_i^2}\right] + \right.$$

$$+ \frac{(\alpha_j e_j)^2}{(R_{ij}^2-\alpha_i^2 e_i^2)\sqrt{(R_{ij}^2-\alpha_i^2 e_i^2)}}\left[\frac{1}{10R_{ij}^2} + \frac{3(\alpha_j e_j)^2}{56(R_{ij}^2-\alpha_i^2 e_i^2)^2} + \frac{(\alpha_j e_j)^4}{144(R_{ij}^2-\alpha_i^2 e_i^2)^4}\left(5R_{ij}^2+4\alpha_i^2 e_i^2\right) + \right.$$

$$+ \frac{(\alpha_j e_j)^6}{1{,}408(R_{ij}^2-\alpha_i^2 e_i^2)^6}\left(35R_{ij}^4+84R_{ij}^2\alpha_i^2 e_i^2+24\alpha_i^4 e_i^4\right) +$$

$$+ \frac{3(\alpha_j e_j)^8}{16{,}640(R_{ij}^2-\alpha_i^2 e_i^2)^8}\left(105R_{ij}^6+504R_{ij}^4\alpha_i^2 e_i^2+432R_{ij}^2\alpha_i^4 e_i^4+64\alpha_i^6 e_i^6\right) + \tag{2.74}$$

$$+ \frac{(\alpha_j e_j)^{10}}{15{,}360(R_{ij}^2-\alpha_i^2 e_i^2)^{10}}\left(231R_{ij}^8+1{,}848R_{ij}^6\alpha_i^2 e_i^2+3{,}168R_{ij}^4\alpha_i^4 e_i^4 + \right.$$

$$\left.\left.\left. +1{,}408R_{ij}^2\alpha_i^6 e_i^6+128\alpha_i^8 e_i^8\right)\right]\right\} +$$

$$+ \frac{9m_i\underset{\sim}{R}_{ij}}{\sqrt{2R_{ij}}\,\pi(\alpha_i e_i)^3}\sum_{m=7}^{\infty}\frac{(2m)!\,(\alpha_j e_j)^{2m}}{2^{2m}(m!)^2(2m+3)}W_m\left(R_{ij},\alpha_i,e_i\right),$$

$$z_j'' = 0\,.$$

Equations (2.73) and (2.74) comprise a system of second-order differential equations for the co-ordinates $x_i, y_i, z_i, x_j, y_j, z_j$, which completely specifies the motion of the centres of mass of the spheroids, and therefore the motion of the spheroids themselves since, due to their symmetry and orientation, they undergo only translation on the XY plane.

We proceed with the evaluation of the potential energy Φ_i of either of the oblate spheroids due to the presence of the other.

According to equations (A6-6) and (2.31), Φ_i is a function of the co-ordinates x_{ij}, y_{ij}, z_{ij} and hence its total differential is given by

$$d\Phi_i = \frac{\partial \Phi_i}{\partial x_{ij}} dx_{ij} + \frac{\partial \Phi_i}{\partial y_{ij}} dy_{ij} + \frac{\partial \Phi_i}{\partial z_{ij}} dz_{ij}, \tag{2.75}$$

which, upon substitution of equations (A6-10) and (A6-11), becomes

$$d\Phi_i = -\left(F_{ix}\,dx_{ij} + F_{iy}\,dy_{ij} + F_{iz}\,dz_{ij}\right). \tag{2.76}$$

On introducing equations (2.36) and (2.42) the foregoing equation takes the form

$$d\Phi_i = F_i\left(\frac{\partial R_{ij}}{\partial x_{ij}} dx_{ij} + \frac{\partial R_{ij}}{\partial y_{ij}} dy_{ij}\right) = F_i\,dR_{ij}. \tag{2.77}$$

Combining equations (2.70) and (2.77) we obtain

$$d\Phi_i = \frac{3m_i m_j}{2}\left\{\frac{1}{2(\alpha_i e_i)^3}\left[\arcsin\left(\frac{\alpha_i e_i}{R_{ij}}\right) - \frac{\alpha_i e_i}{R_{ij}^2}\sqrt{R_{ij}^2-\alpha_i^2 e_i^2}\right] + \right.$$

$$+ \frac{(\alpha_j e_j)^2}{(R_{ij}^2-\alpha_i^2 e_i^2)\sqrt{R_{ij}^2-\alpha_i^2 e_i^2}}\left[\frac{1}{10R_{ij}^2} + \frac{3(\alpha_j e_j)^2}{56(R_{ij}^2-\alpha_i^2 e_i^2)^2} + \frac{(\alpha_j e_j)^4}{144(R_{ij}^2-\alpha_i^2 e_i^2)^4}(5R_{ij}^2+4\alpha_i^2 e_i^2) + \right.$$

$$+ \frac{(\alpha_j e_j)^6}{1{,}408(R_{ij}^2-\alpha_i^2 e_i^2)^6}(35R_{ij}^4+84R_{ij}^2\alpha_i^2 e_i^2+24\alpha_i^4 e_i^4) +$$

$$+ \frac{3(\alpha_j e_j)^8}{16{,}640(R_{ij}^2-\alpha_i^2 e_i^2)^8}(105R_{ij}^6+504R_{ij}^4\alpha_i^2 e_i^2+432R_{ij}^2\alpha_i^4 e_i^4+64\alpha_i^6 e_i^6) + \tag{2.78}$$

$$+\frac{(\alpha_j e_j)^{10}}{15{,}360(R_{ij}^2-\alpha_i^2 e_i^2)^{10}}\Big(231R_{ij}^8+1{,}848R_{ij}^6\alpha_i^2e_i^2+3{,}168R_{ij}^4\alpha_i^4e_i^4+$$

$$\left.\left.+1{,}408R_{ij}^2\alpha_i^6e_i^6+128\alpha_i^8e_i^8\Big)\right]\right\}dR_{ij}^2+$$

$$+\frac{9m_im_j\sqrt{R_{ij}}}{\sqrt{2}\,\pi(\alpha_ie_i)^3}\sum_{m=7}^{\infty}\frac{(2m)!\,(\alpha_je_j)^{2m}}{2^{2m}(m!)^2(2m+3)}W_m(R_{ij},\alpha_i,e_i)\,dR_{ij}\,.$$

Integration of the foregoing expression over R_{ij} leads to the evaluation of the following four types of integrals

$$I_1\equiv\int\frac{x^{2K}dx^2}{(x^2-b^2)^{P+1/2}}\,,\quad K\geqslant 0\,,\ x^2>b^2,\ I_2\equiv\int\frac{dx^2}{x^2(x^2-b^2)^{3/2}}\,,\quad x^2>b^2,$$

$$I_3\equiv\int\arcsin\left(\frac{b}{x}\right)x\,dx\,,\qquad I_4\equiv\int\frac{(x^2-b^2)^{1/2}}{x}\,dx\,,\quad x^2>b^2\,.$$

The evaluation of the first of these integrals becomes trivial by observing that

$$I_1=\int\frac{(t+b^2)^K}{t^{P+1/2}}\,dt\,,$$

and expanding the numerator by the binomial theorem, where $t=x^2-b^2$. In fact

$$\int\frac{x^{2K}dx^2}{(x^2-b^2)^{P+1/2}}=2\sum_{V=0}^{K}\binom{K}{V}\frac{b^{2(K-V)}}{(2V-2p+1)}(x^2-b^2)^{\frac{(2V-2p+1)}{2}}\,,\quad x^2>b^2\,. \qquad (2.79)$$

The other three integrals are well known, so we merely give the results of the integration, which can easily be checked by differentiating both sides of the following relations

$$\int \frac{dx^2}{x^2(x^2-b^2)^{3/2}} = -2\frac{1}{b^2}\left[\frac{1}{b}\arccos\left(\frac{b}{x}\right)+\frac{1}{\sqrt{x^2-b^2}}\right], x^2>b^2,$$

$$2\int x\arcsin\left(\frac{b}{x}\right)dx = \left[x^2\arcsin\left(\frac{b}{x}\right)+b\sqrt{x^2-b^2}\right] \quad , \quad x^2 > b^2, \qquad (2.80)$$

$$\int \frac{(x^2-b^2)^{1/2}}{x}dx = \sqrt{x^2-b^2}+b\arcsin\left(\frac{b}{x}\right), x^2>b^2.$$

Integrating equation (2.78) over R_{ij} and after long and laborious algebraic manipulations one can find, with the help of equations (2.79) and (2.80), the following relation which gives the potential energy of two homogeneous oblate spheroids with coplanar equatorial planes

$$\phi_i = 3m_i m_j\left\{\frac{(5R_{ij}^2-10\alpha_i^2e_i^2+2\alpha_j^2e_j^2)}{20(\alpha_ie_i)^3}\arcsin\left(\frac{\alpha_ie_i}{R_{ij}}\right)-\right.$$

$$-\frac{(5R_{ij}^2-5\alpha_i^2e_i^2+2\alpha_j^2e_j^2)}{20(\alpha_ie_i)^2\sqrt{R_{ij}^2-\alpha_i^2e_i^2}}-\frac{(\alpha_je_j)^4}{8(R_{ij}^2-\alpha_i^2e_i^2)^2\sqrt{R_{ij}^2-\alpha_i^2e_i^2}}\left[\frac{3}{35}+\right.$$

$$+\frac{5(\alpha_je_j)^2}{126(R_{ij}^2-\alpha_i^2e_i^2)}+\frac{(\alpha_je_j)^2}{1{,}584(R_{ij}^2-\alpha_i^2e_i^2)^2}(88\alpha_i^2e_i^2+35\alpha_j^2e_j^2)+$$

$$+\frac{7(\alpha_je_j)^4}{4{,}576(R_{ij}^2-\alpha_i^2e_i^2)^3}(52\alpha_i^2e_i^2+9\alpha_j^2e_j^2)+$$

$$+\frac{(\alpha_j e_j)^4}{8{,}320(R_{ij}^2-\alpha_i^2e_i^2)^4}\left(520\alpha_i^4e_i^4+756\alpha_i^2e_i^2\alpha_j^2e_j^2+77\alpha_j^4e_j^4\right)+ \tag{2.81}$$

$$+\frac{(\alpha_j e_j)^6(\alpha_i e_i)^2}{800(R_{ij}^2-\alpha_i^2e_i^2)^5}\left(135\alpha_i^2e_i^2+77\alpha_j^2e_j^2\right)+\frac{3(\alpha_j e_j)^6(\alpha_i e_i)^4}{320(R_{ij}^2-\alpha_i^2e_i^2)^6}\left(10\alpha_i^2e_i^2+33\alpha_j^2e_j^2\right)+$$

$$\left.+\frac{187(\alpha_j e_j)^8(\alpha_i e_i)^6}{480(R_{ij}^2-\alpha_i^2e_i^2)^7}+\frac{323(\alpha_j e_j)^8(\alpha_i e_i)^8}{1{,}920(R_{ij}^2-\alpha_i^2e_i^2)^8}\right]\Bigg\}+$$

$$+\frac{9m_im_j}{\sqrt{2}\,\pi(\alpha_i e_i)^3}\sum_{m=7}^{\infty}\frac{(2m)!\,(\alpha_j e_j)^{2m}}{2^{2m}(m!)^2(2m+3)}\int\sqrt{R_{ij}}\,W_m(R_{ij},\alpha_i,e_i)\,dR_{ij}+C_1,$$

where C_1 is a constant of integration, whose value can be determined as follows: consider the limit of the potential energy Φ_i as $R_{ij}\to\infty$; by definition

$$\lim_{R_{ij}\to\infty}\Phi_i=0\,. \tag{2.82}$$

On the other hand equation (A7-16) implies the relation

$$\lim_{R_{ij}\to\infty}\int\sqrt{R_{ij}}\,W_m(R_{ij},\alpha_i,e_i)\,dR_{ij}=0\,. \tag{2.83}$$

Taking the limits of both members of equation (2.81) as $R_{ij}\to\infty$ and using the relations (2.82) and (2.83), it can be shown that the constant C_1 has the value

$$C_1=0\,. \tag{2.84}$$

Equations (2.70) and (2.81) complete the solution to the problem of finding the mutual gravitational attraction and potential of two homogeneous oblate spheroids with coplanar equatorial planes.

Figure (2) is the graphical representation of the function

$$\frac{F_i}{m_i m_j} = f(\alpha_i e_i, \alpha_j e_j) ,$$

for $R_{ij}=40$ units of distance (in this particular case $R_{ij}=40\,Kpc$).

The point B corresponds to the force between two particles, and the lines drawn on the surface ABC are the intersections of this surface with the planes described by the relations

$$\alpha_i e_i + \alpha_j e_j = \text{constant} , \quad -\infty < \frac{F_i}{m_i m_j} < \infty .$$

It is evident that as $R_{ij} \to \infty$ the point B moves towards zero and the surface ABC becomes progressively flatter, coinciding with the XY plane in the extreme case when $R_{ij}=\infty$. As $R_{ij} \to 0$ the point B moves towards infinity and the surface ABC rises more and more steeply.

Figure (3) graphically illustrates the relation

$$\frac{F_i}{m_i m_j} = f(\alpha_i e_i, \alpha_j e_j)\Big|_{\alpha_j e_j = 0} ,$$

for different values of R_{ij}.

Non-homogeneous oblate spheroids

Consider a non-homogeneous oblate spheroid (hereafter spheroid A) with semi-major axis α, semi-minor axis c, eccentricity e and its centre located at a point (x_v, y_v, z_v), where the co-ordinates x_v, y_v and z_v are referred to a Cartesian right-handed inertial system of reference with the Z axis

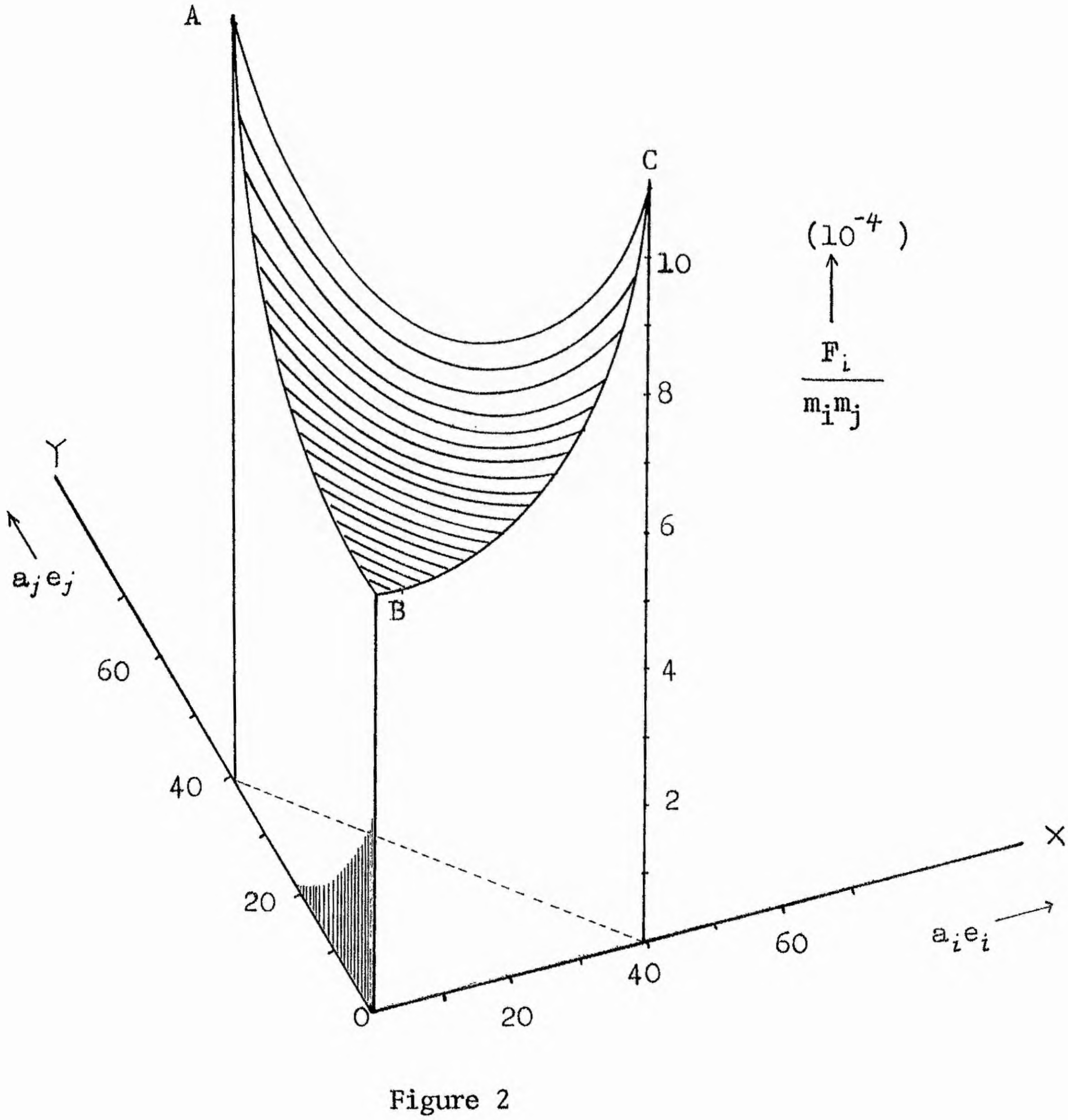


Figure 2

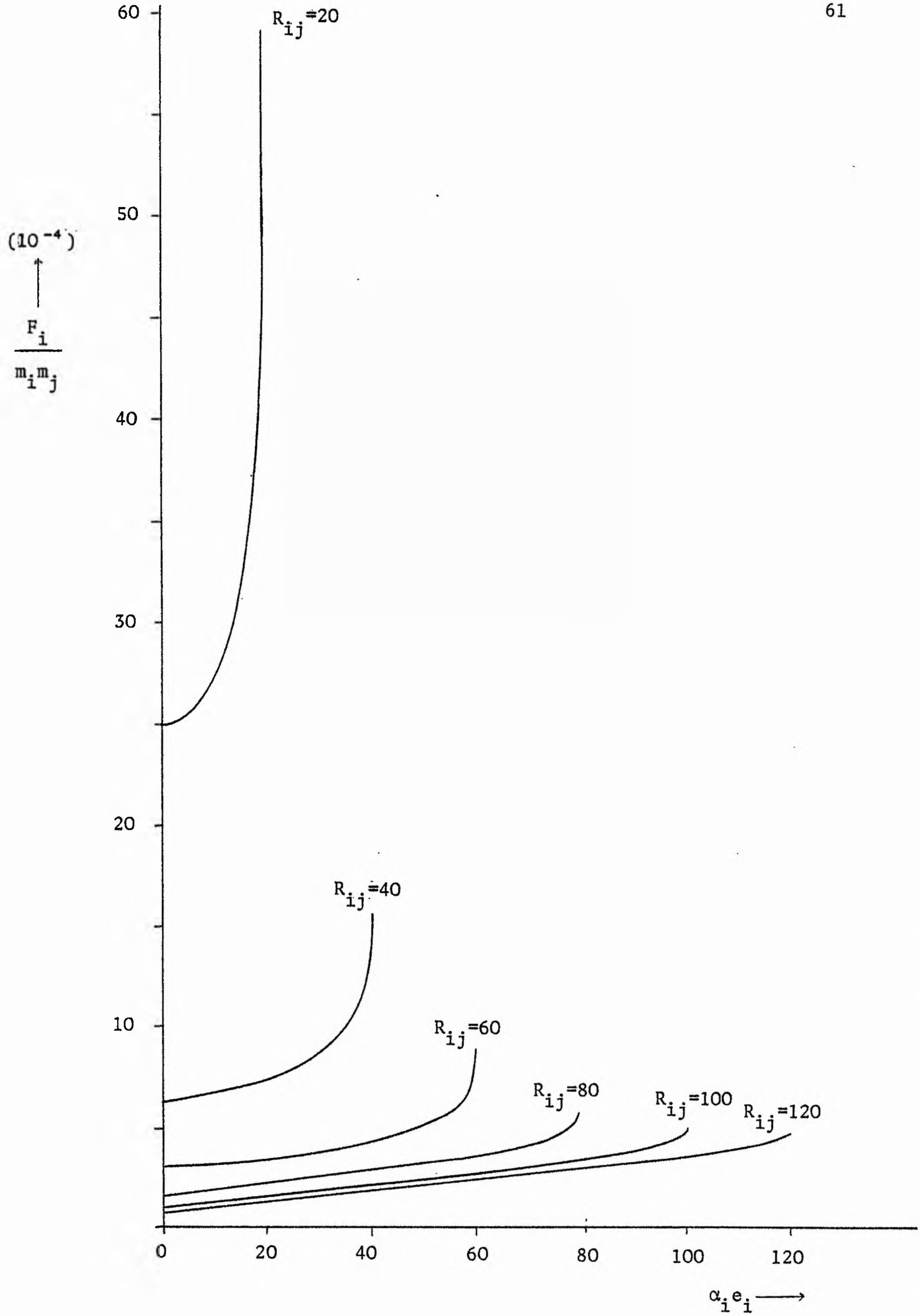

R_{ij}=20
R_{ij}=40
R_{ij}=60
R_{ij}=80
R_{ij}=100
R_{ij}=120
60
50
40
30
20
10
(10^{-4})
$\frac{F_i}{m_i m_j}$
0
20
40
60
80
100
120
$\alpha_i e_i$

Figure 3

parallel to the minor axis of the spheroid and the XY plane parallel to the equatorial plane of the spheroid. The problem of finding the force and potential law of this spheroid varies in difficulty, depending on the density distribution law for the spheroid. In this section we confine ourselves to considering three particular density distribution laws of special physical significance. These laws are described below.

Consider an infinitesimally thin ellipsoidal shell similarly situated with the spheroid A and with semi-major axis $\sigma \leqslant \alpha$, semi-minor axis $c_* \leqslant c$, eccentricity e_*, and assume its density constant. One can build up the spheroid A, using an infinite number of such shells of different density, in at least three ways;

(i) by choosing as shells homoeoids similar to the spheroid,

(ii) by choosing as shells focaloid whose semi-axes satisfy the relations

$$\sigma^2 = \alpha^2 - \lambda \, , \quad c_*^2 = c^2 - \lambda \, , \tag{2.85}$$

where λ is a parameter, and

(iii) by choosing shells with constant semi-major axis

$$\sigma = \alpha ,$$

and eccentricities $1 \geqslant e_* \geqslant e$.

It is obvious that in the first case the density of the spheroid must be a function of the general form

$$\rho = \rho(\sigma) \, , \quad 0 \leqslant \sigma \leqslant \alpha \, , \tag{2.86}$$

and in the second case a function of the form

$$\rho = \rho(\lambda) \, , \quad 0 \leqslant \lambda \leqslant c^2 , \tag{2.87}$$

where λ is the parameter occurring in equations (2.85). In the last case the density distribution law can be expressed by the relation

$$\rho = \rho(e_*) \, , \quad e \leqslant e_* \leqslant 1 . \tag{2.88*}$$

Let a particle of unit mass be located at a point (x_μ, y_μ, z_μ) outside the spheroid A and adopt such units of mass, distance and time that the gravitational constant K has arithmetical value equal to one. The differential homoeoid, whose internal boundary is a spheroid similar to the spheroid A with semi-major axis $\sigma < \alpha$, exerts on the particle (along the X, Y, Z axes) the elementary forces dF_x, dF_y and dF_z . Due to the presence of the homoeoid the particle has potential energy $d\Phi$. The analytical expressions for the forces dF_x, dF_y, dF_z and the potential $d\Phi$ can be found with the aid of equations (2.11), giving the force and potential law of a homogeneous oblate spheroid. Differentiating these equations once with respect to α , using equation (2.17) to reduce their form and changing the symbol α to σ in order to avoid confusion with the semi-major axis of the spheroid A, we obtain

$$
\begin{aligned}
dF_x &= -4\pi e^{-3}(1-e^2)^{1/2} \rho\, x_{\mu\nu} \sin^2 \theta^*_{\mu\nu}\, d\theta^*_{\mu\nu} , \\
dF_y &= -4\pi e^{-3}(1-e^2)^{1/2} \rho\, y_{\mu\nu} \sin^2 \theta^*_{\mu\nu}\, d\theta^*_{\mu\nu} , \\
dF_z &= -4\pi e^{-3}(1-e^2)^{1/2} \rho\, z_{\mu\nu} \tan^2 \theta^*_{\mu\nu}\, d\theta^*_{\mu\nu} , \\
d\Phi &= -4\pi e^{-1}(1-e^2)^{1/2} \rho\, \sigma\, \theta^*_{\mu\nu}\, d\sigma , .
\end{aligned}
\tag{2.88}
$$

where, according to equation (2.17), the parameter $\theta^*_{\mu\nu}$ is related to σ and e through the equation

$$
R^2_{\mu\nu} \sin^2 \theta^*_{\mu\nu} + z^2_{\mu\nu} \tan^2 \theta^*_{\mu\nu} = (\sigma e)^2 . \tag{2.89}
$$

Assume now that the density of the spheroid A, not being uniform is constant in differential shells similar and similarly situated with the spheroid obeying, therefore, a density distribution law of the general form given by equation (2.86); then the force components F_x , F_y , F_z acting along the X, Y, Z axes on the previously mentioned particle and its potential, due to the presence of the spheroid are given by the integrals of equations (2.88) with respect to σ with lower and upper limits 0 and α respectively. That is

$$F_x=-4\pi e^{-3}(1-e^2)^{1/2}x_{\mu\nu}\int_0^{\alpha}\rho(\sigma)\sin^2\theta^*_{\mu\nu}(\sigma)\,d\theta^*_{\mu\nu}(\sigma),$$

$$F_y=-4\pi e^{-3}(1-e^2)^{1/2}y_{\mu\nu}\int_0^{\alpha}\rho(\sigma)\sin^2\theta^*_{\mu\nu}(\sigma)\,d\theta^*_{\mu\nu}(\sigma),$$

(2.90)

$$F_z=-4\pi e^{-3}(1-e^2)^{1/2}z_{\mu\nu}\int_0^{\alpha}\rho(\sigma)\tan^2\theta^*_{\mu\nu}(\sigma)\,d\theta^*_{\mu\nu}(\sigma),$$

$$\phi=-4\pi e^{-1}(1-e^2)^{1/2}\int_0^{\alpha}\rho(\sigma)\sigma\theta^*_{\mu\nu}(\sigma)\,d\sigma.$$

In order to proceed we require the explicit form of the function $\rho=\rho(\sigma)$. From observations in the galactic centre of the rotation curve in the 21 cm line of HI and the intensity distribution of the emission at 2.2μ one can derive, making a few reasonable assumptions, a stellar density distribution described by the formula

$$\rho(\sigma_G)=\frac{7.6\times10^5}{\sigma_G^{1.8}}\,\frac{M_\odot}{pc^3},\quad 0<\sigma_G\leqslant 4\ Kpc, \tag{2.91}$$

where σ_G is the ellipsoidal co-ordinate of the galaxy analogous to the ellipsoidal co-ordinate σ of the spheroid A and it is expressed in pc. If we adopt for the spheroid A a density distribution law of the form (2.91), then the integrals entering the right-hand side of equation (2.90) become insoluble, we therefore choose a density distribution law given by the relation

$$\rho(\sigma)=\frac{C_2}{\sigma^2},\ C_2=\text{constant}, \tag{2.92}$$

which is in reasonable agreement with the formula (2.91) and at the same time facilitates the evaluation of the above mentioned integrals.

Combining equations (2.89), (2.90), (2.92), introducing a new independent variable defined by the relation $u=\beta^*_{\mu\nu}(\sigma)$, and remembering that the particle on which the force components F_x, F_y, F_z are exerted is of unit mass, we obtain

$$X''_\mu=-4\pi e^{-1}(1-e^2)^{1/2}X_{\mu\nu}C_2\int_0^{\gamma_{\mu\nu}}\frac{\cos^2u\,du}{R^2_{\mu\nu}\cos^2u+Z^2_{\mu\nu}},$$

$$Y''_\mu=-4\pi e^{-1}(1-e^2)^{1/2}Y_{\mu\nu}C_2\int_0^{\gamma_{\mu\nu}}\frac{\cos^2u\,du}{R^2_{\mu\nu}\cos^2u+Z^2_{\mu\nu}},$$

$$Z''_\mu=-4\pi e^{-1}(1-e^2)^{1/2}Z_{\mu\nu}C_2\int_0^{\gamma_{\mu\nu}}\frac{du}{R^2_{\mu\nu}\cos^2u+Z^2_{\mu\nu}}, \quad (2.93)$$

$$\Phi=-4\pi e^{-1}(1-e^2)^{1/2}C_2\lim_{\varepsilon\to 0+}\int_\varepsilon^{\alpha}\frac{\beta^*_{\mu\nu}(\sigma)}{\sigma}d\sigma,$$

where the parameter $\gamma_{\mu\nu}$, according to equation (2.89), is given by the relation

$$R^2_{\mu\nu}\sin^2\gamma_{\mu\nu}+Z^2_{\mu\nu}\tan^2\gamma_{\mu\nu}=(\alpha e)^2, \quad (2.94)$$

that is $\gamma_{\mu\nu}=\beta^*_{\mu\nu}(\alpha)$. The potential Φ has been expressed in terms of a limit in order to avoid the singularity of the integrand $\beta^*_{\mu\nu}/\sigma$ at the point $\sigma=0$. It is obvious, from equations (2.93), that when $R_{\mu\nu}=0$

$$X''_\mu=0, \quad Y''_\mu=0, \quad Z''_\mu=-4\pi e^{-1}(1-e^2)^{1/2}\frac{C_2}{Z_{\mu\nu}}\gamma_{\mu\nu}, \quad Z_{\mu\nu}\neq 0, \quad (2.95)$$

while, when $Z_{\mu\nu}=0$,

$$X''_\mu=-4\pi e^{-1}(1-e^2)^{1/2}\frac{C_2X_{\mu\nu}}{R^2_{\mu\nu}}\gamma_{\mu\nu}, \quad Y''_\mu=-4\pi e^{-1}(1-e^2)^{1/2}\frac{C_2Y_{\mu\nu}}{R^2_{\mu\nu}}\gamma_{\mu\nu}, \quad (2.96)$$
$$Z''_\mu=0, \quad R_{\mu\nu}\neq 0.$$

The volume and the mass of a differential spheroidal shell of constant eccentricity e, semi-major axes σ and $\sigma + d\sigma$, and density $\rho(\sigma)$ are given by the formulae

$$dV = 4\pi(1-e^2)^{1/2}\sigma^2 d\sigma, \quad dm = 4\pi(1-e^2)^{1/2}\rho(\sigma)\,\sigma^2 d\sigma . \tag{2.97}$$

It is, therefore, obvious that the mass of the spheroid A is equal to

$$m = 4\pi(1-e^2)^{1/2} C_2 \alpha . \tag{2.98}$$

On the assumption that $R_{\mu\nu} \neq 0$ and $Z_{\mu\nu} \neq 0$, it is not difficult to show that

$$\int_0^{\gamma_{\mu\nu}} \frac{du}{R_{\mu\nu}^2 \cos^2 u + Z_{\mu\nu}^2} = \frac{1}{r_{\mu\nu} Z_{\mu\nu}} \arctan\left(\frac{Z_{\mu\nu}}{r_{\mu\nu}} \tan\gamma_{\mu\nu}\right). \tag{2.99}$$

It is now clear that according to equations (2.98) and (2.99), we can express x''_μ , y''_μ and z''_μ in the form

$$x''_\mu = -\frac{m}{\alpha e}\frac{x_{\mu\nu}}{R_{\mu\nu}^2}\left[\gamma_{\mu\nu} - \frac{Z_{\mu\nu}}{r_{\mu\nu}}\arctan\left(\frac{Z_{\mu\nu}}{r_{\mu\nu}}\tan\gamma_{\mu\nu}\right)\right], \quad R_{\mu\nu} \neq 0,$$

$$y''_\mu = -\frac{m}{\alpha e}\frac{y_{\mu\nu}}{R_{\mu\nu}^2}\left[\gamma_{\mu\nu} - \frac{Z_{\mu\nu}}{r_{\mu\nu}}\arctan\left(\frac{Z_{\mu\nu}}{r_{\mu\nu}}\tan\gamma_{\mu\nu}\right)\right], \quad R_{\mu\nu} \neq 0, \tag{2.100}$$

$$z''_\mu = -\frac{m}{\alpha e}\frac{1}{r_{\mu\nu}}\arctan\left(\frac{Z_{\mu\nu}}{r_{\mu\nu}}\tan\gamma_{\mu\nu}\right) .$$

Before proceeding with the evaluation of the potential Φ we consider the integral

$$I \equiv \int_\varepsilon^\alpha \ln\sigma \, d\beta^*_{\mu\nu}(\sigma) . \tag{2.101}$$

From equation (2.89) it is obvious that

$$\ln\sigma = -\ln e + \frac{1}{2}\ln\left(R_{\mu\nu}^2 \sin^2 \beta_{\mu\nu}^*(\sigma) + Z_{\mu\nu}^2 \tan^2 \beta_{\mu\nu}^*(\sigma)\right) =$$

$$= \frac{1}{2}\ln\frac{r_{\mu\nu}^2 + Z_{\mu\nu}^2}{2e^2} + \ln\tan\beta_{\mu\nu}^*(\sigma) + \frac{1}{2}\ln\left(1 + \frac{R_{\mu\nu}^2}{r_{\mu\nu}^2 + Z_{\mu\nu}^2}\cos 2\beta_{\mu\nu}^*(\sigma)\right), \tag{2.102}$$

$$0 < \sigma \leqslant \alpha .$$

Combining equations (2.101) and (2.102) and introducing a new variable, defined by the relation $\varphi = \beta_{\mu\nu}^*(\sigma)$, we obtain

$$I = \frac{1}{2}\ln\left(\frac{r_{\mu\nu}^2 + Z_{\mu\nu}^2}{2e^2}\right)(\gamma_{\mu\nu} - \delta_{\mu\nu}) + \int_{\delta_{\mu\nu}}^{\gamma_{\mu\nu}} \ln\tan\varphi \, d\varphi +$$

$$+ \frac{1}{4}\int_{\delta_{\mu\nu}}^{\gamma_{\mu\nu}} \ln\left(1 + \frac{R_{\mu\nu}^2}{r_{\mu\nu}^2 + Z_{\mu\nu}^2}\cos 2\varphi\right) d2\varphi , \tag{2.103}$$

where $\delta_{\mu\nu}$ is given by the relation

$$\delta_{\mu\nu} \equiv \beta_{\mu\nu}^*(\varepsilon) . \tag{2.104}$$

The integrals occurring in the right-hand side of equation (2.103) are not elementary; however, their representation by infinite series is possible. The first of these integrals expressed in terms of the Lobatschefsky function takes the form

$$I_1 \equiv \int_{\delta_{\mu\nu}}^{\gamma_{\mu\nu}} \ln\tan\varphi \, d\varphi = \mathcal{L}(\gamma_{\mu\nu}) + \mathcal{L}\left(\frac{\pi}{2} - \gamma_{\mu\nu}\right) - \mathcal{L}(\delta_{\mu\nu}) - \mathcal{L}\left(\frac{\pi}{2} - \delta_{\mu\nu}\right) , \tag{2.105}$$

where, by definition

$$\mathcal{L}(x) = -\int_0^x \ln\cos t \, dt . \tag{2.106}$$

The most common representations of the Lobatschefsky function L(x) by series are the following

$$\mathcal{L}(x) = x\ln 2 - \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin(2kx)}{k^2}, \quad x^2 < \frac{\pi^2}{4},$$

$$\mathcal{L}(x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2^{2k-1}(2^{2k}-1)B_{2k}}{k(2k+1)!}x^{2k+1}, \quad x^2 < \frac{\pi^2}{4} . \tag{2.107}$$

These formulae lend themselves readily to numerical computation; however use of the second formula requires the additional knowledge of the Bernoulli numbers B_{2k} . The conditions

$$|\gamma_{\mu\nu}| < \frac{\pi}{2}, \quad |\delta_{\mu\nu}| < \frac{\pi}{2},$$

necessary for the validity of equation (2.105), can easily be proved as in the case of $\beta_{\mu\nu}$ [see relation (2.20)]. In fact

$$0 < \delta_{\mu\nu} < \gamma_{\mu\nu} \ll \arcsin e < \frac{\pi}{2} . \tag{2.108}$$

The value of the integral I_1 can be obtained in a different form with the aid of the power series expansion

$$\ln\tan\varphi = \ln\varphi + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2^{2k}(2^{2k-1}-1)B_{2k}}{k(2k)!}\varphi^{2k}, \quad 0 < \varphi < \frac{\pi}{2}, \tag{2.109}$$

integration of which over φ , with lower and upper limit $\delta_{\mu\nu}$ and $\gamma_{\mu\nu}$ respectively, yields

$$I_1 = \int_{\delta_{\mu\nu}}^{\gamma_{\mu\nu}} \ln \tan\varphi \, d\varphi = \gamma_{\mu\nu}(\ln \gamma_{\mu\nu} - 1) - \delta_{\mu\nu}(\ln \delta_{\mu\nu} - 1) +$$

$$+ \sum_{K=1}^{\infty} (-1)^{K+1} \frac{2^{2K}(2^{2K-1}-1)B_{2K}}{K(2K+1)!} \left(\gamma_{\mu\nu}^{2K+1} - \delta_{\mu\nu}^{2K+1}\right) . \tag{2.110}$$

We now proceed with the evaluation of the second of the integrals entering the right-hand member of equation (2.103), that is

$$I_2 \equiv \int_{\delta_{\mu\nu}}^{\gamma_{\mu\nu}} \ln\left(1 + \frac{R_{\mu\nu}^2}{r_{\mu\nu}^2 + Z_{\mu\nu}^2} \cos 2\varphi\right) d2\varphi . \tag{2.111}$$

Expanding the integrand according to equation (A3-8) and using term by term integration (since all the series involved converge), we obtain the following relation

$$I_2 = \sum_{K=1}^{\infty} (-1)^{K+1} \frac{R_{\mu\nu}^{2K}}{K(r_{\mu\nu}^2 + Z_{\mu\nu}^2)^K} \int_{\delta_{\mu\nu}}^{\gamma_{\mu\nu}} \cos^K(2\varphi)\, d(2\varphi) , \tag{2.112}$$

which, by introduction of the equations

$$\int \cos^{2n}\omega \, d\omega = \frac{1}{2^{2n}}\binom{2n}{n}\omega + \frac{1}{2^{2n-1}} \sum_{\nu=0}^{n-1} \binom{2n}{\nu} \frac{\sin(2n-2\nu)\omega}{(2n-2\nu)} , \quad n \geq 1 ,$$

$$\int \cos^{2n+1}\omega \, d\omega = \frac{1}{2^{2n}} \sum_{\nu=0}^{n} \binom{2n+1}{\nu} \frac{\sin(2n+1-2\nu)\omega}{(2n+1-2\nu)} , \quad n \geq 0 ,$$

becomes

$$I_2=\sum_{K=1}^{\infty}(-1)^{K+1}\frac{R_{\mu\nu}^{2K}}{K(r_{\mu\nu}^2+Z_{\mu\nu}^2)^K}\Bigg[\zeta^*\frac{\gamma_{\mu\nu}}{2^{K-1}}\binom{K}{\frac{K}{2}}+$$

$$+\frac{1}{2^{K-1}}\sum_{\nu=0}^{\frac{K}{2}-S}\binom{K}{\nu}\frac{\sin 2(K-2\nu)\gamma_{\mu\nu}}{(K-2\nu)}-\zeta^*\frac{\delta_{\mu\nu}}{2^{K-1}}\binom{K}{\frac{K}{2}}- \tag{2.113}$$

$$-\frac{1}{2^{K-1}}\sum_{\nu=0}^{\frac{K}{2}-S}\binom{K}{\nu}\frac{\sin 2(K-2\nu)\delta_{\mu\nu}}{(K-2\nu)}\Bigg].$$

The variables S and ζ^*, appearing in the foregoing equation, are given by the relations

$$\zeta^*=0,\ S=\frac{1}{2}\quad \text{when } K \text{ is odd},$$

$$\zeta^*=1\quad S=1\quad \text{when } K \text{ is even}.$$

The proof of the absolute convergence of the series (2.113) is easy, we can, therefore, re-arrange this series in the extended sense as follows, we consider the two subseries containing the terms of the odd and even indices, that is

$$\sum_{\lambda=0}^{\infty}\frac{R_{\mu\nu}^{4\lambda+2}}{(2\lambda+1)(r_{\mu\nu}^2+Z_{\mu\nu}^2)^{2\lambda+1}}\frac{1}{2^{2\lambda}}\Bigg[\sum_{\nu=0}^{\lambda}\binom{2\lambda+1}{\nu}\frac{\sin 2(2\lambda+1-2\nu)\gamma_{\mu\nu}}{(2\lambda+1-2\nu)}-$$

$$-\sum_{\nu=0}^{\lambda}\binom{2\lambda+1}{\nu}\frac{\sin(2\lambda+1-2\nu)2\delta_{\mu\nu}}{(2\lambda+1-2\nu)}\Bigg],$$

$$\sum_{\lambda=0}^{\infty}\frac{R_{\mu\nu}^{4(\lambda+1)}}{(2\lambda+2)(r_{\mu\nu}^2+Z_{\mu\nu}^2)^{2\lambda+2}}\frac{1}{2^{2\lambda+1}}\Bigg\{\binom{2\lambda+2}{\lambda+1}(\gamma_{\mu\nu}-\delta_{\mu\nu})+$$

$$+\sum_{\nu=0}^{\lambda}\binom{2\lambda+2}{\nu}\frac{1}{2(\lambda+1-\nu)}\Big(\sin 4(\lambda+1-\nu)\gamma_{\mu\nu}-\sin 4(\lambda+1-\nu)\delta_{\mu\nu}\Big)\Bigg\}.$$

Since the index λ denotes a monotonic increasing sequence of different positive integers the foregoing two series are absolutely convergent and their sum is equal to I_2 . That is

$$I_2 = \sum_{\lambda=0}^{\infty} \frac{R_{\mu\nu}^{2(2\lambda+1)}}{(2\lambda+1)(\Gamma_{\mu\nu}^2 + Z_{\mu\nu}^2)^{2\lambda+1}} \frac{1}{2^{2\lambda}} \left\{ \sum_{\nu=0}^{\lambda} \binom{2\lambda+1}{\nu} \frac{\sin 2(2\lambda-2\nu+1)\gamma_{\mu\nu}}{(2\lambda-2\nu+1)} - \right.$$

$$\left. - \sum_{\nu=0}^{\lambda} \binom{2\lambda+1}{\nu} \frac{\sin 2(2\lambda-2\nu+1)\delta_{\mu\nu}}{(2\lambda-2\nu+1)} \right\} +$$

$$+ \sum_{\lambda=0}^{\infty} \frac{R_{\mu\nu}^{4(\lambda+1)}}{(\lambda+1)(\Gamma_{\mu\nu}^2 + Z_{\mu\nu}^2)^{2(\lambda+1)}} \frac{1}{4^{(\lambda+1)}} \left\{ \binom{2\lambda+2}{\lambda+1} (\gamma_{\mu\nu} - \delta_{\mu\nu}) + \right. \tag{2.114}$$

$$\left. + \sum_{\nu=0}^{\lambda} \binom{2\lambda+2}{\nu} \frac{1}{2(\lambda-\nu+1)} \left(\sin 4(\lambda-\nu+1)\gamma_{\mu\nu} - \sin 4(\lambda-\nu+1)\delta_{\mu\nu} \right) \right\} .$$

We can find the value of the integral I_2 in terms of the Lobatschefsky function as follows; equation (2.111) can be re-written in the form

$$I_2 = \int_{\delta_{\mu\nu}}^{\gamma_{\mu\nu}} \ln\left(\frac{2\Gamma_{\mu\nu}^2}{\Gamma_{\mu\nu}^2 + Z_{\mu\nu}^2} \right) d(2\varphi) + \int_{\delta_{\mu\nu}}^{\gamma_{\mu\nu}} 2\ln\left(1 - \sin^2\omega \sin^2\varphi \right) d\varphi , \tag{2.115}$$

where

$$\sin\omega = R_{\mu\nu} \Gamma_{\mu\nu}^{-1} . \tag{2.116}$$

Employing the equation

$$\int_0^u \ln\left(1 - \sin^2\omega \sin^2 x\right) dx = (\pi - 2\theta^*) \ln \operatorname{ctg}\frac{\omega}{2} + 2u \ln\left(\tfrac{1}{2}\sin\omega\right) - \frac{\pi}{2}\ln 2 +$$

$$+ \mathcal{L}(\theta^* + u) - \mathcal{L}(\theta^* - u) + \mathcal{L}\left(\frac{\pi}{2} - 2u\right) , \quad \operatorname{ctg}\theta^* \equiv \cos\omega \cdot \tan u ,$$

$$-\pi \leqslant \omega \leqslant \pi , \; -\pi/2 \leqslant u \leqslant \pi/2 ,$$

and remembering the relations (2.108) and (2.116), we obtain

$$I_2 = 2\ln\left(\frac{2\Gamma_{\mu\nu}^2}{\Gamma_{\mu\nu}^2 + Z_{\mu\nu}^2}\right)(\gamma_{\mu\nu} - \delta_{\mu\nu}) + \Big\{(\pi - 2\theta)\ln\mathrm{ctg}\frac{\omega}{2} - (\pi - 2\theta_1)\ln\mathrm{ctg}\frac{\omega}{2} +$$

$$2(\gamma_{\mu\nu} - \delta_{\mu\nu})\ln\left(\frac{\sin\omega}{2}\right) + L(\theta + \gamma_{\mu\nu}) - L(\theta - \gamma_{\mu\nu}) + L\left(\frac{\pi}{2} - 2\gamma_{\mu\nu}\right) - \tag{2.117}$$

$$- L(\theta_1 + \delta_{\mu\nu}) + L(\theta_1 - \delta_{\mu\nu}) - L\left(\frac{\pi}{2} - 2\delta_{\mu\nu}\right)\Big\}2,$$

where

$$\theta \equiv \mathrm{arcctg}\left(\frac{Z_{\mu\nu}}{\Gamma_{\mu\nu}}\tan\gamma_{\mu\nu}\right), \quad \theta_1 \equiv \mathrm{arcctg}\left(\frac{Z_{\mu\nu}}{\Gamma_{\mu\nu}}\tan\delta_{\mu\nu}\right). \tag{2.118}$$

We now consider the limit of the Lobatschefsky function $L(\varkappa)$ as $\varkappa \to 0$. It is easily verified, with the aid of equations (2.107), that

$$\lim_{\varkappa \to 0} L(\varkappa) = 0. \tag{2.119}$$

Further, from equations (2.89) and (2.104), it is obvious that the limit of $\delta_{\mu\nu}$ as $\varepsilon \to 0$ is equal to zero; that is

$$\lim_{\varepsilon \to 0} \delta_{\mu\nu} \equiv \lim_{\varepsilon \to 0} \beta_{\mu\nu}^{*}(\varepsilon) = 0. \tag{2.120}$$

By virtue of this equation and equation (2.102), the limit

$$\lim_{\varepsilon \to 0}(\delta_{\mu\nu}\ln\varepsilon)$$

reduces to

$$\lim_{\varepsilon \to 0}(\delta_{\mu\nu}\cdot\ln\varepsilon) = \lim_{\varepsilon \to 0}(\delta_{\mu\nu}\cdot\ln\tan\delta_{\mu\nu}),$$

which, by introduction of the power series expansion (2.109) and the well known relation

$$\lim_{\varkappa \to 0} \varkappa\ln\varkappa = 0,$$

yields

$$\lim_{\varepsilon\to 0} (\delta_{\mu\nu} \ln\varepsilon) = 0 \ . \tag{2.121}$$

We now proceed with the evaluation of the potential Φ. The last of equations (2.93), after introducing equation (2.98) and integrating by parts, becomes

$$\Phi = -\frac{m}{(\alpha e)} \lim_{\varepsilon\to 0+} \left[\left\{ b^{*}_{\mu\nu}(\alpha) \ln\alpha - b^{*}_{\mu\nu}(\varepsilon) \ln\varepsilon \right\} + \right.$$

$$\left. + \frac{m}{(\alpha e)} \lim_{\varepsilon\to 0+} \int_{\varepsilon}^{\alpha} \ln\sigma \, d b^{*}_{\mu\nu}(\sigma) \right] . \tag{2.122}$$

In deriving the above equation we made use of the relation $d\ln\sigma = d\sigma/\sigma$. Comparing equations (2.101), (2.103), (2.105), (2.111), (2.117), (2.122) and remembering the relations

$$\gamma_{\mu\nu} \equiv b^{*}_{\mu\nu}(\alpha) , \quad \mathcal{L}\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \ln 2 ,$$

we obtain

$$\Phi = -\frac{m}{(\alpha e)} \left\{ \gamma_{\mu\nu} \ln\frac{2\alpha e}{R_{\mu\nu}} - \mathcal{L}(\gamma_{\mu\nu}) - \mathcal{L}\left(\frac{\pi}{2} - \gamma_{\mu\nu}\right) - \frac{(\pi - 2\theta)}{2} \ln\operatorname{ctg}\frac{\omega}{2} - \right.$$

$$\left. - \frac{1}{2}\mathcal{L}(\theta + \gamma_{\mu\nu}) + \frac{1}{2}\mathcal{L}(\theta - \gamma_{\mu\nu}) - \frac{1}{2}\mathcal{L}\left(\frac{\pi}{2} - 2\gamma_{\mu\nu}\right) + \frac{3}{4}\pi \ln 2 \right\} , \tag{2.123}$$

where θ and ω are given by equations (2.118) and (2.116) respectively.

The parameter $\gamma_{\mu\nu}$, as equation (2.94) denotes, is a function of αe, $x_{\mu\nu}$, $y_{\mu\nu}$ and $z_{\mu\nu}$ whose explicit form, given below, can be found by a process entirely analogous to the process followed for the derivation of equation (2.21)

$$\gamma_{\mu\nu} = \arcsin\left\{\frac{1}{R_{\mu\nu}}\sqrt{\frac{(r_{\mu\nu}^2+\alpha^2e^2)-\sqrt{(r_{\mu\nu}^2+\alpha^2e^2)^2-4R_{\mu\nu}^2\alpha^2e^2}}{2}}\right\}, R_{\mu\nu}\neq 0. \quad (2.124)$$

In the particular case when $R_{\mu\nu}=0$, equation (2.94) reduces to

$$\gamma_{\mu\nu} = \arctan\left(\frac{\alpha e}{|Z_{\mu\nu}|}\right), R_{\mu\nu}=0 . \quad (2.125)$$

Equations (2.100) and (2.123) complete the solution to the problem of finding the force and potential law of a non-homogeneous oblate spheroid whose density distribution law is given by the formula (2.92).

We note that one can express the potential Φ in a different form by merely using equations (2.110) and (2.114) instead of equations (2.105) and (2.117).

We now proceed with the evaluation of the potential energy and the mutual gravitational attraction between two non-homogeneous oblate spheroids with coplanar equatorial planes. The derivation of the required equations becomes easier by considering the following arrangement:

Two homogeneous oblate spheroids, called hereafter the ii^{th} and jj^{th} spheroids, with semi-major axes σ_i , σ_j , eccentricities e_i , e_j and uniform densities ρ_i , ρ_j respectively, have centres located at the points (x_i, y_i, z_i) and (x_j, y_j, z_j), where the co-ordinates $x_i, y_i, z_i, x_j, y_j, z_j$ are referred to a Cartesian right-handed inertial system. Two non-homogeneous oblate spheroids, called hereafter the i^{th} and j^{th} spheroids, with semi-major axes $\alpha_i > \sigma_i$, $\alpha_j > \sigma_j$, masses m_i , m_j and such orientation and geometrical characteristics that they are external to each other and the i^{th} spheroid is similar and similarly situated with the ii^{th} spheroid while the

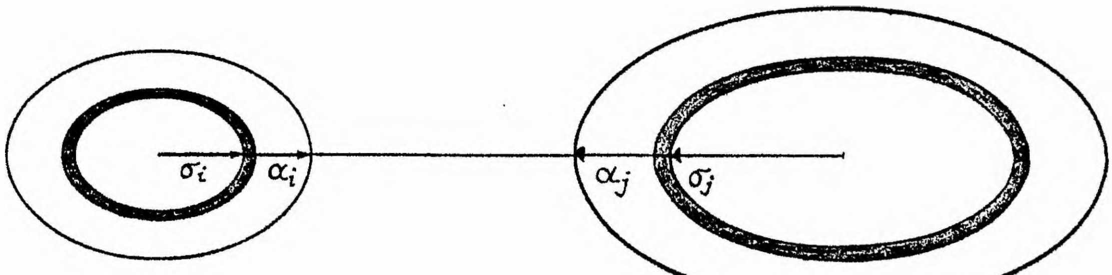


Figure 4

j^{th} spheroid is similar and similarly situated with the jj^{th} spheroid (see figure 4)

The position vectors of the centres of the bodies are

$$\underset{\sim}{r}_i \equiv (x_i, y_i, z_i) \; , \; \underset{\sim}{r}_j \equiv (x_j, y_j, z_j) \, .$$

The vector $\underset{\sim}{r}_{ij}$ and its norm are given by the relations

$$\underset{\sim}{r}_{ij} \equiv (x_{ij}, y_{ij}, z_{ij}) \equiv \underset{\sim}{r}_i - \underset{\sim}{r}_j \; , \; r_{ij} = |\underset{\sim}{r}_{ij}| \, .$$

Under the assumption that the equatorial planes of the spheroids are coplanar (without loss of generality we can assume $z_i=0$, $z_j=0$) the magnitude $F_{i\sigma}$ of the mutual gravitational attraction between the ii^{th} and jj^{th} spheroids can be found with the aid of equations (2.64) and (2.44), and it has the form

$$F_{i\sigma} = \frac{16\pi\sqrt{R_{ij}}}{\sqrt{2}\, e_i^3}(1-e_i^2)^{1/2}(1-e_j^2)^{1/2}\rho_i\rho_j\sum_{m=0}^{\infty}\frac{(2m)!\, e_j^{2m}\sigma_j^{2m+3}}{2^{2m}(m!)^2(2m+3)}W_m(R_{ij},\sigma_i,e_i) \, , \quad (2.126)$$

where R_{ij} denotes the distance between the centres of the spheroids, that is

$$\underset{\sim}{R}_{ij} \equiv (x_{ij}, y_{ij}) \; , \; R_{ij} = |\underset{\sim}{R}_{ij}| \, . \quad (2.127)$$

The functions $W_m(R_{ij},\sigma_i,e_i)$ $m=0,1,2,\ldots\ldots$ entering the right-hand member of equation (2.126) are given, in integral form, by the relation (2.65).

The elementary force $dF_{i\sigma}$ acting on the ii^{th} spheroid, being in the gravitational field of a differential homoeoid of semi-major axis σ_j , eccentricity e_j and density ρ_j, can be obtained by differentiating equation (2.126) once with respect to σ_j and it has the form

$$dF_{i\sigma}=\frac{16\pi\sqrt{R_{ij}}}{\sqrt{2}\,e_i^3}(1-e_i^2)^{1/2}(1-e_j^2)^{1/2}\rho_i\rho_j\sum_{m=0}^{\infty}\frac{(2m)!\,e_j^{2m}\sigma_j^{2(m+1)}}{2^{2m}(m!)^2}W_m(R_{ij},\sigma_i,e_i)\,d\sigma_j. \quad (2.128)$$

Suppose now that the density distribution law of the j^{th} spheroid is described by the relation

$$\rho_j=C_j\sigma_j^{-2}\,,\quad C_j=\text{constant}\,,\quad \sigma_j\in(0,\alpha_j]\,, \quad (2.129)$$

then the mass of this spheroid is equal to

$$m_j=4\pi(1-e_j^2)^{1/2}C_j\alpha_j\,. \quad (2.130)$$

Combining equations (2.128) and (2.129), and integrating the resulting expression over σ_j with lower and upper limits 0 and α_j respectively, we obtain the mutual gravitational attraction between the j^{th} and i^{th} spheroids, whose analytical expression is

$$F_{i\alpha}=\frac{16\pi\sqrt{R_{ij}}}{\sqrt{2}\,e_i^3}(1-e_i^2)^{1/2}(1-e_j^2)^{1/2}\rho_i C_j\sum_{m=0}^{\infty}\frac{(2m)!\,e_j^{2m}\alpha_j^{2m+1}}{2^{2m}(m!)^2(2m+1)}W_m(R_{ij},\sigma_i,e_i)\,.$$

Substituting for $W_m(R_{ij},\sigma_i,e_i)$, $m=0,1,2,3,4,5,6$ according to equations (A5-14), (A5-17), (A5-20), (A5-23), (A5-24) and employing the relation (2.130), we obtain

$$F_{i\alpha}=4\pi(1-e_i^2)^{1/2}\sigma_i^3\rho_i R_{ij}m_j G(R_{ij},\sigma_i e_i,\alpha_j e_j)\,, \quad (2.131)$$

where $G(R_{ij},\sigma_i e_i,\alpha_j e_j)$ is defined by the relation

$$G(R_{ij},\sigma_i e_i,\alpha_j e_j)\equiv\left\{\frac{1}{2(\sigma_i e_i)^3}\left[\arcsin\left(\frac{\sigma_i e_i}{R_{ij}}\right)-\frac{\sigma_i e_i}{R_{ij}^2}\sqrt{R_{ij}^2-\sigma_i^2e_i^2}\right]+\right.$$

$$+\frac{(\alpha_j e_j)^2}{(R_{ij}^2-\sigma_i^2 e_i^2)\sqrt{R_{ij}^2-\sigma_i^2 e_i^2}}\left[\frac{1}{18R_{ij}^2}+\frac{(\alpha_j e_j)^2}{40(R_{ij}^2-\sigma_i^2 e_i^2)^2}+\right.$$

$$+\frac{(\alpha_j e_j)^4}{336(R_{ij}^2-\sigma_i^2 e_i^2)^4}(5R_{ij}^2+4\sigma_i^2 e_i^2)+\frac{(\alpha_j e_j)^6}{3{,}456\,(R_{ij}^2-\sigma_i^2 e_i^2)^6}(35R_{ij}^4+ \tag{2.132}$$

$$+84R_{ij}^2\sigma_i^2 e_i^2+24\sigma_i^4 e_i^4)+\frac{(\alpha_j e_j)^8}{14{,}080(R_{ij}^2-\sigma_i^2 e_i^2)^8}(105R_{ij}^6+504R_{ij}^4\sigma_i^2 e_i^2+$$

$$+432R_{ij}^2\sigma_i^4 e_i^4+64\sigma_i^6 e_i^6)+\frac{(\alpha_j e_j)^{10}}{39{,}936(R_{ij}^2-\sigma_i^2 e_i^2)^{10}}(231R_{ij}^8+$$

$$\left.\left.+1{,}848R_{ij}^6\sigma_i^2 e_i^2+3{,}168R_{ij}^4\sigma_i^4 e_i^4+1{,}408R_{ij}^2\sigma_i^6 e_i^6+128\sigma_i^8 e_i^8)\right]\right\}+$$

$$+\frac{1}{\pi\sqrt{2R_{ij}}\,\sigma_i^3 e_i^3}\sum_{m=7}^{\infty}\frac{(2m)!\,(\alpha_j e_j)^{2m}}{2^{2m}(m!)^2(2m+1)}W_m(R_{ij},\sigma_i,e_i)\,.$$

Differentiating this equation once with respect to σ_i we obtain the magnitude dF_i^* of the force acting on the j^{th} spheroid due to the presence of a differential homoeoid with semi-major axis σ_i , eccentricity e_i and density ρ_i ,

$$dF_i^*=4\pi(1-e_i^2)^{1/2}R_{ij}m_j\rho_i\,d\left[\sigma_i^3\,G(R_{ij},\sigma_i e_i,\alpha_j e_j)\right]. \tag{2.133}$$

We now adopt for the i^{th} spheroid a density distribution law whose mathematical expression is

$$\rho_i=C_i\sigma_i^{-2}\,,\quad C_i=\text{constant}\,,\quad \sigma_i\in(0\;\alpha_i]\,. \tag{2.134}$$

Equation (2.133), upon substitution of equation (2.134) and integration over σ_i with lower and upper limits 0 and α_i respectively, becomes

$$F_i^* = 4\pi(1-e_i^2)^{1/2} R_{ij} m_j C_i \int_0^{\alpha_i} \sigma_i^{-2} d\left[\sigma_i^3 G(R_{ij}, \sigma_i e_i, \alpha_j e_j)\right] .$$

Obviously F_i^* is the magnitude of the mutual gravitational attraction between the i^{th} and j^{th} spheroids; it will be shown below that F_i^* can be expressed in the form of a series expansion. Integration by parts of the right-hand member of the foregoing equation and use of the relation

$$m_i = 4\pi(1-e_i^2)^{1/2} C_i \alpha_i ,$$

yields

$$F_i^* = m_i m_j R_{ij} G(R_{ij}, \alpha_i e_i, \alpha_j e_j) + 2 m_i m_j \frac{R_{ij}}{\alpha_i} \int_0^{\alpha_i} G(R_{ij}, \sigma_i e_i, \alpha_j e_j) d\sigma_i . \quad (2.135)$$

Attempts at further reduction of this formula leads to the evaluation of the following three types of integrals

$$I_1^* = \int \frac{x^{2m} dx}{(\beta^2 - x^2)^{k+1/2}} , \quad I_2^* = \int x^{-3} \arcsin\left(\frac{x}{\beta}\right) dx ,$$

$$I_3^* = \int x^{-2} (\beta^2 - x^2)^{1/2} dx , \quad \beta^2 \geqslant x^2 . \quad (2.136)$$

The first of these integrals, by virtue of the relations

$$d\left(\frac{x^2}{\beta^2 - x^2}\right) = 2x \frac{\beta^2}{(\beta^2 - x^2)^2} dx ,$$

$$\left(\frac{\beta^2}{\beta^2 - x^2}\right)^{k-m-1} = \left(1 + \frac{x^2}{\beta^2 - x^2}\right)^{k-m-1} = \sum_{\nu=0}^{k-m-1} \binom{k-m-1}{\nu} \left(\frac{x^2}{\beta^2 - x^2}\right)^{\nu} , \quad k \geqslant m+1 ,$$

takes the form

$$I_1^* = \int \left\{ \frac{1}{2\beta^{2(K-m)}} \left(\frac{x^2}{\beta^2-x^2}\right)^{m-\frac{1}{2}} \left[\sum_{\nu=0}^{K-m-1} \binom{K-m-1}{\nu} \left(\frac{x^2}{\beta^2-x^2}\right)^{\nu} \right] \right\} d\left(\frac{x^2}{\beta^2-x^2}\right), K \geqslant m+1,$$

from which one readily obtains

$$I_1^* = \frac{1}{\beta^{2(K-m)}} \sum_{\nu=0}^{K-m-1} \frac{1}{(2m+2\nu+1)} \binom{K-m-1}{\nu} \left(\frac{x^2}{\beta^2-x^2}\right)^{m+\nu+\frac{1}{2}}, \quad K \geqslant m+1. \qquad (2.137)$$

The last two of the integrals (2.136) are easily evaluated and they satisfy the relations

$$I_2^* = -\frac{1}{2}\left[x^{-2}\arcsin\left(\frac{x}{\beta}\right) - \frac{\sqrt{\beta^2-x^2}}{\beta^2 x}\right], \beta^2 \geqslant x^2,$$

$$I_3^* = -\frac{\sqrt{\beta^2-x^2}}{x} - \arcsin\left(\frac{x}{\beta}\right), \quad \beta^2 \geqslant x^2. \qquad (2.138)$$

Combining equations (2.132) and (2.135), carrying out the integration with the aid of the relations (2.137) and (2.138), and after long and laborious algebraic manipulations we obtain the following expression for the force F_i^*

$$F_i^* = m_i m_j \, Q(R_{ij}, \alpha_i e_i, \alpha_j e_j), \qquad (2.139)$$

where

$$Q(R_{ij}, \alpha_i e_i, \alpha_j e_j) \equiv \frac{1}{R_{ij}\alpha_i e_i} \arcsin\left(\frac{\alpha_i e_i}{R_{ij}}\right) +$$

$$+ \frac{(\alpha_j e_j)^2}{6R_{ij}^3(R_{ij}^2-\alpha_i^2 e_i^2)\sqrt{R_{ij}^2-\alpha_i^2 e_i^2}} \left[R_{ij}^2 - \frac{2}{3}\alpha_i^2 e_i^2 + \frac{(\alpha_j e_j)^2}{(1-\varphi_{ij}^2)^2}\left(\frac{9}{20} - \frac{7}{10}\varphi_{ij}^2 + \right.\right.$$

$$+\frac{14}{25}\varphi_{ij}^{4}-\frac{4}{25}\varphi_{ij}^{6}\Big)+\frac{(\alpha_j e_j)^4}{R_{ij}^{2}(1-\varphi_{ij}^{2})^{4}}\Big(\frac{15}{56}-\frac{15}{28}\varphi_{ij}^{2}+\frac{33}{35}\varphi_{ij}^{4}-$$

$$-\frac{198}{245}\varphi_{ij}^{6}+\frac{88}{245}\varphi_{ij}^{8}-\frac{16}{245}\varphi_{ij}^{10}\Big)+\frac{(\alpha_j e_j)^6}{R_{ij}^{4}(1-\varphi_{ij}^{2})^{6}}\Big(\frac{35}{192}-\frac{35}{96}\varphi_{ij}^{2}+$$

$$+\frac{49}{40}\varphi_{ij}^{4}-\frac{143}{84}\varphi_{ij}^{6}+\frac{286}{189}\varphi_{ij}^{8}-\frac{52}{63}\varphi_{ij}^{10}+\frac{16}{63}\varphi_{ij}^{12}-\frac{32}{945}\varphi_{ij}^{14}\Big)+$$

$$+\frac{(\alpha_j e_j)^8}{R_{ij}^{6}(1-\varphi_{ij}^{2})^{8}}\Big(\frac{189}{1{,}408}-\frac{147}{704}\varphi_{ij}^{2}+\frac{84}{55}\varphi_{ij}^{4}-\frac{1{,}041}{385}\varphi_{ij}^{6}+\frac{4{,}199}{1{,}155}\varphi_{ij}^{8}-$$

$$-\frac{8{,}398}{2{,}541}\varphi_{ij}^{10}+\frac{5{,}168}{2{,}541}\varphi_{ij}^{12}-\frac{10{,}336}{12{,}705}\varphi_{ij}^{14}+\frac{2{,}432}{12{,}705}\varphi_{ij}^{16}-\frac{256}{12{,}705}\varphi_{ij}^{18}\Big)+ \qquad (2.140)$$

$$+\frac{(\alpha_j e_j)^{10}}{R_{ij}^{8}(1-\varphi_{ij}^{2})^{10}}\Big(\frac{693}{6{,}656}-\frac{231}{3{,}328}\varphi_{ij}^{2}+\frac{1{,}617}{832}\varphi_{ij}^{4}-\frac{10{,}593}{2{,}912}\varphi_{ij}^{6}+$$

$$+\frac{7{,}447}{1{,}092}\varphi_{ij}^{8}-\frac{7{,}429}{858}\varphi_{ij}^{10}+\frac{14{,}858}{1{,}859}\varphi_{ij}^{12}-\frac{29{,}716}{5{,}577}\varphi_{ij}^{14}+\frac{13{,}984}{5{,}577}\varphi^{16}-$$

$$-\frac{1{,}472}{1{,}859}\varphi_{ij}^{18}+\frac{5{,}888}{39{,}039}\varphi_{ij}^{20}-\frac{512}{39{,}039}\varphi_{ij}^{22}\Big)\Bigg]+$$

$$+\frac{\sqrt{R_{ij}}}{\pi\sqrt{2}\,(\alpha_i e_i)^3}\sum_{m=7}^{\infty}\frac{(2m)!\,(\alpha_j e_j)^{2m}}{2^{2m}\,(m!)^2\,(2m+1)}\,W_m(R_{ij},\alpha_i,e_i)+$$

$$+\frac{\sqrt{2R_{ij}}}{\pi\alpha_i e_i^3}\sum_{m=7}^{\infty}\frac{(2m)!\,(\alpha_j e_j)^{2m}}{2^{2m}(m!)^2\,(2m+1)}\int_0^{\alpha_i}\frac{W_m(R_{ij},\sigma_i,e_i)}{\sigma_i^3}\,d\sigma_i\,,$$

and

$$\varphi_{ij}\equiv\frac{\alpha_i e_i}{R_{ij}}\,. \tag{2.141}$$

Formulae (2.69), (2.139), (2.140) and (2.141) imply that the force between two non-homogeneous oblate spheroids, with coplanar equatorial planes and density distribution laws given by the relations (2.129) and (2.134), is completely specified when the distance of their centres and the quantities $(\alpha_i e_i)$ and $(\alpha_j e_j)$ are known. Further, since the semi-axes of confocal spheroids satisfy the relation

$$\alpha_1^2-c_1^2=\alpha_2^2-c_2^2=\text{constant}\,,$$

this force remains unchanged if we substitute one or both of the bodies by any one of their confocal spheroids with the same mass and whose density distribution law obeys the relation (2.92). We note that formula (2.139) must be symmetrical with respect to $(\alpha_i e_i)$ and $(\alpha_j e_j)$, that is

$$F_i^*=m_i m_j\,Q\,(R_{ij}\,,\alpha_j e_j\,,\alpha_i e_i)\,. \tag{2.142}$$

The series expansions (2.139) and (2.142) are equivalent; however the rates of convergence, depending on the values of the parameters $\alpha_i, e_i, \alpha_j, e_j$ and R_{ij}, are not the same; in fact, if $\alpha_i e_i > \alpha_j e_j$ then series (2.139) converges more rapidly than (2.142).

In the numerical applications we truncate the series expansion (2.139) after its ν^{th} term, introducing in this way into the evaluation of the force F_i^* an error e_ν^*, which according to equations (2.139) and (2.140) is given by the relation

$$|e_v^*| = e_v^* = \frac{m_i m_j \sqrt{R_{ij}}}{\pi\sqrt{2}\,(\alpha_i e_i)^3} \sum_{m=v+1}^{\infty} \frac{(2m)!\,(\alpha_j e_j)^{2m}}{2^{2m}(m!)^2(2m+1)} W_m(R_{ij},\alpha_i,e_i) +$$

$$+ \frac{m_i m_j \sqrt{2R_{ij}}}{\pi\,\alpha_i e_i^3} \sum_{m=v+1}^{\infty} \frac{(2m)!\,(\alpha_j e_j)^{2m}}{2^{2m}(m!)^2(2m+1)} \int_0^{\alpha_i} \frac{W_m(R_{ij},\sigma_i,e_i)}{\sigma_i^3}\,d\sigma_i . \tag{2.143}$$

With the aid of equations (A5-24) and (2.65) one can prove that

$$\frac{W_m(R_{ij},\sigma_i,e_i)}{\sigma_i^3} \leqslant \frac{W_m(R_{ij},\alpha_i,e_i)}{\alpha_i^3}, \quad \sigma_i \in (0,\alpha_i), \quad m \geqslant 1 ,$$

and hence

$$\int_0^{\alpha_i} \frac{W_m(R_{ij},\sigma_i,e_i)}{\sigma_i^3}\,d\sigma_i \leqslant \frac{W_m(R_{ij},\alpha_i,e_i)}{\alpha_i^2} . \tag{2.144}$$

Combining equations (2.143), (2.144) and (A7- 9) we readily conclude

$$|e_v^*| \leqslant \frac{3m_i m_j \sqrt{R_{ij}}}{\pi\sqrt{2}\,(\alpha_i e_i)^3 \alpha_j e_j} (R_{ij}-\alpha_i e_i)^{2v+1} W_v(R_{ij},\alpha_i,e_i) \sum_{m=v+1}^{\infty} \frac{(2m)!}{2^{2m}(m!)^2(2m+1)} \left(\frac{\alpha_j e_j}{R_{ij}-\alpha_i e_i}\right)^{2m+1},$$

$$v \geqslant 6,$$

which, upon substitution of the well known series expansion

$$\arcsin x = \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}(m!)^2\,(2m+1)} x^{2m+1} ,$$

becomes

$$|e_v^*| = e_v^* \leqslant \frac{3m_i m_j \sqrt{R_{ij}}}{\sqrt{2}\,\pi\,(\alpha_i e_i)^3 (\alpha_j e_j)} (R_{ij}-\alpha_i e_i)^{2v+1} W_v(R_{ij},\alpha_i,e_i) \left\{ \arcsin\left(\frac{\alpha_j e_j}{R_{ij}-\alpha_i e_i}\right) - \right. \tag{2.145}$$

$$-\sum_{m=0}^{\nu} \frac{(2m)!}{2^{2m}(m!)^2(2m+1)}\left(\frac{\alpha_j e_j}{R_{ij}-\alpha_i e_i}\right)^{2m+1}\Bigg\}, \nu \geqslant 6 .$$

This equation for ν=6 and by introduction of the last of equations (A5-24), yields

$$|e_6^*| = e_6^* \leqslant \frac{m_i m_j R_{ij}(R_{ij}-\alpha_i e_i)^2}{(\alpha_j e_j)(R_{ij}+\alpha_i e_i)^{11}\sqrt{R_{ij}^2-\alpha_i^2 e_i^2}}\Big(R_{ij}^8+8R_{ij}^6\alpha_i^2 e_i^2+\frac{96}{7}R_{ij}^4\alpha_i^4 e_i^4+ \tag{2.146}$$

$$+\frac{128}{21}R_{ij}^2\alpha_i^6 e_i^6+\frac{128}{231}\alpha_i^8 e_i^8\Big)\left\{\arcsin\left(\frac{\alpha_j e_j}{R_{ij}-\alpha_i e_i}\right)-\sum_{m=0}^{6}\frac{(2m)!}{2^{2m}(m!)^2(2m+1)}\left(\frac{\alpha_j e_j}{R_{ij}-\alpha_i e_i}\right)^{2m+1}\right\}.$$

Due to the orientation and the symmetry of the i^{th} and j^{th} spheroids, the magnitude F_i^* of their mutual gravitational attraction must satisfy the relations

$$F_{ix}^* = -\frac{x_{ij}}{R_{ij}}F_i^* , \quad F_{iy}^* = -\frac{y_{ij}}{R_{ij}}F_i^* , \quad F_{iz}^* = -\frac{z_{ij}}{R_{ij}}F_i^* , \tag{2.147}$$

where F_{ix}^* , F_{iy}^* and F_{iz}^* are the force components acting on the i^{th} spheroid along the X, Y and Z axes as a result of the attraction from the j^{th} spheroid. On the other hand, according to theorem (3) given at the beginning of this chapter we have

$$F_{ix}^* = m_i x_i'' , \quad F_{iy}^* = m_i y_i'' , \quad F_{iz}^* = m_i z_i'' . \tag{2.148}$$

Combining equations (2.139), (2.147) and (2.148) and remembering that $z_{ij}=0$, we obtain the relations

$$\underset{\sim}{R}_i'' = -\frac{\underset{\sim}{R}_{ij}}{R_{ij}} m_j \, Q(R_{ij}, \alpha_i e_i, \alpha_j e_j) , \quad z_i'' = 0 . \tag{2.149}$$

Similar equations can be found for x_j, y_j, z_j

$$\underset{\sim}{R}_j'' = \frac{\underset{\sim}{R}_{ij}}{R_{ij}} m_i Q(R_{ij}, \alpha_i e_i, \alpha_j e_j) , \quad z_j'' = 0 \tag{2.150}$$

Equations (2.149) and (2.150) comprise a system of second-order differential equations for the co-ordinates x_i, y_i, z_i, x_j, y_j, z_j, which completely specifies the motion of the centres of mass of the spheroids, and therefore the motion of the spheroids themselves since, due to their symmetry and orientation, undergo only translation on the XY plane.

To obtain the potential energy Φ_i^* of either of the oblate spheroids due to the presence of the other we first notice that, according to equation (2.126), the force between the ii^{th} and jj^{th} spheroids has the general form

$$F_{i\sigma} = \rho_i \rho_j u(\sigma_i, e_i, \sigma_j, e_j, R_{ij}) , \tag{2.151}$$

while their potential energy, according to equations (2.77), (2.81) and (2.84) is given by the relation

$$\Phi_{i\sigma} = \int \rho_i \rho_j \, u(\sigma_i, e_i, \sigma_j, e_j, R_{ij}) \, dR_{ij} .$$

The elementary potential energy of the ii^{th} spheroid, being in the gravitational field of a differential homoeoid of semi-major axis σ_j, eccentricity e_j and density ρ_j, can be obtained by differentiating the foregoing equation once with respect to σ_j and it has the form

$$\frac{\partial \Phi_{i\sigma}}{\partial \sigma_j} d\sigma_j = \left[\int \rho_i \rho_j \frac{\partial u(\sigma_i, e_i, \sigma_j, e_j, R_{ij})}{\partial \sigma_j} dR_{ij} \right] d\sigma_j .$$

Combining this equation with equation (2.129) and integrating the resulting expression over σ_j with lower and upper limits 0 and α_j respectively, we obtain the potential energy $\Phi_{i\alpha}$ of the j^{th} spheroid due to the presence of the i^{th} spheroid; that is

$$\Phi_{i\alpha} = \int_0^{\alpha_j} \int \rho_i \frac{C_j}{\sigma_j^2} \frac{\partial u(\sigma_i, e_i, \sigma_j, e_j, R_{ij})}{\partial \sigma_j} dR_{ij} d\sigma_j \, . \tag{2.152}$$

The differential of the function $\Phi_{i\alpha} = \Phi_{i\alpha}(\sigma_i, e_i, \alpha_j, e_j, R_{ij})$ with respect to σ_i represents the potential energy of the j^{th} spheroid being in the presence of an elementary homoeoid with semi-major axis σ_i, eccentricity e_i and density ρ_i; the analytical expression of this potential energy can easily be obtained from equation (2.152), which after one differention with respect to σ_i and application of the formula (2.38) yields

$$\frac{\partial \Phi_{i\alpha}}{\partial \sigma_i} d\sigma_i = \left[\int_0^{\alpha_j} \int \frac{C_j}{\sigma_j^2} \rho_i \frac{\partial^2 u(\sigma_i, e_i, \sigma_j, e_j, R_{ij})}{\partial \sigma_i \, \partial \sigma_j} dR_{ij} \, d\sigma_j \right] d\sigma_i \, .$$

This equation, upon substitution of equation (2.134) and integration over σ_i with lower and upper limits 0 and α_i respectively, becomes

$$\Phi_i^* = \int_0^{\alpha_i} \int_0^{\alpha_j} \int \frac{C_i C_j}{\sigma_i^2 \, \sigma_j^2} \frac{\partial^2 u(\sigma_i, e_i, \sigma_j, e_j, R_{ij})}{\partial \sigma_i \, \partial \sigma_j} dR_{ij} d\sigma_j \, d\sigma_i \, , \tag{2.153}$$

or interchanging the order of integration

$$\Phi_i^* = \int \left[\int_0^{\alpha_i} \int_0^{\alpha_j} \frac{C_i C_j}{\sigma_i^2 \, \sigma_j^2} \frac{\partial^2 u(\sigma_i, e_i, \sigma_j, e_j, R_{ij})}{\partial \sigma_i \, \partial \sigma_j} d\sigma_j \, d\sigma_i \right] dR_{ij} \, ,$$

which is the analytical expression for the potential energy of the i^{th} and j^{th} spheroids. In order to express the triple integral, occurring in the right-hand member of the above equation, in terms of an infinite series we firstly observe that the mutual gravitational attraction F_i^* between the i^{th} and j^{th} spheroid (as is obvious from the way it was derived and the definition of the function $\mathcal{U}=\mathcal{U}(\sigma_i, e_i, \sigma_j, e_j, R_{ij})$) can take the form

$$F_i^* = \int_0^{\alpha_i}\int_0^{\alpha_j} \frac{C_i C_j}{\sigma_i^2 \sigma_j^2} \frac{\partial^2 \mathcal{U}(\sigma_i, e_i, \sigma_j, e_j, R_{ij})}{\partial\sigma_i \, \partial\sigma_j} d\sigma_j \, d\sigma_i \, .$$

By comparing the foregoing two relations we obtain

$$\Phi_i^* = \int F_i^* dR_{ij} \, . \qquad (2.154)$$

Combining now equations (2.139), (2.140) and (2.154), carrying out the integration, and after long and laborious algebraic manipulations we find the following relation

$$\Phi_i^* = -\frac{m_i m_j}{(\alpha_i e_i)} \left[\sum_{K=0}^{\infty} \frac{(2K)!}{2^{2K}(K!)^2(2K+1)^2} \varphi_{ij}^{2K+1} + \frac{n_{ij}^2}{18}\omega_{ij}\varphi_{ij}^2 + \right.$$

$$+ \frac{n_{ij}^4}{20}\omega_{ij}^5\left(\frac{3}{10} - \frac{\varphi_{ij}^2}{3} + \frac{2\varphi_{ij}^4}{15}\right) - \frac{n_{ij}^6}{7}\omega_{ij}^9\left(\frac{5}{144} - \frac{\varphi_{ij}^2}{10} + \right.$$

$$\left. + \frac{2}{35}\varphi_{ij}^4 - \frac{4}{315}\varphi_{ij}^6\right) + \frac{5}{784} n_{ij}^6 \omega_{ij}^7 + \frac{n_{ij}^8}{45}\omega_{ij}^{13}\left(\frac{77}{128} - \right.$$

$$\left. - \frac{143}{168}\varphi_{ij}^2 + \frac{143}{252}\varphi_{ij}^4 - \frac{13}{63}\varphi_{ij}^6 + \frac{2}{63}\varphi_{ij}^8\right) + \frac{35 n_{ij}^8}{10{,}368}\omega_{ij}^9 -$$

$$- \frac{1}{385}\eta_{ij}^{10}\,\omega_{ij}^{17}\left(\frac{1{,}209}{256} - \frac{221}{18}\varphi_{ij}^{2} + \frac{884}{99}\varphi_{ij}^{4} - \frac{136}{33}\varphi_{ij}^{6} + \frac{544}{495}\varphi_{ij}^{8} - \right.$$

$$\left. - \frac{64}{495}\varphi_{ij}^{10}\right) + \frac{7}{2{,}816}\eta_{ij}^{10}\,\omega_{ij}^{11}\left(\frac{9}{11} + \omega_{ij}^{2} + \frac{169}{25}\omega_{ij}^{4}\right) + \tag{2.155}$$

$$+ \frac{\eta_{ij}^{12}}{117}\omega_{ij}^{21}\left(\frac{103{,}037}{21{,}504} - \frac{323}{44}\varphi_{ij}^{2} + \frac{1{,}615}{286}\varphi_{ij}^{4} - \frac{1{,}292}{429}\varphi_{ij}^{6} + \frac{152}{143}\varphi_{ij}^{8} - \right.$$

$$\left. - \frac{32}{143}\varphi_{ij}^{10} + \frac{64}{3{,}003}\varphi_{ij}^{12}\right) + \frac{11}{3{,}328}\eta_{ij}^{12}\,\omega_{ij}^{13}\left(\frac{21}{52} + \frac{7}{6}\omega_{ij}^{2} + 7\,\omega_{ij}^{4} + \frac{17}{14}\omega_{ij}^{6}\right)\Bigg\} +$$

$$+ \frac{1}{\pi\sqrt{2}\,(\alpha_i e_i)^3}\sum_{m=7}^{\infty}\frac{(2m)!\,(\alpha_j e_j)^{2m}}{2^{2m}(m!)^2(2m+1)}\int\sqrt{R_{ij}}\,W_m(R_{ij},\alpha_i,e_i)\,dR_{ij} +$$

$$+ \frac{\sqrt{2}}{\pi\alpha_i e_i^3}\sum_{m=0}^{\infty}\frac{(2m)!\,(\alpha_j e_j)^{2m}}{2^{2m}(m!)^2(2m+1)}\int\int_0^{\alpha_i}\sqrt{R_{ij}}\,\frac{W_m(R_{ij},\sigma_i,e_i)}{\sigma_i^3}\,d\sigma_i\,dR_{ij},$$

where φ_{ij} is given by equation (2.141) and η_{ij} and ω_{ij} are defined by the relations

$$\eta_{ij} \equiv \frac{\alpha_j e_j}{\alpha_i e_i}, \quad \omega_{ij} \equiv \frac{\alpha_i e_i}{\sqrt{R_{ij}^2 - \alpha_i^2 e_i^2}}. \tag{2.156}$$

In deriving equation (2.155) we made use of equation (2.79) and the relations

$$\int \frac{\sin^m x\,dx}{\cos^n x} = -\frac{1}{m-n}\,\frac{\sin^{m-1} x}{\cos^{n-1} x} + \frac{m-1}{m-n}\int \frac{\sin^{m-2} x}{\cos^n x}\,dx, \quad n \neq m,$$

$$\int \tan^m x\,dx = \frac{\tan^{m-1} x}{m-1} - \int \tan^{m-2} x\,dx, \; m \neq 1, \tag{2.157}$$

$$\int \tan^2 x\,dx = \tan x - x.$$

Equations (2.139), (2.140) and (2.155) complete the solution to the problem of finding the mutual gravitational attraction and potential energy of two non-homogeneous oblate spheroids with coplanar equatorial planes and whose density distribution laws obey the relation (2.92).

Next we give two figures, the first of which (fig. (5)) graphically illustrates the function

$$\frac{F_i^*}{m_i m_j} = f(\alpha_i e_i, \alpha_j e_j),$$

for R_{ij} = 40 units of distance (in this particular case R_{ij} = 40 kpc). The point B' corresponds to the force between two particles and the lines drawn on the surface A'B'C' are the intersections of this surface with the planes described by the relations

$$\alpha_i e_i + \alpha_j e_j = \text{constant} \quad -\infty < \frac{F_i^*}{m_i m_j} < \infty .$$

It is obvious that as $R_{ij} \to \infty$ the point B' moves towards zero and the surface

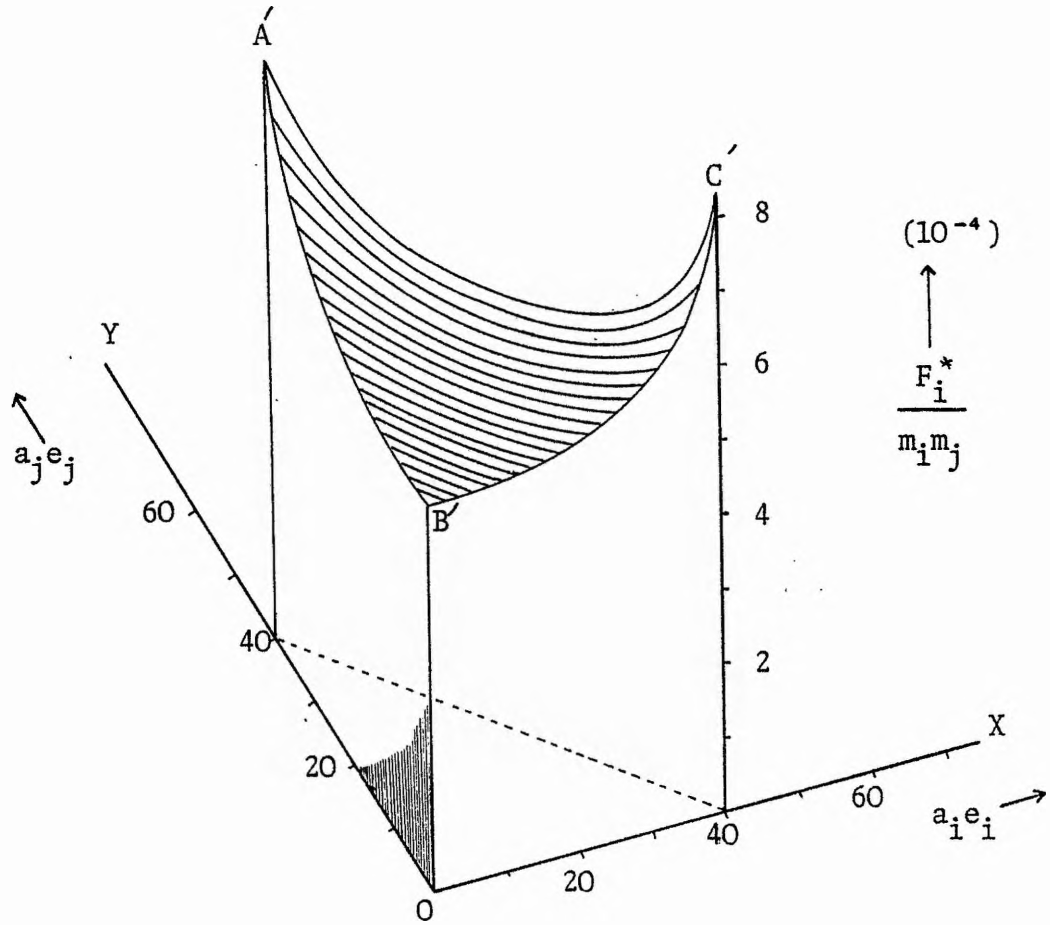


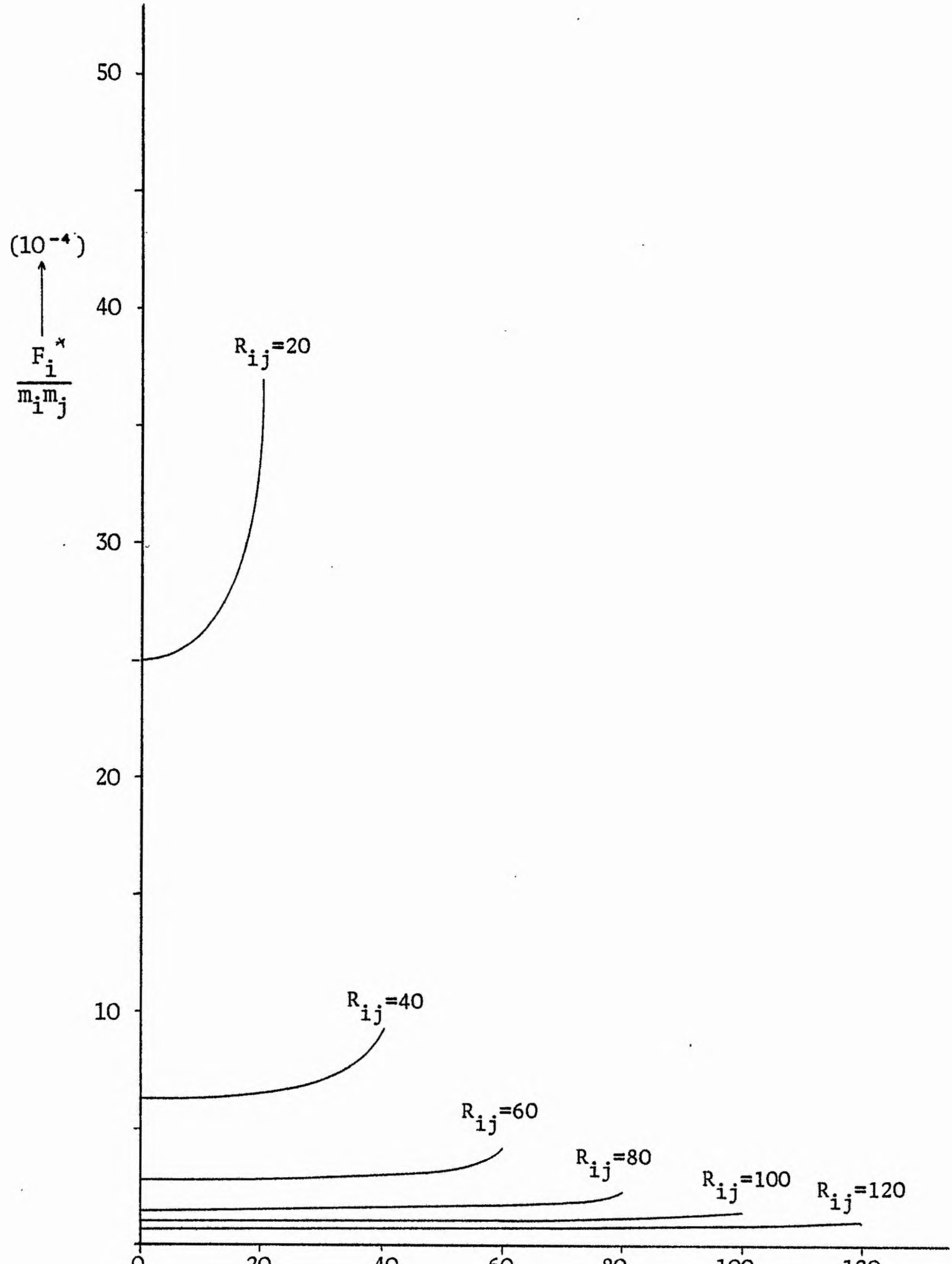
Figure 5

A'B'C' becomes progressively flatter, coinciding with the XY plane in the extreme case when $R_{ij} = \infty$. As $R_{ij} \to 0$ the point B′ moves towards infinity and the surface A'B'C' rises more and more steeply.

Figure (6) is the graphical representation of the relation

$$\frac{F_i^*}{m_i m_j} = f(\alpha_i e_i, \alpha_j e_j)\Big|_{\alpha_j e_j = 10^{-4}}$$

for different values of R_{ij}.



50
40
30
20
10
(10^{-4})
$\frac{F_i^*}{m_i m_j}$
$R_{ij}=20$
$R_{ij}=40$
$R_{ij}=60$
$R_{ij}=80$
$R_{ij}=100$
$R_{ij}=120$
0
20
40
60
80
100
120
$\alpha_i e_i$

Figure 6

The analytical expressions for the potential energy and the mutual gravitational attraction between two oblate spheroids with coplanar equatorial planes, and whose density distribution laws obey relations of the form (2.87) or (2.88*), can be found by a process entirely analogous to the process followed for the derivation of equations (2.139) and (2.155).

Concluding this chapter it is worth noting once more that the formulae involved in the derivation of the equations of the chapter and the reduction of these formulae are extremely long and very laborious.

CHAPTER 3

Solution of the equations of motion of dynamical systems consisting of particles and homogeneous oblate spheroids with coplanar equatorial planes.

In this chapter we consider the equations of motion of a dynamical system consisting of n bodies $(n \geqslant 1)$ - with such internal mass distribution that they may be considered as particles - and K homogeneous oblate spheroids $(K \geqslant 1)$, interacting according to the Newtonian law of gravitation and moving freely in a three dimensional Euclidean space, initially in such a manner that the equatorial planes of the spheroids remain coplanar at any time; we also describe the solution of these equations.

For the sake of simplicity and convenience we adopt such units of mass, distance and time that the gravitational constant k has arithmetical value equal to one and assign to each member of the dynamical system an index, say i; hereafter we shall refer to the particles by the indices $i=1,2,\cdots\cdots n$ and to the spheroids by the indices $i=n+1, n+2,\cdots\cdots n+K$. Thus the coördinates of each particle i in a Cartesian right-handed inertial system are x_i, y_i, z_i $(i=1,\cdots\cdots n)$, while the co-ordinates of the centres of mass of the spheroids are x_i, y_i, z_i $(i=n+1,\cdots\cdots n+K)$. We define the vectors $\underset{\sim}{r}_i$, $\underset{\sim}{R}_i$, $\underset{\sim}{r}_{ij}$ and $\underset{\sim}{R}_{ij}$ as follows:

$$\underset{\sim}{r}_i \equiv (x_i, y_i, z_i), \quad \underset{\sim}{R}_i \equiv (x_i, y_i, 0), \quad (i=1,\cdots\cdots n+K),$$

$$\underset{\sim}{r}_{ij} \equiv \underset{\sim}{r}_i - \underset{\sim}{r}_j \equiv (x_{ij}, y_{ij}, z_{ij}), \quad \underset{\sim}{R}_{ij} \equiv \underset{\sim}{R}_i - \underset{\sim}{R}_j \equiv (x_{ij}, y_{ij}, 0), \quad (i,j=1,\cdots\cdots n+K, \; i \neq j). \tag{3.1}$$

The magnitudes of these vectors are:

$$r_i \equiv |\underset{\sim}{r}_i| \ , \ R_i \equiv |\underset{\sim}{R}_i| \ , (i=1,\cdots\cdots n+K),$$
$$r_{ij} \equiv |\underset{\sim}{r}_{ij}| \ , \ R_{ij} \equiv |\underset{\sim}{R}_{ij}| \ , (i,j=1,\cdots\cdots n+K \ , i \neq j). \tag{3.2}$$

Without loss of generality we can assume that $Z_i = 0 \quad (i=1+n,\cdots\cdots n+K)$, then

$$\underset{\sim}{r}_i = \underset{\sim}{R}_i \quad (i=n+1,\cdots\cdots n+K) \ , \ \underset{\sim}{r}_{ij} = \underset{\sim}{R}_{ij} \quad (i,j=n+1,\cdots\cdots n+K \ , i \neq j \tag{3.3}$$

We, finally, represent the masses of the bodies by $m_i \ (i=1,\cdots\cdots n+K)$ and the semi-major axes and eccentricities of the spheroids by a_i and e_i $(i=n+1,\cdots\cdots n+K)$ respectively.

The acceleration components of the i^{th} particle along the X, Y and Z axes, due to the presence of the other particles and the spheroids, can easily be obtained with the aid of equations (1.6) and (2.15), and they are given by the first two equations of the system (3.4). The last two equations of this system, according to theorem (3) given at the beginning of the second chapter and the relations (2.15) and (2.73), describe the motion of the centre of mass of the i^{th} spheroid.

$$\underset{\sim}{R}''_i = -\sum_{\substack{j=1 \\ j \neq i}}^{n} m_j r_{ij}^{-3} \underset{\sim}{R}_{ij} - \frac{3}{2} \sum_{j=n+1}^{n+K} m_j (a_j e_j)^{-3} \underset{\sim}{R}_{ij} (\theta_{ij} - \sin\theta_{ij} \cos\theta_{ij}) \ , (i=1,\cdots n) \ ,$$

$$Z''_i = -\sum_{\substack{j=1 \\ j \neq i}}^{n} m_j r_{ij}^{-3} Z_{ij} - 3 \sum_{j=n+1}^{n+K} m_j (a_j e_j)^{-3} Z_{ij} (\tan\theta_{ij} - \theta_{ij}) \ , (i=1,\cdots n) \ ,$$

$$\underset{\sim}{R}''_i = -\frac{3}{2} (a_i e_i)^{-3} \sum_{j=1}^{n} m_j \underset{\sim}{R}_{ij} (\theta_{ji} - \sin\theta_{ji} \cos\theta_{ji}) -$$

$$-3\sum_{\substack{j=n+1\\ j\neq i}}^{n+k} m_j \underset{\sim}{R}_{ij}\left\{\frac{1}{2(\alpha_i e_i)^3}\left[\arcsin\left(\frac{\alpha_i e_i}{R_{ij}}\right)-\frac{\alpha_i e_i}{R_{ij}^2}\sqrt{R_{ij}^2-\alpha_i^2 e_i^2}\right]+\right.$$

$$+\frac{(\alpha_j e_j)^2}{(R_{ij}^2-\alpha_i^2 e_i^2)^{3/2}}\left[\frac{1}{10R_{ij}^2}+\frac{3(\alpha_j e_j)^2}{56(R_{ij}^2-\alpha_i^2 e_i^2)^2}+\frac{(\alpha_j e_j)^4}{144(R_{ij}^2-\alpha_i^2 e_i^2)^4}\left(5R_{ij}^2+4\alpha_i^2 e_i^2\right)+\right. \tag{3.4}$$

$$+\frac{(\alpha_j e_j)^6}{1{,}408(R_{ij}^2-\alpha_i^2 e_i^2)^6}\left(35R_{ij}^4+84R_{ij}^2\alpha_i^2 e_i^2+24\alpha_i^4 e_i^4\right)+$$

$$+\frac{3(\alpha_j e_j)^8}{16{,}640\,(R_{ij}^2-\alpha_i^2 e_i^2)^8}\left(105R_{ij}^6+504R_{ij}^4\alpha_i^2 e_i^2+432R_{ij}^2\alpha_i^4 e_i^4+64\alpha_i^6 e_i^6\right)+$$

$$+\frac{(\alpha_j e_j)^{10}}{15{,}360(R_{ij}^2-\alpha_i^2 e_i^2)^{10}}\left(231R_{ij}^8+1{,}848R_{ij}^6\alpha_i^2 e_i^2+3{,}168R_{ij}^4\alpha_i^4 e_i^4+\right.$$

$$\left.\left.\left.+1{,}408R_{ij}^2\alpha_i^6 e_i^6+128\alpha_i^8 e_i^8\right)\right]\right\}-$$

$$-\frac{9}{\sqrt{2}\,\pi(\alpha_i e_i)^3}\sum_{\substack{j=n+1\\ j\neq i}}^{n+k} m_j\frac{\underset{\sim}{R}_{ij}}{\sqrt{R_{ij}}}\sum_{m=7}^{\infty}\frac{(2m)!\,(\alpha_j e_j)^{2m}}{2^{2m}(m!)^2(2m+3)}W_m(R_{ij},\alpha_i e_i)\quad(i=n+1,\ldots\ldots n+k),$$

$$Z_i''=-3(\alpha_i e_i)^{-3}\sum_{j=1}^{n} m_j Z_{ij}\,(\tan\beta_{ji}-\beta_{ji}),\ (i=n+1,\ldots\ldots n+k).$$

The parameter β_{ij} , according to equations (2.17), (2.21) and (2.22) satisfies the relation

$$R_{ij}^2 \sin^2 \beta_{ij} + Z_{ij}^2 \tan^2 \beta_{ij} = (\alpha_j e_j)^2, (i=1,\cdots n), (j=n+1,\cdots n+\kappa), \tag{3.5}$$

or

$$\beta_{ij} = \arcsin\left\{\frac{1}{R_{ij}}\sqrt{\frac{(r_{ij}^2+\alpha_j^2 e_j^2)-\sqrt{(r_{ij}^2+\alpha_j^2 e_j^2)^2+4R_{ij}^2\alpha_j^2 e_j^2}}{2}}\right\}, R_{ij} \neq 0, \tag{3.6}$$

$$(i=1,\cdots\cdots n), (j=n+1,\cdots\cdots\cdots n+\kappa),$$

while, when $R_{ij}=0$

$$\beta_{ij} = \arctan\left(\frac{\alpha_j e_j}{|Z_{ij}|}\right), (i=1,\cdots\cdots n), (j=n+1,\cdots\cdots n+\kappa). \tag{3.7}$$

The function $W_m(R_{ij}, \alpha_i, e_i)$ is given by equation (2.65).

Equations (3.4) constitute a system of $3(n+\kappa)$ second order differential equations for the co-ordinates x_i, y_i, Z_i $(i=1,\cdots n+\kappa)$. This system satisfies the postulates of the theorem known in the theory of differential equations as the "existence theorem"; it, therefore, has a unique solution for every given set of initial conditions. However, the analytical form of this solution cannot be found since the third of equations (3.4) includes an infinite number of terms. In order to proceed, we neglect the series expansion entering the right-hand member of equation (2.73) and construct the following system of differential equations

$$\underset{\sim}{R}_i'' = -\sum_{\substack{j=1\\j\neq i}}^{n} m_j r_{ij}^{-3} \underset{\sim}{R}_{ij} - \frac{3}{2}\sum_{j=n+1}^{n+\kappa} m_j (\alpha_j e_j)^{-3} \underset{\sim}{R}_{ij}(\beta_{ij} - \sin\beta_{ij}\cos\beta_{ij}), (i=1,\cdots n),$$

$$Z_i'' = -\sum_{\substack{j=1\\j\neq i}}^{n} m_j r_{ij}^{-3} Z_{ij} - 3\sum_{j=n+1}^{n+\kappa} m_j (\alpha_j e_j)^{-3} Z_{ij}(\tan\beta_{ij} - \beta_{ij}) \quad (i=1,\cdots n),$$

$$\underset{\sim}{R}_i'' = -\frac{3}{2}(\alpha_i e_i)^{-3}\sum_{j=1}^{n} m_j \underset{\sim}{R}_{ij}(\beta_{ji} - \sin\beta_{ji}\cos\beta_{ji}) -$$

$$-3\sum_{\substack{j=n+1\\ j\neq i}}^{n+K} m_j \underset{\sim}{R}_{ij}\Bigg\{\frac{1}{2(\alpha_i e_i)^3}\left[\arcsin\left(\frac{\alpha_i e_i}{R_{ij}}\right) - \frac{\alpha_i e_i}{R_{ij}^2}\sqrt{R_{ij}^2 - \alpha_i^2 e_i^2}\right] +$$

$$+\frac{(\alpha_j e_j)^2}{(R_{ij}^2 - \alpha_i^2 e_i^2)^{3/2}}\Bigg[\frac{1}{10R_{ij}^2} + \frac{3(\alpha_j e_j)^2}{56(R_{ij}^2 - \alpha_i^2 e_i^2)^2} + \frac{(\alpha_j e_j)^4}{144(R_{ij}^2 - \alpha_i^2 e_i^2)^4}(5R_{ij}^2 + 4\alpha_i^2 e_i^2) +$$

$$+\frac{(\alpha_j e_j)^6}{1{,}408(R_{ij}^2 - \alpha_i^2 e_i^2)^6}(35R_{ij}^4 + 84R_{ij}^2\alpha_i^2 e_i^2 + 24\alpha_i^4 e_i^4) + \tag{3.8}$$

$$+\frac{3(\alpha_j e_j)^8}{16{,}640(R_{ij}^2 - \alpha_i^2 e_i^2)^8}(105R_{ij}^6 + 504R_{ij}^4\alpha_i^2 e_i^2 + 432R_{ij}^2\alpha_i^4 e_i^4 + 64\alpha_i^6 e_i^6) +$$

$$+\frac{(\alpha_j e_j)^{10}}{15{,}360(R_{ij}^2 - \alpha_i^2 e_i^2)^{10}}(231R_{ij}^8 + 1{,}848R_{ij}^6\alpha_i^2 e_i^2 + 3{,}168R_{ij}^4\alpha_i^4 e_i^4 +$$

$$+1{,}408R_{ij}^2\alpha_i^6 e_i^6 + 128\alpha_i^8 e_i^8)\Bigg]\Bigg\},\quad (i = n+1, \cdots\cdots\cdots n+K),$$

$$Z_i'' = -3(\alpha_i e_i)^{-3}\sum_{j=1}^{n} m_j Z_{ij}(\tan\beta_{ji} - \beta_{ji}),\quad (i = n+1, \cdots\cdots n+K).$$

This system has a unique solution and if, for an arbitrary instant of time τ_α, all the position and velocity vectors $\underset{\sim}{\Gamma}_i$ and $\underset{\sim}{\Gamma}_i'$ $(i = 1, \cdots\cdots\cdots n+K)$ are known, then the functions $\underset{\sim}{\Gamma}_i = \underset{\sim}{\Gamma}_i(\tau)$ $(i = 1, \cdots\cdots\cdots n+K)$ are fixed for all times.

The solutions of the differential systems (3.4) and (3.8) are analytically different. Of these two only the solution of the system (3.4) describes precisely the evolution of the dynamical system under consideration. However, this solution cannot be found and hence, in numerical applications, we are forced to solve the system (3.8). It is therefore important to ensure that the difference between the solutions of the true and approximate equations of motion [equations (3.4) and (3.8)] is small and possibly negligible. A direct check of this difference (depending on the values of the parameters R_{ij}, $\alpha_i e_i, \alpha_j e_j$, $(i,j = n+1, \cdots\cdots n+K,\ i\neq j)$ and the independent variable τ) is not possible since the solutions themselves are not known. An indirect check, showing that the neglected series

$$\frac{-9}{\sqrt{2}\pi(\alpha_i e_i)^3}\sum_{\substack{j=n+1\\ j\neq i}}^{n+K} m_j \frac{\underset{\sim}{R}_{ij}}{\sqrt{R_{ij}}}\sum_{m=7}^{\infty}\frac{(2m)!(\alpha_j e_j)^{2m}}{2^{2m}(m!)^2(2m+3)} W_m(R_{ij},\alpha_i,e_i),\ (i=n+1,\cdots n+K), \tag{3.9}$$

indeed has negligible effect on the solution of the system (3.4), is given below.

In appendix (A7) it is shown that, by truncating the series expansion representing the force $\underset{\sim}{F}_{ij}$ between the i^{th} and j^{th} spheroids $(i,j=n+1,\cdots\cdots\cdots\cdots n+K,\ i\neq j)$ after its 7^{th} term, we introduce a relative error $\underset{\sim}{f}_{ij}$ obeying the relation

$$\frac{|\underset{\sim}{f}_{ij}|}{|\underset{\sim}{F}_{ij}|} \ll 2.098\times 10^{-3},\quad (i,j=n+1,\cdots\cdots\cdots n+K,\ i\neq j), \tag{3.10}$$

where

$$\underset{\sim}{f}_{ij} = -\frac{9 m_i m_j \underset{\sim}{R}_{ij}}{\sqrt{2R_{ij}}\,\pi(\alpha_i e_i)^3}\sum_{m=7}^{\infty}\frac{(2m)!(\alpha_j e_j)^{2m}}{2^{2m}(m!)^2(2m+3)} W_m(R_{ij},\alpha_i,e_i). \tag{3.11}$$

On the other hand the third equation of the system (3.4) can be written in the form

$$\underset{\sim}{R}_i'' = \frac{1}{m_i}\left\{\sum_{j=1}^{n}\underset{\sim}{F}_{ij}^{*} + \sum_{\substack{j=n+1\\ j\neq i}}^{n+k}\underset{\sim}{F}_{ij}\right\}, \quad (i=n+1,\cdots\cdots n+k), \tag{3.12}$$

where $\underset{\sim}{F}_{ij}^{*}$ and $\underset{\sim}{F}_{ij}$ represent the forces exerted on the i^{th} $(i=n+1,\cdots\cdots n+k)$ spheroid by the j^{th} $(j=1,\cdots\cdots n)$ particle and the j^{th} $(j=n+1,\cdots\cdots n+k, i\neq j)$ spheroid respectively. Combining the relations (3.10), (3.11) and (3.12) we easily find that the relative error, say e_{ir} , introduced in the right-hand member of the third of equations (3.4) by neglecting the series (3.9), and defined by the relation

$$e_{ir} \equiv \frac{\sum\limits_{\substack{j=n+1\\ j\neq i}}^{n+k}\left|\underset{\sim}{f}_{ij}\right|}{\sum\limits_{j=1}^{n}\left|\underset{\sim}{F}_{ij}^{*}\right| + \sum\limits_{\substack{j=n+1\\ j\neq i}}^{n+k}\left|\underset{\sim}{F}_{ij}\right|}, \quad (i=n+1,\cdots\cdots n+k),$$

satisfies the condition

$$e_{ir} \ll 2.098\times 10^{-3}. \tag{3.13}$$

In fact, in all cases apart from some exceptional ones e_{ir} is considerably smaller than $2.098\ 10^{-3}$, it is therefore clear that the difference between the solutions of the systems (3.4) and (3.8) in numerical applications is negligible.

We now proceed to find the solution of the differential system (3.8). Clearly the equations of this system must be solved simultaneously. There are $3(n+k)$ equations each of the second order; hence, a complete solution requires the knowledge of $6(n+K)$ integrals. Of these, only ten can be found. Using the function

$$
\begin{aligned}
U(x_1, y_1, z_1, x_2, \cdots z_{n+K}) = -\frac{1}{2}\Bigg\{ & \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_i m_j}{r_{ij}} + 3\sum_{i=1}^{n} \sum_{j=n+1}^{n+K} m_i\Big(m_j(\alpha_j e_j)^{-1} b_{ij} - \\
& - m_j(\alpha_j e_j)^{-3}\Big[\frac{R_{ij}^2}{2}(b_{ij} - \sin b_{ij}\cos b_{ij}) + Z_{ij}^2(\tan b_{ij} - b_{ij})\Big]\Big) \\
& + 3\sum_{i=n+1}^{n+K} \sum_{\substack{j=n+1 \\ j\neq i}}^{n+K} m_i m_j\Bigg(\frac{(5R_{ij}^2 - 10\alpha_i^2 e_i^2 + 2\alpha_j^2 e_j^2)}{20(\alpha_i e_i)^3} \arcsin\left(\frac{\alpha_i e_i}{R_{ij}}\right) - \\
& - \frac{(5R_{ij}^2 - 5\alpha_i^2 e_i^2 + 2\alpha_j^2 e_j^2)}{20(\alpha_i e_i)^2\sqrt{R_{ij}^2 - \alpha_i^2 e_i^2}} - \frac{(\alpha_j e_j)^4}{8(R_{ij}^2 - \alpha_i^2 e_i^2)^2\sqrt{R_{ij}^2 - \alpha_i^2 e_i^2}}\Big[\frac{3}{35} + \\
& + \frac{5(\alpha_j e_j)^2}{126(R_{ij}^2 - \alpha_i^2 e_i^2)} + \frac{(\alpha_j e_j)^2}{1{,}584(R_{ij}^2 - \alpha_i^2 e_i^2)^2}(88\alpha_i^2 e_i^2 + 35\alpha_j^2 e_j^2) + \\
& + \frac{7(\alpha_j e_j)^4}{4{,}576(R_{ij}^2 - \alpha_i^2 e_i^2)^3}(52\alpha_i^2 e_i^2 + 9\alpha_j^2 e_j^2) + \\
& + \frac{(\alpha_j e_j)^4}{8{,}320(R_{ij}^2 - \alpha_i^2 e_i^2)^4}(520\alpha_i^4 e_i^4 + 756\alpha_i^2 e_i^2 \alpha_j^2 e_j^2 + 77\alpha_j^4 e_j^4) + \\
& + \frac{(\alpha_i e_i)^2(\alpha_j e_j)^6}{800(R_{ij}^2 - \alpha_i^2 e_i^2)^5}(135\alpha_i^2 e_i^2 + 77\alpha_j^2 e_j^2) + \frac{3(\alpha_i e_i)^4(\alpha_j e_j)^6}{320(R_{ij}^2 - \alpha_i^2 e_i^2)^6}(10\alpha_i^2 e_i^2 + 33\alpha_j^2 e_j^2) + \\
& + \frac{187(\alpha_i e_i)^6(\alpha_j e_j)^8}{480(R_{ij}^2 - \alpha_i^2 e_i^2)^7} + \frac{323(\alpha_i e_i)^8(\alpha_j e_j)^8}{1{,}920(R_{ij}^2 - \alpha_i^2 e_i^2)^8}\Big]\Bigg)\Bigg\} ,
\end{aligned}
\tag{3.14}
$$

one can easily obtain ten integrals which correspond to the ten well-known integrals of the n-body problem. We note that the function $U(x_1, y_1, z_1, x_2, \cdots\cdots z_{n+k})$, according to equations (2.15) and (2.81), is an approximation of the potential energy of the dynamical system under consideration.

In the case of the n-body problem Bruns has proved that, using rectangular co-ordinates there are no new algebraic integrals, while Poincaré has demonstrated that, using the elements of the orbits of the bodies as variables, there are no new uniform transcendental integrals, even when the masses of all the bodies except one are very small. We therefore expect the same to be true for dynamical systems consisting of particles and oblate spheroids and whose equations of motion are much more complicated than those of the n-body problem.

Numerical integration of the differential system (3.8) is feasible with the aid of various methods. We choose to employ the method of Taylor's series for its several advantages. One is that the coefficients of the Taylor's series decrease faster than those of most analogous finite-difference formulae, so that we can use a large step of integration, with many terms, or a small step of integration with few terms. The latter is desirable for problems involving differential equations for which no simple recurrence relations are available and the higher derivatives involve complicated differentiations.

A second advantage is the high accuracy of the method. In the case of the numerical integration of the system (3.8) a prescribed relative error can be obtained for every x_i, y_i and z_i $(i=1, \cdots\cdots n+k)$ using the method developed in appendix (A1).

We proceed by forming the Taylor's series for every function $x_i = x_i(\tau)$, $y_i = y_i(\tau)$, $z_i = z_i(\tau)$ $(i=1, \cdots\cdots n+k)$ in powers of the independent variable τ about some epoch point, say $\tau = 0$. That is

$$x_i=\sum_{\nu=0}^{\infty} x_{io}^{(\nu)} \frac{\tau^\nu}{\nu!}, \quad y_i=\sum_{\nu=0}^{\infty} y_{io}^{(\nu)} \frac{\tau^\nu}{\nu!}, \quad z_i=\sum_{\nu=0}^{\infty} z_{io}^{(\nu)} \frac{\tau^\nu}{\nu!}, \tag{3.15}$$

$$(i=1,\cdots\cdots n+k), \quad \tau\in(-\tau_1,\tau_1),$$

where $(-\tau_1,\tau_1)$ is the common interval of convergence of the foregoing series and $x_{io}^{(0)}$, $y_{io}^{(0)}$, $z_{io}^{(0)}$ $(i=1,\cdots\cdots n+k)$ are taken to mean

$$x_{io}^{(0)} \equiv x_i\Big|_{\tau=0}, \quad y_{io}^{(0)} \equiv y_i\Big|_{\tau=0}, \quad z_{io}^{(0)} \equiv z_i\Big|_{\tau=0}, \quad (i=1\cdots\cdots n+k),$$

while by definition

$$x_{io}^{(\nu)} \equiv \frac{d^\nu x_i}{d\tau^\nu}\Big|_{\tau=0}, \quad y_{io}^{(\nu)} \equiv \frac{d^\nu y_i}{d\tau^\nu}\Big|_{\tau=0}, \quad z_{io}^{(\nu)} \equiv \frac{d^\nu z_i}{d\tau^\nu}\Big|_{\tau=0}, \quad \begin{matrix}(i=1,\cdots n+k)\\ \nu=1,2,3,\cdots\end{matrix}$$

Successive differentiations of equations (3.8) with respect to τ give the derivatives of the functions $x_i=x_i(\tau)$, $y_i=y_i(\tau)$, $z_i=z_i(\tau)$, $(i=1,\cdots n+k)$ of order higher than two. These derivatives at $\tau=0$ are the coefficients of the terms of the power series solution (3.15) and therefore a step by step numerical integration of the system (3.8) is possible.

Differentiating equations (3.8) once with respect to τ we obtain

$$\underset{\sim}{R}_i'''=-\sum_{\substack{j=1\\j\neq i}}^{n} m_j\left[\underset{\sim}{R}_{ij}'-3\underset{\sim}{R}_{ij}\frac{(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}')}{r_{ij}^2}\right]\frac{1}{r_{ij}^3}-$$

$$-\frac{3}{2}\sum_{j=n+1}^{n+k} m_j\,(\alpha_j e_j)^{-3}\left[\underset{\sim}{R}_{ij}'(\theta_{ij}-\sin\theta_{ij}\cos\theta_{ij})+2\underset{\sim}{R}_{ij}\theta_{ij}'\sin^2\theta_{ij}\right], (i=1,\cdots n),$$

$$z_i'''=-\sum_{\substack{j=1\\j\neq i}}^{n} m_j\left[z_{ij}'-3z_{ij}\frac{(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}')}{r_{ij}^2}\right]\frac{1}{r_{ij}^3}-$$

$$-3\sum_{j=n+1}^{n+K} m_j (\alpha_j e_j)^{-3}\left[Z'_{ij}(\tan\theta_{ij}-\theta_{ij})+Z_{ij}\theta'_{ij}\tan^2\theta_{ij}\right],\ (i=1,\cdots\cdots n),$$

$$\underset{\sim}{R}'''_i=-\frac{3}{2}(\alpha_i e_i)^{-3}\sum_{j=1}^{n} m_j\Big[\underset{\sim}{R}'_{ij}\,(\theta_{ji}-\sin\theta_{ji}\cos\theta_{ji})+ \tag{3.16}$$

$$+2\underset{\sim}{R}_{ij}\theta'_{ji}\sin^2\theta_{ji}\Big]-3\sum_{\substack{j=n+1\\ j\neq i}}^{n+K} m_j\Big[\underset{\sim}{R}'_{ij}H(R_{ij},\alpha_i e_i,\alpha_j e_j)+$$

$$+\underset{\sim}{R}_{ij}H'(R_{ij},\alpha_i e_i,\alpha_j e_j)\Big],\quad (i=n+1,\cdots\cdots\cdots n+K),$$

$$Z'''_i=-3(\alpha_i e_i)^{-3}\sum_{j=1}^{n} m_j\left[Z'_{ij}(\tan\theta_{ji}-\theta_{ji})+Z_{ij}\theta'_{ji}\tan^2\theta_{ji}\right],\ (i=n+1,\cdots\cdots n+K),$$

where θ_{ij} $(i=1,\cdots\cdots\cdots n,\ j=n+1,\cdots\cdots\cdots n+K)$ is given by equations (3.6) and (3.7). The derivative θ'_{ij} can be found - by differentiation of equation (3.5) - equal to

$$\theta'_{ij}=-\frac{(\underset{\sim}{R}_{ij}\cdot\underset{\sim}{R}'_{ij})\sin^2\theta_{ij}+Z_{ij}Z'_{ij}\tan^2\theta_{ij}}{R^2_{ij}\sin\theta_{ij}\cos\theta_{ij}+Z^2_{ij}\tan\theta_{ij}(1+\tan^2\theta_{ij})},\ (i=1,\cdots n,\ j=n+1,\cdots\cdots n+K). \tag{3.17}$$

The function $H(R_{ij},\alpha_i e_i,\alpha_j e_j)$ is defined by the relation

$$H(R_{ij},\alpha_i e_i,\alpha_j e_j)\equiv\frac{1}{2(\alpha_i e_i)^3}\left[\arcsin\left(\frac{\alpha_i e_i}{R_{ij}}\right)-\frac{\alpha_i e_i}{R^2_{ij}}\sqrt{R^2_{ij}-\alpha^2_i e^2_i}\right]+$$

$$+\frac{(\alpha_j e_j)^2}{(R_{ij}^2-\alpha_i^2 e_i^2)^{3/2}}\left[\frac{1}{10R_{ij}^2}+\frac{3(\alpha_j e_j)^2}{56(R_{ij}^2-\alpha_i^2 e_i^2)^2}+\frac{(\alpha_j e_j)^4}{144(R_{ij}^2-\alpha_i^2 e_i^2)^4}\left(5R_{ij}^2+4\alpha_i^2 e_i^2\right)+\right.$$

$$+\frac{(\alpha_j e_j)^6}{1,408\,(R_{ij}^2-\alpha_i^2 e_i^2)^6}\left(35R_{ij}^4+84R_{ij}^2\alpha_i^2 e_i^2+24\alpha_i^4 e_i^4\right)+$$

$$+\frac{3(\alpha_j e_j)^8}{16,640(R_{ij}^2-\alpha_i^2 e_i^2)^8}\left(105R_{ij}^6+504R_{ij}^4\alpha_i^2 e_i^2+432R_{ij}^2\alpha_i^4 e_i^4+64\alpha_i^6 e_i^6\right)+ \qquad (3.18)$$

$$+\frac{(\alpha_j e_j)^{10}}{15,360(R_{ij}^2-\alpha_i^2 e_i^2)^{10}}\left(231R_{ij}^8+1,848R_{ij}^6\alpha_i^2 e_i^2+3,168R_{ij}^4\alpha_i^4 e_i^4+\right.$$

$$\left.\left.+1,408R_{ij}^2\alpha_i^6 e_i^6+128\alpha_i^8 e_i^8\right)\right],\ (i,j=n+1,\cdots\cdots n+K,\ i\neq j).$$

We finally obtain $H'(R_{ij},\alpha_i e_i,\alpha_j e_j)$ by differentiation of the foregoing relation once with respect to τ. It can be verified, after long algebraic manipulations, that

$$H'(R_{ij},\alpha_i e_i,\alpha_j e_j)=-\frac{(\underset{\sim}{R}_{ij}\cdot\underset{\sim}{R}'_{ij})}{\sqrt{R_{ij}^2-\alpha_i^2 e_i^2}}\left\{\frac{1}{R_{ij}^4}+\right.$$

$$+\frac{(\alpha_j e_j)^2}{2(R_{ij}^2-\alpha_i^2 e_i^2)^2}\left[\frac{(5R_{ij}^2-2\alpha_i^2 e_i^2)}{5R_{ij}^4}+\frac{3(\alpha_j e_j)^2}{4(R_{ij}^2-\alpha_i^2 e_i^2)^2}+\right. \qquad (3.19)$$

$$+\frac{(\alpha_j e_j)^4}{8(R_{ij}^2-\alpha_i^2 e_i^2)^4}\left(5R_{ij}^2+6\alpha_i^2 e_i^2\right)+\frac{(\alpha_j e_j)^6}{64(R_{ij}^2-\alpha_i^2 e_i^2)^6}\left(35R_{ij}^4+112R_{ij}^2\alpha_i^2 e_i^2+48\alpha_i^4 e_i^4\right)+$$

$$+\frac{3(\alpha_j e_j)^8}{128(R_{ij}^2-\alpha_i^2 e_i^2)^8}\left(21R_{ij}^6+126R_{ij}^4\alpha_i^2e_i^2+144R_{ij}^2\alpha_i^4e_i^4+32\alpha_i^6e_i^6\right)+$$

$$+\frac{(\alpha_j e_j)^{10}}{2{,}560(R_{ij}^2-\alpha_i^2 e_i^2)^{10}}\left(1{,}155R_{ij}^8+11{,}088R_{ij}^6\alpha_i^2e_i^2+23{,}760R_{ij}^4\alpha_i^4e_i^4+\right.$$

$$\left.\left.\left.+14{,}080R_{ij}^2\alpha_i^6e_i^6+1{,}920\alpha_i^8e_i^8\right)\right]\right\},\quad (i,j=n+1\cdots\cdots n+k,\ i\neq j).$$

Differentiating both sides of equations (3.16) with respect to τ and arranging the resulting expressions we find

$$\underset{\sim}{R}_i^{(4)}=-\sum_{\substack{j=1\\ j\neq i}}^{n} m_j\left[\underset{\sim}{R}_{ij}''-6\underset{\sim}{R}_{ij}'\frac{(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}')}{r_{ij}^2}-\right.$$

$$\left.-3\underset{\sim}{R}_{ij}\left(-\frac{5(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}')^2}{r_{ij}^4}+\frac{(\underset{\sim}{r}_{ij}'\cdot\underset{\sim}{r}_{ij}')}{r_{ij}^2}+\frac{(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}'')}{r_{ij}^2}\right)\right]\frac{1}{r_{ij}^3}-$$

$$-\frac{3}{2}\sum_{j=n+1}^{n+k} m_j(\alpha_j e_j)^{-3}\left[\underset{\sim}{R}_{ij}''(\theta_{ij}-\sin\theta_{ij}\cos\theta_{ij})+4\underset{\sim}{R}_{ij}'\theta_{ij}'\sin^2\theta_{ij}+2\underset{\sim}{R}_{ij}\theta_{ij}''\sin^2\theta_{ij}+\right.$$

$$\left.+4\underset{\sim}{R}_{ij}(\theta_{ij}')^2\sin\theta_{ij}\cos\theta_{ij}\right],\quad (i=1,\cdots n),$$

$$Z_i^{(4)}=-\sum_{\substack{j=1\\ j\neq i}}^{n} m_j\left[Z_{ij}''-6Z_{ij}'\frac{(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}')}{r_{ij}^2}-\right. \tag{3.20}$$

$$\left.-3Z_{ij}\left(-\frac{5(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}')^2}{r_{ij}^4}+\frac{(\underset{\sim}{r}_{ij}'\cdot\underset{\sim}{r}_{ij}')}{r_{ij}^2}-\frac{(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}'')}{r_{ij}^2}\right)\right]\frac{1}{r_{ij}^3}-$$

$$-3\sum_{j=n+1}^{n+K} m_j(\alpha_j e_j)^{-3}\Big[Z''_{ij}(\tan\theta_{ij}-\theta_{ij})+2Z'_{ij}\theta'_{ij}\tan^2\theta_{ij}+Z_{ij}\theta''_{ij}\tan^2\theta_{ij}+$$

$$+2Z_{ij}(\theta'_{ij})^2\tan\theta_{ij}(1+\tan^2\theta_{ij}), \quad (i=1,\cdots\cdots n),$$

$$\underset{\sim}{R}_i^{(4)}=-\frac{3}{2}(\alpha_i e_i)^{-3}\sum_{j=1}^{n} m_j\Big[\underset{\sim}{R}''_{ij}(\theta_{ji}-\sin\theta_{ji}\cos\theta_{ji})+4\underset{\sim}{R}'_{ij}\theta'_{ji}\sin^2\theta_{ji}+$$

$$+2\underset{\sim}{R}_{ij}\theta''_{ji}\sin^2\theta_{ji}+4\underset{\sim}{R}_{ji}(\theta'_{ji})^2\sin\theta_{ji}\cos\theta_{ji}\Big]-$$

$$-3\sum_{\substack{j=n+1\\ j\neq i}}^{n+K} m_j\Big[\underset{\sim}{R}''_{ij}H(R_{ij},\alpha_i e_i,\alpha_j e_j)+2\underset{\sim}{R}'_{ij}H'(R_{ij},\alpha_i e_i,\alpha_j e_j)+$$

$$+\underset{\sim}{R}_{ij}H''(R_{ij},\alpha_i e_i,\alpha_j e_j)\Big], \quad (i=n+1,\cdots\cdots n+K),$$

$$Z_i^{(4)}=-3(\alpha_i e_i)^{-3}\sum_{j=1}^{n} m_j\Big[Z''_{ij}(\tan\theta_{ji}-\theta_{ji})+2Z'_{ij}\theta'_{ji}\tan^2\theta_{ji}+Z_{ij}\theta''_{ji}\tan^2\theta_{ji}+$$

$$+2Z_{ij}(\theta'_{ji})^2\tan\theta_{ji}(1+\tan^2\theta_{ji})\Big], \quad (i=n+1,\cdots\cdots n+K).$$

The derivative θ''_{ij} , occurring in these expressions, is obtained by differentiating equation (3.17) once with respect to τ , and it has the form

$$\theta''_{ij} = -\Big[(\underset{\sim}{R}''_{ij}\cdot\underset{\sim}{R}_{ij} + \underset{\sim}{R}'_{ij}\cdot\underset{\sim}{R}'_{ij})\sin^2\theta_{ij} + (Z''_{ij}Z_{ij} + Z'_{ij}Z'_{ij})\tan^2\theta_{ij} +$$

$$+\theta'_{ij}\Big(4\underset{\sim}{R}_{ij}\cdot\underset{\sim}{R}'_{ij}\sin\theta_{ij}\cos\theta_{ij} + 4Z_{ij}Z'_{ij}\tan\theta_{ij}(1+\tan^2\theta_{ij}) +$$

$$+R^2_{ij}\theta'_{ij}(1-2\sin^2\theta_{ij}) + Z^2_{ij}\theta'_{ij}(1+\tan^2\theta_{ij})(1+3\tan^2\theta_{ij})\Big)\Big] \Big/ \tag{3.21}$$

$$\Big/\Big[R^2_{ij}\sin\theta_{ij}\cos\theta_{ij} + Z^2_{ij}\tan\theta_{ij}(1+\tan^2\theta_{ij})\Big], \quad (i=1,\cdots\cdots n,\ j=n+1,\cdots n+K).$$

The function $H''(R_{ij}, \alpha_i e_i, \alpha_j e_j)$ is given by the relation

$$H''(R_{ij}, \alpha_i e_i, \alpha_j e_j) \equiv \frac{dH'(R_{ij}, \alpha_i e_i, \alpha_j e_j)}{d\tau}, \quad (i,j=n+1,\cdots\cdots n+K, i\neq j),$$

which, by introduction of equation (3.19) and after long and laborious algebraic manipulations, becomes

$$H''(R_{ij}, \alpha_i e_i, \alpha_j e_j) = \left(\underset{\sim}{R}''_{ij}\cdot\underset{\sim}{R}_{ij} + \underset{\sim}{R}'_{ij}\cdot\underset{\sim}{R}'_{ij} - \frac{(\underset{\sim}{R}_{ij}\cdot\underset{\sim}{R}'_{ij})^2}{R^2_{ij}}\right)\frac{H'(R_{ij}, \alpha_i e_i, \alpha_j e_j)}{(\underset{\sim}{R}_{ij}\cdot\underset{\sim}{R}'_{ij})} + \tag{3.22}$$

$$+\frac{B_1(R_{ij}, \alpha_i e_i, \alpha_j e_j)}{(R^2_{ij} - \alpha_i^2 e_i^2)^{3/2}}\,\frac{(\underset{\sim}{R}_{ij}\cdot\underset{\sim}{R}'_{ij})^2}{R^2_{ij}}, \quad (i,j=n+1,\cdots\cdots n+K, i\neq j),$$

where the function $B_1(R_{ij}, \alpha_i e_i, \alpha_j e_j)$ $(i,j=n+1,\cdots\cdots n+K, i\neq j)$ is defined by the relation

$$B_1(R_{ij}, \alpha_i e_i, \alpha_j e_j) \equiv \frac{(4R^2_{ij} - 3\alpha_i^2 e_i^2)}{R^4_{ij}} + \frac{(\alpha_j e_j)^2}{2(R^2_{ij} - \alpha_i^2 e_i^2)^2}\Big[\frac{3}{5R^4_{ij}}(10R^4_{ij} - 7R^2_{ij}\alpha_i^2 e_i^2 +$$

$$+2\alpha_i^4 e_i^4) +$$

$$+\frac{3(\alpha_j e_j)^2}{4(R_{ij}^2-\alpha_i^2 e_i^2)^2}(8R_{ij}^2+\alpha_i^2 e_i^2)+\frac{(\alpha_j e_j)^4}{8(R_{ij}^2-\alpha_i^2 e_i^2)^4}(50R_{ij}^4+87R_{ij}^2\alpha_i^2 e_i^2+6\alpha_i^4 e_i^4)+$$

$$+\frac{3(\alpha_j e_j)^6}{64(R_{ij}^2-\alpha_i^2 e_i^2)^6}(140R_{ij}^6+581R_{ij}^4\alpha_i^2 e_i^2+368R_{ij}^2\alpha_i^4 e_i^4+16\alpha_i^6 e_i^6)+ \tag{3.23}$$

$$+\frac{3(\alpha_j e_j)^8}{128(R_{ij}^2-\alpha_i^2 e_i^2)^8}(294R_{ij}^8+2{,}163R_{ij}^6\alpha_i^2 e_i^2+3{,}222R_{ij}^4\alpha_i^4 e_i^4+1{,}072R_{ij}^2\alpha_i^6 e_i^6+$$

$$+32\alpha_i^8 e_i^8)+\frac{(\alpha_j e_j)^{10}}{2{,}560(R_{ij}^2-\alpha_i^2 e_i^2)^{10}}(18{,}480R_{ij}^{10}+209{,}979R_{ij}^8\alpha_i^2 e_i^2+552{,}816R_{ij}^6\alpha_i^4 e_i^4+$$

$$\left.+428{,}560R_{ij}^4\alpha_i^6 e_i^6+88{,}320R_{ij}^2\alpha_i^8 e_i^8+1{,}920\alpha_i^{10}e_i^{10})\right],\ (i,j=n+1,\cdots n+k,\ i\neq j).$$

Differentiation of equations (3.20) gives the fifth order derivatives $\underset{\sim}{R}_i^{(5)}, Z_i^{(5)}$ $(i=1,\cdots\cdots n+k)$, which are

$$\underset{\sim}{R}_i^{(5)}=-\sum_{\substack{j=1\\ j\neq i}}^{n} m_j\left[\underset{\sim}{R}_{ij}'''-9\underset{\sim}{R}_{ij}''\frac{(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}')}{r_{ij}^2}-\right.$$

$$-\left(\frac{9(\underset{\sim}{r}_{ij}'\cdot\underset{\sim}{r}_{ij}')}{r_{ij}^2}+\frac{9(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}'')}{r_{ij}^2}-\frac{45(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}')^2}{r_{ij}^4}\right)\underset{\sim}{R}_{ij}'-\left(\frac{3(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}''')}{r_{ij}^2}+\frac{9(\underset{\sim}{r}_{ij}'\cdot\underset{\sim}{r}_{ij}'')}{r_{ij}^2}-\right.$$

$$\left.\left.-\frac{45(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}')(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}'')}{r_{ij}^4}-\frac{45(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}')(\underset{\sim}{r}_{ij}'\cdot\underset{\sim}{r}_{ij}')}{r_{ij}^4}+\frac{105(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}')^3}{r_{ij}^6}\right)\underset{\sim}{R}_{ij}\right]\frac{1}{r_{ij}^3}-$$

$$-\frac{3}{2}\sum_{j=n+1}^{n+k} m_j(\alpha_j e_j)^{-3}\Big[\underset{\sim}{R}_{ij}'''(\theta_{ij}-\sin\theta_{ij}\cos\theta_{ij})+6\underset{\sim}{R}_{ij}''\theta_{ij}'\sin^2\theta_{ij}+$$

$$+6\underset{\sim}{R}_{ij}'\theta_{ij}''\sin^2\theta_{ij}+12\underset{\sim}{R}_{ij}'(\theta_{ij}')^2\sin\theta_{ij}\cos\theta_{ij}+2\underset{\sim}{R}_{ij}\theta_{ij}'''\sin^2\theta_{ij}+$$

$$+12\underset{\sim}{R}_{ij}\theta_{ij}'\theta_{ij}''\sin\theta_{ij}\cos\theta_{ij}+4\underset{\sim}{R}_{ij}(\theta_{ij}')^3(1-2\sin^2\theta_{ij})\Big],\ (i=1,\cdots\cdots n),$$

$$Z_i^{(5)}=-\sum_{\substack{j=1\\ j\neq i}}^{n} m_j\Big[Z_{ij}'''-Z_{ij}''\frac{9(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}')}{r_{ij}^2}-$$

$$-Z_{ij}'\left(\frac{9(\underset{\sim}{r}_{ij}'\cdot\underset{\sim}{r}_{ij}')}{r_{ij}^2}+\frac{9(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}'')}{r_{ij}^2}-\frac{45(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}')^2}{r_{ij}^4}\right)-Z_{ij}\left(\frac{3(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}''')}{r_{ij}^2}+\frac{9(\underset{\sim}{r}_{ij}'\cdot\underset{\sim}{r}_{ij}'')}{r_{ij}^2}-\right.$$

$$\left.-\frac{45(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}')(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}'')}{r_{ij}^4}-\frac{45(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}')(\underset{\sim}{r}_{ij}'\cdot\underset{\sim}{r}_{ij}')}{r_{ij}^4}+\frac{105(\underset{\sim}{r}_{ij}\cdot\underset{\sim}{r}_{ij}')^3}{r_{ij}^6}\right)\Big]\frac{1}{r_{ij}^3}-$$

$$-3\sum_{j=n+1}^{n+k} m_j(\alpha_j e_j)^{-3}\Big[Z_{ij}'''(\tan\theta_{ij}-\theta_{ij})+3Z_{ij}''\theta_{ij}'\tan^2\theta_{ij}+3Z_{ij}'\theta_{ij}''\tan^2\theta_{ij}+ \tag{3.24}$$

$$+6Z_{ij}'(\theta_{ij}')^2\tan\theta_{ij}(1+\tan^2\theta_{ij})+Z_{ij}\theta_{ij}'''\tan^2\theta_{ij}+6Z_{ij}\theta_{ij}'\theta_{ij}''\tan\theta_{ij}(1+\tan^2\theta_{ij})+$$

$$+2Z_{ij}(\theta_{ij}')^3(1+\tan^2\theta_{ij})(1+3\tan^2\theta_{ij})\Big],\ (i=1,\cdots n),$$

$$\underset{\sim}{R}_i^{(5)}=-\frac{3}{2}(\alpha_i e_i)^{-3}\sum_{j=1}^{n} m_j\Big[\underset{\sim}{R}_{ij}'''(\theta_{ji}-\sin\theta_{ji}\cos\theta_{ji})+$$

$$+6\underset{\sim}{R}''_{ij}b'_{ji}\sin^2 b_{ji}+6\underset{\sim}{R}'_{ij}b''_{ji}\sin^2 b_{ji}+12\underset{\sim}{R}'_{ij}(b'_{ji})^2\cdot\sin b_{ji}\cos b_{ji}+$$

$$+2\underset{\sim}{R}_{ij}b'''_{ji}\sin^2 b_{ji}+12\underset{\sim}{R}_{ij}b'_{ji}b''_{ji}\sin b_{ji}\cos b_{ji}+4\underset{\sim}{R}_{ij}(b'_{ji})^3(1-2\sin^2 b_{ji})\Big]-$$

$$-3\sum_{\substack{j=n+1\\ j\neq i}}^{n+K} m_j\Big[\underset{\sim}{R}'''_{ij}H(R_{ij},\alpha_i e_i,\alpha_j e_j)+3\underset{\sim}{R}''_{ij}H'(R_{ij},\alpha_i e_i,\alpha_j e_j)+3\underset{\sim}{R}'_{ij}H''(R_{ij},\alpha_i e_i,\alpha_j e_j)+$$

$$+\underset{\sim}{R}_{ij}H'''(R_{ij},\alpha_i e_i,\alpha_j e_j)\Big],\quad (i=n+1,\cdots\cdots n+K),$$

$$Z_i^{(5)}=-3(\alpha_i e_i)^{-3}\sum_{j=1}^{n} m_j\Big[Z'''_{ij}(\tan b_{ji}-b_{ji})+$$

$$+3Z''_{ij}b'_{ji}\tan^2 b_{ji}+3Z'_{ij}b''_{ji}\tan^2 b_{ji}+6Z'_{ij}(b'_{ji})^2\tan b_{ji}(1+\tan^2 b_{ji})+Z_{ij}b'''_{ji}\tan^2 b_{ji}+$$

$$+6Z_{ij}b'_{ji}b''_{ji}\tan b_{ji}(1+\tan^2 b_{ji})+2Z_{ij}(b'_{ji})^3(1+\tan^2 b_{ji})(1+3\tan^2 b_{ji})\Big],\ (i=n+1,\cdots n+K).$$

In order to obtain b'''_{ij} $(i=1,\cdots\cdots n,\ j=n+1,\cdots\cdots n+K)$ we differentiate equation (3.21) with respect to τ ; this differentiation yields

$$b'''_{ij}=-\Big\{2\Big[3(\underset{\sim}{R}''_{ij}\cdot\underset{\sim}{R}_{ij}+\underset{\sim}{R}'_{ij}\cdot\underset{\sim}{R}'_{ij})b'_{ij}+3(\underset{\sim}{R}_{ij}\cdot\underset{\sim}{R}'_{ij})b''_{ij}-2R^2_{ij}(b'_{ij})^3\Big]\sin b_{ij}\cos b_{ij}+$$

$$+2\Big[3(Z_{ij}Z''_{ij}+Z'_{ij}Z'_{ij})b'_{ij}+3Z_{ij}Z'_{ij}b''_{ij}+2Z^2_{ij}(b'_{ij})^3(2+3\tan^2 b_{ij})\Big]\tan b_{ij}(1+\tan^2 b_{ij})+$$

$$+3\Big[2(\underset{\sim}{R}_{ij}\cdot\underset{\sim}{R}'_{ij})b'_{ij}+R^2_{ij}b''_{ij}\Big]b'_{ij}(1-2\sin^2 b_{ij})+ \qquad (3.25)$$

$$+3\left[2Z_{ij}Z'_{ij}\theta'_{ij}+Z^2_{ij}\theta''_{ij}\right]\theta'_{ij}(1+\tan^2\theta_{ij})(1+3\tan^2\theta_{ij})+$$

$$\left.\left.+(\underset{\sim}{R}'''_{ij}\underset{\sim}{R}_{ij}+3\underset{\sim}{R}''_{ij}\underset{\sim}{R}'_{ij})\sin^2\theta_{ij}+(Z'''_{ij}Z_{ij}+3Z'_{ij}Z''_{ij})\tan^2\theta_{ij}\right\}\right]\Big/$$

$$\Big/\left[\left\{R^2_{ij}\sin\theta_{ij}\cos\theta_{ij}+Z^2_{ij}\tan\theta_{ij}(1+\tan^2\theta_{ij})\right\}\right],(i=1,\cdots n,\ j=n+1,\cdots\cdots n+K).$$

Finally the derivative $H'''(R_{ij},\alpha_ie_i,\alpha_je_j)$ $(i,j=n+1,\cdots\cdots n+K,\ i\neq j)$ can be obtained from equation (3.22). After long algebraic manipulations we find

$$H'''(R_{ij},\alpha_ie_i,\alpha_je_j)=\left(\underset{\sim}{R}'''_{ij}\underset{\sim}{R}_{ij}+3\underset{\sim}{R}'_{ij}\underset{\sim}{R}''_{ij}-\frac{3(\underset{\sim}{R}_{ij}\underset{\sim}{R}'_{ij})(\underset{\sim}{R}''_{ij}\underset{\sim}{R}_{ij}+\underset{\sim}{R}'_{ij}\underset{\sim}{R}'_{ij})}{R^2_{ij}}+\right.$$

$$\left.+3\frac{(\underset{\sim}{R}_{ij}\cdot\underset{\sim}{R}'_{ij})^3}{R^4_{ij}}\right)\frac{H'(R_{ij},\alpha_ie_i,\alpha_je_j)}{(\underset{\sim}{R}_{ij}\underset{\sim}{R}'_{ij})}+3\frac{(\underset{\sim}{R}_{ij}\cdot\underset{\sim}{R}'_{ij})}{R^2_{ij}}\left(\underset{\sim}{R}''_{ij}\underset{\sim}{R}_{ij}+\underset{\sim}{R}'_{ij}\underset{\sim}{R}'_{ij}-\right. \tag{3.26}$$

$$\left.-\frac{(\underset{\sim}{R}_{ij}\underset{\sim}{R}'_{ij})^2}{R^2_{ij}}\right)\frac{B_1(R_{ij},\alpha_ie_i,\alpha_je_j)}{(R^2_{ij}-\alpha^2_ie^2_i)^{3/2}}-\frac{(\underset{\sim}{R}_{ij}\cdot\underset{\sim}{R}'_{ij})^3}{R^2_{ij}}\frac{B_2(R_{ij},\alpha_ie_i,\alpha_je_j)}{(R^2_{ij}-\alpha^2_ie^2_i)^{5/2}},$$

$$(i,j=n+1,\cdots\cdots n+K,\ i\neq j),$$

where $B_2(R_{ij},\alpha_ie_i,\alpha_je_j)$ $(i,j=n+1,\cdots\cdots n+K,\ i\neq j)$ is given by the following relation

$$B_2(R_{ij},\alpha_ie_i,\alpha_je_j)=\frac{(20R^4_{ij}-29R^2_{ij}\alpha^2_ie^2_i+12\alpha^4_ie^4_i)}{R^6_{ij}}+$$

$$+\frac{(\alpha_je_j)^2}{2(R^2_{ij}-\alpha^2_ie^2_i)^2}\left[\frac{3}{5R^6_{ij}}(70R^6_{ij}-63R^4_{ij}\alpha^2_ie^2_i+36R^2_{ij}\alpha^4_ie^4_i-8\alpha^6_ie^6_i)+\right.$$

$$+\frac{27(\alpha_j e_j)^2}{4(R_{ij}^2-\alpha_i^2 e_i^2)^2}\left(8R_{ij}^2+3\alpha_i^2 e_i^2\right)+\frac{11(\alpha_j e_j)^4}{8(R_{ij}^2-\alpha_i^2 e_i^2)^4}\left(50R_{ij}^4+121R_{ij}^2\alpha_i^2 e_i^2+24\alpha_i^4 e_i^4\right)+$$

$$+\frac{195(\alpha_j e_j)^6}{64(R_{ij}^2-\alpha_i^2 e_i^2)^6}\left(28R_{ij}^6+147R_{ij}^4\alpha_i^2 e_i^2+132R_{ij}^2\alpha_i^4 e_i^4+16\alpha_i^6 e_i^6\right)+ \tag{3.27}$$

$$+\frac{9(\alpha_j e_j)^8}{128(R_{ij}^2-\alpha_i^2 e_i^2)^8}\left(1{,}470R_{ij}^8+13{,}041R_{ij}^6\alpha_i^2 e_i^2+24{,}732R_{ij}^4\alpha_i^4 e_i^4+11{,}800R_{ij}^2\alpha_i^6 e_i^6+\right.$$

$$\left.+960\alpha_i^8 e_i^8\right)+\frac{119(\alpha_j e_j)^{10}}{2{,}560(R_{ij}^2-\alpha_i^2 e_i^2)^{10}}\left(2{,}640R_{ij}^{10}+35{,}079R_{ij}^8\alpha_i^2 e_i^2+111{,}672R_{ij}^6\alpha_i^4 e_i^4+\right.$$

$$\left.\left.+110{,}704R_{ij}^4\alpha_i^6 e_i^6+32{,}960R_{ij}^2\alpha_i^8 e_i^8+1{,}920\alpha_i^{10}e_i^{10}\right)\right], \quad (i,j=n+1,\cdots\cdots n+k,\ i\neq j).$$

Having found the expressions for the derivatives $\underset{\sim}{\Gamma}_i^{(\nu)}$ $(i=1,\cdots\cdots n+k)$, $\nu = 3,4,5$, we now return to the power series solution (3.15) which can be approximated by the polynomial

$$\underset{\sim}{\Gamma}_i^{*}=\sum_{\nu=0}^{5}\underset{\sim}{\Gamma}_{io}^{(\nu)}\frac{\tau^\nu}{\nu!}, \quad (i=1,\cdots\cdots n+k), \quad \tau\in(-\tau_1,\tau_1). \tag{3.28}$$

Differentiation of this relation with respect to τ gives the following approximate formula for the velocity vectors $\underset{\sim}{\Gamma}_i'$ $(i=1,\cdots\cdots n+k)$

$$\underset{\sim}{\Gamma}_i^{\prime *}=\sum_{\nu=1}^{5}\underset{\sim}{\Gamma}_{io}^{(\nu)}\frac{\tau^{\nu-1}}{(\nu-1)!}, \quad (i=1,\cdots\cdots n+k), \quad \tau\in(-\tau_1,\tau_1). \tag{3.29}$$

From a set of initial values $\underset{\sim}{\Gamma}_{io}$ and $\underset{\sim}{\Gamma}_{io}'$ $(i=1,\cdots\cdots n+k)$, and with the aid of equations (3.6), (3.7), (3.17), (3.18) and (3.19), the coefficients $\underset{\sim}{\Gamma}_{io}''$

and $\underset{\sim}{r}_{io}^{(3)}$ $(i=1, \cdots\cdots n+K)$ of the foregoing polynomials can be computed applying equations (3.8) and (3.16). Once the values of these coefficients are known we can obtain $\underset{\sim}{r}_{io}^{(4)}$ and $\underset{\sim}{r}_{io}^{(5)}$ $(i=1, \cdots\cdots n+K)$ using equations (3.20) and (3.24) in combination with equations (3.6), (3.7), (3.17), (3.18), (3.19), (3.21), (3.22), (3.23), (3.25), (3.26) and (3.27). Then, by inserting in the polynomials (3.28) and (3.29) a value for the variable τ , the position and velocity vectors $\underset{\sim}{r}_i$ and $\underset{\sim}{r}_i'$ $(i=1, \cdots\cdots n+K)$ can be calculated at a subsequent moment, say $\Delta\tau$; of course, $\Delta\tau$ must obey the relation

$$|\Delta\tau| < \tau_1 .$$

Taking these new values as initial conditions we repeat the process, thereby detailing the paths of the bodies in space.

The accuracy of the above described method of numerical integration, depending on the length of the used time step, can be predetermined by using the method developed in appendix (A1).

CHAPTER 4

Numerical simulations of models of interacting galaxies

It was pointed out in the introduction that one of the main aims of the present work is to extend the studies previously done on the field of the potentials of homogeneous and heterogeneous ellipsoids and ellipsoidal shells. This was accomplished by deriving the analytical expressions for the potential energy and the mutual gravitational attraction between two oblate spheroids with coplanar equatorial planes, whose densities are either constant or obey the relations (2.129) and (2.134). Once these expressions are known the derivation of similar expressions for a wide range of density distribution laws is a matter of straightforward, though very long and extremely laborious, calculations.

Knowledge of the above mentioned expressions enable us to treat a variety of dynamical problems, for example, to deduce and study the equations of motion of dynamical systems consisting of particles (gravitationally active or not) and homogeneous or heterogeneous oblate spheroids with coplanar equatorial planes. These equations and their solution, for the particular case of homogeneous oblate spheroids, are given in chapter (3). It seems therefore appropriate to conclude the present work by studying dynamical models consisting of particles and homogeneous oblate spheroids and comparing their evolution with the evolution of models consisting only of particles.

We shall give in this chapter details of the numerical computations of different dynamical models - of the above mentioned types - imitating two interactive galaxies. It must, of course, be clear that the present chapter is not a detailed study of the "vast" n-body problem but an accurate numerical integration of the equations of motion of the models under consideration and comparison of their evolution.

The method of integration used is the method of Taylor's series truncated after their sixth term. The required formulae for applying this method are given in chapters (1) and (3).

It is well established that the numerical solutions of dynamical systems are in most cases highly unstable (R.H. Miller 1964, 1971). This instability of the individual orbits of the bodies, while basically a physical phenomenon, depends greatly on the accuracy of the numerical integration; it is therefore obvious that comparison of the evolution of different dynamical models is not feasible unless the errors of the integration are kept to the lowest possible level. In the present work this is achieved in two ways, firstly by maintaining as many terms as practically possible of the series involved in the different stages of the calculations and secondly by using a variable time step, whose length is predetermined by the desired accuracy and is evaluated with the aid of the relations (A1-6a) and (A1-11) where $V=5$. Hence, according to appendix (A1), in every step of the numerical integration a prescribed absolute (or relative) error is obtained for the co-ordinates of every individual body of the dynamical system. Of course, high accuracy of the numerical integration cannot be achieved except at the expense of the required time for the numerical calculations to be carried out, which subsequently means that the number of stars we can treat is limited.

Because of the complexity of the formulae used and the accumulation of the errors of integration it is necessary to check frequently that the overall accuracy is within the desired limits. A suitable way of estimating the overall effects of integration errors lies in watching over the constancy of the ten integrals of motion; any variation of these integrals is due to integration errors only.

The derivation of the formulae given in the previous three chapters is based on the assumption that the system of physical units used is such that the arithmetical value of the gravitational constant K is equal to one. We must therefore scale all units for our models in such a way that ensures the validity of the above assumption. In a system where the astronomical unit, the solar mass and the mean solar day are taken as units the numerical value of K is

$$K = 0.01720209895. \tag{4.1}$$

On the other hand, by definition

$$1 \text{ Kiloparsec (Kpc)} = 2.06264806 \times 10^{8} \text{ Astronomical units (A.u.)}. \tag{4.2}$$

In the present work we adopt as unit of distance the Kiloparsec and as unit of mass the solar mass; it therefore remains to be determined the unit of time (u.t.) so that the gravitational constant K has arithmetical value equal to one. This unit can be found with the aid of the relations (1.2),(1.5), (4.1) and (4.2), comparison of which yields

$$1 \text{ unit of time (u.t.)} = 4.714930759449023 \times 10^{11} \text{ tropical years}. \tag{4.3}$$

The structure and the evolution of dynamical models are completely determined by the masses; their distribution in space and the initial values of the positions and the velocities of the bodies constituting the models. For this reason, the choice of the initial values has far-reaching significance. These values, for every particular model, can be selected on the basis of either empirical or theoretical considerations, or a combination of both.

The number of stars of a typical galaxy is of the order 10^{11}, it is therefore clear that, even if all the necessary data were available, numerical integration of the equations of motion of such systems is impossible. On the other hand recent observations of the rotational curves of galaxies (R.H. Sanders 1977, J.P. Ostriker 1979) suggest that their central spheroidal regions are dynamically distinct from their discs. In view of these facts we can construct mathematically simple models representing two interactive galaxies sufficiently well.

In the present work, where the effects of the shape of interacting galaxies on their evolution is studied it is essential to start the numerical integration of every dynamical model with exactly the same initial conditions. Figure (7) to (13), drawn below, illustrate the initial positions and velocities of the centres of mass of the bodies constituting our models. The perturbing galaxy (point 38) is a massive homogeneous spherical or spheroidal body of mass 8×10^{10} solar masses. The perturbed galaxy consists of a central heavy homogeneous spherical or spheroidal body (point 37) of mass 4×10^{10} solar masses surrounded by 36 particles originally located, in a symmetrical manner, on the surface of an oblate spheroid [see figure (7)] whose centre coincides with the centre of mass of the heavy body. Furthermore, this spheroid (hereafter spheroid S) has such orientation and geometrical characteristics that its surface is similar and similarly situated with the surface of the central heavy body, when this body is an oblate spheroid.

The particles, according to their initial positions in space, can be divided in five groups, where the members of each group lie on a circle. These circles are parallel and equidistant with one of them coinciding with the equatorial plane of the spheroid S . If a and c denote the semi-major and semi-minor axes of this spheroid, the distance between any two adjacent circles is c/3. The radii r_a , r_b , r_c , r_d and r_e of these circles [see figures (8) to (12)] can easily be found with the aid of the equation describing the surface of the spheroid S and they are equal to

$$r_a = r_e = \alpha \frac{\sqrt{5}}{3}, \quad r_b = r_d = \alpha \frac{\sqrt{8}}{3}, \quad r_c = \alpha \qquad (4.4)$$

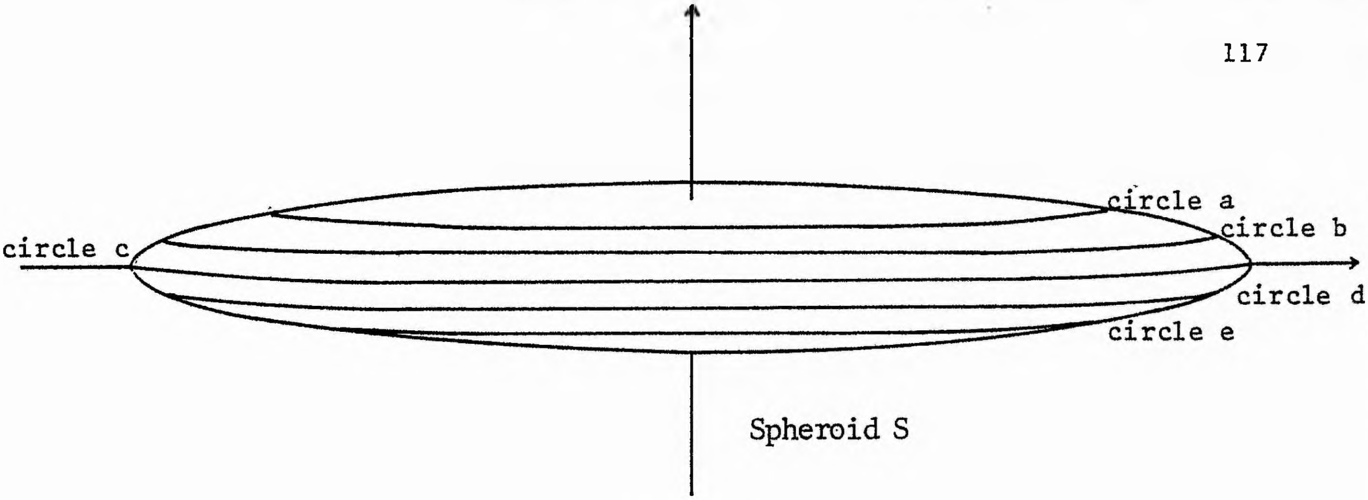


Figure 7

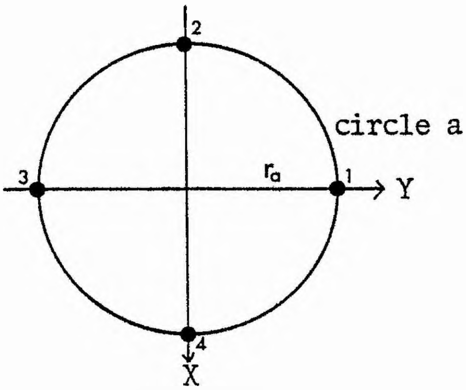


Figure 8

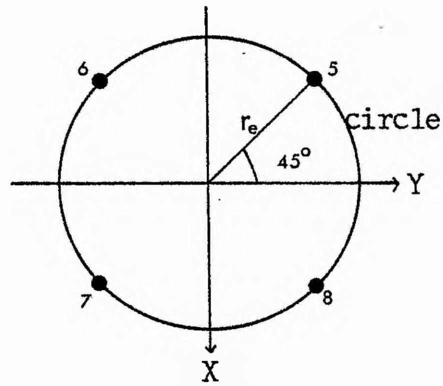


Figure 9

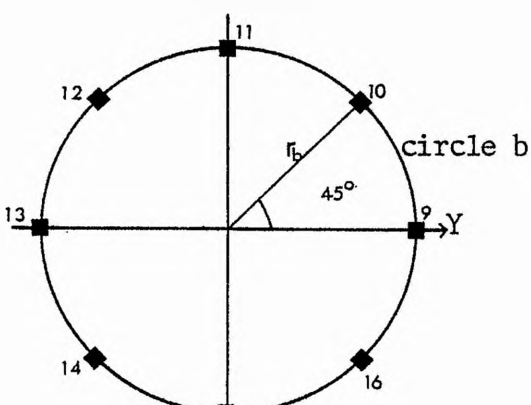


Figure 10

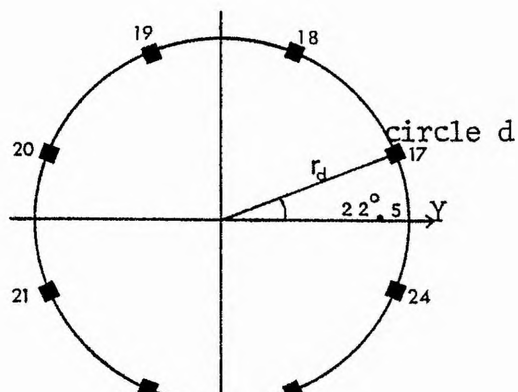


Figure 11

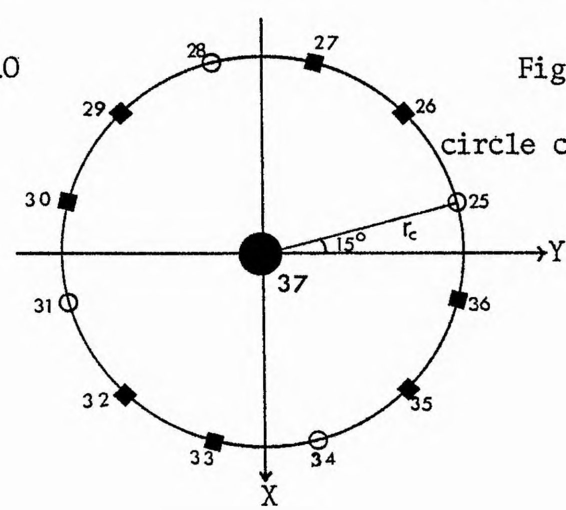


Figure 12

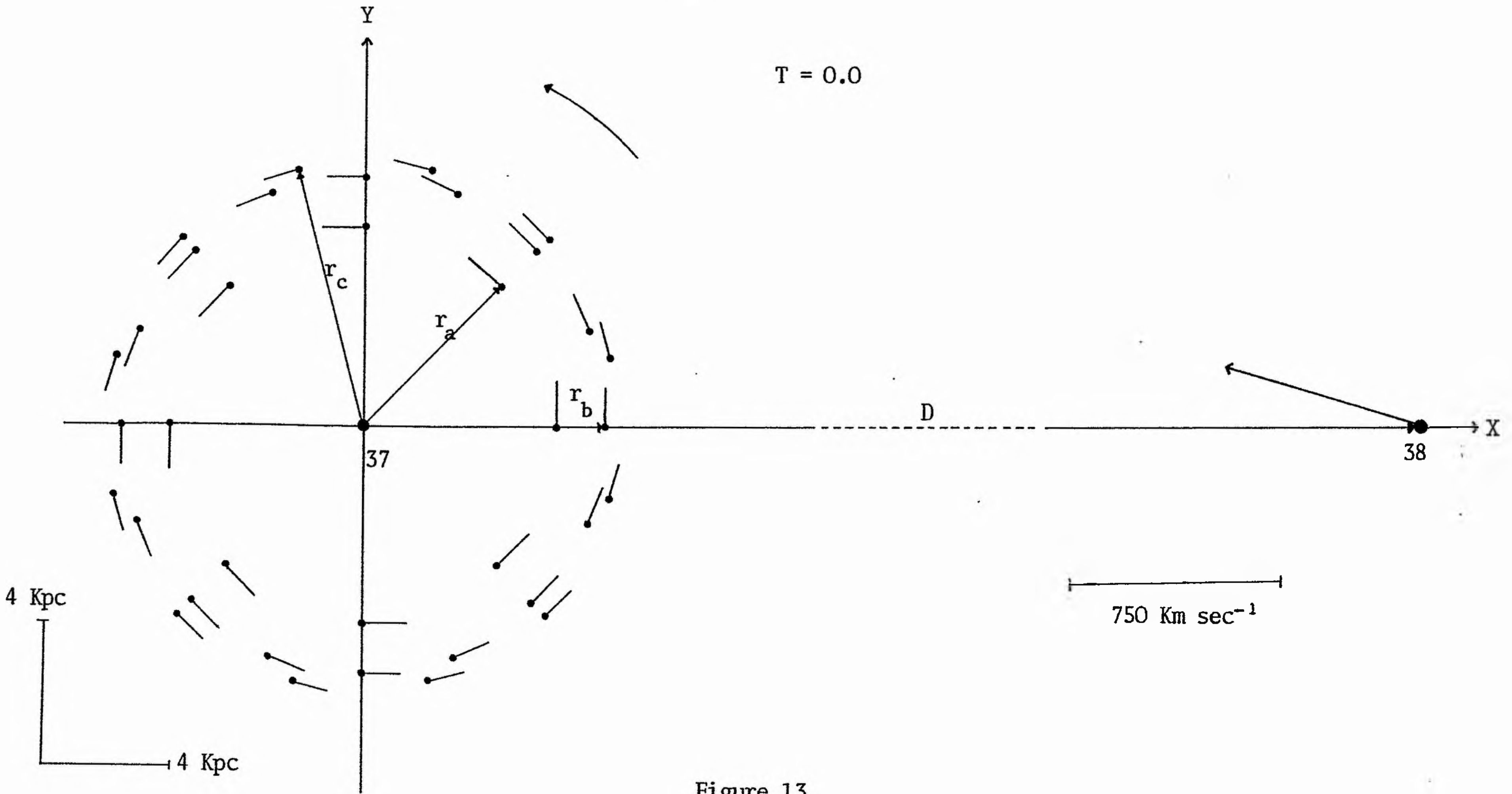


Figure 13

We chose as system of reference a Cartesian right-handed inertial system whose origin at $\tau=0$ is located at the centre of the spheroid S and its Z axis coincides with the direction of the minor axis of this spheroid. The direction of the X axis is chosen to coincide with the line passing through the centres of mass of the two heavy bodies. Figures (8) to (12) are the graphical representation of the projection of the above mentioned circles on the XY plane, while figure (13) illustrates the initial positions (and velocities) of the bodies of our dynamical models projected on this plane. It is obvious that, in order to evaluate these positions, we require the initial values of three parameters, namely the semi-axes a and c of the spheroid S and the distance D between the centres of mass of the two heavy bodies.

Since our main purpose in this chapter is to compare the evolution of different dynamical models, it seems appropriate to avoid "evaporation" of the particles of the models by choosing a rather small value for the parameter a. We take, more or less arbitrarily,

$$a = 8\text{Kpc}. \tag{4.5}$$

This value enables us also to establish that (for rather big impact parameters and relative velocities of interactive galaxies) spiral structure can be developed as a result of tidal effects in distances from the centre of a galaxy as small as 5.3Kpc.

I.D. Karachentsev (1975a) found that the most probable value of the sphericity of the galaxies he studied is 0.15, (see also E.M. Burbidge 1959, 1960, 1961a, 1961b). Adopting this value for the spheroid S, we obtain

$$c = 1.2\text{Kpc}. \tag{4.6}$$

Once the values of the parameters a and c are known we can easily find, using elementary trigonometry, the co-ordinates of the particles of our models with the aid of equations (4.4) and the data given in figures (8),(9),(10),(11) and (12). These co-ordinates are given by the second, third and fourth column of the table (1). The first column of this table indicates the index assigned to each of the bodies constituting our models.

Consider now a particle located at the point where the X axis of the system of reference intersects the surface of the spheroid S. We determine the distance D between the centres of the heavy bodies by seeking that the following requirement is fulfilled: the force acting on the above mentioned particle due to the presence of the mass m_{38} (perturbing galaxy) to be A times smaller than the force exerted on the particle by the mass m_{37} (the massive core of the perturbed galaxy). Mathematically this relation, when the distribution in space of each of the masses m_{37} and m_{38} is spherical, can be expressed as follows:

$$\frac{m_{37}}{a^2} = \frac{m_{38}}{(D-a)^2} A$$

From the data given above it is clear that $m_{37} = 4 \times 10^{10}$ solar masses, $m_{38} = 8 \times 10^{10}$ solar masses and $a = 8$Kpc. Choosing $A = 99.7578$ and solving the foregoing relation with respect to D, we obtain

$$D = 121 \text{ Kpc.} \qquad (4.7)$$

If a plane has any points in common with a quadric, these points form in general a conic. If the quadric is an oblate spheroid and the intersecting plane passes through its centre then the locus of their common points is either a circle or an ellipse whose semi-major axis is equal to the semi-major axis of the spheroid; further, when the intersecting plane passes through the centre of the spheroid and a given point on its surface, then the above mentioned locus has the maximum possible area when its semi-minor axis passes through the given point.

Table 1

INDEX	INITIAL POSITIONS X (Kpc)	Y (Kpc)	Z (Kpc)
1	0.0	5.96284794000	0.8
2	-5.96284794000	0.0	0.8
3	0.0	-5.96284794000	0.8
4	5.96284794000	0.0	0.8
5	-4.21637021356	4.21637021356	-0.8
6	-4.21637021356	-4.21637021356	-0.8
7	4.21637021356	-4.21637021356	-0.8
8	4.21637021356	4.21637021356	-0.8
9	0.0	7.54247233266	0.4
10	-5.33333333333	5.33333333333	0.4
11	-7.54247233266	0.0	0.4
12	-5.33333333333	-5.33333333333	0.4
13	0.0	-7.54247233266	0.4
14	5.33333333333	-5.33333333333	0.4
15	7.54247233266	0.0	0.4
16	5.33333333333	5.33333333333	0.4
17	-2.88637920078	6.96833581267	-0.4
18	-6.96833581267	2.88637920078	-0.4
19	-6.96833581267	-2.88637920078	-0.4
20	-2.88637920078	-6.96833581267	-0.4
21	2.88637920078	-6.96833581267	-0.4
22	6.96833581267	-2.88637920078	-0.4
23	6.96833581267	2.88637920078	-0.4
24	2.88637920078	6.96833581267	-0.4
25	-2.07055236082	7.72740661031	0.0
26	-5.65685424949	5.65685424949	0.0
27	-7.72740661031	2.07055236082	0.0
28	-7.72740661031	-2.07055236082	0.0
29	-5.65685424949	-5.65685424949	0.0
30	-2.07055236082	-7.72740661031	0.0
31	2.07055236082	-7.72740661031	0.0
32	5.65685424949	-5.65685424949	0.0
33	7.72740661031	-2.07055236082	0.0
34	7.72740661031	2.07055236082	0.0
35	5.65685424949	5.65685424949	0.0
36	2.07055236082	7.72740661031	0.0
37	0.0	0.0	0.0
38	121.0	0.0	0.0

We proceed with the determination of the initial velocities of the bodies of our dynamical models. Since the evolution of an isolated, non relativistic, dynamical system depends only on the relative initial positions and velocities of the bodies constituting the system, we can choose the initial velocity of any one of the bodies arbitrarily. In the case of the dynamical models under consideration we choose the mass m_{37} to be initially at rest and we determine the initial velocities of the particles surrounding this mass by requiring their unperturbed orbits to be inside the spheroid S and as close as possible to its surface. This criterion is chosen so that the system consisting of the mass m_{37} and the particles surrounding it, if unperturbed, will maintain its close resemblance to the geometrical characteristics of a real galaxy. Let us now consider one of the above mentioned particles initially located at a point $P \equiv (x_p, y_p, z_p)$ on the surface of the spheroid S and interacting gravitationally only with the mass m_{37}; clearly its orbit, obeying the above criterion, will be either a circle or an ellipse with one of its foci located at the centre of the mass m_{37} (whose distribution in space is assumed spherical). Let us also consider the plane which, passing through the point P and the centre O of the spheroid S, intersects this spheroid in such a way that the locus of their common points has the maximum possible area. According to what is stated in the foregoing paragraph this locus is either a circle or an ellipse (hereafter ellipse E_p) with semi-major axis equal to a and semi-minor axis passing through P. The first is true only when the point P is located on the equatorial plane of the spheroid S. Now, it is easy to prove that the above selected criterion for determining the initial velocity of the particle under consideration is satisfied only when the particle is the circle inscribed to the ellipse E_p. Figure (15) illustrates this ellipse (PFGP) and the inscribed, circular, particle-orbit (PHIP), while figure (16) shows the existing relation between the spheroid S, the ellipse E_p and the point P.

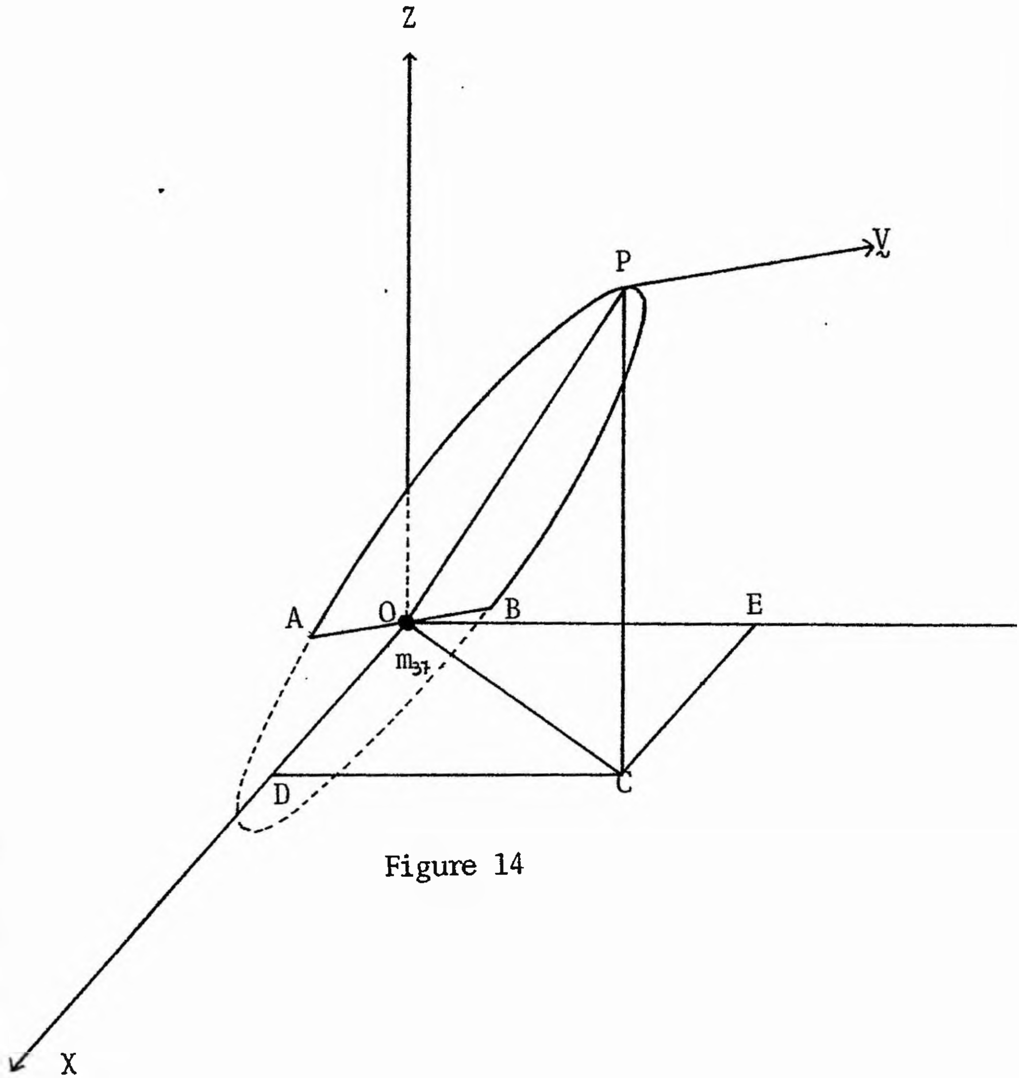


Figure 14

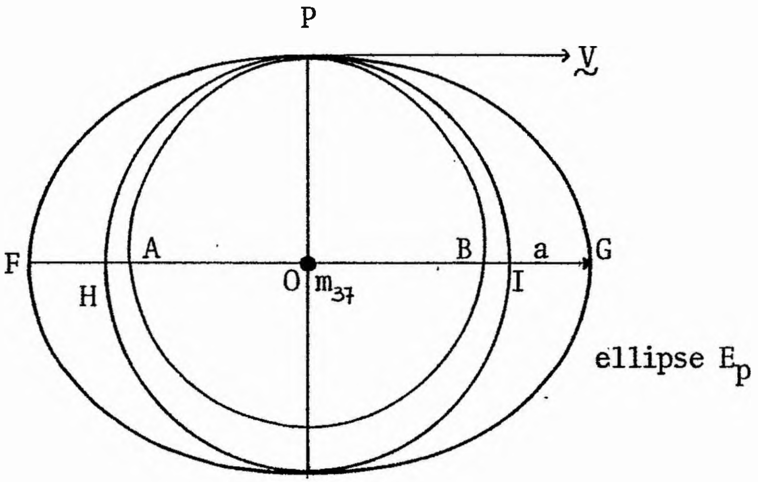


Figure 15

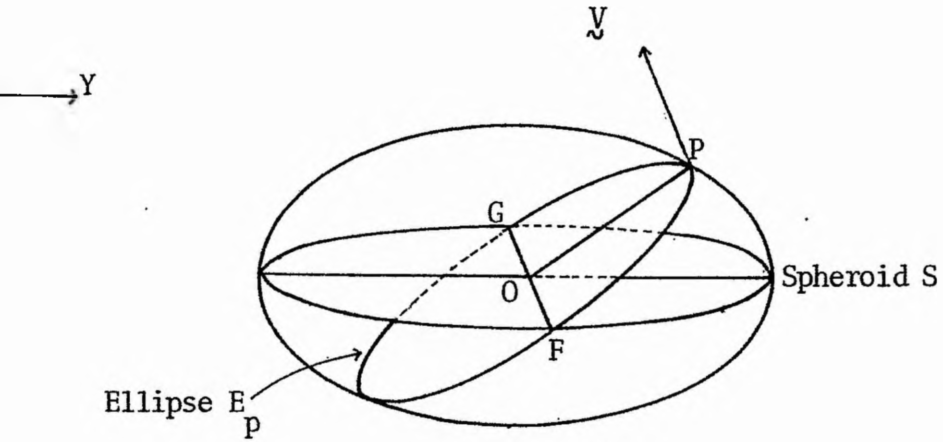


Figure 16

In nature one expects the circular orbits to be the exception rather than the rule; it is also known that the stability of closed orbits against radial perturbations is an increasing function of their eccentricities. We, therefore, depart slightly from the initially chosen criterion, leading to the circular orbit PHIP, and consider an elliptical orbit (PABP) of small, arbitrarily chosen, eccentricity e_p and semi-major axis a_p determined by the relation

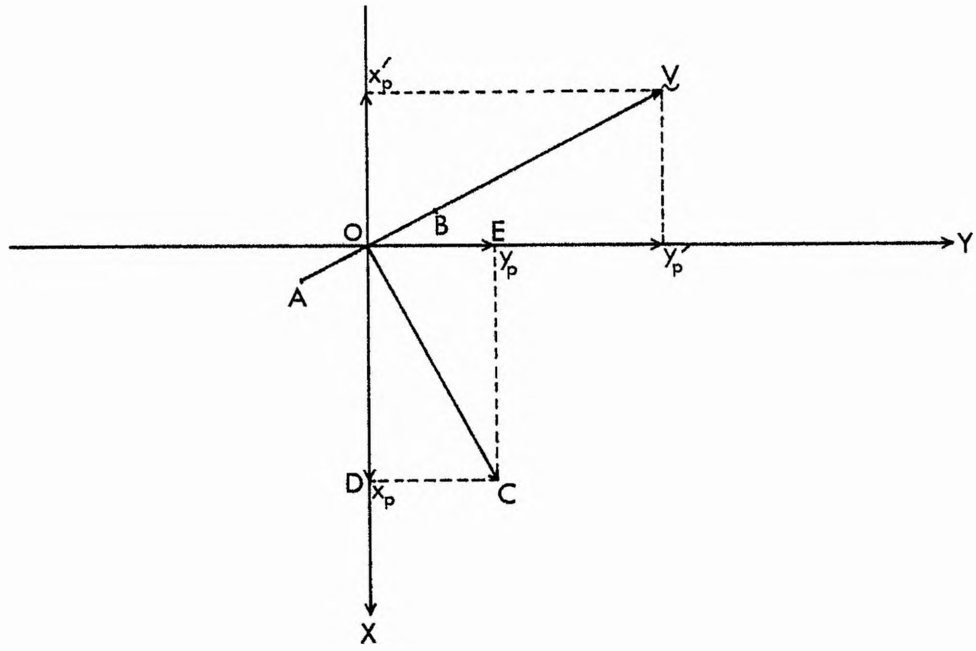
$$a_p(1+e_p)=OP\equiv\left(x_p^2+y_p^2+z_p^2\right)^{\frac{1}{2}}, \tag{4.8}$$

which ensures that the point P is at the apocentre of the elliptic orbit. Figure (14) shows this orbit in perspective and its relation to the system of reference, while figure (15) illustrates the position of the orbit on the plane of the ellipse E_p. For the sake of clarity, in figures (14),(15) and (16), the eccentricities of the elliptical shapes have been purposely altered.

The initial velocity vector $\underset{\sim}{V}\equiv(x_p', y_p', z_p')$ of the particle at its apocentre P is perpendicular to OP since, according to equation (4.8), OP coincides with the direction of the semi-major axis a_p. On the other hand OP and OG, being the semi-axes of the ellipse E_p, are perpendicular. Hence $\underset{\sim}{V}$ is parallel to OG and consequently parallel to the equatorial plane of the spheroid S which coincides with the XY co-ordinate plane. From the above discussion it is clear that

$$z_p' = 0 . \tag{4.9}$$

Projection and parallel shifting of the vector $\underset{\sim}{V}$ on the XY plane results in the following figure



from which we readily conclude

$$x_p' = -|\underset{\sim}{V}| \frac{y_p}{\sqrt{x_p^2 + y_p^2}} , \quad y_p' = |\underset{\sim}{V}| \frac{x_p}{\sqrt{x_p^2 + y_p^2}} . \tag{4.10}$$

We obtain the magnitude of the velocity vector $\underset{\sim}{V}$ with the aid of the energy integral for the unperturbed motion of the particle under consideration; this integral, at $\tau = 0$, has the form

$$\frac{|\underset{\sim}{V}|^2}{2} = (m_{37} + m_p) \left[\frac{1}{(x_p^2 + y_p^2 + z_p^2)^{1/2}} - \frac{1}{2\alpha_p} \right] , \tag{4.11}$$

where m_p is the mass of the particle. The foregoing equation, upon substitution of equation (4.8), becomes

$$|\underset{\sim}{V}|^2 = (m_{37} + m_p) \frac{(1 - e_p)}{(x_p^2 + y_p^2 + z_p^2)^{1/2}} . \tag{4.12}$$

Comparing equation (4.10) and (4.12), and remembering equation (4.9), we readily conclude

$$x_p' = -y_p \sqrt{\frac{(m_{37} + m_p)(1 - e_p)}{(x_p^2 + y_p^2)\sqrt{x_p^2 + y_p^2 + z_p^2}}} , \quad y_p' = x_p \sqrt{\frac{(m_{37} + m_p)(1 - e_p)}{(x_p^2 + y_p^2)\sqrt{x_p^2 + y_p^2 + z_p^2}}} , \tag{4.13}$$

$$z_p' = 0 .$$

In order to employ the foregoing formulae to compute the initial velocities of the 36 particles surrounding the mass m_{37} we need first to determine the masses of these particles. E.E. Salpeter (1955) has shown that the observed luminosity function of stars in the solar neighbourhood and in many galactic clusters indicate a mass spectrum of the form:

$$\frac{dN}{d\log m} \propto m^{-1.35} \qquad (4.14)$$

We, therefore, choose the masses of the particles according to this law, aiming at imitating the mass spectrum of realistic stars. For practical reasons, we do not use a continuous mass spectrum but form three mass groups of particles, where the number of the members in each group is governed by the law (4.14). Table (2) shows these groups and the symbols used in figures (8) to (12) to indicate particles of different mass.

Table 2

Mass m (in solar masses)	Number N_m	Part of the total mass (%)	Assigned symbol
0.5	24	41.66666	■
1.1	8	30.55555	●
2.0	4	27.77777	○

We note here that the initial distribution in space of the particles with different masses is chosen in such a way that allows the equatorial planes of the masses m_{37} and m_{38} (when their distribution in space is spheroidal) to remain coplanar throughout the evolution of the dynamical models. Column (3) of table (3) gives the masses of the particles, while columns (2) and (4) give the eccentricities and periods of the unperturbed particle orbits; the periods were found using the well-known relation, from the two-body problem,

TABLE 3

INDEX	ECCENTRICITY e	MASS m (Solar masses)	SEMI-PERIOD P (Tropical years)x10^8
1	0.125	1.1	0.9159206428
2	0.125	1.1	0.9159206428
3	0.125	1.1	0.9159206428
4	0.125	1.1	0.9159206428
5	0.125	1.1	0.9159206428
6	0.125	1.1	0.9159206428
7	0.125	1.1	0.9159206428
8	0.125	1.1	0.9159206428
9	0.100	0.5	1.3325746932 0
10	0.100	0.5	1.3325746932 0
11	0.100	0.5	1.3325746932 0
12	0.100	0.5	1.3325746932 0
13	0.100	0.5	1.3325746932 0
14	0.100	0.5	1.3325746932 0
15	0.100	0.5	1.3325746932 0
16	0.100	0.5	1.3325746932 0
17	0.100	0.5	1.3325746932 0
18	0.100	0.5	1.3325746932 0
19	0.100	0.5	1.3325746932 0
20	0.100	0.5	1.3325746932 0
21	0.100	0.5	1.3325746932 0
22	0.100	0.5	1.3325746932 0
23	0.100	0.5	1.3325746932 0
24	0.100	0.5	1.3325746932 0
25	0.075	2.0	1.5035486341 9
26	0.075	0.5	1.5035486342 2
27	0.075	0.5	1.5035486342 2
28	0.075	2.0	1.5035486341 9
29	0.075	0.5	1.5035486342 2
30	0.075	0.5	1.5035486342 2
31	0.075	2.0	1.5035486341 9
32	0.075	0.5	1.5035486342 2
33	0.075	0.5	1.5035486342 2
34	0.075	2.0	1.5035486341 9
35	0.075	0.5	1.5035486342 2
36	0.075	0.5	1.5035486342 2
37		4.0×10^{10}	
38	42.59533431982	8.0×10^{10}	

$$P_i = 2\pi\sqrt{\alpha_i^3/(m_{37}+m_i)}\ , \ (i=1,\dots n)\ ,$$

where m_i and α_i is the mass of the i^{th} particle and the semi-major axis of its orbit given, according to equation (4.8), by the relation

$$\alpha_i = (1+e_i)^{-1}\left(x_i^2+y_i^2+z_i^2\right)^{1/2}\ ,$$

x_i, y_i and z_i are the initial co-ordinates of the particle. Finally the first column of the table shows the index assigned to each particle.

Having chosen the masses of the particles and the eccentricities of their unperturbed orbits we can easily compute their initial velocity components using equations (4.13). These components (in Km/sec) are given in table (4).

We complete the set of the necessary initial conditions for our dynamical models by determining the initial velocity of the centre of mass m_{38}. Suppose we know the distance and the relative velocity r_{min} and $\underset{\sim}{r}'_{min}$ of the centres of the masses m_{37} and m_{38} at the instant of their minimum approach. If the distribution of these masses in space is spherical then the energy and angular momentum integrals of their unperturbed orbit ensure the validity of the following relations

$$\underset{\sim}{r}'^2_{min} - 2\left(m_{37}+m_{38}\right)\frac{1}{r_{min}} = \underset{\sim}{r}'^2 - 2\left(m_{37}+m_{38}\right)\frac{1}{r}\ , \tag{4.15}$$

$$\left|\underset{\sim}{r} \times \underset{\sim}{r}'\right|^2 = \left|\underset{\sim}{r}_{min} \times \underset{\sim}{r}'_{min}\right|^2\ , \tag{4.16}$$

where r and $\underset{\sim}{r}'$ is the distance and relative velocity of the masses at any time. Selecting the orbit to lie on the XY co-ordinate plane and remembering that the position and velocity vectors $\underset{\sim}{r}_{min}$ and $\underset{\sim}{r}'_{min}$ are perpendicular,

Table 4

INDEX	INITIAL VELOCITIES		
	X′ (Km/sec)	Y′ (Km/sec)	Z′ (Km/sec)
1	-158.17973376669	0.0	0.0
2	0.0	-158.17973376669	0.0
3	158.17973376669	0.0	0.0
4	0.0	158.17973376669	0.0
5	-111.84996239271	-111.84996239271	0.0
6	111.84996239271	-111.84996239271	0.0
7	111.84996239271	111.84996239271	0.0
8	-111.84996239271	111.84996239271	0.0
9	-143.17595302321	0.0	0.0
10	-101.24068728556	-101.24068728556	0.0
11	0.0	-143.17595302321	0.0
12	101.24068728556	-101.24068728556	0.0
13	143.17595302321	0.0	0.0
14	101.24068728556	101.24068728556	0.0
15	0.0	143.17595302321	0.0
16	-101.24068728556	101.24068728556	0.0
17	-132.27733254594	-54.79106513507	0.0
18	-54.79106513507	-132.27733254594	0.0
19	54.79106513507	-132.27733254594	0.0
20	132.27733254594	-54.79106513507	0.0
21	132.27733254594	54.79106513507	0.0
22	54.79106513507	132.27733254594	0.0
23	-54.79106513507	132.27733254594	0.0
24	-132.27733254594	54.79106513507	0.0
25	-136.23235884217	-36.50335053475	0.0
26	-99.72900830742	-99.72900830742	0.0
27	-36.50335053475	-136.23235884217	0.0
28	36.50335053475	-136.23235884217	0.0
29	99.72900830742	-99.72900830742	0.0
30	136.23235884217	-36.50335053475	0.0
31	136.23235884217	36.50335053475	0.0
32	99.72900830742	99.72900830742	0.0
33	36.50335053475	136.23235884217	0.0
34	-36.50335053475	136.23235884217	0.0
35	-99.72900830742	99.72900830742	0.0
36	-136.23235884217	36.50335053475	0.0
37	0.0	0.0	0.0
38	-695.52427966821	247.93388429673	0.0

equation (4.16) takes the form

$$\left(xy' - x'y\right)^2 = r_{min}^2 r_{min}'^2 \quad . \tag{4.17}$$

At $\tau=0$, when the centre of the mass m_{37} is at rest at the origin of the system of reference and the centre of the mass m_{38} is located on the X axis at a distance D from the origin and moving with a velocity $\underset{\sim}{V}_{38} \equiv (x_{38}' , y_{38}')$, equations (4.15) and (4.17) become

$$\underset{\sim}{V}_{38}^2 \equiv \left(x_{38}'^2 + y_{38}'^2\right) = \left|\underset{\sim}{r}_{min}'\right|^2 - 2\left(m_{37}+m_{38}\right)\left(r_{min}^{-1} - D^{-1}\right) ,$$

$$D^2 y_{38}'^2 = r_{min}^2 \left|\underset{\sim}{r}_{min}'\right|^2 . \tag{4.18}$$

Solving the foregoing elementary system with respect to x_{38}' and y_{38}' and retaining the solution which ensures that the mass m_{38} approaches the system of the mass m_{37} and the particles surrounding it in the same sense as the system rotates (direct passage), we obtain the relations

$$x_{38}' = -\sqrt{\left|\underset{\sim}{r}_{min}'\right|^2 \left(1 - \frac{r_{min}^2}{D^2}\right) - 2\left(m_{37} + m_{38}\right)\left(r_{min}^{-1} - D^{-1}\right)} ,$$

$$y_{38}' = r_{min} \left|\underset{\sim}{r}_{min}'\right| \frac{1}{D} . \tag{4.19}$$

In order to employ these relations to evaluate the velocity components x_{38}' and y_{38}' we require the values of the distance and the relative velocity r_{min} and $\underset{\sim}{r}_{min}'$ of the centres of the masses m_{37} and m_{38} at the instant of their minimum approach. Such values for real systems of interacting galaxies cannot be obtained observationally. However, in the present work an estimate was made based on the data given by P.D. Noerdlinger (1979), I.D. Karachentsev (1975b, 1978, 1979) and E.L. Turner (1976a, 1976b). The chosen values

$$r_{min} = 40 \ Kpc , \left|\underset{\sim}{r}_{min}'\right| = 750 \ Km/sec , \tag{4.20}$$

though significantly greater than those used by other authors, are more realistic. We note here the importance of the careful selection of these values which determine the evolution of the dynamical models under consideration. It will become clear later on that an inappropriate choice may lead to a waste of an enormous amount of computing time.

The condition of the direct passage was imposed because the dynamical results of a retrograde passage are remarkably mild (see also A. Toomre 1972). The reason for this phenomenon is the much longer time the particles of the dynamical models experience the force of the perturbing massive bodies in the direct passage.

Introduction of equations (4.7) and (4.20) into equations (4.19) yields

$$x'_{38} = -695.524\,279\,668\,21 \text{ km·sec}^{-1}, \quad y'_{38} = 247.933\,884\,296\,73 \text{ km·sec}^{-1} \tag{4.21}$$

In a system of reference whose origin is fixed in the centre of the mass m_{37}, the unperturbed orbit of the mass m_{38} (the distribution of both masses is considered spherical) is a conic section whose eccentricity e_{38} is given by the well-known relation

$$e_{38} = \left[1 + 2E\ell^2 / (m_{37} + m_{38})\right]^{1/2}, \tag{4.22}$$

where the constants E and ℓ, expressing the conservation of the total energy and the angular momentum of the system of the two masses, have the form

$$E = \frac{1}{2} m_{37} m_{38} \left[(m_{37} + m_{38})^{-1} \underset{\sim}{r}'^2 - 2r^{-1}\right], \quad \ell = \left|\underset{\sim}{r} \times \underset{\sim}{r}'\right| . \tag{4.23}$$

Combining equations (4.15), (4.16), (4.22) and (4.23) we obtain the relation

$$e_{38} = \left\{1 + \left[\frac{\underset{\sim}{r}'^2_{min}}{(m_{37} + m_{38})} - \frac{2}{r_{min}}\right] \frac{\underset{\sim}{r}'^2_{min}\; r^2_{min}}{(m_{37}\; m_{38})}\right\}^{1/2}, \tag{4.24}$$

which, upon substitution of the values for $|\underset{\sim}{r}'_{min}|$, $\underset{\sim}{r}_{min}$, m_{37} and m_{38}, yields

$$e_{38} = 42.595\,334\,319\,82 .$$

Hence the unperturbed orbit of the centre of the mass m_{38} around m_{37} is a hyperbola.

The value of the impact parameter s of the encounter of the masses m_{37} and m_{38} can easily be found with the aid of the data given above. In fact

$$s = 40.950\,35658 \text{ Kpc.}$$

The positions, the velocities and the masses given in tables (1), (3) and (4) are the initial conditions for each one of our dynamical models. These models are:

Model Ia : It consists of 38 gravitationally interacting particles moving freely in space, initially in the above specified manner.

Model Ib : The only difference between the models Ia and Ib is that in the latter we consider the interaction between any two non-heavy particles as negligible.

Model IIa : This model is generated from model Ia by substituting the 37^{th} and 38^{th} particles by two rigid homogeneous oblate spheroids with coplanar equatorial planes and masses m_{37} and m_{38} respectively. The initial positions and velocities of the centres of the spheroids are identical with the initial positions and velocities of the corresponding particles. The semi-major axes a_{37} , a_{38} and the eccentricities e_{37} , e_{38} of the spheroids are chosen to fulfil the following requirements: the eccentricity e_{37} must be equal to the eccentricity of the spheroid S and the semi-axes a_{37} and a_{38} must have such numerical values that no particle crosses the surfaces of the spheroids at any time. The latter requirement merely reflects the fact that the spheroids, taken as rigid bodies, are not allowed to disrupt, hence no particle "merging" with them at a certain stage of the evolution can be allowed to escape later on. We note that the equations of motion of a particle moving in a gravitational field identical to the field existing in the interior of a homogeneous spheroid can easily be derived with the aid of the relations given in chapter (2).

In accordance with the foregoing two requirements we chose

$$
\begin{aligned}
e_{37} &= [1-(0.15)^2]^{\frac{1}{2}} = 0.988685996664254, \\
a_{37} &= 1.500\,476\,394\,937\,531 \text{ Kpc}, \\
e_{38} &= 0.989, \quad a_{38} = 28 \text{ Kpc}.
\end{aligned} \tag{4.25}
$$

From the relations (2.15), (2.21), (2.47), (2.65), (2.73) and (2.74) it is obvious that the evolution of the dynamical model does not depend explicitly on the numerical values of the above parameters, but rather on the values of the quantities $(a_{37}e_{37})$ and $(a_{38}e_{38})$ which according to equations (4.25) are

$$
(a_{37}e_{37}) = 1.4835 \text{ Kpc}, \quad (a_{38}e_{38}) = 27.692 \text{ Kpc}. \tag{4.26}
$$

This phenomenon implies that one can generate an infinite number of systems dynamically (though not physically) identical with model IIa by simply replacing one or both of the spheroidal bodies of this model by any one of their confocal spheroids with the same mass and density distribution law, and which satisfied the second of the requirements mentioned above.

Model IIb : During the evolution of this model - characterized by initial conditions identical to those of model IIa - the gravitational attraction between any two particles is considered negligible.

Model III : This model differs from model IIb only in the value of the parameter a_{38} which, in this case, is taken to be equal to 6 Kpc; thus

$$
a_{38} = 6 \text{ Kpc}, \quad (a_{38}e_{38}) = 5.934 \text{ Kpc}.
$$

Model IV : This model is the result of the substitution of the spheroidal distribution of the mass m_{38}, in model IIa, by a spherical one.

We now proceed by giving the details concerning the evolution of the foregoing dynamical models.

The significance of the results and the speed of the numerical method is mainly determined by the desired accuracy. In the present work the required high accuracy can be achieved only at the expense of the speed of the numerical integration, this is due to the following fact: The previously mentioned instability of the solutions of equations (1.6) and (3.8) is a principal one and can easily bring a dynamical system to stages unapproachable for the system in reality. To judge the importance of this phenomenon one must study the existing relation between the length of the time step of the integration and the deviation of the ten integrals of motion from their initial values. Given the limited computing time available we were unable to undertake a detailed study of this relation but we were able to establish beyond any doubt that, for any value of the length of the time step exceeding a certain limit, the relative errors of the ten integrals of motion (considered as functions of the independent variable τ) have a significant one-side trend on which only minor fluctuations are superimposed, while for values of the time step smaller than this limit the relative errors are only smoothly fluctuating functions [see figures (17) to (22)] . The above mentioned limit is, obviously, different for different types of models; in practice it was found that for the model Ia it is of the order 6×10^{-7}u.t., while for the models Ib, IIa, IIb, III and IV it is of the order 10^{-6}u.t., 2×10^{-9} u.t., 7×10^{-10} u.t., and 5×10^{-8}u.t. respectively.

The computing time required for the numerical integration of a system of differential equations to be carried out over a certain interval of its independent variable and for a prescribed accuracy depends solely on the structure of the differential equations and the initial conditions. The nature of this dependence is such that even slightly different initial conditions or structure of the equations can lead to remarkably different computing times. It would therefore seem inappropriate to compare the computing times required to follow the evolution of the dynamical models under consideration up to a certain stage. However, figures (23a) to (25h) indicate that the physical states of the models are virtually the same within the interval [0 u.t., 10^{-3} u.t.] of the "model" time τ , and hence a comparison inside this interval is feasible. Table (5) gives the necessary data. The first column of this table gives the type of the model while the next three columns give the model time within which the comparison is attempted, the corresponding computing time and the number of time steps. From these values one can easily obtain the mean time step given in the fifth column. The maximum computing and model times throughout which we followed the evolution of every one of our models are given in the sixth and seventh columns, while the last column gives the type of the computer used to carry out the calculations.

The problem of the tremendous (by any standard) amount of computing time required for the numerical integration of equations (3.8) becomes yet more complicated by the fact that any significant differences between the evolution of models of different types appear only long after the instant of the minimum approach of the two heavy bodies of the models. In fact only two models reached stages which can be successfully compared; that is model Ib and IIb. Note that initially the only difference between these two models is the shape of their heavy bodies.

Next we give six figures which graphically represent, for every model, the relation between the relative error of its total energy and the independent variable τ . The general form of this relation is

$$\frac{|E_\tau - E_{in}|}{E_{in}} \equiv \frac{|\Delta E|}{E_{in}} = f(\tau) \, , \tag{4.27}$$

where E_{in} is the initial total energy of the dynamical model and E_τ is the

Table 5

Type of model	Model time (u.t.)x10^{-3}	Computing time (hours)	Number of time steps	Mean time step (u.t.)x10^{-9}	Maximum model time (u.t.)x10^{-3}	Maximum computing time (hours)	Machine-model
Ia	1.0	10.10	1,980	524.10901	1.111142	11.32	IBM 360/44*
Ib	1.0	0.0467	1,050	952.38095	31.796142	1.45	VAX 11/780
IIa	0.537389	271.97	331,977	1.59002	0.537389	271.97	Honeywell 66/80
IIb	1.0	117.43	1,540,000	0.64935	5.996102	704.08	VAX 11/780
III	0.984858	129.64	1,489,480	0.66121	0.984858	129.64	VAX 11/780
IV	0.611906	50.14	13,034	46.94691	0.611906	50.14	IBM 360/44*

* Note that the speed of a computer IBM 360/44 is ten times smaller than that of a VAX 11/780 or a Honeywell 66/80.

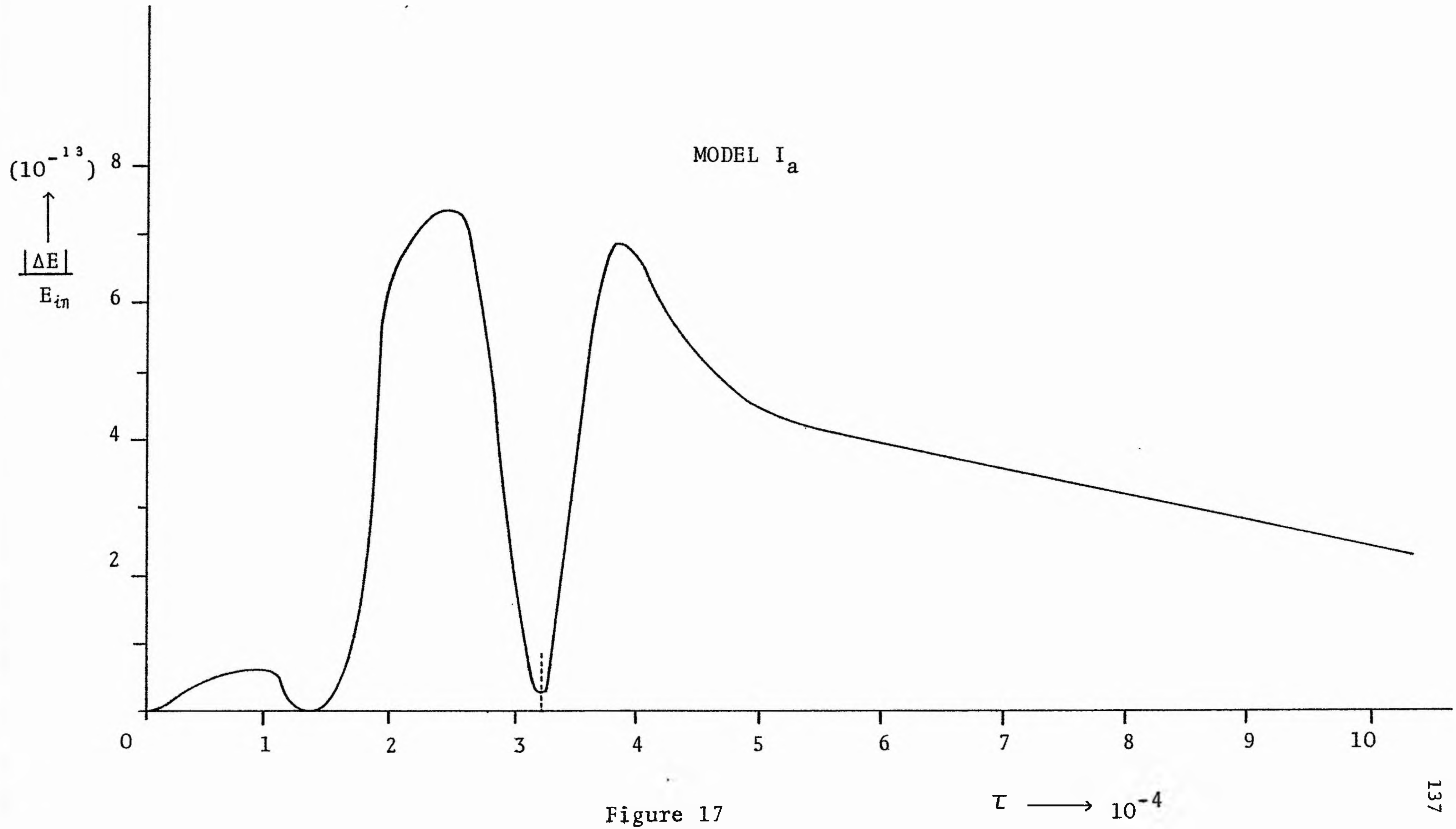

MODEL I_a
(10^{-13})
$\frac{|\Delta E|}{E_{in}}$
8
6
4
2
0
1
2
3
4
5
6
7
8
9
10
$\tau \longrightarrow 10^{-4}$

Figure 17

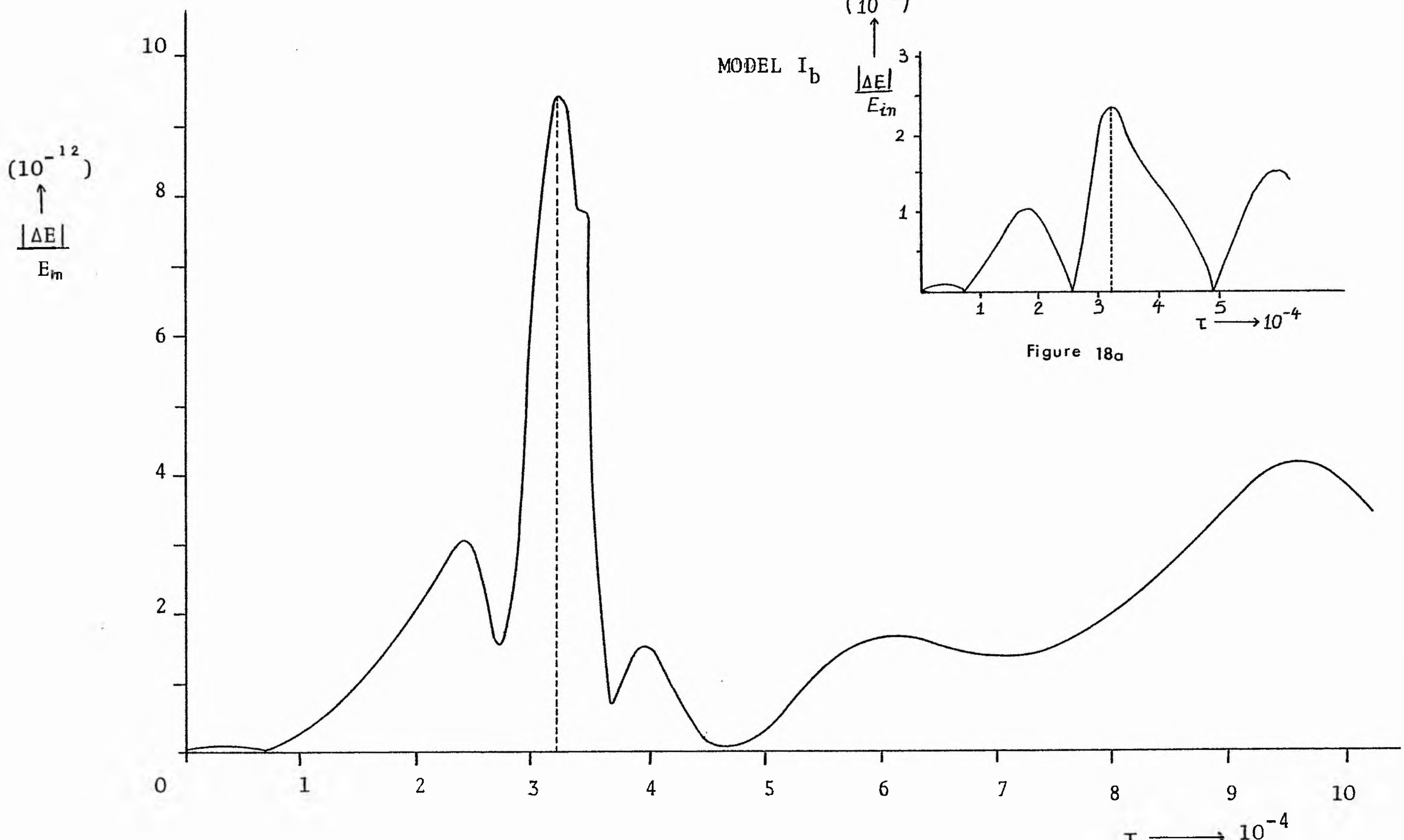


Figure 18a

Figure 18

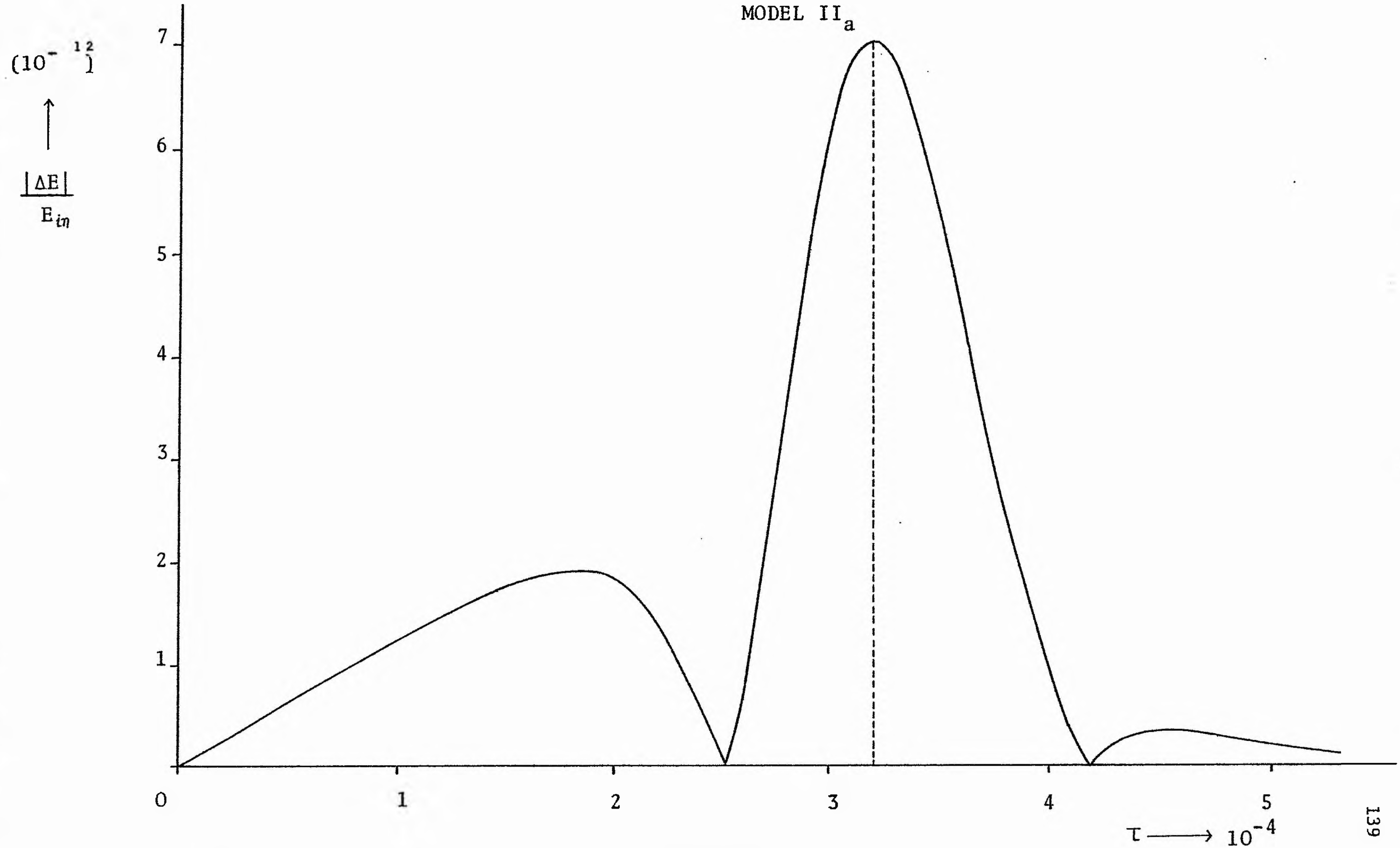

MODEL II$_a$
(10^{-12})
$\frac{|\Delta E|}{E_{in}}$
7
6
5
4
3
2
1
0
1
2
3
4
5
$\tau \longrightarrow 10^{-4}$

Figure 19

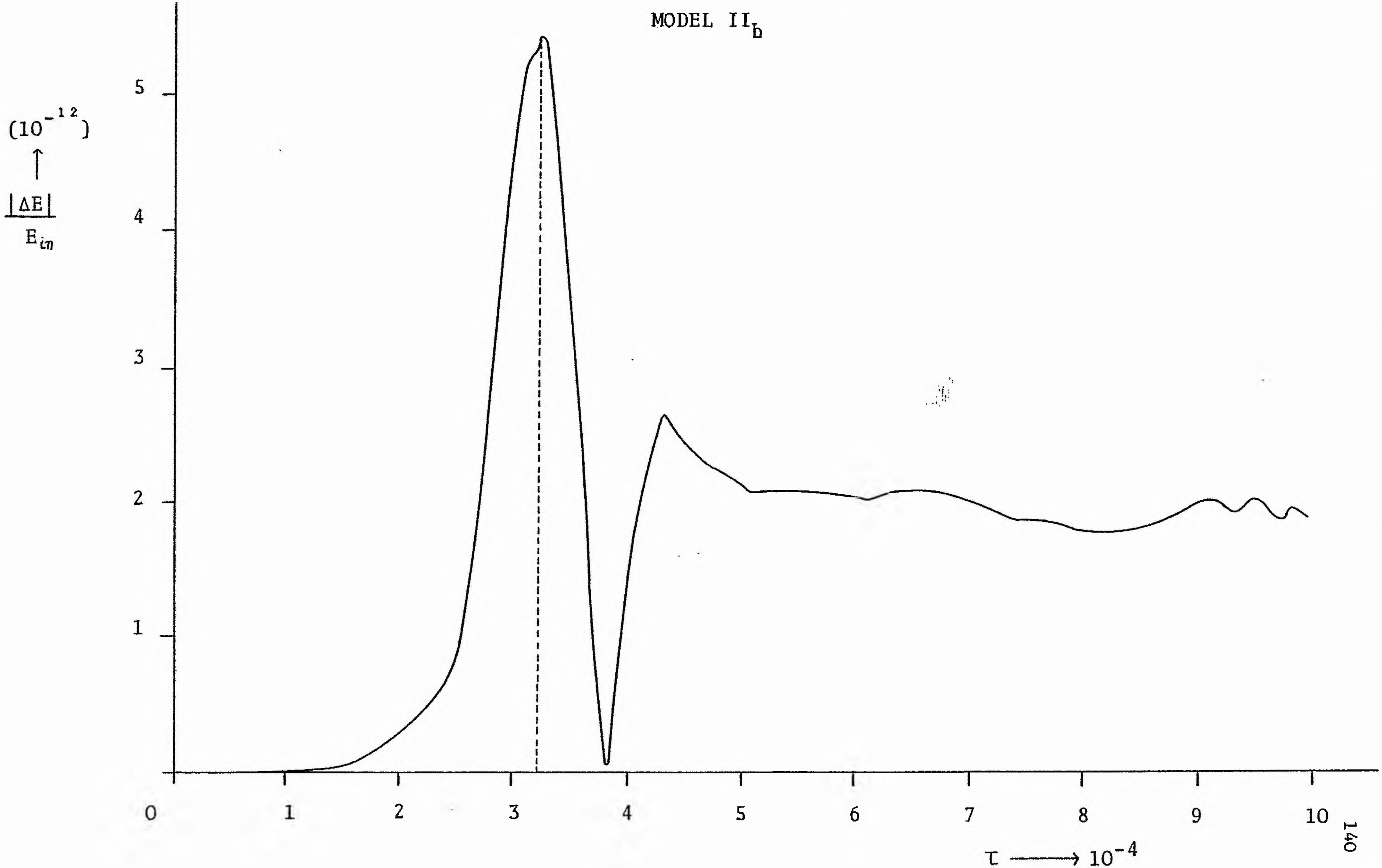

MODEL II$_b$
(10^{-12})
$\frac{|\Delta E|}{E_{in}}$
1
2
3
4
5
0
1
2
3
4
5
6
7
8
9
10
$\tau \longrightarrow 10^{-4}$

Figure 20

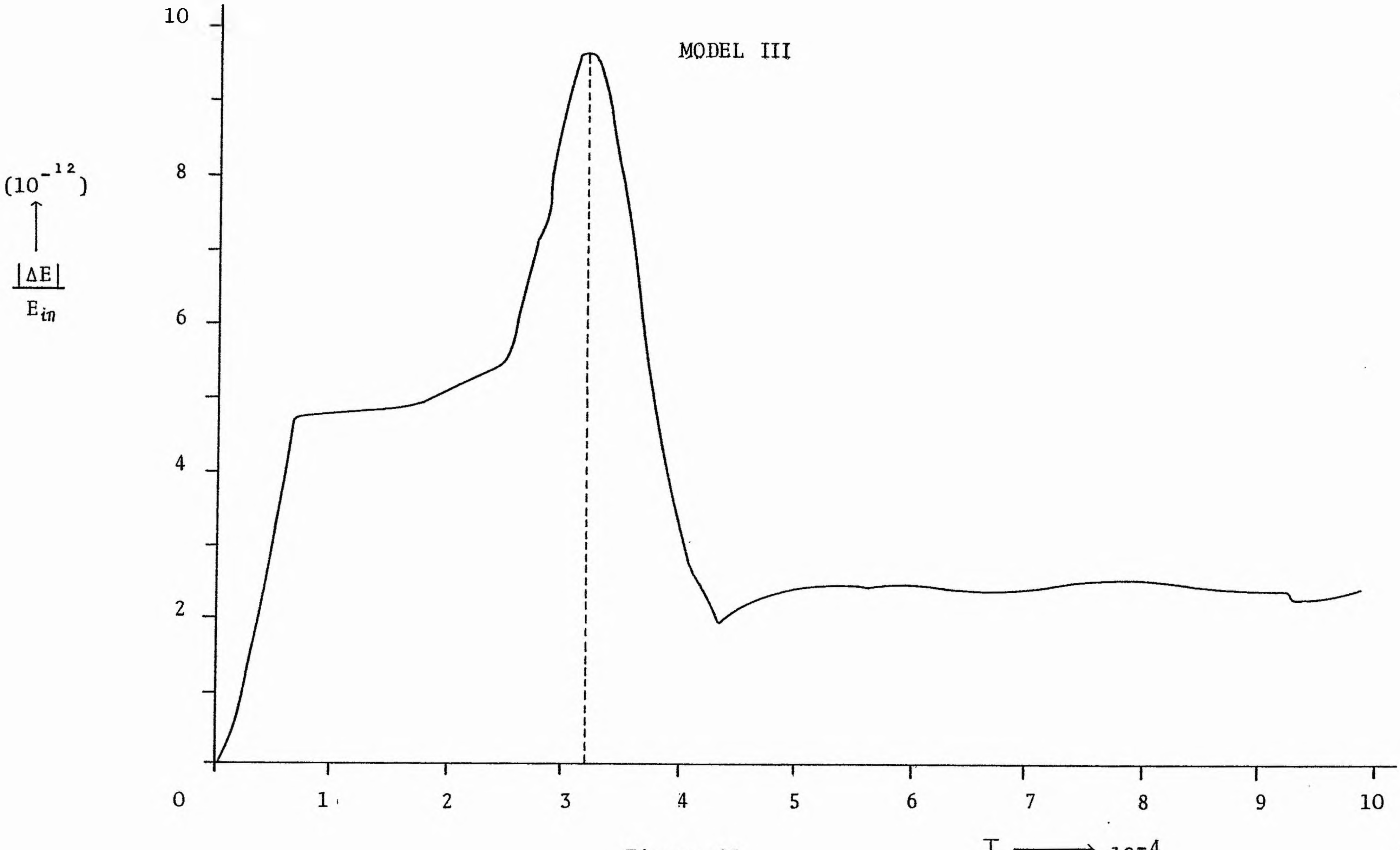
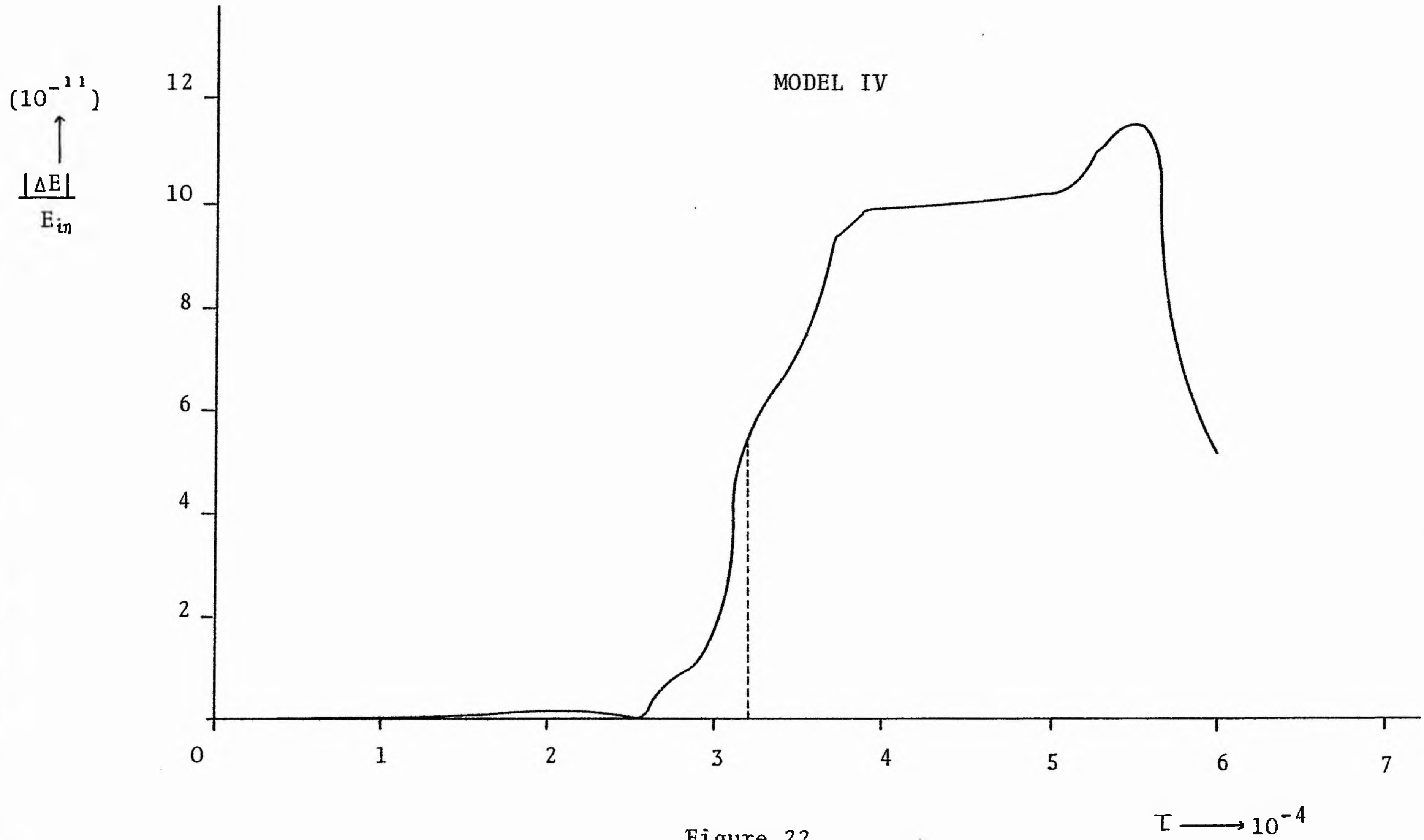


Figure 21



MODEL IV
(10^{-11})
$\frac{|\Delta E|}{E_{in}}$
12
10
8
6
4
2
0
1
2
3
4
5
6
7
$\tau \longrightarrow 10^{-4}$

Figure 22

same quantity at the later time τ . The total potential energy of models consisting of particles and oblate spheroids is given by the relation (3.14).

As it was pointed out above the relative error $|\Delta E| \backslash E_{in}$ depends on the length of the time step used, which is not maintained constant but automatically chosen with the aid of the criteria (A1-6a) and (A1-11). Both criteria are in principle suitable for determining the time step $\Delta\tau$, but it was found in practice that for a particular model one or the other of these criteria is always preferable in terms of speed and accuracy. Table (6) gives the type of the dynamical model, the formula used to determine its time step and the order of the value of the constant (A or μ) associated with this formula. The values of the constants B and C occurring in the relation (A1-13) are the same for every model and equal to

$$B = 10^{-6}\ \text{u.t.}, \quad C = 10^{-10}\ \text{u.t.}$$

Table 6

Type of model	Criterion	A (Kpc)	μ
Ia	(A1-11)	-	10^{8}
Ib	(A1-6a)	10^{-25}	-
IIa	(A1-6a)	2.5×10^{-26}	-
IIb	(A1-6a)	1.0×10^{-26}	-
III	(A1-6a)	5.0×10^{-26}	-
IV	(A1-11)	-	10^{10}

The time of the closest approach of the centres of the heavy bodies of the models Ia and IIa are 3.207×10^{-4} u.t. and 3.194×10^{-4} u.t. respectively, while the times of the closest approach of the heavy bodies for the rest of our dynamical models lie within the interval defined by these two values and are indicated in figures (17) to (22) by dashed lines. From these figures it is apparent that for each of the models Ib, IIa, IIb and III the global maximum of the function (4.27) occurs at the time of closest approach. However, this is not the case for models Ia and IV. The behaviour of the function (4.27) is, in general, determined by the type of the model and not by the type of computer used to integrate the equations of motion nor by the criterion used to evaluate the length of the time step. This can be proved - at least for the model Ib - with the aid of figures (18) and (18a). The first of these figures is the graphical representation of the function $(|\Delta E| / E_{in}) = f(\tau)$, which can be obtained when the integration of the equations of motion of the model Ib is performed by a VAX 11/780 machine and the length of the time step is chosen with the aid of the formula (A1-6a) [see tables (5) and (6)]. The second figure illustrates the same function and is obtained when the calculations are repeated with an IBM 360/44 machine and the criterion used to determine the time step is that given by equation (A1-11), where $\mu = 10^{-8}$. In both cases the general characteristics of the behaviour of the relative error $|\Delta E| / E_{in}$ are the same and most important of all the global maximum occurs at the time of the minimum approach of the heavy bodies of the dynamical model.

One general conclusion from figures (17) to (22) is the high accuracy maintained during the numerical integration of the equations of motion of our dynamical models. This can be seen by noting that the value of the relative error $|\Delta E| / E_{in}$ is always lower than its value at the time of the closest approach of the heavy bodies, at least for the intervals through which we followed the evolution of each model.

Finally figure (21) demonstrates the sensitive dependence of the relative error of the total energy on the length of the time step. The sharply rising first part of the curve is produced when the length of the time step is chosen (during the numerical integration of the equations of motion of model III) with the aid of the criterion (A1-6a), where $A = 1.0 \times 10^{-24}$ Kpc. The rest of the curve is produced using the same criterion where, in this case, $A = 5.0 \times 10^{-26}$ Kpc.

We proceed by considering the impact the close encounter of the heavy bodies in dynamical models Ib, IIb and III has on the evolution of the models.

Figures (23a) to (25h) give, at different times, the projections on the XY co-ordinate plane of the positions of the bodies of the perturbed galaxy; the direction of the X and Y axis; the scale of the drawings; the type of the dynamical model; the co-ordinates of the positions of the centres of the heavy bodies in the perturbing (x',y') and perturbed (x,y) galaxy (or their difference) in Kpc and finally, the corresponding time T expressed in 1.0×10^{-4} u.t. (or $4.714930\,759\,449\,023 \times 10^{7}$ tropical years).

From these figures it becomes apparent that, independent of the model, the distribution in space of the 36 particles surrounding the mass m_{37} remains symmetrical, at least up to the time of minimum approach of the masses m_{37} and m_{38}. The physical states of the models Ib, IIb and III are virtually the same at any time within the interval [0.0 u.t., 9.5×10^{-4} u.t.] which is nearly three times greater than the time required for the masses m_{37} and m_{38} to reach the position of their closest approach.

The first significant stage of the evolution of all three models under consideration is that corresponding to figures (23c),(23d),(23e),(24c),(24d),(24e),(25c),(25d) and (25e) during which the models lose their spheroidal shape becoming triaxial ellipsoids. The next stage in the evolution of the models is marked by a tendency for the particles to concentrate on opposite sides of the mass m_{37} leading to figures (23f),(23g),(24f),(24g),(25f) and (25g).

Though we do not have data to follow the evolution of the model III beyond this stage, we can study what effect the shape of interacting galaxies has on their evolution by following and comparing the evolution of models Ib and IIb. Such a comparison reveals the following:

Model IIb whose heavy bodies are oblate spheroids, after having passed through the two stages mentioned above develops more or less clear spiral structure which though non-permanent appears and reappears in a roughly periodic way. Figures (24f) to (24n) illustrates this structure at the epochs of its clearest appearances; the intervals of time separating these epochs can easily be obtained for the data given in the figures and they are 6.886×10^{-4} u.t. (3.247×10^{8} trop. years), 8.547×10^{-4} u.t. (4.030×10^{8} trop. years), 7.506×10^{-4} u.t. (3.539×10^{8} trop. years), 6.600×10^{-4} u.t. (3.112×10^{8} trop. years), 9.301×10^{-4} u.t. (4.385×10^{8} trop. years) and 9.601×10^{-4} u.t. 4.527×10^{8} trop. years). The intermediate stages in the evolution of model IIb are characterized either by a symmetrical or by a random distribution of the particles constituting the halo of mass m_{37} (see figures 26a and 26b). The disappearance of the spiral structure of the model IIb during these stages, being the result of differential rotation, does not necessarily mean that this is the case in real pairs of interacting galaxies; this phenomenon could merely be the result of the limited number of particles taken to represent the disk of the perturbed galaxy. We must note here that the differential rotation of the particles is not only the reason for the disappearance of the spiral structure, but also the reason for its regular reappearance. The spiral arms once they are formed are rather long-lived structures with an average life 1.350×10^{-4} u.t. (or 6.366×10^{7} trop. years).

Model Ib, whose heavy bodies are particles, does not develop spiral structure during its dynamical evolution, a fact which clearly demonstrates the importance which the shape of the central regions of the galaxies has on their evolution, at least in the case of pairs of interacting galaxies.

The above observations are of extreme importance. Firstly they remove the long-standing difficulty of producing spiral arms as a result of tidal interactions by using realistic distances and velocities and secondly, they prove that the ellipsoidal shapes of galaxies [as figures (2),(3),(5),(6) and (23a) to (25h) indicate] do not have a 10% effect on their dynamics, as had been suggested (J. Einasto 1974a, 1975), but a much greater one. Significant also is the fact that the spiral structure appears repeatedly in a rather regular way for a period of time (5.619×10^{-3} u.t. or 2.649×10^{9} trop. years) comparable to the age of galaxies and that it develops at distances as small as 5.3 Kpc from the centre of the mass m_{37}. Though we do not follow the evolution of model IIb beyond this time all indications suggest that the repeated appearance of the spiral structure will continue if the system consisting of the mass m_{37} and the particles surrounding it remains unperturbed and that such structure can develop even closer than 5.3 Kpc to the centre of the mass m_{37}.

It was mentioned previously that the available data for models Ia, IIa and IV are not sufficient to allow any study of their evolution; we therefore confine ourselves to compare the last computed stages of these models with the corresponding stages of models Ib and IIb. It is easy to establish with the aid of figures (27),(28) and (29) that, up to the time which we followed the evolution of models Ia and IIa their physical states

MODEL I_b

$x = 0.08334$		$x = 0.77141$
$y = 0.00624$		$y = 0.34445$
$x' = 71.16982$		$x' = 4.30350$
$y' = 17.74502$		$y' = 41.28914$
$T = 1.48456$		$T = 3.46807$

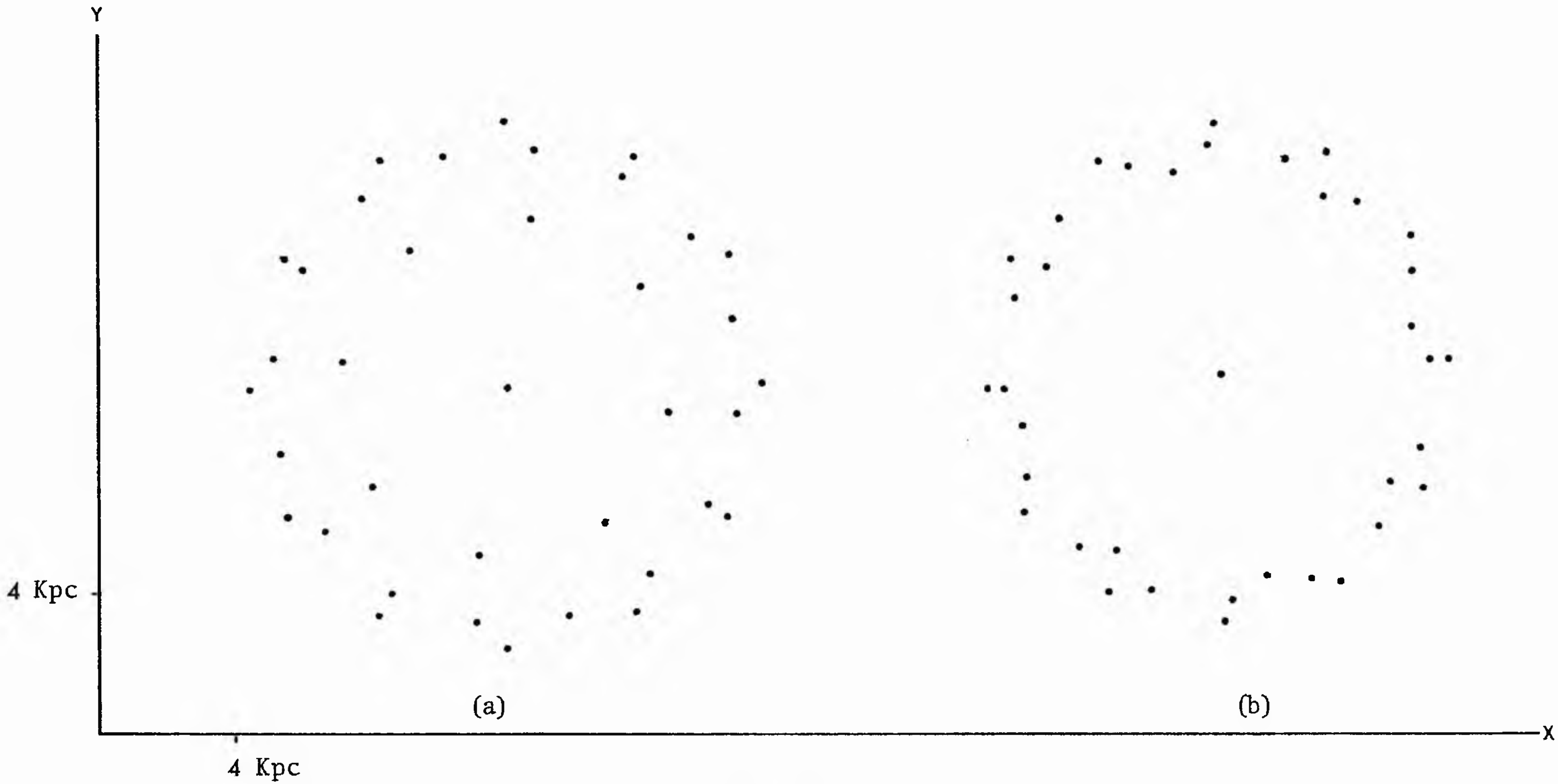


Figure 23

MODEL I_b

x = 1.26641
y = 1.03995

x' = -29.49000
y' = 52.89420

T = 4.46788

x = 1.72214
y = 1.94312

x' = -63.23418
y' = 64.39555

T = 5.40769

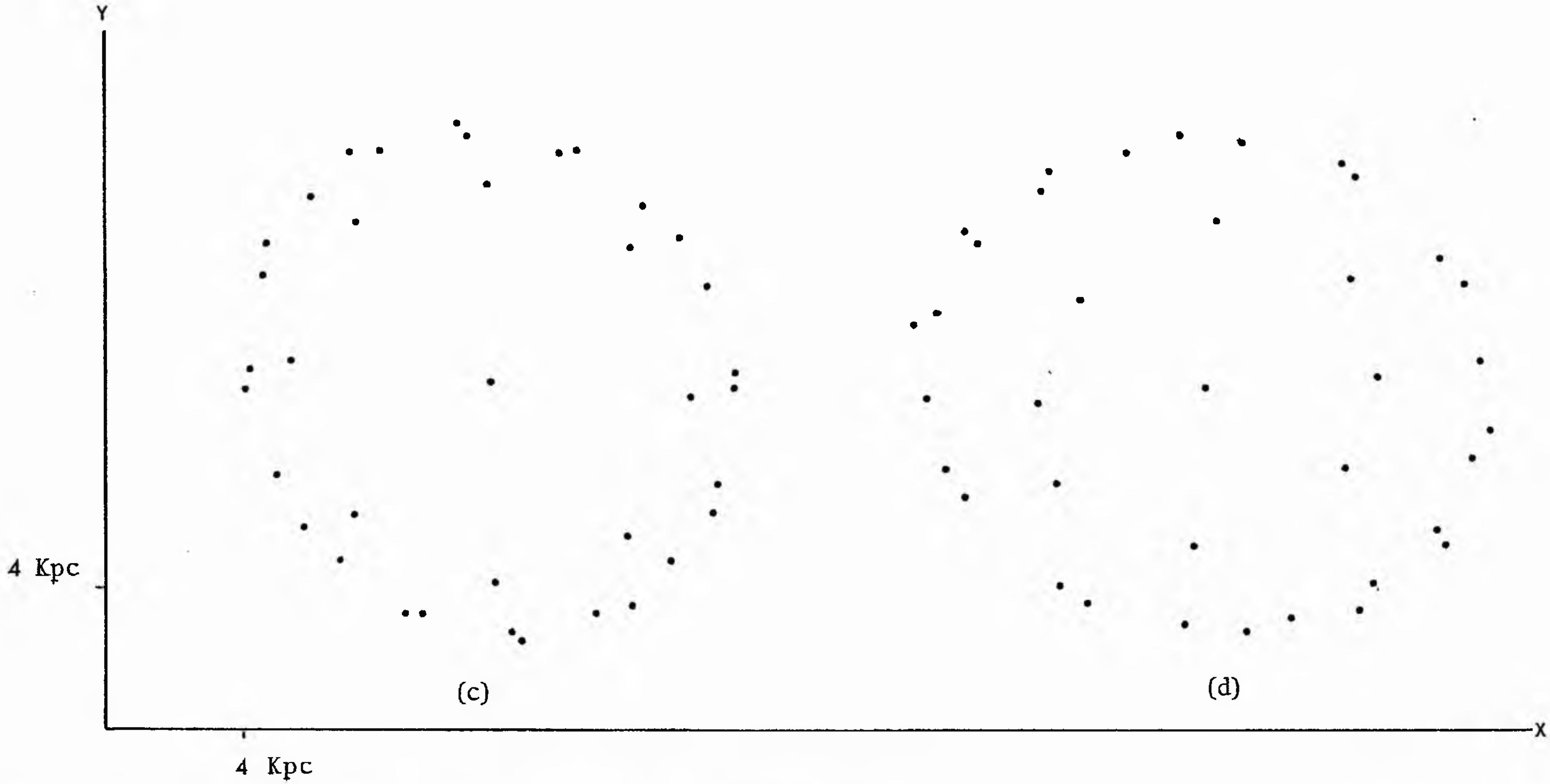


Figure 23

MODEL I_b

$x = 2.07220$
$y = 2.91133$

$x' = -96.61784$
$y' = 75.74933$

$T = 6.45788$

$x = 2.38105$
$y = 3.91842$

$x' = -130.18489$
$y' = 87.15691$

$T = 7.45416$

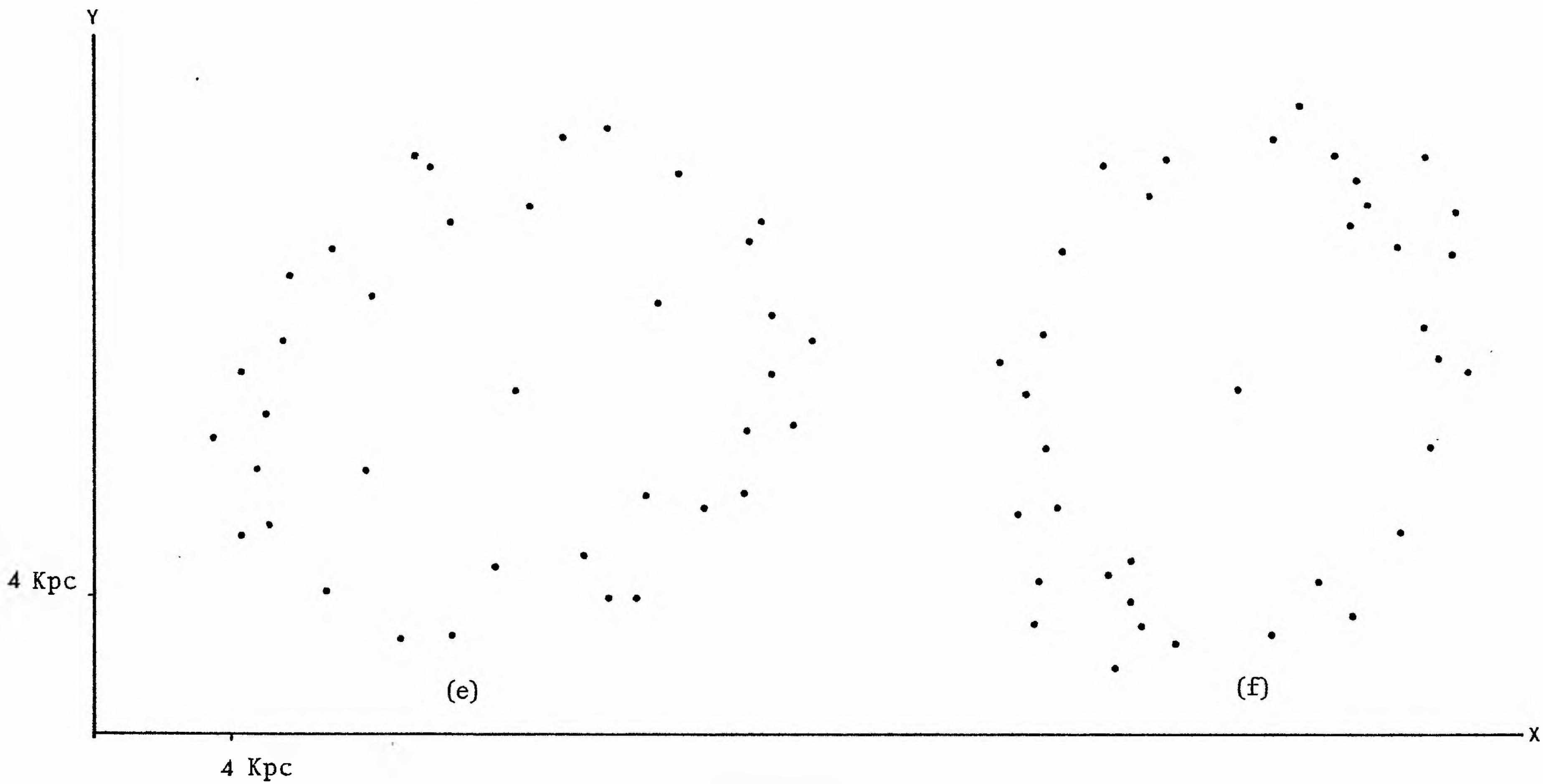


Figure 23

MODEL I_b

$x = 2.66270$
$y = 4.94633$

$x' = -163.83523$
$y' = 98.58761$

$T = 8.45332$

$x = 2.92450$
$y = 5.98320$

$x' = -197.41095$
$y' = 109.99126$

$T = 9.45055$

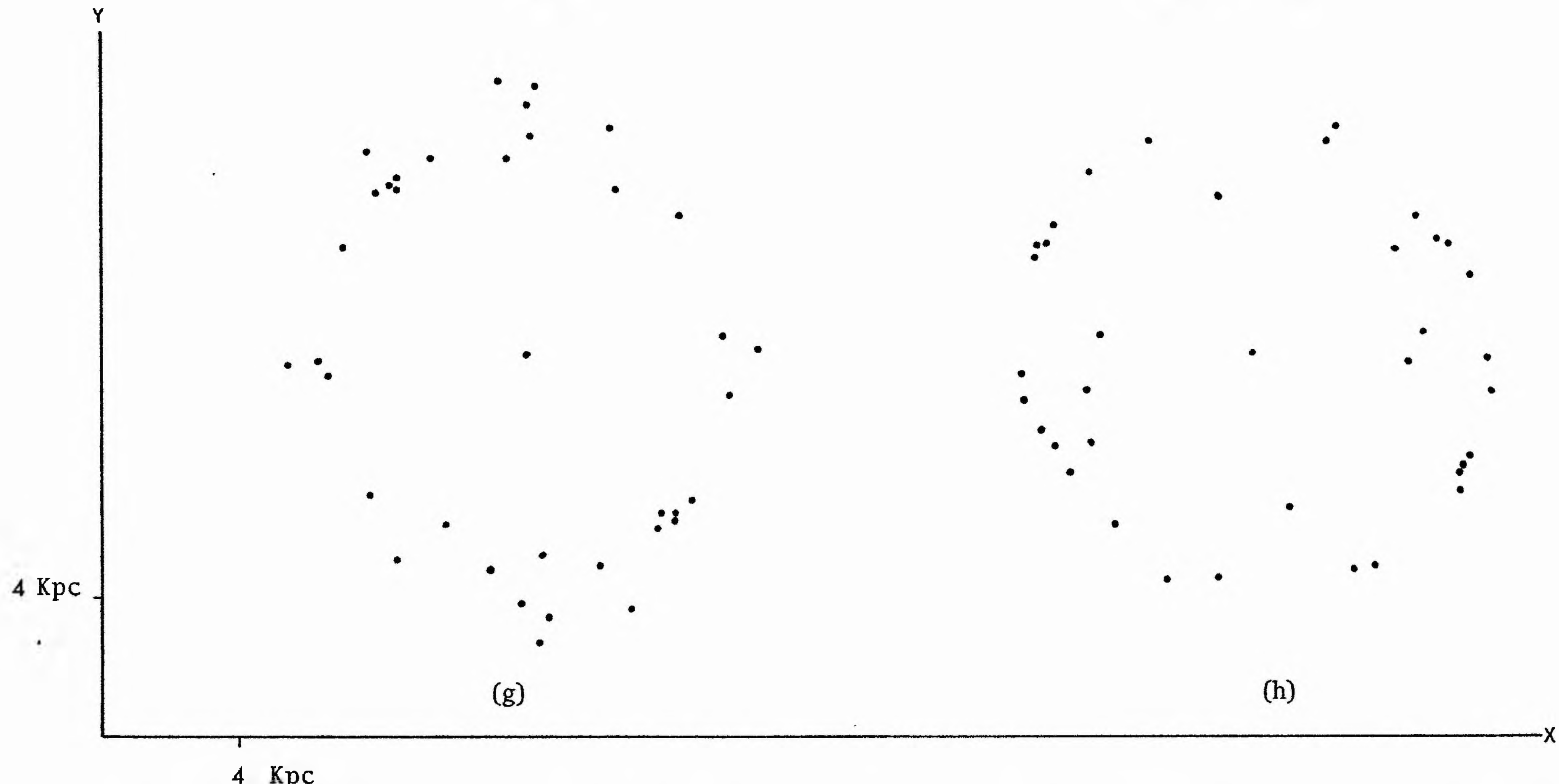


Figure 23

MODEL I_b

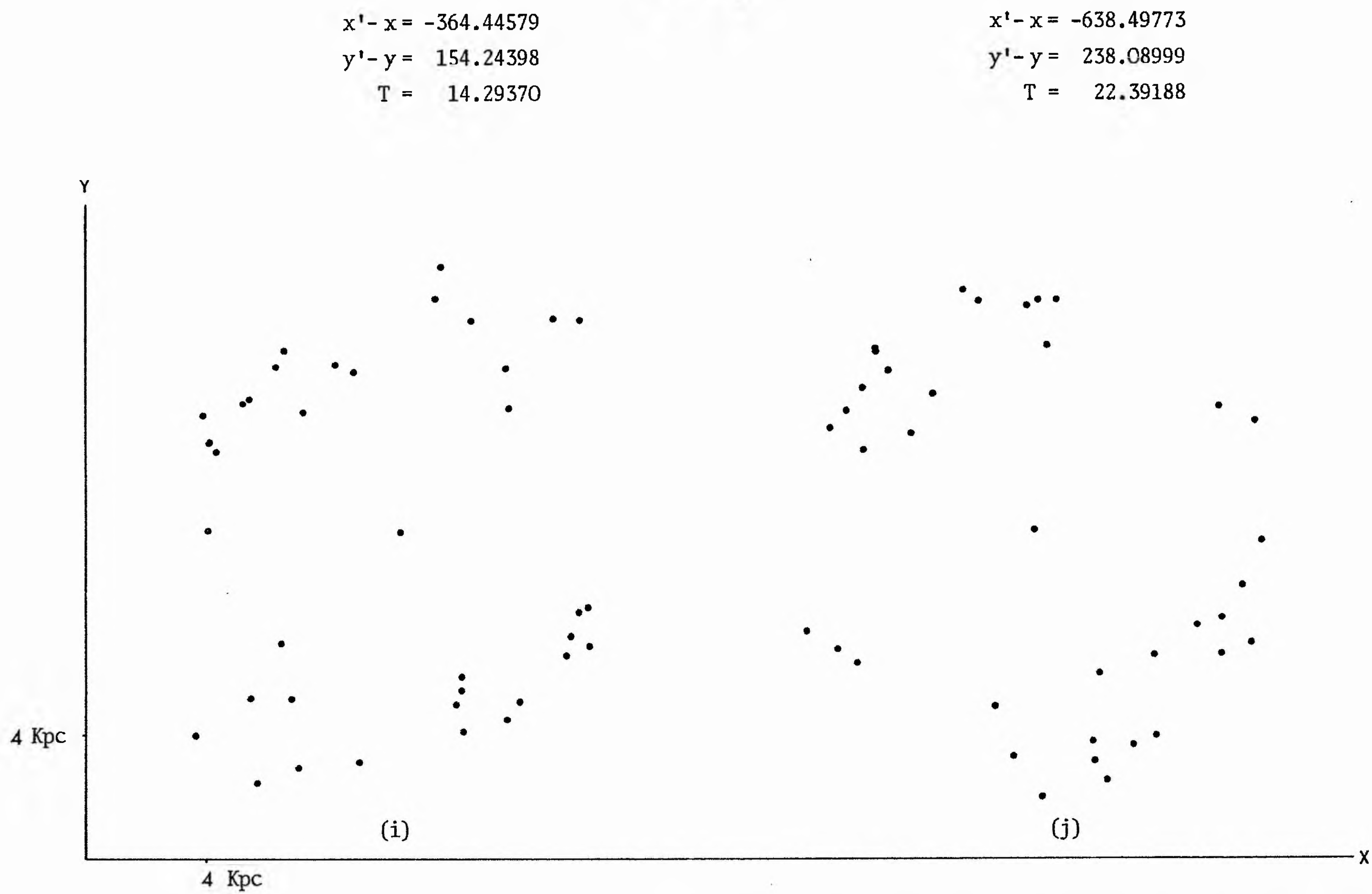


Figure 23

MODEL I_b

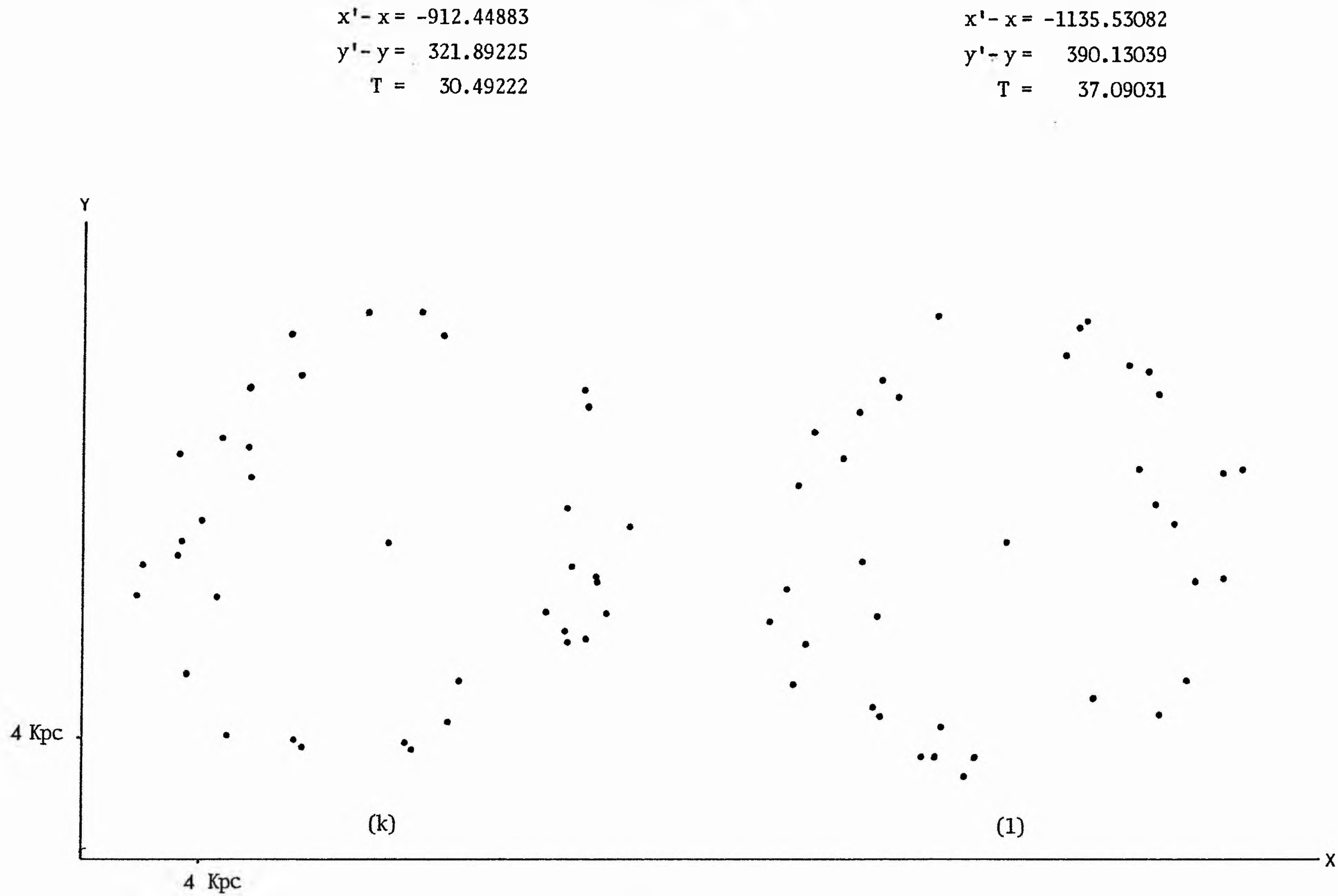
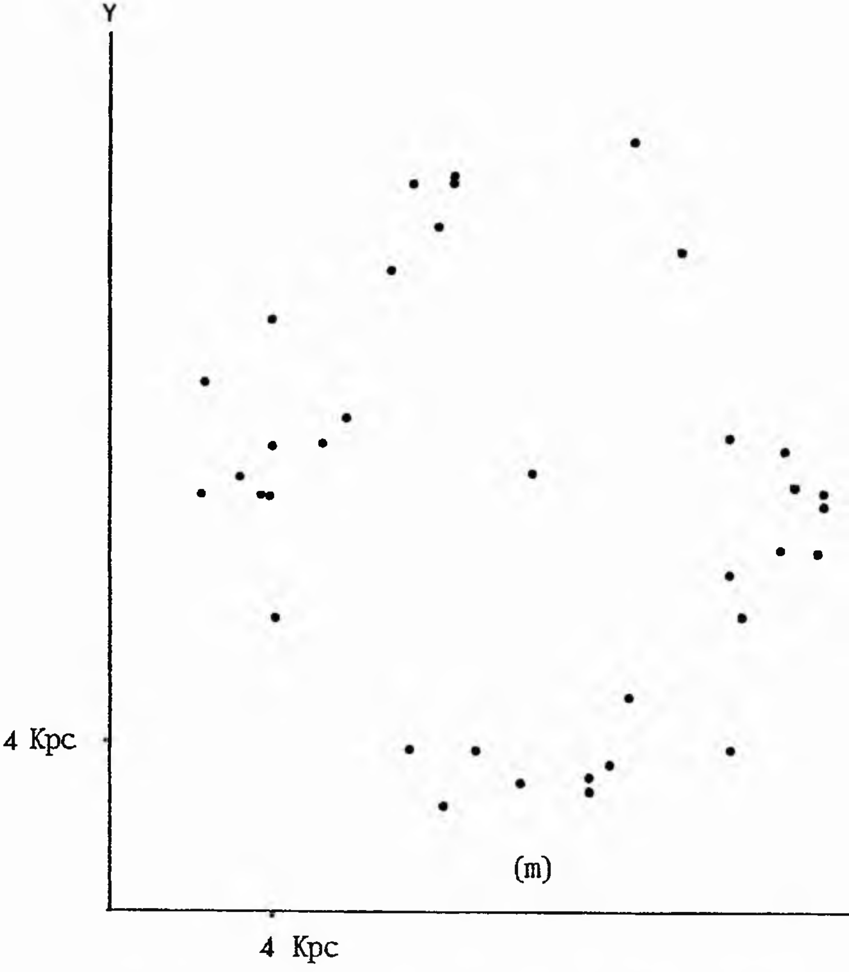


Figure 23

x'- x = -1389.17123
y'- y = 467.71439
T = 44.59345



(m)

MODEL I_b

$x' - x = -1774.55028$
$y' - y = 585.59317$
$T = 55.99530$

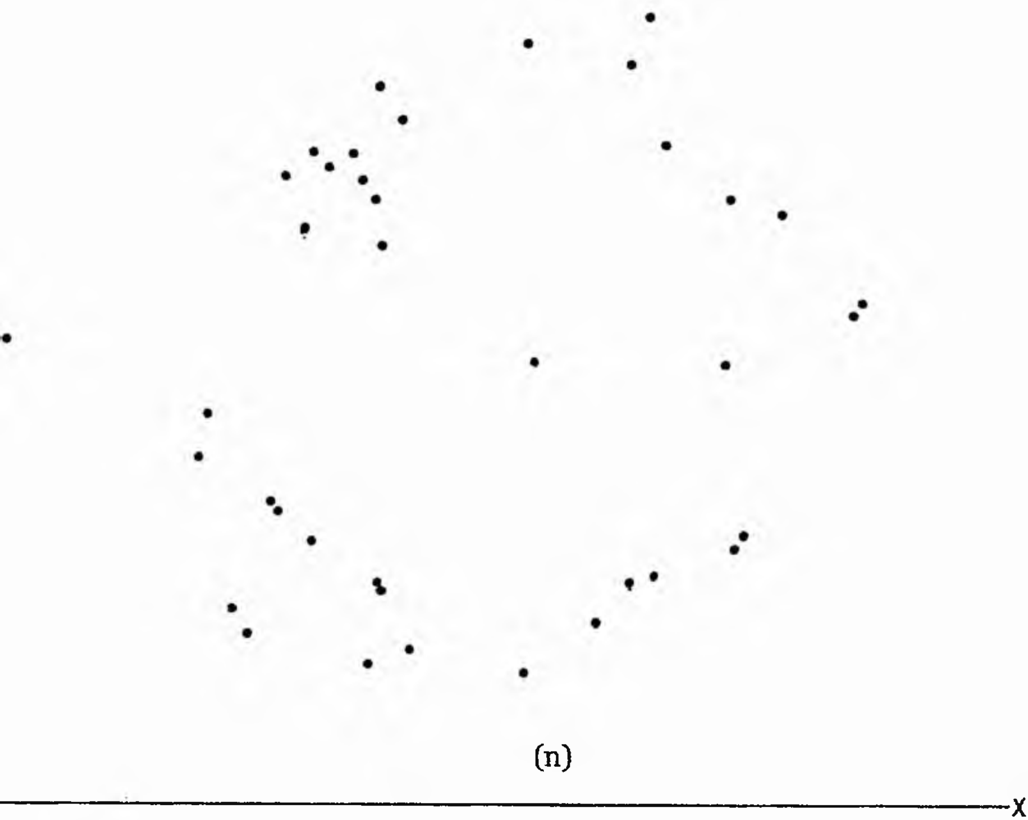


Figure 23

MODEL II_b

$x = 0.08555$
$y = 0.00644$

$x' = 71.12395$
$y' = 17.76087$

$T = 1.48589$

$x = 0.82398$
$y = 0.38569$

$x' = 4.33589$
$y' = 41.24761$

$T = 3.46632$

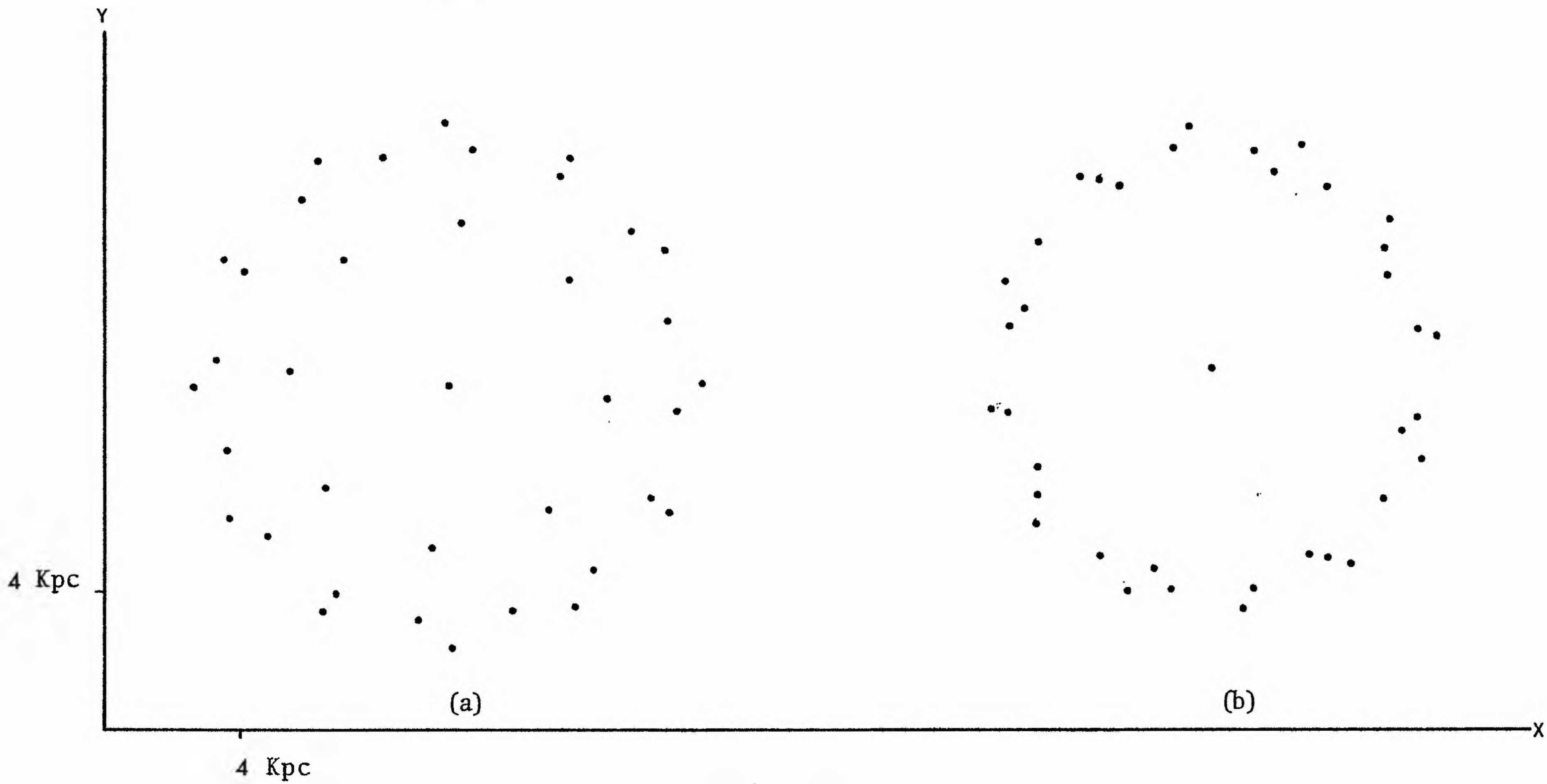


Figure 24

MODEL II_b

x = 1.40274
y = 1.19162

x' = -29.50307
y' = 52.80408

T = 4.46658

x = 1.84819
y = 2.16213

x' = -61.30841
y' = 63.57709

T = 5.40839

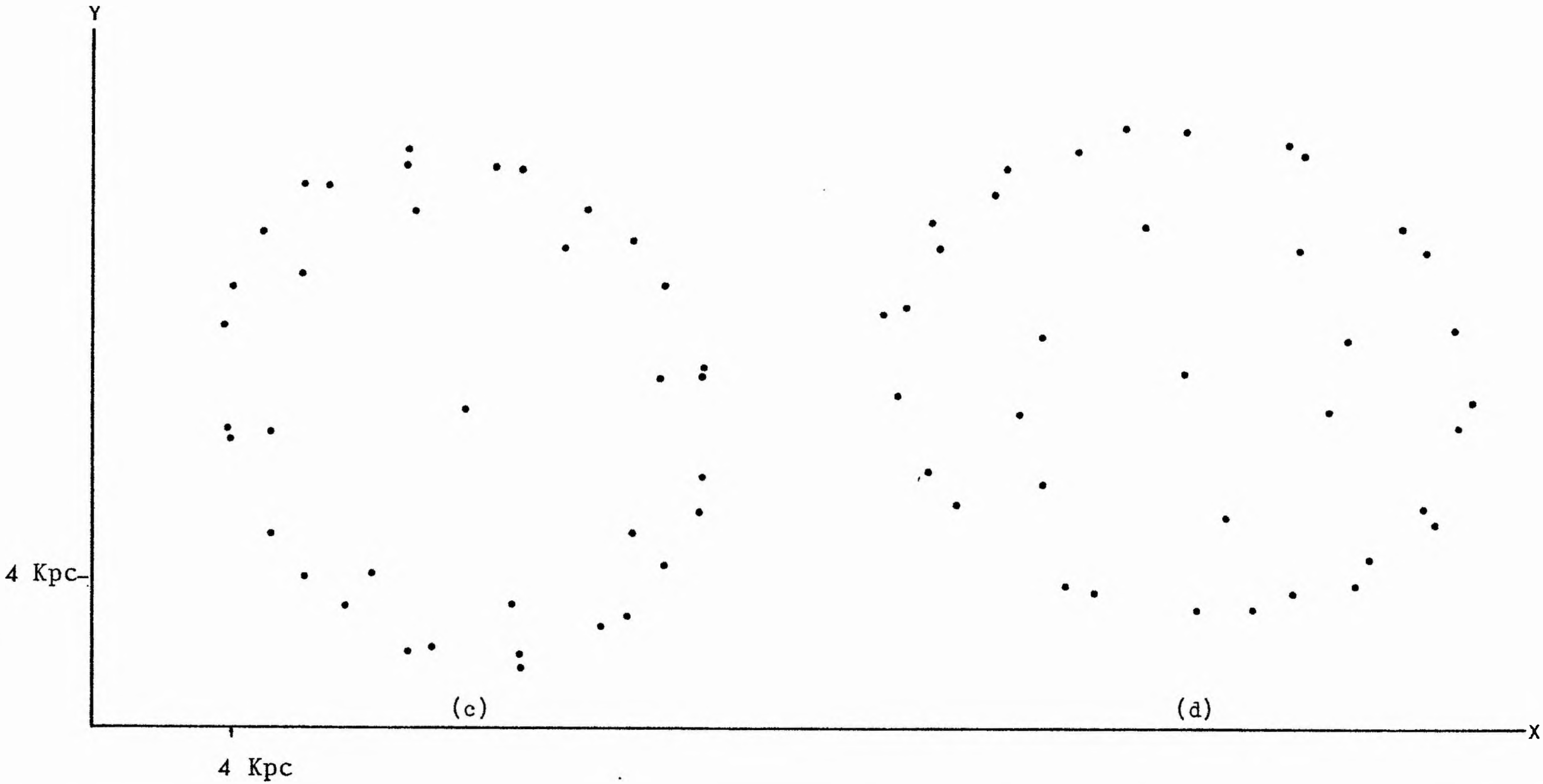


Figure 24

MODEL II_b

x = 2.26439	x = 2.61306
y = 3.32554	y = 4.46415
x' = -96.69568	x' = -130.24517
y' = 75.53604	y' = 86.86368
T = 6.45737	T = 7.45250

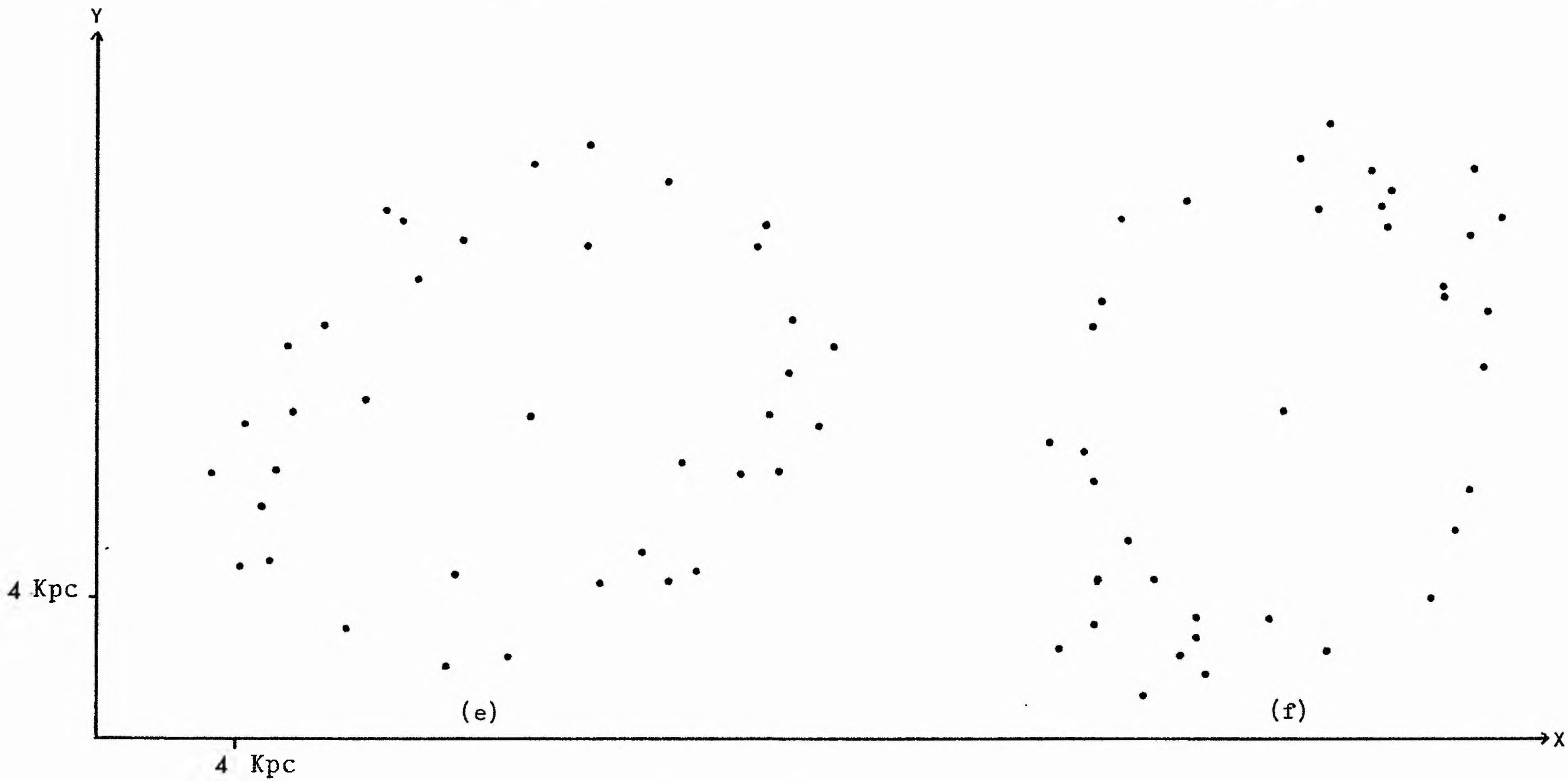


Figure 24

MODEL II_b

x = 2.93451	x = 3.23613
y = 5.62495	y = 6.79569
x' = -163.90473	x' = -197.52552
y' = 98.22462	y' = 109.57031
T = 8.45134	T = 9.44933

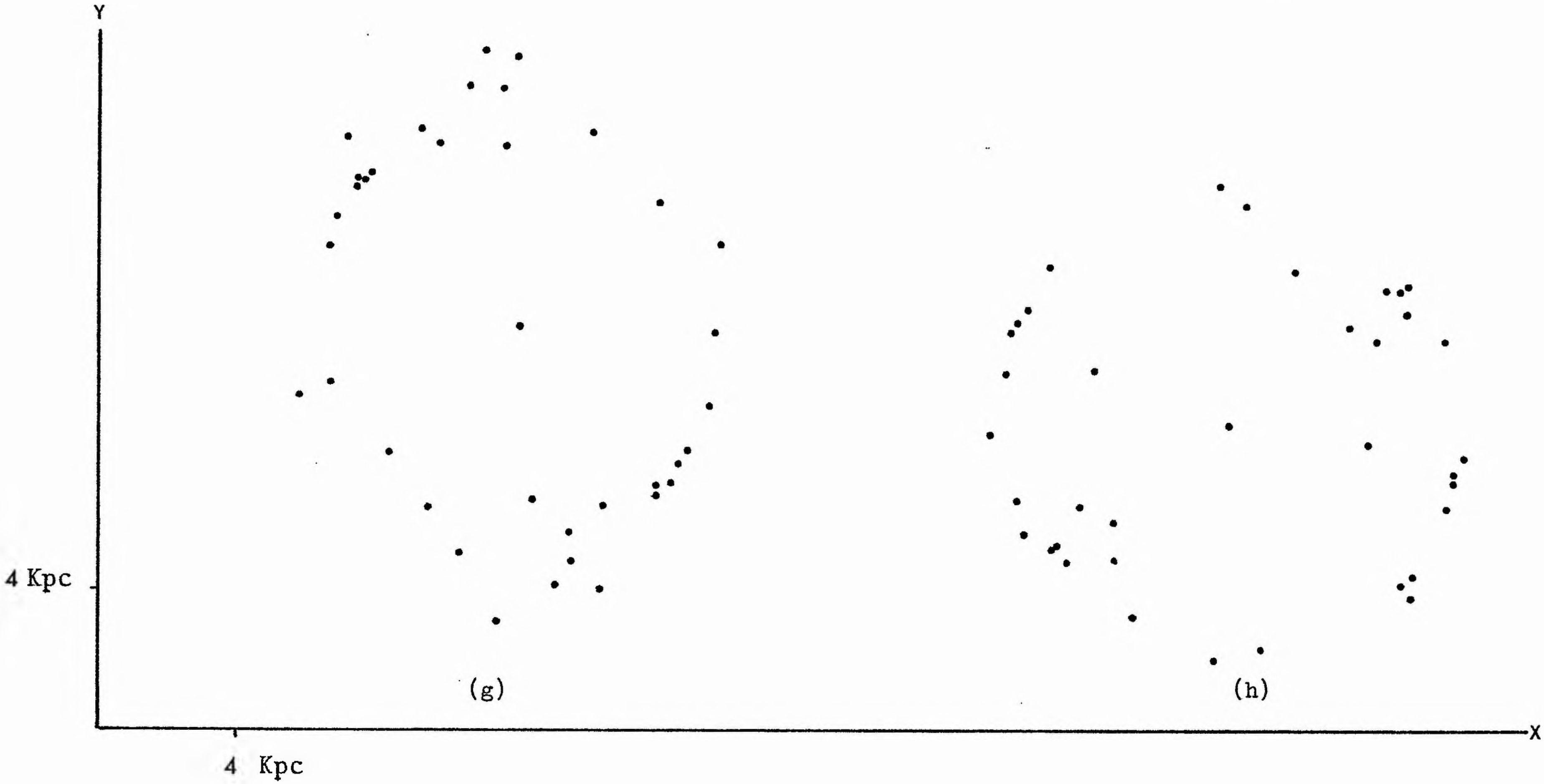


Figure 24

MODEL II_b

$x = 4.56091$	$x = 6.62138$
$y = 12.60600$	$y = 22.87090$
$x' = -362.16059$	$x' = -649.83042$
$y' = 165.11658$	$y' = 262.16269$
$T = 14.33855$	$T = 22.88537$

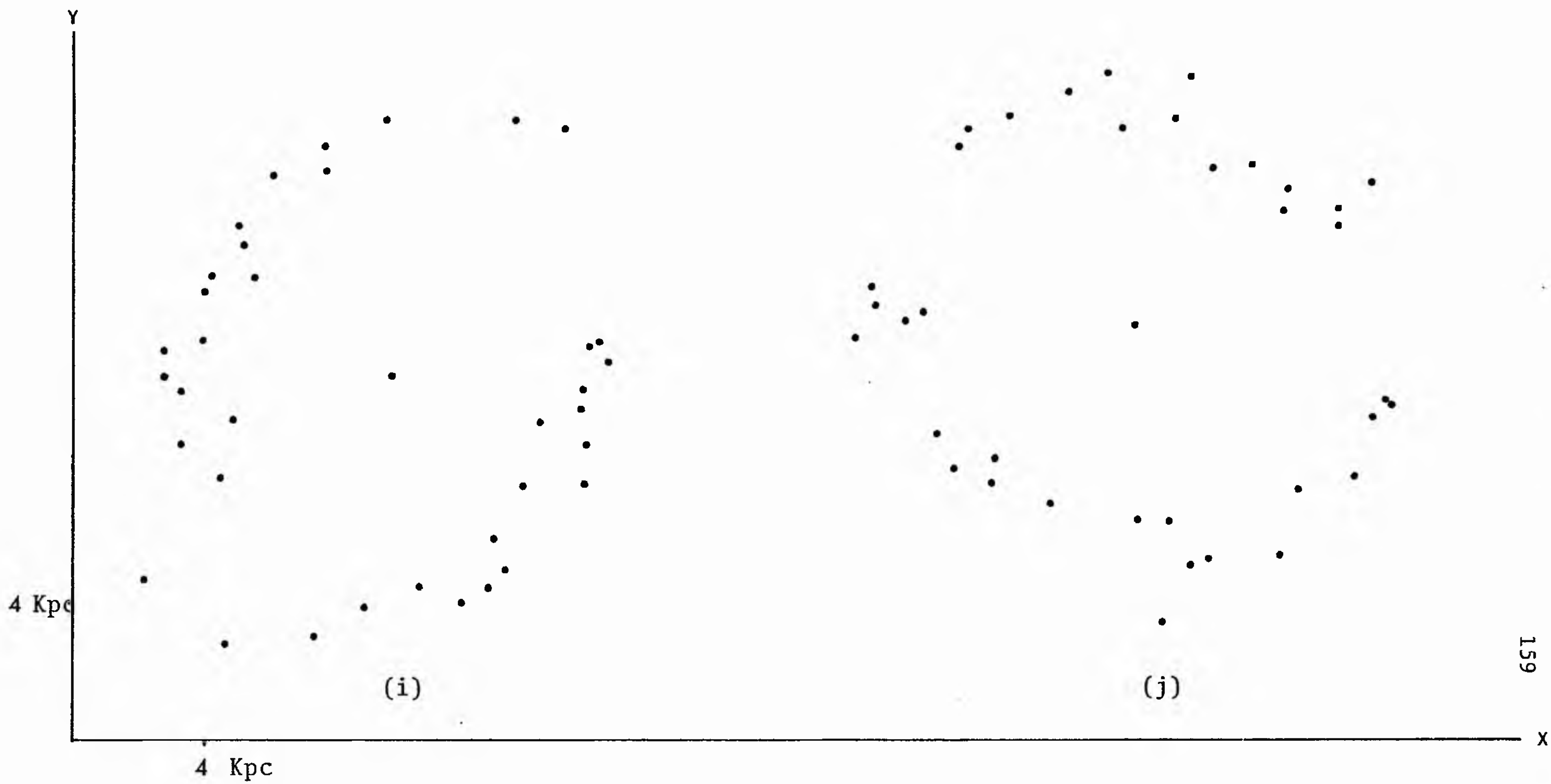
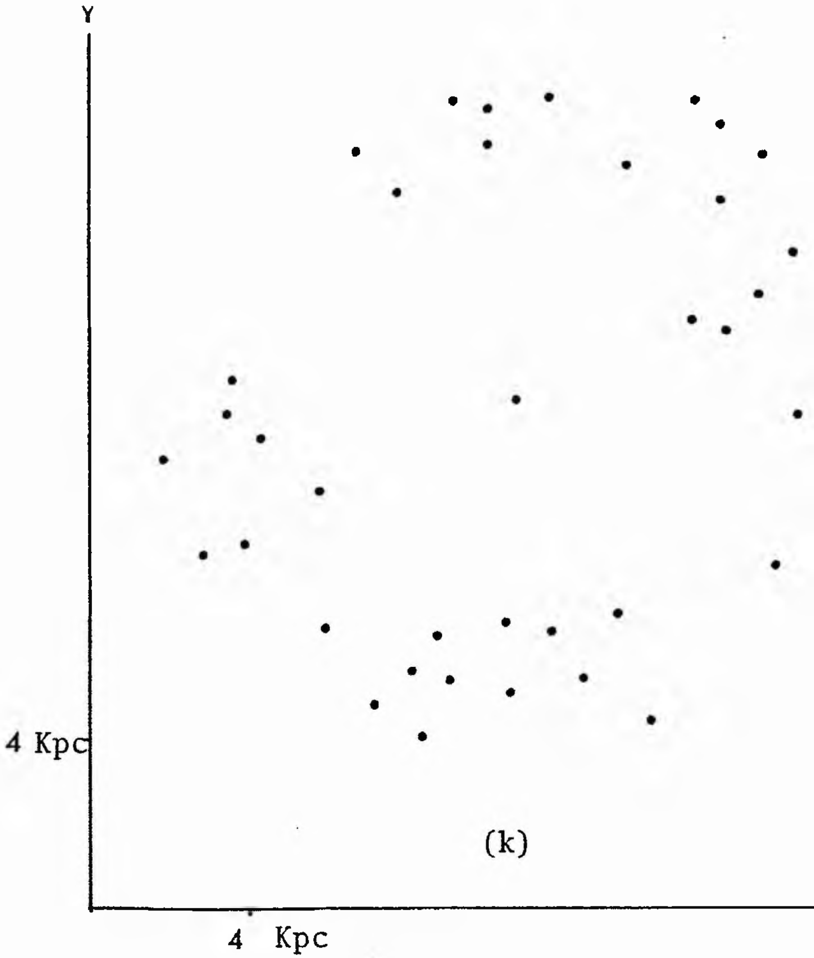


Figure 24

x = 8.32565
y = 31.92501

x'= -902.41636
y'= 347.37131

T = 30.39139



(k)

MODEL II_b

$x = 9.78408$
$y = 39.90023$

$x' = 1124.49359$
$y' = 422.28773$

$T = 36.99139$

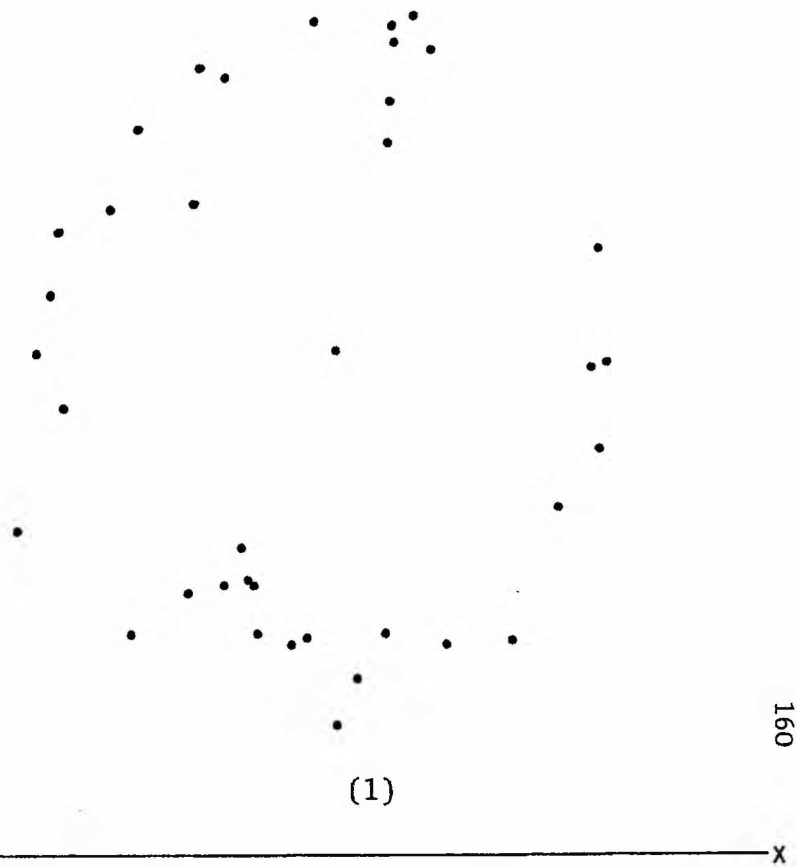


Figure 24

MODEL II_b

(m)	(n)
$x = 11.80119$	$x = 13.85310$
$y = 51.15176$	$y = 62.77591$
$x' = -1437.42599$	$x' = -1760.42991$
$y' = 527.85362$	$y' = 636.81718$
$T = 46.29212$	$T = 55.89263$

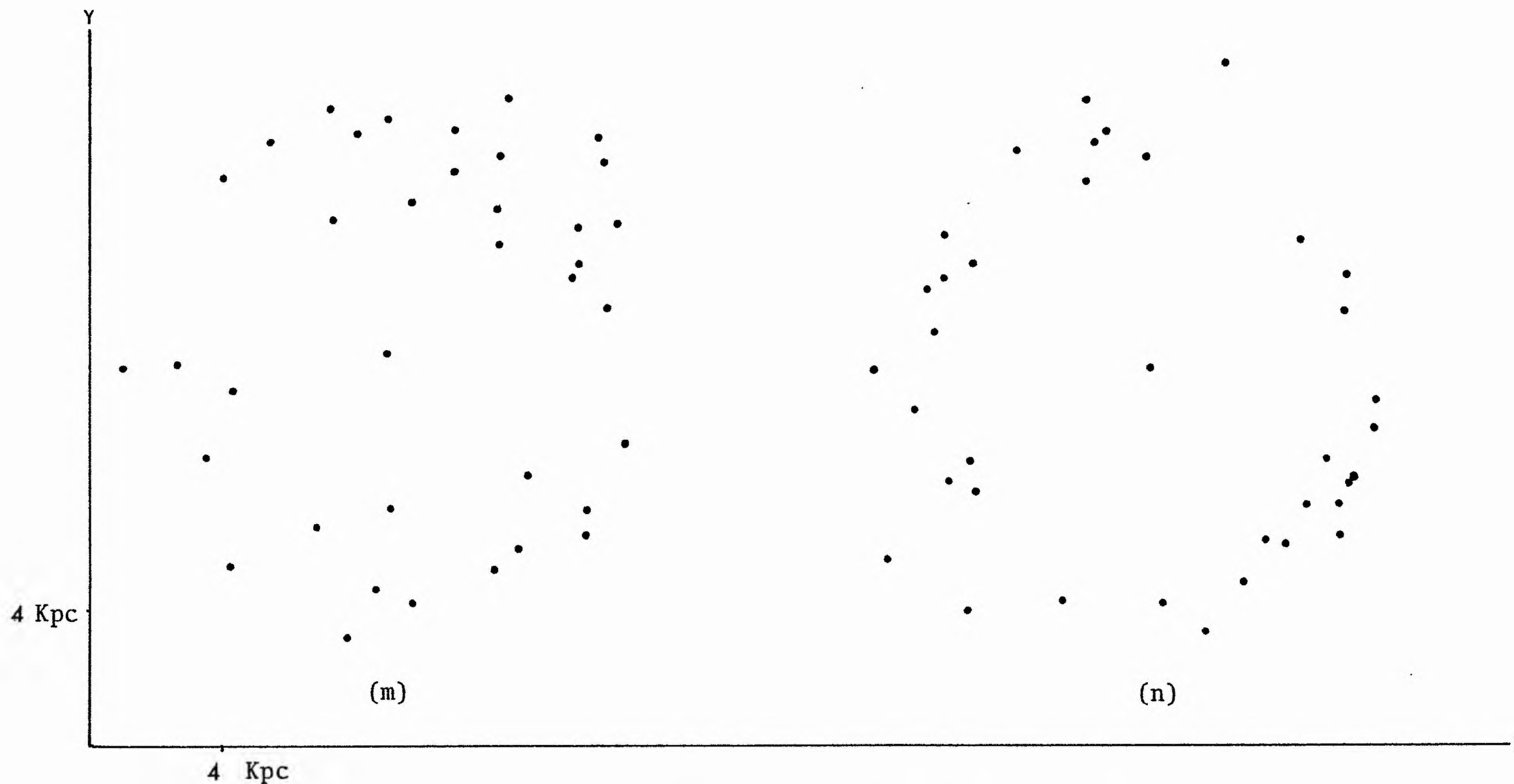


Figure 24

MODEL III

x = 0.08351
y = 0.00625

x' = 71.15120
y' = 17.75162

T = 1.48511

x = 0.77392
y = 0.34636

x' = 4.28411
y' = 41.29465

T = 3.46861

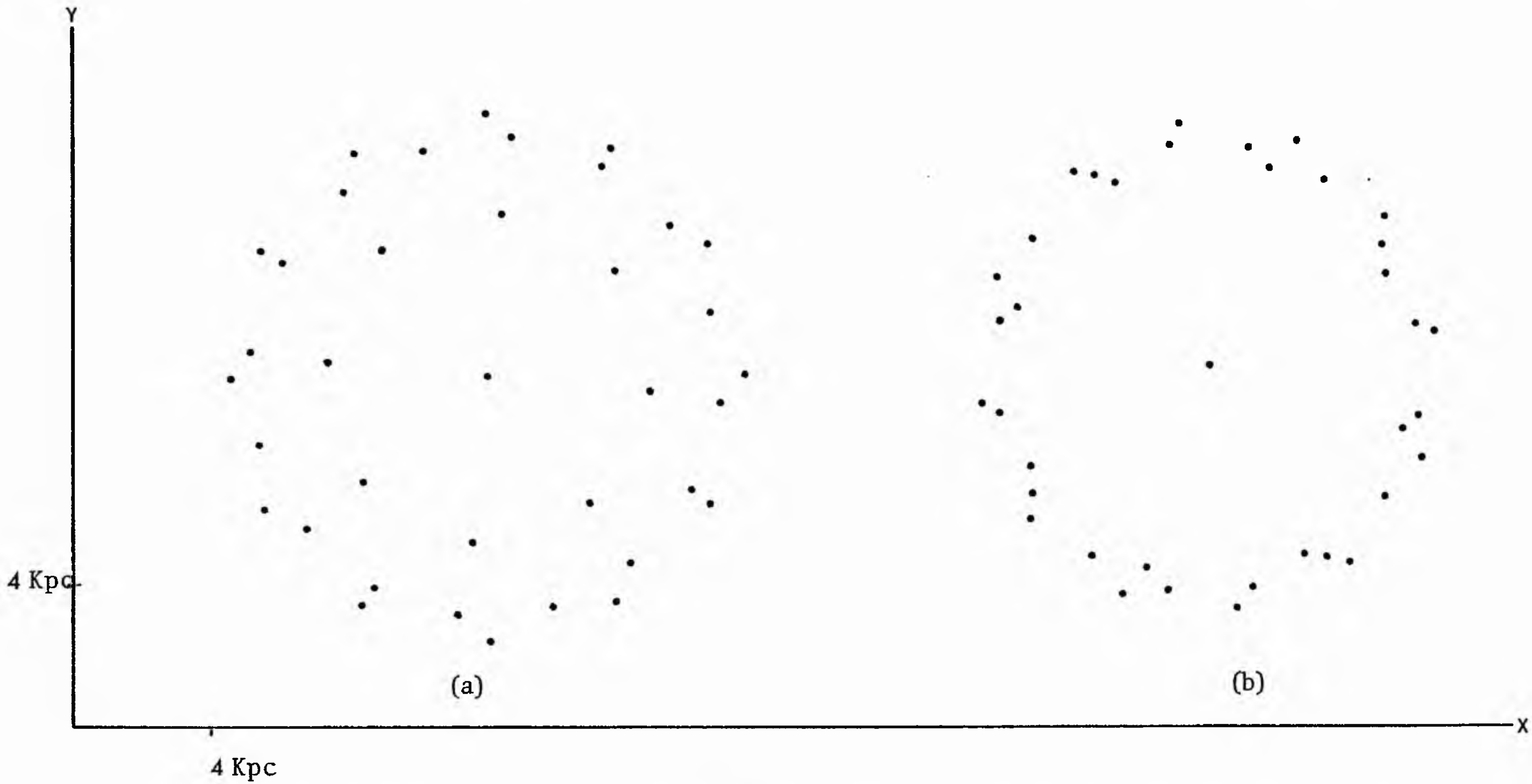


Figure 25

MODEL III

$x = 1.29072$
$y = 1.02821$

$x' = -28.78737$
$y' = 52.65002$

$T = 4.46470$

$x = 1.70491$
$y = 1.89644$

$x' = -61.2108925$
$y' = 63.70071$

$T = 5.40762$

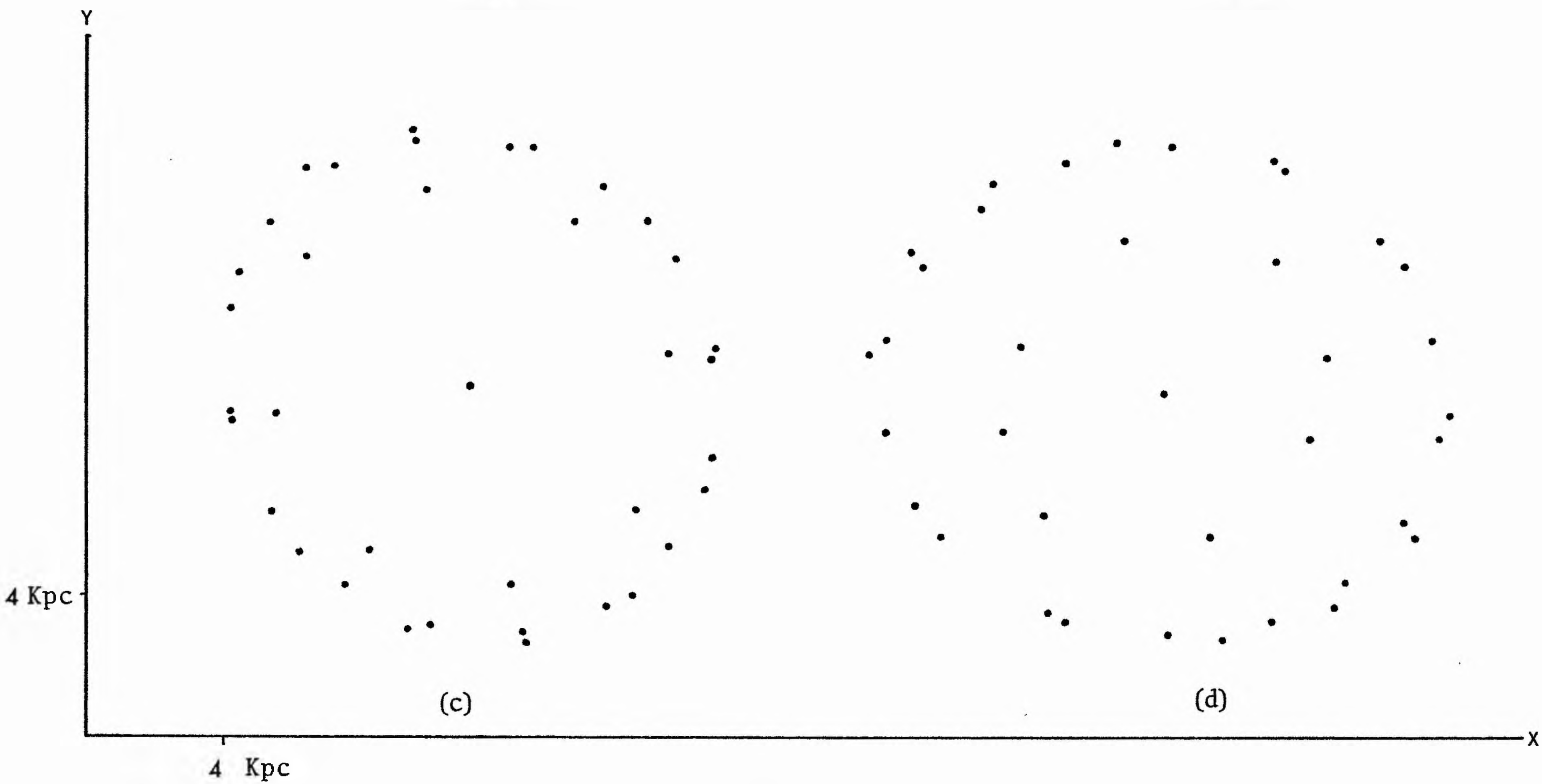


Figure 25

MODEL III

x = 2.08023
y = 2.92829

x' = -96.66564
y' = 75.75646

T = 6.45919

x = 2.39080
y = 3.94129

x' = -130.26080
y' = 87.17030

T = 7.45627

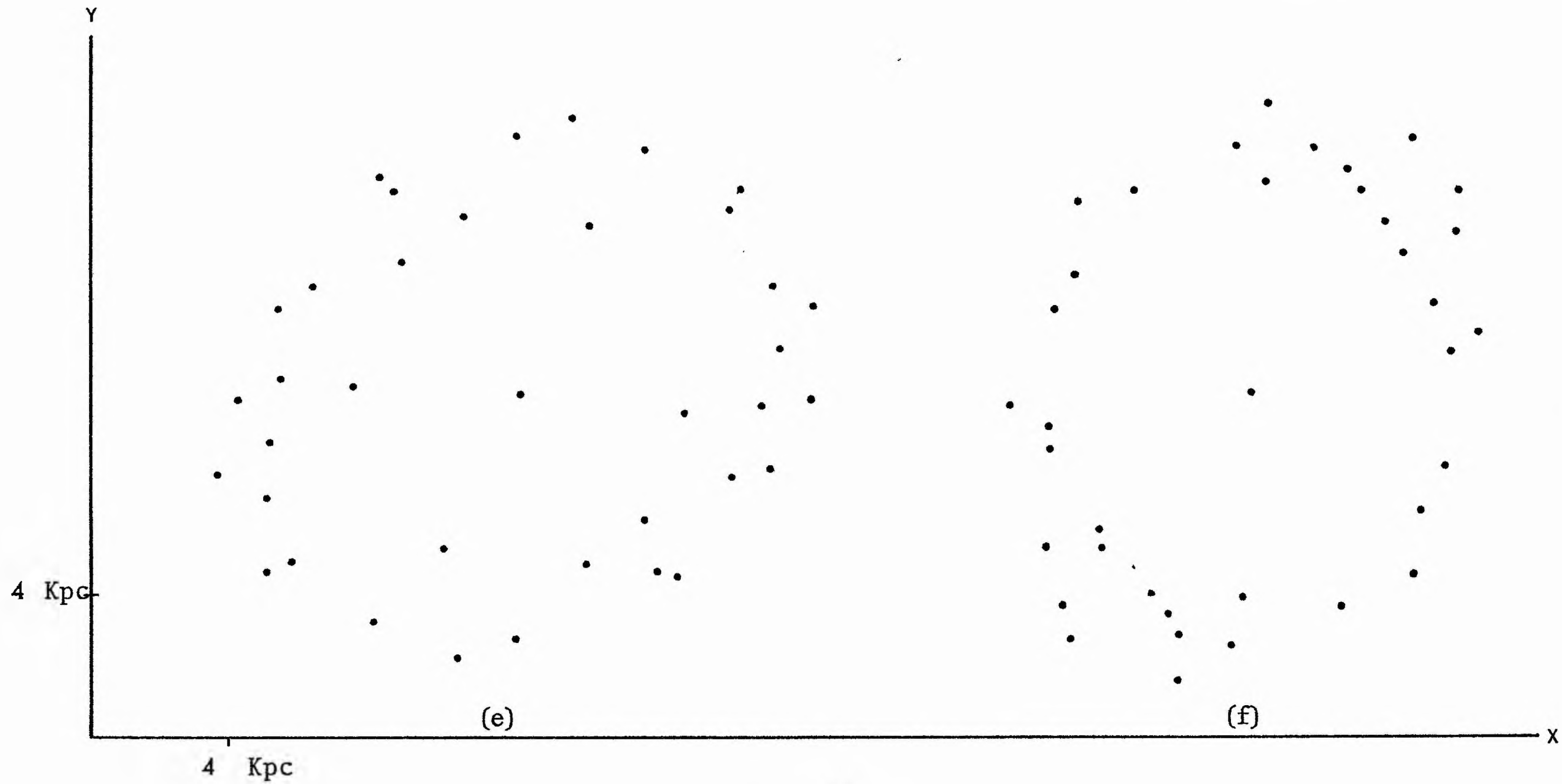
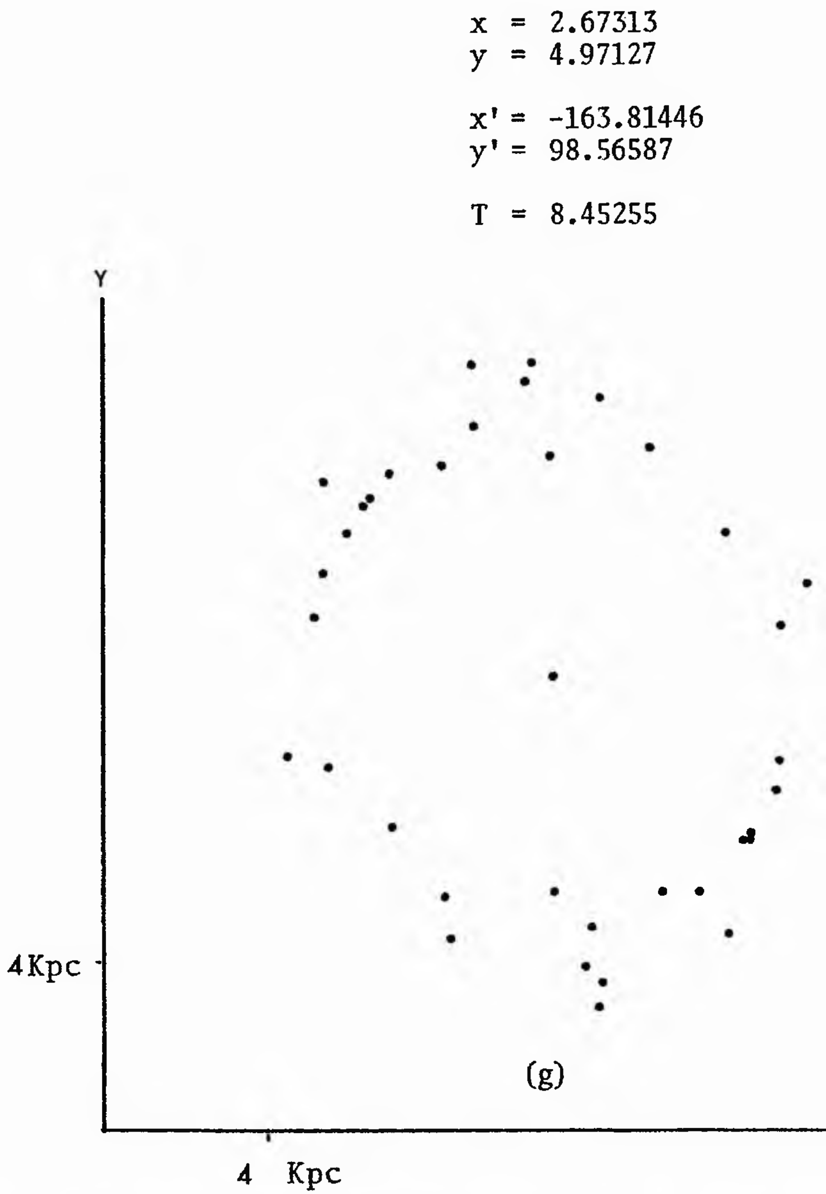


Figure 25

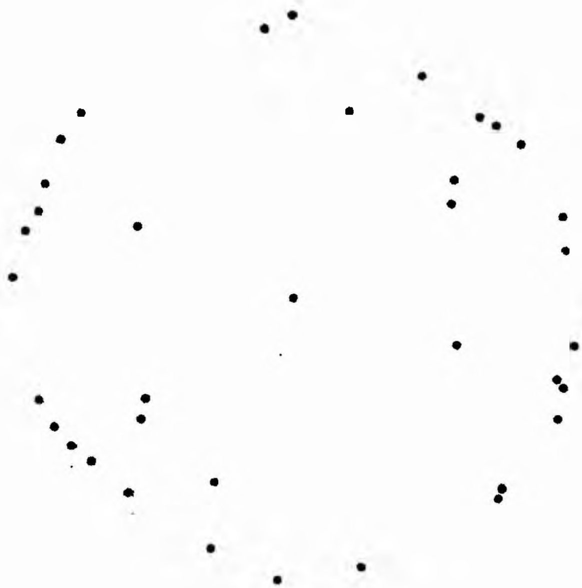

x = 2.67313
y = 4.97127
x' = -163.81446
y' = 98.56587
T = 8.45255
Y
4Kpc
(g)
4 Kpc

MODEL III

$x = 2.93644$
$y = 6.01310$

$x' = -197.38832$
$y' = 109.96612$

$T = 9.44970$



(h)

X

Figure 25

MODEL II_b

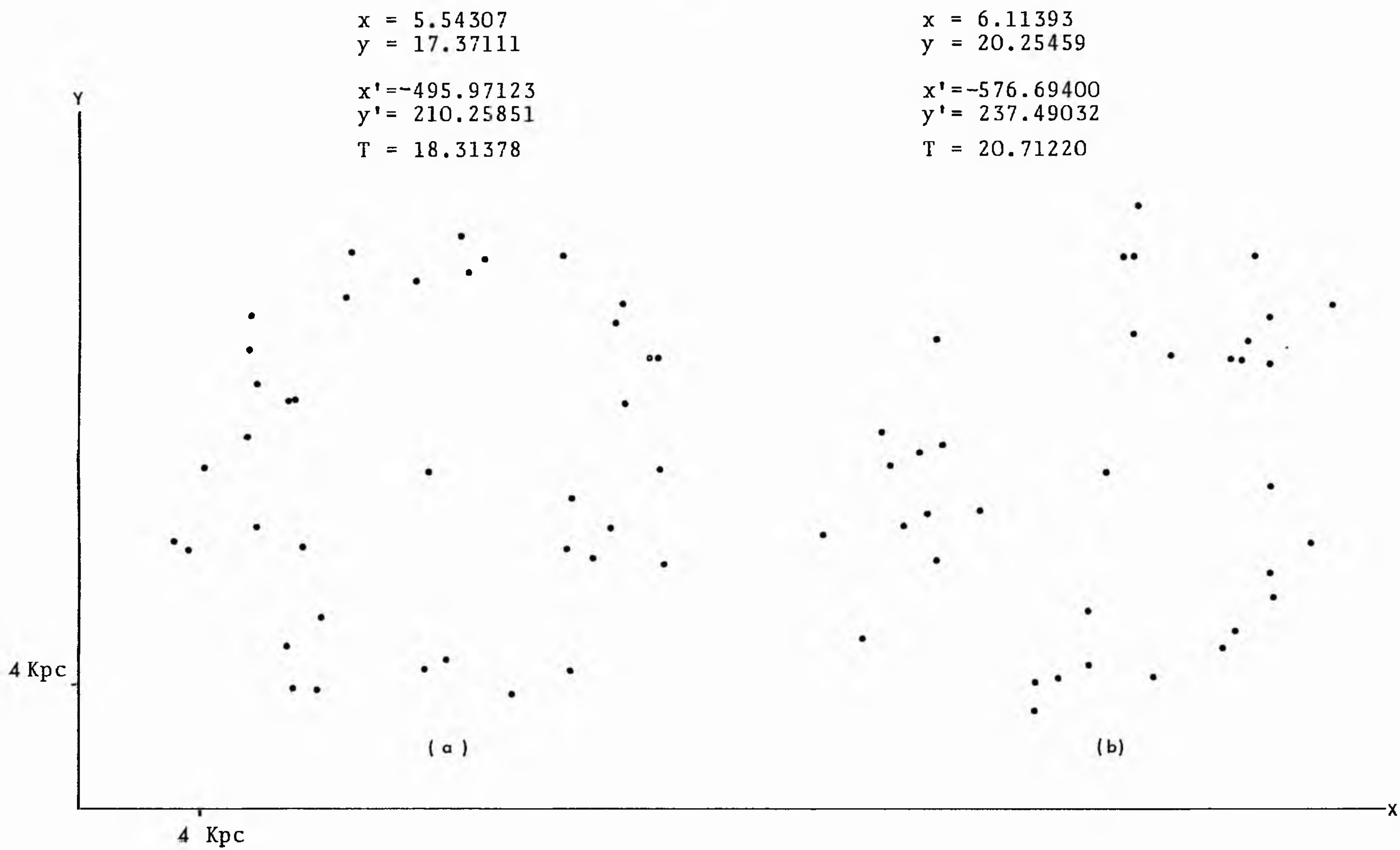


Figure 26

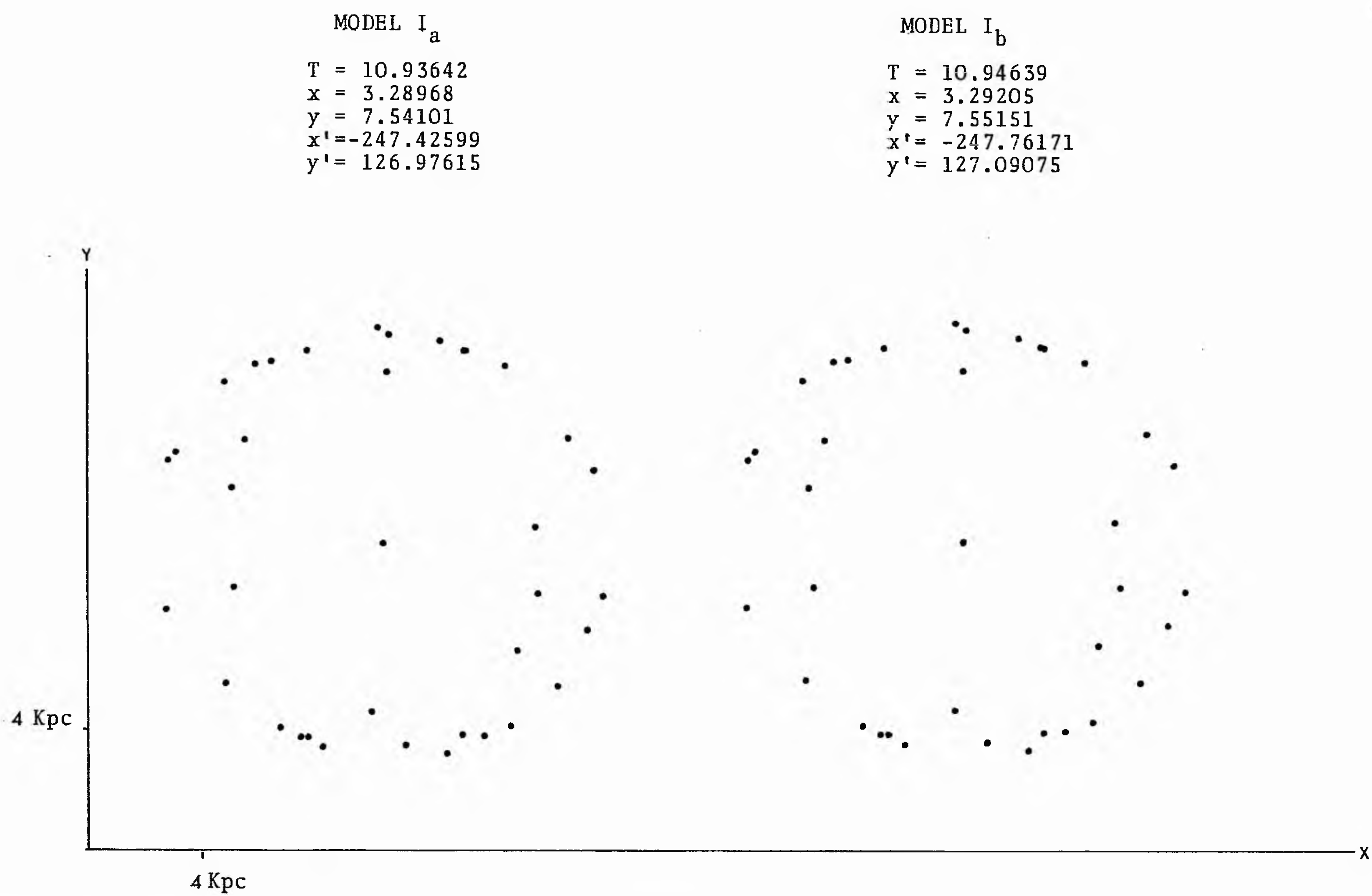

MODEL I_a
T = 10.93642
x = 3.28968
y = 7.54101
x'=-247.42599
y'= 126.97615
MODEL I_b
T = 10.94639
x = 3.29205
y = 7.55151
x'= -247.76171
y'= 127.09075
Y
X
4 Kpc
4 Kpc

Figure 27

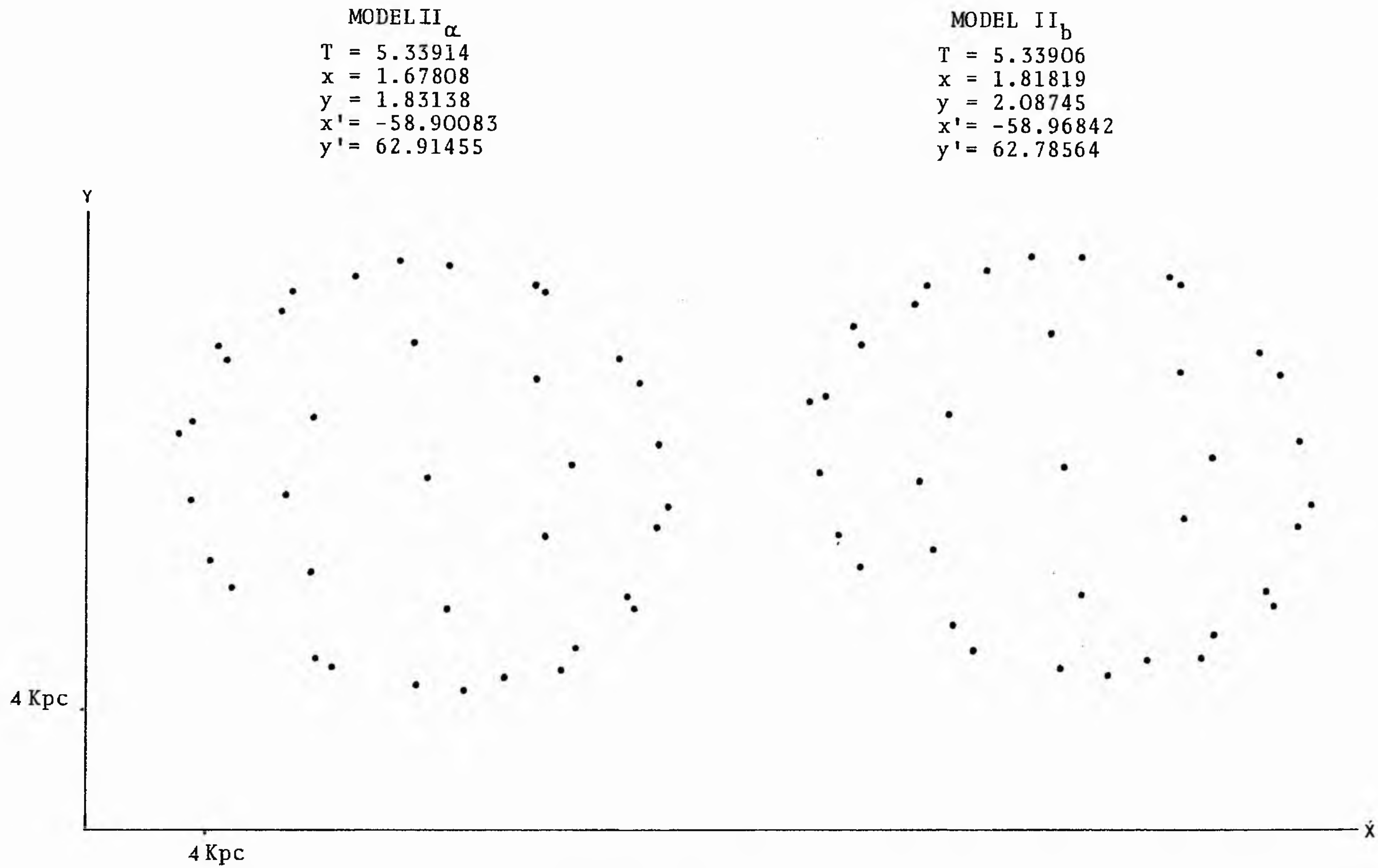

MODEL II$_{\alpha}$
T = 5.33914
x = 1.67808
y = 1.83138
x'= -58.90083
y'= 62.91455
MODEL II$_b$
T = 5.33906
x = 1.81819
y = 2.08745
x'= -58.96842
y'= 62.78564
Y
4 Kpc
4 Kpc
X

Figure 28

x = 1.69948
y = 1.88704
x' = -61.22148
y' = 63.71015
T = 5.40802

MODEL IV

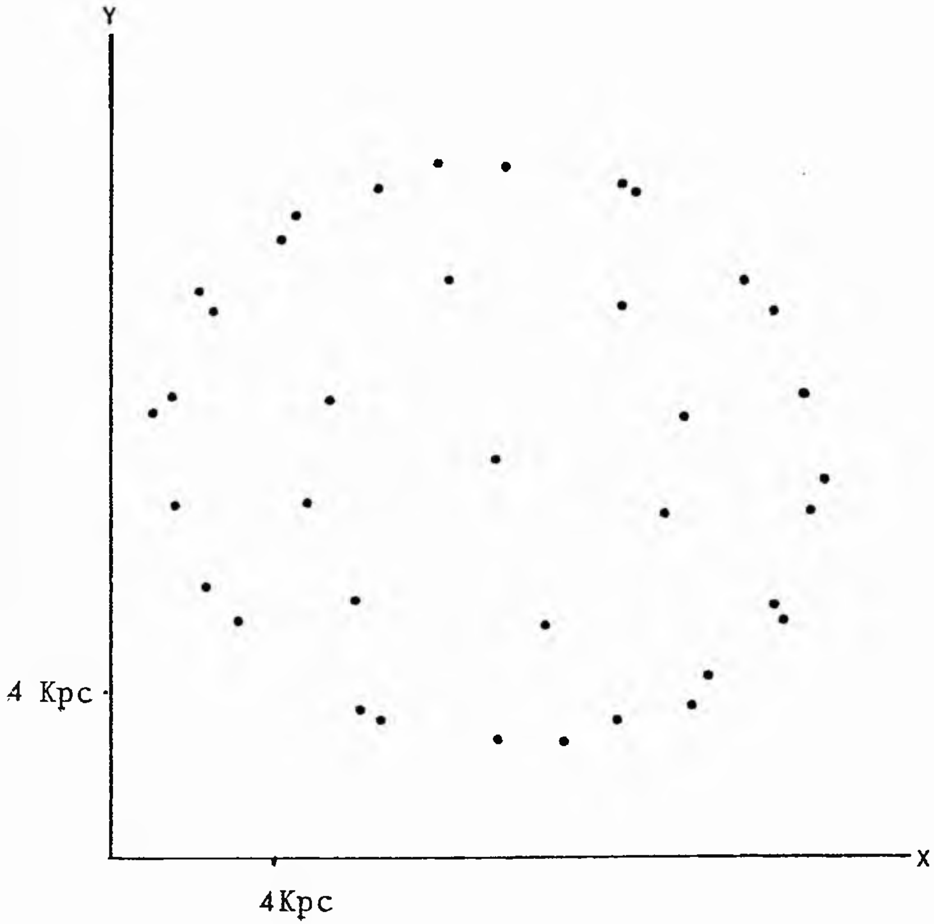


Figure 29

and evolutions are virtually identical with the physical states and evolutions of models Ib and IIb respectively, while the evolution of model IV is very similar to the evolution of model III (see figures 25d and 29).

We now proceed by considering what effect the close encounter of the masses m_{37} and m_{38} in model Ib has on the orbits of the individual particles of this model.

Figures (30) to (35), drawn relative to the centre of the mass m_{37}, graphically represent the functions

$$\left(x_i(\tau),\ y_i(\tau)\right) = \underset{\sim}{R}_i(\tau), \quad (i=25, 26, \cdots 36), \qquad (4.28)$$

where $x_i(\tau)$ and $y_i(\tau)$ are the co-ordinates of the i^{th} particle along the X and Y at the time τ. For the sake of clarity we do not draw these functions as continuous curves but rather as sets of points representing the value of the functions at the instants

$$\tau_n = n \cdot \Delta\tau, \quad n = 0, 1, \cdots\cdots 110,$$

where $\Delta\tau \simeq 2.0 \times 10^{-5}$ u.t. Thus the K^{th} $(0 \leqslant K \leqslant 110)$ point of the drawing, say (30a), is the projection on the XY co-ordinate plane of the position of the 25th particle at the instant $\tau_K = (2 \cdot K) \cdot 10^{-5}$ u.t.

Figures (30) to (35) also give the index assigned to each particle, the direction of the X and Y axis, the scale of the drawings and the position of the particles (indicated by arrows) at the time of the closest approach of the masses m_{37} and m_{38}. The results of the attraction exerted upon each particle by the perturbing mass m_{38} are at once obvious from these figures. The changes in the velocities of the particles - depending on their initial position - induce changes in the shape of the orbits and the periods of rotation of the particles.

Graphical representations of the functions (4.28) drawn relative to the origin of the reference system are given by figures (36) to (41). In these figures part of the orbit of the perturbing mass m_{38} is also illustrated. The position of this mass at the time of its closest approach to the mass m_{37} is indicated in the different drawings by encircled dots.

The changes in the orbits of the rest of the particles are entirely analogous to the changes in the orbits of the particles considered. The same is true for the particles of the rest of the models. Finally, the orbit in space of the mass m_{37} is given by figure (42).

During the evolution of the dynamical models under consideration the equatorial planes of their spheroidal bodies remain coplanar coinciding with the XY plane and except for a slight flattening of the perturbed systems no noticeable changes occur along the direction of the Z axis.

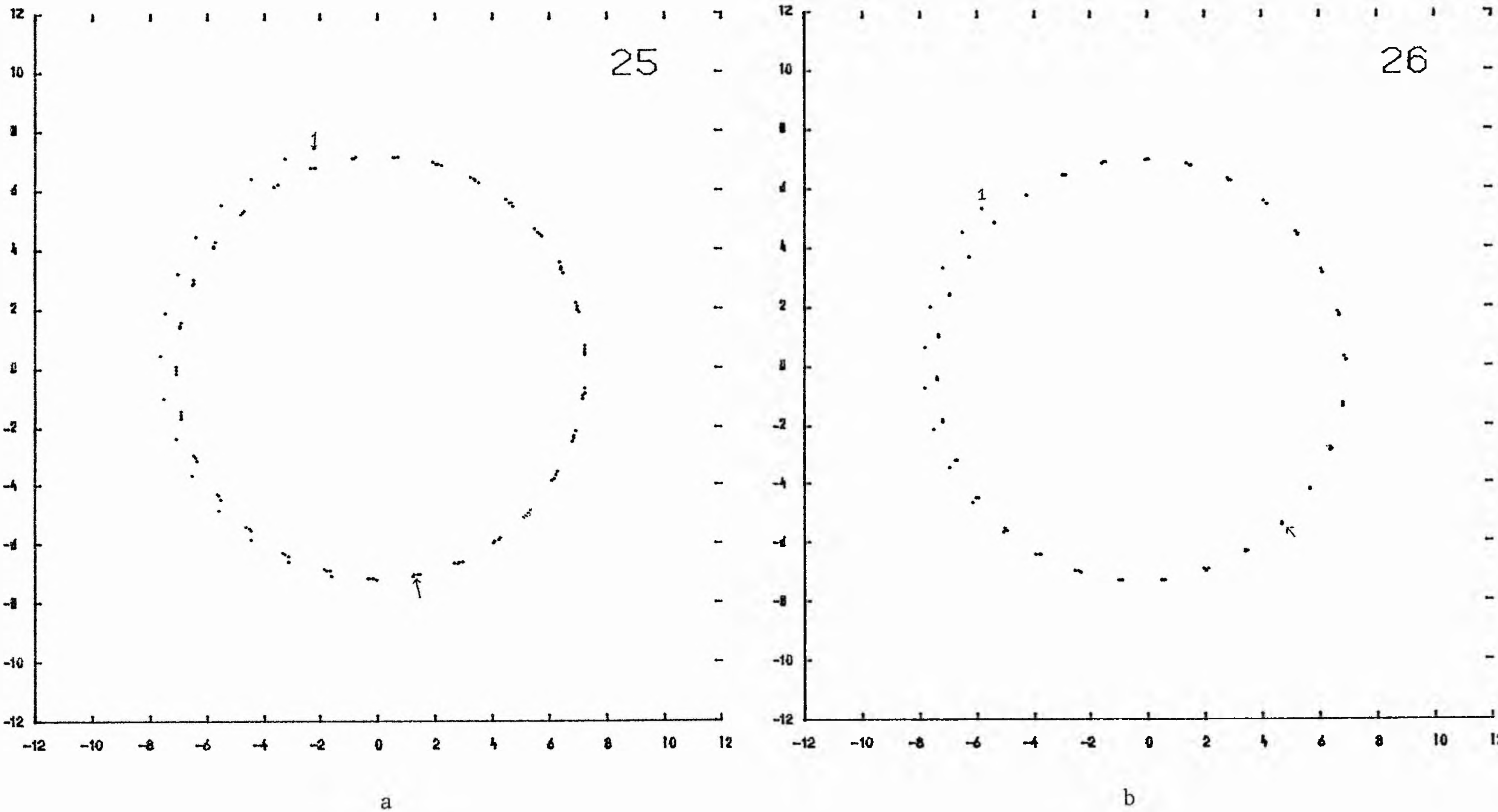
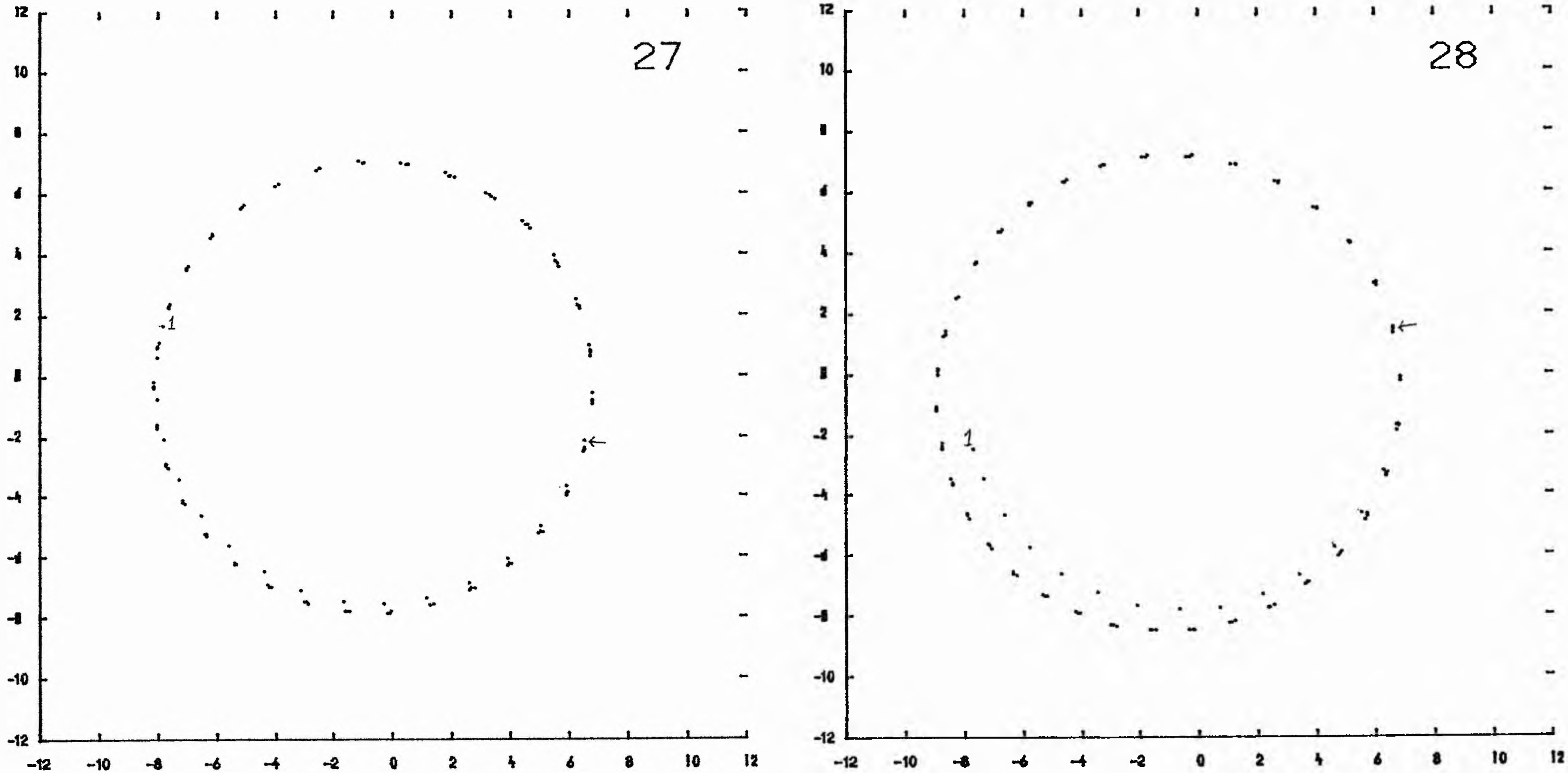
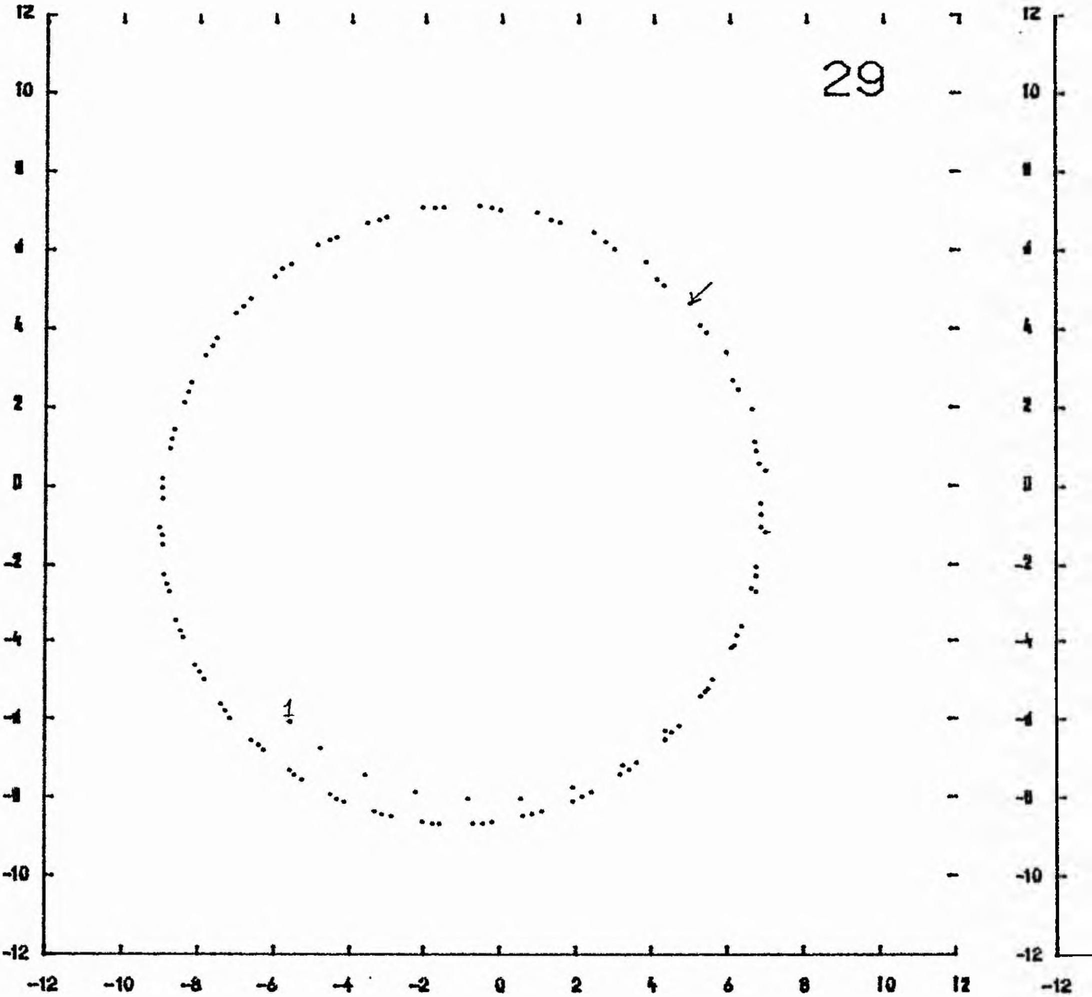


Figure 30



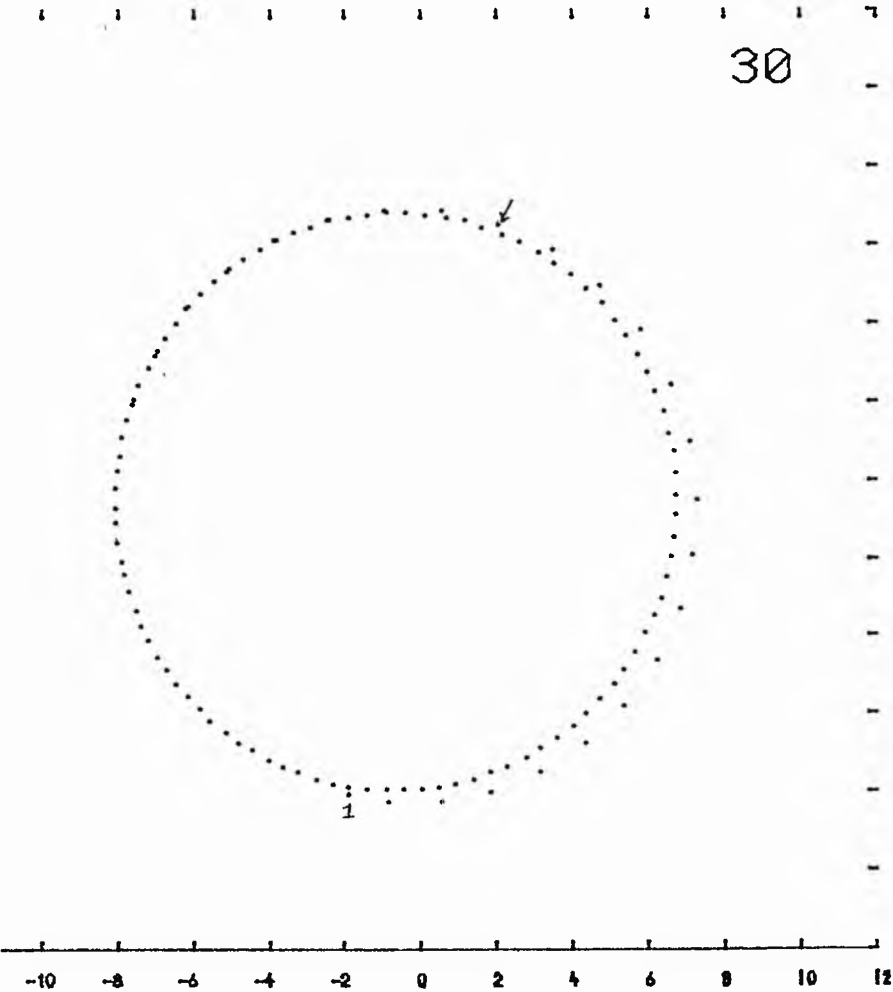
a b

Figure 31



a

Figure 32



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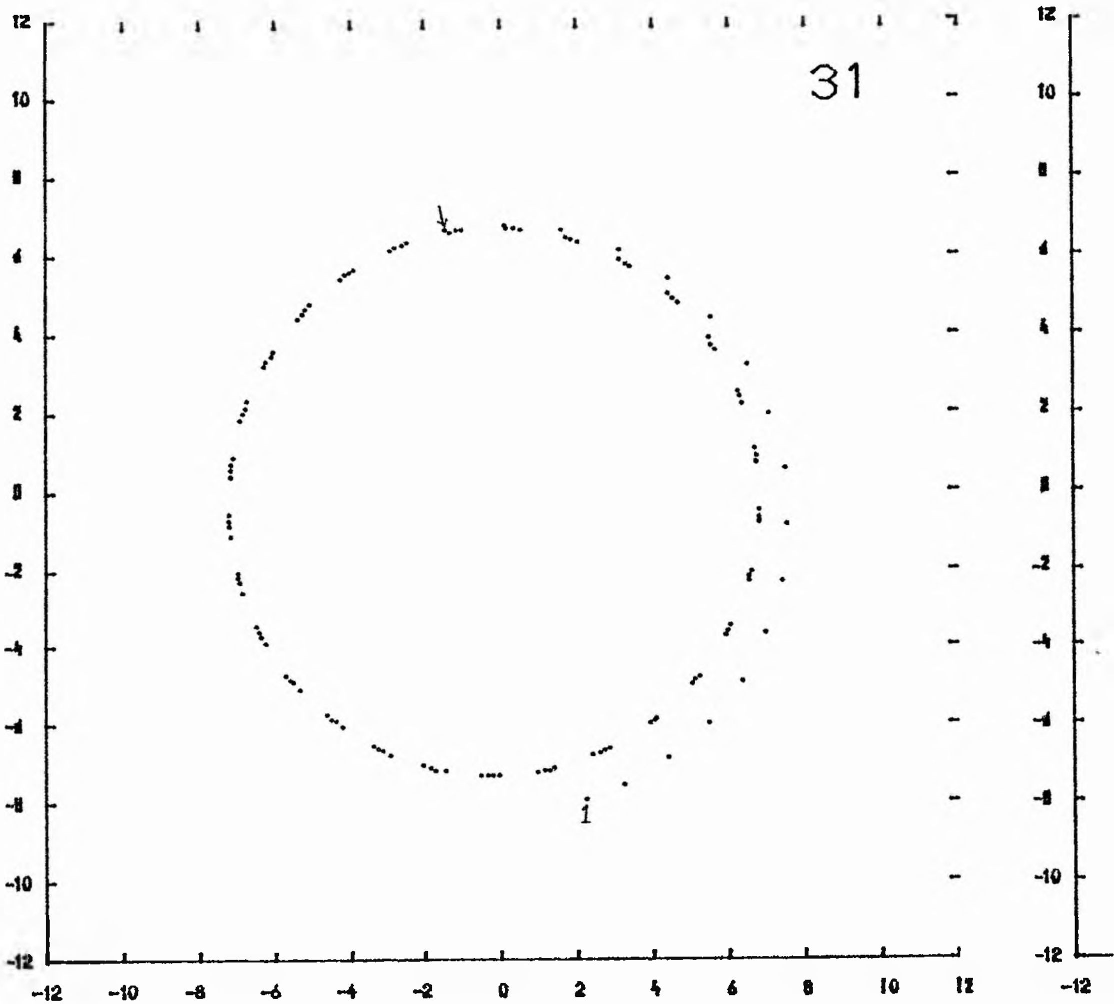
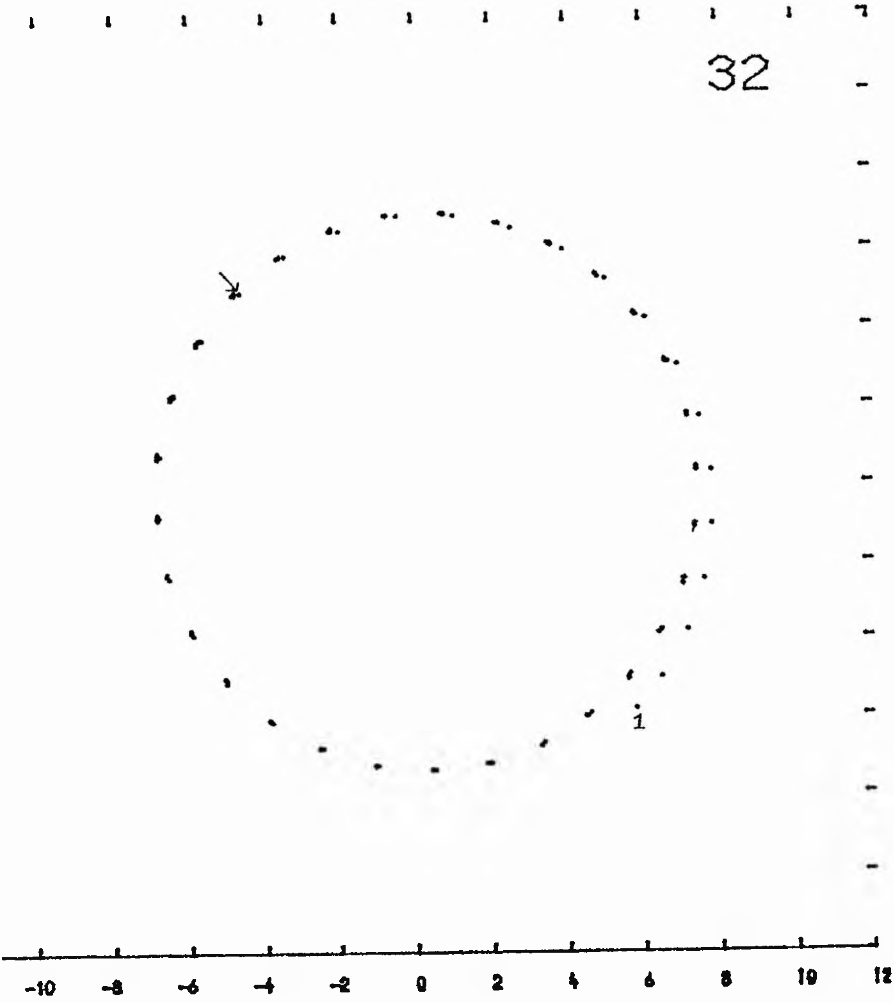


a

Figure 33



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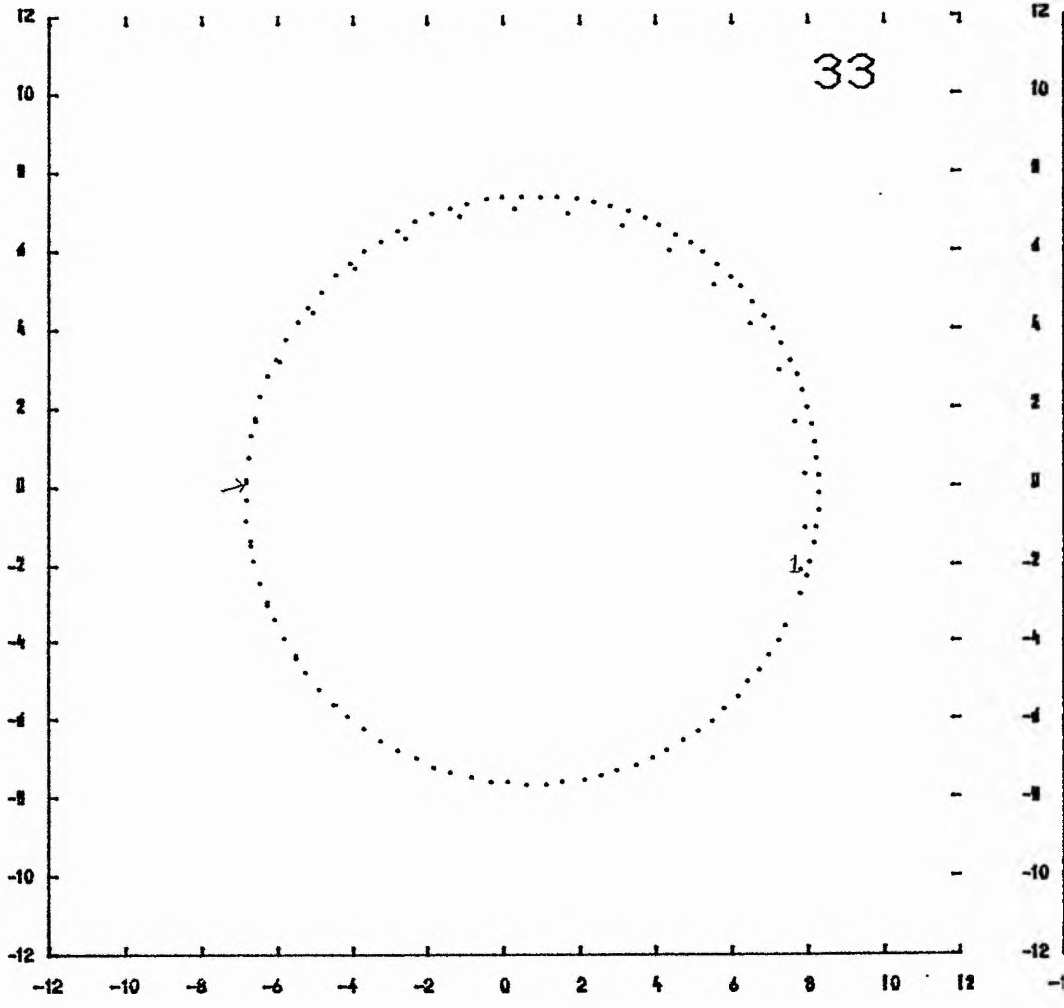
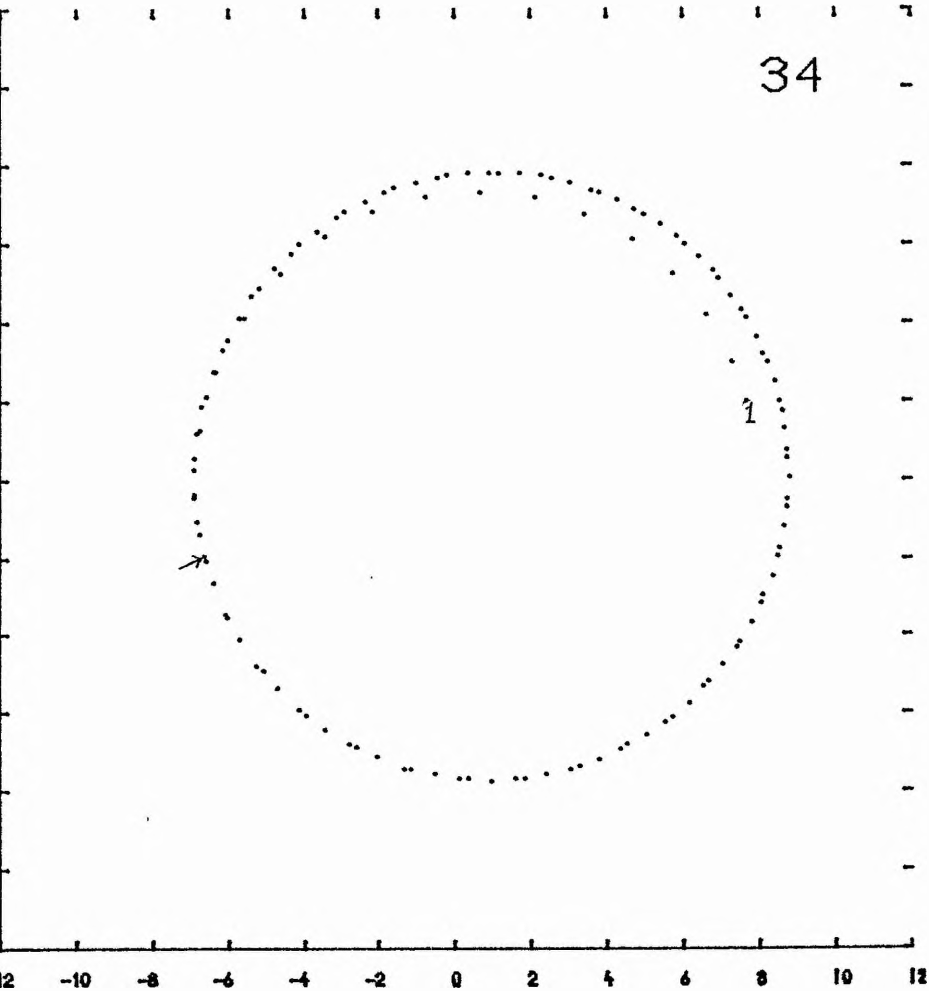


a

Figure 34



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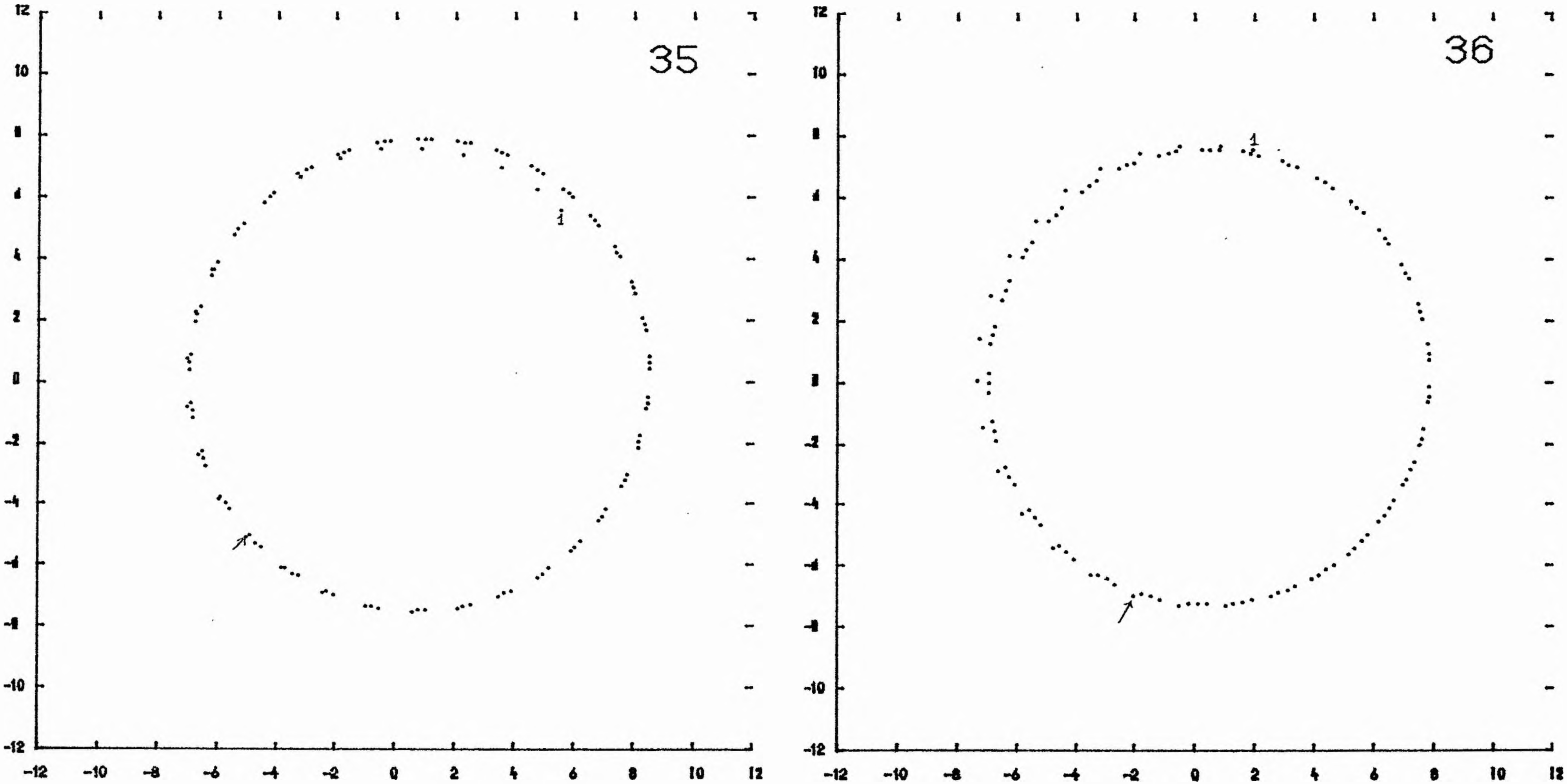
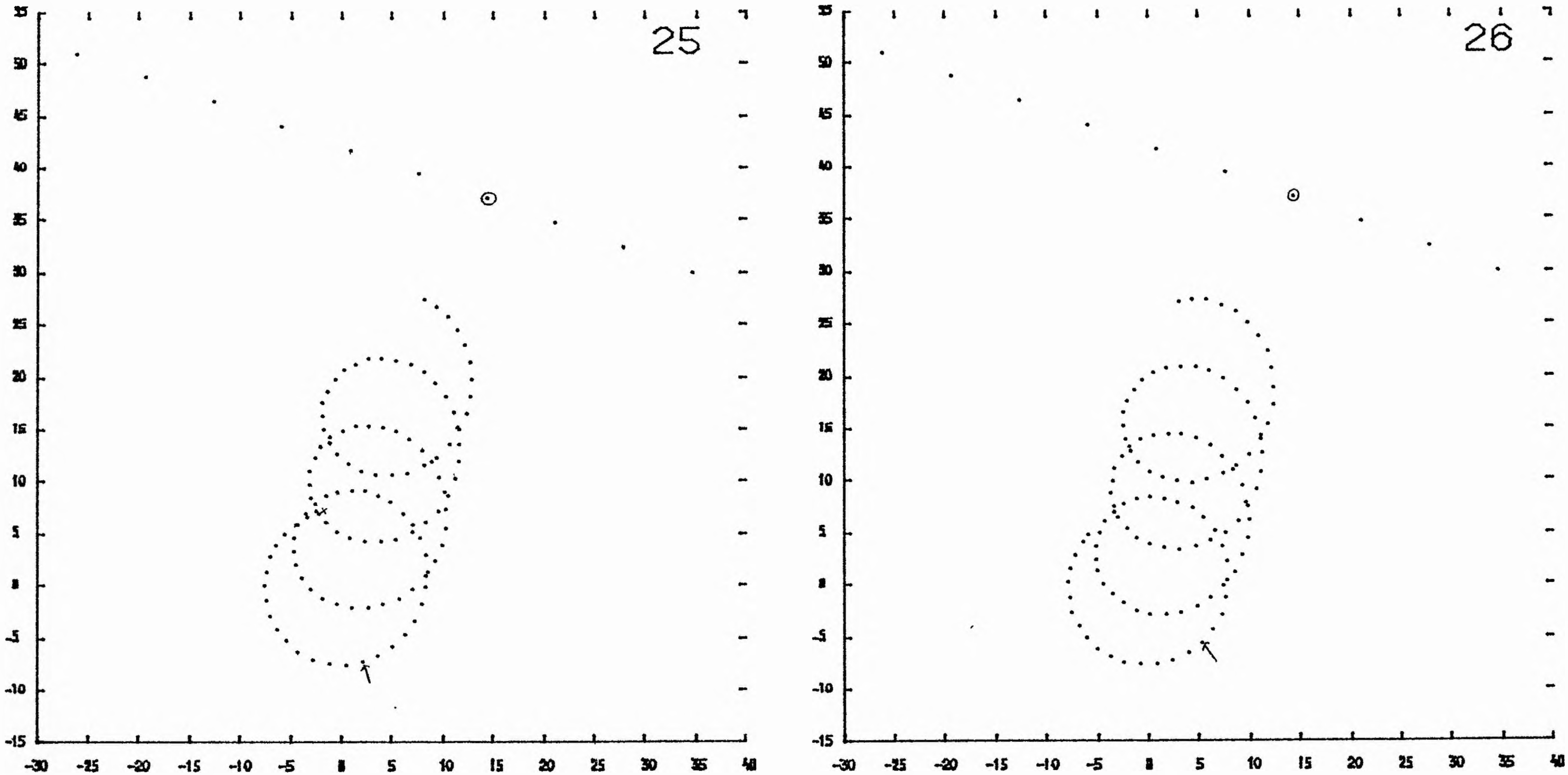


a b

Figure 35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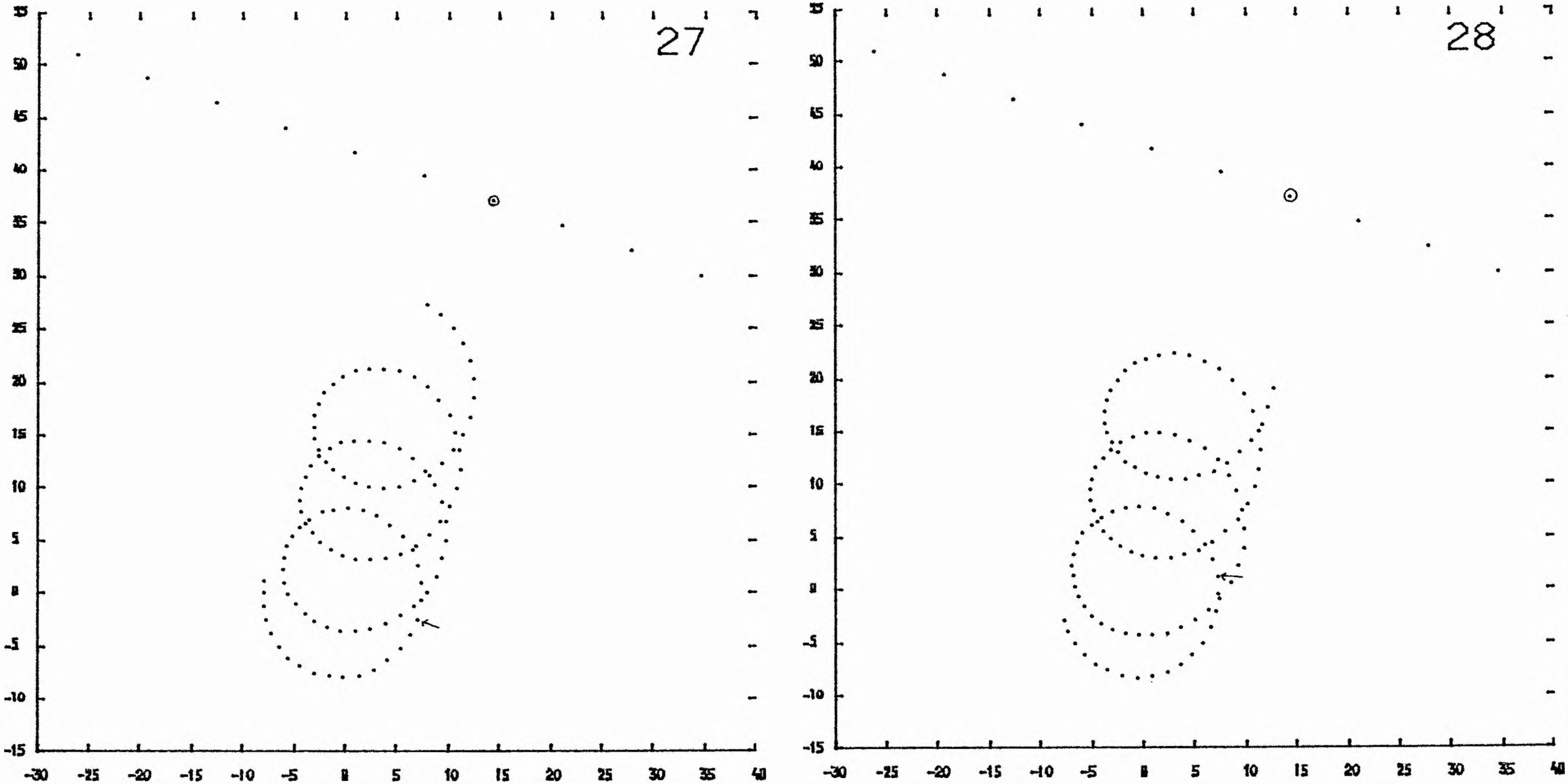
26
-30 -25 -20 -15 -10 -5 0 5 10 15 20 25 30 35 40
-15 -10 -5 0 5 10 15 20 25 30 35 40 45 50 55

a   b

Figure 36



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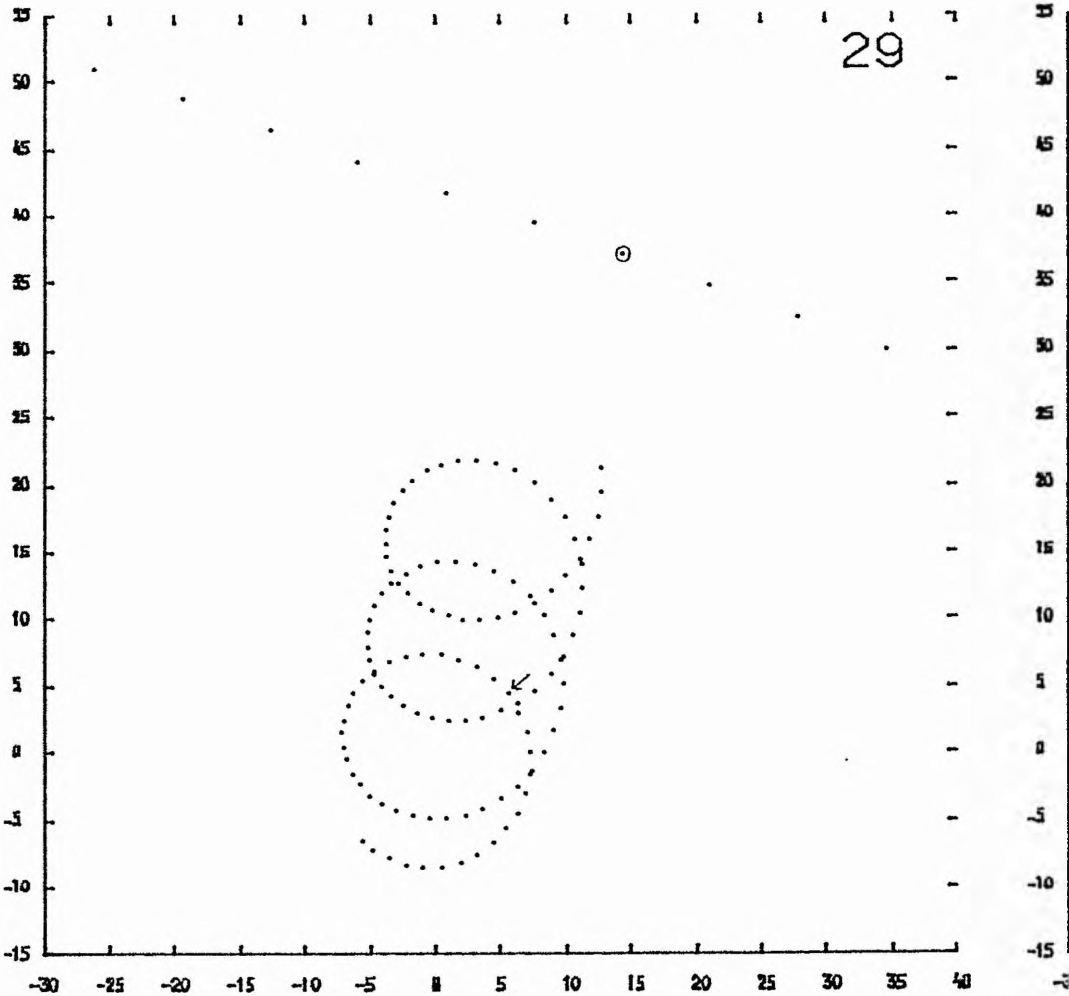
28
-30 -25 -20 -15 -10 -5 0 5 10 15 20 25 30 35 40
-15 -10 -5 0 5 10 15 20 25 30 35 40 45 50

a b

Figure 37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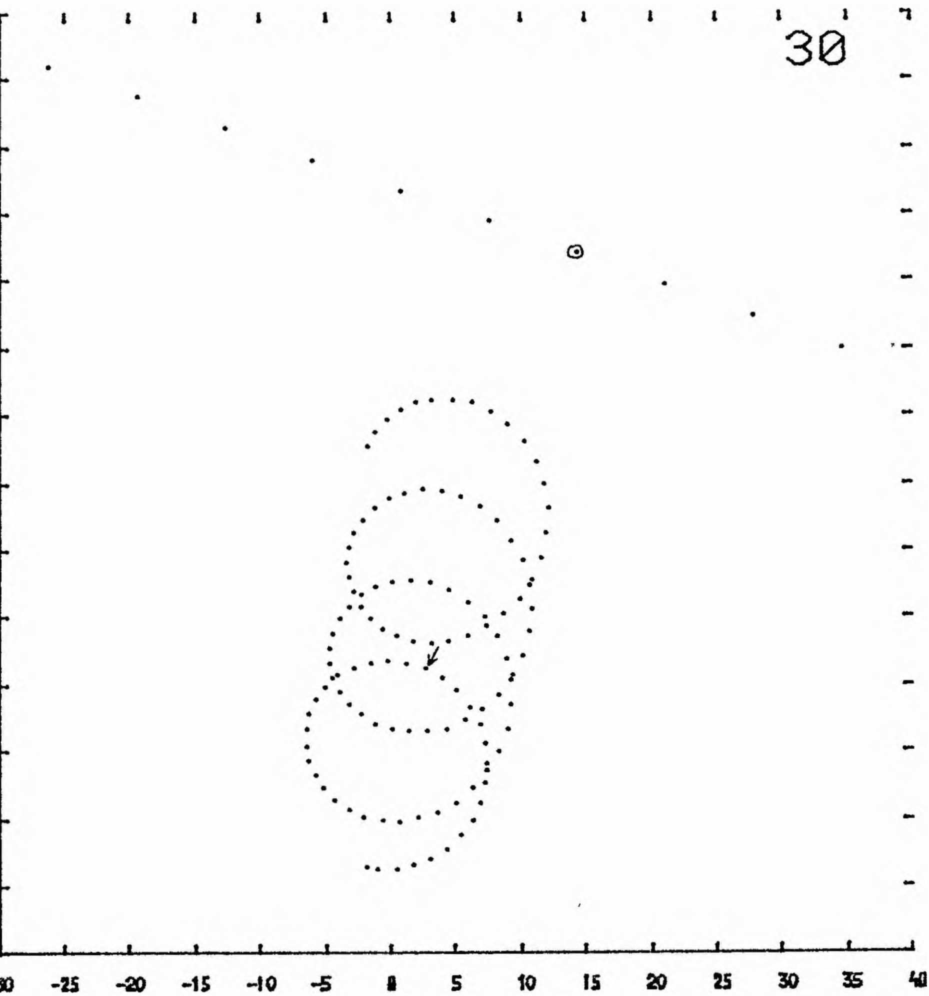
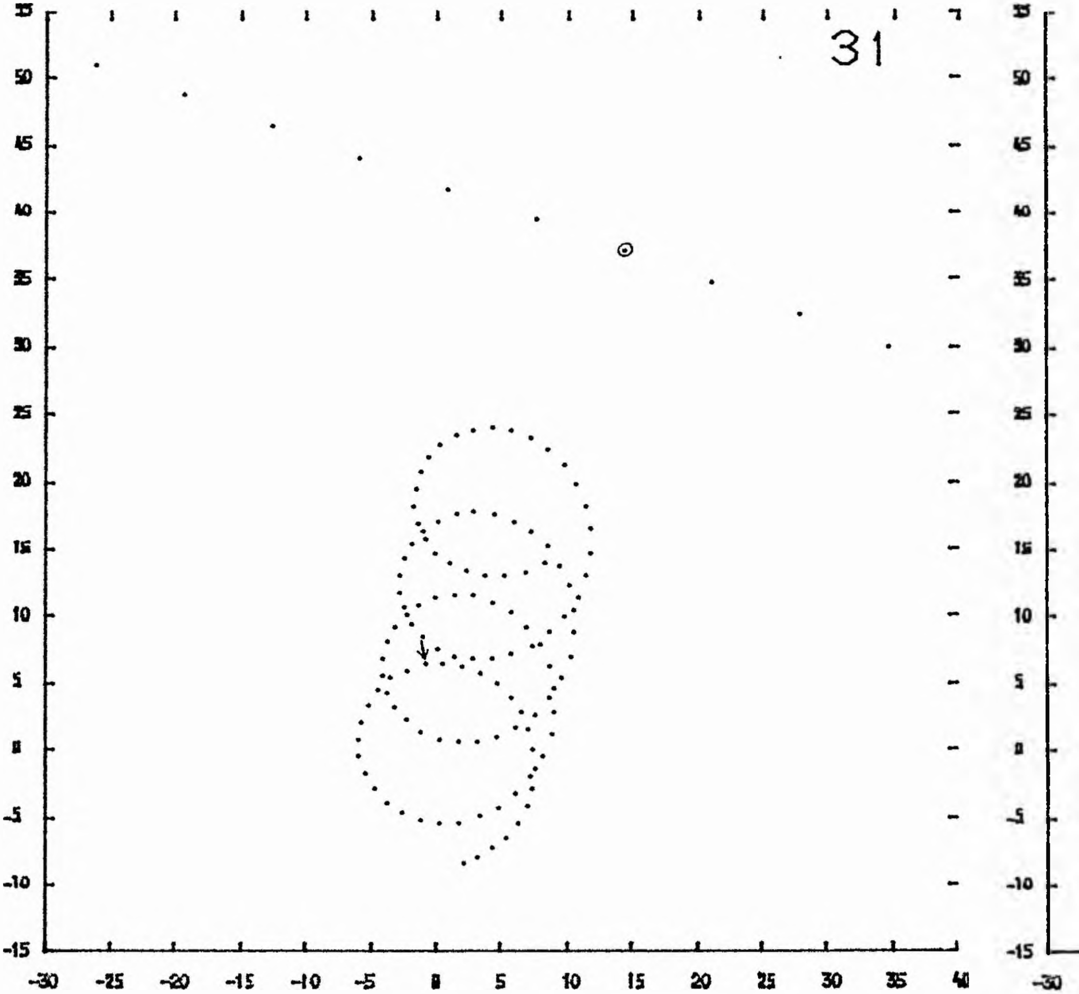
a

Figure 38



b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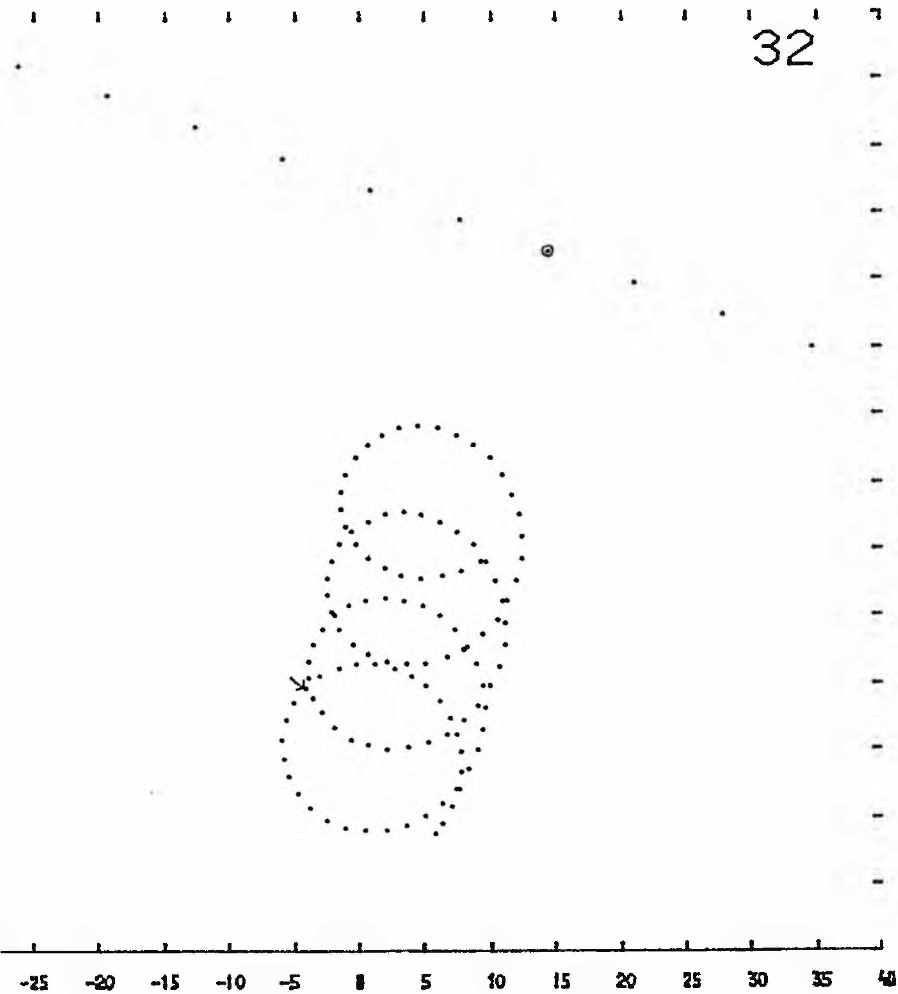
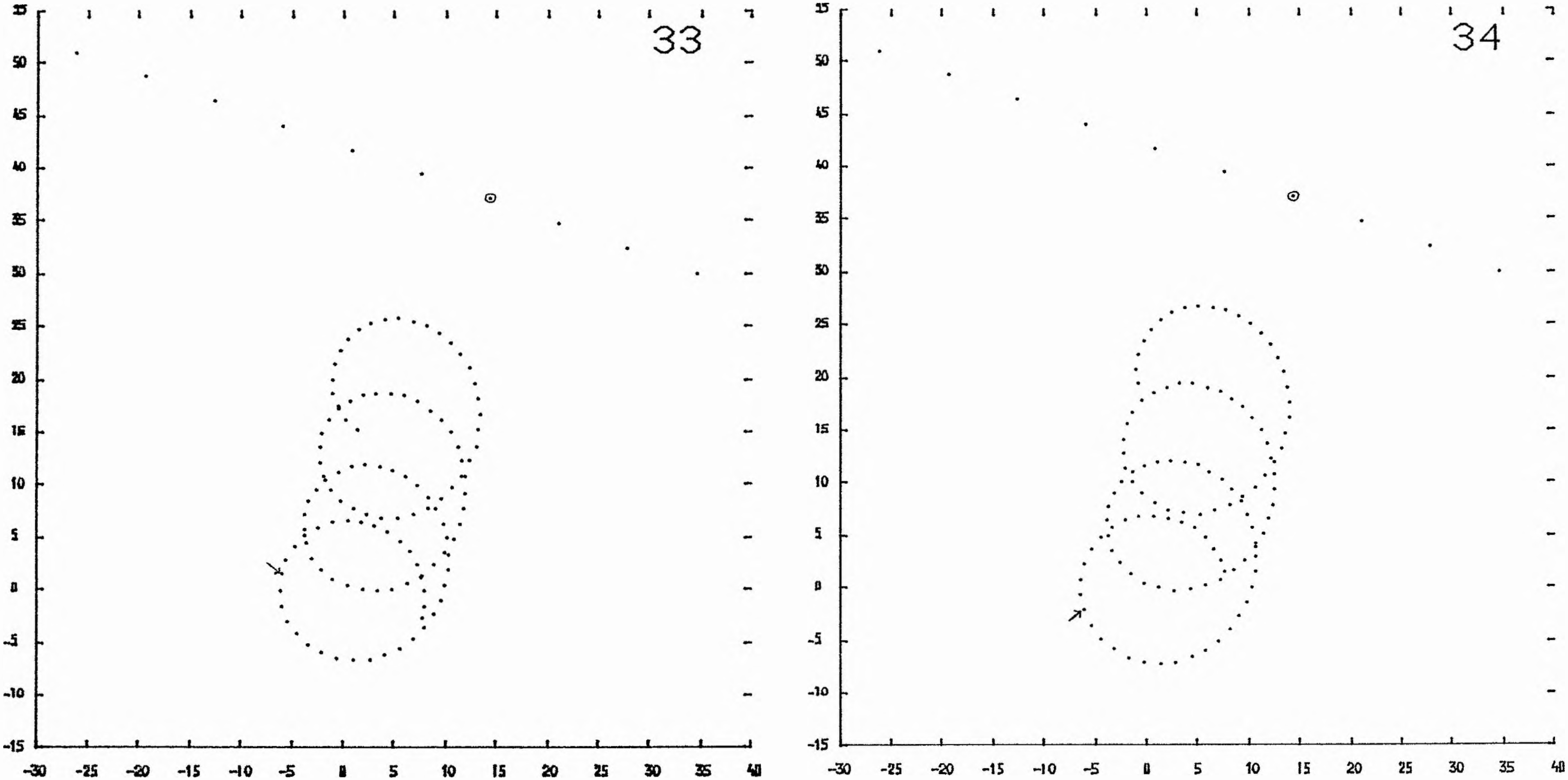
a

Figure 39



b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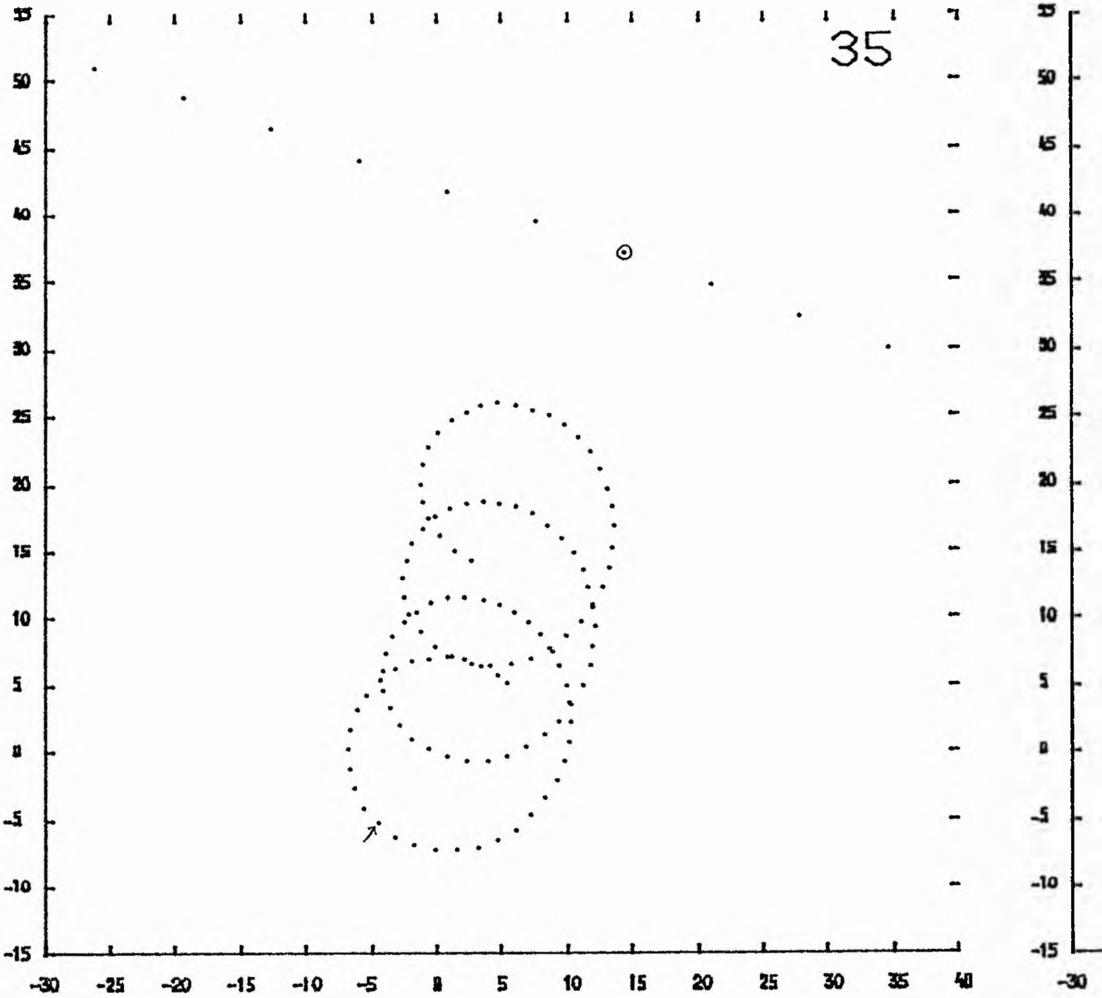
a

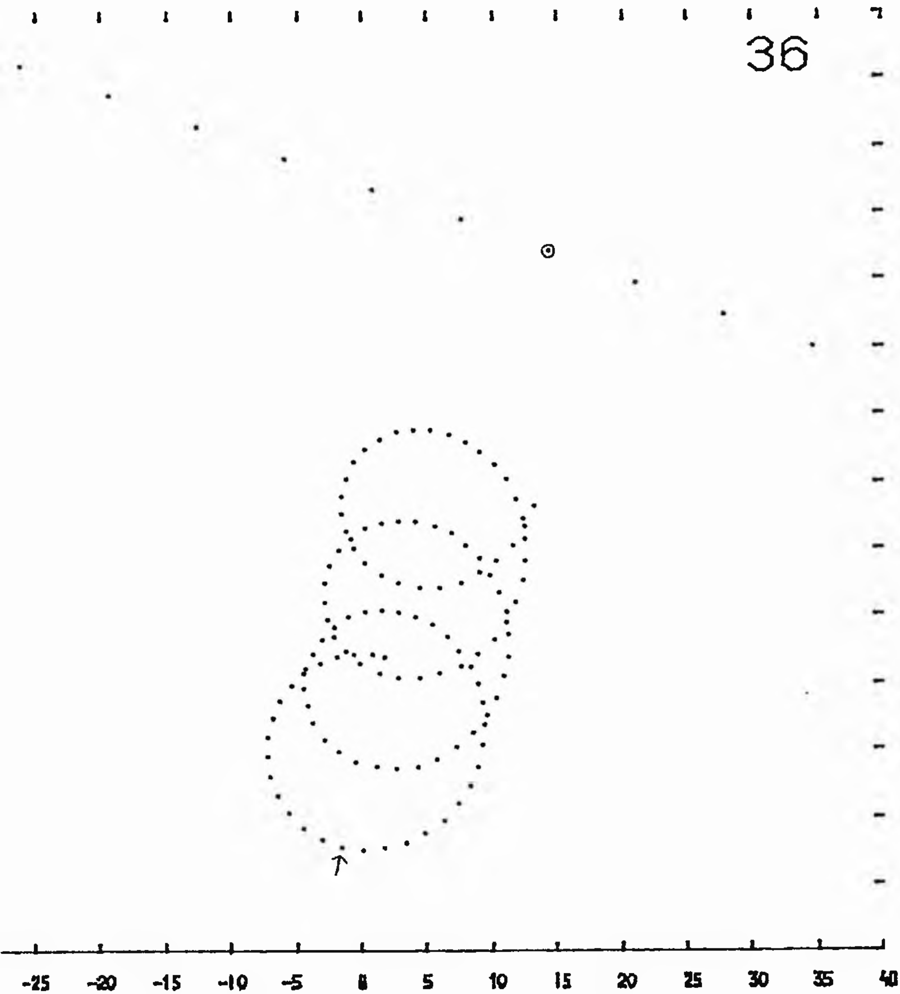
b

Figure 40



a

Figure 41



b

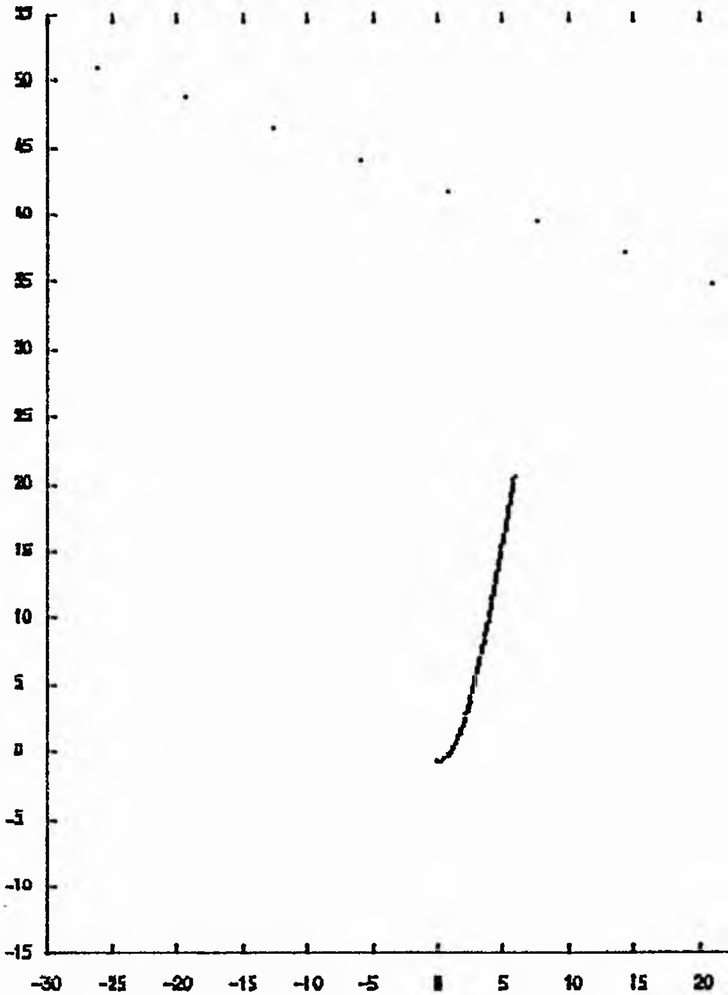


Figure 42


37
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APPENDIX A1

Length of the time step

It is obvious that the greatest disadvantage of methods using series for the numerical solution of the differential equations (1.6) is the restriction of the accuracy of the solution. A quantity directly related with this problem, and as a matter of fact determined by the desired accuracy of the integration, is the length of the time step.

We define the length of the time step by requiring that the difference between the true solution $\underset{\sim}{r}_i \equiv (x_i, y_i, z_i)$ $(i=1,\cdots n)$ and the approximate one $\underset{\sim}{r}_i^* \equiv (x_i^*, y_i^*, z_i^*)$ $(i=1,\cdots n)$ does not exceed a given limit at any step of the integration. Thus the length of the time step is implicitly specified by the relations

$$|\Delta x_i| \equiv |x_i - x_i^*| < A, \ |\Delta y_i| \equiv |y_i - y_i^*| < A, \ |\Delta z_i| \equiv |z_i - z_i^*| < A, \ (i=1,\cdots\cdots n), \quad \text{(A1-1)}$$

where A is a choosen limit.

The position vectors $\underset{\sim}{r}_i$ $(i=1,\cdots n)$ are, of course, unknown and therefore the best way to determine the length of the time step is to use a method based on bounding the remainder of the power series solution.

Let us consider the series

$$x_i = \sum_{\nu=0}^{\infty} x_{i0}^{(\nu)} \frac{\tau^\nu}{\nu!}, \ (i=1,\cdots\cdots n), \ \tau \in (-\tau^*, \tau^*). \quad \text{(A1-2)}$$

The simple convergence of this series secures its absolute convergence in the same open interval of time $(-\tau^*, \tau^*)$. Hence, according to Caushy's ratio test, from some place onwards in the series, say $\nu \gg N$

the following inequality will hold

$$\left|\frac{\dfrac{x_{io}^{(\nu+1)}\tau^{\nu+1}}{(\nu+1)!}}{\dfrac{x_{io}^{(\nu)}\tau^{\nu}}{\nu!}}\right| = \left|\frac{x_{io}^{(\nu+1)}\tau}{x_{io}^{(\nu)}(\nu+1)}\right| \leqslant \alpha < 1 \, , \nu \geqslant N \, , (i=1,\cdots n) \, , \tau \in (-\tau^*, \tau^*) .$$

Dividing both sides of the above inequality by 2 we obtain

$$\left|\frac{x_{io}^{(\nu+1)}\Upsilon}{x_{io}^{(\nu)}(\nu+1)}\right| \leqslant \alpha_* < \frac{1}{2} \, , \nu \geqslant N \, , (i=1,\cdots n) \, , \tau \in (-\tau^*, \tau^*) ,$$

where $\Upsilon \equiv \tau/2$ and $\alpha_* \equiv \alpha/2$. This inequality, by introduction of the conventional notation used up to this point, becomes

$$\left|\frac{x_{io}^{(\nu+1)}\tau}{x_{io}^{(\nu)}(\nu+1)}\right| \leqslant \alpha_* < \frac{1}{2} \, , \nu \geqslant N \, , (i=1,\cdots n) , \tau \in \left(-\frac{\tau^*}{2} \, , \frac{\tau^*}{2}\right) .$$

Application of this relation κ times in sequence results to the following series of inequalities

$$\left|\frac{x_{io}^{(\nu+\kappa)}\tau^{\nu+\kappa}}{(\nu+\kappa)!}\right| \leqslant \left|\frac{x_{io}^{(\nu+\kappa-1)}\tau^{\nu+\kappa-1}}{(\nu+\kappa-1)!}\right|\alpha_* \leqslant \left|\frac{x_{io}^{(\nu+\kappa-2)}\tau^{\nu+\kappa-2}}{(\nu+\kappa-2)!}\right|\alpha_*^2 \leqslant \cdots\cdots \leqslant \left|\frac{x_{io}^{(\nu)}\tau^{\nu}}{\nu!}\right|\alpha_*^{\kappa} ,$$

$$\nu \geqslant N \, , (i=1,\cdots n) \, , \tau \in (-\tau^*/2, \tau^*/2) \, , \kappa = 0,1,2,\cdots\cdots .$$

Comparing the first and the last term we readily conclude

$$\left|\frac{x_{io}^{(\nu+\kappa)}\tau^{\kappa}\nu!}{x_{io}^{(\nu)}(\nu+\kappa)!}\right| \leqslant \alpha_*^{\kappa} \quad \nu \geqslant N \, , (i=1,\cdots n) \, , \tau \in \left(-\frac{\tau^*}{2}, \frac{\tau^*}{2}\right) , \kappa = 0,1,2\cdots . \tag{A1-3}$$

The expansions of the function $x_i = x_i(\tau)$, $(i=1,\cdots n)$, $\tau \in (-\tau^*/2, \tau^*/2)$ in infinite power series and in finite sum with remainder have the form

$$x_i = x_{io} + x'_{io}\tau + x''_{io}\frac{\tau^2}{2!} + \cdots\cdots\cdots + x_{io}^{(\nu-1)}\frac{\tau^{\nu-1}}{(\nu-1)!} + \sum_{\kappa=0}^{\infty} x_{io}^{(\nu+\kappa)}\frac{\tau^{\nu+\kappa}}{(\nu+\kappa)!} ,$$

$$x_i = x_{io} + x'_{io}\tau + x''_{io}\frac{\tau^2}{2!} + \cdots\cdots\cdots + x_{io}^{(\nu-1)}\frac{\tau^{\nu-1}}{(\nu-1)!} + x_i^{(\nu)}(\theta\tau)\frac{\tau^{\nu}}{\nu!} ,$$

$$\nu \geqslant N, \, (i=1,\cdots n) \, , \tau \in \left(-\frac{\tau^*}{2}, \frac{\tau^*}{2}\right) ,$$

where $x_i^{(v)}(\theta\tau)\,\tau^v/v!$ is the remainder and θ satisfies the relation $\theta\in(0,1)$. Comparison of the foregoing two equations yields

$$x_i^{(v)}(\theta\tau)=\sum_{K=0}^{\infty}\frac{x_{io}^{(v+K)}\,\tau^K\,v!}{(v+K)!},\quad v\geq N,\ (i=1,\cdots n),\ \tau\in(-\tau^*/2,\tau^*/2).$$

This is equivalent to the following equation

$$\frac{x_i^{(v)}(\theta\tau)-x_{io}^{(v)}}{x_{io}^{(v)}}=\sum_{K=1}^{\infty}\frac{x_{io}^{(v+K)}\,\tau^K\,v!}{x_{io}^{(v)}(v+K)!},\quad v\geq N,\ (i=1,\cdots n),\tau\in(-\tau^*/2,\tau^*/2),$$

which, upon substitution of inequality (A1-3), admits the relation

$$\left|\frac{x_i^{(v)}(\theta\tau)-x_{io}^{(v)}}{x_{io}^{(v)}}\right|\leq\sum_{K=1}^{\infty}\left|\frac{x_{io}^{(K+v)}\tau^K\,v!}{x_{io}^{(v)}(v+K)!}\right|\leq\sum_{K=1}^{\infty}\alpha_*^K=-1+\sum_{K=0}^{\infty}\alpha_*^K\quad v\geq N,\ (i=1,\cdots\cdots n),$$

$$\tau\in(-\tau^*/2,\tau^*/2),$$

and since $\alpha_*<\frac{1}{2}$ this relation becomes

$$\left|\frac{x_i^{(v)}(\theta\tau)-x_{io}^{(v)}}{x_{io}^{(v)}}\right|\leq-1+\frac{1}{1-\alpha_*}=\frac{\alpha_*}{1-\alpha_*},\alpha_*<\frac{1}{2},\ v\geq N,(i=1,\cdots n),\tau\in(-\frac{\tau^*}{2},\frac{\tau^*}{2}).\quad\text{(A1-4)}$$

We now return to the evaluation of the length of the time step. In the numerical integration we truncate the power series solution after its v^{th} term, introducing in this way an error given by the equation

$$x_i-x_i^*=\left[\sum_{K=0}^{v-1}x_{io}^{(K)}\frac{\tau^K}{K!}+x_i^{(v)}(\theta\tau)\frac{\tau^v}{v!}\right]-\left[\sum_{K=0}^{v-1}x_{io}^{(K)}\frac{\tau^K}{K!}+x_{io}^{(v)}\frac{\tau^v}{v!}\right]=$$

$$=\left(x_i^{(v)}(\theta\tau)-x_{io}^{(v)}\right)\frac{\tau^v}{v!},\ v\geq N,(i=1,\cdots\cdots n),\tau\in(-\frac{\tau^*}{2},\frac{\tau^*}{2}),$$

which, by introduction of inequality (A1-4), becomes

$$|\Delta x_i|=|x_i-x_i^*|\leq\left|\frac{x_{io}^{(v)}\tau^v}{v!}\right|\frac{\alpha_*}{1-\alpha_*},\alpha_*<\frac{1}{2},\ v\geq N,(i=1,\cdots n),\ \tau\in(-\tau^*/2,\tau^*/2).\quad\text{(A1-5)}$$

The expression $\alpha^*/(1-\alpha^*)$ has positive first order derivative, with respect to α_* everywhere in the interval $\left[0,\frac{1}{2}\right)$ and therefore it is absolutely increasing, obeying the relation

$$\frac{\alpha_*}{1-\alpha_*} < \frac{1/2}{1-1/2} = 1 \quad \alpha_* \in \left[0,\frac{1}{2}\right).$$

Inequality (A1-5), upon substitution of the above relation, becomes

$$|\Delta x_i| < \left| \frac{x_{io}^{(\nu)}\tau^\nu}{\nu!} \right| , \quad \nu \geqslant N , \quad (i=1,\cdots\cdots n) , \quad \tau \in \left(-\frac{\tau^*}{2},\frac{\tau^*}{2}\right). \tag{A1-6}$$

Comparing the foregoing inequality with the first of inequalities (A1-1) we readily conclude that a sufficient condition for the validity of inequality $|\Delta x_i| < A$ is the following relation

$$\left| \frac{x_{io}^{(\nu)}\tau^\nu}{\nu!} \right| \leqslant A , \quad \nu \geqslant N , \quad (i=1,\cdots\cdots n), \quad \tau \in \left(-\frac{\tau^*}{2},\frac{\tau^*}{2}\right). \tag{A1-6a}$$

Although relations of this form permit the evaluation of the length of the time step, it seems, it is not sufficient to treat the error limit A as constant, we therefore introduce some dependence of A on the stage of motion of the stars by requiring that the relative error

$$e_{ix} \equiv \left| \frac{\Delta x_i}{x_{io}} \right| , \quad (i=1,\cdots\cdots n), \tag{A1-7}$$

in each step of the integration and for every star, satisfies the relation

$$e_{ix} \leqslant \frac{1}{\mu} , \quad (i=1,\cdots\cdots n) , \tag{A1-8}$$

where μ is a constant however big. Combining the foregoing two relations we obtain

$$|\Delta x_i| \leqslant \frac{|x_{io}|}{\mu} , \quad (i=1,\cdots\cdots n) , \quad \tau \in (-\tau^*/2 , \tau^*/2) , \quad \mu > 0 . \tag{A1-9}$$

Comparing inequalities (A1-6) and (A1-9) we find that the relation (A1-9), and therefore the relation (A1-8), holds for every value of the variable τ obeying the restriction

$$|\tau| \ll \sqrt[\nu]{\left|\frac{x_{io}\,\nu!}{x_{io}^{(\nu)}\,\mu}\right|}\;, \quad \nu \gg N\,, \ (i=1,\cdots\cdots n)\,, \ \tau \in (-\tau^*/2\,, \tau^*/2)\,, \ \mu > 0\,.$$

By an entirely analogous process we can prove that the following relations

$$|\tau| \ll \sqrt[\nu]{\left|\frac{y_{io}\,\nu!}{y_{io}^{(\nu)}\,\mu}\right|}\;,$$

$$\nu \gg N\,, \quad (i=1,\cdots\cdots n)\,, \quad \tau \in (-\tau^*/2\,, \tau^*/2)\,, \ \mu > 0\,.$$

$$|\tau| \ll \sqrt[\nu]{\left|\frac{z_{io}\,\nu!}{z_{io}^{(\nu)}\,\mu}\right|}\;,$$

are sufficient conditions for the validity of the inequalities

$$e_{iy} \equiv \left|\frac{\Delta y_i}{y_{io}}\right| \ll \frac{1}{\mu}\,, \ e_{iz} \equiv \left|\frac{\Delta z_i}{z_{io}}\right| \ll \frac{1}{\mu}\,, \quad (i=1,\cdots\cdots n)\,, \ \mu > 0\,. \qquad \text{(A1-10)}$$

Imposing the following two limitations on the values of τ

$$|\tau| \ll \min\left\{\sqrt[\nu]{\left|\frac{x_{io}\,\nu!}{x_{io}^{(\nu)}\,\mu}\right|}\,, \sqrt[\nu]{\left|\frac{y_{io}\,\nu!}{y_{io}^{(\nu)}\,\mu}\right|}\,, \sqrt[\nu]{\left|\frac{z_{io}\,\nu!}{z_{io}^{(\nu)}\,\mu}\right|}\,, (i=1,\cdots\cdots n)\,, \nu \gg N\,, \mu > 0\right\}\,,$$

$$\tau \in (-\tau^*/2\,, \tau^*/2)\,,$$

one secures the simultaneous validity of inequalities (A1-8) and (A1-10), in each step of the numerical integration.

The interval of convergence $(-\tau^*, \tau^*)$ is not precisely known, we therefore evaluate the length of the time step using the relation

$$|\tau| = \min\left\{ \sqrt[\nu]{\left|\frac{x_{io}\,\nu!}{x_{io}^{(\nu)}\mu}\right|},\ \sqrt[\nu]{\left|\frac{y_{io}\,\nu!}{y_{io}^{(\nu)}\mu}\right|},\ \sqrt[\nu]{\left|\frac{z_{io}\,\nu!}{z_{io}^{(\nu)}\mu}\right|},\ \sqrt[\nu]{\left|\frac{x'_{io}\,(\nu-1)!}{x_{io}^{(\nu)}\mu}\right|},\ \sqrt[\nu]{\left|\frac{y'_{io}\,(\nu-1)!}{y_{io}^{(\nu)}\mu}\right|},\right.$$

$$\left. \sqrt[\nu]{\left|\frac{z'_{io}\,(\nu-1)!}{z_{io}^{(\nu)}\mu}\right|},\ (i=1,\cdots\cdots n),\ \nu \gg N,\ \mu>0 \right\}. \qquad \text{(A1-11)}$$

This equation secures the validity of the following inequalities

$$\left|\frac{\Delta x_i}{x_{io}}\right| \leqslant \frac{1}{\mu},\ \left|\frac{\Delta y_i}{y_{io}}\right| \leqslant \frac{1}{\mu},\ \left|\frac{\Delta z_i}{z_{io}}\right| \leqslant \frac{1}{\mu},\ \left|\frac{\Delta x'_i}{x'_{io}}\right| \leqslant \frac{1}{\mu},\ \left|\frac{\Delta y'_i}{y'_{io}}\right| \leqslant \frac{1}{\mu},\ \left|\frac{\Delta z'_i}{z'_{io}}\right| \leqslant \frac{1}{\mu}, \qquad \text{(A1-12)}$$

$$(i=1,\cdots\cdots n),\ \mu>0.$$

Therefore a prescribed relative error can be obtained by choosing the length of the time step in accordance with equation (A1-11). Numerical experiments show that the values of the length of the time step obtained with the aid of this equation are significantly smaller than those given by the equation

$$\tau = \frac{1}{4(2n-1)A_1^2},$$

(see relation (A2-35)).

In order to avoid significantly small or big values of length of the time step, during the numerical integration an extra condition of the form

$$C \leqslant |\tau| \leqslant B \qquad \text{(A1-13)}$$

must be satisfied, where B and C are constants.

Finally we note that the foregoing conclusions are general and do not depend on the form of the functions $x_i = x_i(\tau)$, $y_i = y_i(\tau)$ and $z_i = z_i(\tau)$

APPENDIX A2

Proof of the existence and convergence of the power series solution of the N-body problem

In this appendix it will be proved that the position vectors $\underset{\sim}{r}_i$ $(i=1,\cdots n)$, being considered as functions of the modified time τ possess derivatives of all orders and so the unique solution of the system (1.6) can be expanded in power series. The proof of the convergence of the series is given in the last section of this appendix.

Let us consider the expressions

$$\alpha_{ij} \equiv r_{ij}^{-1}, \quad (i,j=1,\cdots\cdots n,\, i\neq j),$$

$$\beta_{ij} \equiv \underset{\sim}{r}_i' \cdot \underset{\sim}{r}_j, \quad \gamma_{ij} \equiv \underset{\sim}{r}_i' \cdot \underset{\sim}{r}_j', \quad \delta_{ij} \equiv \underset{\sim}{r}_i \cdot \underset{\sim}{r}_j, (i,j=1,\cdots\cdots n). \tag{A2-1}$$

It is obvious that $\alpha_{ij}=\alpha_{ji}\,(i,j=1,\cdots n,\, i\neq j)$, $\gamma_{ij}=\gamma_{ji}$, $\delta_{ij}=\delta_{ji}$ $(i,j=1,\cdots n)$.

The proof that the derivatives of these expressions with respect to τ are functions of the expressions themselves is given below.

We initially give the formulae for the acceleration vectors $\underset{\sim}{r}_i''$ and $\underset{\sim}{r}_j''$ of the ith and jth particles by changing the indices of equation (1.6), a fact that will help us later to avoid confusion with the indices of the expressions (A2-1)

$$\underset{\sim}{r}_i'' = \sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda \frac{\underset{\sim}{r}_\lambda - \underset{\sim}{r}_i}{r_{i\lambda}^3}, \; (i=1,\cdots\cdots n), \tag{A2-2}$$

$$\underset{\sim}{r}_j'' = \sum_{\substack{\mu=1\\ \mu\neq j}}^{n} m_\mu \frac{\underset{\sim}{r}_\mu - \underset{\sim}{r}_j}{r_{j\mu}^3}, \; (j=1,\cdots n).$$

We proceed now to the evaluation of the derivatives

$$\alpha'_{ij} \quad (i,j=1,\cdots n, i\neq j), \; \beta'_{ij}, \; \gamma'_{ij}, \; \delta'_{ij} \quad (i,j=1,\cdots n). \tag{A2-3}$$

Differentiating the first of equations (A2-1) once with respect to τ we obtain

$$\alpha'_{ij} = -r_{ij}^{-2} r'_{ij} \quad (i,j=1,\cdots n, i\neq j),$$

which by introduction of the relation

$$r'_{ij} = \frac{(\underset{\sim}{r}_j - \underset{\sim}{r}_i)\cdot(\underset{\sim}{r}'_j - \underset{\sim}{r}'_i)}{r_{ij}}, \quad (i,j=1,\cdots n, i\neq j), \tag{A2-4}$$

becomes

$$\alpha'_{ij} = -r_{ij}^{-3}\left(\underset{\sim}{r}_j\cdot\underset{\sim}{r}'_j - \underset{\sim}{r}'_i\cdot\underset{\sim}{r}_j - \underset{\sim}{r}'_j\cdot\underset{\sim}{r}_i + \underset{\sim}{r}'_i\cdot\underset{\sim}{r}_i\right), \quad (i,j=1,\cdots n, i\neq j).$$

This equation, according to expressions (A2-1), takes the form

$$\alpha'_{ij} = -\alpha_{ij}^3(\beta_{jj} - \beta_{ij} - \beta_{ji} + \beta_{ii}) \quad (i,j=1,\cdots n, i\neq j). \tag{A2-5}$$

Similarly, differentiation of the second of the relations (A2-1) gives

$$\beta'_{ij} = \underset{\sim}{r}''_i\cdot\underset{\sim}{r}_j + \underset{\sim}{r}'_i\cdot\underset{\sim}{r}'_j, \quad (i,j=1,\cdots n),$$

which, upon substitution of the first of equations (A2-2), becomes

$$\beta'_{ij} = \sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda r_{i\lambda}^{-3}(\underset{\sim}{r}_j\cdot\underset{\sim}{r}_\lambda) - (\underset{\sim}{r}_i\cdot\underset{\sim}{r}_j)\sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda r_{i\lambda}^{-3} + (\underset{\sim}{r}'_i\cdot\underset{\sim}{r}'_j),$$

$$(i,j=1,\cdots n)$$

Introduction of the expressions (A2-1) yields

$$\beta'_{ij} = \sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda \alpha_{i\lambda}^3 \delta_{j\lambda} - \delta_{ij}\sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda \alpha_{i\lambda}^3 + \gamma_{ij}, \quad (i,j=1,\cdots n). \tag{A2-6}$$

The derivative γ'_{ij} is given by the relation

$$\gamma'_{ij} = \mathbf{r}''_i \cdot \mathbf{r}'_j + \mathbf{r}'_i \cdot \mathbf{r}''_j \quad , \quad (i,j = 1, \cdots\cdots n)$$

which, according to equations (A2-2), takes the form

$$\begin{aligned} \gamma'_{ij} = & \sum_{\substack{\lambda=1 \\ \lambda \neq i}}^{n} m_\lambda r_{i\lambda}^{-3} (\mathbf{r}_\lambda \cdot \mathbf{r}'_j) - (\mathbf{r}'_j \cdot \mathbf{r}_i) \sum_{\substack{\lambda=1 \\ \lambda \neq i}}^{n} m_\lambda r_{i\lambda}^{-3} + \\ & + \sum_{\substack{\mu=1 \\ \mu \neq j}}^{n} m_\mu r_{j\mu}^{-3} (\mathbf{r}'_i \cdot \mathbf{r}_\mu) - (\mathbf{r}'_i \cdot \mathbf{r}_j) \sum_{\substack{\mu=1 \\ \mu \neq j}}^{n} m_\mu r_{j\mu}^{-3} , \end{aligned} \quad (i,j = 1, \cdots n),$$

which, by virtue of the relations (A2-1), becomes

$$\begin{aligned} \gamma'_{ij} = & \sum_{\substack{\lambda=1 \\ \lambda \neq i}}^{n} m_\lambda \alpha_{i\lambda}^3 \beta_{j\lambda} - \beta_{ji} \sum_{\substack{\lambda=1 \\ \lambda \neq i}}^{n} m_\lambda \alpha_{i\lambda}^3 + \\ & + \sum_{\substack{\mu=1 \\ \mu \neq j}}^{n} m_\mu \alpha_{j\mu}^3 \beta_{i\mu} - \beta_{ij} \sum_{\substack{\mu=1 \\ \mu \neq j}}^{n} m_\mu \alpha_{j\mu}^3 . \end{aligned} \quad (i,j = 1, \cdots n), \qquad \text{(A2-7)}$$

Differentiating both sides of the last of the relations (A2-1), with respect τ, we obtain

$$\delta'_{ij} = \mathbf{r}'_i \cdot \mathbf{r}_j + \mathbf{r}'_j \cdot \mathbf{r}_i \quad (i,j = 1, \cdots n). \qquad \text{(A2-8)}$$

Comparing equations (A2-1) and (A2-8) we find

$$\delta'_{ij} = \beta_{ij} + \beta_{ji} \, , (i,j = 1, \cdots n) .$$

For the sake of convenience the results of the above calculations are summarized in the following

$$\alpha'_{ij} = -\alpha^3_{ij}(\beta_{jj}-\beta_{ij}-\beta_{ji}-\beta_{ii}), \quad (i,j=1,\cdots\cdots n, \; i\neq j),$$

$$\beta'_{ij} = \sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda \alpha^3_{i\lambda}\delta_{j\lambda} - \delta_{ij}\sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda \alpha^3_{i\lambda} + \gamma_{ij}, \quad (i,j=1,\cdots\cdots n),$$

$$\gamma'_{ij} = \sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda \alpha^3_{i\lambda}\beta_{j\lambda} - \beta_{ji}\sum_{\substack{\lambda=1\\ \lambda\neq i}}^{n} m_\lambda \alpha^3_{i\lambda} + \sum_{\substack{\mu=1\\ \mu\neq j}}^{n} m_\mu \alpha^3_{j\mu}\beta_{i\mu} - \qquad \text{(A2-9)}$$

$$-\beta_{ij}\sum_{\substack{\mu=1\\ \mu\neq j}}^{n} m_\mu \alpha^3_{j\mu}, (i,j=1,\cdots n),$$

$$\delta'_{ij} = \beta_{ij} + \beta_{ji} \quad (i,j=1,\cdots\cdots n).$$

Differentiation of these equations with respect to τ and introduction of the same equations as soon as the derivatives (A2-3) appear, gives the second-order derivatives of the expressions (A2-1) as functions of the expressions.

In order to find the higher-order derivatives of the expressions (A2-1) we can repeat this differentiation process, each time eliminating the derivatives (A2-3) by means of equations (A2-9) as the case demands.

Our next step is to show, that by making use of expressions (A2-1), we can actually obtain the derivatives $\underset{\sim}{r}_i^{(\nu)}$, $\nu \geqslant 3$, $(i=1,\cdots\cdots n)$ as functions of the position and velocity vectors of bodies. Firstly we notice that the existence of the derivatives $\underset{\sim}{r}'_i$ $\underset{\sim}{r}''_i$ $(i=1,\cdots n)$ is obvious by the nature of the problem, and for the reason that the velocity and acceleration vectors are limited for every value of the time τ. We must, of course, exclude points of singularity, if they exist.

The equations of motion already give the second-order derivatives $\underset{\sim}{r}''_i$ $(i=1,\cdots n)$ in the desired form. On substituting expressions (A2-1) into equation (1.6) we obtain

$$\underset{\sim}{r}_i'' = \sum_{\substack{j=1\\j\neq i}}^{n} m_j \alpha_{ij}^3 \underset{\sim}{r}_j - \underset{\sim}{r}_i \sum_{\substack{j=1\\j\neq i}}^{n} m_j \alpha_{ij}^3 , \quad (i=1,\cdots n). \tag{A2-10}$$

Differentiating this equation once with respect to τ and making use of equations (A2-9) we find

$$\underset{\sim}{r}_i''' = 3 \sum_{\substack{j=1\\j\neq i}}^{n} m_j \alpha_{ij}^5 B_{ij} (\underset{\sim}{r}_i - \underset{\sim}{r}_j) + \sum_{\substack{j=1\\j\neq i}}^{n} m_j \alpha_{ij}^3 (\underset{\sim}{r}_j' - \underset{\sim}{r}_i'), \quad (i=1,\cdots n), \tag{A2-11}$$

where $B_{ij} \equiv b_{jj} - b_{ji} - b_{ij} + b_{ii} \quad (i,j = 1,\cdots n)$.

The same process leads to the following expressions for the fourth and fifth-order derivatives of the position vectors $\underset{\sim}{r}_i$ $(i=1,\cdots n)$

$$\underset{\sim}{r}_i^{(4)} = -15 \sum_{\substack{j=1\\j\neq i}}^{n} m_j \alpha_{ij}^7 B_{ij}^2 (\underset{\sim}{r}_i - \underset{\sim}{r}_j) +$$

$$+3 \sum_{\substack{j=1\\j\neq i}}^{n} \sum_{\substack{\mu=1\\\mu\neq j}}^{n} m_j m_\mu \alpha_{ij}^5 \alpha_{j\mu}^3 (\delta_{j\mu} - \delta_{i\mu} - \delta_{jj} + \delta_{ji}) (\underset{\sim}{r}_i - \underset{\sim}{r}_j) +$$

$$+3 \sum_{\substack{j=1\\j\neq i}}^{n} \sum_{\substack{\lambda=1\\\lambda\neq i}}^{n} m_j m_\lambda \alpha_{ij}^5 \alpha_{i\lambda}^3 (\delta_{i\lambda} - \delta_{j\lambda} - \delta_{ii} + \delta_{ij}) (\underset{\sim}{r}_i - \underset{\sim}{r}_j) + \tag{A2-12}$$

$$+3 \sum_{\substack{j=1\\j\neq i}}^{n} m_j \alpha_{ij}^5 \Gamma_{ij} (\underset{\sim}{r}_i - \underset{\sim}{r}_j) + \sum_{\substack{j=1\\j\neq i}}^{n} \sum_{\substack{\mu=1\\\mu\neq j}}^{n} m_j m_\mu \alpha_{ij}^3 \alpha_{j\mu}^3 (\underset{\sim}{r}_\mu - \underset{\sim}{r}_j) +$$

$$+ \sum_{\substack{j=1\\j\neq i}}^{n} \sum_{\substack{\lambda=1\\\lambda\neq i}}^{n} m_j m_\lambda \alpha_{ij}^3 \alpha_{i\lambda}^3 (\underset{\sim}{r}_i - \underset{\sim}{r}_\lambda) + 6 \sum_{\substack{j=1\\j\neq i}}^{n} m_j \alpha_{ij}^5 B_{ij} (\underset{\sim}{r}_i' - \underset{\sim}{r}_j') , \quad (i=1,\cdots n),$$

where $\Gamma_{ij} \equiv \gamma_{jj} - \gamma_{ij} - \gamma_{ji} + \gamma_{ii}$

$$\underset{\sim}{r}_i^{(5)} = \sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \Bigg\{ 105\,\alpha_{ij}^{9} B_{ij}^{3} - 45 \sum_{\substack{\mu=1 \\ \mu\neq j}}^{n} m_\mu \alpha_{ij}^{7} \alpha_{j\mu}^{3} B_{ij} (\delta_{j\mu} - \delta_{i\mu} - \delta_{jj} + \delta_{ji}) -$$

$$- 45 \sum_{\substack{\lambda=1 \\ \lambda\neq i}}^{n} m_\lambda \alpha_{ij}^{7} \alpha_{i\lambda}^{3} B_{ij} (\delta_{i\lambda} - \delta_{j\lambda} - \delta_{ii} + \delta_{ij}) - 45\,\alpha_{ij}^{7} B_{ij} \Gamma_{ij} -$$

$$- 9 \sum_{\substack{\mu=1 \\ \mu\neq j}}^{n} m_\mu \alpha_{ij}^{5} \alpha_{j\mu}^{5} (\delta_{j\mu} - \delta_{i\mu} - \delta_{jj} + \delta_{ji}) B_{j\mu} +$$

$$+ 3 \sum_{\substack{\mu=1 \\ \mu\neq j}}^{n} m_\mu \alpha_{ij}^{5} \alpha_{j\mu}^{3} (3\gamma_{j\mu} - 3\gamma_{i\mu} - 4\gamma_{jj} + 3\gamma_{ij} + \gamma_{\mu j} - \gamma_{\mu i} + \gamma_{ji}) -$$

$$- 9 \sum_{\substack{\lambda=1 \\ \lambda\neq i}}^{n} m_\lambda \alpha_{ij}^{5} \alpha_{i\lambda}^{5} (\delta_{i\lambda} - \delta_{j\lambda} - \delta_{ii} + \delta_{ij}) B_{i\lambda} + 3 \sum_{\substack{\lambda=1 \\ \lambda\neq i}}^{n} m_\lambda \alpha_{ij}^{5} \alpha_{i\lambda}^{3} (3\gamma_{i\lambda} -$$

$$- 3\gamma_{j\lambda} - 4\gamma_{ii} + 2\gamma_{ji} + 2\gamma_{ij} + \gamma_{\lambda i} - \gamma_{\lambda j}) \Bigg\} (\underset{\sim}{r}_i - \underset{\sim}{r}_j) + \tag{A2-13}$$

$$+ \sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \alpha_{ij}^{5} \Bigg\{ -45\,\alpha_{ij}^{2} B_{ij}^{2} + 9 \sum_{\substack{\mu=1 \\ \mu\neq j}}^{n} m_\mu \alpha_{j\mu}^{3} (\delta_{j\mu} - \delta_{i\mu} - \delta_{jj} + \delta_{ji}) +$$

$$+ 9 \sum_{\substack{\lambda=1 \\ \lambda\neq i}}^{n} m_\lambda \alpha_{i\lambda}^{3} (\delta_{i\lambda} - \delta_{j\lambda} - \delta_{ii} + \delta_{ij}) + 9\,\Gamma_{ij} \Bigg\} (\underset{\sim}{r}_i' - \underset{\sim}{r}_j') +$$

$$+\sum_{\substack{j=1\\j\neq i}}^{n} m_j\alpha_{ij}^3\left\{9\alpha_{ij}^2\left(\sum_{\substack{\lambda=1\\\lambda\neq i}}^{n} m_\lambda\alpha_{i\lambda}^3 B_{ij}(\underset{\sim}{r}_\lambda-\underset{\sim}{r}_i) - \sum_{\substack{\mu=1\\\mu\neq j}}^{n} m_\mu\alpha_{j\mu}^3 B_{ij}(\underset{\sim}{r}_\mu-\underset{\sim}{r}_j)\right)-\right.$$

$$\left.-3\sum_{\substack{\mu=1\\\mu\neq j}}^{n} m_\mu\alpha_{j\mu}^5 B_{j\mu}(\underset{\sim}{r}_\mu-\underset{\sim}{r}_j)+3\sum_{\substack{\lambda=1\\\lambda\neq i}}^{n} m_\lambda\alpha_{i\lambda}^5 B_{i\lambda}(\underset{\sim}{r}_\lambda-\underset{\sim}{r}_i)\right\}+$$

$$+\sum_{\substack{j=1\\j\neq i}}^{n} m_j\alpha_{ij}^3\left\{\sum_{\substack{\mu=1\\\mu\neq j}}^{n} m_\mu\alpha_{j\mu}^3(\underset{\sim}{r}_\mu'-\underset{\sim}{r}_j') - \sum_{\substack{\lambda=1\\\lambda\neq i}}^{n} m_\lambda\alpha_{i\lambda}^3(\underset{\sim}{r}_\lambda'-\underset{\sim}{r}_i')\right\},\ (i=1,\cdots n).$$

In a complete analogous manner, and since the expressions (A2-1) are functions of the position and velocity vectors of the bodies, one can find the derivatives of all orders of the position vector $\underset{\sim}{r}_i$ $(i=1,\cdots n)$ as functions of the vectors $\underset{\sim}{r}_i$ and $\underset{\sim}{r}_i'$ $(i=1,\cdots n)$; it is also evident from equations (A2-1), (A2-9), (A2-10), (A2-11), (A2-12) and (A2-13) that these derivatives have the general form

$$\underset{\sim}{r}_i^{(\nu)}=\sum_{j=1}^{n}\left(F_{ij\nu}\underset{\sim}{r}_j+G_{ij\nu}\underset{\sim}{r}_j'\right),\ (i=1,\cdots n),\ \nu\geqslant 0\ , \qquad \text{(A2-14)}$$

where $F_{ij\nu}$ and $G_{ij\nu}$ are functions of $\alpha_{ij}(i,j=1\cdots\cdots n, i\neq j)$, β_{ij}, γ_{ij}, δ_{ij} $(i,j=1,\cdots n)$. The form of these functions depends on the vectors $\underset{\sim}{r}_i$ $(i=1,\cdots n)$, $\underset{\sim}{r}_j$ $(j=1,\cdots n)$ and the order ν of the derivative.

Comparing equation (A2-14), for $\nu=0,1,2$, with the identities

$$\underset{\sim}{r}_i\equiv\underset{\sim}{r}_i\ ,\ \ \underset{\sim}{r}_i'\equiv\underset{\sim}{r}_i'\quad(i=1,\cdots n),$$

and the relation (A2-10) we readily conclude

$$F_{ij0}=\delta_{ij}^*\ ,\quad G_{ij0}=0\quad(i,j=1,\cdots n),$$

$$F_{ij1} = 0 \;,\; G_{ij1} = \delta^*_{ij} \quad (i,j = 1,\cdots\cdots n),$$

$$F_{ij2} = m_j \alpha^3_{ij} \;,\; (i,j = 1,\cdots\cdots n, i \neq j),$$

(A2-15)

$$F_{ii2} = -\sum_{\substack{\lambda=1 \\ \lambda \neq i}}^{n} F_{i\lambda 2} \;,\; (i=1,\cdots n) \;,\; G_{ij2} = 0 \;,\; (i,j = 1,\cdots\cdots n).$$

where δ^*_{ij} is the Kronecker's delta

Combining the identity $\underset{\sim}{r}_i^{(\nu)} \equiv \left(\underset{\sim}{r}_i^{(\nu-1)}\right)'$, $\nu \geqslant 2$, $(i=1,\cdots n)$ and equation (A2-14) we obtain

$$\sum_{j=1}^{n}\left(F_{ij\nu}\underset{\sim}{r}_j + G_{ij\nu}\underset{\sim}{r}_j'\right) \equiv \sum_{j=1}^{n}\left(F_{ij\nu-1} + G'_{ij\nu-1}\right)\underset{\sim}{r}_j' +$$

$$+\sum_{j=1}^{n} F'_{ij\nu-1}\underset{\sim}{r}_j + \sum_{j=1}^{n} G_{ij\nu-1}\underset{\sim}{r}_j'' \;,\; (i=1,\cdots n) \;,\; \nu \geqslant 3. \qquad \text{(A2-16)}$$

The last of the sums entering the right-hand side the foregoing equation can be rewritten as follows

$$\sum_{j=1}^{n} G_{ij\nu-1}\underset{\sim}{r}_j'' \equiv \sum_{\lambda=1}^{n} G_{i\lambda\nu-1}\underset{\sim}{r}_\lambda'' \;,\; (i=1\cdots n),\; \nu \geqslant 3.$$

On the other hand, with the aid of equation (A2-14) and the last three of equations (A2-15), we find

$$\underset{\sim}{r}_\lambda'' = \sum_{j=1}^{n} F_{\lambda j 2}\underset{\sim}{r}_j \;,\; (\lambda=1,\cdots n).$$

Combining the foregoing two equations we obtain

$$\sum_{j=1}^{n} G_{ijv-1}\Gamma_j'' = \sum_{\lambda=1}^{n}\sum_{j=1}^{n} G_{i\lambda v-1}F_{\lambda j2}\Gamma_j =$$

$$=\sum_{j=1}^{n}\sum_{\lambda=1}^{n} G_{i\lambda v-1}F_{\lambda j2}\Gamma_j\ , \quad (i=1,\cdots n),\ v\geq 3\ ,$$

substitution of which into identity (A2-16), yields

$$\sum_{j=1}^{n}\left[F_{ijv}\Gamma_j + G_{ijv}\Gamma_j'\right] \equiv \sum_{j=1}^{n}\left[\left(F_{ijv-1}' + \sum_{\lambda=1}^{n} G_{i\lambda v-1}F_{\lambda j2}\right)\right]\Gamma_j +$$

$$+\sum_{j=1}^{n}\left(F_{ijv-1} + G_{ijv-1}'\right)\Gamma_j'\ , \quad (i=1,\cdots n),\ v\geq 3\ .$$

From this identity we readily conclude

$$F_{ijv} = F_{ijv-1}' + \sum_{\lambda=1}^{n} G_{i\lambda v-1}F_{\lambda j2}\ , \quad (i=1,\cdots n),\quad v\geq 3\ ,$$

(A2-17)

$$G_{ijv} = F_{ijv-1} + G_{ijv-1}'\ , \quad (i=1,\cdots n),\ v\geq 3.$$

The evaluation of the functions F_{ijv} and G_{ijv} $(i,j=1,\cdots n)$ $v\geq 2$ can be carried out using the foregoing recursive formulae and equations (A2-9), with initial "values" given by the last three of equations (A2-15).

Equations (A2-14), at the point $\tau=0$ takes the form

$$\underset{\sim}{\Gamma}_{io}^{(\nu)} = \nu! \sum_{j=1}^{n} \left\{ f_{ij\nu} \underset{\sim}{\Gamma}_{jo} + g_{ij\nu} \underset{\sim}{\Gamma}_{jo}' \right\}, \ (i=1,\cdots n), \ \nu \geqslant 0, \qquad \text{(A2-18)}$$

where

$$f_{ij\nu} \equiv \left. \frac{F_{ij\nu}}{\nu!} \right|_{\tau=0}, \ g_{ij\nu} \equiv \left. \frac{G_{ij\nu}}{\nu!} \right|_{\tau=0}, \ (i,j=1,\cdots n), \ \nu \geqslant 0. \qquad \text{(A2-19)}$$

Introduction of equations (A2-18) into power series solution (1.7) yields

$$\underset{\sim}{\Gamma}_i = \sum_{\nu=0}^{\infty} \tau^{\nu} \left[\sum_{j=1}^{n} \left(f_{ij\nu} \underset{\sim}{\Gamma}_{jo} + g_{ij\nu} \underset{\sim}{\Gamma}_{jo}' \right) \right] =$$

$$= \sum_{j=1}^{n} \left[\left(\sum_{\nu=0}^{\infty} f_{ij\nu} \tau^{\nu} \right) \underset{\sim}{\Gamma}_{jo} + \left(\sum_{\nu=0}^{\infty} g_{ij\nu} \tau^{\nu} \right) \underset{\sim}{\Gamma}_{jo}' \right], \ (i=1,\cdots n), \ \tau \in (-\tau^*, \tau^*). \qquad \text{(A2-20)}$$

The interchange of the infinite and finite sum in this expression can, of course, take place only under the presupposition of convergence of the series

$$f_{ij} \equiv \sum_{\nu=0}^{\infty} f_{ij\nu} \tau^{\nu}, \quad g_{ij} \equiv \sum_{\nu=0}^{\infty} g_{ij\nu} \tau^{\nu}, \ (i,j=1,\cdots n), \tau \in (-\tau^*, \tau^*). \qquad \text{(A2-21)}$$

This convergence can analytically be proved by a process similar to that, followed below for the proof of the convergence of the power series (1.7). Combining the foregoing two equations we find

$$\underset{\sim}{\Gamma}_i = \sum_{j=1}^{n} \left(f_{ij} \underset{\sim}{\Gamma}_{jo} + g_{ij} \underset{\sim}{\Gamma}_{jo}' \right), \ (i=1,\cdots n), \ \tau \in (-\tau^*, \tau^*). \qquad \text{(A2-22)}$$

We proceed with the proof of the convergence of the series (1.7). The functions

$$\alpha_{ij}=\alpha_{ij}(\tau)\ (i,j=1\cdots n, i\neq j)\ \beta_{ij}=\beta_{ij}\ (\tau),\ \gamma_{ij}=\gamma_{ij}(\tau),\ \delta_{ij}=\delta_{ij}(\tau),$$

$$x_i=x_i(\tau)\ ,\ x'_i=x'_i(\tau),\ (i,j=1,\cdots n), \tag{A2-23}$$

possess first-order derivatives, with respect to τ, in any closed interval $[-\tau_1, \tau_1]$ as it was mentioned previously. They are therefore continuous and bounded in this interval.

Let us define

$$H = \max\left\{\begin{matrix}\text{absolute values of the upper and lower bounds of} \\ \text{the functions (A2-23) in the interval } [-\tau_1, \tau_1]\end{matrix}\right\},$$

and

$$A = \max\left\{H, m_j\ (j=1,\cdots n)\right\}. \tag{A2-24}$$

The right-hand member of the relation

$$x''_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \alpha_{ij}^3 x_j - x_i \sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \alpha_{ij}^3\ ,\ (i=1,\cdots n)\ , \tag{A2-25}$$

contains $2(n-1)$ products, with five factors in each one. Interpreting $Q_1, Q_2 \cdots Q_v \cdots$ to represent any one of the functions (A2-23) and the factors m_j $(j=1,\cdots n)$ the products appearing in the foregoing relation have the general form.

$$Q_1 \cdot Q_2 \cdot Q_3 \cdot Q_4 \cdot Q_5 \tag{A2-26}$$

It is clear, from the definition of A, that in the interval $[-\tau_1, \tau_1]$ the following general relation is valid

$$|Q_1 \cdot Q_2 \cdot Q_3 \cdot Q_4 \cdot Q_5| \leqslant A^5,$$

and therefore

$$|x''_i| \leqslant 2(n-1)A^5 \equiv \alpha_2\ ,\ (i=1,\cdots n)\ ,\ \tau\in[-\tau_1, \tau_1]. \tag{A2-27}$$

Since the second order derivative x_i'' $(i=1,\cdots n)$ contains $2(n-1)$ products of the form (A2-26), the third order derivative x_i''' will contain $2(n-1)5$ products of the general form

$$Q_1 \cdot Q_2 \cdot Q_3 \cdot Q_4 \cdot Q_5' \,, \qquad \text{(A2-28)}$$

where $Q_5' \equiv \dfrac{dQ_5}{d\tau}$ and the meaning of Q_1, Q_2, Q_3, Q_4 and Q_5 is the same as above

We define,

$\theta \equiv$ maximum number of products which appear in the right-hand members of the relations (A2-9) and (A2-25),

$\mu \equiv$ maximum number of factors of these products.

From equations (A2-9) and (A2-25) and the definition of Q_5 it is evident that the derivative Q_5' will be either zero or a sum of θ products, at most, and everyone of them will contain, at most, μ factors. It is obvious, therefore, that the function $x_i'''(\tau)$ $(i=1,\cdots n)$ will be represented by a sum of $2(n-1)5\cdot\theta$ products, at most, of the general form

$$Q_1 \cdot Q_2 \cdot Q_3 \cdot \cdots\cdots Q_\nu \,,$$

where $\nu \leqslant \mu+4$ In the interval $[-\tau_1, \tau_1]$ the foregoing product obeys the relation

$$|Q_1 \cdot Q_2 \cdot Q_3 \cdot \cdots\cdots Q_\nu| \leqslant A^{\mu+4} \,,$$

since $\nu \leqslant \mu+4$. Hence

$$|x_i'''(\tau)| \leqslant 10(n-1)\,\theta A^{\mu+4} \equiv \alpha_3 \,, \quad (i=1,\cdots n) \,, \tau \in [-\tau_1, \tau_1] \,,$$

Working as above we obtain the relations

$$|x_i^{(4)}(\tau)| \leqslant 10(n-1)\,\theta^2(\mu+4)A^{2\mu+3} \equiv \alpha_4 \,,$$

$$\left|x_i^{(5)}(\tau)\right| \leqslant 10\,(n-1)\,\Theta^3\,(\mu+4)\,(2\mu+3)\,A^{3\mu+2} \equiv \alpha_5$$

$$\vdots$$

$$\left|x_i^{(\nu)}(\tau)\right| \leqslant 10\,(n-1)\,\Theta^{\nu-2}(\mu+4)(2\mu+3)(3\mu+2)\cdots\cdots[\nu(\mu-1)-3\mu+8]A^{\nu(\mu-1)-2\mu+7} \equiv \alpha_\nu\,,$$

$$\left|x_i^{(\nu+1)}(\tau)\right| \leqslant 10(n-1)\Theta^{\nu-1}(\mu+4)(2\mu+3)(3\mu+2)\cdots\cdots[\nu(\mu-1)-2\mu+7]A^{\nu(\mu-1)-\mu+6} \equiv \alpha_{\nu+1}\,,$$

$$\vdots$$

$$\nu \geqslant 4\,,\ (i=1,\cdots\cdots n)\,,\ \tau\in[-\tau_1\,,\tau_1]\,.$$

At the point $\tau=0$ we have $x_i^{(\nu)}\Big|_{\tau=0} \equiv x_{i0}^{(\nu)}$, $\nu\geqslant 0$, $(i=1,\cdots\cdots n)$, hence

$$|x_{i0}| \leqslant A\,,$$

$$|x'_{i0}\tau| \leqslant A|\tau|\,,$$

$$\left|x''_{i0}\frac{\tau^2}{2!}\right| \leqslant 2(n-1)A^5\frac{\tau^2}{2!} \equiv \alpha_2\frac{\tau^2}{2!}\,,$$

$$\left|x_{i0}^{(3)}\frac{\tau^3}{3!}\right| \leqslant 10(n-1)\,\Theta\,A^{\mu+4}\frac{|\tau|^3}{3!} \equiv \alpha_3\frac{|\tau|^3}{3!}\,,$$

$$\vdots \tag{A2-29}$$

$$\left|x_{i0}^{(\nu)}\frac{\tau^\nu}{\nu!}\right| \leqslant 10\,(n-1)\,\Theta^{\nu-2}(\mu+4)(2\mu+3)\cdots\cdots[\nu(\mu-1)-3\mu+8]\frac{|\tau|^\nu}{\nu!}A^{\nu(\mu-1)-2\mu+7} \equiv \alpha_\nu\frac{|\tau|^\nu}{\nu!}\,,$$

$$\left|x_{i0}^{(\nu+1)}\frac{\tau^{\nu+1}}{(\nu+1)!}\right| \leqslant 10(n-1)\Theta^{\nu-1}(\mu+4)\,(2\mu+3)\cdots\cdots[\nu(\mu-1)-2\mu+7]\frac{|\tau|^{\nu+1}}{(\nu+1)!}A^{\nu(\mu-1)-\mu+6} \equiv \alpha_{\nu+1}\frac{|\tau|^{\nu+1}}{(\nu+1)!}$$

$$\vdots$$

$$(i=1,\cdots n)\,,\ \tau\in[-\tau_1\,,\tau_1]\,.$$

We now form the series

$$\sum_{\nu=2}^{\infty} \alpha_\nu \frac{|\tau|^\nu}{\nu!}, \tag{A2-30}$$

and take the limit of the ratio of two succesive terms of it as $\nu\to\infty$

$$\lim_{\nu\to\infty} \frac{\alpha_{\nu+1}\frac{|\tau|^{\nu+1}}{(\nu+1)!}}{\alpha_\nu \frac{|\tau|^\nu}{\nu!}} = |\tau| \lim_{\nu\to\infty} \frac{\alpha_{\nu+1}}{\alpha_\nu(\nu+1)} =$$

$$=|\tau| \lim_{\nu\to\infty} \frac{10(n-1)\,\theta^{\nu-1}(\mu+4)\cdots\cdots\cdots[\nu(\mu-1)-3\mu+8][\nu(\mu-1)-2\mu+7]A^{\nu(\mu-1)-\mu+6}}{(\nu+1)\,10(n-1)\,\theta^{\nu-2}(\mu+4)\cdots\cdots\cdots[\nu(\mu-1)-3\mu+8]A^{\nu(\mu-1)-2\mu+7}} =$$

$$=|\tau|\theta A^{\mu-1} \lim_{\nu\to\infty} \frac{[\nu(\mu-1)-2\mu+7]}{(\nu+1)} = |\tau|\theta A^{\mu-1}(\mu-1), \quad \tau\in[-\tau_1,\tau_1].$$

We can choose the interval $[-\tau_1,\tau_1]$ such that

$$|\tau|\theta A^{\mu-1}(\mu-1) < 1, \quad \tau\in[-\tau_1,\tau_1].$$

From the definitions of θ and μ it is obvious that $\theta=4(n-1), n\geqslant 2$ and $\mu=5$
The foregoing relation, by introduction of these values, takes the form

$$|\tau| < \frac{1}{16(n-1)A^4}, \quad \tau\in[-\tau_1,\tau_1]. \tag{A2-31}$$

It is obvious that, in the interval $[-\tau_1,\tau_1]$ the series (A2-30) converges (ratio test). The convergence of the series

$$\sum_{\nu=0}^{\infty} \left|x_{io}^{(\nu)} \frac{\tau^\nu}{\nu!}\right|, (i=1,\cdots n), \tau\in[-\tau_1,\tau_1],$$

follows immediately from the relations (A2-29), and ensures the convergence of the power series

$$\sum_{\nu=0}^{\infty} x_{io}^{(\nu)} \frac{\tau^\nu}{\nu!} \quad , \quad (i=1,\cdots n) \tag{A2-32}$$

in the above mentioned interval of time.

By a similar process it can be proved that the series

$$\sum_{\nu=0}^{\infty} y_{io}^{(\nu)} \frac{\tau^\nu}{\nu!} \, , \; (i=1,\cdots n) \, ,$$

(A2-33)

$$\sum_{\nu=0}^{\infty} z_{io}^{(\nu)} \frac{\tau^\nu}{\nu!} \, , \; (i=1,\cdots n) \, ,$$

converges.

The common interval of convergence $[-\tau^*, \tau^*]$ of the series (A2-32) and (A2-33) is the interval of convergence of the power series solution (1.7).

If instead of the expression (A2-1) we choose the expressions

$$\alpha_{ij}^* \equiv r_{ij}^{-3} \, , \quad \varepsilon_{ij}^* \equiv \frac{r_i^2}{r_{ij}^2} \, , \; (i,j=1,\cdots n \, , \; i \neq j) \, , \tag{A2-34}$$

$$\beta_{ijk}^* \equiv \frac{\underline{r}_i' \cdot \underline{r}_j}{r_k^2} \, , \; \gamma_{ijk}^* \equiv \frac{\underline{r}_i' \cdot \underline{r}_j'}{r_k^2} \, , \; \delta_{ijk}^* \equiv \frac{\underline{r}_i \cdot \underline{r}_j}{r_k^2} \, , \; (i,j=1,\cdots n) \, ,$$

we can prove that the series (1.7) converges for every value of the independent variable τ which obeys the relation

$$|\tau| < \frac{1}{4(2n-1)A_1^2} \tag{A2-35}$$

where $A_1 \equiv \max\{H^*, m_j, (j=1,\cdots n)\}$ and

$H^* \equiv \max\{$ absolute values of the upper and lower bounds of the functions $\alpha^*_{ij}(\tau)$, $\varepsilon^*_{ij}(\tau)$, $(i,j=1\cdots\cdots n,\ i\neq j)$, $\beta^*_{ijk}(\tau)$, $\gamma^*_{ijk}(\tau)$, $\delta^*_{ijk}(\tau)$, $x_i(\tau)$, $x'_i(\tau)$, $y_i(\tau)$, $y'_i(\tau)$, $z_i(\tau)$, $z'_i(\tau)$, $(i,j,k=1,\cdots\cdots n)$ in the interval defined by the relation (A2-35) $\}$.

The singularities of the functions

$$\beta^*_{ijk} = \beta^*_{ijk}(\tau),\ \gamma^*_{ijk} = \gamma^*_{ijk}(\tau),\ \delta^*_{ijk} = \delta^*_{ijk}(\tau),\ (i,j,k=1,\cdots n),$$

occuring when $r_k = 0$, $(k=1,\cdots n)$, can be avoided by choosing a system of reference in which

$$r_k = r_k(\tau) \neq 0,$$

for every $k=1,\cdots n$, and every value of τ in the appropriate closed interval.

APPENDIX A3

Evaluation of the series appearing in equation (1.33)

It will be explained in this section how the series appearing in the right hand side of equation (1.33) can be expressed as functions of the first order.

For the sake of convenience we shall first evaluate the analytical expressions of some series required later on for the rest of the calculations. These series are:

$$\sum_{K=0}^{\infty}(-1)^K \frac{(K+1)(K+2)}{(K+4)}\theta^K, \tag{A3-1}$$

$$\sum_{K=0}^{\infty}(-1)^K \frac{(K+1)(K+2)(K+3)}{(K+4)(K+5)}\theta^K, \tag{A3-2}$$

$$\sum_{K=0}^{\infty}(-1)^{K+1} \frac{(K+1)}{(K+3)}\theta^K, \tag{A3-3}$$

$$\sum_{K=0}^{\infty}(-1)^K \frac{(K+1)(K+2)(K+3)}{(K+5)(K+6)}\theta^K, \tag{A3-4}$$

$$\sum_{K=0}^{\infty}(-1)^{K+1} \frac{(K+1)(K+2)}{(K+3)(K+4)}\theta^K, \tag{A3-5}$$

$$\sum_{K=0}^{\infty}(-1)^K \frac{(K+1)(K+2)(K+3)}{(K+6)(K+7)}\theta^K, \tag{A3-6}$$

$$\sum_{K=0}^{\infty}(-1)^{K+1} \frac{(K+1)(K+2)}{(K+4)(K+5)}\theta^K, \tag{A3-7}$$

$$|\theta| < 1.$$

We start with the well known relation

$$\sum_{K=1}^{\infty}(-1)^{K+1} \frac{\theta^K}{K} = \ln(1+\theta), \quad -1 < \theta \leq 1, \tag{A3-8}$$

which is equivalent to the following equation,

$$\sum_{K=4}^{\infty}(-1)^{K+1}\frac{\theta^{K-2}}{K}=\frac{1}{\theta^2}\left\{\ln(1+\theta)-\theta+\frac{\theta^2}{2}-\frac{\theta^3}{3}\right\},-1<\theta\leqslant 1,\ \theta\neq 0,$$

or

$$\sum_{K=0}^{\infty}(-1)^{K+1}\frac{\theta^{K+2}}{(K+4)}=\frac{\ln(1+\theta)}{\theta^2}-\frac{1}{\theta}+\frac{1}{2}-\frac{\theta}{3},\ -1<\theta\leqslant 1,\ \theta\neq 0.$$

Differentiating both sides of this equation twice with respect to θ and arranging the resulting expression, we obtain

$$\sum_{K=0}^{\infty}(-1)^{K}\frac{(K+1)(K+2)}{(K+4)}\theta^{K}=\frac{2\theta^2+9\theta+6}{(1+\theta)^2\,\theta^3}-\frac{6\ln(1+\theta)}{\theta^4},\ -1<\theta\leqslant 1,\ \theta\neq 0 \qquad \text{(A3-9)}$$

The analytic expression for the series (A3-2) is obtained as follows. We first integrate both sides of the equation (A3-8), which becomes

$$\sum_{K=1}^{\infty}(-1)^{K+1}\frac{\theta^{K+1}}{K(K+1)}=(1+\theta)\ln(1+\theta)-\theta,\ -1<\theta\leqslant 1,\ \theta\neq 0. \qquad \text{(A3-10)}$$

Transferring the first three terms of this series to the right hand side of the relation and dividing both sides by θ^2, we obtain

$$\sum_{K=4}^{\infty}(-1)^{K+1}\frac{\theta^{K-1}}{K(K+1)}=\frac{(1+\theta)\ln(1+\theta)}{\theta^2}-\frac{1}{\theta}-\frac{1}{2}+\frac{\theta}{6}-\frac{\theta^2}{12},\ -1<\theta\leqslant 1,\ \theta\neq 0.$$

This equation, by shifting the origin of the series to $K=0$, becomes

$$\sum_{K=0}^{\infty}(-1)^{K+1}\frac{\theta^{K+3}}{(K+4)(K+5)}=\frac{(1+\theta)\ln(1+\theta)}{\theta^2}-\frac{1}{\theta}-\frac{1}{2}+\frac{\theta}{6}-\frac{\theta^2}{12},\ -1<\theta\leqslant 1,\ \theta\neq 0.$$

It is now easy to obtain an analytic expression for the series (A3-2). It results after three successive differentiations of the foregoing equation with respect to θ and has the form

$$\sum_{K=0}^{\infty}(-1)^K \frac{(K+1)(K+2)(K+3)}{(K+4)(K+5)}\theta^K = \frac{7\theta+6}{(1+\theta)^2\theta^3} + 6\frac{\ln(1+\theta)-4}{\theta^4} + \tag{A3-11}$$

$$+\frac{24\ln(1+\theta)}{\theta^5}, \quad -1<\theta\leqslant 1, \quad \theta\neq 0 .$$

An analogous expression for the series (A3-3) is obtained from equation (A3-8), by transferring the first two terms of the series appearing in this equation to the right hand side, dividing by θ^2 and differentiating once with respect to θ. The derived relation, after shifting the origin of the series to $K=0$ and carrying out some algebraic calculations, takes the form

$$\sum_{K=0}^{\infty}(-1)^{K+1}\frac{(K+1)}{(K+3)}\theta^K = \frac{2\ln(1+\theta)}{\theta^3} - \frac{\theta+2}{(1+\theta)\theta^2}, \quad -1<\theta\leqslant 1, \quad \theta\neq 0 . \tag{A3-12}$$

The processes for the evaluation of the series (A3-4) and (A3-6) are entirely analogous to that followed for the derivation of the expression (A3-11) and they finally lead to the relations

$$\sum_{K=0}^{\infty}(-1)^K \frac{(K+1)(K+2)(K+3)}{(K+5)(K+6)}\theta^K = \frac{3}{\theta^4} - \frac{10\theta+9}{(1+\theta)^2\theta^4} - 12\frac{2\ln(1+\theta)-5}{\theta^5} - \tag{A3.13}$$

$$-\frac{60\ln(1+\theta)}{\theta^6}, \quad -1<\theta\leqslant 1, \quad \theta\neq 0,$$

$$\sum_{K=0}^{\infty}(-1)^K \frac{(K+1)(K+2)(K+3)}{(K+6)(K+7)}\theta^K = \frac{13\theta+12}{(1+\theta)^2\theta^5} + 60\frac{\ln(1+\theta)-2}{\theta^6} + \tag{A3-14}$$

$$+\frac{120\ln(1+\theta)}{\theta^7} + \frac{\theta-12}{\theta^5}, \quad -1<\theta\leqslant 1, \quad \theta\neq 0 .$$

We proceed now with the calculation of the series (A3-5). Modification of equation (A3-10) leads to the relation

$$\sum_{K=3}^{\infty}(-1)^{K+1}\frac{\theta^{K-1}}{K(K+1)}=\frac{(1+\theta)\ln(1+\theta)}{\theta^2}-\frac{1}{\theta}-\frac{1}{2}+\frac{\theta}{6},\ -1<\theta\leqslant 1,\ \theta\neq 0,$$

which is equivalent to the following equation

$$\sum_{K=0}^{\infty}(-1)^{K}\frac{\theta^{K+2}}{(K+3)(K+4)}=\frac{(1+\theta)\ln(1+\theta)}{\theta^2}-\frac{1}{\theta}-\frac{1}{2}+\frac{\theta}{6},\ -1<\theta\leqslant 1,\ \theta\neq 0.$$

Differentiating this equation twice with respect to θ and arranging the resulting expression we find

$$\sum_{K=0}^{\infty}(-1)^{K+1}\frac{(K+1)(K+2)}{(K+4)(K+3)}\theta^{K}=\frac{5\theta+6}{(1+\theta)\theta^3}-\frac{2\ln(1+\theta)}{\theta^3}-\frac{6\ln(1+\theta)}{\theta^4},\ -1<\theta\leqslant 1,\ \theta\neq 0. \quad \text{(A3-15)}$$

The series (A3-7) can be evaluated by a process entirely analogous to the process described above, and it has the form

$$\sum_{K=0}^{\infty}(-1)^{K+1}\frac{(K+1)(K+2)}{(K+4)(K+5)}\theta^{K}=\frac{6\ln(1+\theta)}{\theta^4}-\frac{\theta^2+12\theta+12}{(1+\theta)\theta^4}+\frac{12\ln(1+\theta)}{\theta^5},\quad \text{(A3-16)}$$

$$-1<\theta\leqslant 1,\ \theta\neq 0.$$

We recall now the series appearing in the right hand side of equation (1.33) evaluation of which we are seeking in this appendix. These series have the form

$$\sum_{K=0}^{\infty}B_{ij\,K+2,0}\frac{\tau^{K}}{(K+2)!},\ (i,j=1,\cdots n,\ i\neq j),\ \tau\in(-\tau^*,\tau^*),$$

$$\sum_{K=0}^{\infty}B_{ij\,K+2,0}\frac{\tau^{K}}{(K+2)(K+3)\,K!},\ (i,j=1,\cdots n,\ i\neq j),\ \tau\in(-\tau^*,\tau^*),$$

(A3-17)

$$\sum_{K=0}^{\infty} B_{ijK+2,0} \frac{\tau^K}{(K+3)(K+4)\,K!}\,,\ (i,j=1,\cdots n, i\neq j)\,,\ \tau\in(-\tau^*,\tau^*)\,,$$

$$\sum_{K=0}^{\infty} B_{ijK+2,0} \frac{\tau^K}{(K+4)(K+5)\,K!}\,,(i,j=1,\cdots\cdots n, i\neq j)\ ,\ \tau\in(-\tau^*,\tau^*)\,.$$

Comparing equations (1.26) and (1.29) we readily conclude

$$B_{ijK+2,0} = (-1)^{K+1}\frac{(K+2)!}{2}\left(\frac{\Gamma'_{ijo}}{\Gamma_{ijo}}\right)^K + (-1)^K \frac{K(K-1)}{2!}\,\frac{(K+1)!}{2}\left(\frac{\Gamma'_{ijo}}{\Gamma_{ijo}}\right)^{K-2}\frac{\Gamma''_{ijo}}{\Gamma_{ijo}} +$$

$$+(-1)^{K+1}\frac{K(K-1)(K-2)}{3!}\,\frac{K!}{2}\left(\frac{\Gamma'_{ijo}}{\Gamma_{ijo}}\right)^{K-3}\frac{\Gamma'''_{ijo}}{\Gamma_{ijo}} + Q^*_K\,,\ K\geqslant 0\,,(i,j=1,\cdots n, i\neq j)\,, \qquad \text{(A3-18)}$$

where

$$Q^*_K \equiv Q_K\Big|_{\tau=0}\ . \qquad \text{(A3-19)}$$

The first of the series (A3-17), upon substitution of equation (A3-18), yields

$$\sum_{K=0}^{\infty} B_{ijK+2,0}\frac{\tau^K}{(K+2)!} = -\frac{1}{2}\sum_{K=0}^{\infty}\left(-\frac{\Gamma'_{ijo}\tau}{\Gamma_{ijo}}\right)^K + \frac{\tau^2}{4}\frac{\Gamma''_{ijo}}{\Gamma_{ijo}}\sum_{K=0}^{\infty}(-1)^K\frac{K(K-1)}{(K+2)}\left(\frac{\Gamma'_{ijo}\tau}{\Gamma_{ijo}}\right)^{K-2} +$$

$$+\frac{\tau^3}{12}\frac{\Gamma'''_{ijo}}{\Gamma_{ijo}}\sum_{K=0}^{\infty}(-1)^{K+1}\frac{(K-2)(K-1)K}{(K+1)(K+2)}\left(\frac{\Gamma'_{ijo}\tau}{\Gamma_{ijo}}\right)^{K-3} + \sum_{K=0}^{\infty} Q^*_K\frac{\tau^K}{(K+2)!} =$$

$$= -\frac{1}{2}\sum_{K=0}^{\infty}\left(-\frac{\Gamma'_{ijo}\tau}{\Gamma_{ijo}}\right)^K - \frac{\tau^2}{4}\frac{\Gamma''_{ijo}}{\Gamma_{ijo}}\sum_{K=2}^{\infty}(-1)^K\frac{K(K-1)}{(K+2)}\left(\frac{\Gamma'_{ijo}\tau}{\Gamma_{ijo}}\right)^{K-2} +$$

$$+\frac{\tau^3}{12}\frac{\Gamma'''_{ijo}}{\Gamma_{ijo}}\sum_{K=3}^{\infty}(-1)^{K+1}\frac{(K-2)(K-1)K}{(K+1)(K+2)}\left(\frac{\Gamma'_{ijo}\tau}{\Gamma_{ijo}}\right)^{K-3} + \sum_{K=0}^{\infty} Q^*_K\frac{\tau^K}{(K+2)!}\,,$$

$$(i,j=1,\cdots\cdots n,\ i\neq j),\ \tau\in(-\tau^*\ \tau^*).$$

This relation, by shifting the origin of the series to $K=0$ becomes

$$\sum_{K=2}^{\infty} B_{ijK,0}\frac{\tau^{K-2}}{K!} \equiv \sum_{K=0}^{\infty} B_{ijK+2,0}\frac{\tau^{K}}{(K+2)!} = -\frac{1}{2}\sum_{K=0}^{\infty}\left(-\varphi_{ij}\right)^{K} + \tag{A3-20}$$

$$+\frac{\tau^2}{4}\frac{r''_{ijo}}{r_{ijo}}\sum_{K=0}^{\infty}(-1)^K\frac{(K+1)(K+2)}{(K+4)}\varphi_{ij}^{K} + \frac{\tau^3}{12}\frac{r'''_{ijo}}{r_{ijo}}\sum_{K=0}^{\infty}(-1)^K\frac{(K+1)(K+2)(K+3)}{(K+4)(K+5)}\varphi_{ij}^{K} +$$

$$+\sum_{K=0}^{\infty} Q^{*}_{K}\frac{\tau^{K}}{(K+2)!}\ ,\quad (i,j=1,\cdots n, i\neq j),\ \tau\in(-\tau^{*},\tau^{*}).$$

For the sake of simplicity, we introduced the notation

$$\varphi_{ij} \equiv \frac{r'_{ijo}\,\tau}{r_{ijo}}\ ,\quad (i,j=1,\cdots n,\ i\neq j),\ \tau\in(-\tau^{*},\tau^{*}). \tag{A3-21}$$

With the assumptions that $|\varphi_{ij}|<1$ and $\varphi_{ij}\neq 0$ we can introduce equations (A3-9), (A3-11) and the relation

$$\sum_{\nu=0}^{\infty}\theta^{\nu} = \frac{1}{1-\theta}\qquad |\theta|<1\ ,$$

into equation (A3-20) to obtain

$$\sum_{K=2}^{\infty} B_{ijK,0}\frac{\tau^{K-2}}{K!} = -\frac{1}{2}\frac{1}{(1+\varphi_{ij})} + \frac{\tau^2}{4}\frac{r''_{ijo}}{r_{ijo}}\left(\frac{2\varphi_{ij}^2+9\varphi_{ij}+6}{(1+\varphi_{ij})^2\varphi_{ij}^3} - \frac{6\ln(1+\varphi_{ij})}{\varphi_{ij}^4}\right) +$$

$$+\frac{\tau^3}{12}\frac{r'''_{ijo}}{r_{ijo}}\left(\frac{7\varphi_{ij}+6}{(1+\varphi_{ij})^2\varphi_{ij}^3} + 6\frac{\ln(1+\varphi_{ij})-4\varphi_{ij}}{\varphi_{ij}^4} + 24\frac{\ln(1+\varphi_{ij})}{\varphi_{ij}^5}\right) + \tag{A3-22}$$

$$+\sum_{K=0}^{\infty} Q^{*}_{K}\frac{\tau^{K}}{(K+2)!}\ ,\quad (i,j=1,\cdots n, i\neq j),\ \tau\in(-\tau^{*},\tau^{*}),\ |\varphi_{ij}|<1,\ \varphi_{ij}\neq 0.$$

The second of the series (A3-17), upon substitution of equation (A3-18), becomes

$$\sum_{K=0}^{\infty} B_{ij\,K+2,0}\,\frac{\tau^{K}}{(K+2)(K+3)\,K!} = \frac{1}{2}\sum_{K=0}^{\infty}(-1)^{K+1}\,\frac{(K+1)}{(K+3)}\left(\frac{r'_{ijo}\,\tau}{r_{ijo}}\right)^{K} +$$

$$+\frac{\tau^{2}}{4}\,\frac{r''_{ijo}}{r_{ijo}}\sum_{K=0}^{\infty}(-1)^{K}\,\frac{(K-1)\,K\,(K+1)}{(K+2)(K+3)}\left(\frac{r'_{ijo}\,\tau}{r_{ijo}}\right)^{K-2}+$$

$$+\frac{\tau^{3}}{12}\,\frac{r''_{ijo}}{r_{ijo}}\sum_{K=0}^{\infty}(-1)^{K+1}\,\frac{(K-2)(K-1)K}{(K+2)(K+3)}\left(\frac{r'_{ijo}\,\tau}{r_{ijo}}\right)^{K-3} + \sum_{K=0}^{\infty} Q^{*}_{K}\,\frac{\tau^{K}}{(K+2)(K+3)K!} =$$

$$=\frac{1}{2}\sum_{K=0}^{\infty}(-1)^{K+1}\,\frac{(K+1)}{(K+3)}\left(\frac{r'_{ijo}\,\tau}{r_{ijo}}\right)^{K} + \frac{\tau^{2}}{4}\,\frac{r''_{ijo}}{r_{ijo}}\sum_{K=2}^{\infty}(-1)^{K}\,\frac{(K-1)(K+1)\,K}{(K+2)(K+3)}\left(\frac{r'_{ijo}\,\tau}{r_{ijo}}\right)^{K-2} +$$

$$+\frac{\tau^{3}}{12}\,\frac{r'''_{ijo}}{r_{ijo}}\sum_{K=3}^{\infty}(-1)^{K+1}\,\frac{(K-2)(K-1)K}{(K+2)(K+3)}\left(\frac{r'_{ijo}\,\tau}{r_{ijo}}\right)^{K-3} + \sum_{K=0}^{\infty} Q^{*}_{K}\,\frac{\tau^{K}}{(K+2)(K+3)\,K!}\,,$$

$$(i,j=1,\cdots n,\ i\neq j),\ \tau\in(-\tau^{*},\tau^{*}).$$

This equation, by shifting the origin of the series to $K=0$, takes the form

$$\sum_{K=0}^{\infty} B_{ij\,K+2,0}\,\frac{\tau^{K}}{(K+2)(K+3)\,K!} = \frac{1}{2}\sum_{K=0}^{\infty}(-1)^{K+1}\,\frac{(K+1)}{(K+3)}\left(\frac{r'_{ijo}\,\tau}{r_{ijo}}\right)^{K} +$$

$$+\frac{\tau^{2}}{4}\,\frac{r''_{ijo}}{r_{ijo}}\sum_{K=0}^{\infty}(-1)^{K}\,\frac{(K+1)(K+2)(K+3)}{(K+4)(K+5)}\left(\frac{r'_{ijo}\,\tau}{r_{ijo}}\right)^{K}+$$

$$+\frac{\tau^{3}}{12}\,\frac{r'''_{ijo}}{r_{ijo}}\sum_{K=0}^{\infty}(-1)^{K}\,\frac{(K+1)(K+2)(K+3)}{(K+5)(K+6)}\left(\frac{r'_{ijo}\,\tau}{r_{ijo}}\right)^{K} + \sum_{K=0}^{\infty} Q^{*}_{K}\,\frac{\tau^{K}}{(K+2)(K+3)\,K!}\,,$$

$$(i,j=1,\cdots n\ \ i\neq j),\ \tau\in(-\tau^{*},\tau^{*}),$$

which, by introduction of equations (A3-11), (A3-12), (A3-13) and the notation (A3-21), becomes

$$\sum_{K=2}^{\infty} B_{ijK,0} \frac{\tau^{K-2}}{K(K+1)(K-2)!} = \sum_{K=0}^{\infty} B_{ijK+2,0} \frac{\tau^{K}}{(K+2)(K+3)\,K!} =$$

$$= \frac{1}{2}\left(\frac{2\ln(1+\varphi_{ij})}{\varphi_{ij}^{3}} - \frac{\varphi_{ij}+2}{(1+\varphi_{ij})\,\varphi_{ij}^{2}}\right) + \frac{\tau^2}{4}\frac{r''_{ij0}}{r_{ij0}}\left(\frac{7\varphi_{ij}+6}{(1+\varphi_{ij})^2\varphi_{ij}^{3}} + 6\frac{\ln(1+\varphi_{ij})-4}{\varphi_{ij}^{4}} + \right.$$

$$\left. +24\frac{\ln(1+\varphi_{ij})}{\varphi_{ij}^{5}}\right) + \frac{\tau^3}{12}\frac{r'''_{ij0}}{r_{ij0}}\left(\frac{3}{\varphi_{ij}^{4}} - \frac{10\varphi_{ij}+9}{(1+\varphi_{ij})^2\varphi_{ij}^{4}} - 12\frac{2\ln(1+\varphi_{ij})-5}{\varphi_{ij}^{5}} - \right. \qquad \text{(A3-23)}$$

$$\left. -60\frac{\ln(1+\varphi_{ij})}{\varphi_{ij}^{6}}\right) + \sum_{K=0}^{\infty} Q^{*}_{K}\frac{(K+1)\tau^{K}}{(K+3)!}, \; (i,j=1,\cdots n,\; i\neq j),\; \tau\in(-\tau^{*},\tau^{*}),\; |\varphi_{ij}|<1,\; \varphi_{ij}\neq 0.$$

Following similar steps one can evaluate the third of the series (A3-17). With the help of equations (A3-18) this series takes the form

$$\sum_{K=0}^{\infty} B_{ijK+2,0}\frac{\tau^{K}}{(K+3)(K+4)\,K!} = \frac{1}{2}\sum_{K=0}^{\infty}(-1)^{K+1}\frac{(K+1)(K+2)}{(K+3)(K+4)}\left(\frac{r'_{ij0}\tau}{r_{ij0}}\right)^{K} +$$

$$+\frac{\tau^2}{4}\frac{r''_{ij0}}{r_{ij0}}\sum_{K=0}^{\infty}(-1)^{K}\frac{(K-1)K(K+1)}{(K+3)(K+4)}\left(\frac{r'_{ij0}\tau}{r_{ij0}}\right)^{K-2} +$$

$$+\frac{\tau^3}{12}\frac{r'''_{ij0}}{r_{ij0}}\sum_{K=0}^{\infty}(-1)^{K+1}\frac{(K-2)(K-1)K}{(K+3)(K+4)}\left(\frac{r'_{ij0}\tau}{r_{ij0}}\right)^{K-3} + \sum_{K=0}^{\infty} Q^{*}_{K}\frac{\tau^{K}}{(K+3)(K+4)\,K!} =$$

$$= \frac{1}{2}\sum_{K=0}^{\infty}(-1)^{K+1}\frac{(K+1)(K+2)}{(K+3)(K+4)}\left(\frac{r'_{ij0}\tau}{r_{ij0}}\right)^{K} + \frac{\tau^2}{4}\frac{r''_{ij0}}{r_{ij0}}\sum_{K=2}^{\infty}(-1)^{K}\frac{(K-1)K(K+1)}{(K+3)(K+4)}\left(\frac{r'_{ij0}\tau}{r_{ij0}}\right)^{K-2} +$$

$$+\frac{\tau^3}{12}\frac{r'''_{ij0}}{r_{ij0}}\sum_{K=3}^{\infty}(-1)^{K+1}\frac{(K-2)(K-1)K}{(K+3)(K+4)}\left(\frac{r'_{ij0}\tau}{r_{ij0}}\right)^{K-3}+\sum_{K=0}^{\infty}Q^*_K\frac{\tau^K}{(K+3)(K+4)K!},$$

$$(i,j=1,\cdots\cdots n,\ i\neq j),\ \tau\in(-\tau^*,\tau^*).$$

Introducing equation (A3-21) and shifting the origin of the series to $K=0$, we obtain

$$\sum_{K=0}^{\infty}B_{ijK+2,0}\frac{\tau^K}{(K+3)(K+4)K!}=\frac{1}{2}\sum_{K=0}^{\infty}(-1)^{K+1}\frac{(K+1)(K+2)}{(K+3)(K+4)}\varphi_{ij}^K+$$

$$+\frac{\tau^2}{4}\frac{r''_{ij0}}{r_{ij0}}\sum_{K=0}^{\infty}(-1)^K\frac{(K+1)(K+2)(K+3)}{(K+5)(K+6)}\varphi_{ij}^K+\frac{\tau^3}{12}\frac{r'''_{ij0}}{r_{ij0}}\sum_{K=0}^{\infty}(-1)^K\frac{(K+1)(K+2)(K+3)}{(K+6)(K+7)}\varphi_{ij}^K+$$

$$+\sum_{K=0}^{\infty}Q^*_K\frac{\tau^K}{(K+3)(K+4)K!},\ (i,j=1,\cdots n,\ i\neq j),\ \tau\in(-\tau^*,\tau^*),\ |\varphi_{ij}|<1.$$

This relation, upon substitution of equations (A3-13), (A3-14) and (A3-15), becomes

$$\sum_{K=2}^{\infty}B_{ijK,0}\frac{\tau^{K-2}}{(K+1)(K+2)(K-2)!}=\sum_{K=0}^{\infty}B_{ijK+2,0}\frac{\tau^K}{(K+3)(K+4)K!}=$$

$$=\frac{1}{2}\left(\frac{5\varphi_{ij}+6}{(1+\varphi_{ij})\varphi_{ij}^3}-\frac{2\ln(1+\varphi_{ij})}{\varphi_{ij}^3}-\frac{6\ln(1+\varphi_{ij})}{\varphi_{ij}^4}\right)+$$

$$+\frac{\tau^2}{4}\frac{r''_{ij0}}{r_{ij0}}\left(\frac{3}{\varphi_{ij}^4}-\frac{10\varphi_{ij}+9}{(1+\varphi_{ij})^2\varphi_{ij}^4}-12\frac{2\ln(1+\varphi_{ij})-5}{\varphi_{ij}^5}-\frac{60\ln(1+\varphi_{ij})}{\varphi_{ij}^6}\right)+\quad\text{(A3-24)}$$

$$+\frac{\tau^3}{12}\frac{r'''_{ij0}}{r_{ij0}}\left(\frac{13\varphi_{ij}+12}{(1+\varphi_{ij})^2\varphi_{ij}^5}+60\frac{\ln(1+\varphi_{ij})-2}{\varphi_{ij}^6}+\frac{120\ln(1+\varphi_{ij})}{\varphi_{ij}^7}+\frac{\varphi_{ij}-12}{\varphi_{ij}^5}\right)+$$

$$+\sum_{K=0}^{\infty} Q^{*}_{K} \frac{\tau^{K}}{(K+3)(K+4)\,K!}\,,\ (i,j=1,\dots n, i\neq j)\,,\ \tau\in(-\tau^{*},\tau^{*})\,,\ |\varphi_{ij}|<1\,,\ \varphi_{ij}\neq 0\,.$$

We give finally the evaluation of the last of the series (A3-17). Comparing this series with equation (A3-18), we readily conclude

$$\sum_{K=0}^{\infty} B_{ij\,K+2,0} \frac{\tau^{K}}{(K+4)(K+5)\,K!} = \frac{1}{2}\sum_{K=0}^{\infty}(-1)^{K+1}\frac{(K+1)(K+2)}{(K+4)(K+5)}\left(\frac{r'_{ij0}\tau}{r_{ij0}}\right)^{K} +$$

$$+\frac{\tau^2}{4}\frac{r''_{ij0}}{r_{ij0}}\sum_{K=0}^{\infty}(-1)^{K}\frac{(K-1)K(K+1)}{(K+4)(K+5)}\left(\frac{r'_{ij0}\tau}{r_{ij0}}\right)^{K-2} +$$

$$+\frac{\tau^3}{12}\frac{r'''_{ij0}}{r_{ij0}}\sum_{K=0}^{\infty}(-1)^{K+1}\frac{(K-2)(K-1)K}{(K+4)(K+5)}\left(\frac{r'_{ij0}\tau}{r_{ij0}}\right)^{K-3} + \sum_{K=0}^{\infty} Q^{*}_{K}\frac{\tau^{K}}{(K+4)(K+5)\,K!} =$$

$$=\frac{1}{2}\sum_{K=0}^{\infty}(-1)^{K+1}\frac{(K+1)(K+2)}{(K+4)(K+5)}\left(\frac{r'_{ij0}\tau}{r_{ij0}}\right)^{K} + \frac{\tau^2}{4}\frac{r''_{ij0}}{r_{ij0}}\sum_{K=2}^{\infty}(-1)^{K}\frac{(K-1)K(K+1)}{(K+4)(K+5)}\left(\frac{r'_{ij0}\tau}{r_{ij0}}\right)^{K-2} +$$

$$+\frac{\tau^3}{12}\frac{r'''_{ij0}}{r_{ij0}}\sum_{K=0}^{\infty}(-1)^{K+1}\frac{(K-2)(K-1)K}{(K+4)(K+5)}\left(\frac{r'_{ij0}\tau}{r_{ij0}}\right)^{K-3} + \sum_{K=0}^{\infty} Q^{*}_{K}\frac{\tau^{K}}{(K+4)(K+5)\,K!}\,,$$

$$(i,j=1,\dots n, i\neq j)\,,\ \tau\in(-\tau^{*},\tau^{*})\,.$$

This relation, by shifting the origin of the series to $K=0$ and introducing equation (A3-21) yields

$$\sum_{K=0}^{\infty} B_{ij\,K+2,0}\frac{\tau^{K}}{(K+4)(K+5)\,K!} =$$

$$=\frac{1}{2}\sum_{K=0}^{\infty}(-1)^{K+1}\frac{(K+1)(K+2)}{(K+4)(K+5)}\varphi_{ij}^{K} + \frac{\tau^2}{2}\frac{r''_{ij0}}{r_{ij0}}\sum_{K=0}^{\infty}(-1)^{K}\frac{(K+1)(K+2)(K+3)}{(K+6)(K+7)}\varphi_{ij}^{K} +$$

$$+\frac{\tau^3}{12}\frac{r'''_{ijo}}{r_{ijo}}\sum_{K=0}^{\infty}(-1)^{K+1}\frac{(K-2)(K-1)K}{(K+4)(K+5)}\left(\frac{r'_{ijo}\tau}{r_{ijo}}\right)^{K-3}+\sum_{K=0}^{\infty}Q^*_K\frac{\tau^K}{(K+4)(K+5)K!},$$

$$(i,j=1,\cdots\cdots n,\, i\neq j),\ \tau\in(-\tau^*,\tau^*),\ |\varphi_{ij}|<1,\ \varphi_{ij}\neq 0,$$

which, upon substitution of equations (A3-14) and (A3-16), becomes

$$\sum_{K=2}^{\infty}B_{ijK,0}\frac{\tau^{K-2}}{(K+2)(K+3)(K-2)!}\equiv\sum_{K=0}^{\infty}B_{ijK+2,0}\frac{\tau^K}{(K+4)(K+5)K!}=$$

$$=\frac{1}{2}\left(\frac{6\ln(1+\varphi_{ij})}{\varphi_{ij}^4}-\frac{\varphi_{ij}+12\varphi_{ij}+12}{(1+\varphi_{ij})\varphi_{ij}^4}-\frac{12\ln(1+\varphi_{ij})}{\varphi_{ij}^5}\right)+ \qquad \text{(A3-25)}$$

$$+\frac{\tau^2}{4}\frac{r''_{ijo}}{r_{ijo}}\left(\frac{13\varphi_{ij}+12}{(1+\varphi_{ij})^2\varphi_{ij}^5}+60\frac{\ln(1+\varphi_{ij})-2}{\varphi_{ij}^6}+\frac{120\ln(1+\varphi_{ij})}{\varphi_{ij}^7}+\frac{\varphi_{ij}-12}{\varphi_{ij}^5}\right)+$$

$$+\frac{\tau^3}{12}\frac{r'''_{ijo}}{r_{ijo}}\sum_{K=0}^{\infty}(-1)^{K+1}\frac{(K-2)(K-1)K}{(K+4)(K+5)}\left(\frac{r'_{ijo}\tau}{r_{ijo}}\right)^{K-3}+\sum_{K=0}^{\infty}Q^*_K\frac{\tau^K}{(K+4)(K+5)K!},$$

$$(i,j=1,\cdots\cdots n,\, i\neq j),\ \tau\in(-\tau^*,\tau^*),\ |\varphi_{ij}|<1,\ \varphi_{ij}\neq 0.$$

With this relation we conclude the evaluation of the series appearing in the right hand side of equation (1.33).

APPENDIX A4

Evaluation of the common volume V of the i^{th} and j^{th} spheroids and its partial derivatives $\partial V/\partial R_{ij}$ and $\partial V/\partial \alpha$ (Chapter 2).

The analytical expression of the volume V obviously does not depend on the system of reference; we can, therefore, simplify its evaluation by a suitable choice of axes.

Consider a Cartesian right-handed system of axes X', Y', Z' whose origin is located at the centre O_j of the j^{th} spheroid; the $X'Y'$ plane coincides with the equatorial plane of the i^{th} and j^{th} spheroids and the Y' axis passes through the centre O_i of the i^{th} spheroid. In this system of reference, the positions and orientation of the i^{th}, j^{th}, J^{th} spheroids and the j^{th} homoeoid are shown in figure (A4.1). The internal surface of the infinitesimally thin focaloid, whose semi-major axis is α, intersects the i^{th} spheroid in the curve $E\Gamma Z$, whose projection on the $X'Y'$ plane is the dashed curve EKZ. Obviously, E and Z are the common points of these two curves and the peripheries of the equatorial planes ($E\Delta Z$, EBZ) of the i^{th} and J^{th} spheroids.

We define ξ as the distance from the origin of the system of reference to the plane EAZ perpendicular to Y' axis, and D as the distance between the centres of the i^{th} and j^{th} spheroids. The rest of the notation follows that of chapter 2.

The equations of the i^{th} and J^{th} spheroids, in the above described system of reference, have the form

$$Z^2=\frac{c_i^2}{\alpha_i^2}\left[\alpha_i^2-x^2-(y-D)^2\right],\quad Z^2=\frac{c^2}{\alpha^2}\left[\alpha^2-x^2-y^2\right]. \tag{A4-1}$$

If these two surfaceshave more than one common point, they cut each other in a curve ($E\Gamma Z$), through which the cylinder

$$\frac{c_i^2}{\alpha_i^2}\left[\alpha_i^2-x^2-(y-D)^2\right]=\frac{c^2}{\alpha^2}\left[\alpha^2-x^2-y^2\right],\quad Z\in(-\infty,+\infty) \tag{A4-2}$$

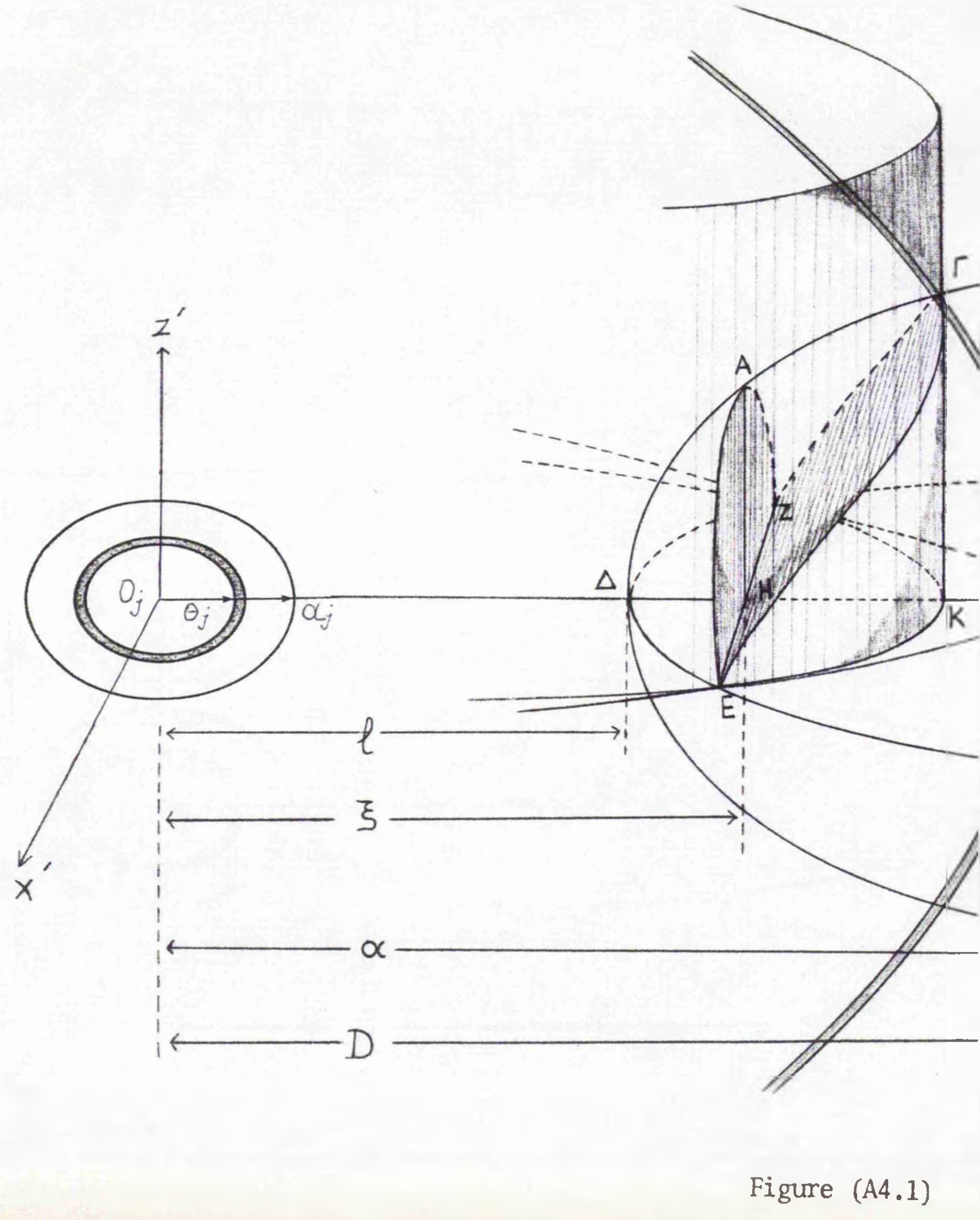
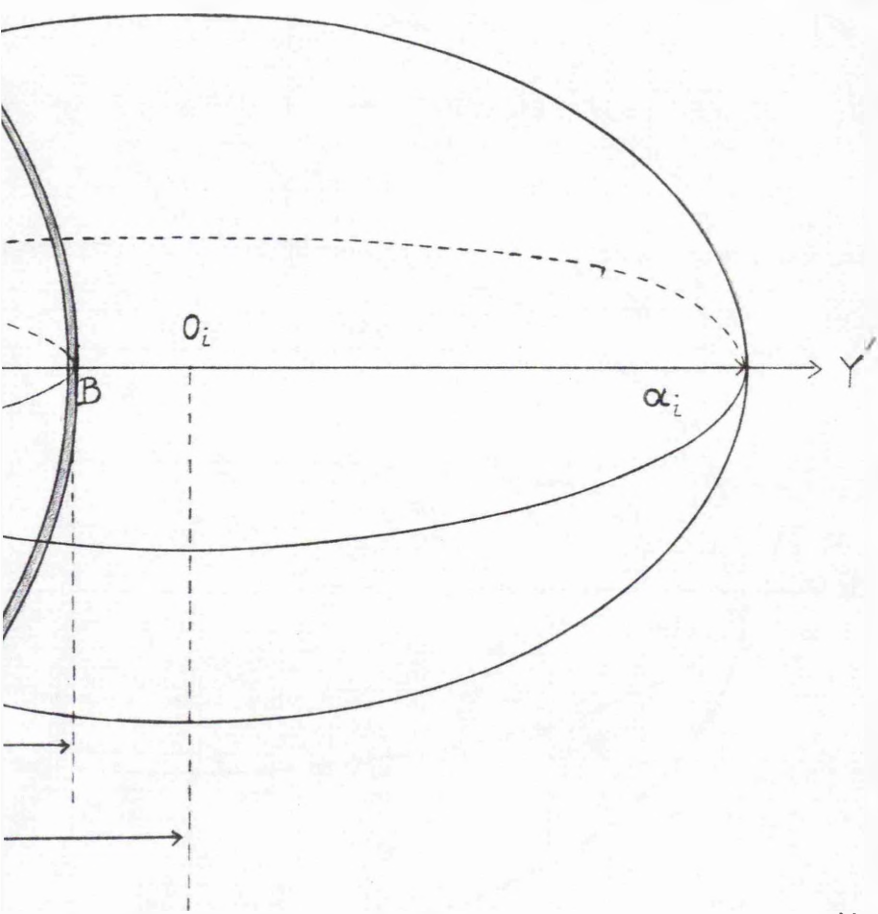


Figure (A4.1)



O_i
B
α_i
Y

passes. The plane $Z=0$ intersects this cylinder in a circle (EKZ) whose equation, according to equation (A4-2), can take the form

$$x^2\left(\frac{c^2}{\alpha^2}-\frac{c_i^2}{\alpha_i^2}\right)+y^2\left(\frac{c^2}{\alpha^2}-\frac{c_i^2}{\alpha_i^2}\right)+2yD\frac{c_i^2}{\alpha_i^2}=(c^2-c_i^2)+D^2\frac{c_i^2}{\alpha_i^2}\cdot \tag{A4-3}$$

With the assumption that

$$\frac{c^2}{\alpha^2}-\frac{c_i^2}{\alpha_i^2}\neq 0,$$

(this implies that the i^{th} and J^{th} spheroids have different eccentricities), equation (A4-3) can be written as follows

$$x^2+y^2+2yD\frac{c_i^2\alpha^2}{c^2\alpha_i^2-c_i^2\alpha^2}=\frac{(c^2-c_i^2)\,\alpha^2\alpha_i^2+D^2\alpha^2c_i^2}{c^2\alpha_i^2-c_i^2\alpha^2}, \tag{A4-4}$$

or more generally $x^2+(y+B)^2=R^2$, where

$$B=\frac{Dc_i^2\alpha^2}{c^2\alpha_i^2-c_i^2\alpha^2},\quad R=\sqrt{\left(\frac{D\alpha c\alpha_i c_i}{c^2\alpha_i^2-c_i^2\alpha^2}\right)^2+\frac{(c^2-c_i^2)\alpha_i^2\alpha^2}{c^2\alpha_i^2-c_i^2\alpha^2}}\,.$$

From this equation it is obvious that if $B>0$, that is to say $\left(\frac{c^2}{\alpha^2}-\frac{c_i^2}{\alpha_i^2}\right)>0$, the circle given by equation (A4-4) has its centre on the left of the origin of the reference system, and on the right if $B<0$. If $\left(\frac{c^2}{\alpha^2}-\frac{c_i^2}{\alpha_i^2}\right)=0$ equation (A4-3) reduces to

$$y=\frac{\alpha_i^2(c^2-c_i^2)}{2c_i^2D}+D=\frac{D^2+\alpha^2-\alpha_i^2}{2D},$$

which represents a line parallel to the X' axis. This line cuts the Y' axis in a point whose distance from the origin of the axis is $(D^2+\alpha^2-\alpha_i^2)/2D$ and equal to ξ, as will be proved below.

In figures (A4.2) and (A4.3) the intersections of the i^{th} and J^{th} spheroids with the $X'Y'$ plane are illustrated for $D-\alpha_i\leqslant\xi\leqslant D$ and $D<\xi\leqslant D+\alpha_i$ respectively

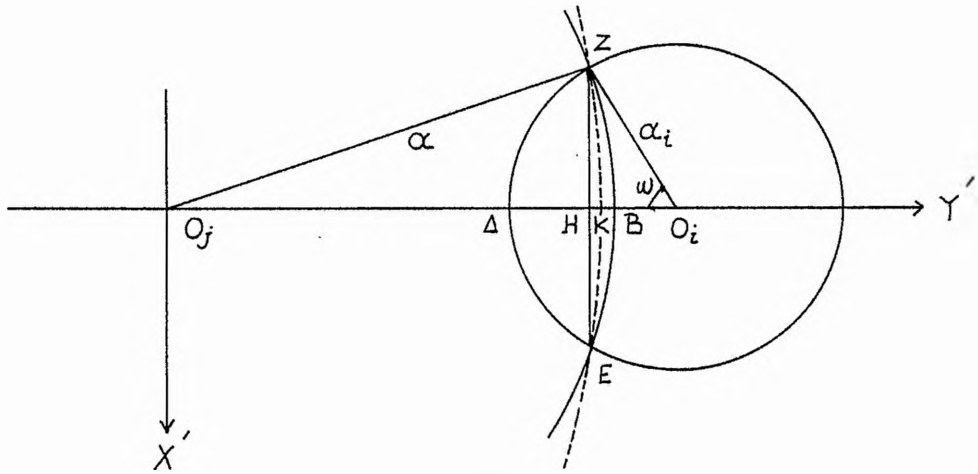


Figure (A4.2)

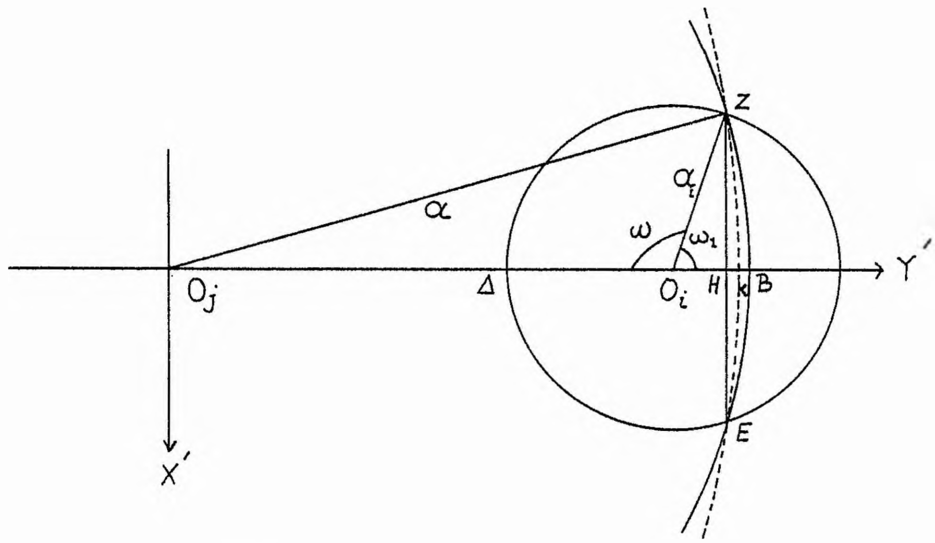


Figure (A4.3)

From these figures it becomes clear that

$$\cos\omega = (D^2-\alpha^2+\alpha_i^2)/2D\alpha_i\,,\quad D-\alpha_i \leqslant \xi \leqslant D,$$

$$\cos\omega_1 = -\cos\omega = -(D^2+\alpha_i^2-\alpha^2)/2D\alpha_i\,,\quad D < \xi \leqslant D+\alpha_i\,,$$

hence

$$\xi = D - O_iH = D - \alpha_i\cos\omega = (D^2+\alpha^2-\alpha_i^2)/2D\,,\quad D-\alpha_i \leqslant \xi \leqslant D,$$

$$\xi = D + O_iH = D + \alpha_i\cos\omega_1 = (D^2+\alpha^2-\alpha_i^2)/2D\,,\quad D < \xi \leqslant D+\alpha_i\,. \tag{A4-5}$$

We thus see that ξ is given by the same expression for every $\alpha \in [D-\alpha_i, D+\alpha_i]$.

The evaluation of the volume V becomes easier by observing that it consists of two parts, V_1 and V_2, where V_1 is the volume confined by the surfaces described by the first of equations (A4-1) and equation (A4-2), and V_2 is the volume confined by the surfaces described by the second of equations (A4-1) and equation (A4-2) . From the geometry and orientation of the spheroids it is obvious that the volume V is symmetrical about $X'Y'$ and $Y'Z'$ co-ordinate planes and therefore it is sufficient to consider only one-fourth of it, namely that being in the space $X \geqslant 0, Z \geqslant 0$. We note also that, due to the position of the i^{th} and J^{th} spheroids, every point on the surface of the volume V satisfies the inequality $y > 0$. Hence the limits of the integrals appearing in the following equation must be positive.

From the preceding it becomes evident that the analytic expressions for the volumes V_1 , V_2 and V are

$$V_1 = 4\int_{\ell}^{\xi}\int_0^{x_1(y)}\int_0^{z_1(x,y)} dz\,dx\,dy + 4\int_{\xi}^{\eta}\int_0^{\mathcal{X}(y)}\int_0^{z_1(x,y)} dz\,dx\,dy \,,$$

$$V_2 = 4\int_{\xi}^{\alpha}\int_0^{x_2(y)}\int_0^{z_2(x,y)} dz\,dx\,dy - 4\int_{\xi}^{\eta}\int_0^{\mathcal{X}(y)}\int_0^{z_2(x,y)} dz\,dx\,dy \,; \qquad \text{(A4-6)}$$

$$V = V_1 + V_2$$

where

$$Z_1(x,y) = \frac{c_i}{\alpha_i}\left(\alpha_i^2 - x^2 - (y-D)^2\right)^{1/2}, \quad Z_2(x,y) = \frac{c}{\alpha}\left(\alpha^2 - x^2 - y^2\right)^{1/2}, \qquad \text{(A4-7)}$$

$$x_1(y) = \left(\alpha_i^2 - (y-D)^2\right)^{1/2}, \quad x_2(y) = \left(\alpha^2 - y^2\right)^{1/2}, \quad \ell = D - \alpha_i \,,$$

and $\mathcal{X}(y)$ is the positive root of equation (A4-3) solved for x. That is,

$$\mathcal{X}(y) = \sqrt{-y^2 - 2Ay + \Gamma} \,, \qquad \text{(A4 -8)}$$

where the parameters $\mathcal{A}$ and Γ occurring in this equation are given by the relations

$$\mathcal{A} \equiv D\frac{c_i^2\alpha^2}{c^2\alpha_i^2 - c_i^2\alpha^2}\,,\quad \Gamma \equiv \frac{(c^2-c_i^2)\,\alpha_i^2\alpha^2 + D^2c_i^2\alpha^2}{c^2\alpha_i^2 - c_i^2\alpha^2}\,. \tag{A4-9}$$

Finally η is the y co-ordinate of the point of intersection of the positive Y' semi-axis with the curve (A4-3). One can easily find that

$$\eta = O_j K = -\frac{D c_i^2\alpha^2}{c^2\alpha_i^2 - c_i^2\alpha^2} + \frac{\sqrt{(D^2-\alpha^2-\alpha_i^2)c^2c_i^2\alpha^2\alpha_i^2 + \alpha^2\alpha_i^2(c^4\alpha_i^2 + c_i^4\alpha^2)}}{c^2\alpha_i^2 - c_i^2\alpha^2}\,. \tag{A4-10}$$

Introducing the first two of equations (A4-7) into the expressions (A4-6), giving the volumes V_1, V_2 and V, we readily conclude that

$$V = 4\frac{c_i}{\alpha_i}\int_{\ell}^{\xi}\int_{0}^{x_1(y)}\sqrt{\alpha_i^2 - x^2 - (y-D)^2}\,dx\,dy + 4\frac{c_i}{\alpha_i}\int_{\xi}^{\eta}\int_{0}^{x(y)}\sqrt{\alpha_i^2 - x^2 - (y-D)^2}\,dx\,dy +$$

$$+ 4\frac{c}{\alpha}\int_{\xi}^{\alpha}\int_{0}^{x_2(y)}\sqrt{\alpha^2 - x^2 - y^2}\,dx\,dy - 4\frac{c}{\alpha}\int_{\xi}^{\eta}\int_{0}^{x(y)}\sqrt{\alpha^2 - x^2 - y^2}\,dx\,dy\,.$$

This equation, upon substitution of the relation

$$2\sqrt{K^2-\varphi^2} = \frac{d\left[\varphi(K^2-\varphi^2)^{1/2} + K^2\arcsin\frac{\varphi}{K}\right]}{d\varphi}\,,$$

becomes

$$V = \pi\frac{c_i}{\alpha_i}\int_{\ell}^{\xi}\left[\alpha_i^2 - (y-D)^2\right]dy + \pi\frac{c}{\alpha}\int_{\xi}^{\alpha}\left[\alpha^2 - y^2\right]dy +$$

$$+2\frac{c_i}{\alpha_i}\int_{\xi}^{\eta}\left[\left\{x(y)\sqrt{\alpha_i^2-(y-D)^2-x^2(y)}+\left[\alpha_i^2-(y-D)^2\right]\arcsin\frac{x(y)}{\sqrt{\alpha_i^2-(y-D)^2}}\right\}dy-\right.$$

(A 4-11)

$$-2\frac{c}{\alpha}\int_{\xi}^{\eta}\left[\left\{x(y)\sqrt{\alpha^2-y^2-x^2(y)}+(\alpha^2-y^2)\arcsin\frac{x(y)}{\sqrt{\alpha^2-y^2}}\right]\right\}dy.$$

Since $x(y)$ is, by definition, one of the roots of equation (A4-3) solved for x, it will satisfy the relation

$$\frac{c_i}{\alpha_i}\sqrt{\alpha_i^2-(y-D)^2-x^2(y)}=\frac{c}{\alpha}\sqrt{\alpha^2-y^2-x^2(y)}. \qquad \text{(A4-12)}$$

We note that equations (A4-2) and (A4-3) are identical.

With the help of equation (A4-12), and after some algebraic manipulation, equation (A4-11) becomes

$$V=\pi c_i\alpha_i(\xi-\ell)-\frac{\pi c_i}{3\alpha_i}(\xi-D)^3-\frac{\pi c_i}{3\alpha_i}(D-\ell)^3+\pi c\alpha(\alpha-\xi)-\frac{\pi c}{3\alpha}(\alpha^3-\xi^3)+$$

$$+2\frac{c_i}{\alpha_i}\int_{\xi}^{\eta}\left[\alpha_i^2-(y-D)^2\right]\arcsin\left(\frac{x(y)}{\sqrt{\alpha_i^2-(y-D)^2}}\right)dy-$$

$$-2\frac{c}{\alpha}\int_{\xi}^{\eta}(\alpha^2-y^2)\arcsin\left(\frac{x(y)}{\sqrt{\alpha^2-y^2}}\right)dy.$$

Introducing in this equation the following elementary relations

$$\xi-D=\frac{(D^2+\alpha_i^2-\alpha^2)}{-2D}, \quad \xi-\ell=\alpha_i-\frac{(D^2+\alpha_i^2-\alpha^2)}{2D}, \quad D-\ell=\alpha_i,$$

$$c_i=\alpha_i(1-e_i^2)^{\frac{1}{2}}, \quad c=(\alpha^2-\theta_j^2e_j^2)^{1/2},$$

and making some minor rearrangments we finally obtain

$$V=\pi(1-e_i^2)^{1/2}\left[\frac{2}{3}\alpha_i^3-\frac{D^2+\alpha_i^2}{2D}\alpha_i^2+\frac{\alpha_i^2\alpha^2}{2D}+\frac{(D^2+\alpha_i^2-\alpha^2)^3}{24D^3}\right]+$$

$$+\pi(\alpha^2-\theta_j^2e_j^2)^{1/2}\left[\frac{2}{3}\alpha^2-\frac{D^2-\alpha_i^2}{2D}\alpha-\frac{\alpha^3}{2D}+\frac{(D^2+\alpha^2-\alpha_i^2)^3}{24\alpha D^3}\right]+$$

$$+2\frac{c_i}{\alpha_i}\int_{\xi}^{\eta}\left[\alpha_i^2-(y-D)^2\right]\arcsin\left(\frac{x(y)}{\sqrt{\alpha_i^2-(y-D)^2}}\right)dy- \qquad \text{(A4-13)}$$

$$-2\frac{c}{\alpha}\int_{\xi}^{\eta}(\alpha^2-y^2)\arcsin\left(\frac{x(y)}{\sqrt{\alpha^2-y^2}}\right)dy.$$

The integrals occuring in this equation, as it will become clear later on, can not be expressed in finite terms of elementary functions

We shall next derive the analytical expression of the partial derivative $\partial V/\partial D$. The function $V=V(D,y)$ has no singularity or discontinuity in the interval $[\xi(D),\eta(D)]$, as we may deduce from equation (A4-13); we can therefore differentiate equation (A4-13) and use the formula (2.38) to obtain

$$\frac{\partial V}{\partial D}=\pi(1-e_i^2)^{1/2}\left[-\alpha_i^2+\alpha_i^2\frac{(D^2+\alpha_i^2-\alpha^2)}{2D^2}+\frac{(D^2+\alpha_i^2-\alpha^2)^2}{4D^2}-\frac{(D^2+\alpha_i^2-\alpha^2)^3}{8D^4}\right]+$$

$$+\pi(\alpha^2-\theta_j^2e_j^2)^{1/2}\left[-\alpha+\alpha\frac{(D^2+\alpha^2-\alpha_i^2)}{2D^2}+\frac{(D^2+\alpha^2-\alpha_i^2)^2}{4D^2\alpha}-\frac{(D^2+\alpha^2-\alpha_i^2)^3}{8\alpha D^4}\right]+$$

$$+2\frac{c_i}{\alpha_i}\left[\alpha_i^2-(\eta-D)^2\right]\arcsin\frac{x(\eta)}{\sqrt{\alpha_i^2-(\eta-D)^2}}\frac{\partial\eta}{\partial D}-2\frac{c_i}{\alpha_i}\left[\alpha_i^2-(\xi-D)^2\right]\arcsin\frac{x(\xi)}{\sqrt{\alpha_i^2-(\xi-D)^2}}\frac{\partial\xi}{\partial D}+ \qquad \text{(A4-14)}$$

$$+4\frac{c_i}{\alpha_i}\int_{\xi}^{\eta}(y-D)\arcsin\frac{x(y)}{\sqrt{\alpha_i^2-(y-D)^2}}dy+$$

$$+2\int_{\xi}^{\eta}\frac{c_i}{\alpha_i}\left[\alpha_i^2-(y-D)^2\right]\frac{\sqrt{\alpha_i^2-(y-D)^2}}{\sqrt{\alpha_i^2-(y-D)^2-x^2(y)}}\frac{\partial\left(\frac{x(y)}{\sqrt{\alpha_i^2-(y-D)^2}}\right)}{\partial D}dy-$$

$$-2\frac{c}{\alpha}(\alpha^2-\eta^2)\arcsin\left(\frac{x(\eta)}{\sqrt{\alpha^2-\eta^2}}\right)\frac{\partial\eta}{\partial D}+2\frac{c}{\alpha}(\alpha^2-\xi^2)\arcsin\left(\frac{x(\xi)}{\sqrt{\alpha^2-\xi^2}}\right)\frac{\partial\xi}{\partial D}-$$

$$-2\int_{\xi}^{\eta}\frac{c}{\alpha}(\alpha^2-y^2)\frac{\sqrt{\alpha^2-y^2}}{\sqrt{\alpha^2-y^2-x^2(y)}}\frac{\partial\left(\frac{x(y)}{\sqrt{\alpha^2-y^2}}\right)}{\partial D}dy.$$

It can easily be verified that

$$\frac{c_i}{\alpha_i}\left[\alpha_i^2-(y-D)^2\right]\frac{\sqrt{\alpha_i^2-(y-D)^2}}{\sqrt{\alpha_i^2-(y-D)^2-x^2(y)}}\frac{\partial\left(\frac{x(y)}{\sqrt{\alpha_i^2-(y-D)^2}}\right)}{\partial D}=$$

$$=\frac{c_i^2}{\alpha_i^2}\left[\alpha_i^2-(y-D)^2-x^2(y)\right]\frac{\partial\left(\frac{x(y)}{\frac{c_i}{\alpha_i}\sqrt{\alpha_i^2-(y-D)^2-x^2(y)}}\right)}{\partial D}, \qquad \text{(A4-15)}$$

$$\frac{c}{\alpha}(\alpha^2-y^2)\frac{\sqrt{\alpha^2-y^2}}{\sqrt{\alpha^2-y^2-x^2(y)}}\frac{\partial\left(\frac{x(y)}{\sqrt{\alpha^2-y^2}}\right)}{\partial D}=\frac{c^2}{\alpha^2}\left[\alpha^2-y^2-x^2(y)\right]\frac{\partial\left(\frac{x(y)}{\frac{c}{\alpha}\sqrt{\alpha^2-y^2-x^2(y)}}\right)}{\partial D}.$$

The values of the function $x(y)$ at the points $y=\xi$ and $y=\eta$ can be found with the help of equation (A4-8), which at these particular points reduces to

$$x(\xi)=\sqrt{\alpha_i^2-(\xi-D)^2}=\left(\alpha^2-\xi^2\right)^{1/2}, \quad x(\eta)=0. \qquad \text{(A4-16)}$$

Combining equations (A4-12), (A4-14), (A4-15) and (A4-16), we have

$$\frac{\partial V}{\partial D}=\pi(1-e_i^2)^{1/2}\left[-\alpha_i^2+\alpha_i^2\frac{(D^2+\alpha_i^2-\alpha^2)}{2D^2}+\frac{(D^2+\alpha_i^2-\alpha^2)^2}{4D^2}-\frac{(D^2+\alpha_i^2-\alpha^2)^3}{8D^4}\right]+$$

$$+\pi(\alpha^2-\theta_j^2e_j^2)^{1/2}\left[-\alpha+\alpha\frac{(D^2+\alpha^2-\alpha_i^2)}{2D^2}+\frac{(D^2+\alpha^2-\alpha_i^2)^2}{4D^2\alpha}-\frac{(D^2+\alpha^2-\alpha_i^2)^3}{8D^4\alpha}\right]-$$

$$-\pi\frac{c_i}{\alpha_i}(\alpha^2-\xi^2)\frac{\partial\xi}{\partial D}+\pi\frac{c}{\alpha}(\alpha^2-\xi^2)\frac{\partial\xi}{\partial D}- \qquad \text{(A4-17)}$$

$$-2\frac{c_i}{\alpha_i}\int_{\xi}^{\eta}\arcsin\left(\frac{x(y)}{\sqrt{\alpha_i^2-(y-D)^2}}\right)d\left[\alpha_i^2-(y-D)^2\right].$$

Differentiation of the formulae (A4-5) with respect to D furnishes the relation

$$\frac{\partial\xi}{\partial D}=\frac{D^2+\alpha_i^2-\alpha^2}{2D^2}.$$

On substituting this relation into equation (A4-17) and arranging the resulting expression we find

$$\frac{\partial V}{\partial D}=\pi(1-e_i^2)^{1/2}(\xi^2-\alpha^2)-2\frac{c_i}{\alpha_i}\int_{\xi}^{\eta}\arcsin\left(\frac{x(y)}{\sqrt{\alpha_i^2-(y-D)^2}}\right)d\left[\alpha_i^2-(y-D)^2\right].$$

To simplify this expression we integrate by parts and introduce equations (A4-16). Then we have

$$\frac{\partial V}{\partial D}=2\frac{c_i}{\alpha_i}\int_{\xi}^{\eta}\frac{\left[\alpha_i^2-(y-D)^2\right]^{3/2}}{\sqrt{\alpha_i^2-(y-D)^2-x^2(y)}}\frac{\partial}{\partial y}\left(\frac{x(y)}{\sqrt{\alpha_i^2-(y-D)^2}}\right)dy,$$

an equation, which can further be simplified with the help of the relation

$$\frac{c_i}{\alpha_i}\frac{\left[\alpha_i^2-(y-D)^2\right]^{3/2}}{\sqrt{\alpha_i^2-(y-D)^2-x^2(y)}}\frac{\partial}{\partial y}\left(\frac{x(y)}{\sqrt{\alpha_i^2-(y-D)^2}}\right)=$$

$$=\frac{c_i}{\alpha_i}\left[\alpha_i^2-(y-D)^2-x^2(y)\right]\frac{\partial}{\partial y}\left(\frac{x(y)}{\sqrt{\alpha_i^2-(y-D)^2-x^2(y)}}\right).$$

Comparison of the last two equations yields

$$\frac{\partial V}{\partial D}=2\frac{c_i}{\alpha_i}\int_{\xi}^{\eta}\left(\alpha_i^2-(y-D)^2-x^2(y)\right)d\left(\frac{x(y)}{\sqrt{\alpha_i^2-(y-D)^2-x^2(y)}}\right).$$

Integrating once more by parts, and using equations (A4-16), we obtain

$$\frac{\partial V}{\partial D}=-4\frac{c_i}{\alpha_i}\int_{\xi}^{\eta}x(y)\,d\left[\alpha_i^2-(y-D)^2-x^2(y)\right]^{1/2},$$

which, upon substitution of equation (A4-12), becomes

$$\frac{\partial V}{\partial D}=-4\frac{c}{\alpha}\int_{\xi}^{\eta}x(y)\,d\left(\alpha^2-y^2-x^2(y)\right)^{1/2}. \qquad \text{(A4-18)}$$

Before proceeding to the evaluation of the foregoing integral we give some equations, useful later on, the validity of which is easily proved by elementary calculations.

According to equation (A4-8) we, clearly, have

$$\alpha^2-y^2-x^2(y)=\alpha^2+2Ay-\Gamma. \qquad \text{(A4-19)}$$

On the other hand, equations (A4-8) and (A4-16) imply the relation

$$2A\xi=\Gamma-\alpha^2. \qquad \text{(A4-20)}$$

Introduction of this equation into equation (A4-19) yields

$$\alpha^2 - y^2 - x^2(y) = 2A(y - \xi) . \qquad \text{(A4-21)}$$

The roots of the quadratic expression $x^2(y) = -y^2 - 2Ay + \Gamma$ obviously, are

$$\rho_1^* = -A + (A^2 + \Gamma)^{1/2}, \quad \rho_2^* = -A - (A^2 + \Gamma)^{1/2},$$

which, upon substitution of equations (A4-9), become

$$\rho_1^* = n, \quad \rho_2^* = -(n + 2A);$$

therefore

$$x(y) = \sqrt{(n - y)(y + n + 2A)} . \qquad \text{(A4-22)}$$

Introduction of equations (A4-8), (A4-21) and (A4-22) into equation (A4-18) and rearrangement of the resulting expression gives

$$\frac{\partial V}{\partial D} = \frac{4c\sqrt{A}}{\alpha\sqrt{2}} \int_{\xi}^{n} \frac{y^2 + 2Ay - \Gamma}{\sqrt{(n - y)(y - \xi)(y - [-n - 2A])}} \, dy . \qquad \text{(A4-23)}$$

Consider now the expression $C_0 t^3 + C_1 t^2 + C_2 t + C_3$ and suppose ρ_1, ρ_2, ρ_3 to be its roots given in descending order, then the validity of the relation

$$\int \frac{t^2 \, dt}{\sqrt{C_0 t^3 + C_1 t^2 + C_2 t + C_3}} \equiv \int \frac{t^2 \, dt}{\sqrt{C_0 (t - \rho_1)(t - \rho_2)(t - \rho_3)}} = \frac{2}{3C_0} \sqrt{(t - \rho_1)(t - \rho_2)(t - \rho_3) C_0} - \qquad \text{(A4-24)}$$

$$- \frac{2C_1}{3C_0} \int \frac{t \, dt}{\sqrt{C_0 (t - \rho_1)(t - \rho_2)(t - \rho_3)}} - \frac{C_2}{3C_0} \int \frac{dt}{\sqrt{C_0 (t - \rho_1)(t - \rho_2)(t - \rho_3)}} + \text{constant}$$

can easily be proved by differentiation of both sides with respect to t. Applying this relation in the case of equation (A4-23), we obtain

$$\frac{\partial V}{\partial D}=\frac{4c\sqrt{A}}{\alpha\sqrt{2}}\left\{\frac{2}{3}(A+\xi)\int_{\xi}^{\eta}\frac{y\,dy}{\sqrt{(\eta-y)(y-\xi)(y-[-\eta-2A])}}-\left[\Gamma-\frac{1}{3}(\eta^2+2A\eta+2A\xi)\right]\int_{\xi}^{\eta}\frac{dy}{\sqrt{(\eta-y)(y-\xi)(y-[-\eta-2A])}}\right\}. \tag{A4-25}$$

Combining equations (A4-8) and (A4-16) we have

$$\eta^2+2\eta A=\Gamma;$$

therefore

$$\Gamma-\frac{1}{3}(\eta^2+2\eta A+2A\xi)=\frac{2}{3}(\Gamma-A\xi),$$

which, by introduction of equation (A4-20), takes the form

$$\Gamma-\frac{1}{3}(\eta^2+2A\eta+2A\xi)=\frac{1}{3}(\Gamma+\alpha^2).$$

On substituting this equation into the relation (A4-25), we obtain

$$\frac{\partial V}{\partial D}=\frac{4c\sqrt{A}}{3\alpha\sqrt{2}}\left\{2(A+\xi)\int_{\xi}^{\eta}\frac{y\,dy}{\sqrt{(\eta-y)(y-\xi)(y-[-\eta-2A])}}-(\Gamma+\alpha^2)\int_{\xi}^{\eta}\frac{dy}{\sqrt{(\eta-y)(y-\xi)(y-[-\eta-2A])}}\right\}. \tag{A4-26}$$

Since the roots of the radicand, occurring in this equation, are all real reduction of the above integrals to Legendre's standard form is accomplished by means of the continuous transformation

$$y=\frac{(\eta+2A)(\eta-\xi)\sin^2\varphi+2(\eta+A)\xi}{2(\eta+A)-(\eta-\xi)\sin^2\varphi}. \qquad \text{(A4-27)}$$

This transformation implies

$$dy=\frac{4(\xi+\eta+2A)(\eta-\xi)(\eta+A)\sin\varphi\cos\varphi}{[2(\eta+A)-(\eta-\xi)\sin^2\varphi]^2}d\varphi ,$$

$$\sqrt{(\eta-y)(y-\xi)(y-[-\eta-2A])}=$$

$$=\frac{2(\xi+\eta+2A)(\eta-\xi)(\eta+A)\sqrt{2(\eta+A)}}{[2(\eta+A)-(\eta-\xi)\sin^2\varphi]^2}\sin\varphi\cdot\cos\varphi\sqrt{1-\frac{(\eta-\xi)}{2(\eta+A)}\sin^2\varphi} , \qquad \text{(A4-28)}$$

and $\varphi\in[0,\frac{\pi}{2}]$ for $y\in[\xi,\eta]$. Further, from equation (A4-27) we obtain

$$y=-(\eta+2A)+2(\eta+A)\left(1-\frac{\eta-\xi}{2(\eta+A)}\sin^2\varphi\right)+(\eta-\xi)(\sin^2\varphi-\cos^2\varphi)-$$
$$-\frac{(\eta-\xi)^2\sin^2\varphi\cos^2\varphi}{2(\eta+A)-(\eta-\xi)\sin^2\varphi}. \qquad \text{(A4-29)}$$

Combining equations (A4-28) and (A4-29), one can find

$$\frac{dy}{\sqrt{(\eta-y)(y-\xi)(y-[-\eta-2A])}}=\frac{2}{\sqrt{2(\eta+A)}}\frac{d\varphi}{\sqrt{1-\frac{(\eta-\xi)}{2(\eta+A)}\sin^2\varphi}} ,$$

$$\frac{y\,dy}{\sqrt{(\eta-y)(y-\xi)(y-[-\eta-2A])}}=\frac{2}{\sqrt{2(\eta+A)}}\left\{-\frac{(\eta+2A)}{\sqrt{1-\frac{\eta-\xi}{2(\eta+A)}\sin^2\varphi}}+\right. \qquad \text{(A4-30)}$$

$$+2(\eta\pm A)\sqrt{1-\frac{\eta-\xi}{2(\eta\pm A)}\sin^2\varphi}\Bigg]d\varphi+2d\left(\frac{(\eta-\xi)\sin\varphi\cdot\cos\varphi}{\sqrt{2(\eta\pm A)-(\eta-\xi)\sin^2\varphi}}\right),\ \varphi\in\left[0,\frac{\pi}{2}\right],$$

$$y\in[\xi,\eta].$$

Introducing the first of equations (A4-9) and equations (A4-30) into equation (A4-26), and remembering also that $C^2=\alpha^2-\theta_j^2e_j^2$, we obtain

$$\frac{\partial V}{\partial D}=-\frac{4c_i\sqrt{D}\sqrt{\alpha^2-\theta_j^2e_j^2}}{3\sqrt{\eta\pm A}\sqrt{(\alpha^2-\theta_j^2e_j^2)\alpha_i^2-c_i^2\alpha^2}}\Bigg\{\Big[2(\xi\pm A)(\eta+2A)+(\Gamma+\alpha^2)\Big]K(\varpi)- \qquad \text{(A4-31)}$$

$$-4(\xi\pm A)(\eta\pm A)E(\varpi)\Bigg\},$$

where the parameter ϖ, called modulus, is related to η, ξ and A by

$$\varpi=\sqrt{\frac{\eta-\xi}{2(\eta\pm A)}}, \qquad \text{(A4-32)}$$

and $K(\varpi)$, $E(\varpi)$ are the complete elliptic integrals of the first and second kind, whose Legendre's standard forms are

$$K(\varpi)\equiv\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-\varpi^2\sin^2\varphi}},\quad E(\varpi)\equiv\int_0^{\pi/2}\sqrt{1-\varpi^2\sin^2\varphi}\,d\varphi, \qquad \text{(A4-33)}$$

while their most common representation by series is

$$K(\varpi)=\frac{\pi}{2}\sum_{\nu=0}^{\infty}\left(\frac{(2\nu)!}{2^{2\nu}(\nu!)^2}\right)^2\varpi^{2\nu},\quad E(\varpi)=-\frac{\pi}{2}\sum_{\nu=0}^{\infty}\left(\frac{(2\nu)!}{2^{2\nu}(\nu!)^2}\right)^2\frac{\varpi^{2\nu}}{(2\nu-1)},\ \varpi\neq 0, \qquad \text{(A4-34)}$$

$$K(0)=E(0)=\frac{\pi}{2}.$$

These formulae lend themselves readily to numerical computation, since $\varpi^2<1$ as is obvious from equations (A4-5), (A4-9), (A4-10) and (A4-32), However, the following two power series

$$K(\varpi)=\frac{\pi}{2}(1+m)\sum_{\nu=0}^{\infty}\left(\frac{(2\nu)!}{2^{2\nu}(\nu!)^2}\right)^2 m^{2\nu}, \quad E(\varpi)=\frac{\pi}{2(1+m)}\sum_{\nu=0}^{\infty}\left(\frac{(2\nu)!}{2^{2\nu}(\nu!)^2}\right)^2\frac{m^{2\nu}}{(2\nu-1)^2}, \quad m\neq 0,$$

$$K(0)=E(0)=\frac{\pi}{2},$$

converge more rapidly by virtue of the relations

$$\varpi'\equiv\left(1-\varpi^2\right)^{1/2}, \quad m^2\equiv\frac{1-\varpi'^2}{1+\varpi'^2}<\varpi^2 .$$

In a completely analogous manner to that followed for the derivation of $\partial V/\partial D$ one can find an analytical expression for the derivative $\partial V/\partial\alpha$.

Differentiating equation (A4-13) partially with respect to α and making use of the relation (2.38) we obtain

$$\frac{\partial V}{\partial\alpha}=\pi(1-e_i^2)^{1/2}\alpha\left\{\frac{\alpha_i^2}{D}-\frac{(D^2+\alpha_i^2-\alpha^2)^2}{4D^3}\right\}+$$

$$+\pi(\alpha^2-\theta_j^2e_j^2)^{1/2}\left\{\frac{4}{3}\alpha-\frac{D^2-\alpha_i^2+3\alpha^2}{2D}-\frac{(D^2-\alpha_i^2+\alpha^2)^3}{24D^3\alpha^2}+\frac{(D^2+\alpha^2-\alpha_i^2)^2}{4D^3}\right\}+$$

$$+\frac{\pi}{(\alpha^2-\theta_j^2e_j^2)^{1/2}}\left\{\frac{2}{3}\alpha^3-\alpha^2\frac{D^2-\alpha_i^2}{2D}-\frac{\alpha^4}{2D}+\frac{(D^2+\alpha^2-\alpha_i^2)^3}{24D^3}\right\}+$$

$$+2\frac{c_i}{\alpha_i}\left(\alpha_i^2-(\eta-D)^2\right)\arcsin\left(\frac{x(\eta)}{\sqrt{\alpha_i^2-(\eta-D)^2}}\right)\frac{\partial\eta}{\partial\alpha}-$$

$$-2\frac{c_i}{\alpha_i}\left(\alpha_i^2-(\xi-D)^2\right)\arcsin\left(\frac{x(\xi)}{\sqrt{\alpha_i^2-(\xi-D)^2}}\right)\frac{\partial\xi}{\partial\alpha}+$$

$$+2\frac{c_i}{\alpha_i}\int_{\xi}^{\eta}\left[\alpha_i^2-(y-D)^2\right]\frac{\sqrt{\alpha_i^2-(y-D)^2}}{\sqrt{\alpha_i^2-(y-D)^2-x^2(y)}}\frac{\partial}{\partial\alpha}\left(\frac{x(y)}{\sqrt{\alpha_i^2-(y-D)^2}}\right)dy- \qquad \text{(A4-35)}$$

$$-2\frac{\theta_j^2 e_j^2}{c\,\alpha^2}\int_{\xi}^{\eta}(\alpha^2-y^2)\arcsin\left(\frac{x(y)}{\sqrt{\alpha^2-y^2}}\right)dy-4c\int_{\xi}^{\eta}\arcsin\frac{x(y)}{\sqrt{\alpha^2-y^2}}\,dy-$$

$$-2\frac{c}{\alpha}\int_{\xi}^{\eta}(\alpha^2-y^2)\frac{\sqrt{\alpha^2-y^2}}{\sqrt{\alpha^2-y^2-x^2(y)}}\frac{\partial}{\partial\alpha}\left(\frac{x(y)}{\sqrt{\alpha^2-y^2}}\right)dy-$$

$$-2\frac{c}{\alpha}(\alpha^2-\eta^2)\arcsin\left(\frac{x(\eta)}{\sqrt{\alpha^2-\eta^2}}\right)\frac{\partial\eta}{\partial\alpha}+2\frac{c}{\alpha}(\alpha^2-\xi^2)\arcsin\left(\frac{x(\xi)}{\sqrt{\alpha^2-\xi^2}}\right)\frac{\partial\xi}{\partial\alpha}\,.$$

It can be verified, as in the case of equations (A4-15), that

$$\frac{c_i}{\alpha_i}\left[\alpha_i^2-(y-D)^2\right]\frac{\sqrt{\alpha_i^2-(y-D)^2}}{\sqrt{\alpha_i^2-(y-D)^2-x^2(y)}}\frac{\partial}{\partial\alpha}\left(\frac{x(y)}{\sqrt{\alpha_i^2-(y-D)^2}}\right)=$$

$$=\frac{c_i^2}{\alpha_i^2}\left[\alpha_i^2-(y-D)^2-x^2(y)\right]\frac{\partial}{\partial\alpha}\left(\frac{x(y)}{\frac{c_i}{\alpha_i}\sqrt{\alpha_i^2-(y-D)^2-x^2(y)}}\right),$$

$$\frac{c}{\alpha}(\alpha^2-y^2)\frac{\sqrt{\alpha^2-y^2}}{\sqrt{\alpha^2-y^2-x^2(y)}}\frac{\partial}{\partial\alpha}\left(\frac{x(y)}{\sqrt{\alpha^2-y^2}}\right)= \qquad \text{(A4-36)}$$

$$=\frac{c^2}{\alpha^2}\left[\alpha^2-y^2-x^2(y)\right]\frac{\partial}{\partial\alpha}\left(\frac{x(y)}{\frac{c}{\alpha}\sqrt{\alpha^2-y^2-x^2(y)}}\right)+\frac{\theta_j^2 e_j^2}{c\cdot\alpha^2}x(y)\sqrt{\alpha^2-y^2-x^2(y)}$$

Comparison of equations (A4-12), (A4-16), (A4-35) and (A4-36) furnishes the following relation

$$\frac{\partial V}{\partial\alpha}=\pi(\alpha^2-\theta_j^2e_j^2)^{1/2}\left\{\frac{4}{3}\alpha-\frac{D^2+\alpha^2-\alpha_i^2}{2D}-\frac{(D^2+\alpha^2-\alpha_i^2)^3}{24D^3\alpha^2}\right]+$$

$$+\frac{\eta}{(\alpha^2-\theta_j^2e_j^2)^{1/2}}\left\{\frac{2}{3}\alpha^3-\alpha^2\frac{(D^2+\alpha^2-\alpha_i^2)}{2D}+\frac{(D^2+\alpha^2-\alpha_i^2)^3}{24D^3}\right\}-$$

(A4-37)

$$-\frac{2(\alpha^2+c^2)}{c}\int_{\xi}^{\eta}\arcsin\frac{x(y)}{\sqrt{\alpha^2-y^2}}dy+2\frac{\theta_j^2e_j^2}{c\cdot\alpha^2}\int_{\xi}^{\eta}y^2\arcsin\frac{x(y)}{\sqrt{\alpha^2-y^2}}dy-$$

$$-2\frac{\theta_j^2e_j^2}{c\,\alpha^2}\int_{\xi}^{\eta}x(y)\sqrt{\alpha^2-y^2-x^2(y)}\,dy.$$

We note that the volume V, considered as function of the parameter α, has one minimum and one maximum at the points $\alpha=D-\alpha_i$ and $\alpha=D+\alpha_i$ respectively, which implies that its first order derivatives with respect to α at these points vanish. That is

$$\left.\frac{\partial V}{\partial\alpha}\right|_{\alpha=D-\alpha_i}=\left.\frac{\partial V}{\partial\alpha}\right|_{\alpha=D+\alpha_i}=0\,. \tag{A4-38}$$

The validity of these equations can also be proved analytically with the help of equation (A4-37) and the relations

$$\eta\big|_{\alpha=D-\alpha_i}=\xi\big|_{\alpha=D-\alpha_i}=D-\alpha_i\,,\quad \eta\big|_{\alpha=D+\alpha_i}=\xi\big|_{\alpha=D+\alpha_i}=D+\alpha_i\,, \tag{A4-39}$$

the proof of which is elementary.

The values of the partial derivative $\partial V/\partial D$, at the points $\alpha=D-\alpha_i$ and $\alpha=D+\alpha_i$, can be found from equations (A4-18) and (A4-39); comparison of these equations yields

$$\left.\frac{\partial V}{\partial D}\right|_{\alpha=D-\alpha_i}=\left.\frac{\partial V}{\partial D}\right|_{\alpha=D+\alpha_i}=0\,. \tag{A4-40}$$

The volume V and its first order partial derivatives $\partial V/\partial D$ and $\partial V/\partial\alpha$ are functions invariant with respect to a change in the reference system and therefore their form, in the system of reference described in chapter 2, is given by equation (A4-13), (A4-31) and (A4-37) where the symbol D has been replaced by R_{ij} defined in equation (2.24). That is,

$$V=\pi(1-e_i^2)^{1/2}\left\{\frac{2}{3}\alpha_i^3-\frac{(R_{ij}^2+\alpha_i^2-\alpha^2)}{2R_{ij}}\alpha_i^2+\frac{(R_{ij}^2+\alpha_i^2-\alpha^2)^3}{24R_{ij}^3}\right\}+$$

$$+\pi(\alpha^2-\theta_j^2e_j^2)^{1/2}\left\{\frac{2}{3}\alpha^2-\frac{(R_{ij}^2-\alpha_i^2+\alpha^2)}{2R_{ij}}\alpha+\frac{(R_{ij}^2-\alpha_i^2+\alpha^2)^3}{24R_{ij}^3\alpha}\right\}+$$

$$+2\frac{c_i}{\alpha_i}\int_{\xi}^{\eta}\left[\alpha_i^2-(y-R_{ij})^2\right]\arcsin\frac{x(y)}{\sqrt{\alpha_i^2-(y-R_{ij})^2}}dy-\qquad\text{(A4-41)}$$

$$-2\frac{c}{\alpha}\int_{\xi}^{\eta}(\alpha^2-y^2)\arcsin\frac{x(y)}{\sqrt{\alpha^2-y^2}}dy,$$

$$\frac{\partial V}{\partial R_{ij}}=-\frac{4c_i\sqrt{R_{ij}}\sqrt{\alpha^2-\theta_j^2e_j^2}}{3\sqrt{\eta+A}\sqrt{(\alpha^2-\theta_j^2e_j^2)\alpha_i^2-c_i^2\alpha^2}}\left\{\left[2(\xi+A)(\eta+2A)+(\Gamma+\alpha^2)\right]K(\omega)-\right.\qquad\text{(A4-42)}$$

$$\left.-4(\xi+A)(\eta+A)E(\omega)\right\},$$

$$\frac{\partial V}{\partial \alpha}=\pi(\alpha^2-\theta_j^2e_j^2)^{1/2}\left\{\frac{4}{3}\alpha-\frac{(R_{ij}^2+\alpha^2-\alpha_i^2)}{2R_{ij}}-\frac{(R_{ij}^2+\alpha^2-\alpha_i^2)^3}{24R_{ij}^3\alpha^2}\right\}+$$

$$+\frac{\pi}{(\alpha^2-\theta_j^2e_j^2)^{1/2}}\left\{\frac{2}{3}\alpha^3-\alpha^2\frac{(R_{ij}^2+\alpha^2-\alpha_i^2)}{2R_{ij}}+\frac{(R_{ij}^2+\alpha^2-\alpha_i^2)^3}{24R_{ij}^3}\right\}-\qquad\text{(A4-43)}$$

$$- \frac{2(\alpha^2+c^2)}{c} \int_{\xi}^{\eta} \arcsin \frac{x(y)}{\sqrt{\alpha^2-y^2}} dy + \frac{2\theta_j^2 e_j^2}{c\alpha^2} \int_{\xi}^{\eta} y^2 \arcsin \frac{x(y)}{\sqrt{\alpha^2-y^2}} dy -$$

$$-2 \frac{\theta_j^2 e_j^2}{c\alpha^2} \int_{\xi}^{\eta} x(y) \sqrt{\alpha^2-y^2-x^2(y)}\, dy \,.$$

In the foregoing three equations we kept the symbols ξ, A, Γ, η and $\tilde{\omega}$, which in this case, according to equations (A4-5), (A4-9), (A4-10) and (A4-32), are taken to mean

$$\xi = \frac{R_{ij}^2 + \alpha^2 - \alpha_i^2}{2R_{ij}} \,, \tag{A4-44}$$

$$A = R_{ij} \frac{c_i^2 \alpha^2}{c^2\alpha_i^2 - c_i^2\alpha^2} \,, \quad \Gamma = \alpha^2 \frac{(c^2-c_i^2)\alpha_i^2 + R_{ij}^2 c_i^2}{c^2\alpha_i^2 - c_i^2\alpha^2} \,, \tag{A4-45}$$

$$\eta = - \frac{R_{ij} c_i^2 \alpha^2}{c^2\alpha_i^2 - c_i^2\alpha^2} + \alpha \cdot \alpha_i \frac{\sqrt{(R_{ij}^2 - \alpha^2 - \alpha_i^2) c^2 c_i^2 + (c^4\alpha_i^2 + c_i^4\alpha^2)}}{c^2\alpha_i^2 - c_i^2\alpha^2} \,, \tag{A4-46}$$

and

$$\tilde{\omega} = \sqrt{\frac{\eta - \xi}{2(\eta + A)}} \,. \tag{A4-47}$$

Finally equations (A4-38) and (A4-40), which hold in any system of reference, take the form

$$\left.\frac{\partial V}{\partial \alpha}\right|_{\alpha=R_{ij}-\alpha_i} = \left.\frac{\partial V}{\partial \alpha}\right|_{\alpha=R_{ij}+\alpha_i} = 0 \,, \tag{A4-48}$$

and

$$\left.\frac{\partial V}{\partial R_{ij}}\right|_{\alpha=R_{ij}-\alpha_i} = \left.\frac{\partial V}{\partial R_{ij}}\right|_{\alpha=R_{ij}+\alpha_i} = 0 \,. \tag{A4-49}$$

We note that the integrals entering the right-hand side of equations (A4-41) and (A4-43) are not elementary; with a process similar to that followed in the case of equation (A4-17) these integrals can be expressed as functions of the complete elliptic integrals of the first and second kind.

APPENDIX A5

Evaluation of the integrals $W_m(R_{ij}, \alpha_i, e_i)$, $m = 0,1,2,3,4,5,6$ (Chapter 2)

In this appendix we consider the integrals appearing in equation (2.65). Before proceeding with the evaluation of these integrals we shall give some general formulae which we are going to use later on in this section.

Let us consider the integral

$$\int \frac{\cos^{\ell}\varphi \, d\varphi}{\sin^{n}\varphi(\beta^2-\gamma^2\cos^2\varphi)^{\kappa}}, \quad \kappa \neq 1, \quad \beta^2 \neq \gamma^2.$$

We seek a relation of the form

$$\int \frac{\cos^{\ell}\varphi \, d\varphi}{\sin^{n}\varphi(\beta^2-\gamma^2\cos^2\varphi)^{\kappa}} = A \frac{\cos^{\ell+1}\varphi}{\sin^{n-1}\varphi\,(\beta^2-\gamma^2\cos^2\varphi)^{\kappa-1}} +$$

$$+B\int \frac{\cos^{\ell}\varphi \, d\varphi}{\sin^{n}\varphi(\beta^2-\gamma^2\cos^2\varphi)^{\kappa-1}} + \Gamma \int \frac{\cos^{\ell}\varphi \, d\varphi}{\sin^{n}\varphi(\beta^2-\gamma^2\cos^2\varphi)^{\kappa-2}}, \quad \kappa \neq 1, \quad \beta^2 \neq \gamma^2 \tag{A5-1}$$

where A, B and Γ are uniquely defined parameters. Evaluation of these parameters is obtained as follows: Differentiating both sides of equation (A5-1) with respect to φ and arranging the resulting expression we find

$$1 = -(\ell+1)\beta^2 A + \beta^2 B + \beta^4 \Gamma + \left[(\ell-2\kappa+3)\gamma^2 + \beta^2(\ell-n+2)\right] A\cos^2\varphi - \gamma^2 B\cos^2\varphi -$$

$$-2\beta^2\gamma^2\Gamma\cos^2\varphi + (2\kappa+n-\ell-4)\gamma^2 A\cos^4\varphi + \gamma^4\Gamma\cos^4\varphi, \quad \kappa \neq 1, \quad \beta^2 \neq \gamma^2.$$

Equating the coefficients of the terms of like powers of $\cos\varphi$ we obtain the following system of elementary equations

$$1=-(\ell+1)\beta^2 A+\beta^2 B+\beta^4\Gamma,$$

$$0=[(\ell-n+2)\beta^2+(\ell-2\kappa+3)\gamma^2]A-\gamma^2 B-2\beta^2\gamma^2\Gamma,$$

$$0=(2\kappa+n-\ell-4)A+\gamma^2\Gamma,\quad \kappa\neq 1,\ \beta^2\neq\gamma^2,$$

which, solved with respect to A, B and Γ , has a unique solution given by the relations

$$A=\frac{\gamma^2}{2\beta^2(\kappa-1)(\beta^2-\gamma^2)},\quad B=\frac{\beta^2(4\kappa+n-\ell-6)-\gamma^2(2\kappa-\ell-3)}{2\beta^2(\kappa-1)(\beta^2-\gamma^2)},$$

$$\Gamma=\frac{-(2\kappa+n-\ell-4)}{2\beta^2(\kappa-1)(\beta^2-\gamma^2)},\ \kappa\neq 1,\ \beta^2\neq\gamma^2.$$

Substitution of these relations into equation (A5-1) yields

$$\int\frac{\cos^\ell\varphi}{\sin^n\varphi(\beta^2-\gamma^2\cos^2\varphi)^\kappa}d\varphi=\frac{\gamma^2\cos^{\ell+1}\varphi}{2\beta^2(\kappa-1)(\beta^2-\gamma^2)\sin^{n-1}\varphi\,(\beta^2-\gamma^2\cos^2\varphi)^{\kappa-1}}+$$

$$+\frac{\beta^2(4\kappa+n-\ell-6)-\gamma^2(2\kappa-\ell-3)}{2\beta^2(\kappa-1)(\beta^2-\gamma^2)}\int\frac{\cos^\ell\varphi\,d\varphi}{\sin^n\varphi\,(\beta^2-\gamma^2\cos^2\varphi)^{\kappa-1}}-\tag{A5-2}$$

$$-\frac{(2\kappa+n-\ell-4)}{2\beta^2(\kappa-1)(\beta^2-\gamma^2)}\int\frac{\cos^\ell\varphi\,d\varphi}{\sin^n\varphi\,(\beta^2-\gamma^2\cos^2\varphi)^{\kappa-2}},\ \kappa\neq 1,\ \beta^2\neq\gamma^2.$$

The following general relations can be found using processes entirely analogous to the process described above

$$\int\frac{\cos^\ell\varphi\,d\varphi}{\sin^n\varphi\,(\beta^2-\gamma^2\cos^2\varphi)^\kappa}=-\frac{\cos^{\ell+1}\varphi}{(n-1)(\beta^2-\gamma^2)\sin^{n-1}\varphi(\beta^2-\gamma^2\cos^2\varphi)^{\kappa-1}}+$$

$$+\frac{(n-\ell-2)\beta^2-(2n+2k-\ell-5)\gamma^2}{(n-1)(\beta^2-\gamma^2)}\int\frac{\cos^{\ell}\varphi\,d\varphi}{\sin^{n-2}\varphi(\beta^2-\gamma^2\cos^2\varphi)^k}+ \tag{A5-3}$$

$$+\frac{\gamma^2(n+2k-\ell-4)}{(n-1)(\beta^2-\gamma^2)}\int\frac{\cos^{\ell}\varphi\,d\varphi}{\sin^{n-4}\varphi(\beta^2-\gamma^2\cos^2\varphi)^k},\quad n\neq 1,\ \beta^2\neq\gamma^2;$$

$$\int\frac{d\varphi}{\cos^{n}\varphi(\beta^2-\gamma^2\cos^2\varphi)^k}=\frac{\sin\varphi}{(n-1)(\beta^2-\gamma^2\cos^2\varphi)^{k-1}\beta^2\cos^{n-1}\varphi}+$$

$$+\frac{\beta^2(n-2)+\gamma^2(n+2k-3)}{(n-1)\beta^2}\int\frac{d\varphi}{\cos^{n-2}\varphi(\beta^2-\gamma^2\cos^2\varphi)^k}+ \tag{A5-4}$$

$$+\frac{\gamma^2(4-n-2k)}{(n-1)\beta^2}\int\frac{d\varphi}{\cos^{n-4}\varphi(\beta^2-\gamma^2\cos^2\varphi)^k},\quad n\neq 1,\quad \beta^2\neq 0;$$

$$\int\frac{d\varphi}{(\beta^2-\varphi^2)^n(\gamma^2-\varphi^2)^{k+1/2}}=-\frac{\varphi}{2\beta^2(n-1)(\beta^2-\gamma^2)(\beta^2-\varphi^2)^{n-1}(\gamma^2-\varphi^2)^{k-1/2}}+$$

$$+\frac{2\beta^2(2n+k-3)-\gamma^2(2n-3)}{2\beta^2(n-1)(\beta^2-\gamma^2)}\int\frac{d\varphi}{(\beta^2-\varphi^2)^{n-1}(\gamma^2-\varphi^2)^{k+1/2}}- \tag{A5-5}$$

$$-\frac{2(n+k-2)}{2\beta^2(n-1)(\beta^2-\gamma^2)}\int\frac{d\varphi}{(\beta^2-\varphi^2)^{n-2}(\gamma^2-\varphi^2)^{k+1/2}},\quad n\neq 1,\ \beta^2\neq\gamma^2,\ \gamma^2>\varphi^2;$$

$$\int\frac{d\varphi}{\varphi^{2n}(\beta^2+\varphi^2)^{1/2}}=-\frac{(\beta^2+\varphi^2)^{1/2}}{(2n-1)\beta^2\varphi^{2n-1}}-\frac{2(n-1)}{(2n-1)\beta^2}\int\frac{d\varphi}{\varphi^{2(n-1)}(\beta^2+\varphi^2)^{1/2}}=$$

$$=-\frac{(\beta^2+\varphi^2)^{\frac{1}{2}}}{\varphi\beta^{2n}}\sum_{\nu=0}^{n-1}(-1)^{\nu}\frac{(n-1)!}{(2n-2\nu-1)(n-1-\nu)!\,\nu!}\left(\frac{\beta^2+\varphi^2}{\varphi^2}\right)^{n-\nu-1}+C. \tag{A5-6}$$

With the help of equations (A5-2), (A5-3), (A5-4), (A5-5), (A5-6) and the relations

$$\int \frac{d\varphi}{(\beta^2-\gamma^2\cos^2\varphi)} = \frac{1}{\beta(\beta^2-\gamma^2)^{1/2}} \arctan\left(\frac{\beta\tan\varphi}{\sqrt{\beta^2-\gamma^2}}\right), \quad \alpha^2 > \beta^2,$$

$$\int \varphi^n \arctan\left(\frac{\varphi}{\beta}\right) d\varphi = \frac{\varphi^{n+1}}{n+1} \arctan\left(\frac{\varphi}{\beta}\right) - \frac{\beta}{n+1}\int \frac{\varphi^{n+1}\, d\varphi}{\beta^2+\varphi^2}, \quad n \neq -1, \qquad \text{(A5-7)}$$

$$\int \frac{d\varphi}{(\beta^2-\gamma^2\cos^2\varphi)^n} = -\int \frac{d(\gamma\cos\varphi)}{(\beta^2-\gamma^2\cos^2\varphi)^n(\gamma^2-\gamma^2\cos^2\varphi)^{1/2}},$$

one can prove the validity of the relations in table (A5.1), given at the end of this appendix.

We proceed now with the evaluation of the integrals $W_m(R_{ij}, \alpha_i, e_i)$. Equation (2.69) for $m = 0$ takes the form

$$W_0(R_{ij}, \alpha_i, e_i) = \sqrt{2R_{ij}} \lim_{\substack{\varepsilon^* \to \frac{\pi}{2}- \\ \varepsilon \to 0+}} \Bigg\{ 2R_{ij} \int_{\varepsilon}^{\varepsilon^*} \frac{\tan^2\varphi}{\cos^2\varphi} d\varphi \int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i} \alpha^{-2} d\alpha -$$

$$- \int_{\varepsilon}^{\varepsilon^*} \frac{\tan^2\varphi}{\cos\varphi} \int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i} \frac{d\alpha}{\sqrt{(R_{ij}^2\tan^2\varphi - \alpha_i^2 e_i^2 \sin^2\varphi) + \alpha^2}} d\varphi - \qquad \text{(A5-8)}$$

$$- \int_{\varepsilon}^{\varepsilon^*} \frac{\tan^2\varphi}{\cos\varphi} \left(R_{ij}^2\tan^2\varphi + \frac{R_{ij}^2}{\cos^2\varphi} - \alpha_i^2 e_i^2\right) \int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i} \alpha^{-2} \frac{d\alpha}{\sqrt{(R_{ij}^2\tan^2\varphi - \alpha_i^2 e_i^2 \sin^2\varphi) + \alpha^2}} d\varphi \Bigg\}.$$

The evaluation of the last two inside integrals in this equation is obtained using equation (A5-6) and the relation

$$\int \frac{dx}{(\beta^2+x^2)^{1/2}} = \ln\left[x+(\beta^2+x^2)^{1/2}\right]+c.$$

These integrals have the form

$$\int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i} \frac{d\alpha}{\sqrt{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)+\alpha^2}} = \ln\left(\frac{R_{ij}+\alpha_i e_i\cos\varphi}{R_{ij}-\alpha_i e_i\cos\varphi}\right),$$

(A5-9)

$$\int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i} \frac{\alpha^{-2}d\alpha}{\sqrt{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)+\alpha^2}} = \frac{2R_{ij}\alpha_i e_i\cos\varphi}{(R_{ij}^2-\alpha_i^2e_i^2)(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)}.$$

Substitution of these equations into equation (A5-8) yields

$$W_0(R_{ij},\alpha_i,e_i)=\sqrt{2R_{ij}}\lim_{\substack{\varepsilon^*\to\frac{\pi}{2}-\\ \varepsilon\to 0+}}\left\{\frac{2R_{ij}\alpha_i e_i}{(R_{ij}^2-\alpha_i^2e_i^2)}\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi}{\cos^2\varphi}\left[2-\frac{\cos^2\varphi}{(R_{ij}^2-\alpha^2e^2\cos^2\varphi)}\left(R_{ij}^2\tan^2\varphi+\right.\right.\right.$$

$$\left.\left.\left.+\frac{R_{ij}^2}{\cos^2\varphi}-\alpha_i^2e_i^2\right)\right]d\varphi-\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi}{\cos\varphi}\ln\left(\frac{1+\frac{\alpha_i e_i}{R_{ij}}\cos\varphi}{1-\frac{\alpha_i e_i}{R_{ij}}\cos\varphi}\right)d\varphi\right\}= \quad \text{(A5-10)}$$

$$=\sqrt{2R_{ij}}\lim_{\substack{\varepsilon^*\to\frac{\pi}{2}-\\ \varepsilon\to 0+}}\left\{2R_{ij}\alpha_i e_i\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi\, d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)}-\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi}{\cos\varphi}\ln\left(\frac{1+\frac{\alpha_i e_i}{R_{ij}}\cos\varphi}{1-\frac{\alpha_i e_i}{R_{ij}}\cos\varphi}\right)d\varphi\right\}.$$

The evaluation of the foregoing limit is obtained as follows: The first of the integrals appearing in this limit can take the form

$$\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi\, d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)}=-\frac{1}{R_{ij}\alpha_i e_i}\int_{\varepsilon}^{\varepsilon^*}\frac{\sin\varphi\, d\left(\frac{\alpha_i e_i}{R_{ij}}\cos\varphi\right)}{\cos^2\varphi\left(1-\frac{\alpha_i^2e_i^2}{R_{ij}^2}\cos^2\varphi\right)},$$

which, on introduction of the relation

$$\frac{1}{2}d\left[\ln\frac{1+x}{1-x}\right]=\frac{dx}{1-x^2}, \quad x^2\neq 1,$$

becomes

$$\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi\, d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)}=-\frac{1}{2R_{ij}\alpha_i e_i}\int_{\varepsilon}^{\varepsilon^*}\frac{\sin\varphi}{\cos^2\varphi}\, d\ln\left(\frac{1+\dfrac{\alpha_i e_i}{R_{ij}}\cos\varphi}{1-\dfrac{\alpha_i e_i}{R_{ij}}\cos\varphi}\right).$$

Integrating by parts and introducing the equation

$$d\left(\frac{\sin\varphi}{\cos^2\varphi}\right)=\left(\frac{1}{\cos^3\varphi}+\frac{\tan^2\varphi}{\cos\varphi}\right)d\varphi,$$

the above integral can be re-written as

$$\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi\, d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)}=\frac{1}{2R_{ij}\alpha_i e_i}\int_{\varepsilon}^{\varepsilon^*}\frac{1}{\cos^3\varphi}\ln\left(\frac{1+\dfrac{\alpha_i e_i}{R_{ij}}\cos\varphi}{1-\dfrac{\alpha_i e_i}{R_{ij}}\cos\varphi}\right)d\varphi+$$

$$+\frac{1}{2R_{ij}\alpha_i e_i}\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi}{\cos\varphi}\ln\left(\frac{1+\dfrac{\alpha_i e_i}{R_{ij}}\cos\varphi}{1-\dfrac{\alpha_i e_i}{R_{ij}}\cos\varphi}\right)d\varphi-\frac{1}{2R_{ij}\alpha_i e_i}\left[\frac{\sin\varphi}{\cos^2\varphi}\ln\left(\frac{1+\dfrac{\alpha_i e_i}{R_{ij}}\cos\varphi}{1-\dfrac{\alpha_i e_i}{R_{ij}}\cos\varphi}\right)\right]_{\varepsilon}^{\varepsilon^*}.$$

Substituting the foregoing relation in equation (A5-10) we have

$$W_o(R_{ij},\alpha_i,e_i)=\sqrt{2R_{ij}}\lim_{\substack{\varepsilon^*\to\frac{\pi}{2}-\\ \varepsilon\to 0+}}\left\{\int_{\varepsilon}^{\varepsilon^*}\frac{1}{\cos^3\varphi}\ln\left(\frac{1+\dfrac{\alpha_i e_i}{R_{ij}}\cos\varphi}{1-\dfrac{\alpha_i e_i}{R_{ij}}\cos\varphi}\right)d\varphi-\right. \tag{A5-11}$$

$$\left.-\left[\frac{\sin\varphi}{\cos^2\varphi}\ln\left(\frac{1+\dfrac{\alpha_i e_i}{R_{ij}}\cos\varphi}{1-\dfrac{\alpha_i e_i}{R_{ij}}\cos\varphi}\right)\right]_{\varepsilon}^{\varepsilon^*}\right\}.$$

Since $\alpha_i e_i \cos\varphi / R_{ij} < 1$, we may employ the well-known series

$$\ln\left(\frac{1+x}{1-x}\right) = 2\sum_{K=0}^{\infty} \frac{x^{2K+1}}{2K+1}, \quad x^2 < 1, \tag{A5-12}$$

to obtain the value of the expression $W_0(R_{ij}, \alpha_i, e_i)$. Combining equations (A5-11) and (A5-12) we have

$$W_0(R_{ij},\alpha_i,e_i) = \sqrt{2R_{ij}} \lim_{\substack{\varepsilon^* \to \frac{\pi}{2}- \\ \varepsilon \to 0+}} \left\{ 2\int_{\varepsilon}^{\varepsilon^*} \frac{1}{\cos^3\varphi} \sum_{K=0}^{\infty} \frac{1}{(2K+1)} \left(\frac{\alpha_i e_i}{R_{ij}}\right)^{2K+1} \cos^{2K+1}\varphi \, - \right.$$

$$\left. -2\left[\frac{\sin\varphi}{\cos^2\varphi} \sum_{K=0}^{\infty} \frac{1}{(2K+1)} \left(\frac{\alpha_i e_i}{R_{ij}}\right)^{2K+1} \cos^{2K+1}\varphi \right]_{\varepsilon}^{\varepsilon^*} \right\}.$$

Since $(\alpha_i e_i / R_{ij}) \cdot \cos\varepsilon$ and $(\alpha_i e_i / R_{ij}) \cdot \cos\varepsilon^*$ are both interior to the interval of convergence of the series appearing in the foregoing equation, we can use the "term by term" integration to verify that this equation is equivalent to

$$W_0(R_{ij},\alpha_i,e_i) = \sqrt{2R_{ij}} \lim_{\substack{\varepsilon^* \to \frac{\pi}{2}- \\ \varepsilon \to 0+}} \left\{ 2\frac{\alpha_i e_i}{R_{ij}} \int_{\varepsilon}^{\varepsilon^*} \frac{d\varphi}{\cos^2\varphi} + 2\sum_{K=0}^{\infty} \left[\frac{1}{(2K+3)} \left(\frac{\alpha_i e_i}{R_{ij}}\right)^{2K+3} \int_{\varepsilon}^{\varepsilon^*} \cos^{2K}\varphi \, d\varphi \right] - \right.$$

$$\left. -2\frac{\alpha_i e_i}{R_{ij}} \left(\tan\varepsilon^* - \tan\varepsilon\right) \right\}. \tag{A5-13}$$

Introducing the relation $d\tan\varphi = d\varphi/\cos^2\varphi$, and since the integral

$$\int_{\varepsilon}^{\varepsilon^*} \cos^{2K}\varphi \, d\varphi$$

converges for all $\varepsilon \in [0, \pi/2]$, $\varepsilon^* \in [0, \pi/2]$ and $K \geq 0$, equation (A5-13) becomes

$$W_o(R_{ij},\alpha_i e_i)=2\sqrt{2R_{ij}}\sum_{K=0}^{\infty}\frac{1}{(2K+3)}\left(\frac{\alpha_i e_i}{R_{ij}}\right)^{2K+3}\int_0^{\frac{\pi}{2}}\cos^{2K}\varphi\, d\varphi ,$$

which, upon substitution of the well-known formula

$$\int_0^{\frac{\pi}{2}}\cos^{m}\varphi\, d\varphi=\frac{\sqrt{\pi}}{2}\,\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}+1\right)},\quad m>-1,$$

takes the form

$$W_o(R_{ij},\alpha_i,e_i)=\sqrt{\pi}\sqrt{2R_{ij}}\sum_{K=0}^{\infty}\frac{1}{(2K+3)}\left(\frac{\alpha_i e_i}{R_{ij}}\right)^{2K+3}\frac{\Gamma\left(K+\frac{1}{2}\right)}{\Gamma(K+1)},$$

where $\Gamma(n)$ is the Gamma function. Using the following properties of the Gamma function

$$\Gamma(n)=(n-1)!,\quad \Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}\,(2n)!}{2^{2n}\,n!}\quad , n \text{ positive integer,}$$

we easily obtain

$$W_o(R_{ij},\alpha_i,e_i)=\pi\sqrt{2R_{ij}}\sum_{K=0}^{\infty}\frac{(2K)!}{2^{2K}(K!)^2(2K+3)}\left(\frac{\alpha_i e_i}{R_{ij}}\right)^{2K+3}.$$

The series occurring in this equation is expressible in finite terms. We can verify that

$$W_o(R_{ij},\alpha_i,e_i)=\pi\sqrt{2R_{ij}}\sum_{K=0}^{\infty}\frac{(2K)!}{2^{2K}(K!)^2(2K+1)}\left(\frac{\alpha_i e_i}{R_{ij}}\right)^{2K+3}-$$

$$-2\pi\sqrt{2R_{ij}}\sum_{K=0}^{\infty}\frac{(2K)!}{2^{2K}(K!)^2(2K+1)(2K+3)}\left(\frac{\alpha_i e_i}{R_{ij}}\right)^{2K+3},$$

$$\left(\frac{\alpha_i e_i}{R_{ij}}\right)^2 \arcsin\left(\frac{\alpha_i e_i}{R_{ij}}\right) = \sum_{K=0}^{\infty} \frac{(2K)!}{2^{2K}(K!)^2(2K+1)}\left(\frac{\alpha_i e_i}{R_{ij}}\right)^{2K+3},$$

$$\sum_{K=0}^{\infty} \frac{(2K)!\;x^{2K+3}}{2^{2K}(K!)^2(2K+1)(2K+3)} = \sum_{K=0}^{\infty} \frac{(2K)!}{2^{2K}(K!)^2(2K+1)} \int x^{2K+2}\,dx$$

$$= \int x \sum_{K=0}^{\infty} \frac{(2K)!}{2^{2K}(K!)^2(2K+1)} x^{2K+1}\,dx = \int x \arcsin x\,dx =$$

$$= \frac{1}{4}\left(2x^2-1\right)\arcsin x + \frac{x}{4}\left(1-x^2\right)^{1/2}, \quad x^2 < 1 .$$

Combining the foregoing three equations and remembering that $\frac{\alpha_i e_i}{R_{ij}} < 1$, we have the result

$$W_0(R_{ij},\alpha_i,e_i) = \frac{\pi}{2}\sqrt{2R_{ij}}\left[\arcsin\left(\frac{\alpha_i e_i}{R_{ij}}\right) - \frac{\alpha_i e_i}{R_{ij}^2}\sqrt{R_{ij}^2-\alpha_i^2 e_i^2}\right]. \qquad \text{(A5-14)}$$

The process of evaluation of the integral $W_1(R_{ij},\alpha_i,e_i)$ is somewhat different. Equation (2.69) for $m = 1$ becomes

$$W_1(R_{ij},\alpha_i,e_i) = \sqrt{2R_{ij}} \lim_{\substack{\varepsilon^* \to \frac{\pi}{2}- \\ \varepsilon \to 0+}} \left[2R_{ij} \int_{\varepsilon}^{\varepsilon^*} \frac{\tan^2\varphi}{\cos^2\varphi}\,d\varphi \int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i} \alpha^{-4}\,d\alpha - \right.$$

$$- \int_{\varepsilon}^{\varepsilon^*} \frac{\tan^2\varphi}{\cos\varphi} \int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i} \frac{d\alpha}{\alpha^2\sqrt{(R_{ij}^2\tan^2\varphi - \alpha_i^2 e_i^2 \sin^2\varphi) + \alpha^2}}\,d\varphi - \qquad \text{(A5-15)}$$

$$-\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi}{\cos\varphi}\left(R_{ij}^2\tan^2\varphi+\frac{R_{ij}^2}{\cos^2\varphi}-\alpha_i^2e_i^2\right)\int_{R_{ij}-\alpha_ie_i}^{R_{ij}+\alpha_ie_i}\frac{d\alpha}{\alpha^4\sqrt{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)+\alpha^2}}\,d\varphi\Bigg\}.$$

We may employ the general relation (A5-6) to evaluate the last of the inside integrals appearing in the above equation. This integral has the form

$$\int_{R_{ij}-\alpha_ie_i}^{R_{ij}+\alpha_ie_i}\frac{d\alpha}{\alpha^4\sqrt{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)+\alpha^2}}=\frac{2}{3}R_{ij}\alpha_ie_i\frac{(3R_{ij}^2+\alpha_i^2e_i^2)}{(R_{ij}^2-\alpha_i^2e_i^2)^3}\frac{\cos\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)\sin^2\varphi}- \tag{A5-16}$$

$$-\frac{2}{3}R_{ij}\alpha_ie_i\frac{(R_{ij}^2+3\alpha_i^2e_i^2)}{(R_{ij}^2-\alpha_i^2e_i^2)^3}\frac{\cos^3\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)\sin^2\varphi}-\frac{4R_{ij}\alpha_ie_i}{3(R_{ij}^2-\alpha_i^2e_i^2)}\frac{\cos^3\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^2\sin^2\varphi},$$

$$\sin\varphi\neq 0.$$

Introducing this relation and the second of the relations (A5-9) in equation (A5-15), and arranging the resulting expression, we obtain

$$W_1(R_{ij},\alpha_i,e_i)=\sqrt{2R_{ij}}\lim_{\substack{\varepsilon^*\to\frac{\pi}{2}- \\ \varepsilon\to 0+}}\Bigg\{\frac{4}{9}R_{ij}\alpha_ie_i\frac{(3R_{ij}^2+\alpha_i^2e_i^2)}{(R_{ij}^2-\alpha_i^2e_i^2)^3}\left(\tan^3\varepsilon^*-\tan^3\varepsilon\right)-$$

$$-\frac{2}{3}R_{ij}(\alpha_ie_i)\frac{(2R_{ij}^4+3\alpha_i^4e_i^4-9R_{ij}^2\alpha_i^2e_i^2)}{(R_{ij}^2-\alpha_i^2e_i^2)^3}\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi\,d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)}-$$

$$-\frac{2R_{ij}^3\alpha_ie_i(3R_{ij}^2+\alpha_i^2e_i^2)}{3(R_{ij}^2-\alpha_i^2e_i^2)^3}\int_{\varepsilon}^{\varepsilon^*}\frac{\sin^2\varphi\,d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)\cos^4\varphi}+\frac{4R_{ij}^3\alpha_ie_i}{3(R_{ij}^2-\alpha_i^2e_i^2)}\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi\,d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^2}-$$

$$-\frac{2R_{ij}^3\alpha_i e_i(3R_{ij}^2+\alpha_i^2e_i^2)}{3(R_{ij}^2-\alpha_i^2e_i^2)^3}\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)\cos^4\varphi}+\frac{4R_{ij}^3\alpha_i e_i}{3(R_{ij}^2-\alpha_i^2e_i^2)}\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^2\cos^2\varphi}+$$

$$+\frac{2}{3}R_{ij}\alpha_i e_i\frac{(R_{ij}^4+6R_{ij}^2\alpha_i^2e_i^2+\alpha_i^4e_i^4)}{(R_{ij}^2-\alpha_i^2e_i^2)^3}\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)\cos^2\varphi}-$$

$$-\frac{2R_{ij}\alpha_i^3e_i^3(R_{ij}^2+3\alpha_i^2e_i^2)}{3(R_{ij}^2-\alpha_i^2e_i^2)^3}\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)}-\frac{4R_{ij}\alpha_i^3e_i^3}{3(R_{ij}^2-\alpha_i^2e_i^2)}\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^2}\Bigg]\Bigg\}.$$

The integrals occurring in this equation are all elementary and they are given in table (A5.1). Substituting the expressions for these integrals into the foregoing equation (and after extensive algebraic manipulations) we obtain

$$W_1(R_{ij},\alpha_i,e_i)=\sqrt{2R_{ij}}\lim_{\varepsilon^*\to\frac{\pi}{2}-}\Bigg\{\frac{8R_{ij}\alpha_i e_i}{3(R_{ij}^2-\alpha_i^2e_i^2)}\frac{\sin\varepsilon^*}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varepsilon^*)\cos\varepsilon^*}-$$

$$-\frac{4R_{ij}\alpha_i e_i(3R_{ij}^2+\alpha_i^2e_i^2)}{9(R_{ij}^2-\alpha_i^2e_i^2)^3}\frac{\sin\varepsilon^*}{\cos^3\varepsilon^*}-\frac{4\alpha_i e_i(3R_{ij}^4-13R_{ij}^2\alpha_i^2e_i^2+6\alpha_i^4e_i^4)}{9(R_{ij}^2-\alpha_i^2e_i^2)^3R_{ij}}\tan\varepsilon^*+$$

$$+\frac{4R_{ij}\alpha_i e_i(3R_{ij}^2+\alpha_i^2e_i^2)}{9(R_{ij}^2-\alpha_i^2e_i^2)^3}\tan^3\varepsilon^*\Bigg\}+\frac{\pi\alpha_i^3e_i^3(R_{ij}^4-2R_{ij}^2\alpha_i^2e_i^2+\alpha_i^4e_i^4)}{3R_{ij}^2(R_{ij}^2-\alpha_i^2e_i^2)^3\sqrt{R_{ij}^2-\alpha_i^2e_i^2}}\sqrt{2R_{ij}}\,.$$

We readily verify that this equation is equivalent to

$$W_1(R_{ij},\alpha_i,e_i)=\sqrt{2R_{ij}}\,\frac{\pi\alpha_i^3e_i^3}{3(R_{ij}^2-\alpha_i^2e_i^2)\,R_{ij}^2\sqrt{R_{ij}^2-\alpha_i^2e_i^2}}\,. \qquad \text{(A5-17)}$$

In an analogous way we may evaluate the integral $W_2(R_{ij},\alpha_i,e_i)$, which, according to equation (2.69), has the following form

$$W_2(R_{ij},\alpha_i,e_i)=\sqrt{2R_{ij}}\lim_{\substack{\varepsilon^*\to\frac{\pi}{2}-\\ \varepsilon\to 0+}}\Bigg[2R_{ij}\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi}{\cos^2\varphi}d\varphi\int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i}\alpha^{-6}d\alpha -$$

$$-\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi}{\cos\varphi}\int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i}\frac{d\alpha}{\alpha^4\sqrt{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)+\alpha^2}}d\varphi- \tag{A5-18}$$

$$-\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi}{\cos\varphi}\left(R_{ij}^2\tan^2\varphi+\frac{R_{ij}^2}{\cos^2\varphi}-\alpha_i^2e_i^2\right)\int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i}\frac{d\alpha}{\alpha^6\sqrt{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)+\alpha^2}}d\varphi\Bigg].$$

The final inside integral in this equation can be evaluated with the help of equation (A5-6) and it takes the form

$$\int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i}\frac{d\alpha}{\alpha^6\sqrt{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)+\alpha^2}}=\frac{16}{15}\frac{R_{ij}\alpha_i e_i}{(R_{ij}^2-\alpha_i^2e_i^2)}\frac{\cos^5\varphi}{\sin^4\varphi(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^3}-$$

$$-\frac{8R_{ij}\alpha_i e_i\cos^3\varphi}{15\sin^4\varphi(R_{ij}^2-\alpha_i^2e_i^2)^3(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^2}\left[3R_{ij}^2+\alpha_i^2e_i^2-(R_{ij}^2+3\alpha_i^2e_i^2)\cos^2\varphi\right]+ \tag{A5-19}$$

$$+\frac{2R_{ij}\alpha_i e_i(5R_{ij}^4+10R_{ij}^2\alpha_i^2e_i^2+\alpha_i^4e_i^4)\cos\varphi}{5(R_{ij}^2-\alpha_i^2e_i^2)^5\sin^2\varphi(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)}-\frac{2R_{ij}\alpha_i e_i(R_{ij}^4+10R_{ij}^2\alpha_i^2e_i^2+5\alpha_i^4e_i^4)\cos^3\varphi}{5(R_{ij}^2-\alpha_i^2e_i^2)^5\sin^2\varphi(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)},$$

$$\sin\varphi\neq 0.$$

Introducing the relations (A5-16) and (A5-19) in equation (A5-18) and after some further reductions, we obtain

$$W_2(R_{ij},\alpha_i,e_i)=\sqrt{2R_{ij}}\lim_{\substack{\varepsilon^*\to\frac{\pi}{2}-\\ \varepsilon\to 0+}}\left\{\frac{4R_{ij}\alpha_i e_i}{15\left(R_{ij}^2-\alpha_i^2e_i^2\right)^5}\left(5R_{ij}^4+\right.\right.$$

$$\left.+10R_{ij}^2\alpha_i^2e_i^2+\alpha_i^4e_i^4\right)\left(\tan^3\varepsilon^*-\tan^3\varepsilon\right)-$$

$$-\frac{4R_{ij}\alpha_i e_i}{15\left(R_{ij}^2-\alpha_i^2e_i^2\right)^5}\left(6R_{ij}^6-35R_{ij}^4\alpha_i^2e_i^2-20R_{ij}^2\alpha_i^4e_i^4+\alpha_i^6e_i^6\right)\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{\left(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi\right)\cos^2\varphi}+$$

$$+\frac{2R_{ij}^3\alpha_i e_i}{15\left(R_{ij}^2-\alpha_i^2e_i^2\right)^5}\left(5R_{ij}^4+2R_{ij}^2\alpha_i^2e_i^2-55\alpha_i^4e_i^4\right)\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{\left(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi\right)}+$$

$$+\frac{4R_{ij}\alpha_i e_i}{15\left(R_{ij}^2-\alpha_i^2e_i^2\right)^3}\left(3R_{ij}^4-16R_{ij}^2\alpha_i^2e_i^2+5\alpha_i^4e_i^4\right)\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{\left(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi\right)^2}-$$

$$-\frac{16R_{ij}^3\alpha_i e_i}{15\left(R_{ij}^2-\alpha_i^2e_i^2\right)}\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{\left(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi\right)^3}+\frac{8R_{ij}^3\alpha_i e_i\left(3R_{ij}^2+\alpha_i^2e_i^2\right)}{15\left(R_{ij}^2-\alpha_i^2e_i^2\right)^3}\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{\left(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi\right)^2\cos^2\varphi}-$$

$$-\frac{2R_{ij}^3\alpha_i e_i}{5\left(R_{ij}^2-\alpha_i^2e_i^2\right)^5}\left(5R_{ij}^4+10R_{ij}^2\alpha_i^2e_i^2+\alpha_i^4e_i^4\right)\int_{\varepsilon}^{\varepsilon^*}\frac{\sin^2\varphi\,d\varphi}{\left(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi\right)\cos^4\varphi}+$$

$$+\frac{2R_{ij}^3\alpha_i e_i}{5\left(R_{ij}^2-\alpha_i^2e_i^2\right)^5}\left(R_{ij}^4+10R_{ij}^2\alpha_i^2e_i^2+5\alpha_i^4e_i^4\right)\int_{\varepsilon}^{\varepsilon^*}\frac{\sin^2\varphi\,d\varphi}{\left(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi\right)\cos^2\varphi}-$$

$$-\frac{16R_{ij}^3\alpha_i e_i}{15\left(R_{ij}^2-\alpha_i^2e_i^2\right)}\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{\left(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi\right)^3\sin^2\varphi}+$$

$$+\frac{8R_{ij}^3\alpha_i e_i}{15\left(R_{ij}^2-\alpha_i^2e_i^2\right)^3}\left(3R_{ij}^2+\alpha_i^2e_i^2\right)\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{\left(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi\right)^2\cos^2\varphi\cdot\sin^2\varphi}-$$

$$-\frac{8R_{ij}\alpha_i e_i}{15(R_{ij}^2-\alpha_i^2e_i^2)^3}\left(R_{ij}^4+6R_{ij}^2\alpha_i^2e_i^2+\alpha_i^4e_i^4\right)\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^2\sin^2\varphi}-$$

$$-\frac{2R_{ij}^3\alpha_i e_i}{5(R_{ij}^2-\alpha_i^2e_i^2)^5}\left(5R_{ij}^4+10R_{ij}^2\alpha_i^2e_i^2+\alpha_i^4e_i^4\right)\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)\cos^4\varphi}+$$

$$\left.+\frac{16R_{ij}\alpha_i^3e_i^3}{15(R_{ij}^2-\alpha_i^2e_i^2)}\int_{\varepsilon}^{\varepsilon^*}\frac{\cos^2\varphi\, d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^3\sin^2\varphi}+\frac{8R_{ij}\alpha_i^3e_i^3(R_{ij}^2+3\alpha_i^2e_i^2)}{15(R_{ij}^2-\alpha_i^2e_i^2)^3}\int_{\varepsilon}^{\varepsilon^*}\frac{\cos^2\varphi\, d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^2\sin^2\varphi}\right] .$$

The values of the integrals entering the right-hand side of the last relation are given in table (A5.1). By use of these values and after extensive algebraic manipulations the foregoing equation reduces to

$$W_2(R_{ij},\alpha_i,e_i)=\sqrt{2R_{ij}}\lim_{\varepsilon^*\to\frac{\pi}{2}-}\left[\frac{4R_{ij}\alpha_i e_i}{15(R_{ij}^2-\alpha_i^2e_i^2)^5}\left(5R_{ij}^4+10R_{ij}^2\alpha_i^2e_i^2+\alpha_i^4e_i^4\right)\left(\tan^3\varepsilon^*-\right.\right.$$

$$\left.-\frac{\sin\varepsilon^*}{\cos^3\varepsilon^*}\right)-\frac{4\alpha_i e_i}{15R_{ij}(R_{ij}^2-\alpha_i^2e_i^2)^5}\left(7R_{ij}^6-30R_{ij}^4\alpha_i^2e_i^2+3R_{ij}^2\alpha_i^4e_i^4+4\alpha_i^6e_i^6\right)\tan\varepsilon^*+$$

$$\left.+\frac{16R_{ij}\alpha_i e_i}{15(R_{ij}^2-\alpha_i^2e_i^2)^3}\left(3R_{ij}^2+\alpha_i^2e_i^2\right)\frac{\tan\varepsilon^*}{(R_{ij}^2-\alpha^2e^2\cos^2\varepsilon^*)}\right\}+$$

$$+\frac{16R_{ij}\alpha_i e_i}{15(R_{ij}^2-\alpha_i^2e_i^2)^2}\sqrt{2R_{ij}}\lim_{\varepsilon\to 0+}\left[\frac{\cos\varepsilon}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varepsilon)\sin\varepsilon}-\frac{(R_{ij}^2-\alpha_i^2e_i^2)\cos\varepsilon}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varepsilon)^2\sin\varepsilon}\right\}+$$

$$+\sqrt{2R_{ij}}\,\frac{\pi\alpha_i^3e_i^3}{3(R_{ij}^2-\alpha_i^2e_i^2)^3\sqrt{R_{ij}^2-\alpha_i^2e_i^2}} .$$

It is easy to prove that the limits occurring in this equation vanish and therefore the integral $W_2(R_{ij}, \alpha_i, e_i)$ takes the form

$$W_2(R_{ij},\alpha_i,e_i)=\sqrt{2R_{ij}}\;\frac{\pi\alpha_i^3e_i^3}{3(R_{ij}^2-\alpha_i^2e_i^2)^3\sqrt{R_{ij}^2-\alpha_i^2e_i^2}}\;. \qquad \text{(A5-20)}$$

We proceed with the evaluation of the integral $W_3(R_{ij},\alpha_i,e_i)$, which, according to equation (2.69) is given by the expression

$$W_3(R_{ij},\alpha_i,e_i)=\sqrt{2R_{ij}}\lim_{\substack{\varepsilon^*\to\frac{\pi}{2}-\\ \varepsilon\to 0+}}\left\{2R_{ij}\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi}{\cos^2\varphi}d\varphi\int_{R_{ij}-\alpha_ie_i}^{R_{ij}+\alpha_ie_i}\alpha^{-8}d\alpha-\right.$$

$$-\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi}{\cos\varphi}\int_{R_{ij}-\alpha_ie_i}^{R_{ij}+\alpha_ie_i}\frac{d\alpha}{\alpha^6\sqrt{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)+\alpha^2}}d\varphi- \qquad \text{(A5-21)}$$

$$\left.-\int_{\varepsilon}^{\varepsilon^*}\frac{\tan^2\varphi}{\cos\varphi}\left(R_{ij}^2\tan^2\varphi+\frac{R_{ij}^2}{\cos^2\varphi}-\alpha_i^2e_i^2\right)\int_{R_{ij}-\alpha_ie_i}^{R_{ij}+\alpha_ie_i}\frac{d\alpha}{\alpha^8\sqrt{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)+\alpha^2}}d\varphi\right\}.$$

Further, by use of equation (A5-6), we can prove that the following formula is valid

$$\int_{R_{ij}-\alpha_ie_i}^{R_{ij}+\alpha_ie_i}\frac{d\alpha}{\alpha^8\sqrt{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)+\alpha^2}}=-\frac{32R_{ij}\alpha_ie_i}{35(R_{ij}^2-\alpha_i^2e_i^2)}\frac{\cos^7\varphi}{(R_{ij}-\alpha_i^2e_i^2\cos^2\varphi)^4\sin^6\varphi}+$$

$$+\frac{16R_{ij}\alpha_ie_i(3R_{ij}^2+\alpha_i^2e_i^2)}{35(R_{ij}^2-\alpha_i^2e_i^2)^3}\frac{\cos^5\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^3\sin^6\varphi}-$$

$$-\frac{16R_{ij}\alpha_i e_i(R_{ij}^2+3\alpha_i^2e_i^2)}{35(R_{ij}^2-\alpha_i^2e_i^2)^3}\frac{\cos^7\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^3\sin^6\varphi}-$$

(A5-22)

$$-\frac{12R_{ij}\alpha_i e_i}{35(R_{ij}^2-\alpha_i^2e_i^2)^5}(5R_{ij}^4+10R_{ij}^2\alpha_i^2e_i^2+\alpha_i^4e_i^4)\frac{\cos^3\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^2\sin^4\varphi}+$$

$$+\frac{12R_{ij}\alpha_i e_i}{35(R_{ij}^2-\alpha_i^2e_i^2)^5}(R_{ij}^4+10R_{ij}^2\alpha_i^2e_i^2+5\alpha_i^4e_i^4)\frac{\cos^5\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^2\sin^4\varphi}+$$

$$+\frac{2R_{ij}\alpha_i e_i}{7(R_{ij}^2-\alpha_i^2e_i^2)^7}(7R_{ij}^6+35R_{ij}^4\alpha_i^2e_i^2+21R_{ij}^2\alpha_i^4e_i^4+\alpha_i^6e_i^6)\frac{\cos\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)\sin^2\varphi}-$$

$$\left.-\frac{2R_{ij}\alpha_i e_i}{7(R_{ij}^2-\alpha_i^2e_i^2)^7}(R_{ij}^6+21R_{ij}^4\alpha_i^2e_i^2+35R_{ij}^2\alpha_i^4e_i^4+7\alpha_i^6e_i^6)\frac{\cos^3\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)\sin^2\varphi}\right] .$$

Introducing the relations (A5-19) and (A5-22) in equation (A5-21) and arranging the resulting expression, we obtain

$$W_3(R_{ij},\alpha_i,e_i)=\sqrt{2R_{ij}}\lim_{\substack{\varepsilon^*\to\frac{\pi}{2}-\\ \varepsilon\to 0+}}\left\{\frac{4R_{ij}\alpha_i e_i}{21(R_{ij}^2-\alpha_i^2e_i^2)^7}(7R_{ij}^6+35R_{ij}^4\alpha_i^2e_i^2+21R_{ij}^2\alpha_i^4e_i^4+\right.$$

$$+\alpha_i^6e_i^6)(\tan^3\varepsilon^*-\tan^3\varepsilon)-\frac{16R_{ij}\alpha_i e_i}{105(R_{ij}^2-\alpha_i^2e_i^2)^3}(4R_{ij}^4-23R_{ij}^2\alpha_i^2e_i^2+$$

$$+7\alpha_i^4e_i^4)\int_{\varepsilon}^{\varepsilon^*}\frac{\cos^2\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^3\sin^2\varphi}d\varphi+$$

$$+\frac{4R_{ij}\alpha_i e_i}{105(R_{ij}^2-\alpha_i^2 e_i^2)^5}(33R_{ij}^6-205R_{ij}^4\alpha_i^2e_i^2-121R_{ij}^2\alpha_i^4e_i^4+5\alpha_i^6e_i^6)\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^2\sin^2\varphi}-$$

$$-\frac{4R_{ij}\alpha_i e_i}{105(R_{ij}^2-\alpha_i^2 e_i^2)^5}(14R_{ij}^6+5R_{ij}^4\alpha_i^2e_i^2-160R_{ij}^2\alpha_i^4e_i^4-3\alpha_i^6e_i^6)\int_{\varepsilon}^{\varepsilon^*}\frac{\cos^2\varphi\, d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^2\sin^2\varphi}-$$

$$-\frac{4R_{ij}\alpha_i e_i}{35(R_{ij}^2-\alpha_i^2 e_i^2)^7}(15R_{ij}^8-70R_{ij}^6\alpha_i^2e_i^2-224R_{ij}^4\alpha_i^4e_i^4-42R_{ij}^2\alpha_i^6e_i^6+$$

$$+\alpha_i^8e_i^8)\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)\cos^2\varphi}+$$

$$+\frac{2R_{ij}^3\alpha_i e_i}{35(R_{ij}^2-\alpha_i^2 e_i^2)^7}(7R_{ij}^6+51R_{ij}^4\alpha_i^2e_i^2-203R_{ij}^2\alpha_i^4e_i^4-175\alpha_i^6e_i^6)\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)}+$$

$$+\frac{32R_{ij}^3\alpha_i e_i}{35(R_{ij}^2-\alpha_i^2 e_i^2)}\int_{\varepsilon}^{\varepsilon^*}\frac{\cos^2\varphi\, d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^4\sin^2\varphi}-\frac{16R_{ij}^3\alpha_i e_i(3R_{ij}^2+\alpha_i^2e_i^2)}{35(R_{ij}-\alpha_i^2e_i^2)^3}\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^3\sin^2\varphi}+$$

$$+\frac{12R_{ij}^3\alpha_i e_i}{35(R_{ij}^2-\alpha_i^2 e_i^2)^5}(5R_{ij}^4+10R_{ij}^2\alpha_i^2e_i^2+\alpha_i^4e_i^4)\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^2\cos^2\varphi}-$$

$$-\frac{12R_{ij}^3\alpha_i e_i}{35(R_{ij}^2-\alpha_i^2 e_i^2)^5}(R_{ij}^4+10R_{ij}^2\alpha_i^2e_i^2+5\alpha_i^4e_i^4)\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^2}-$$

$$-\frac{2R_{ij}^3\alpha_i e_i}{7(R_{ij}^2-\alpha_i^2 e_i^2)^7}(7R_{ij}^6+35R_{ij}^4\alpha_i^2e_i^2+21R_{ij}^2\alpha_i^4e_i^4+\alpha_i^6e_i^6)\int_{\varepsilon}^{\varepsilon^*}\frac{\sin^2\varphi\, d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)\cos^4\varphi}+$$

$$+\frac{2R_{ij}^3\alpha_i e_i}{7(R_{ij}^2-\alpha_i^2 e_i^2)^7}(R_{ij}^6+21R_{ij}^4\alpha_i^2e_i^2+35R_{ij}^2\alpha_i^4e_i^4+7\alpha_i^6e_i^6)\int_{\varepsilon}^{\varepsilon^*}\frac{\sin^2\varphi\, d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)\cos^2\varphi}+$$

$$+\frac{32R_{ij}^3\alpha_i e_i}{35(R_{ij}^2-\alpha_i^2e_i^2)}\int_{\varepsilon}^{\varepsilon^*}\frac{\cos^2\varphi\, d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^4\sin^4\varphi}-$$

$$-\frac{16R_{ij}^3\alpha_i e_i}{35(R_{ij}^2-\alpha_i^2e_i^2)^3}(3R_{ij}^2+\alpha_i^2e_i^2)\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^3\sin^4\varphi}+$$

$$+\frac{16R_{ij}\alpha_i e_i}{35(R_{ij}^2-\alpha_i^2e_i^2)^3}(R_{ij}^4+6R_{ij}^2\alpha_i^2e_i^2+\alpha_i^4e_i^4)\int_{\varepsilon}^{\varepsilon^*}\frac{\cos^2\varphi\, d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^3\sin^4\varphi}+$$

$$+\frac{12R_{ij}^3\alpha_i e_i}{35(R_{ij}^2-\alpha_i^2e_i^2)^5}(5R_{ij}^4+10R_{ij}^2\alpha_i^2e_i^2+\alpha_i^4e_i^4)\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^2\sin^2\varphi\cos^2\varphi}-$$

$$-\frac{2R_{ij}^3\alpha_i e_i}{7(R_{ij}^2-\alpha_i^2e_i^2)^7}(7R_{ij}^6+35R_{ij}^4\alpha_i^2e_i^2+21R_{ij}^2\alpha_i^4e_i^4+\alpha_i^6e_i^6)\int_{\varepsilon}^{\varepsilon^*}\frac{d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)\cos^4\varphi}-$$

$$-\frac{32R_{ij}\alpha_i^3e_i^3}{35(R_{ij}^2-\alpha_i^2e_i^2)}\int_{\varepsilon}^{\varepsilon^*}\frac{\cos^4\varphi\, d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^4\sin^4\varphi}-$$

$$-\frac{16R_{ij}\alpha_i^3e_i^3}{35(R_{ij}^2-\alpha_i^2e_i^2)^3}(R_{ij}^2+3\alpha_i^2e_i^2)\int_{\varepsilon}^{\varepsilon^*}\frac{\cos^4\varphi\, d\varphi}{(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varphi)^3\sin^4\varphi}\Bigg\}.$$

The integrals occurring in this equation are all elementary and they are given in table (A5.1). Substituting the expressions for these integrals into the foregoing equation (and after extensive algebraic manipulations) we obtain

$$W_3(R_{ij},\alpha_i,e_i)=\sqrt{2R_{ij}}\lim_{\varepsilon^*\to\frac{\pi}{2}-}\Bigg\{-\frac{4\alpha_i e_i}{105R_{ij}(R_{ij}^2-\alpha_i^2e_i^2)^7}(90R_{ij}^8-252R_{ij}^4\alpha_i^4e_i^4+$$

$$+144R_{ij}^{2}\alpha_i^6e_i^6+18\alpha_i^8e_i^8)\tan\varepsilon^*+\frac{72R_{ij}\alpha_ie_i}{105\left(R_{ij}^2-\alpha_i^2e_i^2\right)^5}\left(5R_{ij}^4+10R_{ij}^2\alpha_i^2e_i^2+\right.$$

$$\left.+\alpha_i^4e_i^4\right)\frac{\tan\varepsilon^*}{R_{ij}^2-\alpha_i^2e_i^2\cos^2\varepsilon^*}\Bigg]+\sqrt{2R_{ij}}\lim_{\varepsilon\to0+}\Bigg[\frac{16\alpha_ie_i}{105R_{ij}\left(R_{ij}^2-\alpha_i^2e_i^2\right)^6}\left(14R_{ij}^8-20R_{ij}^6\alpha_i^2e_i^2-\right.$$

$$\left.-6R_{ij}^4\alpha_i^4e_i^4+16R_{ij}^2\alpha_i^6e_i^6-4\alpha_i^8e_i^8\right)\frac{\cos\varepsilon}{\left(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varepsilon\right)\sin\varepsilon}-$$

$$-\frac{16\alpha_ie_i}{105R_{ij}\left(R_{ij}^2-\alpha_i^2e_i^2\right)^5}\left(14R_{ij}^8-22R_{ij}^6\alpha_i^2e_i^2-R_{ij}^4\alpha_i^4e_i^4+12R_{ij}^2\alpha_i^6e_i^6-3\alpha_i^8e_i^8\right)\frac{\cos\varepsilon}{\left(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varepsilon\right)^2\sin\varepsilon}-$$

$$-\frac{32\alpha_i^5e_i^5}{105R_{ij}\left(R_{ij}^2-\alpha_i^2e_i^2\right)^2}\frac{\cos\varepsilon}{\left(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varepsilon\right)^3\sin\varepsilon}-\frac{16\alpha_i^3e_i^3}{105R_{ij}\left(R_{ij}^2-\alpha_i^2e_i^2\right)}\frac{\cos\varepsilon}{\left(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varepsilon\right)^3\sin^3\varepsilon}-$$

$$-\frac{32\alpha_ie_i}{105R_{ij}\left(R_{ij}^2-\alpha_i^2e_i^2\right)^2}\frac{\cos\varepsilon}{\left(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varepsilon\right)\sin^3\varepsilon}+$$

$$+\frac{16\alpha_ie_i}{105R_{ij}\left(R_{ij}^2-\alpha_i^2e_i^2\right)^4}\left(2R_{ij}^6-5R_{ij}^4\alpha_i^2e_i^2+4R_{ij}^2\alpha_i^4e_i^4-\alpha_i^6e_i^6\right)\frac{\cos\varepsilon}{\left(R_{ij}^2-\alpha_i^2e_i^2\cos^2\varepsilon\right)^2\sin^3\varepsilon}\Bigg]+$$

$$+\sqrt{2R_{ij}}\frac{\pi\alpha_i^3e_i^3}{105\left(R_{ij}^2-\alpha_i^2e_i^2\right)^6\sqrt{R_{ij}^2-\alpha_i^2e_i^2}}\left(35R_{ij}^4-7R_{ij}^2\alpha_i^2e_i^2-28\alpha_i^4e_i^4\right).$$

It can be proved that the limits entering the right-hand side of the last equation vanish and hence the integral $W_3(R_{ij}, \alpha_i, e_i)$ reduces to

$$W_3(R_{ij},\alpha_i,e_i)=\sqrt{2R_{ij}}\;\frac{\pi\alpha_i^3e_i^3}{105(R_{ij}^2-\alpha_i^2e_i^2)^5\sqrt{R_{ij}^2-\alpha_i^2e_i^2}}\left(35R_{ij}^2+28\alpha_i^2e_i^2\right). \quad \text{(A5-23)}$$

Processes similar to the process followed for the derivation of equation (A5-23) lead to the evaluation of the integrals $W_4(R_{ij},\alpha_i,e_i)$, $W_5(R_{ij},\alpha_i,e_i)$ and $W_6(R_{ij},\alpha_i,e_i)$. The formulae and the reduction of the formulae entering the different stages of the evaluation of these integrals are extremely long, we therefore confine ourselves to giving the final expressions which are

$$W_4(R_{ij},\alpha_i,e_i)=\sqrt{2R_{ij}}\;\frac{\pi\alpha_i^3e_i^3}{315\,(R_{ij}^2-\alpha_i^2e_i^2)^7\sqrt{R_{ij}^2-\alpha_i^2e_i^2}}\left(105R_{ij}^4+252R_{ij}^2\alpha_i^2e_i^2+72\alpha_i^4e_i^4\right),$$

$$W_5(R_{ij},\alpha_i,e_i)=\sqrt{2R_{ij}}\;\frac{\pi\alpha_i^3e_i^3}{3465\,(R_{ij}^2-\alpha_i^2e_i^2)^9\sqrt{R_{ij}^2-\alpha_i^2e_i^2}}\left(1{,}155R_{ij}^6+\right.$$

$$\left.+5{,}544R_{ij}^4\alpha_i^2e_i^2+4{,}752R_{ij}^2\alpha_i^4e_i^4+704\alpha_i^6e_i^6\right),$$

(A5-24)

$$W_6(R_{ij},\alpha_i,e_i)=\sqrt{2R_{ij}}\;\frac{\pi\alpha_i^3e_i^3}{9009(R_{ij}^2-\alpha_i^2e_i^2)^{11}\sqrt{R_{ij}^2-\alpha_i^2e_i^2}}\left(3003R_{ij}^8+24{,}024R_{ij}^6\alpha_i^2e_i^2+\right.$$

$$\left.41{,}184R_{ij}^4\alpha_i^4e_i^4+18{,}304R_{ij}^2\alpha_i^6e_i^6+1{,}664\alpha_i^8e_i^8\right).$$

We note that the integrals needed for the derivation of the above three expressions are given in table (A5.1). Next we give this table. The functions $g_\nu(\varphi)$, $g_\nu^*(\varphi^*)$, $\nu=1,\cdots 35$ occurring in the right-hand-side of the relations given in the table have the property

$$\lim_{\varphi^*\to 0+} g_\nu^*(\varphi^*)=\lim_{\varphi\to\frac{\pi}{2}-} g_\nu(\varphi)=0\;,\;\nu=1,\cdots\cdots 35$$

Table (A5.1)

$$\int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i} \frac{d\alpha}{\alpha^{10}\sqrt{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)+\alpha^2}} = \frac{256\,R_{ij}\alpha_i e_i}{315(R_{ij}^2-\alpha_i^2e_i^2)}\,\frac{\sin^2\varphi}{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)^5\cos\varphi} -$$

$$-\frac{128\,R_{ij}\alpha_i e}{315(R_{ij}^2-\alpha_i^2e_i^2)^3}(3R_{ij}^2+\alpha_i^2e_i^2)\frac{1}{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)^4\cos\varphi}+\frac{128\,R_{ij}\alpha_i e_i}{315(R_{ij}^2-\alpha_i^2e_i^2)^3}(R_{ij}^2+$$

$$+3\alpha_i^2e_i^2)\frac{\cos\varphi}{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)^4}+\frac{96\,R_{ij}\alpha_i e_i}{315(R_{ij}^2-\alpha_i^2e_i^2)^5}(5R_{ij}^4+10R_{ij}^2\alpha_i^2e_i^2+\alpha_i^4e_i^4)\times$$

$$\times\frac{1}{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)^3\cos\varphi}-\frac{96\,R_{ij}\alpha_i e_i}{315(R_{ij}^2-\alpha_i^2e_i^2)^5}(R_{ij}^4+10R_{ij}^2\alpha_i^2e_i^2+5\alpha_i^4e_i^4)\times$$

$$\times\frac{\cos\varphi}{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)^3}-\frac{16\,R_{ij}\alpha_i e_i}{63(R_{ij}^2-\alpha_i^2e_i^2)^7}(7R_{ij}^6+35R_{ij}^4\alpha_i^2e_i^2+21R_{ij}^2\alpha_i^4e_i^4+$$

$$+\alpha_i^6e_i^6)\frac{1}{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)^2\cos\varphi}+\frac{16\,R_{ij}\alpha_i e_i}{63(R_{ij}^2-\alpha_i^2e_i^2)^7}(R_{ij}^6+21R_{ij}^4\alpha_i^2e_i^2+35R_{ij}^2\alpha_i^4e_i^4+$$

$$+7\alpha_i^6e_i^6)\frac{\cos\varphi}{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)^2}+\frac{14\,R_{ij}\alpha_i e_i}{63(R_{ij}^2-\alpha_i^2e_i^2)^9}(9R_{ij}^8+84R_{ij}^6\alpha_i^2e_i^2+$$

$$+126R_{ij}^4\alpha_i^4e_i^4+36R_{ij}^2\alpha_i^6e_i^6+\alpha_i^8e_i^8)\frac{1}{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)\cos\varphi}-\frac{14\,R_{ij}\alpha_i e_i}{63(R_{ij}^2-\alpha_i^2e_i^2)^9}\times$$

$$\times(R_{ij}^8+36R_{ij}^6\alpha_i^2e_i^2+126R_{ij}^4\alpha_i^4e_i^4+84R_{ij}^2\alpha_i^6e_i^6+9\alpha_i^8e_i^8)\frac{\cos\varphi}{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)},$$

$$\int\limits_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i} \frac{d\alpha}{\alpha^{12}\sqrt{\left(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi\right)+\alpha^2}} = \frac{-512R_{ij}\alpha_i e_i \sin^2\varphi}{693\left(R_{ij}^2-\alpha_i^2e_i^2\right)\left(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi\right)^6\cos\varphi}+$$

$$+\frac{256R_{ij}\alpha_i e_i}{693\left(R_{ij}^2-\alpha_i^2e_i^2\right)^3}\left(3R_{ij}^2+\alpha_i^2e_i^2\right)\frac{1}{\left(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi\right)^5\cos\varphi}-\frac{256R_{ij}\alpha_i e_i}{693\left(R_{ij}^2-\alpha_i^2e_i^2\right)^3}\left(R_{ij}^2+\right.$$

$$\left.+3\alpha_i^2e_i^2\right)\frac{\cos\varphi}{\left(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi\right)^5}-\frac{64R_{ij}\alpha_i e_i}{231\left(R_{ij}^2-\alpha_i^2e_i^2\right)^5}\left(5R_{ij}^4+10R_{ij}^2\alpha_i^2e_i^2+\alpha_i^4e_i^4\right)\times$$

$$\times\frac{1}{\left(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi\right)^4\cos\varphi}+\frac{64R_{ij}\alpha_i e_i}{231\left(R_{ij}^2-\alpha_i^2e_i^2\right)^5}\left(R_{ij}^4+10R_{ij}^2\alpha_i^2e_i^2+5\alpha_i^4e_i^4\right)\times$$

$$\times\frac{\cos\varphi}{\left(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi\right)^4}+\frac{160R_{ij}\alpha_i e_i}{693\left(R_{ij}^2-\alpha_i^2e_i^2\right)^7}\left(7R_{ij}^6+35R_{ij}^4\alpha_i^2e_i^2+21R_{ij}^2\alpha_i^4e_i^4+\alpha_i^6e_i^6\right)\times$$

$$\times\frac{1}{\left(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi\right)^3\cos\varphi}-\frac{160R_{ij}\alpha_i e_i}{693\left(R_{ij}^2-\alpha_i^2e_i^2\right)^7}\left(R_{ij}^6+21R_{ij}^4\alpha_i^2e_i^2+35R_{ij}^2\alpha_i^4e_i^4+\right.$$

$$\left.+7\alpha_i^6e_i^6\right)\frac{\cos\varphi}{\left(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi\right)^3}-\frac{140R_{ij}\alpha_i e_i}{693\left(R_{ij}^2-\alpha_i^2e_i^2\right)^9}\left(9R_{ij}^8+84R_{ij}^6\alpha_i^2e_i^2+126R_{ij}^4\alpha_i^4e_i^4+\right.$$

$$\left.+36R_{ij}^2\alpha_i^6e_i^6+\alpha_i^8e_i^8\right)\frac{1}{\left(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi\right)^2\cos\varphi}+\frac{140R_{ij}\alpha_i e_i}{693\left(R_{ij}^2-\alpha_i^2e_i^2\right)^9}\left(R_{ij}^8+36R_{ij}^6\alpha_i^2e_i^2+\right.$$

$$\left.+126R_{ij}^4\alpha_i^4e_i^4+84R_{ij}^2\alpha_i^6e_i^6+9\alpha_i^8e_i^8\right)\frac{\cos\varphi}{\left(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi\right)^2}+\frac{42R_{ij}\alpha_i e_i}{231\left(R_{ij}^2-\alpha_i^2e_i^2\right)^{11}}\left(11R_{ij}^{10}+\right.$$

$$\left.+165R_{ij}^8\alpha_i^2e_i^2+462R_{ij}^6\alpha_i^4e_i^4+330R_{ij}^4\alpha_i^6e_i^6+55R_{ij}^2\alpha_i^8e_i^8+\alpha_i^{10}e_i^{10}\right)\frac{1}{\left(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi\right)\cos\varphi}-$$

$$-\frac{42R_{ij}\alpha_i e_i}{231(R_{ij}^2-\alpha_i^2e_i^2)^{11}}(R_{ij}^{10}+55R_{ij}^8\alpha_i^2e_i^2+330R_{ij}^6\alpha_i^4e_i^4+462R_{ij}^4\alpha_i^6e_i^6+165R_{ij}^2\alpha_i^8e_i^8+11\alpha_i^{10}e_i^{10})\times$$

$$\times\frac{\cos\varphi}{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)},$$

$$\int\limits_{R_{ij}-\alpha_ie_i}^{R_{ij}+\alpha_ie_i}\frac{d\alpha}{\alpha^{14}\sqrt{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)+\alpha^2}}=\frac{2048R_{ij}\alpha_ie_i\sin^2\varphi}{3003(R_{ij}^2-\alpha_i^2e_i^2)(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)^7\cos\varphi}-$$

$$-\frac{1024R_{ij}\alpha_ie_i}{3003(R_{ij}^2-\alpha_i^2e_i^2)^3}(3R_{ij}^2+\alpha_i^2e_i^2)\frac{1}{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)^6\cos\varphi}+\frac{1024R_{ij}\alpha_ie_i}{3003(R_{ij}^2-\alpha_i^2e_i^2)^3}(R_{ij}^2+$$

$$+3\alpha_i^2e_i^2)\frac{\cos\varphi}{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)^6}+\frac{256R_{ij}\alpha_ie_i}{1001(R_{ij}^2-\alpha_i^2e_i^2)^5}(5R_{ij}^4+10R_{ij}^2\alpha_i^2e_i^2+\alpha_i^4e_i^4)\times$$

$$\times\frac{1}{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)^5\cos\varphi}-\frac{256R_{ij}\alpha_ie_i}{1001(R_{ij}^2-\alpha_i^2e_i^2)^5}(R_{ij}^4+10R_{ij}^2\alpha_i^2e_i^2+5\alpha_i^4e_i^4)\times$$

$$\times\frac{\cos\varphi}{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)^5}-\frac{640R_{ij}\alpha_ie_i}{3003(R_{ij}^2-\alpha_i^2e_i^2)^7}(7R_{ij}^6+35R_{ij}^4\alpha_i^2e_i^2+21R_{ij}^2\alpha_i^4e_i^4+\alpha_i^6e_i^6)\times$$

$$\times\frac{1}{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)^4\cos\varphi}+\frac{640R_{ij}\alpha_ie_i}{3003(R_{ij}^2-\alpha_i^2e_i^2)^7}(R_{ij}^6+21R_{ij}^4\alpha_i^2e_i^2+35R_{ij}^2\alpha_i^4e_i^4+$$

$$+7\alpha_i^6e_i^6)\frac{\cos\varphi}{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)^4}+\frac{560R_{ij}\alpha_ie_i}{3003(R_{ij}^2-\alpha_i^2e_i^2)^9}(9R_{ij}^8+84R_{ij}^6\alpha_i^2e_i^2+126R_{ij}^4\alpha_i^4e_i^4+$$

$$+36R_{ij}^2\alpha_i^6e_i^6+\alpha_i^8e_i^8)\frac{1}{(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi)^3\cos\varphi}-\frac{560R_{ij}\alpha_ie_i}{3003(R_{ij}^2-\alpha_i^2e_i^2)^9}(R_{ij}^8+36R_{ij}^6\alpha_i^2e_i^2+$$

$$+126R_{ij}^{4}\alpha_i^4e_i^4+84R_{ij}^{2}\alpha_i^6e_i^6+9\alpha_i^8e_i^8\Big)\frac{\cos\varphi}{\left(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi\right)^3}-\frac{168R_{ij}\alpha_ie_i}{1001\left(R_{ij}^2-\alpha_i^2e_i^2\right)^{11}}\Big(11R_{ij}^{10}+$$

$$+165R_{ij}^{8}\alpha_i^2e_i^2+462R_{ij}^{6}\alpha_i^4e_i^4+330R_{ij}^{4}\alpha_i^6e_i^6+55R_{ij}^{2}\alpha_i^8e_i^8+\alpha_i^{10}e_i^{10}\Big)\frac{\cos^{-1}\varphi}{\left(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi\right)^2}+$$

$$+\frac{168R_{ij}\alpha_ie_i}{1001\left(R_{ij}^2-\alpha_i^2e_i^2\right)^{11}}\Big(R_{ij}^{10}+55R_{ij}^{8}\alpha_i^2e_i^2+330R_{ij}^{6}\alpha_i^4e_i^4+462R_{ij}^{4}\alpha_i^6e_i^6+165R_{ij}^{2}\alpha_i^8e_i^8+11\alpha_i^{10}e_i^{10}\Big)\times$$

$$\times\frac{\cos\varphi}{\left(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi\right)^2}+\frac{154R_{ij}\alpha_ie_i}{1001\left(R_{ij}^2-\alpha_i^2e_i^2\right)^{13}}\Big(13R_{ij}^{12}+286R_{ij}^{10}\alpha_i^2e_i^2+1287R_{ij}^{8}\alpha_i^4e_i^4+$$

$$+1716R_{ij}^{6}\alpha_i^6e_i^6+715R_{ij}^{4}\alpha_i^8e_i^8+78R_{ij}^{2}\alpha_i^{10}e_i^{10}+\alpha_i^{12}e_i^{12}\Big)\frac{1}{\left(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi\right)\cos\varphi}-$$

$$-\frac{154R_{ij}\alpha_ie_i}{1001\left(R_{ij}^2-\alpha_i^2e_i^2\right)^{13}}\Big(R_{ij}^{12}+78R_{ij}^{10}\alpha_i^2e_i^2+715R_{ij}^{8}\alpha_i^4e_i^4+1716R_{ij}^{6}\alpha_i^6e_i^6+1287R_{ij}^{4}\alpha_i^8e_i^8+$$

$$+286R_{ij}^{2}\alpha_i^{10}e_i^{10}+13\alpha_i^{12}e_i^{12}\Big)\frac{\cos\varphi}{\left(R_{ij}^2\tan^2\varphi-\alpha_i^2e_i^2\sin^2\varphi\right)},$$

$$\int_{\varphi^*}^{\varphi}\frac{\sin^2\omega\,d\omega}{(\alpha^2-\beta^2\cos^2\omega)^2\cos^2\omega}=\frac{\tan\varphi}{\alpha^2(\alpha^2-\beta^2\cos^2\varphi)}-\frac{\pi\,(2\alpha^4-5\beta^2\alpha^2+3\beta^4)}{4\alpha^5(\alpha^2-\beta^2)\sqrt{\alpha^2-\beta^2}}+g_1(\varphi)+\overset{*}{g}_1(\varphi^*),$$

$$\int_{0}^{\pi/2}\frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)}=\frac{\pi}{2\alpha\sqrt{\alpha^2-\beta^2}},$$

$$\int_{\varphi^*}^{\varphi}\frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)\cos^2\omega}=\frac{\tan\varphi}{\alpha^2}+\frac{\pi\beta^2}{2\alpha^3\sqrt{\alpha^2-\beta^2}}+g_2(\varphi)+\overset{*}{g}_2(\varphi^*),$$

$$\int_0^{\frac{\pi}{2}} \frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^2} = \frac{\pi(2\alpha^2-\beta^2)}{4(\alpha^2-\beta^2)\alpha^3\sqrt{\alpha^2-\beta^2}},$$

$$\int_{\varphi^*}^{\varphi} \frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)\cos^4\omega} = \frac{1}{3\alpha^2}\frac{\sin\varphi}{\cos^3\varphi} + \frac{(2\alpha^2+3\beta^2)}{3\alpha^4}\tan\varphi + \frac{\pi\beta^4}{2\alpha^5\sqrt{\alpha^2-\beta^2}} + g_3(\varphi) + \overset{*}{g}_3(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi} \frac{\tan^2\omega\, d\omega}{(\alpha^2-\beta^2\cos^2\omega)\cos^2\omega} = \frac{1}{3\alpha^2}\frac{\sin\varphi}{\cos^3\varphi} - \frac{(\alpha^2-3\beta^2)}{3\alpha^4}\tan\varphi - \frac{\pi(\alpha^2-\beta^2)\beta^2}{2\alpha^5\sqrt{\alpha^2-\beta^2}} + g_4(\varphi) + \overset{*}{g}_4(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi} \frac{\tan^2\omega\, d\omega}{(\alpha^2-\beta^2\cos^2\omega)} = \frac{1}{\alpha^2}\tan\varphi - \frac{\pi(\alpha^2-\beta^2)}{2\alpha^3\sqrt{\alpha^2-\beta^2}} + g_5(\varphi) + \overset{*}{g}_5(\varphi^*),$$

$$\int_0^{\frac{\pi}{2}} \frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^3} = \frac{\pi(8\alpha^4-8\alpha^2\beta^2+3\beta^4)}{16\alpha^5(\alpha^2-\beta^2)^2\sqrt{\alpha^2-\beta^2}},$$

$$\int_{\varphi^*}^{\varphi} \frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^2\cos^2\omega} = \frac{\pi\beta^2(4\alpha^2-3\beta^2)}{4\alpha^5(\alpha^2-\beta^2)\sqrt{\alpha^2-\beta^2}} + \frac{\tan\varphi}{\alpha^2(\alpha^2-\beta^2\cos^2\varphi)} + g_6(\varphi) + \overset{*}{g}_6(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi} \frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^3\sin^2\omega} = \frac{\cos\varphi^*}{(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*} + g_7(\varphi) + \overset{*}{g}_7(\varphi^*) -$$

$$- \frac{3\pi\beta^2(8\alpha^4-4\alpha^2\beta^2+\beta^4)}{16\alpha^5(\alpha^2-\beta^2)^3\sqrt{\alpha^2-\beta^2}},$$

$$\int_{\varphi^*}^{\varphi}\frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^2\sin^2\omega}=\frac{\cos\varphi^*}{(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}-$$

$$-\frac{\pi(4\alpha^2-\beta^2)\beta^2}{4\alpha^3(\alpha^2-\beta^2)^2\sqrt{\alpha^2-\beta^2}}+g_8(\varphi)+\overset{*}{g}_8(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi}\frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^2\sin^2\omega\cos^2\omega}=\frac{\cos\varphi^*}{(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}+\frac{\tan\varphi}{\alpha^2(\alpha^2-\beta^2\cos^2\varphi)}-$$

$$-\frac{3\pi\beta^4(2\alpha^2-\beta^2)}{4\alpha^5(\alpha^2-\beta^2)^2\sqrt{\alpha^2-\beta^2}}+g_9(\varphi)+\overset{*}{g}_9(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi}\frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^3\tan^2\omega}=\frac{\cos\varphi^*}{(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}-$$

$$-\frac{\pi(8\alpha^4+8\alpha^2\beta^2-\beta^4)}{16\alpha^3(\alpha^2-\beta^2)^3\sqrt{\alpha^2-\beta^2}}+g_{10}(\varphi)+\overset{*}{g}_{10}(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi}\frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^2\tan^2\omega}=\frac{\cos\varphi^*}{(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}-\frac{\pi(2\alpha^2+\beta^2)}{4\alpha(\alpha^2-\beta^2)^2\sqrt{\alpha^2-\beta^2}}+$$

$$+g_{11}(\varphi)+\overset{*}{g}_{11}(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi}\frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^4\tan^2\omega}=-\frac{\beta^2\cos^3\varphi^*}{6\alpha^2(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin\varphi^*}+g_{12}(\varphi)+\overset{*}{g}_{12}(\varphi^*)+$$

$$+\frac{(10\alpha^2-3\beta^2)\cos\varphi^*}{6\alpha^2(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}-\frac{2\cos\varphi^*}{3\alpha^2(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}-$$

$$-\frac{\pi(16\alpha^6+24\alpha^4\beta^2-6\alpha^2\beta^4+\beta^6)}{32\alpha^5(\alpha^2-\beta^2)^4\sqrt{\alpha^2-\beta^2}},$$

$$\int_{\varphi^*}^{\varphi}\frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^3\sin^4\omega}=\frac{\cos\varphi^*}{3(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}+$$

$$+\frac{(2\alpha^2-9\beta^2)\cos\varphi^*}{3(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}+\frac{\pi\beta^4(48\alpha^4-16\alpha^2\beta^2+3\beta^4)}{16\alpha^5(\alpha^2-\beta^2)^4\sqrt{\alpha^2-\beta^2}}+g_{13}(\varphi)+\overset{*}{g}_{13}(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi}\frac{\cos^2\omega\,d\omega}{(\alpha^2-\beta^2\cos^2\omega)^3\sin^4\omega}=\frac{\cos\varphi^*}{3(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}-$$

$$-\frac{(\alpha^2+6\beta^2)\cos\varphi^*}{3(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}+\frac{\pi\beta^2(24\alpha^4+12\alpha^2\beta^2-\beta^4)}{16\alpha^3(\alpha^2-\beta^2)^4\sqrt{\alpha^2-\beta^2}}+g_{14}(\varphi)+\overset{*}{g}_{14}(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi}\frac{\cos^4\omega\,d\omega}{(\alpha^2-\beta^2\cos^2\omega)^3\sin^4\omega}=\frac{\cos\varphi^*}{3(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}-$$

$$-\frac{(4\alpha^2+3\beta^2)\cos\varphi^*}{3(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}+\frac{\pi(8\alpha^4+24\alpha^2\beta^2+3\beta^4)}{16(\alpha^2-\beta^2)^4\alpha\sqrt{\alpha^2-\beta^2}}+g_{15}(\varphi)+\overset{*}{g}_{15}(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi}\frac{\cos^2\omega\,d\omega}{(\alpha^2-\beta^2\cos^2\omega)^4\sin^4\omega}=-\frac{\beta^2\cos^3\varphi^*}{6\alpha^2(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}-$$

$$-\frac{\cos\varphi^*}{3\alpha^2(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)\sin^3\varphi^*}+\frac{(4\alpha^2-\beta^2)\cos\varphi^*}{6\alpha^2(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}-$$

$$-\frac{(4\alpha^4+23\alpha^2\beta^2-6\beta^4)\cos\varphi^*}{6\alpha^2(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}+\frac{(\alpha^2+4\beta^2)\cos\varphi^*}{3\alpha^2(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}+$$

$$+\frac{\pi\beta^2(64\alpha^6+48\alpha^4\beta^2-8\alpha^2\beta^4+\beta^6)}{32\alpha^5(\alpha^2-\beta^2)^5\sqrt{\alpha^2-\beta^2}}+g_{16}(\varphi)+\overset{*}{g}_{16}(\varphi^*)$$

$$\int_{\varphi^*}^{\varphi}\frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^4\tan^4\omega}=-\frac{\beta^2\cos^5\varphi^*}{6\alpha^2(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}+$$

$$+\frac{(3\alpha^2+2\beta^2)\cos\varphi^*}{3\alpha^2(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}-\frac{(14\alpha^4+10\alpha^2\beta^2-3\beta^4)\cos\varphi^*}{6\alpha^2(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin\varphi^*}-$$

$$-\frac{\cos\varphi^*}{3\alpha^2(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)\sin^3\varphi^*}+\frac{(4\alpha^2-\beta^2)\cos\varphi^*}{6\alpha^2(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}+$$

$$+\frac{\pi(16\alpha^6+72\alpha^4\beta^2+18\alpha^2\beta^4-\beta^6)}{32\alpha^3(\alpha^2-\beta^2)^5\sqrt{\alpha^2-\beta^2}}+g_{17}(\varphi)+\overset{*}{g}_{17}(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi}\frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^4\tan^2\omega\sin^4\omega}=-\frac{\beta^2(2\alpha^2-13\beta^2)\cos^3\varphi^*}{30\alpha^2(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}+$$

$$+\frac{\cos^3\varphi^*}{5(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^5\varphi^*}-\frac{(2\alpha^2-13\beta^2)\cos\varphi^*}{15\alpha^2(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)\sin^3\varphi^*}+$$

$$+\frac{(8\alpha^4-54\alpha^2\beta^2+13\beta^4)\cos\varphi^*}{30\alpha^2(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}-\frac{\cos\varphi^*(8\alpha^6-86\alpha^4\beta^2-207\alpha^2\beta^4+54\beta^6)}{30\alpha^2(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}+$$

$$+\frac{(2\alpha^2-21\alpha^2\beta^2-36\beta^4)\cos\varphi^*}{15\alpha^2(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}-\frac{8\beta^4\cos^3\varphi^*}{30\alpha^2(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin\varphi^*}-$$

$$-\frac{\pi\beta^4(160\alpha^6+80\alpha^4\beta^2-10\alpha^2\beta^4+\beta^6)}{32\alpha^5(\alpha^2-\beta^2)^6\sqrt{\alpha^2-\beta^2}}+g_{18}(\varphi)+\overset{*}{g}_{18}(\varphi^*),$$

$$\int\limits_{\varphi^*}^{\varphi}\frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^4\tan^4\omega\sin^2\omega}=\frac{\cos^3\varphi^*}{5(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^5\varphi^*}+$$

$$+\frac{\beta^2(3\alpha^2+8\beta^2)\cos^3\varphi^*}{30\alpha^2(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}+\frac{(3\alpha^2+8\beta^2)\cos\varphi^*}{15\alpha^2(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)\sin^3\varphi^*}-$$

$$-\frac{(12\alpha^4+29\alpha^2\beta^2-8\beta^4)\cos\varphi^*}{30\alpha^2(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}+\frac{(12\alpha^6+181\alpha^4\beta^2+62\alpha^2\beta^4-24\beta^6)\cos\varphi^*}{30\alpha^2(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}-$$

$$-\frac{(3\alpha^4+36\alpha^2\beta^2+16\beta^4)\cos\varphi^*}{15\alpha^2(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}-\frac{8\beta^4\cos^3\varphi^*}{30\alpha^2(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin\varphi^*}-$$

$$-\frac{\pi\beta^2(64\alpha^6+144\alpha^4\beta^2+24\alpha^2\beta^4-\beta^6)}{32\alpha^3(\alpha^2-\beta^2)^6\sqrt{\alpha^2-\beta^2}}+g_{19}(\varphi)+\overset{*}{g}_{19}(\varphi^*),$$

$$\int\limits_{\varphi^*}^{\varphi}\frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^4\tan^6\omega}=\frac{\cos^3\varphi^*}{5(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^5\varphi^*}-$$

$$-\frac{\beta^2(5\alpha^2+3\beta^2)\cos^3\varphi^*}{30\alpha^2(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin\varphi^*}+\frac{\beta^2(8\alpha^2+3\beta^2)\cos^3\varphi^*}{30\alpha^2(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}+$$

$$+\frac{(8\alpha^2+3\beta^2)\cos\varphi^*}{15\alpha^2(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)\sin^3\varphi^*}-\frac{(32\alpha^4+4\alpha^2\beta^2-3\beta^4)\cos\varphi^*}{30\alpha^2(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}+$$

$$+\frac{(82\alpha^6+161\alpha^4\beta^2-3\alpha^2\beta^4-9\beta^6)\cos\varphi^*}{30\alpha^2(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}-\frac{(18\alpha^4+31\alpha^2\beta^2+6\beta^4)\cos\varphi^*}{15\alpha^2(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}-$$

$$-\frac{\pi(16\alpha^6+120\alpha^4\beta^2+90\alpha^2\beta^4+5\beta^6)}{32\alpha(\alpha^2-\beta^2)^6\sqrt{\alpha^2-\beta^2}}+g_{20}(\varphi)+\overset{*}{g}_{20}(\varphi^*);$$

$$\int_{\varphi^*}^{\varphi}\frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^5\tan^4\omega}=-\frac{\beta^2\cos^5\varphi^*}{8\alpha^2(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos\varphi^*)^4\sin^3\varphi^*}-$$

$$-\frac{(14\alpha^2-3\beta^2)\beta^2\cos^3\varphi^*}{48\alpha^4(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}+\frac{(14\alpha^2-3\beta^2)\beta^2\cos^3\varphi^*}{48\alpha^4(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin\varphi^*}+$$

$$+\frac{(42\alpha^4+19\alpha^2\beta^2-6\beta^4)\cos\varphi^*}{24\alpha^4(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}-\frac{(148\alpha^6+110\alpha^4\beta^2-36\alpha^2\beta^4+9\beta^6)\cos\varphi^*}{48\alpha^4(\alpha^2-\beta^2)^4(\alpha^2\ \beta^2\cos^2\varphi^*)^2\sin\varphi^*}-$$

$$-\frac{(14\alpha^2-3\beta^2)\cos\varphi^*}{24\alpha^4(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)\sin^3\varphi^*}+\frac{(44\alpha^4-14\alpha^2\beta^2+3\beta^4)\cos\varphi^*}{48\alpha^4(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}+$$

$$+\frac{(128\alpha^8+768\alpha^6\beta^2+288\alpha^4\beta^4-32\alpha^2\beta^6-3\beta^8)}{256\alpha^5(\alpha^2-\beta^2)^6\sqrt{\alpha^2-\beta^2}}+g_{21}(\varphi)+\overset{*}{g}_{21}(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi}\frac{\cos^4\omega\,d\omega}{(\alpha^2-\beta^2\cos^2\omega)^5\sin^6\omega}=\frac{(96\alpha^8+2{,}188\alpha^6\beta^2+1{,}046\alpha^4\beta^4-372\alpha^2\beta^6+45\beta^8)\cos\varphi^*}{240\alpha^4(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}+$$

$$+\frac{\beta^4\cos^5\varphi^*}{8\alpha^2(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^3\varphi^*}+\frac{\cos^3\varphi^*}{5(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^5\varphi^*}+$$

$$+\frac{\beta^2(24\alpha^4+134\alpha^2\beta^2-15\beta^4)\cos^3\varphi^*}{240\alpha^4(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}-\frac{\beta^4(134\alpha^2-15\beta^2)\cos^3\varphi^*}{240\alpha^4(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin\varphi^*}+$$

$$+\frac{(24\alpha^4+134\alpha^2\beta^2-15\beta^4)\cos\varphi^*}{120\alpha^4(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)\sin^3\varphi^*}-\frac{(96\alpha^6+452\alpha^4\beta^2-134\alpha^2\beta^4+15\beta^6)\cos\varphi^*}{240\alpha^4(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}-$$

$$-\frac{24\alpha^6+498\alpha^4\beta^2+223\alpha^2\beta^4-30\beta^6}{120\alpha^4(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}\cos\varphi^*-$$

$$-\frac{\pi\beta^2(640\alpha^8+1{,}920\alpha^6\beta^2+480\alpha^4\beta^4-40\alpha^2\beta^6+3\beta^8)}{256\alpha^5(\alpha^2-\beta^2)^7\sqrt{\alpha^2-\beta^2}}+g_{22}(\varphi)+\overset{*}{g}_{22}(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi}\frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^5\tan^6\omega}=\frac{(836\alpha^6+1{,}998\alpha^4\beta^2+316\alpha^2\beta^4-147\beta^6)}{240\alpha^2(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}\cos\varphi^*+$$

$$+\frac{\beta^2\cos^5\varphi^*}{8(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^3\varphi^*}+\frac{\cos^3\varphi^*}{5(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^5\varphi^*}+$$

$$+\frac{\beta^2(94\alpha^2+49\beta^2)\cos^3\varphi^*}{240\alpha^2(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}-\frac{\beta^2(70\alpha^2+49\beta^2)\cos^3\varphi^*}{240\alpha^2(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin\varphi^*}+$$

$$+\frac{(94\alpha^2+49\beta^2)\cos\varphi^*}{120\alpha^2(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)\sin^3\varphi^*}-\frac{(316\alpha^4+162\alpha^2\beta^2-49\beta^4)\cos\varphi^*}{240\alpha^2(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}-$$

$$-\frac{(234\alpha^4+383\alpha^2\beta^2+98\beta^4)\cos\varphi^*}{120\alpha^2(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}-$$

$$-\frac{\pi}{256\alpha^3(\alpha^2-\beta^2)^7\sqrt{\alpha^2-\beta^2}}(128\alpha^8+1{,}280\alpha^6\beta^2+1{,}440\alpha^4\beta^4+160\alpha^2\beta^6-5\beta^8)+g_{23}(\varphi)+\overset{*}{g}_{23}(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi}\frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^6\tan^6\omega}=\frac{(1{,}960\alpha^8+5{,}680\alpha^6\beta^2+2{,}038\alpha^4\beta^4-642\alpha^2\beta^6-27\beta^8)}{\sin\varphi^*\,480\alpha^4(\alpha^2-\beta^2\cos^2\varphi^*)^2(\alpha^2-\beta^2)^6}\cos\varphi^*-$$

$$-\frac{\beta^2\cos^7\varphi^*}{10\alpha^2(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)^5\sin^5\varphi^*}+\frac{3\beta^2(6\alpha^2-\beta^2)\cos^5\varphi^*}{80\alpha^2(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^3\varphi^*}+$$

$$+\frac{(2\alpha^2+\beta^2)\cos^3\varphi^*}{10\alpha^2(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^5\varphi^*}+\frac{(236\alpha^4+184\alpha^2\beta^2+9\beta^4)\beta^2\cos^3\varphi^*}{480\alpha^4(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}-$$

$$-\frac{(188\alpha^4+160\alpha^2\beta^2+9\beta^4)\beta^2\cos^3\varphi^*}{480\alpha^4(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin\varphi^*}+\frac{(236\alpha^4+184\alpha^2\beta^2+9\beta^4)\cos\varphi^*}{240\alpha^4(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}-$$

$$-\frac{(728\alpha^6+752\alpha^4\beta^2-184\alpha^2\beta^4-9\beta^6)\cos\varphi^*}{480\alpha^4(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}-\frac{(612\alpha^6+1{,}072\alpha^4\beta^2+443\alpha^2\beta^4+18\beta^6)\cos\varphi^*}{240\alpha^4(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}-$$

$$-\frac{\pi(256\alpha^{10}+3{,}200\alpha^8\beta^2+4{,}800\alpha^6\beta^4+800\alpha^4\beta^6-50\alpha^2\beta^8-3\beta^{10})}{512\alpha^5(\alpha^2-\beta^2)^8\sqrt{\alpha^2-\beta^2}}+g_{24}(\varphi)+\overset{*}{g}_{24}(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi}\frac{\cos^4\omega\,d\omega}{(\alpha^2-\beta^2\cos^2\omega)^5\sin^8\omega}=\frac{(16\alpha^6-280\alpha^4\beta^2-486\alpha^2\beta^4+35\beta^6)\beta^2\cos^3\varphi^*}{560\alpha^4(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}+$$

$$+\frac{\cos^5\varphi^*}{7(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^7\varphi^*}-\frac{\beta^4(8\alpha^2+7\beta^2)\cos^5\varphi^*}{56\alpha^2(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^3\varphi^*}+$$

$$+\frac{(2\alpha^2-17\beta^2)\cos^3\varphi^*}{35(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^5\varphi^*}+\frac{\beta^4(144\alpha^4+486\alpha^2\beta^2-35\beta^4)\cos^3\varphi^*}{560\alpha^4(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin\varphi^*}+$$

$$+\frac{(16\alpha^6-280\alpha^4\beta^2-486\alpha^2\beta^4+35\beta^6)\cos\varphi^*}{280\alpha^4(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)\sin^3\varphi^*}-$$

$$-\frac{(64\alpha^8-976\alpha^6\beta^2-1{,}684\alpha^4\beta^4+486\alpha^2\beta^6-35\beta^8)\cos\varphi^*}{560\alpha^4(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}+$$

$$+\frac{\cos\varphi^*(64\alpha^{10}-1{,}552\alpha^8\beta^2-11{,}068\alpha^6\beta^4-3{,}742\alpha^4\beta^6+1{,}388\alpha^2\beta^8-105\beta^{10})}{560\alpha^4(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}-$$

$$-\frac{(16\alpha^8-504\alpha^6\beta^2-2{,}290\alpha^4\beta^4-867\alpha^2\beta^6+70\beta^8)\cos\varphi^*}{280\alpha^4(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}+$$

$$+\frac{\pi\beta^4}{256\alpha^5(\alpha^2-\beta^2)^8\sqrt{\alpha^2-\beta^2}}(1{,}920\alpha^8+3{,}840\alpha^6\beta^2+720\alpha^4\beta^4-48\alpha^2\beta^6+3\beta^8)+$$

$$\mathcal{J}_{25}(\varphi)+\overset{*}{\mathcal{J}}_{25}(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi}\frac{\cos^6\omega\,d\omega}{(\alpha^2-\beta^2\cos^2\omega)^5\sin^8\omega}=-\frac{(96\alpha^8+3{,}860\alpha^6\beta^2+5{,}042\alpha^4\beta^4+260\alpha^2\beta^6-249\beta^8)\cos\varphi^*}{336\alpha^2(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}+$$

$$+\frac{\cos^5\varphi^*}{7(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^7\varphi^*}-\frac{15\beta^4\cos^5\varphi^*}{56(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^3\varphi^*}-$$

$$-\frac{(\alpha^2+2\beta^2)\cos^3\varphi^*}{7(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^5\varphi^*}-\frac{\beta^2(24\alpha^4+322\alpha^2\beta^2+83\beta^4)\cos^3\varphi^*}{336\alpha^2(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}+$$

$$+\frac{\beta^4(274\alpha^2+83\beta^2)\cos^3\varphi^*}{336\alpha^2(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin\varphi^*}-\frac{(24\alpha^4+322\alpha^2\beta^2+83\beta^4)\cos\varphi^*}{168\alpha^2(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)\sin^3\varphi^*}+$$

$$+\frac{(96\alpha^6+1{,}084\alpha^4\beta^2+190\alpha^2\beta^4-83\beta^6)\cos\varphi^*}{336\alpha^2(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}+$$

$$+\frac{(24\alpha^6+966\alpha^4\beta^2+989\alpha^2\beta^4+166\beta^6)\cos\varphi^*}{168\alpha^2(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}+$$

$$+\frac{5\pi\beta^2}{256\alpha^3(\alpha^2-\beta^2)^8\sqrt{\alpha^2-\beta^2}}\left(128\alpha^8+640\alpha^6\beta^2+480\alpha^4\beta^4+40\alpha^2\beta^6-\beta^8\right)+$$

$$+g_{26}(\varphi)+\overset{*}{g}_{26}(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi}\frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^5\tan^8\omega}=\frac{(1{,}758\alpha^6+5{,}873\alpha^4\beta^2+2{,}950\alpha^2\beta^4+144\beta^6)\cos\varphi^*}{840\alpha^2(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}+$$

$$+\frac{\cos^5\varphi^*}{7(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^7\varphi^*}-\frac{\beta^2(7\alpha^2+8\beta^2)\cos^5\varphi^*}{56(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^3\varphi^*}-$$

$$-\frac{3(4\alpha^2+\beta^2)\cos^3\varphi^*}{35(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos\varphi^*)^3\sin^5\varphi^*}-\frac{\beta^2(778\alpha^4+1{,}295\alpha^2\beta^2+72\beta^4)\cos^3\varphi^*}{1{,}680\alpha^2(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}-$$

$$-\frac{(778\alpha^4+1{,}295\alpha^2\beta^2+72\beta^4)\cos\varphi^*}{840\alpha^2(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)\sin^3\varphi^*}+\frac{\beta^2(490\alpha^4+1{,}223\alpha^2\beta^2+72\beta^4)\cos^3\varphi^*}{1{,}680\alpha^2(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin\varphi^*}+$$

$$+\frac{(2{,}692\alpha^6+4{,}342\alpha^4\beta^2-527\alpha^2\beta^4-72\beta^6)\cos\varphi^*}{1{,}680\alpha^2(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}-$$

$$-\frac{(6{,}332\alpha^8+27{,}434\alpha^6\beta^2+13{,}436\alpha^4\beta^4-1{,}941\alpha^2\beta^6-216\beta^8)\cos\varphi^*}{1{,}680\alpha^2(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}+$$

$$+\frac{\pi}{256\alpha(\alpha^2-\beta^2)^8\sqrt{\alpha^2-\beta^2}}\left(128\alpha^8+1{,}792\alpha^6\beta^2+3{,}360\alpha^4\beta^4+1{,}120\alpha^2\beta^6+35\beta^8\right)+$$

$$+g_{27}(\varphi)+\overset{*}{g}_{27}(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi}\frac{\cos^6\omega\,d\omega}{(\alpha^2-\beta^2\cos^2\omega)^6\sin^8\omega}=-\frac{(80\alpha^6+1{,}624\alpha^4\beta^2+706\alpha^2\beta^4+21\beta^6)\beta^2\cos^3\varphi^*}{1{,}120\alpha^4(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}+$$

$$+\frac{\beta^4\cos^7\varphi^*}{10\alpha^2(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^5\sin^5\varphi^*}+\frac{\cos^5\varphi^*}{7(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^7\varphi^*}-$$

$$-\frac{3\beta^4(92\alpha^2-7\beta^2)\cos^5\varphi^*}{560\alpha^2(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^3\varphi^*}-\frac{(10\alpha^4+34\alpha^2\beta^2+7\beta^4)\cos^3\varphi^*}{70\alpha^2(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^5\varphi^*}+$$

$$+\frac{\beta^4(1{,}352\alpha^4+650\alpha^2\beta^2+21\beta^4)\cos^3\varphi^*}{1{,}120\alpha^4(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin\varphi^*}-\frac{(80\alpha^6+1{,}624\alpha^4\beta^2+706\alpha^2\beta^4+21\beta^6)\cos\varphi^*}{560\alpha^4(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)\sin^3\varphi^*}+$$

$$+\frac{(320\alpha^8+5{,}312\alpha^6\beta^2+2{,}388\alpha^4\beta^4-706\alpha^2\beta^6-21\beta^8)\cos\varphi^*}{1{,}120\alpha^4(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}-$$

$$-\frac{(320\alpha^{10}+17{,}440\alpha^8\beta^2+30{,}060\alpha^6\beta^4+5{,}622\alpha^4\beta^6-2{,}328\alpha^2\beta^8-63\beta^{10})\cos\varphi^*}{1{,}120\alpha^4(\alpha^2-\beta^2)^7(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}+$$

$$+\frac{(80\alpha^8+4{,}648\alpha^6\beta^2+5{,}798\alpha^4\beta^4+1{,}587\alpha^2\beta^6+42\beta^8)\cos\varphi^*}{560\alpha^4(\alpha^2-\beta^2)^7(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}+$$

$$+\frac{3\pi\beta^2}{512\alpha^5(\alpha^2-\beta^2)^9\sqrt{\alpha^2-\beta^2}}(512\alpha^{10}+3{,}200\alpha^8\beta^2+3{,}200\alpha^6\beta^4+400\alpha^4\beta^6-20\alpha^2\beta^8+\beta^{10})+$$

$$+\mathcal{J}_{28}(\varphi)+\overset{*}{\mathcal{J}}_{28}(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi}\frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^6\tan^8\omega}=\frac{(6{,}056\alpha^6+16{,}104\alpha^4\beta^2+612\alpha^2\beta^4-893\beta^6)\cos\varphi^*}{3{,}360\alpha^2(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}+$$

$$+\frac{\beta^2\cos^7\varphi^*}{10(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^5\sin^5\varphi^*}+\frac{\cos^5\varphi^*}{7(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^7\varphi^*}-$$

$$-\frac{3\beta^2(42\alpha^2+43\beta^2)\cos^5\varphi^*}{560(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^3\varphi^*}-\frac{(24\alpha^2+27\beta^2)\cos^3\varphi^*}{70(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^5\varphi^*}-$$

$$-\frac{\beta^2(1{,}892\alpha^4+4{,}508\alpha^2\beta^2+893\beta^4)\cos^3\varphi^*}{3{,}360\alpha^2(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}+\frac{\beta^2(1{,}316\alpha^4+3{,}860\alpha^2\beta^2+893\beta^4)\cos^3\varphi^*}{3{,}360\alpha^2(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin\varphi^*}-$$

$$-\frac{(1{,}892\alpha^4+4{,}508\alpha^2\beta^2+893\beta^4)\cos\varphi^*}{1{,}680\alpha^2(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)\sin^3\varphi^*}-$$

$$-\frac{(14{,}680\alpha^8+78{,}360\alpha^6\beta^2+64{,}686\alpha^4\beta^4-1{,}894\alpha^2\beta^6-2{,}679\beta^8)\cos\varphi^*}{3{,}360\alpha^2(\alpha^2-\beta^2)^7(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}+$$

$$+\frac{(4{,}524\alpha^6+17{,}164\alpha^4\beta^2+12{,}991\alpha^2\beta^4+1{,}786\beta^6)\cos\varphi^*}{1{,}680\alpha^2(\alpha^2-\beta^2)^7(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}+$$

$$+\frac{\pi}{512\alpha^3(\alpha^2-\beta^2)^9\sqrt{\alpha^2-\beta^2}}(256\alpha^{10}+4{,}480\alpha^8\beta^2+11{,}200\alpha^6\beta^4+5{,}600\alpha^4\beta^6+350\alpha^2\beta^8-$$

$$-7\beta^{10})+g_{29}(\varphi)+g^*_{29}(\varphi^*)\,,$$

$$\int_{\varphi^*}^{\varphi}\frac{\cos^6\omega\,d\omega}{(\alpha^2-\beta^2\cos^2\omega)^6\sin^{10}\omega}=-\frac{\beta^4(2{,}560\alpha^6+26{,}308\alpha^4\beta^2+9{,}380\alpha^2\beta^4+189\beta^6)\cos^3\varphi^*}{10{,}080\alpha^4(\alpha^2-\beta^2\cos^2\varphi^*)^3(\alpha^2-\beta^2)^6\sin\varphi^*}+$$

$$+\frac{\cos^7\varphi^*}{9(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)^5\sin^9\varphi^*}-\frac{\beta^4(10\alpha^2+9\beta^2)\cos^7\varphi^*}{90\alpha^2(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^5\sin^5\varphi^*}+$$

$$+\frac{(2\alpha^2-21\beta^2)\cos^5\varphi^*}{63(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^7\varphi^*}+\frac{\beta^4(320\alpha^4+1{,}358\alpha^2\beta^2-63\beta^4)\cos^5\varphi^*}{1{,}680\alpha^2(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^3\varphi^*}-$$

$$-\frac{(20\alpha^6-310\alpha^4\beta^2-616\alpha^2\beta^4-63\beta^6)\cos^3\varphi^*}{630\alpha^2(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^5\varphi^*}-$$

$$-\frac{\beta^2(160\alpha^8-5{,}040\alpha^6\beta^2-31{,}236\alpha^4\beta^4-9{,}884\alpha^2\beta^6-189\beta^8)\cos^3\varphi^*}{10{,}080\alpha^4(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}-$$

$$+\frac{(640\alpha^{10}-16{,}480\alpha^8\beta^2-107{,}448\alpha^6\beta^4-25{,}352\alpha^4\beta^6+9{,}884\alpha^2\beta^8+189\beta^{10})\cos\varphi^*}{10{,}080\alpha^4(\alpha^2-\beta^2)^7(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}-$$

$$-\frac{(160\alpha^8-5{,}040\alpha^6\beta^2-31{,}236\alpha^4\beta^4-9{,}884\alpha^2\beta^6-189\beta^8)\cos\varphi^*}{5{,}040\alpha^4(\alpha^2-\beta^2)^7(\alpha^2-\beta^2\cos^2\varphi^*)\sin^3\varphi^*}-$$

$$-\frac{(640\alpha^{12}-26{,}720\alpha^{10}\beta^2-410{,}280\alpha^8\beta^4-518{,}040\alpha^6\beta^6-47{,}678\alpha^4\beta^8+31{,}542\alpha^2\beta^{10}+567\beta^{12})\cos\varphi^*}{10{,}080\alpha^4(\alpha^2-\beta^2)^8(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}+$$

$$+\frac{(160\alpha^{10}-9{,}520\alpha^8\beta^2-98{,}892\alpha^6\beta^4-100{,}972\alpha^4\beta^6-21{,}343\alpha^2\beta^8-378\beta^{10})\cos\varphi^*}{5{,}040\alpha^4(\alpha^2-\beta^2)^8(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}-$$

$$-\frac{\pi\beta^4}{512\alpha^5(\alpha^2-\beta^2)^{10}\sqrt{\alpha^2-\beta^2}}(5{,}376\alpha^{10}+22{,}400\alpha^8\beta^2+16{,}800\alpha^6\beta^4+1{,}680\alpha^4\beta^6-70\alpha^2\beta^8+3\beta^{10})+$$

$$+g_{30}(\varphi)+\overset{*}{g}_{30}(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi}\frac{\cos^8\omega\,d\omega}{(\alpha^2-\beta^2\cos^2\omega)^6\sin^{10}\omega}=\frac{\beta^2(560\alpha^6+18{,}936\alpha^4\beta^2+22{,}974\alpha^2\beta^4+3{,}719\beta^6)\cos^3\varphi^*}{10{,}080\alpha^2(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}+$$

$$+\frac{\cos^7\varphi^*}{9(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)^5\sin^9\varphi^*}-\frac{19\beta^4\cos^7\varphi^*}{90(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^5\sin^5\varphi^*}-$$

$$-\frac{(7\alpha^2+12\beta^2)\cos^5\varphi^*}{63(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^7\varphi^*}+\frac{\beta^4(1{,}148\alpha^2+467\beta^2)\cos^5\varphi^*}{1{,}680(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^3\varphi^*}+$$

$$+\frac{(70\alpha^4+526\alpha^2\beta^2+373\beta^4)\cos^3\varphi^*}{630(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^5\varphi^*}+\frac{\cos\varphi^*(560\alpha^6+18{,}936\alpha^4\beta^2+22{,}974\alpha^2\beta^4+3{,}719\beta^6)}{5{,}040\alpha^2(\alpha^2-\beta^2)^7(\alpha^2-\beta^2\cos^2\varphi^*)\sin^3\varphi^*}-$$

$$-\frac{\beta^4(14{,}728\alpha^4+19{,}990\alpha^2\beta^2+3{,}719\beta^4)\cos^3\varphi^*}{10{,}080\alpha^2(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin\varphi^*}-$$

$$-\frac{(2{,}240\alpha^8+61{,}408\alpha^6\beta^2+81{,}132\alpha^4\beta^4-2{,}494\alpha^2\beta^6-3{,}719\beta^8)\cos\varphi^*}{10{,}080\alpha^2(\alpha^2-\beta^2)^7(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}+$$

$$+\frac{\cos\varphi^*(2{,}240\alpha^{10}+180{,}800\alpha^8\beta^2+523{,}860\alpha^6\beta^4+298{,}098\alpha^4\beta^6-23{,}872\alpha^2\beta^8-11{,}157\beta^{10})}{10{,}080\alpha^2(\alpha^2-\beta^2)^8(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}-$$

$$-\frac{(560\alpha^8+50{,}632\alpha^6\beta^2+109{,}242\alpha^4\beta^4+63{,}073\alpha^2\beta^6+7{,}438\beta^8)\cos\varphi^*}{5{,}040\alpha^2(\alpha^2-\beta^2)^8(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}-$$

$$-\frac{\pi\beta^2}{512\alpha^3(\alpha^2-\beta^2)^{10}\sqrt{\alpha^2-\beta^2}}(1{,}536\alpha^{10}+13{,}440\alpha^8\beta^2+22{,}400\alpha^6\beta^4+8{,}400\alpha^4\beta^6+420\alpha^2\beta^8-$$

$$-7\beta^{10})+g_{31}(\varphi)+\overset{*}{g}_{31}(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi}\frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^6\tan^{10}\omega}=\frac{\beta^2(6{,}236\alpha^6+26{,}784\alpha^4\beta^2+12{,}129\alpha^2\beta^4+1{,}040\beta^6)\cos^3\varphi^*}{10{,}080\alpha^2(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}+$$

$$+\frac{\cos^7\varphi^*}{9(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)^5\sin^9\varphi^*}-\frac{\beta^2(9\alpha^2+10\beta^2)\cos^7\varphi^*}{90(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^5\sin^5\varphi^*}-$$

$$-\frac{(16\alpha^2+3\beta^2)\cos^5\varphi^*}{63(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^7\varphi^*}+\frac{(378\alpha^4+1{,}157\alpha^2\beta^2+80\beta^4)\beta^2\cos^5\varphi^*}{1{,}680(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^3\varphi^*}+$$

$$+\frac{(286\alpha^4+553\alpha^2\beta^2+130\beta^4)\cos^3\varphi^*}{630(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^5\varphi^*}-\frac{\beta^2(3{,}948\alpha^6+22{,}360\alpha^4\beta^2+11{,}089\alpha^2\beta^4+1{,}040\beta^6)\cos^3\varphi^*}{10{,}080\alpha^2(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin\varphi^*}+$$

$$+\frac{(6{,}236\alpha^6+26{,}784\alpha^4\beta^2+12{,}129\alpha^2\beta^4+1{,}040\beta^6)\cos\varphi^*}{5{,}040\alpha^2(\alpha^2-\beta^2)^7(\alpha^2-\beta^2\cos^2\varphi^*)\sin^3\varphi^*}-$$

$$-\frac{(20{,}408\alpha^8+91{,}552\alpha^6\beta^2+34{,}656\alpha^4\beta^4-7{,}009\alpha^2\beta^6-1{,}040\beta^8)\cos\varphi^*}{10{,}080\alpha^2(\alpha^2-\beta^2)^7(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}+$$

$$+\frac{(46{,}280\alpha^{10}+371{,}840\alpha^8\beta^2+482{,}838\alpha^6\beta^4+98{,}358\alpha^4\beta^6-26{,}227\alpha^2\beta^8-3{,}120\beta^{10})\cos\varphi^*}{10{,}080\alpha^2(\alpha^2-\beta^2)^8(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}-$$

$$-\frac{(14{,}132\alpha^8+88{,}552\alpha^6\beta^2+96{,}723\alpha^4\beta^4+29{,}458\alpha^2\beta^6+2{,}080\beta^8)\cos\varphi^*}{5{,}040\alpha^2(\alpha^2-\beta^2)^8(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}-$$

$$-\frac{\pi}{512\alpha(\alpha^2-\beta^2)^{10}\sqrt{\alpha^2-\beta^2}}(256\alpha^{10}+5{,}760\alpha^8\beta^2+20{,}160\alpha^6\beta^4+16{,}800\alpha^4\beta^6+3{,}150\alpha^2\beta^8$$

$$+63\beta^{10})+g_{32}(\varphi)+\overset{*}{g}_{32}(\varphi^*),$$

$$\int_{\varphi^*}^{\varphi}\frac{d\omega}{(\alpha^2-\beta^2\cos^2\omega)^7\tan^8\omega}=-\frac{\beta^2(8{,}688\alpha^6+27{,}720\alpha^4\beta^2+10{,}194\alpha^2\beta^4-413\beta^6)\cos^3\varphi^*}{13{,}440\alpha^4(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}-$$

$$-\frac{\beta^2\cos^9\varphi^*}{12\alpha^2(\alpha^2-\beta^2)(\alpha^2-\beta^2\cos^2\varphi^*)^6\sin^7\varphi^*}+\frac{\beta^2(22\alpha^2-3\beta^2)\cos^7\varphi^*}{120\alpha^2(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^5\sin^5\varphi^*}+$$

$$+\frac{(12\alpha^2+7\beta^2)\cos^5\varphi^*}{84\alpha^2(\alpha^2-\beta^2)^3(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^7\varphi^*}-\frac{\beta^2(2{,}072\alpha^4+2{,}360\alpha^2\beta^2+413\beta^4)\cos^5\varphi^*}{6{,}720\alpha^2(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^3\varphi^*}-$$

$$-\frac{(96\alpha^4+234\alpha^2\beta^2-7\beta^4)\cos^3\varphi^*}{280\alpha^2(\alpha^2-\beta^2)^5(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^5\varphi^*}+\frac{\beta^2(6{,}384\alpha^6+22{,}104\alpha^4\beta^2+10{,}362\alpha^2\beta^4-413\beta^6)\cos^3\varphi^*}{13{,}440\alpha^4(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin\varphi^*}-$$

$$-\frac{(8{,}688\alpha^6+27{,}720\alpha^4\beta^2+10{,}194\alpha^2\beta^4-413\beta^6)\cos\varphi^*}{6{,}720\alpha^4(\alpha^2-\beta^2)^7(\alpha^2-\beta^2\cos^2\varphi^*)\sin^3\varphi^*}+$$

$$+\frac{(26{,}464\alpha^8+101{,}040\alpha^6\beta^2+20{,}844\alpha^4\beta^4-10{,}194\alpha^2\beta^6+413\beta^8)\cos\varphi^*}{13{,}440\alpha^4(\alpha^2-\beta^2)^7(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}-$$

$$-\frac{(65{,}440\alpha^{10}+419{,}280\alpha^8\beta^2+489{,}324\alpha^6\beta^4+23{,}938\alpha^4\beta^6-29{,}252\alpha^2\beta^8+1{,}239\beta^{10})\cos\varphi^*}{13{,}440\alpha^4(\alpha^2-\beta^2)^8(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}+$$

$$+\frac{(21{,}456\alpha^8+93{,}912\alpha^6\beta^2+97{,}590\alpha^4\beta^4+18{,}813\alpha^2\beta^6-826\beta^8)\cos\varphi^*}{6{,}720\alpha^4(\alpha^2-\beta^2)^8(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}+$$

$$+\frac{\pi}{2{,}048\alpha^5(\alpha^2-\beta^2)^{10}\sqrt{\alpha^2-\beta^2}}(1{,}024\alpha^{12}+21{,}504\alpha^{10}\beta^2+67{,}200\alpha^8\beta^4+44{,}800\alpha^6\beta^6+$$

$$+4{,}200\alpha^4\beta^8-168\alpha^2\beta^{10}+7\beta^{12})+g_{33}(\varphi)+\overset{*}{g}_{33}(\varphi^*)$$

$$\int_{\varphi^*}^{\varphi}\frac{\cos^8\omega\,d\omega}{(\alpha^2-\beta^2\cos^2\omega)^7\sin^{10}\omega}=-\frac{\beta^4(11{,}152\alpha^6+20{,}896\alpha^4\beta^2+6{,}566\alpha^2\beta^4-177\beta^6)\cos^3\varphi^*}{5{,}760\alpha^4(\alpha^2-\beta^2)^7(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin\varphi^*}+$$

$$+\frac{\cos^7\varphi^*}{9(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^5\sin^9\varphi^*}+\frac{\beta^4\cos^9\varphi^*}{12\alpha^2(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^6\sin^7\varphi^*}-$$

$$-\frac{\beta^4(142\alpha^2-9\beta^2)\cos^7\varphi^*}{360\alpha^2(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^5\sin^5\varphi^*}-\frac{(4\alpha^4+12\alpha^2\beta^2+3\beta^4)\cos^5\varphi^*}{36\alpha^2(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^7\varphi^*}+$$

$$+\frac{\beta^4(952\alpha^4+604\alpha^2\beta^2+59\beta^4)\cos^5\varphi^*}{960\alpha^2(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^3\varphi^*}+\frac{(40\alpha^6+424\alpha^4\beta^2+514\alpha^2\beta^4-9\beta^6)\cos^3\varphi^*}{360\alpha^2(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^5\varphi^*}+$$

$$+\frac{\beta^2(320\alpha^8+14{,}544\alpha^6\beta^2+25{,}008\alpha^4\beta^4+6{,}494\alpha^2\beta^6-177\beta^8)\cos^3\varphi^*}{5{,}760\alpha^4(\alpha^2-\beta^2)^7(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}+$$

$$+\frac{(320\alpha^8+14{,}544\alpha^6\beta^2+25{,}008\alpha^4\beta^4+6{,}494\alpha^2\beta^6-177\beta^8)\cos\varphi^*}{2{,}880\alpha^4(\alpha^2-\beta^2)^8(\alpha^2-\beta^2\cos^2\varphi^*)\sin^3\varphi^*}-$$

$$-\frac{(1{,}280\alpha^{10}+46{,}432\alpha^8\beta^2+89{,}664\alpha^6\beta^4+7{,}508\alpha^4\beta^6-6{,}494\alpha^2\beta^8+177\beta^{10})\cos\varphi^*}{5{,}760\alpha^4(\alpha^2-\beta^2)^8(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}+$$

$$+\frac{(1{,}280\alpha^{12}+131{,}360\alpha^{10}\beta^2+479{,}040\alpha^8\beta^4+380{,}052\alpha^6\beta^6-3{,}382\alpha^4\beta^8-18{,}912\alpha^2\beta^{10}+531\beta^{12})\cos\varphi^*}{5{,}760\alpha^4(\alpha^2-\beta^2)^9(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}-$$

$$-\frac{(320\alpha^{10}+38{,}128\alpha^8\beta^2+102{,}672\alpha^6\beta^4+77{,}866\alpha^4\beta^6+12{,}313\alpha^2\beta^8-354\beta^{10})\cos\varphi^*}{2{,}880\alpha^4(\alpha^2-\beta^2)^9(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}-$$

$$-\frac{7\pi\beta^2}{2{,}048\alpha^5(\alpha^2-\beta^2)^{11}\sqrt{\alpha^2-\beta^2}}(1{,}024\alpha^{12}+10{,}752\alpha^{10}\beta^2+22{,}400\alpha^8\beta^4+11{,}200\alpha^6\beta^6+$$

$$+840\alpha^4\beta^8-28\alpha^2\beta^{10}+\beta^{12})+g_{34}(\varphi)+g^*_{34}(\varphi^*)$$

$$\int_{\varphi^*}^{\varphi} \frac{\cos^{10}\omega\, d\omega}{(\alpha^2-\beta^2\cos^2\omega)^7 \sin^{10}\omega} = \frac{\beta^2(28{,}304\alpha^6+158{,}904\alpha^4\beta^2+122{,}478\alpha^2\beta^6+13{,}637\beta^6)\cos^3\varphi^*}{40{,}320\alpha^2(\alpha^2-\beta^2)^7(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^3\varphi^*}+$$

$$+\frac{\cos^7\varphi^*}{9(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^5\sin^9\varphi^*}+\frac{\beta^2\cos^9\varphi^*}{12(\alpha^2-\beta^2)^2(\alpha^2-\beta^2\cos^2\varphi^*)^6\sin^7\varphi^*}-$$

$$-\frac{\beta^2(66\alpha^2+67\beta^2)\cos^7\varphi^*}{360(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^5\sin^5\varphi^*}-\frac{(64\alpha^2+69\beta^2)\cos^5\varphi^*}{252(\alpha^2-\beta^2)^4(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^7\varphi^*}+$$

$$+\frac{\beta^2(2{,}072\alpha^4+6{,}952\alpha^2\beta^2+2{,}281\beta^4)\cos^5\varphi^*}{6{,}720(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)^4\sin^3\varphi^*}+\frac{(1{,}144\alpha^4+4{,}210\alpha^2\beta^2+1{,}429\beta^4)\cos^3\varphi^*}{2{,}520(\alpha^2-\beta^2)^6(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin^5\varphi^*}-$$

$$-\frac{\beta^2(19{,}152\alpha^6+125{,}224\alpha^4\beta^2+111{,}046\alpha^2\beta^4+13{,}637\beta^6)\cos^3\varphi^*}{40{,}320\alpha^2(\alpha^2-\beta^2)^7(\alpha^2-\beta^2\cos^2\varphi^*)^3\sin\varphi^*}+$$

$$+\frac{(28{,}304\alpha^6+158{,}904\alpha^4\beta^2+122{,}478\alpha^2\beta^4+13{,}637\beta^6)\cos\varphi^*}{20{,}160\alpha^2(\alpha^2-\beta^2)^8(\alpha^2-\beta^2\cos^2\varphi^*)\sin^3\varphi^*}-$$

$$-\frac{(88{,}352\alpha^8+548{,}752\alpha^6\beta^2+387{,}060\alpha^4\beta^4-40{,}558\alpha^2\beta^6-13{,}637\beta^8)\cos\varphi^*}{40{,}320\alpha^2(\alpha^2-\beta^2)^8(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin^3\varphi^*}+$$

$$+\frac{(205{,}280\alpha^{10}+1{,}981{,}040\alpha^8\beta^2+3{,}563{,}412\alpha^6\beta^4+1{,}264{,}206\alpha^4\beta^6-183{,}244\alpha^2\beta^8-40{,}911\beta^{10})\cos\varphi^*}{40{,}320\alpha^2(\alpha^2-\beta^2)^9(\alpha^2-\beta^2\cos^2\varphi^*)^2\sin\varphi^*}-$$

$$-\frac{(66{,}608\alpha^8+484{,}264\alpha^6\beta^2+729{,}738\alpha^4\beta^4+308{,}731\alpha^2\beta^6+27{,}274\beta^8)\cos\varphi^*}{20{,}160\alpha^2(\alpha^2-\beta^2)^9(\alpha^2-\beta^2\cos^2\varphi^*)\sin\varphi^*}-$$

$$-\frac{\pi}{2{,}048\alpha^3(\alpha^2-\beta^2)^{11}\sqrt{\alpha^2-\beta^2}}\left(1{,}024\alpha^{12}+27{,}648\alpha^{10}\beta^2+120{,}960\alpha^8\beta^4+\right.$$

$$\left.+134{,}400\alpha^6\beta^6+37{,}800\alpha^4\beta^8+1{,}512\alpha^2\beta^{10}-21\beta^{12}\right)+g_{35}(\varphi)+\overset{*}{g}_{35}(\varphi^*).$$

APPENDIX A6

Relation between potential energy and mutual gravitational force between two homogeneous oblate spheroids with coplanar equatorial planes

The conventional notation being used in this appendix is the notation introduced in chapter 2.

Let us consider the i^{th} and j^{th} spheroid and the j^{th} homoeoid, having the physical and geometrical properties and orientation described in the first section of chapter 2. The mass dm_i^* included in the infinitesimal volume $dV_i = dx\,dy\,dz$, located inside the i^{th} spheroid at a point, say (x, y, z), is $dm_i^* = \rho_i dx\,dy\,dz$. The force along the X axis acting on dm_i^* and the potential energy of this mass due to the presence of the j^{th} homoeoid, according to equation (2.28), are given by the relations

$$dF_{ix\delta m_j} = -\frac{\partial \varphi_{\delta m_j}}{\partial x} dm_i^* = \frac{\delta m_j}{(\theta_j e_j)} \frac{\partial\left(\arcsin \frac{\theta_j e_j}{\alpha}\right)}{\partial x_{ij}} \frac{\partial x_{ij}}{\partial x} \rho_i \, dx\,dy\,dz,$$

$$d\Phi_{i\delta m_j} = -\frac{\delta m_j}{\theta_j e_j} \arcsin\left(\frac{\theta_j e_j}{\alpha}\right) \rho_i \, dx\,dy\,dz, \qquad \text{(A6-1)}$$

where α is the semi-major axis of the spheroid, confocal with the j^{th} homoeoid, passing through (x, y, z).

An infinitesimal displacement of the i^{th} spheroid does not alter x_j and $(x - x_i)$, if its orientation remains unchanged; it is therefore obvious from the relation $x = x_j + (x - x_i) + x_{ij}$, that

$$\frac{\partial x_{ij}}{\partial x} = 1 \qquad \text{(A6-2)}$$

From equations (2.24), (2.28) and (2.31) we readily conclude that the functions

$$R_{ij}=R_{ij}(x_{ij},y_{ij},z_{ij}) \;,\quad \varphi_{\delta m_j}=\varphi_{\delta m_j}(x_{ij},y_{ij},z_{ij})$$

are continuous and therefore

$$\frac{\partial \varphi_{\delta m_j}}{\partial x_{ij}}=\frac{\partial \varphi_{\delta m_j}}{\partial R_{ij}}\,\frac{\partial R_{ij}}{\partial x_{ij}}\,. \tag{A6-3}$$

Combining equations (A6-1), (A6-2) and (A6-3), we obtain

$$dF_{ix\delta m_j}=\frac{\delta m_j}{\theta_j e_j}\,\frac{\partial\left(\arcsin\frac{\theta_j e_j}{\alpha}\right)}{\partial R_{ij}}\,\frac{\partial R_{ij}}{\partial x_{ij}}\rho_i\,dx\,dy\,dz,$$

$$d\Phi_{i\delta m_j}=-\frac{\delta m_j}{\theta_j e_j}\arcsin\left(\frac{\theta_j e_j}{\alpha}\right)\rho_i\,dx\,dy\,dz.$$

Integration of these equations over the whole volume of the i^{th} spheroid gives the force along the X axis acting on this spheroid and its potential energy, due to the attraction of the j^{th} homoeoid. It is not difficult to prove that

$$F_{ix\delta m_j}=\frac{\partial R_{ij}}{\partial x_{ij}}\left\{8\frac{\delta m_j}{\theta_j e_j}\rho_i\int_{y_i}^{y_i+\alpha_i}\int_{x_i}^{x_i+x_1(y)}\int_{0}^{z_1(x,y)}\frac{\partial\left(\arcsin\frac{\theta_j e_j}{\alpha}\right)}{\partial R_{ij}}\,dz\,dx\,dy\right\},$$

$$\Phi_{i\delta m_j}=-8\frac{\delta m_j}{\theta_j e_j}\rho_i\int_{y_i}^{y_i+\alpha_i}\int_{x_i}^{x_i+x_1(y)}\int_{0}^{z_1(x,y)}\arcsin\left(\frac{\theta_j e_j}{\alpha}\right)dz\,dx\,dy\,, \tag{A6-4}$$

where $x_1(y)=\left[\alpha_i^2-(y-y_i)^2\right]^{1/2}$ and $z_1(x,y)=\frac{c_i}{\alpha_i}\left[\alpha_i^2-(x-x_i)^2-(y-y_i)^2\right]^{\frac{1}{2}}$.

If the thickness of the j^{th} homoeoid is infinitesimal, say $d\theta_j$, its mass is given by the second of equations (2.34); introducing this equation into equations (A6-4) we obtain the potential energy $d\phi_i$ and the force dF_{ix} acting along the X axis on the i^{th} spheroid, being in the gravitational field of the infinitesimally thin homoeoid

$$dF_{ix}=\frac{\partial R_{ij}}{\partial x_{ij}}\left[32\pi\frac{(1-e_j^2)^{\frac{1}{2}}}{e_j}\rho_i\rho_j\theta_j\int_{y_i}^{y_i+\alpha_i}\int_{x_i}^{x_i+x_1(y)}\int_0^{z_1(x,y)}\frac{\partial\left(\arcsin\frac{\theta_j e_j}{\alpha}\right)}{\partial R_{ij}}dz\,dx\,dy\,d\theta_j\right],$$

$$d\phi_i=-32\pi\frac{(1-e_j^2)^{\frac{1}{2}}}{e_j}\rho_i\rho_j\theta_j\int_{y_i}^{y_i+\alpha_i}\int_{x_i}^{x_i+x_1(y)}\int_0^{z_1(x,y)}\arcsin\left(\frac{\theta_j e_j}{\alpha}\right)dz\,dx\,dy\,d\theta_j\ .$$

Integrating these formulae with respect to θ_j, with lower and upper limits of integration o and α_j respectively, we obtain the component F_{ix} along the X axis of the force acting on the i^{th} spheroid and the potential energy of this spheroid due to the presence of the j^{th} spheroid.

$$F_{ix}=-\frac{\partial R_{ij}}{\partial x_{ij}}\left[-32\pi\frac{(1-e_j^2)^{\frac{1}{2}}}{e_j}\rho_i\rho_j\int_0^{\alpha_j}\theta_j\int_{y_i}^{y_i+\alpha_i}\int_{x_i}^{x_i+x_1(y)}\int_0^{z_1(x,y)}\frac{\partial\left(\arcsin\frac{\theta_j e_j}{\alpha}\right)}{\partial R_{ij}}dz\,dx\,dy\,d\theta_j\right], \quad \text{(A6-5)}$$

$$\phi_i=-32\pi\frac{(1-e_j^2)^{\frac{1}{2}}}{e_j}\rho_i\rho_j\int_0^{\alpha_j}\theta_j\int_{y_i}^{y_i+\alpha_i}\int_{x_i}^{x_i+x_1(y)}\int_0^{z_1(y,x)}\arcsin\left(\frac{\theta_j e_j}{\alpha}\right)dz\,dx\,dy\,d\theta_j\ .. \quad \text{(A6-6)}$$

Comparing equations (A6-5) and (2.42) and remembering the relation

$$\frac{\partial R_{ij}}{\partial x_{ij}} = \frac{x_{ij}}{R_{ij}}, \tag{A6-7}$$

we readily conclude that the term of the right-hand side of equation (A6-5), which is inside the brackets, represents the total force between the i^{th} and j^{th} spheroid, and it is therefore independent of the system of reference. The same is valid for the potential energy Φ_i . In a system of reference with its axes parallel to the X, Y and Z axes and its origin at the centre of the i^{th} spheroid, equations (A6-5) and (A6-6) take the form

$$F_{ix} = -\frac{\partial R_{ij}}{\partial x_{ij}}\left\{-32\pi\frac{(1-e_j^2)^{\frac{1}{2}}}{e_j}\rho_i\rho_j\int_0^{\alpha_j}\theta_j\int_0^{\alpha_i}\int_0^{x_2(y)}\int_0^{z_2(x,y)}\frac{\partial\left(\arcsin\frac{\theta_j e_j}{\alpha}\right)}{\partial R_{ij}}\,dz\,dx\,dy\,d\theta_j\right\},$$

(A6-8)

$$\Phi_i = -32\pi\frac{(1-e_j^2)^{\frac{1}{2}}}{e_j}\rho_i\rho_j\int_0^{\alpha_j}\theta_j\int_0^{\alpha_i}\int_0^{x_2(y)}\int_0^{z_2(x,y)}\arcsin\left(\frac{\theta_j e_j}{\alpha}\right)dz\,dx\,dy\,d\theta_j\,.$$

where $x_2(y) = (\alpha_i^2 - y^2)^{1/2}$ and $z_2(x,y) = \frac{c_i}{\alpha_i}(\alpha_i^2 - x^2 - y^2)^{1/2}$. We note that the expression giving the parameter α as function of $\theta_j, e_j, x, y, z, x_i, y_i$ and z_i is different for different systems of reference.

All the limits of integration in equations (A6-8) are independent of R_{ij} ; we can therefore employ equation (2.38) to obtain

$$F_{ix} = -\frac{\partial R_{ij}}{\partial x_{ij}}\frac{\partial}{\partial R_{ij}}\left[-32\pi\frac{(1-e_j^2)^{\frac{1}{2}}}{e_j}\rho_i\rho_j\int_0^{\alpha_j}\theta_j\int_0^{\alpha_i}\int_0^{x_2(y)}\int_0^{z_2(x,y)}\arcsin\left(\frac{\theta_j e_j}{\alpha}\right)dz\,dx\,dy\,d\theta_j\right]. \tag{A6-9}$$

Comparison of this equation with the second of equations (A6-8) yields

$$F_{ix} = -\frac{\partial \phi_i}{\partial R_{ij}} \frac{\partial R_{ij}}{\partial x_{ij}} = -\frac{\partial \phi_i}{\partial x_{ij}} . \tag{A6-10}$$

A similar process results in the following expressions for the force components F_{iy} and F_{iz} acting on the i^{th} spheroid along the Y and Z axes as a result of the attraction from the j^{th} spheroid

$$F_{iy} = -\frac{\partial \phi_i}{\partial R_{ij}} \frac{\partial R_{ij}}{\partial y_{ij}} = -\frac{\partial \phi_i}{\partial y_{ij}} ,$$

$$F_{iz} = -\frac{\partial \phi_i}{\partial R_{ij}} \frac{\partial R_{ij}}{\partial z_{ij}} = -\frac{\partial \phi_i}{\partial z_{ij}} . \tag{A6-11}$$

APPENDIX A7

Estimation of the remainders of the series (2.70) and (2.71)

In numerical applications the series expansion entering the right-hand member of equation (2.70) has to be truncated after a finite number of terms; it is therefore important that an upper bound for the remainder of this expansion be found.

The answer to this problem is based on the following theorem.

Theorem: Let f(x) be a continuous function in the open interval (a,b) obeying

$$m \leq f(x) \leq M, \quad x \in (a,b),$$

where m and M are constants; furthermore let g(x) be a non-negative and integrable function within (a,b). Then there exists at least one point ξ, $\xi \in [a,b]$ so that

$$m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx \leq M\int_a^b g(x)\,dx . \qquad \text{(A7-1)}$$

Consider now the series expansion occurring in the right-hand member of equation (2.70) truncated after its $(\nu+1)^{th}$ term, $\nu \geq 6$; the error e_ν introduced in this way into the evaluation of the force F_i is given by the relation:

$$e_\nu = \frac{9 m_i m_j \sqrt{R_{ij}}}{\sqrt{2}\,\pi (a_i e_i)^3} \sum_{m=\nu+1}^{\infty} \frac{(2m)!\,(a_j e_j)^{2m}}{2^{2m}(m!)^2(2m+3)} W_m(R_{ij}, a_i, e_i), \quad \nu \geq 6. \qquad \text{(A7-2)}$$

The double integral $W_m(R_{ij}, a_i, e_i)$, according to equation (2.65) has the form

$$W_m(R_{ij},\alpha_i,e_i)=\int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i}\frac{1}{\alpha^{2(m+1)}}\int_{\zeta}^{\alpha}\sqrt{\frac{\alpha^2-y^2}{y-\zeta}}\,dy\,d\alpha, \qquad \text{(A7-3)}$$

$$m=0,1,2,\cdots .$$

We define the functions f(a) and g(a) by the relations

$$f(\alpha)\equiv\frac{1}{\alpha^2}, \qquad \text{(A7-4)}$$

$$g(\alpha)\equiv\frac{1}{\alpha^{2(m+1)}}\int_{\zeta}^{\alpha}\sqrt{\frac{\alpha^2-y^2}{y-\zeta}}\,dy, \quad \alpha\in(R_{ij}-\alpha_i e_i, R_{ij}+\alpha_i e_i), \quad 0\leqslant\alpha_i e_i\leqslant R_{ij}, \quad m\geqslant 0.$$

The first of these functions is continuous within $(R_{ij}-\alpha_i e_i, R_{ij}+\alpha_i e_i)$ obeying the relation

$$\frac{1}{(R_{ij}+\alpha_i e_i)^2}<f(\alpha)<\frac{1}{(R_{ij}-\alpha_i e_i)^2}, \quad \alpha\in(R_{ij}-\alpha_i e_i, R_{ij}+\alpha_i e_i), \qquad \text{(A7-5)}$$

while the second is an integrable one whose integral over $\alpha\in(R_{ij}-\alpha_i e_i\ \ R_{ij}+\alpha_i e_i)$, can be found by a process entirely analogous to that followed for the derivation of equations (A5-24). Further, by virtue of equation (2.60), g(a) takes the form

$$g(\alpha)=\frac{4\sqrt{\alpha}}{3\sqrt{2}\,\alpha^{2(m+1)}}\left[(\alpha+\zeta)K(u)-2\zeta E(u)\right], \qquad \text{(A7-6)}$$

$$\alpha\in(R_{ij}-\alpha_i e_i, R_{ij}+\alpha_i e_i), \quad m\geqslant 0 .$$

This form of the function g(a) facilitates the proof of the relation g(a) > o. Comparing equations (A4-34), (A7-6) and the last of equations (2.50) we obtain

$$g(\alpha)=\frac{2\sqrt{\alpha}\pi}{3\sqrt{2}\,\alpha^{2(m+1)}}\sum_{\nu=0}^{\infty}\left(\frac{(2\nu)!}{2^{2\nu}(\nu!)^2}\right)^2\left(\alpha+\frac{2\nu+1}{2\nu-1}\zeta\right)\left(\frac{\alpha-\zeta}{2\alpha}\right)^{\nu},\quad m\geqslant 0,\qquad \text{(A7-7)}$$

$$\alpha\in\left(R_{ij}-\alpha_i e_i\,,\,R_{ij}+\alpha_i e_i\right).$$

On the other hand, the first of equations (2.47) ensures the validity of the relation

$$\alpha-\zeta>0,\quad \alpha\in\left(R_{ij}-\alpha_i e_i\,,\,R_{ij}+\alpha_i e_i\right).$$

From the foregoing two relations we readily conclude

$$g(\alpha)>0,\quad \alpha\in\left(R_{ij}-\alpha_i e_i\,,\,R_{ij}+\alpha_i e_i\right).\qquad \text{(A7-8)}$$

This inequality completes the proof that the functions f(a) and g(a) satisfy the presuppositions of the theorem given at the beginning of this appendix. Application of this theorem in the case of the above functions yields

$$W_{m+1}(R_{ij},\alpha_i,e_i)=\int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i}\frac{1}{\alpha^2}\left(\frac{1}{\alpha^{2(m+1)}}\int_{\zeta}^{\alpha}\sqrt{\frac{\alpha^2-y^2}{y-\zeta}}\,dy\right)d\alpha=\int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i}f(\alpha)g(\alpha)\,d\alpha\leqslant$$

$$\leqslant\frac{1}{\left(R_{ij}-\alpha_i e_i\right)^2}\int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i}\frac{1}{\alpha^{2(m+1)}}\int_{\zeta}^{\alpha}\sqrt{\frac{\alpha^2-y^2}{y-\zeta}}\,dy\,d\alpha,\quad m\geqslant 0,$$

which, upon substitution of equation (A7-3), becomes

$$W_{m+1}(R_{ij},\alpha_i,e_i)\leqslant\frac{W_m(R_{ij},\alpha_i,e_i)}{\left(R_{ij}-\alpha_i e_i\right)^2},\quad m\geqslant 0.$$

Applying this relation $(m-\nu)$, $(m \geqslant \nu \geqslant 0)$, times in sequence we obtain the following series of inequalities

$$W_{m+1}(R_{ij},\alpha_i,e_i) \leqslant \frac{W_m(R_{ij},\alpha_i,e_i)}{(R_{ij}-\alpha_i e_i)^2} \leqslant \frac{W_{m-1}(R_{ij},\alpha_i,e_i)}{(R_{ij}-\alpha_i e_i)^4} \leqslant \cdots\cdots$$

$$\cdots\cdots \leqslant \frac{W_\nu(R_{ij},\alpha_i,e_i)}{(R_{ij}-\alpha_i e_i)^{2(m+1-\nu)}}, \quad m \geqslant \nu \geqslant 0 .$$

Comparing the first and the last term we readily conclude

$$W_{m+1}(R_{ij},\alpha_i,e_i) \leqslant \frac{W_\nu(R_{ij},\alpha_i,e_i)}{(R_{ij}-\alpha_i e_i)^{2(m+1-\nu)}}, \quad m \geqslant \nu \geqslant 0 ,$$

or

$$W_m(R_{ij},\alpha_i,e_i) \leqslant \frac{W_\nu(R_{ij},\alpha_i,e_i)}{(R_{ij}-\alpha_i e_i)^{2(m-\nu)}}, \quad m \geqslant \nu \geqslant 0 . \tag{A7-9}$$

Combining the relations (A7-2) and (A7-9), we obtain

$$e_\nu \leqslant \frac{9 m_i m_j \sqrt{R_{ij}}}{\pi\sqrt{2}(\alpha_i e_i)^3} W_\nu(R_{ij},\alpha_i,e_i) \sum_{m=\nu+1}^{\infty} \frac{(2m!)(\alpha_j e_j)^{2m}}{2^{2m}(m!)^2(2m+3)(R_{ij}-\alpha_i e_i)^{2(m-\nu)}} . \tag{A7-10}$$

The double integrals $W_m(R_{ij},\alpha_i,e_i)$, $m=0,1,\cdots$ satisfy the relation

$$W_m(R_{ij},\alpha_i,e_i) > 0 , \quad m=0,1,2,\cdots , \tag{A7-11}$$

as is obvious from equations (A7-3), (A7-4) and the inequality (A7-8); hence the error e_ν, according to equation (A7-2) is always positive

$$e_\nu = |e_\nu| . \tag{A7-12}$$

In deriving equation (A5-14), we proved the relation

$$\frac{1}{2}\left\{\arcsin\left(\frac{\delta}{\lambda}\right)-\frac{\delta}{\lambda^2}\sqrt{\lambda^2-\delta^2}\right\}=\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}(m!)^2(2m+3)}\left(\frac{\delta}{\lambda}\right)^{2m+3}, \qquad \text{(A7-13)}$$

$$\left|\frac{\delta}{\lambda}\right|<1,$$

with the aid of which, it can easily be verified that

$$\sum_{m=\nu+1}^{\infty}\frac{(2m)!}{2^{2m}(m!)^2(2m+3)}\frac{(\alpha_j e_j)^{2m}}{(R_{ij}-\alpha_i e_i)^{2(m-\nu)}}=$$

$$=\frac{1}{2}\frac{(R_{ij}-\alpha_i e_i)^{2\nu+3}}{(\alpha_j e_j)^3}\left\{\arcsin\left(\frac{\alpha_j e_j}{R_{ij}-\alpha_i e_i}\right)-\frac{(\alpha_j e_j)}{(R_{ij}-\alpha_i e_i)^2}\sqrt{(R_{ij}-\alpha_i e_i)^2-(\alpha_j e_j)^2}-\right.$$

$$\left.-2\sum_{m=0}^{\nu}\frac{(2m)!}{2^{2m}(m!)^2(2m+3)}\left(\frac{\alpha_j e_j}{R_{ij}-\alpha_i e_i}\right)^{2m+3}\right\}, \quad \nu\geqslant 0 .$$

By substituting this expression into the relation (A7-10) and employing equation (A7-12), we obtain

$$e_\nu=|e_\nu|\leqslant S_\nu(R_{ij},\alpha_i e_i,\alpha_j e_j), \qquad \text{(A7-14)}$$

where

$$S_\nu(R_{ij},\alpha_i e_i,\alpha_j e_j)\equiv\frac{9m_i m_j\sqrt{R_{ij}}\,(R_{ij}-\alpha_i e_i)^{2\nu+3}}{2\sqrt{2}\,\pi(\alpha_j e_j)^3(\alpha_i e_i)^3}W_\nu(R_{ij},\alpha_i,e_i)\left\{\arcsin\left(\frac{\alpha_j e_j}{R_{ij}-\alpha_i e_i}\right)-\right.$$

(A7-15)

$$\left.-\frac{\alpha_j e_j}{(R_{ij}-\alpha_i e_i)^2}\sqrt{(R_{ij}-\alpha_i e_i)^2-\alpha_j^2 e_j^2}-2\sum_{m=0}^{\nu}\frac{(2m)!}{2^{2m}(m!)^2(2m+3)}\left(\frac{\alpha_j e_j}{R_{ij}-\alpha_i e_i}\right)^{2m+3}\right\},$$

$$\nu\geqslant 6 .$$

For the particular case $\nu=6$, and using the last of equations (A5-24) to substitute for $W_6(R_{ij},\alpha_i e_i)$, the above two relations take the form:

$$e_6=|e_6|\ll S_6(R_{ij},\alpha_i e_i,\alpha_j e_j), \qquad \text{(A7-16)}$$

$$S_6(R_{ij},\alpha_i e_i,\alpha_j e_j)\equiv\frac{m_i m_j R_{ij}(R_{ij}-\alpha_i e_i)^4}{2002(\alpha_j e_j)^3(R_{ij}+\alpha_i e_i)^{11}\sqrt{R_{ij}^2-\alpha_i^2 e_i^2}}(3003R_{ij}^8+$$

$$+24{,}024R_{ij}^6\alpha_i^2 e_i^2+41{,}184R_{ij}^4\alpha_i^4 e_i^4+18{,}304R_{ij}^2\alpha_i^6 e_i^6+ \qquad \text{(A7-17)}$$

$$+1{,}664\alpha_i^8 e_i^8)\left\{\arcsin\left(\frac{\alpha_j e_j}{R_{ij}-\alpha_i e_i}\right)-\frac{\alpha_j e_j}{(R_{ij}-\alpha_i e_i)^2}\sqrt{(R_{ij}-\alpha_i e_i)^2-(\alpha_j e_j)^2}-\right.$$

$$\left.-2\sum_{m=0}^{6}\frac{(2m)!}{2^{2m}(m!)^2(2m+3)}\left(\frac{\alpha_j e_j}{R_{ij}-\alpha_i e_i}\right)^{2m+3}\right\}.$$

In numerical applications the most important of the errors is the relative (or fractional) error

$$e_{\nu r}\equiv\frac{e_\nu}{F_i}=\frac{1}{1+\dfrac{F_i-e_\nu}{e_\nu}},\quad \nu\geq 0\ , \qquad \text{(A7-18)}$$

which, according to the relations (2.70), (A7-11) and (A7-12), is always positive and, as a consequence of the inequality (A7-14), bounded. In fact

$$e_{\nu r}=|e_{\nu r}|\ll\frac{1}{1+\dfrac{F_i-e_\nu}{S_\nu(R_{ij},\alpha_i e_i,\alpha_j e_j)}}=\frac{1}{1+P_\nu(R_{ij},\alpha_i e_i,\alpha_j e_j)}, \qquad \text{(A7-19)}$$

$$\nu\geq 6,$$

where, by definition

$$P_\nu(R_{ij},\alpha_i e_i,\alpha_j e_j) \equiv \frac{F_i - e_\nu}{S_\nu(R_{ij},\alpha_i e_i,\alpha_j e_j)}, \quad \nu \geqslant 6 . \tag{A7-20}$$

This relation, upon substitution of equations (2.70), (A7-2) and (A7-15), becomes

$$P_\nu(R_{ij},\alpha_i e_i,\alpha_j e_j) = \left[\sum_{m=0}^{\nu} \frac{(2m)!\,(\alpha_j e_j)^{2m+3}}{2^{2m-1}(m!)^2(2m+3)} W_m(R_{ij},\alpha_i e_i)\right] \Big/$$

$$(R_{ij}-\alpha_i e_i)^{2\nu+3} W_\nu(R_{ij},\alpha_i e_i)\left\{\arcsin\left(\frac{\alpha_j e_j}{R_{ij}-\alpha_i e_i}\right) - \right. \tag{A7-21}$$

$$\left. - \frac{\alpha_j e_j}{(R_{ij}-\alpha_i e_i)^2}\sqrt{(R_{ij}-\alpha_i e_i)^2 - (\alpha_j e_j)^2} - \sum_{m=0}^{\nu} \frac{(2m)!}{2^{2m-1}(m!)^2(2m+3)}\left(\frac{\alpha_j e_j}{R_{ij}-\alpha_i e_i}\right)^{2m+3}\right\}, \nu \geqslant 6.$$

Let us now consider the relative error e_{6r} which is of particular importance since in the present work the series expansion (2.64) is used truncated after its 7th term. This error, as the general relation (A7-19) implies, obeys the inequality

$$e_{6r} = |e_{6r}| \leqslant \frac{1}{1+P_6(R_{ij},\alpha_i e_i,\alpha_j e_j)} . \tag{A7-22}$$

The analytical expression of the function $P_6 \equiv P_6(R_{ij},\alpha_i e_i,\alpha_j e_j)$ can be obtained from equation (A7-21) which, for the particular case $\nu=6$ and upon substitution of equations (A5-14), (A5-17), (A5-20) (A5-23) and (A5-24), takes the form

$$P_6\left(R_{ij},\alpha_i e_i,\alpha_j e_j\right)=$$

$$\begin{aligned}
&=6006\left(R_{ij}+\alpha_i e_i\right)^{11}\sqrt{R_{ij}^2-\alpha_i^2e_i^2}\,\left(\alpha_j e_j\right)^3\left\{\frac{1}{2\left(\alpha_i e_i\right)^3}\left[\arcsin\left(\frac{\alpha_i e_i}{R_{ij}}\right)-\right.\right.\\
&\left.-\frac{\alpha_i e_i}{R_{ij}^2}\sqrt{R_{ij}^2-\alpha_i^2e_i^2}\right]+\frac{\left(\alpha_j e_j\right)^2}{\left(R_{ij}^2-\alpha_i^2e_i^2\right)\sqrt{R_{ij}^2-\alpha_i^2e_i^2}}\left[\frac{1}{10R_{ij}^2}+\frac{3\alpha_j^2e_j^2}{56\left(R_{ij}^2-\alpha_i^2e_i^2\right)^2}+\right.\\
&+\frac{\alpha_j^4e_j^4}{144\left(R_{ij}^2-\alpha_i^2e_i^2\right)^4}\left(5R_{ij}^2+4\alpha_i^2e_i^2\right)+\frac{\alpha_j^6e_j^6}{1{,}408\left(R_{ij}^2-\alpha_i^2e_i^2\right)^6}\left(35R_{ij}^4+84R_{ij}^2\alpha_i^2e_i^2+\right.\\
&\left.+24\alpha_i^4e_i^4\right)+\frac{3\alpha_j^8e_j^8}{16{,}640\left(R_{ij}^2-\alpha_i^2e_i^2\right)^8}\left(105R_{ij}^6+504R_{ij}^4\alpha_i^2e_i^2+\right. \qquad \text{(A7-23)}\\
&\left.+432R_{ij}^2\alpha_i^4e_i^4+64\alpha_i^6e_i^6\right)+\frac{\left(\alpha_j e_j\right)^{10}}{15{,}360\left(R_{ij}^2-\alpha_i^2e_i^2\right)^{10}}\left(231R_{ij}^8+1{,}848R_{ij}^6\alpha_i^2e_i^2+3{,}168R_{ij}^4\alpha_i^4e_i^4+\right.\\
&\left.\left.\left.+1{,}408R_{ij}^2\alpha_i^6e_i^6+128\alpha_i^8e_i^8\right)\right]\right\}\Bigg/\left(R_{ij}-\alpha_i e_i\right)^4\left(3{,}003R_{ij}^8+24{,}024R_{ij}^6\alpha_i^2e_i^2+41{,}184R_{ij}^4\alpha_i^4e_i^4+\right.\\
&\left.+18{,}304R_{ij}^2\alpha_i^6e_i^6+1{,}664\alpha_i^8e_i^8\right)\left\{\arcsin\left(\frac{\alpha_j e_j}{R_{ij}-\alpha_i e_i}\right)-\right.\\
&\left.-\frac{\alpha_j e_j}{\left(R_{ij}-\alpha_i e_i\right)^2}\sqrt{\left(R_{ij}-\alpha_i e_i\right)^2-\alpha_j^2e_j^2}-\sum_{m=0}^{6}\frac{(2m)!}{2^{2m-1}(m!)^2(2m+3)}\left(\frac{\alpha_j e_j}{R_{ij}-\alpha_i e_i}\right)^{2m+3}\right\}.
\end{aligned}$$

The numerical value of $P_6\left(R_{ij}, \alpha_i e_i, \alpha_j e_j\right)$ can be determined once the values of the three parameters R_{ij}, $\alpha_i e_i$ and $\alpha_j e_j$ are known; however $P_6\left(R_{ij}, \alpha_i e_i, \alpha_j e_j\right)$ is a function of only two variables, as will be shown below. Let C_i and C_j denote the ratios

$$C_i \equiv \frac{\alpha_i e_i}{R_{ij}}, \quad C_j \equiv \frac{\alpha_j e_j}{R_{ij}}, \tag{A7-24}$$

then it is easy to be verified that

$$P_6\left(R_{ij}, \alpha_i e_i, \alpha_j e_j\right) \equiv P_6\left(C_i, C_j\right) =$$

$$\equiv 6{,}006\left(1+C_i\right)^{11}\sqrt{1-C_i^2}\; C_j^3 \Bigg\{ \Bigg[\frac{1}{2C_i^3}\left(\arcsin(C_i) - C_i\sqrt{1-C_i^2}\right) +$$

$$+ \frac{C_j^2}{(1-C_i^2)\sqrt{1-C_i^2}}\left[\frac{1}{10} + \frac{3C_j^2}{56(1-C_i^2)^2} + \frac{C_j^4}{144(1-C_i^2)^4}\left(5+4C_i^2\right) + \frac{C_j^6}{1{,}408(1-C_i^2)^6}\left(35 + \right.\right.$$

$$\left. +84C_i^2+24C_i^4\right) + \frac{3C_j^8}{16{,}640\,(1-C_i^2)^8}\left(105+504C_i^2+432C_i^4+64C_i^6\right) + \tag{A7-25}$$

$$\left.\left. + \frac{C_j^{10}}{15{,}360(1-C_i^2)^{10}}\left(231+1{,}848C_i^2+3{,}168C_i^4+1{,}408C_i^6+128C_i^8\right)\right]\right] \Bigg/ (1-C_i)^4 \left(3{,}003 + \right.$$

$$\left. +24{,}024C_i^2+41{,}184C_i^4+18{,}304C_i^6+1{,}664C_i^8\right) \left[\arcsin\left(\frac{C_j}{1-C_i}\right) - \right.$$

$$\left. - \frac{C_j}{(1-C_i)^2}\sqrt{(1-C_i)^2-C_j^2} - \sum_{m=0}^{6} \frac{(2m)!}{2^{2m-1}(m!)^2(2m+3)}\left(\frac{C_j}{1-C_i}\right)^{2m+3}\right] \Bigg\} .$$

The parameters C_i and C_j are not absolutely independent; as a result of equations (A7-24) and the relation

$$\alpha_i e_i + \alpha_j e_j \leqslant R_{ij} ,$$

they are subject to the following conditions

$$0 \leqslant c_i \leqslant 1 \ , \ 0 \leqslant c_j \leqslant 1 \ , \ c_i + c_j \leqslant 1 \ . \tag{A7-26}$$

In chapter (2) it has been pointed out that the series (2.70) and (2.71), although they have different rates of convergence, are numerically equivalent and either of these can be used for the evaluation of the force between two oblate spheroids with coplanar equatorial planes; it is therefore important that, as in the case of the series (2.70), to find an upper bound for the remainder of the series (2.71) truncated after its $(\nu+1)^{th}$ term.

By a process entirely analogous to the one followed for the derivation of the relations (A7-14), (A7-16), (A7-19) and (A7-22) it can be proved that the absolute and relative errors e_ν^* and $e_{\nu r}^*$ introduced in the evaluation of the F_i , when the series (2.71) truncated after its $(\nu+1)^{th}$ term is used, obey the relations

$$e_\nu^* = |e_\nu^*| \leqslant S_\nu^*(R_{ij}, \alpha_i e_i, \alpha_j e_j) \ , \quad \nu \geqslant 6 \ ,$$

$$e_{\nu r}^* = |e_{\nu r}^*| \leqslant \frac{1}{1 + P_\nu^*(R_{ij}, \alpha_i e_i, \alpha_j e_j)} \ , \ \nu \geqslant 6 \ , \tag{A7-27}$$

where

$$S_\nu^*(R_{ij}, \alpha_i e_i, \alpha_j e_j) = \frac{9 m_i m_j \sqrt{R_{ij}}\,(R_{ij} - \alpha_j e_j)^{2\nu+3}}{2\sqrt{2}\,\pi (\alpha_i e_i)^3 (\alpha_j e_j)^3} W_\nu(R_{ij}, \alpha_j, e_j) \left\{ \arcsin\left(\frac{\alpha_i e_i}{R_{ij} - \alpha_j e_j}\right) - \right.$$

$$- \frac{\alpha_i e_i}{(R_{ij} - \alpha_j e_j)^2} \sqrt{(R_{ij} - \alpha_j e_j)^2 - \alpha_i^2 e_i^2} - \tag{A7-28}$$

$$\left. - \sum_{m=0}^{\nu} \frac{(2m)!}{2^{2m-1} (m!)^2 (2m+3)} \left(\frac{\alpha_i e_i}{R_{ij} - \alpha_j e_j}\right)^{2m+3} \right\} , \ \nu \geqslant 6 ,$$

and

$$P_\nu^*(R_{ij},\alpha_i e_i,\alpha_j e_j) \equiv \left\{\sum_{m=0}^{\nu} \frac{(2m)!\,(\alpha_i e_i)^{2m+3}}{2^{2m-1}(m!)^2(2m+3)} W_m(R_{ij},\alpha_j,e_j)\right\} \Big/$$

$$\Big/ (R_{ij}-\alpha_j e_j)^{2\nu+3} W_\nu(R_{ij},\alpha_j,e_j)\left\{\arcsin\left(\frac{\alpha_i e_i}{R_{ij}-\alpha_j e_j}\right)- \right. \quad \text{(A7-29)}$$

$$\left. -\frac{\alpha_i e_i}{(R_{ij}-\alpha_j e_j)^2}\sqrt{(R_{ij}-\alpha_j e_j)^2-\alpha_i^2 e_i^2} - \sum_{m=0}^{\nu} \frac{(2m)!}{2^{2m-1}(m!)^2(2m+3)}\left(\frac{\alpha_i e_i}{R_{ij}-\alpha_j e_j}\right)^{2m+3}\right\},$$

$$\nu \gg 6.$$

For the particular case $\nu=6$ the foregoing relations become

$$e_6^* = |e_6^*| \leqslant S_6^*(R_{ij},\alpha_i e_i,\alpha_j e_j), \quad \text{(A7-30)}$$

$$e_{6r}^* = |e_{6r}^*| \leqslant \frac{1}{1+P_6^*(R_{ij},\alpha_i e_i,\alpha_j e_j)}, \quad \text{(A7-31)}$$

$$S_6^*(R_{ij},\alpha_i e_i,\alpha_j e_j) \equiv \frac{m_i m_j R_{ij}(R_{ij}-\alpha_j e_j)^4}{2{,}002\ \alpha_i^3 e_i^3 (R_{ij}+\alpha_j e_j)^{11}\sqrt{R_{ij}^2-\alpha_j^2 e_j^2}}\left(3{,}003 R_{ij}^8 + \right.$$

$$+ 24{,}024 R_{ij}^6 \alpha_j^2 e_j^2 + 41{,}184 R_{ij}^4 \alpha_j^4 e_j^4 + 18{,}304 R_{ij}^2 \alpha_j^6 e_j^6 +$$

$$\left. +1{,}664 \alpha_j^8 e_j^8\right)\left\{\arcsin\left(\frac{\alpha_i e_i}{R_{ij}-\alpha_j e_j}\right) - \frac{\alpha_i e_i}{(R_{ij}-\alpha_j e_j)^2}\sqrt{(R_{ij}-\alpha_j e_j)^2-\alpha_i^2 e_i^2} - \right. \quad \text{(A7-32)}$$

$$\left. -\sum_{m=0}^{6} \frac{(2m)!}{2^{2m-1}(m!)^2(2m+3)}\left(\frac{\alpha_i e_i}{R_{ij}-\alpha_j e_j}\right)^{2m+3}\right\},$$

$$P_6^*(R_{ij}, \alpha_i e_i, \alpha_j e_j) \equiv P_6^*(c_i, c_j) =$$

$$= 6{,}006\,(1+c_j)^{11}\sqrt{1-c_j^2}\; c_i^3 \left\{ \frac{1}{2c_j^3}\left(\arcsin(c_j) - c_j\sqrt{1-c_j^2}\right) + \frac{c_i^2}{(1-c_j^2)\sqrt{1-c_j^2}} \left[\frac{1}{10} + \right.\right.$$

$$+ \frac{3c_i^2}{56\,(1-c_j^2)^2} + \frac{c_i^4}{144\,(1-c_j^2)^4}(5+4c_j^2) + \frac{c_i^6}{1{,}408\,(1-c_j^2)^6}(35+84c_j^2+$$

$$+24c_j^4) + \frac{3c_i^8}{16{,}640\,(1-c_j^2)^8}(105+504c_j^2+432c_j^4+64c_j^6) + \qquad \text{(A7-33)}$$

$$\left.\left. + \frac{c_i^{10}}{15{,}360\,(1-c_j^2)^{10}}(231+1{,}848\,c_j^2+3{,}168c_j^4+1{,}408\,c_j^6+128c_j^8)\right]\right\} \Bigg/$$

$$\Bigg/ (1-c_j)^4(3{,}003+24{,}024c_j^2+41{,}184c_j^4+18{,}304\,c_j^6+1{,}664c_j^8)\left\{ \arcsin\left(\frac{c_i}{1-c_j}\right) - \right.$$

$$\left. - \frac{c_i}{(1-c_j)^2}\sqrt{(1-c_j)^2-c_i^2} - \sum_{m=0}^{6} \frac{(2m)!}{2^{2m-1}(m!)^2(2m+3)}\left(\frac{c_i}{1-c_j}\right)^{2m+3} \right\}.$$

It is now clear that in numerical applications, where the force F_i between two oblate spheroids has to be evaluated with the aid of either of the series expansions (2.70) or (2.71) truncated after the 7th term, we can achieve a minimum relative error by choosing to employ the faster converging of the two series. On this assumption the relative error e_r, introduced in the evaluation of the force F_i satisfies the relations

$$e_r = \min\left(e_{6r}, e_{6r}^*\right) \leqslant f(c_i, c_j), \tag{A7-34}$$

where

$$f(c_i, c_j) \equiv \min\left(\frac{1}{1+P_6(c_i, c_j)}, \frac{1}{1+P_6^*(c_i, c_j)}\right).$$

The analytical expression for the function $e_r = e_r(c_i, c_j)$ is not known, we therefore confine ourselves to study the behaviour of the function $f = f(c_i, c_j)$. This function is graphically illustrated in figure (A7.1). The point A corresponds to the relative error when the two bodies are spherical (or particles). The lines drawn on the surface ABCD are the intersections of this surface with some of the planes described by the relations

$$c_i + c_j = \text{constant}, \quad -\infty < f(c_i, c_j) < +\infty .$$

It is obvious that the point C corresponds to the maximum value of the function $f = f(c_i, c_j)$ occurring when $c_i = c_j = 0.5$ or, according to equations (A7-24), when $e_i = e_j = 1$ and $\alpha_i = \alpha_j = 0.5 R_{ij}$; this value is equal to 2.709×10^{-3}. The surfaces ABCA and ACDA are the graphical representations of the functions

$$S_6^* = \frac{1}{1+P_6^*(c_i, c_j)}, \quad 0 \leqslant c_i \leqslant 0.5, \quad 0 \leqslant c_j \leqslant 1,$$

and

$$S_6 = \frac{1}{1+P_6(c_i, c_j)}, \quad 0 \leqslant c_j \leqslant 0.5, \quad 0 \leqslant c_i \leqslant 1,$$

respectively.

Although the closed analytical expression for the function

$$e_r = e_r(c_i, c_j)$$

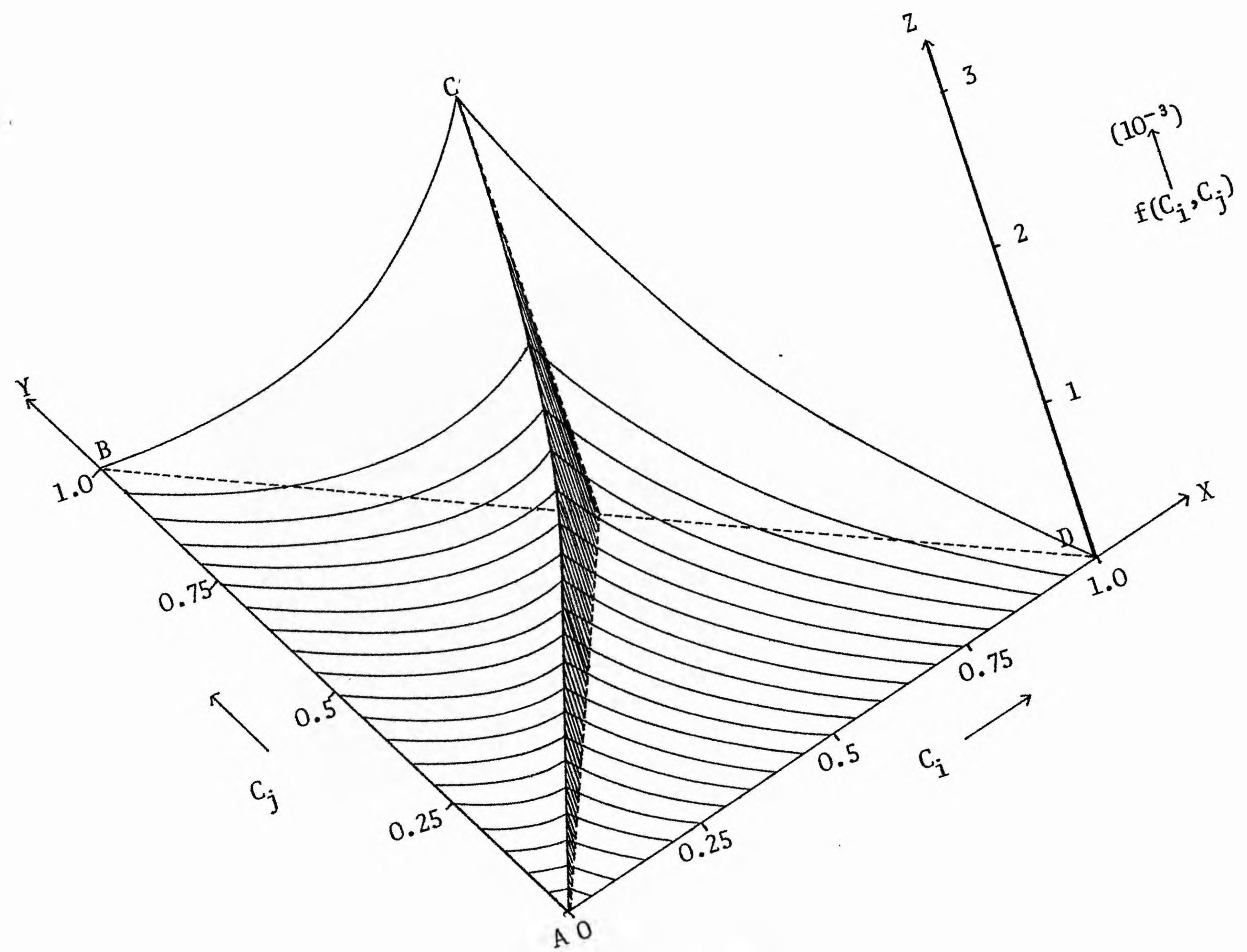


Figure (A7.1)

is not known, its numerical value for every pair of values for the parameters C_i and C_j can be found with the aid of equation (2.55). The relative error e_{6r}, by definition, is given by the relation

$$e_{6r} = \frac{F_i - F_i^*}{F_i}, \tag{A7-35}$$

where F_i^* is the approximate formula used to evaluate F_i ; that is the series (2.70) truncated after its 7^{th} term. Similarly

$$e_{6r}^* = \frac{F_i - F_j^*}{F_i}, \tag{A7-36}$$

where F_j^* is the sum of the first seven terms of the series (2.71). Combining the relations (A7-34), (A7-35) and (A7-36), we find

$$e_r = \min\left(\frac{F_i - F_i^*}{F_i}, \frac{F_i - F_j^*}{F_i}\right).$$

It is now easy to verify, using equations (2.55), (2.70) and (2.71), that

$$e_r = \min\left(\frac{I_i - \pi(\alpha_i e_i)^3(\alpha_j e_j)^3\sqrt{R_{ij}}\, H(R_{ij}, \alpha_i e_i, \alpha_j e_j)}{I_i}, \right.$$

$$\left. \frac{I_i - \pi(\alpha_i e_i)^3(\alpha_j e_j)^3\sqrt{R_{ij}}\, H^*(R_{ij}, \alpha_i e_i, \alpha_j e_j)}{I_i}\right), \tag{A7-37}$$

where

$$I_i \equiv \int_{R_{ij}-\alpha_i e_i}^{R_{ij}+\alpha_i e_i} \alpha^{-1/2}\left(\alpha^2 \arcsin\left(\frac{\alpha_j e_j}{\alpha}\right) - \alpha_j e_j\left(\alpha^2 - \alpha_j^2 e_j^2\right)^{1/2}\right)\Big((\zeta+\alpha)K(u) - 2\zeta E(u)\Big)\, d\alpha,$$

and

$$H^*(R_{ij},\alpha_i e_i,\alpha_j e_j)\equiv\frac{1}{2(\alpha_j e_j)^3}\left[\arcsin\left(\frac{\alpha_j e_j}{R_{ij}}\right)-\frac{\alpha_j e_j}{R_{ij}^2}\left(R_{ij}^2-\alpha_j^2 e_j^2\right)^{1/2}\right]+$$

$$+\frac{\alpha_i^2 e_i^2}{(R_{ij}^2-\alpha_j^2 e_j^2)^{3/2}}\left[\frac{1}{10R_{ij}^2}+\frac{3\alpha_i^2 e_i^2}{56(R_{ij}^2-\alpha_j^2 e_j^2)^2}+\frac{\alpha_i^4 e_i^4}{144(R_{ij}^2-\alpha_j^2 e_j^2)^4}\left(5R_{ij}^2+4\alpha_j^2 e_j^2\right)+\right.$$

$$+\frac{\alpha_i^6 e_i^6}{1{,}408(R_{ij}^2-\alpha_j^2 e_j^2)^6}\left(35R_{ij}^4+84R_{ij}^2\alpha_j^2 e_j^2+24\alpha_j^4 e_j^4\right)+\frac{3\alpha_i^8 e_i^8}{16{,}640(R_{ij}^2-\alpha_j^2 e_j^2)^8}\left(105R_{ij}^6+\right.$$

$$\left.+504R_{ij}^4\alpha_j^2 e_j^2+432R_{ij}^2\alpha_j^4 e_j^4+64\alpha_j^6 e_j^6\right)+\frac{\alpha_i^{10} e_i^{10}}{15{,}360(R_{ij}^2-\alpha_j^2 e_j^2)^{10}}\left(231R_{ij}^8+\right.$$

$$\left.\left.+1{,}848R_{ij}^6\alpha_j^2 e_j^2+3{,}168R_{ij}^4\alpha_j^4 e_j^4+1{,}408R_{ij}^2\alpha_j^6 e_j^6+128\alpha_j^8 e_j^8\right)\right].$$

The function $H\left(R_{ij},\alpha_i e_i,\alpha_j e_j\right)$ is defined by equation (3.18).

Figure (A7.2) graphically represents the function $e_r=e_r\left(c_i,c_j\right)$, whose values were obtained using equation (A7-37). The required values of the integral I_i were found by employing the Simpson's method of numerical integration associated with an $(n\cdot 100+1)$- point division of the interval of integration; n is a positive integer bigger than 7 whose value was determined by the relation

$$\frac{\left|I_{i,(n+1)}-I_{i,n}\right|}{I_{i,n}}\ll 10^{-13}, \quad n=8,9,\cdots\cdots,$$

where $I_{i,n}$ represents the numerical value of I_i computed using an $(100\cdot n+1)$-point division. The values of the elliptic integrals K(u) and E(u) were found with the aid of the series (A4-34) truncated after their ν^{th} term; ν was chosen so that $u^{2\nu} \ll 10^{-25}$.

The lines drawn on the surface A'B'C'D' are the intersections of this surface with the planes

$$C_i + C_j = \mathcal{N} \times 0.05 \ , \quad -\infty < e_r(C_i, C_j) < +\infty \ , \quad \mathcal{N} = 1, 2, 3, \cdots 20 \ .$$

The point C' corresponds to the maximum value of the relative error e_r occurring when $C_i = C_j = 0.5$; this value is equal to 2.098 10^{-3}. The surfaces A'B'C'A' and A'C'D'A' are the graphical representations of the functions

$$e^*_{6r} = \frac{I_i - \pi(\alpha_i e_i)^3(\alpha_j e_j)^3 \sqrt{R_{ij}} \ H^*(R_{ij}, \alpha_i e_i, \alpha_j e_j)}{I_i} \ , \ 0 \ll \alpha_j e_j R_{ij}^{-1} \ll 1 \ ,$$

$$0 \ll \alpha_i e_i R_{ij}^{-1} \ll 0.5 \ ,$$

and

$$e_{6r} = \frac{I_i - \pi(\alpha_i e_i)^3(\alpha_j e_j)^3 \sqrt{R_{ij}} \ H(R_{ij}, \alpha_i e_i, \alpha_j e_j)}{I_i} \ , \ 0 \ll \alpha_i e_i R_{ij}^{-1} \ll 1 \ ,$$

$$0 \ll \alpha_j e_j R_{ij}^{-5} \ll 0.5 \ ,$$

respectively.

It is obvious from figure (A7.2) that the relative error e_r is virtually zero for every value of C_i and C_j , except for values obeying the condition

$$0.95 \ll C_i + C_j \ll 1 \ ,$$

which, upon substitution of equations (A7-24), becomes

$$0.95 R_{ij} \ll \alpha_i e_i + \alpha_j e_j \ll R_{ij} \ .$$

We note that this condition is satisfied only in very rare and extreme cases.

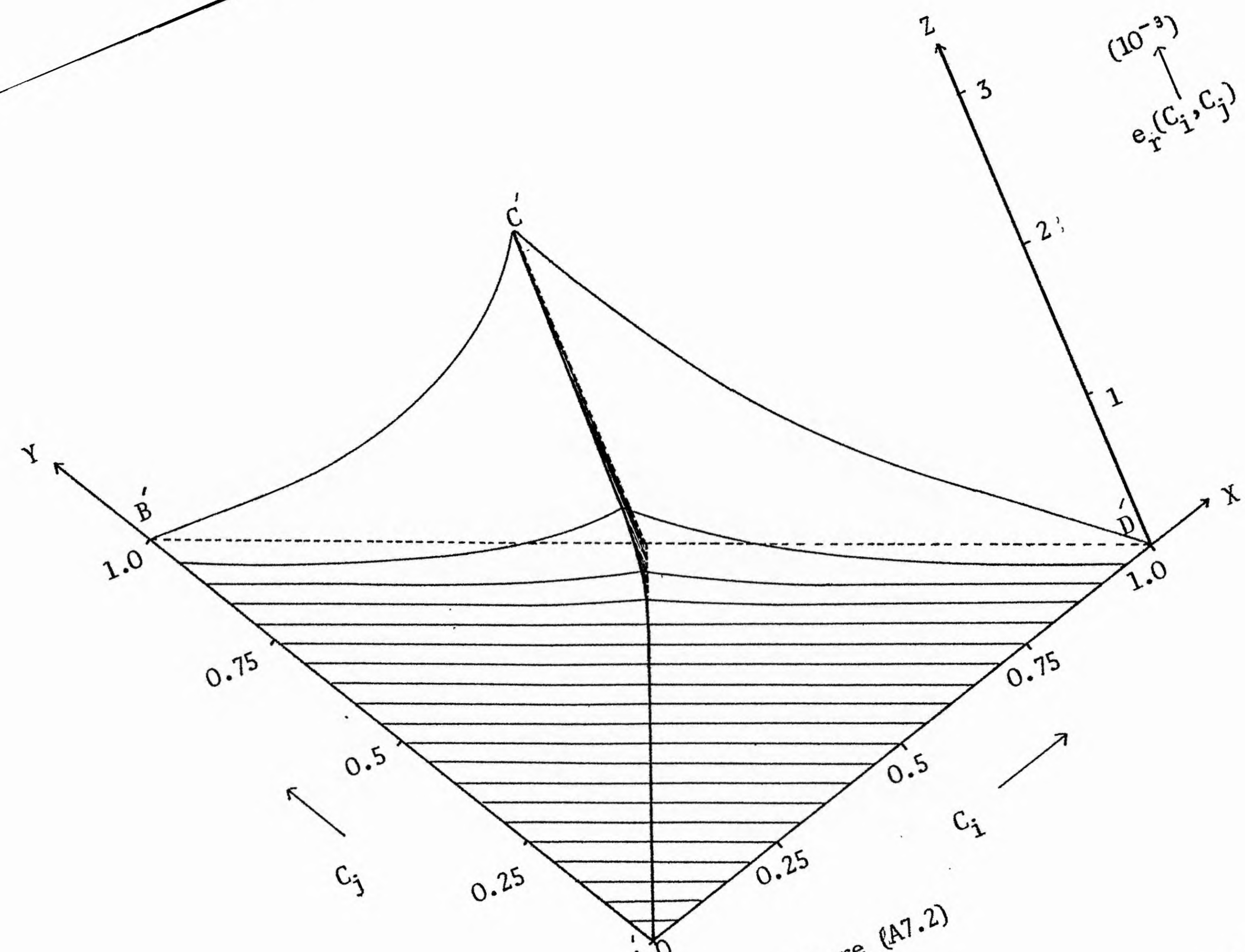


Figure (A7.2)

R E F E R E N C E S

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