ASPECTS OF NATURAL CONVENTION AND OF NON-LINEAR HYDRIDYNAMIC STABILITY

J. R. Usher

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PREFACE

In Part I of this thesis, steady and time-dependent, natural-convection similarity flows with mass transfer are discussed. Similarity flows for natural convection on families of two-dimensional bodies with closed lower ends are enumerated, when both a temperature distribution and a suction velocity distribution are prescribed at the body surface. For steady similarity flow on a heated vertical flat plate, with mass transfer at the surface, a numerical procedure is introduced for determining the velocity and temperature profiles. These results are presented in Figs. 2 and 3. Other similarity flows may be found by the same method.

A simplification, valid for "strong" suction, is discussed.

An extension of Mangler's transformation [1948] is given which reduces the equations governing axisymmetric flow to those for two-dimensional flow in steady natural convection.

In Part II non-linear resonant instability in parallel shear flows is discussed. A.D.D.Craik's (see Usher & Craik [I]) modified version of Bateman's [1956] variational formulation for viscous flows is employed to derive the second-order interaction equations governing the temporal evolution of a resonant wave triad in a shear flow. (An extension of Craik's variational formulation to free surface flows is presented but is not required in the subsequent analysis for the resonance problem). This problem was treated previously using a 'direct' approach (employing the Navier-Stokes equations) by Craik [1971]. The major advantage of the present method over the 'direct' method is the substantial reduction in algebraic complexity. Also, a justification of the validity of Craik's previous analysis is given.

For this same resonance problem, third-order interaction equations are derived by the 'direct' method since, to this order of approximation, little advantage is to be gained from the variational formulation. The resonance theory is thereby developed to the same order of approximation as the non-resonant third-order theory of Stuart [1960, 1962].

An asymptotic analysis for large Reynolds numbers reveals that the magnitudes of the third-order interaction coefficients - like certain of those at second-order - are remarkably large. Such results lead to a discussion of the regions of validity of the perturbation analysis. Also some light is shed on the roles played by resonance and three-dimensionality in the non-linear instability of shear flows.

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PART I

NATURAL-CONVECTION SIMILARITY FLOWS WITH MASS TRANSFER

§1. INTRODUCTION

The analysis and experiments by Schmidt & Beckmann [1930] of the steady natural-convection flow of air on an isothermal, vertical flat plate constitute one of the earliest comprehensive studies of this subject. Others, notably Eckert & Soehngen [1951], later verified and extended their experimental results. There followed various theoretical investigations of steady flows on vertical flat plates. Sparrow & Gregg [1956 , 1958] found similarity solutions for certain prescribed temperature distributions and also for uniform heat flux at the plate. This last case has been investigated experimentally by Dotson [1954]. Ostrach [1953], as part of a general investigation of steady laminar natural convection, examined various similarity flows. The influence of mass transfer at the plate has been considered by Mabuchi [1963] and Brdlik & Mochalov [1965], who found similarity solutions with surface temperature distributions of the form $T_{\infty} + Ax^n$ and suction or blowing velocities proportional to $x^{(n-1)/4}$ where x denotes distance from the leading edge, T_{∞} denotes the ambient temperature and A is a constant. Eichhorn [1960] has obtained numerical solutions to the same problem for cases ranging from strong suction to strong blowing. Sparrow & Cess [1961] treated the case of constant suction (or blowing) and constant wall temperature.

Schuh [1948] first investigated natural-convection similarity flows on curved surfaces. He discussed the flows over classes of two dimensional bodies which are characterized by boundary layers of constant thickness. Later Braun, Ostrach & Heighway [1961] examined similarity flows over further classes of two-dimensional and axisymmetric bodies with closed lower ends.

Yang [1960] has found classes of time-dependent similarity solutions for laminar natural-convection flows on vertical flat plates. Menold &

Yang [1962] and Schetz & Eichhorn [1962] discussed time-dependent flows on a vertical plate whose temperature or heat flux was an arbitrary function of time.

Review articles dealing with natural-convection flows have been written by Ede [1967] and Gebhart [1969 , 1973].

The present work extends these investigations by making a comprehensive study of steady and time-dependent natural-convection similarity flows with mass transfer, on two-dimensional bodies.

In §2 the boundary-layer equations for such flows are derived, their range of validity is examined and necessary conditions for the existence of similarity flows are stated. Time-dependent and steady similarity solutions of the boundary layer equations are discussed in § § 3 and 4 respectively. Several flows are found for which there exist analytic solutions of the equations. Numerical solutions for particular steady similarity flows on a semi-infinite vertical flat plate, with mass transfer, are presented in §5 and the results discussed in §6.

In §7 an approximation to the boundary-layer equations is considered which is appropriate for "strong" suction; and various steady and time-dependent similarity solutions are enumerated.

In §8 an extension of Mangler's transformation [1948] is given whereby the equations governing axisymmetric flow are reduced to those for two-dimensional flow in steady natural convection.

§2. GOVERNING EQUATIONS

Consider the symmetric two-dimensional body shown in fig. (1) on p.22. Its contour is described by the distance $r_o(x)$ from the z-axis. The fluid properties are assumed constant, with the exception that variations in density with temperature are taken into account in determining the body force. That is to say, the Boussinesq approximation is made, which neglects variations in fluid inertia due to density changes. Viscous dissipation of mechanical energy and work done against compression are both neglected.

Assuming the equations of state to be

$$\rho = \rho_\infty\left[1 - \beta\,(T - T_\infty)\right] \quad , \tag{2.1}$$

where ρ is the density of the fluid, T is the absolute temperature, T_∞ is the absolute temperature of the ambient fluid, β and ρ_∞ are the coefficient of cubical expansion and the density measured at temperature T_∞, the dimensionless equations governing the laminar natural-convection flow of a thin boundary-layer along a curved wall, with prescribed suction velocity $v_w(x,t)$ and wall temperature $T_w(x,t)$, are (Ostrach 1953)

$$\begin{aligned}
&\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \quad , \\
&\frac{\partial U}{\partial \tau} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \frac{\partial^2 U}{\partial Y^2} + G\left\{1 - \left(\frac{dR_0}{dX}\right)^2\right\}^{1/2} , \qquad (2.2\ a,b,c)\\
&\frac{\partial G}{\partial \tau} + U\frac{\partial G}{\partial X} + V\frac{\partial G}{\partial Y} = P_r^{-1}\frac{\partial^2 G}{\partial Y^2} \quad ,
\end{aligned}$$

subject to boundary conditions

$$\begin{aligned}
&U = 0 \ , \quad V = l v_w/\nu \ , \quad G = g\beta l^3 (T_w - T_\infty)/\nu^2 \ , \quad (Y = 0) \\
&U = 0 \ , \quad G = 0 \ , \qquad\qquad\qquad\qquad (Y \to +\infty) .
\end{aligned} \tag{2.3 a,b}$$

The dimensionless quantities are defined by

$$\begin{aligned}
&R_0 = r_0/l \ , \ Z = z/l \ , \ X = x/l \ , \ Y = y/l \ , \ \tau = \nu t/l^2 \ , \\
&U = ul/\nu \ , \ V = vl/\nu \ , \ G = g\beta l^3 (T - T_\infty)/\nu^2 \ , \ P_r = \nu/\kappa \ ,
\end{aligned} \tag{2.4}$$

where u, v are the dimensional velocity components in the x- and y-directions respectively, ν, κ are the kinematic viscosity and thermal diffusivity measured at temperature T_∞, t is time, l is a characteristic length scale parallel to the wall, and g is acceleration due to gravity.

Necessary conditions that equations (2.2 a,b,c) are valid approximations of the full Navier-Stokes equations are

$$\bar{t} \gg \delta^2_{max.}/l\bar{u} \quad , \qquad \delta_{max}/l \quad \ll 1 \; , \tag{2.5}$$

where $\bar{t}$ is a characteristic time scale, $\bar{u}$ is a constant such that $u(x,y,t) \sim O(\bar{u})$, $\delta_{max.} = \max(\delta_H, \delta_T)$ and δ_H, δ_T are respectively the hydrodynamic and thermal boundary-layer thicknesses. It is also necessary that one of the following sets of conditions is satisfied:

$$|\bar{v}_w| \ll O\left(\bar{u}\,\delta_{min.}/l\right) , \qquad \delta^2_{max}\frac{\partial\alpha}{\partial x} \ll 1, \quad \alpha l \ll 1 ,$$

$$|\bar{v}_w| \sim O(\bar{u}), \qquad l\,\delta_{max}\frac{\partial\alpha}{\partial x} \ll 1 , \quad \alpha\delta_{max} \ll 1 ,$$

$$|\bar{v}_w| \sim O\left(\bar{u}l/\delta_{max}\right), \qquad l\,\delta_{max}\frac{\partial\alpha}{\partial x} \ll 1 , \quad \alpha\delta_{max} \ll 1 , \tag{2.6 a,b,c,d}$$

$$O\left(\bar{u}l/\delta_{max}\right) < |\bar{v}_w| \ll O\left(\bar{u}l^2/\delta^2_{max}\right), \quad l^2\frac{\partial\alpha}{\partial x} \ll 1 , \quad \alpha\delta_{max} \ll 1 ,$$

where $\delta_{min} = \min(\delta_H, \delta_T)$, α is the curvature of the wall and $\bar{v}_W$ is a constant such that $v_W(x,t) \sim O(\bar{v}_W)$. These conditions are obtained by consideration of the terms omitted from the Navier-Stokes equations. But, for brevity, details are omitted. It should be noted that by consideration of buoyancy and viscous effects (see Le Fevre [1956])

$$\max(\delta_H, \delta_T) = \begin{cases} \delta_H & \text{if } P_r \gg 1 , \\ \delta_T & \text{if } P_r \ll 1 . \end{cases} \tag{2.7}$$

When the magnitude of the suction velocity is comparable with, or greater than the maximum velocity within the boundary-layer, then "strong" suction is said to exist; that is

$$|\bar{v}_w| \geq O(\bar{u}) . \tag{2.8}$$

Returning to equation (2.2 b), it can be seen that the curvature of the wall affects only the buoyancy term $\left[1 - \left(\frac{dR_o}{dX}\right)^2\right]G$ and the condition

$$\left| \frac{dR_o}{dX} \right| \leq 1 \tag{2.9}$$

is necessarily satisfied by all surfaces.

From (2.2 a) there exists a dimensionless stream function ψ such that

$$U = \frac{\partial \psi}{\partial Y} , \quad V = -\frac{\partial \psi}{\partial X} . \tag{2.10}$$

Assuming similarity solutions of the form

$$\psi = f(X,\tau)\,F(\eta) , \quad G = h(X,\tau)\,H(\eta) , \quad \eta = Y/j(X,\tau) , \tag{2.11}$$

substitution of (2.10) and (2.11) in (2.2 b, c) yields

$$(a_2 - a_1)\frac{dF}{d\eta} + (a_4 - a_5)\left(\frac{dF}{d\eta}\right)^2 - (a_1\eta + a_4 F)\frac{d^2F}{d\eta^2} = \frac{d^3F}{d\eta^3} + a_6 H ,$$

$$a_7 H - a_1\eta\frac{dH}{d\eta} + a_3 H\frac{dF}{d\eta} - a_4 F\frac{dH}{d\eta} = P_r^{-1}\frac{d^2H}{d\eta^2} . \tag{2.12a,b}$$

subject to boundary conditions

$$F(0) = -V_w\Big/\frac{\partial f}{\partial X} = a_8 \text{, say, } \frac{dF(0)}{d\eta} = 0 , \; H(0) = 1 , \; \frac{dF(\infty)}{d\eta} = H(\infty) = 0 , \tag{2.13}$$

where

$$V_w = v_w l/\nu, \quad a_1 = j\frac{\partial j}{\partial \tau}, \quad a_2 = \frac{j^2}{f}\frac{\partial f}{\partial \tau}, \quad a_3 = \frac{jf}{h}\frac{\partial h}{\partial X}, \quad a_4 = j\frac{\partial f}{\partial X},$$

$$a_5 = f\frac{\partial j}{\partial X}, \quad a_6 = \frac{j^3 h}{f}\left[1 - \left(\frac{dR_0}{dX}\right)^2\right]^{1/2}, \quad a_7 = \frac{j^2}{h}\frac{\partial h}{\partial \tau}. \tag{2.14}$$

For similarity solutions to exist the a_i ($i = 1, \ldots 8$) must all be constants.

The functions $\frac{dF}{d\eta}$ and H describe the tangential velocity and temperature profiles respectively, while the functions j, $\frac{\partial f}{\partial X}$ and h are a measure of the boundary-layer thickness, the suction velocity and temperature along the wall.

§3. EXISTENCE OF TIME-DEPENDENT SIMILARITY SOLUTIONS

Yang [1960], Menold & Yang [1962] and Schetz & Eichhorn [1962] have found various time-dependent, similarity natural-convection flows on vertical flat plates. The present section extends these investigations to such flows on two-dimensional closed-end bodies.

For time dependent flows, $j(X,\tau)$ is assumed independent of X (i.e. $a_5 = 0$ in (2.12 a)); accordingly the boundary-layer thickness is assumed to depend on time alone. The conditions for such similarity flows to exist are enumerated.

Case 1

A time-dependent similarity flow exists about the body

$$\left[1 - \left(\frac{dr_0}{dx}\right)^2\right]^{1/2} = B x^b, \tag{3.1}$$

(where B,b are constants, $B \geq 0$, $0 \leq b \leq 1$) with $v_w = -v_s(2a_1t + A)^{-\frac{1}{2}}$ and $T_w - T_\infty = Q(a_4x)^{1-b}(2a_1t + A)^{-2}$, (where A, a_1, a_4, v_s and Q are positive constants) when

$$a_2 = -a_1,\ a_3 = a_4(1-b),\ a_5 = 0,\ a_6 = D_1 B\left(\frac{l}{a_4}\right)^b,\ a_7 = -4a_1,\ a_8 = \frac{v_s}{a_4 \nu^{\frac{1}{2}}}\ ,$$

$$j(\tau) = \left(2a_1\tau + \frac{\nu A}{2}\right)^{\frac{1}{2}}, \qquad f(X,\tau) = a_4 X\left(2a_1\tau + \frac{\nu A}{l^2}\right)^{-\frac{1}{2}},$$

$$h(X,\tau) = D_1(a_4 X)^{1-b}\left(2a_1\tau + \frac{\nu A}{l^2}\right)^{-2}, \quad \text{where} \quad D_1 = \frac{Qg\beta}{l^b}.$$

Case 2

A similarity flow exists about the body

$$\left[1 - \left(\frac{dr_o}{dx}\right)^2\right]^{1/2} = a_6 e^{-sx}, \tag{3.2}$$

(where s and a_6 are constants) with $v(x,y,t) \equiv 0$ and $T_w - T_\infty = Qe^{sx}(2a_1 t + A)^{-2}$, (where A, a_1 and Q are positive constants) when $a_2 = -a_1$, $a_3 = sl$, $a_4 = a_5 = 0$, $a_7 = -4a_1$;

$$j(\tau) = \left(2a_1\tau + \frac{\nu A}{l^2}\right)^{\frac{1}{2}}, \qquad f(\tau) = D_1\left(2a_1\tau + \frac{\nu A}{l^2}\right)^{-\frac{1}{2}},$$

$$h(X,\tau) = D_1 e^{slX}\left(2a_1\tau + \frac{\nu A}{l^2}\right)^{-2}, \qquad \text{where} \quad D_1 = \frac{Qg\beta}{l}.$$

It should be noted that for such similarity flows the magnitudes of the right-hand sides of (3.1) and (3.2) cannot exceed unity.

Examples of bodies satisfying relationships (3.1) or (3.2) are now given. In (3.1) if $b = 0$ and $a_6 = 1$, then $R_o(X) = \text{constant}$, which represents a vertical flat plate. If $b = 0$ and $|B| < 1$ then $R_o(X) = \left[1 - B^2\right]^{1/2} X$, which represents a class of wedges. If $b = 1$ and B is non-zero, then $\frac{dR_o}{dX} = \left[1 - (BlX)^2\right]^{\frac{1}{2}}$ which represents a class of parabolic-nosed bodies. These last two classes of bodies also arose in the work of Braun et al. [1961] and Ostrach [1964] where suction at the wall was absent.

In (3.2) if a_6 and s are non-zero then by making use of the relationship $\left(\frac{dZ}{dX}\right)^2 + \left(\frac{dR_o}{dX}\right)^2 = 1$, where Z is the dimensionless axial co-ordinate, and noting that for a closed-end body $R_o(X = 0) = R_o(Z = 0) = 0$, there results

$$R_o = \frac{1}{b_1}\left[\log_e\left\{\left(\frac{1}{Z_1} + \left(\frac{1}{Z_1^2} - 1\right)^{1/2}\frac{a_6}{1+A_1}\right\} - \left\{\left(1 - Z_1^2\right)^{1/2} - A_1\right\}\right], \qquad (3.3)$$

where $b_1 = s\ell$, $Z_1 = a_6 - Zb_1$, $A_1 = (1 - a_6^{\,2})^{\frac{1}{2}}$. The requirements for (3.3) to be real are:-

(i) if $b_1 > 0$ then $0 < Z_1 \leq a_6 \leq 1$,

(ii) if $b_1 < 0$ then $0 < a_6 \leq Z_1 \leq 1$.

Cases (i) and (ii) refer respectively to pointed and parabolic-nosed bodies.

In (3.2) if $s = 0$, $a_6 = 1$ and $|a_6| < 1$ correspond respectively to a vertical flat plate and a class of wedges.

§4. EXISTENCE OF STEADY SIMILARITY SOLUTIONS

For steady similarity flows a_1, a_2 and a_7 are identically zero in equations (2.12 a,b) and the conditions for such flows are now enumerated.

Case 1

A steady similarity flow exists about the body

$$\left[1 - \left(\frac{dr_o}{dx}\right)^2\right]^{1/2} = Bx^b , \qquad (4.1)$$

(where B, b are non-negative constants) with $v_w = -v_s x^s$ and $T_w - T = Qx^{4s+1-b}$, (where s, v_s and Q are constants, $s \geq 0$, $4s+1-b \geq 0$, $v_s, Q > 0$) when $a_1 = a_2 = a_7 = 0$, $a_3 = 1 + 4s-b$,

$$a_4 = 1 + s,\quad a_5 = -s,\quad a_6 = B\ell^b E,\quad a_8 = \frac{v_s \ell^{s+1}}{\nu(1+s)},\quad E \equiv \frac{g\beta\ell^3}{\nu^2} Q\ell^{4s+1-b};$$

$j(X) = X^{-s}$, $f(X) = X^{1+s}$, $h(X) = EX^{1+4s-b}$.

Case 2

A steady similarity flow exists about the body

$$\left[1-\left(\frac{dr_0}{dx}\right)^2\right]^{1/2} = a_6 e^{-bx} , \qquad (4.2)$$

(where a_6 and b are constants, $a_6 \geqslant 0$) with $v_w = -v_s e^{sx}$ and $T_w - T_\infty = Qe^{(b+4s)x}$, (where s, v_s and Q are positive constants) when

$$a_1 = a_2 = a_7 = 0, \quad a_3 = (4s+b)\ell , \quad a_4 = -a_5 = s\ell , \quad a_8 = \frac{v_s c_3}{\nu s} ,$$

$$c_3 \equiv \left(\frac{\nu^2}{Qg\beta \ell^3}\right)^{\frac{1}{4}} ;$$

$$j(X) = c_3 e^{-s\ell X}, \quad f(X) = \frac{e^{s\ell X}}{c_3}, \quad h(X) = \frac{e^{(4s+b)\ell X}}{c_3{}^4} .$$

Case 3

A steady similarity flow exists about the body (4.2), with $v(x,y,t) \equiv 0$ and $T_w - T_\infty = Qe^{bx}$, (where Q and b are positive constants) when

$$a_1 = a_2 = a_4 = a_5 = a_7 = 0, \quad a_3 = c_4 c_2 b\ell ,$$

c_2 an arbitrary positive constant, $\quad c_4 \equiv c_2{}^3 \dfrac{Qg\beta \ell^3}{\nu^2}$;

$$j = c_4, \quad f = c_2, \quad h(X) = \frac{c_4}{c_2{}^3} e^{b\ell X} .$$

Cases 1, 2 and 3 demonstrate that for any prescribed-power law or exponential distribution of temperature and suction velocity at the wall (independent of time), a body can be found about which there exists a steady similarity flow. Examples of bodies satisfying relationships (4.1)-(4.2) were discussed in §3.

§5. NUMERICAL SOLUTION OF THE SIMILARITY EQUATIONS FOR STEADY FLOW

There appear to be no simple analytic, steady, similarity solutions of (2.12 a,b); accordingly these equations must be solved numerically. For brevity, attention is restricted to the similarity flow on a vertical flat plate when the suction rate and wall temperature distributions are

$$v_w = - v_s x^s , \qquad T_w = Q x^{4s + 1} + T_\infty$$

where s and v_s are positive constants and for convenience $Q \equiv \dfrac{\nu^2}{g \beta \ell^{4(s + 1)}}$ (see §4, case 1, with $b = 0$). However, the method to be described is equally applicable to any of the body shapes discussed in §3.

The method of solution involves numerical integration of (2.12 a,b), with appropriate values for the constant a_i's, across the boundary-layer in such a way as to satisfy the boundary conditions both at the outer edge of the boundary layer and at the plate. The methods employed previously on such problems involved numerical integration away from the plate with over-specified boundary conditions and subsequent iteration to determine the solution which satisfied the appropriate boundary conditions. However, since the independent variable η does not appear explicitly in these equations, the origin for numerical integration may be chosen arbitrarily. In the present work the origin was chosen at the outer edge of the boundary-layer and the equations integrated inwards towards the plate. We may introduce the transformations

$$f(\mathcal{X}) = B\,Pr^{\frac{1}{2}}\,F(\eta), \quad h(\mathcal{X}) = B^4 H(\eta), \quad \mathcal{X} = \mathcal{X}_w - \frac{Pr^{\frac{1}{2}}}{B}\eta , \qquad (5.1)$$

(where $\mathcal{X}_w$ and B are constants to be determined later), for which (2.12 a,b), with $a_1 = a_2 = a_7 = 0$, $a_3 = 1 + 4s$, $a_4 = 1 + s$, $a_5 = - s$, $a_6 = 1$, yield

$$(s + 1)\, f \frac{dh}{d\mathcal{X}} - (1 + 4s)\, h \frac{df}{d\mathcal{X}} = \frac{d^2h}{d\mathcal{X}^2} , \qquad (5.2)$$

$$(2s + 1)\left(\frac{df}{d\mathcal{X}}\right)^2 - (1 + s)\, f \frac{d^2f}{d\mathcal{X}^2} = h - Pr \frac{d^3f}{d\mathcal{X}^3} . \qquad (5.3)$$

These equations are clearly independent of the parameter B, which will subsequently be chosen in such a way that a boundary condition at the wall is satisfied.

Using a Runge-Kutta technique, (5.2) and (5.3) were integrated away from the point $\mathcal{X} = 0$, the initial values for $\frac{df}{d\mathcal{X}}$, $\frac{d^2f}{d\mathcal{X}^2}$, h and $\frac{dh}{d\mathcal{X}}$ at $\mathcal{X} = 0$ being determined from the asymptotic solutions of (2.12 a,b), with the values for the constant a_i's given above, for large η . The initial value of f at $\mathcal{X} = 0$ was taken to be some number of order unity. With given initial values, there will be some $\mathcal{X} = \mathcal{X}^*$ for which $\frac{df}{d\mathcal{X}} = 0$, and this value $\mathcal{X}^*$ may be taken to define the location of the plate. Denote by f* the value of f at $\mathcal{X} = \mathcal{X}^*$. Now B is chosen such that $H(\eta = 0) = \frac{1}{B^4} h(\mathcal{X}^*) = 1$, thereby satisfying the temperature boundary conditions at the wall. The resultant solution corresponds to a particular suction rate at the wall, related to the value f^*. Hence a single computation yields an informative result without the need for further iteration.

Graphs of the functions $\frac{dF}{d\eta}$ and $H(\eta)$ for the case $s = 0.5$, $Pr = 0.72$ are shown in figs. (2) and (3) for various suction rates.

To compare the case $Pr = 0.72$, $s = -0.25$ with the results of Eichhorn [1960], consider the relations

$$F(\eta) = 2^{3/2} F_e(\eta_e) \;,\quad H(\eta) = H_e(\eta_e) \;,\quad \eta = 2^{1/2} \eta_e \;,$$

where the suffix e refers to variables used by Eichhorn. The graphs of functions $\frac{dF_e}{d\eta_e}$ and $H_e(\eta_e)$ are shown in fig. (4) using Eichhorn's results and those of the present investigation. Eichhorn's results, which were obtained by integration away from the wall, with iteration, are for the case $Pr = 0.73$ and $F_e(\eta_e = 0) = 0.2$, while the present case is for $Pr = 0.72$, and the low suction rate of $F_e(\eta_e = 0) = 0.218$. Good agreement is obtained, the small difference in Prandtl number and the value of $F_e(\eta_e = 0)$ accounting for the slight differences.

§6. DISCUSSION OF NUMERICAL RESULTS

The suction rate and wall temperature distributions are

$$v = -v_s x^s, \; T = Qx^{4s+1} + T_\infty \qquad (y=0),$$

where v_s, Q are positive constants.

The velocity components in the x and y directions, and the temperature distribution, in terms of dimensionless variables, are

$$u = \frac{\nu}{l} X^{1+2s} \frac{dF}{d\eta}, \; v = \frac{-\nu X^s}{l}\left\{\eta s \frac{dF}{d\eta} + (1+s)F\right\}, \; T = Ql^{1+4s} X^{1+4s} H + T_\infty .$$

The local rate of heat transfer q normal to the surface, owing both to conduction and to convection across the plate by the suction velocity, and τ the skin friction at the wall are

$$q = -k\frac{\partial T}{\partial y} + \rho_\infty c_p (T-T_\infty)v, \qquad (y=0)$$

$$= -kQl^{4s}X^{1+5s}\left\{\frac{dH}{d\eta} + \Pr F(1+s)\right\}, \qquad (\eta=0)$$

(Noting that $H(o) = 1$)

$$\tau = \rho_\infty \nu \frac{\partial u}{\partial y}, \qquad (y=0)$$

$$= \rho_\infty \frac{\nu^2}{l^2} X^{1+3s} \frac{d^2F}{d\eta^2} \qquad (\eta=0)$$

where k and c_p are the thermal conductivity and specific heat at constant pressure, respectively.

For the case $s = 0.5$, $\Pr = 0{,}72$ values of $\frac{d^2F}{d\eta^2}$, $\frac{-dH}{d\eta}$ and $-\left\{\frac{dH}{d\eta} + (1+s)\Pr F\right\}$, all evaluated at $\eta = 0$, are shown in table 1. as a function of the suction rate parameter $F(o)$. These respectively represent the skin friction, the heat transfer by conduction alone and the heat transfer due to both conduction and convection.

Table 1.

$F(o)$	$\frac{d^2F(o)}{d\eta^2}$	$-\frac{dH(o)}{d\eta}$	$-\left\{\frac{dH(o)}{d\eta} + (1+S)Pr\,F(o)\right\}$
1.447	0.542	1.685	0.123
0.856	0.649	1.211	0.595
0.680	0.669	1.087	0.598
-0.207	0.613	0.589	0.738
-3.471	0.189	0.028	2.527

It is seen that as the suction rate at the wall increases ($F(o) > 0$) the skin friction and the heat transfer due to both conduction and convection decrease, whereas the heat transfer due to conduction alone increases. However, as the blowing rate ($F(o) < 0$) at the wall increases the skin friction and the heat transfer due to conduction alone decreases, whereas the heat transfer due to both conduction and convection increases. There exists a particular suction rate which results in a maximum shear stress at a given point on the wall.

As stronger blowing is provided ($F(o) < 0$) a heated element of fluid is pushed further from the wall where the buoyancy forces can act to accelerate it and the influence of viscosity due to the presence of the wall is diminished. This effect tends to increase the maximum velocity within the boundary-layer and to displace the point of maximum velocity away from the wall. The net effect is such that the shear stress at the wall is reduced for increased blowing rates.

As the suction rate increases from zero, there is an increase in mass flow towards the wall, where the influence of viscosity is greater. This effect tends to decrease the maximum velocity within the boundary-layer and to displace the point of maximum velocity towards the wall. In this case, the net effect is first such as to increase the wall shear;

but a suction rate is reached which results in a maximum shear stress at each point on the wall; and further increase of suction rate results in a decrease of shear stress at the wall. In the case considered by Eichhorn [1960] the maximum shear at any given point on the wall, was attained when $F_e(o) \approx 0.1$.

The behaviour of the heat transfer parameter can be reasoned as follows. With increased suction, heat is not able to diffuse so far into the fluid and there is a subsequent decrease in heat transfer from the wall. At sufficiently high suction rates there will be zero net flux of heat from the wall. This statement is substantiated by the results of §7 for "strong" suction. However, with increased blowing, heat is transferred from the wall both by conduction and convection, with a resultant increase in heat transfer from the wall.

§7 AN APPROXIMATION FOR "STRONG" SUCTION

A) The Approximate Equations

If condition (2.8) for "strong" suction is satisfied the dimensionless boundary-layer equations (2.2 a,b,c) may themselves be approximated by the equations

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \quad ,$$

$$\frac{\partial u}{\partial \tau} + V_w \frac{\partial u}{\partial Y} = \frac{\partial^2 u}{\partial Y^2} + G\left[1 - \left(\frac{dR_0}{dx}\right)^2\right]^{1/2} \quad , \qquad (7.1\ a,b,c)$$

$$\frac{\partial G}{\partial \tau} + V_w \frac{\partial G}{\partial Y} = P_r^{-1} \frac{\partial^2 G}{\partial Y^2} \quad ,$$

subject to boundary conditions (2.3 a,b).

For the general steady flow in which $v_w \equiv v_w(x) < 0$ and $T_w \equiv T_w(x)$, the solution of these equations, subject to boundary conditions (2.3 a,b), is given by

$$G = G_W(X)\, e^{\Pr V_W(X) Y}, \quad V = V_W(X) - \int_0^Y \frac{\partial U}{\partial X}\, dY , \qquad (7.2a,b)$$

$$U = \left[\frac{N(X)}{V_W^{\,2}(X) \quad \Pr(\Pr-1)}\right] \left(e^{\Pr V_W(X)Y} - e^{V_W(X)Y}\right), \quad (\Pr \neq 1),$$

$$= \frac{N(X)}{V_W(X)}\; Y\, e^{V_W(X)Y} , \qquad (\Pr = 1), \qquad (7.3a,b)$$

where $N(X) = -\left[1 - \left(\frac{dR_0}{dX}\right)^2\right]^{\frac{1}{2}} G_W(X)$.

In the particular case of parallel flow on a vertical flat plate when T_w and v_w ($v_w < 0$) are constant, the solution given by (7.2a) to (7.3b) is independent of X and the velocity component V is uniform. In general, (7.2a) to (7.3b) is not a similarity solution of equations (7.1a,b,c).

For similarity solutions of the form given by (2.11), equations (7.1b,c) become

$$(a_2 - a_1)\frac{dF}{d\eta} - (a_1\eta + a_9)\frac{d^2F}{d\eta^2} = \frac{d^3F}{d\eta^3} + a_6 H , \qquad (7.4)$$

$$a_7 H - (a_1\eta + a_9)\frac{dH}{d\eta} = \frac{1}{\Pr}\frac{d^2H}{d\eta^2} , \qquad (7.5)$$

subject to the boundary conditions (2.13), where a_1, a_2, a_6 and a_7 are defined by (2.14) and for convenience

$$a_9 = a_4\, F(0) . \qquad (7.6)$$

Since, for similarity solutions to exist, (7.4) and (7.5) must reduce to ordinary differential equations in η alone, it is clear that the a_i must again be constants. The number of constants occuring in (7.4) and (7.5) is six, as compared with eight in (2.12a,b).

B) Steady similarity flows

For steady similarity flows a_1, a_2 and a_7 are identically zero in (7.4) and (7.5).

From expressions (2.14) for a_4 and a_6 it can be seen that, for any prescribed f(X) and h(X) (i.e. for any prescribed velocity and temperature distribution at the wall), there exists a body such that

$$\left[1 - \left(\frac{dR_o}{dX}\right)^2\right]^{\frac{1}{2}} = a_6 \frac{f}{h}\left(\frac{1}{a_4} \frac{\partial f}{\partial X}\right)^3 , \tag{7.7}$$

about which there is a similarity flow. For such a similarity flow, it is clear that the magnitude of the right-hand side of (7.7) cannot exceed unity.

The appropriate similarity solution is readily found to be

$$H(\eta^*) = e^{-Pr\,\eta^*} , \tag{7.8}$$

$$\frac{a_9^3}{a_6} F(\eta^*) = c_1 + (1-e^{-\eta^*}) - \eta^* e^{-\eta^*} , \text{ if } Pr = 1 , \tag{7.9}$$

$$= c_1 + \frac{1}{Pr^2} + \frac{1}{Pr(1-Pr)} \left\{ e^{-\eta^*} - \frac{e^{-Pr\,\eta^*}}{Pr} \right\}, \text{if } Pr \neq 1 \tag{7.10}$$

$$\text{where } \eta^* = a_9\eta \quad \text{and } c_1 = \frac{a_8 a_9^3}{a_6} . \tag{7.11}$$

For the case s = 0.5, Pr = 0.72, F(o) = 1.447 good agreement was found between the numerical results of §5 for the full boundary-layer equations (2.12a,b) and the solution given above with $a_9 = (1+s)F(o)$, $a_6 = 1$, for "strong" suction. The comparison is shown in fig. 5. This result helps to define the range of validity of the "strong" suction solution: for here, the suction velocity at the wall is proportional to $2.17x^{\frac{1}{2}}$ while the maximum value of u is proportional to $0.103x^2$. It is worth noting that for large x the strong suction solution is valid even though $|v_w| \ll u_{max}$.

C. Time-dependent similarity flows

For time-dependent flows $j(X,\tau)$ is assumed to be independent of X, (i.e. $a_5 = 0$). The following analytic solution of (7.4) and (7.5) subject to boundary conditions (2.13) then exists for the case $Pr = 1$, $a_2 = -a_1$, $a_7 = -4a_1$, $(a_6, a_1 \neq 0)$, $a_9^2 \neq 3a_1$ and $a_5 = 0$;

$$H = \frac{(\eta + B)}{B(3-C)} \left[3 - a_1(\eta + B)^2\right] e^{\frac{1}{2}\left[C - a_1(\eta + B)^2\right]}$$

$$F = \frac{D}{2(3-C)} \left[2 - C + a_1(\eta + B)^2\right] e^{\frac{1}{2}\left[C - a_1(\eta + B)^2\right]} \frac{-D}{3-C} + a_8$$

where for convenience $B = a_9 / a_1$, $C = a_9^2 / a_1$, $D = a_6 / a_1 a_9$. For H and $\frac{dF}{d\eta}$ to remain positive for all values of η attention must be restricted to cases for which

$$a_9^2 > 3a_1 > 0 \qquad \text{i.e.} \quad v_w^2 > \frac{3}{j}\frac{\partial j}{\partial \tau} .$$

Graphs for the functions H and $\frac{dF}{d\eta}$ are shown in Fig. 6. for the case $a_9 = 2$, $a_1 = 1$, $a_6 = 1$.

In physical terms the above solution represents a time-dependent motion corresponding to an initially high temperature and "strong" suction at the wall, which subsequently decrease with time. Fluid in the thermal boundary-layer is initially heated by conduction. However the temperature decay at the wall subsequently results in the fluid cooling, which accounts for the peak in H.

§8. TRANSFORMATION FROM AXISYMMETRIC TO TWO-DIMENSIONAL STEADY FLOWS

The following transformation relates steady natural-convection flows over axisymmetric closed-end bodies and those over two-dimensional closed-end bodies. This is an extension of Mangler's transformation [1948] in boundary-layer theory. There does not appear to be a similar transformation for time-dependent flows.

Taking the same co-ordinate system as in Fig. 1., the dimensionless laminar boundary-layer equations for steady natural-convection flow over an axisymmetric body, are

$$\frac{\partial}{\partial X}\left(R_0 \mathcal{U}\right) + \frac{\partial}{\partial Y}\left(R_0 \mathcal{V}\right) = 0 \quad , \tag{8.1}$$

$$\mathcal{U}\frac{\partial \mathcal{U}}{\partial X} + \mathcal{V}\frac{\partial \mathcal{U}}{\partial Y} = \frac{\partial^2 \mathcal{U}}{\partial Y^2} + G\left\{1 - \left(\frac{dR_0}{dX}\right)^2\right\}^{1/2} \quad , \tag{8.2}$$

$$\mathcal{U}\frac{\partial G}{\partial X} + \mathcal{V}\frac{\partial G}{\partial Y} = P_r^{-1}\frac{\partial^2 G}{\partial Y^2} \quad , \tag{8.3}$$

In terms of the transformed variables

$$X^* = \int_0^X R_0^2(X_1)\,dX_1 \quad , \qquad Y^* = R_0 Y \quad , \tag{8.4 a,b}$$

$$\mathcal{U}^* = \mathcal{U} \quad , \qquad \mathcal{V}^* = \frac{1}{R_0}\left[\mathcal{V} + \frac{Y\mathcal{U}}{R_0}\frac{dR_0}{dX}\right] \quad , \tag{8.4 c,d}$$

$$G^* = G \quad , \qquad \frac{dR_0^*(X^*)}{dX^*} = \left[1 - \frac{1}{R_0^4}\left(1 - \left(\frac{dR_0}{dX}\right)^2\right)\right]^{1/2} \quad , \quad (R_0 \neq 0) \tag{8.4 e,f}$$

(8.1) to (8.3) formally become

$$\frac{\partial u^*}{\partial X^*} + \frac{\partial v^*}{\partial Y^*} = 0 ,$$

$$u^* \frac{\partial u^*}{\partial X^*} + v^* \frac{\partial u^*}{\partial Y^*} = \frac{\partial^2 u^*}{\partial Y^{*2}} + G^* \left\{ 1 - \left(\frac{dR_o^*}{dX^*} \right)^2 \right\}^{1/2} ,$$

$$u^* \frac{\partial G^*}{\partial X^*} + v^* \frac{\partial G^*}{\partial Y^*} = P_r^{-1} \frac{\partial^2 G^*}{\partial Y^{*2}} ,$$

which are the dimensionless laminar boundary-layer equations for steady natural-convection flow over a two-dimensional body.

The transformation can only be applied to axisymmetric bodies for which

$$1 - R_o^{\,4} \leq \left(\frac{dR_o}{dX} \right)^2 \leq 1 .$$

Permissible bodies must lie in the shaded region shown in fig. 7, which indicates that they must have sufficiently blunt ends. Consider for example the kettle-shaped axisymmetric body given by the expression

$$R_o^{\,4} = \frac{1}{M} \left\{ 1 - \left(\frac{dR_o}{dX} \right)^2 \right\}^{\frac{1}{2}} , \qquad \text{(M a constant, } 0 < M \leq 1) .$$

This transforms to the two dimensional body given by the expression

$$Ro = X \quad (1-M)^{\frac{1}{2}}$$

which represents a vertical flat plate if $M = 1$ and a wedge if $M < 1$.

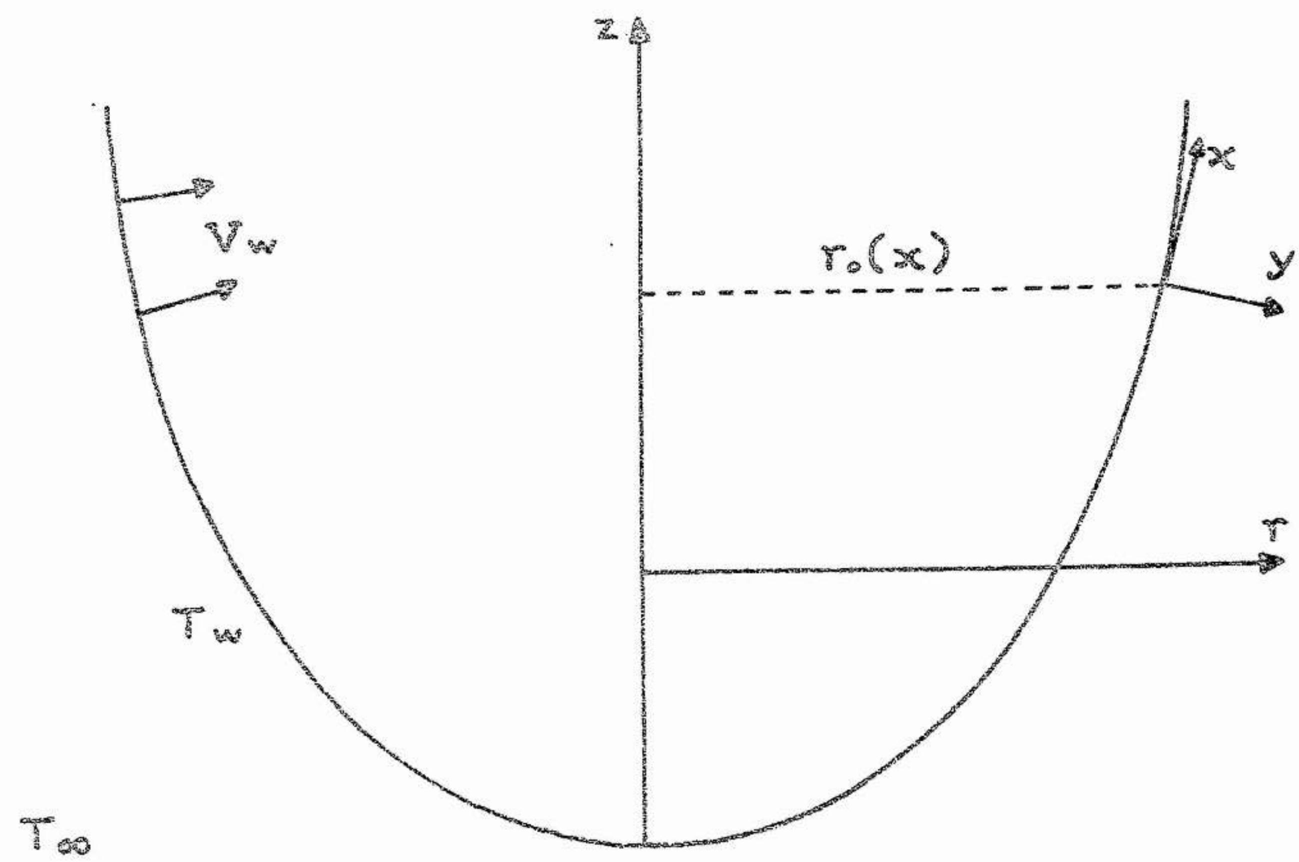
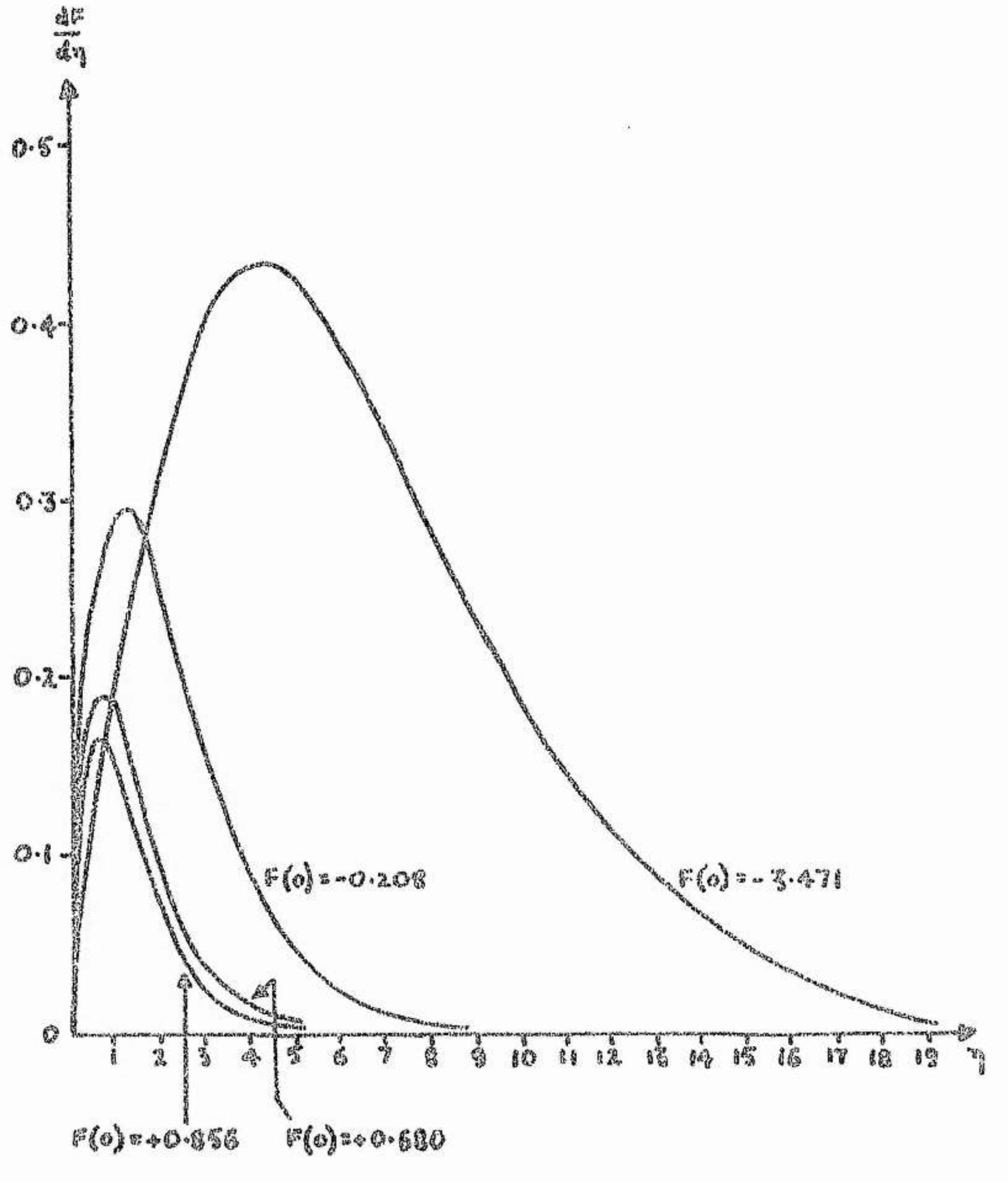


Figure 1.



$\frac{dF}{d\eta}$
0.5
0.4
0.3
0.2
0.1
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 η
F(o) = -0.208
F(o) = -3.471
F(o) = +0.856
F(o) = +0.680

Figure 2.

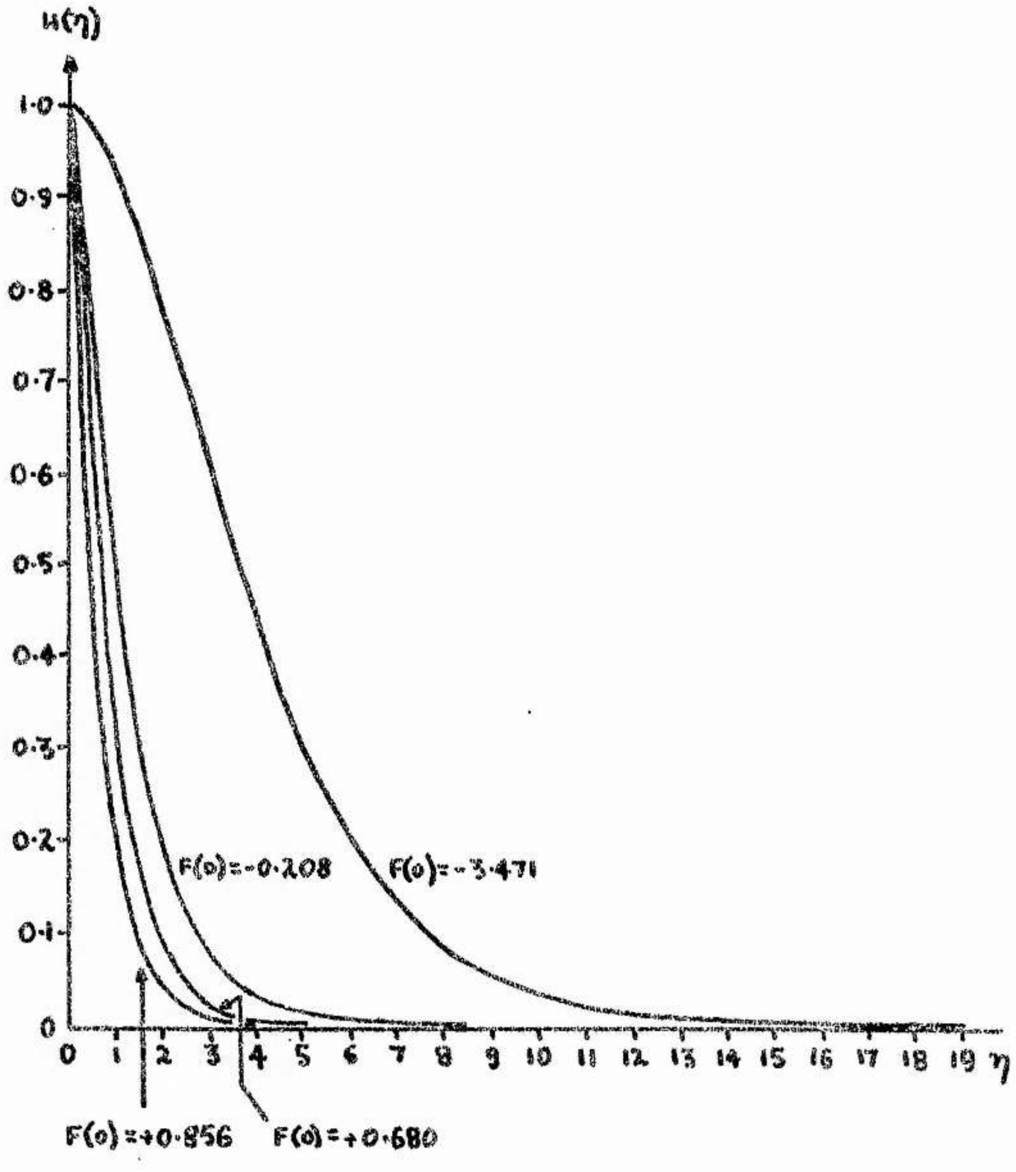
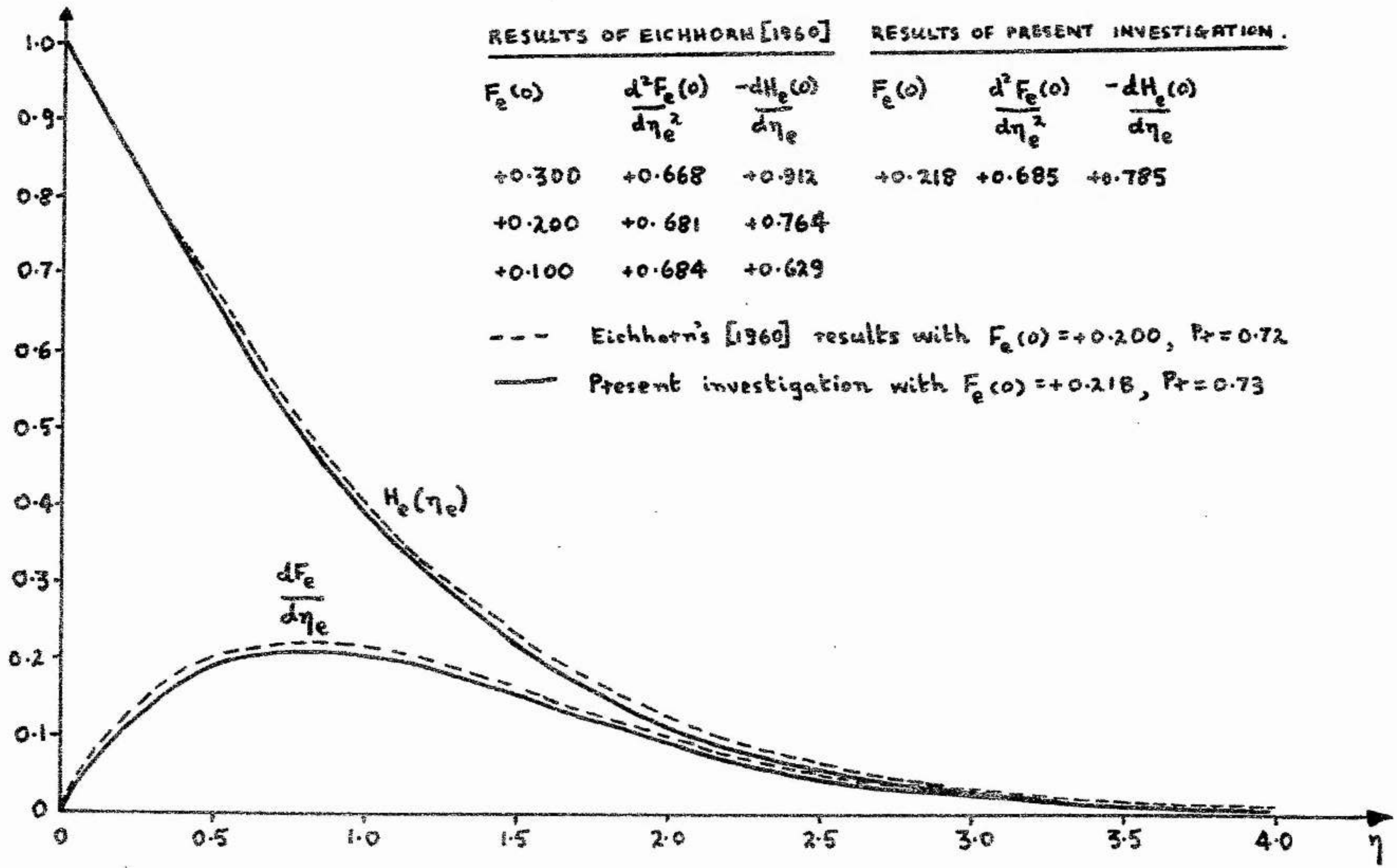

u(η)
1·0
0·9
0·8
0·7
0·6
0·5
0·4
0·3
0·2
0·1
0
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 η
F(0)=-0·208
F(0)=-3·471
F(0)=+0·856
F(0)=+0·680

Figure 3.



RESULTS OF EICHHORN [1960]			RESULTS OF PRESENT INVESTIGATION.		
$F_e(0)$	$\frac{d^2F_e(0)}{d\eta_e^2}$	$-\frac{dH_e(0)}{d\eta_e}$	$F_e(0)$	$\frac{d^2F_e(0)}{d\eta_e^2}$	$-\frac{dH_e(0)}{d\eta_e}$
+0·300	+0·668	+0·912	+0·218	+0·685	+0·785
+0·200	+0·681	+0·764			
+0·100	+0·684	+0·629			

Figure 4.

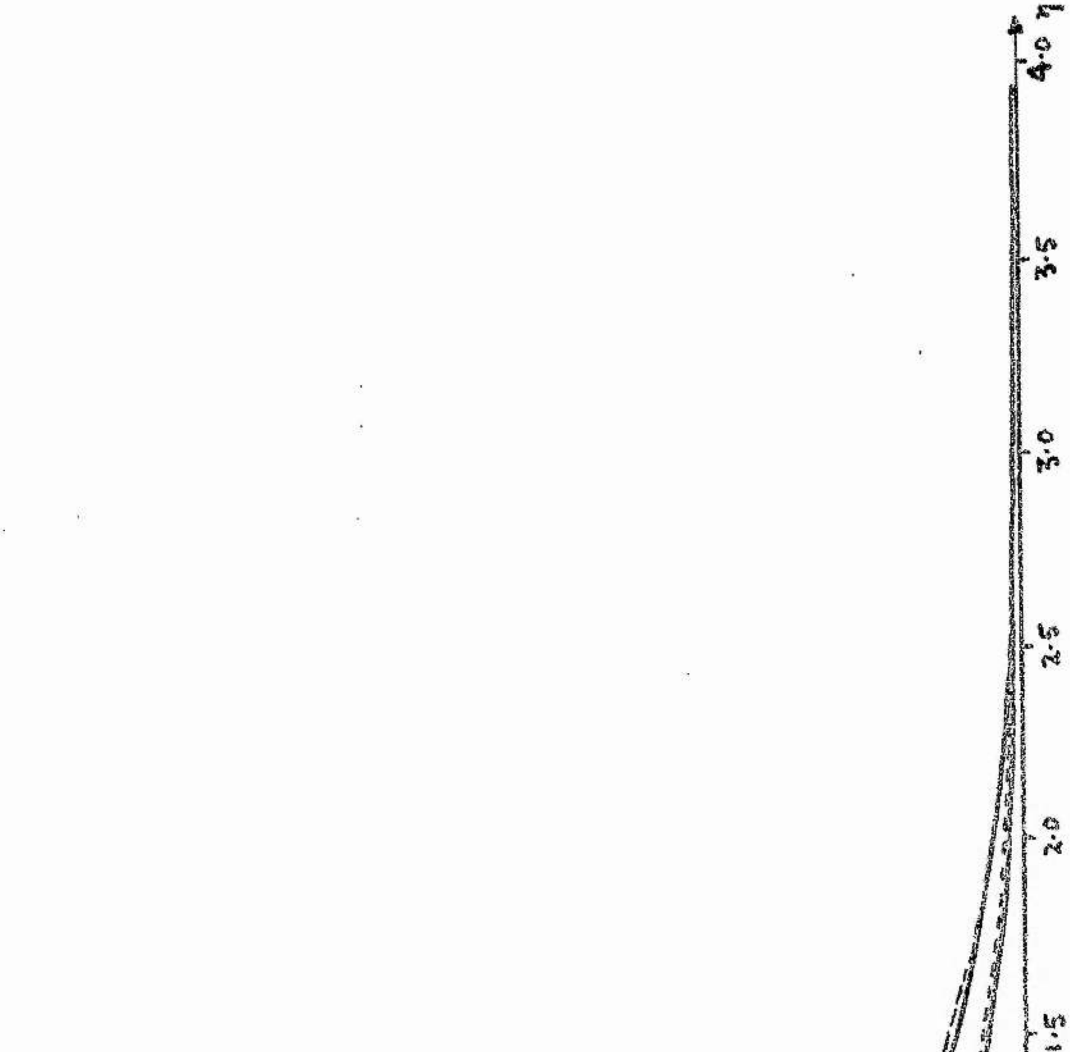


Figure 5.

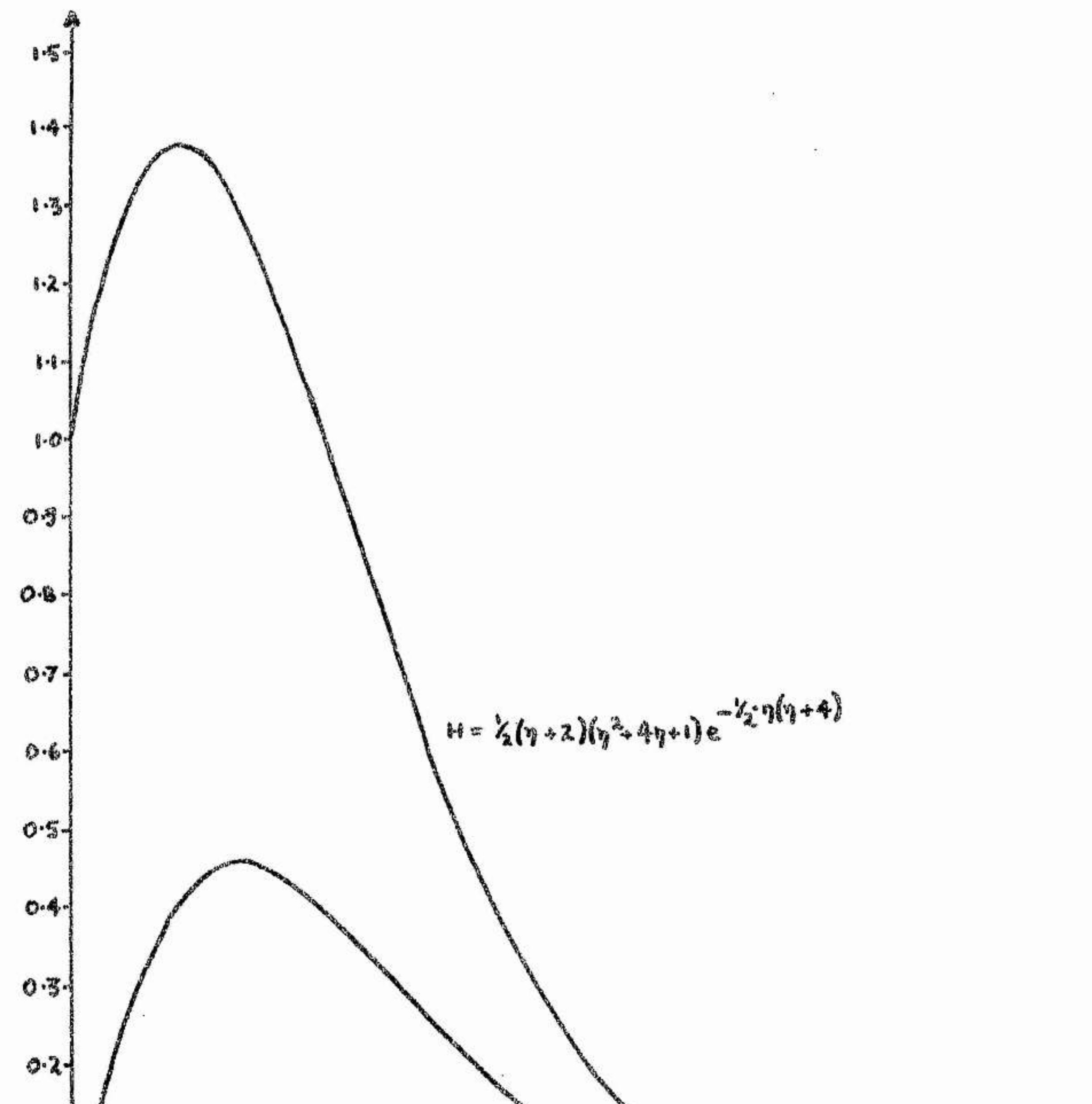

1·5
1·4
1·3
1·2
1·1
1·0
0·9
0·8
0·7
0·6
0·5
0·4
0·3
0·2
0·1
0
0
0·5
1·0
1·5
2·0
2·5
3·0
η
$H = \frac{1}{2}(\eta+2)(\eta^2+4\eta+1)e^{-\frac{1}{2}\eta(\eta+4)}$
$\frac{dF}{d\eta} = \frac{1}{4}(\eta+2)(\eta^2+4)e^{-\frac{1}{2}\eta(\eta+4)}$

Figure 6.

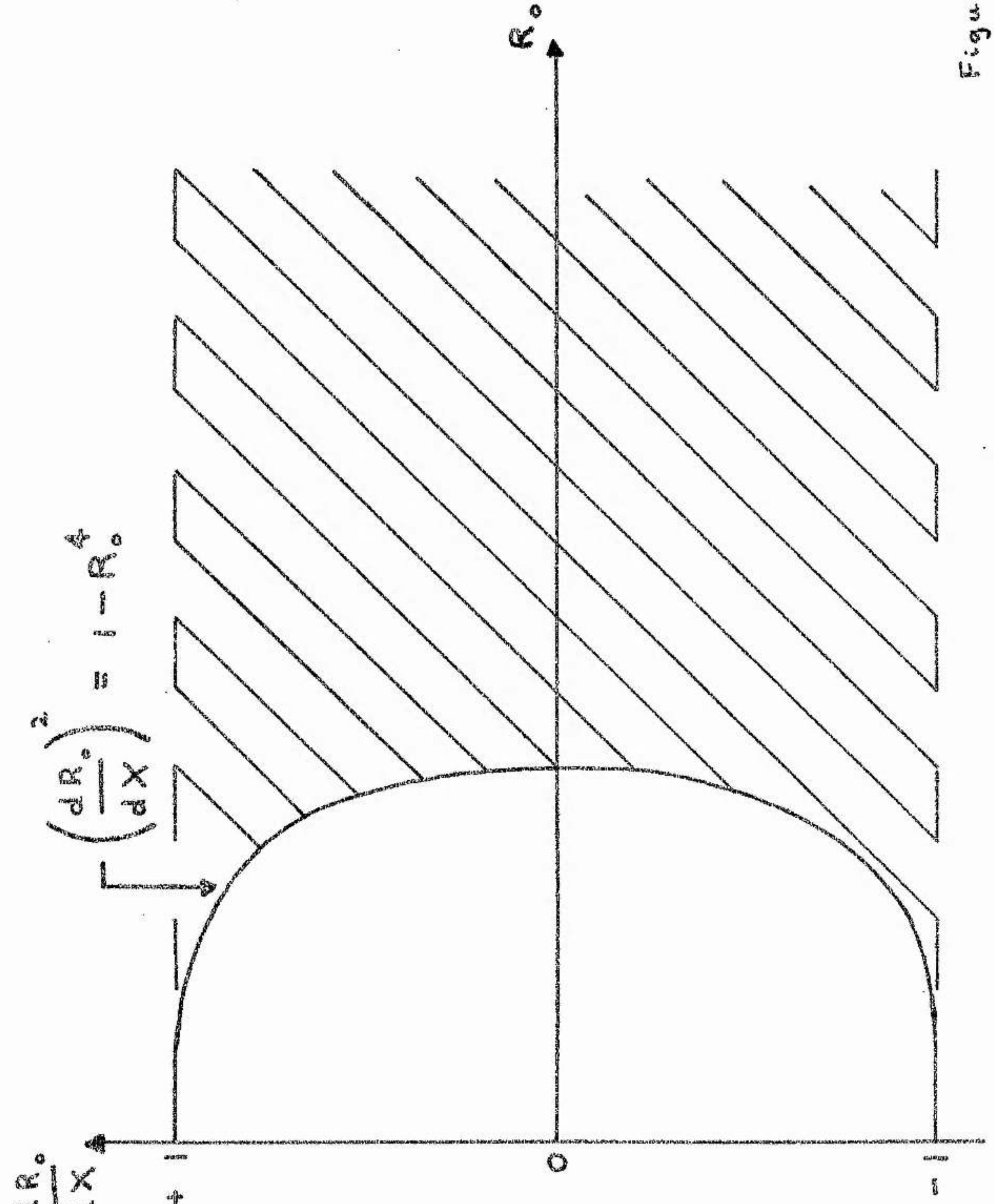


Figure 7.

CAPTIONS

Figure 1. Coordinate system.

Figure 2. Dimensionless velocity distributions $\frac{dF}{d\eta}$ for various values of the suction (or blowing) parameter $F(o)$, in natural convection on a vertical flat plate with $v_w = -v_s x^{\frac{1}{2}}$, $T_w - T_\infty = Qx^3$, $Pr = 0.72$.

Figure 3. Dimensionless temperature distributions $H(\eta)$ for various values of the suction (or blowing) parameter $F(o)$, in natural convection on a vertical flat plate with $v_w = -v_s x^{\frac{1}{2}}$, $T_w - T_\infty = Qx^3$, $Pr = 0.72$.

Figure 4. Dimensionless velocity $\frac{dF_e}{d\eta_e}$ and temperature $H_e(\eta_e)$, comparing Eichhorn's results [1960] ($Pr = 0.72$, $F_e(\eta_e = 0) = +0.2$) with those of the present investigation ($Pr = 0.73$, $F_e(\eta_e = 0) = +0.218$) for natural convection on a vertical flat plate with $v_w = -v_s x^{-\frac{1}{4}}$, $T_w - T_\infty = Q$.

Figure 5. Dimensionless velocity $\frac{dF}{d\eta}$ and temperature $H(\eta)$, comparing the numerical solution of (5.2) and (5.3) (for $F(o) = 1.447$) [denoted by —] and the "strong" suction approximation (7.8) and (7.10) [denoted by ---], for steady natural convection on a vertical flat plate with $v_w = -v_s x^{\frac{1}{2}}$, $T_w - T_\infty = Qx^3$, $Pr = 0.72$.

Figure 6. Time-dependent analytic solutions (7.12 a,b) for strong suction, with $Pr = 1$, $a_1 = 1$, $a_6 = 1$, $a_9 = 2$.

Figure 7. Phase-plane diagram of $\left(\frac{dR_o}{dX}\right)^2 = 1 - R_o^4$.

The class of bodies for which the transformation (8.4a)–(8.4f) is valid lie in the shaded region.

REFERENCES

BRAUN, W.H., OSTRACH, S. & HEIGHWAY, J.E. 1961
Free convection similarity flows about two-dimensional and axisymmetric bodies with closed lower ends.
Int. Jl. of Heat and Mass Transfer. 2, 121-135.

BRDLIK, P.M. & MOCHALOV, V.A. 1965
Porous blowing and suction during free convection at a vertical surface (laminar layer).
Int. Chem. Eng. 5, 603-608.

DOTSON, J.P. 1954 Heat transfer from a vertical plate by free convection.
M. Sc. Dissertation, Purdue University.

ECKERT, E.R.G & SOEHNGEN, E. 1951
Interferometric studies on the stability and transition to turbulence of a free convection boundary layer.
Proc. of General Discussion on Heat Transfer, Inst. Mech. Engs. and A.S.M.E., 321-323.

EDE, A.J. 1967 Article entitled "Advances in free convection" in Advances in Heat Transfer, ed. J.P.Hartnett & T.F.Irvine, Jr. Academic Press, New York.

EICHHORN, R. 1960 The effect of mass transfer on free convection.
Jl. Heat Transfer, Trans. A.S.M.E. C.82, 260-263.

GEBHART, B. 1969 External natural convection flows.
App. Mech. Revs. 22, 691-701.

LE FEVRE, E.J. 1956 Laminar free convection from a vertical plane surface.
Proc. 9th. Int. Cong. App Mech. Brussels 4, 168-174.

MABUCHI, I. 1963 The effect of blowing or suction on heat transfer by free convection from a vertical flat plate.
Bulletin of J.S.M.E. 6, 223-230.

MANGLER, W. 1948 Zusammenhang Zwischen Ebenen U.N.D. Rotationssymmetrischen Grenzschichten in Kompressiblen Flussigkeiten.
Z. Angew. Math. Mech. 28, 97-103.

MENOLD, E.R. & YANG, K.T. 1962
Asymptotic solutions for unsteady laminar free convection on a vertical plate.
J. App. Mech. 29, 124-126.

OSTRACH, S. 1953 An analysis of laminar free-convection flow and heat transfer about a vertical flat plate parallel to the direction of the generating body force.
N.A.C.A. Rep.1111.

OSTRACH, S. 1964 Article entitled "Laminar flows with body forces" in High Speed Aerodynamics and Jet Propulsion VOL. IV, Theory of Laminar Flows ed. F.K.Moore. Princeton University Press, Princeton, New Jersey.

SCHETZ, J.A. & EICHHORN, R. 1962
Unsteady natural convection in the vicinity of a doubly infinite vertical plate.
Jl. of Heat Transfer 84, 334-338.

SCHMIDT, E. & BECKMANN, W. 1930
Das Temperatur und Geschwindigkeitsfeld vor einer Wärme abgebenden senkrechter Platte bei Natürlicher Konvektion.
Tech. Mech. Thermodynamik Bd.1, No.10, 341-349. contd. Bd.1, No.11, 391-406.

SCHUH, H. 1948 Boundary layers of temperature.
Aerodynamische Versuchsanstalt. Reports and Translations 1007, 387-389.

SPARROW, E.M. & GREGG, J.L. 1956
Laminar free convection from a vertical flat plate with uniform surface heat flux.
Trans. A.S.M.E. 78, 435-440.

SPARROW, E.M. & GREGG, J.L. 1958
Similar solutions for convection from a non-isothermal vertical plate.
Trans. A.S.M.E. 80, 379-386.

YANG, K.T. 1960 Possible similarity solutions for laminar free convection on vertical plates and cylinders.
Jl. App. Mech. 82, 230-236.

PART II

NON-LINEAR INSTABILITIES IN SHEAR FLOWS

CHAPTER 1

INTRODUCTION

§1.1. LINEAR STABILITY THEORY

The stability of a laminar flow may be examined by superposing on the basic flow a disturbance of such small amplitude that the governing equations may be linearized. This approach constitutes the linear theory of hydrodynamic stability (see Lin [1955]).

In a Cartesian co-ordinate system x_i $(i = 1,2,3)$ let u_i be the components of velocity. Let $\bar{\underset{\sim}{u}} = (\bar{u}(x_3), 0, 0)$

represent an unperturbed parallel, laminar flow. Attention is restricted to disturbances which may be expressed as a superposition of normal modes. Thus if $A(\underset{\sim}{x},t)$, where $\underset{\sim}{x} = (x_1,x_2,x_3)$ and t is time, represents a typical variable describing such a disturbance and k denotes the set of parameters which distinguish the normal modes then, for a continuous distribution of normal modes,

$$A(\underset{\sim}{x},t) = \int A_{\underset{\sim}{k}}(\underset{\sim}{x},t)\, d\underset{\sim}{k}$$

(For a discrete distribution this would be replaced by a direct summation.) In linear stability theory, products and higher powers of perturbation quantities are neglected. Accordingly, the stability of a laminar flow with respect to such a disturbance can be determined from its stability with respect to the set of normal modes.

Usually, attention is restricted to normal modes of the form

$$A_{\underset{\sim}{k}}(\underset{\sim}{x},t) = \phi_{\underset{\sim}{k}}(x_3)\exp i(\alpha x_1 + \beta x_2 - \alpha c t) \qquad (1.1)$$

where α, β and c are complex quantities. The quantity c may be regarded as a complex eigenvalue, which is a function of α, β and the Reynolds number R, of the flow, and is determined by the governing equations and boundary conditions for the disturbance. For purely temporal amplification (or decay) α, β are real quantities representing the wave numbers appropriate to the directions x_1, x_2 respectively, and $c = c_R + i c_I$ is complex, where c_R represents the wave velocity in the x_1 direction and αc_I the temporal amplification (or decay) of the oscillation. For purely spatial amplification (or decay) $\alpha = \alpha_R + i\alpha_I$, $\beta = \beta_R + i\beta_I$ are complex, where α_I and β_I represent the amplification (or decay) in the directions x_1, x_2 respectively, and the frequency αc is real. For purely temporal amplification it has been shown by Squire [1933] that from the standpoint of a linear analysis the behaviour of a three-dimensional wave-like disturbance is equivalent to that of a two-dimensional disturbance at a lower Reynolds number. As a result, only two-dimensional disturbances $(\beta = 0)$ need then be considered. Thus, restricting attention to the temporal amplification (or decay) of two-dimensional disturbances, if the flow Reynolds number is R_1, the flow is stable if $c_I < 0$ for all α and for all $R \leq R_1$ since in this case the amplitude of each mode decreases exponentially like $\exp \alpha c_I t$; however, if $c_I > 0$ for some α and for some $R \leq R_1$ the amplitude of this mode increases and the flow is linearly unstable. If $c_I = 0$ for a particular mode, that mode neither grows nor decays and is said to be neutrally stable. The requirement that a two-dimensional mode be neutrally stable yields a result of the form $R = R(\alpha)$ which denotes a curve of neutral stability in the $\alpha - R$ plane. Normally, this curve denotes a stability boundary, inside of which $c_I > 0$ and the mode amplifies, and outside of which $c_i < 0$ and the mode decays. On the neutral curve there usually exists a critical value R_c of the Reynolds number below which all infinitesimal disturbances decay.

From linearized theory the differential equation governing a two-dimensional mode is the fourth-order Orr-Sommerfeld equation. In many situations of interest instability occurs only at large Reynolds numbers, and one may employ asymptotic solutions valid for large values of αR. Such asymptotic methods were initiated by Heisenberg [1924] and are discussed at length by Lin [1955] and Reid [1965]. At sufficiently large Reynolds numbers the solution of the Orr-Sommerfeld equation may be adequately represented by the solution of the corresponding inviscid equation (the Rayleigh equation) with $R \to \infty$, except close to a boundary and near the so-called "critical layer" where the velocity of the primary flow equals the phase velocity c_R of the disturbance. The inadequacy near boundaries arises since the Rayleigh equation is of second-order, and therefore not all the boundary conditions at the walls can be satisfied. In the vicinity of the critical layer, the inviscid solutions normally become singular at x_{3c} in the complex x_3-plane where $\bar{u}(x_{3c}) \equiv \bar{u}_c = c$, and when c_i is small this singularity lies close to the real axis. It is known from linear viscous theory that the inviscid solution is a valid asymptotic approximation only in the region

$$-\frac{7}{6}\pi < \arg(x_3 - x_{3c}) < \frac{\pi}{6} \qquad (1.2)$$

of the complex x_3-plane, excluding a small circle of radius $O\left[(\alpha R)^{-1/3}\right]$ with centre at x_{3c}.

The approximate inner viscous solutions in the vicinity of the critical layer are known to be expressible in terms of Airy functions with

$$X_3 = i(\alpha R D\bar{u}_c)^{1/3}(x_3 - x_{3c}) \qquad (1.3)$$

as the independent variable (see for example Reid [1965]), where $D \equiv d/dx_3$.

§1.2. NON-LINEAR ASPECTS OF STABILITY

(a) Introductory remarks

An ultimate objective of non-linear stability theory is to elucidate the evolution of, or transition from, laminar flow to turbulent flow. For boundary-layer flows, this transition process has been the subject of much experimental work, notably by Schubauer & Skramstad [1943], Sato [1956, 1960], Klebanoff, Tidstrom & Sargent [1962] and Ross, Barnes, Burns & Ross [1970]. In particular the experiments of Schubauer & Skramstad [1943] and of Ross, Barnes, Burns & Ross [1970] on Blasius boundary-layer flows confirm that the initial evolution of a small disturbance is successfully described by linearized theory. However there are flows, such as Poiseuille pipe flow, for which linear theory predicts stability at all Reynolds numbers, in contrast to experiments which all yield instability at sufficiently large Reynolds numbers. The source of such instability probably derives from non-linear terms in the governing equations and these should therefore be incorporated in a relevant theory. Also, for flows which are linearly unstable, such as the unstable Blasius boundary-layer, an imposed disturbance will grow in amplitude until non-linear terms in the governing equations become important. Accordingly, linear hydrodynamic stability theory can at most only describe the initial stages of transition to turbulence.

Review articles concerning non-linear effects in shear flows have been written by Stuart [1962(a), 1971], and a review of boundary-layer transition is given by Tani [1969].

(b) Fluctuating flows and the role of Reynolds stresses

When a steady state is perturbed, it is convenient to treat separately the mean and fluctuating parts, where the mean part is an average defined in some suitable way and the fluctuating part has zero mean. For a discussion of methods of averaging, particularly with respect to turbulent flow, see Batchelor [1953]. In non-linear stability theory of parallel shear flows, it is usual to consider an average with respect to distance parallel to the flow. Such an average avoids certain difficulties associated with the fact that the mean flow may vary with time because of energy interchange between the mean and fluctuating parts.

Writing $$u_i = \bar{u}_i + u_i' , \qquad (i=1,2,3)$$

$$p = \bar{p} + p' ,$$

(where for present purposes these are dimensional quantities) where p is the pressure, the overbar denotes mean part and the prime denotes fluctuating part, the mean and fluctuating parts of the Navier-Stokes equations may be derived. The equations governing the mean flow contain the virtual Reynolds stresses $\rho\overline{u_i' u_j'}$. These arise from the inertia terms of the Navier-Stokes equations and represent the mean momentum flux/unit volume carried by the velocity fluctuations. These virtual stresses derive from the fluctuating motion but act on the mean flow. Consequently the fluctuations and mean motion are mutually dependent, and the Reynolds stresses may lead to distortion of the mean flow profile.

Now consider a small wave-like disturbance which initially is growing exponentially with time according to linear theory. As it amplifies it must eventually reach such a size that the mean transport of momentum by the finite fluctuations is appreciable. The resulting distortion of the mean flow will alter the rates of transfer of energy from the mean flow to the disturbance; and, since this energy transfer is the cause of the disturbance growth, the rate of growth of the disturbance is affected. The disturbance is also modified by the generation of harmonics of the fundamental component through the "weak coupling" of the Fourier components due to quadratic interaction in the inertia terms of the Navier-Stokes equation. Meksyn & Stuart [1951], and many others subsequently, have calculated such effects, mainly for flows between fixed, parallel plates. It appears that the influence, upon the wave-like disturbance, of the generation of higher harmonics is usually less important than that of the distortion of the mean profile.

(c) <u>Finite-amplitude stability theory</u>

Landau [1944] suggested, and Stuart [1960] later showed, that for sufficiently weak non-linearity the temporal evolution of the most unstable two-dimensional mode in a parallel shear flow is governed by the equation

$$\frac{dA}{dt} = \alpha c_i A + i\alpha k A|A|^2, \tag{1.4}$$

where, A is the complex disturbance amplitude, α and c_i are as defined in §1.1 and $k = k_r + ik_i$ is a complex constant.

Thus, restricting attention to the most unstable mode, if $c_i < 0$ the flow under investigation is stable to all infinitesimal disturbances. However as pointed out by Meksyn & Stuart [1951], if $k_i < 0$ a disturbance will grow at Reynolds numbers below the critical value R_c given by linear theory, provided its amplitude lies above a threshold value.

In contrast, if $c_r > 0$ the flow is linearly unstable; but, if $k_r > 0$ higher order terms in equation (1.4) may balance the leading term and a finite-amplitude 'super-critical' equilibrium flow can thereby be attained at Reynolds numbers above R_c.

In the special case $c_r = 0$, corresponding to neutral stability in linear theory, the sign of k_r determines whether the disturbance grows or decays with time.

The problem now is the determination of the complex constant k. Stuart [1960] first tackled this problem for plane Poiseuille flow. He derived non-linear equations governing the amplitude functions $\Phi_k(x_2)$ of the fundamental mode and its higher harmonics, neglecting terms of the fourth order in amplitude. By invoking the orthogonality condition necessary for the existence of solutions to his non-homogeneous equations Stuart derived an integral representation for k. Watson [1960] has extended Stuart's theory to purely spatial amplification.

Pekeris & Shkoller [1967] and Reynolds & Potter [1967] have numerically evaluated the constant $a_2 = -2\alpha k_r$ for plane Poiseuille flow. Their results show conclusively that, at least for small values of $R_c - R$, there exists a threshold - amplitude instability phenomenon for $R < R_c$.

Stewartson & Stuart [1971] have generalized equation (1.4), showing that the non-linear evolution, both temporal and spatial, of the most unstable two-dimensional mode in plane Poiseuille flow is governed by an equation of the form

$$\frac{\partial A}{\partial \tau} - a_2 \frac{\partial^2 A}{\partial \xi^2} = \frac{d_1}{d_{1R}} A + k A |A|^2 , \qquad (1.5)$$

where $\tau = \varepsilon t \quad \xi = \varepsilon^{1/2}(x_1 + a_{1R} t) , \quad \varepsilon = (R - R_c) d_{1R}$;

a_{1R} (real constant), a_2 and $d_1 = d_{1R} + i d_{1I}$ (complex constants)

are coefficients in a double-Taylor series expansion for the complex growth rate,

$$-i\alpha c = -i\alpha_c c_{cR} + i a_{1R}(\alpha - \alpha_c) - a_2(\alpha - \alpha_c)^2 + (R - R_c) d_1 + \ldots$$

Here, α_c , R_c denote critical values of α and R for the neutral curve of linear theory, c_{cR} is the real phase velocity of the neutral wave and $-a_{1R}$ is the group velocity.

Equation (1.5) has been derived for non-linear instability waves in one-dimensional flow of a plasma by Watanabe [1969] and in Bénard convection flows by Newell & Whitehead [1969]. In both cases there is no shear flow in the unperturbed state. Di Prima, Eckhaus and Segel [1971] have derived equation (1.5) for a wide class of hydrodynamic flows, which includes Poiseuille flow, by a more general method. The properties of equation (1.5) have been discussed in some detail by Hocking & Stewartson [1972]. The corresponding theory for three-dimensional disturbances is discussed by Hocking & Stewartson [1971] and Hocking, Stewartson & Stuart [1972].

A further interesting aspect of non-linear stability was revealed in the experiments of Klebanoff & Tidstrom [1959] and Klebanoff, Tidstrom & Sargent [1962] who observed that predominantly two-dimensional initial disturbances developed a strong three-dimensional character.

Benney & Lin [1960] and Benney [1961, 1964] have discussed one aspect of this process, namely, the development of spanwise - periodic longitudinal vortices. Their model concerns the interactions at second-order of a plane-wave disturbance proportional to $\exp i\alpha(x_1 - c_1 t)$ and a three-dimensional disturbance proportional to $\exp i\alpha(x_1 - c_2 t)\cos\beta x_2$. With the approximation $c_1 = c_2$ made by Benney & Lin [1960], there develops a secondary spanwise periodic non-oscillatory component resembling the longitudinal vortices observed by Klebanoff, Tidstrom & Sargent [1962].

However, the Lin-Benney theory is incomplete in some important respects. Firstly, Stuart [1962 a,b] has pointed out that according to linear theory the frequencies of the two- and three-dimensional fundamental modes may differ by as much as 15%, which contradicts the assumption made by Lin & Benney that $c_1 = c_2$. Another shortcoming of the theory is its inability to predict any preferred spanwise periodicity, since the periodicity of the three-dimensional wave may be chosen arbitrarily.

In their experiments on boundary-layer transition Klebanoff, Tidstrom & Sargent [1962] also observed a rapid localized onset of turbulence and the formation of a hairpin vortex lifting up from the surface downstream of the primary wave crests. Such breakdown features are discussed in Landahl's [1972] model which concerns a small-scale secondary wave riding 'on the back' of a large scale primary wave.

It should be mentioned that Benney & Bergeron [1969] and Davis [1969] have separately proposed the use of a non-linear theory, rather than the traditional viscous theory described in §1.1, for resolving the singularities associated with the critical layer region in an inviscid parallel shear flow. In particular they showed that viscous theory is applicable for $\lambda = (R\,\epsilon^{3/2})^{-1} \gg 1$, where ϵ is a dimensionless measure of the disturbance amplitude. Their non-linear theory is applicable when $\lambda \ll 1$ and provides a complementary study to linear viscous theory. This non-linear theory has been extended by Kelly & Maslowe [1970] to stratified flows.

Theoretical work has also been done on non-linear aspects of Couette flow between concentric rotating cylinders (Taylor [1923], Stuart [1958], Chandrasekhar [1961], Davey [1962], Reynolds & Potter [1967], Davey, Di Prima & Stuart [1968], Kirchgässner & Sorger [1969]) and of thermal convection (for review articles see Segel [1966] and Busse [1969]).

§1.3. RESONANCE

In weakly interacting systems the governing equations contain non-linear terms which, to a first approximation, may be neglected. The resultant (linear) wave-like solutions are called free modes. To the next approximation, second-order non-linear terms in the governing equation are represented as products of free modes which act essentially as external driving forces. As for linear oscillators, if these driving forces contain components whose spatial and temporal frequency matches, or nearly matches, that of a free mode, a resonance occurs which may lead to systematic energy transfer to, or from, the free mode.

Thus, if three disturbances have periodicities like

$$\exp i\left[\alpha_j x_1 + \beta_j x_2 - \alpha_j c_j t\right] \quad , \quad j = 1,2,3 \text{ (no summation)}, \tag{1.6}$$

where the α_j and β_j are real positive wave numbers and the c_j are complex wave velocities $\left(c_j = c_{jR} + i c_{jI} \; , \; j = 1,2,3 \right)$, resonance occurs at second-order if

$$\alpha_1 \pm \alpha_2 \pm \alpha_3 = 0 \; , \quad \beta_1 \pm \beta_2 \pm \beta_3 = 0 \; , \quad \alpha_1 c_{1R} \pm \alpha_2 c_{2R} \pm \alpha_3 c_{3R} = 0 \; , \qquad (1.7)$$

where the + and − signs correspond. In this circumstance the waves form a resonant triad. The imaginary parts of the wave velocities do not appear in expressions (1.7) since they relate to the growth or decay of the waves.

Although Raetz [1959] applied the fundamental notions of resonance to boundary-layer flows it was the work of Phillips [1960] on resonance among free-surface gravity waves which established the mathematical techniques for evaluating the effects of resonant wave interactions. In fact, Phillips showed that there are no resonant triads for gravity waves, but there are resonant quartets. In this case, a new mode not initially present may be excited through the non-linear interactions of three existing modes. Initially this newly excited wave grows linearly in time (see Phillips [1960] and Longuet-Higgins [1962]); but this representation clearly has a limited duration of validity since the growing wave will eventually violate the chosen ordering scheme. By allowing the complex wave amplitudes to vary slowly with time Benney [1962] was able to extend the duration of validity. He assumed that each wave could be written in the form

$$A(x_3, t) \exp i\left(\alpha x_1 + \beta x_2 - \alpha c_R t\right) \; ,$$

where the complex amplitudes $A(x_3, t)$ are slowly varying functions of time in the sense that

$$\frac{1}{\alpha c_R}\left|\frac{1}{A}\frac{\partial A}{\partial t}\right| << 1 \ . \qquad (1.8)$$

This "two-time" nature of the response discussed by Benney [1962] furnishes a period of modulation as well as the usual wave periods. The interaction equations were shown to possess partition integrals indicating a continuous energy exchange among the participating waves, such that the total energy is conserved, viscous dissipation being neglected (see also Bretherton [1964]). Work similar to that of Phillips and Benney was done by Ball [1964] on coupling between surface and internal waves in a two-fluid system, and by McGoldrick [1965] on capillary gravity wave interactions. In both these cases resonant triads of waves were found to occur.

Investigations by Kelly [1968] and Craik [1968, 1971] have shown that second-order resonant wave interactions can occur in the presence of a primary shear flow. These studies have established that, in addition to the interchange of energy among the wave components owing to their interactions there may also occur a transfer of energy from the primary shear flow to the disturbance (or vice-versa). Kelly [1968] studied resonance among two-dimensional waves in a primary shear flow whereas Craik's work [1968, 1971] concerns three-dimensional waves and emphasizes the role played by viscosity. In particular Craik [1971] studied a resonant triad, consisting of a plane wave propagating in the direction of the primary shear flow and two oblique waves which propagate at equal and opposite angles to this direction, superposed on a general parallel shear flow. He showed that, to second order, the complex amplitude functions for the triad satisfy equations of the form

$$\frac{dA_1}{dt} = \frac{\alpha c_r}{2} A_1 + a_1^{(2)} A_2^* A_3 ,$$

$$\frac{dA_2}{dt} = \frac{\alpha c_r}{2} A_2 + a_2^{(2)} A_1^* A_3 , \qquad (1.9\ a,b,c)$$

$$\frac{dA_3}{dt} = \alpha \tilde{c}_r A_3 + a_3^{(2)} A_1 A_2 ,$$

where A_1 and A_2 denote the complex amplitudes of the two three-dimensional waves and A_3 that of the plane wave, c_r and $\tilde{c}_r$ are the imaginary parts of the complex wave velocities given by linear stability theory, and $a_j^{(2)}$ $(j = 1,2,3)$ are complex interaction constants determined by the analysis. The range of his analysis was defined by the inequalities

$$(\alpha R)^{1/3} \gg 1 , \quad |A_3 R| \ll 1 , \quad \left(|A_1 A_2| / |A_3|\right) \ll 1 . \qquad (1.10)$$

By a method similar to that of Stuart [1960], discussed in §1.2c, Craik derived integral representations for the interaction coefficients $a_j^{(2)}$ $(j = 1,2,3)$. He assumed that all the integrands could be evaluated from linear theory. Using the fact that the inviscid solutions for functions appearing in the integrands of the two integrals appearing in the expression for $a_3^{(2)}$ are valid asymptotic approximations for large αR in the region given by (1.2) (see discussion in §1.1), the integrals may be evaluated by deforming the path of integration to pass beneath the singularity of the integrands at x_{3c} . Thus to a first approximation $|a_3^{(2)}| \sim O(1)$, where in general $a_3^{(2)}$ is complex.

However this device cannot be used to evaluate the interaction integrals occurring in the expressions for $a_1^{(2)}$ and $a_2^{(2)}$.

These integrands involve certain functions together with their complex conjugates and it is readily confirmed that the inviscid estimates of these complex conjugate quantities are valid in the region of the complex plane given by

$$-\frac{\pi}{6} < \arg(x_3 - x_{3c}) < \frac{7}{6}\pi .$$

Accordingly it is not possible to deform the path of integration in such a way that the inviscid estimates of the integrands remain valid approximations. Instead, the path of integration must pass through the singularity at x_{3c} and viscous theory (as discussed in §1.1) must be employed to evaluate the integrands in the vicinity of the critical layer. It turns out that these integrals are dominated by the contributions from this region. In particular, Craik showed that $|a_j^{(2)}| \sim O(R)$, $(j=1,2)$. Evidently a strong non-linear energy-transfer mechanism operates in the critical-layer region. The strength of this mechanism depends crucially on the three-dimensional nature of the waves, for the dominant contributions to the interaction coefficients $a_{1,2}^{(2)}$ derive from terms involving the 'cross-velocity' components of the oblique waves (i.e. the velocity components parallel to the respective wave crests). Such a mechanism could account for the growth of a particular pair of oblique waves, which might then interact in the Lin-Benney manner (see §1.2) to produce longitudinal vortices with a unique spanwise periodicity.

§1.4. VARIATIONAL FORMULATIONS

Simmons [1969] has shown that in previous studies of resonant surface wave interactions in the absence of a shear flow the derivation of the interaction equations was unnecessarily laborious and was better effected by means of a variational method similar to that developed by Whitham [1965 a, 1965 b, 1967] to discuss the slow dispersion of non-linear wave-trains in non-dissipative systems.

Whitham's variational method is concerned with weakly interacting physical systems which support both uniform and non-uniform wave-trains. A uniform wave-train is specified by certain parameters, such as amplitude and wave number, which are obtained from linear theory. Whitham's theory deals with non-uniform wave-trains in which these parameters vary slowly in space and time. Since the properties of non-uniform wave-trains are locally constant, but slowly modulate over a large spatial or temporal scale, a slowly-varying non-uniform wave may be locally well approximated by some uniform wave-train.

Field equations and boundary conditions for the system under consideration are assumed to be given by a variational formulation.

$$\delta \mathcal{L}_1 = \delta \int_{t_0}^{t_1} \int_{\substack{\text{fluid} \\ \text{volume}}} L_1 \, dv \, dt = 0 , \qquad (1.11)$$

where t_0 and t_1 are two distinct instants of time and L_1 is a suitable Lagrangian density. In Whitham's variational method the Lagrangian is averaged over one wave length of a possible uniform wave-train solution of the system. The resultant averaged Lagrangian formulation gives rise to Euler-Lagrange equations governing the slowly-varying quantities, which in Whitham's analysis are the local frequency, amplitude and wave number of the non-uniform wave-train.

However, for resonant wave interactions the frequencies and wave numbers of the participating waves are assumed constant. For such interactions Simmons [1969] reformulated Whitham's variational method in the following manner. The Lagrangian L_1 (see equation 1.11) for the system is determined for a resonant group of waves each with constant wave number and frequency, but with slowly varying amplitude and phase angle. Then, on averaging the Lagrangian over one wave-length, as in Whitham's method, the rapidly-varying terms average to zero and slowly varying terms average locally as constants. The averaged variational formulation then gives rise to Euler-Lagrange equations, which constitute the interaction equations for the given resonant group of waves.

Both Whitham and Simmons were concerned with non-viscous flows. There have been very few studies concerning the use of variational methods in viscous flows. Lee & Reynolds [1967] proposed a numerical method for studying the linear analysis of stability of parallel flows, using a variational formulation for the Orr-Sommerfeld equation and appropriate boundary conditions; and Platten [1971] has made a similar study. A general variational formulation for viscous flows was proposed by Bateman [1956] and such variational methods are discussed by Schechter [1966] and Finlayson [1972 a,b]. Since Millikan [1929] had indicated that it was not possible to derive the Navier-Stokes equations from a variational formulation in which the Lagrangian density depends only upon the physical flow variables, (see also Finlayson [1972 b]), Bateman [1956] discusses the variation of an integral in which the integrand is a function of the physical flow variables together with certain auxiliary functions. Independent variations of the auxiliary functions leads to the Navier-Stokes equations together with boundary conditions appropriate to a bounding surface on which the stress is specified. A shortcoming of Bateman's formulation is that it does not incorporate appropriate boundary conditions on parts of the bounding surface which have prescribed motions rather than prescribed stresses. For steady motion Finlayson [1972 b] merely poses these boundary conditions as additional constraints.

§1.5 AIMS OF THE PRESENT WORK

In Chapter 2, A.D.D.Craik's modified version of Bateman's variational formulation is outlined (see Usher & Craik, [I] (to appear)). This modified variational formulation is further extended to viscous flows over a rigid plane surface where the upper fluid surface may deform (compare with the inviscid variational formulation of Luke [1967]).

A technique similar to that of Simmons [1968] is applied to the modified variational formulation in order to derive interaction equations valid to second-order in wave amplitude for a resonant wave triad in a parallel shear flow. First, the linearized equations are deduced; then a procedure is developed which yields the second-order interaction equations of Craik [1971]. Thereby the value of the variational formulation for viscous flows is demonstrated.

In Chapter 3, an analysis is developed to include third-order interaction equations. In contrast to Chapter 2, the third-order interaction equations are derived directly from the Navier-Stokes equations since little advantage is to be gained from the variational formulation. Integral representations of the third-order interaction coefficients are then derived. Approximate estimates for these integrals are obtained by employing the results of asymptotic theory at large Reynolds numbers. Order-of-magnitude estimates for the third-order interaction coefficients are then deduced.

Finally, the domains of validity of the present analysis and of certain simplified forms of the interaction equations are discussed.

CHAPTER 2

A VARIATIONAL METHOD FOR SECOND-ORDER RESONANT WAVE INTERACTIONS IN VISCOUS FLOWS

§2.1 A VARIATIONAL FORMULATION FOR VISCOUS FLOWS

In this section all quantities will be taken as dimensional. Following Bateman [1956], consider fluid of density ρ in a region R bounded by a surface S and introduce a Cartesian coordinate system x_i $(i = 1,2,3)$. Now let S consist of a part S' on which is prescribed a known distribution F_i of surface force per unit area and a part $S - S'$ on which the velocity vector is given as u_i^p. A known body force X_i per unit mass acts throughout. The motion of the fluid is considered within a time interval (t_0, t_1), say.

Bateman [1956] considers the variation of the integral

$$I \equiv \int_{t_0}^{t_1} \left\{ \int_R L \, d\tau + \int_{S'} M_1 \, dS \right\} dt \,, \tag{2.1}$$

where L and M_1 may be written in Cartesian tensor form as

$$L \equiv \tfrac{1}{2} E_{ij} \left(-p_{ij} + \rho u_i u_j \right) + \rho \left(u_i U_{i,t} + U_j X_j \right) + P u_{i,i} \,, \tag{2.2}$$

$$M_1 \equiv -U_i F_i - \rho l_{1i} u_i U_j u_j \,. \tag{2.3}$$

Here, $p_{ij} \equiv -p \delta_{ij} + \mu e_{ij}$ (the stress tensor),

$$\delta_{ij} = 1 \quad \text{if } i = j, \qquad (\text{Krönecker Delta}) \qquad (2.4)$$
$$= 0 \quad \text{if } i \neq j,$$

$e_{ij} \equiv u_{i,j} + u_{j,i}$ (the rate of deformation tensor), $E_{ij} \equiv U_{i,j} + U_{j,i}$ and μ , $p(x_i,t)$ and $u_j(x_i,t)$ denote respectively the viscosity, pressure and components of velocity, $P(x_i,t)$ and $U_j(x_i,t)$ are auxiliary functions introduced by Bateman and will be referred to respectively as the 'pseudo-pressure' and components of 'pseudo-velocity', l_{1i} denote the direction cosines of the unit outward normal at each point of S'. Bateman supposed the functions $U_j(x_i,t)$ to be analogous to the components of a virtual displacement.

Bateman's analysis yields the correct equations in R and boundary conditions on S' (as is shown below) but it does not yield any boundary conditions on $S - S'$. In order to incorporate these, A. D. D. Craik (see Usher & Craik [I]) has proposed the addition of a further surface integral, namely

$$\int_{t_0}^{t_1} \int_{S-S'} M_2 \, dS \, dt \quad , \qquad (2.5)$$

$$M_2 \equiv p_{ij} l_{2i} U_j - \rho l_{2i} U_j \left[u_i^p u_j + u_j^p (u_i - u_i^p) \right]$$
$$- \mu \left| \underset{\sim}{u} - \underset{\sim}{u}^p \right|_{,k} \varepsilon_{ijk} l_{2i} U_j \quad , \qquad (2.6)$$

where $\left| \underset{\sim}{u} - \underset{\sim}{u}^p \right|$ is the magnitude of the vector $\underset{\sim}{u} - \underset{\sim}{u}^p$, ε_{ijk} is the permutation tensor and l_{2i} are the direction cosines of the unit outward normal at each point of $S - S'$.

Now consider the variational formulation

$$\delta I = 0 \tag{2.7}$$

where u_i, U_i, p and P are allowed to vary independently subject to the restrictions that the variations δu_i vanish on $S - S'$ for all t and that the variations δU_i vanish throughout R at the instants of time t_0 and t_1. Here, I is the integral

$$I \equiv \int_{t_0}^{t_1} \left\{ \int_R L \, d\tau + \int_{S'} M_1 \, dS + \int_{S-S'} M_2 \, dS \right\} , \tag{2.8}$$

where L, M_1, M_2 are the respective expressions (2.2), (2.3), (2.6).

The independent variations of P and U_i lead to the Navier-Stokes equations

$$\delta P: \qquad u_{k,k} = 0 \quad , \tag{2.9}$$

$$(\forall x_i \in R)$$

$$\delta U_i: \quad u_{i,t} + u_j u_{i,j} = -\rho^{-1} p_{,i} + X_i + (\mu/\rho) \nabla^2 u_i \quad , \tag{2.10}$$

and to the boundary conditions on S'.

$$\delta U_i: \qquad F_i = l_{ij} p_{ij} \quad , \qquad (\forall x_i \in S') \tag{2.11}$$

which assert that the stress distribution should balance the given surface force F_i. The variation δP yields no boundary conditions on S'.

The independent variations of p and u_i yield the governing equations for U_i, P, namely

$$\delta p: \qquad \mathcal{U}_{j,j} = 0 \quad , \qquad (2.12)$$

$$(\forall\, x_i \in R)$$

$$\delta u_i: \; \mathcal{U}_{i,t} + u_j\left(\mathcal{U}_{i,j} + \mathcal{U}_{j,i}\right) = \rho^{-1} P_{,i} - (\mu/\rho)\nabla^2 \mathcal{U}_i \;, \qquad (2.13)$$

and also the boundary conditions on S',

$$\delta u_i: \quad l_{1i}\left(P - \rho\,\mathcal{U}_j u_j\right) - l_{1j}\left(\mu E_{ij} + \rho\,\mathcal{U}_i u_j\right) = 0 \,, \qquad (2.14)$$

$$(\forall\, x_i \in S')$$

The variation δp yields no boundary conditions on S'.

The appropriate boundary conditions on $S - S'$ derive from the additional integral of M_2. The variation of P still yields no boundary terms, but variations in U_i give

$$\delta\mathcal{U}_i: \rho\, l_{2j}\left(u_i - u_i^P\right)\left(u_j - u_j^P\right) + \mu\, l_{2j}\, \varepsilon_{ijk}\left|\underset{\sim}{u} - \underset{\sim}{u}^P\right|_{,k} = 0\,, \quad (\forall\, x_i \in S-S') \qquad (2.15)$$

On taking the scalar product with l_{2i} it is seen that

$$l_{2i}\left(u_i - u_i^P\right) = 0 \qquad (\forall\, x_i \in S-S'), \qquad (2.16)$$

independently of whether μ is zero or non-zero. This is the inviscid boundary condition which prescribes the normal velocity component at each point of $S - S'$. Accordingly, the second term of (2.15) must itself vanish. In vector notation, this is just $\mu\left(\underset{\sim}{l}_2 \times \nabla\left|\underset{\sim}{u} - \underset{\sim}{u}^P\right|\right) = 0$ which, if $\mu \neq 0$, asserts that $\left|\underset{\sim}{u} - \underset{\sim}{u}^P\right|$ remains constant on $S - S'$. If we now introduce the further requirement that $\underset{\sim}{u} = \underset{\sim}{u}^p$ at a single point of $S - S'$ we have the boundary conditions for viscous flow, namely

$$u_i = u_i^p \,, \qquad (\forall x_i \in S - S') \qquad (2.17)$$

Variations of p and u_i give the corresponding conditions for U_i on $S - S'$. Variation of p leads immediately to

$$\delta p: \qquad l_{2i} U_i = 0 \,, \qquad (\forall x_i \in S - S') \qquad (2.18)$$

corresponding to the inviscid boundary condition for u_i. Variation of u_i, subject to the requirement that $\delta u_i = 0$ on $S - S'$ (since u_i is equal to u_i^p there), leads to

$$\delta u_i: \quad \mu l_{2i} U_j \{ \delta u_{i,j} + \delta u_{j,i} - \epsilon_{ijk} |\delta \underset{\sim}{u}|_{,k} \} = 0$$
$$(\forall x_i \in S - S') \qquad (2.19)$$

However, since $\delta u_i = 0$ on $S - S'$,

$$(\delta u_i)_{,j} = l_{2j} \frac{\partial}{\partial n} (\delta u_i) \,, \qquad |\delta \underset{\sim}{u}|_{,k} = l_{2k} \frac{\partial}{\partial n} |\delta \underset{\sim}{u}| \,, \qquad (2.20)$$

where n denotes distance along the outwards normal to $S - S'$. This yields

$$\mu \frac{\partial}{\partial n} (\delta u_i) \Big[l_{2i} l_{2j} U_j + U_i \Big] = 0 \,, \qquad (\forall x_i \in S - S') \qquad (2.21)$$

the third term vanishing identically. Using (2.18) we have

$$\delta u_i: \qquad U_i = 0 \,, \qquad (\forall x_i \in S - S') \qquad (2.22)$$

whenever $\mu \neq 0$. The derivation of the equations and boundary conditions is now complete. The analysis involving the integral M_2 is due to A. D. D. Craik.

It is interesting to note that if the body force is independent of time and is conservative (i.e. there exists a potential function $\Omega(x_i)$ such that $X_j = -\Omega_{,j}$), and if U_i and $-P$ are replaced by u_i and p respectively in (2.2), the expression for L is replaced by

$$\frac{D}{Dt}\left(\frac{\rho}{2} u_i u_i\right) - \frac{D}{Dt}(\rho\Omega) - \Phi \quad , \qquad (2.23)$$

$$\text{where} \quad \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} \quad , \quad \Phi = \frac{\mu}{2} e_{ij} e_{ij} \quad . \qquad (2.24)$$

The first two terms of (2.23) may be identified as the respective rates of change of kinetic and potential energy per unit volume. The third term is just the rate of energy dissipation per unit volume due to viscosity (i.e. the Rayleigh dissipation function). In the absence of viscosity, expression (2.23) represents the difference between the rates of change of kinetic and potential energy per unit volume, which is the classical Lagrangian density. It should be noted, however, that the use of expression (2.23) in the integral (2.8) does not give rise to the Navier-Stokes equations.

The variational formulation (2.7) and (2.8) may easily be extended to flows where the bounding surface S may deform. Here, we do so for the case of flow over a lower rigid horizontal plane surface ($S - S''$), where the upper surface (S'') may deform. In fact, this result is not needed for the application discussed in §2.2, but it would be required if the closely related problem discussed by Craik [1966] were to be tackled by variational techniques.

Consider x_1- and x_2-axes in the plane of the undisturbed free surface and x_3-axis vertically upwards. Let $x_3 = -z_0$ be the equation of the rigid plane surface $S - S''$ and let $x_3 = h(x_1,x_2,t)$ be the elevation of the surface S'' above its undisturbed position.

Consider the integral

$$I_0 \equiv \int_{t_0}^{t_1} \left\{ \int_{x_2=c}^{d} \int_{x_1=a}^{b} \int_{x_3=-z_0}^{h(x_1,x_2,t)} L \, dx_3 \, dx_1 \, dx_2 \right.$$

$$\left. + \int_{\substack{x_2=c \\ x_3=-z_0}}^{d} \int_{x_1=a}^{b} M_2 \, dx_1 \, dx_2 + \int_{\substack{x_2=c \\ x_3=h(x_1,x_2,t)}}^{d} \int_{x_1=a}^{b} M_3 \, dx_1 \, dx_2 \right\} dt, \qquad (2.25)$$

where a, b, c, d are constants determining the lateral boundaries and L and M_2 are given by (2.2) and (2.6). In Cartesian tensor form M_3 is defined as

$$M_3 = P\left(\frac{Dh}{Dt} - u_3\right) - U_i F_i - \rho u_j U_j l_{1i} u_i \quad , \qquad (2.26)$$

where F_i now denotes the known distribution of force per unit area prescribed on the upper surface, $x_3 = h(x_1,x_2,t)$ and l_{1i} denote the direction cosines of the outward normal to this surface.

Now consider the variational formulation

$$\delta I_0 = 0$$

where u_i, U_i, p and P vary independently, subject to the same restrictions upon δu_i on the rigid plane surface $S - S''$ and upon δU_i at $t = t_0$, t_1, as in the previous variational formulation (2.7).

The independent variations of P, U_i, p and u_i in the flow regime and on the rigid surface $S - S''$ yield the same equations as before. However the independent variations of P, U_i and u_i on the deformable surface S'' now yield

$$\delta P: \qquad u_3 = \frac{Dh}{Dt} \quad , \tag{2.28}$$

$$\delta U_i: \qquad F_i = + l_{ij} p_{ij} \quad , \tag{2.29}$$

$$\delta u_i: \quad P\left(h_{,i} - \delta_{i3} + l_{1i}\right) = \rho l_{1i} u_j U_j + l_{ij}\left(\rho u_j U_i + \mu E_{ij}\right) \, . \tag{2.30}$$

Clearly, (2.28) is the kinematic deformable-surface condition and (2.29), (2,30) are boundary conditions for the physical and pseudo-flows on the surface S''.

§2.2 APPLICATION OF THE VARIATIONAL FORMULATION TO SECOND-ORDER RESONANT WAVE INTERACTIONS IN SHEAR FLOWS

A modified form of Simmons' [1969] variational method (see §1.4) employing the above variational formulation, is here used to study second-order resonant wave interactions in three-dimensional viscous flows. The motivation for this approach is the same as Simmons', namely, to discover whether the interaction equations may be derived more easily than by the conventional method using the momentum equations. As far as the author is aware, this is the first physical application of Bateman's variational formulation of the Navier-Stokes equations.

In a Cartesian coordinate system x_i consider a parallel primary shear flow

$$\bar{u}_i^{\circ} = (\bar{u}^{\circ}(x_3),\ 0,\ 0), \qquad (0 \leq x_3 \leq l\ ), \tag{2.31}$$

between plane parallel boundaries situated at $x_3 = 0, l$ and of unbounded extent in x_1 and x_2. All quantities are taken to be dimensionless relative to characteristic velocity and length scales of the dimensional primary shear flow; accordingly l may be regarded as unity for channel flows or infinity for boundary layers. The primary pseudo-velocity $\bar{\underset{\sim}{U}}^{\circ}$ will be assumed identically zero; a choice which is consistent with equations (2.12) - (2.14), (2.18), (2.22).

Now consider perturbations of the primary shear flow $\bar{u}_i^{\circ}$ by two oblique waves and a plane wave with periodicities $\exp i\theta_j$ $(j = 1,2,3)$, where the respective phase functions are

$$\theta_1 = \frac{\alpha}{2}x_1 + \beta x_2 - \frac{\alpha}{2}c_R t,\quad \theta_2 = \frac{\alpha}{2}x_1 - \beta x_2 - \frac{\alpha}{2}c_R t,\quad \theta_3 = \alpha x_1 - \alpha c_R t\ , \tag{2.32}$$

with α, β and c_R real constants. These will be referred to as waves 1, 2 and 3 respectively. These three basic waves form a resonant triad (see §1.3) of the type discussed by Craik [1968, 1971]. In particular, Craik [1971] demonstrated the existence of such a resonant triad among Tollmien Schlichting waves in an unstable Blasius boundary layer and we here assume that the primary flow is such that a triad of this kind does indeed exist.

For the perturbed flow the physical and pseudo velocity components u_k and U_k may be written as

$$\begin{aligned}
u_1 &= \bar{u}(x_3) + \sum_{j=1}^{3} u_1^{(j)}(\underset{\sim}{x}, t) \quad , & \mathcal{U}_1 &= \sum_{j=1}^{3} \mathcal{U}_1^{(j)}(\underset{\sim}{x}, t) \quad , \\
u_2 &= \sum_{j=1}^{2} u_2^{(j)}(\underset{\sim}{x}, t) \quad , & \mathcal{U}_2 &= \sum_{j=1}^{2} \mathcal{U}_2^{(j)}(\underset{\sim}{x}, t) \quad , \qquad (2.33) \\
u_3 &= \sum_{j=1}^{3} u_3^{(j)}(\underset{\sim}{x}, t) \quad , & \mathcal{U}_3 &= \sum_{j=1}^{3} \mathcal{U}_3^{(j)}(\underset{\sim}{x}, t) \quad ,
\end{aligned}$$

where, to the order of approximation considered, all other wave components and harmonics are assumed to be negligible. The term $u_k^{(j)}$ represents the velocity component of wave j in the direction x_k; the pseudo velocity component $U_k^{(j)}$ is defined likewise. The overbar of $\bar{u}$ denotes an average taken with respect to the distance x_1 and, since the mean of each of the velocity components of the waves is zero, $\bar{u}_i = \left(\bar{u}(x_3) = \mathring{\bar{u}}(x_3), 0, 0\right)$ represents the mean motion of the perturbed flow to the required order of approximation. Similarly, the physical and pseudo-pressure denoted by p and P, respectively, may be expressed as

$$p = x_1 p^{\circ}(t) + \sum_{j=1}^{3} p_j(\underset{\sim}{x}, t) \quad , \quad \mathcal{P} = \sum_{j=1}^{3} \mathcal{P}_j(\underset{\sim}{x}, t) \,, \qquad (2.34)$$

where $p^{o}(t)$ represents the imposed longitudinal pressure gradient necessary to sustain the primary flow $\bar{u}_i^{o}$ and p_j represents the pressure perturbation associated with wave j.

Presupposing the existence of the continuity equation, the perturbation velocity components $u_1^{(3)}$ and $u_3^{(3)}$ of wave 3 may be expressed in terms of the stream function

$$\Psi_3 = \phi_3(x_3)\, A_3(t) \exp i\,\theta_3 \qquad (2.35)$$

as $$u_1^{(3)} = \mathrm{Re}\left\{D\Psi_3\right\} \quad , \quad u_3^{(3)} = \mathrm{Re}\left\{-i\alpha\Psi_3\right\} \quad , \qquad (2.36)$$

where $D \equiv \frac{\partial}{\partial x_3}$. Henceforth the real parts (denoted by Re) of complex expressions are taken to represent appropriate physical or pseudo quantities. Here, $A_3(t)$ is a complex wave amplitude. Also, the pressure perturbation p_3 associated with wave 3 is of the form

$$p_3 = \mathrm{Re}\left\{\pi_3(x_3)\, A_3(t) \exp i\theta_3\right\} . \qquad (2.37)$$

Because of the form of the phase function θ_1, the velocity components $u_k^{(1)}$ of wave 1 are best written in terms of components parallel and perpendicular to the wave crests, as

$$\gamma u_1^{(1)} = \frac{\alpha}{2}\hat{u}_1^{(1)} - \beta\hat{u}_2^{(1)} \quad , \quad \gamma u_2^{(1)} = \beta\hat{u}_1^{(1)} + \frac{\alpha}{2}\hat{u}_2^{(1)} \qquad (2.38)$$

where $\gamma \equiv \left(\frac{\alpha^2}{4} + \beta^2\right)^{1/2}$ and $\hat{u}_k^{(1)}$ $(k = 1,2)$ are the velocity components in the directions $x_k^{(1)}$, respectively, defined by the expressions

$$\gamma x_1^{(1)} = \frac{\alpha}{2} x_1 + \beta x_2 \quad , \qquad \gamma \hat{x}_2^{(1)} = -\beta x_1 + \frac{\alpha}{2} x_2 \qquad (2.39)$$

Corresponding expressions for the velocity $u_k^{(2)}$ $(k = 1,2)$ of wave 2 are obtained on replacing β by $-\beta$ in (2.38) and (2.39).

The velocity components $\hat{u}_1^{(j)}$ and $u_3^{(j)}$ $(j = 1,2)$ are expressible in terms of the stream functions

$$\psi_j = \phi_j(x_3) A_j(t) \exp i\theta_j \qquad (2.40)$$

$$(j=1,2)$$

as $$\hat{u}_1^{(j)} = \mathrm{Re}\{D\psi_j\} \quad , \qquad u_3^{(j)} = \mathrm{Re}\{-i\gamma\psi_j\} . \qquad (2.41)$$

Further, the functions $\hat{u}_2^{(j)}$ and p_j $(j = 1,2)$ are written as

$$\hat{u}_2^{(j)} = \mathrm{Re}\left\{\hat{v}_j(x_3) A_j(t) \exp i\theta_j\right\} \quad , \qquad (2.42)$$

$$(j=1,2)$$

$$p_j = \mathrm{Re}\left\{\pi_j(x_3) A_j(t) \exp i\theta_j\right\} . \qquad (2.43)$$

A similar analysis applies to the pseudo flow quantities. The pseudo velocity components $U_1^{(3)}$ and $U_3^{(3)}$ of wave 3 may be expressed in terms of a pseudo stream function

$$\mathcal{X}_3 = \left\{\chi_3(x_3) B_3(t) \exp(-i\theta_3)\right\} \qquad (2.44)$$

as $$\mathcal{U}_1^{(3)} = \mathrm{Re}\{D\mathcal{X}_3\} \quad , \qquad \mathcal{U}_3^{(3)} = \mathrm{Re}\{i\alpha\mathcal{X}_3\} \quad , \qquad (2.45)$$

where $B_3(t)$ is a complex amplitude. (The use of $-\theta_3$ in (2.43) in place of the usual $+\theta_3$ turns out to be preferable.) The pseudo pressure perturbation P_3 is similarly written as

$$P_3 = \mathrm{Re}\left\{ \Pi_3(x_3)\, B_3(t) \exp(-i\theta_3) \right\} \tag{2.46}$$

The pseudo velocity components $U_k^{(1)}$ $(k = 1,2)$ of wave 1 may be expressed as

$$\gamma U_1^{(1)} = \frac{\alpha}{2} \hat{U}_1^{(1)} - \beta \hat{U}_2^{(1)} \;, \quad \gamma U_2^{(1)} = \beta \hat{U}_1^{(1)} + \frac{\alpha}{2} \hat{U}_2^{(1)} \tag{2.47}$$

and corresponding expressions for $U_k^{(2)}$ $(k = 1,2)$ of wave 2 have β replaced by $-\beta$. The components $\hat{U}_1^{(j)}$ and $U_3^{(j)}$ $(j = 1,2)$ are given in terms of pseudo stream functions

$$\mathcal{X}_j = \left\{ \chi_j(x_3)\, B_j(t) \exp(-i\theta_j) \right\} \;, \tag{2.48}$$

$$(j = 1,2)$$

$$\text{by} \quad \hat{U}_1^{(j)} = \mathrm{Re}\left\{ D\mathcal{X}_j \right\} \;, \quad U_3^{(j)} = \mathrm{Re}\left\{ i\gamma \mathcal{X}_j \right\} \;. \tag{2.49}$$

Also, $\hat{U}_2^{(j)}$ and P_j $(j = 1,2)$ may be expressed as

$$\hat{U}_2^{(j)} = \mathrm{Re}\left\{ \hat{V}_j(x_3)\, B_j(t) \exp(-i\theta_j) \right\} \;, \tag{2.50}$$

$$(j = 1,2)$$

$$P_j = \mathrm{Re}\left\{ \Pi_j(x_3)\, B_j(t) \exp(-i\theta_j) \right\} . \tag{2.51}$$

It is now necessary to consider the form taken on by the integral (2.8) in the present flow configuration. Since there are no surfaces present on which known surface forces act, the second term vanishes. Also, since the boundary conditions for the physical velocity are

$$\bar{u}^0(x_1, x_2, x_3 = 0, t) = \bar{u}^0(x_1, x_2, x_3 = L, t) = 0 \;,$$

$$u_k^{(j)}(x_1, x_2, x_3 = 0, t) = u_k^{(j)}(x_1, x_2, x_3 = L, t) = 0 \;. \tag{2.52}$$

and the assumed boundary conditions for the pseudo-velocity are

$$\mathcal{U}_k^{(j)}\left(x_1, x_2, x_3 = 0, t\right) = \mathcal{U}_k^{(j)}\left(x_1, x_2, x_3 = l, t\right) = 0 \quad , \tag{2.53}$$

(which are compatible with (2.18), (2.22)), the integrand of the third term in (2.8) is identically zero. Here j and k take on the values 1, 2 and 3. Consequently only the first term of (2.8) is required in the present application of the variational formulation (2.7) and (2.8).

Expressions (2.33), (2.34) for u_j, U_j, p and P, together with (2.35)-(2.51) may be substituted into the dimensionless form of the Lagrangian density L derived from (2.2). Following Simmons [1969], this is averaged over one wavelength, yielding an averaged Lagrangian $\bar{L}$ which suppresses the rapidly-oscillating terms. If $\bar{L}$ is integrated with respect to x_3 from 0 to l, and the boundary conditions (2.52) and (2.53) applied, some manipulation gives the result

$$\begin{aligned}\bar{\bar{L}} \equiv \int_0^l \bar{L}\, dx_3 = 2\,\mathrm{Re}\Big\{ & \sum_{k=1}^{2} A_k B_k \left(I_k + I_{3+k}\right) + A_3 B_3 I_3 + A_1 A_2 B_3 I_6 \\ & + A_2^* A_3 B_1 \left(I_7 + I_9\right) + A_1^* A_3 B_2 \left(I_8 + I_{10}\right) \\ & + \sum_{k=1}^{2} A_k \left(I_{13+k} - I_{10+k}\right) \frac{dB_k}{dt} - A_3 I_{13} \frac{dB_3}{dt} \Big\}\end{aligned} \tag{2.54}$$

where * denotes complex conjugate and the integrals I_j $(j = 1, 2, \ldots, 15)$ are defined as

$$I_k \equiv \int_0^l \tfrac{1}{4} \chi_k L_1[\phi_k]\, dx_3 , \qquad I_3 \equiv \int_0^l \tfrac{1}{4} \chi_3 L_3[\phi_3]\, dx_3 ,$$

$$I_{3+k} \equiv \int_0^l \tfrac{1}{4} \hat{V}_k G_k\, dx_3 , \qquad I_6 \equiv \int_0^l \tfrac{1}{4} \chi_3 F_{12}\, dx_3 ,$$

$$I_7 \equiv \int_0^l \tfrac{1}{4} \chi_1 F_{23}\, dx_3 , \qquad I_8 \equiv \int_0^l \tfrac{1}{4} \chi_2 F_{13}\, dx_3 ,$$

$$I_9 \equiv \int_0^l \tfrac{1}{4} \hat{V}_1 G_{23}\, dx_3 , \qquad I_{10} \equiv \int_0^l \tfrac{1}{4} \hat{V}_2 G_{13}\, dx_3 , \tag{2.55}$$

$$I_{10+k} \equiv \int_0^l \tfrac{1}{4} \chi_k L_4[\phi_k]\, dx_3 , \qquad I_{13} \equiv \int_0^l \tfrac{1}{4} \chi_3 L_5[\phi_3]\, dx_3 ,$$

$$I_{13+k} \equiv \int_0^l \tfrac{1}{4} \hat{V}_k \hat{v}_k\, dx_3 ,$$

where k takes the values 1 and 2 and

$$L_1[\phi_k] \equiv \tfrac{1}{2} i\alpha \left[(\bar{u}^\circ - c_R)(D^2 - \gamma^2) - D^2\bar{u}^\circ \right] \phi_k - R^{-1}(D^2-\gamma^2)^2 \phi_k ,$$

$$L_3[\phi_3] \equiv i\alpha \left[(\bar{u}^\circ - c_R)(D^2 - \alpha^2) - D^2\bar{u}^\circ \right] \phi_3 - R^{-1}(D^2-\alpha^2)^2 \phi_3 ,$$

$$L_4[\phi_k] \equiv (D^2-\gamma^2)\phi_k , \qquad L_5[\phi_3] \equiv (D^2-\alpha^2)\phi_3 ,$$

$$G_k \equiv R^{-1}(D^2-\gamma^2)\hat{v}_k - \tfrac{1}{2} i\alpha(\bar{u}^\circ - c_R)\hat{v}_k + (-1)^k i\beta \phi_k D\bar{u}^\circ$$

$$F_{12} \equiv \frac{1}{4} i\alpha \Big\{ \left(\alpha^2\gamma^{-2} - 2\right)\left[D\phi_1 \left(D^2-\gamma^2\right)\phi_2 + D\phi_2\left(D^2-\gamma^2\right)\phi_1\right] - 4\beta^2\gamma^{-2} D\left(\hat{v}_1\hat{v}_2\right)$$

$$+ 2\alpha\beta\gamma^{-2}\left[\hat{v}_2\left(D^2-\gamma^2\right)\phi_1 - \hat{v}_1\left(D^2-\gamma^2\right)\phi_2 + D\phi_1 D\hat{v}_2 - D\phi_2 D\hat{v}_1\right]$$

$$+ 2\beta\alpha^{-1}D^2\left(\phi_2\hat{v}_1 - \phi_1\hat{v}_2\right) - D\left[\phi_1\left(D^2-\gamma^2\right)\phi_2 + \phi_2\left(D^2-\gamma^2\right)\phi_1\right]\Big\} \qquad (2.56)$$

$$F_{k3} \equiv \frac{1}{4} i\alpha \Big\{ \left(3-\alpha^2\gamma^{-2}\right) D\phi_3\left(D^2-\gamma^2\right)\phi_k^* + \left(2-\alpha^2\gamma^{-2}\right)\phi_3\left(D^2-\gamma^2\right)D\phi_k^*$$

$$+(-1)^{k+1} 2\alpha\beta\gamma^{-2}\left(\phi_3 D^2\hat{v}_k^* + D\phi_3 D\hat{v}_k^* + \gamma^2\phi_3\hat{v}_k^*\right)$$

$$+ \phi_k^*\left(D^2-\alpha^2\right)D\phi_3 + 2D\phi_k^*\left(D^2-\alpha^2\right)\phi_3 \Big\},$$

$$G_{k3} \equiv \frac{1}{4} i\alpha \Big\{ \frac{1}{4}\left(20\beta^2 - 3\alpha^2\right)\gamma^{-2}\hat{v}_k^* D\phi_3 - \frac{1}{2}\left(\alpha^2-4\beta^2\right)\gamma^{-2}\phi_3 D\hat{v}_k^*$$

$$-(-1)^k\left(4\beta^2-3\alpha^2\right)\beta\alpha^{-1}\gamma^{-2} D\phi_k^* D\phi_3$$

$$+(-1)^k 2\alpha\beta\gamma^{-2}\phi_3 D^2\phi_k^* - (-1)^k 2\beta\alpha^{-1}\phi_k^* D^2\phi_3 \Big\}.$$

The Reynolds number R equals $\rho v d/\mu$ where v and d are the chosen scales of velocity and length for non-dimensionalisation.

Attention is now restricted to a time interval (t_0, t_1) which is large compared with the periods of oscillation of the waves but small compared with time scales of amplitude modulations.

The variational formulation (2.7) and (2.8) becomes

$$\delta\left\{ \int_{t_0}^{t_1} \bar{\bar{L}} \; dt \right\} = 0 \qquad (2.57)$$

where the complex amplitude functions $A_j(t)$ and $B_j(t)$ $(j = 1,2,3)$ are allowed to vary independently subject only to the restriction that the variations δB_j vanish at t_0 and t_1 throughout the flow. This yields the Euler-Lagrange equations

$$\frac{\partial \bar{\bar{L}}}{\partial A_j} = 0 , \tag{2.58}$$

$$(j = 1,2,3)$$

$$\frac{\partial \bar{\bar{L}}}{\partial B_j} - \frac{\partial}{\partial t}\left(\frac{\partial \bar{\bar{L}}}{\partial B_{j,t}}\right) = 0 , \tag{2.59}$$

and these constitute the governing equations of the pseudo flow and physical flow, respectively. Variations with respect to the complex conjugate functions $A_j^*(t)$, $B_j^*(t)$ yield equivalent results.

Equations (2.58) furnish the results

$$\left(I_{13+k} - I_{10+k}\right)\frac{dB_k}{dt} = -B_k\left(I_k + I_{3+k}\right) - A_{3-k}B_3 I_6$$

$$- A_3^* B_{3-k}^*\left(I_{9-k}^* + I_{11-k}^*\right) , \tag{2.60}$$

$$I_{13}\frac{dB_3}{dt} = \left\{B_3 I_3 + A_2^* B_1\left(I_7 + I_9\right) + A_1^* B_2\left(I_8 + I_{10}\right)\right\} , \tag{2.61}$$

and equations (2.59) yield

$$\left(I_{13+k} - I_{10+k}\right)\frac{dA_k}{dt} = A_k\left(I_k + I_{3+k}\right) + A_3 A_{3-k}^*\left(I_{6+k} + I_{8+k}\right) , \tag{2.62}$$

$$I_{13}\frac{dA_3}{dt} = -\left(A_3 I_3 + A_1 A_2 I_6\right) . \tag{2.63}$$

Here, there is no summation over repeated subscripts and k takes the values 1 and 2. Equations (2.60)-(2.63) are valid to second-order in wave amplitudes. Results (2.62), (2.63) lead to the second-order interaction equations for the chosen resonant triad, whereas results (2.60), (2.61) determine the corresponding pseudo flow.

§2.3 PROCEDURE FOR DERIVING THE SECOND-ORDER INTERACTION EQUATIONS

a) Linear Theory

In linear theory (see §1.2) products and higher powers of perturbation quantities are assumed negligible and the wave amplitude functions $A_{1,2}(t)$, $A_3(t)$ vary as $\exp \frac{1}{2}\alpha c_I t$ and $\exp \alpha \tilde{c}_I t$ respectively, where $c = c_R + ic_I$ is the complex phase velocity, in the downstream direction, of both waves 1 and 2 and $\tilde{c} = c_R + i\tilde{c}_I$ is the complex phase velocity of wave 3. In such a case

$$\frac{dA_j}{dt} = a_j^{(1)} A_j \quad , \qquad (j=1,2,3) \tag{2.64}$$

where $$a_1^{(1)} = a_2^{(1)} \equiv \frac{\alpha c_I}{2} \quad , \quad a_3^{(1)} \equiv \alpha \tilde{c}_I \quad . \tag{2.65}$$

Equations (2.62), (2.63) then reduce to

$$I_{15+k} + I_{17+k} = 0 \quad , \tag{2.66}$$

and
$$I_{20} = 0 \quad , \tag{2.67}$$

where
$$I_{15+k} \equiv I_k + a_k^{(1)} I_{10+k} \; , \quad I_{17+k} \equiv I_{3+k} - a_k^{(1)} I_{13+k} \; , \tag{2.68a,b}$$

$$I_{20} \equiv I_3 + a_3^{(1)} I_{13} \tag{2.69}$$

Here k takes the values 1 and 2. From expressions (2.55) for I_3 and I_{13} we see that

$$I_{20} = \int_0^l \frac{1}{4} \chi_3 L_6\left[\phi_3\right] dx_3 \tag{2.70}$$

where
$$L_6\left[\phi_3\right] \equiv \left[i\alpha\left\{(\bar{u}^\circ - \tilde{c})(D^2 - \alpha^2) - D^2\bar{u}^\circ\right\} - R^{-1}(D^2 - \alpha^2)^2\right] \phi_3 \tag{2.71}$$

On integration by parts and the application of boundary conditions

$$\phi_3 = D\phi_3 = 0 , \tag{2.72}$$

$$(x_3 = 0, l)$$

$$\chi_3 = D\chi_3 = 0 , \tag{2.73}$$

(which are obtained by substituting (2.35) and (2.36) into boundary conditions (2.52), and (2.44) and (2.45) into boundary conditions (2.53)), it is readily found that

$$\int_0^l \chi_3 L_6\left[\phi_3\right] dx_3 = \int_0^l \phi_3 \tilde{L}_6\left[\chi_3\right] dx_3 \quad , \tag{2.74}$$

where

$$\tilde{L}_6\left[\chi_3\right] \equiv i\alpha\left\{\left(D^2-\alpha^2\right)\left[\chi_3\left(\bar{u}^\circ-\tilde{c}\right)\right]-\chi_3 D^2\bar{u}^\circ\right\}-R^{-1}\left(D^2-\alpha^2\right)^2\chi_3 \qquad (2.75)$$

is the operator adjoint to L_6.

Clearly, equation (2.67) is satisfied on identifying ϕ_3 and χ_3 as solutions of the Orr-Sommerfeld equation and adjoint Orr-Sommerfeld equation, respectively; namely

$$L_3\left[\phi_3\right]=0 \quad , \qquad \tilde{L}_3\left[\chi_3\right]=0 \quad , \qquad (2.76\,a,b)$$

subject to the homogeneous boundary conditions (2.72) and (2.73).

Similarly, equations (2.66) are satisfied by identifying ϕ_k, χ_k, $\hat{v}_k$, and $\hat{V}_k$ $(k = 1,2)$ as the solutions of the equations

$$L_2\left[\phi_k\right] \equiv \left[\frac{i\alpha}{2}\left\{\left(\bar{u}^\circ-c\right)\left(D^2-\gamma^2\right)-D^2\bar{u}^\circ\right\}-R^{-1}\left(D^2-\gamma^2\right)^2\right]\phi_k = 0 \qquad (2.77)$$

$$\tilde{L}_2\left[\chi_k\right] \equiv \frac{i\alpha}{2}\left\{\left(D^2-\gamma^2\right)\left[\chi_k\left(\bar{u}^\circ-c\right)\right]-\chi_k D^2\bar{u}^\circ\right\}-R^{-1}\left(D^2-\gamma^2\right)^2\chi_k = 0 \qquad (2.78)$$

$$K_k\left[\hat{v}_k\right] \equiv \left[R^{-1}\left(D^2-\gamma^2\right)-\frac{i\alpha}{2}\left(\bar{u}^\circ-c\right)\right]\hat{v}_k = -i(-1)^k\beta\,\phi_k D\bar{u}^\circ \qquad (2.79)$$

$$\left[K_k + i(-1)^k\beta\,\phi_k D\bar{u}^\circ\left(\hat{v}_k\right)^{-1}\right]\hat{V}_k = 0 \qquad (2.80)$$

subject to boundary conditions

$$\phi_k = D\phi_k = 0, \quad \chi_k = D\chi_k = 0, \quad \hat{v}_k = 0, \quad \hat{V}_k = 0, \qquad (x_3 = 0, 1) \qquad (2.81\,a,b,c,d)$$

Equations (2.77) and (2.78) are the analogues, for oblique waves, of the Orr-Sommerfeld equation and its adjoint. Equation (2.79) determines the velocity components $\hat{v}_j$ parallel to the respective wave crests. Equation (2.80) determines the corresponding velocity components $\hat{V}_j$ of the pseudo-flow.

The form of the pseudo stream functions (2.44) and (2.48) and the identification of the functions χ_j $(j = 1,2,3)$ adjoint to ϕ_j require that, according to linear theory, $B_{1,2}(t)$ and $B_3(t)$ vary as $\exp -\frac{1}{2}\alpha c_I t$ and $\exp - \alpha \tilde{c}_I t$. The solutions of the linearized forms of equations (2.60), (2.61) are consistent with this fact.

Equations (2.76a) and (2.77) subject to boundary conditions (2.77) and (2.81a) determine linear eigenvalue problems with $\phi_{1,2}$ and ϕ_3 as eigenfunctions together with c and $\tilde{c}$ as corresponding eigenvalues. We have therefore demonstrated how the linearized equations may be derived from the given variational formulation. This demonstration also throws light upon the nature of the pseudo velocities; for, in linear theory these are determined by the equations adjoint to those governing the physical variables.

b. Second-order Theory

The interaction equations (2.62), (2.63) are valid to second-order in wave amplitudes and their form suggests the substitutions

$$\frac{dA_j}{dt} = a_j^{(1)} A_j + \left(1-\delta_{j3}\right) a_j A_3 A_{3-j}^{*} + \delta_{j3}\, a_3 A_1 A_2 \quad ; \quad (j=1,2,3) \qquad (2.82)$$

where δ_{ij} is the Krönecker delta, $a_j^{(1)}$ are constants known from linear theory and the constant interaction coefficients are to be determined.

On substituting (2.82) into (2.62), (2.63) and equating to zero the respective coefficients of $A_j(t)$, the linear equations (2.76a), (2.77), (2.79) are recovered. Likewise, expressions for the interaction coefficients a_j are found by equating to zero the coefficients of the respective products of the amplitude functions $A_j(t)$ and their conjugates. Thus, the terms in $A_3 A^*_{3-j}$ yield

$$a_j = \left(I_{6+j} + I_{8+j}\right) \Big/ \left(I_{13+j} - I_{10+j}\right), \quad (j=1,2) \qquad (2.83)$$

and those in $A_1 A_2$ yield

$$a_3 = -I_6 \Big/ I_{13} \qquad (2.84)$$

To the required order of approximation, the integrands in these equations may be evaluated by employing the solutions of the linear equations (2.76a)-(2.80). In particular, equation (2.80), subject to boundary condition (2.81d) has the solution

$$\hat{v}_j = 0 \qquad (j=1,2) \qquad (0 \le x_3 \le l) \qquad (2.85)$$

In this case $I_{8+j} = I_{13+j} = 0 \ (j = 1,2)$ and the resulting integral representations of $a_j \ (j = 1,2)$ and a_3 given by (2.83) and (2.84) agree with those of Craik 1971 . If second-order corrections to the ~~pseudo-functions~~ functions $\psi_j, \hat{u}_2^{(k)}, \chi_j, \hat{U}_2^{(k)}$ $(j=1,2,3;\ k=1,2)$ are introduced, it is found that they yield no contribution to the interaction equations (see Usher & Craik [I]). In general, these integrals would require numerical evaluation; but, as discussed in §1.4, Craik [1971] has derived asymptotic estimates for them, valid for large Reynolds numbers, R, such that $|a_{1,2}| \sim O(R)$ and $|a_3| \sim O(1)$.

The present derivation of these interaction coefficients, via the given variational formulation, is more immediate than that of Craik [1971], and serves to demonstrate the advantages of the variational method.

C H A P T E R 3

THIRD-ORDER RESONANT WAVE INTERACTIONS IN VISCOUS FLOWS

§3.1 AN ANALYSIS FOR THIRD-ORDER RESONANT WAVE INTERACTIONS

To investigate at third-order the interaction of a resonant wave triad in a viscous shear flow, we here adopt the more conventional approach employing the Navier-Stokes equations, rather than the variational formulation. The reason for this choice is that in the present case it would appear that the advantages of the variational method are outweighed by increased complication due to the introduction of further auxiliary as well as physical variables.

Consider the primary shear flow

$$\bar{u}_i^o = (\bar{u}^o(x_3),\ 0,\ 0), \qquad (0 \leq x_3 \leq l),$$

and the resonant triad of waves discussed in §2.2. Denote by A a dimensionless number characteristic of the wave amplitudes and by $O(A^3)$ those terms with magnitude of third or higher order in the wave amplitudes. The velocity components (u,v,w) and the pressure p for the perturbed flow may then be written as

$$u = \bar{u}^o + \bar{\bar{u}} + 2\,\mathrm{Re}\left[\sum_{j=1}^{3} u_j \exp i\theta_j + u_{jj} \exp 2i\theta_j\right]$$

$$+ 2\,\mathrm{Re}\left[\sum_{k=1}^{2} u_{k3} \exp i(\theta_k + \theta_3) + u_{1-2} \exp i(\theta_1 - \theta_2)\right] + O(A^3),$$

$$v = \bar{\bar{v}} + 2\,\mathrm{Re}\left[\sum_{j=1}^{3} v_j \exp i\theta_j + \sum_{k=1}^{2}\left(v_{kk}\exp 2i\theta_k + v_{k3}\exp i(\theta_k+\theta_3)\right)\right]$$

$$+ 2\,\mathrm{Re}\left[v_{1-2}\exp i(\theta_1-\theta_2)\right] + O(A^3) \;,$$

$$w = 2\,\mathrm{Re}\left[\sum_{j=1}^{3}\left(w_j \exp i\theta_j + w_{jj}\exp 2i\theta_j\right) + \sum_{k=1}^{2} w_{k3}\exp i(\theta_k+\theta_3)\right]$$

$$+ 2\,\mathrm{Re}\left[w_{1-2}\exp i(\theta_1-\theta_2)\right] + O(A^3) \;,$$

$$p = x_1 p_I(t) + \bar{\bar{p}} + 2\,\mathrm{Re}\left[\sum_{j=1}^{3}\left(p_j\exp i\theta_j + p_{jj}\exp 2i\theta_j\right)\right]$$

$$+ 2\,\mathrm{Re}\left[\sum_{k=1}^{2} p_{k3}\exp i(\theta_k+\theta_3) + p_{1-2}\exp i(\theta_1-\theta_2)\right] + O(A^3) \,. \qquad (3.1)$$

Here,

$$\theta_1 = \frac{\alpha}{2}x_1 + \beta x_2 - \frac{\alpha}{2}c_R t, \quad \theta_2 = \frac{\alpha}{2}x_1 - \beta x_2 - \frac{\alpha}{2}c_R t, \quad \theta_3 = \alpha x_1 - \alpha c_R t \qquad (3.2)$$

with α, β and c_R real and Re denotes real parts of complex-valued functions. The terms

$$\left[u_j(x_3,t),\ v_j(x_3,t),\ w_j(x_3,t),\ p_j(x_3,t)\right] \exp i\,\theta_j$$

represent the three waves, $p_I(t)$ is the imposed longitudinal pressure gradient, $\bar{\bar{u}}(x_3,t)$, $\bar{\bar{v}}(x_3,t)$, $\bar{\bar{p}}(x_3,t)$ are modifications of the mean velocity and pressure owing to the non-linear Reynolds stresses. The remaining terms, involving $u_{jj}(x_3,t)$, $u_{k3}(x_3,t)$, $u_{1-2}(x_3,t)$, $v_{kk}(x_3,t)$, $v_{k3}(x_3,t)$, $v_{1-2}(x_3,t)$, $w_{jj}(x_3,t)$, $w_{k3}(x_3,t)$, $w_{1-2}(x_3,t)$, $p_{jj}(x_3,t)$, $p_{k3}(x_3,t)$, $p_{1-2}(x_3,t)$, represent the second-order (sum and difference) harmonics. We omit all $O(A^3)$ terms with periodicities different from those of the basic waves, but retain $O(A^3)$ terms with such periodicities.

Because of the form of the phase functions Θ_1 and Θ_3 (see (3.2)), the velocity terms u_1, v_1, u_{11}, v_{11}, u_{13} and v_{13} are best re-written as

$$u_1 = \gamma^{-1}\left(\frac{\alpha}{2}\hat{u}_1 - \beta\hat{v}_1\right), \qquad v_1 = \gamma^{-1}\left(\beta\hat{u}_1 + \frac{\alpha}{2}\hat{v}_1\right),$$

$$u_{11} = \gamma^{-1}\left(\frac{\alpha}{2}\hat{u}_{11} - \beta\hat{v}_{11}\right), \qquad v_{11} = \gamma^{-1}\left(\beta\hat{u}_{11} + \frac{\alpha}{2}\hat{v}_{11}\right) \qquad (3.3)$$

$$u_{13} = \gamma_0^{-1}\left(\frac{3}{2}\alpha\hat{u}_{13} - \beta\hat{v}_{13}\right) \qquad v_{13} = \gamma_0^{-1}\left(\beta\hat{u}_{13} + \frac{3}{2}\alpha\hat{v}_{13}\right)$$

where $\gamma \equiv \left(\frac{\alpha^2}{4} + \beta^2\right)^{1/2}$ and $\gamma_0 \equiv \left(\frac{9}{4}\alpha^2 + \beta^2\right)^{1/2}$. (3.4)

Quantities with a circumflex represent the velocity components perpendicular and parallel to the respective 'wave crests'. Corresponding transformations for the velocity components u_2, v_2, u_{22}, v_{22}, u_{23} and v_{23} are obtained on replacing β by $-\beta$ in (3.3).

The interaction equations for the chosen resonant triad are derived from the Navier-Stokes equations in Appendix A. Their form suggests the introduction of the following series expansions in terms of the (small but finite) complex wave amplitudes $A_j(t)$ $(j = 1,2,3)$ of the members of the resonant triad.

$$\left.\begin{aligned}\frac{dA_k}{dt} &= \frac{\alpha}{2}c_I A_k + a_k A_3 A_{3-k}^{*} + A_k \sum_{j=1}^{3} a_{kj}|A_j|^2 + o(A^4),\\ \frac{dA_3}{dt} &= \alpha\tilde{c}_I A_3 + a_3 A_1 A_2 + A_3 \sum_{j=1}^{3} a_{3j}|A_j|^2 + o(A^4),\end{aligned}\right]\quad(3.5)$$

$$\left.\begin{aligned}&\bar{\bar{u}} = \sum_{j=1}^{3} f_j |A_j|^2 + o(A^3), && \bar{\bar{v}} = \sum_{k=1}^{2} h_k |A_k|^2 + o(A^3),\\ &p_I(t) = p^{\circ} + \sum_{j=1}^{3} p_{Ij}|A_j|^2 + o(A^3) && \bar{\bar{p}} = \sum_{j=1}^{3} \bar{\bar{p}}_j |A_j|^2 + o(A^3),\end{aligned}\right]\quad(3.6)$$

$$\left.\begin{aligned}&\hat{v}_k = A_k \psi_k^{(1)} + A_3 A_{3-k}^{*} \psi_k^{(2)} + o(A^3), \qquad v_3 = A_1 A_2 \psi_3^{(2)} + o(A^3),\\ &w_k = A_k \chi_k^{(1)} + A_3 A_{3-k}^{*} \chi_k^{(2)} + A_k \sum_{j=1}^{3} \chi_k^{(2+j)} |A_j|^2 + o(A^4),\\ &w_3 = A_3 \chi_3^{(1)} + A_1 A_2 \chi_3^{(2)} + A_3 \sum_{j=1}^{3} \chi_3^{(2+j)} |A_j|^2 + o(A^4),\end{aligned}\right]\quad(3.7)$$

$$\left.\begin{aligned}&w_{jj} = A_j^2 \chi_{jj}^{(2)} + o(A^3),\\ &\hat{v}_{k3} = A_k A_3 \psi_{k3}^{(2)} + o(A^3), && w_{k3} = A_k A_3 \chi_{k3}^{(2)} + o(A^3),\\ &u_{1-2} = A_1 A_2^{*} \phi_{1-2}^{(2)} + o(A^3), && w_{1-2} = A_1 A_2^{*} \chi_{1-2}^{(2)} + o(A^3),\end{aligned}\right]\quad(3.8)$$

where j takes the values 1,2,3 and k takes the values 1,2. The suffices do not denote Cartesian tensor indices and so no summation is implied over repeated indices. p°, p_{Ij} and the interaction coefficients a_j, a_{ij} are constants (p° is specified by the primary flow (see § 2.2)); $\frac{1}{2}\alpha c_I$, $\alpha\tilde{c}_I$ are the respective linear growth (or decay) rates of the waves. All of the remaining quantities introduced in the above expansions are functions of x_3 alone and are to be determined. Corresponding expansions for the velocity terms $\hat{u}_k$, u_3, $\hat{u}_{kk}$, u_{33}, $\hat{u}_{k3}$ and v_{1-2} may be found from the continuity relations (A.6) and an expansion for $\hat{v}_{kk}$ from (A.20). It turns out that an expansion for $\hat{v}_{kk}$ is not required in the following analysis. The interaction equations for the various harmonic components (other than those for the mean motion) are derived in Appendix A by eliminating the pressure terms and so expansions are not required for p_j, p_{jj}, p_{k3} and p_{1-2}.

Boundary conditions (A.31) and (A.32) imply that

$$f_j = h_k = Z_j^{(m)} = DZ_j^{(m)} = \chi_{jj}^{(2)} = D\chi_{jj}^{(2)} = \chi_{k3}^{(2)} = D\chi_{k3}^{(2)} = 0$$
$$\psi_k^{(1)} = \psi_k^{(2)} = \psi_3^{(2)} = \psi_{k3}^{(2)} = \phi_{1-2}^{(2)} = \chi_{1-2}^{(2)} = D\chi_{1-2}^{(2)} = 0 \qquad (3.9)$$

at the boundaries $x_3 = 0, l$.

It is necessary to specify an overall condition on the mean motion. It may be assumed that either the imposed longitudinal pressure gradient remains constant or there is a constant mass flux for the flow. Here, we employ the former condition, although there seems little to choose between these two alternatives. Thus, in expansion (3.6) for $p_I(t)$ we have

$$p_{I_j} = 0 \qquad (3.10)$$

We also assume that no second-order spanwise pressure gradient can occur.

i) Linear Theory

Substitution of (3.5)-(3.8) into the interaction equations (A.14) and (A.15) of Appendix A and linearisation in the $A_j(t)$

yields the equations of linear theory; namely,

$$L_4\left[\chi_k^{(1)}\right] - \frac{\alpha}{2} c_I L_7\left[\chi_k^{(1)}\right] = 0 , \qquad L_5\left[\chi_3^{(1)}\right] - \alpha\tilde{c}_I L_8\left[\chi_3^{(1)}\right] = 0 , \tag{3.11a,b}$$

$$\left[L_6 - \frac{\alpha}{2}c_I\right]\psi_k^{(1)} + (-1)^{k+1}\beta\gamma^{-1}\chi_k^{(1)} D\bar{u}^0 = 0 , \tag{3.12}$$

where $\quad L_4 \equiv \frac{1}{R}\left(D^2-\gamma^2\right)^2 + \frac{i\alpha}{2} D^2\bar{u}^0 - \frac{i\alpha}{2}\left(\bar{u}^0 - c_R\right)\left(D^2-\gamma^2\right) ,$

$$L_5 \equiv \frac{1}{R}\left(D^2-\alpha^2\right)^2 + i\alpha D^2\bar{u}^0 - i\alpha\left(\bar{u}^0 - c_R\right)\left(D^2-\alpha^2\right) ,$$

$$L_6 \equiv \frac{1}{R}\left(D^2-\gamma^2\right) - \frac{i\alpha}{2}\left(\bar{u}-c_R\right) ,$$

$$L_7 \equiv \left(D^2-\gamma^2\right) , \qquad\qquad L_8 \equiv \left(D^2-\alpha^2\right) ,$$

and the boundary conditions at $x_3 = 0, \ell$

$$\chi_j^{(1)} = D\chi_j^{(1)} = \psi_k^{(1)} = 0 \qquad\qquad (j=1,2,3,\ k=1,2) . \tag{3.14}$$

The resultant eigenvalue problems determine the complex phase velocities $c_R + ic_I$ and $c_R + i\tilde{c}_I$, the real parts of which must be equal in order to satisfy the resonance condition. We may normalise these solutions so that

$$\chi_1^{(1)} = \chi_2^{(1)} \quad \text{and} \quad \psi_1^{(1)} = -\psi_2^{(1)} . \tag{3.15}$$

ii) Second-order Theory

Substitution of (3.5)-(3.8) into the interaction equations (A.14) and retention of the second-order terms in $A_3A_2^*$, $A_3A_1^*$ and A_1A_2 yields the second-order equations

$$L_4\left[\chi_k^{(2)}\right] - \left(\frac{\alpha}{2}c_r + \alpha\tilde{c}_r\right)L_7\left[\chi_k^{(2)}\right] = a_k L_7\left[\chi_1^{(1)}\right] - F_1\,, \quad (k=1,2) \tag{3.16}$$

$$L_5\left[\chi_3^{(2)}\right] - \alpha c_r L_8\left[\chi_3^{(2)}\right] = a_3 L_8\left[\chi_3^{(1)}\right] - F_3\,, \tag{3.17}$$

$$F_1 \equiv \frac{1}{2}\Big\{(\alpha^2\gamma^{-2}-2)\,\chi_3^{(1)}(D^2-\gamma^2)D\overset{*}{\chi}{}_1^{(1)} + (\alpha^2\gamma^{-2}-3)D\chi_3^{(1)}(D^2-\gamma^2)\overset{*}{\chi}{}_1^{(1)}$$

$$-2D\overset{*}{\chi}{}_1^{(1)}(D^2-\alpha^2)\chi_3^{(1)} - \overset{*}{\chi}{}_1^{(1)}(D^2-\alpha^2)D\chi_3^{(1)}$$

$$-2i\alpha\beta\gamma^{-1}\left(\chi_3^{(1)}D^2\overset{*}{\psi}{}_1^{(1)} + D\chi_3^{(1)}D\overset{*}{\psi}{}_1^{(1)} + \gamma^2\chi_3^{(1)}\overset{*}{\psi}{}_1^{(1)}\right)\Big\}$$

$$F_3 \equiv -\alpha^2\gamma^{-2}\Big\{D\left[\chi_1^{(1)}(D^2-\gamma^2)\chi_1^{(1)}\right] - (\alpha^2\gamma^{-2}-2)D\chi_1^{(1)}(D^2-\gamma^2)\chi_1^{(1)}$$

$$-2i\alpha\beta\gamma^{-1}\left[\psi_1^{(1)}(D^2-\gamma^2)\chi_1^{(1)} + D\psi_1^{(1)}D\chi_1^{(1)} - \gamma^2\alpha^{-2}D^2\left(\chi_1^{(1)}\psi_1^{(1)}\right)\right] \tag{3.19}$$

$$+4\beta^2\psi_1^{(1)}D\psi_1^{(1)}\Big\}$$

and the boundary conditions at $x_3 = 0, l$

$$\chi_j^{(2)} = D\chi_j^{(2)} = 0 \qquad (j=1,2,3) \tag{3.20}$$

These second-order equations agree with those of Craik [1971] on noting $\chi_1^{(1)} = \chi_2^{(1)}$, $\psi_1^{(1)} = -\psi_2^{(1)}$ and making minor changes in notation. Clearly, F_1 and F_3 may be evaluated once the solutions of the linear system (3.11)-(3.14) are known. The second-order system (3.16)-(3.20) allows determination of the three second-order interaction coefficients a_j (see §3.2).

The remaining equations of second-order theory are derived in Appendix B; but they are not required for the determination of a_j.

We note that result (3.15) ensures that the function $\Psi_3^{(2)}$ satisfies a homogeneous equation, namely (B.6), which, subject to the appropriate boundary conditions (3.9), possesses only the trivial solution $\Psi_3^{(2)} = 0$. Consequently, all terms in $\Psi_3^{(2)}$ may be omitted from the subsequent analysis. Also note the similarity of equations (B.11) and (B.12), governing the functions $\phi_{1-2}^{(2)}$ and $\chi_{1-2}^{(2)}$, with those governing Benney's [1961] longitudinal vortices. The present model includes the effect of such mean flow distortions. However, Benney seeks solutions for neutral waves in which $\chi_{1-2}^{(2)} \propto t$ and $\phi_{1-2}^{(2)} \propto t^2$. The present expansion is rather different and follows, for example, Stuart [1960].

iii) Third-order Theory

On substituting (3.5)-(3.8) into the interaction equations (A.14), retaining third-order terms and equating to zero the respective coefficients of $A_i|A_j|^2$ (i, j = 1,2,3), the following third-order equations are ultimately recovered for the functions $\chi_i^{(2+j)}$:

$$L_4\left[\chi_k^{(2+m)}\right] - \frac{3}{2}\alpha c_x L_7\left[\chi_k^{(2+m)}\right] = a_{km} L_7\left[\chi_k^{(1)}\right] - F_{km}$$

$$L_4\left[\chi_k^{(5)}\right] - \left(\frac{\alpha}{2}c_x + 2\alpha\tilde{c}_x\right) L_7\left[\chi_k^{(5)}\right] = a_{k3} L_7\left[\chi_k^{(1)}\right] - F_{k3}$$

$$L_5\left[\chi_3^{(2+m)}\right] - \left(\alpha c_x + \alpha\tilde{c}_x\right) L_8\left[\chi_3^{(2+m)}\right] = a_{3m} L_8\left[\chi_3^{(1)}\right] - F_{3m} \qquad (3.21\,a,b,c,d)$$

$$L_5\left[\chi_3^{(5)}\right] - 3\alpha\tilde{c}_x L_8\left[\chi_3^{(5)}\right] = a_{33} L_8\left[\chi_3^{(1)}\right] - F_{33}$$

where the operators L_4, L_5, L_7, L_8 are defined by (3.13), and

$$
\begin{aligned}
F_{11} \equiv & -\left(\tfrac{1}{2} i\alpha f_1 + i\beta h_1\right)(D^2-\gamma^2)\chi_1^{(1)} + \chi_1^{(1)} D^2\left(\tfrac{1}{2} i\alpha f_1 + i\beta h_1\right) \\
& + \tfrac{3}{2}\gamma^2\left(\overset{*}{\chi}{}_1^{(1)} D\chi_{11}^{(2)} + 2\chi_{11}^{(2)} D\overset{*}{\chi}{}_1^{(1)}\right) + D\left[\chi_{11}^{(2)} D^2 \overset{*}{\chi}{}_1^{(1)} - \tfrac{1}{2} D\left(\overset{*}{\chi}{}_1^{(1)} D\chi_{11}^{(2)}\right)\right],
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
F_{12} \equiv & -a_3\,(D^2-\gamma^2)\chi_1^{(2)} - \left(\tfrac{1}{2} i\alpha f_2 + i\beta h_2\right)(D^2-\gamma^2)\chi_1^{(1)} + \chi_1^{(1)} D^2\left(\tfrac{1}{2} i\alpha f_2 + i\beta h_2\right) \\
& + i\beta\left[\gamma^{-2}(\alpha^2-\gamma^2) D\left(\psi_3^{(2)} D\overset{*}{\chi}{}_1^{(1)}\right) + D\left(\overset{*}{\chi}{}_1^{(1)} D\psi_3^{(2)}\right) + \gamma^2 \overset{*}{\chi}{}_1^{(1)}\psi_3^{(2)}\right] \\
& + \tfrac{1}{2}\left[3\gamma^2 \overset{*}{\chi}{}_1^{(1)} D\chi_3^{(2)} - D^2\left(\overset{*}{\chi}{}_1^{(1)} D\chi_3^{(2)}\right) + (\alpha^2+2\gamma^2)\chi_3^{(2)} D\overset{*}{\chi}{}_1^{(1)}\right] \\
& + \gamma^{-2}(\gamma^2-2\beta^2) D\left[\chi_3^{(2)} D^2 \overset{*}{\chi}{}_1^{(1)} - \chi_{1-2}^{(2)} D^2 \chi_1^{(1)}\right] \\
& + \alpha\beta\gamma^{-1}\left[\alpha D\left(\psi_1^{(1)}\phi_{1-2}^{(2)}\right) + 2\beta D\left(\psi_3^{(2)}\overset{*}{\psi}{}_1^{(1)}\right)\right] \\
& + \frac{i\alpha}{2}\left[D\left(\chi_1^{(1)} D\phi_{1-2}^{(2)}\right) + \gamma^2\chi_1^{(1)}\phi_{1-2}^{(2)} - \gamma^{-2}(\gamma^2-4\beta^2) D\left(\phi_{1-2}^{(2)} D\chi_1^{(1)}\right)\right] \\
& + (\gamma^2+2\beta^2)\chi_{1-2}^{(2)} D\chi_1^{(1)} + \tfrac{1}{2}\left[3\gamma^2\chi_1^{(1)} D\chi_{1-2}^{(2)} - D^2\left(\chi_1^{(1)} D\chi_{1-2}^{(2)}\right)\right] \\
& - i\alpha\beta\gamma^{-1}\left[\gamma^2\left(\psi_1^{(1)}\chi_{1-2}^{(2)} + \overset{*}{\psi}{}_1^{(1)}\chi_3^{(2)}\right) + D\left(\chi_{1-2}^{(2)} D\psi_1^{(1)} + \chi_3^{(2)} D\overset{*}{\psi}{}_1^{(1)}\right)\right]
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
F_{13} \equiv\; & -a_2^{*}\left(D^2-\gamma^2\right)\chi_1^{(2)} + \frac{i\alpha}{2}\left[\chi_1^{(1)} D^2 f_3 - f_3\left(D^2-\gamma^2\right)\chi_1^{(1)}\right] \\
& + \frac{1}{2}\left[5\gamma^2 \chi_{13}^{(2)} D\chi_3^{\overset{*}{(1)}} + D\left(\chi_{13}^{(2)} D^2 \chi_3^{\overset{*}{(1)}}\right)\right] \\
& + \frac{1}{2}\gamma_0^{-2}\left[\gamma^2\left(3\alpha^2+2\gamma_0^2\right)\chi_3^{\overset{*}{(1)}} D\chi_{13}^{(2)} - \left(\alpha^2+2\gamma^2\right)D\left(\chi_3^{\overset{*}{(1)}} D^2\chi_{13}^{(2)}\right) - 3\gamma^2 D\left(D\chi_{13}^{(2)} D\chi_3^{\overset{*}{(1)}}\right)\right] \\
& + i\alpha\beta\gamma_0^{-1}\left[D\left(\chi_3^{\overset{*}{(1)}} D\psi_{13}^{(2)} + 2\psi_{13}^{(2)} D\chi_3^{\overset{*}{(1)}}\right) + \gamma^2 \chi_3^{\overset{*}{(1)}} \psi_{13}^{(2)}\right] \\
& + \left(\frac{\alpha^2}{2}+\gamma^2\right)\chi_3^{(1)} D\chi_2^{\overset{*}{(2)}} + \gamma^{-2}\left(\gamma^2-2\beta^2\right)D\left(\chi_3^{(1)} D^2 \chi_2^{\overset{*}{(2)}}\right) \\
& + \frac{1}{2}\left[3\gamma^2 \chi_2^{\overset{*}{(2)}} D\chi_3^{(1)} - D^2\left(\chi_2^{\overset{*}{(2)}} D\chi_3^{(1)}\right)\right] + i\alpha\beta\gamma^{-1}\left[D\left(\chi_3^{(1)} D\psi_2^{\overset{*}{(2)}}\right) + \gamma^2 \psi_2^{\overset{*}{(2)}} \chi_3^{(1)}\right],
\end{aligned}
\tag{3.24}
$$

$$
\begin{aligned}
F_{31} \equiv\; & -a_2\left(D^2-\alpha^2\right)\chi_3^{(2)} + i\alpha\left[\chi_3^{(1)} D^2 f_1 - f_1\left(D^2-\alpha^2\right)\chi_3^{(1)}\right] - 2\alpha^2\beta^2\left(\gamma\gamma_0\right)^{-1} D\left(\psi_1^{\overset{*}{(1)}} \psi_{13}^{(2)}\right) \\
& + \alpha^2\left(4\gamma_0^2\gamma^2\right)^{-1}\Big[D\left\{2\gamma_0^2 \chi_{13}^{(2)} D^2 \chi_1^{\overset{*}{(1)}} - \left(3\alpha^2+4\beta^2\right)D\chi_{13}^{(2)} D\chi_1^{\overset{*}{(1)}} - 6\gamma^2 \chi_1^{\overset{*}{(1)}} D^2 \chi_{13}^{(2)}\right\} \\
& \qquad\qquad + 4\gamma_0^2\left(\alpha^2+2\beta^2\right)\chi_{13}^{(2)} D\chi_1^{\overset{*}{(1)}} + 4\gamma^2\left(3\alpha^2+2\beta^2\right)\chi_1^{\overset{*}{(1)}} D\chi_{13}^{(2)}\Big] \\
& + i\alpha\beta\left(4\gamma\gamma_0^2\right)^{-1}\Big[\left(3\alpha^2-4\beta^2\right)D\left(\psi_1^{\overset{*}{(1)}} D\chi_{13}^{(2)}\right) - 4\gamma_0^2\left\{D\left(\chi_{13}^{(2)} D\psi_1^{\overset{*}{(1)}}\right) + \alpha^2 \psi_1^{\overset{*}{(1)}} \chi_{13}^{(2)}\right\}\Big] \\
& - i\alpha\beta\left(4\gamma_0\gamma^2\right)^{-1}\Big[4\gamma^2\left\{D\left(\chi_1^{\overset{*}{(1)}} D\psi_{13}^{(2)}\right) + \alpha^2 \chi_1^{\overset{*}{(1)}} \psi_{13}^{(2)}\right\} \\
& \qquad\qquad + \left(5\alpha^2+4\beta^2\right)D\left(\psi_{13}^{(2)} D\chi_1^{\overset{*}{(1)}}\right)\Big] \\
& + \alpha^2\left(2\gamma^2\right)^{-1}\left[4\beta^2 D\left(\chi_1^{(1)}\chi_2^{(2)}\right) - D\left(\chi_1^{(1)} D^2 \chi_2^{(2)} + \chi_2^{(2)} D^2 \chi_1^{(1)}\right)\right] \\
& + \alpha^2\gamma^{-4}\left(\gamma^2-2\beta^2\right)D\left(D\chi_1^{(1)} D\chi_2^{(2)}\right) + i\alpha\beta\gamma^{-1} D\left(\chi_1^{(1)} D\psi_2^{(2)} - \chi_2^{(2)} D\psi_1^{(1)}\right) \\
& + i\alpha\beta\gamma^{-1}\left[\gamma^{-2}\left(\beta^2-\frac{3}{4}\alpha^2\right)D\left(\psi_2^{(2)} D\chi_1^{(1)} - \psi_1^{(1)} D\chi_2^{(2)}\right) + \alpha^2\left(\psi_2^{(2)}\chi_1^{(1)} - \psi_1^{(1)}\chi_2^{(2)}\right)\right] \\
& + 2\alpha^2\beta^2\gamma^{-2} D\left(\psi_2^{(2)}\psi_1^{(1)}\right),
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
F_{33} = {} & i\alpha\left[\chi_3^{(1)} D^2 f_3 - f_3 (D^2-\alpha^2)\,\chi_3^{(1)}\right] + D\left[\chi_{33}^{(2)} D^2 \overset{*}{\chi}{}_3^{(1)} - \frac{D}{2}\left(\overset{*}{\chi}{}_3^{(1)} D \chi_3^{(2)}\right)\right] \\
& + \frac{3}{2}\alpha^2\left[\overset{*}{\chi}{}_3^{(1)} D \chi_{33}^{(2)} + 2\,\chi_{33}^{(2)} D \overset{*}{\chi}{}_3^{(1)}\right] .
\end{aligned}
\tag{3.26}
$$

The remaining F_{ij} are defined as follows. F_{21} is obtained from (3.23) on replacing $\chi_{1-2}^{(2)}$, $\phi_{1-2}^{(2)}$ by their complex conjugates, β by $-\beta$ and suffix 1 by suffix 2 in the remaining terms, F_{22}, F_{23} and F_{32} are obtained from (3.22), (3.24) and (3.25), respectively, on replacing β by $-\beta$ and suffix 1 by suffix 2. In fact, because the problem is symmetric with respect to the suffices 1 and 2, it turns out that

$$F_{21} = F_{12},\quad F_{22} = F_{11},\quad F_{23} = F_{13},\quad F_{32} = F_{31},$$

provided the first-order solutions are normalised so that

$$\chi_1^{(1)} = \chi_2^{(1)}, \quad \psi_1^{(1)} = -\psi_2^{(1)}.$$

These expressions for F_{mn} contain the functions

$$
\begin{aligned}
& f_j,\quad h_j,\quad \chi_{jj}^{(2)},\quad \chi_j^{(2)},\quad \psi_j^{(2)} \quad (j = 1,2,3), \\
& \chi_{k3}^{(2)},\quad \psi_{k3}^{(2)},\quad \overset{*}{\chi}{}_k^{(2)},\quad \overset{*}{\psi}{}_k^{(2)}, \quad (k = 1,2) \\
& \phi_{1-2}^{(2)},\quad \overset{*}{\phi}{}_{1-2}^{(2)},\quad \chi_{1-2}^{(2)},\quad \overset{*}{\chi}{}_{1-2}^{(2)}.
\end{aligned}
$$

of second-order theory, whose governing equations are derived in Appendix B. Accordingly, the F_{mn} may be determined whenever the appropriate second-order solutions are known.

Equations (3.21 a-d) and the appropriate boundary conditions (3.9) enable determination of the nine third-order interaction coefficients a_{ij} $(i,j = 1,2,3)$.

§3.2 INTEGRAL REPRESENTATIONS FOR THE INTERACTION COEFFICIENTS

Determination of the interaction coefficients involves the following results in the theory of differential equations (see Ince 1956 §9.34).

Consider an n^{th} order inhomogeneous linear differential equation

$$L_n[\phi] = \lambda f - g \tag{3.28}$$

where f and g are known functions of the independent variable y, and λ is a parameter to be determined, subject to the homogeneous boundary conditions

$$\mathcal{L}_j^{(k)}[\phi] = 0, \quad y = y_1, \quad k = 1, 2, \ldots s, \text{ say}$$

$$\mathcal{L}_i^{(m)}[\phi] = 0, \quad y = y_2, \quad m = s+1, \ldots n \;. \tag{3.29a,b}$$

Here, L_n, $\mathcal{L}_p^{(q)}$ are linear differential operators of degree n and $p \leqslant n$, respectively. The superscript q is just for labelling the operators.

A solution of (3.28) subject to boundary conditions (3.29 a,b) will exist and be unique provided there are no non-trivial solutions of the associated homogeneous problem.

$$L_n[w] = 0 \qquad (y_1 \leqslant y \leqslant y_2) \tag{3.30}$$

subject to the same boundary conditions (3.29 a,b) as those imposed on ϕ . However if there exist non-trivial solutions of this homogeneous problem, solutions of (3.28) subject to boundary conditions (3.29 a,b) can be found only for particular values of the parameter λ . These values may be determined with the aid of the function adjoint to w.

The function Ω adjoint to w is defined to be the solution of the differential system

$$\tilde{L}_n[\Omega] = 0 \qquad (3.31)$$

with adjoint boundary conditions

$$B^{(k)}[\Omega] = 0 \quad , \qquad y = y_2 , \; k = 1,\ldots s$$

$$B^{(m)}[\Omega] = 0 \quad , \qquad y = y_1 , \; m = s+1,\ldots n$$

where the differential operator $\tilde{L}_n$ is adjoint to L_n and the adjoint boundary conditions are chosen so that

$$\int_{y_1}^{y_2} \Omega L_n[w]\, dy = \int_{y_1}^{y_2} w \tilde{L}_n[\Omega]\, dy$$

A non-trivial adjoint function Ω exists whenever (3.30), subject to boundary conditions (3.29 a,b), possesses a non-trivial solution w. When this is the case there exists a solution of (3.28), (3.29 a,b) if and only if the parameter λ takes the value

$$\lambda = \left(\int_{y_1}^{y_2} g\Omega\, dy \right) \Bigg/ \left(\int_{y_1}^{y_2} f\Omega\, dy \right) \qquad (3.32)$$

However, this solution is not unique, for a solution kw of (3.30), (3.29 a,b) may always be added, where k is an arbitrary constant.

Returning to the non-homogeneous equations and associated boundary conditions derived in §3.1, it is readily confirmed that the corresponding homogeneous problems (other than those for $\chi_j^{(1)}$ $(j = 1,2,3)$) will usually possess only trivial solutions since the terms in c_I and $\tilde{c}_I$ on the left-hand sides do not correspond to the appropriate linear eigenvalues. However in the special case of neutrally stable (linear) modes $(c_I,\ \tilde{c}_I = 0)$ the homogeneous problems associated with (3.16), (3.21 a,b) and with (3.17), (3.21 c,d) reduce to the linear eigenvalue problems (3.11a) and (3.11b) respectively (also with $c_i = \tilde{c}_i = 0$) and the appropriate boundary conditions. These certainly possess non-trivial solutions. For a resonant triad comprising such neutral waves the interaction coefficients a_{ij} $(i,j = 1,2,3)$ are determined from result (3.32).

The adjoint functions $\Psi_{1,2}^{(1)}$, $\Psi_3^{(1)}$ of linear theory satisfy

$$\tilde{L}_1\left[\Psi_{1,2}^{(1)}\right] \equiv \left[R^{-1}\left(D^2-\gamma^2\right)^2 - \frac{i\alpha}{2}\left(\bar{u}^0 - c\right)D^2 - i\alpha D\bar{u}^0 D\right]\Psi_{1,2}^{(1)} = 0 \qquad (3.33)$$

$$\tilde{L}_3\left[\Psi_3^{(1)}\right] \equiv \left[R^{-1}\left(D^2-\alpha^2\right)^2 - i\alpha\left(\bar{u}^0 - \tilde{c}\right)D^2 - 2i\alpha D\bar{u}^0 D\right]\Psi_3^{(1)} = 0 \qquad (3.34)$$

subject to boundary conditions

$$\Psi_j^{(1)} = D\Psi_j^{(1)} = 0\,, \qquad j = 1,2,3\,, \qquad (x_3 = 0, L) \qquad (3.35)$$

For the given resonant triad the complex phase velocities are

$$c = c_R + ic_I\,, \qquad \tilde{c} = c_R + i\tilde{c}_I\ .$$

When this resonant triad consists of neutrally stable (linear) modes, $c_I = \tilde{c}_I = 0$. In this case, by application of result (3.32) the second- and third-order interaction coefficients are found to be

$$a_k = \left(\int_0^l \Psi_k^{(1)} F_1 \, dx_3\right) \bigg/ \left(\int_0^l \Psi_k^{(1)} L_7\left[\chi_1^{(1)}\right] dx_3\right),$$

$$a_3 = \left(\int_0^l \Psi_3^{(1)} F_3 \, dx_3\right) \bigg/ \left(\int_0^l \Psi_3^{(1)} L_8\left[\chi_3^{(1)}\right] dx_3\right), \qquad (3.36a,b)$$

$$a_{ki} = \left(\int_0^l \Psi_k^{(1)} F_{ki} \, dx_3\right) \bigg/ \left(\int_0^l \Psi_k^{(1)} L_7\left[\chi_k^{(1)}\right] dx_3\right),$$

$$a_{3i} = \left(\int_0^l \Psi_3^{(1)} F_{3i} \, dx_3\right) \bigg/ \left(\int_0^l \Psi_3^{(1)} L_8\left[\chi_3^{(1)}\right] dx_3\right), \quad (k=1,2,\ i=1,2,3) \qquad (3.37a,b)$$

In order that these interaction coefficients and the wave amplitudes A_i are uniquely defined, it is necessary to specify the normalisations imposed on the functions $\chi_i^{(1)}$. One possible choice is

$$-\int_0^l \chi_k^{(1)*} L_7\left[\chi_k^{(1)}\right] dx_3 = \int_0^l \left(\left|D\chi_k^{(1)}\right|^2 + \gamma^2\left|\chi_k^{(1)}\right|^2\right) dx_3 = \gamma^2, \quad (k=1,2),$$

$$-\int_0^l \chi_3^{(1)*} L_8\left[\chi_3^{(1)}\right] dx_3 = \int_0^l \left(\left|D\chi_3^{(1)}\right|^2 + \alpha^2\left|\chi_3^{(1)}\right|^2\right) dx_3 = \alpha^2,$$

which ensures that the (dimensionless) kinetic energy of the two-dimensional wave is just A_3^2. However, other choices may be preferable for particular problems; for instance, Reynolds and Potter [1967] use the normalisations $\chi_1^{(1)} = -i$ and $D\chi_1^{(1)} = -i$ — Pekeris & Shkoller [1967] use $\chi_3^{(1)} = -i\alpha$ and $D\chi_3^{(1)} = -i\alpha$ — at the centre line of the channel (i.e. at $x_3 = 1$, with $l = 2$) for the even and odd modes respectively, of

Poiseuille flow.

Clearly, the normalisations employed for the adjoint functions $\Psi_i^{(1)}$ do not affect the values of a_i and a_{ij}; but we note that expressions (3.36a)-(3.37b) are simplified if we impose the normalisations

$$\int_0^1 \Psi_k^{(1)} L_7\left[\chi_k^{(1)}\right] dx_3 = \int_0^1 \Psi_3^{(1)} L_8\left[\chi_3^{(1)}\right] dx_3 = 1 . \qquad (3.38)$$

Since $\chi_1^{(1)} = \chi_2^{(1)}$ (see 3.15), the solutions of (3.33), (3.35) may be normalised according to (3.38) so that

$$\Psi_1^{(1)} = \Psi_2^{(1)} \qquad (3.39)$$

Then on noting (3.15), we see from (3.36a) that

$$a_1 = a_2 . \qquad (3.40)$$

Also, since the functions F_{ij} are symmetric with respect to suffices 1 and 2 (see (3.27)), the third-order interaction coefficients evaluated from results (3.37 a,b) will possess this symmetry. Consequently

$$a_{11} = a_{22}, \quad a_{13} = a_{23}, \quad a_{12} = a_{21}, \quad a_{31} = a_{32} \qquad (3.41)$$

This in turn means that the solutions of (3.21 a,b) and (3.9) may be normalised so that

$$\chi_1^{(2+j)} = \chi_2^{(2+j)} , \qquad (j=1,2,3) \qquad (3.42)$$

The integral representations (3.36 a,b) of the second-order interaction coefficients agree with those of §2.3b and were in fact obtained by Craik [1971] for non-neutral (linear) waves. Those for the third-order parameters a_{ij} have not been derived previously.

A proper justification of Craik's [1971] treatment is now given which shows that the second-order interaction coefficients for a resonant triad of non-neutral (linear) waves may still be determined from (3.36 a,b) provided the linear growth (or decay) rates are sufficiently small. Indeed, a similar argument follows for the third-order parameters.

Consider the following three pairs of n^{th} order linear differential equations

$$L_{n,i}\left[\chi_i^{(1)}(y)\right] - \delta_i\left(D^2-\alpha_i^2\right)\chi_i^{(1)}(y) = 0 , \tag{3.43}$$

$$L_{n,i}\left[\chi_i^{(2)}(y)\right] - \left(\delta_i+\varepsilon_i\right)\left(D^2-\alpha_i^2\right)\chi_i^{(2)}(y) = \lambda_i\left(D^2-\alpha_i^2\right)\chi_i^{(1)}(y) - F_i , \tag{3.44}$$

$(y_1 \leq y \leq y_2,\ \ i = 1,2,3)$, subject to boundary conditions

$$\left.\begin{aligned} \mathcal{L}_{j,i}^{(k)}\left[\chi_i^{(1,2)}\right] &= 0 \qquad y=y_1 ,\ k=1,\dots,s,\ \text{say}, \\ \mathcal{L}_{l,i}^{(m)}\left[\chi_i^{(1,2)}\right] &= 0 \qquad y=y_2 ,\ m=s+1,\dots,n \end{aligned}\right\} \tag{3.45}$$

$(j,l < n)$. (No summation is implied over repeated indices). Here $L_{n,i}$, $\mathcal{L}_{p,i}^{(q)}$ are linear operators of degree n and $p(< n)$ respectively, F_i is a known function dependent on $\chi_j^{(1)}$ $(j = 1,2,3)$ but independent of $\chi_j^{(2)}$, α_i, ε_i and λ_i are parameters, δ_i is an eigenvalue of (3.43), (3.45). In the context of §3.1 i) and ii), $\alpha_{1,2}$ can be identified with γ, α_3 with α, $\delta_{1,2}$ with $\frac{1}{2}\alpha c_I$, δ_3 with $\alpha\tilde{c}_I$ (see (3.11 a,b)), $\varepsilon_{1,2}$ with $\alpha\tilde{c}_I$, ε_3 with $\alpha(c_i - \tilde{c}_I)$ and λ_i with the second-order interaction coefficients a_i (see (3.16), (3.17)). We shall regard ε_i and δ_i as formally independent.

A non-trivial solution of (3.43), (3.45) exists provided δ_i is an eigenvalue of this homogeneous problem. It is assumed that this solution is known. For $\varepsilon_i \neq 0$, the system (3.44), (3.45) will normally possess a unique solution for each value of λ_i, since the operator on the left-hand side of (3.44) differs from that of the homogeneous problem. The exceptional case occurs when $\delta_i + \varepsilon_i$ is another eigenvalue of the homogeneous problem, but is not considered here. However, as $\varepsilon_i \to 0$ the operators on the left-hand sides of (3.43) and (3.44) tend to equality, and, in this limit, the solution of system (3.44), (3.45) will generally exhibit singular behaviour involving inverse powers of ε_i. We wish to choose the value of λ_i so as to suppress this singularity. Accordingly we let

$$\chi_i^{(2)}(y) = \chi_{i,o}^{(2)}(y) + O(\varepsilon_i) \quad , \tag{3.46}$$

$$\lambda_i = \lambda_{i,o} + O(\varepsilon_i) \; , \qquad (i=1,2,3) \quad . \tag{3.47}$$

Here, $\lambda_{i,o}$ is just the value of λ_i required to yield a regular solution $\chi_{i,o}^{(2)}$ in the limit $\varepsilon_i \to 0$. Assuming that ε_i and δ_i are formally independent (as they are in §3.1) the expansions (3.46), (3.47) in ε_i, holding δ_i fixed, will be valid however small δ_i may be. On substituting these expansions into (3.44) and equating coefficients of ε^o, there results

$$\left[L_{n,i} - \delta_i\left(D^2-\alpha_i^2\right)\right] \chi_{i,o}^{(2)}(y) = \lambda_{i,o}\left(D^2-\alpha_i^2\right)\chi_i^{(1)}(y) - F_i \tag{3.48}$$

If $X_i^{(1)}(y)$ is the function adjoint to $\chi_i^{(1)}(y)$ (see (3.31)) then a unique non-trivial solution of (3.48) subject to boundary conditions of the form (3.45) exists provided

$$\lambda_{i,o} = \left(\int_{y_1}^{y_2} X_i F_i \, dy \right) \Big/ \left(\int_{y_1}^{y_2} X_i \left(D^2 - \alpha_i^2 \right) \chi_i^{(1)} \, dy \right) \qquad (3.49)$$

So for small but finite ϵ_i the correct value of λ_i is given by (3.49) to within $O(\epsilon_i)$.

Returning to equations (3.11 a,b), (3.16), (3.17) and (3.21 a-d), the linear eigenvalues c_R, c_I, $\tilde{c}_I$ and the eigenfunctions $\chi_j^{(1)}$ $(j = 1,2,3)$ are assumed known. There will be unique solutions of the second- and third-order equations (3.16), (3.17) and (3.21 a-d), with boundary conditions (3.20) and (3.9), for each value of the parameters a_i and a_{ij}. But, as discussed above, these will generally exhibit singular behaviour in the limits $\alpha\tilde{c}_I \to 0$ and $\alpha c_I \to \alpha\tilde{c}_I$. There will be just one value for each parameter a_i and a_{ij} which suppresses this singular behaviour, namely that corresponding to result (3.49). Accordingly the representations (3.36a)-(3.37b) for the interaction coefficients a_i, a_{ij} are seen to be valid to order $O(\alpha c_I, \ \alpha\tilde{c}_I)$.

Hereafter it is assumed that the chosen resonant triad consists of waves for which (3.36a)-(3.37b) yield good approximations for the interaction coefficients. (Then $\chi_j^{(1)}$, $\psi_k^{(1)}$ $(j = 1,2,3 \ \ k = 1,2)$ are the linear eigenfunctions, but not necessarily for neutrally stable waves.) In any case, the $O(\alpha c_I, \ \alpha\tilde{c}_I)$ correction to the interaction coefficients a_i, a_{ij} may always be found if necessary. Normally, the present first-order estimates should suffice; but, for example, if the first-order approximation to a_i or a_{ij} were purely real (or imaginary) and the imaginary (or real) part were of crucial importance in the evolution of the disturbances, the higher-order theory would be required. In fact, using the variational formulation , Usher & Craik [I] show results (3.36 a,b) for the second-order parameters to be exact, thus justifying Craik's [1971] second-order theory. It is distinctly possible that results (3.37 a,b) for the third-order parameters are also exact.

The next task is the evaluation of integral representations (3.36a)-(3.37b) for the interaction coefficients. For particular cases, such evaluations may be effected by numerical computation, as was done by Reynolds & Potter [1967] and Pekeris & Shkoller [1967] who computed a_{33} for Poiseuille and Couette-Poiseuille flow. Here, however, we derive approximate estimates for the interaction coefficients by analytical means. An asymptotic theory is developed which yields, under well-defined conditions, valid approximations for sufficiently large values of the Reynolds number R. These estimates should be valid for a wide class of flows.

§3.3 SECOND-ORDER INTERACTION COEFFICIENTS; αR LARGE

We briefly outline Craik's [1971] second-order analysis.

The inviscid estimates for the various first-order functions $\Phi_j^{(1)}$, $\chi_j^{(1)}$, $\Psi_k^{(1)}$ ($j = 1,2,3$ $k = 1,2$) appearing in the integrands of expressions (3.36 a,b) for the second-order parameters a_j, are given by (C.1 a,b,c). These estimates are normally singular at the points in the complex x_3- plane where $\bar{u}(x_3) = c_R + ic_I$ or $c_R + i\tilde{c}_I$. Assuming that the chosen resonant triad consists of waves sufficiently near neutral stability (according to linear theory) these points lie close to the real axis. For the purpose of indicating the strength of the singularities it is sufficient to treat the singular points as if they are coincident on the real axis at $x_3 = x_{3c}$, where $\bar{u}(x_{3c}) = c_R$.

Direct substitution of these estimates in the integrands $\Phi_k^{(1)}F_1$, $\Phi_3^{(1)}F_3$, of (3.36 a,b) yields singularities like $(x_3 - x_{3c})^{-4}$ for strictly neutral waves, and correspondingly large contributions for small but non-zero c_I, $\tilde{c}_I$. Now, it is known from linear viscous theory that for large Reynolds numbers the inviscid estimates for $\Phi_j^{(1)}$, $\chi_j^{(1)}$, $\Psi_k^{(1)}$ are valid asymptotic approximations in the sector

$$-\frac{7}{6}\pi \leq \arg\left(x_3 - x_{3c}\right) \leq \frac{1}{6}\pi$$

of the complex x_3- plane, excluding a small circle of radius $\left[(\alpha R D\bar{u}_c^{\prime})^{-\frac{1}{3}}\right]$ centred at x_{3c}; whereas the corresponding inviscid estimates for the conjugate functions $\overset{*}{\Phi}_j^{(1)}$, $\overset{*}{\chi}_j^{(1)}$, $\overset{*}{\Psi}_k^{(1)}$ are valid in the sector

$$\frac{-\pi}{6} \leq \arg(x_3 - x_{3c}) \leq \frac{7}{6}\pi \quad ,$$

with the same circle of exclusion at x_{3c} (see Lin [1955], p.132 and Craik [1971], p.401). Since the integrands $\Psi_3^{(1)}F_3$, together with those in the denominators of (3.36 a,b), involve no conjugate functions, their paths of integration may be deformed to pass beneath the singularity at x_{3c}; accordingly, in general, the (possibly complex) values of these integrals remain O(1) as $R \to \infty$. Here, we employ normalisations (3.38), which insist that the integrals appearing in the denominators of (3.36 a,b) all take the value unity and are therefore real-valued in this case. Thus, in particular, from (3.36 b) we may deduce that

$$|a_3| \sim O(1) \tag{3.50}$$

In contrast the integrands $\Psi_k^{(1)}F_1$ contain conjugate functions and the paths cannot be deformed so as to avoid the singularity and yet have a uniformly valid inviscid approximation for the integrands. Instead, viscous estimates are required in the critical-layer region when evaluating these integrals. Such estimates are derived in Appendix C. Using these results and noting result (3.39), it is readily shown that the leading-order critical-layer contribution to the integral $\int_0^L \Psi_1^{(1)}F_1 dx_3$ is

$$C_3|C_1|^2\left\{\frac{1}{2}i\pi\alpha^2\beta^2\gamma^{-2}R + O\left[R^{2/3}\log_e R\right]\right\} \tag{3.51}$$

where the constants C_1, C_3 are defined in (C.1 a,b,c). On noting normalisations (3.38) we have the results

$$|a_1| = |a_2| \sim O(R) \tag{3.52}$$

(see Craik [1971], §4). Here, for simplicity, we have discussed the contribution from a single critical layer. When there is more than one such layer, as in Poiseuille flow, the contributions from both layers must of course be retained. Extension of the above analysis to such cases is immediate.

Consideration of the amplitude expansions (3.5) and results (3.50) and (3.52) indicate that, for large R and when the three waves constituting the chosen resonant triad possess amplitudes of comparable size, the second-order interaction terms affecting the oblique waves are much larger than that which influences the two-dimensional wave. This may lead to preferential amplification of the oblique waves. Craik [1971] discusses solutions of the second-order interaction equations in which this growth of three-dimensionality may be rapid (which may have some relevance to the boundary-layer experiments of Klebanoff & Tidstrom [1959] and Klebanoff, Tidstrom & Sargent [1962]) and he shows the possibility of a 'burst' of wave energy at a finite time.

§3.4 THIRD-ORDER INTERACTION COEFFICIENTS; αR LARGE. RESONANT CASE WITH $D^2\bar{u}_c^o \neq 0$.

The derivations of asymptotic estimates for the third-order parameters a_{ij} $(i,j = 1,2,3)$ proceeds similarly. Since the a_{ij} are symmetric with respect to suffices 1 and 2 (see (3.41)) we only require to find estimates for

a_{1j}, a_{3m} $(j = 1,2,3 \quad m = 1,3)$.

In Appendix C it is established that the second-order functions, as well as the first-order functions, occurring in the integrands $\Psi_1^{(1)}F_{1j}$, $\Psi_3^{(1)}F_{3m}$ possess a critical-layer structure. Consequently we must examine contributions to these integrals from both inside and outside the critical-layer (for present purposes we may ignore the thin $O[(\alpha R)^{-\frac{1}{2}}]$ viscous wall layers near $x_3 = 0, l$ since they do not contribute significantly to the interaction integrals). Unlike the integrals for the second-order parameters the contributions from outside the critical-layer are not all O(1) in magnitude since several of the second-order quantities are O(R) or $O(R^2)$ there. In particular the 'longitudinal-vortex' components $\chi_{1-2}^{(2)}$ and $\phi_{1-2}^{(2)}$ possess magnitudes

$$\left|\chi_{1-2}^{(2)}\right| \sim O(R) \ , \quad \left|\phi_{1-2}^{(2)}\right| \sim O(R^2) \ ,$$

both outside and inside the critical layer. The downstream component $\phi_{1-2}^{(2)}$ is particularly large and represents a spanwise-periodic distortion of the primary velocity profile owing to convection of momentum by the x_3-velocity component $\chi_{1-2}^{(2)}$.

i) <u>Resonant Case with $D^2\bar{u}_c^{\bullet} \neq 0$.</u>

For this case the results of Appendix C, for the orders of magnitude of the various functions occurring in the integrands $\Psi_1^{(1)}F_{1j}$, $\Psi_3^{(1)}F_{3m}$ (j = 1,2,3 m= 1,3), are summarized in Table 1, (see end of this chapter).

First we determine the orders of magnitude with respect to R of the contributions to the integrals

$$I_{1j} \equiv \int_0^l \Psi_1^{(1)} F_{1j}\, dx_3 \quad , \qquad I_{3m} \equiv \int_0^l \Psi_3^{(1)} F_{3m}\, dx_3 \quad , \quad (j=1,2,3;\ m=1,3) \qquad (3.53)$$

from outside the critical layer. On employing the outer estimates for the functions occurring in the integrands $\Psi_1 F_{1j}$, $\Psi_3 F_{3m}$, which are derived in Appendix C §§ i) and ii), it turns out that a few terms dominate the integrands for large R. (Note that, for simplicity, outer estimates for the second-order functions f_j, h_k (j = 1,2,3 k = 1,2), $\chi_{1-2}^{(2)}$ and $\phi_{1-2}^{(2)}$ are derived in Appendix C for cases in which

$$|\alpha c_I R \, , \, \alpha \tilde{c}_I R| \ll O(1) \; ,$$

[see (C.6)]). The following results are obtained with the dominant terms stated in brackets.

$$|I_{11}| \sim O(R) \, , \qquad \left[\Psi_1^{(1)} \times \left(\text{terms in } f_1 \, , h_1 \text{ and derivatives}\right)\right] ,$$

$$|I_{12}| \sim O(R^2) \, , \qquad \left[\Psi_1^{(1)} \times \left(\text{terms in } \phi_{1-2}^{(2)} \text{ and derivatives}\right)\right] ,$$

$$|I_{13}| \sim O(R^2) \, , \qquad \left[\Psi_1^{(1)} \times \left(a_1^{*}(D^2-\gamma^2)\chi_1^{(2)} \, , \text{ using result } (3.52)\right)\right] ,$$

$$|I_{31}| \sim O(R) \, , \qquad \left[\Psi_3^{(1)} \times \left(a_1(D^2-\alpha^2)\chi_3^{(2)} \text{ and terms in } f_1 \, , \chi_2^{(2)} \ \psi_2^{(2)} \text{ and derivatives}\right)\right] ,$$

$$|I_{33}| \sim O(R) \, , \qquad \left[\Psi_3^{(1)} \times \left(\text{terms in } f_3 \text{ and derivatives}\right)\right] .$$

Similarly, by employing the inner estimates derived in Appendix C, §§ iii) and iv), which are functions of the 'inner' strained independent variable

$$\xi = \left(x_3 - x_{3c}\right) / \varepsilon \quad ,$$

where $\varepsilon = 1/i(\alpha R D\bar{u}_c^{\circ})^{\frac{1}{3}}$, and are valid for $(\alpha R)^{\frac{1}{3}} \gg 1$, we evaluate the critical-layer contributions of $\Phi_1^{(1)} F_{1j}$, $\Phi_3^{(1)} F_{3m}$ in (3.53). Once again a few terms dominate the integrands for large R. These dominant terms (quoted in brackets below) yield the following leading-order critical-layer contributions.

$$I_{11}:\ iC_1^2 |C_1|^2 D^2\bar{u}_c^{\circ}\left(\frac{1}{4}\alpha R^4 \left(D\bar{u}_c^{\circ}\right)^{-5}\right)^{1/3} \int_{-i\infty}^{+i\infty} \left\{ L_1(\xi) L_1(-\xi) + J_1(\xi) \right\} d\xi + o\left(R \log_e R\right),$$

$$\left[\Phi_1^{(1)} \times \left(\left(\frac{i\alpha}{2} f_1 + i\beta h_1 \right) D^2 \chi_1^{(1)} ,\ \chi_1^{(1)} D^2 \left(\frac{i\alpha}{2} f_1 + i\beta h_1 \right) ,\ \chi_1^{(1)*} D^3 \chi_{11}^{(2)} \right)\right]$$

$$I_{12}:\ R^{8/3} \int_{-i\infty}^{+i\infty} J_2(\xi)\, d\xi + o\left(R^{7/3} \log_e R\right) \ ,$$

$$\left[\Phi_1^{(1)} \times \left(\phi_{1-2}^{(2)} D\psi_2^{(1)} \right)\right] \ ,$$

$$I_{13}:\ |a_1|^2 R^{2/3} \int_{-i\infty}^{+i\infty} J_3(\xi)\, d\xi + o\left(|a_1|^2 R^{1/3} \log_e R\right) \ ,$$

$$\left[\Phi_1^{(1)} \times \left(a_1^* D^2 \chi_1^{(2)} \right)\right] ,$$

$$I_{31}:\ a_1 R^{4/3} \int_{-i\infty}^{+i\infty} J_4(\xi)\, d\xi + o\left(|a_1| R \log_e R\right),$$

$$\left[\Phi_3^{(1)} \times \left(\psi_2^{(2)} D\psi_1^{(1)} ,\ \psi_1^{(1)} D\psi_2^{(2)} ,\ \chi_1^{(1)} D^2\psi_2^{(2)} ,\ a_1 D^2 \chi_3^{(2)} \right)\right] ,$$

(3.57a,b,c,d)

$$I_{33}: \; -iC_3^2 |C_3|^2 D^2\bar{u}_c^0 \left(\alpha R^4 (D\bar{u}_c^0)^{-2}\right)^{1/3} \int_{-i\infty}^{+i\infty} \left\{ L(\xi)L(-\xi) + J_5(\xi) \right\} d\xi + O(R \log_e R)$$

$$\left[\Psi_3^{(1)} \times \left(f_3 D^2 \chi_3^{(1)}, \; \chi_3^{(1)} D^2 f_3 \; , \chi_3^{(1)*} D^3 \chi_{33}^{(2)} \right) \right]. \qquad (3.57e)$$

Here, C_j $(j = 1,2,3)$ is the value of the inviscid estimate $\chi_{j,o}^{(1)}$ at x_{3c} (see C.1a)). The functions $L(\xi)$ and $L_1(\xi)$ satisfy

$$\left(\frac{d^2}{d\xi^2} + \xi \right) L(\xi) = 1 \, , \qquad \left(\frac{d^2}{d\xi^2} + \frac{\xi}{2} \right) L_1(\xi) = 2^{-2/3} \, , \qquad (3.58)$$

with $L(\xi) \to \xi^{-1}$ and $L_1(\xi) \to 2^{\frac{1}{3}} \xi^{-1}$ as $\xi \to \pm i\infty$. In expressions (3.57 a,e) use has been made of result (C.20). The functions $J_m(\xi)$ $(m = 1,2,3,4,5)$, defined by (D.1), are $O(1)$ in magnitude and are analytic in the region

$$-\frac{2}{3}\pi < \arg \xi < \frac{2}{3}\pi \, . \qquad (3.59)$$

For these analytic functions contour integration around the right-hand semi-circle at infinity reveals that

$$\int_{-i\infty}^{+i\infty} J_m(\xi) d\xi = 0 \qquad (3.60)$$

Such integration is tantamount to indenting under the singularity at x_{3c} in cases where outer estimates for the integrands remain valid, which was discussed earlier. However, contour integration cannot be employed directly to evaluate the remaining integrals

$$\int_{-i\infty}^{+i\infty} L_1(\xi) L_1(-\xi) \, d\xi \quad , \qquad \int_{-i\infty}^{+i\infty} L(\xi) L(-\xi) \, d\xi \quad , \qquad (3.61)$$

for the functions $L_1(-\xi)$ and $L(-\xi)$ are defined only on the imaginary ξ-axis. In this case it is best to transform to the real variable Y defined as $\xi = iY$, so that the path of integration is along the real Y-axis from $-\infty$ to $+\infty$. Then on employing results (C.19) expressions (3.57 a,e) simplify to give the following, leading-order, non-zero critical-layer contributions to the integrals I_{11}, I_{33}.

$$I_{11}: \quad -C_1^2|C_1|^2 D^2\bar{u}_c^0\left(\frac{1}{2}\alpha R^4(D\bar{u}_c^0)^{-5}\right)^{1/3}\int_{-\infty}^{+\infty}\left(L_{1R}^2(Y)+L_{1I}^2(Y)\right)dY + o\left(R\log_e R\right), \tag{3.62}$$

$$I_{33}: \quad C_3^2|C_3|^2 D^2\bar{u}_c^0\left(\alpha R^4(D\bar{u}_c^0)^{-2}\right)^{1/3}\int_{-\infty}^{+\infty}\left(L_R^2(Y)+L_I^2(Y)\right)dY + o\left(R\log_e R\right), \tag{3.63}$$

where $L_1(\xi) = L_{1R}(Y) + iL_{1I}(Y)$, $L(\xi) = L_R(Y) + iL_I(Y)$, for $\xi = iY$. Similarly, leading-order non-zero contributions to the integrals I_{13}, I_{31} may be found by employing contour integration, as discussed above, with the following results. The appropriate dominant terms of $\Phi_1^{(1)}F_{13}$ and $\Phi_3^{(1)}F_{31}$ are quoted in brackets.

$$I_{13}: \quad a_1^* R^{4/3}\int_{-i\infty}^{+i\infty} J_6(\xi)\,d\xi + o\left(|a_1| R\log_e R\right),$$

$$\left[\Phi_1^{(1)}\times\left(\chi_3^{(1)} D^2\psi_2^{(2)*}\right)\right], \tag{3.64}$$

$$I_{31}: \quad R^{5/3}\int_{-i\infty}^{+i\infty} J_7(\xi)\,d\xi + o\left(R^{4/3}\log_e R\right),$$

$$\left[\Phi_3^{(1)}\times\left(a_1^*\chi_3^{(2)},\ \psi_1^{(1)*},\ \chi_{13}^{(2)},\psi_{13}^{(2)},\ \chi_2^{(2)},\ \psi_2^{(2)} \text{ and derivatives}\right)\right]. \tag{3.65}$$

The functions $J_{6,7}(\xi)$ are defined in (D.1) and are $O(1)$ in magnitude.

On employing contour integration as discussed above, it is found that there are no non-zero critical-layer contributions to the integrals I_{12} which are of order greater than R^2. Consequently the dominant contribution to this integral comes from the whole flow on account of terms of F_{12} (see (3.23)) which involve the $O(R^2)$ function $\phi^{(2)}_{1-2}$ and its derivatives. Whereas, comparison of results (3.62)-(3.65) with (3.55 a,c,d,e), and noting (3.52), shows that the integrals I_{11}, I_{13}, I_{31} and I_{33} are dominated by their critical-layer contributions.

We note that the orders of magnitude of a_{1j}, a_{3k} ($j = 1,2,3$ $k = 1,2$) are the same as those of the integrals comprising the numerators (3.37 a,b), since the integrals of the denominators may be evaluated by inviscid theory on indenting under the singularity at x_{3c} (as discussed previously for the second-order paramters a_j) and are consequently $O(1)$. Thus results (3.55b), (3.62)-(3.65) yield the following order of magnitude estimates for the interaction coefficients a_{1j}, a_{3k}.

$$|a_{11}|,\ |a_{32}| \sim O\left(R^{4/3}\right), \qquad |a_{12}| \sim O\left(R^{2}\right), \qquad (3.66\,a,b)$$

$$|a_{31}| \sim O\left(R^{5/3}\right), \qquad |a_{13}| \sim O\left(R^{7/3}\right) \qquad (3.67\,a,b)$$

Note that these estimates incorporate undetermined constants (C_1, C_2, C_3, A_2, B_2 etc., which are assumed to be $O(1)$) which can only be calculated by solving the linear eigenvalue problem and the second-order mean flow problem for specified $\bar{u}^{o}(x_3)$.

We may summarize our results by re-expressing the amplitude equations (3.5) for resonance, when $D^2\bar{u}_c \neq 0$, as

$$\frac{dA_1}{dt} = \frac{1}{2}\alpha c_I A_1 + b_1 R A_3 A_2^* + A_1\left[d_1 R^{4/3}|A_1|^2 + d_2 R^2 |A_2|^2 + d_3 R^{7/3}|A_3|^2\right],$$

$$\frac{dA_2}{dt} = \frac{1}{2}\alpha c_I A_2 + b_1 R A_3 A_1^* + A_2\left[d_2 R^2|A_1|^2 + d_1 R^{4/3}|A_2|^2 + d_3 R^{7/3}|A_3|^2\right], \qquad (3.68)$$

$$\frac{dA_3}{dt} = \alpha \tilde{c}_I A_3 + a_3 A_1 A_2 + A_3\left[d_4 R^{5/3}\left(|A_1|^2+|A_2|^2\right) + d_5 R^{4/3}|A_3|^2\right],$$

where the (usually complex) coefficients a_3, b_1, d_i $(i = 1,2,\ldots,5)$ are $O(1)$ in magnitude.

§3.5 NON-RESONANT CASE WITH $D^2\overset{\circ}{u}_c \neq 0$. (THIRD-ORDER COEFFICIENTS)

Since the coefficients a_{11}, a_{33} represent third-order self interactions of the primary waves - a_{11}, a_{33} are just the respective Landau coefficients - they do not depend on resonance. The coefficient a_{12} denotes interaction between the two oblique waves only and so it is also independent of resonance. Consequently, for cases in which $D^2\overset{\circ}{u}_c \neq 0$ results (3.66 a,b) remain valid, even without resonance.

However results (3.67 a,b) for $|a_{31}|$ and $|a_{13}|$ depend on the existence of resonance to the extent that the critical layers for a two-dimensional wave and an oblique wave coincide and also that $|a_1^*| \sim O(R)$. When this is not the case, in order to establish estimates for $|a_{31}|$ and $|a_{13}|$, we must consider contributions to the respective integrals $\Psi_3^{(1)}F_{31}$ and $\Psi_1^{(1)}F_{13}$ from inside the distinct critical layers for wave 1 (oblique wave) and wave 3 (two-dimensional wave) separately and also contributions from outside these critical layers. (We denote the critical layers for wave 1 and wave 3 by C.L.1 and C.L.3 respectively).

Without resonance, terms of F_{13} and F_{31} (see (3.24), (3.25)) containing the subscript 2 must be omitted. Order of magnitude estimates for the remaining terms of $\Psi_3^{(1)}F_{31}$ and $\Psi_1^{(1)}F_{13}$ in the three distinct regions i) C.L.1 ii) C.L.3 iii) outside both C.L.1 and C.L.3, are summarized in Table 2, (see end of this chapter).

Once again, the order of magnitude estimates for $f_{1,3}$ given in Table 2 are valid for cases in which conditions (3.54) are satisfied.

The following results are obtained for the orders of magnitude with respect to R of the contributions to the integrals

$$I_{13} \equiv \int_0^L \Psi_1^{(1)} F_{13}\, dx_3 \quad , \qquad I_{31} \equiv \int_0^L \Psi_3^{(1)} F_{31}\, dx_3 \quad ,$$

from the three distinct regions described below.

i) C.L.1:

$$|I_{13}| \sim O(R) , \qquad \left[\Psi_1^{(1)} \times \left(\chi_1^{(1)} D^2 \overset{+}{f_3} \, , \, \chi_1^{(1)} f_3\right)\right] \quad ,$$

$$|I_{31}| \sim O\left(R^{4/3}\right) , \quad \left[\Psi_3^{(1)} \times \left(\chi_3^{(1)} D^2 \overset{+}{f_1} \, , \, D\left(\psi_1^{(1)*} \psi_{13}^{(2)}\right), \right.\right.$$

$$\left.\left. D\left(\psi_1^{(1)*} D\chi_{13}^{(2)}\right) , D\left(\chi_{13}^{(2)} D\psi_1^{(1)*}\right)\right)\right] ,$$

ii) C.L.3:

$$|I_{13}| \sim O(R) , \qquad \left[\Psi_1^{(1)} \times \left(\chi_1^{(1)} D^2 f_3\right)\right] ,$$

$$|I_{31}| \sim O(R) , \qquad \left[\Psi_3^{(1)} \times \left(\chi_3^{(1)} D^2 \overset{+}{f_1} \, , \, \chi_3^{(1)} f_1\right)\right] ,$$

iii) outside both C.L.1 and C.L.3:

$$|I_{13}| \sim O(R) , \qquad \left[\Psi_1^{(1)} \times \left(\chi_1^{(1)} D^2 \overset{+}{f_3} \, , \, f_3 D^2 \chi_1^{(1)} \, , \, f_3 \chi_1^{(1)}\right)\right] ,$$

$$|I_{31}| \sim O(R) , \qquad \left[\Psi_3^{(1)} \times \left(\chi_3^{(1)} D^2 \overset{+}{f_1} \, , \, f_1 D^2 \chi_3^{(1)} , \, f_1 \chi_3^{(1)}\right)\right] .$$

(For functions marked + see caption for Table 1.).

We note that critical-layer contributions from some apparently-dominant terms in the integrands $\Psi_1^{(1)} F_{13}$ and $\Psi_3^{(1)} F_{31}$ turn out to be zero by contour integration (as previously). The dominant terms giving non-zero critical-layer contributions are quoted in brackets.

These results show that the integral I_{31} is dominated by its critical-layer contributions; while for I_{13}, contributions from the whole flow remain significant. Thus, following the same arguments as given in the previous section, we obtain the results

$$|a_{13}| \sim o(R) , \qquad |a_{31}| \sim o\left(R^{4/3}\right) ,$$

So, without resonance, but of course with the symmetric oblique waves 1 and 2 coupled in phase, if $D^2\bar{u}_c^{\circ} \neq 0$, the amplitude equations (3.5) may be written as

$$\frac{dA_1}{dt} = \frac{1}{2}\alpha c_I A_1 + A_1\left[d_1 R^{4/3}|A_1|^2 + d_2 R^2 |A_2|^2 + d_3' R |A_3|^2\right] ,$$

$$\frac{dA_2}{dt} = \frac{1}{2}\alpha c_I A_2 + A_2\left[d_2 R^2 |A_1|^2 + d_1 R^{4/3} |A_2|^2 + d_3' R |A_3|^2\right] , \qquad (3.70)$$

$$\frac{dA_3}{dt} = \alpha \tilde{c}_I A_3 + A_3\left[d_4' R^{4/3}\left(|A_1|^2 + |A_2|^2\right) + d_5 R^{4/3} |A_3|^2\right] ,$$

Here d_1, d_2 and d_5 are as stated in (3.68). Similarly, d_3', d_4' are complex constants, $O(1)$ in magnitude.

§3.6 CASES WITH $D^2\bar{u}_c^\circ = 0$ (THIRD-ORDER COEFFICIENTS)

Only a few of the estimates for the first- and second-order functions derived in Appendix C require modification when $D^2\bar{u}_c^\circ = 0$ (see the end of Appendix C). It would appear that for such estimates the leading-order contributions are reduced in magnitude. An investigation, similar to that above, reveals the following results for order-of-magnitude estimates for the second-order interaction parameters, when $D^2\bar{u}_c^\circ = 0$.

i) With Resonance

Results (3.66b), (3.67 a,b) for $|a_{12}|$, $|a_{31}|$ and $|a_{13}|$ remain valid since they are insensitive to the value of $D^2\bar{u}^\circ$ at x_{3c}. However $|a_{mm}|$, $(m = 1,3)$ will now be less than $O(R^{4/3})$ since result (3.66a) depends on non-zero curvature at the critical point (i.e. $D^2\bar{u}_c^\circ \neq 0$).

ii) Without Resonance

Result (3.66b) for $|a_{12}|$ remains valid and $|a_{mm}|$ $(m = 1,3)$ is again less than $O(R^{4/3})$, since the appropriate estimates are independent of resonance. Now, $|a_{13}|$ and $|a_{31}|$ are respectively less than $O(R)$ and $O(R^{4/3})$ since results (3.69 a,b) depend on $D^2\bar{u}_c^\circ$ being non-zero.

§3.7 PHYSICAL MECHANISMS

Before proceeding, it must be emphasised that the results of §§ 3.3 - 3.6 concern only the asymptotic form of the interaction coefficients for large values of R, when the other parameters α, β, γ and the first derivative of $\bar{u}^{\circ}$ at $x_3 = x_{3c}$ are regarded as O(1) quantities. Further implicit R-dependence through the eigenvalues c_R, c_I, $\tilde{c}_I$ has been ignored.

The foregoing analysis has clarified four distinct physical mechanisms which dominate the respective third-order interaction coefficients, viz.;

i) a critical-layer mechanism independent of resonance but dependent on non-zero $D^2\bar{u}^{\circ}_c$, (a_{11}, a_{33}),

ii) a critical-layer mechanism dependent on resonance but independent of $D^2\bar{u}^{\circ}_c$ (a_{13}, a_{31}),

iii) a strong effect throughout the flow deriving from the $O(R^2)$ spanwise periodic distortion $\phi^{(2)}_{1-2}$ of the mean flow, which is independent of both $D^2\bar{u}^{\circ}_c$ and resonance but needs waves 1 and 2 to be synchronised (a_{12}),

iv) a further strong effect throughout the flow deriving from the O(R) mean flow terms f_1 and f_3, when $D^2\bar{u}^{\circ}_c \neq 0$ and there is no resonance, (a_{13}).

The analysis has also shown the importance of the role of three-dimensionality. A resonant triad of two-dimensional waves ($\beta = 0$) may sometimes exist but for these, the large interaction coefficients involving β will disappear. Even without resonance three-dimensionality greatly increases the strength of third-order interactions, (e.g. when $D^2\bar{u}^{\circ}_c \neq 0$, $|a_{12}| \sim O(R^2)$ but $|a_{33}| \sim O(R^{4/3})$).

§3.8 VALIDITY CONDITIONS FOR ASYMPTOTIC ANALYSIS

The validity of the above asymptotic analysis depends on the explicit assumptions

$$i) \quad \alpha, \beta, \gamma, |D\bar{u}_c^0|, |D^2\bar{u}_c^0| \sim O(1) \; ,$$

$$ii) \quad \alpha R|c_I|, \alpha R|\tilde{c}_I| \lesssim O(1)$$

$$iii) \qquad R^{1/3} \gg 1 \; ,$$

and the implicit conditions

$$iv) \quad |c_R - \bar{u}^0(0)| \; , \quad |c_R - \bar{u}^0(1)| \gg R^{-1/3} \; , \quad (R \to \infty) \; ,$$

which ensure that the critical layer and viscous wall layers do not overlap. We note that ii) may be relaxed if required provided

$$|c_I| \; , |\tilde{c}_I| \ll (D\bar{u}_c^0)^{2/3}(\alpha R)^{-1/3} \; , \qquad (R \to \infty) \; .$$

This is one of the necessary conditions for the validity of the linear viscous approximations (C.14)-(C.16) to the functions $D^2 \chi_j^{(1)}$, $\Psi_k^{(1)}$, $\Phi_j^{(1)}$ ($j = 1,2,3 \;\; k = 1,2$) in the critical-layer (see Lin [1955], p.136). Another such requirement is

$$|A_j| R^{2/3} \ll O(1) \; , \qquad (j = 1,2,3) \; ,$$

which derives from the condition that viscous effects dominate non-linear terms in the critical layer (see Benney & Bergeron [1969], Davis [1969]). This result was derived for non-resonant waves; with resonance, a more stringent restriction may be required. Conditions i) - iv) must be satisfied for both the resonant and the non-resonant case.

To ensure that the time scales associated with the amplitude modulation of the waves, due to non-linear effects, are large compared with L/V, where L and V are characteristic length and velocity scales used for non-dimensionalisation (typically, L is the channel width or boundary-layer thickness and V is the maximum flow speed) we required that

$$\left| A_j^{-1} \frac{dA_j}{dt} \right| \ll O(1) \, , \qquad (j = 1,2,3) \, . \tag{3.71}$$

For simplicity we assume that A_1 and A_2 are of comparable magnitude, A, say. Then for the second-order terms of (3.68) to be sufficiently small, it is necessary that

$$\text{vi)} \qquad R|A_3| \, , \quad |A^2/A_3| \ll O(1) \, ,$$

and for the third-order terms of (3.68) that

$$\text{vii)} \qquad R^2|A|^2, \quad R^{7/3}|A_3|^2 \ll O(1) \, .$$

Without resonance the corresponding conditions from (3.70) are

$$\text{vii}') \qquad R^2|A|^2, \quad R^{4/3}|A_3|^2 \ll O(1) \, .$$

Conditions vi) - vii$'$) refer to cases in which $D^2\bar{u}_c^0 \neq 0$. When $D^2\bar{u}_c^0 = 0$, both for the resonant and non-resonant case, we require that

$$\text{viii)} \qquad R^2|A|^2 \ll O(1) \, .$$

A corresponding condition on $|A_3|$ (which will be weaker than both conditions vii) and vii$'$)) may be deduced once an order-of-magnitude estimate has been established for the interaction coefficient $|a_{33}|$. We observe that these conditions are necessary but not sufficient to ensure that condition (3.71) is satisfied, since the orders of magnitude with respect to R of the omitted higher-order terms of the perturbation series are unknown and it has not been established that these series are asymptotic. It seems reasonable to expect that no more stringent conditions are required than those above, in order to ensure that the largest non-linear terms have been retained; but some uncertainty remains.

Our approach may be regarded as a first step towards a more formal analysis. By scaling the wave amplitudes A_j and linear growth rates $\frac{1}{2}\alpha c_I$, $\alpha\tilde{c}_I$ as appropriate negative powers of R and introducing multiple time scales as required, we would expect truncated forms of the amplitude equations (3.68), (3.70) to yield formally valid first approximations in the limit $R\to\infty$. However various different scalings are possible and the analysis of each case of interest would probably be far more formidable than the present analysis. In particular, a justification of the retention of certain third-order terms while neglecting all fourth- and higher-order ones would require a pursuit of the analysis at least to fourth-order in the wave amplitudes. This would be a forbidding venture.

§3.9 SIMPLIFIED AMPLITUDE EQUATIONS

We set down some illustrative examples of scaled equations. In these, the stated orders of magnitude refer only to those omitted terms of up to (and including) third-order in the wave amplitudes.

We define $B_k \equiv R^{\alpha_1} . A_k \quad (k = 1,2), \quad B_3 \equiv R^{\alpha_3} . A_3, \; (\tau \equiv R^{\beta} . t)$, regarding $B_i \; (i = 1,2,3)$ and τ as $O(1)$. We assume that

$$|\alpha c_I| \sim O\left(R^{-n}\right) , \quad |\alpha \tilde{c}_I| \sim O\left(R^{-m}\right) ,$$

where $n,m \geqslant 1$ for consistency with condition ii) of §3.8.

If $n = m \geqslant 1$ and we choose $\beta = - n = - m, \quad \alpha_1 = n + \frac{1}{2}$, $\alpha_3 = 1 + n$, equations (3.68) reduce to

$$\frac{dB_k}{d\tau} = \begin{cases} \sigma B_k + b_1 B_3 B^*_{3-k} + d_{3-k} B_k |B_{3-k}|^2 + O\left(R^{-2/3}\right) & \text{if } n = +1, \\ \sigma B_k + b_1 B_3 B^*_{3-k} + O\left(R^{1-n}\right) & \text{if } n > +1, \end{cases} \qquad (3.72\ a,b,c)$$

$$\frac{dB_3}{d\tau} = \tilde{\sigma} B_3 + b_3 B_1 B_2 + O\left(R^{2/3 - n}\right) \qquad \text{if } n \geqslant +1 .$$

Here, $k = 1,2$, $\sigma = \frac{1}{2} \alpha c_I R^n$ and $\tilde{\sigma} = \alpha \tilde{c}_I R^m$. The system (3.72 b,c) was discussed by Craik [1971]. Since $|\sigma|$ and $|\tilde{\sigma}|$ are assumed to be $O(1)$, in the system (3.72 a,c) the oblique waves experience first-, second- and third-order contributions of comparable magnitudes, while the two-dimensional wave only experiences contributions up to second-order and is thus adequately described by second-order theory.

If now $\min (m,n) > 1$, further cases can be considered. On choosing $\alpha_3 = 1 - \beta$ equations (3.68) reduce to the following results dependent on the precise choice for α_1 and β .

$$q < \beta \leq -1,\ \alpha_1 > \tfrac{1}{2} - \beta \quad \text{or} \quad -1 < \beta < -\tfrac{1}{3},\ \alpha_1 > 1 - \tfrac{\beta}{2} \qquad \left\{ \begin{aligned} \frac{dB_k}{d\tau} &= b_1 B_3 B_{3-k}^{*} + O\left(R^{-n-\beta}, R^{2-2\alpha_1-\beta}, R^{\beta+1/3}\right), \\ \frac{dB_3}{d\tau} &= O\left(R^{-m-\beta}, R^{1-2\beta-2\alpha_1}, R^{5/3-2\alpha_1-\beta}, R^{\beta-2/3}\right), \end{aligned} \right. \tag{3.73}$$

$$\beta = -\tfrac{1}{3},\ \alpha_1 > \tfrac{7}{6} \qquad \left\{ \begin{aligned} \frac{dB_k}{d\tau} &= b_1 B_3 B_{3-k}^{*} + d_3 B_k |B_3|^2 + O\left(R^{1/3-n}, R^{7/3-2\alpha_1}\right), \\ \frac{dB_3}{d\tau} &= O\left(R^{1/3-m}, R^{2-2\alpha_1}, R^{-1}\right), \end{aligned} \right. \tag{3.74}$$

$$\beta = -\tfrac{1}{3},\ \alpha_1 = \tfrac{7}{6} \qquad \left\{ \begin{aligned} \frac{dB_k}{d\tau} &= b_1 B_3 B_{3-k}^{*} + B_k\left(d_2 |B_{3-k}|^2 + d_3 |B_3|^2\right) + O\left(R^{-2/3}\right), \\ \frac{dB_3}{d\tau} &= O\left(R^{-1/3}\right). \end{aligned} \right. \tag{3.75}$$

Here, $q = -\min(m,n)$.

In the above systems (3.73)-(3.75) B_3 remains constant on the time scale τ. The corresponding oblique waves experience either second-order contributions alone (see (3.73)) or second- and third-order contributions of comparable magnitudes (see (3.74), (3.75)). In particular for system (3.73), with appropriately chosen initial phases, both $|B_1|$ and $|B_2|$ grow like $\exp |b_1 B_3| \tau$.

§3.10 DISCUSSION

The predictions in the present analysis (and in Craik [1971]) of the sizes of the various interaction coefficients can be tested to some extent by comparison with existing results which have been computed for particular flows. For the second-order coefficients a_i $(i = 1,2,3)$ no published results yet exist, but work is presently in hand by Professor R. E. Kelly to compute these for Blasius flow. Some preliminary results obtained by Professor Kelly and Dr. F. Hendriks are described by Dr. Hendriks in the Appendix to Usher & Craik [II]. These results are shown in Table 3 at the end of this chapter, by kind permission of Professor Kelly and Dr. Hendriks. These are at a fixed Reynolds number of 882 (based on displacement thickness) and concern six separate symmetric resonant triads. It is seen that, at the higher wave numbers the magnitude of the coefficient a_1 $(= a_2)$ for the oblique waves is substantially larger (by a factor of approximately 30 for $\alpha = 0.5$) than that of the two-dimensional coefficient a_3. This is in qualitative agreement with our results. That this is not so markedly the case at small wave numbers is to be expected: for αR is only 88.2 for $\alpha = 0.1$ and the conditions for the validity of the asymptotic theory are not met. Indeed, for Blasius flow, the Reynolds numbers of interest are probably not large enough to encourage great confidence in asymptotic theory; nevertheless the qualitative agreement with our results is most encouraging.

The third-order coefficients a_{ii} $(i = 1,2,3)$ are just the Landau constants for the respective plane waves, for which we have predicted an $O(R^{4/3})$ dependence. No numerical results exist for any other of the a_{ij}, but Reynolds & Potter [1967] tabulate several values of a_{ii} for Poiseuille and for Couette-Poiseuille flow (assuming constant mass flux (see §3.1)). Unfortunately, only a few results are quoted for fixed α but different values of R, and no meaningful comparison can be made for Couette-Poiseuille flow. For plane Poiseuille flow extensive tables of a_{ii} as a function of α and R are given by Pekeris & Shkoller [1967]. Using $\log_e$ scales, we have plotted their data in Fig.1. as curves of $|a_{ii}|$ against $|\alpha R|$ at constant values of α (for normalisation conventions employed in these calculations see §3.2). Reynolds and Potter's few points are in general agreement with these results.

It can be seen that the magnitude of a_{ii} is sensitive to both α and R but exhibits a general increasing trend with R, except at small α. To allow comparison with our predicted $O(R^{4/3})$-dependence, we have superposed dashed curves with gradient $4/3$ in Fig. 1. There is general agreement with the gradients of the computed data when $\alpha \gtrsim 1$. For larger α the departure from a $R^{4/3}$-law is to be expected since the present analysis has ignored the implicit dependence of the eigenvalues c_R, c_I and $\tilde{c}_I$ on R. For smaller α no comparison is valid due to the following limitations on the present analysis. For plane Poiseuille flow an asymptotic analysis (see Reid [1965], p. 299) reveals that the critical layer and the viscous wall layer overlap for the lower branch of the neutral curve but remain distinct for the upper branch. Thus the asymptotic viscous analysis for the critical-layer region, developed in Appendix C, is invalid for the lower branch, where $\alpha \ll 1$, but is valid for values of α and R which lie some distance above this lower branch. However, there are flows - e.g. unbounded ones like $\bar{u}^{o} = \tanh x_3$ - where there are no viscous wall layers, and so no limitations of the above sort. Unfortunately, no numerical data are available for such flows.

The present work may be compared with previous analyses involving non-linear perturbations to parallel shear flows. The validity of the analyses of Di Prima, Eckhaus & Segel [1971] and of Hocking, Stewartson & Stuart [1972] (as corrected by Davey, Hocking & Stewartson [1974]), (see §1.2c), is restricted to the immediate locality of the critical Reynolds number R_c. These analyses concern the non-linear evolution of predominantly two-dimensional disturbances (in the analysis of Hocking et al. the disturbance is dominated by a single plane-wave mode by the time that non-linear effects are felt).

Here, the analysis examines the purely temporal evolution of inherently three-dimensional disturbances where the wave modes remain of comparable importance. (An extension of the present analysis to incorporate spatial as well as temporal evolution of the waves - as was done by Hocking et al. [1974] - appears to be feasible but is not pursued here). The possibility of resonance amongst wave modes is excluded in the above studies but is a major feature of the present work. In the present analysis the amplitude equations (3.68), (3.70) are essentially a large R expansion which hold independently of R_c. (The expansions remain valid even when there is no R_c). For the Blasius boundary-layer, for instance, the values of R of interest may not be large enough for much confidence; whereas other analyses employing $R - R_c$ as a small parameter are formally valid near R_c. However, for plane Poiseuille flow R is large enough.

It is worth noting that the very different "kinematic-wave" analysis by Landahl [1973], of a small-scale secondary wave riding on a large-scale inhomogeneity concerns a type of resonance. Usher & Craik [II] refer to a private communication with Dr. M. A. S. Ross of Edinburgh University in which he shows that, by regarding the secondary wave train as two components with nearly equal wave numbers and frequencies, focussing occurs when the three waves form a resonant triad.

Equations (3.68), (3.70) and the corresponding equations for non-symmetric triads shed light on the roles of resonance and of three-dimensionality in the non-linear instability of parallel shear flows. The following remarks relate to situations where conditions i) - vii′) of §3.8 are met.

The temporal evolution of a single wave component in a parallel shear flow is governed, to third-order in wave amplitude, by an equation of the form

$$\frac{dA}{dt} = \alpha c_x A + \lambda A|A|^2 ,$$

in the usual notation and with λ denoting the Landau constant. For a two-dimensional wave we have shown that $|\lambda| \sim O(R^{4/3})$ for large R; but for a three-dimensional wave with x_1, x_2 and t dependence like $A(t) \cos\beta x_2 \exp i\alpha(x_1 - c_R t)$, we now have $|\lambda| \sim O(R^2)$ for large R. (This is just the special case $A_3 = 0$, $A_1 = A_2$ of (3.70)). More generally third-order interaction coefficients of $O(R^2)$ will arise whenever the first-order disturbance contains amongst its Fourier components a symmetric pair of (not necessarily equal) oblique waves. This large size of the interaction coefficients derives from the fact that the component $\phi^{(2)}_{1-2}$ associated with the second-order spanwise-periodic longitudinal vortex is $O(R^2)$. Clearly three-dimensionality increases the strength of third-order interactions. Whether their effect is to enhance or inhibit the growth of the disturbance energy is not indicated by the present analysis since estimates of the phases of the complex interaction coefficients are not available.

For non-symmetric resonant triads, for which the three critical layers are distinct, one merely adds appropriate second-order terms with $O(1)$ interaction coefficients $|a_i|$ $(i = 1,2,3)$ to equations (3.70). However, substantial changes occur for symmetric resonant triads. For the oblique waves the second-order coefficients $|a_1|$ and $|a_2|$ become $O(R)$ and the third-order coefficients $|a_{13}|$, $|a_{23}|$ increase to $O(R^{7/3})$, essentially because of the superposition of critical layers. For the two-dimensional wave the coefficients $|a_{31}|$, $|a_{32}|$ are also increased, but less dramatically, from $O(R^{4/3})$ to $O(R^{5/3})$. Consequently, not only are strong second-order interactions introduced by such resonances, the strength of the third-order interactions is also enhanced.

That there are larger interaction coefficients for oblique waves than for a two-dimensional wave - both with and without resonance - strongly supports the possibility of rapid and preferential growth of three-dimensional components. However, firm conclusions on this point must await detailed analyses of particular problems incorporating the phases of the interaction coefficients.

The main conclusions to be reached from the present analysis may be summarised as follows.

i) At large R, the influence of non-linearity on the temporal evolution of wavelike disturbances is remarkably strong.

ii) For a three-dimensional disturbance this influence is much greater than for a two-dimensional disturbance of comparable amplitude.

iii) Symmetric resonance at second-order yields even larger non-linear contributions.

iv) Three-dimensionality is likely to develop very rapidly in unstable shear flows at large R.

v) The surprising strength of the non-linear interactions, which increases with R, limits the probable ranges of validity of linear theory and of amplitude expansion techniques to smaller amplitudes than was previously supposed.

FUNCTIONS $(j = 1,2,3\ ;\ k = 1,2)$.	ORDERS OF MAGNITUDE	
	OUTER REGION	INNER REGION
$\chi_j^{(1)}$	1	1
$D\chi_j^{(1)}$	1	$\log_e R$
$D^2\chi_j^{(1)}\ ,\ \psi_k^{(1)}\ ,\ \Phi_j^{(1)}$	1	$R^{1/3}$
$D^3\chi_j^{(1)}\ ,\ D\psi_k^{(1)}$	1	$R^{2/3}$
$D^2\psi_k^{(1)}$	1	R
$\chi_{jj}^{(2)}$	1	$R^{1/3}$
$D\chi_{jj}^{(2)}\ ,\ \chi_3^{(2)}\ ,\ \chi_{k3}^{(2)}$	1	$R^{2/3}$
$D^2\chi_{jj}^{(2)}\ ,\ D\chi_3^{(2)}\ ,\ D\chi_{k3}^{(2)}\ ,\ \psi_{k3}^{(2)}$	1	R
$D^3\chi_{jj}^{(2)}\ \ D^2\chi_3^{(2)}\ \ D^2\chi_{k3}^{(2)}\ \ D\psi_{k3}^{(2)}$	1	$R^{4/3}$
$D^3\chi_3^{(2)}\ ,\ D^3\chi_{k3}^{(2)}\ ,\ D^2\psi_{k3}^{(2)}$	1	$R^{5/3}$
$\chi_k^{(2)}$	R	$R\log_e R$
$D\chi_k^{(2)}$	R	$R^{4/3}$
$D^2\chi_k^{(2)}\ ,\ \psi_k^{(2)}$	R	$R^{5/3}$
$D^3\chi_k^{(2)}\ ,\ D\psi_k^{(2)}$	R	R^2
$D^2\psi_k^{(2)}$	R	$R^{7/3}$
$f_j\ ,\quad h_k\quad \frac{\alpha}{2}f_k + (-1)^{k+1}\beta h_k\ ,\quad \chi_{1-2}^{(2)}$	R	R
$Df_3\ ,\quad D\left(\frac{\alpha}{2}f_k + (-1)^{k+1}\beta h_k\right),\quad D\chi_{1-2}^{(2)}$	R	R
${}^{+}D^2 f_3\ ,\ Df_k\ ,\ Dh_k\ ,\ {}^{+}D^2\left(\frac{\alpha}{2}f_k + (-1)^{k+1}\beta h_k\right),\quad D^2\chi_{1-2}^{(2)}$	R	$R^{4/3}$
${}^{+}D^3 f_k\ \ {}^{+}D^2 h_k\ ,\quad D^3\chi_{1-2}^{(2)}$	R	$R^{5/3}$
$\phi_{1-2}^{(2)}\ ,\ D\phi_{1-2}^{(2)}\ ,\ D^2\phi_{1-2}^{(2)}$	R^2	R^2

Table 1.

FUNCTIONS $(m=1,3)$	ORDERS OF MAGNITUDE: OUTSIDE BOTH C.L.1. & C.L.3.	INSIDE C.L.1.	INSIDE C.L.3
$x_m^{(1)}$	1	1	1
$Dx_1^{(1)}$	1	$\log_e R$	1
$Dx_3^{(1)}$	1	1	$\log_e R$
$D^2 x_1^{(1)}$ $\psi_1^{(1)}$ $\Psi_1^{(1)}$	1	$R^{1/3}$	1
$D\psi_1^{(1)}$	1	$R^{2/3}$	1
$D^2\psi_1^{(1)}$	1	R	1
$D^2 x_3^{(1)}$ $\Psi_3^{(1)}$	1	1	$R^{1/3}$
$x_{13}^{(2)}$	1	$R^{2/3}$	$R^{1/3}$
$D x_{13}^{(2)}$ $\psi_{13}^{(2)}$	1	R	$R^{2/3}$
$D^2 x_{13}^{(2)}$ $D\psi_{13}^{(2)}$	1	$R^{4/3}$	R
$D^3 x_{13}^{(2)}$ $D^2\psi_{13}^{(2)}$	1	$R^{5/3}$	$R^{4/3}$
f_m	R	R	R
Df_1	R	$R^{4/3}$	R
Df_3	R	R	R
$^{+}D^2 f_1$	R	$R^{5/3}$	R
$^{+}D^2 f_3$	R	R	$R^{4/3}$

Table 2.

For functions marked + see caption for Table 1.

← RESONANT TRIAD		→			← INTERACTION COEFFICIENTS	→
α	β	γ	$c_R + i\tilde{c}_I$	$c_R + ic_I$	a_1	a_3
0.1000	0.0617	0.0794	0.2859 -0.0461i	0.2859 -0.0888i	0.5473 +0.7013i	0.6079 +0.5563i
0.2000	0.1209	0.1569	0.3394 +0.0041i	0.3394 -0.0294i	3.7350 +1.1757i	0.0083 -0.2471i
0.2540	0.1480	0.1950	0.3570 +0.0102i	0.3570 -0.0122i	6.0745 +0.6499i	0.3036 -0.3394i
0.3000	0.1705	0.2271	0.3685 +0.0083i	0.3685 -0.0033i	8.8249 -0.1495i	0.4305 -0.3217i
0.4000	0.2098	0.2891	0.3846 -0.0107i	0.3846 +0.0035i	18.8784 -3.7073i	0.4962 -0.4081i
0.5000	0.1911	0.3147	0.3834 -0.0444i	0.3834 +0.0047i	29.5892 -6.0644i	0.1292 -0.9701i

Table 3

(see Hendrik's appendix to Usher & Craik [II])

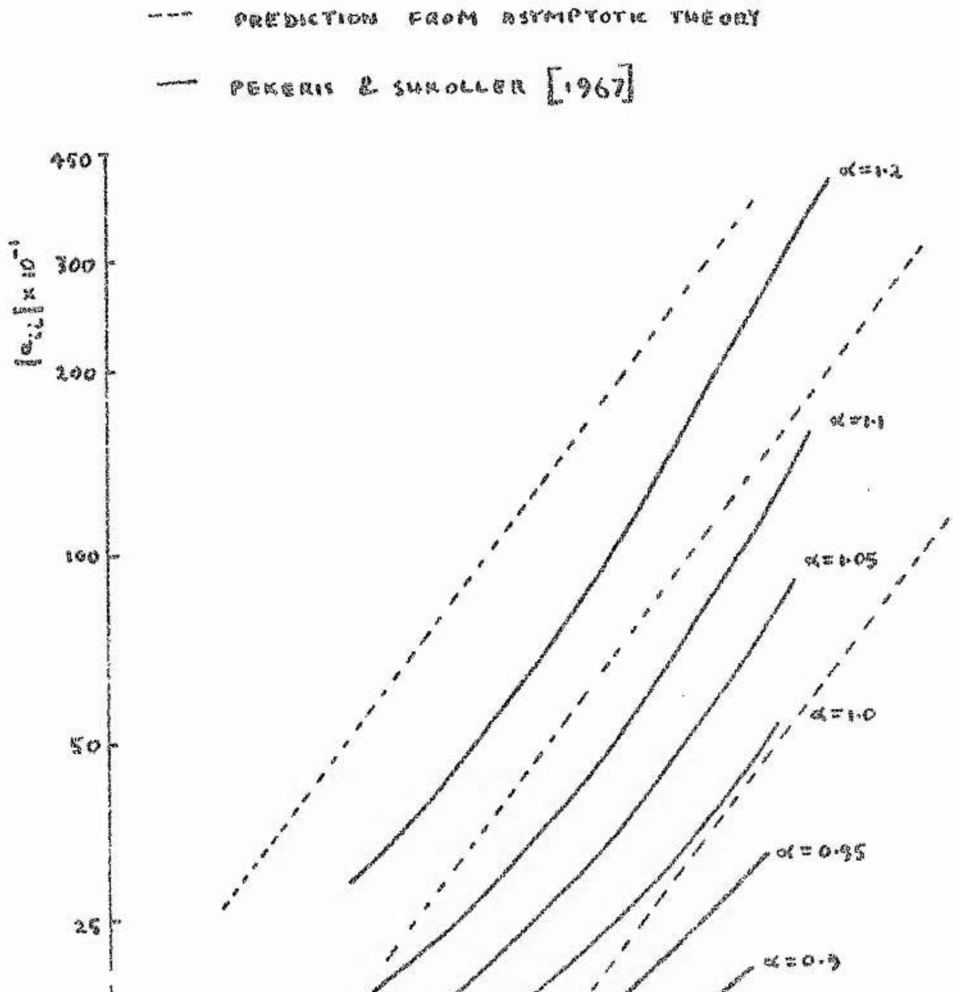

PREDICTION FROM ASYMPTOTIC THEORY
PEKERIS & SHKOLLER [1967]
450
300
200
100
50
25
10
5
α=1·2
α=1·1
α=1·05
α=1·0
α=0·95
α=0·9
α=0·85
α=0·8
α=0·7
α=0·5
α=0·6
5
10
15
20
25
30
αR×10^-3

Figure i.

CAPTIONS

Table 1. Orders of magnitude of functions in 'outer' and 'inner' (critical layer) regions for the resonant case with $D^2\bar{u}_c^{\bullet} \neq 0$ (see §3.4). Functions marked + are zero in outer regions for strictly neutral waves (see Appendix C, §ii)), but are $O(R)$ there if the condition $p_{Ij} = 0$ $(j = 1,2,3)$ and its spanwise counterpart (see §3.1) are not imposed.

Table 2. Orders of magnitude of functions inside the separate critical layers for wave 1 and wave 3 and outside these critical layers for the non-resonant case with $D^2\bar{u}_c^{\bullet} \neq 0$ (see §3.5).

Table 3. Resonant triads, eigenvalues and second-order interaction coefficients for Blasius flow at $R = 882$.

Figure 1 Results of Pekeris & Shkoller [1967] for plane Poiseuille flow.

APPENDIX A

THE MOMENTUM AND VORTICITY EQUATIONS GOVERNING A RESONANT WAVE TRIAD

For the flow configuration discussed in §3.1 we here derive the third-order wave-interaction equations. We start with the dimensionless Navier-Stokes equations

$$\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_2} + \frac{\partial w}{\partial x_3} = 0 \qquad (A.1)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x_1} + v\frac{\partial u}{\partial x_2} + w\frac{\partial u}{\partial x_3} = -\frac{\partial p}{\partial x_1} + \frac{1}{R}\nabla^2 u , \qquad (A.2)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x_1} + v\frac{\partial v}{\partial x_2} + w\frac{\partial v}{\partial x_3} = -\frac{\partial p}{\partial x_2} + \frac{1}{R}\nabla^2 v , \qquad (A.3)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x_1} + v\frac{\partial w}{\partial x_2} + w\frac{\partial w}{\partial x_3} = -\frac{\partial p}{\partial x_3} + \frac{1}{R}\nabla^2 w , \qquad (A.4)$$

and boundary conditions

$$u = v = w = 0, \qquad (x_3 = 0, l)$$

On substituting expressions (3.1) in (A.1), employing (3.3) and then equating to zero the coefficients of

$\exp i\Theta_j$, $\exp 2i\Theta_j$, $\exp i(\Theta_k + \Theta_3)$, $\exp i(\Theta_1 - \Theta_2)$

$(j = 1,2,3 \quad k = 1,2)$ there result

$$i\gamma\hat{u}_k + Dw_k = 0 , \qquad i\alpha u_3 + Dw_3 = 0 ,$$

$$2i\gamma\hat{u}_{kk} + Dw_{kk} = 0 , \qquad 2i\alpha u_{33} + Dw_{33} = 0 , \tag{A.6}$$

$$i\gamma_0\hat{u}_{k3} + Dw_{k3} = 0 , \qquad 2i\beta v_{1-2} + Dw_{1-2} = 0 ,$$

$$\text{where} \quad \gamma \equiv \left(\frac{\alpha^2}{4} + \beta^2\right)^{1/2} , \quad \gamma_0 \equiv \left(\frac{9}{4}\alpha^2 + \beta^2\right)^{1/2} \text{ and } D \equiv \frac{\partial}{\partial x_3} . \tag{A.7}$$

From (A.2)-(A.5), using (3.1), (3.3) and (A.6), equations for the mean flow are found to be

$$\frac{\partial \bar{\bar{u}}}{\partial t} + P_x(t) - \frac{1}{R}D^2\left(\bar{u}^0 + \bar{\bar{u}}\right) = M_1 + O\left(A^3\right) \tag{A.8}$$

$$\frac{\partial \bar{\bar{v}}}{\partial t} - \frac{1}{R}D^2\bar{\bar{v}} = M_2 + O\left(A^3\right) \tag{A.9}$$

$$D\bar{\bar{p}} = M_3 + O\left(A^3\right) \tag{A.10}$$

Here,

$$M_1 \equiv 2\,\mathrm{Re}\left[\sum_{k=1}^{2}\left\{(\beta_k/\gamma)D\left(\hat{v}_k w_k^*\right) - \left(\alpha/2i\gamma^2\right)w_k D^2 w_k^*\right\} - (1/i\alpha)w_3 D^2 w_3^*\right] , \tag{A.11}$$

$$M_2 = -2\,\mathrm{Re}\left[\sum_{k=1}^{2}(\alpha/2\gamma)D\left(\hat{v}_k w_k^*\right) + \left(\beta_k/i\gamma^2\right)w_k D^2 w_k^*\right] , \tag{A.12}$$

$$M_3 \equiv -4\,\mathrm{Re}\left[\sum_{j=1}^{3} D\left(w_j w_j^*\right)\right] , \tag{A.13}$$

where * denotes complex-conjugate and $\beta_1 = -\beta_2 = \beta$. Terms of third or higher order in wave amplitude are denoted by $O(A^3)$, where A is a dimensionless number characteristic of the wave amplitudes.

Similarly, from (A.2)-(A.5) we obtain the governing equations for terms with periodicities $\exp i\,\theta_j$, $\exp 2i\,\theta_j$, $\exp i(\theta_k + \theta_3)$ and $\exp i(\theta_1 - \theta_2)$, $(j = 1,2,3 \quad k = 1,2)$; where appropriate, pressure terms have been eliminated by cross-differentiation. These are

$$L_{8+j}\left[w_j\right] = M_{3+j} + O\left(A^4\right), \qquad (A.14)$$

$$L_{12}\left[\hat{v}_k\right] + \beta_k \gamma^{-1} w_k D\bar{u}^0 = M_{6+k} + O\left(A^3\right), \qquad (A.15)$$

$$L_{13}\left[v_3\right] = M_9 + O\left(A^3\right), \qquad (A.16)$$

$$L_{14}\left[w_{kk}\right] = M_{9+k} + O\left(A^3\right), \qquad (A.17)$$

$$L_{15}\left[w_{33}\right] = M_{12} + O\left(A^3\right), \qquad (A.18)$$

$$L_{16}\left[w_{k3}\right] = M_{12+k} + O\left(A^3\right), \qquad (A.19)$$

$$L_{17}\left[\hat{v}_{kk}\right] + \beta_k \gamma^{-1} w_{kk} D\bar{u}^0 = M_{14+k} + O\left(A^3\right), \qquad (A.20)$$

$$L_{18}\left[\hat{v}_{k3}\right] + \beta_k \gamma_0^{-1} w_{k3} D\bar{u}^0 = M_{16+k} + O\left(A^3\right), \qquad (A.21)$$

$$\left(D^2 - 4\beta^2\right) L_{19}\left[w_{1-2}\right] = M_{19} + O\left(A^3\right), \qquad (A.22)$$

$$L_{19}\left[u_{1-2}\right] - w_{1-2} D\bar{u}^0 = M_{20} + O\left(A^3\right), \qquad (A.23)$$

where

$$L_9 \equiv R^{-1}(D^2-\gamma^2)^2+\frac{i\alpha}{2}\left[D^2(\bar{u}^0+\bar{\bar{u}})-(\bar{u}^0+\bar{\bar{u}}-c_R)(D^2-\gamma^2)\right]$$

$$+\,i\beta\left[D^2\bar{\bar{v}}-\bar{\bar{v}}(D^2-\gamma^2)\right]-\frac{\partial}{\partial t}(D^2-\gamma^2) \quad , \tag{A.24}$$

$$L_{11} \equiv R^{-1}(D^2-\alpha^2)^2+i\alpha\left[D^2(\bar{u}^0+\bar{\bar{u}})-(\bar{u}^0+\bar{\bar{u}}-c_R)(D^2-\alpha^2)\right]-\frac{\partial}{\partial t}(D^2-\alpha^2) \quad ,$$

$$L_{12} \equiv R^{-1}(D^2-\gamma^2)-\frac{i\alpha}{2}(\bar{u}^0-c_R)-\frac{\partial}{\partial t} \quad ,$$

$$L_{13} \equiv R^{-1}(D^2-\alpha^2)-i\alpha(\bar{u}^0-c_R)-\frac{\partial}{\partial t} \quad ,$$

$$L_{14} \equiv R^{-1}(D^2-4\gamma^2)^2+i\alpha\left[D^2\bar{u}^0-(\bar{u}^0-c_R)(D^2-4\gamma^2)\right]-\frac{\partial}{\partial t}(D^2-4\gamma^2) ,$$

$$L_{15} \equiv R^{-1}(D^2-4\alpha^2)^2+2i\alpha\left[D^2\bar{u}^0-(\bar{u}^0-c_R)(D^2-4\alpha^2)\right]-\frac{\partial}{\partial t}(D^2-4\alpha^2) ,$$

$$L_{16} \equiv R^{-1}(D^2-\gamma_0^2)^2+\frac{3i\alpha}{2}\left[D^2\bar{u}^0-(\bar{u}^0-c_R)(D^2-\gamma_0^2)\right]-\frac{\partial}{\partial t}(D^2-\gamma_0^2) \quad ,$$

$$L_{17} \equiv R^{-1}(D^2-4\gamma^2)-i\alpha(\bar{u}^0-c_R)-\frac{\partial}{\partial t} ,$$

$$L_{18} \equiv R^{-1}(D^2-\gamma_0^2)-\frac{3i\alpha}{2}(\bar{u}^0-c_R)-\frac{\partial}{\partial t} ,$$

$$L_{19} \equiv R^{-1}(D^2-4\beta^2)-\frac{\partial}{\partial t} \quad ,$$

$$M_4 \equiv -Dw_3\left\{\left[\frac{3}{2}\gamma^2+\gamma^{-2}\left(\frac{\alpha^2}{8}-\frac{3}{2}\beta^2\right)D^2\right]w_2^*+i\alpha\beta\gamma^{-1}D\hat{v}_2^*\right\}$$

$$+\frac{1}{2}w_2^*D^3w_3+D^2w_3Dw_2^*$$

$$-w_3\left\{\left[\left(\frac{\alpha^2}{2}+\gamma^2\right)D+\left(\frac{\alpha^2}{4}-\beta^2\right)\gamma^{-2}D^3\right]w_2^*+i\alpha\beta\gamma^{-1}\left(\gamma^2+D^2\right)\hat{v}_2^*\right\}$$

$$-i\beta\left(\gamma^2v_3w_2^*+Dw_2^*Dv_3\right)+2\alpha\beta^2\gamma^{-1}D\left(v_3\hat{v}_2^*\right)$$

$$-i\beta\gamma^{-2}\left(\frac{3}{4}\alpha^2-\beta^2\right)D\left(v_3Dw_2^*\right)-w_{11}D^3w_1^*-Dw_1^*\left(3\gamma^2-D^2\right)w_{11}$$

$$-\frac{1}{2}Dw_{11}D^2w_1^*-\frac{1}{2}w_1^*\left(3\gamma^2D-D^3\right)w_{11}$$

$$-\frac{1}{2}w_2\left\{i\alpha\left(\gamma^2+D^2\right)u_{1-2}+\left(3\gamma^2D-D^3\right)w_{1-2}\right\}-i\alpha\beta\gamma^{-1}D\left(w_{1-2}D\hat{v}_2\right)$$

$$-Dw_2\left\{\left[\gamma^2+2\beta^2-D^2\right]w_{1-2}+2i\alpha\beta^2\gamma^{-2}Du_{1-2}\right\}+\alpha^2\beta\gamma^{-1}u_{1-2}D\hat{v}_2$$

$$-\alpha\beta\gamma^{-1}\hat{v}_2\left\{i\gamma^2w_{1-2}-\alpha Du_{1-2}\right\}$$

$$-\frac{\gamma^{-2}}{2}D^2w_2\left\{\left(\beta^2-\frac{3}{4}\alpha^2\right)Dw_{1-2}-i\alpha\left(\frac{\alpha^2}{4}-3\beta^2\right)u_{1-2}\right\}$$

$$+\left(\frac{\alpha^2}{4}-\beta^2\right)\gamma^{-2}w_{1-2}D^3w_2-w_3^*\left\{\left[\gamma^2\left(1+\frac{3\alpha^2}{2\gamma_0^2}\right)D-\gamma_0^{-2}\left(\beta^2+\frac{3}{4}\alpha^2\right)D^3\right]w_{13}\right\}$$

$$-i\alpha\beta\gamma_0^{-1}\left\{D\left(w_3^*\hat{v}_{13}\right)+2D\left(\hat{v}_{13}Dw_3^*\right)\right\}-\frac{5}{2}\gamma^2w_{13}Dw_3^*$$

$$-\frac{1}{2}D\left(w_{13}D^2w_3^*\right)+\frac{3}{2}\gamma^2\gamma_0^{-2}Dw_{13}D^2w_3^*+\frac{1}{2}\left(5\beta^2+\frac{9}{4}\alpha^2\right)\gamma_0^{-2}D^2w_{13}Dw_3^*,$$

(A.25)

$$M_6 \equiv -2\alpha^2\beta^2\gamma^{-2} D(w_1 w_2) + \frac{\alpha^2}{8}\gamma^{-4}(12\beta^2-\alpha^2) D(Dw_1 Dw_2)$$

$$+\frac{\alpha^2}{2}\gamma^{-2}\left(w_2 D^3 w_1 + w_1 D^3 w_2\right) - i\alpha^3\beta\gamma^{-1}\left(w_1\hat{v}_2 - \hat{v}_1 w_2\right) - 2\alpha^2\beta^2\gamma^{-2} D(\hat{v}_1\hat{v}_2)$$

$$-i\alpha\beta\gamma^{-3}\left[\left(\beta^2-\frac{3}{4}\alpha^2\right)D\left(\hat{v}_2 Dw_1 - \hat{v}_1 Dw_2\right) + \gamma^2 D\left(w_1 D\hat{v}_2 - w_2 D\hat{v}_1\right)\right]$$

$$-i\alpha\beta\gamma^{-1}\gamma_0^{-2}\left[\left(\frac{3}{4}\alpha^2-\beta^2\right)D\left(\hat{v}_1^* Dw_{13} - \hat{v}_2^* Dw_{23}\right) - \gamma_0^2 D\left(w_{13}D\hat{v}_1^* - w_{23}D\hat{v}_2^*\right)\right.$$

$$+ i\alpha^3\beta\gamma^{-1}\left(w_{13}\hat{v}_1^* - w_{23}\hat{v}_2^*\right) - \frac{\alpha^2}{2}\gamma^{-2} D\left(w_{13}D^2 w_1^* + w_{23}D^2 w_2^*\right)$$

$$+\frac{\alpha^2}{4}\gamma^{-2}\gamma_0^{-2}\left[6\gamma^2 D\left(w_1^* D^2 w_{13} + w_2^* D^2 w_{23}\right)\right.$$

$$-4\gamma_0^2(\alpha^2+2\beta^2)\left(w_{13}Dw_1^* + w_{23}Dw_2^*\right)$$

$$+(3\alpha^2+4\beta^2)D\left(Dw_1^* Dw_{13} + Dw_2^* Dw_{23}\right)$$

$$\left.-4\gamma^2(3\alpha^2+2\beta^2)\left(w_1^* Dw_{13} + w_2^* Dw_{23}\right)\right]$$

$$+\frac{i\alpha}{4}\beta\gamma_0^{-1}\gamma^{-2}\left[(4\beta^2+5\alpha^2)D\left(\hat{v}_{13}Dw_1^* - \hat{v}_{23}Dw_2^*\right) + 4\gamma^2 D\left(w_1^* D\hat{v}_{13} - w_2^* D\hat{v}_{23}\right)\right]$$

$$+ i\alpha^3\beta\gamma_0^{-1}\left(w_1^*\hat{v}_{13} - w_2^*\hat{v}_{23}\right) + 2\alpha^2\beta^2\gamma_0^{-1}\gamma^{-1} D\left(\hat{v}_1^*\hat{v}_{13} + \hat{v}_2^*\hat{v}_{23}\right)$$

$$+ D\left[\frac{1}{2}D\left(w_3^* Dw_{33}\right) - w_{33}D^2 w_3^*\right] - 3\alpha^2\left[w_{33}Dw_3^* + \frac{1}{2}w_3^* Dw_{33}\right],$$

$$M_7 \equiv \frac{1}{2}\hat{v}_2^* Dw_3 + w_3\gamma^{-3}\left[i\alpha\beta D^2 w_2^* + \gamma\left(\frac{\alpha^2}{4}-\beta^2\right)D\hat{v}_2^*\right] - i\beta\alpha^{-1}\gamma^{-1}w_2^* D^2 w_3, \quad (A.26)$$

$$M_9 \equiv -\frac{\alpha\gamma^{-1}}{2}\left(\hat{v}_1 Dw_2 + \hat{v}_2 Dw_1 - w_1 D\hat{v}_2 - w_2 D\hat{v}_1\right) - i\beta\gamma^{-2}\left(w_1 D^2 w_2 - w_2 D^2 w_1\right),$$

$$M_{10} \equiv 2\left(w_1 D^3 w_1 - Dw_1 D^2 w_1\right), \quad (A.27)$$

$$M_{12} \equiv 2\left(w_3 D^3 w_3 - Dw_3\, D^2 w_3\right),$$

$$M_{13} \equiv i\alpha\beta\gamma^{-1}\left[D\left(w_3 D\hat{v}_1 - 2\,\hat{v}_1 Dw_3\right) + \gamma_0^2\,\hat{v}_1 w_3\right] + \frac{\gamma_0^2}{2}\left[\left(\alpha^2\gamma^{-2}-1\right)w_3 Dw_1 - D(w_1 w_3)\right]$$

$$+\frac{3}{2}D\left(w_1 D^2 w_3\right) + \frac{\gamma^{-2}}{4}\left[\left(3\alpha^2+4\beta^2\right)D\left(w_3 D^2 w_1\right) - 2\gamma_0^2 D\left(Dw_1\, Dw_3\right)\right] \qquad (A\cdot 28)$$

$$M_{15} \equiv w_1 D\hat{v}_1 - \hat{v}_1 Dw_1\,, \qquad (A\cdot 29)$$

$$M_{17} \equiv -i\beta\alpha^{-1}\gamma_0^{-1} w_1 D^2 w_3 + w_3\gamma^{-1}\gamma_0^{-1}\left[i\alpha\beta\gamma^{-1}D^2 w_1 + \left(\frac{3}{4}\alpha^2+\beta^2\right)D\hat{v}_1\right]$$

$$-\frac{3}{2}\,\gamma\gamma_0^{-1}\,\hat{v}_1\, Dw_3\,, \qquad (A\cdot 30)$$

$$M_{19} \equiv D\left[4\beta^2\gamma^{-4}\left(\frac{\alpha^2}{4}-\beta^2\right)Dw_1 Dw_2^* + i\alpha\beta\gamma^{-3}\left(3\beta^2-\frac{\alpha^2}{4}\right)\left(\hat{v}_2^* Dw_1 + \hat{v}_1 D w_2^*\right)\right]$$

$$+D\left[2\beta^2\gamma^{-2}\left(w_1 D^2 w_2^* + w_2^* D^2 w_1\right) - i\alpha\beta\gamma^{-1}\left(w_1 D\hat{v}_2^* + w_2^* D\hat{v}_1\right)\right]$$

$$+2\alpha^2\beta^2\gamma^{-2}D\left(\hat{v}_1\hat{v}_2^* - w_1 w_2^*\right) - 4i\alpha\beta^3\gamma^{-1}\left(w_1\hat{v}_2^* + w_2^*\hat{v}_1\right),$$

$$M_{20} \equiv \beta\gamma^{-1}\left(\hat{v}_1 Dw_2^* - \hat{v}_2^* Dw_1 + w_1 D\hat{v}_2^* - w_2^* D\hat{v}_1\right) + \frac{i\alpha}{2}\gamma^{-2}\left(w_2^* D^2 w_1 - w_1 D^2 w_2^*\right).$$

(No summation is implied over repeated indices).

The remaining operator L_8 and functions M_5, M_8, M_{11}, M_{14}, M_{16}, M_{18} are obtained from the respective expressions (A.24)-(A.30) on replacing β by $-\beta$, suffix 2 by suffix 1, suffix 1 by suffix 2, u_{1-2} and w_{1-2} by $\overset{*}{u}_{1-2}$ and $\overset{*}{w}_{1-2}$.

Equations (A.8), (A.9), (A.14) - (A.23) must be solved subject to conditions

$$\bar{\bar{u}} = \bar{\bar{v}} = w_j = \hat{v}_k = v_3 = w_{jj} = w_{k3} = \hat{v}_{kk} = \hat{v}_{k3} = w_{1-2} = u_{1-2} = 0 \qquad (A\cdot 31)$$

(and by virtue of the continuity relations (A.6))

$$Dw_j = Dw_{jj} = Dw_{k3} = Dw_{1-2} = 0 \qquad (A.32)$$

at the boundaries $x_3 = 0, l$.

These equations and boundary conditions form the basis of the analysis of §3.1 and Appendix B.

APPENDIX B

EQUATIONS GOVERNING THE SECOND-ORDER TERMS OF THE SERIES EXPANSIONS (3.6) - (3.12)

Equations governing the functions

$\chi_j^{(m)}$, $\psi_k^{(1)}$ $(j = 1,2,3 \quad m = 1,\dots,5 \quad k = 1,2)$,

which appear in the series expansions (3.7) are stated in §3.1 (see (3.11 a,b), (3.12), (3.16), (3.17) and (3.21 a-d)). Equations governing the remaining second-order functions in these expansions are as follows.

On employing expansions (3.5)-(3.7) in (A.8), (A.9), noting condition (3.10) on the mean flow, and equating to zero the respective coefficients of $|A_j|^2$ $(j = 1,2,3)$ there result

$$\left(R^{-1}D^2 - \alpha c_1\right) f_k = -2\,\mathrm{Re}\left\{\gamma^{-1}\beta_k D\left(\psi_k^{*(1)}\chi_k^{(1)}\right) + \tfrac{1}{2} i\alpha\gamma^{-2}\chi_k^{(1)} D^2 \chi_k^{*(1)}\right\}, \qquad (B.1)$$

$$\left(R^{-1}D^2 - 2\alpha \bar{c}_1\right) f_3 = 2\,\mathrm{Re}\left\{ i\alpha^{-1}\chi_3^{*(1)} D^2 \chi_3^{(1)}\right\}, \qquad (B.2)$$

$$\left(R^{-1}D^2 - \alpha c_1\right) h_k = 2\,\mathrm{Re}\left\{\tfrac{1}{2}\alpha\gamma^{-1} D\left(\psi_k^{*(1)}\chi_k^{(1)}\right) - i\beta_k\gamma^{-2}\chi_k^{(1)} D^2 \chi_k^{*(1)}\right\} \qquad (B.3)$$

Here, k takes the values 1,2 and $\beta_1 = -\beta_2 = \beta$. No summation is implied over repeated indices. The functions f_j and h_k $(j = 1,2,3 \quad k = 1,2)$ represent second-order modifications of the mean flow.

Estimates for expressions $\frac{1}{2}\alpha f_k + \beta_k h_k$, $\frac{1}{2}\alpha h_k - \beta_k f_k$ are required in §3.3 and Appendix C. The appropriate linear combination of (B.1) and (B.3) reveals that

$$(R^{-1}D^2 - \alpha c_I)\left(\frac{\alpha}{2} h_k - \beta_k f_k\right) = 2\,\mathrm{Re}\left\{\gamma D\left(\psi_k^{(1)*} \chi_k^{(1)}\right)\right\}$$

$$(R^{-1}D^2 - \alpha c_I)\left(\frac{\alpha}{2} f_k + \beta_k h_k\right) = 2\,\mathrm{Im}\left\{\chi_k^{(1)} D^2 \chi_k^{(1)*}\right\} \qquad (B.4a,b)$$

On substituting expansions (3.5)-(3.8) in (A.15)-(A.23), (omitting (A.20) since it turns out that estimates for $\hat{v}_{kk}$ $(k = 1,2)$ are not required in our analysis), and equating to zero the coefficients of $A_3 A^*_{3-k}$, $A_i A_j$, $A_1 A^*_2$ $(i,j = 1,2,3 \quad k = 1,2)$ as appropriate, the following second-order equations are ultimately recovered for the functions $\psi^{(2)}_j$, $\chi^{(2)}_{ij}$, $\psi^{(2)}_{k3}$, $\chi^{(2)}_{1-2}$, $\phi^{(2)}_{1-2}$:

$$L_{20}\left[\psi_k^{(2)}\right] + (-1)^{k+1} \beta \gamma^{-1} \chi_k^{(2)} D\bar{u}^0 = P_k + a_k \psi_k^{(1)}, \qquad (B.5)$$

$$L_{21}\left[\psi_3^{(2)}\right] = 0, \qquad (B.6)$$

$$L_{22}\left[\chi_{kk}^{(2)}\right] = Q_{kk}, \qquad (B.7)$$

$$L_{23}\left[\chi_{33}^{(2)}\right] = Q_{33}, \qquad (B.8)$$

$$L_{24}\left[(D^2 - \gamma_0^2)\chi_{k3}^{(2)}\right] + \frac{3}{2} i\alpha \chi_{k3}^{(2)} D^2 \bar{u}^0 = Q_{k3}, \qquad (B.9)$$

$$L_{24}\left[\psi_{k3}^{(2)}\right] + (-1)^{k+1} \beta \gamma_0^{-1} \chi_{k3}^{(2)} D\bar{u}^0 = P_{k3}, \qquad (B.10)$$

$$L_{25}\left[(D^2 - 4\beta^2)\chi_{1-2}^{(2)}\right] = Q_{1-2}, \qquad (B.11)$$

$$L_{25}\left[\phi_{1-2}^{(2)}\right] - \chi_{1-2}^{(2)} D\bar{u}^0 = P_{1-2}, \qquad (B.12)$$

where

$$L_{20} \equiv R^{-1}\left(D^2-\gamma^2\right) - \tfrac{1}{2} i\alpha\left(\bar{u}^0 - c_R\right) - \alpha\left(\tfrac{1}{2} c_I + \tilde{c}_I\right) ,$$

$$L_{21} \equiv R^{-1}\left(D^2-\alpha^2\right) - i\alpha\left(\bar{u}^0 - c_R\right) - \alpha c_I ,$$

$$L_{22} \equiv \left\{R^{-1}\left(D^2-4\gamma^2\right) - i\alpha\left(\bar{u}^0-c_R\right) - \alpha c_I\right\}\left(D^2-4\gamma^2\right) + i\alpha D^2\bar{u}^0 ,$$

$$L_{23} \equiv \left\{R^{-1}\left(D^2-4\alpha^2\right) - 2i\alpha\left(\bar{u}^0-c_R\right) - 2\alpha\tilde{c}_I\right\}\left(D^2-4\alpha^2\right) + 2i\alpha D^2\bar{u}^0 ,$$

$$L_{24} \equiv R^{-1}\left(D^2-\gamma_0^2\right) - \tfrac{3}{2} i\alpha\left(\bar{u}^0-c_R\right) - \alpha\left(\tfrac{1}{2} c_I + \tilde{c}_I\right) ,$$

$$L_{25} \equiv R^{-1}\left(D^2-4\beta^2\right) - \alpha c_I ,$$

$$P_1 = -P_2 = -\tfrac{1}{2}\psi_1^{(1)*} D\chi_3^{(1)} - \gamma^{-3}\chi_3^{(1)}\left\{i\alpha\beta D^2\chi_1^{(1)*} - \gamma\left(\frac{\alpha^2}{4}-\beta^2\right)D\psi_1^{(1)*}\right\}$$

$$+ i\beta\alpha^{-1}\gamma^{-1}\chi_1^{(1)*}D^2\chi_3^{(1)} , \qquad (B\cdot 13)$$

$$P_{13} = -P_{23} = \tfrac{1}{2}\gamma^{-1}\gamma_0^{-1}\left\{2\left(\tfrac{3}{4}\alpha^2+\beta^2\right)\chi_3^{(1)}D\psi_1^{(1)} - 3\gamma^2\psi_1^{(1)}D\chi_3^{(1)}\right\}$$

$$- i\beta\left(\alpha\gamma_0\gamma^2\right)^{-1}\left\{\gamma^2\chi_1^{(1)}D^2\chi_3^{(1)} - \alpha^2\chi_3^{(1)}D^2\chi_1^{(1)}\right\} ,$$

$$P_{1-2} = \alpha\gamma^{-2}\,\mathrm{Re}\left\{i\chi_1^{(1)*}D^2\chi_1^{(1)}\right\} - 2\beta\gamma^{-1}\,\mathrm{Re}\left\{\chi_1^{(1)*}D\psi_1^{(1)} - \psi_1^{(1)}D\chi_1^{(1)*}\right\}$$

$$Q_{11} = Q_{22} = 2\left(\chi_1^{(1)}D^3\chi_1^{(1)} - D\chi_1^{(1)}D^2\chi_1^{(1)}\right) ,$$

$$Q_{33} = 2\left(\chi_3^{(1)}D^3\chi_3^{(1)} - D\chi_3^{(1)}D^2\chi_3^{(1)}\right) ,$$

$$
\begin{aligned}
Q_{13} = Q_{23} = {} & i\alpha\beta\gamma^{-1}\Big\{\chi_3^{(1)} D^2\psi_1^{(1)} - D\psi_1^{(1)} D\chi_3^{(1)} - \psi_1^{(1)}\left(2D^2-\gamma_0^2\right)\chi_3^{(1)}\Big\} \\
& + \tfrac{3}{2}\chi_1^{(1)} D^3\chi_3^{(1)} - \tfrac{1}{8}\gamma^{-2}\left(3\alpha^2-4\beta^2\right)\left(D^2\chi_1^{(1)} D\chi_3^{(1)} + 2D\chi_1^{(1)} D^2\chi_3^{(1)}\right) \\
& - \tfrac{1}{2}\gamma_0^2\chi_1^{(1)} D\chi_3^{(1)} + \frac{\gamma^{-2}}{4}\chi_3^{(1)}\Big\{\left(3\alpha^2+4\beta^2\right)D^3\chi_1^{(1)} - \gamma_0^2\left(4\beta^2-\alpha^2\right)D\chi_1^{(1)}\Big\}, \\
Q_{1-2} = {} & \beta^2\gamma^{-4} D\Big\{2\gamma^2 D^2\left(|\chi_1^{(1)}|^2\right) - 8\beta^2|D\chi_1^{(1)}|^2 - 2\alpha^2\gamma^2\left(|\psi_1^{(1)}|^2 + |\chi_1^{(1)}|^2\right)\Big\} \\
& + \mathcal{I}m\Big\{8\alpha\beta^3\gamma^{-1}\psi_1^{(1)}\chi_1^{(1)*} - 2\alpha\beta\gamma^{-3} D\Big[\Big(3\beta^2-\frac{\alpha^2}{4}\Big)\psi_1^{(1)} D\chi_1^{(1)*} - \gamma^2\chi_1^{(1)*} D\psi_1^{(1)}\Big]\Big\}.
\end{aligned}
$$

Here, use has been made of results (3.15).

We note the appearance of the second-order interaction coefficients a_k $(k = 1,2)$ on the right-hand side of (B.5). Equations (B.11) and (B.12), for $\chi_{1-2}^{(2)}$ and $\phi_{1-2}^{(2)}$, govern a spanwise-periodic 'longitudinal vortex' distortion of the primary flow, like that studied by Benney & Lin [1960] and Benney [1961, 1964].

Equations (B.5)-(B.12) must be solved subject to boundary conditions (3.9). Approximate solutions valid for large R are derived in Appendix C for use in §3.3.

A P P E N D I X C

ASYMPTOTIC CRITICAL-LAYER ANALYSIS

It is found that the functions appearing in the integral representations (3.36a)-(3.37b) for the interaction coefficients possess a critical-layer structure. Approximations to these functions, which are valid outside the critical-layer region (i.e. 'outer estimates'), may be found in a first attempt to evaluate the required integrals, but these approximations usually require modification in the critical-layer region. Such 'inner estimates', which are valid inside the critical layer, enable evaluation of the critical-layer contribution to the appropriate integrals. Here we determine the inner and outer estimates for the required functions.

i) Inviscid Estimates from Linear Theory

Each of the inviscid forms of equations (3.11a)-(3.12) is normally singular at one of the critical points in the complex x_3- plane where $\bar{u}^{o}(x_3)$ equals the complex phase velocities $c_R + ic_I$ (for the oblique waves) and $c_R + i\tilde{c}_I$ for the two-dimensional wave). For the present purpose of indicating the strength of the singularities (and hence the importance of viscous terms) it is sufficient to treat these critical points as if they were coincident on the real axis at x_{3c}, where $\bar{u}^{o}(x_{3c}) = c_R$. It is emphasised that this assumption does not restrict the validity of the theory given in Chapter 3 to strictly neutral waves. In fact, inviscid estimates are readily obtained for non-neutral waves (with c_I, $\tilde{c}_I$ sufficiently small): however this refinement is unnecessary here, since it turns out that the inner (viscous) estimates are of crucial importance.

Near the critical layer, the inviscid estimates for $\chi_j^{(1)}$, $\Xi_j^{(1)}$, $\psi_k^{(1)}$ $(j = 1,2,3 \quad k = 1,2)$ are (see Reid [1965] and Craik [1971])

$$\chi_{j,o}^{(1)} = C_j\left\{1 + (D^2\bar{u}^o/D\bar{u}^o)_c(x_3 - x_{3c})\log_e(x_3 - x_{3c})\right\} + O\left(|x_3 - x_{3c}|\right),$$

$$\Xi_{j,o}^{(1)} = C_j\,(D\bar{u}_c^o)^{-1}(x_3 - x_{3c})^{-1} + O\left(\log_e|x_3 - x_{3c}|\right), \qquad (C.1\,a,b,c)$$

$$\psi_{k,o}^{(1)} = (-1)^k\,2i\beta\,(\alpha\gamma)^{-1}C_k\,(x_3 - x_{3c})^{-1} + O\left(\log_e|x_3 - x_{3c}|\right),$$

$$\left(x_3 \to x_{3c},\ R \to +\infty,\ D\bar{u}_c^o \neq 0\right).$$

Here the subscript $_o$ denotes outer estimate and the subscript $_c$ denotes evaluation at x_{3c}. The constants C_j are the values of $\chi_{j,o}^{(1)}$ at $x_3 = x_{3c}$. (One possible normalisation scheme is to define the C_j to be unity). We note that the logarithmic singularity of $D\,\chi_{j,o}^{(1)}$ depends on the existence of non-zero profile curvature $(D^2\bar{u}_c \neq 0)$ at the critical point but that $\Xi_{j,o}^{(1)}$ and $\psi_{k,o}^{(1)}$ are singular irrespective of profile curvature provided C_j, C_k are non-zero.

The solutions of (3.11a)-(3.12) with appropriate boundary conditions may be normalised so that $\chi_1^{(1)} = \chi_2^{(1)}$ and $\psi_1^{(1)} = -\psi_2^{(1)}$; in which case $C_1 = C_2$ above.

ii) Outer Estimates From Second-order Theory

The second-order functions fall into two categories: those which have time-periodic components and those which do not. In the former category are the functions

$$\chi_j^{(2)}, \quad \psi_j^{(2)}, \quad \chi_{jj}^{(2)}, \quad \chi_{k3}^{(2)}, \quad \psi_{k3}^{(2)} \quad (k = 1,2 \qquad j = 1,2,3)$$

while the latter category comprises the functions

$$f_j, \quad h_k, \quad \chi_{1-2}^{(2)}, \quad \phi_{1-2}^{(2)} .$$

Functions in the first category are solutions of equations (3.16), (3.17) and (B.5)-(B.10) and boundary conditions (3.9). Inviscid estimates for these functions possess singular points due to the singular nature of both the non-homogeneous forcing terms and the homogeneous terms. These singular points will be treated as if they are coincident on the real axis at x_{3c}, for the same reasons as discussed in the ~~preceeding~~ preceding sub-section.

Near the critical layer use of (C.1 a,c) reveals that the required inviscid estimates are,

$$D^2\chi^{(2)}_{R,0} = 2ia_R C_R \left(D^2\bar{u}^0/(D\bar{u}^0)^2\right)_c \alpha^{-1}(x_3 - x_{3c})^{-2}$$

$$+ O\left(|a_R||x_3-x_{3c}|^{-1}\log_e|x_3-x_{3c}|\right)$$

$$D^2\chi^{(2)}_{3,0} = 8i\beta^2 C_1 C_2 \left(1+2\beta^2\gamma^{-2}\right)\left(\alpha\gamma^2 D\bar{u}^0_c\right)^{-1}(x_3-x_{3c})^{-4}$$

$$+ O\left(|x_3-x_{3c}|^{-3}\log_e|x_3-x_{3c}|\right),$$

$$D^2\chi^{(2)}_{RR,0} = -2iC_R^2\left(D^2\bar{u}^0/(D\bar{u}^0_c)^2\right)_c \alpha^{-1}(x_3-x_{3c})^{-3} \tag{C.2}$$

$$+ O\left(|x_3-x_{3c}|^{-2}\log_e|x_3-x_{3c}|\right),$$

$$D^2\chi^{(2)}_{33,0} = -iC_3^2\left(D^2\bar{u}^0/(D\bar{u}^0)^2\right)_c \alpha^{-1}(x_3-x_{3c})^{-3}$$

$$+ O\left(|x_3-x_{3c}|^{-2}\log_e|x_3-x_{3c}|\right),$$

$$D^2\chi^{(2)}_{R3,0} = 8i\beta^2 C_3 C_R \left(3\alpha\gamma^2 D\bar{u}^0_c\right)^{-1}(x_3-x_{3c})^{-4}$$

$$+ O\left(|x_3-x_{3c}|^{-3}\log_e|x_3-x_{3c}|\right),$$

$$\psi_{k,o}^{(2)} = 4\beta C_k (-1)^{k+1} a_k \left(\alpha^2 \gamma D\bar{u}_c^o\right)^{-1} \left(x_3 - x_{3c}\right)^{-2}$$

$$+ o\left(|a_k|\, |x_3 - x_{3c}|^{-1} \log_e |x_3 - x_{3c}|\right) ,$$

$$\psi_{k3,o}^{(2)} = 4\beta C_3 C_k (-1)^k \left(\tfrac{3}{4}\alpha^2 + \tfrac{7}{9}\beta^2\right) \left(3\alpha^2\gamma^2\gamma_o D\bar{u}_c^o\right)^{-1} \left(x_3 - x_{3c}\right)^{-3} \tag{C.3}$$

$$+ O\left(|x_3 - x_{3c}|^{-2} \log_e |x_3 - x_{3c}|\right)$$

$$\left(x_3 \to x_{3c} ,\ R \to +\infty\right) .$$

In deriving expressions for $D^2 \chi_{k,o}^{(2)}$ and $\psi_{k,o}^{(2)}$, given in (C.2) and (C.3) respectively, terms from F_1 and P_k in expressions (3.18) and (B.13) have been ignored in favour of $a_k D^2 \chi_1^{(1)}$ and $a_k \psi_k^{(1)}$. The ignored terms in F_1 and P_k are respectively $O(x_3 - x_{3c})^{-4}$ and $O(x_3 - x_{3c})^{-3}$ which are both formally larger than $a_k(x_3 - x_{3c})^{-2}$ as $x_3 \to x_{3c}$ with a_k fixed. But since $|a_k| \sim O(R)$ (see (3.52)) the terms $a_k D^2 \chi_1^{(1)}$ and $a_k \psi_1^{(1)}$ dominate in the appropriate intermediate limits for matching purposes at the edge of the critical layer.

As noted in §3.1 ii), the only solution of (B.6) which satisfies the boundary conditions (3.9) is

$$\psi_3^{(2)} = 0 \tag{C.4}$$

Consequently, from henceforth, all terms in $\psi_3^{(2)}$ are omitted.

Direct integration of expressions (C.2) yields the inviscid estimates

$$\chi^{(2)}_{R,0} = -2i\, a_R C_R \left(D^2\bar{u}^0/(D\bar{u}^0)^2 \right)_c \alpha^{-1} \log_e (x_3 - x_{3c})$$

$$+ O\left(|a_R| |x_3 - x_{3c}| \log_e^2 |x_3 - x_{3c}| \right) \quad ,$$

$$\chi^{(2)}_{3,0} = 4i\beta^2 C_1 C_2 \left(1 + 2\beta^2 \gamma^{-2} \right) \left(3\alpha \gamma^2 D\bar{u}^0_c \right)^{-1} (x_3 - x_{3c})^{-2}$$

$$+ O\left(|x_3 - x_{3c}|^{-1} \log_e |x_3 - x_{3c}| \right) \, , \tag{C.5}$$

$$\chi^{(2)}_{RR,0} = -i C_R^2 \left(D^2\bar{u}^0 / (D\bar{u}^0)^2 \right)_c \alpha^{-1} (x_3 - x_{3c})^{-1} + O\left(\log_e |x_3 - x_{3c}| \right) \, ,$$

$$\chi^{(2)}_{33,0} = -i C_3^2 \left(D^2\bar{u}^0 / (D\bar{u}^0)^2 \right)_c (2\alpha)^{-1} (x_3 - x_{3c})^{-1} + O\left(\log_e |x_3 - x_{3c}| \right) ,$$

$$\chi^{(2)}_{R3,0} = 4i\beta^2 C_3 C_R \left(9\alpha\gamma^2 D\bar{u}^0_c \right)^{-1} (x_3 - x_{3c})^{-2}$$

$$+ O\left(|x_3 - x_{3c}|^{-1} \log_e |x_3 - x_{3c}| \right) ,$$

$$\left(x_3 \to x_{3c} \, , \; R \to +\infty \right) \, ,$$

which are required in section iv) of this appendix. Integration gives additional linear terms with arbitrary constants which are all O(1), except in the estimate for $\chi^{(2)}_{k,o}$ where the constants are $O(|a_k|)$. To leading-order, these terms may be neglected.

The remaining second-order functions f_j, h_k, $\chi^{(2)}_{1-2}$, $\phi^{(2)}_{1-2}$ $(j = 1,2,3 \quad k = 1,2)$ which represent non-fluctuating components, satisfy equations (B.1)-(B.3), (B.11), (B.12) and appropriate boundary conditions. For simplicity, we restrict attention to cases for which

$$|\alpha c_I R|, \; |\alpha \tilde{c}_I R| \quad \ll O(1) \;. \tag{C.6}$$

In such cases viscous diffusion is important throughout the flow domain for these non-fluctuating components, and no inviscid approximations exist. Instead, the mean-flow terms f_j, h_k derive from a balance between viscous diffusion and the non-linear Reynolds stresses; acceleration terms are also absent from the 'longitudinal vortex' denoted by $\chi^{(2)}_{1-2}$ and $\phi^{(2)}_{1-2}$. However, these functions still possess a 'critical-layer structure', since the non-homogeneous (Reynolds stress) terms themselves possess inviscid approximations outside the critical layer and viscous approximations within it.

On employing inviscid estimates to evaluate the right-hand sides of equations (B.1)-(B.4b) and (B.11) away from the critical layer, while retaining the viscous terms on the left-hand sides, it is readily seen that outer estimates for

f_j, h_k, $\frac{1}{2}\alpha h_k - \beta_k f_k$, $\frac{1}{2}\alpha f_k + \beta_k h_k$, $\chi^{(2)}_{1-2}$ are all of order O(R). Each of these inviscid estimates will be singular at the singular point of the corresponding forcing terms. Recall that this singular point occurs at one of the critical points in the complex x_3-plane where $\overset{-o}{u}(x_3)$ equals the complex phase velocities $c_R + ic_I$ (for the oblique waves) and $c_R + i\tilde{c}_I$ (for the two-dimensional wave). As previously, we treat the critical points as if they were coincident on the real axis at $x_3 = x_{3c}$. The inviscid estimates have the form

$$
\begin{aligned}
f_{j,o} &= R\left[A_{1j} f_{1j}(x_3) + B_{1j} f_{2j}(x_3)\right], \\
h_{k,o} &= R\left[A_{2k} h_{1k}(x_3) + B_{2k} h_{2k}(x_3)\right], \\
\frac{\alpha}{2} h_{k,o} + (-1)^k \beta f_{k,o} &= R\left[A_{3k} J_{1k}(x_3) + B_{3k} K_{2k}(x_3)\right], \qquad (C.7a,b,c,d,e) \\
\frac{\alpha}{2} f_{k,o} + (-1)^{k+1} \beta h_{k,o} &= R\left[A_{4k} L_{1k}(x_3) + B_{4k} M_{2k}(x_3)\right], \\
\chi^{(2)}_{1-2,o} &= R\left[A_{1-2} K_1(x_3) + B_{1-2} K_2(x_3)\right],
\end{aligned}
$$

where f_{1j}, h_{1k}, J_{1k}, L_{1k}, K_1 are those parts of the solutions which become singular at x_{3c} while f_{2j}, h_{2k}, K_{2k}, M_{2k}, K_2 remain regular at x_{3c}. The A's and B's are complex constants, O(1) in magnitude, which are determined from the boundary conditions at the wall and on matching 'outer' and 'inner' solutions (see section iv) of this appendix).

If condition (C.6) is satisfied, use of inviscid estimates (C.1 a,c) reveals that near x_{3c}, equations (B.1)-(B.4b) and (B.11) reduce to

$$\left.\begin{aligned} D^2 f_k &= 4R\beta^2 |C_k|^2 (\alpha\gamma^2)^{-1}\, \mathcal{I}m\left[(x_3 - x_{3c})^{-2}\right] \\ &\qquad + o\left(|x_3 - x_{3c}|^{-1} \log_e |x_3 - x_{3c}|\right), \\ D^2 h_k &= (-1)^k\, 2\beta R\, |C_k|^2 \gamma^{-2}\, \mathcal{I}m\left[(x_3 - x_{3c})^{-2}\right] \\ &\qquad + o\left(|x_3 - x_{3c}|^{-1} \log_e |x_3 - x_{3c}|\right), \end{aligned}\right] \quad (C.8)$$

$$\left.\begin{aligned} D^2\left(\tfrac{\alpha}{2} h_k + (-1)^k \beta f_k\right) &= (-1)^{k+1}\, 4\beta R\, |C_k|^2 \alpha^{-1}\, \mathcal{I}m\left[(x_3 - x_{3c})^{-2}\right] \\ &\qquad + o\left(|x_3 - x_{3c}|^{-1} \log_e |x_3 - x_{3c}|\right), \\ D^2\left(\tfrac{\alpha}{2} f_k + (-1)^{k+1} \beta h_k\right) &= -2R\, |C_k|^2 \left(D^2\bar{u}^0 / D\bar{u}^0\right)_c\, \mathcal{I}m\left[(x_3 - x_{3c})^{-1}\right] \\ &\qquad + o\left(\log_e |x_3 - x_{3c}|\right), \\ D^2 f_3 &= -2R\, |c_3|^2 \left(D^2\bar{u}^0 / D\bar{u}^0\right)_c\, \alpha^{-1}\, \mathcal{I}m\left[(x_3 - x_{3c})^{-1}\right] \\ &\qquad + o\left(\log_e |x_3 - x_{3c}|\right), \end{aligned}\right] \quad (C.9)$$

$$\begin{aligned} D^4 \chi_{1-2}^{(2)} &= 8\beta^2 R\, C_1 C_2^{*}\, \gamma^{-4} \left(\beta^2 - \frac{\alpha^2}{4}\right)(x_3 - x_{3c})^{-3} \\ &\qquad + o\left(|x_3 - x_{3c}|^{-2} \log_e |x_3 - x_{3c}|\right), \\ &\qquad (x_3 \to x_{3c}). \end{aligned} \quad (C.10)$$

Recall that, in practice the two critical points, corresponding to the chosen symmetric oblique waves and to the two-dimensional wave, do not coincide. For simplicity, consider the critical point $x_3 = x_0$ where $\bar{u}^{o}(x_3)$ equals the complex phase velocity $c = c_R + ic_I$ (c_I small but non-zero) of the oblique waves. As $x_3 \to x_0$,

$$\mathcal{I}m\left[(x_3 - x_0)^{-1}\right] \to \mathcal{I}m\left[D\bar{u}^{o}_{o}(\bar{u}^{o} - c)^{-1}\right] = \frac{c_I D\bar{u}^{o}_{o}}{(\bar{u}^{o} - c_R)^2 + c_I^2},$$

$$\mathcal{I}m\left[(x_3 - x_{3o})^{-2}\right] \to \mathcal{I}m\left[(D\bar{u}^{o}_{o})^2(\bar{u}^{o} - c)^{-2}\right] = \frac{2c_I(\bar{u}^{o} - c_R)(D\bar{u}^{o}_{o})^2}{((\bar{u}^{o} - c_R)^2 + c_I^2)^2},$$

where $D\bar{u}^{o}_{o}$ is the value of $D\bar{u}^{o}$ when $x_3 = x_0$ (we assume $D\bar{u}^{o}_{o} \neq 0$). As $c_I \to 0$ the right-hand terms in these results tend to zero except at the point x_{3c} where $\bar{u}^{o} = c_R$, where they become infinite. However, on making use of the result

$$\operatorname*{Lt}_{a \to 0+} \int_{-L}^{+L} \frac{a}{a^2 + x^2}\, dx = \pi$$

(where L is any +ve constant), we may write

$$\mathcal{I}m\left[(x_3 - x_0)^{-1}\right] = \pi\,\delta(x_3 - x_{3c}),$$

(c.f. footnote on p.121 of Lin [1955]). Here $\delta(x_3 - x_{3c})$ is the Dirac delta function. A similar argument follows for the other critical point corresponding to the two-dimensional wave. Thus, for nearly neutral waves the leading-order terms of expressions (C.8) and (C.9) are $O(Rc_I, R\tilde{c}_I)$, (regarding α, β, γ and the derivatives of $\bar{u}^{o}$ as $O(1)$ and assuming that the linear solutions for $\chi_j^{(1)}$ ($j = 1,2,3$) are normalised so that $|C_j|$ are $O(1)$).

For neutral waves these leading-order terms are zero except precisely at the critical point x_{3c} where they become infinite. This is related to the familiar result that the Reynolds stresses for neutral waves are constant except in the viscous regions (see Lin [1955], pp. 120,121).

Investigation of (B.12) reveals that, due to results (C.7e), the outer solutions for $\phi^{(2)}_{1-2}$ will be $O(R^2)$ and consequently may be written in the form

$$\phi^{(2)}_{1-2,o} = R^2 \left[C_{1-2} M_1(x_3) + D_{1-2} M_2(x_3) \right] , \qquad (C.11)$$

where M_1 becomes singular at x_{3c} while M_2 remains regular at x_{3c}. C_{1-2} and D_{1-2} are complex constants, $O(1)$ in magnitude, which are determined on matching 'outer' and 'inner' solutions.

If $D^2\bar{u}^o_c = 0$ the singular nature of $\Psi^{(1)}_{j,o}$, $\psi^{(1)}_{k,o}$ at $x_3 = x_{3c}$ (see (C.1 b,c)) remains unchanged, but now $\chi^{(1)}_{j,o}$ is non-singular at x_{3c}. The right-hand sides of equations (B.2) and (B.4b) become zero and the only solutions satisfying boundary conditions (3.9) are

$$f_3(x_3) = 0 \ , \qquad \frac{\alpha}{2} f_k(x_3) + (-1)^{k+1} \beta h_k = 0 \qquad (C.12)$$

The leading-order terms in the outer estimates for f_k, h_k, $\frac{1}{2}\alpha h_k + (-1)^k \beta f_k$, $\chi^{(2)}_{1-2}$, $\phi^{(2)}_{1-2}$ remain unchanged. Here, to second-order in wave-amplitude, the fluctuating motion of the two-dimensional wave does not produce any Reynolds stresses and hence to this order does not modify the mean flow. However, Reynolds stresses deriving from the fluctuating motion of the oblique waves modify the mean flow by producing a second-order flow parallel to the respective oblique-wave crests. No such second-order flow is induced perpendicular to the oblique-wave crests.

The singular estimates for the outer solutions, both from linear and second-order theory, require modification in the critical layer.

iii) Viscous Estimates from Linear Theory

On introducing the 'inner'-strained independent variable

$$\xi = (x_3 - x_{3c}) / \varepsilon \ , \qquad (C.13)$$

where $\varepsilon = 1/i(\alpha R D\bar{u}^*_c)^{\frac{1}{3}}$ (see for example Reid [1965]), into equations (3.11a)-(3.12), (3.33) and (3.34), the following viscous estimates, which match onto the corresponding inviscid estimates (C.1 a-c) at the edge of the critical layer, are obtained in the critical layer and are valid as $\varepsilon \to 0$.

$$\left.\begin{aligned} \frac{d^2}{d\xi^2}\chi^{(1)}_{k,I} &= -i\,C_k\left(D^2\bar{u}^\circ/(D\bar{u}^\circ)^{4/3}\right)_c (2\alpha R)^{-1/3}\, L_1(\xi) \\ &\qquad + o\left(R^{-2/3}\log_e R\right) , \\ \frac{d^2}{d\xi^2}\chi^{(1)}_{3,I} &= -i\,C_3\left(D^2\bar{u}^\circ/(D\bar{u}^\circ)^{4/3}\right)_c (\alpha R)^{-1/3}\, L(\xi) \\ &\qquad + o\left(R^{-2/3}\log_e R\right) , \end{aligned}\right\} \tag{C.14}$$

$$\begin{aligned} \psi^{(1)}_{k,I} &= (-1)^{k+1}\beta\, C_k\, \gamma^{-1}\left(4RD\bar{u}^\circ_c/\alpha^2\right)^{1/3} L_1(\xi) \\ &\qquad + o\left(\log_e R\right) , \end{aligned} \tag{C.15}$$

$$\left.\begin{aligned} \Phi^{(1)}_{k,I} &= i\,C_k\left(\alpha R/2(D\bar{u}^\circ_c)^2\right)^{1/3} L_1(\xi) + O\left(\log_e R\right) , \\ \Phi^{(1)}_{3,I} &= i\,C_3\left(\alpha R/(D\bar{u}^\circ_c)^2\right)^{1/3} L(\xi) + o\left(\log_e R\right) . \end{aligned}\right\} \tag{C.16}$$

Here, $L(\xi)$ is the Lommel function (see Benney [1961]) which satisfies

$$\left(\frac{d^2}{d\xi^2} + \xi\right) L(\xi) = 1 , \tag{C.17}$$

and $L(\xi) \to \xi^{-1}$ as $\xi \to \pm i\infty$; while $L_1(\xi)$ satisfies

$$\left(\frac{d^2}{d\xi^2} + \frac{\xi}{2}\right) L_1(\xi) = 2^{-2/3} , \tag{C.18}$$

and $L_1(\xi) \to 2^{\frac{1}{3}}\xi^{-1}$ as $\xi \to \pm i\infty$. We note that ξ, as defined by (C.13), is purely imaginary. On writing $\xi = iY$, where Y is real, it is well known that

$$L(\xi) = L_R(Y) + i\,L_I(Y) \quad , \qquad L_1(\xi) = L_{1R}(Y) + i\,L_{1I}(Y) \;, \qquad (C.19)$$

where L_R, L_{1R} are even and L_I, L_{1I} are odd functions of Y. In such a case

$$\left[L(\xi)\right]^* = L(-\xi) \quad , \qquad \left[L_1(\xi)\right]^* = L_1(-\xi) \;, \qquad (C.20)$$

Normally, the operator $D \equiv d/dx_3$ may be regarded as $O(R^{\frac{1}{3}})$ in the critical layer; but this is not so for the functions $\chi_j^{(1)}$ $(j = 1,2,3)$. In the critical layer $\chi_j^{(1)} \sim O(1)$ on account of the regular part C_j of the inviscid estimate (C.1a), but $D^2\chi_j^{(1)}$ is $O(R^{\frac{1}{3}})$ here, instead of $O(R^{\frac{2}{3}})$. To demonstrate the difficulties consider the function $\chi_1^{(1)}$. Near the critical layer the inviscid estimate for $D\chi_1^{(1)}$ is (from (C.1a))

$$D\chi_{1,0}^{(1)} = C_1\left(D^2\bar{u}^0/D\bar{u}^0\right)_c \log_e(x_3 - x_{3c}) + O(1) \;. \qquad (C.21)$$

The corresponding viscous estimate is obtained on integrating (C.14). Thus

$$\frac{d}{d\xi}\chi_{1,I}^{(1)} = -i\,C_1\left(D^2\bar{u}^0/(D\bar{u}^0)^{4/3}\right)_c (2\alpha R)^{-1/3}\left[\int_0^{\xi} L_1(\xi')\,d\xi' + B\right] \qquad (C.22)$$

where B is a constant of integration. On introducing the intermediate variable

$$\eta = (x_3 - x_{3c})/\delta = \varepsilon\xi/\delta \;, \qquad (C.23)$$

where δ is real and $|\varepsilon| \ll \delta < 1$, into estimates (C.21),(C22) and matching in the limit $\delta \to 0$ (regarding η as $O(1)$), we see that B is $O(\log_e R)$. Thus some care is needed when dealing with such derivatives in the critical-layer region.

iv) Inner Estimates for Second-order Theory

Equations governing the functions

$\chi_j^{(2)}$, $\chi_{jj}^{(2)}$, $\chi_{k3}^{(2)}$ ($j = 1,2,3$ $k = 1,2$) in the critical layer are now obtained.

On making the approximation (C.6) and noting the viscous estimate (C.14) together with the estimate (3.52) for $|a_k|$ ($k = 1,2$) examination of the right-hand side of (3.16) reveals that the term $a_k D^2 \chi_k^{(1)}$ is dominant in the critical layer, being of ~~of~~ order $O(R^{4/3})$. Retaining the highest-order terms of (3.16) in this region, there results

$$\left(\frac{d^2}{d\xi^2} + \frac{\xi}{2}\right)\frac{d^2}{d\xi^2}\chi_k^{(2)} = \mathcal{F}_k(\xi) + o\left(|a_k|\, R^{-1/3}\log_e R\right) , \qquad (C.24)$$

where
$$\mathcal{F}_k(\xi) = i a_k C_k \left(D^2\bar{u}^0 / (D\bar{u}^0)^2\right)_c \left(2^{1/3}\alpha\right)^{-1} L_1(\xi) \qquad (C.25)$$

Similarly, in the critical layer, equations (3.17), (B.7)-(B.9) reduce to

$$\begin{aligned}
&\left(\frac{d^2}{d\xi^2} + \xi\right)\frac{d^2}{d\xi^2}\chi_3^{(2)} = \mathcal{F}_3(\xi) + o\left(R^{1/3}\log_e R\right) ,\\
&\left(\frac{d^2}{d\xi^2} + \xi\right)\frac{d^2}{d\xi^2}\chi_{kk}^{(2)} = \mathcal{F}_{3+k}(\xi) + o\left(\log_e R\right) ,\\
&\left(\frac{d^2}{d\xi^2} + 2\xi\right)\frac{d^2}{d\xi^2}\chi_{33}^{(2)} = \mathcal{F}_6(\xi) + o\left(\log_e R\right) ,\\
&\left(\frac{d^2}{d\xi^2} + \frac{3}{2}\xi\right)\frac{d^2}{d\xi^2}\chi_{k3}^{(2)} = \mathcal{F}_{6+k}(\xi) + o\left(R^{1/3}\log_e R\right) ,
\end{aligned} \qquad (C.26)$$

$$(k = 1,2) ,$$

where

$$\mathcal{Y}_3(\xi) = \left(R^{2/3} i C_1^2 \beta^2 2^{5/3} \gamma^{-2} (\alpha D \bar{u}_c^0)^{-1/3} \right) \times$$

$$\left(\frac{d^2 L_1(\xi)}{d\xi^2} - \beta^2 2^{5/3} \gamma^{-2} L_1(\xi) \frac{dL_1(\xi)}{d\xi} \right) ,$$

$$\mathcal{Y}_{5+R}(\xi) = -C_R^2 (4R/\alpha^2)^{1/3} \left(D^2 \bar{u}^0 / (D\bar{u}^0)^{5/3} \right)_c \frac{dL_1(\xi)}{d\xi} ,$$

$$\mathcal{Y}_6(\xi) = -2 C_3^2 (R/\alpha^2)^{1/3} \left(D^2 \bar{u}^0 / (D\bar{u}^0)^{2/3} \right)_c \frac{dL}{d\xi}(\xi) , \tag{C.27}$$

$$\mathcal{Y}_{6+R}(\xi) = -C_R C_3 \beta^2 \gamma^{-2} i \left(4R^2 / \alpha D \bar{u}_c^0 \right)^{1/3} \frac{d^2 L_1(\xi)}{d\xi^2} .$$

Linear terms in (C.27) arise since, to highest orders, $\chi_j^{(1)}$ $(j = 1,2,3)$ may be regarded as constant in the critical layer.

Solutions of equations (C.24), (C.26) which match onto the corresponding inviscid estimates at the edges of the critical layer (i.e. as $\xi \to \pm i\infty$) are of the form

$$\chi = I_1(\xi) + B\xi + C ,$$

where B, C are complex constants of integration determined by the above matching and

$$I_1(\xi) = \left(\frac{\pi i}{4} \left(\frac{2}{3\delta} \right)^{1/2} \right) \times$$

$$\left(\int_0^{\xi} \int_0^{\xi_3} \left\{ h_\delta^{(2)}(\xi_2) \int_{+i\infty}^{\xi_2} h_\delta^{(1)}(\xi_1) \mathcal{Y}(\xi_1) d\xi_1 - h_\delta^{(1)}(\xi_2) \int_{-i\infty}^{\xi_2} h_\delta^{(2)}(\xi_1) \mathcal{Y}(\xi_1) d\xi_1 \right\} d\xi_2 d\xi_3 \right) . \tag{C.30}$$

with appropriate choices of δ and $\mathcal{F}(\xi_1)$. Here, the modified Hankel functions (see for example Lebedev 1965)

$$h_\delta^{(k)}(z) = \left(\frac{2}{3}\delta^{1/2} z^{3/2}\right)^{1/3} H_{1/3}^{(k)}\left(\frac{2}{3}\delta^{1/2} z^{3/2}\right), \quad (\delta = 1/2, 1, 3/2, 2) \qquad (C.30)$$

are linearly independent solutions of the equation

$$\frac{d^2u}{dz^2} + \delta z u = 0 .$$

The Wronskian of $h_\delta^{(k)}(z)$ is given by

$$W\left[h_\delta^{(1)}(z), h_\delta^{(2)}(z)\right] = -\frac{4i}{\pi}\left(\frac{3}{2}\delta\right)^{1/3} .$$

The inviscid estimates for $\varkappa_3^{(2)}$, $\varkappa_{kk}^{(2)}$, $\varkappa_{33}^{(2)}$, $\varkappa_{k3}^{(2)}$ consist of two parts: one part becomes singular at $x_3 = x_{3c}$ while the other part remains regular at $x_3 = x_{3c}$. At the edges of the critical layer $I_1(\xi)$ matches onto the singular part and $B\xi + C$ matches onto the regular part of the appropriate inviscid estimate. Such matching reveals that for each of the viscous estimates for the functions $\varkappa_3^{(2)}$, $\varkappa_{kk}^{(2)}$, $\varkappa_{33}^{(2)}$, $\varkappa_{k3}^{(2)}$, the integral $I_1(\xi)$ dominates $B\xi + C$ in the critical layer. Accordingly, to highest order each of these functions will possess a viscous estimate of the form

$$\varkappa = I_1(\xi) \qquad (C.31)$$

(see (C.28)).

More care is necessary with $\varkappa_k^{(2)}$. On introducing the intermediate variable $\eta = (x_3 - x_{3c})/\delta$, where δ is real and $|\varepsilon| \ll \delta \ll 1$ (see (C.23)), we may express order-of-magnitude estimates for $\chi_k^{(2)}$, inside and outside the critical layer, as follows.

Inner estimate (see (C.24)):

$$\left|\chi_{R,I}^{(2)}\right| \sim |a_R|\left\{O\left(|\delta\eta/\varepsilon|\right) + O\left(\log_e|\delta\eta|\right) - O\left(\log_e|\varepsilon|\right)\right\}$$

$$+ \left|\left(B_I + B_0\right)\delta\eta/\varepsilon\right| + |C| .$$

Outer estimate (see (C.5)):

$$\left|\chi_{R,0}^{(2)}\right| \sim |a_R|\left\{O\left(\log_e|\delta\eta|\right) + O\left(|\delta\eta|\right)\right\} .$$

$\left(\delta \to 0\right.$ and regarding η as $\left.O(1)\right)$.

Here, B_I, B_0, C are complex constants whose magnitudes are determined by matching the above estimates. Thus,

$$|B_I| \sim |a_R| \quad , \qquad |B_0| \sim |a_R|\, R^{-1/3} \quad ,$$
$$|C| \sim |a_R|\log_e R \quad . \tag{C.32}$$

To highest order we may write

$$\chi_R^{(2)} = \left(\pi i / 2.6^{1/3}\right) \quad \times$$

$$\left(\int_0^{\xi}\int_0^{\xi_3}\left\{h_{1/2}^{(2)}(\xi_2)\int_{+i\infty}^{\xi_2} h_{1/2}^{(1)}(\xi_1)\,\mathcal{Y}_R(\xi_1)\,d\xi_1 - h_{1/2}^{(1)}(\xi_2)\int_{-i\infty}^{\xi_2} h_{1/2}^{(2)}(\xi_1)\,\mathcal{Y}_R(\xi_1)\,d\xi_1\right\}d\xi_2\,d\xi_3\right)$$

$$+ E_1 a_R \xi + E_2 a_R \log_e R \quad , \tag{C.33}$$

in the critical layer. Here, $\mathcal{Y}_k$ is given by (C.25) and E_1, E_2 are complex constants, $O(1)$ in magnitude.

Equations governing the functions $\psi_k^{(2)}$, $\psi_{k3}^{(2)}$ $(k = 1,2)$ in the critical layer are likewise obtained from (B.5) and (B.10). These are

$$\left(\frac{d^2}{d\xi^2} + \frac{\xi}{2}\right)\psi_k^{(2)} = \mathfrak{F}_{8+k}(\xi) + O\left(|a_k| R^{1/3} \log_e R\right),$$

$$\left(\frac{d^2}{d\xi^2} + \frac{3}{2}\xi\right)\psi_{k3}^{(2)} = \mathfrak{F}_{10+k}(\xi) + O\left(R^{2/3} \log_e R\right),$$

where

$$\left.\begin{aligned}
\mathfrak{F}_{8+k}(\xi) &= (-1)^k \beta a_k C_k \gamma^{-1} \left(4R^2/\alpha^4 D \tilde{u}_c^0\right)^{1/3} L_1(\xi) , \\
\mathfrak{F}_{10+k}(\xi) &= R C_3 C_k \left(\beta / \gamma_0 \gamma^2\right)\Big\{ -i 4^{1/3} \alpha^{-1} \left(\tfrac{3}{4}\alpha^2 + \beta^2\right) \frac{dL_1(\xi)}{d\xi} \\
&\qquad + (-1)^k \pi \beta^2 \alpha^{-1} (36)^{1/3} \int_0^{\xi} \int_0^{\xi_2} G_2(\xi_3)\, d\xi_3\, d\xi_2 \Big\} ,
\end{aligned}\right] \quad (C.35)$$

$$G_2(\xi) = h_{3/2}^{(2)}(\xi) \int_{+i\infty}^{\xi} h_{3/2}^{(1)}(\xi_1) \frac{d^2}{d\xi_1^2} L_1(\xi_1)\, d\xi_1 - h_{3/2}^{(1)}(\xi) \int_{-i\infty}^{\xi} h_{3/2}^{(2)}(\xi_1) \frac{d^2}{d\xi_1^2} L_1(\xi_1)\, d\xi_1 . \quad (C.36)$$

We note that the leading-order contribution to $\mathfrak{F}_{10+k}$ derives from the appropriate viscous estimate (C.31) for $\varkappa_{3k}^{(2)}$.

Solutions of (C.34) which match onto the corresponding inviscid estimates (C.3) as $\xi \to \pm i\infty$ are of the form

$$\left(\pi i / 2(12\delta)^{1/3}\right)\left\{ h_\delta^{(2)}(\xi) \int_{+i\infty}^{\xi} h_\delta^{(1)}(\xi_1)\, \mathfrak{F}(\xi_1)\, d\xi_1 - h_\delta^{(1)}(\xi) \int_{-i\infty}^{\xi} h_\delta^{(2)}(\xi_1)\, \mathfrak{F}(\xi_1)\, d\xi_1 \right\} \quad (C.37)$$

with appropriate choices of δ and $\mathfrak{F}(\xi_1)$.

Equations governing the non-fluctuating components f_j, h_k, $\frac{1}{2}\alpha f_k + (-1)^{k+1}\beta h_k$, $\chi^{(2)}_{1-2}$, $\phi^{(2)}_{1-2}$ ($j = 1,2,3$ $k = 1,2$) in the critical layer are now obtained for cases in which condition (C.6) is satisfied. (Viscous estimates for $\frac{1}{2}\alpha h_k + (-1)^k\beta f_k$) are not required in our analysis.)

Using the viscous estimates (C.14), (C.15) and retaining the highest-order terms of (B.1)-(B.3), (B.4b) and (B.11), we obtain

$$\frac{d^2}{d\xi^2} f_k = R\, G_{2+k}(\xi) + O\left(R^{2/3}\log_e R\right),$$

$$\frac{d^2}{d\xi^2} h_k = R\, G_{4+k}(\xi) + O\left(R^{2/3}\log_e R\right),$$

$$\frac{d^2}{d\xi^2}\left(\frac{\alpha}{2} f_k + (-1)^{k+1}\beta h_k\right) = R^{2/3} G_{6+k}(\xi) + O\left(R^{1/3}\log_e R\right), \qquad (C.38)$$

$$\frac{d^2}{d\xi^2} f_3 = R^{2/3}\, G_9(\xi) + O\left(R^{1/3}\log_e R\right),$$

$$\frac{d^4}{d\xi^4}\chi^{(2)}_{1-2} = R^{2/3}\, G_{10}(\xi) + O\left(R^{1/3}\log_e R\right),$$

where, on recalling that ξ is purely imaginary, we have

$$G_{2+k} = -i\left(\beta^2/\alpha\gamma^2\right)\left|C_k\right|^2 2^{5/3}\,\mathrm{Im}\left[\frac{d}{d\xi}L_1(\xi)\right] ,$$

$$G_{4+k} = i(-1)^k\,(\beta/\gamma^2)\,\left|C_k\right|^2 2^{2/3}\,\mathrm{Im}\left[\frac{d}{d\xi}L_1(\xi)\right] ,$$

$$G_{6+k} = \left|C_k\right|^2\left(4/\alpha\right)^{1/3}\left(D^2\bar{u}^0/(D\bar{u}^0)^{4/3}\right)_c \mathrm{Re}\left[\left(L_1(\xi)\right)^*\right] ,$$

$$G_9 = 2\left|C_3\right|^2\alpha^{-4/3}\left(D^2\bar{u}^0/(D\bar{u}^0)^{4/3}\right)_c \mathrm{Re}\left[\,L(\xi)\right] ,$$

$$G_{10} = i\beta^2\left|C_1\right|^2\left(4k^2/\alpha D\bar{u}^0_c\right)^{1/3}\gamma^{-2}\left\{\frac{d^2}{d\xi^2}\left(L_1(\xi)+\left[L_1(\xi)\right]^*\right)\right.$$
$$\left. -\beta^2\gamma^{-2}2^{5/3}\frac{d}{d\xi}\left(L_1(\xi)\left[L_1(\xi)\right]^*\right)\right\} .$$

Integration of (C.38) yields

$$\left.\begin{aligned} f_k &= k\int_0^{\xi}\int_0^{\xi_1} G_{2+k}(\xi_2)\,d\xi_2\,d\xi_1 + E_{2+k}\,\xi + F_{2+k} + O\left(k^{2/3}\log_e k\right) , \\ h_k &= k\int_0^{\xi}\int_0^{\xi_1} G_{4+k}(\xi_2)\,d\xi_2\,d\xi_1 + E_{4+k}\,\xi + F_{4+k} + O\left(k^{2/3}\log_e k\right) \end{aligned}\right] \quad (C.40)$$

$$\frac{\alpha}{2}f_k + (-1)^{k+1}\beta h_k = k^{2/3}\int_0^{\xi}\int_0^{\xi_1} G_{6+k}(\xi_2)\,d\xi_2\,d\xi_1 + E_{6+k}\,\xi + F_{6+k} + O\left(k^{1/3}\log_e k\right)$$

$$f_3 = k^{2/3}\int_0^{\xi}\int_0^{\xi_1} G_9(\xi_2)\,d\xi_2\,d\xi_1 + E_9\,\xi + F_9 + O\left(k^{1/3}\log_e k\right)$$

(C.41 a, b, c)

$$\chi^{(2)}_{1-2} = k^{2/3}\int_0^{\xi}\int_0^{\xi_4}\int_0^{\xi_3}\int_0^{\xi_2} G_{10}(\xi_1)\,d\xi_1\,d\xi_2\,d\xi_3\,d\xi_4 + E_{10}\,\xi^3 + F_{10}\,\xi^2$$

$$+\,C_{10}\,\xi + D_{10} + O\left(k^{1/3}\log_e k\right) .$$

Here, the (complex) constants of integration (E_n, F_n, C_{10}, D_{10}) are obtained from matching with the outer estimates (C.7 a,b,d,e) as $\xi \to \pm i\infty$. In particular some part of these terms will match onto the regular parts of the outer solutions, which are all O(R). Consequently, the regular part of the outer solutions for $\frac{1}{2}\alpha f_k + (-1)^{k+1}\beta h_k$, f_3 and $\chi^{(2)}_{1-2}$ remains dominant in the critical layer; while for f_k and h_k the complete viscous estimates (C.40) are required in the critical-layer region. Accordingly, to highest order we may take $\frac{1}{2}\alpha f_k + (-1)^{k+1}\beta h_k$, f_3 and $\chi^{(2)}_{1-2}$ to be O(R) and constant in the critical layer, but determination of their precise values would require solution of the complete problem. As for $\chi^{(1)}_j$, some care is necessary when dealing with the derivatives of these functions in the critical layer.

Since $\chi^{(2)}_{1-2}$ is constant and O(R) in the critical layer, the critical-layer approximation to (B.12) is

$$\frac{d^2}{d\xi^2}\phi^{(2)}_{1-2} = G\, R^{4/3} + O(R \log_e R) \,, \tag{C.42}$$

where G is a complex constant, O(1) in magnitude, which depends on the value of $\chi^{(2)}_{1-2}$ in the critical layer. Result (C.42) is consistent with the outer solution for the second derivative of the regular part of $\phi_{1-2,0}(x_3)$, which is $O(R^2)$ (see (C.11)).

Integration of (C.42) yields

$$\phi^{(2)}_{1-2} = G R^{4/3}\frac{\xi^2}{2} + H\xi + C + O(R\log_e R) \,, \tag{C.43}$$

with the (complex) constants of integration H, C obtained from matching with the $O(R^2)$ regular outer solution. Clearly $|H| \sim O(R^{5/3})$ and $|C| \sim O(R^2)$. The part of the inner estimate for $\phi^{(2)}_{1-2}$ which matches onto the $O(R^2)$ singular outer solution arises from the first term in (C.41 c), but is dominated in the critical layer by the terms in (C.43). Accordingly, to highest order we may take $\phi^{(2)}_{1-2}$ and $\frac{d^r}{dx_3^r}\phi^{(2)}_{1-2}$ $(r = 1,2)$ to be $O(R^2)$ and constant in the critical layer. So, when $|\xi|$ is $O(1)$,

$$\begin{aligned} \phi^{(2)}_{1-2} &= CR^2 + o(R^{5/3}) , \\ \frac{d}{d\xi}\phi^{(2)}_{1-2} &= IR^{5/3} + o(R^{4/3}) , \end{aligned} \qquad (C.44)$$

where I and C are now (complex) constants, $O(1)$ in magnitude.

A preliminary analysis of the case when $D^2\bar{u}_c = 0$ suggests that the magnitudes of $\chi^{(2)}_k$ and $\chi^{(2)}_{jj}$ are reduced in the critical layer.

The various estimates derived in this appendix are used in §3.3 to determine the orders of magnitude of the interaction coefficients a_i, a_{ij} for large αR.

APPENDIX D

DEFINITION OF FUNCTIONS REQUIRED IN §3.4

$$R_{1,4}(\xi) \equiv \frac{d}{d\xi} h^{(2)}_{1,2}(\xi) . \int_{+i\infty}^{\xi} h^{(1)}_{1,2}(\xi_1) \frac{d}{d\xi_1} L_1(\xi_1) d\xi_1 - \frac{d}{d\xi} h^{(1)}_{1,2}(\xi) . \int_{-i\infty}^{\xi} h^{(2)}_{1,2}(\xi_1) \frac{d}{d\xi_1} L_1(\xi_1) d\xi_1 ,$$

$$R_2(\xi) \equiv h^{(2)}_{1/2}(\xi) \int_{+i\infty}^{\xi} h^{(1)}_{1/2}(\xi_1) L_1(\xi_1) d\xi_1 - h^{(1)}_{1/2}(\xi) \int_{-i\infty}^{\xi} h^{(2)}_{1/2}(\xi_1) L_1(\xi_1) d\xi_1 ,$$

$$R_3(\xi) \equiv h^{(2)}_{1}(\xi) \int_{+i\infty}^{\xi} \left(2^{5/3} \beta^2 \gamma^{-1} L_1(\xi_1) \frac{d}{d\xi_1} L_1(\xi_1) - i \frac{d^2}{d\xi_1^2} L_1(\xi_1) \right) h^{(1)}_{1}(\xi_1) d\xi_1$$

$$- h^{(1)}_{1}(\xi) \int_{-i\infty}^{\xi} \left(2^{5/3} \beta^2 \gamma^{-1} L_1(\xi_1) \frac{d}{d\xi_1} L_1(\xi_1) - i \frac{d^2}{d\xi_1^2} L_1(\xi_1) \right) h^{(2)}_{1}(\xi_1) d\xi_1 ,$$

$$R_5(\xi) \equiv h^{(2)}_{3/2}(\xi) \int_{+i\infty}^{\xi} h^{(1)}_{3/2}(\xi_1) \frac{d^2}{d\xi_1^2} L_1(\xi_1) d\xi_1 - h^{(1)}_{3/2} \int_{-i\infty}^{\xi} h^{(2)}_{3/2}(\xi_1) \frac{d^2}{d\xi_1^2} L_1(\xi_1) d\xi_1 ,$$

$$R_6(\xi) \equiv 2\left(18^{1/3}\right) \left(\frac{3}{4}\alpha^2 + \beta^2 \right) \frac{d}{d\xi_1} L_1(\xi_1) + i\pi\beta^2 \int_0^{\xi_1} \int_0^{\xi_2} R_5(\xi_3) d\xi_3 d\xi_2 ,$$

$$R_7(\xi) \equiv h^{(2)}_{3/2}(\xi) \int_{+i\infty}^{\xi} h^{(1)}_{3/2}(\xi_1) R_6(\xi_1) d\xi_1 - h^{(1)}_{3/2}(\xi) \int_{-i\infty}^{\xi} h^{(2)}_{3/2}(\xi_1) R_6(\xi_1) d\xi_1 ,$$

$$R_8(\xi) \equiv \left(\beta^2/\gamma\gamma_0^2\eta^{1/3}\right)\Big\{\left(\beta^2-\tfrac{3}{4}\alpha^2\right)L_1^*(\xi)\,R_5(\xi)+\gamma_0^2 R_5(\xi)\frac{d^2}{d\xi_1^2}L_1^*(\xi)$$

$$+\left(\tfrac{3}{2}\alpha^2+\beta^2\right)R_5(\xi)\frac{dL_1^*}{d\xi}(\xi) - 18^{-1/3}\frac{d}{d\xi}\left(R_7(\xi)L_1^*(\xi)\right)\Big\}$$

$$-\,i2^{2/3}\pi\frac{d}{d\xi}L_1^*(\xi) \quad ,$$

$$J_1(\xi) \equiv L_1(\xi)\left\{\left(1+F_7|C_1|^{-2}\right)L_1(\xi)-12^{-1/3}i\pi R_1(\xi)\right\} ,$$

$$J_2(\xi) \equiv C\left(2\alpha^5\left(D\bar{u}_c^0\right)^{-1}\right)^{1/3}\beta^2C_1^2\gamma^{-1}L_1(\xi)\frac{d}{d\xi}L_1(\xi) \quad ,$$

$$J_3(\xi) \equiv -\frac{1}{4}C_1^2\pi D^2\bar{u}_c^0\left(\frac{1}{3}\alpha^{-1}\left(D\bar{u}_c^0\right)^{-4}\right)^{1/3}L_1(\xi)R_2(\xi) ,$$

$$J_4(\xi) \equiv \left\{C_1^2C_3\beta^2\pi\gamma^{-1}\left(\frac{1}{12}\alpha\left(D\bar{u}_c^0\right)^{-2}\right)^{1/3}L(\xi)\right\} \times$$

$$\left\{4^{1/3}R_3(\xi)+2^{5/3}\beta^2\gamma^{-1}\frac{d}{d\xi}\left(R_2(\xi)L_1(\xi)\right)-i\frac{d^2}{d\xi^2}R_2(\xi)\right\} ,$$

$$J_5(\xi) \equiv L(\xi)\left\{\left(i\alpha F_9|C_3|^{-2}\left(D\bar{u}_c^0\right)^{-1}+1\right)L(\xi)+\left(i\pi/4.3^{1/3}\right)R_4(\xi)\right\} ,$$

$$J_6(\xi) \equiv \frac{-i}{2}\left(\frac{1}{3}\alpha\left(D\bar{u}_c^0\right)^{-2}\right)^{1/3}\beta^2|C_1|^2C_3\pi\gamma^{-1}L_1(\xi)\frac{d^2}{d\xi^2}\left[R_1(\xi)\right]^* ,$$

$$J_7(\xi) \equiv iC_3^2|C_1|^2\beta^2\pi\left(\alpha^2/D\bar{u}_c^0\right)^{1/3}\gamma^{-1}L(\xi)R_8(\xi) \quad . \qquad (2\ \ )$$

Here, the inner independent variable ξ as defined by (3.56) is purely imaginary and $L(\xi)$, $L_1(\xi)$, $h_{1,2}^{(k)}(\xi)$, $h_{1/2}^{(k)}(\xi)$, $h_{3/2}^{(k)}(\xi)$ $(k = 1,2)$, are known complex-valued functions see (C.17), (C.18), (C.30)), C_j $(j = 1,3)$ is the value of the inviscid estimate for $\chi_j^{(1)}$ at x_{3c} (see (C.1a)), F_7, F_9 and C are complex constants possessing magnitudes of order unity, determined by matching the respective 'inner' and 'outer' estimates for $\frac{1}{2}\alpha f_1 + \beta h_1$, f_3 and $\Phi_{1-2}^{(2)}$ (see (C.41 a,b), (C.44)).

R E F E R E N C E S

BALL, F.K. 1964 Energy transfer between external and internal gravity waves.
J. Fluid Mech. 19, 465-478.

BATCHELOR, G.K. 1953 The Theory of Homogeneous Turbulence.
Cambridge University Press.

BATEMAN, H. 1956 Article entitled "Motion of an incompressible viscous fluid" in Hydrodynamics,
ed. H.L.Dryden, F.D.Murnaghan & H. Bateman.
Dover Publications, New York.

BENNEY, D.J. 1961 A non-linear theory for oscillations in a parallel flow.
J. Fluid Mech. 10, 209-236.

BENNEY, D.J. 1962 Non-linear gravity wave interactions.
J. Fluid Mech. 14, 577-584.

BENNEY, D.J. 1964 Finite amplitude effects in an unstable laminar boundary layer.
Phys. Fluids 7, 319-326.

BENNEY, D.J. & BERGERON, R.F. 1969
A new class of non-linear waves.
Studies in Appl. Math. 48, 181-204.

BENNEY, D.J. & LIN, C.C. 1960
On the secondary motion induced by oscillations in a shear flow.
Phys. Fluids 4, 656-657.

BRETHERTON, F.P. 1964 Resonant interactions between waves.
J. Fluid Mech. 20, 457-480.

BUSSE, F.H. 1969 Stability regions of cellular fluid low.
Proc. I.U.T.A.M. Symposium on "Instability of Continuous Systems", Herrenalb, 41-47.

CHANDRASEKHAR, S. 1961
Hydrodynamic and Hydromagnetic Stability.
Oxford, Clarendon.

CRAIK, A.D.D. 1966 Wind-generated waves in thin liquid films.
J. Fluid Mech. 26, 369-392.

CRAIK, A.D.D. 1968 Resonant gravity-wave interactions in a shear flow.
J. Fluid Mech. 34, 531-549.

CRAIK, A.D.D. 1971 Non-linear resonant instability in boundary layers.
J. Fluid Mech. 50, 393-413.

DAVEY, A. 1962 The growth of Taylor vortices in flows between rotating cylinders.
J. Fluid Mech. 14, 336-368.

DAVEY, A., DIPRIMA, R.C. & STUART, J.T. 1968
On the instability of Taylor vortices.
J. Fluid Mech. 31, 17-52.

DAVEY, A., HOCKING, L.M. & STEWARTSON, K. 1974
On the non-linear evolution of three-dimensional disturbances in plane Poiseuille flow.
J. Fluid Mech. 63, 529-536.

DAVIS, R.E. 1969 On the high Reynolds number flow over a wavy boundary.
J. Fluid Mech. 36, 337-346.

DIPRIMA, R.C., ECKHAUS, W. & SEGEL, L.C. 1971
Non-linear wave-number interaction in near-critical two-dimensional flows.
J. Fluid Mech. 49, 705-744.

FINLAYSON, B.A. 1972(a)
The Method of Weighted Residuals and Variational Principles. Academic Press, New York.

FINLAYSON, B.A. 1972 (b)
Existence of variational principles for the Navier-Stokes equations.
Phys. Fluids 15, 963-967.

HEISENBERG, W. 1924 Über Stabilität und Turbulenz von Flüssigkeitsstromen.
Ann. Phys., Lpz. (4), 74, 577-627.

HOCKING, L.M. & STEWARTSON, K. 1971
On the non-linear response of a marginally unstable plane parallel flow to a three-dimensional disturbance.
Mathematika 18, 219-239.

HOCKING, L.M. & STEWARTSON, K. 1972
On the non-linear response of a marginally unstable plane parallel flow to a two-dimensional disturbance.
Proc. Roy. Soc. A. 326, 289-313.

HOCKING, L.M., STEWARTSON, K. & STUART, J.T. 1972
A non-linear instability burst in plane parallel flow.
J. Fluid Mech. 51, 705-735.

INCE, E.L. 1926 Ordinary Differential Equations.
Dover Publications, New York.

KELLY, R.E. 1968 On the resonant interaction of neutral disturbances in inviscid shear flows.
J. Fluid Mech. 31, 789-800.

KELLY, R.E. & MASLOWE, S.A. 1970
The non-linear critical layer in a slightly stratified shear flow.
Studies in App. Maths. 49, 301-326.

KIRCHGÄSSNER, K. & SORGER, P. 1969
Branching analysis for the Taylor problem
Quart. J. Mech. and App. Maths. 22, 183-209.

KLEBANOFF, P.S. & TIDSTROM, K.D. 1959
Evolution of amplified waves leading to transition in a boundary layer with zero pressure gradient.
N.A.S.A. Tech. Note D - 195.

KLEBANOFF, P.S., TIDSTROM, K.D. & SARGENT, L.M. 1962
The three-dimensional nature of boundary-layer instability.
J. Fluid Mech. 12, 1 -34.

LANDAHL, M.T. 1972 Wave mechanics of breakdown.
J. Fluid Mech. 56, 775-802.

LANDAU, L.D. 1944 On the problem of turbulence.
Compt. rend. acad. sci. U.R.S.S. 44, 311-314.

LEBEDEV, N.N. 1965 Special Functions and their Applications.
Prentice-Hall, New Jersey.

LEE, L.H. & REYNOLDS, W.C. 1967
On the approximate and numerical solution of Orr-Sommerfeld problems.
Quart. J. Mech. and App. Maths. 20, 1-22.

LIN, C.C. 1955 The Theory of Hydrodynamic Stability.
Cambridge University Press.

LONGUET-HIGGINS, M.S. 1962
Resonant interactions between two trains of gravity waves.
J. Fluid Mech. 12, 321-332

LUKE, J.C. 1967 A variational principle for a fluid with a free surface.
J. Fluid Mech. 27, 395-398.

McGOLDRICK, L.F. 1965 Resonant interactions among capillary-gravity waves.
J. Fluid Mech. 21, 305-331.

MEKSYN, D. & STUART, J.T. 1951
Stability of viscous motion between parallel planes for finite disturbances.
Proc. Roy. Soc. A. 208, 517-526.

MILLIKAN, C.B. 1929 Steady motion of viscous incompressible fluids with particular reference to a variational principle.
Phil. Mag. 7, 641-662.

NEWELL, A.C. & WHITEHEAD, J.A. 1969
Finite bandwidth, finite amplitude convection.
J. Fluid Mech. 38, 279-303.

PEKERIS, C.L. & SHKOLLER, B. 1967
Stability of plane Poiseuille flow to periodic disturbances of finite amplitude in the vicinity of the neutral curve.
J. Fluid Mech. 29, 31-38.

PHILLIPS, O.M. 1960 On the dynamics of unsteady gravity waves of finite amplitude. Part.1. The elementary interactions.
J. Fluid Mech. 9, 193-217.

PLATTEN, J.K. 1971 On a variational formulation for hydrodynamic stability.
Int. J. Engng. Sci. 9, 37-48

RAETZ, G.S. 1959 A new theory of the cause of transition in fluid flows.
Norair Rep. NOR - 59 - 383, Hawthorne, Calif.

REID, W.H. 1965 Article entitled "The Stability of Parallel Flows" in Basic Developments in Fluid Mechanics, VOL. I., ed. M. Holt.
Academic Press, New York.

REYNOLDS, W.C. & POTTER, M.C. 1967
Finite amplitude instability of parallel shear flows.
J. Fluid Mech. 27, 465-492.

ROSS, J.A., BARNES, F.H., BURNS, J.G. & ROSS, M.A.S. 1970
The flat plate boundary layer. Part 3. Comparison of theory with experiment.
J. Fluid Mech. 43, 819-832.

SATO, H. 1956 Experimental investigation on the transition of laminar separated flow.
J. Phys. Soc. Jap. 11, 702-709.

SATO, H. 1960 The stability and transition of a two-dimensional jet.
J. Fluid Mech. 7, 53-80.

SCHECHTER, R.S. 1966 Article entitled "Variational principles for continuum systems" in Non-equilibrium Thermodynamics, Variational Techniques and Stability, ed. R.J.Donnelly, R. Herman & I. Prigogine. Chicago University Press.

SCHAUBAUER, G.B. & SKRAMSTAD, H.K. 1943 Laminar boundary layer oscillations and transition on a flat plate. N.A.C.A. Tech. Rep. 909, (1948) originally issued as N.A.C.A.A.C.R. (April 1943).

SEGEL, L.A. 1966 Article entitled "Non-linear hydrodynamic stability theory and its applications to thermal convection and curved flows". in Non-equilibrium Thermodynamics, Variational Techniques and Stability, ed. R.J.Donnelly, R. Herman & I. Prigogine. Chicago University Press.

SIMMONS, W.F. 1969 A variational method for weak resonant wave interactions. Proc. Roy. Soc. A. 309, 551-575.

SQUIRE, H.B. 1933 On the stability for three-dimensional disturbances of viscous fluid flow between parallel walls. Proc. Roy. Soc. A. 142, 621-628.

STEWARTSON, K. & STUART, J.T. 1971 A non-linear instability theory for a wave system in planc Poiseuille flow. J. Fluid Mech. 48, 529-545.

STUART, J.T. 1958 On the non-linear mechanics of hydrodynamic stability. J. Fluid Mech. 4, 1-21.

STUART, J.T. 1960 On the non-linear mechanics of wave disturbances in stable and unstable parallel flows. Part. 1. The basic behaviour in plane Poiseuille flow. J. Fluid Mech. 9, 353-370.

STUART, J.T. 1962(a) On three-dimensional non-linear effects in the stability of parallel flows. Adv. Aero Sci. 3-4, 121-142.

STUART, J.T. 1962(b) Non-linear effects in hydrodynamic stability. Proc. 10th Int. Cong. App. Mech. (Stresa 1960) 63-97. Elsevier.

STUART, J.T. 1971 Non-linear stability theory. Annual Review of Fluid Mech. 3, 347-370.

TANI, I. 1969 Boundary layer transition.
Annual Review of Fluid Mech. 1, 169-196.

TAYLOR, G.I. 1923 Stability of a viscous liquid contained between two rotating cylinders.
Phil. Tran. Roy. Soc. A. 223, 289-343.

USHER, J.R. & CRAIK, A.D.D. I (to appear)
Non-linear wave interactions in shear flows
Part I: a variational formulation.

USHER, J.R. & CRAIK, A.D.D. II (to appear)
Non-linear wave interactions in shear flows
Part II: third-order theory.

WATANABE, T. 1969 A non-linear theory of two-stream instability.
J. Phys. Soc. Jap. 27, 1341-1350.

WATSON, J. 1960 On the non-linear mechanics of wave disturbances in stable and unstable parallel flows. Part. 2. The development of a solution for plane Poiseuille flow and for plane Couette flow.
J. Fluid Mech. 9, 371-390.

WHITHAM, G.B. 1965(a) Non-linear dispersive waves.
Proc. Roy. Soc. A. 283, 238-261.

WHITHAM, G.B. 1965(b) A general approach to linear and non-linear dispersive waves using a Lagrangian
J. Fluid Mech. 22, 273-283.

WHITHAM, G.B. 1967 Non-linear dispersion of water waves.
J. Fluid Mech. 27, 399-412.