STABILITY OF SOME FREE-SURFACE FLOWS

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STABILITY OF SOME FREE-SURFACE FLOWS

by

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DECLARATION

I declare that the following thesis is a record of research work carried out by me, that the thesis is my own composition, and that it has not been presented in application for a higher degree previously.

Frank I. R. Smith

PREFACE

In October 1960, I matriculated at the University of St. Andrews and read for a degree in Applied Mathematics in St. Salvator's College. In June 1964, I graduated with Second Class Honours (Part One) in Applied Mathematics. In October 1964, I was admitted, under Ordinance General No. 12, as a full time research student in the Department of Applied Mathematics of St. Salvator's College under the supervision of Dr. A.D.D.Craik. In October 1966, on appointment to the teaching staff of the department, I transferred to part-time research.

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I am most grateful to my supervisor, Dr A.D.D. Craik, for his patient guidance and helpful criticism throughout the course of this work. My thanks are also due to Mr J. Spark for his assistance in photography and to my wife for her patient erasing of my typographical errors.

CERTIFICATE

I certify that Frank I.P. Smith has satisfied the conditions of the Ordinance and Regulations and is thus qualified to submit the accompanying thesis in application for the degree of Doctor of Philosophy.

CONTENTS

CHAPTER I

INTRODUCTION

The subject matter of this thesis is concerned with the stability of fluid flows; more particularly, with the stability of liquid films which have an interface with air. We will therefore begin by formulating the basic equations and ideas which pertain to this class of problems Later in this chapter, a summary will be given of the topics dealt with in this dissertation.

§1.1 : Basic formulation.

The three-dimensional flow of an incompressible fluid will now be considered. In a Cartesian co-ordinate system x_i $(i = 1,2,3)$ the general equations governing the three-dimensional motion of an incompressible fluid are

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = \rho F_i + \frac{\partial}{\partial x_j}\sigma_{ij}$$

where u_i and F_i $(i = 1,2,3)$ are the velocity and body force (per unit mass) components in the respective co-ordinate directions, ρ is the fluid density, t is the time and σ_{ij} are the components of the stress tensor. The equation of continuity in an incompressible fluid is

$$\frac{\partial u_i}{\partial x_i} = 0.$$

The stress tensor $\sigma_{ij}(x_i,t)$ at any point x_i of the fluid at time t is defined as the i-component of the force per unit area exerted across a plane surface element normal to the j-direction. It satisfies the Newtonian law,

$$\sigma_{ij} = -p\,\delta_{ij} + 2\mu\, e_{ij}$$

where p is the pressure in the fluid, μ is the viscosity of the fluid, δ_{ij} is the 'delta function', and e_{ij} is the rate of strain tensor at that point in the fluid. The rate of strain tensor is related to the velocity of the fluid through the equation

$$e_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

The general equations then become the Navier-Stokes equations for an incompressible fluid

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = \rho F_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right)$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

in which form μ is not assumed to be constant.

It is generally convenient to transform all physical variables into dimensionless form, relative to scales of mass, length and time which are representative of the problem. These may be defined by the constant density ρ of the fluid; the length scale h, which is as yet unspecified, but which is a measure of the effective thickness of the fluid layer; and some convenient velocity V, which in this work will usually be the surface velocity. As viscosity is allowed to vary, the Reynolds number R is defined as $\frac{\rho V h}{\mu_o}$ where μ_o is the (constant) viscosity of particles comprising the liquid surface; and a 'dimensionless viscosity' m is defined as

$$m = \frac{\mu (x_i, t)}{\mu_o}$$

With 'barred' variables referring to the dimensionless form of those 'unbarred' earlier, the Navier- Stokes equations then reduce to the form (c.f. Drazin, 1962)

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial \bar{x}_j} = \bar{F}_i - \frac{\partial \bar{p}}{\partial \bar{x}_i} + \frac{1}{R}\left(\frac{\partial}{\partial \bar{x}_j}\left\{ m \left(\frac{\partial \bar{u}_i}{\partial \bar{x}_j} + \frac{\partial \bar{u}_j}{\partial \bar{x}_i} \right)\right\} \right)$$

$$\frac{\partial \bar{u}_i}{\partial \bar{x}_i} = 0.$$

We now assume that the viscosity of each element of the liquid remains constant throughout its motion: that is,

$$\frac{D\mu}{Dt} = 0. \qquad (1.1)$$

This equation can be taken to be exact provided the viscosity distribution is not subject to diffusion. However, in practice, variations in viscosity are normally due to variations in temperature, T, i.e. $\mu=\mu(T)$: and diffusion of heat may produce a corresponding diffusion of viscosity, i.e.

$$\frac{D\mu}{Dt} = \kappa \nabla^2 \mu$$

(provided $\frac{D^2T}{D\mu^2} / \frac{DT}{D\mu} \ll | \nabla^2 \mu | / | \nabla \mu |^2$) , where κ is the thermal diffusivity of the liquid. In such cases, equation (1.1) is an approximation valid when κ is small.

If the flow experiences a small two-dimensional periodic disturbance of wave-number k, an appropriate dimensionless wave-number is $\alpha = kh$ and we can obtain a more precise condition on κ or Pr where Pr ($= \mu_o/\rho\kappa$) is the Prandtl number. Equation (1.1) may be a good approximation provided

$$\alpha R Pr, \frac{R Pr}{\alpha} \gg 1.$$

Since αR will be taken as $O(1)$ or less in the present work, it is clearly necessary that the Prandtl number of the material in question should be large. This condition appears to be well satisfied, for example, by tektites, molten metals and glass. Restrictions similar to the above exist for other diffusive processes such as molecular diffusion.

In dimensionless form equation (1.1) becomes

$$\frac{Dm}{Dt} = 0. \tag{1.2}$$

A further qualification concerning equation (1.1) must be made. If a small two-dimensional wavelike disturbance propagates with a velocity c' which satisfies the inequality $0 < c' < V$, the linearised form of equation (1.1) may yield a singularity at the 'critical layer' where the liquid velocity equals c'. Such disturbances are outside the scope of this analysis, but have been discussed previously by Lees and Lin (1946) and are also the subject of a paper by Craik (1969). Here, as for uniform films, it is found that the velocity c' of surface waves satisfies the condition $c' > V$, and the question of a singularity does not arise for these waves. However, the possibility of other wave-modes with $0 < c' < V$ is not discounted: if, for example, the viscosity distribution possesses a near-discontinuity at some depth, 'internal' instabilities similar to those discussed by Yih (1967b) may arise.

The assumption of constant density is also an approximation if temperature varies within the fluid. However, in the context of the present work, this is not a serious restriction. The major role of gravity is that due to the large density discontinuity at the liquid surface; and, provided the liquid remains stably stratified, comparatively small changes in density within the liquid are unlikely to be important for the waves under discussion.

§1.2 : Plane Parallel Flows.

If the steady state consists of a one- or two-dimensional plane parallel flow, the dimensionless Cartesian co-ordinate system Oxyz may be chosen so that O lies in the undisturbed fluid interface and Oz is normal to this surface. In this system the dimensionless components of the primary fluid velocity are then $(\bar{U}(z), \bar{V}(z), 0)$ and those of the linearised velocity perturbation caused by the disturbance are $(u'(x,y,z,t), v'(x,y,z,t), w'(x,y,z,t))$. In the primary state, we assume that the viscosity is a function of depth alone i.e. that $m=\bar{m}(z)$. The linearised Navier-Stokes equations including terms which allow for viscosity variation are as follows:

$$\frac{\partial u'}{\partial t} + \bar{U}\frac{\partial u'}{\partial x} + \bar{V}\frac{\partial u'}{\partial y} + w'\frac{\partial \bar{U}}{\partial z} = -\frac{\partial p'}{\partial x} + \frac{\bar{m}}{R}\left[2\frac{\partial^2 u'}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}\right)\right]$$

$$+ \frac{1}{R}\frac{\partial \bar{m}}{\partial z}\left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}\right) + \frac{1}{R}\frac{\partial}{\partial z}\left(m'\frac{\partial \bar{U}}{\partial z}\right)$$

$$\frac{\partial v'}{\partial t} + \bar{U}\frac{\partial v'}{\partial x} + \bar{V}\frac{\partial v'}{\partial y} + w'\frac{\partial \bar{V}}{\partial z} = -\frac{\partial p'}{\partial y} + \frac{\bar{m}}{R}\left[\frac{\partial}{\partial x}\left(\frac{\partial v'}{\partial x} + \frac{\partial u'}{\partial y}\right) + 2\frac{\partial^2 v'}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y}\right)\right]$$

$$+ \frac{1}{R}\frac{\partial \bar{m}}{\partial z}\left(\frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y}\right) + \frac{1}{R}\frac{\partial}{\partial z}\left(m'\frac{\partial \bar{V}}{\partial z}\right)$$

$$\frac{\partial w'}{\partial t} + \bar{U}\frac{\partial w'}{\partial x} + \bar{V}\frac{\partial w'}{\partial y} = -\frac{\partial p'}{\partial z} + \frac{\bar{m}}{R}\left[\frac{\partial}{\partial x}\left(\frac{\partial w'}{\partial x} + \frac{\partial u'}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w'}{\partial y} + \frac{\partial v'}{\partial z}\right) + 2\frac{\partial^2 w'}{\partial z^2}\right]$$

$$+ \frac{2}{R}\frac{\partial \bar{m}}{\partial z}\frac{\partial w'}{\partial z} + \frac{1}{R}\frac{\partial m'}{\partial x}\frac{\partial \bar{U}}{\partial z} + \frac{1}{R}\frac{\partial m'}{\partial y}\frac{\partial \bar{V}}{\partial z}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \qquad (1.3,\ a,b,c,d)$$

The dimensionless pressure and viscosity perturbations are $p'(x,y,z,t)$ and $m'(x,y,z,t)$ respectively.

As is usual in methods of hydrodynamic stability (see Lin (1955), Chapter 1), the perturbation at any instant may be assumed to be resolvable into independent Fourier wave components in x and y, the amplitudes of the waves varying as the real part of an exponential function in time (i.e. $\exp(-i\omega t)$ where ω is the frequency) and both amplitudes and phases varying with depth. Each harmonic component is then a solution of the linearised equations of motion, provided its amplitude is sufficiently small for these equations to hold. Then each of the perturbation velocity components of the disturbance is the real part of an expression of the form

$$q' = \hat{q}(z) \exp\{i(\alpha x + \beta y) - i\alpha c t\}$$

where $\hat{q}(z)$ is a complex function of z, α and β are real positive wave-numbers and c ($= c_r + ic_i$) is a complex phase velocity. More generally, each velocity component of a bounded perturbation which can be prescribed at t=0, can be represented as a double Fourier integral of the form

$$q' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\alpha,\beta,c)\, \hat{q}(z) \exp\{i(\alpha x + \beta y) - i\alpha c t\}\, d\alpha\, d\beta$$

where $A(\alpha,\beta,c)$ is some known function.

The objective is to determine c as a function of α,β and the other properties of the problem. Thus a wave component has a phase velocity c_r and growth factor $\exp(\alpha c_i t)$, being stable or unstable according as c_i is negative or positive. If αc_i is zero, the wave component is 'neutrally stable'. Hence, if αc_i is negative or zero for <u>all</u> harmonic disturbances, the initial steady state is stable to small disturbances: but, if there exists one harmonic mode for which αc_i is positive, the steady state is unstable to disturbances containing this mode as a Fourier component.

If linearised theory remains valid for some time during the amplification process of the initial perturbation, the disturbance may develop into a single dominant wavelike mode due to the selective amplification of the most unstable component. However, owing to non-linear effects linearised theory may only describe the initial stages of amplification, in which the most unstable mode may dominate, but other components remain significant especially those three-dimensional modes neighbouring to this mode (see Benjamin (1961))

Also, since three-dimensional disturbances can be treated in terms of a corresponding two-dimensional problem, only two-dimensional harmonic disturbances need be considered (see Lin 1955, §§3.1,5.2) and that only the component of the primary flow in the direction of the harmonic component need be examined. It follows that, if there is a minimum critical Reynolds number at which neutrally stable disturbances may exist, these disturbances are two-dimensional.

This result is the theorem of Squire (1933). It may here be noted that this theorem does not apply to the three-dimensional flows of Chapter 3. Corresponding relationships between two- and three-dimensional amplified disturbances have been found by Watson (1960) and Michael (1961) subject to certain restrictions.

For the above reasons, in much of the work which follows, the problem can be specified by choosing axes so that $\beta=0$ and by ignoring the y-component of the primary flow.

It should be emphasised that the above formulation is appropriate for the temporal development of spatially uniform disturbances and thereby excludes spatial amplification. The relationship between spatial and temporal amplification has been examined by Gaster (1962). For small growth rates he has shown that the frequencies of these disturbances are approximately equal and that the spatial amplification is approximately equal to the temporal amplification divided by the group velocity.

For a disturbance at the air-liquid interface, with a wind-flow over this interface, of the above periodic form, the stresses exerted by the given airflow may always, in principle, be calculated. The problem may be treated in two parts: the surface stresses exerted by the airflow upon a prescribed disturbance of the boundary may first be evaluated, and, subsequently, the stability problem for the liquid film may be solved subject to these stresses. As indicated by Benjamin (1959), this 'divided' method has considerable advantages, especially in simplicity, over one in which the complete stability

produced
problem is posed for the system. The surface stresses/by the interaction of the mean airflow with small periodic perturbations of the bounding surface (i.e. the surface of the film) have been evaluated by Miles (1957, 1959, 1962) and by Benjamin (1959) for mean air velocity profiles of boundary layer type. Turbulent fluctuations in the airflow are neglected although small disturbances at the surface may become dynamically unstable under the linearised stresses exerted by a laminar or 'quasi-laminar' flow.

These estimates derived by Benjamin are sufficient for the purposes of this work and are introduced at an advanced stage in the analysis. The precise estimates required are for the pressure stress component in phase with the wave elevation, for which the 'quasi-laminar model' appears to be adequate,and for the tangential stress component in phase with the wave slope. For this latter the 'quasi-laminar model' is less accurate but of the correct order of magnitude.

Plate, Chang and Hidy (1969) have made an experimental investigation into the generation of small water waves by wind. Under the shearing action of the airflow, the growth rate of the waves was found to follow the Miles-Benjamin theoretical predictions although at a slightly higher value.

Thus, for any small periodic disturbance, precise boundary conditions satisfied by the liquid film at the mobile interface can be formulated with due allowance for surface contamination (see later).

The equations of motion can then be solved subject to the boundary conditions at the interface together with those at the rigid wall.

They constitute a complete boundary value problem whose solution yields a secular relationship between the properties of the primary flow and those of the disturbance. From this relationship can be deduced the occurrence of instability in the liquid film.

The arrangement of this dissertation is as follows:

In Chapter 2, a review of previous work on the stability of thin liquid films is given. In Chapter 3, the stability problem for a liquid film flowing down an inclined plane subject to an oblique airflow is solved and previous work discussed in relation to the results. Chapter 4 deals with the stability of liquid films whose viscosity varies and the results found are shown to be similar to those known for thin liquid films.

In Chapters 5 and 6 the mechanism of surface contamination is investigated in relation to liquid films and the theoretical models formulated compared with some experiments on contaminated liquid films.

CHAPTER II

PREVIOUS WORK ON THIN LIQUID FILM INSTABILITY.

There have been several previous theoretical investigations into the stability of thin liquid films; however, only one of these has been concerned with variation of viscosity.

One of the earliest rigorous formulations of the hydrodynamic stability problem for the uniform film flowing down an inclined plane was given by Yih (1954). His solution of the governing equation was by " a power series expansion of αR, α being the wave number and R the Reynolds number. The numerical results, however, were not accurate enough." (quoted Yih (1963)) Later Benjamin (1957) performed a power series expansion in y, the co-ordinate measuring the depth of the film, and obtained the neutral stability criterion analytically. In this work surface tension effects had been introduced.

Much later Yih (1963) returned to the problem and gave a very neat analytical solution to his power series expansion in αR, in agreement with Benjamin. He extended the work from small α to small αR with any α and to large α with finite αR.

Similar techniques have been applied to the problem of wind generated waves on a thin liquid film. In 1960, Zaitsev examined the instability of wind generated waves in a horizontal liquid film. The wind stress was represented by a constant tangential stress acting on the plane surface. No allowance was made for the action of surface perturbation on the wind stress. A series solution was obtained in powers of Reynolds and wave numbers. Pairs of neutral curves for different values of the ratio of fluid to air viscosities and curves of Reynolds number against surface tension were obtained. However, the results and approximations were not discussed. In 1961, Lyubutskaya extended the work to inclined plane flows. Here again, a series solution in powers of Reynolds and wave numbers was utilised and a neutral stability criterion established which reduced to that of Benjamin and Yih for the case of no wind.

Benjamin (1963) subsequently examined the effect of insoluble contaminants on the stability of flow down an inclined plane. His results will be discussed in Chapter 5 where surface contamination is dealt with in more detail.

Semenov (1964) has considered the stability of a horizontal flow in which viscosity increases exponentially with depth below the mean liquid surface, and where the motion is due entirely to a constant tangential stress exerted at the liquid surface. This model is again an attempt to introduce wind stress. He assumed that the viscosity remained constant in horizontal planes even when the flow was given a small perturbation. Hence, for this model, fluid particles do not have constant viscosity. In contrast, the work following in Chapter 4 takes into account the fluctuations in viscosity at a given location due to the perturbed motion. The viscosity of individual particles of fluid remain constant whilst the variation in viscosity which occurs at a certain level in the fluid is due to the wave motion in the fluid.

Also, as with the model of Zaitsev and Lyubutskaya, that of Semenov is not suited to an examination of wind-generated waves since the periodic stresses at the air-liquid interface, which arise due to the interaction of the air-flow and the perturbed liquid surface, are not adequately formulated. Such stress perturbations are incorporated in the analysis later in the same manner as was employed by Craik (1966; 1968); these stresses are represented by suitable parameters and appropriate estimates for these may be substituted if desired.

Drazin (1962) has discussed the stability of parallel flows where both viscosity and density may vary, but he restricted the study to flows at large Reynolds numbers. His method was that of Tollmein and Schlichting, where asymptotic solutions for large αR are introduced. Some aspects of his work- particularly the initial formulation of the stability problem - are relevant to the present investigation and these are mentioned later.

Craik (1966) has investigated the stability of a thin uniform liquid film on a horizontal boundary subject to a concurrent airflow over its surface. The stress perturbations at the air-liquid surface due to the wind were represented in an appropriate parametric form, in accordance with the theoretical estimates of Miles (1957) and Benjamin (1959). The latter estimates involve factors which depend upon the air-velocity profile; but, for simplicity, Craik treated these as given constant parameters, the values of which may be calculated, if desired, for particular airflows. This was a better representation for wind-generated waves than those of Zaitsev and Lyubutskaya, but not so precise as that of Cohen and Hanratty (1965) in their investigation of films of larger Reynolds numbers. As with the case of a film on an inclined plane previously mentioned, instability is found to exist at rather small liquid Reynolds numbers, the wavelengths of the unstable disturbances being large compared to the film thickness. A surprising aspect of the results is that instability occurs in sufficiently thin films. This is caused by the component of tangential stress perturbation which is in phase with the wave slope.

Figure 2.1: Sketch of effect of tangential stress component in phase with wave slope.

The destabilising effect of this stress component becomes dominant for sufficiently thin films; and, as shown in figure 2.1, its effect is such as to displace liquid away from the troughs and towards the crests of a wave, thereby increasing its amplitude.

Cohen and Hanratty (1965) have made investigations similar in many respects to that of Craik. Their experimental work was confined to thicker films. In their theory the surface stresses were determined by the method developed by Miles (1962) and their determination involved the integration of a Ricatti-type equation for a prescribed velocity profile. Their theory agrees well with experiment and their results seem to support the 'quasi-laminar' assumption, that the turbulence of the airflow does not contribute significantly to the systematic stresses in phase with the wave.

CHAPTER III

STABILITY OF LIQUID FILM FLOW DOWN AN INCLINED PLANE WITH OBLIQUE AIRFLOW.

In this chapter we will examine the hydrodynamic stability of a liquid film of uniform thickness (and constant viscosity) flowing down an inclined plane, subject to a wind-stress in some direction on its surface and having insoluble surface contaminants present. Here, the viscosity of the liquid film will be kept constant whereas in the succeeding chapter the effect of variation in viscosity will be examined. Our purpose is to discover how these two instabilities - that due to gravity and that due to wind stress - are related for a three-dimensional primary flow and to find criteria for the onset of instability. The previous remarks concerning Squire's theorem in Chapter 1 must be remembered. One physical problem to which the succeeding theory could be applied is that of the flow of fluid over turbine blades.

§3.1 : Basic formulation.

The three-dimensional motion whose stability is to be investigated is illustrated in figure 3.1. The liquid, having constant density ρ and viscosity μ ($=\rho\nu$) flows in a uniform film of thickness h upon a solid plane inclined at an angle θ to the horizontal. It is convenient to adopt a co-ordinate system (x,y,z) in which the undisturbed liquid surface is represented by the plane $z=0$, z increases along the outward normal to the liquid surface and to choose the

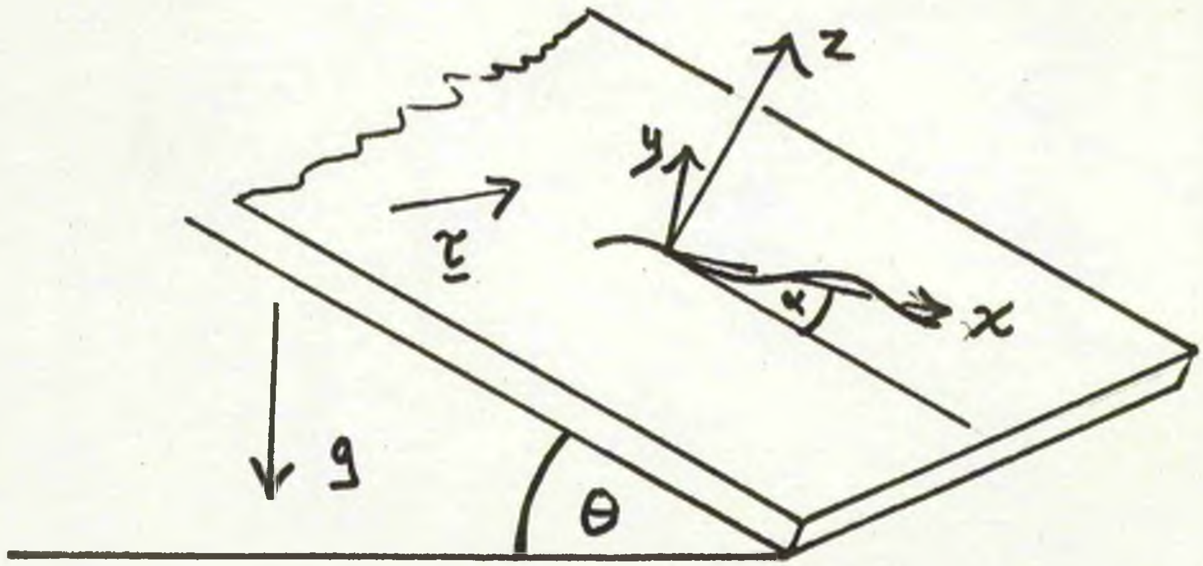


Figure 3.1: Sketch of flow configuration and surface disturbance.

x-direction in the wave direction.

The appropriate equations of motion in dimensional form are from Chapter 1

$$\rho \frac{\partial \underline{u}}{\partial t} + \rho (\underline{u} . \nabla) \underline{u} = - \nabla p + \mu \nabla^2 \underline{u} + \rho \underline{F} \qquad (3.1\ a,b)$$
$$\nabla . \underline{u} = 0.$$

where the fluid has pressure p and velocity $\underline{u}$ and experiences a body force $\underline{F}$ due to gravity. $\underline{F}$ is expressible in the form

$$\underline{F} = (g_x \sin\theta,\ g_y \sin\theta,\ g \cos\theta)$$

where $g_x = g \cos\alpha$ and $g_y = -g \sin\alpha$, θ is the angle of inclination of the plane to the horizontal and α is the angle the x-direction makes with the line of steepest descent down the plane.

The primary velocity profile is of the form $\underline{\bar{U}} = (\bar{U}(z), \bar{V}(z), 0)$ where

$$\mu D^2 \underline{\bar{U}} = -\rho \underline{F} + \nabla \bar{P} ,$$

$D \equiv \frac{d}{dz}$ and $\bar{P}$ is the hydrostatic pressure. Also the boundary conditions are

$$\mu D \underline{\bar{U}} = \underline{\tau} \quad \text{at} \quad z=0$$
$$\underline{\bar{U}} = 0 \quad \text{at} \quad z=-h$$

where $\tau = (\tau_x, \tau_y, 0)$ is a 2-dimensional vector in the x-y plane denoting mean wind-stress. This gives for the basic velocity profile

$$\underline{\bar{U}} = -\frac{\rho}{2\mu} (z^2-h^2)\ \underline{F} + \frac{1}{\mu} (z+h)\ \underline{\tau} \qquad (3.2)$$

Here, only the x- and y-components of $\underline{F}$ taken, as the z-component is balanced by the hydrostatic pressure.

§3.2: The stability problem.

Because the basic problem is three-dimensional, the perturbations of the liquid stream in directions both normal and parallel to the plane surface must be considered. Let the normal displacement of the surface be represented by

$$z = \eta(x,y,t) = \delta \exp(ik(x-ct)) \tag{3.3}$$

where k is the wave number and c is the wave velocity. With $\underline{u} = \bar{\underline{U}} + \underline{u}'$ and $P = \bar{\bar{P}} + p'$ as in the notation of Chapter 1, (but where terms are dimensional) the linearised equations of motion in vector notation are

$$\begin{aligned} \rho\frac{\partial \underline{u}'}{\partial t} + \rho(\bar{\underline{U}} \cdot \nabla)\underline{u}' + \rho(\underline{u}' \cdot \nabla)\bar{\underline{U}} &= -\nabla p' + \mu\nabla^2\underline{u}' \\ \nabla \cdot \underline{u}' &= 0. \end{aligned} \tag{3.4a,b}$$

Let the displacement of the surface parallel to the plane be $\underline{\xi} = \bar{\bar{\underline{X}}}\eta$ where both $\underline{\xi}$ and $\bar{\underline{X}}$ are 2-dimensional vectors in the x-y plane. Then the time derivative of $\underline{\xi}$ following the motion of the undisturbed liquid surface is

$$\begin{aligned} \frac{D\underline{\xi}}{Dt} &= \bar{\underline{X}}\left[\frac{\partial\eta}{\partial t} + \bar{U}(0)\frac{\partial\eta}{\partial x}\right] \\ &= ik\,\bar{\underline{X}}\left[-c + \frac{\rho h^2 g\sin\theta}{2\mu} + \frac{h\tau_x}{\mu}\right]\eta \end{aligned} \tag{3.5}$$

The linearised kinematic surface conditions are

$$\frac{D\eta}{Dt} = w'(0)\,, \qquad \frac{D\underline{\xi}}{Dt} = \underline{u}'(0) + \left(\frac{d\bar{\underline{U}}}{dz}\right)_{z=0}\eta \tag{3.6a,b}$$

Therefore from equations (3.5) and (3.6b) we obtain

$$-ik\bar{\underline{X}}\left(\frac{\rho h^2 g\sin\theta}{2\mu} + \frac{h\tau_x}{\mu} - c\right)\eta = \underline{u}'(0) + \frac{\underline{\tau}}{\mu}\eta \tag{3.7}$$

Also, the linearised boundary condition (3.6a) becomes

$$(w')_{z=0} = ik\left[\frac{h\tau_x}{\mu} + \frac{\rho h^2 g\sin\theta}{2\mu} - c\right]\eta \tag{3.8}$$

At a solid boundary, the perturbation velocity must vanish

$$\underline{u}'(-h) = 0$$

and by continuity at $z = -h$

$$\frac{\partial w'}{\partial z} = 0 \qquad (3.9)$$

The two remaining boundary conditions concern the normal and tangential stresses at the air-liquid interface. The wind stress may be represented by the parameters $\underline{\sigma}_n$ and $\underline{\sigma}_t$ which are related to the airflow perturbation through

$$\underline{\sigma}_n = \underline{\pi}\eta \quad \text{and} \quad \underline{\sigma}_t = \underline{\Sigma}\eta \qquad (3.10a,b)$$

$\underline{\pi}$ is in the direction of the z-axis perpendicular to the plane and $\underline{\Sigma}$ lies in the plane in the x-direction and depends on the component of the windh velocity profile in the direction of the wavenumber vector. $\underline{\pi}$ and $\underline{\Sigma}$ are usually complex.

In the tangential stress condition allowance will be made for insoluble surface contamination. The present representation is that used by Benjamin (1963). He identified the surface properties of a liquid with those of a visco-elastic membrane whose deformation produces quasi-elastic forces proportional to strains and quasi-viscous forces proportional to temporal rates of strain: this model thereby incorporates both surface elasticity and surface viscosity. This simple model is re-examined and compared with other models in a later chapter.

The stresses due to surface elasticity and surface viscosity are obtained in the form

$$\underline{\sigma}_c = \left(\gamma_1 + \kappa \frac{D}{Dt} \right) \nabla_2 (\nabla_2 \cdot \underline{\xi})$$

where ∇_2 operates only on the x and y co-ordinates, γ_1 is the sum of elastic moduli of surface dilatation and shear and κ is the sum of surface dilatation and shear viscosities ;

$$\text{i.e. } \underline{\sigma}_c = -k^2 \bar{\underline{X}} \left(\gamma_1 + \kappa \frac{D}{Dt} \right) \eta$$

$$= -k^2 \bar{\underline{X}} \left(\gamma_1 + ik\kappa \left(\bar{U}(0) - c \right) \right) \eta \qquad (3.11)$$

The tangential stress just inside the liquid surface must equal the sum of the tangential stresses exerted by the contaminated surface and by the air stream. The boundary condition equating tangential stress at $z=\eta$ is therefore

$$\underline{\sigma} = \underline{\sigma}_t + \underline{\sigma}_c$$

The shear stress at $z=\eta$ is

$$\underline{\sigma} = \mu \left(\frac{\partial w'}{\partial x} + \frac{\partial u'}{\partial z} , \frac{\partial w'}{\partial y} + \frac{\partial v'}{\partial z} , 0 \right)$$

Using a Taylor's series expansion about $z=0$, this is

$$\underline{\sigma} = \mu \left(\frac{\partial w'}{\partial x} + \frac{\partial u'}{\partial z} + \frac{d^2\bar{U}}{dz^2} \eta , \frac{\partial w'}{\partial y} + \frac{\partial v'}{\partial z} + \frac{d^2\bar{V}}{dz^2} \eta, 0 \right) \qquad (3.12)$$

evaluated at $z=0$.

The condition that the capillary pressure and the normal stress on either side of the surface should be in equilibrium is, in the linearised approximation,

$$-p' + 2\mu \frac{\partial w'}{\partial z} = \Gamma_0 (\nabla_2{}^2 \eta) + | \underline{\sigma}_n |$$

evaluated at $z=\eta$. Here, Γ_0 is the surface tension coefficient. At $z=0$ the above equation reduces to

$$(-p')_{z=0} - \frac{\partial \bar{P}}{\partial z} \eta + 2\mu \frac{\partial w'}{\partial z} + \Gamma_0 k^2 \eta = \varpi \eta$$

where ϖ is the magnitude of $\underline{\sigma}_n$. Using the fact that the basic pressure $\bar{P}$ is hydrostatic, $\frac{\partial \bar{P}}{\partial z} = -\rho g \cos\theta$ and we further obtain

$$\eta\ (\rho g \cos\theta + \Gamma_\theta k^2)\ +\ (2\mu \frac{\partial w'}{\partial z} - p')\ =\ \varpi\eta \qquad (3.13)$$

§3.3: The stream function.

For convenience the x-axis was chosen to lie in the direction of propagation of the wave. Thus, for a given wave perturbation, velocity perturbations in the y-direction do not enter the problem. From equation (3.4b) a stream function of the form

$$\psi\ =\ \phi(z)\ \eta$$

where $\quad u' = \frac{\partial\psi}{\partial z} = (D\phi)\eta \quad ; \quad w' = \frac{-\partial\psi}{\partial x} = -ik\phi\eta \qquad (3.14a,b)$

can be introduced. D in this chapter alone denotes differentiation with respect to z. Substitution of this stream function into the equations of motion and elimination of the pressure terms gives the well-known Orr-Sommerfeld equation

$$\nu\ (\ D^4 - 2\ k^2\ D^2 + k^4\ )\ \phi = ik\ \left\{ (\bar{U}-c)(D^2-k^2)\phi - (D^2\bar{U})\phi \right\} \qquad (3.15)$$

The boundary conditions at $z = -h$ yield

$$\phi(-h) = D\phi(-h) = 0. \qquad (3.16a,b)$$

The kinematic surface condition (3.8) becomes

$$\phi(0) = c - \frac{h\tau}{\mu}x - \frac{\rho\sin\theta\ g_x\ h^2}{2\mu} = c - \bar{U}(0) \qquad (3.17)$$

The two stress conditions at $z = 0$ are obtained from equations (3.10 - 3.13) :-

$$\mu\ (\ D^2\phi + k^2\phi\ ) = \Sigma - k^2\bar{X}_x\{\ \gamma_1 + ik\kappa(\bar{U}(0)-c)\} + \rho\sin\theta g_x \qquad (3.18)$$

and $\quad (\bar{U}(0)-c)\ D\phi - (D\bar{U})\phi = \frac{\nu}{ik}\ (\ D^3\phi - 3k^2 D\phi\ )$

$$+\ (\ \varpi - \rho g \cos\theta - \Gamma_\theta k^2\ )\frac{\phi(0)}{\rho(c-\bar{U}(0))} \qquad (3.19)$$

in the tangential and normal directions respectively. $\bar{X}_x$ denotes the x-component of $\bar{X}$

§3.4: Solution for small wave numbers.

a) The first approximation.

The quantities kh and $\frac{h^2\tau}{\omega\mu}$ are taken to be small and to a first approximation are made zero. This approach was first used by Yih (1963). From equations (3.15) - (3.19) we must solve

$$D^4\phi = 0 \tag{3.20}$$

subject to

$$\begin{aligned} \phi &= D\phi = 0 && \text{at } z = -h \\ \phi &= c - \bar{U}(0) && \text{at } z = 0 \\ \mu D^2\phi &= \rho \sin\theta \, g_x && \text{at } z = 0 \\ D^3\phi &= 0 && \text{at } z = 0 \end{aligned} \tag{3.21a-e}$$

Solving the equation (3.20), subject to conditions (3.21 a,b,e) we obtain

$$\phi = A \,(z + h)^2$$

where A is an arbitrary multiplicative constant. Using the further conditions (3.21 c,d) we invoke the condition that the equations are consistent i.e. have a unique solution. This yields

$$\mu D^2\phi = \rho \sin\theta \, g_x \frac{\phi}{c - U(0)} \quad \text{at } z = 0$$

i.e.

$$c = \bar{U}(0) + \frac{\sin\theta \, g_x h^2}{2\nu}$$

$$= \frac{\tau_x h}{\mu} + \frac{\sin\theta \, g_x h^2}{\nu} \tag{3.22}$$

This means that when $g_x > 0$, waves travel down the plane with a velocity greater than the x-component of surface velocity of the liquid film by an amount which is proportional to the x-component of gravitational force. Clearly, the greater the slope of the inclined plane the faster the waves travel and the thinner the film the slower they travel relative to the surface velocity.

It is interesting to note the case when waves travel directly up the plane. This occurs when the wind stress acting up the plane

$$\tau_x > \rho \sin\theta \, g \, h$$

Also the wave speed is

$$c = \bar{U}(0) + \frac{\sin\theta g h^2}{2\nu}$$

This means that if the waves are travelling up the plane then so must the film surface itself. Also, from this equation, with a wind stress acting up the plane and the film surface travelling up the plane, waves can still travel down. Indeed, a balance may be obtained and a stationary train of waves set up in the film if

$$\tau_x = \rho \sin\theta \, gh.$$

It is also of interest to note that the velocity of wave propagation is that of kinematic waves. If v denotes the mean velocity of the liquid film, averaged over the film thickness h, the velocity of propagation of kinematic waves is $c_k = \frac{d}{dh}(vh)$. From the x-component of equation (3.2) in the equation

$$v = \frac{1}{h} \int_{-h}^{0} \bar{U}(z) \, dz,$$

it is found that $c_k = c$, where c is given by equation (3.22). This means that the disturbances are dominated by viscous rather than gravity or pressure forces and are thus governed by continuity. The theory of kinematic waves was developed by Lighthill & Whitham (1955).

h) The second approximation.

Following Yih's technique, we examine the change in the value of c as kh departs from zero. Let

$$\phi = \phi_0 + \frac{ikh^2\overline{U}(0)}{\nu}\phi_1$$

$$c = c_0 + \frac{ikh^2\overline{U}(0)}{\nu}c_1$$

where the subscript 0 refers to the first approximation. From equation (3.15), this gives

$$h^2\overline{U}(0)\, D^4\phi_1 = (\overline{U} - c)\, D^2\phi_0 - (D^2\overline{U})\phi_0$$

$$= 2A\left(\frac{\tau}{\mu}x + \frac{\rho\sin\theta g_x h}{\mu} \right)$$

which solves to

$$h^2\overline{U}(0)\, \phi_1 = 2A\left(\frac{\tau}{\mu}x + \frac{g_x\rho\sin\theta h}{\mu} \right)\frac{z^5}{5!} + A_1z^3 + B_1z^2 + C_1z + D_1 \tag{3.23}$$

In the ensuing evaluation, it is found that the terms in ϕ_1 involving D_1 are $D_1\,(z + h)^2$ which is proportional to ϕ_0. D_1 will therefore be taken to be zero, for otherwise another first approximation would be introduced within the second. This does not affect the c_1 which is to be found.

The boundary conditions reduce to

$$\phi_1 = D\phi_1 = 0 \qquad \text{at } z = -h \tag{3.24a,b}$$

$$h^2\overline{U}(0)\, D^2\phi_1 = -\frac{2i\nu}{\rho\sin\theta g_x}\frac{\Sigma}{kh^2}\phi_0 - \frac{4\nu h^2\overline{U}(0)c_1}{\sin\theta g_x h^4}\phi_0$$

$$+ \frac{2i\,\overline{X}_0 k\,\{\gamma_1 + ik\kappa(\overline{U} - c)\}_0}{\rho\sin\theta\, g_x h^2}\phi_0 \tag{3.24c}$$

$$\text{at } z = 0$$

and $$h^2\overline{U}(0)\, D^3\phi_1 = (\overline{U} - c_0)\, D\phi_0 - (D\overline{U})\,\phi_0 - \frac{\rho g\cos\theta + \Gamma_0 k^2 - \pi}{\rho(\overline{U} - c_0)}\phi_0$$

$$\text{at } z = 0 \tag{3.24d}$$

In condition (3.24c) $\bar{X}_x$ is eliminated using equation (3.7). Combining these conditions the value of c is found to be

$$c = \bar{U}(0) + \frac{\sin\theta g_x}{2\nu} h^2 + \frac{\Sigma h^2}{2\mu} - \frac{ik}{3\mu}(\rho\ g \cos\theta + \Gamma_0 k^2 - \pi\ ) h^3$$

$$+\frac{2\ ik \sin\theta\ g_x}{15}\left(\frac{\tau_x}{\mu} + \frac{\rho\sin\theta g_x}{\mu} h\ \right) h^5 \tag{3.25}$$

$$-\frac{ik}{\nu}\left(\gamma_1 - \frac{ik \sin\theta\ g_x\ h^2}{2\nu}\kappa\right)\left(h + \frac{\tau\nu}{\mu \sin\theta\ g_x}\right)$$

On separating this equation into real and imaginary parts and setting $c = c_r + ic_i$, $\pi = \pi_r + i\pi_i$ and $\Sigma = \Sigma_r + i\Sigma_i$ the following expressions for c_r and c_i are obtained

$$c_r = \bar{U}(0) + \frac{\sin\theta g_x}{2\nu} h^2 + \frac{\Sigma_r}{2\mu} h^2 - \frac{k\pi_i}{3\mu} h^3 - \frac{k^2 \sin\theta g_x \kappa}{2\nu^2}\left(h^3 + \frac{\tau\nu h^2}{\mu\sin\theta g_x}\right) \tag{3.26}$$

$$c_i = \frac{\Sigma_i}{2\mu} h^2 - k(\ \rho g\cos\theta + \Gamma_0 k^2 - \pi_r)\ \frac{h^3}{3\mu} + \frac{2k\sin\theta g_x}{15\nu^2}\left(\frac{\tau_x}{\mu} + \frac{\rho\sin\theta g_x h}{\mu}\right) h^5$$

$$- \frac{k}{\nu}\gamma_1\left(h + \frac{\tau_x\ \nu}{\mu\sin\theta g_x}\right) \tag{3.27}$$

The stability of the flow may now be investigated. When $c_i \neq 0$ the amplification rate or the rate of damping is given by kc_i. Instability occurs when

$$\frac{3\Sigma_i}{2kh} + \pi_r - \frac{3\rho}{h^3}\left(\ h + \frac{\tau_x}{\rho\sin\theta g_x}\right)\gamma_1 + \frac{2}{5\nu^2}\sin\theta g_x\ h^2\ (\tau_x + \rho\sin\theta g_x h) >$$

$$\rho\ g\ \cos\theta + \Gamma_0 k^2 \tag{3.28}$$

In order to proceed further, we must introduce actual expressions for π_r and π_i. The expressions to be used are taken from Benjamin's (1959) paper as mentioned in Chapter 1. These expressions are only estimates. However, it has been shown by Craik (1965) that these estimates may be applicable, in a physical situation, to turbulent airflows in a channel of finite height provided this height is large compared with the wavelength of the disturbance.

With the subscript a denoting a property of the airflow, the primary motion of the liquid is related to that of the air by the equation

$$\mu\,(D\bar{U})_{z=0} = \mu_a\,(DU_a) = \tau_x \tag{3.29}$$

where $U_a(z)$ is the mean air velocity at the interface in the x-direction. The normal stress component in phase with the wave displacement is as given by Craik (1965)

$$\tau_r = \frac{\rho\bar{U}(0)\,k}{h}\;\frac{I}{c_f}, \qquad I = \int_0^\infty \left(\frac{U_a}{U_o}\right) e^{-kz}\,k\,dz \tag{3.30}$$

where the limits of the integral, I, denote the boundaries of the air flow and U_o is max. $|U_a|$. c_f is a friction coefficient defined by

$$\nu_a\,(DU_a) = c_f\,U_o^{\,2}$$

The tangential stress in phase with the wave slope is given by $\Sigma_i\eta$ where

$$\Sigma_i = \frac{\beta I}{c_f}\,\rho\left(\frac{\nu^4\bar{U}(0)^2k^5}{h^2}\right)^{\frac{1}{3}}$$

where the quantities $\frac{I}{c_f}$ and β may be regarded as constants which depend on the properties of the air flow profile such as its shape but not its magnitude. Craik (1965) has shown that these estimates are accurate enough for such a situation as this. He demonstrated their form for the case where the velocity profile U_a obeyed the $\frac{1}{7}$th power law - a situation occurring in turbulent flow. Better methods for calculating these surface stresses were given by Miles (1962). However, the estimates used here are sufficient for our present purposes.

Equation (3.28) for the neutral case, $c_i = 0$, then becomes

$$g\cos\theta + \frac{\Gamma_0 k^2}{\rho} = \frac{I}{c_f}\left\{ \frac{\nu k}{h}\bar{U}(0) + \frac{3}{2}\beta\left(\frac{\nu^4\bar{U}(0)^2k^2}{h^5}\right)^{\frac{1}{3}}\right\} - \frac{3}{h}\left(h + \frac{\tau x}{\rho\sin\theta g_x}\right)\gamma_1 + \frac{2}{5\nu^2\rho}\sin\theta g_x h^2(\tau_x + \rho\sin\theta g_x h)$$

which may be re-written in terms of k as

$$\frac{3}{2}\frac{I\beta}{c_f}\left(\frac{\nu^4\bar{U}(0)^2}{h^5}\right)^{\frac{1}{3}} = k^{-2/3}\left\{ G - \frac{I\nu\bar{U}(0)}{c_f\, h}k + \frac{\Gamma_0 k^2}{\rho}\right\} \tag{3.32}$$

where $G = g\cos\theta + \frac{3}{h^3}\left(h + \frac{\tau}{\rho\sin\theta g_x}\right)\gamma_1 - \frac{2\sin\theta g_x h^2}{5\nu^2}\left(\frac{\tau}{\rho}x + \sin\theta h g_x\right)$

$\frac{\partial h}{\partial k}$ vanishes when

$$\frac{4\Gamma_0}{\rho}k^2 - \frac{I}{c_f}\frac{\nu\bar{U}(0)}{h}k - 2G = 0. \tag{3.33}$$

Only one root of the quadratic (3.33) is positive. Thus, for a prescribed surface velocity, $\bar{U}(0)$, there is a wavenumber given by (3.33) for which h has a critical value. Depending on the value of $\bar{U}(0)$ and G, this critical value h denotes a film thickness immediately above or below which disturbances are amplified. The root of equation (3.33) which is greater than zero is

$$k_c = \left(\frac{G\rho}{2\Gamma_0}\right)^{\frac{1}{2}}\left\{\phi + \sqrt{1-\phi^2}\right\} \tag{3.34}$$

where $$\phi = \left(\frac{I\nu\bar{U}(0)}{8c_f\, h}\right)\sqrt{\frac{2\rho}{G\Gamma_0}} \tag{3.35}$$

On substituting for k_c in equation (3.32), the expression for h_c becomes

$$\frac{h}{\beta}\left(\frac{c_f G}{2I\nu^2}\right)^{\frac{1}{3}} = \frac{\phi^{2/3}\left(\phi + \sqrt{1+\phi^2}\right)^{2/3}}{1 - \phi^2 - \phi\sqrt{(1+\phi^2)}} = \mathcal{F}(\phi) \tag{3.36}$$

This function $\mathcal{F}(\phi)$ of the dimensionless parameter ϕ is the same as that obtained by Craik (1966), Fig. 11, and the values of this function computed there may be used.

The results shown in figures (3.2) and (3.3) for neutral stability were obtained from equations (3.34), (3.35) and (3.36) for $\gamma_1 = 0$, $\beta = 0{\cdot}76$ and $I/c_f = 220$,

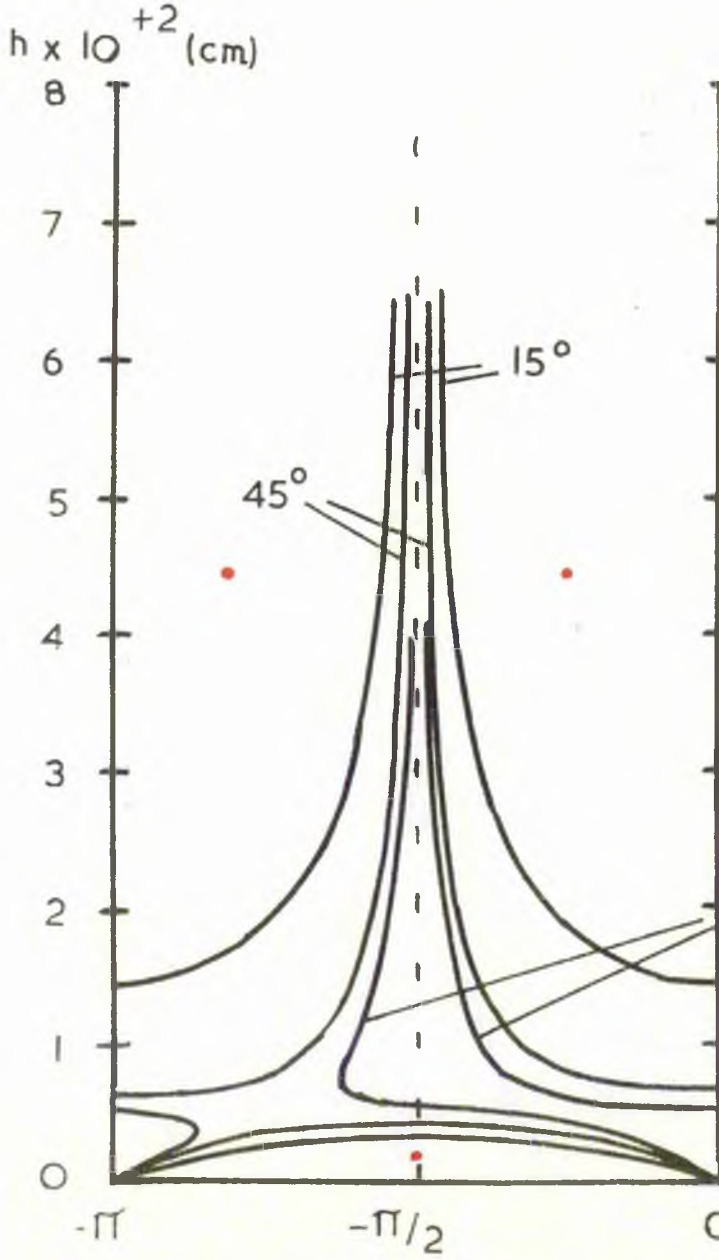
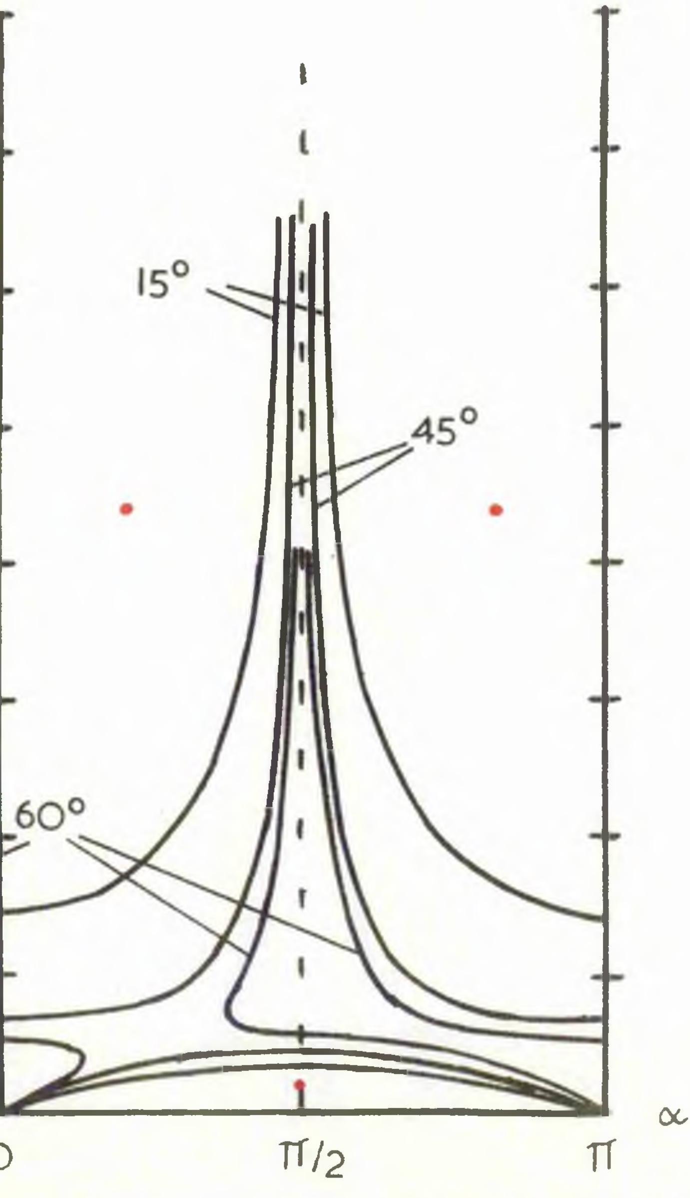


Figure 3.2: Neutral stability curves
constant wind stress.



for different angles of inclination with

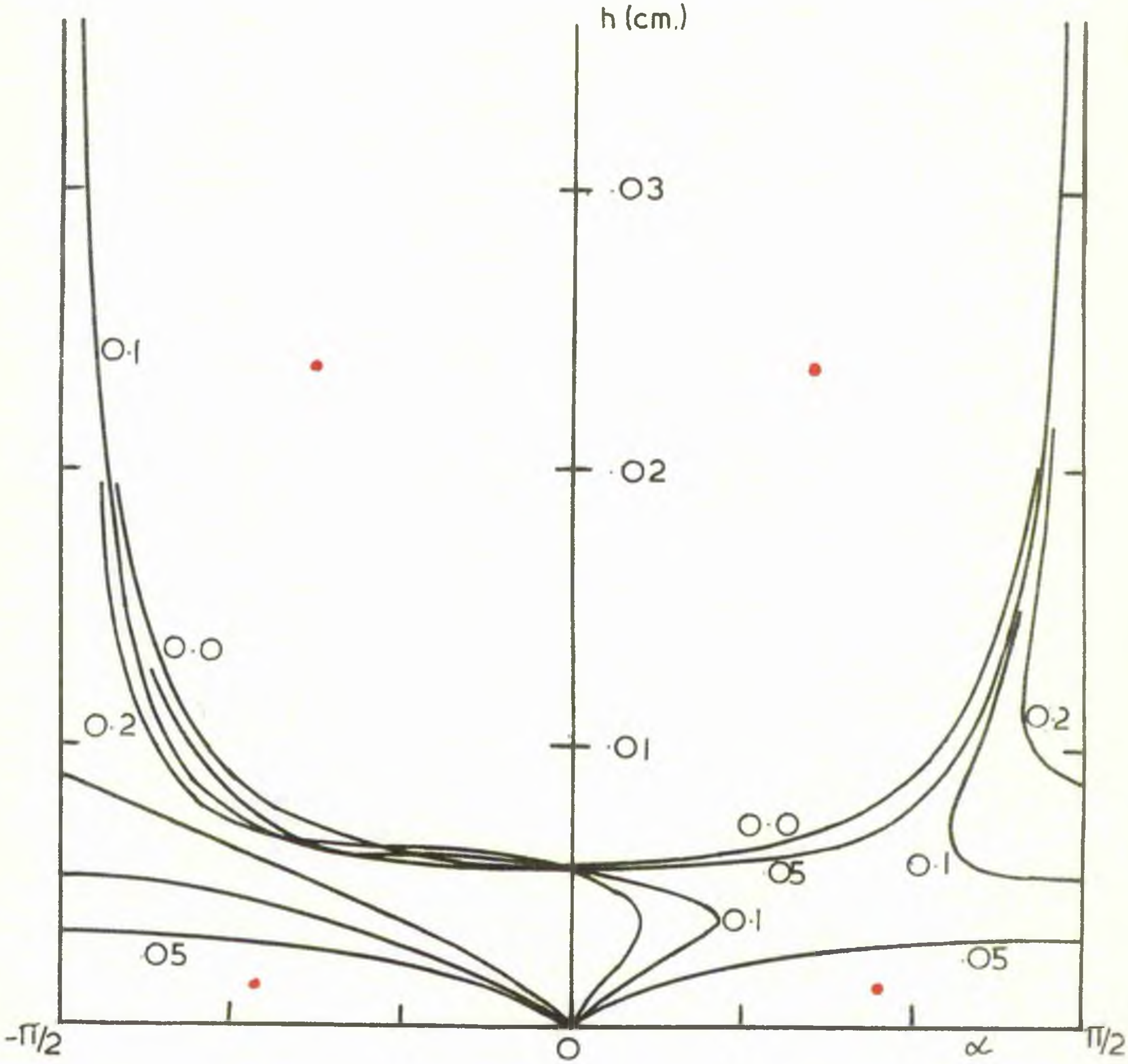


Figure 3.3: Neutral stability curves for different values of wind stress with constant angle of inclination.

using the I.B.M. 1620 of St. Andrews University. In these results the wind stress was taken perpendicular to the component of gravity down the plane. This was done for brevity. However, curves can easily be calculated for any angle of the air flow to the line of greatest slope.

In figure (3.2), the film thickness, h_c, is plotted against the angle of wave propagation, α, for different angles of inclination ($\theta = 15^\circ, 45^\circ, 60^\circ$) with constant wind stress $\tau = 0.1$. The two unlabelled curves nearest to the α-axis depict the other parts of the 45° and 15° neutral curves, the one reaching the higher value of h being that for 45°.

In figure (3.3), the film thickness is plotted against the angle of wave propagation for different values of the wind stress ($\tau = 0, 0.05, 0.1, 0.2$) with constant angle of inclination ($\theta = 60^\circ$). The unstable regions for each curve in both figures are those which contain the red spot.

The following features are of interest. A wave with wave number in the direction $\alpha + \pi$ and velocity $-c_r$ is identical to one with wavenumber in the direction α and velocity c_r thus giving the symmetry in the figures. The interaction of the gravity and wind stress forces produce asymmetrical curves about $\alpha = \pi/2$ which are almost symmetrical when either θ or τ is small but quite asymmetrical when either θ or τ is large. In this latter case the unstable regions overlap each other.

It is thus found that for films thicker than 0.02 cm., the instability is a gravitational one, whilst under 0.004 cm. the instability is primarily due to the wind stress forces. Thus the instability mechanism found by Craik to exist at extremely thin films only becomes dominant at these extremely thin films. Between these two regions i.e. for values of h near to 0.006 cm., the instability curves may link up with each other, increasing the region of instability. For those cases where the effect of gravity and wind stress are comparable, there are no values of the film thickness for which the situation will be stable.

It is also of interest to examine how previous results are incorporated into the above theory.

§3.5: A comparison with previous work.

(i) Flow down an inclined plane without surface contamination.

Benjamin (1957); Yih (1963).

For this case, $\underline{\tau} = 0$, $\Sigma = \pi = \gamma_1 = \kappa = 0$ and equation (3.2) yields the velocity profile

$$\bar{U} = \frac{\sin\theta}{2\nu} (h^2 - z^2) g_x$$

From equation (3.26) it is found that

$$c_r = \bar{U}(0) + \frac{\sin\theta h^2}{2\nu} g_x$$

$$= \frac{\sin\theta h^2}{\nu} g_x = 2\bar{U}(0)$$

i.e. the wave's velocity is twice that of the surface- the result found by both Benjamin and Yih.

Again, from equation (3.27) it is found that

$$c_i = \frac{2k}{15\nu^3} (\sin\theta g_x h^3)^2 - \frac{kh}{3\mu} (\rho g\cos\theta + \Gamma_0 k^2)$$

$$= \frac{8kh^2}{15\nu} \bar{U}(0)^2 - \frac{kh^3}{3\nu} (g\cos\theta + \frac{\Gamma_0}{\rho} k^2)$$

which is the result obtained previously by Benjamin. The stability criterion follows immediately i.e. the flow is unstable when

$$\frac{8}{5}\,\bar{U}(0)^2 \;>\; h\,(\,g\cos\theta + \frac{\Gamma_0}{\rho}\,k^2\,)$$

The equality condition of this equation (the neutral stability case) for the restricted case $\theta = 60^\circ$, $a = 0^\circ$ is represented in the figure (3.3) by the point where the curves cut the h-axis.

<u>(ii) Flow down an inclined plane with a constant tangential force acting on the surface.</u>

Lyubutskaya (1961).

Here, we will extract the results of Lyubutskaya. She took constant τ on the surface $z = 0$, not on the perturbed surface. In this case $\Sigma = v = \gamma_1 = \kappa = \Gamma_0 = 0$ and the basic velocity profile

$$\bar{U} = \frac{\sin\theta}{2\nu}\,(\,h^2 - z^2)g_x + (\,z + h\,)\,\frac{\tau_x}{\mu}$$

is obtained from equation (3.2) when only 2-dimensional motion in the x-direction is considered. From equation (3.26) the wave velocity

$$\begin{aligned} c_r &= \frac{\sin\theta h^2 g_x}{\nu} + \frac{h\tau_x}{\mu} \\ &= \bar{U}(0) + \frac{\sin\theta h^2 g_x}{2\nu} \end{aligned}$$

is obtained. The wave moves with a speed greater than the surface but less than twice the surface speed. This agrees with the result of Lyubutskaya which she obtained in a dimensionless system of co-ordinates.

From equation (3.27) it follows that

$$c_i = \frac{2kh^5\sin\theta g_x}{15\nu^2\mu} (\tau_x + \rho\sin\theta g_x h) - \frac{k\cos\theta h^3 g_x}{3\nu}$$

and instability occurs when

$$\frac{2\tan\theta h^2}{5\nu^2\rho} (\tau_x + \rho\sin\theta g_x h) > 1$$

again in agreement with Lyubutskaya. It is to be noted that this result cannot be represented within the figures (3.2) and (3.3) since the representation used for the perturbation stresses is inadequate.

<u>(iii) Wind-generated waves in horizontal liquid films.</u>

Craik (1966)

The liquid surface is taken to be horizontal and uncontaminated; thus,

$$g_x = \gamma_1 = \kappa = 0,$$

and

$$\bar{U} = \frac{(z + h)}{\mu} \tau_x$$

is the basic velocity profile obtained from equation (3.2) when only 2-dimensional motion in the x-direction is considered. Equations (3.26) and (3.27) reduce to

$$c_r = \bar{U}(0) + \frac{h^2}{2\mu} \Sigma_r - \frac{kh^3}{3\mu} \pi_i$$

$$c_i = \frac{h^2}{2\mu} \Sigma_i + \frac{kh^3}{3\mu} \pi_r - \frac{k}{3\mu} (\rho g + \Gamma_0 k^2)$$

in agreement with the previous results of Craik. Instability occurs when

$$\frac{3\Sigma_i}{2kh} + \pi_r > g\rho + \Gamma_0 k^2$$

(iv) Flow down an inclined plane with surface contamination.

Benjamin (1963)

Here, $\underline{\tau} = 0$, $\Sigma = \tau = 0$ hold. The basic velocity profile is as in (i) and equations (3.26) and (3.27) yield

$$c_r = \frac{\sin\theta h^2}{\nu} g_x - \frac{k^2 h}{\nu}\left(\frac{\sin\theta h^2 g_x}{2\nu} \right)\kappa$$

$$= 2\left(1 - \frac{k^2 h}{\nu}\kappa \right)\bar{U}(0)$$

$$c_i = \frac{kh^2}{\nu}\left(\frac{8}{15}\bar{U}(0)^2 - \frac{\gamma_1}{h} - \frac{h}{3}\left(g\cos\theta + \frac{\Gamma_0 k^2}{\rho} \right) \right)$$

and instability results when

$$\frac{8}{5}\bar{U}(0)^2 > \frac{3\gamma_1}{h} + h\left(g\cos\theta + \frac{\Gamma_0 k^2}{\rho} \right)$$

Agreement is made with Benjamin's dimensionless results.

(v) Wind-generated waves in contaminated liquid films.

Craik (1968)

In this case only $g_x = 0$. The velocity profile obtained in (iii) is unchanged and from equation (3.26) we have

$$c_r = \bar{U}(0).$$

Unfortunately this means that the analysis for the evaluation of c_i breaks down due to a factor $\frac{1}{U(0) - c}$ contained within the surface contaminating term, and, accordingly, the analysis has to be taken to the next approximation as in the work of Craik.

CHAPTER IV

STABILITY OF FREE SURFACE FLOWS WITH VISCOSITY STRATIFICATION.

In this chapter the stability of a liquid film which flows steadily down an inclined plane and wave generation by a concurrent air flow at a horizontal liquid surface are investigated. The liquid viscosity is allowed to vary with distance below its free surface. It is assumed that the viscosity becomes indefinitely large with distance from the surface, and that there is an 'effective depth' within which most of the motion occurs; but, otherwise, the viscosity distribution is arbitrary. It is shown that instabilities may occur which are similar to those known for thin liquid films discussed in the last section. This type of situation where viscosity can vary may be regarded as a model of a melting surface.

Although the fluid is not enclosed by a rigid lower boundary it is generally possible to define a length-scale h which, for hydrodynamic purposes, is a measure of the 'effective thickness' of the liquid layer. Such a length-scale may be specified in terms of the distribution of viscosity with depth. If the viscosity becomes very large at comparatively small depths - as is usually the case at melting surfaces - this length-scale is small, and the liquid might be expected to behave to some extent like a thin film on a rigid boundary.

Dimensionless variables are defined relative to the velocity V of the liquid surface, the constant density ρ of the liquid, and the length scale h which is as yet unspecified, but which is a measure of the 'effective thickness' of the liquid layer.

In the notation of Chapter 1, we may ignore the y-dependence as we are dealing with a 2-dimensional problem. Thus, dimensionless co-ordinates (x,z) are chosen such that the primary flow is in the x-direction, and z denotes the depth below the undisturbed free surface. (u,w) denote the dimensionless velocity components in the (x,z) directions. A sketch of this configuration is given in figure (4.1).

From equations (1.3), the appropriate equations for 2-dimensional motion then become

$$\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + F_x + R^{-1}\left[\frac{\partial}{\partial x}\left(2m\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left\{m\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right\}\right],$$

$$\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + F_z + R^{-1}\left[\frac{\partial}{\partial x}\left\{m\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right\} + \frac{\partial}{\partial z}\left(2m\frac{\partial w}{\partial z}\right)\right],$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \ ,$$

$$\frac{Dm}{Dt} = 0 \ , \qquad (4.1a,b,c,d)$$

where $\frac{D}{Dt}$ denotes the (dimensionless) material time derivative and equation (4.1d) expresses the constancy in viscosity of a fluid particle derived in equation (1.2).

If the direction of the primary flow is inclined at an angle θ to the horizontal, the dimensionless body force (F_x,F_z) is $(G\sin\theta,\ G\cos\theta)$ where $G = gh/V^2$ and g is gravitational acceleration.

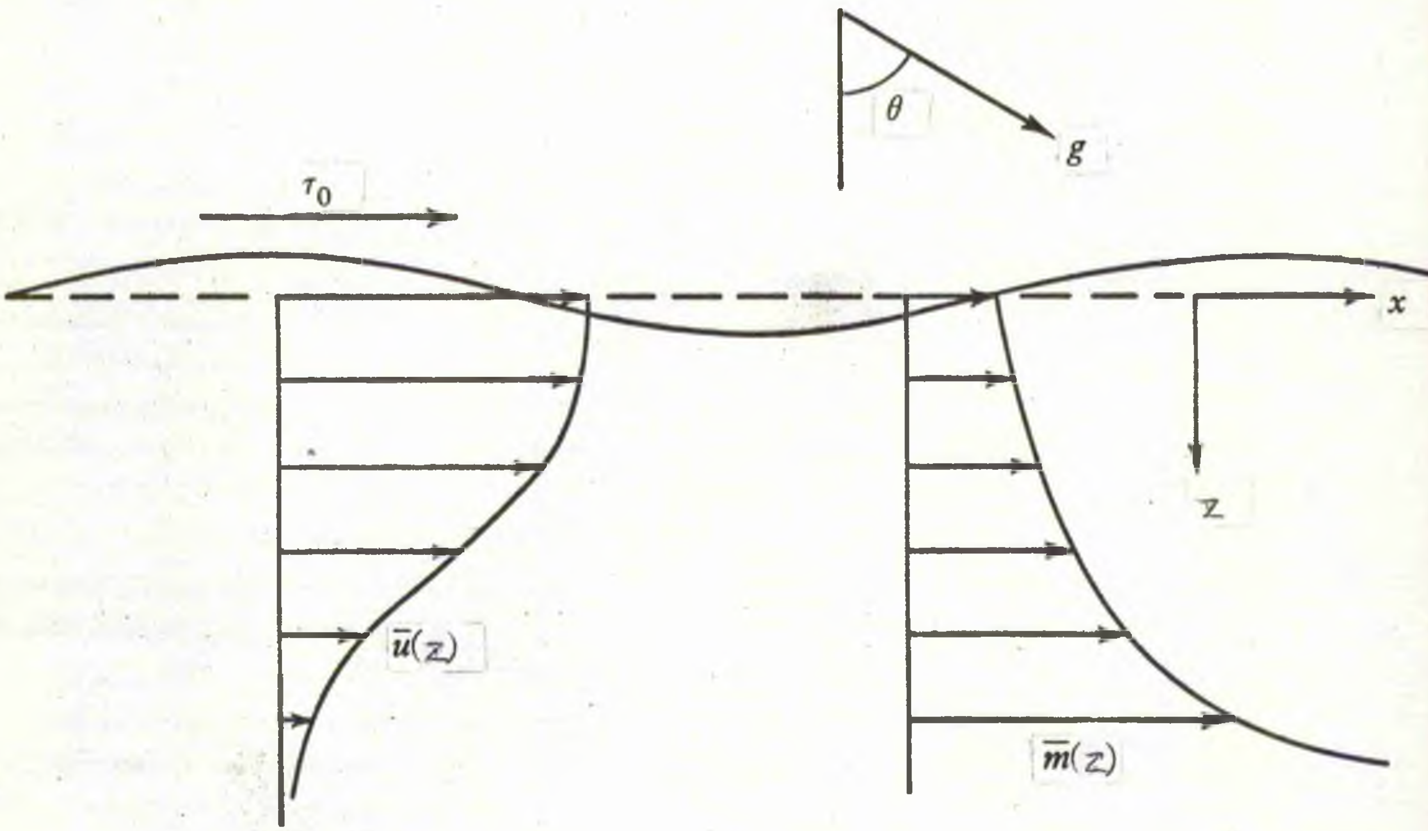


Figure 4.1: Sketch of flow configuration.

In the absence of any pressure gradient in the direction of motion, the primary flow is specified by

$$u = \bar{u}(z),\ v = 0,\ m = \bar{m}(z),\ p = \bar{p}(z) \quad ,$$

where

$$(\bar{m}\,\bar{u}')' = -\,GR\sin\theta \ , \quad \bar{p}' = G\cos\theta \qquad , \qquad (4.2a,b)$$

and where here the prime denotes differentiation with respect to z. (For convenience the "D" notation of Chapter 3 has been dropped.)

As the viscosity distribution $\bar{m}(z)$ can be regarded as a given property of the liquid, equation (4.2a) may be integrated using the boundary conditions

$$\bar{m}(0) = 1, \quad \bar{u}(0) = 1, \quad \bar{u}(\infty) = 0.$$

The corresponding velocity profile is then found to be

$$\bar{u}(z) = \left\{1 - GR\sin\theta \int_0^\infty \frac{\xi d\xi}{\bar{m}(\xi)}\right\} \int_z^\infty \frac{d\xi}{\bar{m}(\xi)} \Bigg/ \int_0^\infty \frac{d\xi}{\bar{m}(\xi)} + GR\sin\theta \int_z^\infty \frac{\xi d\xi}{\bar{m}(\xi)}$$

(4.3)

Now, if the liquid surface experiences a mean dimensional shear stress τ_0, we require that

$$\bar{u}'(0) = -\,R(\tau_0/\rho V^2) \quad , \qquad (4.4)$$

which yields the result

$$1 - GR\sin\theta \int_0^\infty \frac{\xi d\xi}{\bar{m}(\xi)} = R\,\frac{\tau_0}{\rho V^2} \int_0^\infty \frac{d\xi}{\bar{m}(\xi)} \ .$$

In dimensional form, this expression relates the surface velocity V in terms of the other flow quantities. Thus, for a horizontal film, where θ is zero,

$$V = \frac{\tau_0 h}{\mu_0} \int_0^\infty \frac{d\xi}{\bar{m}(\xi)} \qquad (4.5a)$$

while, for inclined flow under gravity with $\tau_0 = 0$,

$$V = \frac{\rho g h^2 \sin\theta}{\mu_0} \int_0^\infty \frac{\xi d\xi}{m(\xi)} \tag{4.5b}$$

§4.2: The stability problem.

We now consider that the primary flow experiences a small two-dimensional disturbance which is periodic in x. An extension of this analysis to three-dimensional disturbances can be effected by Squire's transformation as mentioned earlier in the first chapter. (also, see Yih (1955))

The normal disturbance of the liquid surface is represented by

$$z = \eta(x,t) = \delta e^{i\alpha(x-ct)} ,$$

where α is the dimensionless wave-number, taken to be real, and c the dimensionless wave velocity, which may be complex. The wave amplitude δ is assumed to be sufficiently small for the problem to be linearised.

The continuity condition (4.1c) permits the introduction of a perturbation stream function

$$\psi(x,y,t) = f(z)\eta(x,t)$$

such that the velocity components are

$$u = \bar{u}(z) + f'(z)\eta \qquad w = -i\alpha f(z)\eta \tag{4.6a,b}$$

Also the dimensionless pressure and viscosity are of the form

$$p = \bar{p}(z) + \hat{p}(z)\eta ,$$

$$m = \bar{m}(z) + \hat{m}(z)\eta .$$

On substituting these quantities into equations (4.1a,b,d) linearising and then eliminating the pressure term it is found that

$$\hat{p} = \bar{u}'f - (\bar{u} - c)f' + (i\alpha R)^{-1}\left\{-2\alpha^2\bar{m}f' + \left[\bar{m}(f'' - \alpha^2 f) + \hat{m}\bar{u}'\right]'\right\}, \quad (4.7a)$$

$$\begin{aligned} i\alpha R\,[(\bar{u} - c)(f'' - \alpha^2 f) - \bar{u}''f] = {} & \bar{m}(f^{iv} - 2\alpha^2 f'' + \alpha^4 f) + 2\bar{m}'(f''' - \alpha^2 f') \\ & + \bar{m}''(f'' + \alpha^2 f) + \hat{m}(\bar{u}''' + \alpha^2\bar{u}') \\ & + 2\hat{m}'\bar{u}'' + \hat{m}''\bar{u}' \quad , \end{aligned} \quad (4.7b)$$

$$(\bar{u} - c)\hat{m} = \bar{m}'f \quad . \quad (4.7c)$$

Equations similar to these were found by Drazin. When $\bar{m}(z) = 1$, the above equations reduce to those for a homogeneous fluid and, in particular, equation (4.7b) reduces to the Orr–Sommerfeld equation.

These equations are subject to five boundary conditions. The two boundary conditions at $z = \infty$ are

$$f(\infty) = f'(\infty) = 0 \; . \quad (4.8a,b)$$

In fact, if the flow is bounded by a rigid plane at some finite depth H, the adjustment necessary is to let $\bar{m}(z)$ become infinite for all $z \geq H$. Three boundary conditions remain at the liquid surface: two of these derive from the normal and tangential stress conditions at the surface and the third is the linearised kinematic condition which relates w and the surface displacement η. This last is

$$w = \frac{D\eta}{Dt} \quad \text{at } z = 0,$$

or,

$$f(0) = c - 1. \quad (4.9)$$

The dimensionless normal and tangential stress perturbations exerted by an air stream on the perturbed liquid surface are of the form

$$\sigma_{zz} = -\Pi\eta \quad , \qquad \sigma_{xz} = \Sigma\eta \quad , \tag{4.10a,b}$$

where Π and Σ are quantities, generally complex, which depend on the properties of the air flow and the surface disturbance. For present purposes, it is convenient to regard them as complex parameters, whose values may be estimated in particular cases. They assume the same role as those used by Craik (1966), the only difference being the sign of (4.10a) due to the opposite z direction used here.

The pressure p_i just inside the liquid surface is, to linearized approximation,

$$p_i = (\hat{p} + \bar{p}')\eta \qquad \text{at } z = 0 \tag{4.11}$$

and the linearized condition that the capillary pressure and normal stresses on either side of the surface should be in equilibrium is

$$-p_i + \frac{2}{R}\frac{\partial v}{\partial z} = -T\frac{\partial^2\eta}{\partial x^2} + \sigma_{zz} \quad \text{at } z = 0$$

where $T = \gamma(\rho h V^2)^{-1}$ and γ is the coefficient of surface tension at the liquid surface. Using the results (4.2b), (4.6b), (4.10a) and (4.11), this boundary condition becomes

$$-\hat{p}(0) - 2i\alpha R^{-1}f'(0) = (G\cos\theta - \Pi + \alpha^2 T) \quad , \tag{4.12}$$

where $\hat{p}(0)$ is given by result (4.7a).

The dimensionless shear stress in the liquid is

$$\tau = \frac{m}{R}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\right) .$$

Consequently, to linearized approximation, the tangential stress perturbation τ_i just inside the liquid surface satisfies

$$R\,\tau_i = \left[(\overline{mu}')' + \hat{m}\bar{u}' + \bar{m}(f'' + \alpha^2 f)\right]\eta, \text{ at } z = 0.$$

Also, τ_i must equal the tangential stress perturbation σ_{xz} exerted by the air flow; and on using results (4.7c),(4.9) and (4.10), this condition becomes

$$f'''(0) + [\,\bar{u}''(0) - RE\,(c - 1)^{-1} + \alpha^2\,]\,f(0) = 0. \qquad (4.13)$$

The linearized characteristic-value problem is now completely specified by the equations (4.7a,b,c), together with the five boundary conditions (4.8a,b),(4.9),(4.12) and (4.13). Its solution yields an eigenvalue equation for c, the complex wave velocity, in terms of α and the other parameters of the problem.

§4.3: The long-wave approximation.

At this stage, it is convenient to introduce some approximations, similar to those made in analogous work on uniform films. We assume that

$$\alpha^2 \ll 1, \qquad \alpha R, \quad \alpha R|c| \ll 1 \qquad (4.14a,b)$$

Later in the chapter, relaxation of the conditions (4.14b) is discussed. These conditions require that the wavelength of the disturbance is large compared with the 'effective depth' h, and that, in the equations of motion, the inertia terms are small compared with the viscous terms. A first approximation to equation (4.7b) is then

$$(\,\bar{m}f'' + \bar{u}'\hat{m}\,)'' = 0.$$

On integrating twice and using result (4.7c) this becomes

$$\bar{m}f'' + \frac{\bar{u}'\bar{m}'}{\bar{u} - c}\,f = Az + B \qquad (4.15)$$

where A and B are constants of integration which may be determined from the boundary conditions. The corresponding approximations to the boundary conditions (4.12) and (4.13) are

$$(\bar{m}f'' + \frac{\bar{u}'\bar{m}'}{\bar{u} - c} f)' = -i\alpha R(G \cos\theta - H + \alpha^2 T) \qquad (z = 0)$$

$$f'' = RI - \bar{u}'' \qquad (z = 0)$$

where results (4.7a) and (4.9) have been used. It is important to note that, although α^2 and αR are small, the term on the right hand side of the former boundary condition must be retained, since the parameters G, H, and T may be large. It follows from these boundary conditions that the constants A and B are

$$A = -i\alpha R(G \cos\theta - H + \alpha^2 T) \, , \tag{4.16a}$$

$$B = RI - (\overline{mu'})'_{z=0} = RI + RG \sin\theta \, . \tag{4.16b}$$

With these values, it remains to solve equation (4.15) subject to the boundary conditions (4.8a,b) and (4.9). This is done in the remainder of this chapter for two particular problems; namely, the onset of wind-generated waves on horizontal flows and the stability of inclined flows under gravity. The solution of the former problem yields the stability criterion to good approximation; but, for the latter, a better approximation is required which incorporates the highest order inertia terms from the equations of motion.

An equation similar to (4.15) but with A = B = 0 is discussed by Drazin. Two linearly independent solutions of this equation may be expressed as series of multiple integrals; but these are not required in the present work. Drazin also noted that the equation has a simple solution when $\overline{mu'}$ is constant; and this case is examined in the next section.

4.4: Wind-generated waves on a horizontal flow.

The problem of wind-generated waves on a horizontal flow is now examined.

If the primary flow is horizontal, $\theta = 0$ and the motion is entirely due to the mean tangential stress τ_0 exerted by the air flow at the liquid surface. Then, from results (4.2) and (4.4)

$$\overline{m}\overline{u}' = - R(\tau_0/\rho V^2) \equiv D \ , \tag{4.17}$$

where D is a constant. For this case,

$$\frac{\overline{m}'\overline{u}'}{\overline{m}(\overline{u} - c)} = - \frac{\overline{u}''}{\overline{u} - c} \ ,$$

and equation (4.15) becomes

$$(\overline{u} - c)f'' - \overline{u}''f = \frac{\overline{u}'(\overline{u} - c)}{D}(Az + B) \equiv F(z) \ .$$

First and second integrals of this equation, which satisfy the boundary conditions (4.8a,b) are

$$(\overline{u} - c)f' - \overline{u}'f = - \int_z^{\infty} F(z_1)dz_1 \ ,$$

$$f = (\overline{u} - c)\int_z^{\infty}(\overline{u} - c)^{-2}dz_1\int_{z_1}^{\infty} F(z_2)dz_2 \ .$$

This latter expression, together with the remaining boundary condition (4.9) yields

$$\int_0^{\infty}(\overline{u} - c)^{-2}dz\int_z^{\infty} F(z_1)dz_1 = -1 \ ,$$

and integrations by parts lead to the result

$$AI_1 + BI_2 = 2D \ , \tag{4.18}$$

where

$$I_1(c) = \int_0^{\infty} (\bar{u} - c)^{-2} \left\{ \int_z^{\infty} [(\bar{u} - c)^2 - c^2]\, dz - m [(\bar{u} - c)^2 - c^2] \right\} dz \; ,$$

$$I_2(c) = \int_0^{\infty} (\bar{u} - c)^{-2} [(\bar{u} - c)^2 - c^2]\, dz \; .$$

Since A, B and D are known, and $\bar{u}(z)$ may be found for each prescribed viscosity distribution $\bar{m}(z)$, the complex wave velocity c may be determined from this equation. In the following, the real and imaginary parts of complex quantities will be denoted by the subscripts r and i.

The main interest is to determine the conditions for neutral stability when c is real. However, if c is real and less than unity, the integrals I_1 and I_2 are singular at $z = z_c$, where $\bar{u}(z_c) = c$. As mentioned before, this possibility is not dealt with here (see Craik (1969)). Instead, attention is restricted to those disturbances with $c_r > 1$, for which I_1 and I_2 are real when $c_i = 0$.

When $c_i = 0$, the real and imaginary components of equation (4.18) become

$$\begin{aligned} A_r I_1(c_r) + B_r I_2(c_r) &= 2D \; , \\ A_i I_1(c_r) + B_i I_2(c_r) &= 0 \; . \end{aligned} \qquad (4.19a,b)$$

Also, equations (4.16a,b) reveal that

$$\begin{aligned} A_r &= -\alpha R H_i , & A_i &= -\alpha R(G - H_r + \alpha^2 T), \\ B_r &= R I_r \; , & B_i &= R I_i \end{aligned} \qquad (4.20)$$

where the subscripts denote real and imaginary parts.

Particular estimates for H and I, which derive from earlier work of Benjamin (1959), are given by Craik (1966), where their range of validity is also discussed. Here, we need only mention that $R|\Sigma|$ and $\alpha R|\Pi_i|$ are typically small compared with unity, that Π_i is negative and that Π_r, Σ_r, and Σ_i are positive. In the following it is therefore permissible to assume that

$$R|\Sigma|, \quad \alpha R|\Pi_i| << 1, \qquad (4.21a,b)$$

as was done by Craik (1966). These assumptions again enable a simple solution to be found.

With these assumptions, $|A_r|$ and $|B_r|$ are small compared with unity; whereas, with an appropriate choice of length scale h, the value of D is O(1). Therefore, if equation (4.19) is to be satisfied, either I_1 or I_2 must be large compared with unity. However, for disturbances with $c_r > 1$, it is easily verified that the magnitudes of I_1 and I_2 are O(1) except when c_r is close to unity. (It should be recalled that, with an appropriate length scale h, $\bar{u}$ is very small for depths z greater than O(1), due to the large values of the viscosity there: the main contributions to the integrals I_1 and I_2 then derive from a layer near the surface whose dimensionless depth is O(1).) If $c_r = 1 + \epsilon$, say where ϵ is a small positive quantity, the integrands become very large near z = 0, and their contributions to I_1 and I_2 are correspondingly increased. Therefore, in order that I_1 or I_2 may be sufficiently large to satisfy equation (4.19a) it is necessary that $c_r = 1+$. This result is similar to that for uniform thin films (see Craik 1966, equation 7.2), that the phase velocity of infinitesimal waves is nearly equal to the velocity of the liquid surface.

This result enables simplification of equation (4.19b). On writing $c = c_r = 1 + \varepsilon$ in I_1 and I_2, and examining the contributions to these integrals from the vicinity of $z = 0$, it is easily verified that

$$I_1(1 + \varepsilon) = \frac{1}{\bar{u}'(0)\varepsilon} \int_0^\infty \bar{u}(2 - \bar{u})dz + O(\log\varepsilon) ,$$

$$I_2(1 + \varepsilon) = \frac{1}{\bar{u}'(0)\varepsilon} + O(1)$$

On retaining only the highest order terms in ε, equation (4.19b) becomes

$$\frac{B_i}{A_i} = \int_0^\infty \bar{u}(\bar{u} - 2)dz$$

or, from equation (4.20),

$$(G - \Pi_r + \alpha^2 T) \int_0^\infty \bar{u}(2 - \bar{u})dz = \Sigma_i/\alpha .$$

To good approximation, this equation represents the condition for neutral stability. In terms of dimensional quantities, it becomes

$$(\rho g - P_r + k^2\gamma) \int_0^\infty U(2V - U)dz' = (kh)^{-1}V^2hT_i , \qquad (4.22)$$

where P_r and T_i are dimensional stress parameters defined by

$$P_r = (\rho V^2h^{-1})\Pi_r , \qquad T_i = (\rho V^2h^{-1})\Sigma_i ,$$

and $U(z')$ denotes the dimensional primary velocity profile as a function of the actual depth z'.

The corresponding result for a uniform film of thickness h is (Craik, equation 7.1b)

$$\rho g - P_r + k^2\gamma = \tfrac{1}{2}(kh)^{-1}T_i \quad , \qquad (4.23)$$

and this may be recovered from (4.22) by setting

$$U = V(1 - z'/h) \; , \qquad 0 \leq z' \leq h \; ,$$

$$U = 0 \qquad , \qquad z' > h.$$

The similarity between results (4.22) and (4.23) permits a very convenient definition of the 'effective depth' h of liquids with viscosity stratification. (Up till now, h has not been precisely defined.) For, if h is taken to be

$$h = \tfrac{1}{2} V^{-2} \int_0^\infty U(2V - U)dz' \quad , \qquad (4.24)$$

these equations become identical. Then, the criterion for neutral stability with viscosity stratification is precisely the same as that for uniform films.

When c_i is non-zero, but is sufficiently small that

$$|c_i| \ll |c_r - 1|, 1 \; ,$$

an analysis similar to that above yields the approximate result

$$\alpha R c_i = \tfrac{1}{2}(\alpha R)^2 \left\{ (\Sigma_i/\alpha) - (G - \Pi_r - \alpha^2 T) \int_0^\infty (2 - \bar{u})\bar{u}\, dz \right\}$$

with the above choice of h, this equation is also identical to the corresponding result (Craik, equation 7.3) for uniform films.

From (4.24) and (4.17), the 'effective depth' h may be expressed in terms of the dimensional viscosity distribution $\bar{\mu}(z')$, as

$$h = \frac{1}{2} \int_0^\infty G(z')dz' ,$$

$$G(z') = 1 - \left\{ \int_0^{z'} [\bar{\mu}(\xi)]^{-1} d\xi \Bigg/ \int_0^\infty [\bar{\mu}(\xi)]^{-1} d\xi \right\}^2 .$$

With this value of h, and with V given by (4.5a), the conditions (4.14a,b) may be expressed in terms of the wave-number k, the mean shear stress τ_0 and the viscosity distribution $\bar{\mu}(z')$.

It is clear that all the results obtained by Craik (1966) for uniform films are directly applicable to the present situation. Accordingly, further details need not be presented here: instead the reader is referred to §§7 – 10 of Craik's paper, which examine the neutral case, the stability curves and the range of validity of the approximations. Also, following Craik (1968), the above analysis may easily be extended to include the effects of contamination by insoluble surface-active agents.

Finally, it should be recalled that the stability criterion examined here refers only to 'surface-wave' modes. As mentioned in the introduction, there remains the possibility of unstable 'internal-wave' modes with $0 < c_r < 1$, similar to those examined by Yih (1967) for two superposed liquid layers.

§4.5: Inclined flow under gravity.

In this section we consider the primary flow to be entirely due to the body force component $G\sin\theta$ in the direction of motion. Since the air flow is absent, the mean shear stress τ_0 and the stress perturbations σ_{xz} and σ_{zz} are zero. From results (4.2a) and (4.4)

$$\bar{m}\bar{u}' = -GR\sin\theta\, z\ ,$$

and

$$\frac{\bar{m}'\bar{u}'}{\bar{m}(\bar{u}-c)} = \frac{(\bar{u}'/z) - \bar{u}''}{\bar{u}-c}\ .$$

Equation (4.15) is therefore

$$f'' + \frac{(\bar{u}'/z) - \bar{u}''}{\bar{u}-c}f = -\frac{(\bar{u}'/z)}{GR\sin\theta}(Az + B)\ , \qquad (4.25)$$

and the appropriate values of A and B are, from (4.16a,b)

$$A = i\alpha R(\,G\cos\theta + \alpha^2 T)\ , \qquad B = RG\sin\theta\ .$$

We now require the solution $f(z)$ and the wave velocity c which satisfy (4.25) and the boundary conditions (4.8a,b) and (4.9), for a prescribed velocity profile $\bar{u}(z)$.

For the present problem, the depth scale h is not yet precisely defined. Without loss of generality, it may be chosen such that

$$\int_0^{\infty} \frac{\xi d\xi}{\bar{m}(\xi)} = 1\ ; \qquad (4.26)$$

and the result (4.5b) then becomes

$$V = \frac{\rho g h^2 \sin\theta}{\mu_0}\ ,$$

or, in dimensionless form,

$$RG\,\sin\theta = 1\ . \tag{4.27}$$

It follows that B may be taken as unity in equation (4.25); this step being equivalent to defining the length scale h.

When α is sufficiently small, $|A|$ is small compared with $|B|$, and a first approximation to equation (4.25)- which is itself an approximation to equation (4.7b) - may be obtained on setting A equal to zero. Denoting the approximate solutions by the subscript zero, it is required to find $f_0(z)$ and c_0 which satisfy the equation

$$f_0'' + \left[\frac{\bar{u}'/z - \bar{u}''}{\bar{u} - c_0}\right] f_0 = -(\bar{u}'/z) \tag{4.28}$$

and the boundary conditions

$$f_0(\infty) = f_0'(\infty) = 0\ ,$$

$$f_0(0) = c_0 - 1\ .$$

The associated viscosity variation $\hat{m}_0(z)$ is ob tained from equation (4.7c) as

$$\hat{m}_0(z) = \frac{\bar{m}' f_0}{\bar{u} - c_0} = \frac{\bar{u}''z - \bar{u}'}{\bar{u}'^2(\bar{u} - c)}\, f_0\ . \tag{4.29}$$

For prescribed values of $\bar{u}(z)$, the appropriate solutions for $f_0(z)$ and c_0 may be obtained numerically, using an iteration procedure. To this approximation, c_0 is real and the solution represents a neutrally stable wave.

In order to derive the stability criterion, second approximations for f(z) and c to second order are required. For this, the term in A and the highest-order inertia terms must be included in the analysis. The inclusion of the latter requires that equations (4.7) be re-examined. After the method of Yih (1963), the following are written

$$f(z) = f_0(z) + i\alpha R f_1(z) \ ,$$

$$\hat{m}(z) = \hat{m}_0(z) + i\alpha R \hat{m}_1(z) \ ,$$

$$c = c_0 + i\alpha R c_1 \ .$$

These expressions may be regarded as the leading terms in a power series expansion in terms of the small parameters αR and α^2, when α^2 is also small compared with αR. The retention of only those terms of equation (4.7b) which are $O(\alpha R)$ means that

$$(\bar{m} f_1'' + \bar{u}' \hat{m}_1)' = (\bar{u} - c_0) f_0'' - \bar{u}'' f_0 \ , \tag{4.30}$$

$$\hat{m}_1 = \frac{\bar{m}' f_1}{\bar{u} - c_0} + \frac{\bar{m}' f_0 c_1}{(\bar{u} - c_0)} \ . \tag{4.31}$$

First and second integrals of (4.30) are

$$(\bar{m} f_1'' + \bar{u}' \hat{m}_1)' = (\bar{u} - c_0) f_0' - \bar{u}' f_0 + C \tag{4.32}$$

$$\bar{m} f_1'' + \bar{u}' \hat{m}_1 = \int_0^z \left[(\bar{u} - c_0) f_0' - \bar{u}' f_0 \right] dz_1 + Cz + D \ , \tag{4.33}$$

where C and D are disposable constants. To the same order of approximation the boundary conditions (4.12), (4.13) and (4.9) yield

$$(\bar{m} f_1'' + \bar{u}' \hat{m}_1)' = (1 - c_0) f_0' + G \cos\theta + \alpha^2 T \qquad (z = 0)$$

$$\bar{m} f_1'' + \bar{u}' \hat{m}_1 = 0 \qquad (z = 0)$$

$$f_1 = c_1 \qquad (z = 0)$$

The first two of these boundary conditions, together with equations (4.32) and (4.33) determine C and D to be

$$C = G \cos\theta + \alpha^2 T, \qquad D = 0.$$

On substituting in equation (4.33) for C, D, $\bar{m}$ and $\hat{m}_1$, the following results from an integration by parts

$$f_1'' + \left\{ \frac{(\bar{u}'/z) - \bar{u}''}{\bar{u} - c_0} \right\} f_1 = - c_1 \left\{ \frac{(\bar{u}'/z) - \bar{u}''}{(\bar{u} - c_0)^2} \right\} f_0 - (\bar{u}'/z) I - \bar{u}'(G \cos\theta + \alpha^2 T) \ , \tag{4.34}$$

where

$$I \equiv \int_0^z [(\bar{u} - c_0) f_0' - \bar{u}' f_0] \, dz_1$$

$$= (\bar{u} - c_0) f_0 + (1 - c_0)^2 - 2 \int_0^z \bar{u}' f_0 \, dz_1 \tag{4.35}$$

Also, from equation (4.28),

$$(\bar{u} - c_0) z f_0' + (\bar{u} - z\bar{u}'') f_0 = - \bar{u}'(\bar{u} - c_0) \ ;$$

and this equation may be integrated from 0 to z, to find, after integration by parts, that

$$3 \int_0^z \bar{u} f_0 \, dz_1 = (\bar{u} - c_0 + \bar{u}' z) f_0 - (\bar{u} - c_0) z f_0' - \tfrac{1}{2}(\bar{u} - c_0)^2 + \frac{3}{2}(1 - c_0)^2 \ . \tag{4.36}$$

On using results (4.35) and (4.36), equation (4.34) becomes

$$f_1'' + \left\{ \frac{(\bar{u}'/z) - \bar{u}''}{\bar{u} - c_0} \right\} f_1 = - c_1 \left\{ \frac{(\bar{u}'/z) - \bar{u}''}{(\bar{u} - c_0)^2} \right\} f_0 - \bar{u}'(G \cos\theta + \alpha^2 T) + H(z), \tag{4.37}$$

where

$$H(z) \equiv \tfrac{1}{3}(\bar{u}'/z) [(\bar{u} - c_0 - 2\bar{u}' z) f_0 + 2(\bar{u} - c_0) z f_0' + (\bar{u} - c_0)^2] \ .$$

In addition, $f_1(z)$ and c_1 must be such that the boundary conditions

$$f_1(\infty) = f_1'(\infty) = 0, \qquad f_1(0) = c_1 \tag{4.38}$$

are satisfied.

With a given dimensionless velocity profile $\bar{u}(z)$, the first approximations c_0 and $f_0(z)$ may be calculated numerically from equation (4.28), as mentioned above. When these are known, a similar iterative procedure yields solutions c_1, $f_1(z)$ to the above problem, which correspond to chosen values of the parameter $(G \cos\theta + \alpha^2 T)$. It is clear that c_1 must be real; and, since the imaginary part of the wave velocity c is $i\alpha R c_1$, the sign of c_1 determines whether the wave is stable or unstable.

Such calculations have been carried out using the I. B. M. 1620 of the University of St. Andrews for the particular velocity profile

$$\bar{u}(z) = (1 + z)e^{-z} \qquad (4.39)$$

which occurs in the interesting case where the viscosity increases exponentially with depth. The corresponding dimensional and dimensionless viscosity distributions are

$$\mu(z') = \mu_0 \, e^{z'/h}, \qquad \bar{m}(z) = e^{z} \qquad (z, z' \geq 0) \,,$$

(It may be verified that the constant h used here is such that condition (4.27) is satisfied.) The value of c_0 was found to be

$$c_o = 1.81 \,,$$

which represents a neutral wave travelling with a velocity somewhat less than twice that of the liquid surface. This contrasts with the result of Benjamin and Yih for a uniform film, that the wave velocity is just twice that of the liquid surface. The solution $f_0(z)$ is shown in Figure (4.2).

In the next approximation, values of c_1 were found which correspond to several chosen values of $(G \cos\theta + \alpha^2 T)$. These results are shown in Figure (4.3). The curve of c_1 against

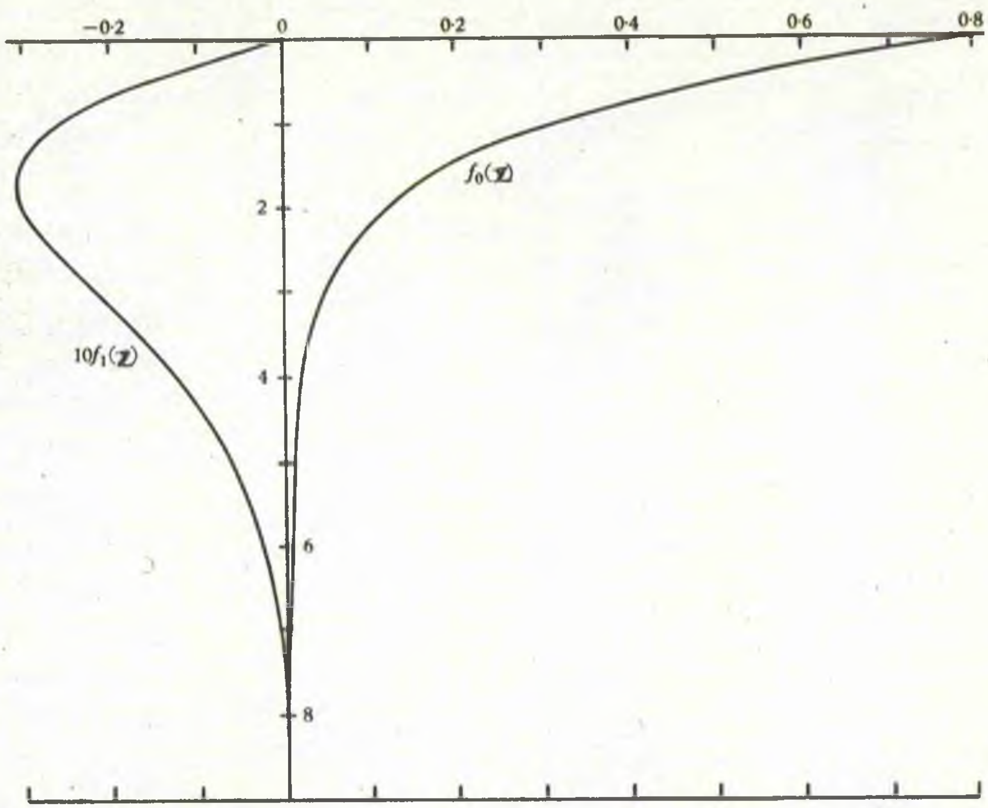


Figure 4.2: The functions $f_0(z)$ and $10f_1(z)$ for the viscosity distribution $\bar{m}(z) = e^z$.

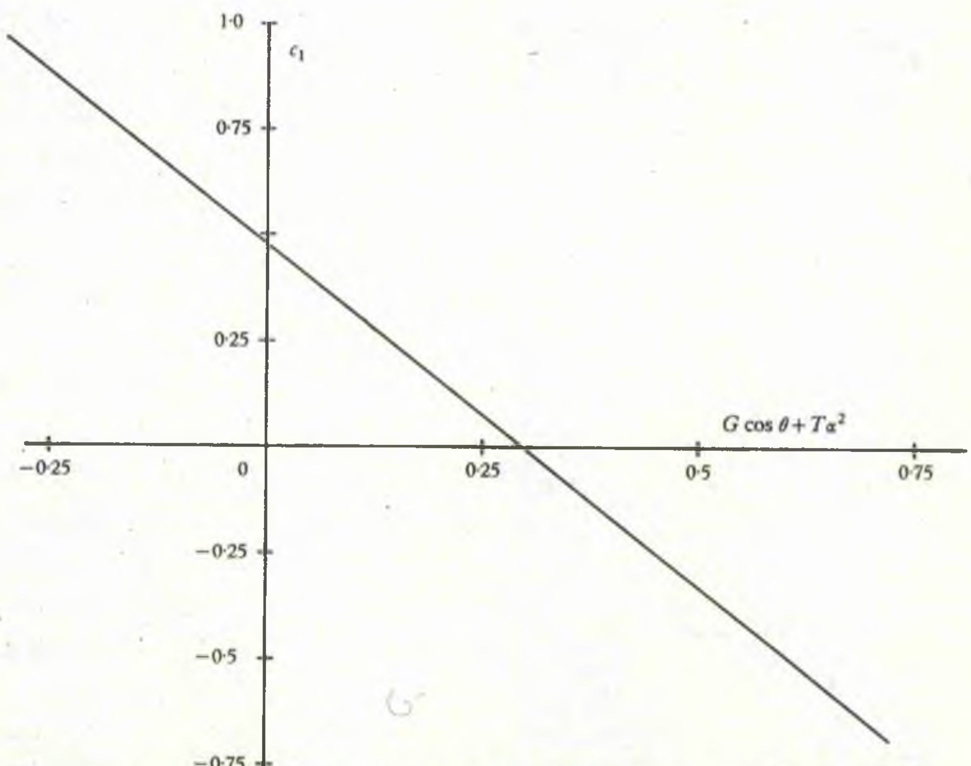


Figure 4.3: Curve of $c_1 (=c_i/\alpha R)$ against $(G\cos\theta + T\alpha^2)$ for $\bar{m}(z) = e^z$

$(G\cos\theta + \alpha^2 T)$ resembles a straight line, but no satisfactory explanation for this has been found. A disturbance of wave-number α is stable or unstable according as $(G\cos\theta + \alpha^2 T)$ is greater or less than 0.295. The function $f_1(z)$ corresponding to the neutral case $c_i = 0$ is shown in Figure (4.2). In the range $0 \leq z \leq 2$, where most of the motion takes place, $|f_1(z)|$ is considerably smaller than $|f_0(z)|$, and the same is true of their first derivatives. This fact suggests that condition (4.14b) may be unnecessarily severe, and that the present approximate theory may hold for all values of αR less than $O(1)$.

When

$$G\cos\theta < 0.295 ,$$

very long waves $(\alpha \to 0)$ are unstable. Also, on using result (4.27), this instability condition becomes

$$R > 3.39 \cot\theta .$$

Now, for the velocity profile (4.39), the volumetric flow rate per unit span Q is

$$Q \equiv Vh \int_0^\infty \bar{u}(z)dz = 2Vh ;$$

therefore, in terms of Q, the instability condition is

$$(\rho/\mu_0)Q > 6.78 \cot\theta . \qquad (4.40)$$

This may be compared with the result

$$(\rho/\mu\ )Q > \tfrac{5}{6} \cot\theta ,$$

which was found for uniform films by Benjamin and Yih. Once again, instability is predicted whenever the liquid surface is vertical, for $\cot\theta$ is then zero. For $\theta < 90°$, the flow is stable provided Q is sufficiently small, but instability occurs when Q is large enough to satisfy condition (4.40).

The above analysis may readily be extended to include the effects of surface contamination, by following the method of Benjamin (1963). It is clear that the instability examined in this chapter is very similar to that occurring in uniform films which was discussed in the previous chapter, and further discussion is therefore unnecessary.

The work of this chapter has been published in Craik and Smith (1968).

CHAPTER V

THE EFFECTS OF SURFACE CONTAMINATION

This chapter is concerned with the effect of surface contamination on the stability of a thin liquid film which is subject to an airflow over its upper free surface.

The content of the chapter is as follows.

Previous work on surface contamination will first be described and the various terms such as solubility and elastic modulus introduced. During this resumé the various models used to describe the surface contaminating effect will be emphasised for comparison with later theory.

This theory is first developed along general lines where the diffusion of the contaminating substance from and to the bulk of the solution or liquid beneath the surface is governed by the diffusion equation. This model is later solved for two cases:- one, where the diffusion effect extends throughout the film and the other, where the effect is only significant in a thin layer near the surface. However, before these two cases are solved, a simpler model based on a straight-forward dependence between the mean and the perturbed concentration at the surface is examined.

The results are summarised at the end of the chapter for comparison with the experimental work of the next chapter.

§5.1:Introduction and previous work.

Some organic substances, when added to water behave in such a way that their molecules on reaching the water's surface accumulate there forming a very thin (sometimes monomolecular) layer. The tendency of these substances or surface-active agents to accumulate on the surface is called adsorption. This type of physical reaction will be occuring in much of the succeeding work.

It had been observed many centuries ago that the addition of a surface-active agent could cause waves in liquids to decay more quickly than was observed previously. The observations of Pliny the Elder, in the first century A.D., are cited by many authors including Levich (1962) and Craik (1965). Benjamin Franklin (1774) suggested that waves were more difficult to excite in the presence of oil on the surface because the oil lubricated the surface so that the wind could not "easily catch on it .... but slides over it".

The first satisfactory explanations appear to have been made by Reynolds (1880) and Aitken (1883) who realised that local variations in surface tension, due to changes in surface-active agent (or surfactant) concentration, were more important than any overall change in surface tension. Lamb in "Hydrodynamics" (1895) provided the mathematical analysis in support of their model but abbreviated the work to only inextensible films in editions subsequent to this.

Levich (1941, 1962) considered theoretically both insoluble and soluble surfactants and found that the rates of damping of capillary and gravity waves on deep water are greatly increased by the presence of surface elasticity. Surface elasticity is a measure of the resistance to extensions and contractions of a surface during the passage of a wave. His work does not give too clear a picture of the relations between the various surface parameters although it must be remembered that it was the first really extensive investigation into surface contamination effects.

However, a surface film does not behave as a truly elastic membrane but may exhibit elastico-viscous properties. This leads to the introduction of a surface viscosity coefficient which can be related to the diffusion of surfactant either along the surface or into the bulk liquid. Dorrestein (1951) has examined both theoretically and experimentally the effects of surface elasticity and surface viscosity on waves in deep water but considered only insoluble surfactants.

Keulegan (1951) examined experimentally the effect of surfactants on wave damping in closed basins subject to a wind stress over the surface. He found that without contamination waves were produced with wind speeds of 300 - 400 cm. per sec. whereas with contamination waves were still not apparent at speeds of 1200 cm. per sec., the maximum wind capacity of the apparatus.

Tempel and Riet (1965) showed that damping values, much higher than those obtained previously, may be obtained and that they were in agreement with experiment. This difference in their theoretical work they attributed to previous "use of an incorrect boundary condition for the tangential stress". However, little difference in their boundary condition is apparent and direct comparison with previous work is impossible due to their numerical method of solution.

Hansen and Mann in three papers (1963, 1964a,b) treated both soluble and insoluble films. Their results are presented as the reduction of the velocity of wave propagation and the damping rate against varying degrees of contaminant. Both theory and experiment are outlined and their work quotes several useful values of surface properties of different monolayers.

Davies and Vose (1965) have performed experimental work on capillary wave damping by soluble and insoluble films. They have compared their results with the theories of Lamb (1932), Levich (1942) and Dorrestein (1951). They found that the damping coefficient for clean water was in agreement with that given theoretically by Lamb and that that for insoluble surfactant did vary with increasing concentration as predicted by Dorrestein. Also Dorrestein's calculations on the effect of surface viscosity were corroborated. When soluble surfactants were added they found that the damping coefficient became smaller provided that the relaxation frequency (or the time scale characteristic of the diffusion or transfer of contaminant from the surface into the bulk liquid or vice-versa) was comparable or less than the

the period of the wave. This was not in complete agreement with Levich who assumed that damping would increase monotonically with the amount of contaminant.

All the above work was concerned with the effect of contamination on waves in deep liquid layers. A mathematical model, based on the effect of increased viscous dissipation was adopted by Miles (1962) in examining the damping of surface waves in closed basins. This increased viscous dissipation occurs in a thin layer near the surface. The work included a short section on wind-generated waves and an extensive derivation of the modifications to the tangential stress condition due to surface effects. Miles concluded that "surface contamination .... may contribute significantly to the damping of surface waves in closed basins" but that more observations are needed before a reliable comparison can be made between theory and experiment. Miles also included the effect of capillary hysteresis which is not relevant to this work.

Benjamin (1963) extended his previous work on stability of flow down inclined planes, described earlier in this work, to include surface contamination effects. He found that surface viscosity had a surprisingly insignificant effect but that the damping influence of surface elasticity was considerable to an extent "that the Reynolds number at which observable waves will arise may be increased by a factor of ten or more." Also, the lowering in the equilibrium value of the surface tension was of relatively little influence. The mathematical representation used is adopted later in this work.

Yih (1967b) examined the stability of a horizontal layer of viscous liquid in which the motion was due to a gradient of surface tension. Plane Couette flow with zero surface velocity was chosen as a convenient example. However the linearisation procedure in the boundary condition governing surface concentration (equation (28) of the paper) was incorrectly performed. In a corrected paper Yih (1968) found two modes of instability which could both be unstable.

Craik (1968) extended his previous work on wind-generated waves to include a surface contaminating effect and examined not only fairly small but also moderately large Reynolds number approximations. At large liquid Reynolds numbers the influence of the contamination on the roles of variable tangential stress, Σ, and the bulk viscosity, R^{-1}, were examined. It was found that the effect upon the role of Σ could almost always be neglected in comparison with that on R^{-1}. In such films instability occurs when the viscous dissipation within the liquid is insufficient to balance the energy transfer from the mean air flow to a neutral wave. This dissipation occurs in two layers - one at the wall and the other at the surface. The magnitude of the latter dissipation increased from $O(\alpha R|c_r - 1|^{-1})$ to $O(\alpha^{\frac{1}{2}}R^{-\frac{1}{2}}|c_r - 1|^{-\frac{1}{2}})$ in the presence of surface contamination. (Here, as before, α is the dimensionless wavenumber and $c = c_r + ic_i$ is the dimensionless wave velocity where $c_r > 1$.). Thus the surface elasticity forces modify the motion so as to increase the rate at which energy is dissipated within the surface layer.

At small liquid Reynolds numbers, a new class of disturbances was found for which $c_r - 1$ is a small negative quantity (c.f. a small positive quantity predicted for uncontaminated films) and for these surface elasticity fulfils a destabilizing role. This component reinforces the tangential stress component due to the air flow and together they cause liquid to be dragged towards the crests and away from the troughs of a small wave-like disturbance (see figure 2.1). The approximate condition for instability is then a simple kinematic one: namely, that there should be a net horizontal volume flux towards the crests and away from the troughs of this disturbance. This destabilizing role of surface elasticity is due to the presence of a non-zero velocity gradient at the liquid surface. In the case of Benjamin's flow down an inclined plane, this velocity gradient is zero and this eliminates the possibility of contamination promoting instability. As in Benjamin's work, surface viscosity was found to have a minor role. The possibility was also mentioned that the visible waves in such thin films were non-linear (i.e. their amplitude being comparable with film thickness) with c_r significantly less than one; in which case surface elasticity would have a much smaller effect.

The present work re-examines Craik's results for thin films and includes solubility and diffusion. The theoretical work will not be concerned with thicker films.

In a more recent paper Gottifredi and Jamieson (1968a) have examined theoretically the effect of an adsorbed layer over a deep liquid. Their theory is performed in terms of the surface elasticity parameter, both surface viscosity and solubility being ignored. They found that "even a film of very low elasticity will have a considerable effect at very short wavelengths" and that an optimum concentration of the surfactant existed for wave damping. The critical wind speed for wave generation was increased by a factor of up to ten, depending on the surface elasticity of the monolayer present and the wavelength. For the shortest wavelengths, a very small value of surface elasticity would increase the wind speed required by a factor of two.

They also state that " as far as the liquid and the surface film are concerned, the theory is on relatively firm ground". However, in their equation for tangential stress (equation 2.3) the effect of the shear flow generated at the liquid surface has not been considered and this would result in misleading stability curves. Craik found a similar omission in his early work on contamination which altered results quite considerably (compare the figures Craik (1966) figure 10.1 and Craik (1968) figure 2).

In a short sequel Gottifredi and Jamieson (1968b) compared the results of the first paper with experimental work of Davies et al. (1968) on wind-generated waves on a liquid surface covered by a surface film. This was an attempt to show that the experimental results could be explained by their theory. However, the diagrams representing the growth rate of a wave against the air flow for both theory and experiment are so sparse of experimental points that what agreement there is is only "qualitative".

§5.2: Basic equations.

Before deriving the equations of motion for the contaminant, we must distinguish between the cases when the contaminant is soluble and when it is insoluble and emphasise what the various terms imply. Contamination by an insoluble substance which does not diffuse along the liquid surface is identified by the surface elasticity coefficient. Where there is diffusion along the surface, this is covered by the surface viscosity coefficient. Benjamin (1963) and Craik (1968) who were concerned with insoluble contaminants which diffuse along the surface only made use of these two coefficients.

With soluble contaminants where only diffusion along the surface is modelled this too is identified by the surface viscosity coefficient. However, when the soluble contaminant diffuses normal to the surface into and out of the bulk of the liquid this case is governed by a diffusion equation with a boundary condition which relates the concentration perturbation at the surface to that in the bulk. For thin films this process would be important and it is this latter case with which we will at first be concerned.

The diffusion equation will now be derived. When a surface is disturbed by a wave the accompanying liquid motion causes periodic contraction and expansion of the elements of the surface. If there is some adsorbed surface-active material or surfactant on the surface, the concentration will exhibit corresponding periodic deviations from its equilibrium value. If the solubility of the surfactant is quite appreciable, there may also be a periodic transfer of surfactant from the surface to the liquid bulk and vice-versa. These periodic fluctuations in surfactant concentration in the layer immediately beneath the surface may be represented in a diffusion 'mass balance' equation

$$\frac{Dc_0}{Dt} = \left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x} \right) c_0 = \Delta \left(\frac{\partial^2 c_0}{\partial x^2} + \frac{\partial^2 c_0}{\partial z^2} \right) \tag{5.1}$$

where c_0 is the dimensionless concentration of solute or contaminant in the bulk solution, Δ is the dimensionless diffusivity constant and $\underline{\bar{U}} = (\bar{U}(z), 0, 0)$ is the dimensionless basic velocity profile. All quantities have been made dimensionless with respect to the velocity, V, of the liquid surface, the film thickness h and the liquid density ρ. Equation (5.1) represents diffusion of soluble contaminant in directions both normal and parallel to the surface.

As shown in figure (5.1), the liquid occupies the space contained within the rigid boundary $z = 0$ and the free surface $z = 1$, the origin of co-ordinates being chosen at some position on the rigid boundary. The basic velocity components are defined above. The perturbed velocity components are $\underline{u} = (u, v, w)$ where $\underline{U} = \underline{\bar{U}} + \underline{u}$ in the cartesian system.

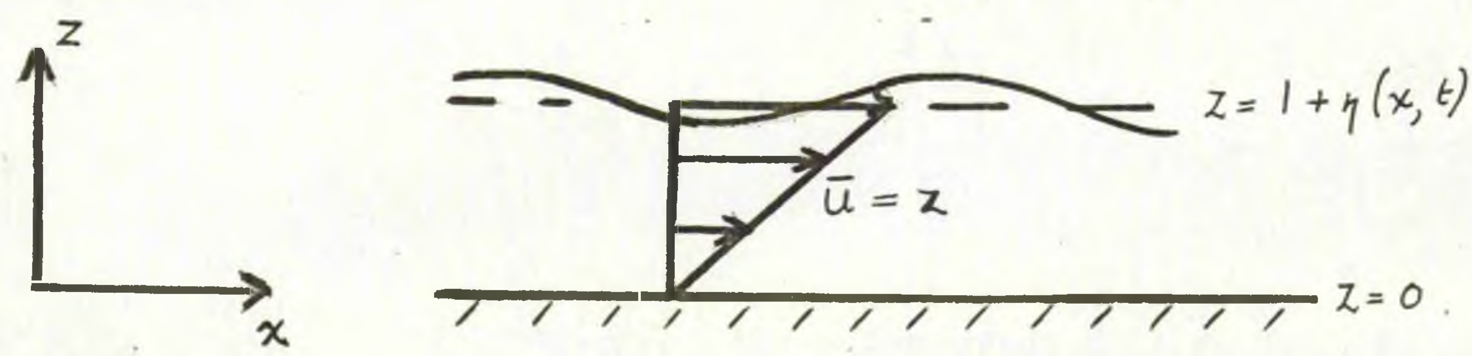


Figure 5.1: Sketch of shear flow and surface disturbance.

An air stream is assumed to exert a constant mean tangential stress on the liquid surface; also, we take the pressure gradient in the direction of motion to produce negligible curvature of the liquid velocity profile whenever the depth of the air phase is large compared with that of the liquid. This is certainly the case for the experimental apparatus used later (see also Craik (1965) figure 4). We therefore take the mean velocity profile of the liquid to be linear. The dimensionless profile $\bar{U}(z)$ then is

$$\bar{U}(z) = z \tag{5.2}$$

This is shown in figure (5.1).

Two boundary conditions govern equation (5.1). At the rigid boundary, there is no diffusion of solute and so

$$\frac{\partial c_0}{\partial z} = 0 \qquad \text{at } z = 0 \tag{5.3}$$

At the upper boundary, there is a free surface condition. The amount of contaminant adsorbed at a surface element of area A (where A may change) is ΓA where Γ is the contaminant concentration and this amount varies with time due to diffusion of contaminant normal to the surface (assuming diffusion along the surface negligible in comparison). Thus

$$\frac{D}{Dt}(\Gamma A) = -\Delta A\left(\frac{\partial c_0}{\partial n}\right)_s$$

where n is the direction normal to the surface and s denotes evaluation at the surface. This relationship was first proposed by Boussinesq (1913). Linearisation gives

$$\frac{D\Gamma}{Dt} = \frac{\partial\Gamma}{\partial t} + \bar{U}\frac{\partial\Gamma}{\partial x} = -\frac{\Gamma}{A}\frac{DA}{Dt} - \Delta\left(\frac{\partial c_o}{\partial z}\right)_s$$

The rate of dilation of the surface element to the same approximation is

$$\frac{1}{A}\frac{DA}{Dt} = \left(\frac{\partial u}{\partial x}\right)_s$$

Therefore to a linearised approximation, we obtain

$$\frac{D\Gamma}{Dt} = -\Gamma\frac{\partial u}{\partial x} - \Delta\frac{\partial c_o}{\partial z} \qquad (5.4)$$

evaluated at s. It is necessary to reduce this equation to terms in contaminant, c_o, alone. Small perturbations in surface tension T, Γ and c_o are related according to

$$\nabla T = \left(\frac{DT}{D\Gamma}\right)_{z=0} \nabla(\Gamma - \Gamma_{un}) \qquad (5.5)$$

$$\Gamma - \Gamma_{un} = \lambda(c_o - c_{o_{un}})_{z=0}, \qquad \lambda = \left(\frac{D\Gamma}{Dc_o}\right)_{un} \qquad (5.6)$$

where λ is a frequency independent length and the subscript un denotes evaluation on the unperturbed surface. These two relationships will be derived in Appendix I.

On substituting condition (5.6) into equation (5.4), we obtain the surface boundary condition in its final form

$$\lambda\frac{Dc_o}{Dt} = -\Gamma\frac{\partial u}{\partial x} - \Delta\frac{\partial c_o}{\partial z} \qquad (5.7)$$

evaluated at the liquid surface.

§5.3: A simplified model.

Before attempting to obtain a full solution to the diffusion equation (5.1), it is of interest to investigate a more simple representation of the surfactant diffusion. Let us examine the equation

$$\frac{D\Gamma}{Dt} = \left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x}\right)\Gamma = -\kappa\Gamma - c_0\frac{\partial}{\partial x}(u + \bar{U}'\eta) \qquad (5.8)$$

evaluated at $z = 1$, where c_0 is still the mean concentration, Γ is the perturbation of this mean concentration and κ represents the constant of proportionality of the rate of diffusion of the contaminant to the bulk solution. This latter condition is proportional to the deviation of the value of the concentration from its mean.

The problem with which we are concerned with in this chapter is the effect of contamination on the stability of a horizontal liquid film subject to an air flow on its upper boundary.

The vertical displacement of the liquid surface due to a small two-dimensional perturbation is represented by

$$z = 1 + \eta(x,t) = 1 + \delta e^{i\alpha(x-ct)} \qquad (5.9)$$

as in previous chapters, but now the unperturbed surface is at $z = 1$.

The perturbation in concentration is assumed to have the same frequency and wavelength as the surface perturbation i.e.

$$c_e \propto e^{i\alpha(x-ct)} .$$

Because of continuity, a perturbation stream function may be introduced of the form

$$\psi(x,z,t) = -\phi(z)\,\eta(x,t)$$

such that the perturbation velocity components are

$$u = \psi_z = -\phi'\eta \;; \qquad w = \psi_x = i\alpha\phi\eta \qquad (5.10)$$

where 'dashed' variables again denote differentiation with respect to z. Their substitution into the linearised equations of motion yields

$$\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha R(z-c)\,(\phi''-\alpha^2\phi) \qquad (5.11)$$

The boundary conditions governing this equation are, with one exception, those derived in a different configuration in Chapter III, and by Craik (1968). The boundary conditions at the rigid wall $z = 0$ give

$$\phi(0) = \phi'(0) = 0 \qquad (5.12a,b)$$

The kinematic surface condition at $z = 1$ yields

$$\phi(1) = 1 - c \qquad (5.13)$$

The normal stress condition at $z = 1$ gives

$$(1 - c)\phi' - \phi - (i\alpha R)^{-1}(\phi''' - 3\alpha^2\phi') - (T\alpha^2 + G - \Pi)(1 - c)^{-1}\phi = 0 \qquad (5.14)$$

Equation (5.8) can now be reduced to the form

$$\Gamma = \frac{-\,c_o\,(\phi' + \phi/(c - 1))}{1 - c - i\kappa/\alpha} \quad \text{at } z = 1 \qquad (5.15)$$

This equation is used in the tangential stress condition which now relates directly the stress to the rate of change of concentration in the x-direction i.e. the stress due to the contamination is

$$\bar{\sigma}_{xz} = -\beta \frac{\partial \Gamma}{\partial x} \qquad (5.16)$$

where β is some constant (>0). The tangential boundary condition at the surface then becomes (c.f. Craik (1968))

$$\tau = \sigma_{xz} + \bar{\sigma}_{xz} \qquad \text{at } z = 1$$

i.e.
$$\frac{1}{R}(\phi'' + \alpha^2\phi) = \sigma_{xz} + \bar{\sigma}_{xz}$$
$$= \frac{\Sigma\phi}{c-1} + \frac{i\alpha\beta c_o(\phi' + \phi/(c-1))}{1 - c - i\kappa/\alpha} \qquad \text{at } z = 1 \qquad (5.17)$$

In a similar manner to before we take a first approximation to the Orr-Sommerfeld equation applied to the linear velocity profile

i.e $\qquad \phi^{iv} = 0$

giving $\qquad \phi = A_0z^3 + A_1z^2 + A_2z + A_3$

and apply the boundary conditions

$$\phi = \phi' = 0 \qquad \text{at } z = 0$$

on the solid boundary,

$$\phi''' = \frac{i\alpha R\phi}{c-1}(T\alpha^2 + G - \Pi) \qquad \text{on } z = 1$$

the ~~tangential~~ normal stress condition, together with the tangential stress condition and equation (5.17). The resulting equations give

$$[illegible] - A_3 = 0$$

$$3A_0 + 2A_1 + A_2 = 0$$

$$6A_0 - \frac{i\alpha R(T\alpha^2 + G - \Pi)}{c-1}(A_0 + A_1) = 0$$

$$6A_0 + 2A_1 - \frac{i\alpha\beta Rc_o}{1-c-i\kappa/\alpha}(3A_0 + 2A_1) + \left\{\frac{i\alpha\beta Rc_o}{(1-c)(1-c-i\kappa/\alpha)} + \frac{R\Sigma}{1-c}\right\}(A_0 + A_1) = 0$$

The determinantal equation of the above yields

$$4(T\alpha^2 + G - \Pi) + \frac{6i\Sigma}{\alpha} - \frac{12(1-c)}{i\alpha R}$$
$$= \frac{i\alpha R\beta c_o}{1-c-i\kappa/\alpha}\left\{T\alpha^2 + G - \Pi + \frac{6}{i\alpha R}(2c-1)\right\} \qquad (5.18)$$

We may now examine this equation for various approximations and compare the results with the previous results of Craik (1968).

<u>Case (a)</u>

Letting $|1-c| >> \kappa/\alpha$ we obtain

$$3(1-c)^2 - i\alpha R(1-c)(T\alpha^2 + G - \Pi + \frac{3}{2}\frac{i\Sigma}{\alpha} + 3\beta c_o)$$
$$+ i\alpha R\beta c_o\left\{\frac{3}{2} + \frac{i\alpha R}{4}(T\alpha^2 + G - \Pi)\right\} = 0.$$

This equation holds when the diffusing effect is quite small or, in other words, the contaminant is virtually insoluble. This we may compare with the work of Craik - his equation (8.2) is the above where $\beta c_o = T_1$, where T_1 is the dimensionless sum of the elastic moduli of surface dilatation and shear. Thus, Craik's work and comments on the above equation hold.

<u>Case (b)</u>

Next, if the contaminating effect is large, the terms containing c_o become dominant and equation (5.18) becomes

$$T\alpha^2 + G - \Pi + \frac{6}{i\alpha R}(2c-1) = 0$$

The real and imaginary parts of this equation yield

$$c_r = \tfrac{1}{2} - \frac{\alpha R\Pi_i}{12} = \tfrac{1}{2} \text{ when } \alpha R|\Pi_i| << 1$$

$$c_i = -\frac{\alpha R}{12}(T\alpha^2 + G - \Pi_r)$$

and the motion is stable when $Ta^2 + G > \Pi_r$.

We may again compare this with Craik (1968), equation 8.2. His equation reduces to the above under the assumption

$$\frac{\alpha R\left|T_1 + i\alpha K(1-c)\right|}{1-c} \gg 1 \text{ and } |R\Sigma| << 1$$

where K is the sum of the surface dilatation and shear viscosities. This assumption implies that the presence of contaminant completely suppresses the destabilizing influence of the tangential stress Σ_i.

Thus, in this case when a large amount of contaminant is present, the flow experiences its effect on the surface as one of an elastic membrane, completely suppressing the influence of the wind.

Case (c)

Finally, the case where diffusion is dominant is examined.

i.e. [illegible] is large at the surface.

This implies that $\kappa/\alpha \gg |1-c|$ and in equation (5.18) this gives

$$4(Ta^2 + G - \Pi) + \frac{6i\Sigma}{\alpha} - \frac{12(1-c)}{i\alpha R} = -\frac{\alpha^2 R\beta c_0}{\kappa}\left\{Ta^2 + G - \Pi + \frac{6}{i\alpha R}(2c-1)\right\}$$

The imaginary part of this equation is

$$c_i = \frac{\alpha R}{3}\left(\frac{(\Pi_r - Ta^2 - G)(1 + (\alpha^2 R\beta c_0)/4\kappa) + (3\Sigma_i/2\alpha)}{1 + \frac{\alpha^2 R\beta c_0}{\kappa}}\right)$$

From this the motion is stable if

$$(Ta^2 + G - \Pi_r)\left(1 + \frac{\alpha^2 R\beta c_0}{4\,\kappa}\right) > \frac{3\Sigma_i}{2\alpha}.$$

The real part yields

$$c_r = \frac{1}{2} + \frac{1}{2(1+\frac{\alpha^2 R\beta c_0}{\kappa})}$$

i.e. $\frac{1}{2} < c_r < 1$, when $\infty > \frac{\alpha^2 R\beta c_0}{\kappa} > 0$.

This again can be compared with the equations of Craik. This same condition on c_r can be extracted from his equation (8.2) when $\frac{\alpha R|T_1 + i\alpha\kappa(1-c)|}{1 - c} = O(1)$ in his notation. It is found from his work that the following result:-

(i) when $\frac{\alpha RT_1}{|1 - c|} \gg 1$, $c_r = \frac{1}{2}$.

(ii) when $\frac{\alpha RT_1}{|1 - c|} \ll 1$, $c_r = 1$.

Thus, in this section the results of Craik (1968) have been re-established and their regions of validity emphasised.

§5.4: The full diffusion model.

We return to the equations of §5.2. The vertical displacement is again assumed to be of the form (5.9) and the surface condition (5.7) reduces to

$$\lambda \frac{Dc_0}{Dt} = - \Gamma \left(\frac{\partial u}{\partial x} + \bar{U}' \frac{\partial \eta}{\partial x}\right) - \Delta \frac{\partial c_0}{\partial z} \quad \text{at } z = 1. \tag{5.19}$$

The perturbation in concentration is again assumed to have the same frequency and wavelength as the surface perturbation

$$\text{i.e.} \quad c_0 = \delta f(z)\, e^{i\alpha(x-ct)} = f(z)\eta \tag{5.20}$$

where now $f(z)$ is some function of z.

For the fluid flow the equations (5.10) - (5.14) are still valid.

For the tangential stress boundary condition a modification must again be introduced to allow for the tangential stress exerted by the contaminated surface. This modification is given by

$$\bar{\sigma}_{xz} = \frac{\partial T}{\partial x} = \left(\frac{dT}{d\Gamma}\right)_{un} \frac{\partial \Gamma}{\partial x}$$

$$= \left(\frac{dT}{d\Gamma}\right)_{un} \lambda \frac{\partial c_o}{\partial x} \tag{5.21}$$

where, again, the subscript un denotes that the quantity is evaluated on the unperturbed surface. This equation is directly related to the expression used for the elastic moduli of surface dilatation and shear in equation (3.11) earlier, and to equation (5.16).

The change in the value of surface tension is related to the change in surface contaminant concentration through equation (5.5) in the form

$$T_1 = - \Gamma_{un} \left(\frac{dT}{d\Gamma}\right)_{un}$$

where $T_1 = \gamma_1(\rho V^2 h)^{-1}$, V is the surface film velocity and γ_1 is the sum of the elastic moduli of surface dilatation and shear. Equation (5.21) then becomes

$$\bar{\sigma}_{xz} = - \frac{T_1}{\Gamma} \lambda \frac{\partial c_o}{\partial x} \tag{5.22}$$

The boundary condition expressing continuity of tangential stress at the surface is therefore c.f. Craik (1968)

$$\tau = \sigma_{xz} + \bar{\sigma}_{xz} \quad \text{at } z = 1$$

$$(\phi'' + \alpha^2\phi + \frac{RI\phi}{1-c}) - \frac{T_1}{\Gamma} \lambda \, i\alpha Rf = 0 \quad \text{at } z = 1 \tag{5.23}$$

Substitution of equation (5.20) into equation (5.1) yields the differential equation

$$f'' - \frac{i\alpha}{\Delta}(z - c)\, f - \alpha^2 f = 0 \tag{5.24}$$

and into condition (5.3) gives

$$f'(0) = 0 \tag{5.25}$$

Substitution of both equation (5.20) and equation (5.10) into (5.19) yields

$$\lambda\,(1 - c)\, f(1) + \frac{\Delta}{i\alpha} f'(1) + \Gamma(1 - \phi(1)) \quad \text{at } z = 1 \tag{5.26}$$

The stability problem is now specified. We must solve equations (5.11) and (5.24) subject to the conditions (5.12, 5.13, 5.14, 5.23, 5.25 and 5.26). This is a characteristic value problem specified by a second order and a fourth order equation subject to seven boundary conditions. Their solution yields an eigenvalue problem for the complex wave velocity c in terms of α and the other parameters of the problem.

We will now investigate these equations for two different models; firstly, a model where the diffusion effect of the contaminant extends throughout the film, and, secondly, one where the diffusion effect is only significant in a thin layer near the liquid surface. It must be remembered that the liquid's kinematic viscosity will always be much larger than the dimensional diffusion coefficient for each model.

§5.5: Solution for large diffusive effect.

Here we examine the case where diffusion is important throughout the film.

If $\frac{\alpha c}{\Delta}$ and α^2 are small the function $f(z)$ may be expressed by a series

$$f(z) = \sum_{n=0}^{\infty} A_n z^n$$

and introduced into equation (5.24) in the form

$$f'' - \frac{i\alpha}{\Delta}(z - c)\, f = 0 \tag{5.27}$$

to give

$$\begin{aligned} f = B_0 &\left\{ 1 + \tfrac{1}{2}\left(\alpha^2 - \frac{i\alpha c}{\Delta}\right)z^2 + \frac{i\alpha}{\Delta} z^3 + \frac{1}{24}\left(- \frac{i\alpha c}{\Delta} + \alpha^2\right)^2 z^4 + \ldots\ldots\ldots \right\} \\ &+ B_1 \left\{ z + \frac{1}{6}\left(\alpha^2 - \frac{i\alpha c}{\Delta}\right)z^3 + \frac{i\alpha}{12\Delta} z^4 + \ldots\ldots\ldots \right\} \end{aligned} \tag{5.28}$$

From equation (5.25), $B_1 = 0$.

It is to be noted that equation (5.24) does have an exact solution expressed in Hankel functions but these functions are rather unwieldy and the approximations made to obtain a solution are sufficient for the purposes of this work.

The solution to equation (5.11) to a similar approximation was given by Craik (1968) in the form

$$\begin{aligned} \phi(z) = A_0 + A_1 z + A_2&\left(z^2 + \frac{\beta}{12} z^4 + \frac{q}{60} z^5\right) \\ &+ A_3\left(z^3 + \frac{\beta}{20} z^5 + \frac{q}{60} z^6\right) \end{aligned} \tag{5.29}$$

where $\qquad \beta = i\alpha R(1 - c) + 2\alpha^2$

$\qquad\qquad q = i\alpha R.$

The rigid wall conditions (5.12) imply $A_0 = A_1 = 0$. Conditions (5.26) and (5.13) give

$$\lambda(1-c)\, f(1) + \frac{\Delta}{i\alpha} f'(1) + \Gamma\left(\frac{\phi(1)}{1-c} - \phi'(1)\right) = 0 \quad \text{at } z = 1 \tag{5.30}$$

On neglecting β, δ and α^2 in equation (5.29) and substituting this and equation (5.28) into (5.14),(5.28) and (5.30) we obtain

$$A_2\left(2 + \frac{R\Sigma}{1-c}\right) + A_3\left(6 + \frac{R\Sigma}{1-c}\right) + B_0\frac{T_1\lambda i\alpha R}{\Gamma}\left(1 + \frac{i\alpha}{\Delta} - \frac{i\alpha c}{2\Delta}\right) = 0$$

$$A_2\left(2 + \frac{1}{c-1}\right) + A_3\left(3 + \frac{1}{c-1}\right) - \frac{B_0}{\Gamma}\left((3-c-i\alpha\Delta) + \lambda(1-c)\right) = 0$$

$$A_2 i\alpha R\left(1 + \frac{T\alpha^2 + G - H}{1-c}\right) + A_3\left(6 + i\alpha R\left[1 + \frac{T\alpha^2 + G - H}{1-c}\right]\right) = 0$$

Excluding the trivial case $A_2 = A_3 = B_0 = 0$, it is evident that these three equations are consistent only when the determinant of the coefficients of A_2, A_3, B_0 is zero. In the evaluation of this determinant, terms of second and higher order in αR, $\frac{\alpha}{\Delta}$, etc. are neglected. This gives as the condition for consistency

$$4\left(T\alpha^2 + G - H + \frac{3}{2}\frac{i\Sigma}{\alpha} - \frac{3(1-c)}{i\alpha R}\right) = \frac{T_1\lambda\left(1 + \frac{i\alpha}{\Delta}\left(1-\frac{c}{2}\right)\right)}{3-c-i\alpha\Delta + \lambda(1-c)}\left(i\alpha R\,(T\alpha^2 + G - H) + 6(2c-1)\right) \tag{5.31}$$

We now examine various approximations to this equation.

<u>Case (a)</u>

On taking $|1-c|\,\lambda \gg 1$ but $\frac{\alpha}{\Delta}\left|1-\frac{c}{2}\right| \ll 1$ we obtain

$$T\alpha^2 + G - H + \frac{3}{2}\frac{i\Sigma}{\alpha} - \frac{3(1-c)}{i\alpha R} + 3T_1 = \frac{T_1}{4}\left(\frac{i\alpha R(T\alpha^2 + G - H) + 6}{1-c}\right) \tag{5.32}$$

in agreement with the result obtained by Craik (1968).

This helps to distinguish when the Craik model is valid.

The condition, $\alpha(1-\frac{c}{2}) \ll \Delta$, implies that the rate of contaminant diffusion into and off the surface happens quickly in comparison with the period of the wave. This gives a very stringent condition for the Craik model when it is applied to soluble films.

Case (b)

On taking the approximations $\Delta\alpha \ll |3-c|$, $\frac{\alpha}{\Delta}\,|1-\frac{c}{2}| \ll 1$ but $\lambda\,|1-c| \ll |3-c|$ we obtain a quadratic in c.

$$c^2 - c\left(4 - \frac{i\alpha R}{3}(T\alpha^2 + G - \Pi) + \frac{R\Sigma}{2} - i\alpha R\lambda T_1\right)$$
$$+ 3 - i\alpha R(T\alpha^2 + G - \Pi) + \frac{3R\Sigma}{2} - \frac{i\alpha RT_1\lambda}{2} = 0.$$

On separating this equation into real and imaginary parts and letting $c = c_r + ic_i$, $\Pi = \Pi_r + i\Pi_i$ and $\Sigma = \Sigma_r + i\Sigma_i$ we obtain

$$c_r - \left(4 + \frac{R\Sigma_r}{2} - \frac{\alpha R\Pi_i}{3}\right)c_r + \left(\frac{R\Sigma_i}{2} - \frac{\alpha R}{3}(T\alpha^2 + G - \Pi_r) - RT_1\lambda\right)c_i$$
$$+ 3 + \frac{3}{2}R\Sigma_r - \alpha R\Pi_i = 0 \qquad (5.33)$$

$$2c_rc_i - \left(\frac{R\Sigma_i}{2} - \frac{\alpha R}{3}(T\alpha^2 + G - \Pi_r) - \alpha R\lambda T_1\right)c_r - \left(4 + \frac{R\Sigma_r}{2} - \frac{\alpha R\Pi_i}{3}\right)c_i$$
$$- \alpha R(T\alpha^2 + G - \Pi_r) + \frac{3}{2}R\Sigma_i - \frac{\alpha RT_1\lambda}{2} = 0 \qquad (5.34)$$

When $\frac{\alpha R|\Pi_i|}{3}$, $\frac{R\Sigma_r}{2}$ and αR are small compared with unity equation (5.33) yields

$$c_r^2 - 4c_r + 3 = 0$$

giving

$$c_r = 1, 3$$

To the same approximation with $c_r = 3$ equation (5.34) gives

$$c_i = -\frac{5}{4}\,\alpha R T_1 \lambda < 0 \ .$$

This means that a disturbance with $c_r = 3$ is stable.

With $c_r = 1$ in equation (5.34) we obtain

$$c_i = \frac{\alpha R T_1 \lambda}{4} + \frac{R\Sigma_i}{2} - \frac{\alpha R}{3}(T\alpha^2 + G - \Pi_r)$$

Instability occurs when

$$\Pi_r + \frac{3}{2}\frac{\Sigma_i}{\alpha} + \frac{3}{4}\,\alpha R T_1 \lambda \ > \ G + T\alpha^2 \ .$$

In this case the contamination through the term containing λT_1 promotes instability. This may be compared with the corresponding result for clean films

$$\Pi_r + \frac{3}{2}\frac{\Sigma_i}{\alpha} > \ G + T\alpha^2$$

which is the case (iii) of §3.5 .

The contaminated case examined by Craik (1966) was sensitive to c having values close to unity. In this case, the result is not so sensitive. Diffusion has been responsible for the removal of the singularity.

§5.6: Solution for thin diffusion layers.

We return to equation (5.24) and obtain two approximate solutions valid for large values of $\frac{\alpha}{\Delta}$. From Lin (1955), the simplest approximations are

$$f_{1,2} = \text{const.}\ (z - c)^{-\frac{1}{4}} \exp\left[\ \mp \frac{2}{3}\left(\frac{i\alpha}{\Delta}\right)^{\frac{1}{2}}(z - c)^{\frac{3}{2}}\right]\left[1 + \left(\frac{\alpha}{\Delta}\right)^{-\frac{1}{2}} g(z) + \ldots\right] \tag{5.35}$$

where $i^{\frac{1}{2}} = \exp\ (\frac{1}{4}i\pi)$ and, for nearly real values of c, $z - c = (c - z)\exp\ (-i\pi)$ when $z < c_r$. Both these solutions fluctuate rapidly: the amplitude of the fluctuations associated with f_1 decreases exponentially with distance from the solid wall $z = 0$, while the amplitude of those associated with f_2 increases exponentially towards the surface. As we are investigating a quantity whose gradient is zero at $z = 0$ and whose value is a positive quantity at the surface, we confine interest on the solution f_2 and allow $f_1 = 0$. The solution f_2 remains valid when

$$\left(\frac{\alpha}{\Delta}\right)^{\frac{1}{2}} |1 - c|^{\frac{3}{2}} \gg 1$$
$$\left(\frac{\alpha}{\Delta}\right)^{\frac{1}{2}} |c|^{\frac{3}{2}} \gg 1 \qquad (5.36)$$

A perturbation stream function $\psi(x,z,t)$ may be introduced into the problem and a series solution $\phi = \sum_{n=0}^{3} A_n z^n$ may be sought subject to the same assumptions as before, namely

$$\alpha R,\ \alpha R|c|,\ \alpha^2 \ll 1.$$

$\phi(z)$ is the z-dependent part of ψ.

The boundary conditions governing the problem remain as before viz. equations (5.12),(5.13),(5.14),(5.23),(5.26). From equations (5.12) we again find $A_0 = A_1 = 0$ and the remaining conditions (5.14),(5.23) (5.26) yield to the same approximation in α, αR,

$$6A_3 + i\alpha R\left(1 + \frac{T\alpha^2 + G - H}{1 - c}\right)\left(A_2 + A_3\right) = 0$$

$$2A_2 + 6A_3 + \frac{R\Sigma}{1 - c}(A_2 + A_3) - \frac{\lambda T_1}{\Gamma} i\alpha R\, f_{21} = 0$$

$$2A_2 + 3A_3 - \frac{(A_2 + A_3)}{1 - c} - \frac{\Delta}{i\alpha\Gamma} f_{21}' - \frac{\lambda(1 - c)}{\Gamma} f_{21} = 0$$

where f_{21} denotes the function f_2 evaluated at $z = 1$.

The evaluation of the resulting determinantal equation gives

$$T_1\lambda\left\{ -\frac{(\alpha R)^2 G}{1-c} - 12i\alpha R + \frac{6i\alpha R}{1-c} \right\}$$

$$- \left\{ \frac{\Delta}{i\alpha}\frac{f'_{21}}{f_{21}} + \lambda(1-c)\right\}\left\{ \frac{4i\alpha RG}{1-c} - \frac{6RI}{1-c} - 12 \right\} = 0$$

where $G = T\alpha^2 + G - \Pi + (1-c)$.

Also from the asymptotic approximation f_2 we obtain

$$\frac{f'_{21}}{f_{21}} = \left(\frac{i\alpha}{\Delta}\right)^{\frac{1}{2}}(1-c)^{\frac{1}{2}} - \frac{1}{4(1-c)} + 0\left\{\left(\frac{\alpha}{\Delta}\right)^{-\frac{1}{2}}\right\}$$

This gives

$$T_1\lambda\left\{ \frac{6i\alpha R}{1-c} - 12i\alpha R - \frac{(\alpha R)^2 G}{1-c} \right\}$$

$$- \left\{ \left(\frac{\Delta}{i\alpha}\right)^{\frac{1}{2}}(1-c)^{\frac{1}{2}} - \frac{\Delta}{4i\alpha(1-c)} + \lambda(1-c)\right\}\left\{ \frac{4i\alpha RG}{1-c} \right.$$

$$\left. - \frac{6RI}{1-c} - 12 \right\} = 0 \qquad (5.37)$$

This is the stability condition. It will now be examined for various approximations.

<u>Case (a)</u>

On letting $\Delta = 0$, the result of Craik (1968) for films without diffusion processes is obtained i.e.

$$T\alpha^2 + G - \Pi + \frac{3}{2}\frac{iI}{\alpha} = (1-c)\left(\frac{3}{i\alpha R} - 1\right) - T_1\frac{3}{2}(c-1)^{-1} + 3$$

$$- \frac{i\alpha R}{4}\left\{1 + \frac{T\alpha^2 + G - \Pi}{1-c}\right\} = 0 \qquad (5.38)$$

This is the same equation as equation (5.32) found in the previous section when diffusion rate was fast.

Case (b)

To the next approximation, it is assumed that

$$\lambda\ |1 - c|^{\frac{1}{2}} \ll \left| \frac{\Delta}{\alpha} \right|^{\frac{1}{2}} \ll |1 - c|^{\frac{3}{2}}$$

$$\rightarrow \quad \lambda \ll |1 - c| \ .$$

Equation (5.37) then gives

$$6 - 12\,(c - 1)(1 - \Omega) + 6R\Sigma\Omega + i\alpha RG\,(1 - 4\Omega) = 0, \qquad (5.39)$$

where $$\Omega = \left(\frac{\Delta(1 - c)}{i\alpha\lambda^2} \right)^{\frac{1}{3}} \frac{1}{T_1 i\alpha R} \ .$$

With the further assumptions that

$$\alpha R\,|G| \ll \frac{1 - \Omega}{1 - 4\Omega}\ , \qquad R\Sigma\Omega \ll 1 - \Omega$$

this equation becomes

$$6 + 12\,(c - 1)(1 - \Omega) = 0\ .$$

Separating this into real and imaginary parts gives

$$c_r - 1 = \frac{1 - \Omega_r}{2(\,(1 - \Omega_r)^2 + \Omega_i^{\,2})} = \frac{\Omega_r - 1}{2|\,\Omega - 1\,|^2} \qquad (5.40)$$

and $$c_i = \frac{(c_r - 1)\Omega_i}{1 - \Omega_r} = \frac{-\,\Omega_i}{2|\,\Omega - 1\,|^2} \qquad (5.41)$$

For a situation where $\Omega_i \ll |1 - \Omega_r|$ i.e. Δ is small but λ, T_1 are large these equations give

$$c_r = 1 - \tfrac{1}{2}\,\frac{1}{1 - \Omega_r}$$

$$= \tfrac{1}{2}\ , \text{ when } \Omega \text{ is small,}$$

and $$c_i = \frac{-\Omega_i}{2(1 - \Omega_r)} = -\frac{\Omega_i}{2}$$

Now Ω_i is proportional to $-\,(1 - c_r)^{\frac{1}{2}}$ and so Ω_i is negative. This implies that c_i is positive and that wave disturbances are unstable for this case. They travel with a velocity half that of the free surface.

Case (c)

For a larger diffusion effect with $\Delta \gg 1$ and $|\Omega_r| \gg 1$ the equations (5.40) and (5.41) become

$$c_r = 1 + \frac{1}{2\Omega_r} \simeq 1,$$

and

$$c_i = -\frac{\Omega_i}{4\Omega_r^2} .$$

Again Ω_i is < 0 and c_i is thus positive showing that disturbances are unstable and travel with a velocity almost equal to that of the liquid surface. This agrees with the result of §5.5 .

The effect of increasing diffusion into and out of a thin layer near the surface of the film seems to be a quickening in the velocity of the disturbance. This model does not yield any 'new' stabilizing phenomenon.

It must be noted that the results may break down when $c_r = 1$ due to a singularity in the approximations.

§5.7: Summary of results.

It is appropriate here to summarise the results obtained in this chapter.

I. With the simplified model for diffusion the following were found:

(a) when the diffusion effect is small (i.e. the contaminant is virtually insoluble, $\alpha|1 - c| \gg \kappa$), the flow becomes more or less stable than the corresponding clean film according as the wave velocity is greater or less than the surface velocity. This is due to the surface elasticity term, (βc_o) .

(b) when the contaminant effect, c_o, is large it completely suppresses the destabilizing influence of tangential stress which is present with clean films. Unless the normal stress due to the wind is extremely large, the flow would be unstable.

(c) when the diffusion effect is dominant (i.e. the contaminant is soluble, $\alpha|1 - c| << \kappa$), the flow is again more stable. However, if the flow becomes unstable, then the greater the diffusion effect the slower the waves travel relative to the surface (in fact, $V/2 < c' < V$).

II. With the model where diffusion extends throughout the film the following occurred:

(a) when the rate of diffusion into and off the surface happens quickly relative to the period of a wave, the case I(a) above is recovered. (Here, $\alpha(1 - \frac{c}{2}) << \Delta$, $|1 - c|\ \lambda >> 1$.)

(b) when the diffusion rate happens more slowly relative to the wave period, two wave modes are found :- the one, whose wave velocity is three times that of the surface, is damped; the other, whose wave velocity is that of the surface, is amplified. Thus instability is expected in this latter case. (Here, $\alpha(1 - \frac{c}{2}) << \Delta$, $\Delta\alpha$, $\lambda|1 - c| << 1$.)

III. With the final situation where diffusion was only important in a thin layer near the free surface, the following resulted:

(a) when diffusion is unimportant ($\Delta = 0$), the results of the cases (a) above are recovered.

(b) when there is little diffusion but a large elasticity effect (Δ small but T_1 large), an instability is found whose velocity is half that of the surface.

(c) when the effect of diffusion is dominant ($\Delta >> 1$), the motion is again unstable and waves travel with the surface velocity as in case II(b) above.

In conclusion, it can be noted that as well as the agreement shown in cases (a) above, there is also some agreement which could be expected between cases I(b) and (c) and cases III(b) and (c). If there is an instability present in these cases it travels at between one half and the full value of the surface velocity depending on the strength of the diffusion.

CHAPTER VI

EXPERIMENTAL WORK

ON

CONTAMINATION EFFECTS

Experiments relating to the effect of surface contaminants on the generation of waves by wind are now described. The first part of the chapter describes the experimental apparatus and the results whilst the second part compares these results with the earlier theory.

§6.1: Experimental procedure and apparatus.

The apparatus was, in the main, that used by Craik (1965) in his investigations into uncontaminated flow. It was set up by Mr. D. Niven who reproduced the results found by Craik. A diagram of the apparatus is given in Figure 6.1.

The experiments were conducted in a closed channel 46 in. long, 11.4 in. wide and 1 in. high. The channel's bottom consisted of a single sheet of plate glass whilst the sides and top were made of Plexiglass. Throughout the experiments, the plate glass bottom was horizontal.

Air was drawn through the apparatus by an extraction fan. The air flow was straightened by means of two honeycomb grids, situated at each end of the channel, and the flow could be regulated by unmasking part of the downstream honeycomb grid which was open to

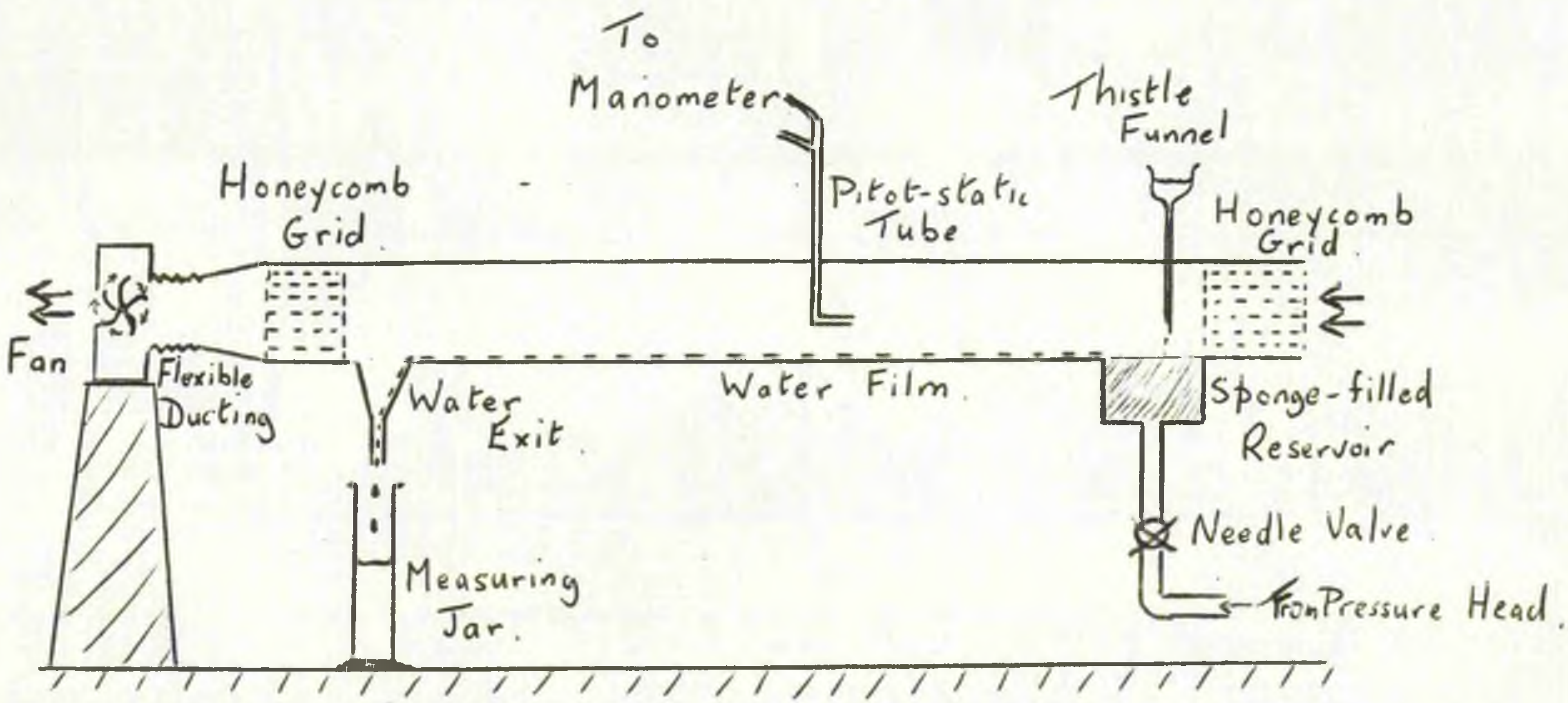


Figure 6.1: Sketch of apparatus.

the air. The fan was connected to the channel by a flexible duct to prohibit the transmission of vibrations to the test area. Air flow measurements were made by means of a Pitot-static tube inserted through the top of the channel. The tube was made by Mr. Niven and was constructed out of two lengths of metal tubing of extremely small diameter and bore (about 1 mm. bore). The tube was connected by flexible tubing to a "Mercury" Greer Manometer (a pressure head measuring device using reflexions of a light beam from a membrane onto a photo-electric cell). The manometer was extremely sensitive after it had been completely masked from any stray light - the casing supplied on the equipment was not light-tight. The tube could be moved up and down inside the channel by means of a screw device acting on a toothed bar. From the height of the tube in the air flow and the reading on the manometer, velocity profiles could be obtained which were the same as those obtained previously for the clean film work (see Craik (1965)). It may be remarked that the tube used was much smaller than that used by Craik and measurements could be made very close to the film surface.

Water entered the channel through sponge rubber which was closely fitted into a reservoir. This reservoir was situated in front of the glass plate but behind the upstream honeycomb grid. The top surface of the sponge was made absolutely level. The water flow rate was regulated by means of a needle valve. At the downstream end of the channel, the water flowed off the plate into a V-shaped collection reservoir and was then led into a measuring cylinder. Hence the flow rate could be calculated.

Lengths of the channel were marked off by pieces of measuring tape fixed across the underside of the glass plate. Lycopodium dust could be sprinkled onto the water film surface and individual dust particles timed as they traversed one or more lengths of the channel. The surface film velocity could then be found at several points across the width of the channel and the mean surface velocity determined graphically.

The surface of the uniform film was horizontal and so its motion was independent of gravity. Also, the ratio of film thickness to channel height was small and so the pressure gradient required to maintain the air flow had negligible effect upon the velocity profile in the water film. Since the film was subject to a uniform shear stress this velocity profile was very nearly linear and the mean velocity of the film was half its surface velocity. Since the surface velocity, V, the volumetric flow rate, Q, and the channel width, w, are all known, the mean film thickness, h, may be found from the expression $h = 2Q/Vw$.

Two different types of contaminant were used:- "Teepol" and camphor.

Teepol is an industrial cleansing solution which easily spreads over and diffuses with water. Before use it was filtered to remove suspended particles. It was allowed to drip from a thistle funnel of very narrow bore onto the sponge rubber at the entry section. The Teepol was used in two concentrations: 1 part Teepol to 2 parts and to 4 parts of water. The drip rate was reasonably constant and was measureable (= 0.11 cc./min. for 1:2 and = 0.15 cc./min. for 1:4).

Camphor ($C_{10}H_{17}Cl$; Molecular weight = 172.69) is a solid which spreads quickly over the surface of a water film but does not dissolve or diffuse into the water. It was applied by placing solid blocks on the sponge rubber and allowed to spread naturally onto the surface of the water film.

The surface properties of the contaminants, such as the lowering in the value of surface tension were not measured. Difficulty would be experienced if an attempt were made to measure them within the apparatus.

By careful adjustment of the water flow rate and scrupulous cleaning of the glass plate, very thin, uniform water films could be maintained in the presence of the air flow. When Teepol, the cleansing agent, was used thin films were easily attained but trouble was found with camphor. In this latter case, it was found that the water film would not leave the plate at the outlet end and reversal in the flow would occur. This was due to the lowering in the value of surface tension causing a smaller contact angle between the liquid and the plate's edge. The trouble was overcome by suspending damp pieces of foam rubber from thin wires onto the edge of the plate. This eliminated the sharp break in surface afforded to the film and the film then left the plate quite easily. A better procedure might have been to bevel the edge of the plate but this would have meant complete dissembly of the apparatus and some sophisticated glass working.

§6.2: Observations.

As was to be expected, the apparatus when used by Mr. Niven reproduced the results of Craik. It was proposed therefore to investigate what difference, if any, contaminant made and to observe any new phenomena.

With a constant air flow and uncontaminated water films, the reduction in the film thickness caused the following sequence of events, as recorded by Craik (1966).

" (i) a 'pebbled' surface occurred for thick films.

(ii) regular waves travelling down the channel were obtained on decreasing the film thickness. (fast waves)

(iii) these waves disappeared for still thinner films, leaving an essentially smooth surface.

(iv) for very thin films, the surface again exhibited disturbances..... (slow waves)

(v) when the flow rate was decreased still further, dry patches formed on the plate. "

(a) "Teepol" contaminant.

With one drop of Teepol, waves in cases (i) and (ii) were eliminated instantaneously to reappear after several seconds when all the Teepol had been carried out of the apparatus.

In fact, the effect of one drop was so great that no waves of this type could be formed within the range of wind speeds available. On reduction of water flow with a constant drip rate waves of type (iv) could be obtained but only at film thicknesses less than those obtained before for similar conditions. Therefore, the detergent, in general, widened the region of stability. Results for critical film thickness against the maximum air velocity for both concentrations are compared in Figure 6.2 with the uncontaminated results of Craik (1968). The region containing the red spot is that of stability. Due to the sudden overall drying up of the film at extremely small depths no results could be taken at film thicknesses less than 0(0.02 cm.). This phenomenon can be compared to the dry patches which formed at small film thicknesses in the experiments without contaminant and which also meant that no results could be taken. No results for transition to fast waves could, of course, be given in the figure. Thus, the stability region has been widened in both 'directions'.

However, one feature occurred with very thin contaminated films which was absent in the uncontaminated case. During case (iii) streaks formed lengthwise down the channel. They would start at the entry section and continue varying distances downstream, sometimes indeed to the exit section. They were quite difficult to photograph - the channel was illuminated by floodlights placed to one side and then the streak cast a 'shadow' on a sheet of white paper attached to the underside of the glass plate. Naturally,

Figure 6.2 Maximum air velocity plotted against thickness of water film with different contaminants.

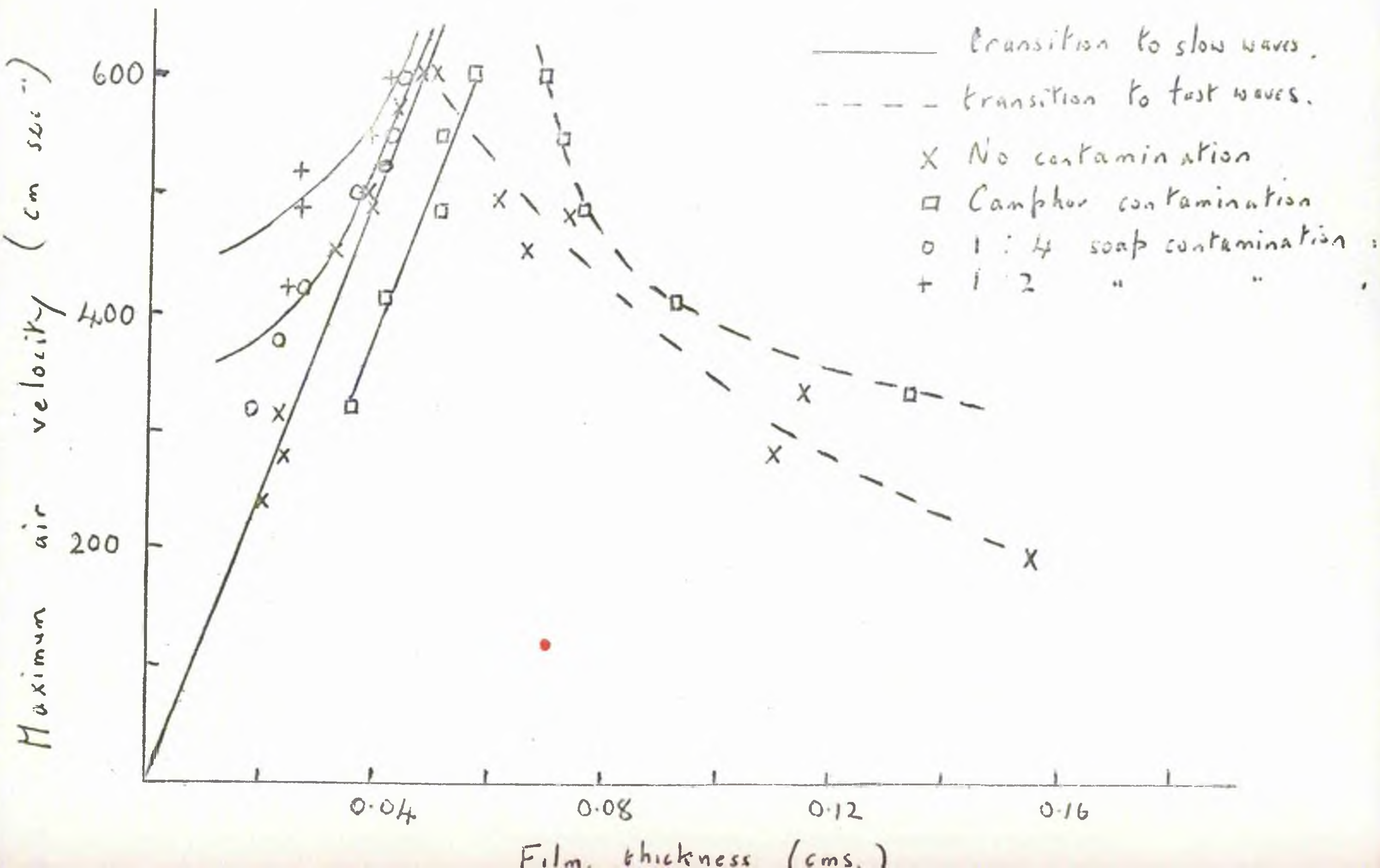


Plate 1: Photograph of typical streak.

when this phenomenon occurred no results could be obtained for 'slow' waves but the streaks could be eliminated by careful smoothing of the sponge rubber of the entry section. A photograph of a typical streak is shown in Plate 1.

The explanation of this occurrence is simple but interesting. If there is a slight indentation in the sponge rubber entry section the film leaves with the same indented profile. When the surface is contaminated with Teepol, it has greater resistance to deformation and the liquid is either prevented or greatly impeded from flowing into this indentation. In this way the streak extends down the channel with the flow:- the fluid flow in the direction of the wind being much greater than that which can eventually flow cross-wise. A derivation of the liquid surface profile is given in Appendix II.

(b) Camphor contaminant.

Here again the fast waves of case (ii) were affected by the contaminant but not so spectacularly as before. The transition from 'fast' waves to the stable film could now be easily measured. The film was made more stable than when uncontaminated and results can be compared in Figure 6.2. The region of stability can be seen to have been extended.

When the trouble with the film at the downstream end had been overcome, results for the transition from the stable film to 'slow' waves could be taken. The transition was found to occur at larger film thicknesses than when uncontaminated. The results are again shown in Figure 6.2. The region of stability in this case has been decreased.

§6.3: Discussion and comparison with theory.

The results of the experiments with detergent are now compared with the previous theory.

With the detergent, a contaminant which diffuses easily normal to the surface, the fast waves were found to be completely annihilated in the experiments. The flow rate was extremely fast and one would expect that diffusion would not have had too important a role even though the phase velocity was large. This result can be compared with the theory of Craik (1968) for thicker films. His results were given in terms of an elasticity parameter and the mean shear stress. They showed that the presence of contaminant on thicker films had a stabilizing effect on the flow.

For the thinner films, the stabilizing nature of this diffusive contaminant on the 'slow' waves was apparent in the experiments. This can be compared to the case I(c) in the results of the previous chapter when the simple diffusion on and off the surface was considered to be more important than the diffusion within the bulk of the liquid film. In case I(c) the diffusion effect was taken to be larger than the effect of the presence of contaminant on the surface. The result was that the flow was more stable and travelled at a speed less than that of the surface - the more contaminant the slower they travelled. It must be noted that the waves seen inthe experiments would be influenced by non-linear effects and so the results cannot be fully corroborated with the theory.

The 'fast' wave instability again experienced a stabilising effect when camphor contaminant was applied. This contaminant does not diffuse rapidly normal to the surface into the bulk liquid but does diffuse readily along the surface. Again, the role of surface elasticity was one of stabilisation in agreement with the work of Craik (1968) on thicker films.

In the case of 'slow' waves, camphor contamination tended to destabilise. In the cases (a) of the previous chapter, for small diffusion effect, the surface elasticity term of the tangential stress is the critical factor and is a stabilising or destabilising influence depending on whether the wave velocity is greater or less than the surface velocity. In the observations the waves did move more slowly than the surface i.e. $c_r' < V$, and so surface elasticity theoretically would provide a destabilizing effect. The film thickness is of the required order of magnitude, $O(10^{-2}$ cm.) i.e. 4.10^{-2} cm. With relation to the size of the surface elasticity coefficient and this type of instability, Craik (1968) states that "values of γ_1 as small as $O(10^{-2}$ dyne per cm.) may have a significant effect". In the experiments, the values for the surface elasticity would be much larger than this.

APPENDIX I

RELATIONSHIPS

GOVERNING SURFACE CONCENTRATION

In 1916, Langmuir considered adsorption, when applied to a gas-liquid interface, as due to the condensation of gas molecules arriving at the surface from the gas followed by evaporation after a certain time. Those molecules reaching the surface may either rebound immediately or condense for a finite time and evidence shows that many do actually condense. This same mechanism is directly applicable to adsorption of liquid surfactant molecules to and from the surface of a liquid from its bulk solution.

Consider unit area of a surface with a fraction β of it covered with adsorbed surfactant. Let the rate of evaporation of adsorbed surfactant be $n\beta$ molecules per second and the rate of molecules striking the surface be γ. On supposing that a fraction α of them condense and that no condensation occurs on an adsorbed layer the rate of condensation will be

$$\alpha\gamma\,(1-\beta)\,.$$

In equilibrium this must equal the rate of evaporation i.e. $n\beta$, and then

$$\alpha\gamma = \frac{n\beta}{1-\beta} \qquad (A.1)$$

The rate of striking varies proportionally to the concentration i.e. $\alpha\gamma = kc_o$. On letting the number of molecules in a complete adsorbed layer be N_o and the number adsorbed at pressure p be η, then

$n = N_o\beta$ and equation (A.1) becomes

$$\frac{n}{N} = \beta = \frac{kc_o}{n + kc_o}$$

Now the surfactant adsorption, $\Gamma = \beta\Gamma_\infty$ where Γ_∞ is the saturation adsorption and the equation becomes

$$\Gamma = \frac{\Gamma_\infty c_o}{\frac{n}{k} + c_o} = \frac{\Gamma_\infty c_o}{a + c_o} \qquad (A.2)$$

where a is a constant. Equation (A.2) is known as the Langmuir adsorption isotherm equation (ref. Adam, 1941, p117).

It is required to obtain a relationship between surface tension, the concentration and the surfactant adsorption. This is obtained from a form of Gibbs' adsorption equation (an equation derived from energy considerations by an analytical proof: see Gibbs' Collected Papers, 1957). When θ is the temperature and μ_n is the chemical poential of the surfactant, Gibbs' equation in its basic form is

$$dT = -(\Gamma)\, d\mu_n \qquad (A.3)$$

for a dilute solution of this single surfactant. The chemical potential is related to the temperature and concentration of the surfactant through the equation

$$d\mu_n = R_g\theta\, d(\ln c_o) \qquad (A.4)$$

where R_g is the gas constant. Using this equation and (A.2) in equation (A.3), we obtain Gibbs' equation in the form

$$dT = -R_g\theta \frac{\Gamma_\infty c_o}{a + c_o}\, d(\ln c_o) \qquad (A.5)$$

or, $$dT = -R_g \theta \Gamma_\infty \frac{dc_o}{a + c_o}$$

An integration yields the result

$$T_{pure} - T_{contam.} = R_g \theta \Gamma_\infty \ln(1 + \frac{c}{a}), \qquad (A.6)$$

an equation found empirically by Szyszkowski in 1908 (ref. Adam 1941, p116).

From equation (A.4) we obtain

$$\frac{R_g \theta}{c_o \frac{\partial \mu_n}{\partial \Gamma}} = \frac{\partial \Gamma}{\partial c_o} = \left(\frac{D\Gamma}{Dc_o}\right)_{un} = \lambda,$$

which forms part of equation (5.6), where the subscript 'un' represents evaluation in a steady state.

From equation (A.5) we also obtain

$$\begin{aligned} \nabla T &= - R_g \theta \Gamma \frac{\nabla c_o}{c_o} \\ &= - \lambda \Gamma \frac{\partial \mu_n}{\partial \Gamma} \nabla c_o \\ &= \left(\frac{dT}{d\Gamma}\right)_{un} \nabla(\Gamma) \end{aligned}$$

where $\Gamma = \lambda c_o$

i.e. $$\nabla T = \left(\frac{dT}{d\Gamma}\right)_{z=0} \nabla(\Gamma - \Gamma_{un})$$

where $\Gamma - \Gamma_{un} = \lambda (c_o - c_{o_{un}})_{z=0}$

This latter expression has $z = 0$ as the reference level with respect to which perturbed quantities are measured.

The remaining parts of equations (5.5) and (5. 6) have now been derived.

APPENDIX II

DERIVATION OF THE PERTURBED SURFACE PROFILE FOR CONTAMINATION BY DETERGENT

Cartesian co-ordinates (x, y, z) are chosen in the unperturbed free surface as in Figure A.1. (The y-co-ordinate is into the page.)

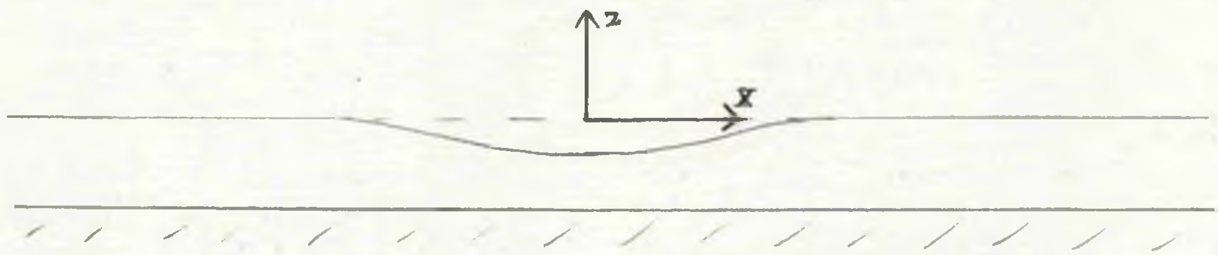


Figure A.1

In this two-dimensional cross-section the equation of the liquid-gas interface is $z = \eta(x)$ and the principal curvatures of the interface are

$$\frac{1}{R_1} = 0 , \qquad \frac{1}{R_2} = \frac{\eta''}{(1 + \eta'^2)^{\frac{3}{2}}}$$

where the dashes denote differentiation with respect to x, and R_1 and R_2 are the principal radii of curvature of the intercepts of $z = \eta(x)$ by the two orthogonal planes Oxz, Oyz.

The condition for equilibrium at any point on the interface is

$$\rho g z - \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) = \text{const.}$$

where ρ is the liquid density, g is the gravitational force and γ is the coefficient of surface tension. Using the expressions for R_1 and R_2 this equation becomes

$$\frac{\rho g}{\gamma}\eta - \frac{\eta''}{(1+\eta'^2)^{\frac{3}{2}}} = 0 ,$$

the constant on the right-hand side being zero because the interface becomes plane far from the origin, where $\eta = 0$. One integration gives

$$\frac{\rho g}{\gamma}\frac{\eta^2}{2} + \frac{1}{(1+\eta'^2)^{\frac{1}{2}}} = C$$

and the same boundary condition shows that $C = 1$. This equation can be transformed into

$$x = \frac{1}{d}\int_0^\theta \frac{\cos^2\theta \, d\theta}{\sin\theta(1+\cos^2\theta)^{\frac{1}{2}}} + \frac{D}{d}$$

where $d^2 = \rho g/2\gamma$ and $\eta d = \sin\theta$, and this has the solution

$$xd = -\frac{1}{\sqrt{2}}\cosh^{-1}\left(\frac{\sqrt{2}}{\eta d}\right) + \left(2-(\eta d)^2\right)^{\frac{1}{2}} + D$$

At $x = 0$, $\eta' = \infty$, $\eta d = 1$. This means that at $x = 0$, there is a cusp and near $x = 0$ the surface profile cannot be given by the above equation. In fact, there would be a slow transverse flow of the fluid towards the η-axis, and the surface profile can only be estimated.

Finally, the equation of the surface profile is, in part

$$|xd| = \frac{1}{\sqrt{2}}\left\{\cosh^{-1}\sqrt{2} - \cosh^{-1}\left(\frac{\sqrt{2}}{\eta d}\right)\right\} + (2-(\eta d)^2)^{\frac{1}{2}} - 1$$

This profile is depicted in Figure A.2. by the full line - the broken is estimat[ed]

An analysis, similar to this, has been performed by Rothrock (1968) in his investigations into the formation of rivulets when a thin liquid film flows down the underside of an inclined plane.

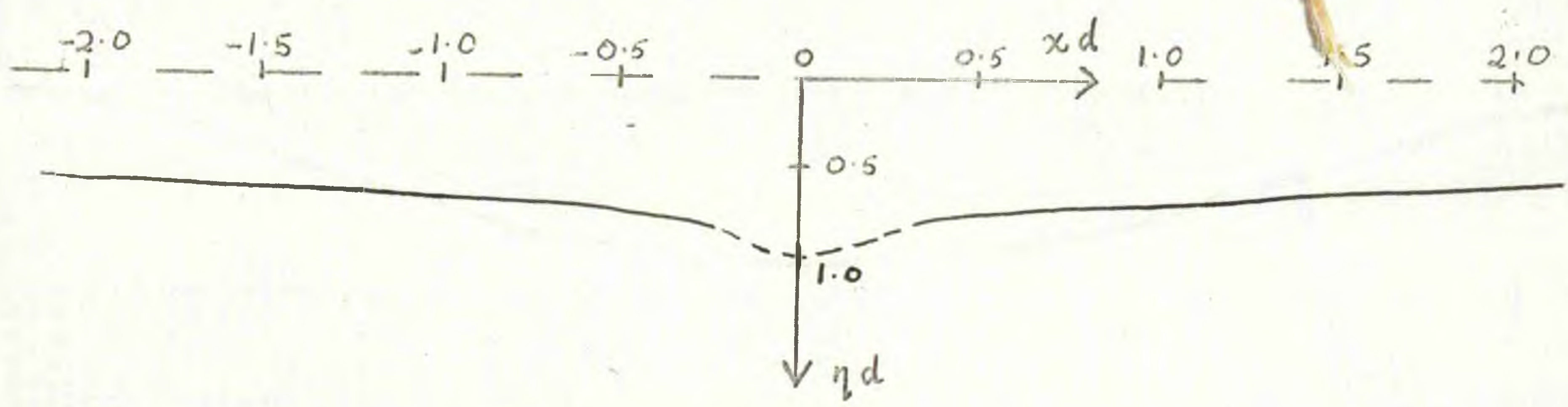


Figure A.2: Theoretical surface profile for contaminated film.

REFERENCES

ADAM, N.K. (1941) The physics and chemistry of surfaces. Oxford University Press.

AITKEN, J. (1883) On the effect of oil on a stormy sea. Proc. Roy. Soc. Edin. 12, 56.

BENJAMIN, T. BROOKE (1957) Wave formation in laminar flow down an inclined plane. Jour. Fluid Mech. 2, 554.

" " (1959) Shearing flow over a wavy boundary. Jour. Fluid Mech. 6, 161.

" " (1961) The development of three-dimensional disturbances in an unstable film of liquid flowing down an inclined plane. Jour. Fluid Mech. 10, 401.

" " (1963) Effects of surface contamination on wave formation in falling liquid films. Arch. Mech. Stos. 16, 615.

COHEN, L.S. & HANRATTY, T.J. (1965) Generation of waves in the concurrent flow of air and a liquid. Amer. Inst. Chem. Eng. Jour. 11, 138.

CRAIK, A.D.D. (1965) Wind-generated waves in liquid films. Ph. D. Thesis. (Cambridge)

" (1966) Wind-generated waves in thin liquid films. Jour. Fluid Mech. 26, 369.

" (1968) Wind-generated waves in contaminated liquid films. Jour. Fluid Mech. 31, 141.

" (1969) The stability of plane Couette flow with viscosity stratification. Jour. Fluid Mech. 36, 685.

CRAIK, A.D.D. & SMITH, F.I.P. (1968) The stability of free-surface flows with viscosity stratification. Jour. Fluid Mech. 34, 393.

DAVIES, J.T. & VOSE, R.W. (1965) On the damping of capillary waves by surface films. Proc. Roy. Soc. Lon. Ser.A 286, 218.

DORRESTEIN, R. (1951) General linearised theory of the effect of surface films on water ripples I. Proc. Kon. Ak. v. Wet. Amst. B 54, 260.

DRAZIN, P.G. (1962) On stability of parallel flow of an incompressible fluid of variable density and viscosity. Proc. Camb. Phil. Soc. 58, 646.

FRANKLIN, B. (1774) On the stilling of waves by means of oil. Phil. Trans. Roy. Soc. Lon. 64, 445.

GASTER, M. (1962) A note on the relation between temporally-increasing and spatially-increasing disturbances in hydrodynamic stability. Jour. Fluid Mech. 14, 222.

GOTTIFREDI, J.C. & JAMIESON, G.J. (1968a) The suppression of wind-generated waves by a surface film. Jour. Fluid Mech. 32, 609.

" (1968b) The interpretation of measurements of the amplitude of capillary waves generated by a wind. Chem. Engng. Sci. 23, 331.

HANSEN, R.S. & MANN, J.A. (1964a) Propagation characteristics of capillary ripples I. Jour. App. Phys. 35, 152.

" (1964b) Effect of gravity on plane capillary waves on visco-elastic films. Jour. App. Phys. 35, 384.

KEULEGAN, G.H. (1951) Wind tides in small closed channels. J. Res. Nat. Bur. Stands. 46, 358.

LAMB, H. (1895,1932,1945) Hydrodynamics. 1st,5th,6th editions. Cambridge Univ. Press (2); Dover Public

LEES, L. & LIN, C.C. (1946) Investigation of the stability of the laminar boundary layer in a compressible fluid. NACA Tech. Note No. 1115.

LEVICH, V.G. (1962) Physicochemical Hydrodynamics.
Prentice Hall, New York.
(translation of Fisico-khimicheskaya Gidrodinamika. Moscow, 1941.)

LIGHTHILL, M.J. & WHITHAM, G.B. (1955) On kinematic waves. I. Flood movements in long rivers.
Proc. Roy. Soc. Lon. Ser. A 229, 281.

LIN, C.C. (1955) The Theory of Hydrodynamic Stability.
Cambridge Univ. Press.

LYUBUTSKAYA, L.V. (1961) The stability of a plane parallel flow of viscous liquid under the influence of constant tangential tension and inclined force of gravity.
Zh. Vych. Math. 1, 1139.

MANN, J.A. & HANSEN, R.S. (1963) Measurement of capillary ripples on visco-elastic films.
J. Colloid Sci. 18, 805.

MICHAEL, D.H. (1961) Note on the stability of plane parallel flows.
Jour. Fluid Mech. 10, 525.

MILES, J.W. (1957) On the generation of surface waves by shear flows.
Jour. Fluid Mech. 3, 185.

" (1959) On the generation of surface waves by shear flows. Parts 2& 3.
Jour. Fluid Mech. 6, 568 & 583.

" (1962a) On the generation of surface waves by shear flows. Part 4.
Jour. Fluid Mech. 13, 433.

" (1962b) Surface-wave damping in closed basins.
Proc. Roy. Soc. Lon. Ser. A 297, 459.

PLATE, E.J., CHANG, P.C. & HIDY, G.M. (1969) Experiments on the generation of small water waves by wind.
Jour. Fluid Mech. 35, 625.

REYNOLDS, O. (1880) On the effect of oil in destroying waves on the surface of water.
Brit. Ass. Rep. (or 'Collected Papers' p409)

ROTHROCK, A. (1968) Ph. D. Thesis (Cambridge)

SEMENOV, E.V. (1964) On a problem of the hydrodynamic theory of stability in the case of variable viscosity.
Izv. Akad. Nauk. SSSR Mekh. i. Mash inéstr 4 161.

SQUIRE, H.B. (1933) On the stability for three-dimensional disturbances of viscous fluid flow between parallel walls.
Proc. Roy. Soc. A, 142, 621.

TEMPEL, M.v.d. & RIET, R.P.v.d. (1965) Damping of waves by surface-active materials.
Jour. Chem. Phys. 42, 2769.

WATSON, J. (1960) Three-dimensional disturbances in flow between parallel planes.
Proc. Roy. Soc. Lon. Ser. A, 254, 562.

YIH, C-S. (1954) Stability of parallel laminar flow with a free surface.
Proc. 2nd U.S. Congr. App. Math, 623.

" (1955) Stability of two-dimensional flows for three-dimensional disturbances.
Quart. App. Math. 12, 434.

" (1963) Stability of liquid flow down an inclined plane.
Phys. Fluids 6, 321.

" (1967a) Instability due to viscosity stratification.
Jour. Fluid Mech. 27, 337.

" (1967b) Instability of laminar flows due to a film of adsorption.
Jour. Fluid Mech. 28, 493.

" (1968) Fluid motion induced by surface tension variation.
Phys. Fluids 11, 477.

ZAITSEV, A.A. (1960) On the stability of a viscous film on a solid in a gas.
Vestn. Mosk. Univ. Ser. Mekh. 2, 53.