INVESTIGATIONS ON CLASSICAL SYMMETRIES THEORY OF QUANTIZATION

P. B. Guest

A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

1972

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INVESTIGATIONS ON CLASSICAL SYMMETRIES.

THEORY OF QUANTIZATION.

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I hereby declare that the accompanying thesis is my own composition, that it is based upon research carried out by me and that no part of it has previously been presented in application for a higher degree.

Under Ordinance General No. 12 I was admitted as a research student in the Department of Theoretical Physics, University of St. Andrews, on October 1, 1966 and as a candidate for the Degree of Doctor of Philosophy on October 1, 1967.

I hereby certify that Paul Barrie Guest, B.Sc., M.Sc., has spent nine terms as a research student under my supervision in the Department of Theoretical Physics of the United College of St. Salvator and St. Leonard in the University of St. Andrews, has fulfilled the conditions of Resolution of the University Court, 1967, No. 1 and is qualified to submit the accompanying thesis in application for the Degree of Doctor of Philosophy.

Supervisor

PREFACE

The main ideas for the work included in this volume were formulated while I was holding an S.R.C. Research Studentship in the Department of Theoretical Physics, University of St. Andrews, during the period October 1967 - March 1970. The tidying up of these ideas (the hardest part) was completed under the auspices of the Department of Mathematics, University of Canterbury, New Zealand.

The thesis divides naturally into two parts. Part I raises, and in some cases answers, questions concerning symmetry in classical mechanics. §1-§4 is largely an attempt to tidy-up some left-overs from my earlier M.Sc. thesis. §5 and §6 treat the problem of the correspondence between Lie groups of transformations and realizations of Lie algebras in terms of infinitely-differentiable functions on phase space. The main result (Theorem *6.4*) shows that the assumption of the existence of a realization puts an upper limit on the rank of the algebra.

The heart of the thesis (covering three-quarters of the volume) is section II on the quantization of classical systems. §1 lists axioms desirable in any quantization rule for the 'functions of the q's'. The momentum observables are introduced in §2 prior to their quantization in §4. §5 essentially shows how conventional quantum mechanics fits into this scheme of things. By progressive specialization from a general manifold to a vector space, from a general quantization scheme to one which is linear on the linear momentum functions, and finally to an entirely well-behaved (admissible) quantization rule, into which conventional quantum mechanics fits nicely, we obtain in §7-§9 results which become progressively more and more powerful. The final theorem (Theorem *9.2*) is perhaps the most significant of all. This result states that there exists a class of functions, which contains all functions of the q's and functions of the p's and all momentum observables and which is closed with respect to any linear canonical transformation L; a rule Λ assigning a unique self-adjoint operator to each such function f; a unitary operator W_L corresponding to L and an equation $\Lambda(f \circ L) = W_L^{-1} \Lambda f W_L$.

I am indebted to the Science Research Council for the financial support for this work and to Professor R.B. Dingle for the gaining of that support in the first place and for numerous other things. Finally I wish to express my great appreciation for the generous help given by my supervisor, Dr. J.F. Cornwell, from the very inception of this work right to the very end.

Christchurch, New Zealand

March 4, 1972

P. B. GUEST

CONTENTS

ABSTRACT

The thesis divides naturally into two parts. Part I raises, and in some cases answers, questions concerning symmetry in classical mechanics. §1-§4 is largely an attempt to tidy-up some left-overs from my earlier M.Sc. thesis. §5 and §6 treat the problem of the correspondence between Lie groups of transformations and realizations of Lie algebras in terms of infinitely-differentiable functions on phase space. The main result (Theorem *6.4*) shows that the assumption of the existence of a realization puts an upper limit on the rank of the algebra.

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And I proceeded to give my heart to knowing wisdom and to knowing madness, and I have come to know folly, that this too is a striving after wind. For in the abundance of wisdom there is an abundance of vexation, so that he that increases knowledge increases pain.

King Solomon (*Ecclesiastes* 1: 17-18)

I

SYMMETRY IN CLASSICAL MECHANICS

The concept of symmetry in classical mechanics is not new. Apart from the well-known three-dimensional rotational symmetry of a Hamiltonian with a spherically symmetric potential, the idea that other hidden symmetries may be present was recognized as long ago as 1933 when Klein (see Hulthen [1]) showed that the Hamiltonian for the Kepler problem possesses a symmetry group (locally) isomorphic to the four-dimensional rotation group. The Hamiltonian for the three-dimensional harmonic oscillator has been shown to have $SU(3)$ as a symmetry group (Bargmann [1]; Fradkin [1]). A representative review of these and other questions of classical symmetries will be found in the author's M.Sc. thesis (Guest [1]).

An attempt to generalize these results to a wider class of Hamiltonians and to more general groups has been the cause of much conjecture recently. In particular, a number of papers have been written on the problem of finding canonical realizations of Lie groups and their corresponding Lie algebras. A detailed general theory has been given by Pauri and Prosperi [1] concerning general Lie groups and by Mukunda [1] concerning semi-simple Lie algebras. Specializations to the rotation group (Pauri and Prosperi [2]) and the Galilei group (Pauri and Prosperi [3]) have also been given. Almost all of the literature, however, is implicitly concerned with local properties; indeed, it is a topic of great concern of deciding when a given local realization of a Lie algebra can be extended and hence defined globally. See, for example, Guest and Bors [1] on the problem of the $O(4)$ symmetry of the hydrogen atom.

For the present we shall restrict ourselves to the case of a single particle moving throughout three-dimensional space $\mathbf{R}^3$ under the influence of a central potential V. The classical Hamiltonian in Cartesian coordinates will be

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + V(r) \tag{1}$$

and in polar coordinates

$$H = \frac{1}{2m}(p_r^2 + \frac{p_\theta{}^2}{r^2} + \frac{p_\phi{}^2}{r^2 \sin^2\theta}) + V(r) \tag{2}$$

Here m is the mass of the particle, (q_1, q_2, q_3) and (p_1, p_2, p_3) are the position and momentum coordinates; (r, θ, ϕ), (p_r, p_θ, p_ϕ) are defined by

$$\begin{aligned} q_1 &= r \sin\theta \cos\phi \\ q_2 &= r \sin\theta \sin\phi \\ q_3 &= r \cos\theta, \end{aligned} \tag{3}$$

$$p_r = m\frac{dr}{dt}, \quad p_\theta = mr^2\frac{d\theta}{dt}, \quad p_\phi = mr^2 \sin^2\theta\frac{d\phi}{dt}, \tag{4}$$

where t is the time. We have the following relations among the variables:

$$\begin{aligned} rp_r &= p_1q_1 + p_2q_2 + p_3q_3, \\ \sqrt{q_1^2 + q_2^2}\; p_\theta &= r(q_3p_r - rp_3) \\ p_\phi &= q_1p_2 - q_2p_1. \end{aligned} \tag{5}$$

The Poisson Bracket of two suitably differentiable functions f and g of the six variables $(q_1, q_2, q_3, p_1, p_2, p_3)$ can be defined by

$$[\![f, g]\!] = \sum_{i=1}^{3}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right). \tag{6}$$

§1. Definition of the vectors $\underline{B}$ and $\underline{C}$.

The fact that higher symmetries in classical mechanics are connected with extra constants of motion suggests that, in the case of a single particle at least, we attempt to analyse the set of all constants of motion possessed by a given system. Since a function f is a constant of motion iff $[\![f, H]\!] = 0$, this equation will, in principle, provide us with all the information we require. With H given by *(2)* we obtain

$$p_r \frac{\partial f}{\partial r} + \frac{p_\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{p_\phi}{r^2 \sin^2\theta}\frac{\partial f}{\partial \phi} + \left[\frac{{p_\theta}^2}{r^3} + \frac{{p_\phi}^2}{r^3\sin^2\theta} - mV'(r)\right]\frac{\partial f}{\partial p_r} + \frac{{p_\phi}^2\cos\theta}{r^2\sin^3\theta}\frac{\partial f}{\partial p_\theta} = 0\ . \tag{7}$$

We now briefly review the properties of the vectors $\underline{B}$ and $\underline{C}$ which are constants of motion obtained from the solutions of this equation. Define the angular momentum vector $\underline{L}$ by

$$\underline{L} = \underline{r} \times \underline{p}; \tag{8}$$

then

$$\underline{B} = \frac{1}{r}[(\underline{r} \times \underline{L})\cos G - \underline{r}L\sin G], \tag{9}$$

$$\underline{C} = \frac{1}{r}[(\underline{r} \times \underline{L})\sin G + \underline{r}L\cos G], \tag{10}$$

where $L = \sqrt{L_1^2 + L_2^2 + L_3^2}$ and

$$G = \int \frac{dr}{r\sqrt{\frac{2m}{L^2}(E - V(r))r^2 - 1}}\ , \tag{11}$$

L and $E = H$ being considered as constants in the integration. (Guest [1])*.

The following relations exist between $\underline{L}$, $\underline{B}$ and $\underline{C}$:

* $\underline{B}$ as defined here is equal to minus that defined in this reference.

$$\frac{1}{L}(\underline{B} \times \underline{C}) = \underline{L},$$
$$\frac{1}{L}(\underline{C} \times \underline{L}) = \underline{B}, \qquad (12)$$
$$\frac{1}{L}(\underline{L} \times \underline{B}) = \underline{C}.$$

In addition, all rows and columns of the matrix

$$\frac{1}{L}\begin{pmatrix} L_1 & L_2 & L_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} \qquad (13)$$

are orthonormal. It can be verified that the following Poisson Bracket commutation relations are satisfied:

$$[\![L_i, L_j]\!] = L_k,$$

$$[\![L_i, B_j]\!] = B_k, \qquad [\![B_i, L_j]\!] = B_k, \qquad (14)$$

$$[\![B_i, B_j]\!] = -L_k,$$

$$[\![L_i, C_j]\!] = C_k, \qquad [\![C_i, L_j]\!] = C_k, \qquad (15)$$

$$[\![C_i, C_j]\!] = -L_k,$$

where (i, j, k) is in cyclic order. Thus the sets $\{L_1, L_2, L_3, B_1, B_2, B_3\}$ and $\{L_1, L_2, L_3, C_1, C_2, C_3\}$ are each realizations of D_2, the Lie algebra of the inhomogeneous Lorentz group. Furthermore, using the relations *(12)*, it can be verified that

$$[\![B_i, L]\!] = C_i, \qquad [\![C_i, L]\!] = -B_i,$$

$$[\![B_i, C_i]\!] = L, \tag{16}$$

$$[\![B_i, C_j]\!] = 0, \ (i \neq j)$$

showing that (formally, at least) we have a ten-dimensional algebra spanned by the set $\{L, L_1, L_2, L_3, B_1, B_2, B_3, C_1, C_2, C_3\}$ and containing two copies of D_2 as subalgebras.

§2. The Case of Circular Orbits.

All of this argument is purely formal; indeed, when we look more closely at the actual functions B_i, C_i, we find that they are not at all well-behaved functions in general; in one case, at least, the component B_3 possesses a discontinuity at each of an infinite set of points in phase space. For an algebra of functions defined on phase space to generate a group of transformations, it is necessary that the functions be real-valued and infinitely differentiable and that their respective vector fields be complete. While the vectors $\underline{B}$ and $\underline{C}$ are certainly real where they are defined, the second condition does not hold while the third is entirely fortuitous at this stage.

It is found that the main difficulty occurs with Hamiltonians for which circular motion is possible: the integrand of G is $L/(r^2 p_r)$ which is not defined for circular orbits $(p_r = 0)$. Consequently, we are led to define a new vector constant of motion which is closely related to the vectors $\underline{B}$ and $\underline{C}$ and which vanishes for circular motion. The simplest form for such a vector will be

$$\underline{K} = \psi_1(E, L)\underline{B} + \psi_2(E, L)\underline{C}. \qquad (17)$$

We require also that the components of this vector shall satisfy the same commutation relations as those of $\underline{B}$ or $\underline{C}$. This last condition puts the following restriction on the functions ψ_1, ψ_2:

$$\psi_1^2 + \psi_2^2 = 1 + \frac{h(E)}{L^2} \qquad (18)$$

for some function h of E. With this understanding, the set $\{L_1, L_2, L_3, K_1, K_2, K_3\}$ is a realization of the Lie algebra D_2. If we introduce another vector $\underline{J}$, linearly independent of $\underline{K}$, and of the form

$$\underline{J} = \phi_1(E, L)\underline{B} + \phi_2(E, L)\underline{C},$$

with $\phi_1^2 + \phi_2^2 = 1 + \dfrac{l(E)}{L^2}$,

for some function l of E, and demand that the set $\{L, L_1, L_2, L_3, K_1, K_2, K_3, J_1, J_2, J_3\}$ shall be a ten-dimensional algebra with the additional commutation relations the analogues of *(16)*, we find that we cannot do so unless $h = 0$. The first and second of equations *(16)*, for example, would have the analogues

$$[\![K_i,\ L]\!] = J_i, \quad [\![J_i,\ L]\!] = -K_i.$$

For these equations to hold, it is straightforward to verify that we must have

$$\phi_1 = -\psi_2,\ \phi_2 = \psi_1.$$

However, using these values for ϕ_1 and ϕ_2, we find that

$$[\![K_i,\ J_i]\!] = \left(\psi_1^2 + \psi_2^2 + (B_i^2 + C_i^2)\frac{\partial}{\partial L^2}(\psi_1^2 + \psi_2^2)\right)L$$

$$= (1 + \frac{hL_i^{\,2}}{L^4})L.$$

This equation is the analogue of the third of equations *(16)*. We obtain an algebra iff $h = 0$. Since ψ_1 and ψ_2 are each required to vanish for circular orbits, it is clear that we cannot take h to be zero. Thus the generators of the symmetry group have been made continuous at the expense of destroying the supposed higher symmetry given by equations *(16)*. It may be remarked here that for Hamiltonians for which circular motion is not possible, we are free to take $h = 0$ in *(18)* and in this case the higher symmetry is preserved. The most obvious examples are the free particle and a particle in a repulsive Coulomb potential; in each case $\underline{B}$ and $\underline{C}$ may be evaluated explicitly and are found to be infinitely differentiable (almost) everywhere.

The actual determination of h in the general case is difficult. Since for a circular orbit $p_r = 0$, we must have

$$H_c = \frac{L_c^2}{2mr_c^2} + V(r_c). \tag{19}$$

Also, Hamilton's equations of motion give

$$mV'(r_c) = \frac{L_c^2}{r_c^3}, \tag{20}$$

where the subscript c refers to the case of a circular orbit. Substitution of (20) into (19) gives

$$H_c = \tfrac{1}{2}r_c V'(r_c) + V(r_c). \tag{21}$$

Formally, we can argue as follows: Elimination of r_c between (20) and (21) will give a relation between H_c and L_c^2 from which can be obtained, on solving for L_c^2, an expression involving H_c only. If we now write E for H_c and $h(E)$ for $-L_c^2$ we obtain an expression $h(E)$ which is equal to $-L^2$ in the case of circular orbits. Thus $1 + h(E)/L^2$ will be a continuous function which vanishes for circular orbits. Identifying the function h in this manner, we can now look upon r_c as a parameter ω having range $(0, \infty)$; (20) and (21) can therefore be written

$$h = -m\omega^3 V'(\omega) \tag{22}$$

$$E = \tfrac{1}{2}\omega V'(\omega) + V(\omega), \tag{23}$$

which gives a parametric representation of $h(E)$. A necessary and sufficient condition for $h(E)$ to be well-defined is that whenever $E(\omega_1) = E(\omega_2)$ for some values ω_1, ω_2 of ω, then $h(\omega_1) = h(\omega_2)$. This is clearly satisfied if $E(\omega)$ is a strictly increasing or strictly decreasing function. Furthermore, $h(E)$, as given by (22) and (23), is defined only for those values of E assumed by the right side of (23). These values may not exhaust the totality of possible values as given by the Hamiltonian. In this case, we may extend h to these values also, at the same time preserving if possible the properties of continuity, differentiability, etc..

We note that

$$\begin{aligned}\frac{dh}{d\omega} &= -m\omega^2\left[\omega V''(\omega) + 3V'(\omega)\right]\\ &= -2m\omega^2\frac{dE}{d\omega},\end{aligned}$$

so that

$$\frac{dh}{dE} = -2m\omega^2$$

or

$$\omega = \sqrt{\frac{d\rho}{dE}}, \tag{24}$$

where $\rho = -h/2m$.

Substitution in *(22)* and *(23)* gives differential equations for ρ:

$$\rho = \tfrac{1}{2}\left(\frac{d\rho}{dE}\right)^{3/2} V'\left(\sqrt{\frac{d\rho}{dE}}\right) \tag{25}$$

$$E = \tfrac{1}{2}\sqrt{\frac{d\rho}{dE}}\, V'\left(\sqrt{\frac{d\rho}{dE}}\right) + V\left(\sqrt{\frac{d\rho}{dE}}\right), \tag{26}$$

which, however, are of more academic than practical interest.

Solving *(23)* for V, we find

$$V(\omega) = \frac{2}{\omega^2}\int \omega E(\omega)\,d\omega + \frac{k}{\omega^2}, \tag{27}$$

where k is arbitrary. Choosing $E(\omega)$ to be the 1 - 1 function

$$E(\omega) = a(1 + \tfrac{s}{2})\omega^s, \tag{28}$$

where $s \in (-\infty, -2) \cup (-2, 0) \cup (0, \infty)$ and $a \neq 0$, so that we can be sure that $h(E)$ is defined, we obtain from *(27)*

$$V(\omega) = a\omega^s + \frac{k}{\omega^2}\ . \tag{29}$$

(22) now gives

$$h(\omega) = -mas\omega^{s+2} + 2mk,$$

or, from *(28)*,

$$h(E) = -mas\left[\frac{E}{a(1 + \frac{s}{2})}\right]^{1 + \frac{2}{s}} + 2mk. \tag{30}$$

If we let

$$E(\omega) = a/2\omega^2, \qquad (a \neq 0) \tag{31}$$

we obtain from (27)

$$V(\omega) = \frac{a}{\omega^2}\ln\omega + \frac{k}{\omega^2}, \tag{32}$$

and from (22),

$$h(\omega) = 2ma\ln\omega - ma + 2mk,$$

i.e.

$$h(E) = ma\ln(a/2E) - ma + 2mk. \tag{33}$$

In the case

$$E(\omega) = 0, \tag{34}$$

we obtain from (27)

$$V(\omega) = k/\omega^2. \tag{35}$$

Substitution in (22) now gives

$$h(\omega) = 2mk,$$

so that $h(E)$ can be defined in this case also:

$$h(E) = 2mk. \tag{36}$$

With the values $k = 0$, $s = 2$, (30) gives the function h in the case of the oscillator:

$$V(\omega) = a\omega^2,$$

$$h(E) = \frac{-mE^2}{2a} \tag{37}$$

and in this case the range of the right side of *(28)* is equal to that of the Hamiltonian. Similarly, the values $k = 0$, $s = -1$, give

$$V(\omega) = a/\omega$$

$$h(E) = ma^2/2E. \qquad (38)$$

The right side of *(28)* is in this case equal to $a/2\omega$ and if $a < 0$ its range is equal to $(-\infty, 0)$ so that *(38)* is defined initially only for negative values of E. However, there is no continuous extension of h for positive values of E; indeed, it is well-known that the hydrogen atom possesses two distinct symmetry groups according as the energy is less than or greater than zero.

§3. The Coulomb Potential.

With $V(r) = -\alpha^2/r$, we obtain from *(11)*

$$G = -\sin^{-1}\left(\frac{L^2 - m\alpha^2 r}{r\sqrt{m^2\alpha^2 + 2mEL^2}}\right),$$

and substitution in *(9)* gives

$$\underline{B} = \frac{L(\underline{p} \times \underline{L} - \frac{m\alpha^2}{r}\underline{r})}{\sqrt{m^2\alpha^2 + 2mEL^2}} \quad . \qquad (39)$$

Multiplication by $\sqrt{1 + \frac{m\alpha^4}{2EL^2}}$, obtained from *(38)*, gives the expression

$$\frac{1}{\sqrt{2mE}}\ (\underline{p} \times \underline{L} - \frac{m\alpha^2}{r}\ \underline{r}).$$

As a function depending on E, this vector is pure imaginary for $E < 0$ and real for $E > 0$. If we define

$$\underline{B} = \frac{1}{\sqrt{2m|E|}}\ (\underline{p} \times \underline{L} - \frac{m\alpha^2}{r}\ \underline{r}), \tag{40}$$

$$\underline{C} = \frac{1}{\sqrt{2m|E|}}\ [L\underline{p} - \frac{m\alpha^2}{rL}\ (\underline{L} \times \underline{r})], \tag{41}$$

where $E \neq 0$, and $\underline{C} = \frac{1}{L}(\underline{L} \times \underline{B})$, we obtain for $E > 0$ two copies of the algebra D_2, namely $\{L_1, L_2, L_3, B_1, B_2, B_3\}$ and $\{L_1, L_2, L_3, C_1, C_2, C_3\}$, the generators of which are continuous functions differentiable almost everywhere. In the region $E < 0$ we obtain two copies of the compact form of D_2 (the Lie algebra of O_4). $\underline{B}$ as given by *(40)* is of course the Lenz vector (Lenz [1]).

For future use we remark here upon the commutation relations between $\underline{B}$ and $\underline{C}$. With $\psi_1 = \sqrt{1 + m\alpha^4/2EL^2}$, $\psi_2 = 0$ in *(17)*, it follows from the discussion following *(18)* that the relations

$$[\![B_i, L]\!] = C_i, \quad [\![C_i, L]\!] = -B_i \tag{42}$$

are satisfied.

§4. The Two-Body Problem.

The Hamiltonian for a system of two particles of masses m_1 and m_2, moving under a potential depending only upon the distance apart of the particles is

$$H = \frac{1}{2\tilde{m}} \left(p_{\tilde{r}}^2 + \frac{p_{\tilde{\theta}}^2}{\tilde{r}^2} + \frac{p_{\tilde{\phi}}^2}{\tilde{r}^2 \sin^2 \tilde{\theta}}\right)$$

$$+ \frac{1}{2m} \left(p_r + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta}\right) + V(r) \; . \qquad (43)$$

Here r, θ, ϕ, etc. refer to the relative coordinates and $\tilde{r}$, $\tilde{\theta}$, $\tilde{\phi}$, etc. to the centre of mass coordinates; $m = m_1 m_2/(m_1 + m_2)$ is the reduced mass; $\tilde{m} = m_1 + m_2$ is the total mass.

The constants of motion can be found by solving an equation similar to *(7)*; they can be conveniently written in the following form:

$$\left.\begin{array}{l} l_1,\ l_2,\ l_3;\quad b_1,\ b_2,\ b_3;\quad c_1,\ c_2,\ c_3;\ \mathcal{E}; \\ \tilde{l}_1,\ \tilde{l}_2,\ \tilde{l}_3;\quad \tilde{b}_1,\ \tilde{b}_2,\ \tilde{b}_3;\quad \tilde{c}_1,\ \tilde{c}_2,\ \tilde{c}_3;\ \tilde{\mathcal{E}}; \end{array}\right. \eta . \qquad (44)$$

For the relative motion, l_1, l_2, l_3 are the angular momentu components; b_1, b_2, b_3 and c_1, c_2, c_3 are the components of the vectors $\underline{b}$ and $\underline{c}$ analogous to the vectors $\underline{B}$ and $\underline{C}$ defined in *(9)* and *(10)*, respectively; $\mathcal{E}$ is the energy. Similarly, the components $\tilde{l}_1$, $\tilde{l}_2$, $\tilde{l}_3$, etc. are the corresponding quantities for the centre of mass motion.

The scalar constant of motion η is peculiar to the two-body problem:

$$\eta = \frac{\tilde{r}p_{\tilde{r}}}{2\tilde{\mathcal{E}}} - \frac{m}{l}\int \frac{r\,dr}{\sqrt{\frac{2m}{l^2}[\mathcal{E} - V(r)]r^2 - 1}}\,, \tag{45}$$

where $\mathcal{E}$ and l are considered as constants in the integration. The commutation relations between $\underline{l}$, $\underline{b}$, $\underline{c}$ and $\mathcal{E}$ and between $\underline{\tilde{l}}$, $\underline{\tilde{b}}$, $\underline{\tilde{c}}$ and $\tilde{\mathcal{E}}$ are the same as those in §1; in addition, any quantity from the first set commutes with any quantity from the second set. Also, we have

$$\begin{aligned} [\![\eta, l_i]\!] = [\![\eta, b_i]\!] = [\![\eta, c_i]\!] = [\![\eta, \tilde{l}_i]\!] = [\![\eta, \tilde{b}_i]\!] \\ = [\![\eta, \tilde{c}_i]\!] = 0, \end{aligned} \tag{46}$$

$$[\![\mathcal{E}, \eta]\!] = -[\![\tilde{\mathcal{E}}, \eta]\!] = 1.$$

There are several ways in which η can be incorporated in a Lie algebra containing some or all of these functions. We could define

$$\eta_1 = \sin\eta, \qquad \eta_2 = \cos\eta,$$

to obtain

$$[\![\mathcal{E}, \eta_1]\!] = -[\![\tilde{\mathcal{E}}, \eta_1]\!] = \eta_2,$$

$$[\![\mathcal{E}, \eta_2]\!] = -[\![\tilde{\mathcal{E}}, \eta_2]\!] = -\eta_1.$$

Alternatively, we could define

$$\rho_1 = \&\eta, \quad \rho_2 = \tilde{\&}\eta$$

to obtain

$$[\![\&, \rho_1]\!] = -[\![\tilde{\&}, \rho_1]\!] = \&,$$
$$[\![\&, \rho_2]\!] = -[\![\tilde{\&}, \rho_2]\!] = \tilde{\&}$$

The question whether a semi-simple symmetry higher than D_2 exists for the general two-body problem must remain unsolved because of the difficulty of dealing with the function η. In particular, it would be interesting to see if globally-defined (except perhaps for zero relative energy) generators of a higher algebra could be constructed for the Coulomb two-body problem in a way analogous to that of §2. It is, however, doubtful that this would be the case; in fact, it seems plausible that almost all Hamiltonians have only the symmetry groups generated by themselves (see, for example Abraham [1], p.112). Notable exceptional cases are the Hamiltonians for several non-interacting particles and the Hamiltonians derived from these by canonical transformations. In this connection, we notice that for $V(r) = 0$ we have, from *(45)*,

$$\eta = \eta_0 = \tfrac{1}{2}\left(\frac{\tilde{r}\cdot\tilde{p}}{\tilde{\&}} - \frac{r\cdot p}{\&}\right) . \qquad (47)$$

η_0 is clearly a constant of motion which connects the relative and centre-of-mass motions and for non-interacting particles it is globally-defined and differentiable almost everywhere.

§5. Symmetry in Classical Mechanics.

The manifold of prime importance in classical mechanics is phase space. Geometrically, this has the structure of a differentiable sympletic $2n$-manifold which allows the occurrence of phase variables in canonically conjugate pairs. One of the most important operations admitted by this special structure of phase space is that of the Poisson Bracket.

In the usual applications of classical symmetries we consider a group of canonical transformations of phase space that leaves invariant a distinguished real-valued function called the Hamiltonian. The infinitesimal generators of this group are vector fields each of which defines (up to a constant) a real-valued function on phase space. The vector space spanned by the set of infinitesimal generators is, of course, a Lie algebra (with respect to the commutator) and the space spanned by the corresponding set of real-valued functions is a Lie algebra with respect to the Poisson Bracket. These two Lie algebras are isomorphic.

Let us state the situation more precisely. Here a manifold will always mean a Hausdorff, second countable C^{∞} differentiable manifold of constant dimension. In the rest of this section M will denote an arbitrary fixed symplectic $2n$-manifold.

Let $\mathbf{F}(M)$ denote the vector space over $\mathbf{R}$ (reals) of all C^∞ real-valued functions on M. A diffeomorphism of M is a 1 - 1 C^∞ function $M \to M$ whose inverse is also C^∞. D_M will denote the group of all diffeomorphisms of M. We shall say that $\phi : \mathbf{G} \to D_M$ is an action of a (finite dimensional) Lie group $\mathbf{G}$ on M if ϕ is a homomorphism and the mapping $\mathbf{G} \times M \to M$ given by $(g, m) \to \phi(g)m$ is C^∞.

5.1. *Definition.*

Let ϕ be an action of a Lie group $\mathbf{G}$ on M. If $H \in \mathbf{F}(M)$, $\mathbf{G}$ is a symmetry group of H iff $\phi(g)$ is symplectic for all $g \in \mathbf{G}$ (i.e. $\phi(g)$ is a homogeneous canonical transformation) and $H \circ \phi(g) = H$ for all $g \in \mathbf{G}$.

5.2. *Theorem*

Let ϕ be essential (i.e. 1 - 1 and into) and symplectic (i.e. $\phi(g)$ is symplectic for all $g \in \mathbf{G}$) and $l(\mathbf{G})$ the Lie algebra of $\mathbf{G}$. Then, for each point $m \in M$, there exists an open neighbourhood U of m and a monomorphism $\phi' : l(\mathbf{G}) \to \mathbf{F}(U)$ such that

$$[\![\phi'(x_1), \phi'(x_2)]\!] = -[x_1, x_2],$$

for all $x_1, x_2 \in l(\mathbf{G})$. (On the left we have the Poisson bracket operation in $\mathbf{F}(U)$ and on the right the bracket operation in $l(\mathbf{G})$.)

Proof. The proof is direct but will not be given here. See, for example Abraham [1], p. 148 et.seq..

This theorem shows that we can transfer the problem of finding groups of transformations to the problem of finding Lie algebras of functions in the sense that the existence of a particular group implies the existence of a set of functions that satisfy (up to a sign) the same commutation relations as those of the Lie algebra of the group. This is the attitude usually taken when one talks of symmetries in classical mechanics.

§6. Realizations of Lie algebras.

6.1. Definition

Let $\mathbf{A}$ *be a Lie algebra over* $\mathbf{R}$.

A realization ω *of* $\mathbf{A}$*, on* M*, is an isomorphism*

$$\omega : \mathbf{A} \to \omega(\mathbf{A}) \subset \mathbf{F}(M),$$

where

$$\omega[a_1, a_2] = [\![\omega a_1, \omega a_2]\!]$$

for all $a_1, a_2 \in \mathbf{A}$.

We shall write $G = \omega(\mathbf{A})$. Note that G is a real vector space having the same dimension as $\mathbf{A}$, thus ruling out the possibility that $\omega a = 0$ for some $a \neq 0$ and in particular the trivial 'representation' $\omega a = 0$ for all $a \in \mathbf{A}$.

If we assume that there exists a realization of a Lie algebra and if we restrict ourselves to those which are semi-simple, then a number of interesting results can be obtained. Necessary conditions of existence are given by Simoni, et.al. [1], for example, and the present section is devoted to the presentation of a rigorous proof of a theorem already given by these authors and conjectured by Mukunda [1]. It is shown that the rank of a semi-simple Lie algebra such that there exists a realization on a symplectic $2n$-manifold M is at most equal to n.*

It is perhaps intuitively plausible from the considerations given by Simoni, et. al. (which amount to no more than four lines) that the theorem in true, but it is by no means obvious. In the rigorous formulation of classical mechanics (Abraham [1]), Poisson Brackets are defined only for real functions, and a careful consideration of the 'complexification' of the algebra is an important step in the proof. Our treatment of realizations of Lie algebras is based upon a differential -geometry-theoretic approach. It will be seen how many obscurities in the classical treatment can be

* The present author constructed the proof to be given here while he was unaware of the work by Simoni referred to above.

cleared up when the whole is rephrased in the language of modern mathematics (Guest [2]).

We shall be interested in functions that are 'regular' in the sense that they are not constant anywhere.

6.2. Definition

Let N be a manifold.

$f \in \mathbf{F}(N)$ is regular on N iff there does not exist a nonempty open set $U \subset N$ such that $f|U$ is constant.

6.3. Definition

A realization ω of $\mathbf{A}$ on M is regular iff ωa is regular for all $a \in \mathbf{A}$, $a \neq 0$.

Let us now consider the case when $\mathbf{A}$ is semi-simple. We shall show that if $\mathbf{A}$ is semi-simple and finite-dimensional, then there exists no realization of $\mathbf{A}$ on M if the rank of $\mathbf{A}$ exceeds n. Otherwise stated, if ϕ is an essential symplectic action of a finite-dimensional, semi-simple Lie group $\mathbf{G}$ on M, we obtain, by *5.2*, for each $m \in M$, an open neighbourhood U of m and a realization (apart from a sign) of $l(\mathbf{G})$ on U. Clearly, in the theory that follows, we can, without loss of generality, take U to be the whole space M. $\mathbf{G}$, then, is of necessity locally isomorphic to one of the following:

SO_3, SO_5, ..., SO_{2n+1};

SO_4, SO_6, ..., SO_{2n};

SU_2, SU_3, ..., SU_{n+1};

Sp_1, Sp_2, ..., Sp_n;

E_i; $(i = 6,7,8;\ n \geqslant i)$

F_4; $(n \geqslant 4)$

G_2. $(n \geqslant 2)$

The reason for this limitation on the rank of **G** is connected with the symplectic structure of phase space. If is certainly not surprising, any more than the common knowledge that any group of linear transformations that leaves invariant the quadratic form $x_1^2 + x_2^2 + \ldots + x_m^2$ in m-dimensional Euclidean space and having unit determinant must be a subgroup of SO_m (and hence have dimension at most equal to $\frac{1}{2}m(m-1)$). In phase space, we do not have a quadratic form but an exterior form $dq^1 \wedge dp_1 + \ldots + dq^n \wedge dp_n$ which is left invariant by our Lie transformation group.

We consider the complex extension $\tilde{G}$ of G defined by

$$\tilde{G} = \mathbf{C} \otimes G \tag{48}$$

where **C** is the real space of complex numbers.

If $\{1, i\}$, $\{e_j : j = 1, 2, \ldots, \dim G\}$ are bases of **C** and G, respectively, a general element x of $\tilde{G}$ can be written

$$x = \sum_j [a^j (1 \otimes e_j) + b^j (i \otimes e_j)] \tag{49}$$

with a^j, $b^j \in \mathbf{R}$. From the normal rules governing tensor products, we see that this is the same as

$$\begin{aligned} x &= 1 \otimes (\sum_j a^j e_j) + i \otimes (\sum_j b^j e_j) \\ &= 1 \otimes f_1 + i \otimes f_2, \end{aligned} \tag{50}$$

where f_1, $f_2 \in G$.

$\tilde{G}$ so constructed is a real vector space which can be made into a complex space by defining, for $\lambda = \lambda_1 + i\lambda_2 \in \mathbf{C}$, λ_1, $\lambda_2 \in \mathbf{R}$,

$$\begin{aligned} \lambda x &= \lambda \otimes f_1 + \lambda i \otimes f_2 \\ &= 1 \otimes (\lambda_1 f_1 - \lambda_2 f_2) + i \otimes (\lambda_1 f_2 + \lambda_2 f_1). \end{aligned} \tag{51}$$

$\tilde{G}$ also has a natural Lie algebra structure over $\mathbf{C}$:

$$[\![x, y]\!]' \equiv [\![1 \otimes f_1 + i \otimes f_2,\ 1 \otimes g_1 + i \otimes g_2]\!]' \equiv$$

$$1 \otimes ([\![f_1, g_1]\!] - [\![f_2, g_2]\!]) + i \otimes ([\![f_1, g_2]\!] + [\![f_2, g_1]\!]). \tag{52}$$

Furthermore, $\tilde{G}$ is semi-simple iff G is semi-simple, and if Ω is a Cartan subalgebra of G, then $\tilde{\Omega} \equiv \mathbf{C} \otimes \Omega \subset \tilde{G}$ is a Cartan subalgebra of $\tilde{G}$ (Jacobson [1], pp. 70, 61). Thus if

$$\{H_i : i = 1, 2, \ldots, \dim \Omega\}$$

is a basis of Ω,

$$\{\tilde{H}_i \equiv 1 \otimes H_i : i = 1, 2, \ldots, \dim \Omega\}$$

is a basis of $\tilde{\Omega}$.

Now let **A** be finite dimensional and $\dim_{\mathbf{R}} \Omega = \dim_{\mathbf{C}} \tilde{\Omega} = l$: l is called the rank of G or $\tilde{G}$. Recall that a root of $\tilde{G}$ is a linear mapping

$$\alpha: \tilde{\Omega} \to \mathbf{C}$$

with components $\alpha_j \equiv \alpha(\tilde{H}_j)$ which are eigenvalues of $\mathrm{Ad}\,\tilde{H}_j$ acting in $\tilde{G}$ with eigenvector $E_\alpha (j = 1, 2, \ldots, l)$.
(Ad X, for $X \in \tilde{G}$ is the linear mapping $\tilde{G} \to \tilde{G}$ defined by $\mathrm{Ad}\, X.f = [\![X, f]\!]'$ for all $f \in \tilde{G}$.) Thus

$$[\![\tilde{H}_j, E_\alpha]\!]' = \alpha_j E_\alpha . \qquad (53)$$

It is also true that there are l linearly independent roots α, β, ... with l corresponding linearly independent eigenvectors E_α, E_β, $\ldots \in \tilde{G}$.

Let

$$E_\alpha = 1 \otimes E_\alpha^1 + i \otimes E_\alpha^2 . \qquad (54)$$

Then *(53)* becomes

$$\begin{aligned} 1 \otimes [\![H_j, E_\alpha^1]\!] &+ i \otimes [\![H_j, E_\alpha^2]\!] \\ &= \alpha_j (1 \otimes E_\alpha^1 + i \otimes E_\alpha^2). \end{aligned} \qquad (55)$$

We shall need one further property of the tensor product. For any bilinear (over **R**) mapping

$$\psi: \mathbf{C} \times G \to V$$

where V is a real vector space, there exists a unique linear (over **R**) mapping

$$\phi: \tilde{G} \to V$$

such that

$$\phi(\lambda \otimes f) = \psi(\lambda, f) \tag{56}$$

for all $\lambda \in \mathbf{C}$, $f \in G$.

Let $V = \mathbf{C}$ and, for $m \in M$, let

$$\psi(\lambda, f) = \lambda f(m).$$

It is easily seen that ψ is bilinear, for

$$\begin{aligned}\psi(a\lambda + b\mu, f) &= (a\lambda + b\mu)f(m) = a[\lambda f(m)] + b[\mu f(m)]\\ &= a\psi(\lambda, f) + b\psi(\mu, f)\end{aligned}$$

and

$$\begin{aligned}\psi(\lambda, af + bg) &= \lambda(af + bg)(m) = \lambda[(af)(m) + (bg)(m)]\\ &= \lambda[af(m) + bg(m)] = a\lambda f(m) + b\lambda g(m)\\ &= a\psi(\lambda, f) + b\psi(\lambda, g),\end{aligned}$$

for $a, b \in \mathbf{R}$; $\lambda, \mu \in \mathbf{C}$; $f, g \in G \subset \mathbf{F}(M)$. The second and third steps in the second part are a consequence of the fact that $\mathbf{F}(M)$ is a vector space over $\mathbf{R}$ with corresponding definitions of addition and scalar multiplication. Hence

$$\phi(\lambda \otimes f) = \lambda f(m).$$

Equation *(55)* now gives on operating on each side with ϕ:

$$[\![H_j, E_\alpha^1]\!](m) + i[\![H_j, E_\alpha^2]\!](m) = \alpha_j[E_\alpha^1(m) + iE_\alpha^2(m)]. \tag{57}$$

The Poisson bracket of two functions $f, g \in \mathbf{F}(M)$ is defined in a coordinate-free manner by

$$[\![f, g]\!](m) = df(m)\cdot[X_g(m)], \tag{58}$$

where X_g is the unique vector field on M associated with g and df is the unique differential 1-form associated with f. The left side of *(57)* is therefore

$$dH_j(m)\cdot[X_{E_\alpha^1}(m)] + i\,dH_j(m)\cdot[X_{E_\alpha^2}(m)]. \tag{59}$$

Now let $\xi^j\ (j = 1,2,\ldots,l)$ be real numbers such that

$$\sum_{j=1}^{l} \xi^j dH_j(m) = 0\ . \tag{60}$$

Multiplying the expression *(59)* by ξ^j and summing from 1 to l, we obtain

$$\left(\sum_{j=1}^{l} \xi^j dH_j(m)\right)\cdot[X_{E_\alpha^1}(m)] + i\left(\sum_{j=1}^{l} \xi^j dH_j(m)\right)\cdot[X_{E_\alpha^2}(m)],$$

which is zero. Thus, from *(57)*,

$$\sum_{j=1}^{l} \xi^j \alpha_j[E_\alpha^1(m) + iE_\alpha^2(m)] = 0. \tag{61}$$

6.4 Theorem

Let ω be a realization on a symplectic 2n-manifold M of a finite-dimensional semi-simple Lie algebra $\mathbf{A}$. *Then rank $(\mathbf{A}) \leqslant n$.*

We shall first prove several lemmas. $\mathbf{A}$ will be as in the statement of the theorem.

6.5 Lemma

Let $f \in G = \omega(\mathbf{A})$, $f \neq 0$, and let $U \subset M$ be a nonempty open set such that $f|U$ is constant. Then $g|U = 0$ for all $g \in G$.

Proof. Let $G_1 \subset \mathbf{F}(U)$ be the Lie algebra spanned by the set $\{g|U\colon g \in G\}$. Define the mapping $\tau\colon G \to G_1\colon g \to g|U$. Since

$$[\![\tau g,\ \tau h]\!] = [\![g|U,\ h|U]\!] = [\![g,\ h]\!]|U = \tau[\![g,\ h]\!]$$

for all g, $h \in G$, τ is a homomorphism. (For the second step see Abraham [1], pp. 99, 48). We have

$$0 = [\![f|U,\ g|U]\!] = \tau[\![f,\ g]\!]$$

for all $g \in G$.

We shall assume that G is simple; the extension to the case where G is only semi-simple is straightforward and follows from the fact that a semi-simple algebra is the direct sum of simple algebras. If G is simple, $[\![f,\ h]\!] \neq 0$ for some $h \in G$, since $[\![f,\ g]\!] = 0$ for all $g \in G$ implies $\{f\}$ is a non-trivial ideal in G.

Thus $\tau^{-1}\{0\} \neq \{0\}$ and is an ideal in G. Therefore $\tau^{-1}\{0\} = G$; i.e. $\tau g = 0$ for all $g \in G$; i.e. $g|U = 0$ for all $g \in G$.

6.6. Lemma.

Let $f \in G$, $f \neq 0$. Then there exists a nonempty open set $c(\omega) \subset M$ such that $f|c(\omega) \in \mathbf{F}(c(\omega))$ is regular on $c(\omega)$.

Proof. Let $W = \cup\{U\colon U \subset M$ is open and $f|U$ is constant$\}$. By 6.5, $f|W = 0$. Let $\bar{W}$ denote the closure of W. By continuity, $f|\bar{W} = 0$. Thus $c(\omega) = M - \bar{W}$ is open (since $\bar{W}$ is closed) and nonempty (for otherwise $M = \bar{W}$ and $f|M = f = 0$). It should be clear that $f|c(\omega)$ is regular.

6.7. Lemma

$g|c(\omega)$ is regular on $c(\omega)$ for all $g \in G$, $g \neq 0$.

Proof. If there is an $h \in G$, $h \neq 0$, such that $h|c(\omega)$ is not regular, then there exists a nonempty open set $V \subset c(\omega)$ such that $h|V$ is constant. By *6.5*, this implies that $g|V = 0$ for all $g \in G$; in particular, $f|V = 0$, which contradicts *6.6*.

It is obvious from this result that the set $c(\omega)$ of *6.6* is independent of choice of f.

6.8. Lemma

$$\omega_1 : \mathbf{A} \to \mathbf{F}(c(\omega)) : a \to (\omega a)|c(\omega)$$

is a regular realization of $\mathbf{A}$ *on the symplectic 2n-manifold* $c(\omega)$.

Proof. Since $\omega_1 a = (\omega a)|c(\omega) \neq 0$ if $a \neq 0$, ω_1 is 1 - 1 (see *6.7*). Hence, since $[\![,]\!]$ and linear operations in G commute with restrictions, ω_1 is in fact a realizatio That it is regular follows from *6.7*. Since $c(\omega)$ is an open subset of M it follows that $c(\omega)$ is symplectic and of dimension $2n$.

6.9 Lemma

Let N be a manifold. If r functions $f_1, f_2, \ldots, f_r \in \mathbf{F}(N)$, r finite, are regular on N, then there is a nonempty open set $U \subset N$ such that $f_i(u) \neq 0$ for all $u \in U$ and each $i = 1, 2, 3, \ldots, r$.

Proof. It is clear that there is an $m \in N$ such that $f_1(m) \neq 0$. Hence $U_1 = f_1^{-1}(\mathbf{R} - \{0\})$ is nonempty and since $\mathbf{R} - \{0\}$ is open in $\mathbf{R}$ and f_1 is continuous, $f_1^{-1}(\mathbf{R} - \{0\})$ is open in N.

Similarly, there is an $m \in U_1$ such that $f_2(m) \neq 0$. Hence $U_2 = U_1 \cap f_2^{-1}(\mathbf{R} - \{0\})$ is nonempty and open. We can repeat the argument and obtain after r steps the nonempty open set $U = U_r \subset U_{r-1} \subset \ldots \subset U_1 \subset N$ on which no member of the set $\{f_i : i = 1, 2, \ldots, r\}$ vanishes.

6.10 Lemma

Let N be a symplectic $2n$-manifold and $g_1, g_2, \ldots, g_k \in \mathbf{F}(N)$. We say that $\{g_1, g_2, \ldots, g_k\}$ is in involution on N iff

(i) For each $m \in N$, $\{dg_i(m) : i = 1, 2, \ldots, k\}$ is a linearly independent set in $T_m{}^*(N)$, where $T_m{}^*(N)$ is the cotangent space at m;

(ii) $[\![g_i, g_j]\!] = 0$ for $i, j = 1, 2, \ldots, k$.

Then if $\{g_1, g_2, \ldots, g_k\}$ is in involution on N, $k \leqslant n$.

Proof. The method of proof makes use of the symplectic structure of N (see Abraham [1], p. 112).

Proof of Theorem *6.4*

Since $E_\alpha \in \tilde{G}$ is a member of a linearly independent set, E_α is not null, and, therefore, from *(54)*, not both of $E_\alpha{}^1$ and $E_\alpha{}^2$ are null. Assume that $E_\alpha{}^{k_1}$ ($k_1 = 1$ or 2) is not null. Apply *6.9* with $N = c(\omega)$ and the set of l functions $\{E_\alpha{}^{k_1}, E_\beta{}^{k_2}, \ldots\}$. Each element of this set is a member of G and so is regular on $c(\omega)$. By *6.9*, there is a nonempty open set $U \subset c(\omega)$ such that $E_\alpha{}^{k_1}(u) \neq 0$, $E_\beta{}^{k_2}(u) \neq 0$, ... for all l roots α, β, ... and for all $u \in U$.

(61) now gives, for $m = u \in U$,

$$\begin{aligned} &\sum_{j=1}^{l} \xi^j \alpha_j = 0, \\ &\sum_{j=1}^{l} \xi^j \beta_j = 0, \qquad\qquad (62) \\ &\ldots\ldots\ldots\ldots . \end{aligned}$$

Since α, β, ... are linearly independent, we have

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_l \\ \beta_1 & \beta_2 & \cdots & \beta_l \\ \cdot & \cdot & \cdots & \cdot \end{vmatrix} \neq 0$$

and hence

$$\xi^j = 0 \quad (j = 1, 2, \ldots, l). \qquad (63)$$

Thus, from *(60)*, $\{dH_j(u) : j = 1, 2, \ldots, l\}$ is a linearly independent set in $T_u{}^*U$ for all $u \in U$. Further, $[\![H_i|U, H_j|U]\!] = [\![H_i, H_j]\!]|U = 0$, and so $\{H_i : i = 1, 2, \ldots, l\}$ is in involution on U. Since U is an open subset of M, U is a symplectic $2n$-manifold. Hence, applying *6.10* with $N = U$, we obtain $l \leqslant n$. This completes the proof of the theorem.

The set $c(\omega) \subset M$ described in *6.6* will be called the *carrier* of the realization ω. It is clear that $c(\omega)$ has the following properties:

(i) $c(\omega)$ is open and nonempty;

(ii) $g|(M - c(\omega)) = 0$ for all $g \in G$;

(iii) $g|c(\omega)$ is regular on $c(\omega)$ for all $g \in G$, $g \neq 0$.

$c(\omega)$ may be connected or not connected, a proper subset of M or M itself. It is not known whether the fact that $\mathbf{A}$ is semi-simple puts extra restrictions on $c(\omega)$.

6.11 Theorem

Let ω be a realization on a symplectic $2n$-manifold M of a finite-dimensional semi-simple Lie algebra $\mathbf{A}$. Let $H \in \mathbf{F}(M)$ be nonconstant on a connected component of the carrier of ω and let $[\![H, g]\!] = 0$ for all $g \in G = \omega(\mathbf{A})$. Then rank $(\mathbf{A}) \leqslant n - 1$.

Proof. First note that $dH|D \neq 0$ for some connected component D of $c(\omega)$; for $dH|D = 0$ implies that $H|D$ is constant. Note that D is open (manifolds are locally connected; each component of a locally connected space is open).

Thus there is a point $m \in D$ such that $dH(m) \neq 0$; $D \cap dH^{-1}(T^*M - \{0\})$ is therefore nonempty (T^*M is the cotangent bundle over M) and, since dH is continuous, is open in M. Since $D \subset c(\omega)$, g is regular on D for all $g \in G$, $g \neq 0$. Applying 6.9, we are assured of the existence of a nonempty open set $U \subset D \cap dH^{-1}(T^*M - \{0\})$ on which no member of the set $\{E_\alpha^{k_1}, E_\beta^{k_2}, \ldots\}$ vanishes.

Now let ξ, ξ^1, ξ^2, ..., ξ^l be real numbers such that

$$\xi dH(u) + \sum_{j=1}^{l} \xi^j dH_j(u) = 0 \qquad (64)$$

where $u \in U$. We now conclude, as before, since H commutes with E_α^1, E_α^2, E_β^1, E_β^2, ..., that $\xi^j = 0$ $(j = 1, 2, \ldots, l)$. Hence, since $dH(u) \neq 0$, we must have $\xi = 0$. Thus the set $\{dH(u), dH_j(u) : j = 1, 2, \ldots, l)$ is linearly independent in $T_u{}^*U$ for each $u \in U$ and is also in involution. Hence, by 6.10, $l + 1 \leqslant n$. This completes the proof of the theorem.

Finally, consider the following consequence of 6.4.

6.12 Theorem

Let ϕ be an essential action of a finite-dimensional, semi-simple Lie group $\mathbf{G}$ *on a p-manifold N (not necessarily symplectic). Then rank* $(\mathbf{G}) \leqslant p$.

Proof. For $g \in \mathbf{G}$, $[\phi(g)]^*$ is an essential action of $\mathbf{G}$ on the symplectic $2p$-manifold T^*N (see II §2). Moreover, $[\phi(g)]^*$ is symplectic (see Abraham [1], p. 97). Also, rank $[\phi(\mathbf{G})]^*$ = rank $\phi(\mathbf{G})$. Hence by *6.4*, rank $(\mathbf{G}) \leqslant p$.

In more familiar terms (and considering the case where $N = \mathbf{R}^p$): if there exists a faithful (not necessarily linear) representation of $\mathbf{G}$ by a group of C^∞ transformations of $\mathbf{R}^p$, then rank $(\mathbf{G}) \leqslant p$.

Obvious generalizations of *6.11* to the case where G commutes with two or more 'functionally independent' functions can be made. However, the statements of the corresponding theorems become more complicated the greater the number of commuting functions.

It is important to note that *6.4* and *6.11* give only necessary conditions for the existence of a realization of a particular Lie algebra. An example is given by Mukunda [1] which shows that no realizations exist on a *6*-manifold of the semi-simple algebras B_2 or G_2 such that $\omega(B_2)$ or $\omega(G_2)$ commutes with a regular Hamiltonian. It is clear that such realizations are not ruled out by *6.11*.

This concludes our discussion of classical symmetries. In Part II we shall see how such symmetries become important when we consider the quantization of classical systems.

II

THE QUANTIZATION OF CLASSICAL SYSTEMS.

A classical (holonomic) system is popularly described by the giving of its configuration space M and a real-valued function on TM (the Lagrangian on the tangent bundle of M or 'state space') or on T^*M (the Hamiltonian on the cotangent bundle of M or 'phase space'). For example, for a free particle moving throughout physical space, M is equal to $\mathbf{R}^3$ or three-dimensional Euclidean space; for a particle moving in a Coulomb field of force M is equal to $\mathbf{R}^3 - \{0\}$ ($\mathbf{R}^3$ with the origin removed); for a double pendulum M is equal to the torus (the Cartesian product $S^1 \times S^1$ of two unit circles); for a spinning top M can be taken to be $O^+(3)$ (the space of all orthogonal 3×3 matrices with positive determinant). In each case M has the structure of a C^∞ differentiable manifold. The phase space T^*M, in particular, is then uniquely determined.

We shall be concerned exclusively with Hamiltonian systems and with the cases where M is finite dimensional. A classical observable is then a real-valued Borel function on T^*M.

Many attempts have been made and are still being made to set up a 1-1 correspondence between a subset of the set of all classical observables and a subset of the set of all quantum observables. By a quantum observable we mean, of course, any self-adjoint operator in a complex, infinite-dimensional, separable Hilbert space (the Hilbert space will depend upon the quantum system under consideration). Most of the quantization procedures given in the past have been purely formal in character: not only have they been mathematically non-rigorous but also little attempt has been made to incorporate even simple physical principles into the scheme.

We shall for the moment write $\tilde{A}$ for the quantum observable corresponding to the classical A. Von Neumann's rules are as follows (Von Neumann [1]):

(i) $[f(A)]^{\sim} = f(\tilde{A})$,

(ii) $(A + B)^{\sim} = \tilde{A} + \tilde{B}$,

where A and B are classical observables and f is a real-valued function (presumably well-behaved: we shall not stop here to explain in detail what is meant by a function of an operator.). These rules are inconsistent since in certain cases it can be shown that they lead to different operators for the same classical observable. In other words, the assignment $A \to \tilde{A}$ is not a mapping in the mathematical sense.

The most remarkable quantization procedure is probably that due to Weyl [1]. If $A(q, p)$ is a classical observable, we form the $2n$-dimensional 'Fourier transform' γ defined by

$$\gamma(x,y) = (2\pi)^{-n} \int_{\mathbf{R}^{2n}} e^{-i[(x|q)+(y|p)]} A(q,p)\,dq\,dp$$

where $(q,p) = (q_1,q_2,\ldots,q_n, p_1,p_2,\ldots,p_n)$, $(x,y) = (x_1,x_2,\ldots,x_n, y_1,y_2,\ldots,y_n)$, $(x|q) = \sum_{i=1}^{n} x_i q_i$, $(y|p) = \sum_{i=1}^{n} y_i p_i$. We then take the inverse Fourier transform of γ but in operator form, i.e. we write

$$\tilde{A} = (2\pi)^{-n} \int_{\mathbf{R}^{2n}} e^{i[(Q|x)+(P|y)]} \gamma(x,y)\,dx\,dy,$$

where $Q = (Q_1,Q_2,\ldots,Q_n)$, $P = (P_1,P_2,\ldots,P_n)$ and $(Q|x)$ and $(P|y)$ have their obvious meanings. Q_j is the operator $f \to q_j f$, P_j the operator $f \to \frac{\hbar}{i} \frac{\partial f}{\partial x_j}$ where f is an arbitrary function in the Hilbert space. Such an integral of operators that appears in the definition of $\tilde{A}$ can of course be defined for certain functions γ (notably, if γ is integrable over $\mathbf{R}^{2n}$) but apart from this the main difficulty is in the definition of γ when A is not integrable. However, it can be shown

(McCoy [1]) that Weyl's rule leads to several undesirable features, notably the nonexistence of a correspondence between classical and quantum constants of motion. Nevertheless, Weyl's rule is interesting because of its elegance and because no mention is made of symmetrisation. Several other quantization rules involving integrals have been given (Rivier[1]).

Among the plethora of other quantization schemes, mention may be made of simplicity rules (Yvon [1]), symmetrisation rules (Tolman [1, p.206]) and of course Dirac's rule (Dirac [1,2]). Dirac's rule is the Poisson bracket-commutator correspondence.

As an illustration, we give an example to show how inconsistencies may arise due to the adoption of some of the rules mentioned above. With obvious notation, we wish to form the operator corresponding to the classical function q^3p^2 for a system consisting of one degree of freedom. Depending upon how we write q^3p^2, we can arrive at several different expressions for $(q^3p^2)^\sim$. Thus writing Q for $\tilde{q}$, P for $\tilde{p}$, we have

$$\begin{aligned}(q^3p^2)^\sim = [q.(qp)^2]^\sim &= \tfrac{1}{8}[Q(QP+PQ)^2+(QP+PQ)^2Q] \\ &= -\hbar^2\left(q^3\frac{\partial^2}{\partial q^2}+3q^2\frac{\partial}{\partial q}+\frac{5q}{4}\right); \qquad (i)\end{aligned}$$

$$(q^3p^2)^\sim = (q^3.p^2)^\sim = \tfrac{1}{2}(Q^3P^2 + P^2Q^3)$$

$$= -\hbar^2\left(q^3\frac{\partial^2}{\partial q^2} + 3q^2\frac{\partial}{\partial q} + 3q\right); \qquad (ii)$$

$$(q^3p^2)^\sim = [q^2.(qp^2)]^\sim = \tfrac{1}{4}[Q^2(QP^2 + P^2Q) + (QP^2 + P^2Q)Q^2]$$

$$= -\hbar^2\left(q^3\frac{\partial^2}{\partial q^2} + 3q^2\frac{\partial}{\partial q} + 2q\right); \qquad (iii)$$

$$(q^3p^2)^\sim = [(q^3p).p]^\sim = \tfrac{1}{4}[(Q^3P + PQ^3)P + P(Q^3P + PQ^3)]$$

$$= -\hbar^2\left(q^3\frac{\partial^2}{\partial q^2} + 3q^2\frac{\partial}{\partial q} + \frac{3q}{2}\right). \qquad (iv)$$

Lest it be thought that the simplest classical form (say *(ii)*) would surely give the 'correct' operator, one need only notice that if we write

$$(q^3p^2)^\sim = 3S_2 - 2S_1$$

where S_1 and S_2 are the operators in *(ii)* and *(iii)*, respectively, we arrive at

$$(q^3p^2)^\sim = -\hbar^2\left(q^3\frac{\partial^2}{\partial q^2} + 3q^2\frac{\partial}{\partial q}\right),$$

surely the 'simplest' operator of them all, and yet this operator results from writing the classical expression in the form

$$3[q^2.(qp^2)] - 2(q^3.p^2)$$

and is in fact the operator

$$\tfrac{1}{4}(3Q^2P^2Q + 3QP^2Q^2 - Q^3P^2 - P^2Q^3).$$

Here we have used the conventional association

$Q: f \to qf, \quad P: f \to \frac{\hbar}{i}\,\frac{\partial f}{\partial q}$.

A major error in all of the rules of quantization discussed above is the attempt to quantize *too large* a set of classical observables. We quote from Mackey [4]: "... there is no reason to suppose that every classical observable has a quantum counterpart or that if it has one it has only one ... they [the quantum observables] will not correspond in a one-to-one fashion to classical ones. Moreover as classical mechanics is a limiting case of quantum mechanics it would not be surprising to have a number of different quantum observables coincide in the classical limit. On the other hand, we shall find natural correspondences between the *basic* classical observables and corresponding quantum ones."

§1. Prequantization Schemes.

If N is a topological space $\mathbf{B}(N)$ will denote the set of all Borel subsets of N, i.e. the σ-algebra generated by the open subsets of N (this is the smallest collection of sets which includes the open sets and is closed with respect to the formation of differences and countable unions). In this work we shall refer to a mapping f of N into a topological space N' as *measurable* if $f^{-1}(\Delta') \in \mathbf{B}(N)$ whenever $\Delta' \in \mathbf{B}(N')$. Continuous functions and all pointwise limits of continuous functions are measurable.

We shall consider the configuration space and phase space of a classical system. Since, however, the former is nothing more than a manifold M' (in which case the latter is T^*M'), it will be convenient in view of later developments to formulate everything in terms of an arbitrary manifold M.

The mapping $\tau: T^*M \to M$ will denote the cotangent bundle projection of M (i.e. it sends (q,p) into q). If f is a real-valued measurable function on M, then $f \circ \tau$ is a real-valued measurable function. We shall write $\mathfrak{a} = \mathfrak{a}(M) = \{f \circ \tau : f$ is a real-valued measurable function on $M\}$. (Elements of $\mathfrak{a}$ are functions of the q's only.) A classical observable will be defined as a real-valued measurable function on T^*M.

We shall for the present restrict ourselves to $\mathfrak{a}(M)$ and shall set up a quantization scheme which successfully 'quantizes' these functions. But first a digression on terminology is in order.

An *operator in* a Hilbert space $\mathcal{H}$ with norm $\|\cdot\|$ and inner product $\langle\ ,\ \rangle$ linear in the first variable is a linear map A from a linear subset $D(A)$, called the *domain* of A into $\mathcal{H}$. A is said to be *closed* if whenever $x_n \in D(A)$ and both $x_n \to x$ and $Ax_n \to y$ for some y in $\mathcal{H}$, then $x \in D(A)$ and $Ax = y$. An operator A' in $\mathcal{H}$ is called an *extension* of A, symbolically, $A \subset A'$, in case $D(A) \subset D(A')$

and the two operators agree on their common domain. Suppose A and B are operators in $\mathcal{H}$ and let c be a scalar. Then

(1) $A + B$ is defined as the operator whose domain is $D(A) \cap D(B)$ and which has the value $Ax + Bx$ for any vector x in this domain;

(2) AB is defined as the operator whose domain consists of all vectors x in $D(B)$ such that $Bx \in D(A)$ and which has the value $A(Bx)$ on any such vector;

(3) cA is defined as the operator with domain $D(A)$ and having the value $c(Ax)$ for any vector x in this domain.

We shall say that a sequence $\{A_n: n = 1,2,3,\ldots\}$ of partially-defined operators in $\mathcal{H}$ *converges* to another such operator A if the following conditions hold: There exists an integer n_0 such that

$$(i)\quad D(A) = \{x: x \in D(A_n) \text{ for all } n > n_0, \lim_{n > n_0} A_n x \text{ exists in } \mathcal{H}\},$$

$$(ii)\quad Ax = \lim_{n > n_0} A_n x \text{ for all } x \in D(A).$$

The convergence is of course taken in the topology of $\mathcal{H}$. We shall write $A_n \to A$.

If A is an operator in $\mathcal{H}$ with dense domain $D(A)$ the *adjoint* A^* of A is defined as the operator whose domain consists of all vectors y in $\mathcal{H}$ for which there exists another vector y' in $\mathcal{H}$ such that $\langle Ax,y\rangle = \langle x,y'\rangle$ for all x in $D(A)$, and which has the value y' on such a vector y (the value y' is unique in view of the density of $D(A)$). If A, B and $A+B$ are all densely defined, then

$$A^* + B^* \subset (A+B)^*.$$

Similarly, if A, B and AB are densely defined, then

$$B^*A^* \subset (AB)^*.$$

If A is densely defined and $A \subset B$, then $B^* \subset A^*$.
An operator A is said to be *self-adjoint* if A is densely defined and $A = A^*$. An operator A is called *symmetric* in case it is densely defined and has the property that $\langle Ax,y\rangle = \langle x,Ay\rangle$ for all vectors x and y in its domain.
A densely-defined operator A is symmetric iff $A \subset A^*$.
An operator is said to be *closable* if it has a closed extension. A is closable iff

$$x_n \in D_A,\ \lim_{n\to\infty} x_n = 0 \text{ and } \lim_{n\to\infty} Ax_n = y$$

imply

$$y = 0.$$

The minimal closed extension $\bar{A}$ of a closable operator A (in the sense that any closed extension of A is also an extension of $\bar{A}$) is called the *closure* of A. A vector x

belongs to $D(\bar{A})$ iff there exists a sequence (x_n) of elements of $D(A)$ such that both $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} Ax_n$ exist and $\lim_{n\to\infty} x_n = x$. In this case we have $\bar{A}x = \lim_{n\to\infty} Ax_n$. If $\bar{A}$ is self-adjoint, then $\bar{A}$ is the unique self-adjoint extension of A. The adjoint A^* of any densely defined operator A is closed; consequently any symmetric operator is closable. The closure of a symmetric operator A is equal to A^{**} and is symmetric. A self-adjoint operator is symmetric and closed.

The symbols O and I will represent the null and identity operators respectively, each having domain $\mathcal{H}$.

The first quantization scheme we shall employ will comprise five axioms. It is to be understood that all of our considerations are nonrelativistic. Furthermore, we shall pretend that identical particles are distinguishable. (The only difference here is that our Hilbert space is larger than the correct one, which is the symmetric or anti-symmetric subspace of a multiple tensor product according as the particles are bosons or fermions, respectively.)

1.1 Definition.

A prequantization scheme on M is a mapping Λ of $\mathfrak{A}(M)$ into the set of all self-adjoint operators in a complex, infinite-dimensional, separable Hilbert space $\mathcal{H}$

such that the following conditions (1)-(5) are satisfied.

If $f,\ g \in \mathfrak{a}$, $c \in \mathbf{R}$, *then*

(1) $\Lambda f + \Lambda g$ *is closable and* $\Lambda(f+g) = \overline{\Lambda f + \Lambda g}$;

(2) $\Lambda(cf) = \overline{c\Lambda f}$;

(3) $\Lambda f \Lambda g$ *is closable and* $\Lambda(fg) = \overline{\Lambda f \Lambda g}$.

(4) *If* $h(x) = 1$ *for all* $x \in T^*M$, *then* $\Lambda h = I$.

(5) *If* $f,\ f_n \in \mathfrak{a}$ $(n = 1,2,3,\ldots)$, *if* $|f_n(x)| \leqslant |f(x)|$ *and* $\lim\limits_{n\to\infty} f_n(x) = f(x)$ *for all* $x \in T^*M$, *then* $\Lambda f_n \to \Lambda f$.

Apart from the fact that the self-adjoint operators involved may not be everywhere defined, axioms *(1)-(3)* can be considered as describing an algebra homomorphism. More precisely, if $\mathfrak{a}'$ is that subset of $\mathfrak{a}$ consisting of all $f \in \mathfrak{a}$ for which Λf is everywhere defined, then $\mathfrak{a}'$ is an algebra (i.e. is closed under addition, scalar multiplication and formation of products; this can be verified from axioms *(1)-(3)*) and $\Lambda : \mathfrak{a}' \to \Lambda\mathfrak{a}'$ is an algebra homomorphism. It is clear that $\Lambda\mathfrak{a}'$ is a commutative algebra. Axioms *(1)-(3)* then describe the generalization to the whole of $\mathfrak{a}$ of the restriction of Λ to $\mathfrak{a}'$. Axiom *(4)* is a normalisation condition. Loosely, axiom *(5)* says that Λ satisfies a kind of continuity condition with respect to dominated convergence of sequences in $\mathfrak{a}$.

However, $\mathfrak{a}'$ is not closed with respect to formation of these sequences; in fact the closure of $\mathfrak{a}'$ with respect to formation of dominated sequences is $\mathfrak{a}$ itself. It will not be necessary to pursue these matters in the sequel.

A number of non-trivial results can be deduced from the axioms. If $\Delta \in \mathbf{B}(M)$ and χ_Δ is the characteristic function of Δ(i.e. $\chi_\Delta(x) = 1$ if $x \in \Delta$ and is zero otherwise), then clearly χ_Δ is a measurable function on M. We shall write $P(\Delta) = \Lambda(\chi_\Delta \circ \tau)$.

1.2 Theorem

For every $\Delta \in \mathbf{B}(M)$, $P(\Delta)$ is a projection.

Proof: We have

$$P(\Delta) = \Lambda(\chi_\Delta \circ \tau) = \Lambda(\chi_\Delta^2 \circ \tau) = \Lambda[(\chi_\Delta \circ \tau).(\chi_\Delta \circ \tau)]$$

$$= \overline{\Lambda(\chi_\Delta \circ \tau)\Lambda(\chi_\Delta \circ \tau)} = \overline{P(\Delta)^2} = P(\Delta)^2$$

by axiom *(3)* and the fact that $(f \circ \tau).(g \circ \tau) = (fg) \circ \tau$ for real-valued measurable functions f, g on M.

Since $P(\Delta)^* = P(\Delta)$ and $D(P(\Delta)) = D(P(\Delta)^2)$ we have, if $x \in D(P(\Delta))$,

$$\|P(\Delta)x\|^2 = \langle P(\Delta)x, P(\Delta)x \rangle = \langle x, P(\Delta)^2 x \rangle$$

$$= \langle x, P(\Delta)x \rangle \leqslant \|x\| \|P(\Delta)x\|$$

so that

$$\|P(\Delta)x\| \leqslant \|x\|.$$

Hence $P(\Delta)$ is continuous at every point of its domain. Let $z \in \mathcal{H}$, $x_n \in D(P(\Delta))$ $(n = 1, 2, 3, \ldots)$ and $\|x_n - z\| \to 0$. Then (x_n) is a Cauchy sequence and

$$\|P(\Delta)x_n - P(\Delta)x_m\| = \|P(\Delta)(x_n - x_m)\| \leqslant \|x_n - x_m\|.$$

Therefore $(P(\Delta)x_n)$ is a Cauchy sequence also and hence converges to a vector $y \in \mathcal{H}$. Since $P(\Delta)$ is closed we have $z \in D(P(\Delta))$ and $P(\Delta)z = y$. Thus $D(P(\Delta)) = \mathcal{H}$ and since $P(\Delta)^2 = P(\Delta)$, $P(\Delta)$ is a projection.

Recall that a projection-valued measure on a set X is a map S from a σ-algebra of subsets of X to the projections on a complex Hilbert space which satisfies the following conditions *(i)*-*(iv)*:

(i) $S(E \cap F) = S(E)S(F)$

and $S(E)S(F) = 0$ if E and F are disjoint.

(ii) $S(E)^* = S(E) = S(E)^2$.

(iii) $S(X) = I$.

(iv) If $E = \bigcup_i E_i$, where the E_i are mutually disjoint, then

$$S(E) = \sum_i S(E_i)$$

in the sense that

$$S(E)x = \lim_{n\to\infty} \sum_{i=1}^{n} S(E_i)x$$

for all x in the Hilbert space, where the limit is taken in the Hilbert space topology.

1.3 Theorem

The map $\Delta \to P(\Delta)$ of $\mathbf{B}(M)$ *into the projections on* $\mathcal{H}$ *is a projection-valued measure on M.*

Proof. We must verify that the above conditions *(i)-(iv)* are satisfied.

For *(i)*, we have

$$P(\Delta_1 \cap \Delta_2) = \Lambda(\chi_{\Delta_1 \cap \Delta_2} \circ \tau) = \Lambda((\chi_{\Delta_1} \cdot \chi_{\Delta_2}) \circ \tau)$$
$$= \Lambda[(\chi_{\Delta_1} \circ \tau) \cdot (\chi_{\Delta_2} \circ \tau)] = \overline{\Lambda(\chi_{\Delta_1} \circ \tau)\Lambda(\chi_{\Delta_2} \circ \tau)}$$
$$= \overline{P(\Delta_1)P(\Delta_2)} = P(\Delta_1)P(\Delta_2)$$

from *1.1 (3)*. If $\Delta_1 \cap \Delta_2 = \phi$, then $P(\Delta_1)P(\Delta_2) = P(\phi) = \Lambda(\chi_\phi \circ \tau) = \Lambda(0.f)$ for any $f \in \mathfrak{A}$. By *1.1 (2)* this is equal to $\overline{0.\Lambda f} = \overline{0|D(\Lambda f)} = 0$.

(ii) follows immediately from *1.2*.

For *(iii)*, we have $P(M) = \Lambda(\chi_M \circ \tau) = I$ by *1.1 (4)*.

For *(iv)*, we note that if $\Delta_1 \cap \Delta_2 = \phi$ then $P(\Delta_1 \cup \Delta_2) = P(\chi_{\Delta_1 \cup \Delta_2} \circ \tau) = \Lambda((\chi_{\Delta_1} + \chi_{\Delta_2}) \circ \tau) = \Lambda(\chi_{\Delta_1} \circ \tau + \chi_{\Delta_2} \circ \tau) = \overline{\Lambda(\chi_{\Delta_1} \circ \tau) + \Lambda(\chi_{\Delta_1} \circ \tau)} = \overline{P(\Delta_1) + P(\Delta_2)} = P(\Delta_1) + P(\Delta_2)$ by *1.1 (1)*. Extending to any finite collection $\{\Delta_i : i = 1, 2, .., n\}$ of pairwise disjoint sets, we have

$$P\left(\bigcup_{i=1}^{n} \Delta_i\right) = \sum_{i=1}^{n} P(\Delta_i).$$

If (Δ_n) is a sequence of pairwise disjoint sets in $\mathbf{B}(M)$, let $s_n = \chi_{\bigcup_{i=1}^{n} \Delta_i} \circ \tau$, $s = \chi_{\bigcup_{i=1}^{n} \Delta_i} \circ \tau$; then $s_n \to s$ pointwise and $s_n \leqslant s$; hence by *1.1 (5)*, $\Lambda s_n \to \Lambda s$. Thus

$$\begin{aligned} P\left(\bigcup_{i=1}^{\infty} \Delta_i\right)x &= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} \Delta_i\right)x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} P(\Delta_i)x \end{aligned}$$

for all $x \in \mathcal{H}$.

We shall now appeal to the general theory of representations of C*-algebras (the algebra of bounded operators on $\mathcal{H}$ generated by the $P(\Delta)$ for $\Delta \in \mathbf{B}(M)$ is such an algebra) as developed by Bochner, Godement, Gelfand and Neumark. We shall quote a standard result of that theory *(1.4)*. The relevant theory leading to this result is not readily found in the literature; see, for example, Dieudonne [2, chap. 15], Mackey [2]; perhaps Halmos [1, chap. 3] is the nearest classical exposition.

A *cyclic vector* for a set S of everywhere defined linear operators on a topological linear space L is a vector z in L such that the finite linear combinations of the vectors of the form Az, $A \in S$, form a dense subset of L.

Let N denote the set of integers $\geqslant 1$ and for each $k \in \mathrm{N}$ let N_k be the set $\{n \in \mathrm{N}: 1 \leqslant n \leqslant k\}$. Define $\mathrm{N}_\infty = \mathrm{N}$.

1.4 Theorem

Let $S = \{P(\Delta): \Delta \in \mathbf{B}(M)\}$. Then $\mathcal{H}$ is a direct sum *

$$\mathcal{H} = \bigoplus_{k \in \mathrm{N} \cup \{\infty\}} \mathcal{H}_k$$

of closed subspaces $\mathcal{H}_k$ and each $\mathcal{H}_k$ is a direct sum

$$\mathcal{H}_k = \bigoplus_{i \in \mathrm{N}_k} \mathcal{H}_k^i$$

of closed subspaces $\mathcal{H}_k^i$ such that

(i) each $\mathcal{H}_k^i$ is left invariant by all members of S and has a cyclic vector for the relative action of S restricted to that subspace,

(ii) for each k, for each $i, j \in \mathrm{N}_k$, there is a unitary mapping of $\mathcal{H}_k^i$ onto $\mathcal{H}_k^j$ which carries $P(\Delta)|\mathcal{H}_k^i$ into $P(\Delta)|\mathcal{H}_k^j$,

(iii) if $k \neq l$, if $i \in \mathrm{N}_k$, $j \in \mathrm{N}_l$, and if $\mathcal{H}_k^i \neq \{0\}$, no unitary mapping exists of $\mathcal{H}_k^i$ onto $\mathcal{H}_l^j$ such that $P(\Delta)|\mathcal{H}_k^i$ is carried into $P(\Delta)|\mathcal{H}_l^j$.

The subspaces $\mathcal{H}_k$ are uniquely determined by P.

For the further development of the theory we give the following.

* By *direct sum* we shall always mean a *Hilbert sum* (cf. Dieudonne [1, p.123]).

1.5 *Definition*

Let N be a topological space and ν a (nonnegative, countably additive) measure on $\mathbf{B}(N)$.

$\mathcal{L}_2(N,\nu)$ *will denote the set of all complex-valued, measurable functions f on N for which* $\int_N |f|^2 d\nu$ *is finite.* $\tilde{\mathcal{L}}_2(N,\nu)$ *will denote the set of all equivalence classes of such functions, two functions being equivalent iff they are equal except on a set in* $\mathbf{B}(N)$ *of ν-measure zero. As is known,* $\tilde{\mathcal{L}}_2(N,\nu)$ *is a Hilbert space with inner product* $\langle f,g\rangle \equiv \int_N f\bar{g}d\nu$. *We shall generally suppress the distinction between* $\mathcal{L}_2(N,\nu)$ *and* $\tilde{\mathcal{L}}_2(N,\nu)$.

Let h be a complex-valued measurable function on N. By $\mathrm{m}(h)$ *we shall mean the operator defined by*

$$D(\mathrm{m}(h)) = \{f \in \mathcal{L}_2(N,\nu) : hf \in \mathcal{L}_2(N,\nu)\},$$

$$\mathrm{m}(h)f = hf \quad \text{for all} \quad f \in D(\mathrm{m}(h)),$$

where $(hf)(x) = h(x)f(x)$ *for all* $x \in N$. $\mathrm{m}(h)$ *is self-adjoint in* $\tilde{\mathcal{L}}_2(N,\nu)$ *iff h is real-valued.*

For $r = 1,2,\ldots,$ *define* $\mathbf{C}^r = \mathbf{C} \times \mathbf{C} \times \ldots \times \mathbf{C}$ *(r times) together with its obvious vector space structure. With the norm* $\|x\|_r = \sqrt{\sum_{i=1}^{r} |x_i|^2}$ *and inner product* $\langle x,y\rangle_r = \sum_{i=1}^{r} x_i\bar{y}_i$, $\mathbf{C}^r$ *is a Hilbert space. Define* $\mathbf{C}^\infty$ *to be the set of all sequences* $n \to x_n$ *for which* $\sum_{i=1}^{\infty} |x_i|^2 < \infty$. *With the* *norm* $\|x\|_\infty = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$ *and inner product* $\langle x,y\rangle_\infty = \sum_{i=1}^{\infty} x_i \bar{y}_i$, $\mathbf{C}^\infty$ *is a Hilbert space.*

For $r = \infty, 1, 2, \ldots,$ *define* $\mathcal{L}_2(N, \mathbf{C}^r, \nu)$ *to be the set of all* $\mathbf{C}^r$*-valued measurable functions* f *on* N *for which* $\int_N \|f(x)\|_r^2 d\nu$ *is finite. With the norm* $\|f\|_r = \int_N \|f(x)\|_r^2 d\nu$ *and inner product* $\langle f,g\rangle_r = \int_N \langle f(x),\ g(x)\rangle_r d\nu$, *the set* $\tilde{\mathcal{L}}_2(N, \mathbf{C}^r, \nu)$ *of equivalence classes of such functions is a Hilbert space*

If h *is a complex-valued measurable function on* N, $\mathrm{m}(h)$ *will be the operator defined by*

$$D(\mathrm{m}(h)) = \{f \in \mathcal{L}_2(N, \mathbf{C}^r, \nu),\ hf \in \mathcal{L}_2(N, \mathbf{C}^r, \nu)\},$$

$$\mathrm{m}(h)f = hf \quad \text{for all } f \in D(\mathrm{m}(h)),$$

where $(hf)(x) = h(x)f(x)$ *for all* $x \in N$. $\mathrm{m}(h)$ *is self-adjoint in* $\mathcal{L}_2(N, \mathbf{C}^r, \nu)$ *iff* h *is real-valued.*

(Whenever $\mathrm{m}(h)$ is mentioned it will be generally clear from the context which space $\mathcal{L}_2(N, \mathbf{C}^r, \nu)$ is referred to. If $r = 1$, the second definition agrees with the earlier one.)

We shall also need

1.6 Lemma

For $r = 1, 2, \ldots,$ *(resp.* $r = \infty$*), let* S *denote the direct sum of* r *copies (resp.* $\aleph_0$ *copies) of* $\tilde{\mathcal{L}}_2(N, \nu)$.

Then S is unitarily equivalent to $\tilde{\mathcal{L}}_2(N, \mathbf{C}^r, \nu)$.

Proof. For $r = 1, 2, \ldots,$ let

$\check{f}(x) = (f_1(x), f_2(x), \ldots, f_r(x))$ for all $f = (f_1, f_2, \ldots, f_r) \in S$. Then $\check{f}$ is measurable from N into $\mathbf{C}^r$ and we have

$$\int_N \|\check{f}(x)\|_r^2 d\nu = \int_N \left(\sum_{i=1}^{r} |f_i(x)|^2 \right) d\nu$$

$$= \sum_{i=1}^{r} \int_N |f_i(x)|^2 d\nu = \|f\|_S^2 < \infty,$$

(where $\|\cdot\|_S$ is the norm in S), so that $\check{f} \in \mathcal{L}_2(N, \mathbf{C}^r, \nu)$; hence we can write $\|\check{f}\|_r = \|f\|_S$; since $f \to \check{f}$ is linear and clearly onto, it is unitary.

If $r = \infty$, let $\check{f}(x) = (f_1(x), f_2(x), \ldots)$ for all $f = (f_1, f_2, \ldots) \in S$. The same conclusions hold as in the finite case if we note that $\|f\|_S^2 = \sum_{i=1}^{\infty} \int_N |f_i(x)|^2 d\nu < \infty$ (by definition of direct (Hilbert) sum); hence we can interchange the order of integration and summation to obtain $\|f\|_S^2 = \int_N \|\check{f}(x)\|_\infty^2 d\nu$.

It is clear that the operator S which is the direct sum of the operators $\mathrm{m}(h)$ in each summand $\mathcal{L}_2(N, \nu)$ is carried into the operator $\mathrm{m}(h)$ in $\mathcal{L}_2(N, \mathbf{C}^r, \nu)$ by the unitary mapping $f \to \check{f}$.

Of great importance is the following

1.7 Theorem

With the notation of 1.4, for each $\Delta \in \mathbf{B}(M)$ and each $k \in \mathrm{N} \cup \{\infty\}$, $i \in \mathrm{N}_k$, $P(\Delta)$ on $\mathcal{H}_k^i$ is unitarily equivalent to $\mathrm{m}(\chi_\Delta)$ on $\tilde{\mathcal{L}}_2(M, \nu_k)$ for any σ-finite measure ν_k on $\mathbf{B}(M)$ whose null sets Δ are those for which $P(\Delta)|\mathcal{H}_k = 0$.

Proof. Let Z_k^i be the set of all finite linear combinations $\sum_j c^j P(\Delta_j) z_k^i$, where z_k^i is a cyclic vector in $\mathcal{H}_k^i$ for $\{P(\Delta)|\mathcal{H}_k^i\colon \Delta \in \mathbf{B}(M)\}$. Then Z_k^i is dense in $\mathcal{H}_k^i$.

It is clear that $\nu_k^i\colon \Delta \to \langle P(\Delta) z_k^i, z_k^i\rangle$ is a measure on $\mathbf{B}(M)$ and that the null sets for ν_k^i are those for which $P(\Delta)|\mathcal{H}_k^i = 0$. For $P(\Delta) z_k^i = 0$ implies $\sum_j c^j P(\Delta_j) P(\Delta) z_k^i = 0$ which implies $P(\Delta) \sum_j c^j P(\Delta_j) z_k^i = 0$; hence if $P(\Delta) z_k^i = 0$ then $P(\Delta)$ vanishes on Z_k^i, i.e. $P(\Delta)|\mathcal{H}_k^i = 0$. Conversely, $P(\Delta)|\mathcal{H}_k^i = 0$ implies $P(\Delta) z_k^i = 0$ trivially. Hence $P(\Delta) z_k^i = 0$ iff $P(\Delta)|\mathcal{H}_k^i = 0$. But $\nu_k^i(\Delta) = \|P(\Delta) z_k^i\|^2$ which is zero iff $P(\Delta) z_k^i$ is zero iff $P(\Delta)|\mathcal{H}_k^i$ is zero.

Let S be the set of all measurable, complex-valued step functions on M. Then S is dense in $\mathcal{L}_2(M, \nu_k^i)$. Define $U_k^i\colon \mathrm{S} \to Z_k^i$ by

$$U_k^i \left(\sum_{j=1}^{n} c^j \chi_{\Delta_j}\right) = \sum_{j=1}^{n} c^j P(\Delta_j) z_k^i$$

where the step function $\sum_{j=1}^{n} c^j \chi_{\Delta_j}$ is written so that

$\Delta_j \cap \Delta_{j'} = \phi$ if $j \neq j'$. We have

$$\|\sum_{j=1}^{n} c^j P(\Delta_j) z_k^i\|^2 = \left\langle \sum_{j=1}^{n} c^j P(\Delta_j) z_k^i, \sum_{j'=1}^{n} c^{j'} P(\Delta_{j'}) z_k^i \right\rangle$$

$$= \sum_{j,j'=1}^{n} c^j \overline{c^{j'}} \langle P(\Delta_j \cap \Delta_{j'}) z_k^i, z_k^i \rangle$$

$$= \sum_{j=1}^{n} |c^j|^2 \langle P(\Delta_j) z_k^i, z_k^i \rangle = \sum_{j=1}^{n} |c^j|^2 \nu_k^i(\Delta_j)$$

$$= \sum_{j=1}^{n} \int_M |c^j|^2 \chi_{\Delta_j} d\nu_k^i = \int_M \left(\sum_{j=1}^{n} |c^j|^2 \chi_{\Delta_j} \right) d\nu_k^i$$

$$= \int_M \left(\sum_{j=1}^{n} |c^j \chi_{\Delta_j}|^2 \right) d\nu_k^i = \|\sum_{j=1}^{n} c^j \chi_{\Delta_j}\|^2$$

where the last norm is taken in $\mathcal{L}_2(M, \nu_k^i)$. Hence U_k^i is a norm-preserving linear map of S onto Z_k^i. U_k^i extends uniquely to a unitary map $U_k^i\colon \mathcal{L}_2(M, \nu_k^i) \to \mathcal{H}_k^i$. Further

$$P(\Delta) U_k^i \sum_{j=1}^{n} c^j \chi_{\Delta_j} = P(\Delta) \sum_{j=1}^{n} c^j P(\Delta_j) z_k^i$$

$$= \sum_{j=1}^{n} c^j P(\Delta \cap \Delta_j) z_k^i = U_k^i \sum_{j=1}^{n} c^j \chi_{\Delta \cap \Delta_j}$$

$$= U_k^i \chi_\Delta \sum_{j=1}^{n} c^j \chi_{\Delta_j},$$

i.e. ${U_k^i}^{-1} P(\Delta) U_k^i f = \chi_\Delta f$ for all $f \in$ S. $m(\chi_\Delta)$ is the unique continuous extension to $\mathcal{L}_2(M, \nu_k^i)$ of its restriction to S and hence equals ${U_k^i}^{-1} (P(\Delta)|\mathcal{H}_k^i) U_k^i$.

Now let ν_k be any σ-finite measure whose null sets Δ are those for which $P(\Delta)|\mathcal{H}_k = 0$. By *1.4*, $P(\Delta)|\mathcal{H}_k^i = 0$ iff $P(\Delta)|\mathcal{H}_k^{i'} = 0$ $(i, i' \in \mathrm{N}_k)$; hence for any $i \in \mathrm{N}_k$, $P(\Delta)|\mathcal{H}_k^i = 0$ iff $P(\Delta)|\mathcal{H}_k = 0$. Thus the null sets of ν_k are the same as those of ν_k^i and the mapping $f \to \sqrt{\dfrac{d\nu_k^i}{d\nu_k}}\, f$ is unitary from $\mathfrak{L}_2(M,\nu_k^i)$ onto $\mathfrak{L}_2(M,\nu_k)$ and carries $\mathrm{m}(\chi_\Delta)$ in $\mathfrak{L}_2(M,\nu_k^i)$ into $\mathrm{m}(\chi_\Delta)$ in $\mathfrak{L}_2(M,\nu_k)$. This completes the proof of the theorem.

If $f \in \mathfrak{a}(M)$ we shall denote by f' that real-valued measurable function on M such that $f = f' \circ \tau$.

We now have

1.8 Lemma

With the notation of 1.1, 1.4 and 1.7, for each $k \in \mathrm{N} \cup \{\infty\}$, $i \in \mathrm{N}_k$, $\mathcal{H}_k^i$ *is invariant under* Λf *for any* $f \in \mathfrak{a}(M)$ *and there exists a unitary mapping* $U_k^i \colon \mathfrak{L}_2(M,\nu_k) \to \mathcal{H}_k^i$ *such that for all* $f \in \mathfrak{a}(M)$,

$$(\Lambda f)|\mathcal{H}_k^i = U_k^i \,\mathrm{m}(f')\, U_k^{i^{-1}}$$

Proof. Let $f \in \mathfrak{a}(M)$. There exists a sequence $(f_n{}')$ of measurable, real-valued step functions on M such that $|f_n{}'| \leqslant |f'|$ and $\lim\limits_{n\to\infty} f_n{}'(x) = f'(x)$ for all $x \in M$ (Hewitt and Stromberg [1, p.159]). Hence if $f_n = f_n{}' \circ \tau$, *1.1 (5)* gives $\Lambda f_n \to f$.

Now $f_n{}'$ can be written as a finite sum

$$f_n{}' = \sum_j c_n^j \chi_{\Delta_j} \qquad (\Delta_j \in \mathbf{B}(M)).$$

Hence

$$\Lambda f_n = \sum_j c_n^j P(\Delta_j)$$

by *1.1 (1)* and *1.1 (2)* since each term is everywhere defined. Thus each $\mathcal{H}_k^i$ is invariant under Λf_n by *1.4* and since, for $y \in D(\Lambda f)$, $(\Lambda f)y$ is the limit of the sequence $((\Lambda f_n)y)$, $\mathcal{H}_k^i$ is invariant under Λf.

By *1.7*, there is a unitary mapping $U_k^i\colon \tilde{\mathcal{L}}_2(M,\nu_k) \to \mathcal{H}_k^i$ such that $P(\Delta)|\mathcal{H}_k^i = U_k^i \mathrm{m}(\chi_\Delta) U_k^{i^{-1}}$. Hence we have

$$(\Lambda f_n)|\mathcal{H}_k^i = U_k^i (\sum_j c_n^j \mathrm{m}(\chi_{\Delta_j})) U_k^{i^{-1}}$$

$$= U_k^i \mathrm{m}(f_n{}') U_k^{i^{-1}}.$$

We shall next show that $\mathrm{m}(f_n{}') \to \mathrm{m}(f')$ in $\mathcal{L}_2 = \mathcal{L}_2(M,\nu_k)$. Let

$$D = \{h \in \mathcal{L}_2 \colon \lim_{n\to\infty} f_n{}'h \text{ exists in } \mathcal{L}_2\}.$$

$h \in D$ implies that there is $h_1 \in \mathcal{L}_2$ such that $\|f_n{}'h - h_1\| \to 0$ (norm in $\mathcal{L}_2$). Since also $(f_n{}'h)(x) \to (f'h)(x)$ for every $x \in M$, we have $f'h \in \mathcal{L}_2$ and $f'h = h_1$ ν_k- *a.e.*. Hence $h \in D$ implies $h \in D(\mathrm{m}(f'))$ and $\lim_{n\to\infty} \mathrm{m}(f_n{}')h = \mathrm{m}(f')h$.

Conversely, if $h \in D(\mathrm{m}(f'))$, then since $f_n'h \to f'h$ pointwise and $|f_n'h| \leqslant |f'h|$ so that $|f_n'h - f'h|^2 \to 0$ pointwise and $|f_n'h - f'h|^2 \leqslant 4|f'h|^2$, the dominated convergence theorem shows that $\|f_n'h - f'h\| \to 0$. Hence $h \in D(\mathrm{m}(f'))$ implies $h \in D$. Thus $D = D(\mathrm{m}(f'))$ and we have $\mathrm{m}(f_n') \to \mathrm{m}(f')$. Finally, by continuity, we have

$$(\Lambda f)|\mathcal{H}_k^i = U_k^i\ \mathrm{m}(f')U_k^{i^{-1}} \quad .$$

1.9 *Theorem*

Let Λ *be a prequantization scheme on* M *with values the self-adjoint operators in* $\mathcal{H}$. *Then* $\mathcal{H}$ *is the direct sum*

$$\mathcal{H} = \bigoplus_{k \in \mathrm{N} \cup \{\infty\}} \mathcal{H}_k$$

of uniquely determined closed subspaces $\mathcal{H}_k$ *invariant under every* Λf, $f \in \mathfrak{Q}(M)$, *such that there exist* σ*-finite measures* ν_k *on* $\mathbf{B}(M)$ *and unitary mappings* $U_k\colon \mathcal{L}_2(M, \mathbf{C}^k, \nu_k) \to \mathcal{H}_k (k \in \mathrm{N} \cup \{\infty\})$ *with the property that for each* $f \in \mathfrak{Q}(M)$

$$(\Lambda f)|\mathcal{H}_k = U_k \mathrm{m}(f')U_k^{-1} \ .$$

Conversely, if such a decomposition of $\mathcal{H}$, *measures* ν_k *and unitary mappings* U_k *are given, then the map*

$$f \to \bigoplus_{k \in \mathrm{N} \cup \{\infty\}} U_k \mathrm{m}(f')U_k^{-1}$$

of $\mathfrak{Q}(M)$ *into the self-adjoint operators in* $\mathcal{H}$ *is a prequantization scheme on* M.

Proof. With the notation of *1.8*, the map from the direct sum of k copies (if $k \in \mathrm{N}$) or $\aleph_0$ copies (if $k = \infty$) of $\tilde{\mathcal{L}}_2(M, \nu_k)$ onto $\mathcal{H}_k$ given by

$$(f_1, f_2, \ldots) \rightarrow (U_k^1 f_1, U_k^2 f_2, \ldots)$$

is unitary and carries the direct sum of the operators $\mathrm{m}(f')$ in each copy of $\tilde{\mathcal{L}}_2(M, \nu_k)$ into Λf. By *1.6* and the remark following *1.6* the first conclusion follows.

For the converse, we must verify that axioms *1.1 (1)* - *1.1 (5)* are satisfied. It will clearly suffice to prove this for the mapping $\Lambda: f \rightarrow \mathrm{m}(f')$ of $\mathbf{Q}$ into the self-adjoint operators in some $\tilde{\mathcal{L}}_2 = \tilde{\mathcal{L}}_2(M, \nu)$.

If $f_n \in \mathbf{Q}$, $f \in \mathbf{Q}$, $|f_n| \leqslant |f|$ and $f_n \rightarrow f$ pointwise then a similar argument to that used in the proof of *1.8* shows that $\mathrm{m}(f_n') \rightarrow \mathrm{m}(f')$, i.e. $\Lambda f_n \rightarrow \Lambda f$, and *1.1 (5)* is satisfied.

1.1 (4) follows immediately since if $h(x) = 1$ for all $x \in M$, then $\mathrm{m}(h) = I$.

1.1 (2) follows from the fact that $\mathrm{m}(ch) = c\mathrm{m}(h)$ if $c \in \mathbf{R}$ and $c \neq 0$ and $\mathrm{m}(0.h) = \mathrm{m}(0) = 0 = \overline{0|D(\mathrm{m}(h))} = \overline{0.\mathrm{m}(h)}$ for any real-valued measurable function h on M.

To verify *1.1 (1)* we proceed as follows. Let f, g be real-valued, measurable functions on M. If $h \in D(\mathrm{m}(f) + \mathrm{m}(g))$, then $fh \in \mathcal{L}_2$, $gh \in \mathcal{L}_2$ and hence $fh + gh = (f+g)h \in \mathcal{L}_2$, i.e. $h \in D(\mathrm{m}(f+g))$. Hence $\mathrm{m}(f) + \mathrm{m}(g) \subset \mathrm{m}(f+g)$. This proves that $\mathrm{m}(f) + \mathrm{m}(g)$ is closable. Now $m(f)+m(g)$ is densely defined, for let χ_n be the characteristic function of the set $\{x \in M: |f(x)| \leqslant n, |g(x)| \leqslant n\}$. Then for any $h \in \mathcal{L}_2$ we have $|f\chi_n h| \leqslant n|h|$, $|g\chi_n h| \leqslant n|h|$ so that $f\chi_n h \in \mathcal{L}_2$ and $g\chi_n h \in \mathcal{L}_2$, i.e. $\chi_n h \in D(m(f)) \cap D(m(g)) = D(m(f)+m(g))$. Since $\|\chi_n h - h\| \to 0$ as $n \to \infty$ (norm in $\mathcal{L}_2$), which follows easily from the dominated convergence theorem, $D(m(f)+m(g))$ is dense in $\mathcal{L}_2$. It follows that the adjoint $(m(f)+m(g))^*$ is defined and we have $(m(f)+m(g))^* \supset m(f)^* + m(g)^* = m(f)+m(g)$ so that $m(f)+m(g)$ is symmetric.

Let now $h \in D[(m(f)+m(g))^*]$. Then there is $h_1 \in \mathcal{L}_2$ such that

$$\langle (f+g)y,\ h\rangle = \langle y, h_1\rangle$$

for all $y \in D(m(f)+m(g))$. *Replacing* y *by* $\chi_n y$ *gives*

$$\int_M (f+g)\chi_n y\bar{h}\,d\nu = \int_M \chi_n y\bar{h}_1\,d\nu$$

or

$$\int_M y\cdot\overline{\chi_n(f+g)h}\,d\nu = \int_M y\cdot\overline{\chi_n h_1}\,d\nu \ .$$

Now $|\chi_n(f+g)h| = \chi_n|(f+g)h| \leqslant (\chi_n|f| + \chi_n|g|)|h| \leqslant 2n|h|$ so that $\chi_n(f+g)h \in \mathcal{L}_2$. Since also $\chi_n h_1 \in \mathcal{L}_2$, the last equation can be written

$$\langle y,\ \chi_n(f+g)h\rangle = \langle y,\ \chi_n h_1\rangle.$$

This holds for all y in the dense set $D(\mathrm{m}(f)+\mathrm{m}(g))$. Hence

$$(f+g)\chi_n h = \chi_n h_1 \qquad \nu\text{-}a.e..$$

The fact that $\chi_n h \in D(\mathrm{m}(f+g))$, $\chi_n h \to h$, $(f+g)\chi_n h = \chi_n h_1 \to h_1$ in the topology of $\mathcal{L}_2$ and $\mathrm{m}(f+g)$ is closed shows that $h \in D(\mathrm{m}(f+g))$ and that $h_1 = (\mathrm{m}(f)+\mathrm{m}(g))^* h = (f+g)h$.

Thus $(\mathrm{m}(f)+\mathrm{m}(g))^* \subset \mathrm{m}(f+g)$ and by a previous relation $\mathrm{m}(f+g) \subset (\mathrm{m}(f)+\mathrm{m}(g))^*$; hence

$$\mathrm{m}(f+g) = (\mathrm{m}(f)+\mathrm{m}(g))^*.$$

Since $\mathrm{m}(f)+\mathrm{m}(g)$ is symmetric, we have finally

$$\overline{\mathrm{m}(f)+\mathrm{m}(g)} = (\mathrm{m}(f)+\mathrm{m}(g))^{**} = \mathrm{m}(f+g).$$

This proves *1.1 (1)*.

In a similar way it can be proved that *1.1 (3)* holds. Again, let f, g be real-valued measurable functions on M and let $h \in D(\mathrm{m}(f)\mathrm{m}(g))$. Then $gh \in \mathcal{L}_2$ and $fgh \in \mathcal{L}_2$; hence $h \in D(\mathrm{m}(fg))$ and $\mathrm{m}(f)\mathrm{m}(g) \subset \mathrm{m}(fh)$ so that $\mathrm{m}(f)\mathrm{m}(g)$ is closable. $\mathrm{m}(f)\mathrm{m}(g)$ is densely defined, for if $h \in \mathcal{L}_2$ then since $|g\chi_n h| \leqslant n|h|$ and $|fg\chi_n h| \leqslant n^2|h|$ we have $\chi_n h \in D(\mathrm{m}(f)\mathrm{m}(g))$. The fact that $\|\chi_n h - h\| \to 0$ means that $D(\mathrm{m}(f)\mathrm{m}(g))$ is dense in $\mathcal{L}_2$.

Let now $h \in D[(\mathrm{m}(f)\mathrm{m}(g))^*]$. Then there is $h_1 \in \mathcal{L}_2$ such that

$$\langle fgy, h\rangle = \langle y, h_1\rangle$$

for all $y \in D(\mathrm{m}(f)\mathrm{m}(g))$. Replacing y by $\chi_n y$ gives

$$\int_M fg\chi_n y\bar{h}d\nu = \int_M \chi_n y\bar{h}_1 d\nu$$

or

$$\int_M y\cdot\overline{\chi_n fgh}d\nu = \int_M y\cdot\overline{\chi_n h_1}d\nu.$$

Now $|\chi_n fgh| \leqslant n^2|h|$ so that $\chi_n fgh \in \mathcal{L}_2$. Since also $\chi_n h_1 \in \mathcal{L}_2$, the last equation can be written

$$\langle y, \chi_n fgh\rangle = \langle y, \chi_n h_1\rangle$$

for all y in the dense set $D(\mathrm{m}(f)\mathrm{m}(g))$. Hence

$$fg\chi_n h = \chi_n h_1 \qquad \nu - a.e..$$

As before we conclude that $h \in D(\mathrm{m}(fg))$ and that $h_1 = (\mathrm{m}(f)\mathrm{m}(g))^*h = fgh$.

Thus $(\mathrm{m}(f)\mathrm{m}(g))^* \subset \mathrm{m}(fg)$ and by a previous result we obtain

$$\mathrm{m}(fg) = (\mathrm{m}(f)\mathrm{m}(g))^*.$$

Since $(\mathrm{m}(f)\mathrm{m}(g))^* = \mathrm{m}(fg) = \mathrm{m}(gf) = (\mathrm{m}(g)\mathrm{m}(f))^* \supset \mathrm{m}(f)^*\mathrm{m}(g)^* = \mathrm{m}(f)\mathrm{m}(g)$, $\mathrm{m}(f)\mathrm{m}(g)$ is symmetric and we have finally

$$\overline{\mathrm{m}(f)\mathrm{m}(g)} = (\mathrm{m}(f)\mathrm{m}(g))^{**} = \mathrm{m}(fg).$$

This proves *1.1 (3)*.

We shall say that a prequantization scheme is of *multiplicity m* if for each subspace $\mathcal{H}_k$ of *1.9* we have

$\mathcal{H}_k = \{0\}$ if $k \neq m$ $(m \in \mathbb{N} \cup \{\infty\})$. We now add the following assumption.

† *If M is the configuration space of a classical system consisting of N particles, the nth particle having spin j_n and isotopic spin k_n, and if Λ is a prequantization scheme on M, then Λ is of multiplicity $\prod_{n=1}^{N}(2j_n + 1)(2k_n + 1)$.*

Note that we are not asserting a converse assumption; indeed, if the nth particle had spin j_n' and isotopic spin k_n' Λ would have the same multiplicity provided that $\prod_{n=1}^{N}(2j_n + 1)(2k_n + 1) = \prod_{n=1}^{N}(2j_n' + 1)(2k_n' + 1)$. The precise specification of the actual system cannot be done at this stage: it would appear as an extension of the theory to be given in succeeding sections. Let us note parenthetically that the most general quantum system that can be 'described classically' in some sense would correspond to the most general decomposition *1.9*. It has not yet been ascertained by the author to what systems such a general decomposition would correspond. Note also that † could clearly be generalised to the case of supermultiplets.

If Λ is a prequantization scheme of multiplicity *1* on a manifold M, and if S is the set of all $P(\Delta)$ for

$\Delta \in \mathbf{B}(M)$, then it can be shown by the theory that leads to *1.4* that S is maximal Abelian in the set of all projections on $\mathcal{H}$, i.e. if J is a projection which commutes with every element of S then $J \in S$. This condition is equivalent to the statement that if A is a bounded operator on $\mathcal{H}$ which commutes with every member of the algebra S'' of bounded operators generated by S, then $A \in S''$ (in the terminology of Dirac, S'' is a complete commuting set of operators). Clearly, a single particle of spin zero is described by a prequantization scheme of multiplicity *1*. It must be considered an axiom that † is the correct generalisation to the case of higher spins.

§2 Momentum Observables

In this section we introduce the notion of momentum observables. We shall have occasion to use some basic notions in the theory of differentiable manifolds. For the details see Abraham [1], Sternberg [1] or Helgason [1].

Let M be a manifold. The *tangent bundle* TM of M can be considered to be M with a vector space of the same dimension as M attached at each point, i.e.

$$TM = \bigcup_{q \in M} \{q\} \times V_q,$$

where for each $q \in M$, V_q is a finite-dimensional real vector space.

For any finite-dimensional vector space V, let us write $T^r_s(V)$ for the space of tensors on V of contravariant order r and covariant order $s (r,s = 0,1,2,\ldots)$. We write

$$T^r_s M = \bigcup_{q\in M} \{q\} \times T^r_s(V_q).$$

We have $T^1_0 M = TM$; $T^*M = T^0_1 M = \bigcup_{q\in M} \{q\} \times V_q{}^*$ is called the *cotangent bundle* of M.* In the case where M is an open subset of a finite-dimensional vector space V, we can identify V_q with V for every q and we have

$$TM = M \times V, \quad T^*M = M \times V^* \quad .$$

In general, V_q can be thought of as the set of possible 'velocities' of the system when in the position q; $V_q{}^*$ will then be the corresponding set of 'momenta'.

Let $\phi: M \to M$ be a diffeomorphism. Then we can define $T^r_s \phi$: $T^r_s M \to T^r_s M$ (Abraham [1, §6]. $T\phi = T^1_0 \phi$ is called the *tangent* of ϕ. For $T^0_1 \phi$ we write ϕ^*. If M is an open subset of V, then $T\phi$ works out to be

$$T\phi(q,v) = (\phi(q), \mathrm{D}\phi(q).v)$$

where $(q,v) \in M \times V$, and ϕ^*:

$$\phi^*(q,p) = (\phi(q), p \circ \mathrm{D}\phi^{-1}(\phi(q)))$$

where $(q,p) \in M \times V^*$. [In terms of coordinates: if

* If V is a finite-dimensional real vector space, the *dual space* V^* is the space of linears maps $V \to \mathbf{R}$.

$q = (q^i)_{i=1}^n$, $v = (v^i)_{i=1}^n$ with respect to a basis of V and $p = (p_i)_{i=1}^n$ with respect to the dual basis, then writing $\phi(q) = (\phi^i(q))_{i=1}^n$, $\phi^{-1}(q) = (\phi^{-1,i}(q))_{i=1}^n$, we have

$$T\phi(q,v) = \left[(\phi^i(q))_{i=1}^n, \left(\sum_{j=1}^n v^j \frac{\partial\phi^i}{\partial q^j}\right)_{i=1}^n\right]$$

and

$$\phi^*(q,p) = \left[(\phi^i(q))_{i=1}^n, \left(\sum_{j=1}^n p_j \left(\frac{\partial\phi^{-1,j}}{\partial q^i}\right)_{\phi(q)}\right)_{i=1}^n\right].]$$

As a simple example, let $M = V = \mathbf{R}$ and $\phi : \mathbf{R} \to \mathbf{R}$, $q \to \sinh q$. Then

$$T\phi(q,v) = (\sinh q,\ v \cosh q),$$

$$\phi^*(q,p) = (\sinh q,\ p/\cosh q).$$

The point about ϕ^* in general is that if M is the configuration space of a classical system then T^*M will be phase space and ϕ^* will be a canonical transformation.

For a manifold M, let $\mathbf{X}(M)$ denote the set of all vector fields on M, i.e. the set of all C^∞ maps $X: M \to \bigcup_{q\in M} V_q$ for which $X(q) \in V_q$ for every $q \in M$*. $\mathbf{X}(M)$ can be identified with the set of all $\mathbf{R}$-linear derivations of $\mathbf{F}(M)$. For each $q_0 \in M$, $X \in \mathbf{X}(M)$ defines a

* This definition of a vector field differs from the one tacitly employed in I §6 and the usual definition as a map $M \to TM$. Our purpose is to simplify the statements of results to be given below. The same applies to our definition of df which is different from the usual $df: M \to T^*M$ used in I §6. The tangent space at $q \in M$ is $T_qM = \{q\} \times V_q$, the cotangent space $T_q{}^*M = \{q\} \times V_q^*$.

local C^∞ one-parameter group $\{F_t\colon |t| < a\}$ for some $a \in \mathbf{R}$, $a > 0$, of diffeomorphisms of a neighbourhood of q_0. If M is an open subset of V, this local group is given by the solution of the equation

$$Dc(t) = X(c(t))$$

where $c(t) = F_t q_0$ (c is a *curve* at q_0). [In classical notation and writing $X(q) = (X^i(q))_{i=1}^n$ the above equation becomes the system of differential equations

$$\frac{dq^i(t)}{dt} = X^i(q^1, q^2, \ldots, q^n) \quad (1 \leqslant i \leqslant n)$$

with the initial condition $q(0) = q_0$.]

We say that X is *complete on M* if the local group can be extended to a *group* of diffeomorphisms of M. This is the same as saying that the solution of the above equations with arbitrary initial point q_0 exists for all $t \in \mathbf{R}$. In case X is complete, we write $\{F_t : t \in \mathbf{R}\}$ for the one-parameter group and call F the *flow* of X on M. Then we have

$$F\colon \mathbf{R} \times M \to M, \qquad (t,q) \to F_t q$$

and F is C^∞ on $\mathbf{R} \times M$ (see Abraham [1, §7] for details).

For $f \in \mathbf{F}(T^*M)$, we shall denote by X_f the vector field in $\mathbf{X}(T^*M)$ generated by f (Abraham [1, p.99]). In case M is an open subset of V we have

$$X_f^i(q) = \begin{cases} \dfrac{\partial f}{\partial p_i} & (1 \leqslant i \leqslant n) \\[2ex] -\dfrac{\partial f}{\partial q^i} & (n+1 \leqslant i \leqslant 2n) \end{cases}.$$

If X_f is complete on T^*M we shall write $F^f\colon \mathbf{R} \times T^*M \to T^*M$, $(t,x) \to F^f_t x$ for the flow of X_f on T^*M. In this case f is a 'generating function' for the group $\{F^f_t\colon t \in \mathbf{R}\}$.

For $f \in \mathbf{F}(M)$, we shall denote by $\mathrm{d}f\colon M \to \bigcup_{q \in M} V_q{}^*$ the differential 1-form associated with f.* If M is an open subset of V then $\mathrm{d}f = \mathrm{D}f$ and $\mathrm{d}f(q) = \left(\frac{\partial f}{\partial q^i}\right)_{i=1}^{n}$ with respect to the dual basis.

2.1 Definition

If $X \in \mathbf{X}(M)$, define $\Gamma(X)\colon T^*M \to \mathbf{R}$ by

$$\Gamma(X)(q,p) = p.X(q)$$

(observe that $p \in V_q{}^*$, $X(q) \in V_q$). $\Gamma(X)$ is called the *momentum* of X.

If M is an open subset of V and $X(q) = (X^i(q))_{i=1}^{n}$, then $\Gamma(X)(q,p) = \sum_{i=1}^{n} X^i(q)p_i$.

The following results are of importance:

2.2 Theorem

(i) $\Gamma(X) \in \mathbf{F}(T^*M)$ for all $X \in \mathbf{X}(M)$;

(ii) if $\phi\colon M \to M$ is a diffeomorphism, then $\phi^*\colon T^*M \to T^*M$ is a symplectic diffeomorphism;

* See footnote on page 67.

(iii) if $X \in \mathbf{X}(M)$ is complete on M with flow ϕ, then $X_{\Gamma(X)}$ is complete on T^*M with flow ϕ^*, where $(\phi^*)_t = (\phi_t)^*$;

(iv) $\Gamma(\mathbf{X}(M))$ is a Lie subalgebra of $\mathbf{F}(T^*M)$ and Γ is an anti-homomorphism, i.e.

$$[\![\Gamma(X), \Gamma(Y)]\!] = -\Gamma([X,Y]),$$

where $[X,Y]$ is the Lie bracket of the vector fields X, Y ($XY - YX$ when X, Y are identified with derivations (differential operators) of $\mathbf{F}(M)$).

Proof. See Marsden [1, p. 351] (statement *(iv)* is included here for completeness only and will not be needed in the sequel).

In view of *2.2(iii)*, for such ϕ we shall write $\phi_t^* = (\phi^*)_t = (\phi_t)^*$.

Let M be a manifold. We shall write

$$\mathbf{P}(M) = \{\Gamma(X) : X \in \mathbf{X}(M) \text{ is complete on } M\}.$$

Elements of $\mathbf{P}(M)$ will be called *momentum observables on M*. By *2.2 (i)* they are C^∞ real-valued functions on T^*M. If M is an open subset of V, $\mathbf{P}(M)$ consists of those functions of the form $(q,p) \to \sum_{i=1}^{n} a^i(q)p_i$ with C^∞ a_i's for which the vector field $X: q \to (a^i(q))_{i=1}^n$ is complete on M. Examples for the case $M = V = \mathbf{R}^3$ are the linear momentum components p_i, the angular momentum components $q^i p_j - q^j p_i$ and the scalar $\sum_{i=1}^{3} q^i p_i$.

As an illustration of the foregoing definitions we consider the following example. Let $M = \mathbf{R} - \{0\}$ so that $T^*M = M \times \mathbf{R}^*$ and let $f\colon T^*M \to \mathbf{R}$ be given by

$$f(q,p) = \ln|q|\; p.q$$

for all $(q,p) \in T^*M$. Then f is C^∞ on T^*M and the vector field $X_f\colon M \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}^*$ is given by

$$X_f(q,p) = (q \ln|q|,\; -(1 + \ln|q|)p)$$

for all $(q,p) \in T^*M$. X_f is complete on T^*M with flow $F^f\colon \mathbf{R} \times T^*M \to T^*M$ given by

$$F_t^f(q,p) = (\operatorname{sgn}(q)\,|q|^{e^t},\; e^{-t}\,|q|^{1-e^t}p),$$

where $\operatorname{sgn}(q) = 1$ if $q > 0$, -1 if $q < 0$. f is also in $\mathbf{P}(M)$ for it is of the form $\Gamma(X)$ where $X \in \mathbf{X}(M)$ is given by

$$X(q) = q \ln|q|$$

for all $q \in M$ (see 2.1). X is complete on M with flow $\phi\colon \mathbf{R} \times M \to M$ given by

$$\phi_t q = \operatorname{sgn}(q)\,|q|^{e^t}.$$

According to 2.2 *(iii)*, the flow F^f of X_f is also given by $F_t^f = \phi_t{}^*$ where

$$\phi_t^*(q,p) = (\phi_t q,\; p \circ D\phi_{-t}(\phi_t q)).$$

An easy computation shows that this is the same as

$F_t^f(q,p)$. f is an example of a momentum observable on $M = \mathbf{R} - \{0\}$.

§3 The set $\mathbf{L}(M)$

We shall be particularly interested in those classical observables which are sums of momentum observables and position observables. We define

$$\mathbf{L}(M) = \mathbf{Q}(M) + \mathbf{P}(M)$$

$$= \{f_1 + f_2 : f_1 \in \mathbf{Q}(M),\ f_2 \in \mathbf{P}(M)\}.$$

Observe that every $f \in \mathbf{L}(M)$ has a unique decomposition $f = f_1 + f_2$ where $f_1 \in \mathbf{Q}(M)$, $f_2 \in \mathbf{P}(M)$. For $f_2 \in \mathbf{P}(M)$ implies $f_2(q,0) = 0$ for all $q \in M$ (see *2.1*), hence $f_1(q,p) = f_1(q,0) = f(q,0) - f_2(q,0) = f(q,0)$ and so f_1 is uniquely determined by f. If $f \in \mathbf{L}(M)$ we shall denote by f_1, f_2 the components of f in $\mathbf{Q}(M)$, $\mathbf{P}(M)$ respectively. We shall denote by X^f that unique vector field in $\mathbf{X}(M)$ such that $f_2 = \Gamma(X^f)$. $\phi^f : \mathbf{R} \times M \to M$, $(t,q) \to \phi_t^f q$ will denote the flow of X^f on M.

Since the function identically zero is in both $\mathbf{Q}(M)$ and $\mathbf{P}(M)$, it follows that $\mathbf{Q}(M) \subset \mathbf{L}(M)$ and $\mathbf{P}(M) \subset \mathbf{L}(M)$.

Let $\mathbf{L}^\infty(M) = \mathbf{L}(M) \cap \mathbf{F}(T^*M)$, i.e. $\mathbf{L}^\infty(M)$ is the set of C^∞ members of $\mathbf{L}(M)$. Clearly $\mathbf{P}(M) \subset \mathbf{L}^\infty(M) \subset \mathbf{L}(M)$.

The next theorem is of importance.

3.1 Theorem

Let $f \in \mathbf{L}^{\infty}(M)$. Then X_f is complete on T^*M; the flow F^f of X_f is given by

$$F_t^f(q,p) = \left(\phi_t^f q,\ \mathrm{P}_2 \circ \phi_t^{f*}(q,p) - \int_0^t \mathrm{d}(f_1' \circ \phi_{s-t}^f)(\phi_t^f q)ds\right)$$

for all $(q,p) \in T^*M$, where P_2 is the projection onto the second factor: $(q,p) \to p$.

Proof. We shall construct an integral curve of X_f at $(q_0,p_0) \in T^*M$. By means of a diffeomorphism of an open set $T^*U \subset T^*M$ (U is open in M) with $(q_0,p_0) \in T^*U$ onto $U' \times \mathbf{R}^{n*}$ where $U' \subset \mathbf{R}^n$ is open, we can suppose everything to take place in $U' \times \mathbf{R}^{n*}$. Thus we have $X^f\colon U' \to \mathbf{R}^n$, $X^f \in \mathbf{X}(U')$, $f_1'\colon U' \to \mathbf{R}$, $f_2\colon U' \times \mathbf{R}^{n*} \to \mathbf{R}$, $(q,p) \to p.X^f(q)$ and $f_1' \in \mathbf{F}(U')$.

Let $c\colon (-a,a) \to U' \times \mathbf{R}^{n*}$, $t \to (c_1(t), c_2(t))$ for some $a > 0$ be an integral curve of X_f at (q_0,p_0). The classical equations of motion are

$$\mathrm{D}c_1(t) = X^f(c_1(t)), \tag{i}$$

$$\mathrm{D}c_2(t) = -\mathrm{D}f_1'(c_1(t)) - \mathrm{D}(c_2(t) \circ X^f)(c_1(t)) \tag{ii}$$

for all $t \in (-a,a)$. We have

$$\begin{aligned}\mathrm{D}(c_2(t) \circ X^f)(c_1(t)) &= \mathrm{D}(c_2(t))(X^f c_1(t)) \circ \mathrm{D}X^f(c_1(t)) \\ &= c_2(t) \circ \mathrm{D}X^f(c_1(t))\end{aligned}$$

since $c_2(t)\colon \mathbf{R}^n \to \mathbf{R}$ is linear. Hence by *(ii)*

$$Dc_2(t) = -Df_1'(c_1(t)) - c_2(t) \circ DX^f(c_1(t)). \qquad (iii)$$

By assumption the solution of *(i)* is known:

$$c_1(t) = \phi_t^f q_0; \qquad (iv)$$

Hence *(iii)* can be written

$$Dc_2(t) = -Df_1'(\phi_t^f q_0) - c_2(t) \circ DX^f(\phi_t^f q_0). \qquad (v)$$

(v) is of the form

$$Dc_2(t) = A(t) \cdot c_2(t) + B(t) \qquad (vi)$$

where $A(t)\colon \mathbf{R}^{n*} \to \mathbf{R}^{n*}$, $x \mapsto -x \circ DX^f(\phi_t^f q_0)$ is linear and $B(t) = -Df_1'(\phi_t^f q_0) \in \mathbf{R}^{n*}$. The solution of *(vi)* can be expressed in terms of the solution of the corresponding homogeneous equation

$$Dc_2(t) = A(t) \cdot c_2(t). \qquad (vii)$$

By assumption the solution of *(vii)* is known:

$$c_2(t) = P_2 \circ \phi_t^{f*}(q_0, p') \qquad (viii)$$

where $p' \in \mathbf{R}^{n*}$ is arbitrary. We refer to Dieudonne [1, p.305, 10.8.4]. If $|s| < a$, our need is for a particular solution *(viii)* with the property $c_2(s) = p_0$. The choice $p' = P_2 \circ \phi_{-s}^{f*}(\phi_s^f q_0, p_0)$ in *(viii)* gives

$$c_2(t) = \mathrm{P}_2 \circ \phi_t^{f*}(q_0, \mathrm{P}_2 \circ \phi_{-s}^{f*}(\phi_s^f q_0, p_0))$$

$$= \mathrm{P}_2 \circ \phi_t^{f*} \circ \phi_{-s}^{f*}(\phi_s^f q_0, p_0)$$

$$= \mathrm{P}_2 \circ \phi_{t-s}^{f*}(\phi_s^f q_0, p_0)$$

as the unique solution of *(vii)* with the property $c_2(s) = p_0$. The solution of *(vi)* having the property $c_2(0) = p_0$ is consequently

$$c_2(t) = \mathrm{P}_2 \circ \phi_t^{f*}(q_0, p_0) - \int_0^t \mathrm{P}_2 \circ \phi_{t-s}^{f*}(\phi_s^f q_0, \mathrm{D}f_1'(\phi_s^f q_0))ds$$

(Dieudonne, loc. cit, p.306, 10.8.6). Since

$$\phi_{t-s}^{f*}(\phi_s^f q_0, \mathrm{D}f_1'(\phi_s^f q_0)) = (\phi_t^f q_0, \mathrm{D}f_1'(\phi_s^f q_0) \circ \mathrm{D}\phi_{s-t}^f(\phi_t^f q_0)),$$

we can write for the integral curve c:

$$c(t) = (\phi_t^f q_0, \mathrm{P}_2 \circ \phi_t^{f*}(q_0, p_0) - \int_0^t \mathrm{D}f_1'(\phi_s^f q_0) \circ \mathrm{D}\phi_{s-t}^f(\phi_t^f q_0)ds)$$

$$= (\phi_t^f q_0, \mathrm{P}_2 \circ \phi_t^{f*}(q_0, p_0) - \int_0^t \mathrm{D}(f_1' \circ \phi_{s-t}^f)(\phi_t^f q_0)ds) \qquad (ix)$$

with $c(0) = (q_0, p_0)$, $|t| < a$. On transferring via the inverse diffeomorphism to T^*U $c(t)$ can be written finally in an invariant form:

$$c(t) = F_t^f(q_0, p_0) = (\phi_t^f q_0, \mathrm{P}_2 \circ \phi_t^{f*}(q_0, p_0) - \int_0^t \mathrm{d}(f_1' \circ \phi_{s-t}^f)(\phi_t^f q_0)ds) \qquad (x)$$

where $(q_0, p_0) \in T^*U$ and $|t| < a$.

Clearly, for each $(q_0, p_0) \in T^*M$, the local flow given by (x) can be extended to a flow $F^f : \mathbf{R} \times T^*M \to T^*M$ (define F^f by (x) for any $(q_0, p_0) \in T^*M$ and any $t \in \mathbf{R}$). Thus X_f is complete on T^*M.

Both *3.1* and the next theorem are important for the theory to be developed. If $X \in \mathbf{X}(M)$ is complete on M with flow F and $\phi : M \to M$ is a diffeomorphism, we define $\phi X \in \mathbf{X}(M)$ to be that (unique) vector field having the flow $(t, q) \to \phi \circ F_t \circ \phi^{-1}(q)$. If M is an open subset of V then ϕX is given by $(\phi X)(q) = \mathrm{D}\phi(\phi^{-1} q).X(\phi^{-1} q)$.

3.2 Theorem

Let $f \in \mathbf{L}^\infty(M)$, $g \in \mathbf{L}(M)$. *Then for all* $t \in \mathbf{R}$,

$g \circ F^f_t \in \mathbf{L}(M)$,

$$(g \circ F^f_t)'_1(q) = g'_1(\phi^f_t q) - \int_0^t \mathrm{d}(f'_1 \circ \phi^f_{s-t})(\phi^f_t q).X^g(\phi^f_t q)ds$$

for all $q \in M$ *and*

$$(g \circ F^f_t)_2 = \Gamma(\phi^f_{-t} X^g).$$

If $g \in \mathbf{L}^\infty(M)$ *then* $g \circ F^f_t \in \mathbf{L}^\infty(M)$. *If* $g \in \mathbf{Q}(M)$ *then* $g \circ F^f_t \in \mathbf{Q}(M)$.

Proof. By *3.1*, for $g \in \mathbf{L}(M)$,

$$(g \circ F^f_t)(q,p) = g(F^f_t(q,p)) = g'_1(\phi^f_t q) +$$

$$\mathrm{P}_2 \circ \phi^{f*}_t(q,p).X^g(\phi^f_t q) - \int_0^t \mathrm{d}(f'_1 \circ \phi^f_{s-t})(\phi^f_t q).X^g(\phi^f_t q)ds$$

for all $(q,p) \in T^*M$. By means of a local diffeomorphism into $\mathbf{R}^n$ we can write

$$P_2 \circ \phi_t^{f*}(q,p).X^g(\phi_t q) = p.[D\phi_{-t}^f(\phi_t^f q).X^g(\phi_t^f q)]$$
$$= p.(\phi_{-t}^f X^g)(q)$$
$$= \Gamma(\phi_{-t}^f X^g)(q,p).$$

Thus $g \circ F_t^f$ is the sum of a function in $\mathbf{Q}(M)$ and a function in $\mathbf{P}(M)$ and hence is in $\mathbf{L}(M)$. If $g \in \mathbf{Q}(M)$ then $g_2 = 0$, $X^g = 0$ and $(g \circ F_t^f)_2 = 0$. That $g \circ F_t^f \in \mathbf{L}^\infty(M)$ if $g \in \mathbf{L}^\infty(M)$ is obvious.

§4 Quantization Schemes

If N is a topological space we shall say that a real-valued function f on N is *locally bounded* if for each compact set $S \subset N$, $f|S$ is bounded. A continuous function on N is locally bounded.

If M is a manifold $\mathbf{L}_\circ(M)$ will denote the set of locally bounded members of $\mathbf{L}(M)$. We have the inclusions $\mathbf{P}(M) \subset \mathbf{L}^\infty(M) \subset \mathbf{L}_\circ(M) \subset \mathbf{L}(M)$. It is clear that if $g \in \mathbf{L}_\circ(M)$, $f \in \mathbf{L}^\infty(M)$, then $g \circ F_t^f \in \mathbf{L}_\circ(M)$ for every $t \in \mathbf{R}$.

Put $\mathbf{U}(M) = \mathbf{Q}(M) \cup \mathbf{L}_\circ(M)$.

The object of study in this section is summarised in

4.1 Definition

Let $\hbar > 0$. *A quantization scheme on a manifold* M *is a mapping* Λ *of* $\mathbf{U}(M)$ *into the set of all self-adjoint operators in a complex, infinite-dimensional, separable Hilbert space* $\mathcal{H}$ *such that* $\Lambda|\mathbf{Q}(M)$ *is a prequantization scheme on* M *and if* $f \in \mathbf{L}^{\infty}(M)$, *then for all* $g \in \mathbf{U}(M)$ *and* $t \in \mathbf{R}$,

$$\Lambda(g \circ F_t^f) = U_t^f \Lambda g U_{-t}^f$$

where

$$U_t^f = \exp\left(\frac{it}{\hbar}\Lambda f\right).$$

We shall say that Λ *is of multiplicity* m *if* $\Lambda|\mathbf{Q}(M)$ *is of multiplicity* m.

4.1 is seen to embody a global version of the Poisson bracket-commutator relations, for proceeding formally and ignoring questions about domains of operators we have

$$\Lambda\left(\frac{d}{dt}(g \circ F_t^f)\right)_{t=0} = \Lambda([\![g,f]\!]) =$$

$$\left[U_t^f\left(\frac{i}{\hbar}\Lambda f\right)\Lambda g U_{-t}^f + U_t^f \Lambda g\left(-\frac{i}{\hbar}\Lambda f\right)U_{-t}^f\right]_{t=0}$$

$$= \left[\frac{1}{i\hbar}U_t^f(\Lambda g \Lambda f - \Lambda f \Lambda g)U_{-t}^f\right]_{t=0}$$

$$= \frac{1}{i\hbar}(\Lambda g \Lambda f - \Lambda f \Lambda g),$$

i.e $\Lambda([\![g,f]\!]) = \frac{1}{i\hbar}[\Lambda g, \Lambda f]$.

On the other hand, *4.1* can be looked upon as the mathematical expression of the principle of equivalence between observers displaced with respect to each other *in phase space*. For instance, if $M = \mathbf{R}^n$ and g is the ith coordinate function $q \to q^i$, then an observer translated by an amount t in the direction of the negative x^i axis with respect to a second observer will observe the function $q \to q^i + t$. Contained in *4.1* is the statement that the corresponding quantum observable defined by the first observer will be connected with that defined by the second observer by a unitary equivalence and furthermore that unitary equivalence will be expressible directly in terms of the 'generating function' of the displacement. The same will apply for rotations and in general for any one-parameter group of C^∞ transformations of M. A similar line of argument holds for an observer moving with constant velocity t in the direction of the negative x^i axis. He will observe the ith linear momentum coordinate p_i of a particle of mass m to be $p_i + mt$ and *4.1* contains the statement that the corresponding quantum observable shall be unitarily equivalent to that defined by a stationary observer.

Again, if we take $M = \mathbf{R}$, $f(q,p) = p.1$, $g(q,p) = q$ in *4.1* we obtain the requirement

$$\Lambda(g \circ F_t^f) = \Lambda(g + t) = \overline{\Lambda g + \Lambda t} \qquad \text{(by } 1.1\ (1)\text{)}$$

$$= \overline{\Lambda g + tI} \qquad \text{(by } 1.1\ (4) \text{ and } 1.1\ (2)\text{)}$$

$$= \Lambda g + tI,$$

where we have simply written t for the constant function with value t, and

$$\exp\ (is(\Lambda g + tI)) = e^{ist}e^{is\Lambda g}$$

$$= e^{it/\hbar\Lambda f}e^{is\Lambda g}e^{-it/\hbar\Lambda f},$$

i.e. $$e^{it/\hbar\Lambda f}e^{is\Lambda g} = e^{its}e^{is\Lambda g}e^{it/\hbar\Lambda f}.$$

The last equation is Weyl's form of the Heisenberg commutation relations (Weyl [2]), all solutions for Λg, Λf of which were obtained by von Neumann [2].

§5 An Example

We shall give an example of a quantization scheme; the following terminology will be necessary.

If M is a manifold we shall say that a measure μ on $\mathbf{B}(M)$ is *quasi-invariant under a diffeomorphism* ϕ of M if the measure $\mu_\phi : \Delta \to \mu(\phi(\Delta))$, $\Delta \in \mathbf{B}(M)$, is equivalent to μ, i.e. $\mu_\phi(\Delta) = 0$ iff $\mu(\Delta) = 0$. Any manifold admits a measure which is quasi-invariant under *all* diffeomorphisms of M (an example is Lebesgue measure in $\mathbf{R}^n$). If μ is quasi-invariant under ϕ, we can define the Radon-Nikodym derivative $\frac{d\mu_\phi}{d\mu}$. This is a real-valued function on M and has the property $\frac{d\mu_\phi}{d\mu}(x) > 0$ for μ-almost all $x \in M$.

A measure which is quasi-invariant under all C^∞ one-parameter groups of diffeomorphisms of M will be simply called *quasi-invariant*.

Let μ be a σ-finite quasi-invariant measure on $\mathbf{B}(M)$. If $f \in \mathbf{L}(M)$ let $\mu_t^f = \mu_{\phi_t^f}$. Let $\mathcal{H} = \tilde{\mathcal{L}}_2(M, \mathbf{C}^m, \mu)$. Suppose $f \in \mathbf{L}_\circ(M)$. Then f_1' is locally bounded on M. For each $x \in M$ the map $\mathbf{R} \to M$, $s \to \phi_s^f x$ is continuous; hence the sets $\{\phi_s^f x\colon 0 \leqslant s \leqslant t\}$ if $t \geqslant 0$ and $\{\phi_s^f x\colon t \leqslant s \leqslant 0\}$ if $t \leqslant 0$ are compact. Since f_1' are measurable and moreover bounded on these sets, the integral $\int_0^t f_1'(\phi_s^f x)ds$ exists and is finite for each $t \in \mathbf{R}$.

If $f \in \mathbf{Q}(M)$, then $\phi_t^f x = x$ for all $x \in M$ and $t \in \mathbf{R}$ so that $\int_0^t f_1'(\phi_s^f x)ds$ exists in this case also. For $h \in \mathcal{L}_2(M, \mathbf{C}^m, \mu)$ and $f \in \mathbf{U}(M)$ let $U_t^f h\colon M \to \mathbf{C}$ be given by

$$(U_t^f h)(x) = \exp\left(i/\hbar \int_0^t f_1'(\phi_s^f x)ds\right). \sqrt{\frac{d\mu_t^f(x)}{d\mu}}\; h(\phi_t^f x) \qquad (1)$$

for all $x \in M$. Then we have

5.1 Lemma

$U^f\colon t \to U_t^f$ is a strongly continuous unitary representation on $\tilde{\mathcal{L}}_2(M, \mathbf{C}^m, \mu)$ of the additive group of real numbers.

Proof. If f is any measurable, complex-valued function on M for which $\int_M f d\mu$ is defined and if ϕ is any diffeomorphism of M, then $\int_M f\circ\phi\, d\mu_\phi$ is defined and

$$\int_M f(x)\,d\mu(x) = \int_M f(\phi x)\,d\mu_\phi(x). \tag{i}$$

Hence

$$\int_M \|(U_t^f h)(x)\|_m^2 d\mu(x) = \int_M \|h(\phi_t^f x)\|_m^2 d\mu_t^f(x)$$

$$= \int_M \|h(x)\|_m^2 d\mu(x) = \|h\|_m^2 < \infty.$$

Thus for each t, U_t^f is a unitary mapping onto $\mathcal{L}_2(M,\mathbf{C}^m,\mu)$.

We next prove that for all s, $t\in\mathbf{R}$

$$\frac{d\mu_{s+t}^f}{d\mu}(x) = \frac{d\mu_s^f}{d\mu}(x)\ \frac{d\mu_t^f}{d\mu}(\phi_s^f x) \tag{ii}$$

for μ-almost all $x\in M$. We have, for $\Delta\in\mathbf{B}(M)$,

$$\mu_{s+t}^f(\Delta) = \mu(\phi_{s+t}^f\Delta) = \mu(\phi_t^f\circ\phi_s^f\Delta) = \mu_t^f(\phi_s^f\Delta)$$

$$= \int_{\phi_s^f\Delta} d\mu_t^f(x) = \int_M \chi_{\phi_s^f\Delta}(x)\frac{d\mu_t^f}{d\mu}(x)\,d\mu(x)$$

$$= \int_M \chi_{\phi_s^f\Delta}(\phi_s^f x)\ \frac{d\mu_t^f}{d\mu}(\phi_s^f x)\,d\mu_s^f(x) \qquad \text{(by } (i))$$

$$= \int_M \chi_\Delta(x)\frac{d\mu_t^f}{d\mu}(\phi_s^f x)\ \frac{d\mu_s^f}{d\mu}(x)\,d\mu(x).$$

But $\mu^f_{s+t}(\Delta) = \int_M \chi_\Delta(x)\frac{d\mu^f_{s+t}}{d\mu}(x)d\mu(x)$; hence *(ii)* follows.

Furthermore, if

$$A^f_t(x) = \exp\ (i/\hbar \int_0^t f'_1(\phi^f_u x)du), \qquad (iii)$$

then

$$A^f_s(x)A^f_t(\phi^f_s s) = A^f_{s+t}(x) \qquad (iv)$$

for all $s,\ t \in \mathbf{R}$ and

$$A^f_0(x) = 1 \quad . \qquad (v)$$

By *(ii)*, *(iv)* and *(v)* it follows easily that $U^f_s U^f_t = U^f_{s+t}$, i.e. U^f is a representation of the additive group $\mathbf{R}$.

It remains to be shown that U^f is strongly continuous, i.e. for each $h \in \tilde{\mathcal{L}}_2(M,\mathbf{C}^m,\mu)$, the map $\mathbf{R} \to \tilde{\mathcal{L}}_2(M,\mathbf{C}^m,\mu)$, $t \to U^f_t h$ is continuous. For this it is sufficient to show that the map $\mathbf{R} \to \mathbf{C}$, $t \to \langle U^f_t \mathrm{f}, \mathrm{g}\rangle_m$ is measurable for all $\mathrm{f},\ \mathrm{g} \in \mathcal{L}_2(M,\mathbf{C}^m,\mu)$.

We first note that we may assume the map $\mathbf{R} \times M \to \mathbf{R}$, $(t,x) \to \frac{d\mu^f_t}{d\mu}(x)$ to be measurable (Mackey [1, p.317], Varadarajan [1, p.16]). Next we shall show that the map $\mathbf{R} \times M \to \mathbf{R}$, $(t,x) \to \xi^f(t,x) = \int_0^t f'_1(\phi^f_s x)ds$ is measurable. If $t \geqslant 0$ we have

$$\xi^f(t,x) = \int_{\mathbf{R}} \chi_{[0,\,t]}(s) f'_1(\phi^f_s x)ds.$$

The function η: $[0,\infty) \times \mathbf{R} \to \mathbf{R}$, $(t,s) \to \chi_{[0,t]}(s)$ is measurable; for if $a \leqslant 0$ (resp. $a > 1$) $\eta^{-1}((a,\infty)) = [0,\infty) \times \mathbf{R}$ (resp. ϕ) while if $0 < a \leqslant 1$ $\eta^{-1}((a,\infty)) = \{(t,s): 0 \leqslant s \leqslant t\}$ which is of the form $\{y \in [0,\infty) \times \mathbf{R}: 0 \leqslant \mathrm{pr}_2(y) \leqslant \mathrm{pr}_1(y)\}$ where pr_i is the ith projection. Hence $\eta^{-1}((a,\infty)) \in \mathbf{B}([0,\infty) \times \mathbf{R})$ (Hewitt and Stromberg [1, p.152]). It follows that the function $[0,\infty) \times M \times \mathbf{R} \to \mathbf{R}$, $(t,x,s) \to \chi_{[0,t]}(s)$ is measurable.

The function $M \times \mathbf{R} \to \mathbf{R}$, $(x,s) \to f_1'(\phi_s^f x)$ is measurable; for $(x,s) \to \phi_s^f x$ is continuous and f_1' is measurable. Hence the function $[0,\infty) \times M \times \mathbf{R} \to \mathbf{R}$, $(t,x,s) \to f_1'(\phi_s^f x)$ is measurable. It follows that the function $[0,\infty) \times M \times \mathbf{R} \to \mathbf{R}$, $(t,x,s) \to \chi_{[0,t]}(s) f_1'(\phi_s^f x)$ is measurable and by writing this function as the difference of its positive and negative parts it follows from the Fubini theorem that the function $[0,\infty) \times M \to \mathbf{R}$, $(t,x) \to \xi^f(t,x)$ is measurable.

A similar argument holds if $t < 0$, that $(-\infty,0) \times M \to \mathbf{R}$, $(t,x) \to \xi^f(t,x)$ is measurable. Hence the map $\mathbf{R} \times M \to \mathbf{R}$, $(t,x) \to \xi^f(t,x)$ is measurable. Clearly, the same is true for the function $\mathbf{R} \times M \to \mathbf{R}$, $(t,x) \to A_t^f(x) = \exp\,(i/\hbar\ \xi^f(t,x))$.

Since for all f, $g \in \mathcal{L}_2(M,\mathbf{C}^m,\mu)$ the maps $(t,x) \to f(\phi_t^f x)$ and $(t,x) \to g(x)$ are both measurable, it follows by considering components that $(t,x) \to \langle f(\phi_t^f x), g(x)\rangle_m$ is measurable.

Thus we have proved that the mapping $\mathbf{R} \times M \to \mathbf{C}$ given by

$$(t,x) \to A_t^f(x) \sqrt{\frac{d\mu_t^f}{d\mu}(x)}\ \langle \mathrm{f}(\phi_t^f x), \mathrm{g}(x) \rangle_m$$

is measurable. But this mapping is just the mapping $(t,x) \to \langle (U_t^f \mathrm{f})(x), \mathrm{g}(x) \rangle_m$. By the Fubini theorem it now follows that

$$t \to \langle U_t^f \mathrm{f}, \mathrm{g} \rangle_m = \int_M \langle (U_t^f \mathrm{f})(x), \mathrm{g}(x) \rangle_m d\mu(x)$$

is measurable. This completes the proof.

If $\{U_t : t \in \mathbf{R}\}$ is a strongly continuous one-parameter group of unitary operators on a Hilbert space then there exists a unique self-adjoint operator A such that $U_t = \exp(itA)$ for all $t \in \mathbf{R}$. A is called the *self-adjoint generator* of the group.

5.2 Theorem

For each $f \in \mathbf{U}(M)$ *let* Λf *in* $\mathfrak{L}_2(M, \mathbf{C}^m, \mu)$ *be* $\hbar$ *times the self-adjoint generator of the group* $\{U_t^f : t \in \mathbf{R}\}$, *where* U_t^f *is given by (1).*

Then Λ *is a quantization scheme of multiplicity* m *on* M.

Proof. First observe that if $f \in \mathbf{Q}(M)$ then $f_2 = 0$, $\phi_t^f x = x$ for all $x \in M$ and $t \in \mathbf{R}$ and $\mu_t^f = \mu$. Hence

$$(U^f_t h)(x) = \exp\ (i/\hbar \int_0^t f_1'(x)ds)h(x) = \exp\ (it/\hbar f_1'(x))h(x)$$

for all $h \in \mathcal{L}_2(M,\mathbf{C}^m,\mu)$, i.e. $U^f_t = \mathrm{m}(\exp\ (it/\hbar f_1'))$. The self-adjoint generator of the group $\{U^f_t\colon t \in \mathbf{R}\}$ is then $\mathrm{m}(f_1'/\hbar) = 1/\hbar\mathrm{m}(f_1')$ and $\Lambda f = \mathrm{m}(f_1')$. By *1.9* it is immediate that $\Lambda|\mathbf{Q}(M)$ is a prequantization scheme of multiplicity m on M.

It suffices to prove that

$$U^f_t U^g_s U^f_{-t} = U^g_s{}^{\circ F^f_t}$$

for all $f \in \mathbf{L}^\infty(M)$, $g \in \mathbf{U}(M)$, s, $t \in \mathbf{R}$.

We shall first prove that for all $f \in \mathbf{L}^\infty(M)$, $g \in \mathbf{L}(M)$,

$$\frac{d\mu^f_t}{d\mu}(x)\ \frac{d\mu^f_{-t}}{d\mu}(\phi^g_s \circ \phi^f_t x)\ \frac{d\mu^g_s}{d\mu}\ (\phi^f_t x) = \frac{d\mu^{g \circ F^f_t}_s}{d\mu}(x) \qquad (i)$$

for μ-almost all $x \in M$. By *3.2*

$$X^{g \circ F^f_t} = \phi^f_{-t} X^g;$$

hence the flow $\phi^{g \circ F^f_t}$ of $X^{g \circ F^f_t}$ is given by

$$\phi^{g \circ F^f_t}_s = \phi^f_{-t} \circ \phi^g_s \circ \phi^f_t. \qquad (ii)$$

If $\Delta \in \mathbf{B}(M)$,

$$\mu_s^g \circ F_t^f(\Delta) = \mu_{\phi_s^g \circ F_t^f}(\Delta) = \mu(\phi_s^g \circ F_t^f \Delta) = \mu(\phi_{-t}^f \circ \phi_s^g \circ \phi_t^f \Delta)$$

$$= \mu_{-t}^f(\phi_s^g \circ \phi_t^f \Delta) = \int_{\phi_s^g \circ \phi_t^f \Delta} d\mu_{-t}^f(x)$$

$$= \int_M \chi_{\phi_s^g \circ \phi_t^f \Delta}(x) \frac{d\mu_{-t}^f}{d\mu}(x) d\mu(x)$$

$$= \int_M \chi_\Delta(\phi_{-t}^f \circ \phi_{-s}^g x) \frac{d\mu_{-t}^f}{d\mu}(x) d\mu(x)$$

$$= \int_M \chi_\Delta(\phi_{-t}^f x) \frac{d\mu_{-t}^f}{d\mu}(\phi_s^g x) \frac{d\mu_s^g}{d\mu}(x) d\mu(x)$$

$$= \int_M \chi_\Delta(x) \frac{d\mu_{-t}^f}{d\mu}(\phi_s^g \circ \phi_t^f x) \frac{d\mu_s^g}{d\mu}(\phi_t^f x) \frac{d\mu_t^f}{d\mu}(x) d\mu(x)$$

(by *(i)* of 5.1).

But $\mu_s^g \circ F_t^f(\Delta) = \int_M \chi_\Delta(x) \frac{d\mu_s^g \circ F_t^f}{d\mu}(x) d\mu(x)$; hence *(i)* follows.

Now suppose that $f \in \mathbf{L}^{\infty}(M)$, $g \in \mathbf{U}(M)$. A straightforward calculation gives

$$(U^f_t U^g_s U^f_{-t} h)(x) =$$

$$\exp\left[i/\hbar \left(\int_0^t f_1'(\phi^f_u x)\,du + \int_0^s g_1'(\phi^g_u \circ \phi^f_t x)\,du + \int_0^{-t} f_1'(\phi^f_u \circ \phi^g_s \circ \phi^f_t x)\,du \right) \right] \cdot$$

$$\cdot \sqrt{\frac{d\mu^f_t}{d\mu}(x)\ \frac{d\mu^f_{-t}}{d\mu}(\phi^g_s \circ \phi^f_t x)\frac{d\mu^g_s}{d\mu}(\phi^f_t x)}\, h(\phi^f_{-t} \circ \phi^g_s \circ \phi^f_t x). \qquad (iii)$$

For any $q \in M$ let $c_q(s) = \phi^g_s q$; then by means of a local diffeomorphism into $\mathbf{R}^n$ we can write

$$\mathrm{D}c_q(s) = X^g(\phi^g_s q)$$

for all sufficiently small s. Let $x = \phi^f_{-t} q$. Then

$$\mathrm{D}c_{\phi^f_t x}(s) = X^g(\phi^g_s \circ \phi^f_t x);$$

hence

$$\mathrm{D}(f_1' \circ \phi^f_{-v})(\phi^g_s \circ \phi^f_t x) \cdot X^g(\phi^g_s \circ \phi^f_t x)$$

$$= \mathrm{D}(f_1' \circ \phi^f_{-v})(\phi^g_s \circ \phi^f_t x) \cdot \mathrm{D}c_{\phi^f_t x}(s)$$

$$= \mathrm{D}(f_1' \circ \phi^f_{-v})(c_{\phi^f_t x}(s)) \cdot \mathrm{D}c_{\phi^f_t x}(s)$$

$$= \mathrm{D}(f_1' \circ \phi_{-v}^f \circ c_{\phi_t^f x})(s)$$

$$= \frac{\partial}{\partial s}(f_1' \circ \phi_{-v}^f \circ c_{\phi_t^f x}(s))$$

$$= \frac{\partial}{\partial s} f_1'(\phi_{-v}^f \circ \phi_s^g \circ \phi_t^f x).$$

Integration with respect to v gives, in invariant form,

$$\int_0^t [\frac{\partial}{\partial s} f_1'(\phi_{-v}^f \circ \phi_s^g \circ \phi_t^f x)]\, dv =$$

$$\int_0^t \mathrm{d}(f_1' \circ \phi_{-v}^f)(\phi_s^g \circ \phi_t^f x).X^g(\phi_s^g \circ \phi_t^f x)\, dv. \qquad (iv)$$

The functions $(v,s) \to f_1'(\phi_{-v}^f \circ \phi_s^g \circ \phi_t^f x)$ and $(v,s) \to \frac{\partial}{\partial s} f_1'(\phi_{-v}^f \circ \phi_s^g \circ \phi_t^f x)$ are both continuous on $[0,t] \times \mathbf{R}$ $(t \geqslant 0)$ or $[t,0] \times \mathbf{R}$ $(t \leqslant 0)$ (remember that $f \in \mathbf{L}^\infty(M)$). Since also the closed interval with endpoints 0, t is compact we may interchange the order of integration and differentiation in the first integral to obtain

$$\int_0^t f_1'(\phi_{-v}^f \circ \phi_s^g \circ \phi_t^f x)\, dv =$$

$$\int_0^s \int_0^t \mathrm{d}(f_1' \circ \phi_{-v}^f)(\phi_u^g \circ \phi_t^f x).\, X^g(\phi_u^g \circ \phi_t^f x)\, dv du + k^f(t,x).$$

k^f is obtained by putting $s = 0$:

$$k^f(t,x) = \int_0^t f_1'(\phi_{t-v}^f x)\, dv.$$

Simple changes of variable in all three integrals gives finally

$$\int_0^{-t} f_1'(\phi_u^f \circ \phi_s^g \circ \phi_t^f x)du + \int_0^t f_1'(\phi_u^f x)du =$$

$$- \int_0^s \int_0^t d(f_1' \circ \phi_{v-t})(\phi_u^g \circ \phi_t^f x) \cdot X^g(\phi_u^g \circ \phi_t^f x)dvdu.$$

The inner integral on the right is

$$\int_0^t d(f_1' \circ \phi_{v-t})(\phi_t^f \circ \phi_{-t}^f \circ \phi_u^g \circ \phi_t^f x) \cdot X^g(\phi_t^f \circ \phi_{-t}^f \circ \phi_u^g \circ \phi_t^f x)dv$$

which by *3.2* is equal to

$$g_1'(\phi_u^g \circ \phi_t^f x) - (g \circ F_t^f)_1'(\phi_{-t}^f \circ \phi_u^g \circ \phi_t^f x).$$

Hence

$$\int_0^t f_1'(\phi_u^f x)du + \int_0^s g_1'(\phi_u^g \circ \phi_t^f x) + \int_0^{-t} f_1'(\phi_u^f \circ \phi_s^g \circ \phi_t^f x)du$$

$$= \int_0^s (g \circ F_t^f)_1'(\phi_{-t}^f \circ \phi_u^g \circ \phi_t^f x)du$$

$$= \int_0^s (g \circ F_t^f)_1'(\phi_u^{g \circ F_t^f} x)du \qquad (v)$$

by *(ii)*. Using *(i)*, *(ii)* and *(v)*, *(iii)* can now be written

$$(U_t^f U_s^g U_{-t}^f h)(x) =$$

$$\exp\left(i/\hbar \int_0^s (g \circ F_t^f)_1'(\phi_u^{g \circ F_t^f} x)du\right) \cdot \sqrt{\frac{d\mu_s^{g \circ F_t^f}}{d\mu}}(x) h(\phi_s^{g \circ F_t^f} x)$$

$$= (U_s^{g \circ F_t^f} h)(x).$$

Hence

$$U_t^f U_s^g U_{-t}^f = U_s^{g \circ F_t^f}.$$

This completes the proof of the theorem.

A quantization scheme of the type described in *5.2* will be called *canonical*. The variables describing all such canonical schemes are the multiplicity m, the measure μ and the manifold M. We stress here the hypotheses on μ of σ-finiteness and quasi-invariance. If M is a finite-dimensional normed space V then there is only one such measure μ (apart from equivalence) so that we may speak in this case of *the* canonical quantization scheme on V of given multiplicity.

Let us remark at this point that the operator correspondences given by the canonical schemes are in fact the ones employed in conventional quantum mechanics. If the reader will take M to be $\mathbf{R}^n$, μ to be Lebesgue measure and $f(q,p) = a(q) + \sum_{j=1}^{n} b^j(q)p_j$ and evaluate formally $\hbar/i\left(\frac{\partial U_t^f}{\partial t}\right)_{t=0}$ he will find that Λf turns out to be something like

$$a + \frac{\hbar}{i}\sum_{j=1}^{n}\left(b^j \frac{\partial}{\partial x^j} + \frac{1}{2}\frac{\partial b^j}{\partial x^j}\right).$$

Of course this can be done rigorously. As in *5.2* the operators Λf are best defined in terms of the groups they generate, at least for non-linear manifolds.

These and other topics will be discussed in §7.

5.2 is important as an existence theorem: without it we would not know that quantization schemes exist at all.

§6 Systems of Imprimitivity

The material of this section is largely the work of Mackey. We shall deal with a small portion of the theory, simplifying it somewhat by transcribing it in our context of differentiable manifolds and unitary representations of the additive group $\mathbf{R}$ (the general theory deals with metrically standard Borel spaces and projective representations of separable, locally-compact groups).

6.1 Definition.

Let M be a manifold and P a projection-valued measure on $\mathbf{B}(M)$ with values the projections on a separable Hilbert space $\mathcal{H}$. Let U be a strongly continuous, unitary representation on $\mathcal{H}$ of the additive group $\mathbf{R}$. Let $\phi\colon \mathbf{R}\times M\to M$, $(t,x)\to\phi_t x$ be measurable and such that for all s, $t\in\mathbf{R}$, $\phi_t\colon M\to M$ is 1-1, onto and measurable, $\phi_{s+t}=\phi_s\circ\phi_t$ and

$$U_t P(\Delta) U_{-t} = P(\phi_{-t}\Delta)$$

for all $\Delta \in \mathbf{B}(M)$.

Then the pair (P,ϕ) *is called a system of imprimitivity for* U *(*Mackey [3, p.279]*).*

*6.2 Theorem (*Mackey [3, p.283]*)*

With the notation of 6.1, let μ *be a* σ*-finite measure on* $\mathbf{B}(M)$ *quasi-invariant under every* $\phi_t (t \in \mathbf{R})$*, let* $\mathcal{H} = \mathcal{L}_2(M, \mathbf{C}^m, \mu)$ $(m \in \mathrm{N} \cup \{\infty\})$ *and* $P(\Delta) = \mathrm{m}(\chi_\Delta)$ *for all* $\Delta \in \mathbf{B}(M)$.

Then for each $t \in \mathbf{R}$ *there is a mapping* A_t *of* M *into the unitary operators on* $\mathbf{C}^m$ *such that*

$$(U_t h)(x) = A_t(x) \sqrt{\frac{d\mu\phi_t}{d\mu}(x)}\, h(\phi_t x)$$

for all $h \in \mathcal{L}_2(M, \mathbf{C}^m, \mu)$, $x \in M$. *A has the following properties:*

(i) for all $s,\ t \in \mathbf{R}$

$$A_{s+t}(x) = A_s(x) A_t(\phi_s x)$$

for μ*-almost all* $x \in M$*;*

(ii) $A_0(x)$ *is the identity operator on* $\mathbf{C}^m$ *for* μ*-almost all* $x \in M$*;*

(iii) for all $z,\ z' \in \mathbf{C}^m$ *the mapping*

$$\mathbf{R} \times M \to \mathbf{C},\quad (t,x) \to \langle A_t(x)z,\ z'\rangle_m$$

is measurable and for all $t \in \mathbf{R}$ *the mapping*

$$M \to \mathbf{C},\ x \to \langle A_t(x)z, z'\rangle_m$$

is measurable.

Let now Λ be an arbitrary quantization scheme of multiplicity m on M, with values the self-adjoint operators in $\mathcal{H}$. For each $\Delta \in \mathbf{B}(M)$, put $g_\Delta = \chi_\Delta \circ \tau$, $P(\Delta) = \Lambda g_\Delta$. Then $g_\Delta \in \mathbf{U}(M)$ and P is a projection-valued measure on M. If $f \in \mathbf{L}^\infty(M)$,

$$g_\Delta \circ F_t^f(q,p) = g_\Delta(F_t^f(q,p)) = \chi_\Delta(\phi_t^f q)$$

$$= \chi_{\phi_{-t}^f\Delta}(q) = g_{\phi_{-t}^f\Delta}(q,p)$$

by *3.1*; hence

$$g_\Delta \circ F_t^f = g_{\phi_{-t}^f\Delta}.$$

By *4.1* we then have

$$P(\phi_{-t}^f\Delta) = U_t^f P(\Delta) U_{-t}^f \tag{2}$$

where

$$U_t^f = \exp\ (it/\hbar\Lambda f). \tag{3}$$

Since ϕ^f is continuous on $\mathbf{R} \times M$ and for each t ϕ_t^f is continuous on M, (P, ϕ^f) is a system of imprimitivity for U^f.

By *1.9* we may assume that if $f \in \mathbf{Q}(M)$ then $\Lambda f = \mathrm{m}(f')$ in $\tilde{\mathcal{L}}_2(M, \mathbf{C}^m, \mu)$ for some σ-finite measure μ and in particular $\Lambda g_\Delta = P(\Delta) = \mathrm{m}(\chi_\Delta)$. Furthermore it is clear that the null sets of μ are the same as those of P; hence by *(2)*, $\mu(\Delta) = 0$ iff $\mu(\phi_t^f \Delta) = \mu_t^f(\Delta) = 0$, i.e. μ is quasi-invariant under every $\phi_t^f (t \in \mathbf{R},\ f \in \mathbf{L}^\infty(M))$. The following theorem is now immediate.

6.3 Theorem

Every quantization scheme of multiplicity m on M is unitarily equivalent to a quantization scheme Λ on M with values the self-adjoint operators in $\tilde{\mathcal{L}}_2(M, \mathbf{C}^m, \mu)$ for some σ-finite quasi-invariant measure μ and having the following properties:

(i) $\Lambda f = m(f')$ *for all* $f \in \mathbf{Q}(M)$;

(ii) for each $f \in \mathbf{L}^\infty(M)$ *there is a mapping* A^f *of* $\mathbf{R} \times M$ *into the unitary operators on* $\mathbf{C}^m$ *such that*

$$(a) \quad (U_t^f h)(x) = A^f(t,x) \sqrt{\frac{d\mu_t^f}{d\mu}(x)}\, h(\phi_t^f x) \qquad (4)$$

for all $h \in \mathcal{L}_2(M, \mathbf{C}^m, \mu)$, *where* $U_t^f = \exp\,(it/\hbar \Lambda f)$,

(b) for all $s,\ t \in \mathbf{R}$

$$A^f(s+t,x) = A^f(s,x) A^f(t, \phi_s^f x) \qquad (5)$$

for μ*-almost all* $x \in M$,

(c) $A^f(0,x)$ *is the identity operator on* $\mathbf{C}^m$ *for* μ*-almost all* $x \in M$,

(d) for all z, $z' \in \mathbf{C}^m$ the mapping

$$\mathbf{R} \times M \to \mathbf{C}, \quad (t,x) \to \langle A^f(t,x)z, z' \rangle_m$$

is measurable and for all $t \in \mathbf{R}$ the mapping

$$M \to \mathbf{C}, \quad x \to \langle A^f(t,x)z, z' \rangle_m$$

is measurable,

(e) if $g \in \mathbf{L}^\infty(M)$ then for all s, $t \in \mathbf{R}$

$$A^{g \circ F^f_t}(s,x) = A^f(t,x)A^g(s,\phi^f_t x)A^f(t,\phi^f_{-t} \circ \phi^g_s \circ \phi^f_t x)^{-1} \qquad (6)$$

for μ-almost all $x \in M$.

Proof. The quasi-invariance of μ follows from the fact that every C^∞ one-parameter group of differomorphisms of M is of the form $\{\phi^f_t\colon t \in \mathbf{R}\}$ for some $f \in \mathbf{P}(M)$.

(i) and *(iia)-(iid)* follow from previous remarks and *6.2*. To prove *(iie)* substitute *(4)* and a similar equation involving g into the basic identity given in *4.1* to obtain

$$A^{g \circ F^f_t}(s,x) = A^f(t,x)A^g(s,\phi^f_t x)A^f(-t,\phi^g_s \circ \phi^f_t x).$$

That

$$A^f(-t,\phi^g_s \circ \phi^f_t x) = A^f(t,\phi^f_{-t} \circ \phi^g_s \circ \phi^f_t x)^{-1}$$

follows from *(iib)* and *(iic)* on replacing s by t, t by $-t$ and x by $\phi^f_{-t} \circ \phi^g_s \circ \phi^f_t x$.

The problem of determining all quantization schemes involves the determination of all functions A^f satisfying *(b)-(e)* of *6.3*. This apparently is very difficult.

(5) can in a certain sense be 'solved' for particular functions f, notably when for each $x_0 \in M$ the mapping $t \to \phi_t^f x_0$ of $\mathbf{R}$ onto the orbit $\Theta_{x_0} = \{\phi_t^f x_0 : t \in \mathbf{R}\}$ is a diffeomorphism, or at least when the set of x_0 for which this is not the case has μ-measure zero. In the former case *(5)* gives

$$A^f(t, \phi_s^f x_0) = A^f(s, x_0)^{-1} A^f(s+t, x_0).$$

Let $\gamma(t) = \phi_t^f x_0$ and put $s = \gamma^{-1} x$; then

$$A^f(t,x) = A^f(\gamma^{-1}x, x_0)^{-1} A^f(\gamma^{-1}x + t, x_0).$$

Since $\gamma(\gamma^{-1}x + t) = \phi^f_{\gamma^{-1}x+t} x_0 = \phi_t^f \phi^f_{\gamma^{-1}x} x_0 = \phi_t^f x$,

we have

$$A^f(t,x) = A^f(\gamma^{-1}x, x_0)^{-1} A^f(\gamma^{-1}(\phi_t^f x), x_0).$$

For each $x \in \Theta_{x_0}$ write $B^f_{x_0}(x) = A^f(\gamma^{-1}x, x_0)$. Then

$$A^f(t,x) = B^f_{x_0}(x)^{-1} \; B^f_{x_0}(\phi_t^f x)$$

for all $x \in \Theta_{x_0}$. We may choose an x_0 in each orbit and write $B^f(x) = B^f_{x_0}(x)$ if $x \in \Theta_{x_0}$, thus defining B^f almost everywhere and we have

$$A^f(t,x) = B^f(x)^{-1} B^f(\phi_t^f x) \tag{7}$$

for μ-almost all $x \in M$. Conversely, if B^f is any map from M into the unitary operators on $\mathbf{C}^m$, then $A^f(t,x)$

defined by *(7)* satisfies *(5)*.

§7 Quantization Schemes on Vector Spaces

If M is a finite-dimensional real normed space V the results of the preceding section can be made more precise. First of all, the measure μ may be taken to be Haar measure (and hence Lebesgue measure when $V = \mathbf{R}^n$).

7.1 Theorem

Any quasi-invariant, σ-finite measure μ on V is equivalent to Haar measure.

Proof. If $\alpha \in V$, $t \in \mathbf{R}$, the mapping $V \to V$, $x \to x + t\alpha$ is a diffeomorphism and the set of all such mappings for fixed α and varying t is a C^∞ one-parameter group of diffeomorphisms of V. By hypothesis, for each $\Delta \in \mathbf{B}(V)$, we have for each $\alpha \in V$ (putting $t = 1$)

$$\mu(\Delta) = 0 \text{ iff } \mu(\Delta + \alpha) = 0.$$

The function $V \times V \to \mathbf{R}$, $(x,y) \to \chi_\Delta(x + y)$ is measurable since it is the composition of the continuous function $(x,y) \to x + y$ with the measurable function $x \to \chi_\Delta(x)$. Let λ denote Haar measure on the locally compact additive group V. By the Fubini theorem

$$\int_{V\times V}\chi_\Delta(x+y)\,d(\lambda\otimes\mu)(x,y) = \int_V\left[\int_V\chi_\Delta(x+y)\,d\lambda(x)\right]d\mu(y)$$

$$= \int_V\left[\int_V\chi_\Delta(x)\,d\lambda(x)\right]d\mu(y) = \int_V\lambda(\Delta)\,d\mu(y) = \lambda(\Delta)\mu(V)$$

(with the usual convention that $0.\infty = 0$, $\infty.\infty = \infty$). Also

$$\int_{V\times V}\chi_\Delta(x+y)\,d(\lambda\otimes\mu)(x,y) = \int_V\left[\int_V\chi_\Delta(x+y)\,d\mu(y)\right]d\lambda(x)$$

$$= \int_V\left[\int_V\chi_{\Delta-x}(y)\,d\mu(y)\right]d\lambda(x) = \int_V\mu(\Delta-x)\,d\lambda(x).$$

Hence

$$\int_V\mu(\Delta-x)\,d\lambda(x) = \lambda(\Delta)\mu(V).$$

Suppose that $\lambda(\Delta) = 0$. Then $\int_V\mu(\Delta-x)\,d\lambda(x) = 0$; hence $\mu(\Delta-x) = 0$ for λ-almost all $x\in V$ and in particular (since $\lambda \neq 0$) $\mu(\Delta-x) = 0$ for some $x\in V$. Hence $\mu(\Delta) = 0$. Conversely, suppose that $\mu(\Delta) = 0$; then $\mu(\Delta-x) = 0$ for all $x\in V$ so that $\int_V\mu(\Delta-x)\,d\lambda(x) = \lambda(\Delta)\mu(V) = 0$. Since $\mu \neq 0$ (the Hilbert space $\tilde{\mathcal{L}}_2(V,\mathbf{C}^m,\mu)$ is infinite-dimensional) we have $\mu(V) \neq 0$, hence $\lambda(\Delta) = 0$. Thus $\mu(\Delta) = 0$ iff $\lambda(\Delta) = 0$ and μ and λ are equivalent.

The mapping $h \to \sqrt{\frac{d\mu}{d\lambda}}h$ is unitary from $\tilde{\mathcal{L}}_2(V,\mathbf{C}^m,\mu)$ onto $\tilde{\mathcal{L}}_2(V,\mathbf{C}^m,\lambda)$ and carries $\mathrm{m}(f)$ in the first space into $\mathrm{m}(f)$ in the second space for any complex measurable function f. Hence in the case $M = V$ the measure μ of *6.3* may be taken to be Haar measure.

More generally, it is almost obvious that a result analogous to *7.1* could be proved in the case where M is an arbitrary *Lie group* or even where M is an *open subset* of a Lie group. For reasons of space we shall not pursue these matters here.

For ease of calculation in applications we note here that in the case where M is an open subset of V and λ is the appropriate restriction of Haar measure, the function $\frac{d\lambda_t^f}{d\lambda}$ for $f \in \mathbf{U}(M)$ can be given an explicit form in terms of the determinant of ϕ_t^f. The result is

$$\frac{d\lambda_t^f}{d\lambda} = |\det \phi_t^f| \qquad \lambda\text{-}a.e.. \qquad (8)^*$$

We next show that, with certain restrictions, the quantized linear momentum components can always be taken to be those of conventional quantum mechanics. I_m will denote the identity operator on $\mathbf{C}^m$.

7.2 Theorem

Let the notation be as in 6.3 with $M = V$, $u = \lambda$. Let $(e_k)_{k=1}^n$ be a linearly independent family of vectors in V. For each k let $f_k \in \mathbf{P}(V)$ be given by

$$f_k(q,p) = p.e_k$$

* By det S, for any differentiable map $S: V \to V$, we mean the function $x \to \det\,[DS(x)]$.

for all $(q,p) \in T^*V$. *Then there exists a unitary mapping* W *of* $\tilde{\mathfrak{L}}_2(V,\mathbf{C}^m,\lambda)$ *onto itself which commutes with all multiplication operators and for which*

$$W \exp\,(it/\hbar\Lambda f_k)W^{-1} = U_k(t) \qquad (1 \leqslant k \leqslant n)$$

for all $t \in \mathbf{R}$, *where*

$$(U_k(t)h)(x) = h(x + te_k)$$

for all $h \in \mathfrak{L}_2(V,\mathbf{C}^m,\lambda)$, $t \in \mathbf{R}$ *and* λ*-almost all* $x \in V$.

Proof. Let $1 \leqslant k \leqslant n$. The flow F^{f_k} of X_{f_k} is given by

$$F_t^{f_k}(q,p) = (q + te_k, p),$$

and ϕ^{f_k} by

$$\phi_t^{f_k} q = q + te_k.$$

Clearly $\dfrac{d\lambda_t^{f_k}}{d\lambda}(x) = 1$ for λ-almost all $x \in V$.

By *(4)*, we have

$$(\exp(it/\hbar\Lambda f_k)h)(x) = A_k(t,x)h(x + te_k)$$

for all $h \in \mathfrak{L}_2(V,\mathbf{C}^m,\lambda)$, where $A_k = A^{f_k}$; *(5)* gives

$$A_j(s + t,x) = A_j(s,x)A_j(t,x + se_j) \quad (1 \leqslant j \leqslant n) \quad (i)$$

for all $s,t \in \mathbf{R}$ and λ-almost all $x \in V$. Put $f = f_j$, $g = f_k$ in *(6)* to obtain

$$A_k(s,x) = A_j(t,x)A_k(s,x + te_j)A_j(t,x + se_k)^{-1}$$

$$(1 \leqslant j,k \leqslant n) \qquad (ii)$$

for all $s, t \in \mathbf{R}$ and λ-almost all $x \in V$. Replace t by s, s by t and x by $x - se_j$ in *(ii)* to obtain

$$A_k(t, x - se_j) = A_j(s, x - se_j)A_k(t, x)A_j(s, x - se_j + te_k)^{-1} \qquad (1 \leqslant j, k \leqslant n) \quad (iii)$$

for all s, $t \in \mathbf{R}$ and λ-almost all $x \in V$.

We shall use the following convention for a product of operators. If $S_1, S_2, \ldots, S_m$ are operators we shall write

$$\prod_{i=1}^{m} S_i = S_1 S_2 \ldots S_m,$$

$$\prod_{i=m}^{1} S_i = S_m S_{m-1} \ldots S_1,$$

etc..

We may adjoin to the set $\{e_k : 1 \leqslant k \leqslant n\}$ a set $\{e_k : n+1 \leqslant k \leqslant n'\}$ of elements of V so that $\{e_k : 1 \leqslant k \leqslant n'\}$ is a basis of V. For $x \in V$ write $x = \sum_{i=1}^{n'} x^i e_i$ where $x^i \in \mathbf{R}$. Let

$$\hat{W}(x) = \prod_{j=1}^{n} A_j\left(x^j, \sum_{i=1}^{j-1} x^i e_i\right) \qquad (iv)$$

(the summation is taken to be 0 when $j=1$). If $1 \leqslant k \leqslant n$ we have

$$\hat{W}(x)A_k(t,x)\hat{W}(x+te_k)^{-1} = [\prod_{j=1}^{n} A_j(x^j, \sum_{i=1}^{j-1} x^i e_i)]A_k(t,x).$$

$$\cdot \prod_{j=n}^{1} A_j(x^j + t\delta_k^j, \sum_{i=1}^{j-1}(x^i + t\delta_k^i)\ e_i)^{-1} =$$

$$[\prod_{j=1}^{k-1} A_j(x^j, \sum_{i=1}^{j-1} x^i e_i)]Z_k(t,x) \prod_{j=k-1}^{1} A_j(x^j, \sum_{i=1}^{j-1} x^i e_i)^{-1}$$

where

$$Z_k(t,x) = [\prod_{j=k}^{n} A_j(x^j, \sum_{i=1}^{j-1} x^i e_i)]A_k(t,x).$$

$$\cdot[\prod_{j=n}^{k+1} A_j(x^j, \sum_{i=1}^{j-1} x^i e_i + te_k)]A_k(x^k + t, \sum_{i=1}^{k-1} x^i e_i)^{-1}.$$

Repeated use of *(iii)* shows that

$$[\prod_{j=k+1}^{n} A_j(x^j, \sum_{i=1}^{j-1} x^i e_i)]A_k(t,x) \prod_{j=n}^{k+1} A_j(x^j, \sum_{i=1}^{j-1} x^i e_i + te_k)^{-1}$$

$$= A_k(t, \sum_{i=1}^{k} x^i e_i)$$

so that

$$Z_k(t,x) = A_k(x^k, \sum_{i=1}^{k-1} x^i e_i)A_k(t, \sum_{i=1}^{k} x^i e_i)A_k(x^k + t, \sum_{i=1}^{k-1} x^i e_i)^{-1}$$

$$= I_m$$

by *(i)*. Thus

$$\hat{W}(x)A_k(t,x)\hat{W}(x+te_k)^{-1} = I_m. \quad (1 \leqslant k \leqslant n) \qquad (v)$$

Define W on $\mathcal{L}_2(V, \mathbf{C}^m, \lambda)$ to itself by $(Wh)(x) = \hat{W}(x)h(x)$. Then W is unitary and we have

$$W \exp (it/\hbar\Lambda f_k)W^{-1} = U_k(t) \qquad \text{by } (v).$$

Clearly W commutes with all multiplication operators.

7.2 is not true in general if the e_k are not linearly independent. To see this observe first that if Λ is a quantization scheme of multiplicity m on V and $c \in \mathbf{R}$, $c \neq 0$, then Λ' defined by $\Lambda' f = \Lambda f$ if $f \in \mathbf{Q}(V)$, $\Lambda' f = \Lambda f + cI$ if $f \in \mathbf{U}(V) - \mathbf{Q}(V)$ is a quantization scheme of multiplicity m on V. For suppose that $g \in \mathbf{Q}(V)$; then $g \circ F_t^f \in \mathbf{Q}(V)$ for all $f \in \mathbf{L}^\infty(V)$. Hence $\exp (it/\hbar\Lambda' f)\Lambda' g \exp (-it/\hbar\Lambda' f) = \exp (it/\hbar\Lambda f)\Lambda g \exp (-it/\hbar\Lambda f) = \Lambda(g \circ F_t^f) = \Lambda'(g \circ F_t^f)$. Next suppose that $g \in \mathbf{U}(V) - \mathbf{Q}(V)$; then $g \circ F_t^f \in \mathbf{U}(V) - \mathbf{Q}(V)$ for all $f \in \mathbf{L}^\infty(V)$. Hence $\exp (it/\hbar\Lambda' f)\Lambda' g \exp (-it/\hbar\Lambda' f) = \exp (it/\hbar\Lambda f)\Lambda g \exp (-it/\hbar\Lambda f) + cI = \Lambda(g \circ F_t^f) + cI = \Lambda'(g \circ F_t^f)$.
Now let $f_\alpha(q,p) = p.\alpha$ for all $(q,p) \in T^*V$, $\alpha \in V$ and $(U_\alpha(t)h)(x) = h(x + t\alpha)$ for all $t \in \mathbf{R}$, $h \in \mathcal{L}_2(V, \mathbf{C}^m, \lambda)$ and λ-almost all $x \in V$. Let $\alpha \in V$, $a \in \mathbf{R}$, $a \neq 1$. Let Λ be as in *6.3* with $M = V$, $\mu = \lambda$. If there is no unitary mapping W on $\tilde{\mathcal{L}}_2(V, \mathbf{C}^m, \lambda)$ such that

$$W \exp (it/\hbar\Lambda f_\alpha)W^{-1} = U_\alpha(t)$$

and

$$W \exp (it/\hbar\Lambda f_{a\alpha})W^{-1} = U_{a\alpha}(t)$$

for all $t \in \mathbf{R}$, there is nothing to prove. Otherwise, we can consider the quantization scheme Λ such that $\exp(it/\hbar \Lambda f_\alpha) = U_\alpha(t)$, $\exp(it/\hbar \Lambda f_{a\alpha}) = U_{a\alpha}(t)$ for all $t \in \mathbf{R}$. Suppose there is a unitary mapping W such that

$$W \exp(it/\hbar \Lambda' f_\alpha) W^{-1} = U_\alpha(t)$$

and

$$W \exp(it/\hbar \Lambda' f_{a\alpha}) W^{-1} = U_{a\alpha}(t)$$

for all $t \in \mathbf{R}$. Then we will have

$$e^{itc/\hbar} W U_\alpha(t) W^{-1} = U_\alpha(t)$$

and

$$e^{itc/\hbar} W U_{a\alpha}(t) W^{-1} = U_{a\alpha}(t),$$

i.e.

$$e^{itc/\hbar} W U_{a\alpha}(t) W^{-1} = U_{a\alpha}(t) = U_\alpha(at) = e^{iatc/\hbar} W U_\alpha(at) W^{-1}$$

$$= e^{iatc/\hbar} W U_{a\alpha}(t) W^{-1}$$

giving

$$e^{itc/\hbar} = e^{iatc/\hbar}$$

for all $t \in \mathbf{R}$, or

$$a = 1,$$

a contradiction. Thus no such mapping W exists. Since the set $\{\alpha, a\alpha\}$ is linearly dependent, our assertion is proved.

We next introduce a set of classical observables which will be important in the sequel.

For $z = (\alpha, \beta) \in V \times V^*$ let $f_z \in \mathbf{U}(V)$ be defined by

$$f_z(q,p) = p.\alpha - \beta.q \tag{9}$$

for all $(q,p) \in T^*V$. Note that f_z is linear in z, i.e. $f_{z+z'} = f_z + f_{z'}$ and $f_{cz} = cf_z$ for any z, $z' \in V \times V^*$, $c \in \mathbf{R}$. Let

$$\mathbf{I}(V) = \{f_z : z \in V \times V^*\}.$$

Let G be an inner product in V. Associated with G is an isomorphism, also denoted by G, of V onto V^* and defined by $Gx.y = G(x,y)$ for all x, $y \in V$. Thus we have $G(x,y) = G(y,x) = Gx.y = Gy.x$ for all $x,y \in V$. If L is a linear mapping of V into V^* define the linear map $L^*: V \to V^*$ by $L^*x.y = Ly.x$ for all $x,y \in V$. Clearly $G = G^*$.

Let $\alpha \in V$, $\alpha \neq 0$, $\beta \in V^*$. Define $K: V \to V^*$ by

$$Kx = \frac{G(\alpha,x)}{G(\alpha,\alpha)} \left[\beta - \frac{(\beta.\alpha)}{2G(\alpha,\alpha)}\, G\alpha \right]$$

for all $x \in V$. Then K has the property that

$$K\alpha + K^*\alpha = \beta.$$

We can now prove

7.3 Theorem

Let Λ be a quantization scheme of multiplicity m on V, with values the self-adjoint operators in $\tilde{\mathcal{L}}_2(V,\mathbf{C}^m,\lambda)$. Let $(e_k)_{k=1}^n$ be a basis of V. Then there exists a unitary mapping W of $\tilde{\mathcal{L}}_2(V,\mathbf{C}^m,\lambda)$ onto itself such that for all $z = (\alpha,\beta) \in V \times V^*$,

$$(W \exp (it/\hbar\Lambda \mathrm{f}_z)W^{-1}h)(x) =$$

$$\exp [-i/\hbar((\beta.x)t + \tfrac{1}{2}(\beta.\alpha)t^2)]A_\alpha(t)h(x + t\alpha)$$

for all $t \in \mathbf{R}$, $h \in \mathcal{L}_2(V,\mathbf{C}^m,\lambda)$ and λ-almost all $x \in V$, where $A_\alpha(t)$ is a unitary mapping on $\mathbf{C}^m$ with the properties

$$A_\alpha(s + t) = A_\alpha(s)A_\alpha(t)$$

and

$$A_0(t) = I_m, \; A_{e_k}(t) = I_m \qquad (1 \leqslant k \leqslant n)$$

for all s, $t \in \mathbf{R}$.

Proof. First observe that $\phi_t^{\mathrm{f}_z}q = q + t\alpha$ for all $q \in V$, $t \in \mathbf{R}$, so that by *(8)* and *6.3* there is a unitary mapping W on $\tilde{\mathcal{L}}_2(V,\mathbf{C}^m,\lambda)$ such that

$$(W \exp (it/\hbar\Lambda \mathrm{f}_z)W^{-1}h)(x) = A^{\mathrm{f}_z}(t,x)h(x + t\alpha)$$

for all $h \in \mathcal{L}_2(V,\mathbf{C}^m,\lambda)$, $t \in \mathbf{R}$ and λ-almost all $x \in V$ and

$$W\Lambda f W^{-1} = \mathrm{m}(f')$$

for all $f \in \mathbf{Q}(V)$.

If $\alpha = 0$ then $f_z \in \mathbf{\Omega}(V)$ and the result is obvious.

Suppose that $\alpha \neq 0$ and put $f_\alpha = f_{(\alpha,0)}$, $A_\alpha = A^{f_\alpha}$, $A_k = A_{e_k}$ $(1 \leqslant k \leqslant n)$. Let $K\colon V \to V^*$ be any linear map with the property $K\alpha + K^*\alpha = \beta$ and let

$$g(q,p) = Kq.q$$

for all $(q,p) \in T^*V$. Then $g \in \mathbf{\Omega}(V) \cap \mathbf{L}^\infty(V)$ and calculation shows that

$$Dg'(q).x = Kq.x + Kx.q = (Kx + K^*x).q$$

for all q, $x \in V$. Hence

$$Dg'(q).\alpha = \beta.q$$

for all $q \in V$. Since $\phi^g_t q = q$ for all $q \in V$, $t \in \mathbf{R}$, *3.2* gives $(f_\alpha \circ F^g_t)'_1(q) = -\int_0^t Dg'(q).\alpha ds = -t\beta.q$ and $(f_\alpha \circ F^g_t)_2 = \Gamma(X^{f_\alpha}) = f_\alpha$, i.e.

$$(f_\alpha \circ F^g_t)(q,p) = p.\alpha - t\beta.q$$

for all $(q,p) \in T^*V$, $t \in \mathbf{R}$. Hence

$$f_\alpha \circ F^g_1 = f_z$$

for all $z = (\alpha,\beta) \in V \times V^*$, $\alpha \neq 0$.

Clearly $A^g(t,x) = \exp\,(it/\hbar Kx.x)I_m$ for all $t \in \mathbf{R}$ and λ-almost all $x \in V$. Hence by *6.3(e)* we have

$$A^{f_z}(s,x) = \exp\,[\,i/\hbar(Kx.x - K(x+s\alpha).(x+s\alpha))]\,A_\alpha(s,x)$$

for all $s \in \mathbf{R}$ and λ-almost all $x \in V$. The argument of the exponential can be written $-i/\hbar[s(Kx.\alpha + K\alpha.x) + s^2K\alpha.\alpha] = -i/\hbar[s(K\alpha + K^*\alpha).x + s^2/2(K\alpha + K^*\alpha).\alpha] = -i/\hbar(s\beta.x + \frac{1}{2}s^2\beta.\alpha)$ so that finally

$$A^{f_z}(s,x) = \exp\,[-i/\hbar((\beta.x)s + \tfrac{1}{2}(\beta.\alpha)s^2)]A_\alpha(s,x)$$

for all $s \in \mathbf{R}$ and λ-almost all $x \in V$.

6.3 (iib) gives

$$A_\alpha(s + t,x) = A_\alpha(s,x)A_\alpha(t,x + s\alpha).$$

analogous to *(iii)* of *7.2* we find

$$A_\alpha(t,x - se_k) = A_k(s,x - se_k)A_\alpha(t,x)A_k(s,x - se_k + t\alpha)^{-1}$$

$$(1 \leqslant k \leqslant n).$$

By *7.2* we may suppose that $A_k(t,x) = I_m$ for all $t \in \mathbf{R}$ and λ-almost all $x \in V$ $(1 \leqslant k \leqslant n)$ so that the last equation becomes

$$A_\alpha(t,x - se_k)=A_\alpha(t,x) \qquad (1 \leqslant k \leqslant n)$$

for all $t \in \mathbf{R}$ and λ-almost all $x \in V$. Since the e_k form a basis of V, $A_\alpha(t,x)$ is therefore *independent of* x λ-*a.e.*. Write $A_\alpha(t,x)$ simply as $A_\alpha(t)$. This completes the proof of the theorem.

The remark following *7.2* shows that we cannot assume in general that $A_\alpha(t) = I_m$ for *all* $\alpha \in V$.

If we take $V = \mathbf{R}^n$ and $(e_k)_{k=1}^n$ to be the standard basis then if Λ is a quantization scheme of multiplicity m on $\mathbf{R}^n$ *7.3* shows that we may assume Λf_z to be the operator whose formal expression turns out to be

$$\Lambda f_z = \hbar/i \sum_{j=1}^{n} \alpha^j \frac{\partial}{\partial x^j} - \beta . x + c_\alpha$$

in $\mathcal{L}_2(\mathbf{R}^n, \mathbf{C}^m, \lambda)$, where $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^n)$, $x = (x^1, x^2, \ldots, x^n)$ and c_α is a $m \times m$ *constant matrix* with the property $c_\alpha = 0$ whenever $\alpha = 0,\ e_k\ (1 \leqslant k \leqslant n)$. Again, we cannot assume in general that $c_\alpha = 0$ for *all* $\alpha \in \mathbf{R}^n$.

Results analogous to *7.2* and *7.3* can be given for other observables by making use of the fundamental equations.

§8 Standard Quantization Schemes

The physicist reader will no doubt complain that the theory so far covered has given a rather asymmetric treatment of 'q' and 'p'. He is accustomed to having the option of working in 'momentum space' or 'configuration space' as he pleases, thus placing 'p' on an equal standing with 'q'. There is no reason to suppose that for an arbitrary manifold such a symmetric treatment of coordinates and momenta is possible. On the other hand for vector spaces a symmetric treatment is forthcoming in certain cases. In this and the next section we propose to carry out this procedure and in fact to give a far-reaching extension of the quantization schemes so far discussed. V will continue to denote a finite-dimensional real normed space. The topology on V^* is understood to be that induced on V^* when V^* is considered a subset of T^*V; it is of course the same as that defined by any norm on V^*. λ will denote Haar measure on V.

Let S be a subset of a real vector space. A map L of S into the self-adjoint operators in a Hilbert space will be called *linear* if the following conditions are satisfied:

(i) if $x, y, x+y \in S$ then $Lx + Ly$ is closable and $L(x+y) = \overline{Lx + Ly}$;

(ii) if $c \in \mathbf{R}$, x, $cx \in S$ then $L(cx) = \overline{cLx}$.

A prequantization scheme is linear.

We shall write $f_\alpha = f_{(\alpha, 0)}$ for $\alpha \in V$ and put

$$\mathbf{I}_1(V) = \{f_\alpha : \alpha \in V\}.$$

We shall call a quantization scheme Λ on V *standard* if $\Lambda | \mathbf{I}_1(V)$ is *linear*.

For each inner product G on V there is $\kappa > 0$ such that the mapping $T : \tilde{\mathcal{L}}_2(V, \mathbf{C}^m, \lambda) \to \tilde{\mathcal{L}}_2(V, \mathbf{C}^m, \lambda)$ where

$$(Th)(x) = \kappa\int_V e^{-iG(x,y)}h(y)d\lambda(y) \tag{10}$$

for all $h \in \mathcal{L}_1(V,\mathbf{C}^m,\lambda) \cap \mathcal{L}_2(V,\mathbf{C}^m,\lambda)$ is unitary and has the property $(T^{-1}h)(x) = (Th)(-x)$ for all $h \in \mathcal{L}_2(V,\mathbf{C}^m,\lambda)$. [If $V = \mathbf{R}^n$ and λ is Lebesgue measure then we can take $\kappa = (2\pi)^{-n/2}\sqrt{|\det G_0^{-1}G|}$ where G_0 is the Euclidean inner product: $G_0(e_i,e_j) = \delta_{ij}$ where $(e_i)_{i=1}^n$ is the standard basis; in this case T is the Fourier (Plancherel) transform when $G = G_0$.]

Our first result says that if Λ is standard then the factor $A_\alpha(t)$ of *7.3* may be assumed to be the identity for *all* $\alpha \in V$.

8.1 Theorem

Let Λ be a standard quantization scheme of multiplicity m on V with values the self-adjoint operators in $\tilde{\mathcal{L}}_2(V,\mathbf{C}^m,\lambda)$. Then there exists a unitary mapping W of $\tilde{\mathcal{L}}_2(V,\mathbf{C}^m,\lambda)$ onto itself such that for all $z = (\alpha,\beta) \in V \times V^*$,

$$(W \exp(it/\hbar\Lambda f_z)W^{-1}h)(x) =$$
$$\exp[-i/\hbar((\beta.x)t + \tfrac{1}{2}(\beta.\alpha)t^2)]\,h(x+t\alpha) \tag{i}$$

for all $t \in \mathbf{R}$, $h \in \mathcal{L}_2(V,\mathbf{C}^m,\lambda)$ and λ-almost all $x \in V$.

Conversely, if Λ is such a quantization scheme having the property (i) for some unitary map W, then Λ is standard.

Proof: By *7.3* with $\beta = 0$ there exists W such that for all $\alpha \in V$

$$(W \exp (it/\hbar \Lambda f_\alpha) W^{-1} h)(x) = A_\alpha(t) h(x + t\alpha)$$

for all $t \in \mathbf{R}$, $h \in \mathcal{L}_2(V, \mathbf{C}^m, \lambda)$ and λ-almost all $x \in V$, where

$$A_{e_k}(t) = I_m \qquad (1 \leqslant k \leqslant n)$$

for all $t \in \mathbf{R}$.

It follows easily that

$$(TW \exp (it/\hbar \Lambda f_\alpha) W^{-1} T^{-1} h)(x) = \exp (itG(x, \alpha)) A_\alpha(t) h(x) \tag{ii}$$

for all t, h, α and λ-almost all x. If Λ is standard so is Λ': $f \to TW \Lambda f W^{-1} T^{-1}$ $(f \in \mathbf{U}(V))$ and we have $\Lambda' f_k = \mathrm{m}(\hbar G e_k)$ where $f_k = f_{e_k}$ $(1 \leqslant k \leqslant n)$. We shall show that for all $\alpha \in V$, $\Lambda' f_\alpha = \mathrm{m}(\hbar G\alpha)$.

Let $\alpha = \sum_{i=1}^{n} \alpha^i e_i$ $(\alpha^i \in \mathbf{R})$ and put $\gamma_k = \sum_{i=1}^{k} \alpha^i e_i$ $(1 \leqslant k \leqslant n)$. Suppose that

$$\Lambda' f_{\gamma_k} = \mathrm{m}(\hbar \sum_{i=1}^{k} \alpha^i G e_i) \tag{iii}$$

for some $k < n$. Then

$$\Lambda' f_{\gamma_{k+1}} = \Lambda' f_{\gamma_{k+\alpha}k+1_{e_{k+1}}} = \Lambda'(f_{\gamma_k} + \alpha^{k+1} f_{k+1})$$

$$= \overline{\Lambda' f_{\gamma_k} + \overline{\alpha^{k+1}\Lambda' f_{k+1}}} = \overline{m(\hbar \sum_{i=1}^{k} \alpha^i Ge_i) + m(\hbar\alpha^{k+1} Ge_{k+1})}$$

$$= m(\hbar \sum_{i=1}^{k+1} \alpha^i Ge_i)$$

where we have used the fact that $\overline{m(f) + m(g)} = m(f+g)$ for measurable functions f, g on V (cf. the proof of *1.9* (p.62)). Thus *(iii)* is true for $k+1$. Since $\Lambda' f_{\gamma_1} = \Lambda' f_{\alpha^1 e_1} = \Lambda'(\alpha^1 f_1) = \overline{\alpha^1 \Lambda' f_1} = \overline{\alpha^1 m(\hbar Ge_1)} = m(\hbar\alpha^1 Ge_1)$, *(iii)* is true for $k = 1$. By induction it follows that *(iii)* is true for $k = n$, i.e. $\Lambda' f_\alpha = m(\hbar G\alpha)$.

Hence

$$(TW \exp\,(it/\hbar\Lambda f_\alpha)W^{-1}T^{-1}h)(x) = \exp\,(itG(x,\alpha))h(x) \qquad (iv)$$

for all t, h, α and λ-almost all x. Comparing *(ii)* and *(iv)* we see that

$$A_\alpha(t) = I_m$$

for all $\alpha \in V$, $t \in \mathbf{R}$. The first result now follows from *7.3*.

For the converse, let Λ have the property *(i)* for some W. Then $\Lambda' f_\alpha = m(\hbar G\alpha)$ for all $\alpha \in V$. Hence

$\Lambda' f_\alpha + \Lambda' f_{\alpha'} = m(\hbar G\alpha) + m(\hbar G\alpha')$ is closable and $\overline{\Lambda' f_\alpha + \Lambda' f_{\alpha'}} = m(\hbar G(\alpha + \alpha')) = \Lambda' f_{\alpha+\alpha'}$, for any $\alpha, \alpha' \in V$. Also $\Lambda' f_{c\alpha} = m(\hbar G(c\alpha)) = m(c\hbar G\alpha) = \overline{cm(\hbar G\alpha)} = \overline{c\Lambda' f_\alpha}$ for any $\alpha \in V$, $c \in \mathbf{R}$. Hence Λ', and hence Λ, is standard.

It is relatively easy to prove from *8.1* that Λ is standard iff $\Lambda | \mathbf{I}(V)$ is linear, in fact, iff the restriction of Λ to the larger set $\{f_z + c \colon f_z \in \mathbf{I}(V),\ c \in \mathbf{R}\}$ is linear, though the latter result requires further study.

By the remark immediately following *7.3* it follows that not every quantization scheme on V is standard. By taking $M = V$, $\mu = \lambda$, $f = f_z$ in *5.2* it follows that the canonical scheme of multiplicity m on V is standard.

Let Λ be a quantization scheme of multiplicity m on V and let $c \in \mathbf{R}$, $c \neq 0$. Put $\Lambda' f = \Lambda f$ if $f \in \mathbf{Q}(V) \cup \mathbf{L}^\infty(V)$, $\Lambda' f = \Lambda f + cI$ if $f \in \mathbf{U}(V) - \mathbf{Q}(V) \cup \mathbf{L}^\infty(V)$. If $g \in \mathbf{Q}(V) \cup \mathbf{L}^\infty(V)$ (resp $g \in \mathbf{U}(V) - \mathbf{Q}(V) \cup \mathbf{L}^\infty(V)$) then $g \circ F^f_t \in \mathbf{Q}(V) \cup \mathbf{L}^\infty(V)$ (resp. $g \circ F^f_t \in \mathbf{U}(V) - \mathbf{Q}(V) \cup \mathbf{L}^\infty(V)$) for all $f \in \mathbf{L}^\infty(V)$, $t \in \mathbf{R}$. It follows easily that $\exp(it/\hbar\Lambda' f)\Lambda' g \exp(-it/\hbar\Lambda' f) = \Lambda'(g \circ F^f_t)$ for any $f \in \mathbf{L}^\infty(V)$, $g \in \mathbf{U}(V)$, $t \in \mathbf{R}$, i.e. Λ' is a quantization scheme of multiplicity m on V.

Now suppose that Λ is *canonical*. Then Λ' is standard. Since $X_{-f} = -X_f$ and hence $F^{-f}_t = F^f_{-t}$ for all

$f \in \mathbf{F}(T^*V)$, $t \in \mathbf{R}$ (Abraham [1, p.40]), *(1)* and *(8)* show that if $f \in \mathbf{U}(V)$ then

$$(\exp\,(it/\hbar\Lambda(-f))h)(x)$$

$$= \exp\,(-i/\hbar\int_0^t f_1'\,(\phi^f_{-s}x)ds)\sqrt{|\det\phi^f_{-t}(x)|}\,h(\phi^f_{-t}x)$$

$$= \exp\,(i/\hbar\int_0^{-t} f_1'(\phi^f_s x)ds)\sqrt{|\det\phi^f_{-t}(x)|}\,h(\phi^f_{-t}x)$$

$$= (\exp\,(-it/\hbar\Lambda f)h)(x)$$

for all $h \in \mathcal{L}_2(V, \mathbf{C}^m, \lambda)$, $t \in \mathbf{R}$ and λ-almost all $x \in V$. Hence $\Lambda(-f) = -\Lambda f$. If there exists a unitary mapping W such that $W\Lambda' f W^{-1} = \Lambda f$ for all $f \in \mathbf{U}(V)$ then we have $\Lambda'(-f) = -\Lambda' f$ and so if $f \in \mathbf{U}(V) - \mathbf{Q}(V) \cup \mathbf{L}^\infty(V)$

$$0 = \overline{\Lambda' f - \Lambda' f} = \overline{\Lambda' f + \Lambda'(-f)}$$

$$= \overline{\Lambda f + \Lambda(-f) + 2cI} = \overline{\Lambda f - \Lambda f + 2cI}$$

$$= 2cI,$$

i.e. $c = 0$, a contradiction. Thus no such W exists and we have shown that not every standard scheme of multiplicity m on V is equivalent to the canonical one.

Obviously, a *linear* quantization scheme on V is standard, but not every standard scheme is linear. For if Λ is linear then Λ' as defined above is standard but if $f \in \mathbf{U}(V) - \mathbf{Q}(V) \cup \mathbf{L}^\infty(V)$ and $a \neq 0$, $a \neq 1$, then

$$\begin{aligned}\Lambda'(af) &= \Lambda(af) + cI = a\Lambda f + cI \\ &= a(\Lambda f + cI) + c(1-a)I \\ &= a\Lambda' f + c(1-a)I \neq a\Lambda' f = \overline{a\Lambda' f},\end{aligned}$$

so that Λ' is not linear.

Having made these remarks we shall now proceed with the development of the theory. Throughout the rest of this section Λ will denote a standard quantization scheme of multiplicity m on V with values the self-adjoint operators in a Hilbert space $\mathcal{H}$. For each $z = (\alpha, \beta) \in V \times V^*$ let

$$S(z) = \exp\,(i/\hbar\Lambda f_z). \tag{11}$$

By *8.1* there is a unitary mapping W of $\mathcal{H}$ onto $\tilde{\mathcal{L}}_2(V, \mathbf{C}^m, \lambda)$ such that for each z

$$WS(z)W^{-1} = \mathring{S}(z) \tag{12}$$

where

$$(\mathring{S}(z)h)(x) = \exp\,(-i/\hbar(\beta.x + \tfrac{1}{2}\beta.\alpha))h(x+\alpha) \tag{13}$$

for all $h \in \mathcal{L}_2(V, \mathbf{C}^m, \lambda)$ and λ-almost all $x \in V$. For each $\alpha \in V$, $\beta \in V^*$ put

$$\begin{aligned} Q(\alpha) &= S(\alpha, 0), & R(\beta) &= S(0, \beta), \\ \mathring{Q}(\alpha) &= \mathring{S}(\alpha, 0), & \mathring{R}(\beta) &= \mathring{S}(0, \beta). \end{aligned} \tag{14}$$

Let $\eta\colon (V \times V^*) \times (V \times V^*) \to \mathbf{R}$ be defined by

$$\eta(z, z') = \beta.\alpha' - \beta'.\alpha \tag{15}$$

where $z = (\alpha, \beta)$, $z' = (\alpha', \beta')$. We then have

8.2 Lemma

(i) for all z, $z' \in V \times V^*$

$$S(z)S(z') = \exp\left(\frac{i}{2\hbar}\eta(z, z')\right)S(z + z'); \qquad (16)$$

(ii) for all $\alpha \in V$, $\beta \in V^*$

$$Q(\alpha)R(\beta) = \exp(-i/\hbar\beta.\alpha)R(\beta)Q(\alpha); \qquad (17)$$

(iii) Q (resp. R) is a strongly continuous unitary representation on $\mathcal{H}$ of the locally compact additive group V (resp. V^*).

Proof. Equation *(16)* with S replaced by $\mathring{S}$ follows immediately from the definition *(13)*. *(16)* itself then follows from *(12)*.

Put $z = (\alpha, 0)$, $z' = (0, \beta)$ in *(16)* to obtain $Q(\alpha)R(\beta) = \exp\left(-\frac{i}{2\hbar}\beta.\alpha\right)S(\alpha, \beta)$. Similarly $z = (0, \beta)$, $z' = (\alpha, 0)$ in *(16)* leads to $R(\beta)Q(\alpha) = \exp\left(\frac{i}{2\hbar}\beta.\alpha\right)S(\alpha, \beta)$ whence $Q(\alpha)R(\beta) = \exp(-i/\hbar\beta.\alpha)R(\beta)Q(\alpha)$ and *(17)* is proved.

To prove *(iii)* first observe that $z = (\alpha, 0)$, $z' = (\alpha', 0)$ in *(16)* gives

$$Q(\alpha)Q(\alpha') = Q(\alpha + \alpha'),$$

while $z = (0,\beta)$, $z' = (0,\beta')$ in (16) leads to

$$R(\beta)R(\beta') = R(\beta+\beta').$$

Hence Q (resp. R) is a unitary representation of V (resp. V^*) on $\mathcal{H}$.

Now, by (13),

$$(\overset{\circ}{Q}(\alpha)h)(x) = h(x+\alpha),$$
$$(\overset{\circ}{R}(\beta)h)(x) = \exp\,(-i/\hbar\beta.x)h(x)$$

for all $h\in\mathcal{L}_2(V,\mathbf{C}^m,\lambda)$, $\alpha\in V$, $\beta\in V^*$, λ-almost all $x\in V$. Since, with T defined by (10), we have $(T\overset{\circ}{Q}(\alpha)T^{-1}h)(x) = e^{iG\alpha.x}h(x)$ for all $h\in\mathcal{L}_2(V,\mathbf{C}^m,\lambda)$, and in view of (12), to prove that Q and R are strongly continuous representations it is sufficient to show that the map $V^*\to\tilde{\mathcal{L}}_2(V,\mathbf{C}^m,\lambda)$, $\beta\to \mathrm{m}(e^{i\beta})h$, where $e^{i\beta}$ denotes the function $x\to e^{i\beta.x}$, is continuous for all $h\in\mathcal{L}_2(V,\mathbf{C}^m,\lambda)$. To do this it is clearly sufficient to prove continuity at 0. Without loss of generality we may assume the norm on V^* to be given by $\|\beta\| = \sup\limits_{\substack{x\in V\\ \|x\|\leqslant 1}}|\beta.x|$ for all $\beta\in V^*$. We have

$$\|\mathrm{m}(e^{i\beta})h - h\|^2 = \int_V |e^{i\beta.x}-1|^2|h(x)|^2 d\lambda(x)$$

$$= \int_V 4\sin^2\left(\frac{\beta.x}{2}\right)|h(x)|^2 d\lambda(x);$$

since $4\sin^2(\frac{\beta.x}{2})|h(x)|^2 \leqslant (\beta.x)^2|h(x)|^2 \leqslant \|\beta\|^2\|x\|^2|h(x)|^2$ the integrand tends to 0 with β. Since also $4\sin^2(\frac{\beta.x}{2})|h(x)|^2 \leqslant 4|h(x)|^2$, the dominated convergence theorem applies to give

$$\lim_{\beta\to 0} \|m(e^{i\beta})h - h\| = 0$$

for any $h \in \mathcal{L}_2(V, \mathbf{C}^m, \lambda)$. This completes the proof of the theorem.

A 1-1 linear mapping $L\colon V \times V^* \to V \times V^*$ is symplectic (regarded as a mapping of T^*V onto T^*V) iff it leaves the antisymmetric bilinear form η invariant, i.e.

$$\eta(Lz, Lz') = \eta(z, z') \tag{18}$$

for all z, $z' \in V \times V^*$ (Abraham [1, p.96, 14.13]). The set $Sp(V)$ of all symplectic automorphisms of $V \times V^*$ is a group under composition, called the *symplectic group*. It is a locally compact Lie group of dimension $2n^2 + n$, where n is the dimension of V.

If $L \in Sp(V)$ and $f_z \in \mathsf{I}(V)$, $z = (\alpha, \beta)$, then

$$\begin{aligned} f_z(q,p) &= p.\alpha - \beta.q \\ &= \eta((q,p), z) \end{aligned}$$

for all $(q,p) \in T^*V$, so that

$$\begin{aligned} (f_z \circ L)(q,p) &= f_z(L(q,p)) = \eta(L(q,p), z) \\ &= \eta((q,p), L^{-1}z) = f_{L^{-1}z}(q,p) \end{aligned}$$

for all $(q,p) \in T^*V$. Hence

$$f_z \circ L = f_{L^{-1}z} \tag{19}$$

for all $z \in V \times V^*$. (Thus the map $f_z \to f_z \circ L$ leaves $\mathbb{I}(V)$ invariant.)

We shall write

$$S_L(z) = S(L^{-1}z), \; \mathring{S}_L(z) = \mathring{S}(L^{-1}z), \tag{20}$$

$$\begin{aligned} Q_L(\alpha) &= S_L(\alpha, 0), \; R_L(\beta) = S_L(0,\beta), \\ \mathring{Q}_L(\alpha) &= \mathring{S}_L(\alpha, 0), \; \mathring{R}_L(\beta) = \mathring{S}_L(0,\beta), \end{aligned} \tag{21}$$

for all $z \in V \times V^*$, $\alpha \in V$, $\beta \in V^*$. We then have, in analogy with *8.2*,

8.3 Theorem

Let $L \in Sp(V)$. Then

(i) for all z, $z' \in V \times V^*$

$$S_L(z)S_L(z') = \exp\left(\frac{i}{2\hbar}\eta(z,z')\right)S_L(z+z'); \tag{22}$$

(ii) for all $\alpha \in V$, $\beta \in V^*$

$$Q_L(\alpha)R_L(\beta) = \exp(-i/\hbar\,\beta.\alpha)R_L(\beta)Q_L(\alpha); \tag{23}$$

(iii) Q_L (resp. R_L) is a strongly continuous unitary representation on $\mathcal{H}$ of the additive group V (resp. V^*).

Proof. By *8.2(i)*,

$$S_L(z).S_L(z') = S(L^{-1}z)S(L^{-1}z')$$
$$= \exp\left(\frac{i}{2\hbar}\,\eta(L^{-1}z, L^{-1}z')\right)S(L^{-1}(z+z'))$$
$$= \exp\left(\frac{i}{2\hbar}\,\eta(z,z')\right)S_L(z+z').$$

Put $z=(\alpha,0)$, $z'=(0,\beta)$ in (22) to obtain $Q_L(\alpha)R_L(\beta) = \exp\left(-\frac{i}{2\hbar}\,\beta.\alpha\right)S_L(\alpha,\beta)$. Similarly, $z=(0,\beta)$, $z'=(\alpha,0)$ in (22) leads to $R_L(\beta)Q_L(\alpha) = \exp\left(\frac{i}{2\hbar}\beta.\alpha\right)S_L(\alpha,\beta)$ whence $Q_L(\alpha)R_L(\beta) = \exp(-i/\hbar\,\beta.\alpha)R_L(\beta)Q_L(\alpha)$ and (23) is proved.

To prove (iii) first observe that $z=(\alpha,0)$, $z'=(\alpha',0)$ (resp. $z=(0,\beta)$, $z'=(0,\beta')$) in (22) gives $Q_L(\alpha)Q_L(\alpha') = Q_L(\alpha+\alpha')$ (resp. $R_L(\beta)R_L(\beta') = R_L(\beta+\beta')$). Hence Q_L (resp. R_L) is a unitary representation of V (resp. V^*) on $\mathcal{H}$.

Since $S(\alpha,\beta) = \exp\left(\frac{i}{2\hbar}\,\beta.\alpha\right)Q(\alpha)R(\beta)$, we have

$$Q_L(\alpha) = S_L(\alpha,0) = S(L^{-1}(\alpha,0))$$
$$= \exp\left(\frac{i}{2\hbar}\,L_2^{-1}(\alpha,0).L_1^{-1}(\alpha,0)\right)Q(L_1^{-1}(\alpha,0))R(L_2^{-1}(\alpha,0)) \qquad (i)$$

and similarly

$$R_L(\beta) = \exp\left(\frac{i}{2\hbar}\,L_2^{-1}(0,\beta).L_1^{-1}(0,\beta)\right)Q(L_1^{-1}(0,\beta))R(L_2^{-1}(0,\beta)) \qquad (ii)$$

where we have written $L^{-1}(z) = (L_1^{-1}(z), L_2^{-1}(z))$ for any $z\in V\times V^*$. Since the maps $V\to V$, $\alpha\to L_1^{-1}(\alpha,0)$; $V\to V^*$,

$\alpha \to L_2^{-1}(\alpha, 0)$; $V^* \to V$, $\beta \to L_1^{-1}(0, \beta)$; $V^* \to V^*$, $\beta \to L_2^{-1}(0, \beta)$ are continuous, it follows from equations *(i)* and *(ii)* and *8.2 (iii)* that Q_L and R_L are strongly continuous.

We next introduce the *character group* V'. This is the set of all continuous homomorphisms w of V into the multiplicative group of all complex numbers of modulus 1. V' is a group relative to the usual pointwise product of functions and is equipped with the topology defined by the set of basic neighbourhoods $\{N(w_0, C, \varepsilon): w_0 \in V', C \subset V$ is compact, $\varepsilon > 0\}$ where

$$N(w_0, C, \varepsilon) = \{w \in V': |w(x) - w_0(x)| < \varepsilon \text{ for all } x \in C\}.$$

With respect to this topology V' is a locally compact topological Abelian group. For our work the following result is of importance.

8.4 Theorem

The mapping $J: V' \to V^*$, $w \to i\hbar Dw(0)$ *(considering* w *as a map into the two dimensional real space of complex numbers* $\mathbf{C}_0$*) is a homomorphic homeomorphism onto* V^**, whose inverse is given by* $(J^{-1}\beta)(x) = \exp(-i/\hbar \beta . x)$ *for all* $\beta \in V^*$, $x \in V$.

Proof. We shall show first that every $w \in V'$ is a continuously differentiable mapping into $\mathbf{C}_0$ and that

$$w(x) = \exp\,(Dw(0).x) \tag{i}$$

for all $x \in V$.

Let $(e_k)_{k=1}^n$ be a basis of V and put $E_k = \{te_k : t \in \mathbf{R}\}$, $w_k = w|E_k$ and define $\theta_k : \mathbf{R} \to \mathbf{C}_0$ by $\theta_k(t) = w_k(te_k)$. By definition $w(x+y) = w(x)w(y)$ for all $x, y \in V$. Hence

$$\begin{aligned}\theta_k(s+t) &= w_k(se_k + te_k) = w_k(se_k)w_k(te_k) \\ &= \theta_k(s)\theta_k(t)\end{aligned} \tag{ii}$$

for all $s, t \in \mathbf{R}$. Let

$$g_k(t) = \int_0^t \theta_k(s)ds.$$

Clearly there exists $a \in \mathbf{R}$ such that $g_k(a) \neq 0$; hence

$$\begin{aligned}g_k(a)\theta_k(t) &= \int_0^a \theta_k(t)\theta_k(s)ds \\ &= \int_0^a \theta_k(s+t)ds = \int_t^{t+a} \theta_k(s)ds \\ &= g_k(t+a) - g_k(t),\end{aligned}$$

i.e.

$$\theta_k(t) = (g_k(t+a) - g_k(t))/g_k(a). \tag{iii}$$

Since θ_k is continuous g_k is differentiable in $\mathbf{R}$ and, by *(iii)*, we conclude that θ_k (and hence w_k) is differentiable in $\mathbf{R}$. Differentiating *(ii)* with respect to s and setting $s = 0$ we have $D\theta_k(t) = \theta_k(t)D\theta_k(0)$ with solution $\theta_k(t) = \exp\,(D\theta_k(0).t)$ for all $t \in \mathbf{R}$. In terms of w_k these formulas read

$$\mathrm{D}w_k(x) = w_k(x)\mathrm{D}w_k(0) \tag{iv}$$

and

$$w_k(x) = \exp\,(\mathrm{D}w_k(0).x) \tag{v}$$

for all $x \in E_k$.

For $x \in V$ put $x = \sum_{k=1}^{n} x_k$ where $x_k \in E_k$. Then (considering V as the product $E_1 \times E_2 \times \ldots \times E_n$) we have

$$\begin{aligned} \mathrm{D}_k w(x) &= (\prod_{i \neq k} w_i(x_i))\mathrm{D}w_k(x_k) \\ &= (\prod_{i \neq k} w_i(x_i))w_k(x_k)\mathrm{D}w_k(0) \qquad \text{by } (iv) \\ &= w(x)\mathrm{D}w_k(0) \qquad (1 \leqslant k \leqslant n) \qquad (vi) \end{aligned}$$

for all $x \in V$. By (vi) for each k the map $x \to \mathrm{D}_k w(x)$ of V into the normed space of $\mathbf{R}$-linear maps of V into $\mathbf{C}_0$ is continuous. Hence w is continuously differentiable in V and

$$\mathrm{D}w(x).y = \sum_{k=1}^{n} \mathrm{D}_k w(x).y_k = \sum_{k=1}^{n} w(x)\mathrm{D}w_k(0).y_k \tag{vii}$$

for all x, $y \in V$. Thus in particular, since $w(0) = 1$,

$$\mathrm{D}w(0).x = \sum_{k=1}^{n} \mathrm{D}w_k(0).x_k \tag{viii}$$

for all $x \in V$. Finally (v) and $(viii)$ give

$$\begin{aligned} w(x) &= \prod_k w_k(x_k) \\ &= \exp\,(\sum_{k=1}^{n} \mathrm{D}w_k(0).x_k) \\ &= \exp\,(\mathrm{D}w(0).x) \end{aligned}$$

for all $x \in V$. This proves *(i)*.

Since $w(x)$ must lie on the unit circle in the complex plane $Dw(0).x$ is pure imaginary, hence $i\hbar Dw(0).x$ is real. This proves that $i\hbar Dw(0)$ is an $\mathbf{R}$-linear mapping of V into $\mathbf{R}$, i.e. $Jw = i\hbar Dw(0) \in V^*$.

For $w_1, w_2 \in V'$ let $w_1 w_2$ denote the mapping $x \to w_1(x) w_2(x)$. Then $J(w_1 w_2) = i\hbar D(w_1 w_2)(0) = i\hbar(Dw_1(0) + Dw_2(0)) = Jw_1 + Jw_2$. Thus J is a homomorphism.

By *(vii)* and *(viii)* we have $Dw(x) = w(x)Dw(0)$ for all $w \in V'$, $x \in V$. Hence $Jw = Dw(0) = 0$ implies $Dw(x) = 0$ for all $x \in V$, i.e. $w(x) = c$ for all $x \in V$ (c a constant). Thus $c = w(0) = w(0+0) = w(0)w(0) = c^2$, i.e. $c = 1$. Thus the kernel of J consists entirely of the identity in V'; hence J is 1 - 1.

Let $\beta \in V^*$. Then the map $w^\beta: V \to \mathbf{C}$, $x \to \exp(-i/\hbar \beta . x)$ is in V' and $Jw^\beta = \beta$. This proves that J is onto V^* and consequently that $J^{-1}\beta = w^\beta$.

It remains to be proved that J and J^{-1} are continuous. Let $w_0 \in V'$. Let $\varepsilon > 0$ and suppose that $w \in N(w_0, C_1, r)$ where $C_1 = \{x \in V: \|x\| \leqslant 1\}$ and $r = 2 \sin(\min(\frac{\varepsilon}{4\hbar}, \frac{\pi}{3}))$. Then $|w(x) - w_0(x)| < r$ for all $x \in C_1$, i.e.

$$|\exp(Dw(0).x - Dw_0(0).x) - 1| < r$$

for all $x \in C_1$. But this is the same as

$$\left|\sin\left(\frac{1}{2\hbar}(Jw - Jw_0).x\right)\right| < r/2 \qquad (ix)$$

for all $x \in C_1$. Thus for each $x \in C_1$ there exists an integer n_x such that

$$n_x\pi - s < \frac{1}{2\hbar}\,(Jw - Jw_0).x < n_x\pi + s$$

where $s = \min\,(\varepsilon/(4\hbar), \pi/3)$. Suppose that $n_x > 0$. Then

$$\frac{\pi}{2} - \frac{s}{2n_x} < \frac{1}{2\hbar}(Jw - Jw_0).y < \frac{\pi}{2} + \frac{s}{2n_x}$$

where $y = x/(2n_x) \in C_1$. Since $s/(2n_x) \leqslant s/2 \leqslant \pi/6$, we have

$$\frac{\pi}{3} < \frac{1}{2\hbar}(Jw - Jw_0).y < \frac{2\pi}{3};$$

hence

$$\sin\left(\frac{1}{2\hbar}(Jw - Jw_0).y\right) > \sin\left(\frac{\pi}{3}\right) \geqslant \sin s = \frac{r}{2}$$

which contradicts (ix). A similar argument holds if we assume $n_x < 0$. Thus $n_x = 0$ and we have

$$\left|\frac{1}{2\hbar}\,(Jw - Jw_0).x\right| < s$$

for all $x \in C_1$. Hence

$$\|Jw - Jw_0\| = \sup_{x \in C_1} |(Jw - Jw_0).x| \leqslant 2\hbar s$$

$$\leqslant \varepsilon/2 < \varepsilon$$

and J is continuous at w_0, for every $w_0 \in V'$.

Let $\beta_0 \in V^*$. Let $\varepsilon > 0$ and suppose that $\beta \in V^*$ and $\|\beta - \beta_0\| < \hbar\varepsilon$. Then

$$\sup_{x \in C_1} |(\beta - \beta_0).x| < \hbar\varepsilon;$$

hence $|(\beta - \beta_0).x| < \hbar\varepsilon$ for all $x \in C_1$. Now $|J^{-1}\beta(x) - J^{-1}\beta_0(x)| = |\exp(-i/\hbar(\beta - \beta_0).x) - 1|$ $= 2\ |\sin \frac{1}{2\hbar}(\beta - \beta_0).x| \leqslant \frac{1}{\hbar}|(\beta - \beta_0).x| < \varepsilon$ for all $x \in C_1$. Thus $J^{-1}\beta \in N(J^{-1}\beta_0, C_1, \varepsilon)$ and J^{-1} is continuous at β_0, for every $\beta_0 \in V^*$. This completes the proof of the theorem.

If we put $R_L' = R_L \circ J$ then by *8.3 (ii)*, *8.3 (iii)* and *8.4* we have

8.5 Theorem

Let $L \in S_p(V)$. Then Q_L (resp. R_L') is a strongly continuous unitary representation on $\mathcal{H}$ of V (resp. V') and for each $\alpha \in V$, $w \in V'$ we have

$$Q_L(\alpha)R_L'(w) = w(\alpha)R_L'(w)Q_L(\alpha). \tag{24}$$

We now obtain the following fundamental result.

8.6 Theorem

Let Λ be a standard quantization scheme of multiplicity m on V. For each $L \in S_p(V)$ there exists a unitary mapping $W_L \colon \mathcal{H} \to \mathcal{H}$ such that for all $f \in \mathbf{I}(V)$,

$$\Lambda(f \circ L) = W_L^{-1} \Lambda f\ W_L.$$

Proof. By *8.5, 1.6* and Mackey [1, p.314, Theorem 1] or Hewitt and Ross [1, p.323], there is a unitary mapping W_1 of $\mathcal{H}$ onto $\tilde{\mathcal{L}}_2(V,\mathbf{C}^n,\lambda)$ for some $n \in \mathrm{N} \cup \{\infty\}$ such that

$$(W_1 Q_L(\alpha) W_1^{-1} h)(x) = h(x+\alpha) \qquad (i)$$

and

$$(W_1 R_L'(w) W_1^{-1} h)(x) = w(x) h(x) \qquad (ii)$$

for all $\alpha \in V$, $w \in V'$, $h \in \mathcal{L}_2(V,\mathbf{C}^n,\lambda)$ and λ-almost all $x \in V$.

It follows from *(22)* that if $z = (\alpha,\beta)$ then $S_L(z) = \exp\left(\frac{i}{2\hbar}\beta.\alpha\right) Q_L(\alpha) R_L(\beta)$. Consequently, by *(i)*, *(ii)* and *8.4*, we have

$$W_1 S_L(z) W_1^{-1} = \mathring{S}'(z) \qquad (iii)$$

for all $z \in V \times V^*$, where $\mathring{S}'(z) : \tilde{\mathcal{L}}_2(V,\mathbf{C}^n,\lambda) \to \tilde{\mathcal{L}}_2(V,\mathbf{C}^n,\lambda)$ is given by $(\mathring{S}'(z)h)(x) = \exp(-i/\hbar(\beta.x + \tfrac{1}{2}\beta.\alpha))h(x+\alpha)$ for all $h \in \mathcal{L}_2(V,\mathbf{C}^n,\lambda)$. However, by *(12)* we have $WS_L(z)W^{-1} = WS(L^{-1}z)W^{-1} = \mathring{S}(L^{-1}z)$, so that *(iii)* gives

$$\mathring{S}(L^{-1}z) = W_2 \mathring{S}'(z) W_2^{-1} \qquad (iv)$$

where $W_2 = WW_1^{-1}$. W_2 is a unitary mapping of $\tilde{\mathcal{L}}_2(V,\mathbf{C}^n,\lambda)$ onto $\tilde{\mathcal{L}}_2(V,\mathbf{C}^m,\lambda)$.

We shall show that $n = m$. By *1.6*, $\tilde{\mathcal{L}}_2(V,\mathbf{C}^p,\lambda)$, where $p = m$ (resp. $p = n$), is a direct sum of closed subspaces $(\mathcal{H}_k)_{k=1}^p$ invariant under all $\mathring{S}(z)$ (resp. $\mathring{S}'(z)$), each of which is identifiable with $\tilde{\mathcal{L}}_2(V,\lambda)$. Since $\mathring{S}(z)$ can be expressed in terms of the product of $\mathring{Q}(\alpha)$ and $\mathring{R}(\beta)$ and since it is well-known that the set of operators $\{\mathring{Q}(\alpha),\ \mathring{R}(\beta):\ \alpha\in V,\ \beta\in V^*\}$ acts *irreducibly* on $\tilde{\mathcal{L}}_2(V,\lambda)$, each $\mathcal{H}_k$ $(1\leq k\leq m)$ is *irreducible* with respect to the operators $\{\mathring{S}(z):\ z\in V\times V^*\}$. An analogous result holds for the set $\{\mathring{S}'(z):\ z\in V\times V^*\}$ acting on each $\mathcal{H}_k$ $(1\leq k\leq n)$.

Let $1\leq k\leq n,\ 1\leq j\leq m$. Then by *(iv)*, $\mathcal{H}_j\cap W_2(\mathcal{H}_k)$ is invariant under all $\mathring{S}(z)$; as this set of operators acts irreducibly on $\mathcal{H}_j$ we must have $\mathcal{H}_j\cap W_2(\mathcal{H}_k) = \{0\}$ or $\mathcal{H}_j$. Hence for each k $(1\leq k\leq n)$ there exists k' $(1\leq k'\leq m)$ such that $\mathcal{H}_{k'}\cap W_2(\mathcal{H}_k) = \mathcal{H}_{k'}$. Now $W_2^{-1}(\mathcal{H}_{k'}) = \mathcal{H}_k\cap W_2^{-1}(\mathcal{H}_{k'}) \neq \{0\}$ and by the argument just given but applied to W_2^{-1} we conclude that $\mathcal{H}_k\cap W_2^{-1}(\mathcal{H}_{k'}) = \mathcal{H}_k$; thus $W_2(\mathcal{H}_k) = \mathcal{H}_{k'}$. The mapping $\mathrm{N}_n\to\mathrm{N}_m$, $k\to k'$ is clearly an injection. Hence $n\leq m$. Similarly, it may be proved that $m\leq n$. Hence $m = n$.

We can now write $\mathring{S}(z)$ for $\mathring{S}'(z)$ and *(iii)* and *(12)* give $S_L(z) = W_1^{-1}\mathring{S}(z)W_1 = W_1^{-1}WS(z)W^{-1}W_1$, i.e.

$$S_L(z) = W_L^{-1}S(z)W_L \tag{v}$$

for all $z\in V\times V^*$, where $W_L = W^{-1}W_1$.

By the remark following *8.1* or directly from *8.1* itself it follows that exp $(it/\hbar\Lambda f_z) = \exp(i/\hbar\Lambda f_{tz})$ for all $t \in \mathbf{R}$; hence

$$\exp(it/\hbar\Lambda(f_z \circ L)) = \exp(it/\hbar\Lambda f_{L^{-1}z})$$

$$= \exp(i/\hbar\Lambda f_{tL^{-1}z}) = S(tL^{-1}z)$$

$$= S(L^{-1}tz) = S_L(tz) = W_L^{-1}S(tz)W_L \qquad \text{(by } (v))$$

$$= W_L^{-1} \exp(i/\hbar\Lambda f_{tz})W_L$$

$$= W_L^{-1} \exp(it/\hbar\Lambda f_z)W_L$$

for all $t \in \mathbf{R}$, $z \in V \times V^*$. Hence $\Lambda(f_z \circ L) = W_L^{-1}\Lambda f_z W_L$ for all $z \in V \times V^*$. This completes the proof.

It should be noted that the requirement that Λ be standard is also a *necessary* condition for the existence of *8.6*. For if α, $\alpha' \in V$, there is always a $L \in S_p(V)$ such that $L^{-1}(\alpha, 0) = (0, \beta)$, $L^{-1}(\alpha', 0) = (0, \beta')$ for some β, $\beta' \in V^*$ (for example, let G be an inner product on V and define L by $L(q,p) = (G^{-1}p, -Gq)$). Hence if *8.6* holds we have

$$\overline{\Lambda f_\alpha + \Lambda f_{\alpha'}} = W_L\overline{(\Lambda(f_\alpha \circ L) + \Lambda(f_{\alpha'} \circ L))}W_L^{-1}$$

$$= W_L\overline{(\Lambda f_{(0,\beta)} + \Lambda f_{(0,\beta')})}W_L^{-1}$$

$$= W_L\Lambda(f_{(0,\beta)} + f_{(0,\beta')})W_L^{-1} = W_L\Lambda f_{(0,\beta+\beta')}W_L^{-1}$$

$$= W_L\Lambda(f_{\alpha+\alpha'} \circ L)W_L^{-1}$$

$$= \Lambda f_{\alpha+\alpha'} = \Lambda(f_\alpha + f_{\alpha'}),$$

since $f_{(0,\beta)}$, $f_{(0,\beta')} \in \mathbf{Q}(V)$ and $\Lambda|\mathbf{Q}(V)$ is linear. Similarly we may prove that $\Lambda(cf_\alpha) = \overline{c\Lambda f_\alpha}$ for all $c \in \mathbf{R}$. Thus $\Lambda|\mathbf{I}_1(V)$ is linear.

Because of the use of the uniqueness theorem of Mackey in the proof, *8.6* has been obtained with essentially little work. On the other hand, the proof of *8.6* is a nonconstructive proof, giving no prescription for finding the operator W_L for given L. An alternative and semi-constructive proof which makes no use of Mackey's theorem can be obtained as follows.

If $z' \in V \times V^*$, the mapping $L_{z'}: V \times V^* \to V \times V^*$, $z \to z + \eta(z,z')z'$ is called a *symplectic transvection.* We have $L_{z'} \in Sp(V)$. If we define

$$W_{L_{z'}} = \exp\left(\frac{-i}{2\hbar}(\Lambda f_{z'})^2\right), \tag{25}$$

then $W_{L_{z'}}$ has the property that $\Lambda(f \circ L_{z'}) = W_{L_{z'}}^{-1} \Lambda f W_{L_{z'}}$ for all $f \in \mathbf{I}(V)$. This is proved by using *4.1* and the relation

$$e^{itA^2} = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{e^{\pm i\pi/4}}{\sqrt{4\pi|t|}} \int_{\mathbf{R}} e^{-(i\pm\varepsilon)s^2/(4t)} e^{isA} ds$$

for any self-adjoint operator A and $t \in \mathbf{R}$ $(t \neq 0)$ (upper sign to be taken if $t > 0$, lower if $t < 0$). The interpretation of this equation is that

$$\langle e^{itA^2}x,y\rangle = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{e^{\pm i\pi/4}}{\sqrt{4\pi|t|}} \int_{\mathbf{R}} e^{-(i\pm\varepsilon)s^2/(4t)} \langle e^{isA}x,y\rangle ds$$

for all x, y in the Hilbert space. Since any $L\in Sp(V)$ is a product of symplectic transvections (Artin [1, p.139]), this provides an alternative proof of *8.6*.

For given $L\in Sp(V)$, the operator W_L of *8.6* is not unique. However, if W'_L is another such operator then we must have $W'_L = C_L W_L$ where C_L is a unitary operator on $\mathcal{H}$ that commutes with every $\exp(it/\hbar\Lambda f_z)$ (i.e. with every $S(z)$, $z\in V\times V^*$). The following lemma shows that C_L must be of a special form.

8.7 Lemma

Let C be any unitary operator on $\tilde{\mathcal{L}}_2(V,\mathbf{C}^m,\lambda)$ which commutes with every $\mathring{S}(z)$ $(z\in V\times V^*)$. Then there exists a unitary operator $\hat{C}$ on $\mathbf{C}^m$ such that $(Ch)(x) = \hat{C}h(x)$ for all $h\in\mathcal{L}_2(V,\mathbf{C}^m,\lambda)$ and λ-almost all $x\in V$.

Proof. With $z = (0,\beta)$ we see that C commutes with all multiplication operators $m(e^{i\beta})$ (where $e^{i\beta}$ is the function $x\to e^{i\beta.x}$). By standard arguments we conclude that C commutes with every $m(\chi_\Delta)$ $(\Delta\in\mathbf{B}(V))$. Hence C must be of the form $(Ch)(x) = \hat{C}(x)h(x)$ where for each $x\in V$, $\hat{C}(x)$ is a unitary operator on $\mathbf{C}^m$ (Mackey [4, p.282]). The fact that C commutes also with all translation operators

$\mathring{S}(\alpha, 0)$ shows that $\hat{C}(x)$ must be independent of x for λ-almost all x.

8.6 further shows that if L, $L' \in Sp(V)$ then $W_{LL'} = C(L,L')W_L W_{L'}$, where $C(L,L')$ commutes with every $S(z)$. If $m = 1$ then by *8.7* $C(L,L')$ will just be a complex number of modulus 1 and in addition we shall have in this case

$$C(L_1L_2, L_3)C(L_1, L_2) = C(L_1, L_2L_3)C(L_2, L_3)$$

for all $L_1, L_2, L_3 \in Sp(V)$ This follows on writing $L_1L_2L_3$ in the two forms $(L_1L_2)L_3$ and $L_1(L_2L_3)$. Indeed, if $m = 1$, C will be a *multiplier* for $Sp(V)$ and $W: L \to W_L$ will be a *projective representation* of $Sp(V)$. It would be interesting to find the conditions under which C is a multiplier if $m \neq 1$ and under which W is strongly continuous (for any m).

For properties of the symplectic group see Helgason [1] or Artin [1].

§9. Admissible Quantization Schemes.

If Λ is standard and $g \in \mathbf{U}(V) - \mathbf{I}(V)$, then if $L \in Sp(V)$, $g \circ L$ need not lie in $\mathbf{U}(V)$ and when it does we cannot be sure that $\Lambda(g \circ L)$ will equal $W_L^{-1}\Lambda g W_L$.

A quantization scheme Λ of multiplicity m on V will be called *admissible* if Λ is standard and if, in the notion of *8.6*, for each $L \in Sp(V)$, $g \in \mathbf{U}(V)$,

(i) $W_L^{-1}\Lambda g W_L$ is independent of choice of W_L and

(ii) $g \circ L \in \mathbf{U}(V)$ implies $\Lambda(g \circ L) = W_L^{-1}\Lambda g W_L$.

9.1 Theorem

Let Λ be a canonical quantization scheme on V. Then Λ is admissible.

Proof. The only proof that the author has been able to find for general V is exceedingly long and complicated. For reasons of space, therefore, we shall give the proof for the relatively simple case of one-dimensional V, identified to $\mathbf{R}$.

By a remark following *8.1* we know that Λ must be standard. Let λ be Haar measure on $\mathbf{R}$. By *(1)* and *(8)* we have

$$(\exp\,(it/\hbar\Lambda f)h)(x) = \exp\,(i/\hbar \int_0^t f_1'(\phi_s^f x)ds)\sqrt{|\det\,\phi_t^{f\prime}(x)|}\,h(\phi_t^f x) \tag{i}$$

for all $f \in \mathbf{U}(\mathbf{R})$, $t \in \mathbf{R}$, $h \in \mathcal{L}_2(\mathbf{R}, \mathbf{C}^m, \lambda)$ and λ-almost all $x \in \mathbf{R}$. If $\Lambda(f \circ L) = W_L^{-1}\Lambda f W_L$ for all $f \in \mathbf{I}(\mathbf{R})$ then any other such operator W_L' is such that $W_L' = C_L W_L$ where, by *8.7*, $(C_L h)(x) = \hat{C}_L h(x)$ for all $h \in \mathcal{L}_2(\mathbf{R}, \mathbf{C}^m, \lambda)$, where $\hat{C}_L$ is a unitary operator on $\mathbf{C}^m$. Hence if $f \in \mathbf{U}(\mathbf{R})$,

$W_L'^{-1} \exp (it/\hbar\Lambda f) W_L' = W_L^{-1} C_L^{-1} \exp (it/\hbar\Lambda f) C_L W_L$
$= W_L^{-1} \exp (it/\hbar\Lambda f) W_L$, since it is easily seen from *(i)* that C_L commutes with $\exp (it/\hbar\Lambda f)$.

We next identify the set $\Theta(L)$ of functions $g \in \mathbf{U}(\mathbf{R})$ such that $g \circ L \in \mathbf{U}(\mathbf{R})$. To do this we identify $\mathbf{R}^*$ with $\mathbf{R}$ via the mapping $p \to p.1$ of $\mathbf{R}^*$ onto $\mathbf{R}$ and write L in the form $L: (q,p) \to (aq + bp,\ cq + dp)$ where $a,\ b,\ c,\ d \in \mathbf{R}$. In order that $L \in Sp(V)$ it is necessary and sufficient that $ad - bc = 1$. Let $g \in \mathbf{U}(\mathbf{R})$ be written as $g(q,p) = A(q) + p.X(q)$, where $A = g_1'$, $X = X^g$ and suppose that $g \circ L \in \mathbf{U}(\mathbf{R})$. Then we must have

$$A(aq + bp) + (cq + dp)X(aq + bp) = B(q) + pY(q) \qquad (ii)$$

for all $(q,p) \in \mathbf{R} \times \mathbf{R}^*$ and some measurable function B: $\mathbf{R} \to \mathbf{R}$ and $C^\infty Y: \mathbf{R} \to \mathbf{R}$.

If $b = 0$ then $a \neq 0$ and clearly $B(q) = A(aq) + cqX(aq)$, $Y(q) = 1/aX(aq)$ satisfy *(ii)* for *any* A and X. Hence in this case $\Theta(L) = \mathbf{U}(\mathbf{R})$. Let $f_{a,c} \in \mathbf{U}(\mathbf{R})$ be defined by

$$f_{a,c}(q,p) = \ln |a|.qp - c\gamma(a)q^2$$

where $\gamma(a) = \ln |a|/(a - 1/a)$ if $|a| \neq 1$, $\gamma(\pm 1) = \pm 1/2$.
Then calculation shows that

$$F_1^{f_{a,c}}(q,p) = (|a|q,\ p/|a| \pm cq)$$

(the upper sign is taken if $a > 0$, the lower if $a < 0$).

Hence $L: (q,p) \to (aq, p/a + cq)$ can be written as $L = \pm F_1^{f_{a,c}}$. If we let $K: \mathbf{R} \times \mathbf{R}^* \to \mathbf{R} \times \mathbf{R}^*$, $z \to -z$, this is the same as $L = F_1^{f_{a,c}}$ $(a > 0)$, $L = F_1^{f_{a,c}} \circ K (a < 0)$.

If $g \in \mathbf{U}(\mathbf{R})$, then $g \circ K(q,p) = g_1'(-q) - p.X^g(-q)$, i.e. $g \circ K \in \mathbf{U}(\mathbf{R})$ and $(g \circ K)_1' = g_1' \circ K_1$, $X^{g \circ K} = K_1 \circ X^g \circ K_1$, where $K_1: \mathbf{R} \to \mathbf{R}$, $x \to -x$. Let $(\check{K}h)(x) = h(-x)$ for all $h \in \mathcal{L}_2(\mathbf{R}, \mathbf{C}^m, \lambda)$. We shall show first that $\Lambda(g \circ K) = \check{K}\Lambda g \check{K}$ for all $g \in \mathbf{U}(\mathbf{R})$. By (i),

$$(\check{K} \exp (it/\hbar \Lambda g)\check{K}h)(x) =$$
$$\exp \left[i/\hbar \int_0^t g_1'(\phi_s^g(-x))ds\right] \sqrt{|\det \phi_t^g(-x)|}\, h(-\phi_t^g(-x)). \tag{iii}$$

The flow $\phi^{g \circ K}$ of $X^{g \circ K}$ on $\mathbf{R}$ is given by $\phi_t^{g \circ K} = K_1 \circ \phi_t^g \circ K_1$ (see the remark immediately prior to 3.2). Thus $D\phi_t^{g \circ K}(x) = DK_1(\phi_t^g \circ K_1 x) \circ D\phi_t^g(K_1 x) \circ DK_1(x) = K_1 \circ D\phi_t^g(-x) \circ K_1$; hence $\det \phi_t^{g \circ K}(x) = \det [D\phi_t^{g \circ K}(x)] = (\det K_1)^2 \det[D\phi_t^g(-x)]$ $= \det [D\phi_t^g(-x)] = \det \phi_t^g(-x)$. The right side of (iii) now becomes

$$\exp \left[i/\hbar \int_0^t g_1' \circ K_1(\phi_s^{g \circ K}x)ds\right] \sqrt{|\det \phi_t^{g \circ K}(x)|}\, h(\phi_t^{g \circ K}x),$$

i.e. $\check{K}\Lambda g \check{K} = \Lambda(g \circ K)$.

Let $g \in \mathbf{U}(\mathbf{R})$. Then if $a > 0$, $\Lambda(g \circ L) = \Lambda(g \circ F_1^{f_{a,c}}) =$ $\exp (i/\hbar \Lambda f_{a,c})\Lambda g \exp (-i/\hbar \Lambda f_{a,c})$ by 4.1. This in particular holds if $g \in \mathbf{I}(\mathbf{R})$ so that we may write $W_L = \exp (-i/\hbar \Lambda f_{a,c})$; this shows that $\Lambda(g \circ L) = W_L^{-1} \Lambda g W_L$

for all $g \in \Theta(L)$. If $a < 0$ we have $\Lambda(g \circ L) = \Lambda(g \circ F_1^{f_{a,c}} \circ K)$ $= \check{K}\Lambda(g \circ F_1^{f_{a,c}})\check{K} = \check{K} \exp(i/\hbar\Lambda f_{a,c})\Lambda g \exp(-i/\hbar\Lambda f_{a,c})\check{K}$ so that if we write $W_L = \exp(-i/\hbar\Lambda f_{a,c}) \circ \check{K}$ then $\Lambda(g \circ L) = W_L^{-1}\Lambda g W_L$ for all $g \in \Theta(L)$ in this case also.

If $b \neq 0$, again the first step is to identify $\Theta(L)$. Put $q = 0,\ p = (aq + bp)/b$ in *(ii)* to obtain $A(aq + bp) + (ad/bq + dp)X(aq + bp) = B(0) + (a/bq + p)Y(0)$. As $ad/b = c + 1/b$ this is $A(aq + bp) + (cq + dp)X(aq + bp)$ $+ q/bX(aq + bp) = B(0) + (a/bq + p)Y(0)$ and comparison with *(ii)* yields $q/bX(aq + bp) = B(0) - B(q) + aq/bY(0) + p(Y(0) - Y(q))$. The substitution $q = b,\ p = q/b - a$ gives $X(q) = B(0)$ $- B(b) + aY(b) + 1/b(Y(0) - Y(b))q$. Now put $q = 0,\ p = q/b$ in *(ii)* to obtain $A(q) = B(0) + q/b(Y(0) - dX(q))$ and consequently from the last result, $A(q) = B(0) - 1/b[d(B(0)$ $- B(b) + aY(b)) - Y(0)]q - d/b^2(Y(0) - Y(b))q^2$. Thus we have proved that for $g \circ L$ to lie in $\mathbf{U}(\mathbf{R})$ we must have $X(q) = \alpha_0 + \alpha_1 q,\ A(q) = \alpha_2 + \alpha_3 q - d\alpha_1/bq^2$ for some α_0, α_1, α_2, $\alpha_3 \in \mathbf{R}$. Substitution in *(ii)* shows that this condition is also sufficient and we obtain further that if

$$g(q,p) = \alpha_2 + \alpha_3 q - d\alpha_1/bq^2 + p(\alpha_0 + \alpha_1 q) \qquad (iv)$$

then

$$(g \circ L)(q,p) = \alpha_2 + (c\alpha_0 + a\alpha_3)q - a\alpha_1/bq^2 + p(d\alpha_0 + b\alpha_3 - \alpha_1 q). \qquad (v)$$

$\Theta(L)$ is the set of functions g of the form *(iv)*.

Any self-adjoint operator is the closure of its restriction to any domain which is dense in the Hilbert space and invariant under the one-parameter group which it generates. Thus if $\dot{H} = \Lambda g | C_0^\infty(\mathbf{R})$ where $C_0^\infty(\mathbf{R})$ is the set of $\mathbf{C}^m$-valued functions on $\mathbf{R}$ with compact support, we have $\Lambda g = \bar{\dot{H}}$. Calculation shows that

$$(\dot{H}h)(x) = (\alpha_2 + \hbar\alpha_1/(2i) + \alpha_3 x - d\alpha_1/bx^2)h(x)$$
$$+ \hbar/i(\alpha_0 + \alpha_1 x)Dh(x) \qquad (vi)$$

for all $h \in C_0^\infty(\mathbf{R})$. If we write $H_1 = \Lambda f_{(1,0)}$, $H_2 = \Lambda f_{(0,-1)}$, $\dot{H}_1 = H_1 | C_0^\infty(\mathbf{R})$, $\dot{H}_2 = H_2 | C_0^\infty(\mathbf{R})$, *(vi)* can be written $\dot{H} = \alpha_2 I + \alpha_0 \dot{H}_1 + \alpha_3 \dot{H}_2 - d\alpha_1/b\dot{H}_2^2 + \alpha_1/2(\dot{H}_1\dot{H}_2 + \dot{H}_2\dot{H}_1)$. Let

$$H = \alpha_2 I + \alpha_0 H_1 + \alpha_3 H_2 - d\alpha_1/bH_2^2$$
$$+ \alpha_1/2(H_1H_2 + H_2H_1).$$

H is obviously densely defined and has the property $H^* \supset H$. Thus H is symmetric and hence so is its closure $\bar{H}$. Since $H \supset \dot{H}$ and so $\bar{H} \supset \bar{\dot{H}} = \Lambda g$ we have $\bar{H} \subset \bar{H}^* \subset \Lambda g$, i.e. $\bar{H} = \Lambda g$.

Let $H_1^L = W_L^{-1} H_1 W_L$, $H_2^L = W_L^{-1} H_2 W_L$, $H^L = W_L^{-1} H W_L$. Then $W_L^{-1} \Lambda g W_L$ is the closure of H^L and $H^L = \alpha_2 I + \alpha_0 H_1^L + \alpha_3 H_2^L - d\alpha_1/b(H_2^L)^2 + \alpha_1/2(H_1^L H_2^L + H_2^L H_1^L)$. Since also $H_1^L = \Lambda(f_{(1,0)} \circ L) = \Lambda f_{(d,-c)}$,

$H_2^L = \Lambda(\mathrm{f}_{(0,-1)} \circ L) = \Lambda \mathrm{f}_{(b,-a)}$ the respective restrictions $\dot{H}_1^L$ and $\dot{H}_2^L$ of these operators to $C_0^\infty(\mathbf{R})$ are given by $(\dot{H}_1^L h)(x) = cxh(x) + \hbar/id\mathrm{D}h(x)$ and $(\dot{H}_2^L h)(x) = ax + \hbar/ib\mathrm{D}h(x)$, respectively, for all $h \in C_0^\infty(\mathbf{R})$. Hence the restriction $\dot{H}^L$ of H^L to $C_0^\infty(\mathbf{R})$ is given by

$$(\dot{H}^L h)(x) = (\alpha_2 - \hbar\alpha_1/(2i) + (c\alpha_0 + a\alpha_3)x$$

$$- a\alpha_1/bx^2)h(x) + \hbar/i\ (d\alpha_0 + b\alpha_3 - \alpha_1 x)\mathrm{D}h(x). \qquad (vii)$$

However, *(v)* and the same argument that led to *(vi)* shows that $\dot{H}^L$ as given by *(vii)* is equal to $\Lambda(g \circ L)|C_0^\infty(\mathbf{R})$ and moreover $\Lambda(g \circ L) = \bar{\dot{H}}^L$. Thus $W_L^{-1}\Lambda g W_L = \bar{H}^L \supset \bar{\dot{H}}^L = \Lambda(g \circ L)$. Since $W_L^{-1}\Lambda g W_L$ and $\Lambda(g \circ L)$ and both self-adjoint, it follows that $W_L^{-1}\Lambda g W_L = \Lambda(g \circ L)$. This completes the proof of the theorem.

Let Λ be admissible and put

$$\bar{\mathbf{U}}(V) = \{f \circ L\colon f \in \mathbf{U}(V),\ L \in Sp(V)\}.$$

Clearly $\mathbf{U}(V) \subset \bar{\mathbf{U}}(V)$. We *extend* Λ from $\mathbf{U}(V)$ to $\bar{\mathbf{U}}(V)$ by defining, in the notation of *8.6*, $\Lambda(f \circ L)$ to be $W_L^{-1}\Lambda f W_L$ $(f \in \mathbf{U}(V))$. To show that this definition is possible it is only necessary to check that $W_{L_1}^{-1}\ \Lambda f^{(1)} W_{L_1} = W_{L_2}^{-1}\ \Lambda f^{(2)} W_{L_2}$ whenever $f^{(1)}, f^{(2)} \in \mathbf{U}(V)$ and $f^{(1)} \circ L_1 = f^{(2)} \circ L_2$ for some $L_1, L_2 \in Sp(V)$. But $f^{(1)} \circ L_1 = f^{(2)} \circ L_2$ implies $f^{(2)} = f^{(1)} \circ L_1 \circ L_2^{-1}$ and by the hypothesis of admissibility $\Lambda f^{(2)} = W_{L_1 L_2^{-1}}^{-1} \Lambda f^{(1)} W_{L_1 L_2^{-1}}$.

Since, however, $W_{L_1L_2^{-1}}^{-1}\Lambda g W_{L_1L_2^{-1}} = \Lambda(g \circ L_1 \circ L_2^{-1}) =$ $W_{L_2}\Lambda(g \circ L_1)W_{L_2}^{-1} = W_{L_2}W_{L_1}^{-1}\Lambda g W_{L_1}W_{L_2}^{-1}$ for all $g \in \mathbf{I}(V)$, by the hypothesis of admissibility of Λ we also have $\Lambda f^{(2)} = W_{L_2}W_{L_1}^{-1}\Lambda f^{(1)}W_{L_1}W_{L_2}^{-1}$. Hence $W_{L_2}^{-1}\Lambda f^{(2)}W_{L_2} = W_{L_1}^{-1}\Lambda f^{(1)}W_{L_1}$.

If $f \in \overline{\mathbf{U}}(V)$ then $f \circ L \in \overline{\mathbf{U}}(V)$ for any $L \in Sp(V)$. Also, $f = g \circ L'$ for some $g \in \mathbf{U}(V)$, $L' \in Sp(V)$. Thus $\Lambda(f \circ L) = \Lambda(g \circ L' \circ L) = W_{L'L}^{-1}\Lambda g W_{L'L}$, which, by an argument similar to that just given, is the same as $W_L^{-1}W_{L'}^{-1}\Lambda g W_{L'}W_L = W_L^{-1}\Lambda(g \circ L')W_L = W_L^{-1}\Lambda f W_L$. Thus the following theorem has been proved.

9.2 Theorem

Let Λ be an admissible quantization scheme on V with values the self-adjoint operators in $\mathcal{H}$. Then Λ can be extended to $\overline{\mathbf{U}}(V)$ in such a way that for each $L \in Sp(V)$ there exists a unitary mapping $W_L\colon \mathcal{H} \to \mathcal{H}$ with the property $\Lambda(f \circ L) = W_L^{-1}\Lambda f W_L$ for all $f \in \overline{\mathbf{U}}(V)$. This extension is the unique one with this property.

(Observe also that $\overline{\mathbf{U}}(V)$ is completely symmetric with respect to 'q' and 'p'.)

This result has obvious theoretical significance. We are not concerned here with the determination of W_L in the general case, nor with seeing what the set of functions $\overline{\mathbf{U}}(V)$ looks like (simple examples will show that the form of these functions is exceedingly diverse). Nor are we concerned with possible extensions or generalisations of our entire theory. There is no space even to hint at these.

We shall conclude by working out two examples. First, let us take $V = \mathbf{R}$ and identify $\mathbf{R}^*$ with $\mathbf{R}$ as in the proof of *9.1*. Suppose, for some reason, we wish to attach an operator to the classical observable

$$f(q,p) = (p+q)^5 + (p^2 - q^2) \text{ sech } (q+p).$$

As it stands, f is not a member of $\mathbf{U}(\mathbf{R})$. However, if there is $L \in Sp(\mathbf{R})$ such that $f \circ L \in \mathbf{U}(\mathbf{R})$, then we shall have $f \in \overline{\mathbf{U}}(\mathbf{R})$ and we shall be able to incorporate f within the scope of an extended admissible quantization scheme. Let $a,\ b \in \mathbf{R},\ b \neq 0$, and define $L(q,p) = (aq - p/b, (b-a)q + p/b)$. Then $L \in Sp(\mathbf{R})$ and we have

$$(f \circ L)(q,p) = b^5q^5 + [b(b-2a)q^2 + 2qp] \text{ sech } (bq).$$

It can be shown that $f \circ L \in \mathbf{U}(\mathbf{R})$, so that if Λ is any admissible quantization scheme on $\mathbf{R}$ we can write $\Lambda(f \circ L) = W_L^{-1}\Lambda f W_L$ or $\Lambda f = W_L \Lambda(f \circ L) W_L^{-1}$. Given Λ on $\mathbf{U}(\mathbf{R})$ we can be assured that the operator Λf we shall obtain in this way will be independent of choice of L (i.e. of a and b) and, for fixed L, will be independent of choice of W_L. This said, we might as well take

$b = 1$, $a = \frac{1}{2}$, so that $(f \circ L)(q,p) = q^5 + 2qp \operatorname{sech} q$.
If Λ is canonical, we have, formally

$$\Lambda(f \circ L) = x^5 + \hbar/i \; [\, 2x \operatorname{sech} x\frac{\partial}{\partial x} +$$

$$\tfrac{1}{2}\,(2 \operatorname{sech} x - 2x \operatorname{sech} x \tanh x)\,].$$

Hence

$$\Lambda f = W_L[x^5 + \hbar/i \;(2x \operatorname{sech} x\frac{\partial}{\partial x} + \operatorname{sech} x - x \operatorname{sech} x \tanh x)]\, W_L^{-1},$$

where $L(q,p) = (q/2 - p,\; q/2 + p)$.

As a more familiar example we shall 'quantize' the components of the vector commonly denoted $\underline{p} \times \underline{N}$, where $\underline{p}$ is the linear momentum vector and $\underline{N}$ the angular momentum vector of a single particle. The case in point is $V = \mathbf{R}^3$ and

$$(N_1(q,p), N_2(q,p), N_3(q,p)) =$$

$$(q^2p_3 - q^3p_2, q^3p_1 - q^1p_3, q^1p_2 - q^2p_1),$$

relative to the standard bases of $\mathbf{R}^3$ and $\mathbf{R}^{3*}$. Let $\underline{p} \times \underline{N} = \underline{S}$. Then, for example,

$$S_3(q,p) = p_1N_2(q,p) - p_2N_1(q,p) =$$

$$-p_1p_3q^1 - p_2p_3q^2 + (p_1^2 + p_2^2)q^3.$$

Let G_0 be the ordinary Euclidean inner product on $\mathbf{R}^3$, i.e. $G_0(e_i, e_j) = \delta_{ij}$ where $(e_i)_{i=1}^{3}$ is the standard basis. Then L defined by $L(q,p) = (G_0^{-1}p, -G_0q)$ is in $Sp(\mathbf{R}^3)$.
In component form we have

$$L(q^1,q^2,q^3,p_1,p_2,p_3) = (p_1,p_2,p_3,-q^1,-q^2,-q^3),$$

so that $S_3 \circ L$ is given by

$$(S_3 \circ L)(q,p) = -q^1q^2p_1 - q^2q^3p_2 + ({q^1}^2 + {q^2}^2)p_3.$$

It is easily seen that $S_3 \circ L \in \mathbf{U}(\mathbf{R}^3)$; hence $S_3 \in \overline{\mathbf{U}}(\mathbf{R}^3)$. If Λ is any admissible quantization scheme on $\mathbf{R}^3$, then $\Lambda(S_3 \circ L) = W_L^{-1}\Lambda S_3 W_L$ or $\Lambda S_3 = W_L\Lambda(S_3 \circ L)W_L^{-1}$. We shall assume that Λ is canonical and λ is Lebesque measure on $\mathbf{R}^3$. Calculation shows that W_L is the unitary extension to all of $\tilde{\mathcal{L}}_2(\mathbf{R}^3,\mathbf{C}^m,\lambda)$ of its restriction to $\mathcal{L}_1(\mathbf{R}^3,\mathbf{C}^m,\lambda) \cap \mathcal{L}_2(\mathbf{R}^3,\mathbf{C}^m,\lambda)$ given by

$$(W_Lh)(x) = (2\pi\hbar)^{-3/2} \int_{\mathbf{R}^3} e^{-i/\hbar G_0(x,y)} h(y)\,dy$$

for all h in this domain. For it easily verified that $W_L^{-1}\mathring{S}(z)W_L = \mathring{S}(L^{-1}z)$. Following immediately from this are the formal results $W_L^{-1}x^kW_L = \hbar/i \frac{\partial}{\partial x}k$, $W_L^{-1}\hbar/i \frac{\partial}{\partial x}k\; W_L = -x^k$, with obvious notation $(k = 1,2,3)$.

Formally, we have (see the remarks at the end of §5)

$$\Lambda(S_3 \circ L) = \hbar/i\,[-x^1x^3\frac{\partial}{\partial x^1} - x^2x^3\frac{\partial}{\partial x^2} + ({x^1}^2 + {x^2}^2)\,\frac{\partial}{\partial x^3} + \tfrac{1}{2}(-x^3 - x^3)]$$

$$= \hbar/i\,[({x^1}^2 + {x^2}^2 + {x^3}^2)\frac{\partial}{\partial x^3} - x^3(x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + 1)],$$

so that

$$\Lambda(S_3) = W_L\Lambda(S_3 \circ L)W_L^{-1} = \hbar^2[\frac{\partial}{\partial x^3}(\frac{\partial}{\partial x^1}x^1 + \frac{\partial}{\partial x^2}x^2 + \frac{\partial}{\partial x^3}x^3 - 1) - \nabla^2 x^3]$$

$$= \hbar^2[(x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + 1)\frac{\partial}{\partial x^3} - x^3\nabla^2]$$

and, in general,

$$\Lambda(S_k) = \hbar^2[(x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + 1)\frac{\partial}{\partial x^k} - x^k\nabla^2] \quad (k = 1,2,3).$$

Note that the formal expression for $\Lambda(S_3 \circ L)$ can be written

$$\Lambda(S_3 \circ L) = -\tfrac{1}{2}\{[x^1.\hbar/i(x^3\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^3})$$

$$-x^2.\hbar/i(x^2\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^2})] - [\hbar/i(x^2\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^2})x^2$$

$$-\hbar/i(x^3\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^3})x^1]\}$$

so that

$$\Lambda(S_3) = \tfrac{1}{2}\{[\hbar/i\frac{\partial}{\partial x^1}.\hbar/i(x^3\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^3})$$

$$-\hbar/i\frac{\partial}{\partial x^2}.\hbar/i(x^2\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^2})] - [\hbar/i(x^2\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^2}).\hbar/i\frac{\partial}{\partial x^2}$$

$$-\hbar/i(x^3\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^3}).\hbar/i\frac{\partial}{\partial x^1}]\},$$

i.e. the same as obtained in conventional quantum mechanics by 'symmetrisation'. (Let us not be led into thinking that the purpose of our whole programme has been to prove this.)

APPENDIX

For reference purposes we give here an explicit description of the elements of $Sp(V)$. If $A: V \to V$, $B: V^* \to V$, $C: V \to V^*$, $D: V^* \to V^*$ are linear maps, define $A^*: V^* \to V^*$, $B^*: V^* \to V$, $C^*: V \to V^*$, $D^*: V \to V$ by $A^*p.q = p.Aq$, $p.B^*p' = p'.Bp$, $C^*q.q' = Cq'.q$, $p.D^*q = Dp.q$ for all q, $q' \in V$, p, $p' \in V^*$. Let I be the identity on V.

It follows from *(18)* that a linear map $L: V \times V^* \to V \times V^*$, $(q,p) \to (Aq + Bp, Cq + Dp)$ is in $Sp(V)$ iff

$$D^*B = B^*D, \qquad C^*A = A^*C,$$

$$D^*A - B^*C = I, \qquad A^*D - C^*B = I^*.$$

Since L^{-1} is the map $(q,p) \to (D^*q - B^*p, -C^*q + A^*p)$ and $L^{-1} \in Sp(V)$, it follows from the above relations that

$$CD^* = DC^* \qquad AB^* = BA^*,$$

$$AD^* - BC^* = I, \qquad DA^* - CB^* = I^*$$

also hold.

REFERENCES

Abraham, R.: [1], Foundations of Mechanics (Benjamin, 1967).

Artin, E.: [1], Geometric Algebra (Interscience, 1957).

Bargmann, V.: [1], Z. Phys., 99, 576 (1936).

Dieudonne, J.: [1], Foundations of Modern Analysis (Academic Press, 1969);
[2], Treatise on Analysis, Vol. II (Academic Press, 1970).

Dirac, P.A.M.: [1], Proc. Roy. Soc., A110, 561 (1926);
[2], The Principles of Quantum Mechanics, 3rd Edition (O.U.P., London, 1947).

Fradkin, D.M.: [1], Amer. Journ. Phys., 33, 207 (1965).

Guest, P.B.: [1], M.Sc. Thesis, University of St. Andrews (1967);
[2], Nuovo Cimento 61A, 593 (1969).

Guest, P.B. and A. Bors,: [1], Proc. Phys. Soc., 92, 525 (1967).

Halmos, P.R.: [1], Introduction to Hilbert space and the Theory of Spectral Multiplicity (Chelsea, 1957).

Helgason, S.: [1], Differential Geometry and Symmetric Spaces (Academic Press, 1962).

Hewitt, E. and K.A. Ross,: [1], Abstract Harmonic Analysis, Vol. II (Springer-Verlag, 1970).

Hewitt, E. and K. Stromberg,: [1], Real and Abstract Analysis (Springer-Verlag, 1965).

Hulthen, L.: [1], Z. Phys., 86, 21 (1933).

Jacobson, N.: [1], Lie Algebras (Interscience, 1962).

Lenz, W.: [1], Z. Phys., 24, 197 (1924).

Mackey, G.W.: [1], Duke Math. J., 16, 313 (1949);
[2], Ann. Math., 58, 193 (1953);
[3], Acta Mathematica, 99, 265 (1958);
[4], The Mathematical Foundations of Quantum Mechanics (Benjamin, 1963).

Marsden, J.E.: [1], Arch. Rat. Mech. Anal., 28, 5, 323 (1968).

McCoy, N.H.: [1], Proc. Natl.Acad. Sci., 18, 674, (1932).

Mukunda, N.: [1], Journ. Math. Phys., 8, 5, 1069 (1967).

Pauri, M. and G.M. Prosperi,: [1], Journ. Math. Phys., 7, 366 (1966);
[2], Journ. Math. Phys., 8, 2256 (1967);
[3], Journ. Math. Phys., 9, 1146 (1968).

Rivier, D.C.: [1], Phys. Rev. 83, 862 (L) (1957).

Simoni, A., Zaccaria, F. and B. Vitale,: [1], Nuovo Cimento, 51A, 448 (1967).

Sternberg, S.: [1], Lectures on Differential Geometry (Prentice Hall, 1964).

Tolman, R.C.: [1], The Principles of Statistical Mechanics (O.U.P., London, 1938).

Varadarajan, V.S.: [1], Geometry of Quantum Theory, Vol. II (Van Nostrand, 1970).

Von Neumann, J.:[1], Nachr. Akad. Wiss. Gottingen, Math.-Physik. Kl., 252 (1927);
[2], Math. Ann., 104, 570 (1931).

Weyl, H.: [1], Z. Phys., 46,1(1927);
[2], Zeits. fur Phys., 46, 1 (1928).

Yvon, J.: [1], Cahiers de Physique, 33, 25 (1948).