POLYNOMIAL INTERPOLATION ON A TRIANGULAR REGION

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A Thesis Submitted for the Degree of PhD
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POLYNOMIAL INTERPOLATION
ON A TRIANGULAR REGION

BY

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ABSTRACT

It is well known that given f there is a unique polynomial of degree at most n which interpolates f on the standard triangle with uniform nodes (i, j), $i, j \geq 0$, $i + j \leq n$. This leads us to the study of polynomial interpolation on a "triangular" domain with the nodes $S = \{([i], [j]): i, j \geq 0, i + j \leq n\}$, $[k] = [k]_q = \frac{1 - q^k}{1 - q}$, $q > 0$, which includes the standard triangle as a special case. In Chapter 2 of this thesis we derive a forward difference formula (of degree at most n) in the x and y directions for the interpolating polynomial P_n on S. We also construct a Lagrange form of an interpolating polynomial which uses hyperbolas (although its coefficients are of degree up to 2n) and discuss a Neville-Aitken algorithm. In Chapter 3 we derive the Newton formula for the interpolating polynomial P_n on the set of distinct points $\{(x_i, y_j): i, j \geq 0, i + j \leq n\}$. In particular if $x_i = [i]_p$ and $y_j = [j]_q$, we show that Newton's form of P_n reduces to a forward difference formula. We show further that we can express the interpolating polynomial on S itself in a Lagrange form and although its coefficients $L^n_{i,j}$ are not as simple as those of the first Lagrange form, they all have degree n. Moreover, $L^n_{i,j}$ can all be expressed in terms of $L^m_{0,0}$, $0 \leq m \leq n$. In Chapter 4 we show that P_n has a limit when both $p, q \to 0$. We then verify that the interpolation properties of the limit form depend on the appropriate partial derivatives of f(x, y). In Chapter 5 we study integration rules I_n of interpolatory type on the triangle $S_n = \{(x, y): 0 \leq x \leq y \leq [n]\}$. For $1 \leq n \leq 5$, we calculate the weights $w^n_{i,j}$ for I_n in terms of the parameter q and study certain general properties which govern $w^n_{i,j}$ on S_n. Finally, Chapter 6 deals with the behaviour of the Lebesgue functions $\lambda_n(x, y; q)$ and the corresponding Lebesgue constant. We prove a property concerning where λ_n takes the value 1 at points other than the interpolation nodes. We also analyse the discontinuity of the directional derivative of λ_n on S_n.

DECLARATION

I DAUD YAHAYA hereby certify that this thesis has been composed by myself, that it is a record of my own work, and that it has not been accepted in partial or complete fulfilment of any other degree of professional qualification.

Signed Date 14-4-94

I was admitted to the Faculty of Science of the University of St. Andrews under Ordinance General No. 12 on April 1991 and as a candidate for the Degree of Ph.D. on April 1992.

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I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate to the degree of Ph. D. of the University of St. Andrews and that he is qualified to submit this thesis in application for that degree.

Signature of Supervisor Date 15 April 1994

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Chapter 1

Interpolation and preliminary results

1.1 The interpolating polynomial

One of the most common methods of approximating functions is by polynomials. The use of polynomials is justified by the following classical theorem given by Weierstrass in 1885. (See [20].)

Theorem 1.1 Let $f(x) \in C[a, b]$. For any $\varepsilon > 0$, there exists a polynomial $P(x)$ such that

$$| f(x) - P(x) | < \varepsilon \qquad \text{for all } x \in [a, b].$$

Perhaps the most direct way in which a polynomial of degree n can be fitted to a given function f(x) is by interpolation at prescribed points. We know that a straight line can be passed through any two points having distinct abscissas. Similarly a parabola can be made to pass through any three points having distinct abscissas. The following is a generalization of these results.

Theorem 1.2 Given the values of f(x) at n + 1 distinct points $x_0, x_1, \ldots, x_n$, there exists a unique polynomial $P_n(x)$ of degree at most n such that

$$P_n(x_i) = f(x_i), \qquad i = 0, 1, 2, \ldots, n. \tag{1.1}$$

This result may be proved in a number of ways; see for example [6]. Since the polynomial is of degree $\le n$, it may be expressed as

$$P_n(x) = \sum_{i=0}^{n} c_i\, x^i .$$

Written out in matrix form, (1.1) becomes

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \cdot \\ \cdot \\ c_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \cdot \\ \cdot \\ f_n \end{bmatrix}$$

where we have denoted $f(x_i) = f_i$. The determinant of this matrix is Vandermonde's determinant, and has the value

$$\prod_{0 \le i < j \le n} (x_j - x_i)$$

which is non zero if the points are distinct.

The interpolating polynomial is often represented by the Lagrange formula where the polynomial is expressed explicitly in terms of the ordinates involved. Let $x_0, x_1, \ldots, x_n$ be distinct points in [a, b] and for each $i = 0, 1, 2, \ldots, n$, introduce the following product:

$$(x - x_0) \ldots (x - x_{i-1})(x - x_{i+1}) \ldots (x - x_n) .$$

For $i = 0, 1, 2, \ldots, n$, the normalized version of this product

$$L_i(x) = \prod_{\substack{v=0 \\ v \neq i}}^{n} \frac{x - x_v}{x_i - x_v}$$

is a polynomial of degree n and satisfies the condition

$$L_i(x_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} .$$

It follows that the function

$$P_n(x) = \sum_{i=0}^{n} f_i \, L_i(x) \tag{1.2}$$

is a polynomial of degree at most n and satisfies the required interpolation conditions (1.1). Formula (1.2) is called the Lagrange interpolation formula. Note that, since the interpolation problem (1.1) has a unique solution, (1.2) gives the only polynomial $P_n(x)$ of degree at most n which satisfies (1.1).

The accuracy with which the interpolating polynomial approximates the function f(x) depends on the choice of $x_0, x_1, \ldots, x_n$. If $f(x) \in C^n[a, b]$ and $f^{(n+1)}(x)$ exists for $a < x < b$, we can estimate the error $f(x) - P_n(x)$ in terms of the $(n + 1)$th derivative of f(x). (See [7].)

Theorem 1.3 For any $x \in [a, b]$,

$$f(x) - P_n(x) = (x - x_0)(x - x_1) \ldots (x - x_n) \frac{f^{(n+1)}(\xi_x)}{(n + 1)!} \tag{1.3}$$

where $\min \{x, x_0, x_1, \ldots, x_n\} < \xi_x < \max \{x, x_0, x_1, \ldots, x_n\}$. The point ξ_x depends upon x.

The interpolating polynomial can also be constructed recursively by an algorithm without reference to the Lagrangian formula (1.2). The basic tool is a lemma which enables us to represent an interpolating polynomial of degree m+1 in terms of two such polynomials of degree m. If S is any non empty subset of all interpolating points $\{x_0, x_1, \ldots, x_n\}$, we denote by $P_S(x)$ the polynomial that interpolates f(x) at those points x in S. Thus if S contains k+1 points, $P_S(x)$ is the unique polynomial of degree $\leq k$ such that $P_S(x_i) = f_i$, for all $x_i \in S$. We have (see [9])

Lemma 1.1 Let S and T be two proper subsets of $\{x_0, x_1, \ldots, x_n\}$, having all but the two points $x_i \in S$ and $x_j \in T$ in common. Then

$$P_{S\cup T}(x) = \frac{(x - x_i)\, P_T(x) - (x - x_j)\, P_S(x)}{x_j - x_i}.$$

The expression on the right is called a linear cross-mean. Lemma 1.1 enables us to generate interpolating polynomials of higher degree successively from polynomials of lower degree. Two standardised choices of the sets S and T used to obtain the polynomial $P_{S\cup T}$ have become widely utilized, one named after Neville, the other after Aitken. In Neville's scheme, a triangular array of polynomials $P_{k,k+d}(x)$ are generated. Here $P_{k,k+d}(x)$ is a certain polynomial of degree d that interpolates f(x) on a set of d + 1 points depending on k. Neville's scheme is as follows.

Neville's algorithm For d = 0, 1, ..., n construct the polynomial $P_{k,k+d}$ as follows:

$$P_{k,k}(x) = f_k, \qquad k = 0, 1, \ldots, n$$

$$P_{k,k+d}(x) = \frac{(x - x_k)\, P_{k+1,k+d}(x) - (x - x_{k+d})\, P_{k,k+d-1}(x)}{x_{k+d} - x_k},$$

$$k = 0, 1, \ldots, n - d.$$

Table 1.1 indicates the order of the calculations. The entry $P_{1,3}(x)$ is computed by linear cross-mean multiplication of the bold entries.

$x - x_i$	$d = 0$	$d = 1$	$d = 2$	$d = 3$
$x - x_0$	$P_{0,0} = f_0$			
		$P_{0,1}(x)$		
$\mathbf{x - x_1}$	$P_{1,1} = f_1$		$P_{0,2}(x)$	
		$\mathbf{P_{1,2}(x)}$		$P_{0,3}(x)$
$x - x_2$	$P_{2,2} = f_2$		$P_{1,3}(x)$	
		$\mathbf{P_{2,3}(x)}$		
$\mathbf{x - x_3}$	$P_{3,3} = f_3$			

Table 1.1

Continuing in this way, the final value is $P_{0,n}(x)$, which is the polynomial that interpolates on the set of all points $x_0, x_1, \ldots, x_n$.

1.2 The Newton formula and q-differences

Another method of evaluating the interpolating polynomial uses divided differences. Let us attempt to express the desired polynomial in the form

$$P_n(x) = a_0 + (x - x_0)\, a_1 + \ldots + (x - x_0)(x - x_1) \ldots (x - x_{n-1})\, a_n \tag{1.4}$$

for some values of $a_0, a_1, \ldots, a_n$. To determine the constants a_i, set $x = x_0$, $x = x_1$, ..., successively, and solve the resulting linear equations. We obtain

$$a_0 = f(x_0)$$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$a_2 = \frac{1}{x_2 - x_1}\left(\frac{f(x_2) - f(x_0)}{x_2 - x_0} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}\right)$$

for the first three coefficients. Further investigation shows that $P_n(x)$ can indeed be written uniquely in form of (1.4). Note that for a fixed set of points $x_0, x_1, \ldots, x_n$, each a_i is a linear combination of the $f(x_i)$ and that, furthermore, a_i involves only $f(x_0), \ldots, f(x_i)$. Thus a_i can be designated by $a_i = f[x_0, x_1, \ldots, x_i]$, say. The constant $f[x_0, x_1, \ldots, x_i]$ is called a divided difference of order i. In this notation the interpolating polynomial is written as

$$P_n(x) = f[x_0] + (x - x_0)\, f[x_0, x_1] + \ldots + (x - x_0) \ldots (x - x_{n-1})\, f[x_0, \ldots, x_n]. \qquad (1.5)$$

This is called the Newton interpolation formula of f(x) on the set $\{x_0, x_1, \ldots, x_n\}$.

A compact formula for a_i can be found by comparing (1.5) with the Lagrange formula (1.2) with which it must coincide. Hence

$$a_k = f[x_0, x_1, \ldots, x_k] = \sum_{i=0}^{k} \{ f(x_i) \Big/ \prod_{\substack{\nu=0 \\ \nu \neq i}}^{k} (x_i - x_\nu) \} \qquad (1.6)$$

Divided differences can be expressed in terms of lower order divided differences, as follows:

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}. \qquad (1.7)$$

This is easily verified using the symmetric form (1.6). Formula (1.7) provides a standard procedure for calculating divided differences, as indicated in Table 1.2 .

x_0	$f[x_0]$			
x_1	$f[x_1]$	$f[x_0, x_1]$		
x_2	$f[x_2]$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$	
x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$

Table 1.2

It is often convenient to think of the divided difference $f[x_0, x_1, \dots, x_n]$ as a value of the nth derivative of the function f(x) divided by the factor n!, provided this derivative exists. Let us write

$$f[x, x_0, \dots, x_k] = \frac{f[x_0, x_1, \dots, x_k] - f[x, x_0, \dots, x_{k-1}]}{x_k - x}.$$

Then it can be shown that (see [18])

$$f(x) = f[x_0] + (x - x_0)\, f[x_0, x_1] + \dots + (x - x_0) \dots (x - x_{n-1})\, f[x_0, \dots, x_n]$$

$$+ (x - x_0) \dots (x - x_n)\, f[x, x_0, \dots, x_n]. \tag{1.8}$$

On using (1.5) and comparing with the error formula (1.3), we find that

$$f[x, x_0, \dots, x_n] = \frac{f^{(n+1)}(\xi_x)}{(n + 1)!} \tag{1.9}$$

where $\min\{x, x_0, x_1, \dots, x_n\} < \xi_x < \max\{x, x_0, x_1, \dots, x_n\}$. In particular, on retaining the divided difference notation and reducing the number of points used, we have

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\eta_0)}{n!}$$

where $\min\{x_0, x_1, \ldots, x_n\} < \eta_0 < \max\{x_0, x_1, \ldots, x_n\}$.

In the case of interpolation abscissas $x_0, x_1, \ldots, x_n$ that are spaced evenly, where $x_k = x_0 + kh$, $h > 0, k = 0, 1, \ldots, n$, the divided differences may be given an elegant expression in terms of forward differences. We define the forward operators Δ on f(x) by $\Delta f(x) = f(x+h) - f(x)$ and higher differences by

$$\Delta^m f(x) = \Delta(\Delta^{m-1} f(x)) = \Delta^{m-1} f(x+h) - \Delta^{m-1} f(x), \qquad m = 2, 3, \ldots$$

Lemma 1.2 We have

$$\Delta^m f_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} f_{k+m-i}, \quad k = 0, 1, \ldots, n-m.$$

As a consequence of taking the points x_i to be equally spaced, we have

Lemma 1.3 $$f[x_k, x_{k+1}, \ldots, x_{k+m}] = \frac{\Delta^m f_k}{m!\, h^m}, \quad \text{for } k = 0, 1, \ldots, n.$$

Thus the interpolating polynomial in (1.5) can now be written as

$$P_n(x) = f_0 + (x - x_0)\frac{\Delta f_0}{h} + \ldots + (x - x_0)(x - x_1) \ldots (x - x_{n-1}) \frac{\Delta^n f_0}{n!\, h^n} \tag{1.10}$$

and each forward difference can be calculated systematically using a similar scheme to that in Table 1.2 .

If we change the variable x by putting $x - x_0 = sh$, then we can simplify $x - x_k = (s-k)h$ and

$$(x - x_0)(x - x_1) \ldots (x - x_{k-1}) \frac{\Delta^k f_0}{k!\ h^k} = h^k s(s-1) \ldots (s-k+1) \frac{\Delta^k f_0}{k!\ h^k}$$

$$= \binom{s}{k} \Delta^k f_0 .$$

Thus the polynomial (1.10) becomes

$$P_n(x) = f_0 + \binom{s}{1} \Delta f_0 + \ldots + \binom{s}{n} \Delta^n f_0 = \sum_{m=0}^{n} \binom{s}{m} \Delta^m f_0 \qquad (1.11)$$

which is called the forward difference formula for $P_n(x)$.

Thus Newton's interpolation formula (1.5) simplifies considerably, to give (1.11), in the case where the points $x_k = x_0 + kh$ form an arithmetic progression, for the divided differences reduce to ordinary differences. In [22] I. J. Schoenberg emphasized that a simplification also occurs in the case when the points of interpolation form a geometric progression. The problem of polynomial interpolation in one dimension at the points of a geometric progression was originally proposed by J. Stirling [25]. Specifically, let

$$x_k = aq^k, \quad k = 0, 1, \ldots, \quad a \neq 0,\ q \neq 1,\ q > 0. \qquad (1.12)$$

be a geometric progression. Let us consider Newton's formula (1.5) for this case. Following Schoenberg [22] we define q-differences $\mathfrak{D}^m f(x)$ recursively by

$$\mathfrak{D}^0 f(x_k) = f(x_k) ,$$

$$\mathfrak{D}^m f(x_k) = \mathfrak{D}^{m-1} f(x_{k+1}) - q^{m-1} \mathfrak{D}^{m-1} f(x_k),$$

$m = 1, 2, \ldots$. In terms of these differences we have the following result.

Lemma 1.4 For the interpolation points (1.12) the divided differences are

$$f[a, aq, \ldots, aq^n] = \frac{1}{a^n (q^n - 1)(q^n - q) \ldots (q^n - q^{n-1})} \mathfrak{D}^n f(a) .$$

Hence, using (1.5) we obtain the q-forward difference formula for $P_n(x)$

$$P_n(x) = f(a) + \sum_{i=1}^{n} \frac{(x-a)(x-aq) \ldots (x-aq^{i-1})}{a^i (q^i - 1)(q^i - q) \ldots (q^i - q^{i-1})} \mathfrak{D}^i f(a) .$$

In particular, for x = 0 we obtain

$$P_n(0) = f(a) + \sum_{i=1}^{n} \frac{1}{(1-q)(1-q^2) \ldots (1-q^i)} \mathfrak{D}^i f(a)$$

which Schoenberg calls the Stirling-Schellbach formula.

Schoenberg [22] further pointed out that the Romberg algorithm is equivalent to the Neville-Aitken algorithm when the interpolation nodes form a geometric progression. Let r be a constant such that $|r| > 1$. Starting from the column of values $R_0^{(m)}$ we form the Romberg triangular array

$$\begin{array}{cccc}
R_0^{(0)} & & & \\
 & R_1^{(0)} & & \\
R_0^{(1)} & & R_2^{(0)} & \\
 & R_1^{(1)} & & R_3^{(0)} \\
R_0^{(2)} & & R_2^{(1)} & \\
 & R_1^{(2)} & & \\
R_0^{(3)} & & &
\end{array}$$

the general recursive definition being

$$R_m^{(k)} = \frac{r^m R_{m-1}^{(k+1)} - R_{m-1}^{(k)}}{r^m - 1}.$$

Let us apply the Neville algorithm for a geometric progression $x_k = aq^k$ and for interpolation at x = 0, and form the Neville triangular array. On taking $p_{k,k}(0) = R_0^{(k)}$ $k = 0, 1, \ldots, n$ and assuming that $p_{k,k+d}(0) = R_d^{(k)}$ we obtain

$$p_{k,k+d+1}(0) = \frac{(-aq^k)\, p_{k+1,k+d+1}(0) - (-aq^{k+d+1})\, p_{k,k+d}(0)}{aq^{k+d+1} - aq^k}$$

$$= \frac{-R_d^{(k+1)} + q^{d+1} R_d^{(k)}}{q^{d+1} - 1}.$$

Now, write $r = 1/q$ to give $p_{k,k+d+1}(0) = R_{d+1}^{(k)}$. Hence these two algorithms are equivalent.

1.3 Two-dimensional polynomial interpolation

The result of Theorem 1.2 can be extended to polynomial interpolation in two dimensions at certain finite sets of points defined in a certain region of the X-Y plane. Here, we shall deal with the question of the existence, uniqueness and representation of the interpolating polynomial using a triangular network of interpolating points. A function P(x, y) in x and y is said to be a polynomial of degree not greater than n if

$$P(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} c_{i,j}\, x^i\, y^j .$$

In [14], A. R. Mitchell and G. M. Phillips showed the existence and uniqueness of the interpolating polynomial at the $(n + 1)(n + 2)/2$ uniform nodes over the standard triangle with vertices at $(0, 0)$, $(1, 0)$, and $(0, 1)$.

Theorem 1.4 Let $f(x, y)$ be a function defined over the standard triangle. There exists a unique polynomial $P_n(x, y)$ of degree not greater than n in x and y which takes the same values as $f(x, y)$ at $(i/n, 1 - j/n)$, $0 \leq i \leq j \leq n$.

Let us consider a function $f(x, y)$ defined over the standard triangle with vertices at $(0, 0)$, $(n, 0)$ and $(0, n)$. For brevity let $f(i, j)$ be denoted by $f_{i,j}$. It is possible to represent the interpolating polynomial at the uniform nodes (i, j), $i, j \geq 0$, $i + j \leq n$, in a Lagrangian form. We have

$$P_n(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} L_{i,j}^{n}(x, y)\, f_{i,j} \tag{1.13}$$

where

$$L_{i,j}^{n}(x, y) = \frac{1}{i!\,j!\,(n-i-j)!} \prod_{\nu=0}^{i-1} (x - \nu) \prod_{\nu=0}^{j-1} (y - \nu) \prod_{\nu=0}^{n-i-j-1} (n - \nu - x - y) \tag{1.14}$$

are polynomials satisfying

$$L_{i,j}^{n}(s, t) = \begin{cases} 1 & \text{if } (s, t) = (i, j) \\ 0 & \text{if } (s, t) \neq (i, j) \end{cases}$$

where $s, t \geq 0$, $s + t \leq n$.

This interpolating polynomial can also be computed by an iterative process. In [12], S. L. Lee and G. M. Phillips presented an efficient algorithm for the evaluation of the polynomial (1.13).

Let

$$f^0_{i,j}(x, y) = f_{i,j} \qquad \text{for } i, j \ge 0, \ i + j \le n$$

and for $m = 1, 2, \ldots, n$, define $f^m_{i,j}(x, y)$ recursively by

$$m\, f^m_{i,j}(x, y) = (m + i + j - x - y)\, f^{m-1}_{i,j}(x, y) + (x - i)\, f^{m-1}_{i+1,j}(x, y) + (y - j)\, f^{m-1}_{i,j+1}(x, y)$$

for $i, j \ge 0$, $i + j \le n - m$. (1.15)

Lemma 1.5 Let $P_n(x, y)$ be the polynomial interpolating the function $f(x, y)$ at (i, j), $0 \le i + j \le n$. Then

$$P_n(x, y) = f^n_{0,0}(x, y).$$

Note that this algorithm is a generalization of the Neville-Aitken algorithm for computing the one-dimensional interpolating polynomial.

Since the interpolation nodes chosen on the standard triangle are uniform nodes, it is appropriate to introduce forward differences Δ_x, Δ_y in the x and y directions respectively. On defining

$$\Delta^0_x f_{i,j} = f_{i,j}, \quad \Delta^0_y f_{i,j} = f_{i,j}$$

and for $m = 1, 2, \ldots$

$$\Delta^m_x f_{i,j} = \Delta^{m-1}_x f_{i+1,j} - \Delta^{m-1}_x f_{i,j}$$

$$\Delta^m_y f_{i,j} = \Delta^{m-1}_y f_{i,j+1} - \Delta^{m-1}_y f_{i,j},$$

the authors derived a forward difference formula for $P_n(x, y)$. This is a two-dimensional analogue of the forward difference representation of the interpolating polynomial for a function defined on equally spaced points on the real line.

Theorem 1.5 For $m = 0, 1, \ldots, n$ and $i, j \geq 0$, $i + j \leq n - m$

$$f_{i,j}^{m}(x, y) = \sum_{k=0}^{m} \sum_{r=0}^{k} \binom{x-i}{r} \binom{y-j}{k-r} \Delta_x^r \Delta_y^{k-r} f_{i,j}\,.$$

In particular, let $P_n(x, y)$ be the interpolating polynomial of $f(x, y)$ at the points (i, j), $0 \leq i + j \leq n$. Then

$$P_n(x, y) = \sum_{k=0}^{n} \sum_{r=0}^{k} \binom{x}{r} \binom{y}{k-r} \Delta_x^r \Delta_y^{k-r} f_{0,0}\,.$$

It is interesting to consider the problem of finding an interpolating polynomial on a triangle where the nodes are not necessarily uniformly spaced, for example a system of nodes which are in geometric progression. J. Stirling [25] was the first to propose polynomial interpolation in one dimension at the points of a geometric progression. As we have encountered earlier, I. J. Schoenberg [22] discussed various works with this setting and give a unified version of the problem. He showed the connection between the results of J. Stirling (1749), K. H. Schellbach (1864) and C. Runge (1891) on this problem. Schoenberg pointed out that the Stirling-Schellbach formula may be regarded as a q-forward difference formula. He also showed that the Romberg algorithm leads to this formula.

In [13], S. L. Lee and G. M. Phillips extend these results to the two-dimensional case, for a triangle domain, using as nodes the data points obtained by intersections of lines parallel to the axes in geometric spacing. They introduced a real parameter $q > 0, q \neq 1$, and used q-integers defined by

$$[n] = \frac{1-q^n}{1-q} ,$$

where n is an integer. Note that the q-integers satisfy:

(i) $[n] = \begin{cases} 1 + q + \dots + q^{n-1} & \text{if } n \text{ is positive integer} \\ 0 & \text{if } n = 0 \end{cases}$

(ii) $[-n] = \frac{-[n]}{q^n} = -q^{-n} - q^{-n+1} - \dots - q^{-1}$, where n is a positive integer

(iii) $[n] - [k] = q^k[n-k]$, for $0 \leq k \leq n$

(iv) If $k \mid n$ then $[k] \mid [n]$

(v) $\lim_{q \to 1} [n] = n$. (1.16)

G. E. Andrews [1] mentioned that C. F. Gauss (1863) was the first to study the polynomials of the form

$$G(n-k, k; q) = \frac{(1-q^n)(1-q^{n-1}) \dots (1-q^{n-k+1})}{(1-q^k)(1-q^{k-1}) \dots (1-q)} .$$

These polynomials are known as Gaussian polynomials and involve the use of the q-integers. In fact, letting $[k]! = [k][k-1] \dots [1]$, we have

$$G(n-k, k; q) = \frac{[n]!}{[k]!\,[n-k]!} = \begin{bmatrix} n \\ k \end{bmatrix}$$

which are the q-binomial coefficients. For $0 \leq k \leq n$, we see that the q-binomial coefficients satisfy the following relations.

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}$$

$$\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^{k} \begin{bmatrix} n-1 \\ k \end{bmatrix}. \tag{1.17}$$

Specifically, Lee and Phillips [13] considered the interpolating polynomial $P_n(x, y)$ for $f(x, y)$ on the triangular geometric mesh points $\{([i], [j]) : 0 \le i \le j \le n\}$. Note that all of the nodes lie on the union of the straight lines $y = q^{\nu} x + [\nu]$, $0 \le \nu \le n$, which meet at the point $(\frac{1}{1-q}, \frac{1}{1-q})$, see Figure 1.1 . Then the authors constructed the Lagrange form of the unique interpolating polynomial $P_n(x, y)$ and derived a forward difference formula for $P_n(x, y)$.

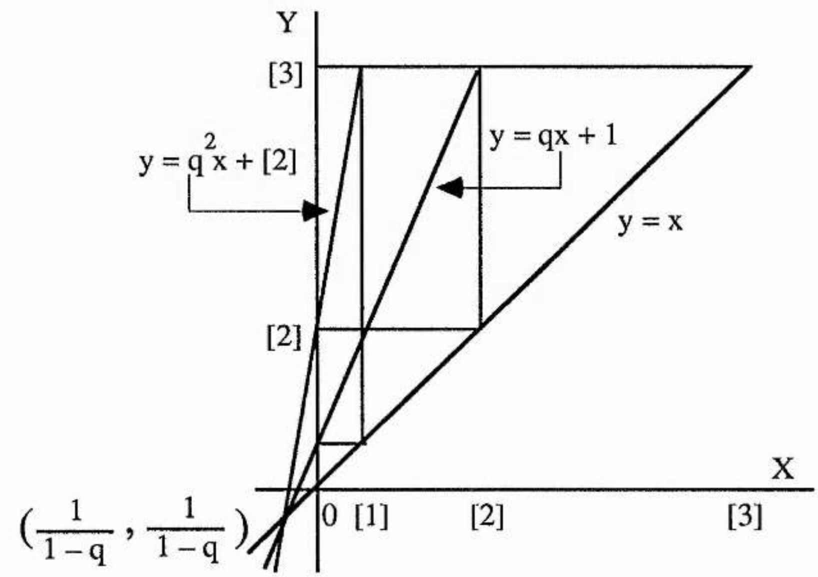

Figure 1.1

Theorem 1.6 Given a function f(x, y) defined on $\{([i], [j]):\ 0 \le i \le j \le n\}$, there exists a unique polynomial $P_n(x, y)$ of degree at most n such that

$$P_n([i], [j]) = f_{i,j} \qquad \text{for all } 0 \le i \le j \le n ,$$

where we have written $f_{i,j}$ to denote f([i], [j]). Furthermore $P_n(x, y)$ may be written in the Lagrangian form

$$P_n(x, y) = \sum_{j=0}^{n} \sum_{i=0}^{j} L_{i,j}^{n}(x, y)\, f_{i,j} \tag{1.18}$$

where

$$L_{ij}^{n}(x, y) = \frac{q^{-(2n-j-1)j/2}}{[i]!\,[j-i]!\,[n-j]!} \prod_{\nu=0}^{i-1} (x - [\nu]) \prod_{\nu=j+1}^{n} ([\nu] - y) \prod_{\nu=0}^{j-i-1} (y - q^{\nu}x - [\nu]) \tag{1.19}$$

are polynomials satisfying the conditions

$$L_{ij}^{n}([k], [m]) = \begin{cases} 1 & ([k], [m]) = ([i], [j]) \\ 0 & ([k], [m]) \ne ([i], [j]) \end{cases} .$$

Since all the points $([i], [j]):\ 0 \le i \le j \le n$ lie on the lines $y = q^{\nu}x + [\nu]$ and $y = [\nu]$, $\nu = 0, 1, \ldots, n$, it is appropriate to define q-differences along these lines. The differences along the y-direction are defined by

$$\mathfrak{D}_y^0 f_{i,j} = f_{i,j}$$

and

$$\mathfrak{D}_y^m f_{i,j} = \mathfrak{D}_y^{m-1} f_{i,j+1} - q^{m-1} \mathfrak{D}_y^{m-1} f_{i,j} ,$$

for m = 1, 2, … . The differences along diagonal-direction are defined by

$$\mathfrak{D}_z^0 f_{i,j} = f_{i,j}$$

and

$$\mathfrak{D}_z^m f_{i,j} = \mathfrak{D}_z^{m-1} f_{i+1,j+1} - q^{m-1} \mathfrak{D}_z^{m-1} f_{i,j},$$

for m = 1, 2, … . Mixed q-differences $\mathfrak{D}_z^m \mathfrak{D}_y^n$ are defined in a similar manner by

$$\mathfrak{D}_z^m \mathfrak{D}_y^n f_{i,j} = \mathfrak{D}_z^{m-1} \mathfrak{D}_y^n f_{i+1,j+1} - q^{m+n-1} \mathfrak{D}_z^{m-1} \mathfrak{D}_y^n f_{i,j}. \tag{1.20}$$

Using this notation we have the following q-forward difference formula for $P_n(x, y)$.

Theorem 1.7 Let $P_n(x, y)$ be the polynomial of degree n which interpolates the function f(x, y) at ([i], [j]): $0 \le i \le j \le n$. Then

$P_n(x, y) =$

$$\sum_{k=0}^{n} q^{-k(k-1)/2} \sum_{m=0}^{k} \prod_{\nu=0}^{m-1} \frac{(x - [\nu])}{[\nu + 1]} \prod_{\nu=0}^{k-m-1} \frac{(y - q^{\nu} x - [\nu])}{[\nu + 1]} \mathfrak{D}_z^m \mathfrak{D}_y^{k-m} f_{0,0} \tag{1.21}$$

where the void product is taken to be 1.

As a special case of Theorem 1.7, set x = 0 in (1.21). This gives

$$P_n(0, y) = \sum_{k=0}^{n} q^{-k(k-1)/2} \prod_{\nu=0}^{k-1} \frac{(y - [\nu])}{[\nu + 1]} \mathfrak{D}_y^k f_{0,0}$$

which is the one-dimensional analogue of equation (1.21), which Schoenberg calls the Stirling-Schellbach formula.

The authors further evolved an iterative process similar to the Neville-Aitken algorithm for evaluating the interpolating polynomial $P_n(x,y)$ efficiently. This algorithm can be given as follows. Let

$$f^0_{i,j}(x, y) = f_{i,j}$$

for $i \le j \le n$ and, for $m = 1, 2, \ldots, n$, define $f^m_{i,j}(x, y)$ recursively by

$$q^j\,[m]\, f^m_{i,j}(x, y) = ([m + j] - y)\, f^{m-1}_{i,j}(x, y) + (y - q^{j-i} x - [j - i])\, f^{m-1}_{i,j+1}(x, y)$$

$$+ \; q^{j-i}(x - [i])\, f^{m-1}_{i+1,j+1}(x, y)\,,$$

$0 \le i \le j \le n - m$. Then $f^m_{i,j}(x, y)$ is a polynomial in x and y of total degree m. Furthermore $f^m_{i,j}(x, y)$ interpolates the function $f(x, y)$ at $([i + s], [j + t])$, $0 \le s \le t \le m$. In particular $f^n_{0,0}(x, y)$ interpolates $f(x, y)$ at $([s], [t])$, $0 \le s \le t \le n$ and hence

$$f^n_{0,0}(x, y) = P_n(x, y).$$

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Chapter 2

Interpolating polynomial on a q-triangle

2.1 Introduction

It is known that for a function f(x, y) defined over the standard triangle with vertices at (0, 0), (n, 0) and (0, n), there exists a unique polynomial $P_n(x, y)$ of degree at most n in x and y which interpolates f(x, y) at the $(n + 1)(n + 2)/2$ points of the mesh (i, j), $i, j \geq 0$, $i + j \leq n$. In [12], S. L. Lee and G. M. Phillips derived a forward difference for the polynomial $P_n(x, y)$ and represent it in the x and y directions. In a subsequent paper [13] the authors extend the results on polynomial interpolation at the points of geometric progression to the two-dimensional case. They considered the interpolating polynomial $P_n(x, y)$ for f(x, y) on the "triangular" mesh points $\{([i], [j]): 0 \leq i \leq j \leq n\}$ and gave a forward difference formula in the y and "diagonal" directions.

In view of this, we ask whether a forward difference formula can be derived for the interpolating polynomial on a "triangular" domain with the nodes $S = \{([i], [j]): i, j \geq 0, i + j \leq n\}$, which includes the standard triangle as a special case. In this chapter we shall derive a forward difference formula in the x and y directions for the interpolating polynomial at the nodes of S. We also construct a Lagrange form of an interpolating polynomial on S and discuss a Neville-Aitken algorithm.

2.2 Successive q-forward differences

We begin by extending the definition of the q-integers to q-real. For any $t \in \mathbb{R}$ the q-real t, denoted by [t], is defined by

$$[t] = \begin{cases} \dfrac{1-q^t}{1-q}, & q \neq 1 \text{ and } q \geq 0 \\ t, & q = 1 \end{cases} .$$

We see that the numbers q-real satisfy the following properties.

(a) For each $t \in \mathbb{R}$, [t] is a continuous function of q.

(b) $[s] - [t] = q^t [s-t]$ for any $s, t \in \mathbb{R}$.

(c) For any $q \neq 1$, given any real number z satisfying $1 - z(1-q) > 0$ there exists $t = \ln\{1 - z(1-q)\}/\ln q$ such that $z = [t]$. Thus for $q > 1$ this holds for any $z > \dfrac{1}{1-q}$ and for $0 < q < 1$ this holds for any $z < \dfrac{1}{1-q}$. See Figure 2.1. For $q = 1$, then for any real z, we simply choose $t = z$.

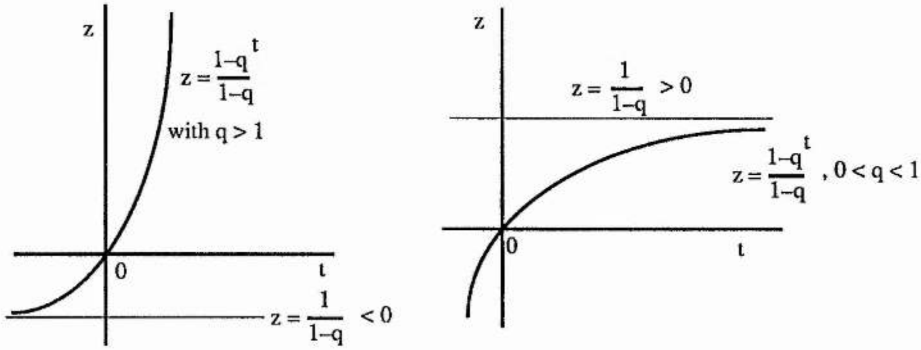

Figure 2.1

If t is an integer we will also refer to the q-real [t] as a q-integer. We note that the q-integers satisfy the properties mentioned in (1.16). Furthermore for any $t \in \mathbb{R}$ and $k \in \mathbb{Z}$, with $0 \leq k \leq t$, the q-binomial coefficient is denoted and defined by

$$\begin{bmatrix} t \\ k \end{bmatrix} = \frac{[t][t-1] \ldots [t-k+1]}{[k]!} = \prod_{\nu=0}^{k-1} \frac{[t-\nu]}{[\nu+1]}$$

where $[0]! = 1$. If $t < k$ we define $\begin{bmatrix} t \\ k \end{bmatrix} = 0$. We note that if $t = n$, a positive integer, then the q-binomial coefficients satisfy the relations given in (1.17).

Given a positive integer n, let us consider the triangular array of $(n+1)(n+2)/2$ points: $([i], [j])$ where $i, j \geq 0$, and $i + j \leq n$, formed by the lines $x = [i]$ and $y = [j]$, as shown in Figure 2.2 . If $q = 1$ then the nodes $([i], [j]) = (i, j)$ become the lattice points with integer coordinates. This array of nodes is bounded by the X-axis, the Y-axis and the hyperbola $x + y - (1-q)xy = [n]$. We shall call this region a standard q-triangle of order n. The derivation of the hyperbola equation will be shown in section 2.4. We note that for any (x, y) within the q-triangle there exist $\bar{x}, \bar{y} \in \mathbb{R}$ such that $(x, y) = ([\bar{x}], [\bar{y}])$.

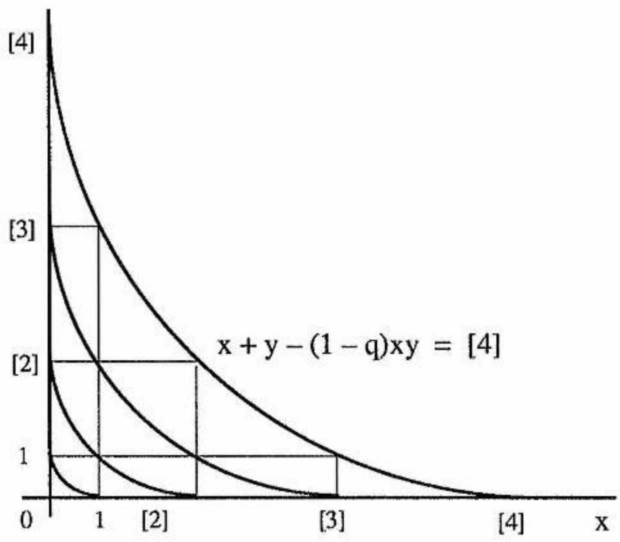

Figure 2.2

Given a function f(x, y), let us denote $f_{i,j} = f([i], [j])$ where $i, j \geq 0$, $i + j \leq n$. Then the forward difference operators $\mathfrak{D}_x f_{i,j}$, $\mathfrak{D}_y f_{i,j}$ are defined as follows.

Let $\mathfrak{D}_x^0 f_{i,j} = f_{i,j}$ and for m = 1, 2, 3, … , define recursively

$$\mathfrak{D}_x^m f_{i,j} = \mathfrak{D}_x^{m-1} f_{i+1,j} - q^{m-1} \mathfrak{D}_x^{m-1} f_{i,j} .$$

Similarly let $\mathfrak{D}_y^0 f_{i,j} = f_{i,j}$ and for n = 1, 2, 3, … , define recursively

$$\mathfrak{D}_y^n f_{i,j} = \mathfrak{D}_y^{n-1} f_{i,j+1} - q^{n-1} \mathfrak{D}_y^{n-1} f_{i,j} .$$

It follows that for m = 1, 2, 3, … , and n = 0, 1, 2, … , the mixed q-differences $\mathfrak{D}_x \mathfrak{D}_y f_{i,j}$ satisfy

$$\mathfrak{D}_x^m \mathfrak{D}_y^n f_{i,j} = \mathfrak{D}_x^{m-1} \mathfrak{D}_y^n f_{i+1,j} - q^{m-1} \mathfrak{D}_x^{m-1} \mathfrak{D}_y^n f_{i,j}$$

and for m = 0,1, 2, … , and n = 1, 2, 3, … ,

$$\mathfrak{D}_y^n \mathfrak{D}_x^m f_{i,j} = \mathfrak{D}_y^{n-1} \mathfrak{D}_x^m f_{i,j+1} - q^{n-1} \mathfrak{D}_y^{n-1} \mathfrak{D}_x^m f_{i,j}$$

From the definition above we see that divided differences $\mathfrak{D}_x^m f_{i,j}$ and $\mathfrak{D}_y^n f_{i,j}$ and mixed divided differences $\mathfrak{D}_x^m \mathfrak{D}_y^n f_{i,j}$ can be expressed in terms of function values. We need the following Lemmas.

Lemma 2.1 Let q be a positive number. Then for $n \geq 1$ and all real x

$$(x-1)(x-q) \ldots (x-q^{n-1}) = \sum_{\nu=0}^{n} (-1)^\nu \begin{bmatrix} n \\ \nu \end{bmatrix} q^{\nu(\nu-1)/2} x^{n-\nu} .$$

Proof The proof is by induction on n. The above relation clearly holds if n = 1. Suppose it is true for n. Then

$$(x-1)(x-q)\ldots(x-q^n)$$

$$= x(x-1)(x-q)\ldots(x-q^{n-1}) - q^n(x-1)(x-q)\ldots(x-q^{n-1})$$

$$= \sum_{\nu=0}^{n}(-1)^{\nu}\begin{bmatrix} n \\ \nu \end{bmatrix} q^{\nu(\nu-1)/2}\, x^{n+1-\nu} + \sum_{\nu=0}^{n}(-1)^{\nu+1}\begin{bmatrix} n \\ \nu \end{bmatrix} q^{n+\nu(\nu-1)/2}\, x^{n-\nu} \qquad (2.1)$$

$$= \begin{bmatrix} n \\ 0 \end{bmatrix} x^{n+1} + (-1)^{n+1}\begin{bmatrix} n \\ n \end{bmatrix} q^{n+n(n-1)/2}$$

$$+ \sum_{k=1}^{n}(-1)^k\left\{\begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-1)/2} + \begin{bmatrix} n \\ k-1 \end{bmatrix} q^{n+(k-1)(k-2)/2}\right\} x^{n-k+1}$$

where we have written k = ν in the first summation of (2.1) and k = ν + 1 in the second summation. Factorising the terms in the last summation above and using the q-binomial property (1.17), we obtain

$$(x-1)(x-q)\ldots(x-q^n) = \begin{bmatrix} n+1 \\ 0 \end{bmatrix} x^{n+1} + (-1)^{n+1}\begin{bmatrix} n+1 \\ n+1 \end{bmatrix} q^{n(n+1)/2}$$

$$+ \sum_{k=1}^{n}(-1)^k\left\{\begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix} q^{n+1-k}\right\} q^{k(k-1)/2}\, x^{n-k+1}$$

$$= \sum_{k=0}^{n+1}(-1)^k\begin{bmatrix} n+1 \\ k \end{bmatrix} q^{k(k-1)/2}\, x^{n-k+1}$$

and this complete the proof.

Corollary Taking x = −1 in Lemma 2.1 we see that

$$\sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix} q^{\nu(\nu-1)/2} = \prod_{\nu=0}^{n-1} (1 + q^{\nu}) .$$

For q = 1 this reduces to the familiar identity

$$\sum_{\nu=0}^{n} \binom{n}{\nu} = 2^n .$$

Lemma 2.2 For all real x and y,

$$\prod_{\alpha=0}^{r-1}(x - q^{\alpha}) \prod_{\beta=0}^{s-1}(y - q^{\beta}) = \sum_{\alpha=0}^{r} \sum_{\beta=0}^{s} (-1)^{\alpha+\beta} \begin{bmatrix} r \\ \alpha \end{bmatrix} \begin{bmatrix} s \\ \beta \end{bmatrix} q^d x^{r-\alpha} y^{s-\beta}$$

where $d = \alpha(\alpha - 1)/2 + \beta(\beta - 1)/2$.

Proof Lemma 2.1 implies

$$\prod_{\alpha=0}^{r-1} (x - q^{\alpha}) \prod_{\beta=0}^{s-1} (y - q^{\beta})$$

$$= \sum_{\alpha=0}^{r} (-1)^{\alpha} \begin{bmatrix} r \\ \alpha \end{bmatrix} q^{\alpha(\alpha-1)/2} x^{r-\alpha} \sum_{\beta=0}^{s} (-1)^{\beta} \begin{bmatrix} s \\ \beta \end{bmatrix} q^{\beta(\beta-1)/2} y^{s-\beta}$$

$$= \sum_{\alpha=0}^{r} \sum_{\beta=0}^{s} (-1)^{\alpha+\beta} \begin{bmatrix} r \\ \alpha \end{bmatrix} \begin{bmatrix} s \\ \beta \end{bmatrix} q^{\alpha(\alpha-1)/2 + \beta(\beta-1)/2} x^{r-\alpha} y^{s-\beta} .$$

We can now express $\mathfrak{D}_x^r \mathfrak{D}_y^s f_{i,j}$ in terms of function values.

Lemma 2.3 The mixed q-forward differences of order (r, s) satisfy

$$\mathfrak{D}_x^r \mathfrak{D}_y^s f_{i,j} = \sum_{\alpha=0}^{r} \sum_{\beta=0}^{s} (-1)^{\alpha+\beta} \begin{bmatrix} r \\ \alpha \end{bmatrix} \begin{bmatrix} s \\ \beta \end{bmatrix} q^d f_{i+r-\alpha, j+s-\beta} \tag{2.2}$$

where $d = \alpha(\alpha - 1)/2 + \beta(\beta - 1)/2$.

Proof Let E_x and E_y denote the q-forward shift operators defined by

$$E_x f_{i,j} = f_{i+1,j} \text{ and } E_y f_{i,j} = f_{i,j+1}$$

then the operators E_x and $\mathfrak{D}_x$ satisfy

$$E_x^r f_{i,j} = f_{i+r,j} \text{ and } \mathfrak{D}_x = E_x - I,$$

where I is the identity operator. Similarly

$$E_y^s f_{i,j} = f_{i,j+s} \text{ and } \mathfrak{D}_y = E_y - I.$$

Since

$$\mathfrak{D}_x^{m+1} f_{i,j} = \mathfrak{D}_x^m f_{i+1,j} - q^m \mathfrak{D}_x^m f_{i,j} = \mathfrak{D}_x^m E_x f_{i,j} - \mathfrak{D}_x^m q^m I f_{i,j} = \mathfrak{D}_x^m (E_x - q^m I) f_{i,j}$$

we also have

$$\mathfrak{D}_x^r = (E_x - I)(E_x - q I) \dots (E_x - q^{r-1} I) \quad \text{for } r = 1, 2, \dots,$$

and similarly

$$\mathfrak{D}_y^s = (E_y - I)(E_y - q I) \dots (E_y - q^{s-1} I) \quad \text{for } s = 1, 2, \dots, \tag{2.3}$$

with $\mathfrak{D}_x^0 = \mathfrak{D}_y^0 = I$.

Now using (2.3) and Lemma 2.2, we can write the mixed divided difference as

$$\mathfrak{D}_x^r \mathfrak{D}_y^s f_{i,j} = \prod_{\alpha=0}^{r-1} (E_x - q^\alpha I) \prod_{\beta=0}^{s-1} (E_y - q^\beta I) f_{i,j}$$

$$= \sum_{\alpha=0}^{r} \sum_{\beta=0}^{s} (-1)^{\alpha+\beta} \begin{bmatrix} r \\ \alpha \end{bmatrix} \begin{bmatrix} s \\ \beta \end{bmatrix} q^d E_x^{r-\alpha} E_y^{s-\beta} f_{i,j}$$

where $d = \alpha(\alpha-1)/2 + \beta(\beta-1)/2$ and hence

$$\mathfrak{D}_x^r \mathfrak{D}_y^s f_{i,j} = \sum_{\alpha=0}^{r} \sum_{\beta=0}^{s} (-1)^{\alpha+\beta} \begin{bmatrix} r \\ \alpha \end{bmatrix} \begin{bmatrix} s \\ \beta \end{bmatrix} q^d f_{i+r-\alpha, j+s-\beta} \, .$$

Note that, taking $r = 0$ or $s = 0$ in (2.2), we have the special cases

(i) $\mathfrak{D}_x^r f_{i,j} = \sum_{\alpha=0}^{r} (-1)^\alpha \begin{bmatrix} r \\ \alpha \end{bmatrix} q^{\alpha(\alpha-1)/2} f_{i+r-\alpha, j}$

(ii) $\mathfrak{D}_y^s f_{i,j} = \sum_{\beta=0}^{s} (-1)^\beta \begin{bmatrix} s \\ \beta \end{bmatrix} q^{\beta(\beta-1)/2} f_{i, j+s-\beta}$.

Corollary It follows immediately from Lemma 2.3 that

$$\mathfrak{D}_x^r \mathfrak{D}_y^s f_{i,j} = \mathfrak{D}_y^s \mathfrak{D}_x^r f_{i,j} \qquad \text{for } r, s = 1, 2, 3, \ldots ,$$

that is, the mixed q-differences are commutative.

2.3 Forward difference formula on the q-triangle

Let $P_n(x, y)$ be the polynomial of degree n which interpolates a function $f(x, y)$ at nodes $([i], [j])$, $i, j \geq 0$, $i + j \leq n$. First we derive a representation of such a polynomial. For the cases $n = 0$ and $n = 1$, we can take

$$P_0(x, y) = f_{0,0}$$

and

$$P_1(x, y) = f_{0,0} + x\mathfrak{D}_x f_{0,0} + y\,\mathfrak{D}_y f_{0,0}$$

respectively since $P_1([0], [0]) = f_{0,0}$, $P_1([1], [0]) = f_{1,0}$ and $P_1([0], [1]) = f_{0,1}$.

Now for $n = 2$ let us try to write $P_2(x, y)$ in the form

$$P_2(x, y) = P_1(x, y) + \frac{Ax(x - [1])}{q[2]!} + Bxy + \frac{Cy(y - [1])}{q[2]!} .$$

Note that the last three terms of $P_2(x, y)$ are multiples of the Lagrange coefficients for which $i + j = 2$. We require that $P_2([i], [j]) = f_{i,j}$ for $i, j \geq 0$, $i + j \leq 2$ and we see immediately that $P_2([i], [j]) = P_1([i], [j]) = f_{i,j}$ for $i, j \geq 0$, $i + j \leq 1$. Therefore the constants A, B and C must satisfy the following conditions. First

$$f_{2,0} = f_{0,0} + [2]\mathfrak{D}_x f_{0,0} + A = f_{1,0} + q\mathfrak{D}_x f_{0,0} + A$$

and this gives

$$A = \mathfrak{D}_x f_{1,0} - q\mathfrak{D}_x f_{0,0} = \mathfrak{D}_x^2 f_{0,0} .$$

To determine B, we have

$$f_{1,1} = f_{0,0} + \mathfrak{D}_x f_{0,0} + \mathfrak{D}_y f_{0,0} + B = f_{1,0} + \mathfrak{D}_y f_{0,0} + B$$

and therefore

$$B = \mathfrak{D}_y f_{1,0} - \mathfrak{D}_y f_{0,0} = \mathfrak{D}_x \mathfrak{D}_y f_{0,0} .$$

Finally,

$$f_{0,2} = f_{0,0} + [2]\mathfrak{D}_y f_{0,0} + C = f_{0,1} + q\mathfrak{D}_y f_{0,0} + C$$

and hence

$$C = \mathfrak{D}_y f_{0,1} - q\mathfrak{D}_y f_{0,0} = \mathfrak{D}_y^2 f_{0,0} .$$

Thus $P_2(x, y)$ can be written as

$$P_2(x, y) = f_{0,0} + x\mathfrak{D}_x f_{0,0} + y\,\mathfrak{D}_y f_{0,0} + \frac{x(x-[1])}{q[2]!}\mathfrak{D}_x^2 f_{0,0} + xy\,\mathfrak{D}_x\mathfrak{D}_y f_{0,0}$$

$$+ \frac{y(y-[1])}{q[2]!}\mathfrak{D}_y^2 f_{0,0}\ .$$

Similarly for $n = 3$ we begin by writing

$$P_3(x, y) = P_2(x, y) + \frac{Ax(x-[1])(x-[2])}{q^3[3]!} + \frac{Bx(x-[1])y}{q[2]!} + \frac{Cxy(y-[1])}{q[2]!}$$

$$+ \frac{Dy(y-[1])(y-[2])}{q^3[3]!}\ .$$

and thus $P_3([i], [j]) = P_2([i], [j])$ for all $i, j \geq 0$, $i + j \leq 2$. We determine A, B, C and D as follows. On evaluating $P_3(x, y)$ at $([3], [0])$ we have

$$f_{3,0} = f_{0,0} + [3]\mathfrak{D}_x f_{0,0} + [3]\mathfrak{D}_x^2 f_{0,0} + A\ .$$

Therefore

$$A = f_{3,0} - f_{0,0} - (1+q+q^2)\,\mathfrak{D}_x f_{0,0} - (1+q)(\mathfrak{D}_x f_{1,0} - q\mathfrak{D}_x f_{0,0}) - q^2\mathfrak{D}_x^2 f_{0,0}$$

$$= f_{3,0} - f_{1,0} - \mathfrak{D}_x f_{1,0} - q\mathfrak{D}_x f_{1,0} - q^2\mathfrak{D}_x^2 f_{0,0}$$

$$= \mathfrak{D}_x f_{2,0} - q\mathfrak{D}_x f_{1,0} - q^2\mathfrak{D}_x^2 f_{0,0}$$

$$= \mathfrak{D}_x^2 f_{1,0} - q^2\mathfrak{D}_x^2 f_{0,0} = \mathfrak{D}_x^3 f_{0,0}\ .$$

Since

$$f_{2,1} = f_{0,0} + [2]\mathfrak{D}_x f_{0,0} + \mathfrak{D}_y f_{0,0} + \mathfrak{D}_x^2 f_{0,0} + [2]\mathfrak{D}_x\mathfrak{D}_y f_{0,0} + B$$

we obtain

$$B = f_{2,1} - f_{0,0} - \mathfrak{D}_y f_{0,0} - (1+q)\mathfrak{D}_x f_{0,0} - \mathfrak{D}_x^2 f_{0,0} - (1+q)(\mathfrak{D}_x f_{0,1} - \mathfrak{D}_x f_{0,0})$$

$$= f_{2,1} - f_{0,1} - \mathfrak{D}_x f_{0,1} - q\mathfrak{D}_x f_{0,1} - \mathfrak{D}_x^2 f_{0,0}$$

$$= \mathfrak{D}_x f_{1,1} - q\mathfrak{D}_x f_{0,1} - \mathfrak{D}_x^2 f_{0,0}$$

$$= \mathfrak{D}_x^2 f_{0,1} - \mathfrak{D}_x^2 f_{0,0} = \mathfrak{D}_x^2 \mathfrak{D}_y^1 f_{0,0} .$$

Similarly we have

$$f_{1,2} = f_{0,0} + \mathfrak{D}_x f_{0,0} + [2]\mathfrak{D}_y f_{0,0} + [2]\mathfrak{D}_x\mathfrak{D}_y f_{0,0} + \mathfrak{D}_y^2 f_{0,0} + C$$

which gives

$$C = \mathfrak{D}_x^1\mathfrak{D}_y^2 f_{0,0}$$

and

$$f_{0,3} = f_{0,0} + [3]\mathfrak{D}_y f_{0,0} + [3]\mathfrak{D}_y^2 f_{0,0} + D$$

which yields

$$D = \mathfrak{D}_y^3 f_{0,0} .$$

Hence we obtain an interpolating polynomial

$$P_3(x, y) = f_{0,0} + \{ x\mathfrak{D}_x f_{0,0} + y\,\mathfrak{D}_y f_{0,0} \}$$

$$+ \{ \frac{x(x-[1])}{q[2]!} \mathfrak{D}_x^2 f_{0,0} + xy\,\mathfrak{D}_x\mathfrak{D}_y f_{0,0} + \frac{y(y-[1])}{q[2]!} \mathfrak{D}_y^2 f_{0,0} \}$$

$$+ \{ \frac{x(x-[1])(x-[2])}{q^3\,[3]!} \mathfrak{D}_x^3 f_{0,0} + \frac{x(x-[1])y}{q[2]!} \mathfrak{D}_x^2\mathfrak{D}_y^1 f_{0,0}$$

$$+ \frac{xy(y-[1])}{q[2]!} \mathfrak{D}_x^1\mathfrak{D}_y^2 f_{0,0} + \frac{y(y-[1])(y-[2])}{q^3[3]!} \mathfrak{D}_y^3 f_{0,0} \}$$

which is of degree at most three. Alternatively we can write $P_3(x, y)$ as

$$P_3(x, y) = \sum_{r=0}^{3} \sum_{s=0}^{r} \prod_{\nu=0}^{r-s-1} \frac{(x - [\nu])}{q^\nu[\nu + 1]} \prod_{\nu=0}^{s-1} \frac{(y - [\nu])}{q^\nu[\nu + 1]} \mathfrak{D}_x^{r-s} \mathfrak{D}_y^{s} f_{0,0} .$$

We conjecture that $P_n(x, y)$ is a polynomial of degree at most n which may be written as

$$P_n(x, y) = \sum_{r=0}^{n} \sum_{s=0}^{r} \prod_{\nu=0}^{r-s-1} \frac{(x - [\nu])}{q^\nu[\nu + 1]} \prod_{\nu=0}^{s-1} \frac{(y - [\nu])}{q^\nu[\nu + 1]} \mathfrak{D}_x^{r-s} \mathfrak{D}_y^{s} f_{0,0} . \qquad (2.4)$$

We know that for a given point (x, y) in the q-triangle there exist $\bar{x}, \bar{y} \in \mathbb{R}$ such that $x = [\bar{x}]$ and $y = [\bar{y}]$. Then the double products in (2.4) become

$$\prod_{\nu=0}^{r-s-1} \frac{(x - [\nu])}{q^\nu[\nu + 1]} \prod_{\nu=0}^{s-1} \frac{(y - [\nu])}{q^\nu[\nu + 1]} = \prod_{\nu=0}^{r-s-1} \frac{([\bar{x}] - [\nu])}{q^\nu[\nu + 1]} \prod_{\nu=0}^{s-1} \frac{([\bar{y}] - [\nu])}{q^\nu[\nu + 1]}$$

$$= \prod_{\nu=0}^{r-s-1} \frac{[\bar{x} - \nu]}{[\nu + 1]} \prod_{\nu=0}^{s-1} \frac{[\bar{y} - \nu]}{[\nu + 1]}$$

$$= \begin{bmatrix} \bar{x} \\ r - s \end{bmatrix} \begin{bmatrix} \bar{y} \\ s \end{bmatrix}$$

since $[\bar{x}] - [\nu] = q^\nu [\bar{x} - \nu]$ and $\begin{bmatrix} t \\ k \end{bmatrix} = \frac{[t][t-1] \ldots [t-k+1]}{[k]!}$.

Hence, using these q-binomial coefficients, the polynomial in (2.4) may be written more simply as

$$P_n(x, y) = \sum_{r=0}^{n} \sum_{s=0}^{r} \begin{bmatrix} \bar{x} \\ r - s \end{bmatrix} \begin{bmatrix} \bar{y} \\ s \end{bmatrix} \mathfrak{D}_x^{r-s} \mathfrak{D}_y^{s} f_{0,0} .$$

We are now ready to justify the conjecture expressed in (2.4).

Theorem 2.1 Let f(x, y) be defined at all points of the q-triangle of order n. For any (x, y) in the q-triangle, let

$$P_n(x, y) = \sum_{r=0}^{n} \sum_{s=0}^{r} \begin{bmatrix} \bar{x} \\ r-s \end{bmatrix} \begin{bmatrix} \bar{y} \\ s \end{bmatrix} \mathfrak{D}_x^{r-s} \mathfrak{D}_y^{s} f_{0,0} \tag{2.5}$$

where $x = [\bar{x}]$ and $y = [\bar{y}]$. Then the polynomial $P_n(x, y)$ interpolates f(x, y) at the nodes $([i], [j])$, $i, j \ge 0$, $i + j \le n$.

Proof Clearly the theorem is true for n = 0. Suppose the theorem is true for n. Now for all points ([i], [j]) with $i, j \ge 0$ and $i + j \le n$,

$$P_{n+1}([i], [j]) = \sum_{r=0}^{n} \sum_{s=0}^{r} \begin{bmatrix} i \\ r-s \end{bmatrix} \begin{bmatrix} j \\ s \end{bmatrix} \mathfrak{D}_x^{r-s} \mathfrak{D}_y^{s} f_{0,0}$$

$$+ \sum_{s=0}^{n+1} \begin{bmatrix} i \\ n+1-s \end{bmatrix} \begin{bmatrix} j \\ s \end{bmatrix} \mathfrak{D}_x^{n+1-s} \mathfrak{D}_y^{s} f_{0,0}$$

The second summation is zero because, if $s > j$ then $\begin{bmatrix} j \\ s \end{bmatrix}$ is zero and similarly if $s \le j$ then $i \le n - j \le n - s < n + 1 - s$ and therefore $\begin{bmatrix} i \\ n+1-s \end{bmatrix}$ is zero. Thus

$$P_{n+1}([i], [j]) = P_n([i], [j]) = f_{i,j}, \quad 0 \le i + j \le n .$$

To complete the proof by induction, we have to show that $P_{n+1}([i], [j]) = f_{i,j}$ for all points ([i], [j]) with $i + j = n + 1$. We obtain

$$P_{n+1}([i], [j]) = \sum_{r=0}^{n+1} \sum_{s=0}^{r} \begin{bmatrix} i \\ r-s \end{bmatrix} \begin{bmatrix} j \\ s \end{bmatrix} \mathfrak{D}_x^{r-s} \mathfrak{D}_y^{s} f_{0,0}$$

$$= \sum_{s=0}^{n+1} \sum_{r=s}^{n+1} \begin{bmatrix} i \\ r-s \end{bmatrix} \begin{bmatrix} j \\ s \end{bmatrix} \mathfrak{D}_x^{r-s} \mathfrak{D}_y^s f_{0,0}$$

$$= \sum_{s=0}^{n+1} \sum_{t=0}^{n+1-s} \begin{bmatrix} i \\ t \end{bmatrix} \begin{bmatrix} j \\ s \end{bmatrix} \mathfrak{D}_x^t \mathfrak{D}_y^s f_{0,0}$$

where we have changed the order of summation and written $t = r - s$. We now use the fact that the q-binomial coefficient $\begin{bmatrix} j \\ s \end{bmatrix} = 0$ if $j < s$, and that $i = n + 1 - j \le n + 1 - s$, to obtain

$$P_{n+1}([i], [j]) = \sum_{s=0}^{j} \sum_{t=0}^{i} \begin{bmatrix} i \\ t \end{bmatrix} \begin{bmatrix} j \\ s \end{bmatrix} \mathfrak{D}_x^t \mathfrak{D}_y^s f_{0,0} .$$

We then apply Lemma 2.3 to give

$$P_{n+1}([i], [j]) = \sum_{s=0}^{j} \sum_{t=0}^{i} \begin{bmatrix} i \\ t \end{bmatrix} \begin{bmatrix} j \\ s \end{bmatrix} \sum_{\alpha=0}^{t} \sum_{\beta=0}^{s} (-1)^{\alpha+\beta} \begin{bmatrix} t \\ \alpha \end{bmatrix} \begin{bmatrix} s \\ \beta \end{bmatrix} q^d f_{t-\alpha,s-\beta}$$

where $d = \alpha(\alpha - 1)/2 + \beta(\beta - 1)/2$. If we substitute $\alpha = t - k$ and $\beta = s - m$ and rearrange the summations, the last equation above becomes

$$P_{n+1}([i], [j])$$

$$= \sum_{t=0}^{i} \sum_{s=0}^{j} \sum_{k=0}^{t} \sum_{m=0}^{s} (-1)^{t-k+s-m} \begin{bmatrix} t \\ t-k \end{bmatrix} \begin{bmatrix} s \\ s-m \end{bmatrix} \begin{bmatrix} i \\ t \end{bmatrix} \begin{bmatrix} j \\ s \end{bmatrix} q^d f_{k,m} . \qquad (2.6)$$

In (2.6) we write

$$\begin{bmatrix} i \\ t \end{bmatrix} \begin{bmatrix} t \\ t-k \end{bmatrix} = \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} i-k \\ t-k \end{bmatrix}, \quad \begin{bmatrix} j \\ s \end{bmatrix} \begin{bmatrix} s \\ s-m \end{bmatrix} = \begin{bmatrix} j \\ m \end{bmatrix} \begin{bmatrix} j-m \\ s-m \end{bmatrix}$$

and change the order of the summation, writing

$$\sum_{t=0}^{i} \sum_{k=0}^{t} = \sum_{k=0}^{i} \sum_{t=k}^{i} \quad \text{and} \quad \sum_{s=0}^{j} \sum_{m=0}^{s} = \sum_{m=0}^{j} \sum_{s=m}^{j} .$$

This gives

$P_{n+1}([i], [j])$

$$= \sum_{k=0}^{i} \sum_{m=0}^{j} \sum_{t=k}^{i} \sum_{s=m}^{j} (-1)^{t-k+s-m} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} i-k \\ t-k \end{bmatrix} \begin{bmatrix} j \\ m \end{bmatrix} \begin{bmatrix} j-m \\ s-m \end{bmatrix} q^d f_{k,m}$$

$$= \sum_{k=0}^{i} \sum_{m=0}^{j} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} j \\ m \end{bmatrix} f_{k,m} \sum_{\nu=0}^{i-k} \sum_{\mu=0}^{j-m} \begin{bmatrix} i-k \\ \nu \end{bmatrix} \begin{bmatrix} j-m \\ \mu \end{bmatrix} (-1)^{\nu+\mu} q^{\nu(\nu-1)/2+\mu(\mu-1)/2}$$

where we have written $\nu = t - k$ and $\mu = s - m$.

If we choose $k < i$, it follows from Lemma 2.2 on setting $x = y = 1$, that

$$\sum_{\nu=0}^{i-k} \sum_{\mu=0}^{j-m} \begin{bmatrix} i-k \\ \nu \end{bmatrix} \begin{bmatrix} j-m \\ \mu \end{bmatrix} (-1)^{\nu+\mu} q^{\nu(\nu-1)/2+\mu(\mu-1)/2} = 0$$

and similarly if $k = i$ and $m < j$ this sum is also zero. Thus, taking $k = i$ and $m = j$, we have $P_{n+1}([i], [j]) = f_{i,j}$ and the theorem follows.

Remark When $q = 1$, (2.5) reduces to Theorem 1.5. a forward difference formula for polynomial interpolation on the standard triangle:

$$P_n(x, y) = \sum_{r=0}^{n} \sum_{s=0}^{r} \binom{x}{r-s} \binom{y}{s} \Delta_x^{r-s} \Delta_y^{s} f_{0,0} .$$

Example 2.1 To establish the forward difference formula given in (2.5), we consider the function f(x) = exp(0.003x + 0.01y) defined on the standard q-triangle of order 5 with q = 3, and then evaluate the interpolating polynomial $P_5(x, y)$ at the point (2.5, 6.0). From Theorem 2.1, the interpolating polynomial P_5 can be written as

$$P_5(2.5, 6.0) = \sum_{m=0}^{5} \sum_{k=0}^{m} \begin{bmatrix} \bar{x} \\ k \end{bmatrix} \begin{bmatrix} \bar{y} \\ m-k \end{bmatrix} \mathfrak{D}_x^k \mathfrak{D}_y^{m-k} f_{0,0}$$

where

$$[\bar{x}] = 2.5 \quad \text{and} \quad [\bar{y}] = 6.0$$

and

$$\begin{bmatrix} \bar{x} \\ k \end{bmatrix} = \frac{1-q^{\bar{x}}}{1-q^k} \cdot \frac{1-q^{\bar{x}-1}}{1-q^{k-1}} \cdots \frac{1-q^{\bar{x}-k+1}}{1-q} .$$

In Appendix 2, we give a detailed Pascal computer program, created to evaluated $P_5(2.5, 6.0)$. The result shows that

$$f(2.5, 6.0) = 1.0698302596$$
$$P_5(2.5, 6.0) = 1.0698302625.$$

We obtain a good estimate to f(x, y) at the point (2.5, 6.0), where the error is less than 3.0×10^{-9}.

2.4 Lagrange interpolation formula using hyperbolas

Let f(x, y) be a function defined on the q-triangle of order n with the set of nodes $S = \{([i], [j]): i, j \geq 0, i + j \leq n\}$. For q = 1 it has been shown (see, for example, Lee and Phillips [12]) that the interpolating polynomial of degree n on such a triangle can be written in Lagrangian form (1.13) and (1.14). In this section we shall construct the Lagrange form of an interpolation polynomial at the points S on the q-triangle. On a system where q = 1, the diagonal nodes (i, j), where $i, j \geq 0$ and $i + j = k$, lie on a straight line. For $q \neq 1$, let us consider a curve that passes through the "diagonal nodes" $\{([i], [j]):\ i, j \geq 0,\ i + j = k\}$. Its parametric equations are given by

$$x(1-q) = 1 - q^i \quad \text{and} \quad y(1-q) = 1 - q^j, \quad \text{with } i + j = k.$$

This curve satisfies

$$\{1-(1-q)x\}\{1-(1-q)y\} = 1-(1-q)x-(1-q)y+(1-q)^2xy = q^k$$

and it is a hyperbola

$$x + y - (1-q)xy = [k],$$

(see also Figure 2.2) which we will write as $\gamma(x, y) = [k]$.

We now recognize that the nodes in the set S are formed by two linear systems $x = [\nu]$, $y = [\nu]$ and the system of hyperbolas $\gamma(x, y) = [\nu]$ where $\nu = 0, 1, \ldots, n$. So given any point $([i], [j])$ on the triangle, the union of the hyperbolas $\gamma(x, y) = [n - \nu]$ for $\nu = 0, 1, \ldots, n - i - j - 1$, the straight lines $x = [\nu]$ for $\nu = 0, 1, \ldots, i - 1$ and $y = [\nu]$ for $\nu = 0, 1, \ldots, j - 1$ contain all nodes on the triangle except the point $([i], [j])$ itself. Thus the product

$$\prod_{\nu=0}^{i-1}(x - [\nu]) \prod_{\nu=0}^{j-1}(y - [\nu]) \prod_{\nu=0}^{n-i-j-1}([n-\nu] - \gamma(x, y))$$

vanishes at all nodes $([h], [k])$, $h, k \geq 0$, $h + k \leq n$, except at the point $([i], [j])$ where its value is

$$\omega(i, j) = [i]!\, q^{(i-1)i/2}\, [j]!\, q^{(j-1)j/2}\, ([n] - [i+j])([n-1] - [i+j]) \ldots ([i+j+1] - [i+j])$$

$$= [i]!\, [j]!\, [n-i-j]!\, q^{(i+j)(2n-1-i-j)/2 - ij}.$$

Note that $\gamma([i], [j]) = [i + j]$. It follows that, for $i, j \geq 0$, $i + j \leq n$, the polynomial

$$M^n_{i,j}(x, y) = \frac{1}{\omega(i, j)} \prod_{\nu=0}^{i-1}(x - [\nu]) \prod_{\nu=0}^{j-1}(y - [\nu]) \prod_{\nu=0}^{n-i-j-1}([n-\nu] - \gamma(x, y)) \tag{2.7}$$

satisfies the conditions

$$M^{n}_{i,j}([h],[k]) = \begin{cases} 1 & ([h],[k])=([i],[j]) \\ 0 & \text{at all other nodes in } S \end{cases} .$$

We also note that, in the above expression for $M^{n}_{i,j}(x, y)$, an empty product (when $i = 0$ or $j = 0$ or $i + j = n$) is taken to have value 1.

Thus we obtain a Lagrangian form of an interpolating polynomial which uses hyperbolas and two linear systems. The polynomial can be expressed as

$$P(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} M^{n}_{i,j}(x, y)\, f_{i,j} . \tag{2.8}$$

In this case the degree of $P(x, y)$ is at most $2n$, since the degree of any $M^{n}_{i,j}(x, y)$ is at most $2n - i - j$. However, letting q tend to 1, the polynomial $P(x, y)$ in (2.8) reduces to (1.13), the interpolating polynomial of degree n on the standard triangle. Later we will obtain an interpolating polynomial of possibly lower degree than that of $P(x, y)$ for the above system of points.

By following the method in Lee and Phillips [13], we now give an algorithm for evaluating the Lagrange coefficients $M^{n}_{i,j}(x, y)$ in (2.8).

Let $a^{0}_{0,0}(x, y) = 1$ and for $m = 1, 2 \ldots, n$ define $a^{m}_{i,j}(x, y)$, $i, j \geq 0$, $i + j \leq m$, recursively by

$$[n - m + 1]\, a^{m}_{i,j}(x, y) = q^{-i-j}\{[n - m + 1 + i + j] - \gamma(x, y)\}\, a^{m-1}_{i,j}(x, y)$$

$$+ q^{1-i}(x - [i - 1])\, a^{m-1}_{i-1,j}(x, y) + q^{1+i-j}(y - [j - 1])\, a^{m-1}_{i,j-1}(x, y) \tag{2.9}$$

and $a^m_{i,j}(x, y) = 0$ otherwise.

Then, as we will see from Lemma 2.4 below,

$$a^n_{i,j} = M^n_{i,j},$$

where $i, j \geq 0$, $i + j \leq n$.

Lemma 2.4 For $m = 0, 1, \ldots, n$ and $i, j \geq 0$, $0 \leq i + j \leq m$,

$$a^m_{i,j}(x, y) = \tau^m_{i,j}\, b^m_{i,j}(x, y) \tag{2.10}$$

where

$$\tau^m_{i,j} = q^{ij-(i+j)(2m-1-i-j)/2} \frac{1}{[i]!\,[j]!\,[m-i-j]!} \Big/ \begin{bmatrix} n \\ m \end{bmatrix}$$

and

$$b^m_{i,j}(x, y) = \prod_{\nu=0}^{i-1} (x - [\nu]) \prod_{\nu=0}^{j-1} (y - [\nu]) \prod_{\nu=0}^{m-i-j-1} ([n - \nu] - \gamma(x, y)) .$$

Proof By definition $a^0_{0,0}(x, y) = 1$. Suppose (2.10) is true for $m - 1$. Then

$$[n - m + 1]\, a^m_{i,j}(x, y) = q^{-i-j}\, \tau^{m-1}_{i,j} \{[n - m + 1 + i + j] - \gamma(x, y)\}\, b^{m-1}_{i,j}(x, y)$$

$$+ q^{1-i}\, \tau^{m-1}_{i-1,j}\, (x - [i - 1])\, b^{m-1}_{i-1,j}(x, y)$$

$$+ q^{1+i-j}\, \tau^{m-1}_{i,j-1}\, (y - [j - 1])\, b^{m-1}_{i,j-1}(x, y)$$

$$= \{q^{-i-j}\, \tau^{m-1}_{i,j} + q^{1-i}\, \tau^{m-1}_{i-1,j} + q^{1+i-j}\, \tau^{m-1}_{i,j-1}\}\, b^m_{i,j}(x, y)$$

$$= \left\{ \frac{q^d}{[i]!\,[j]!\,[m-1-i-j]!} + \frac{q^{d+m-i-j}}{[i-1]!\,[j]!\,[m-i-j]!} \right.$$

$$\left. + \frac{q^{d+m-j}}{[i]!\,[j-1]!\,[m-i-j]!} \right\} b^m_{i,j}(x, y) \Big/ \begin{bmatrix} n \\ m-1 \end{bmatrix}$$

where $d = ij - (i + j)(2m - 1 - i - j)/2$. Thus

$$[n - m + 1]\, a^m_{i,j}(x, y) = \left\{ \frac{[m-i-j] + [i]q^{m-i-j} + [j]q^{m-j}}{[i]!\,[j]!\,[m-i-j]!} \right\} q^d\, b^m_{i,j}(x, y) \Big/ \begin{bmatrix} n \\ m-1 \end{bmatrix}$$

and so

$$a^m_{i,j}(x, y) = \frac{[m]}{[i]!\,[j]!\,[m-i-j]!} \; \frac{[m-1]!\,[n-m+1]!}{[n-m+1]\,[n]!} \; q^d\, b^m_{i,j}(x, y)$$

$$= \tau^m_{i,j}\, b^m_{i,j}(x, y) ,$$

since $[m]\, q^{n-m} = [n] - [n - m]$. Thus completes the proof by induction.

Corollary For $i, j \geq 0$, $0 \leq i + j \leq n$,

$$a^n_{i,j} = M^n_{i,j} .$$

2.5 Neville-Aitken Algorithm

We also construct a Neville-Aitken algorithm for an interpolating polynomial on the q-triangle of order n. For each $m = 1, 2, \ldots, n$, the algorithm generates polynomials $f^m_{i,j}(x, y)$: $i, j \geq 0$, $i + j \leq n - m$, which interpolate $f(x, y)$ on $T^m_{i,j}$, $i, j \geq 0$, $i + j \leq n - m$ respectively. Here we have used the notation $T^m_{i,j}$ to mean the set of nodes

$$T^m_{i,j} = \{([i + s], [j + t]):\; s, t \geq 0 \text{ and } s + t \leq m\}.$$

These are in the q-triangle bounded by the lines $x = [i]$, $y = [j]$ and the hyperbola $\gamma(x, y) = [m + i + j]$.

Lemma 2.5 Let $f^{0}_{i,j}(x, y) = f_{i,j}$, $i, j \geq 0$, $i + j \leq n$. For $m = 1, 2, \ldots, n$, we define $f^{m}_{i,j}(x, y)$, $i, j \geq 0$, $i + j \leq n - m$, recursively by

$$\begin{aligned} q^{i+j}[m]\, f^{m}_{i,j}(x, y) &= \{[m + i + j] - \gamma(x, y)\}\, f^{m-1}_{i,j}(x, y) \\ &+ \{\gamma(x, y) - q^{i} y - [i]\}\, f^{m-1}_{i+1,j}(x, y) \\ &+ q^{i}(y - [j])\, f^{m-1}_{i,j+1}(x, y) . \end{aligned} \tag{2.11}$$

Then $f^{m}_{i,j}(x, y)$ interpolates $f(x, y)$ on $T^{m}_{i,j}$.

Lemma 2.5 is a special case of Lemma 2.6 which we state and prove below. Note that the recurrence relation in (2.11) gives us an interpolating polynomial $f^{n}_{0,0}(x, y)$ which satisfies $f^{n}_{0,0}([s], [t]) = f_{s,t}$, $0 \leq s + t \leq n$. However, as we will see, this polynomial is not the same as $P(x, y)$ in (2.8). We also see that there is an asymmetry in the last two coefficients on the right side of (2.11). By modifying these coefficients, we can derive a one-parameter family of polynomials $f^{n}_{0,0}(x, y)$ which satisfy the above interpolating property.

Lemma 2.6 Let $f^{0}_{i,j}(x, y) = f_{i,j}$, $i, j \geq 0$, $i + j \leq n$. For $m = 1, 2, \ldots, n$, we define $f^{m}_{i,j}(x, y)$, $0 \leq i + j \leq n - m$, recursively by

$$q^{i+j}[m]\, f^{m}_{i,j}(x, y) = \{[m + i + j] - \gamma(x, y)\} f^{m-1}_{i,j}(x, y)$$

$$+ (x - [i])\{q^j - \lambda(1-q)(y - [j])\}\, f^{m-1}_{i+1,j}(x, y)$$

$$+ (y - [j])\{q^i - (1-\lambda)(1-q)(x - [i])\}\, f^{m-1}_{i,j+1}(x, y) \qquad (2.12)$$

where λ is an arbitrary real number. Then $f^m_{i,j}(x, y)$ interpolates $f(x, y)$ on $T^m_{i,j}$.

Proof First we note that the coefficient of $f^{m-1}_{i,j}(x, y)$ in (2.12) may be expressed as

$$[m+i+j] - \gamma(x, y) = q^{i+j}[m] - q^j(x - [i]) - q^i(y - [j]) + (1-q)(x - [i])(y - [j])\,.$$

This is easily verified and this alternative form makes the rest of this proof easier. When $\lambda = 1$, (2.12) simplifies to give (2.11), because

$$(x - [i])\,\{q^j - (1-q)(y - [j])\} = (x - [i])\,\{1 - (1-q)y\} = \gamma(x, y) - q^i y - [i]\,.$$

The proof is by induction. Clearly the above result holds for $m = 0$ where $f^0_{i,j}$, $0 \le i + j \le n$, interpolates f at the single point $([i], [j])$. Suppose that (2.12) holds for some $m - 1$. Therefore the polynomials $f^{m-1}_{i,j}$, $f^{m-1}_{i+1,j}$ and $f^{m-1}_{i,j+1}$ interpolate f on $T^{m-1}_{i,j}$, $T^{m-1}_{i+1,j}$ and $T^{m-1}_{i,j+1}$ respectively, as shown in Figure 2.3 . For any integers $i, j \ge 0$, $0 \le i + j \le n - m$ consider the function $f^m_{i,j}(x, y)$ at the nodes

$$T^m_{i,j} = T^{m-1}_{i,j} \cup T^{m-1}_{i+1,j} \cup T^{m-1}_{i,j+1} = \{([i+s], [j+t]),\ s, t \ge 0,\ 0 \le s + t \le m\}\,.$$

We now show that polynomial $f^m_{i,j}(x, y)$ interpolates f on $T^m_{i,j}$. First we see that if the node $([h], [k]) \in T^{m-1}_{i,j} \cap T^{m-1}_{i+1,j} \cap T^{m-1}_{i,j+1}$ then

$$f^{m-1}_{i,j}([h], [k]) = f^{m-1}_{i+1,j}([h], [k]) = f^{m-1}_{i,j+1}([h], [k]) = f_{h,k}\,,$$

and hence

$$q^{i+j}\,[m]\,f^{m}_{i,j}([h],[k]) \;=\; \{\, q^{i+j}\,\{[m]-[h-i]-[k-j] \;+\; (1-q)[h-i][k-j]\}$$

$$+\; q^{i+j}\,[h-i]\{1-\lambda(1-q)[k-j]\}$$

$$+\; q^{i+j}\,[k-j]\{1-(1-\lambda)(1-q)[h-i]\}\,\}\; f_{h,k}$$

$$=\; q^{i+j}\,[m]\,f_{h,k}\,.$$

We see also that on the three extreme points $([i],[j])$, $([i+m],[j])$ and $([i],[j+m])$

$$q^{i+j}\,[m]\,f^{m}_{i,j}([i],[j]) \;=\; q^{i+j}\,[m]\,f^{m-1}_{i,j}([i],[j]) \;=\; q^{i+j}\,[m]\,f_{i,j}$$

and similarly

$$f^{m}_{i,j}([i+m],[j]) \;=\; f_{i+m,j}\,,\;\text{ and }\; f^{m}_{i,j}([i],[j+m]) \;=\; f_{i,j+m}\,.$$

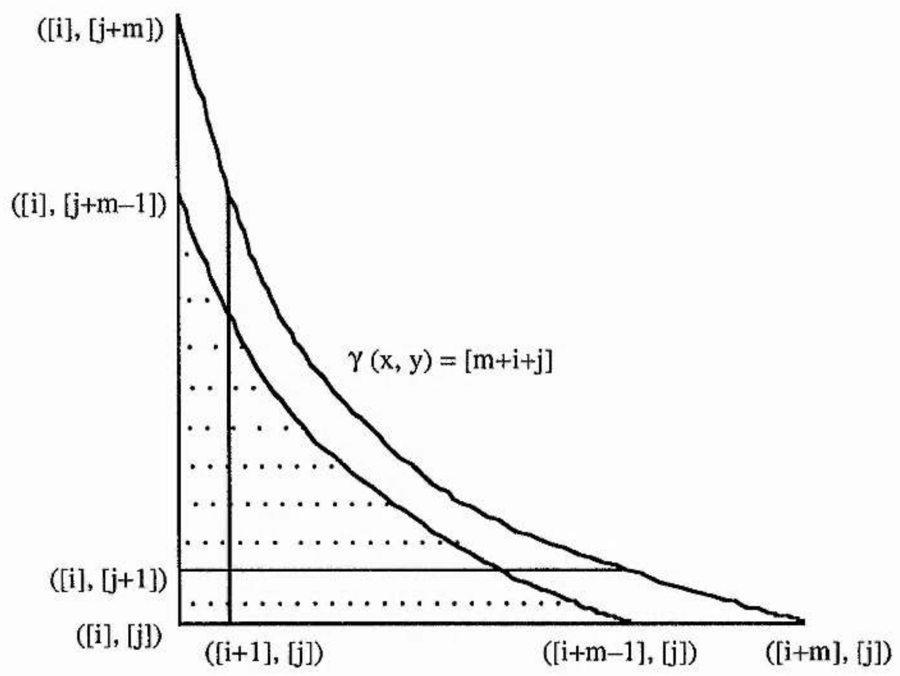

Figure 2.3

To complete the proof we consider the rest of the nodes, which are on the hyperbola $\gamma(x, y) = [m + i + j]$ or one of the straight lines $x = [i]$ and $y = [j]$. On the hyperbola $\gamma(x, y) = [m + i + j]$, at the nodes $([h], [k])$ such that $h < i + m$, $k < j + m$, we have

$$f^{m-1}_{i+1,j}([h], [k]) = f^{m-1}_{i,j+1}([h], [k]) = f_{h,k}$$

and thus

$$\begin{aligned} q^{i+j}[m]\, f^{m}_{i,j}([h], [k]) &= \{\, q^{i+j}\{[h-i] - \lambda(1-q)[h-i][k-j]\} \\ &\quad + q^{i+j}\{[k-j] - (1-\lambda)(1-q)[h-i][k-j]\}\,\} f_{h,k} \\ &= q^{i+j}\{[h-i] + [k-j] - (1-q)\,[h-i][k-j]\} f_{h,k} \\ &= q^{i+j}[m]\, f_{h,k}\,, \end{aligned}$$

where $h, k \geq 0$, $h + k = m + i + j$.

On the line $x = [i]$, with $j < k < j + m$, we obtain

$$\begin{aligned} q^{i+j}[m]\, f^{m}_{i,j}([i], [k]) &= \{\, q^{i+j}\{[m] - [k-j]\} + q^{i+j}[k-j]\,\} f_{i,k} \\ &= q^{i+j}[m]\, f_{i,k}\,. \end{aligned}$$

Similarly, on the line $y = [j]$ with $i < h < i + m$,

$$f^{m}_{i,j}([h], [j]) = f_{h,j}\,.$$

Thus, by induction, the formula is true for all m, $0 \leq m \leq n$.

We note further that the recurrence relation (2.12) cannot be written in the form

$$q^{i+j}\,[m]\,f^{m}_{i,j}(x, y) = \{[m + i + j] - \gamma(x, y)\}\,f^{m-1}_{i,j}(x, y)$$

$$+ \; q^{j}\,(x - [i])\,f^{m-1}_{i+1,j}(x, y) \; + \; q^{i}\,(y - [j])\,f^{m-1}_{i,j+1}(x, y)$$

as we might expect, in seeking to generalize the case of q = 1. This is because, at the points of $T^{m-1}_{i,j} \cap T^{m-1}_{i+1,j} \cap T^{m-1}_{i,j+1}$, we would require

$$q^{i+j}\,[m]\,f^{m}_{i,j}([i + s], [j + t]) = (\,q^{i+j+s+t}\,[m - s - t] + q^{i+j}\,[s] + q^{i+j}\,[t]\,)\,f_{i+s,j+t}$$

$$= q^{i+j}\,\{[m] - [s + t] + [s] + [t]\}\,f_{i+s,j+t}\,,$$

which means

$$f^{m}_{i,j}([i + s], [j + t]) = \frac{[m] + (1 - q)[s][t]}{[m]}\,f_{i+s,j+t}\,.$$

Thus we would require

$$\frac{[m] + (1 - q)[s][t]}{[m]} = 1$$

and, unless q = 1, the above equality holds only for s = 0 or t = 0.

We note that $f^{n}_{0,0}(x, y)$ in (2.12) and P(x, y) in (2.8) are two interpolating polynomials on the same q-triangle of order n. Thus it is natural to ask, are these polynomials identical ? We know that the degree of P(x, y) is at most 2n. In fact some of the Lagrange coefficients $M^{n}_{i,j}$ (those for which i + j = n) are of degree precisely n. Similarly the class of polynomials $f^{n}_{0,0}(x, y)$ given by (2.12) are of degree at most 2n. However none of the Neville-Aitken algorithms of the form (2.12) generate the interpolating polynomial defined in (2.8). This is shown in the following counter example.

Example 2.2 Consider the two interpolating polynomials $P(x, y)$ and $f^{1}_{0,0}(x, y)$ defined by (2.8) and (2.12) respectively on a q-triangle of order 1. From (2.8) we have

$$P(x, y) = M^{1}_{0,0}(x, y)\, f_{0,0} + M^{1}_{1,0}(x, y)\, f_{1,0} + M^{1}_{0,1}(x, y)\, f_{0,1}$$

$$= \{1 - \gamma(x, y)\}\, f_{0,0} + x\, f_{1,0} + y\, f_{0,1}\,.$$

Now let us consider the recurrence relation (2.12). We have

$$f^{0}_{0,0}(x, y) = f_{0,0}\,,\ f^{0}_{1,0}(x, y) = f_{1,0} \text{ and } f^{0}_{0,1}(x, y) = f_{0,1}$$

and thus

$$f^{1}_{0,0}(x, y) = \{1 - \gamma(x, y)\}\, f^{0}_{0,0}(x, y) + x\{1 - \lambda(1 - q)y\}\, f^{0}_{1,0}(x, y)$$

$$+ y\{1 - (1 - \lambda)(1 - q)x\}\, f^{0}_{0,1}(x, y)\,.$$

Hence

$$P(x, y) - f^{1}_{0,0}(x, y) = (1 - q)\{\lambda\, f_{1,0} + (1 - \lambda)\, f_{0,1}\}xy$$

which is identically zero only for $q = 1$. As a result the polynomial $P(x, y)$ in (2.8) can not be written in the form

$$\sum_{i=0}^{n-m} \sum_{j=0}^{n-m-i} f^{m}_{i,j}(x, y)\, a^{n-m}_{i,j}(x, y)$$

where $a^{n-m}_{i,j}(x, y)$ is defined recursively in (2.9). For if $m = 0$ and $m = n$ the above expression reduces to

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i} f_{i,j}^{0}(x, y)\, M_{i,j}^{n}(x, y) = P(x, y) \quad \text{and} \quad f_{0,0}^{n}(x, y)$$

respectively.

2.6 Generalised Neville-Aitken Algorithm

Having shown that none of the Neville-Aitken algorithms of the form (2.12) generate the interpolating polynomial defined in (2.8), it is interesting to explore whether there exists some other Neville-Aitken algorithm which generates the interpolating polynomial defined in (2.8).

Let $f_{i,j}^{0}(x, y) = f_{i,j}$, where $i, j \geq 0$ and $i + j \leq n$. For $m = 1, 2, \ldots, n$, we define $f_{i,j}^{m}(x, y)$, $0 \leq i + j \leq n - m$, recursively by

$$f_{i,j}^{m}(x, y) = c_{i,j}^{m}(x, y)\, f_{i,j}^{m-1}(x, y) + d_{i,j}^{m}(x, y)\, f_{i+1,j}^{m-1}(x, y) + e_{i,j}^{m}(x, y)\, f_{i,j+1}^{m-1}(x, y) \tag{2.13}$$

where

$$c_{i,j}^{m}(x, y) + d_{i,j}^{m}(x, y) + e_{i,j}^{m}(x, y) = 1 . \tag{2.14}$$

We shall call (2.13) a generalised Neville-Aitken algorithm. It includes the class of algorithms given in (2.12) as a special case. An examination of equations (2.13) and (2.14) shows that

(i) the three coefficients cannot be of the form

$$c_{i,j}^{m}(x, y) = \frac{[m + i + j] - \gamma(x, y)}{q^{i+j}\,[m]}, \quad d_{i,j}^{m}(x, y) = \frac{(x - [i])}{q^{i}\,[m]}, \quad e_{i,j}^{m}(x, y) = \frac{(y - [j])}{q^{j}\,[m]}$$

which extends the case q = 1 in an obvious way,

(ii) if the degrees of $c^m_{i,j}(x, y)$, $d^m_{i,j}(x, y)$ and $e^m_{i,j}(x, y)$ are r, s and t respectively, then the degree of $f^m_{i,j}(x, y)$ is not greater than $m \,.\, \max\{r, s, t\}$.

We observe further that the recurrence relation (2.13) cannot give (2.8). For let P(x, y) and $f^1_{0,0}(x, y)$ be the two interpolating polynomials on a q-triangle of order 1 defined by (2.8) and (2.13) respectively. Following the argument used in Example 2.2, we see that the polynomial

$$f^1_{0,0}(x, y) = c^1_{0,0}(x, y)\, f^0_{0,0}(x, y) + d^1_{0,0}(x, y)\, f^0_{1,0}(x, y) + e^1_{0,0}(x, y)\, f^0_{0,1}(x, y)$$

is equal to P(x, y) if and only if

$$c^1_{0,0}(x, y) = 1 - \gamma(x, y),\ \ d^1_{0,0}(x, y) = x \ \text{ and } \ e^1_{0,0}(x, y) = y.$$

This contradicts the fact that $c^1_{0,0}(x, y) + d^1_{0,0}(x, y) + e^1_{0,0}(x, y) = 1$.

The following example shows that, even if we relax the condition (2.14) so that it holds only for points in $T^m_{i,j}$ and not for all x and y, we still cannot find a Neville-Aitken algorithm of the form (2.13) which generates P(x, y) in (2.8).

Example 2.3 Consider the polynomial in (2.8) which interpolates f(x, y) on the set $T^2_{0,0}$,

$$P(x, y) = \frac{1}{[2]}([2] - \gamma(x, y))(1 - \gamma(x, y))\, f_{0,0} + \frac{1}{q}x([2] - \gamma(x, y))\, f_{1,0}$$

$$+ \frac{1}{q}y([2] - \gamma(x, y))\, f_{0,1} + xy\, f_{1,1} + \frac{1}{q[2]}x(x - 1)\, f_{2,0} + \frac{1}{q[2]}y(y - 1)\, f_{0,2}\,.$$

Suppose that the polynomial can be expressed in the form of (2.13) such that the condition (2.14) holds on $T^2_{0,0}$. So for some coefficient functions $c^2_{0,0}(x, y)$, $d^2_{0,0}(x, y)$ and $e^2_{0,0}(x, y)$, we can write

$$P(x, y) = c^2_{0,0}(x, y)\, P^{0,0}(x, y) + d^2_{0,0}(x, y)\, P^{1,0}(x, y) + e^2_{0,0}(x, y)\, P^{0,1}(x, y)$$

where

$$P^{0,0}(x, y) = (1 - \gamma(x, y))\, f_{0,0} + x\, f_{1,0} + y\, f_{0,1}$$

$$P^{1,0}(x, y) = \frac{1}{q}([2] - \gamma(x, y))\, f_{1,0} + \frac{1}{q}(x - 1)\, f_{2,0} + y\, f_{1,1}$$

$$P^{0,1}(x, y) = \frac{1}{q}([2] - \gamma(x, y))\, f_{0,1} + x\, f_{1,1} + \frac{1}{q}(y - 1)\, f_{0,2}$$

are the interpolating polynomials on $T^1_{0,0}$, $T^1_{1,0}$ and $T^1_{0,1}$ respectively. However on comparing the coefficients of $f_{0,0}$, $f_{2,0}$ and $f_{0,2}$, we obtain

$$c^2_{0,0}(x, y) = \frac{1}{[2]}([2] - \gamma(x, y)), \quad d^2_{0,0}(x, y) = \frac{1}{[2]}x \quad \text{and} \quad e^2_{0,0}(x, y) = \frac{1}{[2]}y$$

on $T^2_{0,0}$. This implies that on $T^2_{0,0}$

$$c^2_{0,0}(x, y) + d^2_{0,0}(x, y) + e^2_{0,0}(x, y) = \frac{[2] + (1 - q)xy}{[2]} \neq 1$$

unless $q = 1$.

Now, given a generalised Neville-Aitken algorithm (2.13) which generates the polynomial $f^n_{0,0}(x, y) = \tilde{P}(x, y)$, say, we can always define the corresponding Lagrange coefficients $a^n_{i,j}(x, y)$ for $\tilde{P}(x, y)$ as follows.

Let $a^0_{0,0}(x, y) = 1$ and for $m = n - 1, \ldots, 0$ define $a^{n-m}_{i,j}$, $i, j \geq 0$, $i + j \leq n - m$, recursively by

$$a_{i,j}^{n-m+1}(x, y) = c_{i,j}^{m}(x, y)\, a_{i,j}^{n-m}(x, y) + d_{i-1,j}^{m}(x, y)\, a_{i-1,j}^{n-m}(x, y)$$

$$+ e_{i,j-1}^{m}(x, y)\, a_{i,j-1}^{n-m}(x, y) \qquad (2.15)$$

where $a_{i,j}^{m}(x, y) = 0$ if $i, j < 0$ or $i + j > m$. Then we shall see that $\tilde{P}(x, y)$ can be written in terms of both $f_{i,j}^{m}(x, y)$ and $a_{i,j}^{n-m}(x, y)$ for any m satisfying $0 \le m \le n$.

Theorem 2.2 Let $\tilde{P}(x, y)$ be the interpolating polynomial on a q-triangle of order n generated by the generalised Neville-Aitken algorithm. Then, for $m = 0, 1, \ldots, n$,

$$\tilde{P}(x, y) = \sum_{j=0}^{n-m} \sum_{i=0}^{n-m-j} f_{i,j}^{m}(x, y)\, a_{i,j}^{n-m}(x, y),$$

where $a_{i,j}^{n-m}(x, y)$ is defined recursively by (2.15).

Proof The formula is true for $m = n$ since $a_{0,0}^{0}(x, y) = 1$ and $f_{0,0}^{n}(x, y)$, which we have denoted by $\tilde{P}(x, y)$, is the polynomial generated by (2.13) and interpolates f on $T_{0,0}^{n}$. Suppose the formula is true for some $m > 0$. Then using (2.13) we show that it is also true for $m - 1$. First we see that

$$\tilde{P}(x, y) = \sum_{j=0}^{n-m} \sum_{i=0}^{n-m-j} f_{i,j}^{m}(x, y)\, a_{i,j}^{n-m}(x, y)$$

$$= \sum_{j=0}^{n-m} \sum_{i=0}^{n-m-j} \{ c_{i,j}^{m}(x, y)\, f_{i,j}^{m-1}(x, y) + d_{i,j}^{m}(x, y)\, f_{i+1,j}^{m-1}(x, y)$$

$$+ e_{i,j}^{m}(x, y)\, f_{i,j+1}^{m-1}(x, y)\}\; a_{i,j}^{n-m}(x, y)$$

$$= \sum_{j=0}^{n-m} \sum_{i=0}^{n-m-j} c_{i,j}^{m}(x, y)\, a_{i,j}^{n-m}(x, y)\, f_{i,j}^{m-1}(x, y)$$

$$+ \sum_{j=0}^{n-m} \sum_{h=1}^{n-m-j+1} d_{h-1,j}^{m}(x, y)\, a_{h-1,j}^{n-m}(x, y)\, f_{h,j}^{m-1}(x, y)$$

$$+ \sum_{k=1}^{n-m+1} \sum_{i=0}^{n-m-k+1} e_{i,k-1}^{m}(x, y)\, a_{i,k-1}^{n-m}(x, y)\, f_{i,k}^{m-1}(x, y)$$

where we have written $h = i + 1$ and $k = j + 1$ in the last two double summations. Thus

$$\tilde{P}(x, y) = \sum_{j=0}^{n-m+1} \sum_{i=0}^{n-m-j+1} c_{i,j}^{m}(x, y)\, a_{i,j}^{n-m}(x, y)\, f_{i,j}^{m-1}(x, y)$$

$$+ \sum_{j=0}^{n-m+1} \sum_{h=0}^{n-m-j+1} d_{h-1,j}^{m}(x, y)\, a_{h-1,j}^{n-m}(x, y)\, f_{h,j}^{m-1}(x, y)$$

$$+ \sum_{k=0}^{n-m+1} \sum_{i=0}^{n-m-k+1} e_{i,k-1}^{m}(x, y)\, a_{i,k-1}^{n-m}(x, y)\, f_{i,k}^{m-1}(x, y)$$

where the added terms in each double summation are all zero. This follows, since by definition $a_{i,j}^{r}(x, y) = 0$ if $i, j < 0$ or $i + j > r$. Finally we obtain

$$\tilde{P}(x, y) = \sum_{j=0}^{n-m+1} \sum_{i=0}^{n-m+1-j} \{\, c_{i,j}^{m}(x, y)\, a_{i,j}^{n-m}(x, y) + d_{i-1,j}^{m}(x, y)\, a_{i-1,j}^{n-m}(x, y)$$

$$+ e_{i,j-1}^{m}(x, y)\, a_{i,j-1}^{n-m}(x, y) \,\}\, f_{i,j}^{m-1}(x, y)$$

$$= \sum_{j=0}^{n-m+1} \sum_{i=0}^{n-m+1-j} f_{i,j}^{m-1}(x, y)\, a_{i,j}^{n-m+1}(x, y)\,.$$

Therefore by induction the formula is true for all $m = 0, 1, \ldots, n$.

In particular, for m = 0, the interpolating polynomial in Theorem 2.2 reduces to

$$\widetilde{P}(x, y) = \sum_{j=0}^{n} \sum_{i=0}^{n-j} f_{i,j}^{0} \, a_{i,j}^{n}(x, y)$$

and thus $a_{i,j}^{n}(x, y)$, $i, j \geq 0$, $i + j \leq n$, are the Lagrange coefficients for $\widetilde{P}(x, y)$. As a special case, we note that the Lagrange coefficients for the interpolating polynomial generated by the Neville-Aitken algorithm (2.12) can be obtained from the recurrence relation

$$\begin{aligned} q^{i+j}\,[m]\, a_{i,j}^{n-m+1}(x, y) &= \{[m + i + j] - \gamma(x, y)\}\, a_{i,j}^{n-m}(x, y) \\ &\quad + (x - [i-1])\{q^{j} - \lambda(1-q)(y - [j])\}\, a_{i-1,j}^{n-m}(x, y) \\ &\quad + (y - [j-1])\{q^{i} - (1-\lambda)(1-q)(x - [i])\}\, a_{i,j-1}^{n-m}(x, y). \end{aligned}$$

Hence

$$\begin{aligned} f_{0,0}^{n}(x, y)\, a_{0,0}^{0}(x, y) &= \sum_{j=0}^{n-m} \sum_{i=0}^{n-m-j} f_{i,j}^{m}(x, y)\, a_{i,j}^{n-m}(x, y), \quad \text{for } m = n-1, n-2, \ldots, 1, \\ &= \sum_{j=0}^{n} \sum_{i=0}^{n-j} f_{i,j}\, a_{i,j}^{n}(x, y) \end{aligned}$$

where $f_{0,0}^{n}(x, y)$ is the interpolating polynomial on $T_{0,0}^{n}$ generated by the recurrence relation (2.12).

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Chapter 3

Newton formula and Lagrange coefficients for the interpolating polynomial

3.1 Introduction

Let $x_0, x_1, \dots, x_n$ be $n + 1$ distinct points in $[a, b]$. From (1.5) we see that, the Newton form of the polynomial that interpolates $f(x)$ at $x_0, x_1, \dots, x_n$ can be written as

$$P_n(x) = \sum_{i=0}^{n} f[x_0, x_1, \dots, x_i] \prod_{v=0}^{i-1} (x - x_v) ,$$

where $f[x_0, x_1, \dots, x_i]$ is the divided difference defined by

$$f[x_0, x_1, \dots, x_i] = \frac{f[x_1, x_2, \dots, x_i] - f[x_0, x_1, \dots, x_{i-1}]}{x_i - x_0} .$$

While the above is not the only notation for divided differences, it may be the one most commonly used. Another notation uses $[x_0, x_1, \dots, x_i]f$ in place of the previous notation $f[x_0, x_1, \dots, x_i]$ and this is more suitable for extending divided differences to two dimensions. We shall modify the notation of one-dimensional divided differences to suit a scheme for a higher dimension, using suffices to denote divided differences with respect to x and y respectively.

Let f(x, y) be a function defined on some region containing the set of distinct points $S = \{(x_i, y_j): i, j \geq 0, i + j \leq n\}$. Note that, if we choose $x_0 < x_1 < \ldots < x_n$ and $y_0 < y_1 < \ldots < y_n$, these points will lie in a triangular formation. In the first section however, we will derive a two-dimensional Newton's formula for f(x, y) without imposing such a restriction . In section two, we discuss the error of the above interpolating polynomial. We will also study the Newton form of the interpolating polynomial when the nodes are arranged in particular ways. Specifically, we consider the following three systems of nodes in triangular formation. If x_i and y_i are chosen to be equally spaced on the X and Y axes respectively, we will show that the Newton's formula reduces to the forward difference formula. In another case, we will let x_i and y_i be two different q-integers on the X and Y axes respectively and derive the forward difference formula on a q-triangle. As a comparison, we also will include the system of nodes on a triangle considered by Lee and Phillips [13]. We find that the Newton's formula simplifies to give the backward difference formula.

In the last section, we answer the question raised in Chapter 2. We will prove that there is a Lagrange form of an interpolating polynomial of degree lower than 2n, (in fact of degree at most n) on the q-triangle of order n. We also study the properties of the Lagrange coefficients $L^n_{i,j}(x, y) = 0$, n = 2, 3, 4, 5 for general values of q. In this case a significant property of the Lagrange coefficients has been found: each $L^n_{i,j}$ contains i + j linear factors and the remaining factor is a transformation of the function $L^{n-i-j}_{0,0}$.

3.2 Two-dimensional Newton interpolation formula

Let $x_0, x_1, \ldots, x_n$ be any n + 1 distinct points. Then for a fixed value of y, we denote x-divided differences by

$$[x_0]_x \, f(\cdot, y) = f(x_0, y)$$

and

$$[x_0, x_1, \dots, x_n]_x \, f(\cdot, y)$$

$$= \frac{[x_1, x_2, \dots, x_n]_x \, f(\cdot, y) - [x_0, x_1, \dots, x_{n-1}]_x \, f(\cdot, y)}{x_n - x_0} \; .$$

Similarly let $y_0, y_1, \dots, y_m$ be any $m + 1$ distinct points. Then for a fixed x, we denote y-divided differences by

$$[y_0]_y \, f(x, \cdot) = f(x, y_0)$$

and

$$[y_0, y_1, \dots, y_m]_y \;\; f(x, \cdot)$$

$$= \frac{[y_1, y_2, \dots, y_m]_y \, f(x, \cdot) - [y_0, y_1, \dots, y_{m-1}]_y \, f(x, \cdot)}{y_m - y_0} \; .$$

We define, in an obvious way, the mixed divided difference

$$[y_0, y_1, \dots, y_m]_y [x_0, x_1, \dots, x_n]_x \, f = [y_0, \dots, y_m]_y \, ([x_0, \dots, x_n]_x \, f(\cdot, y)) \; .$$

Since the extended form of formula (1.6) gives

$$[x_0, x_1, \dots, x_n]_x \, f(\cdot, y) = \sum_{i=0}^{n} \frac{f(x_i, y)}{\prod_{\substack{k=0 \\ k \neq i}}^{n} (x_i - x_k)} = \phi_n(y),$$

say, then

$$[y_0, y_1, \dots, y_m]_y \left([x_0, x_1, \dots, x_n]_x f(\cdot, y) \right) = \sum_{j=0}^{m} \frac{\phi_n(y_j)}{\prod\limits_{\substack{k=0 \\ k \neq j}}^{m} (y_j - y_k)}$$

and thus

$$[y_0, y_1, \dots, y_m]_y [x_0, x_1, \dots, x_n]_x f = \sum_{j=0}^{m} \sum_{i=0}^{n} \frac{f(x_i, y_j)}{\prod\limits_{\substack{k=0 \\ k \neq j}}^{m} (y_j - y_k) \prod\limits_{\substack{k=0 \\ k \neq i}}^{n} (x_i - x_k)} .$$

From this, the following result is obvious.

Lemma 3.1 Mixed divided differences commute, that is,

$$[y_0, y_1, \dots, y_m]_y [x_0, x_1, \dots, x_n]_x f = [x_0, x_1, \dots, x_n]_x [y_0, y_1, \dots, y_m]_y f .$$

Now let us extend the one-dimensional Newton interpolation formula to two dimensions, using the method of D. D. Stancu [24]. For any fixed y, f(x, y) may be regarded as a function of the single variable x. Apply (1.8), the one-dimensional Newton formula with remainder to $f(\cdot, y)$ at the nodes $x = x_i$, $i = 0, 1, \dots, n$, to give

$$f(x, y) = \sum_{i=0}^{n} [x_0, x_1, \dots, x_i]_x f(\cdot, y) \prod_{\nu=0}^{i-1} (x - x_\nu)$$

$$+ [x, x_0, \dots, x_n]_x f(\cdot, y) \prod_{\nu=0}^{n} (x - x_\nu) . \tag{3.1}$$

Apply the Newton formula once again to the function

$$A_i(y) = [x_0, x_1, \dots, x_i]_x f(\cdot, y)$$

at the nodes $y = y_j$, $j = 0, 1, \dots, n-i$ to give

$$A_i(y) = \sum_{j=0}^{n-i} [y_0, y_1, \dots, y_j]_y [x_0, x_1, \dots, x_i]_x f \prod_{\nu=0}^{j-1} (y - y_\nu) + R_i(y) \qquad (3.2)$$

where

$$R_i(y) = [y, y_0, \dots, y_{n-i}]_y [x_0, x_1, \dots, x_i]_x f \prod_{\nu=0}^{n-i} (y - y_\nu) .$$

Substitute (3.2) into (3.1) to obtain

$$f(x, y) = \sum_{i=0}^{n} \{ \sum_{j=0}^{n-i} [y_0, \dots, y_j]_y [x_0, \dots, x_i]_x f \prod_{\nu=0}^{j-1} (y - y_\nu) + R_i(y) \} \prod_{\nu=0}^{i-1} (x - x_\nu)$$

$$+ [x, x_0, \dots, x_n]_x f(\cdot, y) \prod_{\nu=0}^{n} (x - x_\nu) .$$

We will write this as

$$f(x, y) = P_n(x, y) + R(x, y)$$

where

$$P_n(x, y) = \sum_{i=0}^{n} \{ \sum_{j=0}^{n-i} [y_0, \dots, y_j]_y [x_0, \dots, x_i]_x f \prod_{\nu=0}^{j-1} (y - y_\nu) \} \prod_{\nu=0}^{i-1} (x - x_\nu)$$

and

$$R(x, y) = \sum_{i=0}^{n} R_i(y) \prod_{\nu=0}^{i-1} (x - x_\nu) + [x, x_0, \dots, x_n]_x f(\cdot, y) \prod_{\nu=0}^{n} (x - x_\nu) . \qquad (3.3)$$

We note that each of the terms

$$R_i(y) \prod_{\nu=0}^{i-1} (x - x_\nu)$$

is zero when $x = x_0, x_1, \dots, x_{i-1}$ or $y_0, y_1, \dots, y_{n-i}$, which includes all points in the set $S = \{(x_r, y_s): r, s \geq 0, r + s \leq n\}$. Also the term

$$[x, x_0, \dots, x_n]_x f(\cdot, y) \prod_{v=0}^{n} (x - x_v)$$

is zero at all points of S. Thus R(x, y) is zero at all points in the set S and so we have shown the following.

Lemma 3.2 Let $x_0, x_1, \dots, x_n$ be distinct and let $y_0, y_1, \dots, y_n$ be distinct. Then

$$P_n(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} [x_0, \dots, x_i]_x [y_0, \dots, y_j]_y f \prod_{v=0}^{i-1} (x - x_v) \prod_{v=0}^{j-1} (y - y_v) . \quad (3.4)$$

interpolates f(x, y) at all points in the set $\{(x_i, y_j): i, j \geq 0, i + j \leq n\}$.

3.3 The error in polynomial interpolation

Let $\mathbb{P}_n$ be the set of one variable polynomials of degree at most n. The following lemmas show how divided difference operators reduce the degree of a polynomial.

Lemma 3.3 If $P(x) \in \mathbb{P}_n$, then $[x, x_0, \dots, x_k]P(x) \in \mathbb{P}_{n-k-1}$ where $0 \leq k \leq n - 1$ and $x_0, x_1, \dots, x_k$ are any $k + 1$ distinct real numbers.

Proof The proof is by induction on k. It is easily verified that the statement is true for $k = 0$. Assume that the divided difference $[x, x_0, \dots, x_{k-1}]P(x)$ is a polynomial of degree $n - k$, for any $k \geq 1$. Then

$$[x, x_0, \ldots, x_k]P(x) = \frac{[x, x_0, \ldots, x_{k-1}]P(x) - [x_0, x_1, \ldots, x_k]P(x)}{x - x_k}.$$

Since $[x_0, x_1, \ldots, x_k]P(x)$ is a constant, the numerator is a polynomial of degree $n-k$ and is zero when $x = x_k$. Hence $(x - x_k)$ is a factor of the numerator. This implies that $[x, x_0, \ldots, x_k]P(x)$ is of degree $n-k-1$. Hence the statement is true for all $0 \le k \le n-1$.

Lemma 3.4 If $P(x) \in \mathbb{P}_n$, then $[x, x_0, \ldots, x_n]P(x) = 0$.

Proof Taking $k = n-1$ in Lemma 3.3 we see that $[x, x_0, \ldots, x_{n-1}]P(x) \in \mathbb{P}_0$ and so is independent of x and has the value $[x_n, x_0, \ldots, x_{n-1}]P(x)$, say, on putting $x = x_n$. Thus we have,

$$[x, x_0, \ldots, x_n]P(x) = \frac{[x, x_0, \ldots, x_{n-1}]P(x) - [x_0, x_1, \ldots, x_n]P(x)}{x - x_n} = 0.$$

Now we will verify that if $f(x, y)$ is in $\mathcal{P}_n$, the set of all polynomials of degree at most n in x and y, then the error $R(x, y)$, defined above, is identically zero. Consider the repeated divided difference of (3.3),

$$[y, y_0, \ldots, y_{n-i}]_y \{ [x_0, x_1, \ldots, x_i]_x f(\cdot, y) \}.$$

We may write any $f(x, y) \in \mathcal{P}_n$ in the form

$$f(x, y) = a_n(y) + a_{n-1}(y)\, x + a_{n-2}(y)\, x^2 + \ldots + a_0(y)\, x^n,$$

where $a_j(y)$ is a polynomial in y only, of degree $\le j$. Then, for a fixed value of y,

$$[x_0, x_1, \ldots, x_i]_x\, f(\cdot, y) \;=\; [x_0, x_1, \ldots, x_i]_x \left\{ \sum_{j=0}^{n} a_{n-j}(y)\, x^j \right\}$$

$$= \sum_{j=0}^{n} a_{n-j}(y)\, [x_0, x_1, \ldots, x_i]_x\, x^j .$$

Since $[x_0, x_1, \ldots, x_i]_x\, x^j = 0$ if $j < i$, we have

$$[x_0, x_1, \ldots, x_i]_x\, f(\cdot, y) = \sum_{j=i}^{n} a_{n-j}(y)\, [x_0, x_1, \ldots, x_i]_x\, x^j$$

Thus $[x_0, x_1, \ldots, x_i]_x\, f(\cdot, y)$ is a polynomial in the variable y only, of degree not greater than $n - i$ and so

$$[y, y_0, \ldots, y_{n-i}]_y \left\{ [x_0, x_1, \ldots, x_i]_x\, f(\cdot, y) \right\} = 0.$$

It follows that

$$\sum_{i=0}^{n} [y, y_0, \ldots, y_{n-i}]_y \left\{[x_0, x_1, \ldots, x_i]_x\, f(\cdot, y)\right\} \prod_{\nu=0}^{i-1} (x - x_\nu) \prod_{\nu=0}^{n-i} (y - y_\nu) = 0.$$

Since the remaining error term in (3.3),

$$[x, x_0, \ldots, x_n]_x\, f(\cdot, y) \prod_{\nu=0}^{n} (x - x_\nu)$$

is also zero, we deduce that $f(x, y) \in \mathcal{P}_n$ implies $R(x, y) = 0$.

It will also be convenient to express the error (3.3) in terms of partial derivatives of f(x, y). We know from (1.9) that if $\frac{\partial^{k+1}}{\partial x^{k+1}} f(x, y)$, exists then for any y

$$[x, x_0, \ldots, x_k]_x\, f(\cdot, y) = \frac{1}{(k+1)!} \frac{\partial^{k+1}}{\partial x^{k+1}} f(\xi, y) \tag{3.5}$$

where ξ lies between the minimum and maximum values of $x, x_0, \ldots, x_k$. A similar result holds for $\frac{\partial^{k+1}}{\partial y^{k+1}} f(x, y)$. Suppose that $f(x, y)$ possess all partial derivatives up to order $n + 1$. Apply (3.5) repeatedly to the mixed divided differences in $R(x, y)$, to give

$$[y, y_0, \ldots, y_{n-i}]_y \left\{ [x_0, x_1, \ldots, x_i]_x\, f(\cdot, y) \right\}$$

$$= [y, y_0, \ldots, y_{n-i}]_y \frac{1}{i!} \frac{\partial^i}{\partial x^i} f(\xi_i, y)$$

$$= \frac{\partial^{n+1}}{\partial x^i\, \partial y^{n-i+1}} \frac{1}{(n-i+1)!\; i!} f(\xi_i, \eta_i),$$

say, where ξ_i and η_i lie between the minimum and maximum values of $x_0, \ldots, x_i$ and $y, y_0, \ldots, y_{n-i}$ respectively. Hence the error of (3.3) becomes

$$R(x, y) = \prod_{\nu=0}^{n} (x - x_\nu) \frac{1}{(n+1)!} \frac{\partial^{n+1}}{\partial x^{n+1}} f(\xi, y)$$

$$+ \sum_{i=0}^{n} \frac{1}{i!\,(n-i+1)!} \prod_{\nu=0}^{i-1} (x - x_\nu) \prod_{\nu=0}^{n-i} (y - y_\nu) \frac{\partial^{n+1}}{\partial x^i\, \partial y^{n-i+1}} f(\xi_i, \eta_i)$$

where ξ lies between the minimum and maximum values of $x, x_0, \ldots, x_n$. Note that in the above error term, each η_i depends on y. From this form of the error of interpolation it is easy to see that $R(x, y) = 0$ whenever $f(x, y) \in \mathcal{P}_n$, as we verified above from the divided difference form of the error.

We may deduce from the error property discussed above that the polynomial $P_n(x, y)$ in (3.4) is unique. Suppose there is another polynomial $P^*(x, y) \in \mathcal{P}_n$ which agrees with $f(x, y)$ at $\{([i], [j]): i, j \geq 0, i + j \leq n\}$. Then $P_n(x, y)$ is an interpolating polynomial for $P^*(x, y)$ on this set of points. Since $P^*(x, y) \in \mathcal{P}_n$ the error $R(x, y)$ is identically zero and thus $P^*(x, y) = P_n(x, y)$.

3.4 Newton formula on a triangle with equally spaced nodes

The Newton form of the polynomial $P_n(x, y)$ in Lemma 3.2 can be simplified further if the nodes are arranged in particular ways. First let x_i and y_j be equally spaced so that $x_i = x_0 + ih$, $h > 0$, $0 \leq i \leq n$, and $y_j = y_0 + jk$, $k > 0$, $0 \leq j \leq n$. For a fixed value of y, let the forward difference operator in the x-direction be defined by

$$\Delta_x f(x, y) = f(x + h, y) - f(x, y).$$

Then we have from Lemma 1.3 that

$$[x_0, x_1, \dots, x_i]_x f(\cdot, y) = \frac{1}{h^i\, i!} \Delta_x^i f(x_0, y) .$$

Similarly for a fixed value of x we have

$$[y_0, y_1, \dots, y_j]_y f(x, \cdot) = \frac{1}{k^j\, j!} \Delta_y^j f(x, y_0)$$

where $\Delta_y f(x, y) = f(x, y + k) - f(x, y)$ is the forward difference operator in the y-direction. The mixed divided differences simplify to give

$$[x_0, x_1, \dots, x_i]_x [y_0, y_1, \dots, y_j]_y f = [x_0, x_1, \dots, x_i]_x \frac{1}{k^j\, j!} \Delta_y^j f(x, y_0)$$

$$= \frac{1}{h^i\, k^j\, i!\, j!} \Delta_x^i \Delta_y^j f(x_0, y_0) .$$

Given a function $f(x, y)$ then, from Lemma 3.2, the Newton form of the interpolating polynomial on $S = \{(x_0 + ih, y_0 + jk): i, j \geq 0, i + j \leq n\}$ can be written as

$$P_n(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \prod_{\nu=0}^{i-1} (x - x_\nu) \prod_{\nu=0}^{j-1} (y - y_\nu) \frac{1}{h^i k^j i! j!} \Delta_x^i \Delta_y^j f(x_0, y_0) .$$

On putting $x = x_0 + sh$ and $y = y_0 + tk$, we obtain

$$\prod_{\nu=0}^{i-1} (x - x_\nu) = h^i s(s-1) \ldots (s-i+1)$$

$$\prod_{\nu=0}^{j-1} (y - y_\nu) = k^j t(t-1) \ldots (t-j+1)$$

and therefore

$$P_n(x_0 + sh, y_0 + tk) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{s}{i} \binom{t}{j} \Delta_x^i \Delta_y^j f(x_0, y_0) . \quad (3.6)$$

Thus for the triangular grid defined above, where the spacing of points in the x and y-directions is not necessarily the same, the Newton interpolation formula (3.4) reduces to the forward difference formula (3.6).

3.5 Newton interpolation formula on a q-triangle

In this section we consider set of points $S = \{(x_i, y_j): i, j \geq 0, i + j \leq n\}$, where

$$x_i = [i]_p = \frac{1 - p^i}{1 - p} \quad \text{and} \quad y_j = [j]_q = \frac{1 - q^j}{1 - q} , \quad (3.7)$$

and $p, q > 0$. Now for a fixed y define the forward difference operator in the x-direction by

$$\mathcal{D}_x^0 f([i]_p, y) = f([i]_p, y) ,$$

$$\mathfrak{D}_x^n f([i]_p, y) = \mathfrak{D}_x^{n-1} f([i+1]_p, y) - p^{n-1} \mathfrak{D}_x^{n-1} f([i]_p, y), \qquad n = 1, 2, \ldots .$$

Similarly, for a fixed value of x, we define the forward difference operator in the y-direction by

$$\mathfrak{D}_y^0 f(x, [j]_q) = f(x, [j]_q)$$

$$\mathfrak{D}_y^m f(x, [j]_q) = \mathfrak{D}_y^{m-1} f(x, [j+1]_q) - q^{m-1} \mathfrak{D}_y^{m-1} f(x, [j]_q) \qquad m = 1, 2, \ldots .$$

On taking divided differences in the x-direction we obtain, for a given value of y,

$$\big[[0]\big]_x f(\cdot, y) = f([0]_p, y),$$

and

$$\big[[0], [1], \ldots, [i]\big]_x f(\cdot, y)$$

$$= \frac{\big[[1], [2], \ldots, [i]\big]_x f(\cdot, y) - \big[[0], [1], \ldots, [i-1]\big]_x f(\cdot, y)}{[i]_p - [0]_p},$$

for $i = 1, 2, \ldots, n$.

Above, we have written the notation $\big[[0], [1], \ldots, [i]\big]_x$ where strictly we should have written $\big[[0]_p, [1]_p, \ldots, [i]_p\big]_x$. The omission of the subscript p should cause no confusion. We obtain similar results on taking divided differences in the y-direction. We now show how divided differences are related to forward differences.

Lemma 3.5 For a fixed value of y

$$\big[[i], [i+1], \ldots, [i+n]\big]_x f(\cdot, y) = \frac{1}{[n]_p!\, p^{in+(n-1)n/2}} \mathfrak{D}_x^n f([i]_p, y).$$

Proof The proof is by induction on n and the case n = 0 is trivial. Suppose it is true for $n = k \geq 0$. Then for $n = k + 1$ we have

$$\big[[i], [i+1], \dots, [i+k+1]\big]_x f(\cdot, y)$$

$$= \frac{\big[[i+1], [i+2], \dots, [i+k+1]\big]_x f(\cdot, y) - \big[[i], [i+1], \dots, [i+k]\big]_x f(\cdot, y)}{[i+k+1]_p - [i]_p}$$

$$= \frac{1}{p^i\,[k+1]_p} \left\{ \frac{\mathfrak{D}_x^k f([i+1]_p, y)}{[k]_p!\; p^{(i+1)k+(k-1)k/2}} - \frac{\mathfrak{D}_x^k f([i]_p, y)}{[k]_p!\; p^{ik+(k-1)k/2}} \right\}.$$

Thus

$$\big[[i], [i+1], \dots, [i+k+1]\big]_x f(\cdot, y)$$

$$= \frac{1}{[k+1]_p!\; p^{i+(i+1)k+(k-1)k/2}} \left\{ \mathfrak{D}_x^k f([i+1]_p, y) - p^k\, \mathfrak{D}_x^k f([i]_p, y) \right\}$$

$$= \frac{1}{[k+1]_p!\; p^{i(k+1)+k(k+1)/2}}\, \mathfrak{D}_x^{k+1} f([i]_p, y).$$

Following the same argument as above we obtain for a fixed x that

$$\big[[j], [j+1], \dots, [j+m]\big]_y f(x, \cdot) = \frac{1}{[m]_q!\; q^{jm+(m-1)m/2}}\, \mathfrak{D}_y^m f(x, [j]_q).$$

In particular we have

$$\big[[0], [1], \dots, [n]\big]_x f(\cdot, y) = \frac{\mathfrak{D}_x^n f([0]_p, y)}{[n]_p!\; p^{(n-1)n/2}}$$

and

$$\big[[0], [1], \dots, [m]\big]_y f(x, \cdot) = \frac{\mathfrak{D}_y^m f(x, [0]_q)}{[m]_q!\; q^{(m-1)m/2}}.$$

Thus, for mixed divided differences, we obtain

$$[[0],\dots,[i]]_x\,[[0],\dots,[j]]_y\, f = \frac{1}{[i]_p!\; p^{(i-1)i/2}\; [j]_q!\; q^{(j-1)j/2}}\, \mathfrak{D}_x^i\, \mathfrak{D}_y^j\, f([0]_p, [0]_q)\,. \tag{3.8}$$

We can now express the polynomial $P_n(x, y)$ in terms of forward differences. We have from (3.4) that

$$P_n(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \prod_{\nu=0}^{i-1} (x - [\nu]_p) \prod_{\nu=0}^{j-1} (y - [\nu]_q) \;\; [[0], [1], \dots, [i]]_x\, [[0], [1], \dots, [j]]_y\, f\,. \tag{3.9}$$

Let $\bar{x}, \bar{y} \in \mathbb{R}$ satisfy $x = [\bar{x}]_p$ and $y = [\bar{y}]_q$. Then we may write

$$P_n(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \prod_{\nu=0}^{i-1} \frac{[\bar{x} - \nu]_p}{[\nu + 1]_p} \prod_{\nu=0}^{j-1} \frac{[\bar{y} - \nu]_q}{[\nu + 1]_q} \;\; \mathfrak{D}_x^i\, \mathfrak{D}_y^j\, f([0]_p, [0]_q)\,.$$

Thus

$$P_n(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \begin{bmatrix} \bar{x} \\ i \end{bmatrix}_p \begin{bmatrix} \bar{y} \\ j \end{bmatrix}_q \mathfrak{D}_x^i \mathfrak{D}_y^j\, f([0]_p, [0]_q)\,. \tag{3.10}$$

When $p = q$ we note that (3.10) reduces to Theorem 2.1 .

3.6 Newton formula on another set of nodes

In the above discussion we gave two-dimensional Newton's formula for interpolating f on the set $\{(x_i, y_j): i, j \geq 0, i + j \leq n\}$. Let us write this formula for interpolation on the set $S' = \{(x_r, y_s): 0 \leq r \leq s \leq n\}$. We therefore consider the set $S = \{(x_i, y_{n-j}): i, j \geq 0, i + j \leq n\}$ where we kept the order of $\{x_0, \dots, x_n\}$ and reversed the order of $\{y_0, y_1, \dots, y_n\}$ to give $\{y_n, y_{n-1}, \dots, y_0\}$. Then the restriction $i + j \leq n$ becomes $i + (n - j) \leq n$, that is $i \leq j$ and S is changed to S'. Thus (3.4) becomes

$$P_n(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} [x_0, x_1, \dots, x_i]_x [y_n, y_{n-1}, \dots, y_{n-j}]_y f$$

$$\times \prod_{\nu=0}^{i-1} (x - x_\nu) \prod_{\nu=0}^{j-1} (y - y_{n-\nu}) . \qquad (3.11)$$

We now consider (3.11) for the case where $x_i = [i]_p$, $y_j = [j]_q$, as defined in (3.7) above. We need to simplify the divided differences in the y-direction, as follows. We have

$$\big[[n]\big]_y f(x, \cdot) = f(x, [n]_q)$$

and

$$\big[[n], [n-1], \dots, [n-k]\big]_y f(x, \cdot)$$

$$= \frac{\big[[n], [n-1], \dots, [n-k-1]\big]_y f(x, \cdot) - \big[[n-1], [n-2], \dots, [n-k]\big]_y f(x, \cdot)}{[n]_q - [n-k]_q} .$$

It is helpful to define

$$\mathfrak{B}_y^0 f(x, [j]_q) = f(x, [j]_q) ,$$

$$\mathfrak{B}_y^m f(x, [j]_q) = q\, \mathfrak{B}_y^{m-1} f(x, [j]_q) - q^m\, \mathfrak{B}_y^{m-1} f(x, [j-1]_q) , \qquad m = 1, 2, \dots, n.$$

We will called these backward difference operators and we now examine their relation to divided differences.

Lemma 3.6 For $k = 0, 1, 2, \dots, n$

$$\big[[n], [n-1], \dots, [n-k]\big]_y f(x, \cdot) = \frac{1}{[k]_q!\, q^{nk-k(k-1)/2}} \mathfrak{B}_y^k f(x, [n]_q). \qquad (3.12)$$

Proof Clearly (3.12) holds for $k = 0$. Suppose that (3.12) is true for any $k \geq 0$. Then

$$[[n], [n-1], \dots, [n-k-1]]_y f(x, \cdot)$$

$$= \frac{[[n], [n-1], \dots, [n-k]]_y f(x, \cdot) - [[n-1], [n-2], \dots, [n-k-1]]_y f(x, \cdot)}{[n]_q - [n-k-1]_q}$$

$$= \frac{1}{[k+1]_q q^{n-k-1}} \left\{ \frac{\mathfrak{B}_y^k f(x, [n]_q)}{[k]_q!\, q^{kn-k(k-1)/2}} - \frac{\mathfrak{B}_y^k f(x, [n-1]_q)}{[k]_q!\, q^{k(n-1)-k(k-1)/2}} \right\}$$

$$= \frac{1}{[k+1]_q!\, q^{n-k+kn-k(k-1)/2}} \{ q\, \mathfrak{B}_y^k f(x, [n]_q) - q^{k+1}\, \mathfrak{B}_y^k f(x, [n-1]_q) \}$$

$$= \frac{1}{[k+1]_q!\, q^{(k+1)n-k(k+1)/2}} \mathfrak{B}_y^{k+1} f(x, [n]_q) .$$

Thus by induction (3.12) holds for all $k = 0, 1, \dots, n$. In particular, if we take $k = n$,

$$[[n], [n-1], \dots, [0]]_y f(x, \cdot) = \frac{1}{[n]_q!\, q^{n(n+1)/2}} \mathfrak{B}_y^n f(x, [n]_q) .$$

On defining $\bar{x}, \bar{y} \in \mathbb{R}$ so that $x = [\bar{x}]_p$ and $y = [\bar{y} + n]_q$, the Newton interpolation formula (3.11) for $f(x, y)$ on $S = \{([i]_p, [j]_q) : 0 \leq i \leq j \leq n\}$ can be written as

$$P_n(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} [[0], [1], \dots, [i]]_x [[n], [n-1], \dots, [n-j]]_y f$$

$$\times \prod_{\nu=0}^{i-1} p^{\nu} [\bar{x} - \nu]_p \prod_{\nu=0}^{j-1} q^{n-\nu} [\bar{y} + \nu]_q . \qquad (3.13)$$

Using Lemmas 3.5 and 3.6, we have

$$[[0],[1],\dots,[i]]_x\,[[n],[n-1],\dots,[n-j]]_y\, f = \frac{\mathfrak{D}_x^i\,\mathfrak{B}_y^j\, f([0]_p,[n]_q)}{[i]_p!\,p^{(i-1)i/2}\,[j]_q!\,q^{jn-(j-1)j/2}}$$

and thus (3.13) simplifies to give

$$P_n(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\begin{bmatrix}\bar{x}\\ i\end{bmatrix}_p\begin{bmatrix}\bar{y}+j-1\\ j\end{bmatrix}_q \mathfrak{D}_x^i\,\mathfrak{B}_y^j\, f([0]_p,[n]_q)\,.$$

We see that finally the polynomial is expressed in terms of mixed differences at the point $([0]_p, [n]_q)$.

Alternatively, let us redefine the backward difference operators in a similar way to the forward difference operators so that, for a fixed x,

$$\mathfrak{B}_y^0 f(x,[j]_q) = f(x,[j]_q)\,,$$

$$\mathfrak{B}_y^m f(x,[j]_q) = \mathfrak{B}_y^{m-1} f(x,[j]_q) - q^{m-1}\,\mathfrak{B}_y^{m-1} f(x,[j-1]_q)\,,\quad m = 1, 2, \dots, n.$$

Then, by following the same argument as above, we obtain

$$[[n],[n-1],\dots,[n-k]]_y\, f(x,\cdot) = \frac{1}{[k]_q!\,q^{nk-k(k+1)/2}}\,\mathfrak{B}_y^k f(x,[n]_q),$$

$k = 0, 1, 2, \dots, n$. Thus we obtain the interpolating polynomial in the form

$$P_n(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i}(-1)^j q^{(-2\bar{y}+j+1)j/2}\begin{bmatrix}\bar{x}\\ i\end{bmatrix}_p\begin{bmatrix}\bar{y}\\ j\end{bmatrix}_q \mathfrak{D}_x^i\,\mathfrak{B}_y^j\, f([0]_p,[n]_q)$$

where $\bar{x},\bar{y} \in \mathbb{R}$, such that $x = [\bar{x}]_p$ and $y = [n-\bar{y}]_q$. The simplification of the definition of the backward differences is offset by an increase in the complication of the latter formula.

3.7 Lagrange coefficients for the interpolating polynomial

Let f(x, y) be defined on the q-triangle bounded by the lines x = 0, y = 0 and the hyperbola $\gamma(x, y) = x + y - (1-q)xy = [n]$. We see from Theorem 2.1 that there is an interpolating polynomial $P_n(x, y)$ of degree at most n which interpolates f(x, y) on the set $S = \{([i], [j]): i, j \geq 0, i + j \leq n\}$ and

$$P_n(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \begin{bmatrix} \bar{x} \\ i \end{bmatrix} \begin{bmatrix} \bar{y} \\ j \end{bmatrix} \mathfrak{D}_x^i \mathfrak{D}_y^j f_{0,0} \tag{3.14}$$

where $\bar{x}, \bar{y} \in \mathbb{R}$ such that $x = [\bar{x}]$, $y = [\bar{y}]$. Let us rewrite the polynomial $P_n(x, y)$ in the Lagrangian form

$$P_n(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} L_{i,j}^n(x, y)\, f_{i,j}$$

where

$$L_{i,j}^n([k], [m]) = \begin{cases} 1 & \text{if } ([k], [m]) = ([i], [j]) \\ 0 & \text{if } ([k], [m]) \neq ([i], [j]) \end{cases} \qquad 0 \leq i + j \leq n.$$

From (1.13) and (1.14) we know that $L_{i,j}^n(x, y)$ is a product of n linear forms when q = 1. Thus it is interesting to explore the properties of $L_{i,j}^n(x, y)$ for general values of q. For the simplest case n = 1, we can write

$$P_1(x, y) = (1 - x - y)\, f_{0,0} + x\, f_{1,0} + y\, f_{0,1} = \sum_{i=0}^{1} \sum_{j=0}^{1-i} L_{i,j}^1(x, y)\, f_{i,j}\,.$$

In this case each Lagrange coefficient $L_{i,j}^1(x, y)$, $0 \leq i + j \leq 1$, is linear and is independent of q.

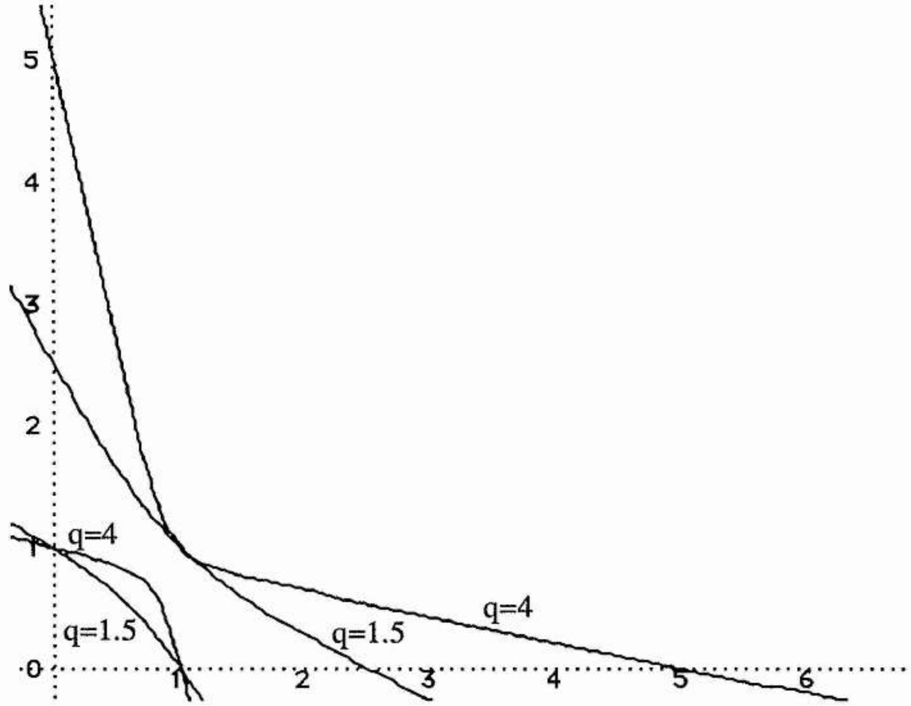


Figure 3.1 Graphs of $L^{2}_{0,0}(x, y) = 0$ where $(q > 1)$ $q = 1.5$ and $q = 4$

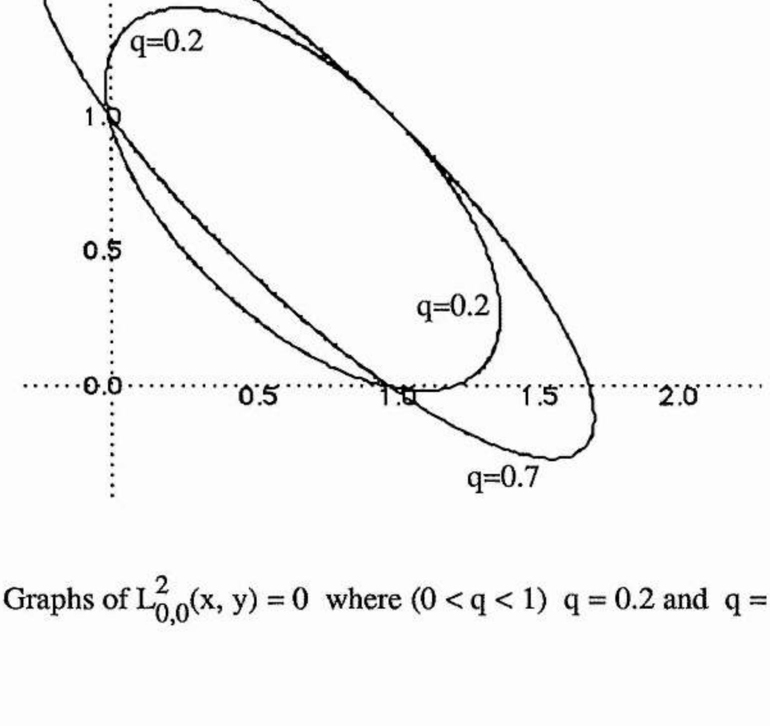


Figure 3.2 Graphs of $L^{2}_{0,0}(x, y) = 0$ where $(0 < q < 1)$ $q = 0.2$ and $q = 0.7$

The interpolating of degree two can be written as

$$P_2(x, y) = P_1(x, y) + \frac{[\bar{x}][\bar{x}-1]}{[2]!}\mathfrak{D}_x^2 f_{0,0} + [\bar{x}][\bar{y}]\mathfrak{D}_x^1\mathfrak{D}_y^1 f_{0,0} + \frac{[\bar{y}][\bar{y}-1]}{[2]!}\mathfrak{D}_y^2 f_{0,0}$$

$$= \{1 - x - y + \frac{x(x-1)}{[2]!} + xy + \frac{y(y-1)}{[2]!}\} f_{0,0} + x(1 - y - \frac{x-1}{q}) f_{1,0}$$

$$+ y(1 - x - \frac{y-1}{q}) f_{0,1} + \frac{x(x-1)}{q[2]!} f_{2,0} + xy f_{1,1} + \frac{y(y-1)}{q[2]!} f_{0,2}$$

$$= \sum_{i=0}^{2} \sum_{j=0}^{2-i} L_{i,j}^2(x, y) f_{i,j} .$$

All of the Lagrange coefficients are products of two linear forms except $L_{0,0}^2(x, y)$. In Figures 3.1 and 3.2 we plot the graphs of $L_{0,0}^2(x, y) = 0$ for various values of q.

Lemma 3.7 Let

$$L_{0,0}^2(x, y) = 1 - x - y + \frac{x(x-1)}{[2]!} + xy + \frac{y(y-1)}{[2]!}$$

be the Lagrange coefficient of $f_{0,0}$ of the interpolating polynomial on the set $S = \{([i], [j]) : i, j \geq 0, i + j \leq 2\}$ and let us consider the conic $L_{0,0}^2(x, y) = 0$. We find that

(i) if $q = 1$, this conic is the pair of straight lines $x + y = 2$ and $x + y = 1$,

(ii) if $0 < q < 1$, it is an ellipse with axes $x = y$ and $x + y = \frac{2(2+q)}{3+q}$,

(iii) if $q > 1$, it is a hyperbola with axes $x = y$ and $x + y = \frac{2(2+q)}{3+q}$.

Proof Setting the coefficient $L_{0,0}^2(x, y)$ to zero, we have

$$x^2 + y^2 + (1+q)xy - (2+q)x - (2+q)y + (1+q) = 0 .$$

On reducing this conic to standard form, we obtain

$$\left(x + y - \frac{4 + 2q}{3 + q}\right)^2 + \frac{1 - q}{3 + q}(x - y)^2 = \left(\frac{2}{3 + q}\right)^2 .$$

On examining the last equation, we see that the conic $L^2_{0,0}(x, y) = 0$ is indeed a pair of straight lines, an ellipse or a hyperbola for $q = 1$, $0 < q < 1$ or $q > 1$ respectively, as given in the statement of this lemma.

For $P_3(x, y)$, we have

$$P_3(x, y) = P_2(x, y) + \frac{[\bar{x}][\bar{x} - 1][\bar{x} - 2]}{[3]!} \mathfrak{D}^3_x f_{0,0} + \frac{[\bar{x}][\bar{x} - 1][\bar{y}]}{[2]!} \mathfrak{D}^2_x \mathfrak{D}^1_y f_{0,0}$$

$$+ \frac{[\bar{x}][\bar{y}][\bar{y} - 1]}{[2]!} \mathfrak{D}^1_x \mathfrak{D}^2_y f_{0,0} + \frac{[\bar{y}][\bar{y} - 1][\bar{y} - 2]}{[3]!} \mathfrak{D}^3_y f_{0,0}$$

$$= \sum_{i=0}^{3} \sum_{j=0}^{3-i} L^3_{i,j}(x, y)\, f_{i,j} .$$

Since $L^3_{i,j}(x, y) = L^3_{j,i}(y, x)$ we will not write down all ten Lagrange coefficients $L^3_{i,j}(x, y)$, but only the six for which $i \geq j$. We have

$$L^3_{0,0}(x, y) = 1 - x - y + \frac{x(x - 1)}{[2]!} + xy + \frac{y(y - 1)}{[2]!} - \frac{x(x - 1)(x - [2])}{[3]!}$$

$$- \frac{x(x - 1)y}{[2]!} - \frac{xy(y - 1)}{[2]!} - \frac{y(y - 1)(y - [2])}{[3]!} ,$$

$$L^3_{1,0}(x, y) = x \left\{1 - y - \frac{x - 1}{q} + \frac{(x - 1)(x - [2])}{q^2 [2]!} + \frac{(x - 1)y}{q} + \frac{y(y - 1)}{[2]!} \right\},$$

$$L^3_{2,0}(x, y) = \frac{x(x - 1)}{q[2]!} \left\{1 - \frac{x - [2]}{q^2} - y \right\},$$

$$L^3_{1,1}(x, y) = xy\,\{1 - \frac{x-1}{q} - \frac{y-1}{q}\},$$

$$L^3_{3,0}(x, y) = \frac{x(x-1)(x-[2])}{q^3\,[3]!} \qquad \text{and} \qquad L^3_{2,1}(x, y) = \frac{x(x-1)y}{q[2]!}\ .$$

We observe that the functions $L^3_{i,j}(x, y)$, $2 \le i + j \le 3$, are all products of three linear forms, whereas for $q \ne 1$, $L^3_{1,0}(x, y)$ and $L^3_{0,1}(x, y)$ are both products of one linear form and one conic (ellipse or hyperbola). Making the substitution $x = 1 + qX$ and $y = Y$ in $L^3_{1,0}(x, y)$, we obtain

$$L^3_{1,0}(x, y) = (1 + qX)\,\{1 - X - Y + \frac{X(X-1)}{[2]!} + XY + \frac{Y(Y-1)}{[2]!}\}\ .$$

We deduce from Lemma 3.7 that the quadratic factor in coefficient $L^3_{1,0}(x, y)$ is elliptical if $0 < q < 1$ and is hyperbolic if $q > 1$.

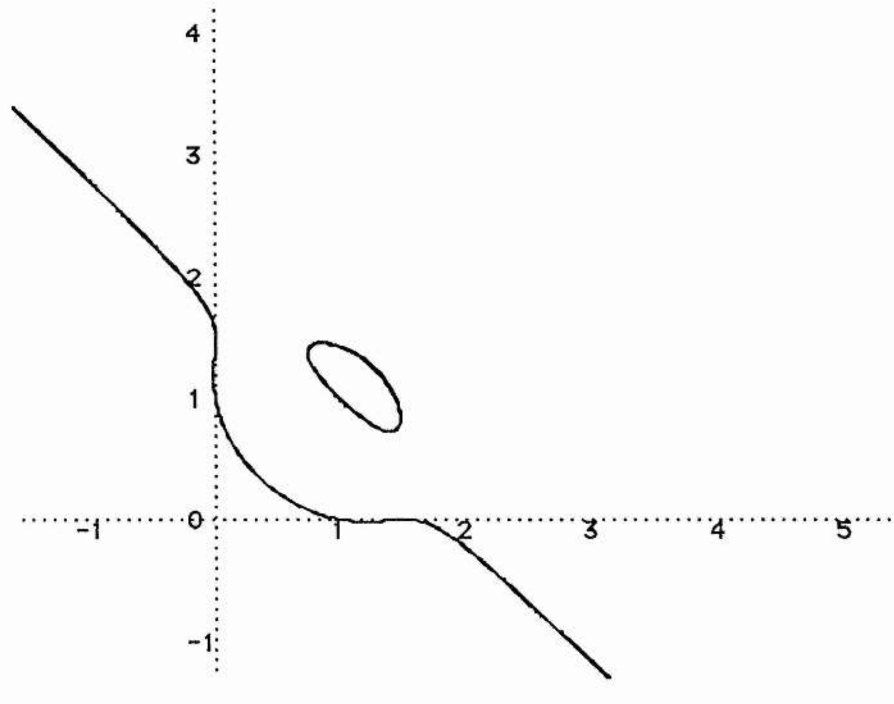
The remaining coefficient $L^3_{0,0}(x, y)$ has no obvious factor. However, on examining the graph of $L^3_{0,0}(x, y) = 0$ for various values of q (see Figures 3.3 to 3.6), $L^3_{0,0}(x, y)$ appears to have a linear factor for the value of q for which the points ([3], 0), (0, [3]) and ([1], [1]) lie in a straight line. This occurs when ([1], [1]) lies on the line $x + y = [3]$ and hence $[3] = 2$, giving $q = (\sqrt{5} - 1)/2$. We can obviously construct a conic to pass through the remaining six interpolating points and so $L^3_{0,0}(x, y)$ does factorize for $q = (\sqrt{5} - 1)/2$. To complete the details we state a lemma which is readily verified.

Lemma 3.8 If a and b are distinct non-zero real numbers, the conic

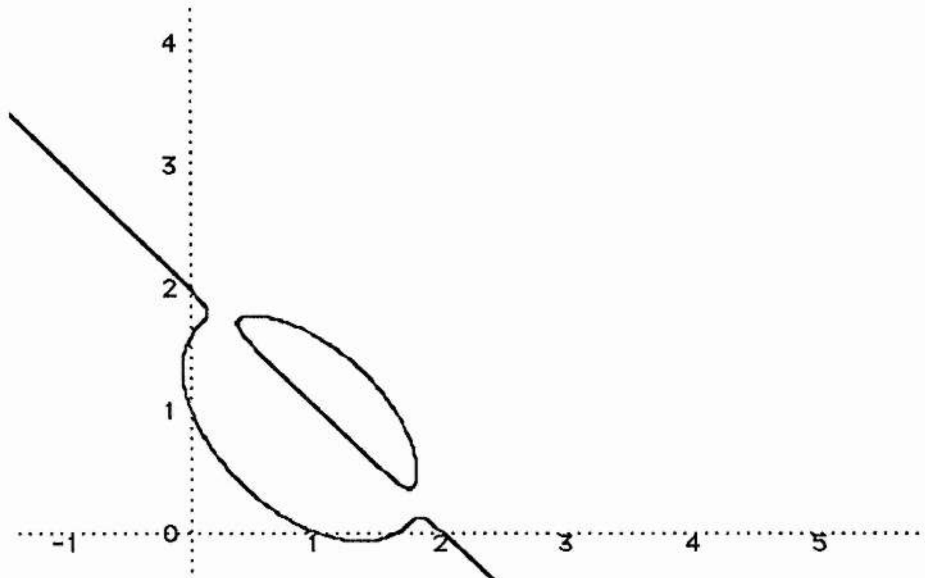
$$x^2 + y^2 + xy - (a + b)(x + y) + ab = 0$$

passes through the six points (a, 0), (b, 0), (a, b) and (0,a), (0, b), (b, a). This conic is an ellipse, which may be written as

$$(x + y - \tfrac{2}{3}(a + b))^2 + \tfrac{1}{3}(x - y)^2 = \tfrac{4}{9}(a^2 - ab + b^2)\ .$$

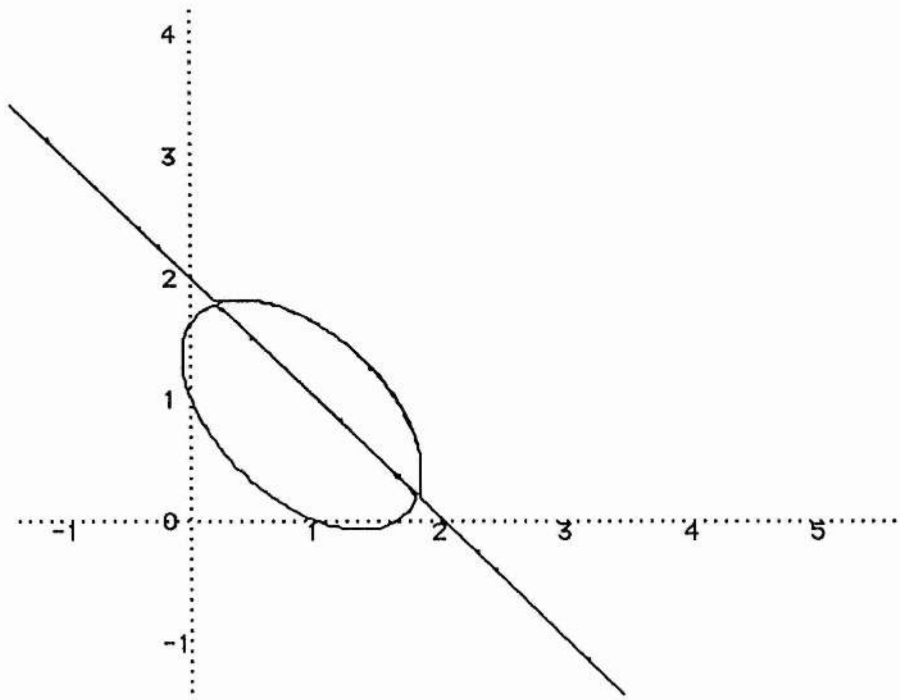


Graph of $L^3_{0,0}(x, y) = 0$ where $q = 0.40$



Graph of $L^3_{0,0}(x, y) = 0$ where $q = 0.61$

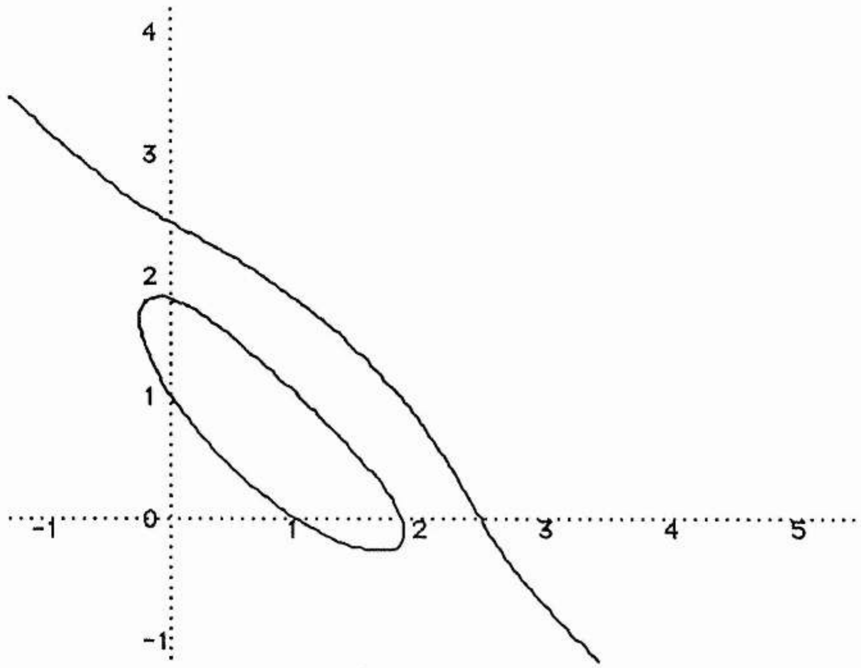
Figure 3.3



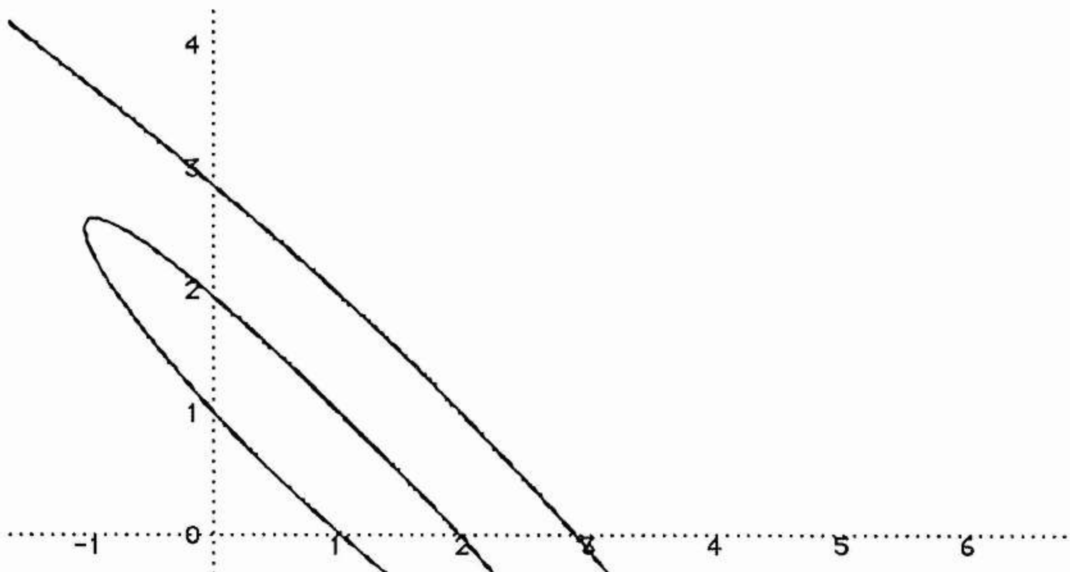
Graph of $L_{0,0}^{3}(x, y) = 0$ where $q = 0.62$

Graph of $L_{0,0}^{3}(x, y) = 0$ where $q = 0.63$

Figure 3.4

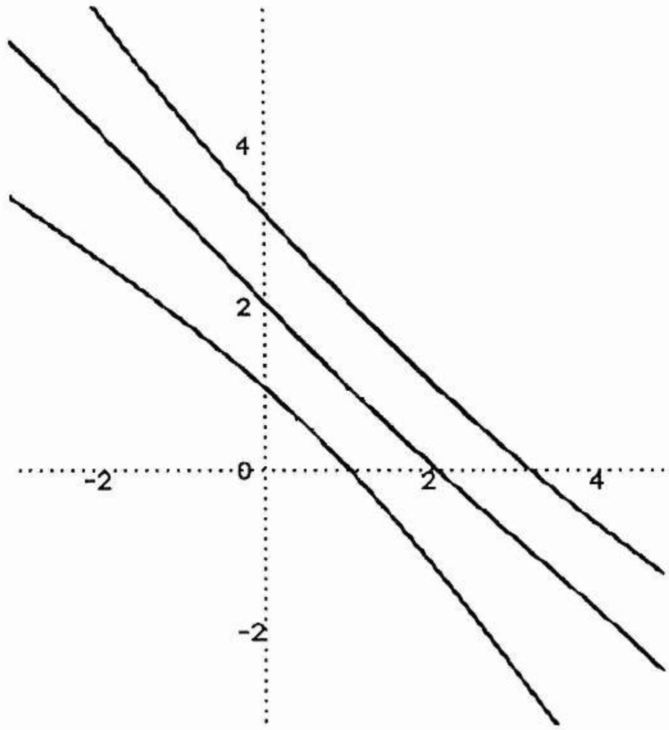


Graph of $L_{0,0}^{3}(x, y) = 0$ where $q = 0.80$

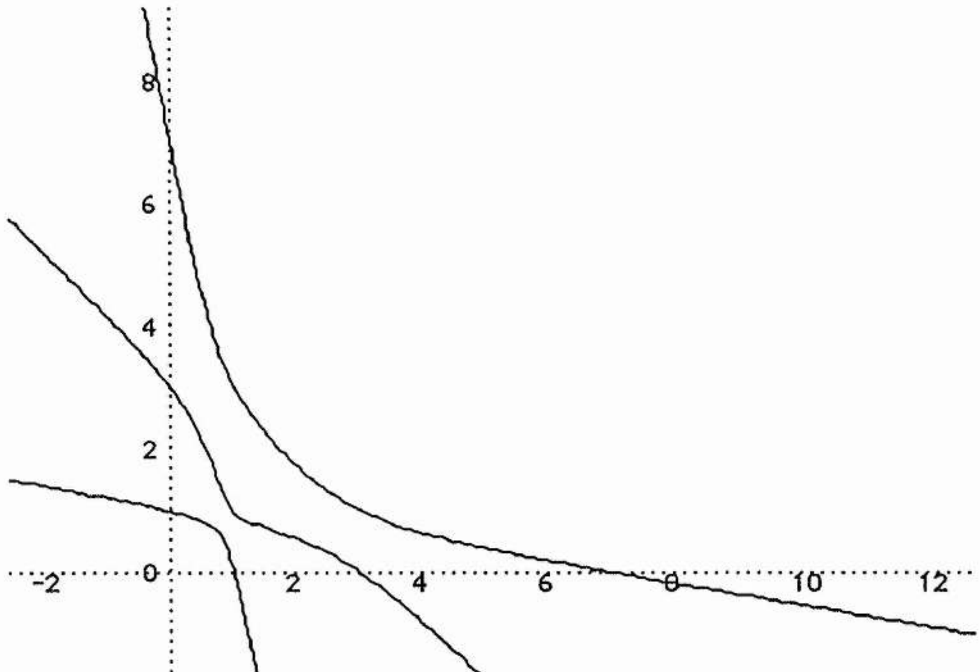


Graph of $L_{0,0}^{3}(x, y) = 0$ where $q = 0.95$

Figure 3.5



Graph of $L^3_{0,0}(x, y) = 0$ where $q = 1.05$



Graph of $L^3_{0,0}(x, y) = 0$ where $q = 2.00$

Figure 3.6

Let us apply Lemma 3.8 to the case where $a = [1]$ and $b = [2]$. It follows that

$$L^3_{0,0}(x, y) = K(x + y - 2)(x^2 + y^2 + xy - (2 + q)(x + y) + (1 + q))$$

for $q = (\sqrt{5} - 1)/2$, and the condition $L^3_{0,0}(0, 0) = 1$ gives the value $K = (1 - \sqrt{5})/4$. Note that the ellipse which is part of the graph of $L^3_{0,0}(x, y) = 0$ has its centre at $(\frac{3 + \sqrt{5}}{6}, \frac{3 + \sqrt{5}}{6})$ and has semi-major axis $\frac{2}{\sqrt{3}}$ and semi-minor axis $\frac{2}{3}$.

Having shown that there is one value of q for which $L^3_{0,0}(x, y)$ factorizes, we have looked at $L^4_{0,0}(x, y; q) = 0$ and $L^5_{0,0}(x, y; q) = 0$, but have found no value of q for which either of these Lagrange coefficients factorizes. (See Appendix 3.)

In (2.8) we obtained a Lagrange form of an interpolating polynomial on the q-triangle where the Lagrange coefficients have degrees between n and 2n. We now obtain the interpolating polynomial on the q-triangle of smallest possible degree, where each Lagrangian coefficient is of degree at most n.

Theorem 3.1 Let us rewrite the interpolating polynomial (3.14) in the Lagrange form

$$P_n(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} L^n_{i,j}(x, y) f_{i,j} .$$

Then for $i, j \geq 0$, $0 \leq i + j \leq n$, the Lagrange coefficients may be expressed in the form

$$L^n_{i,j}(x, y) = \sum_{r=i}^{n-j} \sum_{s=j}^{n-r} (-1)^{r+s-i-j} \begin{bmatrix} r \\ i \end{bmatrix} \begin{bmatrix} s \\ j \end{bmatrix} \begin{bmatrix} \bar{x} \\ r \end{bmatrix} \begin{bmatrix} \bar{y} \\ s \end{bmatrix} q^{(r-i)(r-i-1)/2 + (s-j)(s-j-1)/2}, \tag{3.15}$$

where $x = [\bar{x}]$, $y = [\bar{y}]$ for some $\bar{x}$, $\bar{y}$ in $\mathbb{R}$.

Proof First we expand the forward differences in (3.14) to give (see Lemma 2.3)

$$P_n(x, y) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \begin{bmatrix} \bar{x} \\ r \end{bmatrix} \begin{bmatrix} \bar{y} \\ s \end{bmatrix} \sum_{\alpha=0}^{r} \sum_{\beta=0}^{s} (-1)^{\alpha+\beta} \begin{bmatrix} r \\ \alpha \end{bmatrix} \begin{bmatrix} s \\ \beta \end{bmatrix} q^{\alpha(\alpha-1)/2 + \beta(\beta-1)/2} f_{r-\alpha, s-\beta} .$$

On putting $\mu = r - \alpha$, $\nu = s - \beta$ we obtain

$$P_n(x, y) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \begin{bmatrix} \bar{x} \\ r \end{bmatrix} \begin{bmatrix} \bar{y} \\ s \end{bmatrix} \sum_{\mu=0}^{r} \sum_{\nu=0}^{s} (-1)^{r-\mu+s-\nu} \begin{bmatrix} r \\ r-\mu \end{bmatrix} \begin{bmatrix} s \\ s-\nu \end{bmatrix} q^{d} f_{\mu,\nu} ,$$

where $d = (r - \mu)(r - \mu - 1)/2 + (s - \nu)(s - \nu - 1)/2$. Thus

$$P_n(x, y) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \sum_{\mu=0}^{r} \sum_{\nu=0}^{s} (-1)^{r-\mu+s-\nu} \begin{bmatrix} r \\ \mu \end{bmatrix} \begin{bmatrix} s \\ \nu \end{bmatrix} \begin{bmatrix} \bar{x} \\ r \end{bmatrix} \begin{bmatrix} \bar{y} \\ s \end{bmatrix} q^{d} f_{\mu,\nu} .$$

On picking out the coefficients of $f_{i,j}$, for fixed values of i and j, we obtain

$$L^n_{i,j}(x, y) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} (-1)^{r+s-i-j} \begin{bmatrix} r \\ i \end{bmatrix} \begin{bmatrix} s \\ j \end{bmatrix} \begin{bmatrix} \bar{x} \\ r \end{bmatrix} \begin{bmatrix} \bar{y} \\ s \end{bmatrix} q^{(r-i)(r-i-1)/2 + (s-j)(s-j-1)/2} .$$

Since $\begin{bmatrix} r \\ i \end{bmatrix} = 0$ if $0 \le r < i$ and similarly $\begin{bmatrix} s \\ j \end{bmatrix} = 0$ if $0 \le s < j$, we obtain the required result (3.15).

We now verify directly that

$$L^n_{i,j}([k], [m]) = \begin{cases} 1 & \text{if } ([k], [m]) = ([i], [j]) \\ 0 & \text{if } ([k], [m]) \neq ([i], [j]) \end{cases} \qquad i, j \ge 0,\ 0 \le i + j \le n .$$

We have

$$L^n_{i,j}([i], [j]) = \sum_{r=i}^{n-j} \sum_{s=j}^{n-r} (-1)^{r-i} q^{(r-i)(r-i-1)/2} \begin{bmatrix} r \\ i \end{bmatrix} \begin{bmatrix} i \\ r \end{bmatrix} (-1)^{s-j} q^{(s-j)(s-j-1)/2} \begin{bmatrix} s \\ j \end{bmatrix} \begin{bmatrix} j \\ s \end{bmatrix} .$$

Since

$$\begin{bmatrix} r \\ i \end{bmatrix}\begin{bmatrix} i \\ r \end{bmatrix}\begin{bmatrix} s \\ j \end{bmatrix}\begin{bmatrix} j \\ s \end{bmatrix} = 1$$

for $r = i$ and $s = j$ and is zero otherwise, it follows that $L^n_{i,j}([i], [j]) = 1$. For $k, m \geq 0$, $k + m \leq n$ and $([k], [m]) \neq ([i], [j])$, we have

$$L^n_{i,j}([k], [m]) = \sum_{r=i}^{n-j} \sum_{s=j}^{n-r} (-1)^{r-i} q^{(r-i)(r-i-1)/2} \begin{bmatrix} r \\ i \end{bmatrix}\begin{bmatrix} k \\ r \end{bmatrix} (-1)^{s-j} q^{(s-j)(s-j-1)/2} \begin{bmatrix} s \\ j \end{bmatrix}\begin{bmatrix} m \\ s \end{bmatrix}.$$

Since $\begin{bmatrix} k \\ r \end{bmatrix} = 0$ for $r > k$ and $\begin{bmatrix} m \\ s \end{bmatrix} = 0$ for $s > m$, we have

$$L^n_{i,j}([k], [m]) = \sum_{r=i}^{k} (-1)^{r-i} q^{(r-i)(r-i-1)/2} \begin{bmatrix} r \\ i \end{bmatrix}\begin{bmatrix} k \\ r \end{bmatrix}$$

$$\times \sum_{s=j}^{m} (-1)^{s-j} q^{(s-j)(s-j-1)/2} \begin{bmatrix} s \\ j \end{bmatrix}\begin{bmatrix} m \\ s \end{bmatrix} \qquad (3.16)$$

for $k \geq i$ and $m \geq j$, and $L^n_{i,j}([k], [m]) = 0$ otherwise. Writing $t = s - j$ in the second sum on right hand side of (3.16), we obtain

$$\sum_{t=0}^{m-j} (-1)^t q^{(s-j)(s-j-1)/2} \begin{bmatrix} j+t \\ j \end{bmatrix}\begin{bmatrix} m \\ j+t \end{bmatrix} = \begin{bmatrix} m \\ j \end{bmatrix} \sum_{t=0}^{m-j} (-1)^t q^{t(t-1)/2} \begin{bmatrix} m-j \\ t \end{bmatrix}.$$

From Lemma 2.1 we see that this last summation is zero. (Similarly we note that the first sum in (3.16) is also zero.) Thus $L^n_{i,j}([k], [m]) = 0$ for $k, m \geq 0$, $k + m \leq n$ and $([k], [m]) \neq ([i], [j])$.

For a fixed value of n, there are $\binom{n+2}{2}$ Lagrange coefficients $L^n_{i,j}(x, y)$ for $i, j \geq 0$, $i + j \leq n$. However the following theorem shows that each Lagrange coefficient $L^n_{i,j}$ contains $i + j$ linear factors. The remaining factor is a transformation of the function $L_{0,0}^{n-i-j}$. Since the coefficients $L_{0,0}^{n}(x, y)$ are of such importance, we will write them down explicitly. We have

$$L_{0,0}^{n}(x, y) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} (-1)^{r+s}\, q^d \begin{bmatrix} \bar{x} \\ r \end{bmatrix} \begin{bmatrix} \bar{y} \\ s \end{bmatrix} \tag{3.17}$$

where $2d = r(r-1) + s(s-1)$.

Theorem 3.2 Let $L_{i,j}^{n}(x, y)$ be the Lagrange coefficient of $f_{i,j}$ where $i, j \geq 0$, $i + j \leq n$. Then

$$L_{i,j}^{n}(x, y) = \begin{bmatrix} \bar{x} \\ i \end{bmatrix} \begin{bmatrix} \bar{y} \\ j \end{bmatrix} L_{0,0}^{n-i-j}(X,Y)$$

where $x = [\bar{x}]$, $y = [\bar{y}]$ and $X = [\bar{x} - i]$, $Y = [\bar{y} - j]$.

Proof In (3.15) for $r \geq i$ we write

$$\begin{bmatrix} r \\ i \end{bmatrix} \begin{bmatrix} \bar{x} \\ r \end{bmatrix} = \begin{bmatrix} \bar{x} \\ i \end{bmatrix} \begin{bmatrix} \bar{x} - i \\ r - i \end{bmatrix}$$

and, for $s \geq j$,

$$\begin{bmatrix} s \\ j \end{bmatrix} \begin{bmatrix} \bar{y} \\ s \end{bmatrix} = \begin{bmatrix} \bar{y} \\ j \end{bmatrix} \begin{bmatrix} \bar{y} - j \\ s - j \end{bmatrix}.$$

Thus we may rewrite (3.15) in the form

$$L_{i,j}^{n}(x, y) = \begin{bmatrix} \bar{x} \\ i \end{bmatrix} \begin{bmatrix} \bar{y} \\ j \end{bmatrix} \sum_{r=i}^{n-j} \sum_{s=j}^{n-r} (-1)^{r+s-i-j} \begin{bmatrix} \bar{x} - i \\ r - i \end{bmatrix} \begin{bmatrix} \bar{y} - j \\ s - j \end{bmatrix} q^d$$

where $2d = (r-i)(r-i-1) + (s-j)(s-j-1)$. We now write $X = [\bar{x} - i] = [\bar{X}]$, $Y = [\bar{y} - j] = [\bar{Y}]$ and $\mu = r - i$, $\nu = s - j$, to give

$$L^n_{i,j}(x, y) = \begin{bmatrix} \bar{x} \\ i \end{bmatrix} \begin{bmatrix} \bar{y} \\ j \end{bmatrix} \sum_{\mu=0}^{n-i-j} \sum_{\nu=0}^{n-i-j-\mu} (-1)^{\mu+\nu} \begin{bmatrix} \bar{X} \\ \mu \end{bmatrix} \begin{bmatrix} \bar{Y} \\ \nu \end{bmatrix} q^d$$

where $2d = \mu(\mu - 1) + \nu(\nu - 1)$. On comparison with (3.17) we obtain

$$L^n_{i,j}(x, y) = \begin{bmatrix} \bar{x} \\ i \end{bmatrix} \begin{bmatrix} \bar{y} \\ j \end{bmatrix} L^{n-i-j}_{0,0}(X,Y) .$$

As a consequence of Theorem 3.2, we see that the curve $L^n_{i,j}(x, y) = 0$ consists of the straight lines

$$x = [\mu], \ 0 \le \mu \le i-1,$$

$$y = [\nu], \ 0 \le \nu \le j-1,$$

together with the curve $L^{n-i-j}_{0,0}(X,Y) = 0$, where $X = q^{-i}(x - [i])$, $Y = q^{-j}(y - [j])$. This result implies that in order to be able to write down all Lagrange coefficients of the form $L^n_{i,j}(x, y)$, for a fixed value of n, we need only concern ourselves with the (non-linear) coefficients $L^m_{0,0}(x, y)$, for $2 \le m \le n$.

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Chapter 4

On the limit of the interpolating polynomial

4.1 Introduction

Let $f(x, y)$ be a function defined on some region containing the set of distinct nodes $S = \{([r]_p, [s]_q): 0 \le r + s \le n,\ r, s \ge 0\},\ p, q > 0$. We will assume that $f(x, y)$ possesses partial derivatives of appropriate order at (1, 0), (0, 1) and (1, 1). We know from (3.10) that the Newton form of the interpolating polynomial at the nodes of S can be written as

$$P_n(x, y) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \begin{bmatrix} \bar{x} \\ r \end{bmatrix}_p \begin{bmatrix} \bar{y} \\ s \end{bmatrix}_q \mathfrak{D}_x^r\, \mathfrak{D}_y^s\, f_{0,0} \tag{4.1}$$

where $x = [\bar{x}]_p$ and $y = [\bar{y}]_q$. If we let p tend to 0, $[0]_p$ converges to zero and the other distinct numbers $[1]_p, \ldots, [n]_p$ all converge to 1. Hence for $r \ge 1$ the expression

$$p^{(r-1)r/2} \begin{bmatrix} \bar{x} \\ r \end{bmatrix}_p = \prod_{\mu=0}^{r-1} \frac{x - [\mu]_p}{[\mu + 1]_p}$$

simply reduces to $x(x - 1)^{r-1}$ as p tends to 0.

In this chapter we show that the polynomial $P_n(x, y)$ has a limit when both $p \to 0$ and $q \to 0$. This task depends on the simplification of the limit of the expression $\mathfrak{D}_x^r f_{0,0} / p^{(r(r-1)/2)}$. To simplify our notation we will write

$$Q_n(x, y) = \lim_{p \to 0} \lim_{q \to 0} P_n(x, y)$$

provided this repeated limits exists. Then we will study the interpolation properties satisfied by the polynomial $Q_n(x, y)$. We will verify that these properties depend on the appropriate partial derivatives of $f(x, y)$ evaluated at the three points $(1, 0)$, $(0, 1)$ and $(1, 1)$. In the last section we show that $Q_n(x, y)$ can also be derived as a limiting form of the interpolating polynomial on a simpler system of nodes.

4.2 Note on taking the limit of the q-forward differences directly

First we consider the polynomial $P_1(x, y)$ which interpolates $f(x, y)$ at the three nodes $(0, 0)$, $(0, 1)$ and $(1, 0)$,

$$P_1(x, y) = (1 - x - y) f_{0,0} + x f_{1,0} + y f_{0,1} .$$

Since $P_1(x, y)$ is independent of p and q there are no difficulties in taking the limit as $p \to 0$ and $q \to 0$. Note that, in the rest of this chapter, we may drop the subscripts p and q. Thus $[k]$ will mean $[k]_p$ when it is associated with the variable x and $[k]_q$ when associated with the variable y. In particular we may write the ordered pair $([\mu]_p, [\nu]_q)$ simply as $([\mu], [\nu])$.

Let $P_2(x, y)$ be the interpolating polynomial of degree two which interpolates $f(x, y)$ at the nodes $\{([r], [s]): r, s \geq 0, r + s \leq 2\}$. Then

$$P_2(x, y) = f_{0,0} + x\mathfrak{D}_x^1 f_{0,0} + y\mathfrak{D}_y^1 f_{0,0} + \frac{x(x-1)}{p[2]_p}\mathfrak{D}_x^2 f_{0,0} + xy\mathfrak{D}_x^1 \mathfrak{D}_y^1 f_{0,0}$$

$$+ \frac{y(y-1)}{q[2]_q}\mathfrak{D}_y^2 f_{0,0} .$$

Since

$$\lim_{p\to 0} \frac{1}{p} \mathfrak{D}_x^2 f_{0,0} = \lim_{p\to 0} \frac{f(1+p, 0) - f(1, 0)}{p} - f_{1,0} + f_{0,0}$$

$$= f_x(1, 0) - f_{1,0} + f_{0,0}$$

and similarly

$$\lim_{q\to 0} \frac{1}{q} \mathfrak{D}_y^2 f_{0,0} = f_y(0, 1) - f_{0,1} + f_{0,0} , \tag{4.2}$$

we have

$$\lim_{p\to 0} \lim_{q\to 0} P_2(x, y) = f_{0,0} + x(f_{1,0} - f_{0,0}) + y(f_{0,1} - f_{0,0}) + xy(f_{1,1} - f_{0,1} - f_{1,0} + f_{0,0})$$

$$+ x(x-1) \lim_{p\to 0} \frac{1}{p} \mathfrak{D}_x^2 f_{0,0} + y(y-1) \lim_{q\to 0} \frac{1}{q} \mathfrak{D}_y^2 f_{0,0} .$$

Hence

$$Q_2(x, y) = (1 - 2x - 2y + xy + x^2 + y^2) f_{0,0} + x(2 - x - y) f_{1,0}$$

$$+ y(2 - x - y) f_{0,1} + xy f_{1,1} + x(x-1) f_x(1, 0) + y(y-1) f_y(0, 1) .$$

We see that $Q_2(x, y)$ satisfies the following conditions.

$$Q_2([k], [m]) = f_{k,m} , \quad 0 \le k , m \le 1 ,$$

$$\frac{\partial}{\partial x} Q_2(1, 0) = f_x(1, 0)$$

and

$$\frac{\partial}{\partial y} Q_2(0, 1) = f_y(0, 1) .$$

We now consider the interpolating polynomial (4.1) of order 3,

$$P_3(x, y) = P_2(x, y) + \frac{x(x-1)(x-[2])}{[3]_p!} \frac{1}{p^3} \mathfrak{D}_x^3 f_{0,0} + \frac{x(x-1)y}{[2]_p!} \frac{1}{p} \mathfrak{D}_x^2 \mathfrak{D}_y^1 f_{0,0}$$

$$+ \frac{xy(y-1)}{[2]_q!} \frac{1}{q} \mathfrak{D}_x^1 \mathfrak{D}_y^2 f_{0,0} + \frac{y(y-1)(y-[2])}{[3]_q!} \frac{1}{q^3} \mathfrak{D}_y^3 f_{0,0} .$$

Here we need to simplify the limits involving the third order differences in $P_3(x, y)$. First we consider

$$\lim_{p\to 0} \lim_{q\to 0} \frac{1}{p^3} \mathfrak{D}_x^3 f_{0,0} = \lim_{p\to 0} \lim_{q\to 0} \{\frac{1}{p^3} \mathfrak{D}_x^2 f_{1,0} - \frac{1}{p} \mathfrak{D}_x^2 f_{0,0}\} .$$

Using the definition

$$\mathfrak{D}_x^2 f_{1,0} = \mathfrak{D}_x^1 f_{2,0} - p\, \mathfrak{D}_x^1 f_{1,0}$$

and then applying L'Hospital's rule, we have

$$\lim_{p\to 0} \lim_{q\to 0} \frac{1}{p^3} \mathfrak{D}_x^2 f_{1,0} = \lim_{p\to 0} \frac{1}{p^3} \{f(1+p+p^2, 0) - (1+p)\, f(1+p, 0) + p\, f(1, 0)\}$$

$$= \lim_{p\to 0} \frac{1}{3p^2} \{(1+2p)\, f_x(1+p+p^2, 0) - (1+p)\, f_x(1+p, 0)$$

$$- f(1+p, 0) + f(1, 0)\} .$$

On applying L'Hospital's rule repeatedly, we obtain

$$\lim_{p,q\to 0}\frac{1}{p^3}\mathfrak{D}_x^2 f_{1,0} = \lim_{p\to 0}\frac{1}{6p}\{(1+2p)^2 f_{xx}(1+p+p^2,0) + 2f_x(1+p+p^2,0)$$

$$-(1+p)f_{xx}(1+p,0) - 2f_x(1+p,0)\}$$

$$= \lim_{p\to 0}\frac{1}{6}\{(1+2p)^3 f_{xxx}(1+p+p^2,0) + 6(1+2p)f_{xx}(1+p+p^2,0)$$

$$-(1+p)f_{xxx}(1+p,0) - 3f_{xx}(1+p,0)\}$$

$$= \frac{1}{2}f_{xx}(1,0)\ .$$

Thus we obtain

$$\lim_{p\to 0}\lim_{q\to 0}\frac{1}{p^3}\mathfrak{D}_x^3 f_{0,0} = \frac{1}{2}f_{xx}(1,0) - f_x(1,0) + f_{1,0} - f_{0,0}\ ,$$

and similarly

$$\lim_{p\to 0}\lim_{q\to 0}\frac{1}{q^3}\mathfrak{D}_y^3 f_{0,0} = \frac{1}{2}f_{yy}(0,1) - f_y(0,1) + f_{0,1} - f_{0,0}\ .$$

Also on using (4.2), we obtain

$$\lim_{p\to 0}\lim_{q\to 0}\frac{1}{p}\mathfrak{D}_x^2\mathfrak{D}_y^1 f_{0,0} = \lim_{p\to 0}\lim_{q\to 0}\{\frac{1}{p}\mathfrak{D}_x^2 f_{0,1} - \frac{1}{p}\mathfrak{D}_x^2 f_{0,0}\}$$

$$= \{f_x(1,1) - f_{1,1} + f_{0,1}\} - \{f_x(1,0) - f_{1,0} + f_{0,0}\},$$

and similarly

$$\lim_{p\to 0}\lim_{q\to 0}\frac{1}{q}\mathfrak{D}_x^1\mathfrak{D}_y^2 f_{0,0} = \{f_y(1,1) - f_{1,1} + f_{1,0}\} - \{f_y(0,1) - f_{0,1} + f_{0,0}\}\ .$$

Thus we have

$$Q_3(x, y) = Q_2(x, y) + x(x-1)^2 \{\tfrac{1}{2} f_{xx}(1, 0) - f_x(1, 0) + f_{1,0} - f_{0,0}\}$$

$$+ x(x-1)y \{f_x(1, 1) - f_{1,1} + f_{0,1} - f_x(1, 0) + f_{1,0} - f_{0,0}\}$$

$$+ xy(y-1) \{f_y(1, 1) - f_{1,1} + f_{1,0} - f_y(0, 1) + f_{0,1} - f_{0,0}\}$$

$$+ y(y-1)^2 \{ \tfrac{1}{2} f_{yy}(0, 1) - f_y(0, 1) + f_{0,1} - f_{0,0}\}$$

and after some simplification we obtain

$$Q_3(x, y) = \{-1 + xy(3 - x - y) - (x-1)^3 - (y-1)^3\} f_{0,0} + xy(3 - x - y) f_{1,1}$$

$$+ (3 - 3x - 3y + x^2 + xy + y^2)(x f_{1,0} + y f_{0,1}) + x(x-1)(2 - x - y) f_x(1, 0)$$

$$+ y(y-1)(2 - x - y) f_y(0, 1) + x(x-1)y f_x(1, 1) + xy(y-1) f_y(1, 1)$$

$$+ \tfrac{1}{2} x(x-1)^2 f_{xx}(1, 0) + \tfrac{1}{2} y(y-1)^2 f_{yy}(0, 1) .$$

We now verify the interpolation properties of $Q_3(x, y)$. We differentiate the function $Q_3(x, y)$ with respect to x to give

$$\frac{\partial}{\partial x} Q_3(x, y) = \{y(3 - 2x - y) - 3(x-1)^2\}(f_{0,0} - f_{1,0}) - y(3 - 2x - y)(f_{0,1} - f_{1,1})$$

$$+ (2x-1)y f_x(1, 1) + (6x - 3x^2 - 2xy + y - 2) f_x(1, 0)$$

$$+ (y - y^2)\{f_y(0, 1) - f_y(1, 1)\} + \tfrac{1}{2} (3x^2 - 4x + 1) f_{xx}(1, 0)$$

and differentiate again to give

$$\frac{\partial^2}{\partial x^2} Q_3(x, y) = (6 - 6x - 2y)\{f_{0,0} - f_{1,0} + f_x(1, 0)\} + 2y\{f_{0,1} - f_{1,1} + f_x(1, 1)\}$$

$$+ (3x - 2)\, f_{xx}(1, 0) \, .$$

Since the polynomial is symmetric in x and y, partial derivatives of $Q_3(x, y)$ with respect to y can be obtained similarly. We may verify that $Q_3(x, y)$ satisfies the following ten conditions.

(i) $Q_3([k], [m]) = f_{k,m}, \quad 0 \le k, m \le 1$

(ii) $\frac{\partial}{\partial x} Q_3(1,k) = f_x(1, k)$ and $\frac{\partial}{\partial y} Q_3(k, 1) = f_y(k, 1)$ for $k = 0, 1$

(iii) $\frac{\partial^2}{\partial x^2} Q_3(1, 0) = f_{xx}(1, 0)$ and $\frac{\partial^2}{\partial y^2} Q_3(0, 1) = f_{yy}(0, 1)$.

4.3 The limit of $P_n(x, y)$ as p, q tend to zero

It is clear that the derivation of $Q_n(x, y)$ by the methods used above becomes progressively more tedious as n increases. We therefore seek an alternative approach. If g is a function of one variable and $g^{(r)}(x)$ exists, then we have from (1.9)

$$[x_0, x_1, \ldots, x_r]\, g = \frac{g^{(r)}(\zeta)}{r!} \tag{4.3}$$

where ζ lies in the interval $(\min x_i, \max x_i)$. We now use this result to derive limits of q-differences. From Lemma 3.5 we have for a fixed y

$$\left[[i], [i+1], \ldots, [i+r]\right]_x f(\cdot, y) = \frac{1}{[r]!\, p^{ir+(r-1)r/2}} \mathfrak{D}_x^r f([i], y)\,. \tag{4.4}$$

Then from (4.3) and (4.4) we obtain

$$\lim_{p\to 0} \frac{1}{p^{(r+1)r/2}} \mathfrak{D}_x^r f(1, y) = \lim_{p\to 0} \frac{f^{(r)}(\zeta, y)}{r!}, \qquad 1 < \zeta < [1+r],$$

$$= \frac{1}{r!} \frac{\partial^r}{\partial x^r} f(1, y)\,. \tag{4.5}$$

Note that in (4.4) $\lim_{p\to 0} [r]! = 1$.

Similarly

$$\lim_{q\to 0} \frac{1}{q^{(s+1)s/2}} \mathfrak{D}_y^s f(x, 1) = \frac{1}{s!} \frac{\partial^s}{\partial y^s} f(x, 1)$$

and hence

$$\lim_{p\to 0} \lim_{q\to 0} \frac{1}{p^{(r+1)r/2}\, q^{(s+1)s/2}} \mathfrak{D}_x^r \mathfrak{D}_y^s f(1, 1) = \lim_{p\to 0} \frac{1}{p^{(r+1)r/2}} \mathfrak{D}_x^r \left\{\frac{1}{s!} \frac{\partial^s}{\partial y^s} f(1, 1)\right\}$$

$$= \frac{1}{r!\ s!} \frac{\partial^{r+s}}{\partial x^r\, \partial y^s} f(1, 1)\,. \tag{4.6}$$

As we have already noted, as $p \to 0$, $[r]_p \to 1$ for all $r > 0$ and $[0]_p \to 0$. Thus it is appropriate to express the forward difference operator $\mathfrak{D}_x^r f(0, y)$ in terms of $f(0, y)$ and $\mathfrak{D}_x^t f(1, y)$, $0 \le t \le r-1$. We will show that, for $r \ge 0$,

$$\mathfrak{D}_x^r f(0, y) = (-1)^r p^{(r-1)r/2} f(0, y) + \sum_{t=0}^{r-1} (-1)^{r-t-1} p^{(r-t-1)(r+t)/2} \mathfrak{D}_x^t f(1, y) \tag{4.7}$$

and similarly

$$\mathfrak{D}_y^s f(x, 0) = (-1)^s q^{(s-1)s/2} f(x, 0) + \sum_{u=0}^{s-1} (-1)^{s-u-1} q^{(s-u-1)(s+u)/2} \mathfrak{D}_y^u f(x, 1). \tag{4.8}$$

We will verify (4.7) by induction, and this will also verify (4.8). First, (4.7) evidently holds for $r = 0$, where the sum is empty and has the value zero. Suppose that (4.7) is true for any $r \geq 0$. Then

$$\mathfrak{D}_x^{r+1} f(0, y) = \mathfrak{D}_x^r f(1, y) - p^r \mathfrak{D}_x^r f(0, y)$$

$$= \mathfrak{D}_x^r f(1, y) + (-1)^{r+1} p^{r+(r-1)r/2} f(0, y) + \sum_{t=0}^{r-1} (-1)^{r-t} p^{r+(r-t-1)(r+t)/2} \mathfrak{D}_x^t f(1, y),$$

and this simplifies to give

$$\mathfrak{D}_x^{r+1} f(0, y) = (-1)^{r+1} p^{(r+1)r/2} f(0, y) + \sum_{t=0}^{r} (-1)^{r-t} p^{(r-t)(r+t+1)/2} \mathfrak{D}_x^t f(1, y),$$

showing that (4.7) holds for $r + 1$. This completes the proof.

Now divide both sides of (4.7) by $p^{(r-1)r/2}$ and note that $(r - t - 1)(r + t)/2 - r(r - 1)/2 = - t(t+1)/2$. Let $p \to 0$ and apply (4.5) to give

$$\lim_{p \to 0} \frac{1}{p^{(r-1)r/2}} \mathfrak{D}_x^r f(0, y) = (-1)^r f(0, y) + (-1)^r \sum_{t=0}^{r-1} (-1)^{t+1} \frac{1}{t!} \frac{\partial^t}{\partial x^t} f(1, y).$$

Similarly we have

$$\lim_{q \to 0} \frac{1}{q^{(s-1)s/2}} \mathfrak{D}_y^s f(x, 0) = (-1)^s f(x, 0) + (-1)^s \sum_{u=0}^{s-1} (-1)^{u+1} \frac{1}{u!} \frac{\partial^s}{\partial y^s} f(x, 1). \qquad (4.9)$$

We now consider limits involving mixed forward differences.

Lemma 4.1 Suppose that all derivatives of f(x, y) of order up to n exist. Then

$$\lim_{p\to 0}\lim_{q\to 0}\frac{1}{p^{(r-1)r/2}\,q^{(s-1)s/2}}\,\mathfrak{D}_x^r\,\mathfrak{D}_y^s\,f_{0,0}$$

$$= (-1)^{r+s}\ \{\ f_{0,0}+\sum_{t=0}^{r-1}\ (-1)^{t+1}\frac{1}{t!}\frac{\partial^t}{\partial x^t}f_{1,0}+\sum_{u=0}^{s-1}\ (-1)^{u+1}\frac{1}{u!}\frac{\partial^u}{\partial y^u}f_{0,1}$$

$$+\sum_{t=0}^{r-1}\ \sum_{u=0}^{s-1}\ (-1)^{t+u}\frac{1}{t!\ u!}\frac{\partial^{t+u}}{\partial x^t\,\partial y^u}f_{1,1}\ \}\ . \qquad (4.10)$$

Proof Using (4.7), we apply the operator $\mathfrak{D}_x^r$ to each term of $\mathfrak{D}_y^s$ f(x, 0) in (4.8), after putting x = 0. This gives

$$\mathfrak{D}_x^r\,\mathfrak{D}_y^s\,f_{0,0} = (-1)^s\,q^{(s-1)s/2}\ \{\ (-1)^r\,p^{(r-1)r/2}\,f(0, 0)$$

$$+\sum_{t=0}^{r-1}\ (-1)^{r-t-1}\,p^{(r-t-1)(r+t)/2}\,\mathfrak{D}_x^t\,f(1, 0)\ \}$$

$$+\sum_{u=0}^{s-1}\ (-1)^{s-u-1}\,q^{(s-u-1)(s+u)/2}\ \{\ (-1)^r\,p^{(r-1)r/2}\,\mathfrak{D}_y^u\,f(0, 1)$$

$$+\sum_{t=0}^{r-1}\ (-1)^{r-t-1}\,p^{(r-t-1)(r+t)/2}\,\mathfrak{D}_x^t\,\mathfrak{D}_y^u\,f(1, 1)\ \}\ .$$

Hence, abbreviating the notation in an obvious way,

$$\mathfrak{D}_x^r\,\mathfrak{D}_y^s\,f_{0,0} = (-1)^{r+s}\,p^{(r-1)r/2}\,q^{(s-1)s/2}\,f_{0,0}$$

$$+\ (-1)^s\,q^{(s-1)s/2}\sum_{t=0}^{r-1}\ (-1)^{r-t-1}\,p^{(r-t-1)(r+t)/2}\,\mathfrak{D}_x^t\,f_{1,0}$$

$$+ (-1)^r p^{(r-1)r/2} \sum_{u=0}^{s-1} (-1)^{s-u-1} q^{(s-u-1)(s+u)/2} \mathfrak{D}_y^u f_{0,1}$$

$$+\sum_{t=0}^{r-1} \sum_{u=0}^{s-1} (-1)^{r+s-t-u} p^{(r-t-1)(r+t)/2} q^{(s-u-1)(s+u)/2} \mathfrak{D}_x^t \mathfrak{D}_y^u f_{1,1} . \tag{4.11}$$

Now divide both sides of (4.11) by $p^{(r-1)r/2} q^{(s-1)s/2}$ and let $p \to 0$, $q \to 0$ in turn to give

$$\lim_{p\to 0} \lim_{q\to 0} \frac{1}{p^{(r-1)r/2} q^{(s-1)s/2}} \mathfrak{D}_x^r \mathfrak{D}_y^s f_{0,0} = (-1)^{r+s} f_{0,0} + (-1)^s \sum_{t=0}^{r-1} (-1)^{r-t-1} \frac{1}{t!} \frac{\partial^t}{\partial x^t} f_{1,0}$$

$$+ (-1)^r \sum_{u=0}^{s-1} (-1)^{s-u-1} \frac{1}{u!} \frac{\partial^s}{\partial y^s} f_{0,1}$$

$$+ \sum_{t=0}^{r-1} \sum_{u=0}^{s-1} (-1)^{r+s-t-u} \frac{1}{t!\ u!} \frac{\partial^{t+u}}{\partial x^t \partial y^u} f_{1,1} .$$

Finally the powers of (−1) can be written more neatly to give (4.10). We note that (4.10) is valid even for $r = 0$ or $s = 0$, so that it applies to "single" forward differences also.

We are now ready to obtain the limit of the interpolating polynomial $P_n(x, y)$ as $p, q \to 0$. Since

$$\sum_{r=0}^{n} \sum_{s=0}^{n-r} = \sum_{r=0}^{0} \sum_{s=0}^{0} + \sum_{r=1}^{n} \sum_{s=0}^{0} + \sum_{r=0}^{0} \sum_{s=1}^{n} + \sum_{r=1}^{n-1} \sum_{s=1}^{n-r} ,$$

we can write the polynomial (4.1) in the form

$$P_n(x, y) = f_{0,0} + \sum_{r=1}^{n} \prod_{\mu=0}^{r-1} \frac{x - [\mu]}{p^{\mu}\,[\mu+1]} \mathfrak{D}_x^r f_{0,0} + \sum_{s=1}^{n} \prod_{\nu=0}^{s-1} \frac{y - [\nu]}{q^{\nu}\,[\nu+1]} \mathfrak{D}_y^s f_{0,0}$$

$$+ \sum_{r=1}^{n-1} \sum_{s=1}^{n-r} \prod_{\mu=0}^{r-1} \frac{x - [\mu]}{p^{\mu}\,[\mu+1]} \prod_{\nu=0}^{s-1} \frac{y - [\nu]}{q^{\nu}\,[\nu+1]} \mathfrak{D}_x^r \mathfrak{D}_y^s f_{0,0}\,.$$

As $p \to 0$ and $q \to 0$, $[\mu] \to 0$ if $\mu = 0$ and $[\mu] \to 1$ if $\mu \geq 1$. We obtain the following results concerning $Q_n(x, y)$, the limit of $P_n(x, y)$ as $p \to 0$, $q \to 0$.

Theorem 4.1 Let $f(x, y)$ possess partial derivatives up to order n. Then

$$Q_n(x, y) = a(x, y)\, f_{0,0} + \sum_{t=0}^{n-1} b_t(x, y) \frac{(-1)^t}{t!} \frac{\partial^t}{\partial x^t} f_{1,0}$$

$$+ \sum_{u=0}^{n-1} b_u(y, x) \frac{(-1)^u}{u!} \frac{\partial^s}{\partial y^s} f_{0,1} + \sum_{t=0}^{n-2} \sum_{u=0}^{n-2-t} c_{t,u}(x, y) \frac{(-1)^{t+u}}{t!\,u!} \frac{\partial^{t+u}}{\partial x^t\, \partial y^u} f_{1,1}\,,$$

where

$$a(x, y) = 1 - x \sum_{r=1}^{n} (1 - x)^{r-1} - y \sum_{s=1}^{n} (1 - y)^{s-1} + xy \sum_{r=1}^{n-1} \sum_{s=1}^{n-r} (1 - x)^{r-1} (1 - y)^{s-1}\,,$$

$$b_t(x, y) = x \sum_{r=t+1}^{n} (1 - x)^{r-1} - xy \sum_{r=t+1}^{n-1} \sum_{s=1}^{n-r} (1 - x)^{r-1} (1 - y)^{s-1}$$

and

$$c_{t,u}(x, y) = xy \sum_{r=t+1}^{n-1-u} \sum_{s=u+1}^{n-r} (1 - x)^{r-1} (1 - y)^{s-1}\,.$$

Proof Let $p \to 0$, $q \to 0$ in turn to give

$$Q_n(x, y) = f_{0,0} + \sum_{r=1}^{n} x(x-1)^{r-1} \lim_{p\to 0} \frac{1}{p^{(r-1)r/2}} \mathfrak{D}_x^r f_{0,0}$$

$$+ \sum_{s=1}^{n} y(y-1)^{s-1} \lim_{q\to 0} \frac{1}{q^{(s-1)s/2}} \mathfrak{D}_y^s f_{0,0}$$

$$+ \sum_{r=1}^{n-1} \sum_{s=1}^{n-r} xy(x-1)^{r-1}(y-1)^{s-1} \lim_{p\to 0} \lim_{q\to 0} \frac{1}{p^{(r-1)r/2}\, q^{(s-1)s/2}} \mathfrak{D}_x^r \mathfrak{D}_y^s f_{0,0}\,.$$

Applying (4.10) to the limits involving the forward differences, we have

$$Q_n(x, y) = f_{0,0} + \sum_{r=1}^{n} x(x-1)^{r-1} (-1)^r \{ f_{0,0} + \sum_{t=0}^{r-1} (-1)^{t+1} \frac{1}{t!} \frac{\partial^t}{\partial x^t} f_{1,0} \}$$

$$+ \sum_{s=1}^{n} y(y-1)^{s-1} (-1)^s \{ f_{0,0} + \sum_{u=0}^{s-1} (-1)^{u+1} \frac{1}{u!} \frac{\partial^u}{\partial y^u} f_{0,1} \}$$

$$+ \sum_{r=1}^{n-1} \sum_{s=1}^{n-r} xy(x-1)^{r-1}(y-1)^{s-1} (-1)^{r+s} \{ f_{0,0} + \sum_{t=0}^{r-1} (-1)^{t+1} \frac{1}{t!} \frac{\partial^t}{\partial x^t} f_{1,0}$$

$$+ \sum_{u=0}^{s-1} (-1)^{u+1} \frac{1}{u!} \frac{\partial^u}{\partial y^u} f_{0,1} + \sum_{t=0}^{r-1} \sum_{u=0}^{s-1} (-1)^{t+u} \frac{1}{t!\, u!} \frac{\partial^{t+u}}{\partial x^t\, \partial y^u} f_{1,1} \}. \tag{4.12}$$

We now rearrange the expression (4.12), collecting together all terms involving $f_{0,0}$, those involving $f_{1,0}$ and its partial derivatives, and so on, to give

$$Q_n(x, y) = \{1 - x\sum_{r=1}^{n} (1-x)^{r-1} - y\sum_{s=1}^{n} (1-y)^{s-1} + xy \sum_{r=1}^{n-1} \sum_{s=1}^{n-r} (1-x)^{r-1}(1-y)^{s-1}\} f_{0,0}$$

$$+ \sum_{t=0}^{n-1} \{ \sum_{r=t+1}^{n} x(1-x)^{r-1} - \sum_{r=t+1}^{n-1} \sum_{s=1}^{n-r} xy(1-x)^{r-1} (1-y)^{s-1}\} \frac{(-1)^t}{t!} \frac{\partial^t}{\partial x^t} f_{1,0}$$

$$+ \sum_{u=0}^{n-1} \{ \sum_{s=u+1}^{n} y(1-y)^{s-1} - \sum_{s=u+1}^{n-1} \sum_{r=1}^{n-s} xy(1-x)^{r-1} (1-y)^{s-1} \} \frac{(-1)^u}{u!} \frac{\partial^u}{\partial y^u} f_{0,1}$$

$$+ \sum_{t=0}^{n-2} \sum_{u=0}^{n-2-t} \{ xy \sum_{r=t+1}^{n-1-u} \sum_{s=u+1}^{n-r} (1-x)^{r-1} (1-y)^{s-1} \} \frac{(-1)^{t+u}}{t!\, u!} \frac{\partial^{t+u}}{\partial x^t \partial y^u} f_{1,1} .$$

The last line, involving the coefficients of $\frac{\partial^{t+u}}{\partial x^t \partial y^u} f_{1,1}$, was obtained by first changing

$$\sum_{r=1}^{n-1} \sum_{s=1}^{n-r} \sum_{t=0}^{r-1} \sum_{u=0}^{s-1} \quad \text{to} \quad \sum_{r=1}^{n-1} \sum_{t=0}^{r-1} \sum_{s=1}^{n-r} \sum_{u=0}^{s-1} .$$

Then we rearrange the first two and the last two summations as a pair to give

$$\sum_{t=0}^{n-2} \sum_{r=t+1}^{n-1} \sum_{u=0}^{n-r-1} \sum_{s=u+1}^{n-r} .$$

Finally we rearrange the second and third summations to give

$$\sum_{t=0}^{n-2} \sum_{u=0}^{n-2-t} \sum_{r=t+1}^{n-1-u} \sum_{s=u+1}^{n-r} .$$

This completes the proof of the theorem.

4.4 Interpolation properties of $Q_n(x, y)$

Before analysing the properties of the functions $a(x, y)$, $b_t(x, y)$ and $c_{t,u}(x, y)$ which will determine the properties of the polynomial $Q_n(x, y)$, we need the following lemma.

Lemma 4.2 For any integer $n \geq 2$

$$(1-X)(1-Y)\sum_{r=1}^{n-1}\sum_{s=1}^{n-r} X^{r-1}Y^{s-1} = 1 - X^n - Y^n - \frac{X^n - Y^n}{X - Y} + \frac{X^{n+1} - Y^{n+1}}{X - Y}.$$

Proof First we simplify the geometric sum to give

$$\sum_{r=1}^{n-1}\sum_{s=1}^{n-r} X^{r-1}Y^{s-1}$$

$$= (1 + Y + \ldots + Y^{n-2}) + X(1 + Y + \ldots + Y^{n-3}) + X^2(1 + Y + \ldots + Y^{n-4})$$

$$+ \ldots + X^{n-3}(1 + Y) + X^{n-2}$$

$$= \frac{1 - Y^{n-1}}{1 - Y} + X\frac{1 - Y^{n-2}}{1 - Y} + X^2\frac{1 - Y^{n-3}}{1 - Y} + \ldots + X^{n-2}$$

$$= \frac{1}{1 - Y}(1 + X + \ldots + X^{n-2}) - \frac{Y^{n-1}}{1 - Y}\left(1 + \frac{X}{Y} + \left(\frac{X}{Y}\right)^2 + \ldots + \left(\frac{X}{Y}\right)^{n-2}\right)$$

$$= \frac{1}{1 - Y}\left(\frac{1 - X^{n-1}}{1 - X}\right) - \frac{Y}{1 - Y}\left(\frac{Y^{n-1} - X^{n-1}}{Y - X}\right).$$

Hence, we obtain

$$(1-X)(1-Y)\sum_{r=1}^{n-1}\sum_{s=1}^{n-r} X^{r-1}Y^{s-1}$$

$$= 1 - X^{n-1} - \frac{Y(1 - X)(Y^{n-1} - X^{n-1})}{Y - X}$$

$$= \frac{Y - X - Y^n + X^n + Y^nX - YX^n}{Y - X}$$

$$= 1 - \frac{Y^n - X^n}{Y - X} + \frac{Y^nX - Y^{n+1} - YX^n + X^{n+1} + Y^{n+1} - X^{n+1}}{Y - X}$$

$$= 1 - X^n - Y^n - \frac{X^n - Y^n}{X - Y} + \frac{X^{n+1} - Y^{n+1}}{X - Y}.$$

This completes the proof.

We can now simplify the first coefficient $a(x, y)$. Since

$$1 - (1 - X) \sum_{r=1}^{n} X^{r-1} - (1 - Y) \sum_{s=1}^{n} Y^{s-1} = 1 - (1 - X^n) - (1 - Y^n)$$

we have

$$a(x, y) = \frac{X^{n+1} - Y^{n+1}}{X - Y} - \frac{X^n - Y^n}{X - Y}$$

where $X = 1 - x$ and $Y = 1 - y$. To investigate the properties of $a(x, y)$, that it has the value 1 at $(x, y) = (0, 0)$ and is zero, and appropriate partial derivatives are zero, at $(1, 0)$, $(0, 1)$ and $(1, 1)$, let us write $a(x, y)$ in the form

$$a(x, y) = X^n + X^{n-1} Y + \ldots + X Y^{n-1} + Y^n$$

$$- (X^{n-1} + X^{n-2} Y + \ldots + X Y^{n-2} + Y^{n-1}).$$

When $x = y = 0$, $X = Y = 1$ and we see that $a(0, 0) = 1$. It is also easy to see that $a(1, 1) = 0$ and all partial derivatives of order not greater than $n - 2$ are zero at $x = 1$, $y = 1$.

Let us now write $a(x, y)$ in the form

$$a(x, y) = X^n - (1 - Y)(X^{n-1} + X^{n-2} Y + \ldots + X Y^{n-2} + Y^{n-1}).$$

We see that all partial derivatives with respect to X of order $\leq n-1$ are zero at $x = 1$ and $y = 0$ and also $a(1, 0) = 0$. Similarly we may write

$$a(x, y) = Y^n - (1 - X)(X^{n-1} + X^{n-2}\,Y + \ldots + X\,Y^{n-2} + Y^{n-1})$$

and likewise deduce that

$$a(0, 1) = 0 \quad \text{and} \quad \frac{\partial^k}{\partial y^k}\,a(0, 1) = 0, \;\; \text{for } 0 < k \leq n-1.$$

Let us now consider the second coefficient

$$b_t(x, y) = (1 - X)\sum_{r=t+1}^{n} X^{r-1} - (1 - X)(1 - Y)\sum_{r=t+1}^{n-1}\;\sum_{s=1}^{n-r} X^{r-1}\,Y^{s-1}\,.$$

We can extract the common factor X^t from each sum to give

$$b_t(x, y) = X^t(1 - X)\sum_{r=1}^{n-t} X^{r-1} - X^t(1 - X)(1 - Y)\sum_{r=1}^{n-t-1}\;\sum_{s=1}^{n-t-r} X^{r-1}\,Y^{s-1}\,.$$

Using Lemma 4.2 we obtain

$$b_t(x, y) = X^t - X^n - X^t\left\{1 - X^{n-t} - Y^{n-t} - \frac{X^{n-t} - Y^{n-t}}{X - Y} + \frac{X^{n-t+1} - Y^{n-t+1}}{X - Y}\right\}.$$

This simplifies to give

$$b_t(x, y) = X^t\,(1 - X)(X^{n-t-1} + X^{n-t-2}\,Y + \ldots + Y^{n-t-1})\,.$$

Since each term is of "total degree" $n - 1$ or n, all partial derivatives of $b_t(x, y)$ of degree $\leq n-2$ are zero at $x = y = 1$ and also $b_t(1, 1) = 0$. Thus there is no contribution to $P_n(1, 1)$ from $b_t(1, 1)$ or $\frac{\partial^{t+u}}{\partial x^t \partial y^u} b(1, 1)$. We note also that, at $x = 0$, $b_t(x, y)$ and all partial derivatives with respect to y are zero. When $x = 1$, $y = 0$, $b_0(1, 0) = 1$.

The lowest power of X in $b_t(x, y)$ is X^t. Hence, for $0 \leq k < t$,

$$\frac{\partial^k}{\partial x^k} b_t(1, 0) = 0 .$$

For $t < k \leq n-1$, when $x = 1$, we have

$$\frac{\partial^k}{\partial x^k} b_t(x, y) = \frac{\partial^k}{\partial x^k} (X^k Y^{n-1-k} - X^k Y^{n-k}) = 0, \text{ for } y = 0$$

and finally, when $x = 1$,

$$\frac{\partial^t}{\partial x^t} b_t(x, y) = \frac{\partial^t}{\partial x^t} (1-x)^t Y^{n-1-t} = (-1)^t t! , \quad \text{for } y = 0.$$

Now we consider the remaining coefficient

$$c_{t,u}(x, y) = (1-X)(1-Y) \sum_{r=t+1}^{n-1-u} \sum_{s=u+1}^{n-r} X^{r-1} Y^{s-1} .$$

We may remove the common factor $X^t Y^u$ to give

$$c_{t,u}(x, y) = X^t Y^u (1-X)(1-Y) \sum_{r=1}^{n-1-t-u} \sum_{s=1}^{n-t-u-r} X^{r-1} Y^{s-1} .$$

On applying Lemma 4.2, with n replaced by $n-t-u$, we obtain

$$c_{t,u}(x, y) = X^tY^u\left\{1 - X^{n-t-u} - Y^{n-t-u} - \frac{X^{n-t-u} - Y^{n-t-u}}{X - Y} + \frac{X^{n+1-t-u} - Y^{n+1-t-u}}{X - Y}\right\}.$$

We may write this as

$$c_{t,u}(x, y) = X^tY^u - (X^{n-u-1}Y^u + X^{n-u-2}Y^{u+1} + \dots + X^{t+1}Y^{n-t-2} + X^tY^{n-t-1})$$

$$+ (X^{n-u-1}Y^{u+1} + X^{n-u-2}Y^{u+2} + \dots + X^{t+2}Y^{n-t-2} + X^{t+1}Y^{n-t-1}).$$

Note that each term in the first bracket is of total degree $n - 1$ and each term in the second bracket is of total degree n. So for all partial derivatives of $c_{t,u}(x, y)$ of order $\leq n - 2$ at $x = 1$, $y = 1$, there is zero contribution from all terms in these two brackets. Thus, for $0 \leq k + m \leq n - 2$,

$$\frac{\partial^{k+m}}{\partial x^k \partial y^m} c_{t,u}(x, y) = \frac{\partial^k}{\partial x^k}(1 - x)^t \frac{\partial^m}{\partial y^m}(1 - y)^u \quad \text{for } x = 1, y = 1$$

and this is clearly zero unless $k = t$ and $m = u$, when

$$\frac{\partial^{t+u}}{\partial x^t \partial y^u} c_{t,u}(x, y) = (-1)^{t+u}\, t!\, u! \qquad \text{for } x = y = 1.$$

We also have $c_{t,u}(0, 0) = 0$ for all t, u. Moreover, by writing

$$c_{t,u}(x, y) = X^t(Y^u - Y^{n-t-1}) - (1 - Y)(X^{n-u-1}Y^u + X^{n-u-2}Y^{u+1} + \dots + X^{t+1}Y^{n-t-2})$$

we find that $c_{t,u}(x, y)$ and all its derivatives with respect to x are zero when $y = 0$. Similarly, on writing

$$c_{t,u}(x, y) = (X^t - X^{n-u-1})Y^u - (1 - X)(X^{n-u-2}Y^{u+1} + X^{n-u-3}Y^{u+2} + \dots + X^tY^{n-t-1})$$

we see that $c_{t,u}(x, y)$ and all its derivatives with respect to y are zero when $x = 0$.

As a consequence of the properties satisfied by $a(x, y)$, $b_t(x, y)$ and $c_{t,u}(x, y)$ at (0, 0), (1, 0), (0, 1) and (1, 1), we have the following theorem.

Theorem 4.2 The polynomial $Q_n(x, y)$ satisfies the following $(n + 1)(n + 2)/2$ interpolation properties.

(a) $Q_n(0, 0) = f_{0,0}$.

(b) $Q_n(1, 0) = f_{1,0}$ and, for $1 \le k \le n - 1$, $\dfrac{\partial^k}{\partial x^k} Q_n(1, 0) = \dfrac{\partial^k}{\partial x^k} f(1, 0)$.

(c) $Q_n(0, 1) = f_{0,1}$ and, for $1 \le m \le n - 1$, $\dfrac{\partial^m}{\partial y^m} Q_n(0, 1) = \dfrac{\partial^m}{\partial y^m} f(0, 1)$.

(d) $Q_n(1, 1) = f_{1,1}$,

$$\frac{\partial^k}{\partial x^k} Q_n(1, 1) = \frac{\partial^k}{\partial x^k} f(1, 1) \quad \text{for } 1 \le k \le n - 2,$$

$$\frac{\partial^m}{\partial y^m} Q_n(1, 1) = \frac{\partial^m}{\partial y^m} f(1, 1) \quad \text{for } 1 \le m \le n - 2,$$

$$\frac{\partial^{k+m}}{\partial x^k \, \partial y^m} Q_n(1, 1) = \frac{\partial^{k+m}}{\partial x^k \, \partial y^m} f(1, 1) \quad \text{for } k, m \ge 1,\ k + m \le n - 2.$$

4.5 An alternative derivation of $Q_n(x, y)$

In Theorem 4.1, we established the existence of the approximating polynomial $Q_n(x, y)$ as a limiting form of the interpolating polynomial based on the q-triangle. We now show that $Q_n(x, y)$ can also be derived as the limiting form of the interpolating polynomial on a simpler system of nodes. Specifically let p, q > 0 and define a new "arithmetic mesh"

$$S = \{(x_i, y_j): i, j \geq 0, i + j \leq n\}$$

where

$$x_0 = 0, \;\; x_i = 1 + (i - 1)p, \;\; 1 \leq i \leq n$$

$$y_0 = 0, \; y_j = 1 + (j - 1)q, \quad 1 \leq j \leq n. \tag{4.13}$$

If f(x, y) is a function defined on a region which includes S then from Lemma 3.2 there exists a polynomial of degree at most n which interpolates f on S.

Since $x_1, \ldots, x_n$ are equally spaced we know from Lemma 1.3 that

$$[x_1, \ldots, x_i]\, f(\cdot, y) = \frac{1}{(i-1)!\, p^{i-1}} \Delta_x^{i-1} f(1, y) \tag{4.14}$$

where

$$\Delta_x^1 f(x, y) = f(x + p, y) - f(x, y)$$

and higher order differences are defined in the usual way.

We cannot extend relation (4.14) to include the point x_0 since, unless p = 1, the points $x_0, x_1, \ldots, x_n$ are not equally spaced. The following lemma will be used to overcome this difficulty.

Lemma 4.3 Let $x_1, x_2, \ldots$ be distinct non-zero real numbers. Then for $i \geq 1$ and a fixed value of y

$$[0, x_1, \ldots, x_i] f(\cdot, y) = \frac{(-1)^i}{x_i x_{i-1} \cdots x_1} f(0, y)$$

$$+ \sum_{k=1}^{i} \frac{(-1)^{k-1}}{x_i x_{i-1} \cdots x_{i-k+1}} [x_1, x_2, \ldots, x_{i-k+1}] f(\cdot, y) .$$

Proof We have by definition

$$[0, x_1] f(\cdot, y) = \frac{f(x_1, y)}{x_1} - \frac{f(0, y)}{x_1} ,$$

which shows the above statement holds for $i = 1$. Suppose the formula is true for a given $i \geq 1$. Then we have

$$[0, x_1, \ldots, x_{i+1}] f(\cdot, y)$$

$$= \frac{[x_1, x_2, \ldots, x_{i+1}] f(\cdot, y) - [0, x_1, \ldots, x_i] f(\cdot, y)}{x_{i+1}}$$

$$= \frac{[x_1, x_2, \ldots, x_{i+1}] f(\cdot, y)}{x_{i+1}}$$

$$- \frac{1}{x_{i+1}} \left\{ \frac{(-1)^i}{x_i x_{i-1} \cdots x_1} f(0, y) + \sum_{k=1}^{i} \frac{(-1)^{k-1}}{x_i x_{i-1} \cdots x_{i-k+1}} [x_1, \ldots, x_{i-k+1}] f(\cdot, y) \right\} .$$

We may combine the first term above with the summation to give

$$[0, x_1, \ldots, x_{i+1}] f(\cdot, y) = \frac{(-1)^{i+1}}{x_{i+1} x_i \cdots x_1} f(0, y)$$

$$+ \sum_{k=1}^{i+1} \frac{(-1)^{k-1}}{x_{i+1}\, x_i \cdots x_{i-k+2}} [x_1, x_2, \dots, x_{i-k+2}]\, f(\cdot, y)$$

where, in the original summation, k has been replaced by k − 1. Hence by the induction principle, the formula is true for all $i \geq 1$.

Let $x_1, x_2, \dots$ be defined as in (4.13) and let y be fixed. From (4.14) and Lemma 4.3 we have, for any $i \geq 1$,

$$[0, x_1, \dots, x_i]\, f(\cdot, y) = \frac{(-1)^i}{x_i\, x_{i-1} \cdots x_1} f(0, y)$$

$$+ \sum_{k=0}^{i-1} \frac{(-1)^{i-k-1}}{x_i\, x_{i-1} \cdots x_{k+1}} \frac{1}{k!\, p^k} \Delta_x^k f(1, y).$$

(In the summation in Lemma 4.3 we have replaced k by i − k.) Similarly, let $y_1, y_2, \dots$ be defined as in (4.13). Then for a fixed x and any integer $j \geq 1$

$$[0, y_1, \dots, y_j]\, f(x, \cdot) = \frac{(-1)^j}{y_j\, y_{j-1} \cdots y_1} f(x, 0)$$

$$+ \sum_{m=0}^{j-1} \frac{(-1)^{j-m-1}}{y_j\, y_{j-1} \cdots y_{m+1}} \frac{1}{m!\, q^m} \Delta_y^m f(x, 1)\,. \qquad (4.15)$$

We can now express mixed divided differences in terms of forward differences. First, apply the operator $[0, x_1, \dots, x_i]$ to each term of (4.15). We obtain

$$[0, x_1, \dots, x_i]\, [0, y_1, \dots, y_j]\, f$$

$$= \sum_{m=0}^{j-1} \frac{(-1)^{j-m-1}}{y_j\, y_{j-1} \cdots y_{m+1}} \frac{1}{m!\, q^m} \{\frac{(-1)^i}{x_i\, x_{i-1} \cdots x_1} \Delta_y^m f(0, 1)$$

$$+ \sum_{k=0}^{i-1} \frac{(-1)^{i-k-1}}{x_i\, x_{i-1} \cdots x_{k+1}} \frac{1}{k!\, p^k} \Delta_x^k \Delta_y^m f(1,1)\}$$

$$+ \frac{(-1)^j}{y_j\, y_{j-1} \cdots y_1} \{\frac{(-1)^i}{x_i\, x_{i-1} \cdots x_1} f(0,0)$$

$$+ \sum_{k=0}^{i-1} \frac{(-1)^{i-k-1}}{x_i\, x_{i-1} \cdots x_{k+1}} \frac{1}{k!\, p^k} \Delta_x^k f(1,0)\}.$$

Hence we can rearrange the summations to give

$$[0, x_1, \dots, x_i]\,[0, y_1, \dots, y_j]\, f$$

$$= \sum_{k=0}^{i-1} \sum_{m=0}^{j-1} \frac{(-1)^{i-k+j-m}}{(x_i\, x_{i-1} \cdots x_{k+1})\,(y_j\, y_{j-1} \cdots y_{m+1})} \frac{1}{k!\, m!\, p^k q^m} \Delta_x^k \Delta_y^m f(1,1)$$

$$+ \frac{(-1)^j}{y_j\, y_{j-1} \cdots y_1} \sum_{k=0}^{i-1} \frac{(-1)^{i-k-1}}{x_i\, x_{i-1} \cdots x_{k+1}} \frac{1}{k!\, p^k} \Delta_x^k f(1,0)$$

$$+ \frac{(-1)^i}{x_i\, x_{i-1} \cdots x_1} \sum_{m=0}^{j-1} \frac{(-1)^{j-m-1}}{y_j\, y_{j-1} \cdots y_{m+1}} \frac{1}{m!\, q^m} \Delta_y^m f(0,1)$$

$$+ \frac{(-1)^{i+j}}{(x_i\, x_{i-1} \cdots x_1)\,(y_j\, y_{j-1} \cdots y_1)} f(0,0).$$

We know from the one-dimensional case that

$$\lim_{p \to 0} \frac{1}{p^r} \Delta_x^r f(1,y) = \frac{\partial^r}{\partial x^r} f(1,y) \quad \text{and} \quad \lim_{q \to 0} \frac{1}{q^s} \Delta_y^s f(x,1) = \frac{\partial^s}{\partial x^s} f(x,1)\,.$$

It follows that

$$\lim_{p\to 0}\lim_{q\to 0}\frac{1}{p^r\,q^s}\;\Delta_x^r\,\Delta_y^s\,f(1,1) \;=\; \lim_{p\to 0}\;\{\frac{1}{p^r}\Delta_x^r\;\{\lim_{q\to 0}\frac{1}{q^s}\Delta_y^s\,f(1,1)\}\}$$

$$= \frac{\partial^{r+s}}{\partial x^r \partial y^s}\,f(1,1)\;.$$

We deduce that

$$\lim_{p\to 0}\;[0, x_1\,,\,\dots\,,x_i]\;f(\cdot, y) \;=\; (-1)^i\,f(0, y) \;+\; (-1)^i\sum_{k=0}^{i-1}\frac{(-1)^{k+1}}{k!}\frac{\partial^k}{\partial x^k}\,f(1, y), \tag{4.16}$$

$$\lim_{q\to 0}\;[0, y_1\,,\,\dots\,,y_j]\;f(x, \cdot) \;=\; (-1)^j\,f(x, 0) \;+\; (-1)^j\sum_{m=0}^{j-1}\frac{(-1)^{m+1}}{m!}\frac{\partial^m}{\partial y^m}\,f(x, 1) \tag{4.17}$$

and

$$\lim_{p\to 0}\lim_{q\to 0}\;[0, x_1\,,\,\dots\,,x_i]\;[0, y_1\,,\,\dots\,,y_j\,]\;f$$

$$= (-1)^{i+j}\,f(0,0) \;+\; (-1)^{i+j}\sum_{k=0}^{i-1}\;\sum_{m=0}^{j-1}\;\frac{(-1)^{k+m}}{k!\;m!}\;\frac{\partial^{k+m}}{\partial x^k\,\partial y^m}\,f(1,1)$$

$$+ \;(-1)^{i+j}\sum_{k=0}^{i-1}\;\frac{(-1)^{k+1}}{k!}\frac{\partial^k}{\partial x^k}\,f(1,0) \;+\; (-1)^{i+j}\sum_{m=0}^{j-1}\;\frac{(-1)^{m+1}}{m!}\frac{\partial^m}{\partial y^m}\,f(0,1). \tag{4.18}$$

We are now ready to find the limit of the interpolating polynomial $P_n(x, y)$ on the set S defined in (4.13). From Lemma 3.2 the polynomial can be written as

$$P_n(x, y) \;=\; \sum_{i=0}^{n}\;\sum_{j=0}^{n-i}\;\prod_{\mu=0}^{i-1}(x - x_\mu)\;\prod_{\nu=0}^{j-1}(y - y_\nu)\;\;[x_0, x_1\,,\,\dots\,,x_i]\;[y_0, y_1\,,\,\dots\,,y_j]\;f$$

$$= \;f(0,0) \;+\; \sum_{i=1}^{n}\;\prod_{\mu=0}^{i-1}(x - x_\mu)\;[0, x_1\,,\,\dots\,,x_i]\;\;f(\cdot, 0)$$

$$+ \sum_{j=1}^{n} \prod_{\nu=0}^{j-1} (y - y_\nu)\, [0, y_1, \dots, y_j]\, f(0, \cdot)$$

$$+ \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \prod_{\mu=0}^{i-1} (x - x_\mu) \prod_{\nu=0}^{j-1} (y - y_\nu)\ [0, x_1, \dots, x_i]\, [0, y_1, \dots, y_j]\, f.$$

Let $p \to 0$ and $q \to 0$ and apply (4.16), (4.17) and (4.18). We obtain

$$\lim_{p\to 0} \lim_{q\to 0} P_n(x, y) = f_{0,0} + \sum_{i=1}^{n} x(x-1)^{i-1} (-1)^i \{ f_{0,0} + \sum_{k=0}^{i-1} (-1)^{k+1} \frac{1}{k!} \frac{\partial^k}{\partial x^k} f_{1,0} \}$$

$$+ \sum_{j=1}^{n} y(y-1)^{j-1} (-1)^j \{ f_{0,0} + \sum_{m=0}^{j-1} (-1)^{m+1} \frac{1}{m!} \frac{\partial^m}{\partial y^m} f_{0,1} \}$$

$$+ \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} xy(x-1)^{i-1}(y-1)^{j-1} (-1)^{i+j} \{ f_{0,0} + \sum_{k=0}^{i-1} (-1)^{k+1} \frac{1}{k!} \frac{\partial^k}{\partial x^k} f_{1,0}$$

$$+ \sum_{m=0}^{j-1} (-1)^{m+1} \frac{1}{m!} \frac{\partial^m}{\partial y^m} f_{0,1} + \sum_{k=0}^{i-1} \sum_{m=0}^{j-1} (-1)^{k+m} \frac{1}{k!\ m!} \frac{\partial^{k+m}}{\partial x^k \partial y^m} f_{1,1} \} .$$

This is indeed the same polynomial as the polynomial $Q_n(x, y)$ defined in (4.12).

*****

Chapter 5

Integration rules of interpolatory type on a triangle

5.1 Introduction

In this chapter we shall use interpolating polynomials to study integration rules on the triangle $S_n = \{(x, y): 0 \leq x \leq y \leq [n]\}$. Many formulae for approximating double integrals have the form (see [23])

$$\iint_B K(x, y)\, f(x, y)\, dx\, dy \approx \sum_{i=0}^{N} w_i\, f(a_i, b_i)$$

where B is a given closed region in $\mathbb{R}^2$, where $K(x, y)$ is a fixed positive weight function (often $K(x, y) = 1$), where (a_i, b_i) are points which lie in B and where w_i are constants. R. Lauffer [11] obtained certain integration rules on the general simplex. Special cases of Lauffer's rules (taking the dimension of the simplex to be 2) are special cases of the rules to be obtained here (taking the parameter q to be 1). Given a function $f(x, y)$ on S_n we shall find an integration rule I_n which uses the approximation

$$\iint_{S_n} f(x, y)\, dx\, dy \approx \iint_{S_n} P_n(x, y)\, dx\, dy\,.$$

Here $P_n(x, y)$ is the interpolating polynomial on S_n constructed at the nodes ([i], [j]), $0 \le i \le j \le n$. Note that the rule I_n is exact if the function f is a polynomial of degree at most n in x and y.

We know from Lee and Phillips [13] (see Theorem 1.6) that this interpolating polynomial exists and

$$P_n(x, y) = \sum_{j=0}^{n} \sum_{i=0}^{j} L_{i,j}^{n}(x, y)\, f_{i,j} \tag{5.1}$$

where the Lagrange coefficient $L_{i,j}^{n}(x, y)$ takes the form

$$L_{i,j}^{n}(x, y) = \frac{q^{-(2n-j-1)j/2}}{[i]!\,[j-i]!\,[n-j]!} \prod_{\nu=0}^{i-1} (x - [\nu]) \prod_{\nu=j+1}^{n} ([\nu] - y) \prod_{\nu=0}^{j-i-1} (y - q^{\nu}x - [\nu]). \tag{5.2}$$

Integrating (5.1) over S_n, we obtain

$$\iint_{S_n} P_n(x, y)\, dx\, dy = \sum_{j=0}^{n} \sum_{i=0}^{j} w_{i,j}^{n} f_{i,j} = I_n(f) \tag{5.3}$$

say, where $w_{i,j}^{n}$ is called the weight at the node ([i], [j]) and is given by

$$w_{i,j}^{n} = \iint_{S_n} L_{i,j}^{n}(x, y)\, dx\, dy\,. \tag{5.4}$$

Note that, taking f = 1, the weights (5.4) clearly satisfy

$$\sum_{j=0}^{n} \sum_{i=0}^{j} w_{i,j}^{n} = \iint_{S_n} dx\, dy = A\,,$$

the area of the triangle S_n. The first aim of this chapter is to use this method to calculate the weights for the integration rule I_n, n = 1, 2, ..., 5 directly from formula (5.4). Later we will discuss an alternative method of obtaining these weights $w^n_{i,j}$. We also study certain properties which govern the weights $w^n_{i,j}$ on the triangle. We will verify that all the weights of I_1, I_3 and I_5 are positive for certain values of q including q = 1. We will also consider the expression $w^n_{i,j}(1/q)$, and show that these weights satisfy a kind of symmetric property. In the last section we study a relation between integration rules over certain triangles of the same order.

5.2 The integration rules - in terms of a parameter q

If we let q = 1, the integration rule I_n in (5.3) reduces to the Lauffer rule. We tabulate below the relative weights for the Lauffer rules with n = 1, 2 and 3. For a given n, the actual weight $w^n_{i,j}$ is obtained on multiplying the relative weight by the factor α_n displayed below, where A denotes the area of the triangle. See also Phillips [16].

						4	9	9	4
		0	1	0		9	54	9	
1	1	1	1			9	9		
1		0				4			
$\alpha_1 = A/3$		$\alpha_2 = A/3$				$\alpha_3 = A/120$			
(a)		(b)				(c)			

Figure 5.1

We now give an integration rule on the triangle S_n for a general value of q > 0. For n = 1, 2, 3, 4, 5 we use Maple to calculate directly the weights $w^n_{i,j}$. See Appendix 5A for the details. Note that if n = 1, all three weights $w^1_{i,j}$ are independent of q and thus we simply have the Lauffer rules given in Figure 5.1(a). Using the same format as in Figure 5.1, the weights required for I_n, n = 2 and 3, are given in Figures 5.2 and 5.3 below.

	[0]	[1]	[2]
[2]	$-(-1+q)^2$	$[2]^2$	$2(-1+q)$
[1]	$[2]^2$	$[2]^2$	
[0]	$-2q(-1+q)$		

$$\alpha_2 = \frac{A}{12q}$$

Figure 5.2 Weights on the triangle S_2

	[0]	[1]	[2]	[3]
[3]	$2[2]\,F(q)$	$-[3]^2\,(2q^3-2q^2-2q+1)$	$[3]^2\,H(q)$	$2\,G(q)$
[2]	$-[3]^2\,(q^3-2q^2-2q+2)$	$[2][3]^3$	$[3]^2\,H(q)$	
[1]	$-q[3]^2\,(3q^2-2q-2)$	$-q[3]^2\,(3q^2-2q-2)$		
[0]	$2q^3\,G(q)$			

$$\alpha_3 = \frac{A}{60[2]q^3}.$$

Figure 5.3 Weights on the triangle S_3

In order to write the above weights in a compact form, we have written

$$F(q) = q^6 - q^5 - 2q^4 + 5q^3 - 2q^2 - q + 1$$
$$G(q) = 3q^4 + q^3 - 6q^2 + q + 3$$
$$H(q) = 2q^2 + 2q - 3.$$

Calculation for the rules I_n, n = 4 and 5 involves polynomials of higher degree. However the weights for these rules satisfy

$$w^4_{4-j,4-i}(1/q) = q^{-6}\, w^4_{i,j}(q), \qquad w^5_{5-j,5-i}(1/q) = q^{-8}\, w^5_{i,j}(q).$$

We will show this "symmetric" property later. Let $C^n_{i,j}$ be such that $w^n_{i,j} = \alpha_n \times C^n_{i,j}$, where α_n is a factor chosen so that the relative weights $C^n_{i,j}$ are polynomials in q. Then the rules I_n, n = 4, 5 can be summarised as follows. (Because the expressions for the relative weights $C^4_{i,j}$ and $C^5_{i,j}$ are somewhat lengthy, we have had to give up the triangular lay-out which we have used above for n =2 and 3.)

$$C^4_{0,0} = -6\,[2]\,q^6\,(q-1)\,(2q^6 + q^5 + q^4 - 3q^3 + q^2 + q + 2)$$

$$C^4_{0,1} = C^4_{1,1} = 3\,[2]\,[4]\,q^3\,(q^2+1)\,(2q^5 - q^4 - 2q^3 + q + 1)$$

$$C^4_{0,2} = q\,[3]\,(q^2+1)^2\,(2q^7 - 5q^6 - 3q^5 - q^4 + 9q^3 + q^2 - q - 3)$$
$$C^4_{1,2} = -q\,[3]\,[4]^2\,(q^2+1)\,(2q^3 - q^2 - q - 1)$$
$$C^4_{2,2} = -q\,[3]\,(q^2+1)^2\,(4q^6 + 2q^5 - 11q^3 + 2q + 4)$$

$$C^4_{0,3} = [2]\,[4]\,(q^2+1)\,(q^8 - q^7 - 2q^6 + 2q^5 + 5q^4 + q^3 - 4q^2 - 2q + 3)$$
$$C^4_{1,3} = -[3]\,[4]^2\,(q^2+1)\,(q^4 - q^3 - q^2 - q + 1)$$

$$C^4_{0,4} = -[2]^2\,(3q^{10} + q^9 + 3q^8 - 5q^7 + 9q^6 + 3q^5 + 9q^4 - 5q^3 + 3q^2 + q + 3)\,(q-1)^2$$

$$\alpha_4 = \frac{A}{180[3]q^6}.$$

Table 5.1 Weights on the triangle S_4

$$C^5_{0,0} = 6q^{10}(10q^{16} + 26q^{15} + 30q^{14} + 11q^{13} - 35q^{12} - 54q^{11} - 27q^{10} + 30q^9 + 73q^8 + 30q^7 - 27q^6 - 54q^5 - 35q^4 + 11q^3 + 30q^2 + 26q + 10)$$

$$C^5_{0,1} = C^5_{1,1} = -3q^6[5]^2(10q^{12} + 16q^{11} + 4q^{10} - 19q^9 - 32q^8 - 2q^7 + 22q^6 + 26q^5 + 10q^4 - 12q^3 - 17q^2 - 12q - 4)$$

$$C^5_{0,2} = -q^3[5]^2(1+q^2)(10q^{13} - 14q^{12} - 44q^{11} - 17q^{10} + 4q^9 + 59q^8 + 56q^7 - 19q^6 - 40q^5 - 49q^4 - 9q^3 + 18q^2 + 18q + 12)$$

$$C^5_{1,2} = q^3[4][5]^3(10q^8 + 6q^7 - 12q^6 - 9q^5 - 6q^4 + 5q^3 + 9q^2 + 6q + 3)$$

$$C^5_{2,2} = q^3[5]^2(1+q^2)(20q^{12} + 32q^{11} + 8q^{10} - 10q^9 - 64q^8 - 53q^7 + 37q^6 + 52q^5 + 55q^4 + 4q^3 - 27q^2 - 24q - 15)$$

$$C^5_{0,3} = -q[5]^2(1+q^2)(5q^{15} - 2q^{14} - 17q^{13} + 4q^{12} + 19q^{11} + 48q^{10} + 23q^9 - 40q^8 - 49q^7 - 46q^6 + 19q^5 + 39q^4 + 12q^3 - 18q - 12)$$

$$C^5_{1,3} = q[3][5]^3(1+q^2)(5q^9 - 7q^8 - 5q^7 - 3q^6 - q^5 + 12q^4 + 3q^3 + q^2 - q - 3)$$

$$C^5_{2,3} = -q[3][5]^2(1+q^2)(5q^{12} + 8q^{11} + 9q^{10} + 8q^9 - 9q^8 - 15q^7 - 17q^6 - 15q^5 - 9q^4 + 8q^3 + 9q^2 + 8q + 5)$$

$$C^5_{0,4} = -[5](3q^{22} + 7q^{21} + 5q^{20} - 5q^{19} - 15q^{18} + 2q^{17} + 28q^{16} + 30q^{15} - 10q^{14} - 80q^{13} - 120q^{12} - 106q^{11} - 44q^{10} + 27q^9 + 65q^8 + 55q^7 + 11q^6 - 33q^5 - 42q^4 + 30q^2 + 30q + 12)$$

$$C^5_{1,4} = [2][4][5]^3(3q^{10} - 2q^9 - 4q^8 + q^7 - q^6 + 12q^5 - q^4 + q^3 - 4q^2 - 2q + 3)$$

$$C^5_{0,5} = 6(2q^{26} + 5q^{25} + 5q^{24} - 10q^{22} - 9q^{21} + 5q^{20} + 20q^{19} + 20q^{18} - 15q^{17} - 33q^{16} - 15q^{15} + 25q^{14} + 55q^{13} + 25q^{12} - 15q^{11} - 33q^{10} - 15q^9 + 20q^8 + 20q^7 + 5q^6 - 9q^5 - 10q^4 + 5q^2 + 5q + 2)$$

$$\alpha_5 = \frac{A}{1260\, q^{10}\, [4]!}.$$

Table 5.2 Weights on the triangle S_5

It is easily verified that for q = 1 the rules given in Figures 5.2 and 5.3 do indeed coincide with Lauffer's rules for n = 2 and 3 given in Figure 5.1. On putting q = 1 in the above rules given for n = 4 and 5, we obtain the rules given below. These are not given in Lauffer [11].

0	4	−1	4	0
4	8	8	4	
−1	8	−1		
4	4			
0				

$\alpha_4 = A/45$

11	25	25	25	25	11
25	200	25	200	25	
25	25	25	25		
25	200	25			
25	25				
11					

$\alpha_5 = A/1008$

Figure 5.4

5.3 Positive weights on the triangle S_n

For each of the rules discussed above we will determine whether there are values of q for which all weights are positive or at least all non-negative. The case n = 1 is trivial: the weights are independent of q and are all positive. However the weights on triangles S_2 and S_4 do not possess this property. In fact

$$w_{0,2}^2 = -\frac{A}{12q}(q-1)^2 < 0 \quad \text{for all } q > 0,\, q \neq 1.$$

Thus the weights of I_2 are never all positive and they are all non-negative only for q = 1. For the rule I_4, let us examine the weight

$$w_{2,2}^4 = -\frac{A}{180q^5}(q^2+1)^2(4q^6+2q^5-11q^3+2q+4).$$

Since $q^{2m} \geq 2\,q^m - 1$ then for all $q > 0$

$$4q^6 + 2q^5 - 11q^3 + 2q + 4 \;\geq\; 4(2q^3 - 1) + 2(2q^3 - q) - 11q^3 + 2q + 4$$
$$= q^3 \;>\; 0.$$

This shows that the weights of I_4 are never all non-negative.

For $n = 3$ and $n = 5$ we have already seen that the weights are all positive when $q = 1$. We know that if there is a q such that all weights of I_n are positive then, by continuity, there is an interval containing this value of q for which all weights of I_n are positive.

Let us consider the rule I_3. Since (see Figure 5.3) $\alpha_3 > 0$ for all $q > 0$, it suffices to examine the positivity of the relative weights given in Figure 5.3 . First it is clear that

$$w^3_{1,2} = \alpha_3\,[2]\,[3]^3 > 0 \text{ for all } q > 0.$$

Next consider the function G(q) which is quoted in Figure 5.3 . We have

$$G(q) \;=\; 3q^4 + q^3 - 6q^2 + q + 3 \;=\; 3(q^2 - 1)^2 + q^3 + q \;>\; 0 \quad \text{for all } q > 0$$

and thus $w^3_{0,0}$ and $w^3_{3,3}$ are positive for all $q > 0$. We also have (again see Figure 5.3)

$$\frac{1}{q^3}F(q) \;=\; \frac{1}{q^3}(q^6 - q^5 - 2q^4 + 5q^3 - 2q^2 - q + 1).$$

On putting $\sigma = q + \frac{1}{q} - 2$

$$\frac{1}{q^3}F(q) \;=\; \sigma^3 + 5\,\sigma^2 + 3\,\sigma + 1 > 0.$$

for all $q > 0$, since $\sigma \geq 0$ for all $q > 0$. Hence $w^3_{0,3} > 0$ for all $q > 0$. We also see that (with $q > 0$) the polynomial $-3q^2 + 2q + 2$ is positive for $q < (\sqrt{7} + 1)/3$ and thus the polynomial $2q^2 + 2q - 3$ is positive for $q > 3/(\sqrt{7} + 1) = (\sqrt{7} - 1)/2$. Thus the weights $w^3_{0,1}$ and $w^3_{2,2}$ are positive for

$$\frac{\sqrt{7} - 1}{2} < q < \frac{\sqrt{7} + 1}{3} \tag{5.5}$$

and we note that $w^3_{1,1} = w^3_{0,1}$ and $w^3_{2,3} = w^3_{2,2}$. It remains to examine the weights $w^3_{0,2}$ and $w^3_{1,3}$. These are positive if and only if $-q^3 + 2q^2 + 2q - 2$ and

$$-2q^3 + 2q^2 + 2q - 1 = q^3(-q^{-3} + 2q^{-2} + 2q^{-1} - 2)$$

are both positive. By direct calculation we find that this holds in the intersection of the two intervals $(0.68889, 2.48119)$ and $(1/2.48119, 1/0.68889)$, approximately, where the numbers have been given to five decimal places. Since both these intervals contain the interval defined by (5.5) above, it follows that all weights for the rule I_3 are positive if q satisfies the inequality in (5.5).

Now let us consider the weights of the rule I_5. In this case more extensive calculations are necessary because the degrees of the polynomials involved are higher. To find intervals for which the weights of I_5 are positive, we need only consider the polynomials obtained by dividing each $C^5_{i,j}$ by its obvious positive factors. By using the Maple "fsolve" command to find the roots of the polynomials and comparing with their graphs, first we find that

$$w^5_{0,0},\ w^5_{1,2},\ w^5_{1,4} \text{ and } w^5_{0,5} \quad \text{are positive for all } q > 0.$$

(See Table 5.5 and Figures 5.5 and 5.6 .) This is also true for $w^5_{3,4}$ and $w^5_{5,5}$, since if a function $f(q)$ is positive for $0 < a < q < b$ then $f(1/q)$ is positive for $1/b < q < 1/a$. Intervals on which the other weights of I_5 are positive are obtained similarly and these results are shown in Tables 5.3 and 5.4 below. (See also Appendix 5B for details.)

Weights $w^5_{i,j}$	A factor of $w^5_{i,j}$ considered	Interval on which $w^5_{i,j} > 0$
$w^5_{0,1}$, $w^5_{1,1}$	$-\frac{C^5_{0,1}}{3q^6[5]^2}$	(0, 1.05974)
$w^5_{0,2}$	$-\frac{C^5_{0,2}}{q^3[5]^2(1 + q^2)}$	(0.92351, 2.96624)
$w^5_{2,2}$	$\frac{C^5_{2,2}}{q^3[5]^2(1 + q^2)}$	(0.94273, ∞)
$w^5_{0,3}$	$-\frac{C^5_{0,3}}{q[5]^2(1 + q^2)}$	(0, 1.07591)
$w^5_{1,3}$	$\frac{C^5_{1,3}}{q[3][5]^3(1 + q^2)}$	(0.73974, 1.05853)$\cup$(1.91397, ∞)
$w^5_{2,3}$	$-\frac{C^5_{2,3}}{q[3][5]^2(1 + q^2)}$	(0.90951, 1.09949)
$w^5_{0,4}$	$-\frac{C^5_{0,4}}{[5]}$	(0.88838, 1.38397)

<u>Table 5.3</u>

As we will verify later, the weights $w^5_{i,j}$ satisfy

$$w^5_{5-j,5-i}\,(1/q) = \frac{1}{q^8}\; w^5_{i,j}\,(q).$$

This allow us to determine intervals for which the remaining weights $w^5_{i,j}$ are positive. These intervals, which can be deduced from Table 5.3, are as follows.

Weights $w^5_{i,j}$	Interval for $w^5_{i,j} > 0$	Weights $w^5_{i,j}$	Interval for $w^5_{i,j} > 0$
$w^5_{2,4}$	$(0, 0.52247) \cup (0.94471, 1.35183)$	$w^5_{1,5}$	$(0.72256, 1.12564)$
$w^5_{3,3}$	$(0, 1.06075)$	$w^5_{2,5}$	$(0.92945, \infty)$
$w^5_{4,4}$, $w^5_{4,5}$	$(0.94363, \infty)$	$w^5_{3,5}$	$(0.33713, 1.08283)$

Table 5.4

Thus all weights of the rule I_5 are positive if $0.94471 < q < 1/0.94471 = 1.05853$, approximately.

Thus we have found that for $q = 1$ the weights of I_3 and I_5 are positive and have derived the largest interval around $q = 1$, in each case, for which the weights are still all positive. We might be tempted to conjecture that this will hold for all I_n, with n odd. Let us consider the weights associated with the interpolating polynomial on equally spaced nodes over the triangle $\{(x, y): 0 \le x \le y \le 7\}$. This is the integration rule I_7 in (5.3) with $q = 1$. From (5.2) we have

$$P_7(x, y) = \sum_{j=0}^{7} \sum_{i=0}^{j} L^7_{i,j}(x, y)\, f_{i,j}$$

where

$$L^n_{i,j}(x, y) = \frac{1}{i!\,(j-i)!\,(n-j)!} \prod_{v=0}^{i-1} (x - v) \prod_{v=j+1}^{n} (v - y) \prod_{v=0}^{j-i-1} (y - x - v)\,.$$

We use Maple again to calculate the weights $I^7_{i,j}$ and these are shown in Appendix 5C. We find that the weight $I^7_{1,3}$ (and also $I^7_{2,3}$, $I^7_{1,5}$, $I^7_{2,6}$, $I^7_{4,5}$, $I^7_{4,6}$) is negative. This shows the above conjecture to be false.

- **Roots for the weight W02 (n = 3).**

```
W02 = A [3]^2 (-q^3 + 2*q^2 + 2*q - 2) / (60 [2] q^3)
```

```
• fsolve(-q^3 + 2*q^2 + 2*q - 2, q);
```

$$-1.170086487,\ .6888921825,\ 2.481194304$$

- **Roots for the weights W00,W12,W14 and W05 (n = 5).**
Let Bij denote the polynomial factor considered in Table 5.2 (obtained by dividing Cij by its obvious positive factors).

•

$$B00 := 10 q^{16} + 26 q^{15} + 30 q^{14} + 11 q^{13} - 35 q^{12} - 54 q^{11} - 27 q^{10} + 30 q^{9} + 73 q^{8} + 30 q^{7} - 27 q^{6} - 54 q^{5} - 35 q^{4} + 11 q^{3} + 30 q^{2} + 26 q + 10$$

$$B12 := 10 q^{8} + 6 q^{7} - 12 q^{6} - 9 q^{5} - 6 q^{4} + 5 q^{3} + 9 q^{2} + 6 q + 3$$

$$B14 := 3 q^{10} - 2 q^{9} - 4 q^{8} + q^{7} - q^{6} + 12 q^{5} - q^{4} + q^{3} - 4 q^{2} - 2 q + 3$$

$$B05 := 2 + 5 q - 9 q^{5} - 10 q^{22} - 9 q^{21} + 5 q^{20} + 20 q^{19} + 20 q^{18} - 15 q^{17} - 33 q^{16} - 15 q^{15} + 25 q^{14} + 55 q^{13} + 25 q^{12} - 15 q^{11} - 33 q^{10} - 15 q^{9} + 20 q^{8} + 20 q^{7} + 5 q^{6} - 10 q^{4} + 5 q^{2} + 5 q^{25} + 2 q^{26} + 5 q^{24}$$

```
• fsolve(B00, q);
```

```
• fsolve(B12, q);
```

$$-1.281205350,\ -.7259561389$$

```
• fsolve(B05, q);
```

```
• fsolve(B14, q);
```

$$-1.469135153,\ -.6806725696$$

Table 5.5

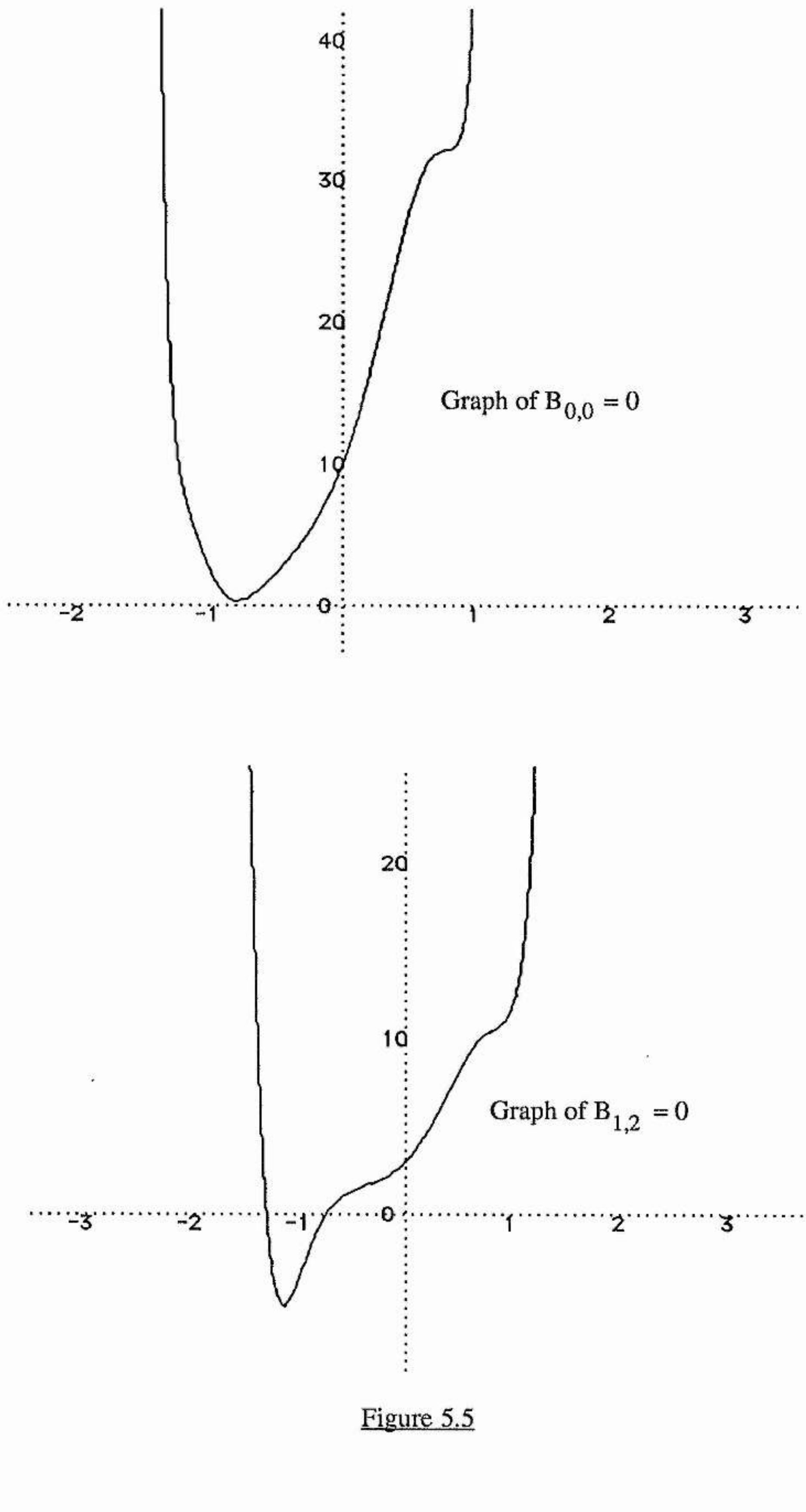
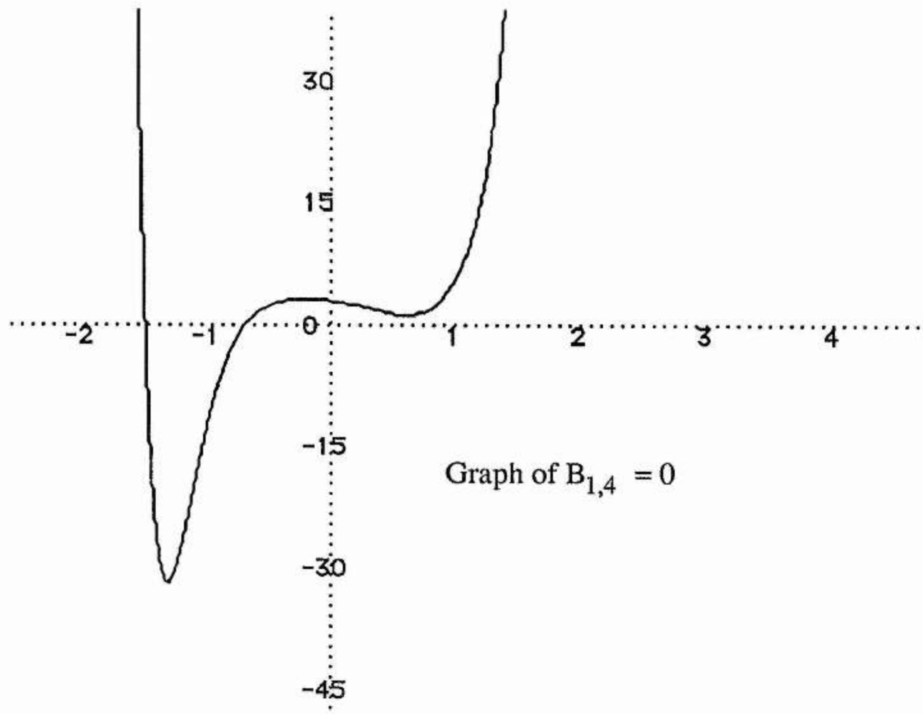


Figure 5.5


30
15
-2
-1
0
1
2
3
4
-15
Graph of $B_{1,4} = 0$
-30
-45

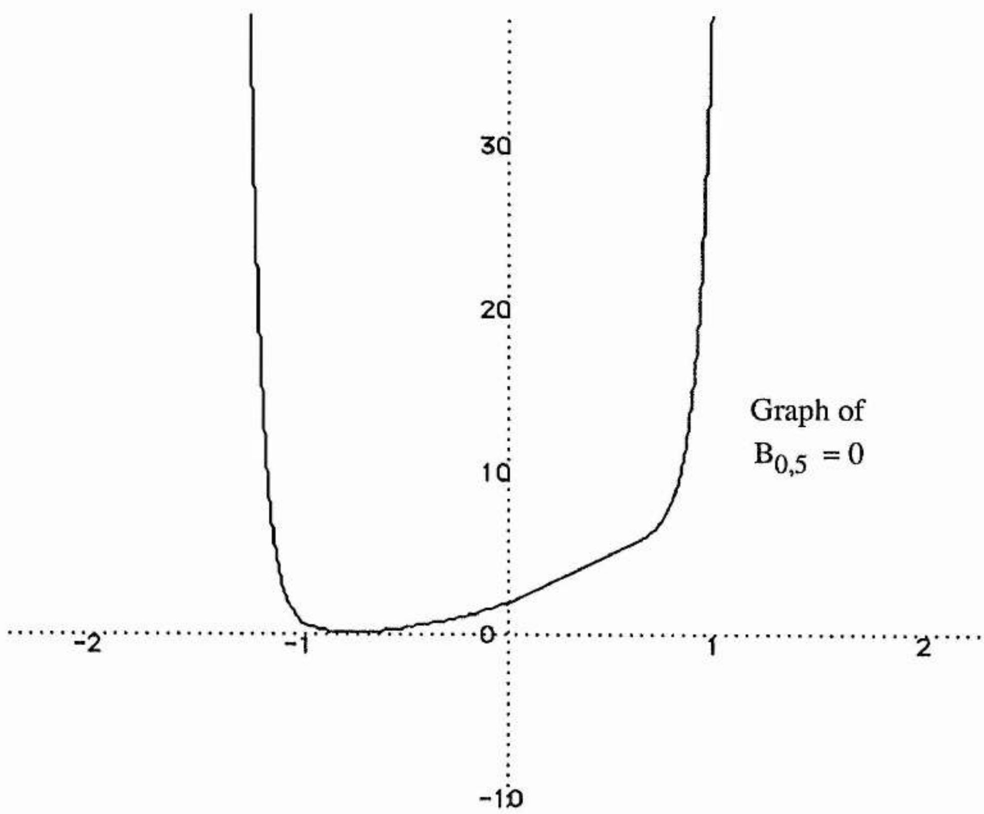

30
20
Graph of
$B_{0,5} = 0$
10
-2
-1
0
1
2
-10

Figure 5.6

5.4 Symmetric weights

Referring to Figures 5.2 and 5.3, we see that the weights on the triangles S_2 and S_3 possess some kind of symmetric property. By using a command in Maple: "normal (subs (q = 1/q, $w^2_{i,j}$))", we calculate $w^2_{i,j}(1/q)$ and normalise it. We find that

$$w^2_{2,2}\left(\frac{1}{q}\right) = \frac{1}{q^2}\, w^2_{0,0}\,(q), \qquad w^2_{1,2}\left(\frac{1}{q}\right) = \frac{1}{q^2}\, w^2_{0,1}\,(q)$$

$$w^2_{1,1}\left(\frac{1}{q}\right) = \frac{1}{q^2}\, w^2_{1,1}\,(q) \quad \text{and} \quad w^2_{0,2}\left(\frac{1}{q}\right) = \frac{1}{q^2}\, w^2_{0,2}\,(q).$$

We show further that this property holds in general.

Theorem 5.1 Let $w^n_{i,j}(q)$ be the weight at the node ([i], [j]) for the rule I_n. Then

$$w^n_{n-j,n-i}\left(\frac{1}{q}\right) = \frac{1}{q^{2(n-1)}}\, w^n_{i,j}\,(q)\,.$$

In order to prove Theorem 5.1 above we need the following lemma. First let us write $L^n_{i,j}(x, y; q)$, for $0 \le i \le j \le n$, to denote the Lagrange coefficients of the interpolating polynomial $P_n(x, y)$ on the triangle S_n, where we have emphasised the dependence of $L^n_{i,j}$ on the parameter q.

Lemma 5.1 For $0 \le x \le y \le q^{1-n}\,[n]$, $0 \le \xi \le \eta \le [n]$ and $0 \le i \le j \le n$,

$$L^n_{n-j,n-i}\left(x, y; \frac{1}{q}\right) = L^n_{i,j}\,(\xi, \eta; q)$$

where

$$\xi = [n] - q^{n-1}\, y \quad \text{and} \quad \eta = [n] - q^{n-1}\, x\,.$$

Proof First we note from (5.2) that the constant which precedes the three products is a normalising factor. We may rewrite $L^n_{i,j}(x, y; q)$ in the form

$$L^n_{i,j}(x, y; q) = \prod_{\nu=0}^{i-1}\left(\frac{x-[\nu]}{[i]-[\nu]}\right)\prod_{\nu=j+1}^{n}\left(\frac{[\nu]-y}{[\nu]-[j]}\right)\prod_{\nu=0}^{j-i-1}\left(\frac{y-q^{\nu}x-[\nu]}{[j]-q^{\nu}[i]-[\nu]}\right)$$

$$= \prod_{\nu=0}^{i-1}\left(\frac{x-[\nu]}{q^{\nu}[i-\nu]}\right)\prod_{\nu=j+1}^{n}\left(\frac{[\nu]-y}{q^{j}[\nu-j]}\right)\prod_{\nu=0}^{j-i-1}\left(\frac{y-q^{\nu}x-[\nu]}{q^{i+\nu}[j-i-\nu]}\right).$$

Then for $0 \le x \le y \le [n]$ we have

$$L^n_{n-j,n-i}(x, y; q)$$

$$= \prod_{\nu=0}^{n-j-1}\left(\frac{x-[\nu]}{q^{\nu}[n-j-\nu]}\right)\prod_{\nu=n-i+1}^{n}\left(\frac{[\nu]-y}{q^{n-i}[\nu-n+i]}\right)\prod_{\nu=0}^{j-i-1}\left(\frac{y-q^{\nu}x-[\nu]}{q^{n-j+\nu}[j-i-\nu]}\right).$$

By making a change of variable $q \to \frac{1}{q}$ the domain changes to $0 \le x \le y \le q^{1-n}[n]$ which is where $L^n_{n-j,n-i}(x, y; 1/q)$ is defined. Thus for $0 \le x \le y \le q^{1-n}[n]$

$$L^n_{n-j,n-i}(x, y; 1/q) = \prod_{\nu=0}^{n-j-1}\left(\frac{x-q^{1-\nu}[\nu]}{q^{-n+j+1}[n-j-\nu]}\right)\prod_{\nu=n-i+1}^{n}\left(\frac{q^{1-\nu}[\nu]-y}{q^{1-\nu}[\nu-n+i]}\right)$$

$$\times \prod_{\nu=0}^{j-i-1}\left(\frac{y-q^{-\nu}x-q^{1-\nu}[\nu]}{q^{-n+i+1}[j-i-\nu]}\right). \tag{5.6}$$

Now consider a transformation $(x, y) \to (\xi, \eta)$ given by

$$x = \frac{[n]-\eta}{q^{n-1}}, \quad y = \frac{[n]-\xi}{q^{n-1}}.$$

Then each point $([n-j], [n-i])$ which changed into $\left(\frac{[n-j]}{q^{n-j-1}}, \frac{[n-i]}{q^{n-i-1}}\right)$ by $q \to \frac{1}{q}$, is transformed into $([i], [j])$ and the inequalities $0 \le x \le y \le q^{1-n}[n]$ correspond to $0 \le \xi \le \eta \le [n]$. Each product of $L^n_{n-j,n-i}(x, y; 1/q)$ is thus transformed as follows.

$$\prod_{\nu=0}^{n-j-1} \frac{x - q^{1-\nu}[\nu]}{q^{-n+j+1}[n-j-\nu]} = \prod_{\nu=0}^{n-j-1} \frac{q^{n-1}x - q^{n-\nu}[\nu]}{q^{j}[n-j-\nu]}$$

$$= \prod_{\nu=0}^{n-j-1} \frac{[n-\nu]-\eta}{q^{j}[n-j-\nu]} = \prod_{\mu=j+1}^{n} \frac{[\mu]-\eta}{q^{j}[\mu-j]}$$

and

$$\prod_{\nu=n-i+1}^{n} \frac{q^{1-\nu}[\nu]-y}{q^{1-\nu}[\nu-n+i]} = \prod_{\nu=n-i+1}^{n} \frac{q^{n-\nu}[\nu]-q^{n-1}y}{q^{n-\nu}[\nu-n+i]}$$

$$= \prod_{\nu=n-i+1}^{n} \frac{\xi-[n-\nu]}{q^{n-\nu}[\nu-n+i]} = \prod_{\mu=0}^{i-1} \frac{\xi-[\mu]}{q^{\mu}[i-\mu]}$$

where we have denoted $\mu = n - \nu$. Finally

$$\prod_{\nu=0}^{j-i-1} \frac{y - q^{-\nu}x - q^{1-\nu}[\nu]}{q^{-n+i+1}[j-i-\nu]} = \prod_{\nu=0}^{j-i-1} \frac{q^{n-1}y - q^{n-1-\nu}x - q^{n-\nu}[\nu]}{q^{i}[j-i-\nu]}$$

$$= \prod_{\nu=0}^{j-i-1} \frac{\eta - q^{\nu}\xi - [n] + q^{\nu}[n-\nu]}{q^{i+\nu}[j-i-\nu]}$$

$$= \prod_{\mu=0}^{j-i-1} \frac{\eta - q^{\mu}\xi - [\mu]}{q^{i+\mu}[j-i-\mu]}$$

where we have taken $\mu = \nu$. Hence, by substituting the above transformed products into (5.6), we obtain

$$L^{n}_{n-j,n-i}(x, y; 1/q) = \prod_{\mu=j+1}^{n}\left(\frac{[\mu]-\eta}{q^{j}[\mu-j]}\right) \prod_{\mu=0}^{i-1}\left(\frac{\xi-[\mu]}{q^{\mu}[i-\mu]}\right) \prod_{\mu=0}^{j-i-1}\left(\frac{\eta-q^{\mu}\xi-[\mu]}{q^{i+\mu}[j-i-\mu]}\right)$$

$$= L^{n}_{i,j}(\xi, \eta; q).$$

Proof of Theorem 5.1 Let the triangle $S_n = S_n(q)$ be transformed into the triangle $S_n(1/q)$ under $q \to \frac{1}{q}$. From (5.4) we have

$$w^n_{n-j,n-i}(1/q) = \iint_{S_n(1/q)} L^n_{n-j,n-i}(x, y; \frac{1}{q})\, dx\, dy.$$

Under the transformation $x = ([n] - \eta)/q^{n-1}$ and $y = ([n] - \xi)/q^{n-1}$ the region $S_n(1/q)$ is mapped onto $S_n(q)$ and, on using Lemma 5.1, we obtain

$$w^n_{n-j,n-i}(1/q) = \iint_{S_n(q)} L^n_{i,j}(\xi, \eta; q) \left| \frac{\partial(x, y)}{\partial(\xi, \eta)} \right| d\xi\, d\eta$$

$$= \frac{1}{q^{2(n-1)}}\, w^n_{i,j}(q).$$

This completes the proof. Note that if we consider $\omega^n_{i,j} = w^n_{i,j}$ / Area, a relative weight on the triangle, then we simply have

$$\omega^n_{n-j,n-i}(1/q) = \omega^n_{i,j}(q).$$

5.5 Alternative method of obtaining the weights

Consider any integration rule of the form

$$I_n(f) = \sum_{j=0}^{n} \sum_{i=0}^{j} w^n_{i,j}\, f([i], [j]).$$

Suppose this rule is exact, that is

$$I_n(f) = \iint_{S_n} f(x, y)\, dx\, dy,$$

for all $f \in \mathcal{P}_n$. In particular, let us choose $f = L^n_{i,j}$, the Lagrange coefficient on S_n at the point $([i], [j])$, $0 \le i \le j \le n$. As $L^n_{i,j} \in \mathcal{P}_n$, we obtain $I_n(L^n_{i,j}) = w^n_{i,j}$ and so there is a unique rule of this form. Since the rule is exact for all monomials in $\mathcal{P}_n$, we also have

$$I_n(x^\alpha y^\beta) = \int_0^{[n]} \int_0^y x^\alpha y^\beta \, dx \, dy = \sum_{j=0}^{n} \sum_{i=0}^{j} w^n_{i,j} [i]^\alpha [j]^\beta$$

so that

$$\sum_{j=0}^{n} \sum_{i=0}^{j} w^n_{i,j} [i]^\alpha [j]^\beta = \frac{[n]^{\alpha+\beta+2}}{(\alpha + 1)(\alpha + \beta + 2)},$$

for each $\alpha, \beta \ge 0$, $\alpha + \beta \le n$. This is a system of $(n + 1)(n + 2)/2$ linear equations in the $(n + 1)(n + 2)/2$ unknowns $w^n_{i,j}$, $0 \le i \le j \le n$. We can determine $w^n_{i,j}$ by solving these linear equations as an alternative to the method, which we used above, of integrating the Lagrange coefficients. Note that, since the above linear equations have a unique solution, the matrix must be non-singular.

5.6 Relation between integration rules over certain triangles of the same order

Consider another triangular formation of nodes, formed by the set of points

$$\{([i - j]_p, -[-j]_p) : 0 \le j \le i \le n\}, \tag{5.7}$$

where $p > 0$. This set of points lies in the union of the three pencils of lines

$$x = [i]_p, \quad 0 \le i \le n,$$
$$y = -[-j]_p, \quad 0 \le j \le n,$$
$$x + p^k y = [k]_p, \quad 0 \le k \le n.$$

If we let $q = 1/p$, we have

$$-[-j]_p = -\frac{1 - p^{-j}}{1 - p} = q \, \frac{1 - q^j}{1 - q} = q \, [j]_q \, ,$$

and using the fact that $\{(t_{i-j}, u_j): 0 \le j \le i \le n\} = \{(t_i, u_j): i, j \ge 0, i + j \le n\}$ the set of points (5.7) can be expressed in the form

$$\{([i]_p, q[j]_q): i, j \ge 0, i + j \le n\}. \tag{5.8}$$

Let f(x, y) be a function defined on a triangle

$$\Delta_n = \{(x, y): x, y \ge 0, \; x + p^n y \le [n]_p\}.$$

For simplicity, let us denote $q[j]_q$ by y_j, $j = 0,1, \ldots, n$. Then from Lemma 3.2, the interpolating polynomial at the set of nodes in (5.8) can be written as

$$P_n(x, y) =$$

$$\sum_{i=0}^{n} \sum_{j=0}^{n-i} \prod_{\nu=0}^{i-1} (x - [\nu]_p) \prod_{\nu=0}^{j-1} (y - q[\nu]_q) \; [[0], \ldots, [i]]_x \, [y_0, \ldots, y_j]_y \, f \tag{5.9}$$

Here, we have used the notations $[[0], [1], \ldots, [i]]_x \, f(\cdot, y)$ and $[y_0, y_1, \ldots, y_j]_y \, f(x, \cdot)$, defined earlier in Chapter 3, to denote divided differences in the x and y directions respectively. First we shall express the polynomial (5.9) in terms of forward difference operators. By virtue of Lemma 3.5, we only need to find the relation between the divided difference $[y_0, y_1, \ldots, y_j]_y \, f(x, \cdot)$ and the corresponding forward difference. For a fixed value of x and any $m \ge 0$, we have

$$[y_j, y_{j+1}, \ldots, y_{j+m}]_y \, f(x, \cdot) = \frac{1}{[m]_q! \, q^{(2j+m+1)m/2}} \, \mathfrak{D}_y^m \, f(x, y_j) \, . \tag{5.10}$$

where the differences $\mathfrak{D}_y^m \, f(x, y_j)$ are defined by

$$\mathfrak{D}_y^0 f(x, y_j) = f(x, y_j),$$

$$\mathfrak{D}_y^m f(x, y_j) = \mathfrak{D}_y^{m-1} f(x, y_{j+1}) - q^{m-1} \mathfrak{D}_y^{m-1} f(x, y_j), \qquad m = 1, 2, \dots .$$

This result is similar to Lemma 3.5. We give a brief proof of (5.10). The proof is by induction. Obviously (5.10) holds for $m = 0$. If (5.10) is true for any $m \geq 0$ then

$$[y_j, y_{j+1}, \dots, y_{j+m+1}]_y f(x, \cdot)$$

$$= \frac{[y_{j+1}, \dots, y_{j+m+1}]_y f(x, \cdot) - [y_j, \dots, y_{j+m}]_y f(x, \cdot)}{y_{j+m+1} - y_j}$$

$$= \frac{q^{-j-1}}{[m+1]_q} \left\{ \frac{1}{[m]_q! \, q^{(2j+m+3)m/2}} \mathfrak{D}_y^m f(x, y_{j+1}) - \frac{1}{[m]_q! \, q^{(2j+m+1)m/2}} \mathfrak{D}_y^m f(x, y_j) \right\}$$

$$= \frac{1}{q^{(2j+m+2)(m+1)/2} \, [m+1]_q!} \mathfrak{D}_y^{m+1} f(x, y_j) ,$$

which means (5.10) is also true for $m + 1$. Hence (5.10) holds for all $m \geq 0$.

As a special case of (5.10) we obtain

$$[y_0, y_1, \dots, y_m]_y f(x, \cdot) = \frac{1}{q^{(m+1)m/2} \, [m]_q!} \mathfrak{D}_y^m f(x, y_0) . \qquad (5.11)$$

Now apply Lemma 3.5 and (5.11) to the mixed divided difference in (5.9), to give

$$P_n(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{p^{-(i-1)i/2} \, q^{-(j+1)j/2}}{[i]_p! \, [j]_q!} \prod_{\nu=0}^{i-1} (x - [\nu]_p) \prod_{\nu=0}^{j-1} (y - q[\nu]_q) \, \mathfrak{D}_x^i \mathfrak{D}_y^j f_{0,0}$$

If we let $x = [\bar{x}]_p$ and $y = q[\bar{y}]_q$ for some $\bar{x}, \bar{y} \in \mathbb{R}$ then the above polynomial simplifies to

$$P_n(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{1}{[i]_p!} \prod_{\nu=0}^{i-1} [\bar{x} - \nu]_p \frac{1}{[j]_q!} \prod_{\nu=0}^{j-1} [\bar{y} - \nu]_q \; \mathfrak{D}_x^i \mathfrak{D}_y^j f_{0,0},$$

or, in the q-binomial notation,

$$P_n(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \begin{bmatrix} \bar{x} \\ i \end{bmatrix}_p \begin{bmatrix} \bar{y} \\ j \end{bmatrix}_q \mathfrak{D}_x^i \mathfrak{D}_y^j f_{0,0}. \tag{5.12}$$

In (5.12), if we take the limit as $p \to 1$ then, since $q = 1/p$, this means that $q \to 1$ also. Then

$$[i]_p! \to i!, \qquad [j]_q! \to j!,$$

$$[\bar{x} - \nu]_p \to x - \nu, \qquad [\bar{y} - \nu]_q \to y - \nu,$$

$$\mathfrak{D}_x^i \mathfrak{D}_y^j f_{0,0} \to \Delta_x^i \Delta_y^j f_{0,0}$$

and hence $P_n(x, y)$ in (5.12) tends to

$$P_n(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{x}{i} \binom{y}{j} \Delta_x^i \Delta_y^j f_{0,0},$$

which agrees with (3.6).

Let J_n be the integration rule over Δ_n. To obtain J_n we need the interpolating polynomial $P_n(x, y)$ expressed in the Lagrange form. First we write

$$y_j = q\,[j]_q = \frac{1}{p}\,\frac{1 - p^{-j}}{1 - 1/p} = \frac{[j]_p}{p^j}$$

and for simplicity we will drop the subscript p in the notation of $[k]_p$. We see also that the interpolating nodes $\{([i], [j]/p^j): \ i, j \geq 0, \ i + j \leq n\}$ lie in the union of the three pencils of lines

$$x = [i], \ 0 \leq i \leq n,$$

$$y = -[-j], \ 0 \leq j \leq n,$$

$$x + p^k y = [k], \ 0 \leq k \leq n.$$

Hence following section 2.4 , we have

$$P_n(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{i} M^n_{i,j}(x, y)\, f([i], \frac{[j]}{p^j})$$

where the Lagrange coefficient $M^n_{i,j}(x, y)$ takes the form

$$M^n_{i,j}(x, y) = \prod_{\nu=0}^{i-1} \frac{x - [\nu]}{[i] - [\nu]} \ \prod_{\nu=0}^{j-1} \frac{y - p^{-\nu}[\nu]}{p^{-j}[j] - p^{-\nu}[\nu]} \ \prod_{\nu=0}^{n-i-j-1} \frac{[n-\nu] - x - p^{n-\nu}y}{[n-\nu] - [i] - p^{n-\nu-j}[j]}\ .$$

Thus $M^n_{i,j}(x, y)$ has the value 1 at ([i], [j]) and is zero at all other nodes. See also Figure 5.7 .

Let $V^n_{i,j}$ be the weight at the node $([i], \frac{[j]}{p^j})$. Then we have

$$V^n_{i,j} = \int\int_{\Delta_n} M^n_{i,j}(x, y)\, dx\, dy.$$

It is interesting to compare the weights $V^n_{i,j}$ on Δ_n and the weights $w^n_{i,j}$ for the integration rule on the triangle $S_n = \{(x, y): 0 \leq x \leq y \leq [n]\}$. For the case n = 1, we obtain

$$w^1_{0,0} = p\, V^1_{0,1}, \quad w^1_{0,1} = p\, V^1_{0,0} \quad \text{and} \quad w^1_{1,1} = p\, V^1_{1,0}$$

and, for the case n = 2,

$$w^2_{0,0} = p^2\, V^2_{0,2} \qquad w^2_{0,2} = p^2\, V^2_{0,0}$$

$$w^2_{0,1} = p^2\, V^2_{0,1} \qquad w^2_{1,2} = p^2\, V^2_{1,0}$$

$$w^2_{1,1} = p^2\, V^2_{1,1} \qquad w^2_{2,2} = p^2\, V^2_{2,0}.$$

These results and those for the other small values of n suggest the following lemma, where we show that there is a relation between the integration rules I_n on S_n and J_n on Δ_n, for all values of n.

Lemma 5.2 Let $w^n_{i,j}$ be the weight at ([i], [j]) on S_n. Then the transformation

$$\xi = x, \qquad \eta = \frac{[n] - y}{p^n} \tag{5.13}$$

maps S_n onto $\Delta_n = \{(\xi, \eta):\ \xi, \eta \geq 0,\ \xi + p^n \eta \leq [n]\}$ and the weights $w^n_{i,n-j}$ into $V^n_{i,j}$, the weight at $([i], \frac{[j]}{p^j})$ on Δ_n, such that

$$w^n_{i,n-j} = p^n\, V^n_{i,j}\ .$$

Proof First we see that the transformation (5.13) maps: each node ([i], [n − j]) into $([i], \frac{[j]}{p^j})$, and each line $y = p^{n-\nu} x + [n - \nu]$ onto $\xi + p^\nu \eta = [\nu]$. Hence the region S_n is transformed onto the region Δ_n. Note that the (n + 1)(n + 2)/2 interpolation nodes $([i], \frac{[j]}{p^j})$ on Δ_n still lie on the lines

$$\xi = [k] \quad \text{or} \quad \eta = \frac{[k]}{p^k} \quad \text{or} \quad \xi + p^k \eta = [k], \qquad 0 \leq k \leq n.$$

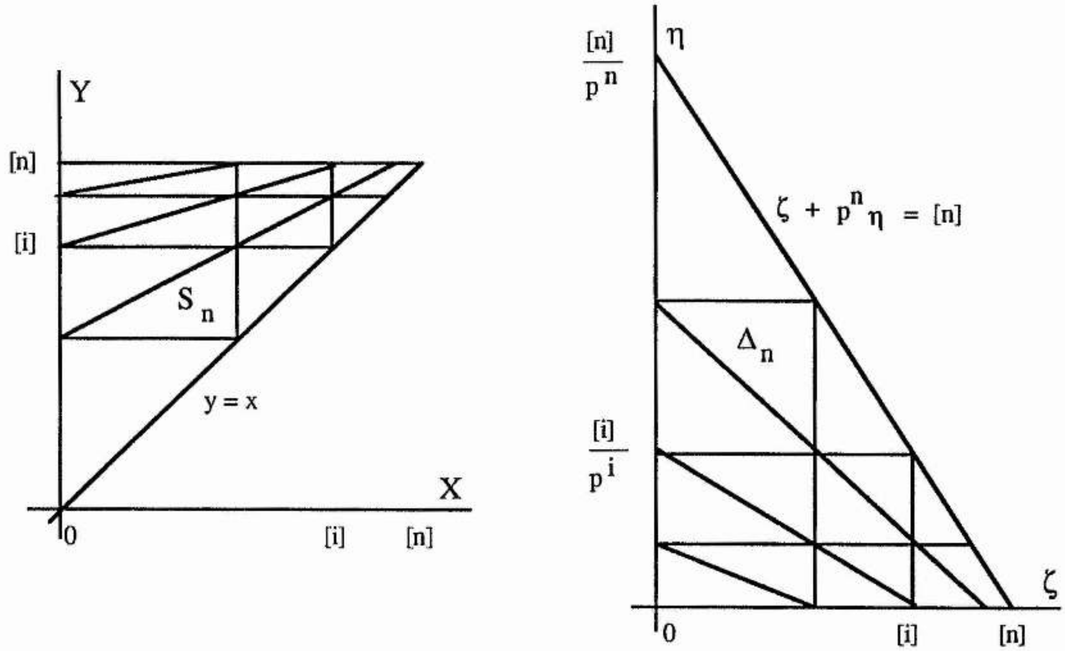


Figure 5.7

Now let us consider the Lagrange coefficient $L^n_{i,n-j}(x, y)$ on S_n. From (5.2) we have

$$L^n_{i,n-j}(x, y) = \prod_{\nu=0}^{i-1} \frac{x - [\nu]}{[i] - [\nu]} \prod_{\nu=n-j+1}^{n} \frac{[\nu] - y}{[\nu] - [n-j]} \prod_{\nu=0}^{n-i-j-1} \frac{y - p^\nu x - [\nu]}{[n-j] - p^\nu [i] - [\nu]} .$$

If we apply the transformation (5.13) to $L^n_{i,n-j}(x, y)$, the first product requires no work. On applying the transformation (5.13) to the second product in $L^n_{i,n-j}(x, y)$, we obtain

$$\prod_{\nu=n-j+1}^{n} \frac{[\nu] - y}{[\nu] - [n-j]} = \prod_{\mu=0}^{j-i} \frac{[n-\mu] - [n] + p^n\eta}{[n-\mu] - [n-j]}$$

on putting $\mu = n - \nu$. Thus

$$\prod_{\nu=n-j+1}^{n} \frac{[\nu] - y}{[\nu] - [n-j]} = \prod_{\mu=0}^{j-i} \frac{p^n\eta - p^{n-\mu}[\mu]}{p^{n-j}[j-\mu]}$$

$$= \prod_{\mu=0}^{j-i} \frac{\eta - p^{-\mu}[\mu]}{p^{-j}[j] - p^{-\mu}[\mu]} .$$

The third product in $L^n_{i,n-j}(x, y)$ is transformed to give

$$\prod_{\nu=0}^{n-i-j-1} \frac{y - p^{\nu}x - [\nu]}{[n-j] - p^{\nu}[i] - [\nu]} = \prod_{\nu=0}^{n-i-j-1} \frac{[n] - p^n\eta - p^{\nu}\xi - [\nu]}{p^{\nu}[n-j-\nu] - p^{\nu}[i]}$$

$$= \prod_{\nu=0}^{n-i-j-1} \frac{[n-\nu] - p^{n-\nu}\eta - \xi}{-[n-\nu] + [n-j-\nu] + [n-\nu] - [i]}$$

$$= \prod_{\mu=0}^{n-i-j-1} \frac{[n-\mu] - \xi - p^{n-\mu}\eta}{[n-\mu] - [i] - p^{n-j-\mu}[j]} ,$$

on writing $\mu = \nu$. Hence $L^n_{i,n-j}(x, y)$ is transformed into

$$M^n_{i,j}(\xi, \eta) = \prod_{\mu=0}^{i-1} \frac{\xi - [\mu]}{[i] - [\mu]} \prod_{\mu=0}^{j-i} \frac{\eta - p^{-\mu}[\mu]}{p^{-j}[j] - p^{-\mu}[\mu]}$$

$$\times \prod_{\mu=0}^{n-i-j-1} \frac{[n-\mu] - \xi - p^{n-\mu}\eta}{[n-\mu] - [i] - p^{n-j-\mu}[j]} ,$$

the Lagrange coefficient on Δ_n.

Finally, we establish the relation between the weights $w^n_{i,n-j}$ and $V^n_{i,j}$ of the rules I_n and J_n respectively. We obtain

$$w^n_{i,n-j} = \iint_{S_n} L^n_{i,n-j}(x, y)\, dx\, dy = \iint_{\Delta_n} M^n_{i,j}(\xi, \eta) \left| \frac{\partial(x, y)}{\partial(\xi, \eta)} \right| d\xi\, d\eta$$

$$= p^n \iint_{\Delta_n} M^n_{i,j}(\xi, \eta)\, d\xi\, d\eta = p^n V^n_{i,j} .$$

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Chapter 6

Lebesgue functions and Lebesgue constants of the interpolating polynomial

6.1 Introduction

Given values of f(x) at n + 1 distinct points $x_0, x_1, \ldots, x_n$ in [–1, 1], we know from Theorem 1.2 that there exists a unique polynomial $P_n(x)$ of degree at most n such that $P_n(x_j) = f(x_j)$, j = 0, 1, …, n. Also from (1.2), the polynomial can be written in the Lagrangian form

$$P_n(x) = \sum_{i=0}^{n} f(x_i)\, L_i^n(x) \tag{6.1}$$

where

$$L_i^n(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}.$$

Consider the error in approximating to f(x) by the polynomial (6.1), given by

$$G_n = \max_{-1 \le x \le 1} | f(x) - P_n(x) | .$$

It is well known that G_n is not in general the smallest error that can be achieved in approximating f(x) by polynomials. Let $\mathbb{P}_n$ denotes the class of polynomials p(x) of degree at most n. Given f(x) in C[−1, 1], there exists a polynomial p_n^* in $\mathbb{P}_n$ such that

$$E_n = \max_{-1\le x\le 1} | f(x) - p_n^*(x) | \le \max_{-1\le x\le 1} | f(x) - p(x) |, \quad \text{for all } p(x) \text{ in } \mathbb{P}_n .$$

If one compares G_n with the least error E_n, we have (see [21])

$$G_n \le E_n \left(1 + \max_{-1\le x\le 1} \sum_{i=0}^{n} | L_i^n(x) | \right).$$

The function

$$\lambda_n(x) = \sum_{i=0}^{n} | L_i^n(x) |$$

which appears in the above comparison is called the Lebesgue function of order n and the quantity Λ_n defined by

$$\Lambda_n = \max_{-1\le x\le 1} \lambda_n(x)$$

is called the Lebesgue constant of order n. We see that

$$\lambda_n(x_j) = | L_j^n(x_j) | = 1, \quad \text{for all } j = 0,1, \ldots, n.$$

Also, on taking f(x) = 1 in (6.1), we have

$$\sum_{i=0}^{n} L_i^n(x) = 1 \tag{6.2}$$

and hence

$$\lambda_n(x) \ge \left| \sum_{i=0}^{n} L_i^n(x) \right| = 1, \quad \text{for all } x \text{ in } [-1, 1]. \tag{6.3}$$

Let us determine whether $\lambda_n(x) = 1$ at more than the n + 1 points noted above. First we consider the Lebesgue function $\lambda_4(x)$ on [−1, 1], constructed at the points

$$x_0 = -0.9,\ x_1 = -0.2,\ x_2 = 0,\ x_3 = 0.3 \text{ and } x_4 = 0.6 .$$

Note that the Lagrange coefficients of $\lambda_4(x)$ take the form

$$L_0^4(x) = \frac{x}{1.134}(x + .2)(x - .3)(x - .6)$$

$$L_1^4(x) = \frac{-x}{0.056}(x + .9)(x - .3)(x - .6)$$

$$L_2^4(x) = \frac{(x + .9)}{0.0324}(x + .2)(x - .3)(x - .6)$$

$$L_3^4(x) = \frac{-x}{0.054}(x + .9)(x + .2)(x - .6)$$

$$L_4^4(x) = \frac{x}{0.216}(x + .9)(x + .2)(x - .3)\ .$$

The graph in Figure 6.1 reveals that $\lambda_4(x) = 1$ at x_0, x_1, x_2, x_3 and x_4 and $\lambda_4(x) > 1$ at all other points of [−1, 1].

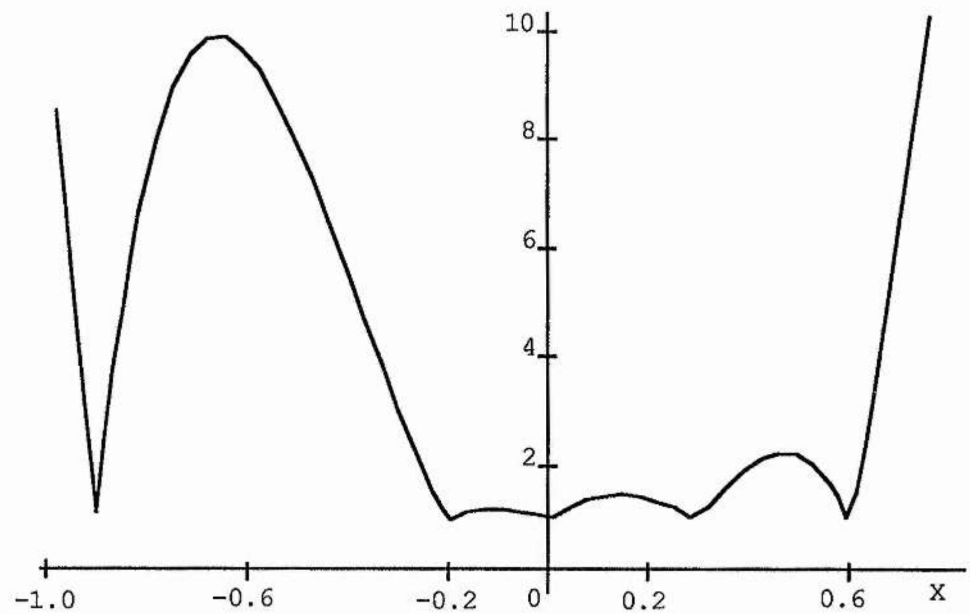


Figure 6.1

In the following lemma, which is well known, we see that, for all $n \geq 2$, $\lambda_n(x)$ cannot assume the value 1 other than at the interpolation points $x_0, x_1, \ldots, x_n$.

Lemma 6.1 Let $-1 \le x_0 < x_1 < \ldots < x_n \le 1$ and let $\lambda_n(x)$ be the Lebesgue function defined on $[-1, 1]$. Then

(i) $\lambda_1(x) = 1$ on $[x_0, x_1]$ and $\lambda_1(x) > 1$ on $[-1, 1] \setminus [x_0, x_1]$.

(ii) For $n \ge 2$, $\lambda_n(x) > 1$ on $[-1, 1] \setminus \{x_0, x_1, \ldots, x_n\}$.

Proof We begin with the case $n = 1$. For any $x \in [x_0, x_1]$, we have

$$L_0^1(x) = \frac{x_1 - x}{x_1 - x_0} \ge 0 \quad \text{and} \quad L_1^1(x) = \frac{x - x_0}{x_1 - x_0} \ge 0$$

and thus, for $x_0 \le x \le x_1$,

$$\lambda_1(x) = |L_0^1(x)| + |L_1^1(x)| = L_0^1(x) + L_1^1(x) = 1.$$

For any $-1 \le x < x_0$, $L_0(x) > 1$ and $L_1(x)$ is negative, therefore $\lambda_1(x) > 1$. Similarly if $x_1 < x \le 1$ then $L_0(x) < 0$ and $L_1(x) > 1$, therefore $\lambda_1(x) > 1$.

Next, consider the case $n \ge 2$ and suppose that $\lambda_n(x) = 1$ for some $x \in [x_0, x_n] \setminus \{x_0, x_1, \ldots, x_n\}$. (We will consider $x \in [-1, x_0] \cup [x_n, 1]$ later.) Then from (6.2) we have

$$\sum_{i=0}^{n} |L_i^n(x)| = \left| \sum_{i=0}^{n} L_i^n(x) \right|.$$

This means that all $L_i^n(x)$ have the same sign. Now each $L_i^n(x)$ has the value 1 at x_i and has simple zeros at x_j, $j \ne i$. Therefore the function satisfies

$$L_i^n(x) \ge 0 \text{ on } [x_{i-1}, x_{i+1}] \text{ and } L_i^n(x) \le 0 \text{ on } [x_{i-2}, x_{i-1}] \cup [x_{i+1}, x_{i+2}]. \quad (6.4)$$

Similarly, we have

$$L^n_{i-1}(x) \geq 0 \text{ on } [x_{i-2}, x_i] \text{ and } L^n_{i-1}(x) \leq 0 \text{ on } [x_{i-3}, x_{i-2}] \cup [x_i, x_{i+1}]$$

and

$$L^n_{i+1}(x) \geq 0 \text{ on } [x_i, x_{i+2}] \text{ and } L^n_{i+1}(x) \leq 0 \text{ on } [x_{i-1}, x_i] \cup [x_{i+2}, x_{i+3}].$$

On examining the three Lagrange coefficients $L^n_{i-1}(x)$, $L^n_i(x)$ and $L^n_{i+1}(x)$ on the interval $(x_{i-1}, x_i) \cup (x_i, x_{i+1})$, we see that either L^n_{i-1} or L^n_{i+1} is negative. See Figure 6.2 . This gives a contradiction. Thus

$$\lambda_n(x) = \sum_{i=0}^{n} | L^n_i(x) | > 1 \text{ on } [x_0, x_n] \setminus \{x_0, x_1, \ldots, x_n\}$$

because we know from (6.3) that $\lambda_n(x) \geq 1$ for all $x \in [x_0, x_n]$.

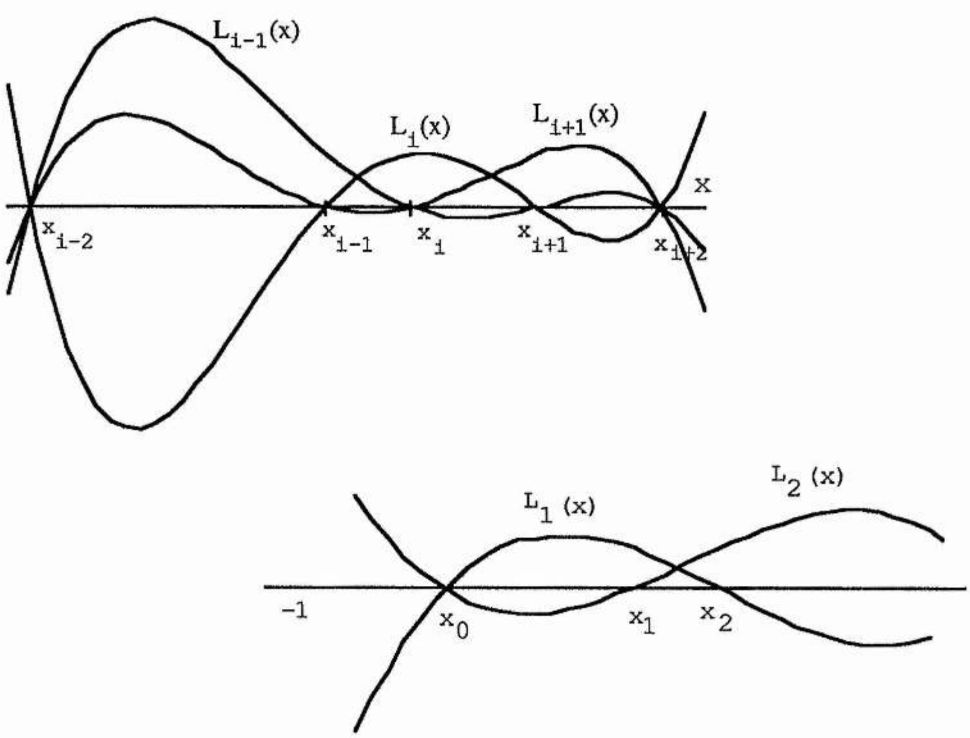


Figure 6.2

To complete the proof, we consider the value of $\lambda_n(x)$ on the end-intervals $[-1, x_0)$ and $(x_n, 1]$. From (6.4) we see that $L_1^n(x) > 0$ and $L_2^n(x) < 0$ in (x_0, x_1) and neither coefficient has any zeros in $[-1, x_0)$. This means that $L_1^n(x) < 0$ and $L_2^n(x) > 0$ on $[-1, x_0)$. Therefore $\lambda_n(x) \neq 1$ for all $x \in [-1, x_0)$ for otherwise all $L_i^n(x)$ would need to have the same sign at some point on this interval. Hence $\lambda_n(x) > 1$ on $[-1, x_0)$. Similarly $\lambda_n(x) > 1$ on the other end-interval $(x_n, 1]$.

It is interesting to extend this idea to consider a two-dimensional Lebesgue function and Lebesgue constant of the interpolating polynomial $P_n(x, y; q)$. Specifically, given a function f defined on a triangle $S_n = \{(x, y): 0 \le x \le y \le [n]\}$, we consider the polynomial

$$P_n(x, y; q) = \sum_{j=0}^{n} \sum_{i=0}^{j} L_{i,j}^{n}(x, y; q)\, f_{i,j}\,.$$

Then $P_n([i], [j]) = f_{i,j}$, $0 \le i \le j \le n$, and the Lagrange coefficients take the form

$$L_{i,j}^{n}(x, y; q)$$

$$= \frac{q^{-(2n-j-1)j/2}}{[i]!\,[j-i]!\,[n-j]!} \prod_{v=0}^{i-1} (x - [v]) \prod_{v=j+1}^{n} ([v] - y) \prod_{v=0}^{j-i-1} (y - q^{v}x - [v]). \qquad (6.5)$$

We now define, in an obvious way, a two-dimensional Lebesgue function of order n by

$$\lambda_n(x, y; q) = \sum_{j=0}^{n} \sum_{i=0}^{j} \left| L_{i,j}^{n}(x, y; q) \right|. \qquad (6.6)$$

Clearly, $\lambda_n(x, y; q) = 1$ at the interpolating nodes $\{([i], [j]): 0 \le i \le j \le n\}$ and, similarly to (6.2), we obtain

$$\sum_{j=0}^{n} \sum_{i=0}^{j} L_{i,j}^{n}(x, y; q) = 1 \qquad (6.7)$$

and hence $\lambda_n(x, y; q) \geq 1$ for all $(x, y) \in S_n$. We also define the Lebesgue constant of order n

$$\Lambda_n(q) = \max_{0 \leq x \leq y \leq [n]} \lambda_n(x, y; q).$$

The simplest case to consider is the function $\lambda_1(x, y; q)$ defined on S_1. Since all points on S_1 satisfy $0 \leq x \leq y \leq 1$, we have for all $q > 0$

$$\lambda_1(x, y; q) = |1 - y| + |y - x| + |x| = 1.$$

This means that

$$\Lambda_1(q) = \max_{0 \leq x \leq y \leq 1} \lambda_1(x, y; q) = 1 \quad \text{for all } q > 0.$$

The other cases are not trivial due to the complexity of $L^n_{i,j}(x, y; q)$.

In the rest of this chapter, we will investigate further the behaviour of the Lebesgue function defined in (6.6) and its corresponding Lebesgue constant. To obtain conjectures concerning properties of $\lambda_n(x, y; q)$, for a fixed value of $q > 0$, we shall plot the surface of this function using a Unimap package [26]. We will write a program (again in Pascal) to create a data point of $\lambda_n(x, y; q)$ for the package. Then we will examine the existence of maximum points of $\lambda_n(x, y; q)$ over S_n. We also will prove an analogue to the Lemma 6.1 concerning the value of $\lambda_n = 1$ at points other than the interpolation nodes. In section 6.5 we proceed with the study of Lebesgue constant and show that $\Lambda_n(q)$ are symmetric. In the last section we analyse the discontinuity of the directional derivative of λ_n at $x = [k]$, $y = [k]$, $y = q^k x + [k]$, $k = 1, \ldots, n-1$.

6.2 Surface plots of Lebesgue functions

We begin by writing a Pascal program to evaluate the Lagrange coefficients $L^n_{i,j}(x, y; q)$, where n is fixed positive integer. Let $q > 0$ and $(x, y) \in S_n$. For any pair (i, j), $0 \leq i \leq j \leq n$, we evaluate each product

$$\prod_{\nu=0}^{i-1} (x - [\nu]) , \quad \prod_{\nu=j+1}^{n} ([\nu] - y) \quad \text{and} \quad \prod_{\nu=0}^{j-i-1} (y - q^{\nu}x - [\nu])$$

of $L^n_{i,j}(x, y; q)$ and add all absolute values of $L^n_{i,j}(x, y; q)$ to give $\lambda_n(x, y; q)$. The resulting Lebesgue function is denoted by the Pascal function LN(x, y: real): real. Then we choose a regular grid on S_n with the grid spacing d, say, so that the set of grid nodes includes the interpolating points $\{([i], [j]): 0 \le i \le j \le n\}$. To create a data point for $z = \lambda_n(x, y; q)$, we evaluate the function LN(x, y) at these grid nodes and the arrange values of x, y and z in three separate columns. A program for the evaluation of $\lambda_n(x, y; q)$ is given in Appendix 6A.

We are now ready to plot the surface of $\lambda_n(x, y; q)$, using the Unimap package. Unimap reads the data points as irregular data and generates a new set of regular data before plotting the surface. This is done under a Unimap interpolation method (which we chose as bilinear). The method replaces the original data points by a more dense regular grid of nodes. To plot the surface of $\lambda_n(x, y; q)$ over a triangular area, we need to define the region. Otherwise the new regular data points and the surface plotted are based on a rectangular area. A region is defined as a number of border descriptions which are sets of (x, y) coordinates. The region chosen here overlaps S_n, each side being increased by an amount d, the grid spacing. Note that, if the region is chosen to be precisely S_n, Unimap plots the surface only in the interior of S_n.

First we plot the surface of $z = \lambda_4(x, y; 2)$ over S_4 using the grid node spacing $d = 0.2$. A brief Unimap plotting instruction sequence is given as follows.

DATA / IRREGULAR / READ /..,	To read a data points stored in the file data.dat .
INTERPOLATE / GRID CELL /..,	To interpolate the data points into a specified number of grid cells.
METHOD /..,	To choose an interpolation method.
DATA / REGION / READ /..,	To read a region data stored in the file reg.dat .

MAP / GALLERY /..,	To choose a map type required, e.g. 3D-contour.
LAYOUT / AXES /..,	To add primary or vertical axes and scale.
STYLE / 3D / VIEWPOINT /..,	To get 3D-viewpoint from a position above the X-Y plane by specifying Elevation and Azimuth.

Figure 6.3 shows the 2D grid representation of $\lambda_4(x, y; 2)$ where the surface has been plotted over the whole triangular region. Also, on using the same data for $\lambda_4(x, y; 2)$ and for the region, Figures 6.4 and 6.5 give 3D-contour representation of the surfaces with the viewpoint 75; 70 and 90; 320 respectively. Note that, if we view the diagonal side of the surface from the horizon, we see that the surface covers both ends of the diagonal boundary. These Figures also indicate that $\lambda_4(x, y; 2)$ has minimum values at the interpolation nodes of f(x, y). From (6.5), we know that these values are 1.

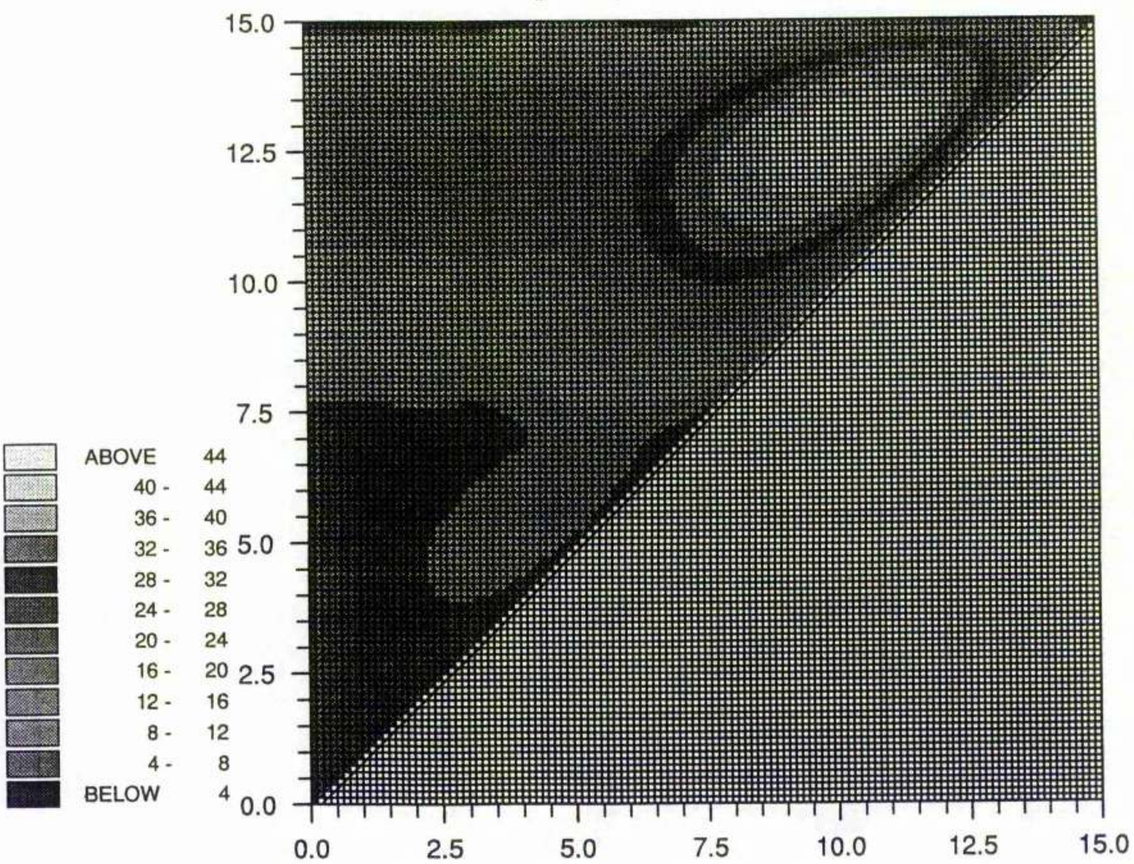


Figure 6.3 : The map (2D-grid) of $\lambda_4(x, y; 2)$.

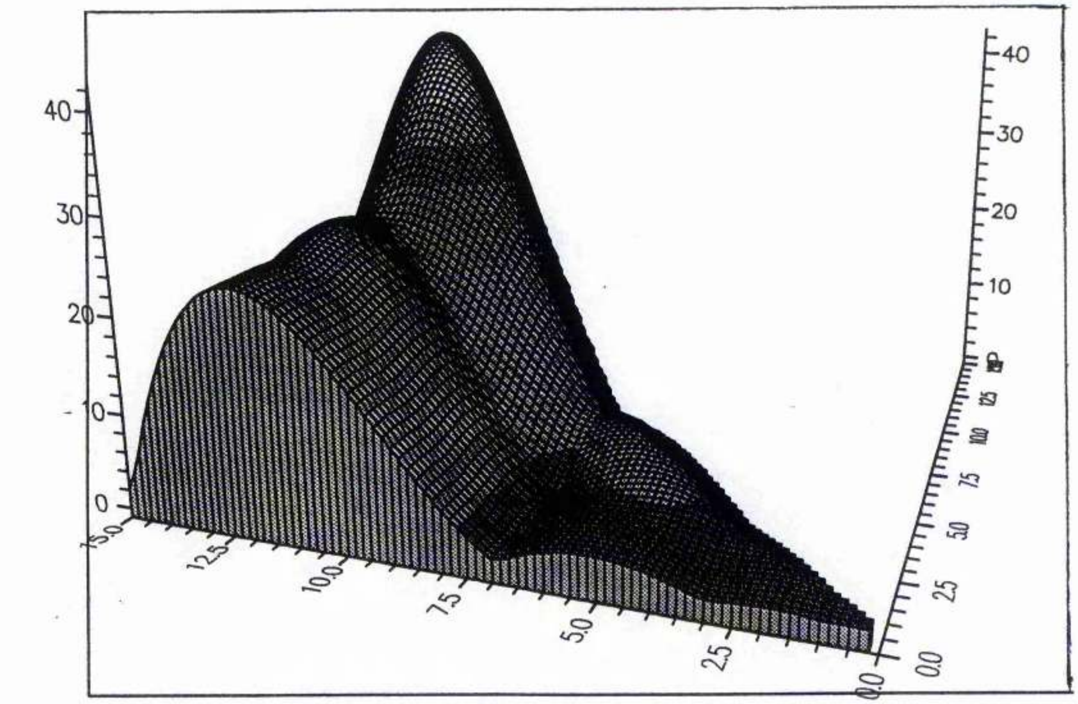


Figure 6.4 : The map (3D-contour) of $\lambda_4(x, y; 2)$, viewpoint 75; 70.

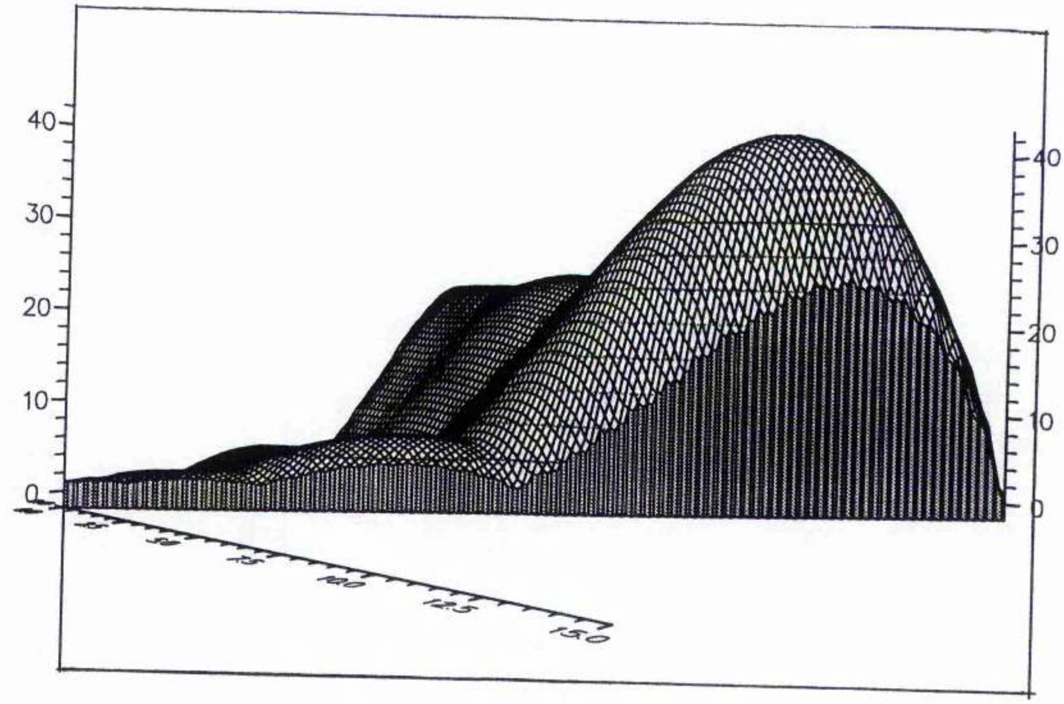


Figure 6.5 : The map (3D-contour) of $\lambda_4(x, y; 2)$, viewpoint 90; 320.

6.3 Peaks of the Lebesgue functions

We noticed from the surface of $\lambda_4(x, y; 2)$ over S_4, that the contour of a higher peak hides away some contour of the lower one. Since the value of q = 2 is large, only the higher peaks of $\lambda_4(x, y; 2)$ are well plotted. To overcome such a disadvantage, we choose q close to 1. This reduces the height of the higher peaks considerably and reveals more features of the surface in the vicinity of the lower peaks. Let us consider the Lebesgue functions $\lambda_n(x, y; q)$ where n = 2, 3 and q = 1.1, 0.5. In Table 6.1, we summarise some of the plotting information used in the mapping of these surfaces.

Order n	Grid mesh d	Number of data	Number of grid cells	Surface viewpoint	Map type	$\lambda_n(x, y; q)$ Surface
q = 1.1						
2	0.1	253	80×80	60; 120	3D-LINE	Figure 6.6
3	0.05	2278	100×100	35; 70	3D-LINE	Figure 6.7
3	0.03	6216	160×160		2D-LINE	Figure 6.8
q = 0.5						
2	0.05/ 0.025	1281	120×120	75; 238	3D-LINE + PROJ	Figure 6.9
3	.1/.05/ .025	1036	140×140		2D-LINE	Figure 6.10
3	.1/.05/ .025	1036	140×140	50; 215	3D-LINE	Figure 6.11

Table 6.1

Note that, we have used a different grid spacing for the case where q = 0.5 . On S_2, we chose d = 0.05 and 0.025 for $0 \le y \le 1$ and $1 \le y \le 1.5$ respectively. On S_3, we chose d = 0.1, 0.05 and 0.025 for $0 \le y \le 1$, $1 \le y \le 1.5$ and $1.5 \le y \le 1.75$ respectively. We see that $\lambda_2(x, y; 1.1)$ has only one maximum, and $\lambda_3(x, y; 1.1)$ has six maxima and one minimum point, ([1], [2]). The surface of $\lambda_2(x, y; 0.5)$ possesses

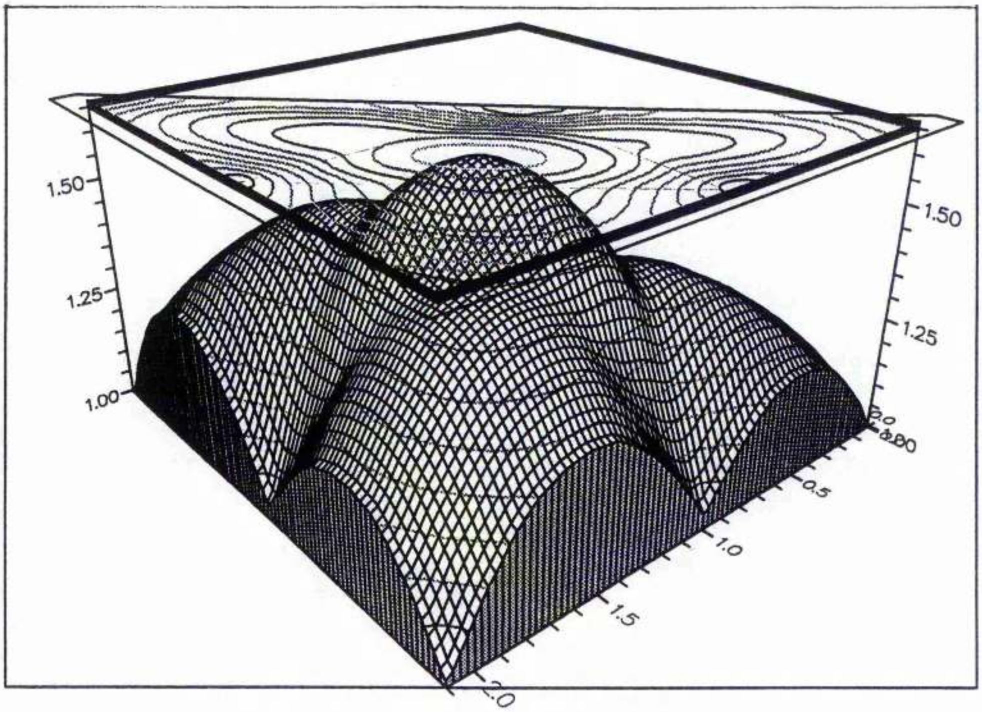


Figure 6.6 : The map (3D-line) of $\lambda_2(x, y; 1.1)$.

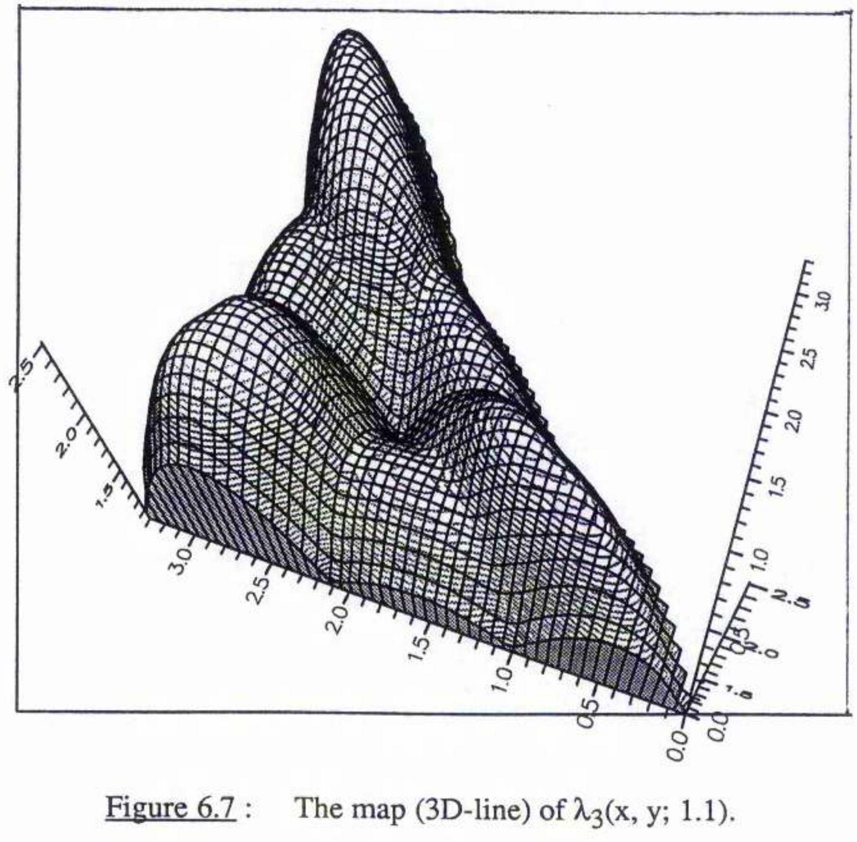


Figure 6.7 : The map (3D-line) of $\lambda_3(x, y; 1.1)$.

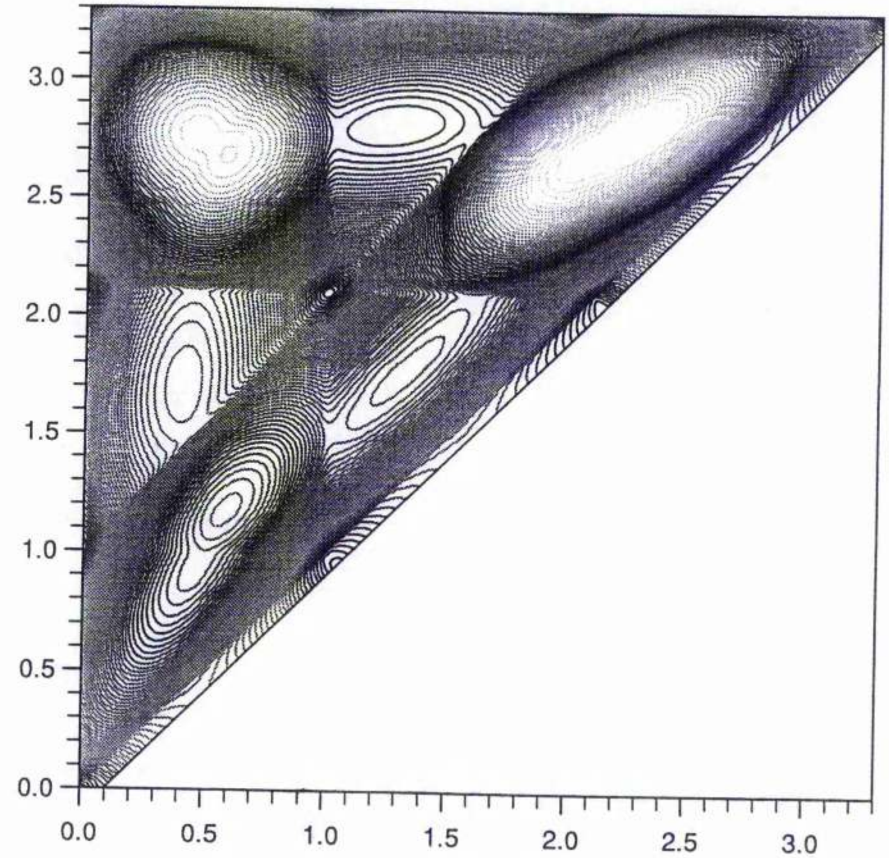


Figure 6.8 : The map (2D-line) of $\lambda_3(x, y; 1.1)$.

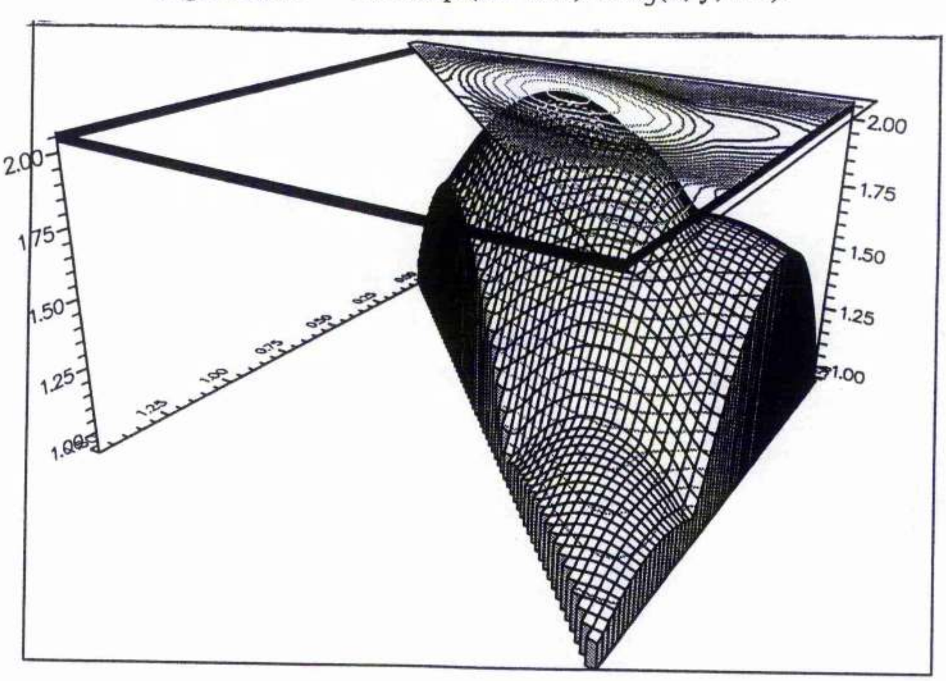


Figure 6.9 : The map (3D-line) of $\lambda_2(x, y; 0.5)$.

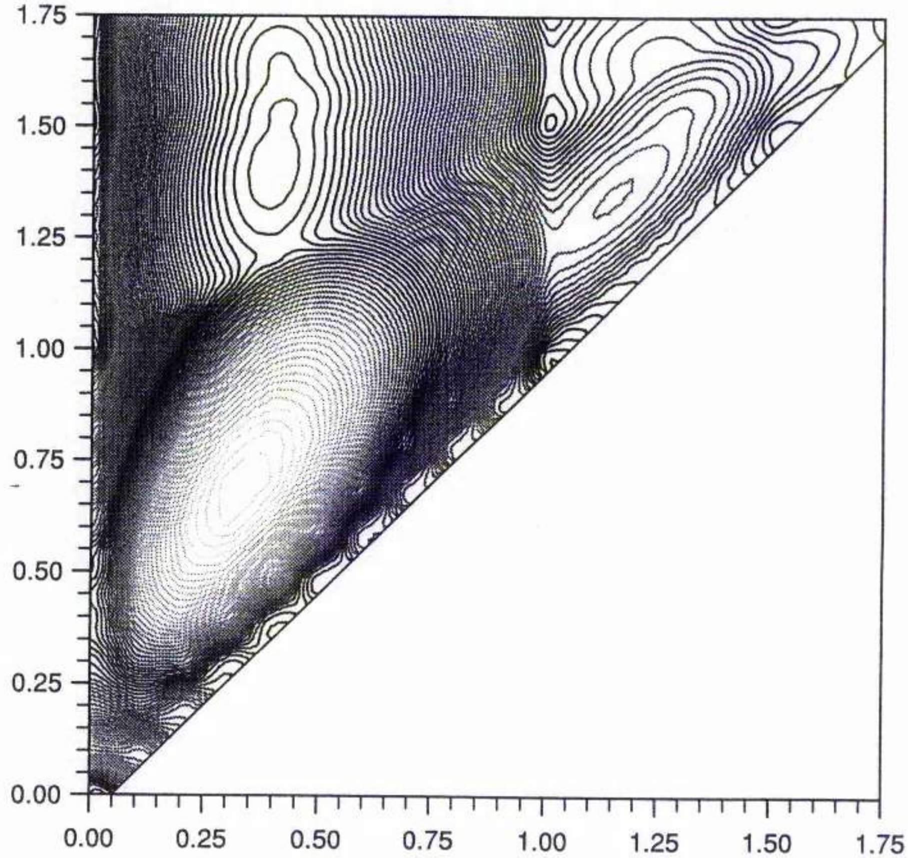


Figure 6.10 : The map (2D-line) of $\lambda_3(x, y; 0.5)$.

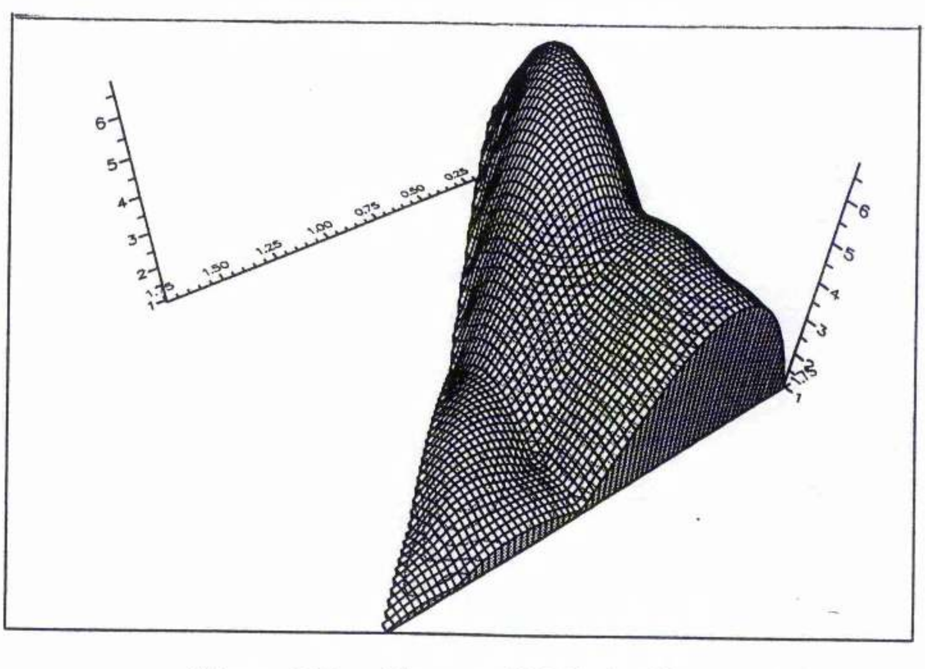


Figure 6.11 : The map (3D-line) of $\lambda_3(x, y; 0.5)$.

one maximum, and $\lambda_3(x, y; 0.5)$ has three maxima and one minimum. Note also that, in Figures 6.6 and 6.8, there are substantial gradient changes at the lines $x = [i]$, $y = [j]$ and at the interpolation nodes respectively.

Let us examine the existence of all maximum points of $\lambda_3(x, y; 1.1)$ in the subregions of S_3. Divide S_3 along the lines $y = [\nu]$, $y = q^{\nu}x + [\nu]$ and $x = [\nu]$, where $\nu = 1$ and 2, to give nine subregions $A_1, A_2, \ldots, A_9$, say as shown in Figure 6.12. We see that the six maxima points of Figure 6.8 lie in A_2, A_3, A_4, A_6, A_7 and A_9.

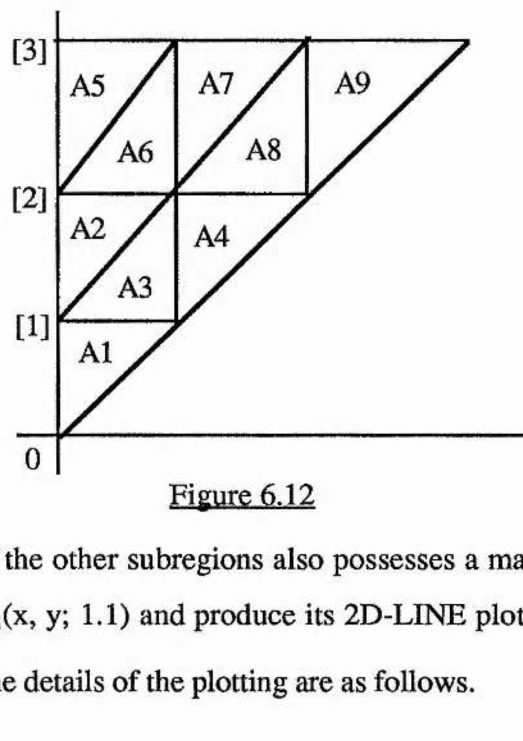


Figure 6.12

To determine whether each of the other subregions also possesses a maximum point, we refine the data points of $\lambda_3(x, y; 1.1)$ and produce its 2D-LINE plot on a suitably small area in the subregion. The details of the plotting are as follows.

Subregion	Area chosen	spacing d	No. of grid cells	Surface
A_1	$0.25 \leq x \leq y - 0.25$ $0.8 \leq y \leq 1.1$	0.01	110×54	Figure 6.13
A_5	$0 \leq x \leq y$ $2.5 \leq y \leq 2.9$	0.03	200×27	Figure 6.14
A_8	$1.9 \leq x \leq y$ $2.6 \leq y \leq 2.8$	0.005	150×75	Figure 6.15

Table 6.2

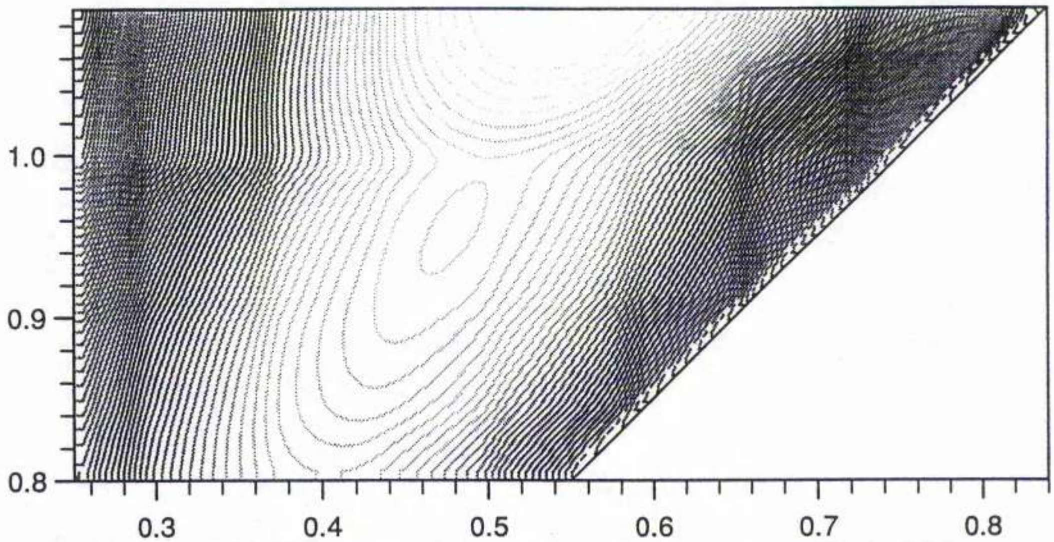


Figure 6.13 : The map (2D-line) of $\lambda_3(x, y; 1.1)$ over subregion A_1.

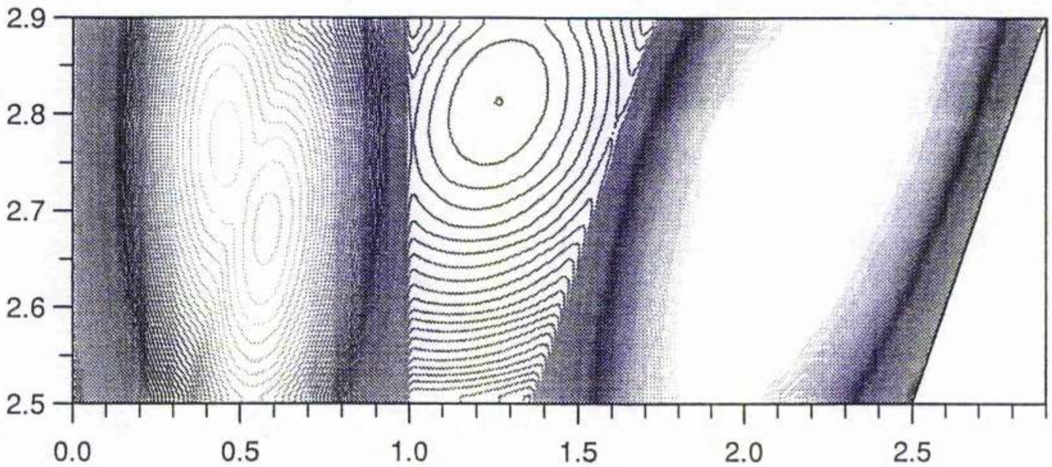


Figure 6.14 : The map (2D-line) of $\lambda_3(x, y; 1.1)$ over subregion A_5.

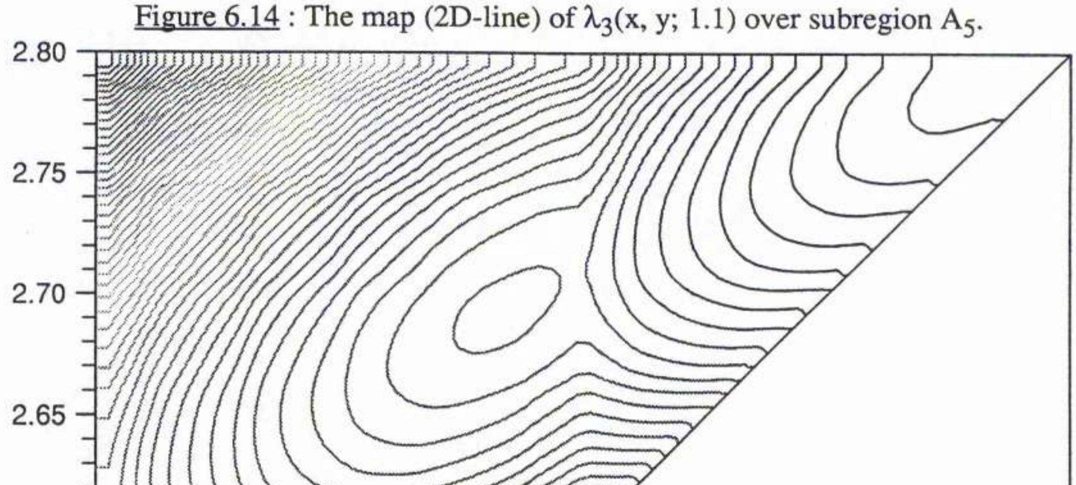


Figure 6.15 : The map (2D-line) of $\lambda_3(x, y; 1.1)$ over subregion A_8.

The information in Table 6.2 suggests that $\lambda_3(x, y; 1.1)$ possesses at least one maximum point in each of its subregions. If we choose $q \geq 2$ then some contour-lines on the lower peaks are lost due to the very large contours difference on the highest peak. The surface of $\lambda_4(x, y; 2)$ in Figure 6.4 already exhibited such a phenomenon. However $\lambda_n(x, y; q)$ does not always possess a maximum in every subregion of S_n. This is shown in the example below.

Example 6.1 Consider the function $\lambda_2(x, y; q)$, $q > 0$ and divide $S_2 = \Delta ABC$ into four subregions A_1, A_2, A_3 and A_4 by lines joining the middle nodes D, E and F. See Figure 6.16. We seek maximum points of $\lambda_2(x, y; q)$ on these subregions.

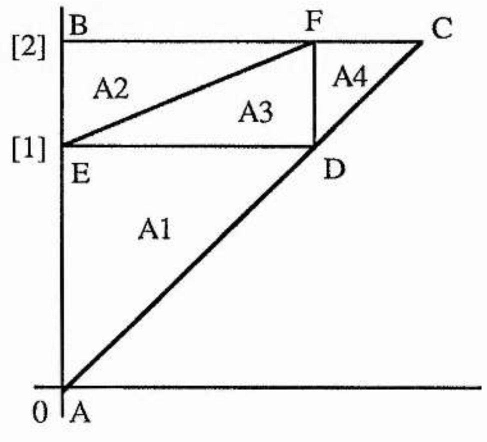


Figure 6.16

We obtain from (6.5) and (6.6) that

$$z = \lambda_2(x, y; q) = \frac{|1-y|\,|[2]-y|}{[2]} + \frac{|[2]-y|\,|y-x|}{q} + \frac{|x|\,|[2]-y|}{q}$$

$$+ \frac{|y-x|\,|y-qx-1|}{q[2]} + \frac{|x|\,|y-x|}{q} + \frac{|x|\,|x-1|}{q[2]}. \qquad (6.8)$$

Let $z_k(x, y)$ be the surface section of $\lambda_2(x, y; q)$ on subregion A_k, $k = 1, 2, 3$ and 4, and write

$$z_0(x, y) = \frac{([2] - y)(y - x)}{q} + \frac{x([2] - y)}{q} + \frac{x(y - x)}{q}.$$

Then each function $z_k(x, y)$, k = 1, 2, 3 and 4, can be written as follows.

$$z_1(x, y) = \frac{(1 - y)([2] - y)}{[2]} - \frac{(y - x)(y - qx - 1)}{q[2]} - \frac{x(x - 1)}{q[2]} + z_0(x, y)$$

$$z_2(x, y) = -\frac{(1 - y)([2] - y)}{[2]} + \frac{(y - x)(y - qx - 1)}{q[2]} - \frac{x(x - 1)}{q[2]} + z_0(x, y)$$

$$z_3(x, y) = -\frac{(1 - y)([2] - y)}{[2]} - \frac{(y - x)(y - qx - 1)}{q[2]} - \frac{x(x - 1)}{q[2]} + z_0(x, y)$$

$$z_4(x, y) = -\frac{(1 - y)([2] - y)}{[2]} - \frac{(y - x)(y - qx - 1)}{q[2]} + \frac{x(x - 1)}{q[2]} + z_0(x, y).$$

We now determine the relative maxima of $\lambda_2(x, y; q)$ and the results are summarised in the Tables 6.3 and 6.4 below.

	$z_1(x, y)$	$z_2(x, y)$
$\frac{\partial z}{\partial x}$	$\frac{2}{q}(y - 2x)$	$-\frac{2}{q[2]}(2x - 1)$
$\frac{\partial z}{\partial y}$	$-\frac{2}{q[2]}(2y - [2]x - 1)$	$-\frac{2}{[2]}(2y - 2 - q)$
Critical point (x_0, y_0)	$\left(\frac{1}{3 - q}, \frac{2}{3 - q}\right)$	$\left(\frac{1}{2}, \frac{2 + q}{2}\right)$
Condition on q such that $(x_0, y_0) \in A_k$	$0 \le x_0 \le y_0$ $y_0 \le 1$ $\Rightarrow 0 < q < 1$	$x_0 \ge 0$ $qx_0 + 1 \le y_0 \le [2]$ $\Rightarrow q > 0$
Δ	$\frac{4(3 - q)}{q^2[2]} > 0$	$\frac{16}{q[2]^2} > 0$

Table 6.3

	$z_3(x, y)$	$z_4(x, y)$
$\frac{\partial z}{\partial x}$	$\frac{2}{q}(y-2x)$	$\frac{2}{q[2]}([2]y-2qx-1)$
$\frac{\partial z}{\partial y}$	$-\frac{2}{q}(2y-x-[2])$	$-\frac{2}{q}(2y-x-[2])$
Critical point (x_0, y_0)	$(\frac{[2]}{3}, \frac{2[2]}{3})$	$(\frac{q^2+2q-1}{3q-1}, \frac{2q^2+2q-1}{3q-1})$
Condition on q such that $(x_0, y_0) \in A_k$	$x_0 \le 1$ $1 \le y_0 \le qx_0 + 1$ $\Rightarrow 1/2 \le q \le 2$ $(q \neq 1)$	$x_0 \ge 1$ $x_0 \le y_0 \le [2]$ $\Rightarrow q > 1$
Δ	$\frac{12}{q^2} > 0$	$\frac{4(3q-1)}{q^2(1+q)} > 0$

Table 6.4

In these tables Δ denotes the value of

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\, \partial y}\right)^2$$

evaluated at (x_0, y_0).

We see that the existence of a local maximum in each subregion depends on varying conditions on q. Hence the Lebesgue function does not necessarily have a local maximum in every subregion of S_2. We note further that, if $q \to 1$, then the local maxima points of $\lambda_2(x, y; q)$ tend to (1/2, 1), (1/2, 3/2), (1, 3/2) and (2/3, 4/3). The first three are the middle points of the boundaries of A_3 and the last point is the centroid of A_3. We also notice that if a local maximum of $\lambda_2(x, y; q)$ exists in a subregion, then it is unique.

6.4 The characteristic nature of the Lebesgue functions

In the case of the one-dimensional Lebesgue function, we showed in Lemma 6.1 that $\lambda_n(x) > 1$ for all $x \in [-1, 1] \setminus \{x_0, x_1, \ldots, x_n\}$. We now show that this property can be extended, in the two-dimensional case, to $\lambda_n(x, y; q)$. First we state:

Lemma 6.2 Let $\lambda_2(x, y; q)$ be the Lebesgue function defined on S_2. Then

$$\lambda_2(x, y; q) > 1$$

if (x, y) is not one of the interpolation nodes on S_2.

Proof Suppose that for some point (x, y) in $R_2 = S_2 \setminus \{([i], [j]) : 0 \leq i \leq j \leq 2\}$,

$$\lambda_2(x, y; q) = \sum_{j=0}^{2} \sum_{i=0}^{j} | L_{i,j}^2(x, y; q) | = 1.$$

Then, on using property (6.7), we have

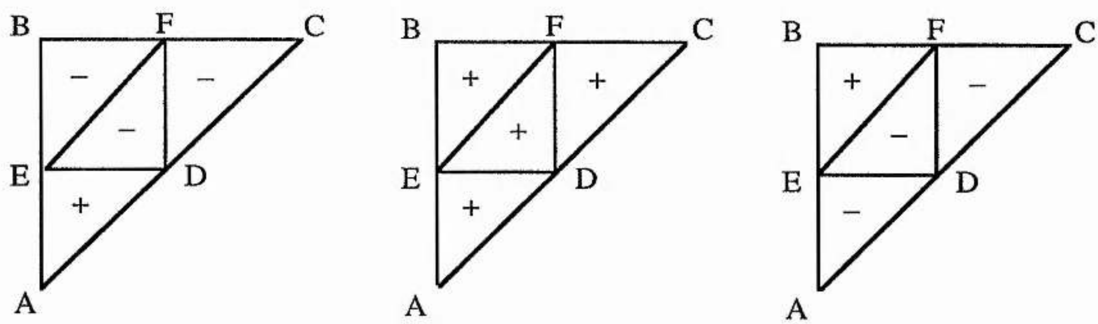
$$\sum_{j=0}^{2} \sum_{i=0}^{j} | L_{i,j}^2(x, y; q) | = | \sum_{j=0}^{2} \sum_{i=0}^{j} L_{i,j}^2(x, y; q) | .$$

This implies that each $L_{i,j}^2(x, y; q)$, $0 \leq i \leq j \leq 2$, has the same sign at these points.

Now divide $S_2 = \Delta ABC$ into four subtriangles as shown in Figure 6.16. Let us verify the sign of $L_{i,j}^2(x, y; q)$ on each of the subregions. We begin with the first three Lagrange coefficients L_A, L_E and L_B. Here we have used $L_K = L_K^2(x, y; q)$ to denote the Lagrange coefficient with respect to the interpolation node K on ΔABC. We see that each L_K is expressible as a product of two linear forms, so that

$$L_K(x) = c\,(a_1x + b_1y + c_1)\,(a_2x + b_2y + c_2)$$

and therefore L_K is zero only on the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ and changes sign on these lines. See also Figures 6.17, 6.18 and 6.19 (in Appendix 6B) which show the surfaces of Lagrange coefficients L_A, L_E and L_B respectively. Note that each L_K has the value one at the node K and is zero at all other interpolation nodes. Hence the single value at the node K determines the sign of L_K on ΔABC and the lines (which contains the other interpolation nodes) separate ΔABC into regions where L_K has a constant sign.



Sign of L_A Sign of L_E Sign of L_B

Figure 6.20

In particular, let us determine the sign of $L_A = (1 - y)([2] - y)/[2]$. Since L_A is positive at A, the same sign is maintained in the interior of subtriangle ADE and, in the interior of the lines AE and AD. This means the sign of L_A changes to negative in the interior of subtriangles BEF, DEF and CDF, and in the interiors of the lines BE, EF, DF and CD because these parts lie on the other side of the line DE. Note that, the signs of L_E and L_B are determined similarly and both are shown in Figure 6.20. We see that L_A, L_E and L_B do not have the same sign in the interior of

(i) the subtriangles AED, BEF, DEF and CDF,

(ii) the lines DE, EF and FD,

(iii) the outer lines AE and BE.

To complete the proof, we consider the sets $\{L_B, L_F, L_C\}$ and $\{L_A, L_D, L_C\}$. By symmetry, each set does not have the same sign on the sets (i) and (ii) above and also on the lines BF, CF and AD, CD respectively. Hence the six Lagrange coefficients do not have the same sign at any point of R_2. Thus no point in R_2 where λ_2 has the value one and, since $\lambda_2 \geq 1$ on ΔABC, this means $\lambda_2(x, y; q) > 1$ on R_2.

Lemma 6.3 Let $R_n = S_n \setminus \{([i], [j]): 0 \leq i \leq j \leq n\}$ where $n \geq 3$. Then

$$\lambda_n(x, y; q) > 1 \quad \text{for all } (x, y) \in R_n.$$

Proof Lemma 6.2 dealt with the case where $n = 2$. For $n \geq 3$, suppose that $\lambda_n(x, y; q) = 1$ for some point $(x, y) \in R_n$. Using the same argument as in Lemma 6.2, we see that all the Lagrange coefficients $L^n_{i,j}(x, y; q)$, $0 \leq i \leq j \leq n$, must have the same sign at these points.

Let us divide S_n into triangular subregions by drawing the lines $x = [\nu]$, $y = [\nu]$ and $y = q^\nu x + [\nu]$ where $\nu = 1, 2, \ldots, n-1$. We will determine the sign of $L^n_{i,j}(x, y; q)$ at all subregions of S_n. However it suffices to find the sign on any subtriangle ABC of S_n of the form

$$\Delta ABC = \{(x, y):\ x \geq [i],\ y \leq [j+2],\ y \geq q^{j-i}x + [j-i]\}$$

where $0 \leq i \leq j \leq n-2$. We see that the subtriangle ABC contains six interpolation nodes $\{([i+r], [j+s]): 0 \leq r \leq s \leq 2\}$. See Figure 6.21. The lines joining the middle nodes of ΔABC, that is, $x = [i+1]$, $y = [j+1]$ and $y = q^{j-i+1}x + [j-i+1]$ divide the subtriangle further into four subregions ADE, BEF, DEF and CDF.

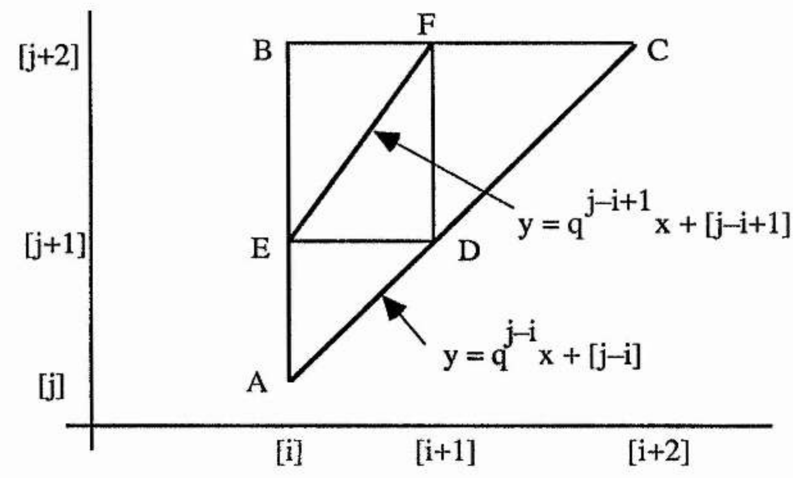


Figure 6.21

We know that each Lagrange coefficient, $L^n_{i,j}(x, y; q)$ in (6.5) consists of n linear forms which pass through all interpolation nodes except the node ([i], [j]). Therefore the sign of each $L^n_{i,j}(x, y; q)$ remains unchanged in any subregion formed by the above linear forms until a boundary is crossed. Of the $(n + 1)(n + 2)/2$ Lagrange coefficients $L^n_{i,j}(x, y)$, we will restrict our attention to the six Lagrange coefficients associated with the nodes on the subtriangle ABC. We see that the previous argument in Figure 6.20 of Lemma 6.2, also applies to the above ΔABC. Therefore we have shown that not even six $L^n_{i,j}(x, y)$ named above have the same sign on

$$\Delta ABC \setminus \{([i + r], [j + s]): 0 \le r \le s \le 2\}.$$

This implies that all $L^n_{i,j}(x, y)$, $0 \le i \le j \le n$ do not have the same sign on any point in R_n. Thus, there are no points in the region R_n such that $\lambda_n(x, y; q) = 1$. Finally, since $\lambda_n(x, y; q) \ge 1$ for all $(x, y) \in S_n$ then $\lambda_n(x, y; q) > 1$ on R_n.

6.5 Lebesgue constants

In this section, we shall investigate two properties: first, the minimum value of the Lebesgue constants and second, the symmetric property of the constant. We know that the Lebesgue function on S_n is bounded below by 1 and

$$\min_{(x,y)\in S_n} \lambda_n(x, y; q) = 1$$

holds for all n and q. On taking the maximum value of the Lebesgue function we shall investigate the variation of

$$\Lambda_n(q) = \max_{(x,y)\in S_n} \lambda_n(x, y; q)$$

with respect to n and q. The case n = 1 is exceptional, since $\Lambda_1(q) = 1$ for all $q > 0$.

We now evaluate $\Lambda_n(q)$ for n = 2, 3 and 4, and compare $\Lambda_n(q)$ for different values of q. In our numerical experiment we have taken values of q between 0.3 and 4.0 . For each q, we chose a small grid spacing for the mesh on S_n so that the value of $\Lambda_n(q)$ is correct to three decimal places. Note that, for a given q

$$\max_{(x,y)\in G} \lambda_n(x, y; q) \leq \max_{(x,y)\in S_n} \lambda_n(x, y; q)$$

where G is the set of grid nodes on S_n. Hence the approximate value of $\Lambda_n(q)$ does not only depend on the smaller grid spacing, d, it is also depends on how close is the grid mesh to a maximum point of $\lambda_n(x, y; q)$. We now give the approximate values of $\Lambda_n(q)$ for n = 2, 3 and 4.

q	$\Lambda_2(q)$	$\Lambda_3(q)$	$\Lambda_4(q)$
4.0	3.326	37.602	1826.140
3.0	2.687	17.733	362.349
2.0	2.067	6.884	43.465
1.5	1.778	3.967	12.142
1.25	1.700	2.983	6.333
1.1	1.673	2.552	4.374
1.05	1.668	2.387	3.890
1.005	1.667	2.279	3.513
0.95	1.668	2.394	3.914
0.9	1.674	2.553	4.491
0.8	1.700	2.983	6.332
0.7	1.752	3.656	10.064
0.5	2.067	6.884	43.466
0.4	2.374	11.357	133.970
0.3	2.899	23.213	650.651

Table 6.5

We see from Table 6.5 that the values of $\Lambda_n(q)$ decrease as q approaches 1. We conjecture that $\Lambda_n(q)$ attains its minimum at q = 1. In Appendix 6C, we give a detailed calculation of the minimum of $\Lambda_n(q)$ through the sequences of q-values

0.957, 0.962, ..., 0.997 and 1.052, 1.047, ..., 1.002.

Although this result strengthens our belief that the conjecture is correct, we have no proof of this.

If we seek the minimum values of $\Lambda_n(q)$ for n = 1 , 2, 3, 4 and 5, we obtain:

Degree n	1	2	3	4	5
$\min_{q>0} \Lambda_n(q)$	1.000	1.667	2.274	3.890	6.638

We note that, for the values of n considered, $\min_{q>0} \Lambda_n(q)$ increases as n increases.

Let us refer again to Table 6.5. On comparing $\Lambda_n(q)$ with $\Lambda_n(1/q)$ for n = 2, 3 and 4, we obtain

$$\Lambda_n(0.95) \cong \Lambda_n(1.05), \qquad \Lambda_n(0.90) \cong \Lambda_n(1.10),$$

$$\Lambda_n(0.80) \cong \Lambda_n(1.25), \qquad \Lambda_n(0.50) \cong \Lambda_n(2.00).$$

The result suggests that $\Lambda_n(q)$ is invariant under a transformation $q \to 1/q$. We show that this conjecture is correct.

Lemma 6.4 For n = 1, 2, 3, ... the Lebesgue constant satisfies $\Lambda_n(q) = \Lambda_n(1/q)$.

Proof. Let $L^n_{i,j}(x, y; 1/q)$ be the Lagrange coefficient of the interpolating polynomial on the triangle $S_n(1/q) = \{(x, y): 0 \le x \le y \le [n]/q^{n-1}\}$. From Lemma 5.1 we have

$$L^n_{i,j}(x, y; 1/q) = L^n_{n-j,n-i}(\xi, \eta; q).$$

where

$$x = ([n] - \eta)/q^{n-1} \quad \text{and} \quad y = ([n] - \xi)/q^{n-1}.$$

Hence, the Lebesgue constant on $S_n(1/q)$ is transformed into

$$\Lambda_n(1/q) = \max_{0 \le x \le y \le [n]/q^{n-1}} \sum_{j=0}^{n} \sum_{i=0}^{j} | L^n_{i,j}(x, y; 1/q) |$$

$$= \max_{0 \le \xi \le \eta \le [n]} \sum_{j=0}^{n} \sum_{i=0}^{j} \mid L^n_{n-j,n-i}(\xi, \eta; q) \mid . \tag{6.9}$$

Now substitute $r = n - j$, $s = n - i$ and change the order of summation in (6.9), to give

$$\Lambda_n(1/q) = \max_{0 \le \xi \le \eta \le [n]} \sum_{s=0}^{n} \sum_{r=0}^{s} \mid L^n_{r,s}(\xi, \eta; q) \mid = \Lambda_n(q).$$

This complete the proof.

6.6 The discontinuity of partial derivatives of the Lebesgue functions

Let us consider the surface of $z = \lambda_2(x, y; q)$ over the subtriangles A_i, $i = 1, 2, 3$ and 4, as shown in Figure 6.16. We know from Lemma 6.2 that, on any of the subtriangles A_i, the surface of z is always above the plane $Z = 1$ and it assumes the value one only at the vertices of the subtriangle. If we restrict $z = \lambda_2(x, y; q)$ to the interior of A_i, then z reduces to a polynomial and hence all its partial derivatives exist and are continuous. We notice also that the gradient of z changes considerably at the boundaries of the subtriangles.

First we find partial derivative of z at the adjacent boundary of A_3 and A_4. Here it is appropriate to consider the function z in the subregion $A_3 \cup A_4$. Using (6.8) and the fact that $1 \le y \le [2]$ and $x \le y \le qx + 1$, then z can be written as

$$z = g(x, y) + \frac{|x|\ |x-1|}{q[2]}$$

where $g(x, y)$ is a polynomial that does not change sign in $A_3 \cup A_4$. For any point $(x, y) \in$ the interior of A_3, we have

$$z = g(x, y) + \frac{x(1-x)}{q[2]}$$

and hence

$$\frac{\partial z}{\partial x} = \frac{\partial g(x, y)}{\partial x} + \frac{1-2x}{q[2]}.$$

Similarly, if $(x, y) \in A_4$

$$z = g(x, y) + \frac{x(x-1)}{q[2]}$$

and

$$\frac{\partial z}{\partial x} = \frac{\partial g(x, y)}{\partial x} + \frac{2x-1}{q[2]}.$$

On taking the limit of $\partial z/\partial x$ as x approach the boundary, we obtain

$$\frac{\partial z}{\partial x} \to \frac{\partial g(1, y)}{\partial x} - \frac{1}{q[2]}, \text{ as } x \to 1^- \text{ from } A_3, \quad (6.10)$$

and

$$\frac{\partial z}{\partial x} \to \frac{\partial g(1, y)}{\partial x} + \frac{1}{q[2]}, \text{ as } x \to 1^+ \text{ from } A_4. \quad (6.11)$$

Hence $\partial z/\partial x$ is discontinuous across the boundary. On the other hand, there is no discontinuity of $\partial z/\partial y$ along the boundary since $\partial z/\partial y = \partial g(1, y)/\partial y$ for all $1 \le y \le [2]$.

We note also that the slope of z on the opposite side of the boundary may not necessarily of different sign. In particular, since

$$g(x, y) = -\frac{(1-y)([2]-y)}{[2]} + \frac{([2]-y)(y-x)}{q} + \frac{([2]-y)x}{q}$$

$$- \frac{(y-x)(y-qx-1)}{q[2]} + \frac{x(y-x)}{q}$$

then

$$\frac{\partial g}{\partial x} = \frac{2[2]y - 4qx - 2x - 1}{q[2]}.$$

From (6.10) and (6.11), we see that

$$-\frac{2}{q} \le \lim_{x\to 1^-} \frac{\partial z}{\partial x} \le \frac{2}{q}(q-1) \quad \text{for all } 1 \le y \le [2]$$

and

$$-\frac{2}{[2]} \le \lim_{x\to 1^+} \frac{\partial z}{\partial x} < 0 \quad \text{for all } 1 \le y < 1 + \frac{q}{[2]},$$

$$0 \le \lim_{x\to 1^+} \frac{\partial z}{\partial x} \le \frac{2q}{[2]} \qquad \text{for all } 1 + \frac{q}{[2]} \le y \le [2].$$

Therefore if $0 < q < 1$, the slope on both sides of the boundary remains negative for all $1 \le y < 1 + \frac{q}{[2]}$ and x is sufficiently closed to the boundary.

We now generalise the above result to the partial derivative of any Lebesgue function $\lambda_n(x, y; q)$ at adjacent boundaries of subtriangles of S_n. We obtain:

Theorem 6.1 Let S_n be divided into subtriangles by the lines

$$x = [\nu], \; y = [\nu] \text{ and } y = q^\nu x + [\nu], \; \nu = 1, 2, \ldots, n-1.$$

Then the directional derivative of $\lambda_n(x, y; q)$ at the interior boundary of any two adjacent subtriangles is discontinuous in all directions except along the boundary.

Proof The subdivision of S_n produces n^2 subtriangles. However it suffices to prove the discontinuity at the following $n(n-1)/2$ subtriangles ABC of the form

$$\Delta ABC = \{(x, y): \; y \ge [j], \; x \le [i] \text{ and } y \le q^{j-i+1}x + [j-i+1]\}$$

where $j = 1, 2, \ldots, n-1$ and $i = 1, 2, \ldots, j$.

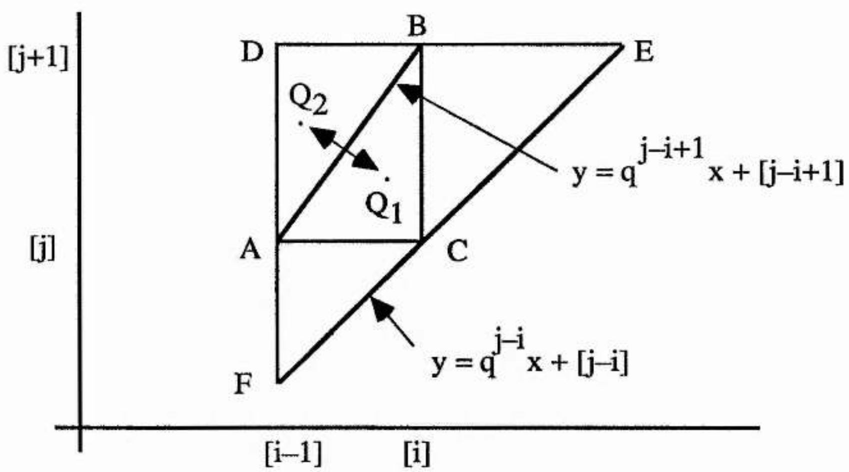


Figure 6.22

First we consider the Lebesgue function $\lambda_n(x, y; q)$, defined in (6.6) over the region $\Delta ABC \cup \Delta ABD$. Here we have the segment AB, the boundary of two adjacent subtriangles ΔABC and ΔABD and it lies on the line $y = q^{j-i+1}x + [j-i+1]$. We see that each $L^n_{i,j}(x, y; q)$, $0 \le i \le j \le n$, does not change sign within a subtriangle, but only when a boundary is crossed. Hence, the Lebesgue function on $\Delta ABC \cup \Delta ABD$ can be written as two sums,

$$\lambda_n(x, y; q) = \sum\nolimits_1 |L^n_{i,j}(x, y; q)| + \sum\nolimits_2 |L^n_{i,j}(x, y; q)|,$$

where the second sum has all terms which contain the form $y - q^{j-i+1}x - [j-i+1]$. The first sum is a polynomial and for simplicity we let

$$\sum\nolimits_1 |L^n_{i,j}(x, y; q)| = g(x, y).$$

On factoring the second sum, we obtain

$$\sum\nolimits_2 |L^n_{i,j}(x, y; q)| = |y - q^{j-i+1}x + [j-i+1]|\ h(x, y)$$

for some polynomial h(x, y). Thus the Lebesgue function can be written as

$$\lambda_n(x, y; q) = g(x, y) + | y - q^{j-i+1}x + [j - i + 1] | \; h(x, y).$$

Note that, g(x, y) and h(x, y) do not change sign and both are positive in the interior of $\Delta ABC \cup \Delta ABD$. The reason for the latter is that each function is a sum of moduli and hence can only be zero if each term is zero.

Now let us find the directional derivative at any two points Q_1 and Q_2 in ΔABC and ΔABD respectively. For all $(x, y) \in$ the interior of ΔABC,

$$\lambda_n(x, y; q) = g(x, y) - (y - q^{j-i+1}x - [j - i + 1]) \, h(x, y).$$

This gives

$$\frac{\partial \lambda_n}{\partial x} = \frac{\partial g}{\partial x} - (y - q^{j-i+1}x - [j - i + 1]) \frac{\partial h}{\partial x} + q^{j-i+1}h,$$

$$\frac{\partial \lambda_n}{\partial y} = \frac{\partial g}{\partial y} - (y - q^{j-i+1}x - [j - i + 1]) \frac{\partial h}{\partial y} - h.$$

Then the directional derivative at Q_1 in the direction of θ is given by

$$\frac{d\lambda_n}{ds} = \frac{\partial \lambda_n}{\partial x} \cos \theta + \frac{\partial \lambda_n}{\partial y} \sin \theta$$

$$= \frac{\partial g}{\partial x} \cos \theta + \frac{\partial g}{\partial y} \sin \theta - (y - q^{j-i+1}x - [j - i + 1])(\frac{\partial h}{\partial x} \cos \theta + \frac{\partial h}{\partial y} \sin \theta)$$

$$+ q^{j-i+1}h \cos \theta - h \sin \theta ,$$

evaluated at the point Q_1 in ΔABC. Similarly, we have for any $(x, y) \in$ interior ΔABD

$$\lambda_n(x, y; q) = g(x, y) + (y - q^{j-i+1}x - [j - i + 1])\, h(x, y)$$

and

$$\frac{\partial \lambda_n}{\partial x} = \frac{\partial g}{\partial x} + (y - q^{j-i+1}x - [j - i + 1])\frac{\partial h}{\partial x} - q^{j-i+1}h,$$

$$\frac{\partial \lambda_n}{\partial y} = \frac{\partial g}{\partial y} + (y - q^{j-i+1}x - [j - i + 1])\frac{\partial h}{\partial y} + h.$$

Hence the directional derivative at Q_2 takes the form

$$\frac{d\lambda_n}{ds} = \frac{\partial g}{\partial x}\cos\theta + \frac{\partial g}{\partial y}\sin\theta + (y - q^{j-i+1}x - [j - i + 1])\left(\frac{\partial h}{\partial x}\cos\theta + \frac{\partial h}{\partial y}\sin\theta\right)$$

$$- q^{j-i+1}h\cos\theta + h\sin\theta,$$

evaluated at the point Q_2 in ΔABD.

Let $Q = (a, b)$ be any point in the interior of AB. On taking the limit of $d\lambda_n/ds$ as $Q_1 \to Q$, we obtain

$$\lim_{Q_1\to Q}\frac{d\lambda_n}{ds} = \frac{\partial g(a, b)}{\partial x}\cos\theta + \frac{\partial g(a,b)}{\partial y}\sin\theta + q^{j-i+1}h(a, b)\cos\theta - h(a, b)\sin\theta.$$

Similarly, if $Q_2 \to Q$ then

$$\lim_{Q_2\to Q}\frac{d\lambda_n}{ds} = \frac{\partial g(a, b)}{\partial x}\cos\theta + \frac{\partial g(a,b)}{\partial y}\sin\theta - q^{j-i+1}h(a, b)\cos\theta + h(a, b)\sin\theta.$$

Thus directional derivative at Q is continuous if and only if

$$q^{j-i+1}h(a, b)\cos\theta - h(a, b)\sin\theta = 0 \qquad (6.12)$$

Since $h \neq 0$ on the interior of AB, then the directional derivative at Q is continuous if and only if $\theta = \tan^{-1}(q^{j-i+1})$, the direction of AB itself. This completes the proof for the discontinuity on the boundary AB.

Finally, we note that the discontinuity at the other boundaries of ΔABC can be dealt with similarly. In fact, if Q is in the interior of BC or interior of AC then equation (6.12) reduces to $h \cos \theta = 0$ or $h \sin \theta = 0$ respectively.

*****

Appendix 2 Evaluation of forward difference formula of the interpolating polynomial on q-triangle

```
program pned;

const
nmax = 8;
type  DDfij = array[0..nmax, 0..nmax, 0..nmax, 0..nmax] of real;

var
f: text;                                        n, m, k, r, s, i, j:  integer;
q, fvlue, x, y, x1, y1, psum, pp:  real;        d: DDfij;

function  F(i, j: integer):  real;          {to calculate f([i], [j])}
var  fij: real;
begin
fij:= 0.003*(1–exp(i*ln(q)))/(1–q);    fij:= fij + 0.01*(1–exp(j*ln(q)))/(1–q);
fij:= exp(fij);    F:= fij;
end;

function bmial(x: real;  s: integer):  real;         {to calculate the q-binomial coefficient}
var
i, r:  integer;              B: real;
begin
if  s > 0  then
      begin
      B:= 1;  r:= s;   for  i:= 1  to  s  do
            begin
            B:= B*((1–exp(x*ln(q)))/(1–exp(r*ln(q))));
            x:= x–1;          r:= r–1;
            end;
      bmial:= B;
      end
else  bmial:= 1;
end;

begin   {main program}
rewrite(f,'pned.out');
writeln('Enter the degree of the polynomial and the value for the q-integer');
readln(n, q);

for  m:= 0  to  n  do
      for  k:= 0  to  m  do
            d[0, 0, k, m–k]:= F(k, m–k);
writeln('The list of D(0)D(0)f(ij)  is   ');       writeln(f, 'The list of D(0)D(0)f(ij)  is   ');

for  m:= 0  to  n  do
      for  k:= 0  to  m  do
            begin   writeln(d[0, 0, k, m–k]);   writeln(f, d[0, 0, k, m–k]);   end;
```

```
for r:= 0 to n–1 do
      for s:= 0 to n–1–r–1 do
            for i:= 0 to n–1–r–s do
                  for j:= 0 to n–1–r–s–i do
                        begin
                        d[r+1, s, i, j]:= d[r, s, i+1, j] – exp(r*ln(q))*d[r, s, i, j];
                        d[r, s+1, i, j]:= d[r, s, i, j+1] – exp(s*ln(q))*d[r, s, i, j];
                        end;

writeln('The list of D(r)D(s)f(0, 0) is:  ');
writeln(f, 'The list of D(r)D(s)f(0, 0) is:  ');
for m:= 0 to n do
   for k:= 0 to m do
      begin   writeln(d[k, m–k, 0, 0]);
      writeln(f, d[k, m–k, 0, 0]);
      end;

writeln('Enter the chosen point (x, y)');        readln(x, y);
fvlue:= exp(0.003*x + 0.01*y);
writeln('The value of x is  ', x);        writeln('The value of y is  ', y);
writeln(f, 'The value of x is  ', x);     writeln(f, 'The value of y is  ', y);
x1:= ln(1–(1–q)*x)/ln(q);
y1:= ln(1–(1–q)*y)/ln(q);

psum:= 0;
for m:= 0 to n do
      for k:= 0 to m do
            begin
            pp:= bmial(x1, k);
            pp:= pp*bmial(y1, m–k);
            pp:= pp*d[k, m–k, 0, 0];
            psum:= psum + pp;
            end;

writeln('The value of Pn(x, y) is   ', psum);
writeln(f, 'The value of Pn(x, y) is   ', psum);
writeln('The value of f(x, y) is   ', fvlue);
writeln(f, 'The value of f(x, y) is   ', fvlue);
writeln('The error is   ', psum – fvlue);
writeln(f, 'The error is   ', psum – fvlue);
end.
```

***

Appendix 3 On the reducibility of the Lagrange coefficients

Let $L^n_{0,0}(x, y)$ be the Lagrange coefficient of the interpolating polynomial on the q-triangle as given in (3.17). From Lemma 3.7 and Lemma 3.8, we showed that the Lagrange coefficients $L^n_{0,0}(x, y) = 0$, $n = 2, 3$ are reducible. To discuss whether this property holds in general let us consider

$$L^4_{0,0}(x, y) = L^3_{0,0}(x, y) + \frac{x}{[4]!}(x-1)(x-[2])(x-[3]) + \frac{xy}{[3]!}(x-1)(x-[2])$$

$$+ \frac{xy}{[2][2]}(x-1)(y-1) + \frac{xy}{[3]!}(y-1)(y-[2]) + \frac{y}{[4]!}(y-1)(y-[2])(y-[3]).$$

We use MacTutor package [2] to plot some graphs of $L^4_{0,0}(x, y) = 0$ and these are shown in Figures 3.7 to 3.12. We see that all the graphs of $L^4_{0,0}(x, y) = 0$ are symmetric with respect to the line $y = x$ and when $q > 1$, the graphs splits into four branches of "hyperbolic" form. When $0 < q < 1$, the graphs split into two closed curves in two different configurations, one inside the other if $0 < q \le 0.71$ and two separate closed curves if $0.73 \le q < 1$. The transition between these two phases is not clear, the best possible picture available (see Figure 3.9) shows that the graph might look like two ellipses. We examine this idea further algebraically.

Consider the graph of $L^4_{0,0}(x, y) = 0$ with $q = 0.72$. Suppose that the graphs split into two ellipses ABCDEF and GHIJKL. Then each ellipse must satisfy

$$(x + y - a)^2 + b(x - y)^2 = c^2, \quad \text{for some constants } a, b, c$$

where the axes are $y - x = 0$ and $y + x = a$ respectively. The ellipses must pass through 14 interpolation nodes of the q-triangle. Using just three nodes ([1], [0]), ([2], [0]) and ([2], [1]) from the six interpolation nodes of the first ellipse ABCDEF, (the others are ([0], [1]), ([0], [2]) and ([1], [2])), we obtain

$$b = 1/3, \quad a = (4 + 2q)/3 \quad \text{and} \quad c^2 = 4(1 + q + q^2)/9 .$$

Therefore the equation of the first ellipse ABCDEF is

$$\{x + y - 2(1 + [2])/3\}^2 + (x - y)^2 / 3 = 4[3]/9 .$$

The other eight nodes must lie on the second ellipse GHIJKL. Substitute the first two nodes ([1], [1]), ([2], [2]) to gives $a = 2 + q$ and $c = q$. Using the second two nodes ([4], [0]) and ([3], [1]) we obtain two values for b and hence

$$(-q^6 - 2q^5 - q^4 + 2q^3 + 3q^2 - 1) / [4]^2 = (1 - q^2) / [2]^2 .$$

Thus $(q - 1)^2 (q + 1) (q^2 + q + 1) = 0$ which means b has no solution. Therefore $L^4_{0,0}(x, y) = 0$ does not factorizes into two ellipses.

Now let us consider the equation $L^5_{0,0}(x, y) = 0$. See Figures 3.13 to 3.20. If $0 < q < 0.744$ the graphs of $L^5_{0,0}(x, y) = 0$ consist of curve 1: a "hyperbolic curve" which interpolates $\{([k], [0]), ([0], [k]): k = 1, 2, \ldots, 5\}$ and, curve 2 and curve 3: two separate closed curves which interpolate $\{([i], [j]): 1 \le i, j \le 2, 2 \le i + j \le 3\}$ and $\{([i], [j]): 1 \le i, j \le 4, 4 \le i + j \le 5\}$ respectively. If $0.745 < q < 0.781$, curve 3 breaks up: the upper part of it forms a "hyperbolic curve", curve 4 say, which interpolates $\{([i], [j]): i, j \ge 0, i + j = 5\}$. The lower part of curve 3 connects to the curve 1 to form a closed curve which we shall call curve 5. Note that curve 5 interpolates the nodes $\{([k], [0]), ([0], [k]): 1 \le k \le 4\} \cup \{([i], [j]): i, j \ge 1, \ i + j = 4\}$ and it contains curve 2. If $0.782 \le q < 1$ both curve 2 and 5 break up: their upper parts combine to form a closed curved which interpolates $\{([i], [j]): i, j \ge 0, 3 \le i + j \le 4\}$ and their lower parts form another closed curved which interpolates $\{([i], [j]): i, j \ge 0, 1 \le i + j \le 2\}$. However curve 4 does not change form. If $q > 1$, the graphs separate into five "hyperbolic curves".

As q varies from 0.50 to 0.9, the graphs of $L^5_{0,0}(x, y) = 0$ approach two critical values, that is when q close to 0.7447 and 0.7812. In both cases, the two closed curves cannot be ellipses since the remaining curve is not linear. Hence $L^5_{0,0}(x, y) = 0$ is not reducible.

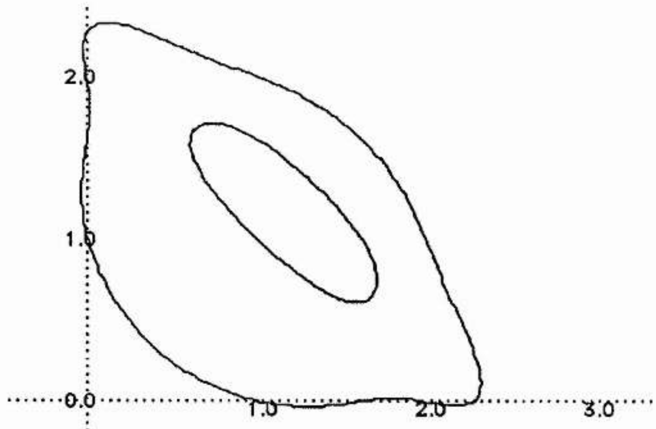


Figure 3.7 Graph of $L^4_{0,0}$ (x, y) = 0 with q = 0.62

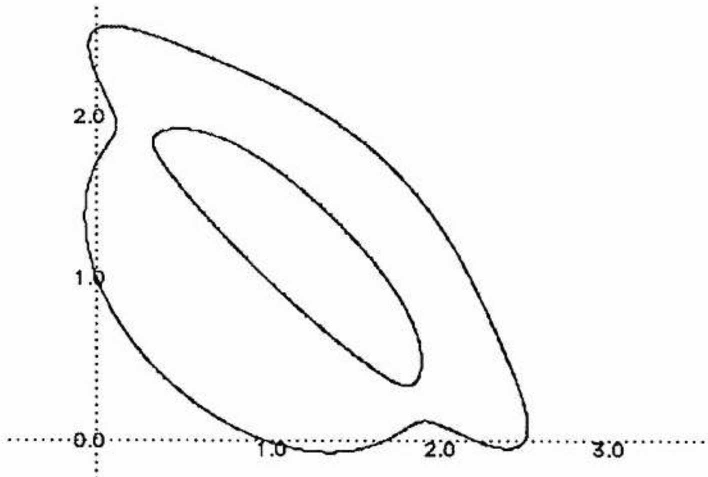


Figure 3.8 Graph of $L^4_{0,0}$ (x, y) = 0 with q = 0.71

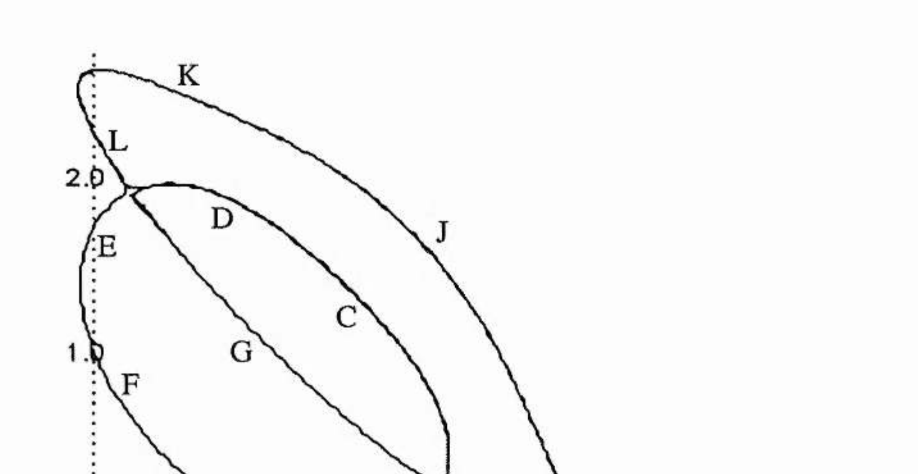


Figure 3.9 Graph of $L^4_{0,0}$ (x, y) = 0 with q = 0.72

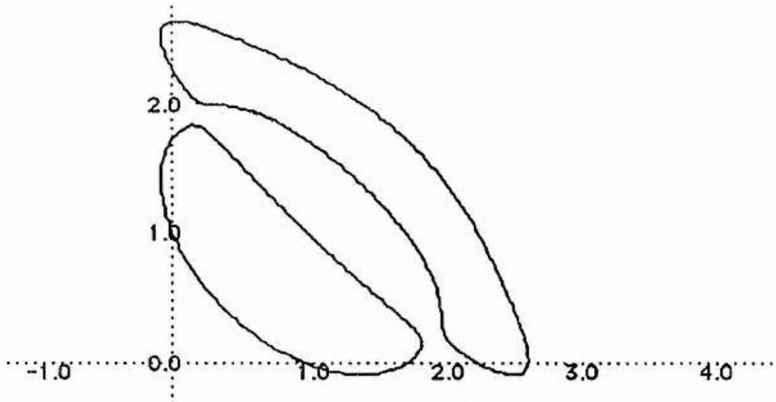


Figure 3.10 Graph of $L^{4}_{0,0}$ (x, y) = 0 with q = 0.73

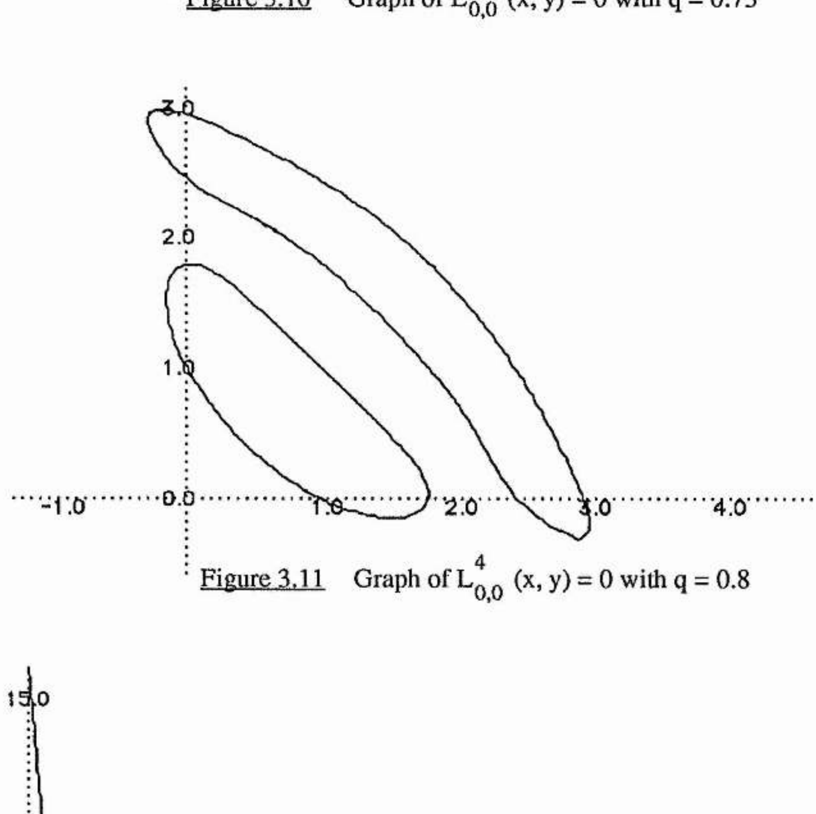
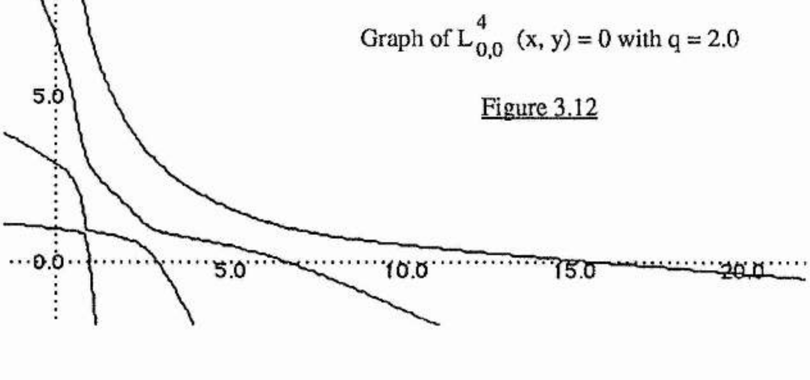


Figure 3.11 Graph of $L^{4}_{0,0}$ (x, y) = 0 with q = 0.8



Graph of $L^{4}_{0,0}$ (x, y) = 0 with q = 2.0

Figure 3.12

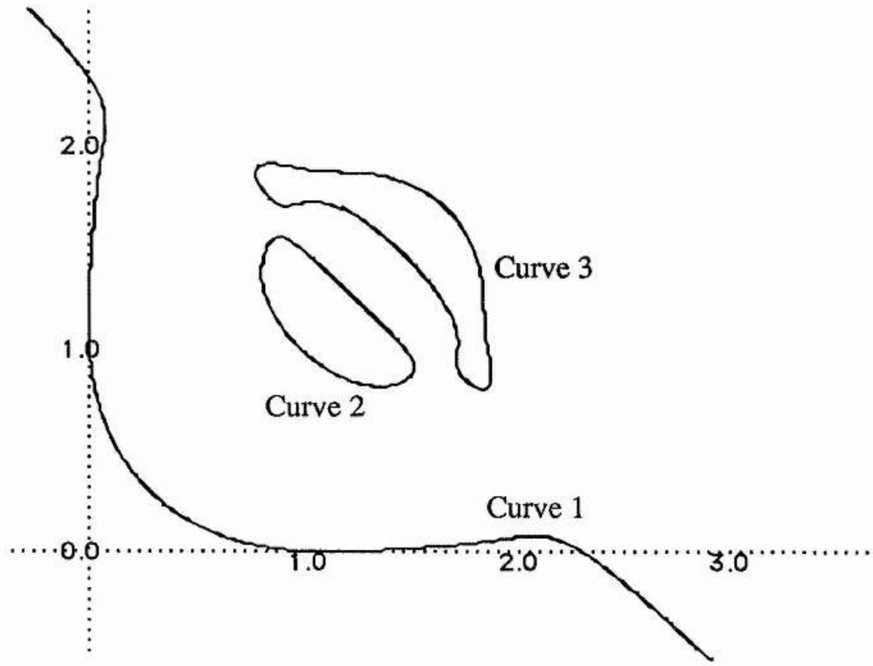


Figure 3.13 Graph of $L^{5}_{0,0}(x, y) = 0$ with q = 0.5

Figure 3.14 Graph of $L^{5}_{0,0}(x, y) = 0$ with q = 0.74

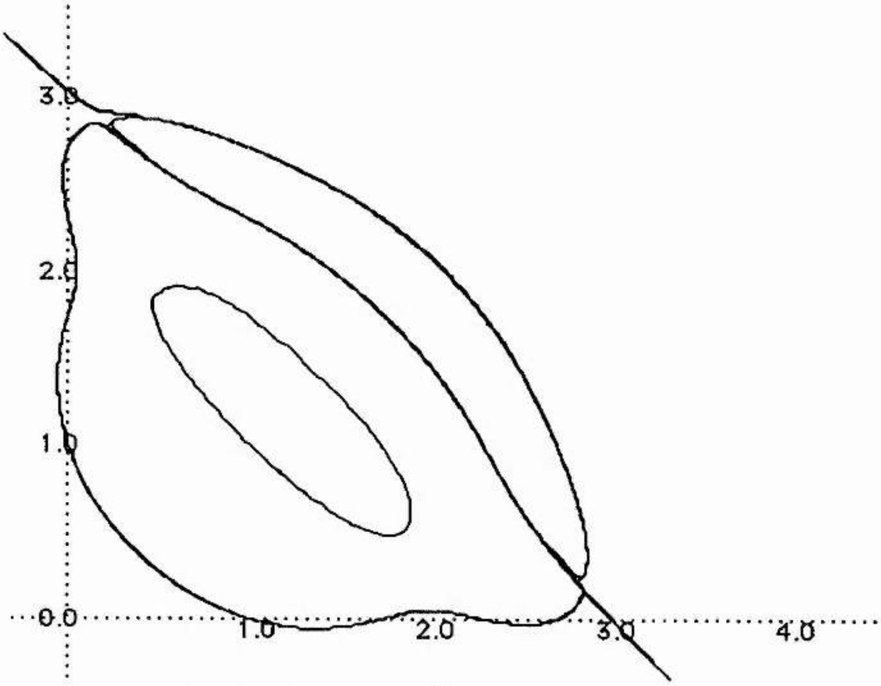


Figure 3.15 Graph of $L^5_{0,0}(x, y) = 0$ with $q = 0.7447$

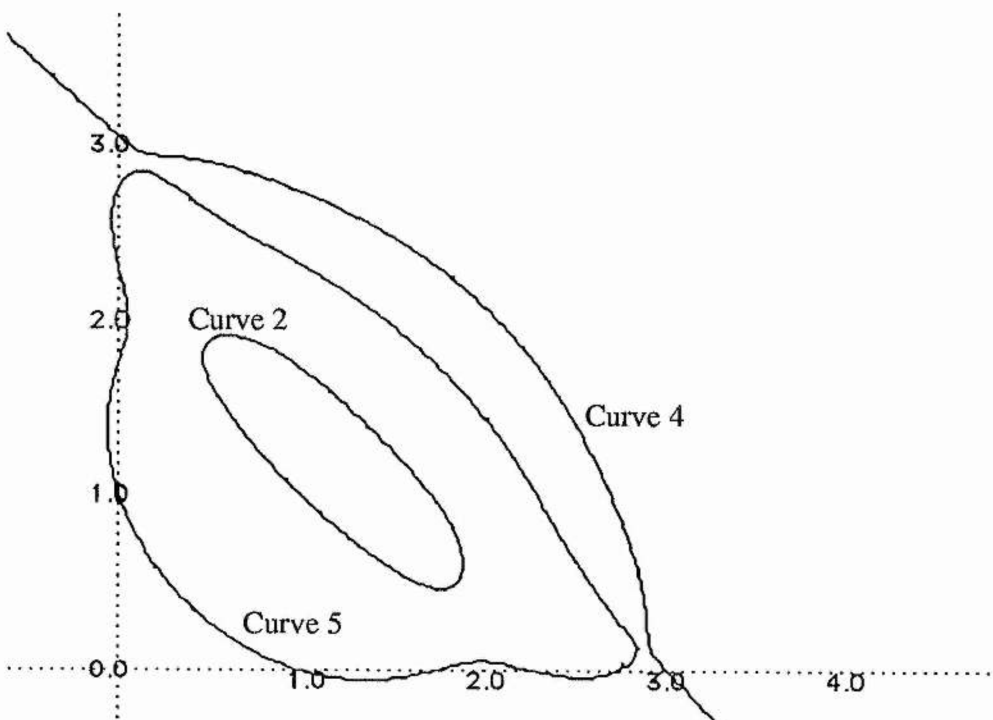


Figure 3.16 Graph of $L^5_{0,0}(x, y) = 0$ with $q = 0.75$

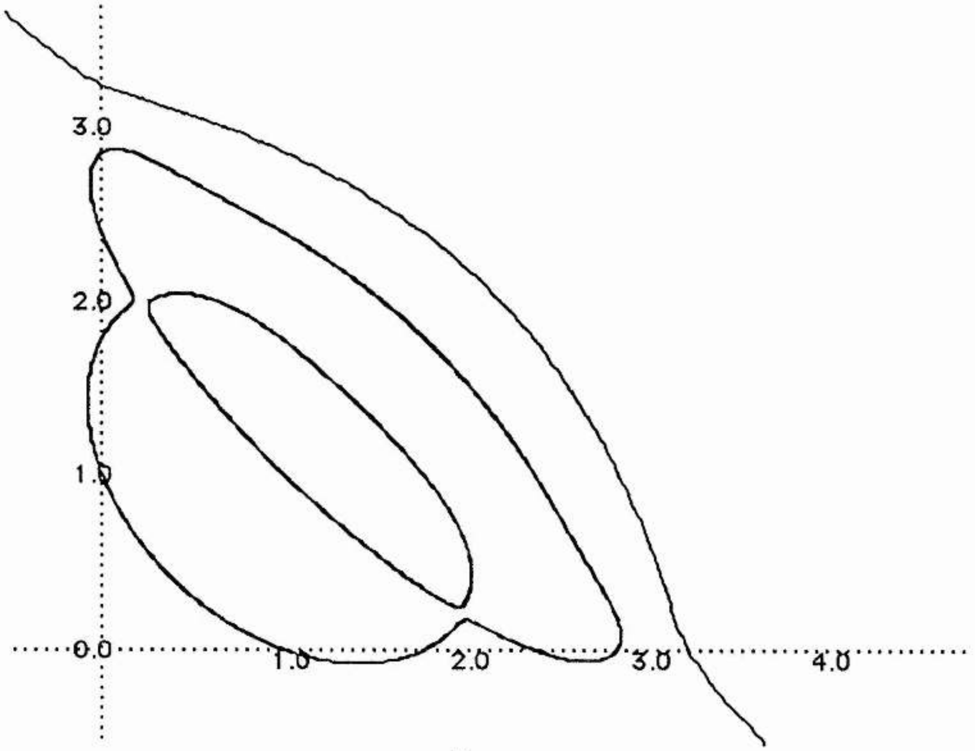


Figure 3.17 Graph of $L^5_{0,0}(x, y) = 0$ with q = 0.78

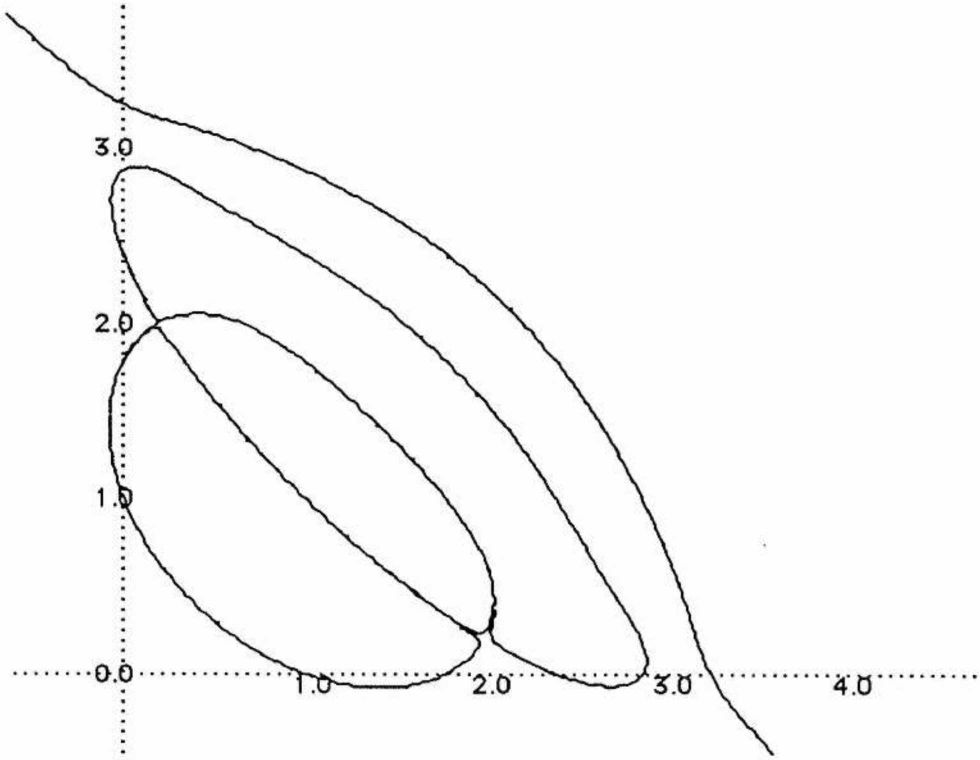
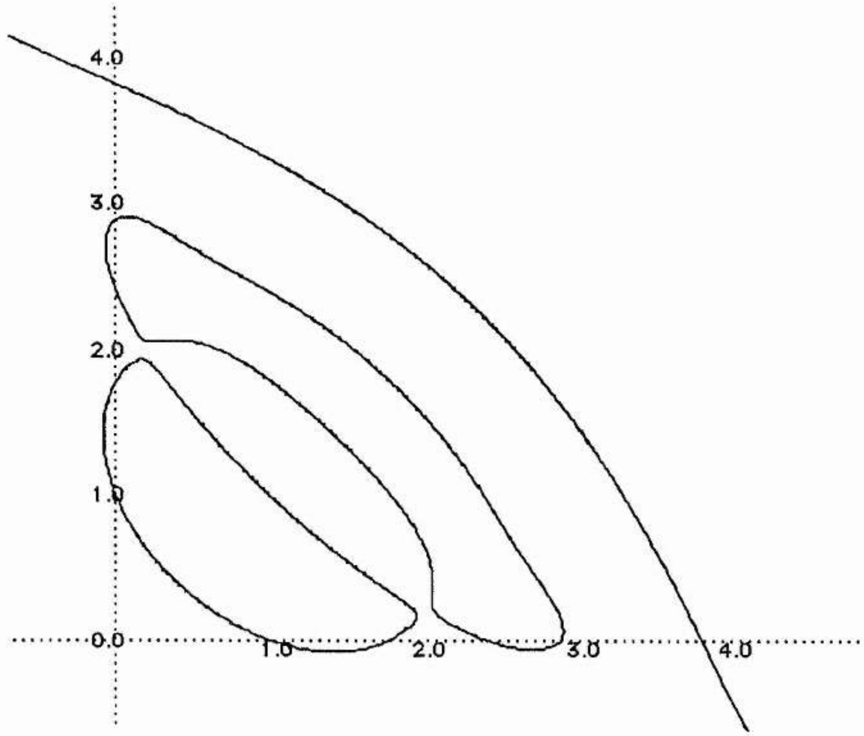
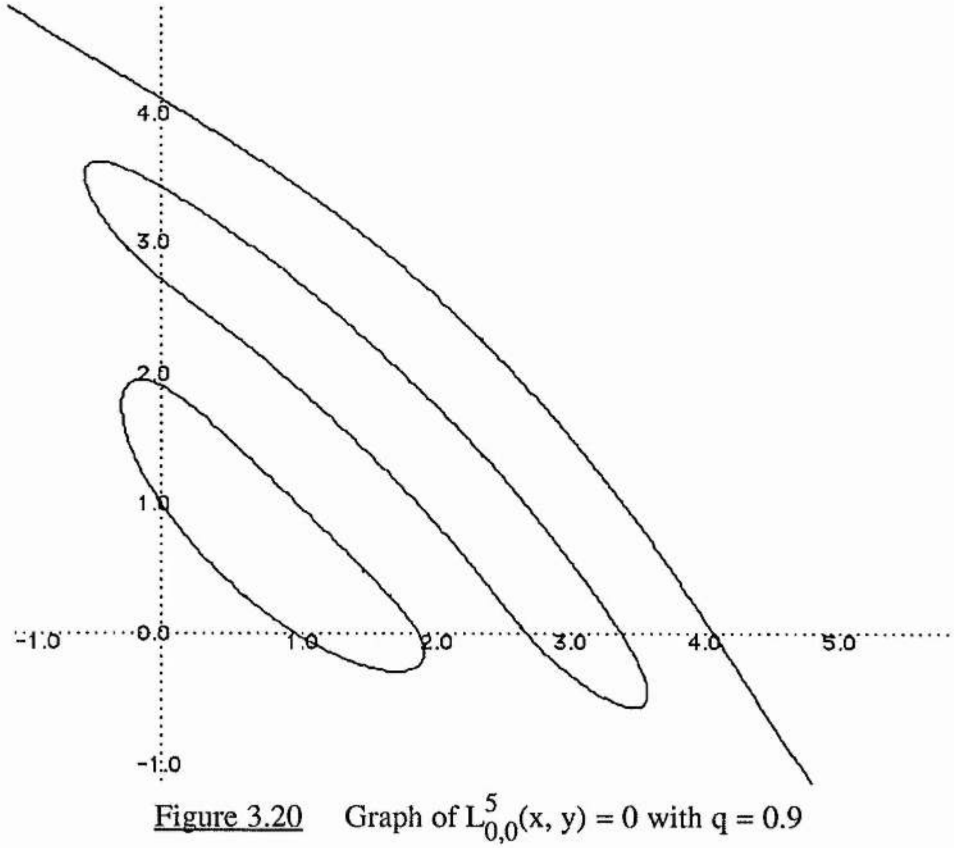


Figure 3.18 Graph of $L^5_{0,0}(x, y) = 0$ with q = 0.7812



<u>Figure 3.19</u> Graph of $L^5_{0,0}(x, y) = 0$ with $q = 0.782$



<u>Figure 3.20</u> Graph of $L^5_{0,0}(x, y) = 0$ with $q = 0.9$

***

APPENDIX 5A Calculation of the integration rules I_n

In Figures 5.2 and 5.3, and Tables 5.1 and 5.2, the weights are calculated directly from formula (5.4)

$$w_{i,j}^n = \int_0^{[n]} \int_0^{y} L_{i,j}^n(x, y)\, dx\, dy$$

where the Lagrange coefficients $L_{i,j}^n(x, y)$ are defined as in (5.2). Here we give a detailed Maple calculation of the integration rules I_n, n = 1, 2, ..., 5. First we comment on some of the Maple notations used in the calculation, which are as follows:

Sij := int ($L_{i,j}^n$(x, y), x = 0..y) for evaluating $\int_0^{y} L_{i,j}^n(x, y)\, dx$,

Iij := int (Sij , y = 0..[n]) for evaluating $\int_0^{[n]} S_{i,j}^n\, dy = \int_0^{[n]} \int_0^{y} L_{i,j}^n(x, y)\, dx\, dy$,

limit (Iij , q = 1) for evaluating $\lim_{q \to 1} w_{i,j}^n\, (q)$.

For each n = 1, 2, 3, 4, 5 , we also check that the sum of all the weights is equal to

$$A_n = \int_0^{[n]} \int_0^{y} dx\, dy\ ,$$

the area of the triangle S_n. Furthermore, letting q tend to 1, we see that the limits of $w_{i,j}^n$ are in agreement with the results of Lauffer [11] for $n \leq 3$.

- **Integration rule on a triangle {(x,y): 0 ≤ x ≤ y ≤ [1]}**
  Weights Iij at the nodes ([i],[j]), 0 ≤ i ≤ j ≤ 1

```
• S1:=int(1,x=0..y);
                         S1 := y
```

```
• A1:=int(S1,y=0..1);
                         A1 := 1/2
```

```
• S00:=int(1-y,x=0..y);
                   S00 := - (- 1 + y) y
```

```
• I00:=int(S00,y=0..1);
                        I00 := 1/6
```

```
• S01:=int(y-x,x=0..y);
                                    2
                      S01 := 1/2 y
```

```
• I01:=int(S01,y=0..1);
                        I01 := 1/6
```

```
• S11:=int(x,x=0..y);
                                    2
                      S11 := 1/2 y
```

```
• I11:=int(S11,y=0..1);
                        I11 := 1/6
```

- **Integration rule on a triangle {(x,y): 0 ≤ x ≤ y ≤ [2]}**
  Weights Iij at the nodes ([i],[j]), 0 ≤ i ≤ j ≤ 2

```
n[2]:=(1-q^2)/(1-q):
```

```
• S00:=int((1-y)*(n[2]-y)/n[2],x=0..y):
  I00:=int(S00,y=0..n[2]);
                                        2
                I00 := - 1/12 (q + 1)  (- 1 + q)
```

```
• S01:=int((n[2]-y)*(y-x)/q,x=0..y):
  I01:=int(S01,y=0..n[2]);
                                        4
                                 (q + 1)
                    I01 := 1/24 --------
                                    q
```

```
• S11:=int(x*(n[2]-y)/q,x=0..y):
  I11:=int(S11,y=0..n[2]);
                                        4
                                 (q + 1)
                    I11 := 1/24 --------
                                    q
```

```
• S02:=int((y-x)*(y-q*x-1)/(q*n[2]),x=0..y):
  I02:=int(S02,y=0..n[2]);
                                     2    2
                             (q + 1)  (q  - 2 q + 1)
              I02 := - 1/24 ----------------------
                                       q
```

```
• S12:=int(x*(y-x)/q,x=0..y):
  I12:=int(S12,y=0..n[2]);
```

$$I12 := 1/24\,\frac{(q+1)^4}{q}$$

```
• S22:=int(x*(x-1)/(q*n[2]),x=0..y):
  I22:=int(S22,y=0..n[2]);
```

$$I22 := 1/12\,\frac{(q+1)^2\,(-1+q)}{q}$$

- **Integration rule on a triangle {(x,y): 0 ≤ x ≤ y ≤ [3]}**
  Weights Iij at the nodes ([i],[j]), 0 ≤ i ≤ j ≤ 3

```
n[3]:=(1-q^3)/(1-q):
f[3]:=n[3]*n[2]:
```

```
• S00:=int((1-y)*(n[2]-y)*(n[3]-y)/f[3],x=0..y):
  I00:=int(S00,y=0..n[3]);
```

$$I00 := 1/60\,\frac{(3q^6+4q^5-2q^4-2q^2-4q^3+4q+3)\,(q^2+q+1)}{q+1}$$

```
• S01:=int((n[2]-y)*(n[3]-y)*(y-x)/(n[2]*q^2),x=0..y):
  I01:=int(S01,y=0..n[3]);
```

$$I01 := -1/120\,\frac{(3q^4+q^3-q^2-4q-2)\,(q^2+q+1)^3}{q^2\,(q+1)}$$

```
• S11:=int(x*(n[2]-y)*(n[3]-y)/(n[2]*q^2),x=0..y):
  I11:=int(S11,y=0..n[3]);
```

$$I11 := -1/120\,\frac{(3q^4+q^3-q^2-4q-2)\,(q^2+q+1)^3}{q^2\,(q+1)}$$

```
• S02:=int((n[3]-y)*(y-x)*(y-q*x-1)/(n[2]*q^3),x=0..y):
  I02:=int(S02,y=0..n[3]);
```

$$I02 := -1/120\,\frac{(q^5-q^4-3q^3-2q^2+2)\,(q^2+q+1)^3}{q^3\,(q+1)}$$

```
• S12:=int(x*(n[3]-y)*(y-x)/q^3,x=0..y):
  I12:=int(S12,y=0..n[3]);
```

$$I12 := 1/120\,\frac{(q^2+q+1)^5}{q^3}$$

```
• S22:=int(x*(x-1)*(n[3]-y)/(n[2]*q^3),x=0..y):
  I22:=int(S22,y=0..n[3]);
```

$$I22 := 1/120\,\frac{(2q^4+4q^3+q^2-q-3)\,(q^2+q+1)^3}{q^3\,(q+1)}$$

```
• S03:=int((y-x)*(y-q*x-1)*(y-x*q^2-n[2])/(f[3]*q^3),x=0..y):
  I03:=int(S03,y=0..n[3]);
```

$$I03 := 1/60\,\frac{(q^2+q+1)^2\,(q^6-q^5-2q^4+5q^3-2q^2-q+1)}{q^3}$$

```
• S13:=int(x*(y-x)*(y-q*x-1)/(n[2]*q^3),x=0..y):
  I13:=int(S13,y=0..n[3]);
```

$$I13 := -1/120\,\frac{(2q^3-2q^2-2q+1)\,(q^2+q+1)^4}{q^3\,(q+1)}$$

```
• S23:=int(x*(x-1)*(y-x)/(n[2]*q^3),x=0..y):
  I23:=int(S23,y=0..n[3]);
```

$$I23 := 1/120\,\frac{(2q^2+2q-3)\,(q^2+q+1)^4}{q^3\,(q+1)}$$

```
• S33:=int(x*(x-1)*(x-n[2])/(f[3]*q^3),x=0..y):
  I33:=int(S33,y=0..n[3]);
```

$$I33 := 1/60\,\frac{(3q^4+q^3-6q^2+3+q)\,(q^2+q+1)^2}{q^3\,(q+1)}$$

• **Integration rule on a triangle {(x,y): 0 ≤ x ≤ y ≤ [4]}**
Weights Iij at the nodes ([i],[j]), 0 ≤ i ≤ j ≤ 4

```
  n[4]:=(1-q^4)/(1-q):
  f[4]:=n[4]*f[3]:
```

```
• S00:=int((1-y)*(n[2]-y)*(n[3]-y)*(n[4]-y)/f[4],x=0..y):
  I00:=int(S00,y=0..n[4]);
```

$$I00 := -1/60\,(q^3+q^2+q+1)\,\left(2q^{11}+3q^{10}+2q^9-2q^8-4q^7-q^6+q^5+4q^4+2q^3-2q^2-3q-2\right)\Big/(q^2+q+1)$$

```
• S01:=int((n[2]-y)*(n[3]-y)*(n[4]-y)*(y-x)/(f[3]*q^3),x=0..y):
  I01:=int(S01,y=0..n[4]);
```

$$I01 := 1/120\,\frac{(q^3+q^2+q+1)^3\,(2q^8+q^7-q^6-q^5-2q^4+2q^2+2q+1)}{(q^2+q+1)\,q^3}$$

```
• S11:=int(x*(n[2]-y)*(n[3]-y)*(n[4]-y)/(f[3]*q^3),x=0..y):
  I11:=int(S11,y=0..n[4]);
```

$$I11 := 1/120\,\frac{(q^3+q^2+q+1)^3\,(2q^8+q^7-q^6-q^5-2q^4+2q^2+2q+1)}{(q^2+q+1)\,q^3}$$

```
• S02:=int((n[3]-y)*(n[4]-y)*(y-x)*(y-q*x-1)/(n[2]*n[2]*q^5),x=0..y):
  I02:=int(S02,y=0..n[4]);
```

$$I02 := 1/360\,(q^2+1)\,(q^3+q^2+q+1)^2\,(2q^9-5q^8-q^7-6q^6+6q^5+8q^3-2q^2-q-3)\,/\,q^5$$

```
• S12:=int(x*(n[3]-y)*(n[4]-y)*(y-x)/(n[2]*q^5),x=0..y):
  I12:=int(S12,y=0..n[4]);
```

$$I12 := -\,1/360\,\frac{(q^3+q^2+q+1)^4\,(2q^5-q^4+q^3-2q^2-q-1)}{q^5}$$

```
• S22:=int(x*(x-1)*(n[3]-y)*(n[4]-y)/(n[2]*n[2]*q^5),x=0..y):
  I22:=int(S22,y=0..n[4]);
```

$$I22 := -\,1/360\,(q^2+1)\,(q^3+q^2+q+1)^2\,(4q^8+2q^7+4q^6-9q^5-9q^3+4q^2+2q+4)\,/\,q^5$$

```
• S03:=int((n[4]-y)*(y-x)*(y-q*x-1)*(y-x*q^2-n[2])/(f[3]*q^6),x=0..y):
  I03:=int(S03,y=0..n[4]);
```

$$I03 := 1/360\,\frac{(q^3+q^2+q+1)^3\,(q^{11}-2q^9+4q^7+6q^6+4q^5-2q^3-3q^2+q+3)}{(q^2+q+1)\,q^6}$$

```
• S13:=int(x*(n[4]-y)*(y-x)*(y-q*x-1)/(n[2]*q^6),x=0..y):
  I13:=int(S13,y=0..n[4]);
```

$$I13 := -\,1/360\,\frac{(q^3+q^2+q+1)^4\,(q^6-q^5-2q^3-q+1)}{q^6}$$

```
• S23:=int(x*(x-1)*(n[4]-y)*(y-x)/(n[2]*q^6),x=0..y):
  I23:=int(S23,y=0..n[4]);
```

$$I23 := 1/360\,\frac{(q^3+q^2+q+1)^4\,(q^5+q^4+2q^3-q^2+q-2)}{q^6}$$

```
• S33:=int(x*(x-1)*(x-n[2])*(n[4]-y)/(f[3]*q^6),x=0..y):
  I33:=int(S33,y=0..n[4]);
```

$$I33 := 1/120\,\frac{(q^3+q^2+q+1)^3\,(q^8+2q^7+2q^6-2q^4-q^3-q^2+q+2)}{(q^2+q+1)\,q^6}$$

```
• S04:=int((y-x)*(y-q*x-1)*(y-x*q^2-n[2])*(y-x*q^3-n[3])/(f[4]*q^6),x=0..y):
  I04:=int(S04,y=0..n[4]);
```

$$I04 := -1/360\,(q^3+q^2+q+1)^2\,(3q^{14}+q^{13}-3q^{12}-7q^{11}+6q^{10}+14q^9-6q^8-16q^7-6q^6+14q^5+6q^4-7q^3-3q^2+q+3)\Big/\left(q^6\,(q^2+q+1)\right)$$

```
• S14:=int(x*(y-x)*(y-q*x-1)*(y-x*q^2-n[2])/(f[3]*q^6),x=0..y):
  I14:=int(S14,y=0..n[4]);
```

I14 := 1/360

$$\frac{(q^3+q^2+q+1)^4\,(3q^8-2q^7-4q^6+q^5+5q^4+2q^3-2q^2-q+1)}{q^6\,(q^2+q+1)}$$

```
• S24:=int(x*(x-1)*(y-x)*(y-q*x-1)/(n[2]*n[2]*q^6),x=0..y):
  I24:=int(S24,y=0..n[4]);
```

I24 := - 1/360

$$\frac{(q^2+1)^2\,(q^3+q^2+q+1)^2\,(3q^7+q^6-q^5+3q^2-9q^4-2+q^3+5q)}{q^6}$$

```
• S34:=int(x*(x-1)*(x-n[2])*(y-x)/(f[3]*q^6),x=0..y):
  I34:=int(S34,y=0..n[4]);
```

$$I34 := 1/120\,\frac{(q^3+q^2+q+1)^4\,(q^5+q^4-2q^2-q+2)}{q^6\,(q^2+q+1)}$$

```
• S44:=int(x*(x-1)*(x-n[2])*(x-n[3])/(f[4]*q^6),x=0..y):
  I44:=int(S44,y=0..n[4]);
```

$$I44 := 1/60\,\frac{(q^3+q^2+q+1)^2\,(2q^8+q^7-q^6-4q^5+4q^3+q^2-q-2)}{q^6\,(q^2+q+1)}$$

- **Integration rule on a triangle {(x,y): 0 ≤ x ≤ y ≤ [5]}**
  Weights Iij at the nodes ([i],[j]), 0 ≤ i ≤ j ≤ 5

```
n[5]:=(1-q^5)/(1-q):
f[5]:=n[5]*f[4]:
```

```
S00:=int((1-y)*(n[2]-y)*(n[3]-y)*(n[4]-y)*(n[5]-y)/f[5],x=0..y):
I00:=int(S00,y=0..n[5]);
```

$$I00 := 1/420\,(10 + 36 q + 66 q^2 + 77 q^3 + 42 q^4 - 22 q^5 - 75 q^{13} + 77 q^{17} - 13 q^{12} + 42 q^{16} + 52 q^{11} - 13 q^8 + 79 q^{10} + 36 q^{19} - 75 q^{14} + 66 q^{18} - 22 q^{15} + 10 q^{20} - 75 q^7 - 75 q^6 + 52 q^9)\,(q^4 + q^3 + q^2 + q + 1) \Big/ ((q + 1)(q^2 + q + 1)(q^3 + q^2 + q + 1))$$

```
S01:=int((n[2]-y)*(n[3]-y)*(n[4]-y)*(n[5]-y)*(y-x)/(f[4]*q^4),x=0..y):
I01:=int(S01,y=0..n[5]);
```

$$I01 := - 1/840\,(- 4 - 16 q - 33 q^2 - 45 q^3 - 35 q^4 - 5 q^5 + 26 q^{15} - 33 q^{11} + 11 q^{13} - 21 q^{12} + 10 q^{16} + 44 q^7 + 30 q^{14} + 24 q^8 - 27 q^{10} + 29 q^6 - 5 q^9)\,(q^4 + q^3 + q^2 + q + 1) \Big/ ((q + 1)(q^2 + q + 1)(q^3 + q^2 + q + 1)\, q^4)$$

```
S11:=int(x*(n[2]-y)*(n[3]-y)*(n[4]-y)*(n[5]-y)/(f[4]*q^4),x=0..y):
I11:=int(S11,y=0..n[5]);
```

$$I11 := - 1/840\,(- 4 - 16 q - 33 q^2 - 45 q^3 - 35 q^4 - 5 q^5 + 26 q^{15} - 33 q^{11} + 11 q^{13} - 21 q^{12} + 10 q^{16} + 44 q^7 + 30 q^{14} + 24 q^8 - 27 q^{10} + 29 q^6 - 5 q^9)\,(q^4 + q^3 + q^2 + q + 1) \Big/ ((q + 1)(q^2 + q + 1)(q^3 + q^2 + q + 1)\, q^4)$$

```
S02:=int((n[3]-y)*(n[4]-y)*(n[5]-y)*(y-x)*(y-q*x-1)/      (f[3]*n[2]*q^7),x=0..y):
I02:=int(S02,y=0..n[5]);
```

$$I02 := - 1/2520\,(12 + 30 q + 48 q^2 + 39 q^3 - 10 q^4 - 62 q^5 + 60 q^9 + 7 q^8 - 48 q^{15} - 99 q^6 - 12 q^{12} - 4 q^{16} - 61 q^{13} + 10 q^{17} + 58 q^{11} - 65 q^{14} - 61 q^7 + 83 q^{10})\,(q^4 + q^3 + q^2 + q + 1) \Big/ (q^7 (q + 1)^2 (q^2 + q + 1))$$

```
• S12:=int(x*(n[3]-y)*(n[4]-y)*(n[5]-y)*(y-x)/(f[3]*q^7),x=0..y):
  I12:=int(S12,y=0..n[5]);
```

$$I12 := 1/2520\,(3 + 9q + 18q^2 + 23q^3 + 17q^4 + 5q^5 - 5q^9 - 11q^8 + 10q^{12} + 16q^{11} - 16q^7 - 13q^6 + 4q^{10})\,(q^4 + q^3 + q^2 + q + 1)^4 \Big/ \left(q^7 (q+1)(q^2 + q + 1)\right)$$

```
• S22:=int(x*(x-1)*(n[3]-y)*(n[4]-y)*(n[5]-y)/(f[3]*n[2]*q^7),x=0..y):
  I22:=int(S22,y=0..n[5]);
```

$$I22 := 1/2520\,(-15 - 39q - 66q^2 - 62q^3 - 7q^4 + 60q^5 + 121q^6 - 38q^9 + 52q^{15} - 82q^{10} + 50q^{13} - 14q^{12} + 20q^{16} - 87q^{11} + 60q^{14} + 27q^8 + 95q^7)\,(q^4 + q^3 + q^2 + q + 1)^3 \Big/ \left(q^7 (q+1)^2 (q^2 + q + 1)\right)$$

```
• S03:=int((n[4]-y)*(n[5]-y)*(y-x)*(y-q*x-1)*(y-x*q^2-n[2])/     (f[3]*n[2]*q^9),x=0..y):
  I03:=int(S03,y=0..n[5]);
```

$$I03 := -1/2520\,(-12 - 30q - 30q^2 - 18q^3 + 21q^4 + 52q^5 + 24q^6 + 77q^{13} + 9q^{15} + 54q^{12} - 10q^{16} + q^{11} + 5q^{19} + 52q^{14} + 3q^{18} - 93q^9 - 14q^{17} - 25q^7 - 64q^{10} - 77q^8)\,(q^4 + q^3 + q^2 + q + 1)^3 \Big/ \left(q^9 (q+1)^2 (q^2 + q + 1)\right)$$

```
• S13:=int(x*(n[4]-y)*(n[5]-y)*(y-x)*(y-q*x-1)/(n[2]*n[2]*q^9),x=0..y):
  I13:=int(S13,y=0..n[5]);
```

$$I13 := 1/2520\,(-3 - 4q - 3q^2 + 12q^4 + 14q^5 - 11q^9 - 4q^8 - 10q^{10} - 2q^{12} - 7q^{11} + 5q^{13} + 6q^7 + 12q^6)\,(q^4 + q^3 + q^2 + q + 1)^4 \Big/ \left((q+1)^2 q^9\right)$$

```
• S23:=int(x*(x-1)*(n[4]-y)*(n[5]-y)*(y-x)/(n[2]*n[2]*q^9),x=0..y):
  I23:=int(S23,y=0..n[5]);
```

$$I23 := -1/2520\,(5 + 8q + 9q^2 + 8q^3 - 9q^4 - 15q^5 + 5q^{12} + 8q^{11} + 8q^9 - 9q^8 - 15q^7 - 17q^6 + 9q^{10})\,(q^4 + q^3 + q^2 + q + 1)^4 \Big/ \left((q+1)^2 q^9\right)$$

```
• S33:=int(x*(x-1)*(x-n[2])*(n[4]-y)*(n[5]-y)/(f[3]*n[2]*q^9),x=0..y):
  I33:=int(S33,y=0..n[5]);
```

$$I33 := -1/2520\,(-20 - 52q - 60q^2 - 50q^3 + 14q^4 + 87q^5 + 15q^{16} - 60q^{11} + 39q^{15} + 66q^{14} + 7q^{12} - 95q^9 - 27q^8 + 62q^{13} + 38q^7 + 82q^6 - 121q^{10})\,(q^4 + q^3 + q^2 + q + 1)^3 \Big/ \left((q^2 + q + 1)\,(q + 1)^2\,q^9\right)$$

```
• S04:=int((n[5]-y)*(y-x)*(y-q*x-1)*(y-x*q^2-n[2])*(y-x*q^3-n[3])/     (f[4]*q^10),x=0..y):
  I04:=int(S04,y=0..n[5]);
```

$$I04 := -1/2520\,(12 + 30q + 30q^2 - 42q^4 - 33q^5 + 27q^9 + 65q^8 - 44q^{10} - 10q^{14} - 15q^{18} - 80q^{13} + 2q^{17} + 30q^{15} + 5q^{20} - 106q^{11} + 3q^{22} + 55q^7 + 11q^6 - 120q^{12} + 28q^{16} - 5q^{19} + 7q^{21})\,(q^4 + q^3 + q^2 + q + 1)^3 \Big/ \left(q^{10}\,(q + 1)^2\,(q^2 + q + 1)\,(q^3 + q^2 + q + 1)\right)$$

```
• S14:=int(x*(n[5]-y)*(y-x)*(y-q*x-1)*(y-x*q^2-n[2])/(f[3]*q^10),x=0..y):
  I14:=int(S14,y=0..n[5]);
```

$$I14 := 1/2520\,(q^4 + q^3 + q^2 + q + 1)^4\,(3q^{14} + q^{13} - 3q^{12} - 2q^{11} - 3q^{10} + 6q^9 + 7q^8 + 12q^7 + 7q^6 + 6q^5 - 3q^4 - 2q^3 - 3q^2 + q + 3) \Big/ \left(q^{10}\,(q^2 + q + 1)\right)$$

```
• S24:=int(x*(x-1)*(n[5]-y)*(y-x)*(y-q*x-1)/(n[2]*n[2]*q^10),x=0..y):
  I24:=int(S24,y=0..n[5]);
```

$$I24 := -1/2520\,(-5 + 2q + 7q^2 + 10q^3 + 11q^4 + 4q^5 - 14q^8 + 3q^{13} + 4q^{12} + 3q^{11} - 12q^9 - 12q^7 - 6q^6)\,(q^4 + q^3 + q^2 + q + 1)^4 \Big/ \left(q^{10}\,(q + 1)^2\right)$$

```
• S34:=int(x*(x-1)*(x-n[2])*(n[5]-y)*(y-x)/(f[3]*q^10),x=0..y):
  I34:=int(S34,y=0..n[5]);
```

$$I34 := 1/2520\,(10 + 16q + 4q^2 - 5q^3 - 11q^4 - 16q^5 + 18q^{10} + 9q^{11} - 13q^6 + 23q^9 + 3q^{12} + 5q^7 + 17q^8)\,(q^4 + q^3 + q^2 + q + 1)^4 \Big/ \left(q^{10}\,(q + 1)\,(q^2 + q + 1)\right)$$

```
• S44:=int(x*(x-1)*(x-n[2])*(x-n[3])*(n[5]-y)/(f[4]*q^10),x=0..y):
  I44:=int(S44,y=0..n[5]);

                                     2      3      4      5      8
  I44 := 1/840 (- 10 - 26 q - 30 q  - 11 q  + 21 q  + 33 q  - 24 q

           13       15      16      11       14      6      9      7
       + 45 q   + 16 q   + 4 q   + 5 q   + 33 q   + 27 q  - 44 q  + 5 q

           12        10    4    3    2        3
       + 35 q   - 29 q  ) (q  + q  + q  + q + 1)

         /  10         2             3    2
        /  (q   (q + 1)  (q  + q + 1)  (q  + q  + q + 1))
       /
```

```
• S05:=int((y-x)*(y-q*x-1)*(y-x*q^2-n[2])*(y-x*q^3-n[3])*(y-x*q^4-n[4])
  /(f[5]*q^10),x=0..y):
  I05:=int(S05,y=0..n[5]);

                              5       4      2       9       8       10
  I05 := 1/420 (2 + 5 q - 9 q  - 10 q  + 5 q  - 15 q  + 20 q  - 33 q

          7      6       22      21       15       20       14       19
      + 20 q  + 5 q  - 10 q   - 9 q   - 15 q   + 5 q   + 25 q   + 20 q

          13       18       12       17       11       16      24
      + 55 q   + 20 q   + 25 q   - 15 q   - 15 q   - 33 q   + 5 q

         26       25    4    3    2         2
      + 2 q   + 5 q  ) (q  + q  + q  + q + 1)

        /  10   3    2             2
       /  (q   (q  + q  + q + 1) (q  + q + 1)  (q + 1))
      /
```

```
• S15:=int(x*(y-x)*(y-q*x-1)*(y-x*q^2-n[2])*(y-x*q^3-n[3])     /(f[4]*q^10),x=0..y):
  I15:=int(S15,y=0..n[5]);

                              4       14              5       15       3      2
  I15 := - 1/2520 (3 - 10 q  - 42 q   + 4 q + 20 q  - 30 q   - 10 q  - 2 q

          8       18       6       9       10       13       12       17
     - 50 q  + 12 q   + 30 q  - 80 q  - 20 q   + 21 q   + 62 q   + 18 q

          11   4    3    2        4
     + 44 q  ) (q  + q  + q  + q + 1)

       /  10          2             3    2
      /  (q   (q + 1)  (q  + q + 1)  (q  + q  + q + 1))
     /
```

```
• S25:=int(x*(x-1)*(y-x)*(y-q*x-1)*(y-x*q^2-n[2])/(f[3]*n[2]*q^10),x=0..y):
  I25:=int(S25,y=0..n[5]);

                               15              5       4      3       2       8
  I25 := 1/2520 (- 5 + 12 q   + 2 q - 48 q  - 19 q  - 4 q  + 17 q  + 49 q

          14       12       9       10       6       11       7
      + 18 q   - 12 q   + 46 q  - 19 q   - 23 q  - 39 q   + 40 q )

       4    3    2         4  /  10         2   2
     (q  + q  + q  + q + 1)  /  (q   (q + 1)  (q  + q + 1))
                            /
```

```
• S35:=int(x*(x-1)*(x-n[2])*(y-x)*(y-q*x-1)/(f[3]*n[2]*q^10),x=0..y):
  I35:=int(S35,y=0..n[5]);
```

$$I35 := -1/2520\,(10 - 14q + 59q^5 + 4q^4 - 17q^3 - 44q^2 + 56q^6 + 18q^{11} + 12q^{13} - 49q^9 - 40q^8 - 9q^{10} - 19q^7 + 18q^{12})\,(q^4 + q^3 + q^2 + q + 1) \Big/ \left(q^{10}\,(q+1)^2\,(q^2 + q + 1)\right)$$

```
• S45:=int(x*(x-1)*(x-n[2])*(x-n[3])*(y-x)/(f[4]*q^10),x=0..y):
  I45:=int(S45,y=0..n[5]);
```

$$I45 := 1/840\,(-10 - 16q + 2q^5 + 32q^4 + 19q^3 - 4q^2 + 12q^{11} + 12q^9 - 10q^8 - 22q^6 + 17q^{10} - 26q^7 + 4q^{12})\,(q^4 + q^3 + q^2 + q + 1) \Big/ \left(q^{10}\,(q+1)^2\,(q^2 + q + 1)\,(q^3 + q^2 + q + 1)\right)$$

```
• S55:=int(x*(x-1)*(x-n[2])*(x-n[3])*(x-n[4])/(f[5]*q^10),x=0..y);
```

$$S55 := -1/60\,(-30 - 90q - 90q^5 - 150q^4 - 180q^3 - 150q^2 + 80y + 180yq + 20q^6y - 30q^6 - 105y^2q^3 - 135y^2q - 90y^2 - 135y^2q^2 + 240yq^2 + 160yq^4 + 240yq^3 - 10y^4 + 12y^3q^3 + 36y^3q - 15q^5y^2 - 45y^2q^4 + 48y^3 + 24y^3q^2 + 80q^5y)\,y^2 \Big/ \left(q^{10}\,(q^4 + q^3 + q^2 + q + 1)\,(q^3 + q^2 + q + 1)\,(q^2 + q + 1)\,(q + 1)\right)$$

```
• I55:=int(S55,y=0..n[5]);
```

$$I55 := 1/420\,(10 + 26q^{15} + 26q - 54q^5 - 35q^4 + 11q^3 + 30q^2 + 10q^{16} - 27q^{10} + 30q^7 + 30q^9 + 73q^8 + 11q^{13} - 35q^{12} - 54q^{11} + 30q^{14} - 27q^6)\,(q^4 + q^3 + q^2 + q + 1)^2 \Big/ \left(q^{10}\,(q^3 + q^2 + q + 1)\,(q^2 + q + 1)\,(q + 1)\right)$$

•

APPENDIX 5B Interval on which Wij > 0 (n=5)

Let Bij denote the polynomial factor (considered in Table 5.3) obtained by dividing Cij by its obvious positive factors.

- $$B01 := 10q^{12} + 16q^{11} + 4q^{10} - 19q^{9} - 32q^{8} - 2q^{7} + 22q^{6} + 26q^{5} + 10q^{4} - 12q^{3} - 17q^{2} - 12q - 4$$
- $$B02 := 10q^{13} - 14q^{12} - 44q^{11} - 17q^{10} + 4q^{9} + 59q^{8} + 56q^{7} - 19q^{6} - 40q^{5} - 49q^{4} - 9q^{3} + 18q^{2} + 18q + 12$$
- $$B22 := 20q^{12} + 32q^{11} + 8q^{10} - 10q^{9} - 64q^{8} - 53q^{7} + 37q^{6} + 52q^{5} + 55q^{4} + 4q^{3} - 27q^{2} - 24q - 15$$
- $$B03 := 5q^{15} - 2q^{14} - 17q^{13} + 4q^{12} + 19q^{11} + 48q^{10} + 23q^{9} - 40q^{8} - 49q^{7} - 46q^{6} + 19q^{5} + 39q^{4} + 12q^{3} - 18q - 12$$
- $$B13 := 5q^{9} - 7q^{8} - 5q^{7} - 3q^{6} - q^{5} + 12q^{4} + 3q^{3} + q^{2} - q - 3$$
- $$B23 := 5q^{12} + 8q^{11} + 9q^{10} + 8q^{9} - 9q^{8} - 15q^{7} - 17q^{6} - 15q^{5} - 9q^{4} + 8q^{3} + 9q^{2} + 8q + 5$$
- $$B04 := 3q^{22} + 7q^{21} + 5q^{20} - 5q^{19} - 15q^{18} + 2q^{17} + 28q^{16} + 30q^{15} - 10q^{14} - 80q^{13} - 120q^{12} - 106q^{11} - 44q^{10} + 27q^{9} + 65q^{8} + 55q^{7} + 11q^{6} - 33q^{5} - 42q^{4} + 30q^{2} + 30q + 12$$

- **Roots of Bij > 0 (n = 5)**

- fsolve(B01, q);

  -1.291881517, 1.059740995

- fsolve(B02, q);

  -.7950764185, .9235112258, 2.966237082

- fsolve(B22, q);

  -.7740048931, .9427304614

- fsolve(B03, q);

  -1.702916753, -1.058809519, 1.075906744

- fsolve(B13, q);

  -.9660981586, -.7352266109, .7397382670, 1.058533923, 1.913972906

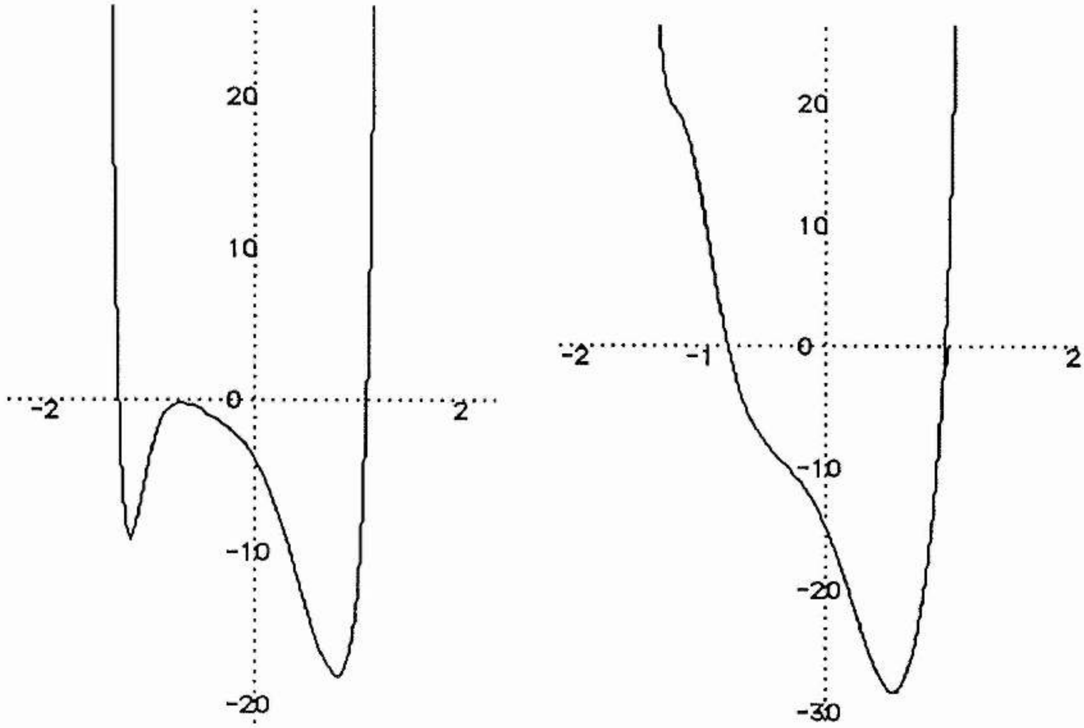
- fsolve(B23, q);

  -1.430757608, -.6989304090, .9095109318, 1.099492007

- fsolve(B04, q);

  -1.503733574, -.7309178485, .8883820534, 1.383973508

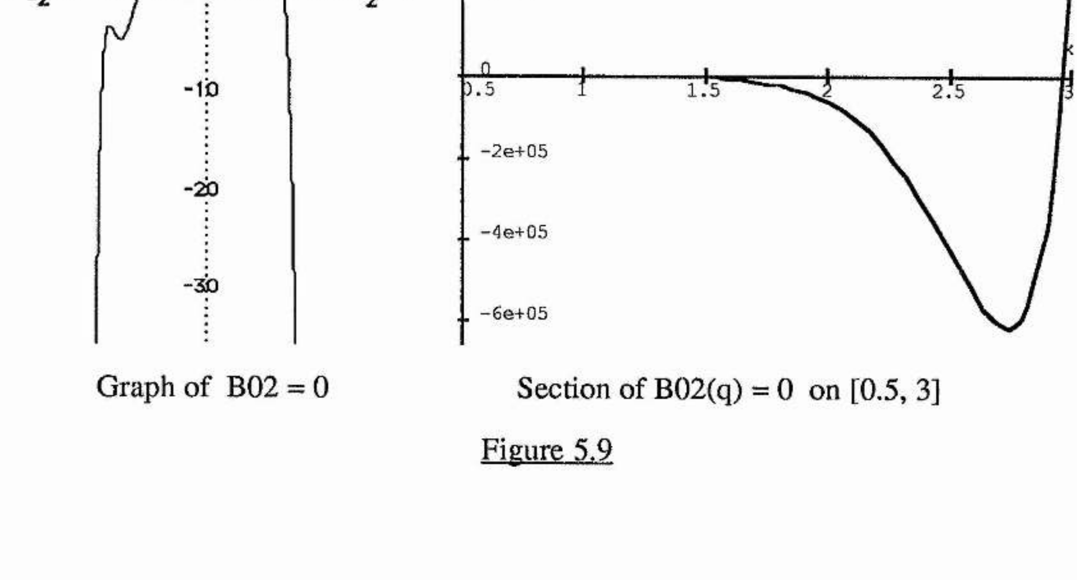
-



Graph of B01 = 0

Graph of B22 = 0

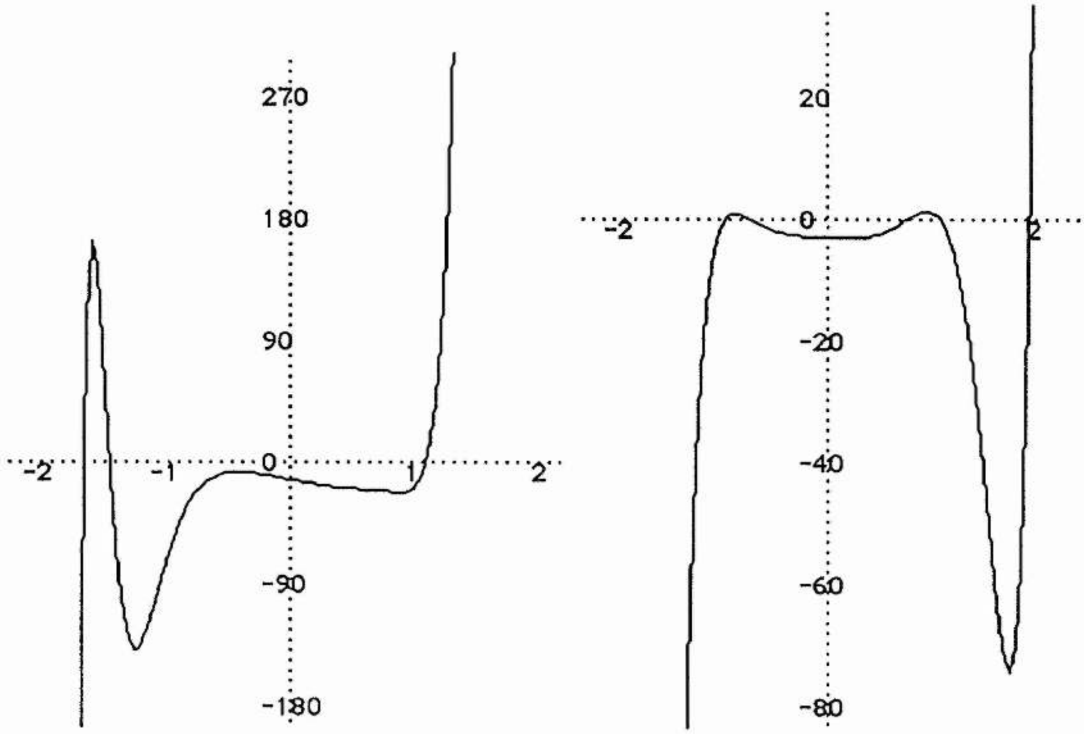
Figure 5.8



Graph of B02 = 0

Section of B02(q) = 0 on [0.5, 3]

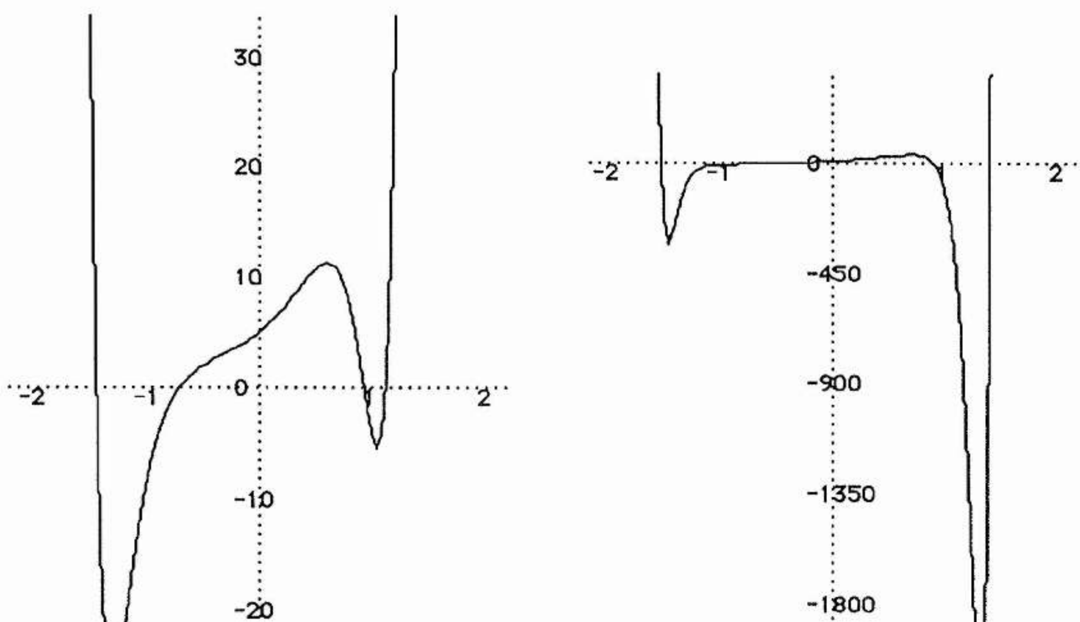
Figure 5.9



Graph of B03 = 0

Graph of B13 = 0

Figure 5.10



Graph of B23 = 0

Graph of B04 = 0

Figure 5.11

APPENDIX 5C Integration rule on odd order standard triangle $\{(x, y): 0 \le x \le y \le 7\}$

```
• S7:=int(1,y=x..7):
  A7:=int(S7,x=0..7);                                  A7 := 49/2
```

For brevity,the commands Iij:=int(Sij,x=0..7) will not be displayed.

```
• S00:=int((7-y)*(6-y)*(5-y)*(4-y)*(3-y)*(2-y)*(1-y)/7!,y=x..7):

                    I00 := 8183 / 64800

• S01:= int((7-y)*(6-y)*(5-y)*(4-y)*(3-y)*(2-y)*(y-x)/6!,y=x..7):

                    I01 := 146461 / 518400

• S11:= int(x*(7-y)*(6-y)*(5-y)*(4-y)*(3-y)*(2-y)/6!,y=x..7):

                    I11 := 146461 / 518400

•S02:=int((7-y)*(6-y)*(5-y)*(4-y)*(3-y)*(y-x)*(y-x-1)/ (2!*5!),y=x..7):

                    I02 := 175273 / 518400

• S12:= int(x*(7-y)*(6-y)*(5-y)*(4-y)*(3-y)*(y-x)/5!,y=x..7):

                    I12 := 789929 / 259200

• S22:= int(x*(x-1)*(7-y)*(6-y)*(5-y)*(4-y)*(3-y)/(2!*5!),y=x..7):

                    I22 := 175273 / 518400

•S03:=int((7-y)*(6-y)*(5-y)*(4-y)*(y-x)*(y-x-1)*(y-x-2)/ (3!*4!),y=x..7):

                    I03 := 26411 / 103680

• S13:= int(x*(7-y)*(6-y)*(5-y)*(4-y)*(y-x)*(y-x-1)/(2!*4!),y=x..7):

                    I13 := - 16807 / 25920

• S23:= int(x*(x-1)*(7-y)*(6-y)*(5-y)*(4-y)*(y-x)/(2!*4!),y=x..7):

                    I23 := - 16807 / 25920

• S33:= int(x*(x-1)*(x-2)*(7-y)*(6-y)*(5-y)*(4-y)/(3!*4!),y=x..7):

                    I33 := 26411 / 103680

•S04:=int((7-y)*(6-y)*(5-y)*(y-x)*(y-x-1)*(y-x-2)*(y-x-3)/ (3!*4!),y=x..7):

                    I04 := 26411 / 103680

•S14:=int(x*(7-y)*(6-y)*(5-y)*(y-x)*(y-x-1)*(y-x-2)/ (3!*3!),y=x..7):

                    I14 := 218491 / 51840

•S24:=int(x*(x-1)*(7-y)*(6-y)*(5-y)*(y-x)*(y-x-1)/ (2!*2!*3!),y=x..7):

                    I24 := 16807 / 51840

• S34:= int(x*(x-1)*(x-2)*(7-y)*(6-y)*(5-y)*(y-x)/(3!*3!),y=x..7):

                    I34 := 218491 / 51840

• S44:= int(x*(x-1)*(x-2)*(x-3)*(7-y)*(6-y)*(5-y)/(3!*4!),y=x..7):

                    I44 := 26411 / 103680

•S05:=int((7-y)*(6-y)*(y-x)*(y-x-1)*(y-x-2)*(y-x-3)*(y-x-4)/ (2!*5!),y=x..7):

                    I05 := 175273 / 518400

•S15:=int(x*(7-y)*(6-y)*(y-x)*(y-x-1)*(y-x-2)*(y-x-3)/ (2!*4!),y=x..7):

                    I15 := - 16807 / 25920
```

```
•S25:=int(x*(x-1)*(7-y)*(6-y)*(y-x)*(y-x-1)*(y-x-2)/ (2!*2!*3!),y=x..7):
```

I25 := 16807 / 51840

```
•S35:=int(x*(x-1)*(x-2)*(7-y)*(6-y)*(y-x)*(y-x-1)/ (2!*2!*3!),y=x..7):
```

I35 := 16807 / 51840

```
• S45:= int(x*(x-1)*(x-2)*(x-3)*(7-y)*(6-y)*(y-x)/(2!*4!),y=x..7):
```

I45 := - 16807 / 25920

```
• S55:= int(x*(x-1)*(x-2)*(x-3)*(x-4)*(7-y)*(6-y)/(2!*5!),y=x..7):
```

I55 := 175273 / 518400

```
•S06:=int((7-y)*(y-x)*(y-x-1)*(y-x-2)*(y-x-3)*(y-x-4)*(y-x-5) /6!,y=x..7):
```

I06 := 146461 / 518400

```
• S16:= int(x*(7-y)*(y-x)*(y-x-1)*(y-x-2)*(y-x-3)*(y-x-4)/5!,y=x..7):
```

I16 := 789929 / 259200

```
•S26:=int(x*(x-1)*(7-y)*(y-x)*(y-x-1)*(y-x-2)*(y-x-3)/ (2!*4!),y=x..7):
```

I26 := - 16807 / 25920

```
•S36:=int(x*(x-1)*(x-2)*(7-y)*(y-x)*(y-x-1)*(y-x-2)/ (3!*3!),y=x..7):
```

I36 := 218491 / 51840

```
• S46:= int(x*(x-1)*(x-2)*(x-3)*(7-y)*(y-x)*(y-x-1)/(2!*4!),y=x..7):
```

I46 := - 16807 / 25920

```
• S56:= int(x*(x-1)*(x-2)*(x-3)*(x-4)*(7-y)*(y-x)/5!,y=x..7):
```

I56 := 789929 / 259200

```
• S66:= int(x*(x-1)*(x-2)*(x-3)*(x-4)*(x-5)*(7-y)/6!,y=x..7):
```

I66 := 146461 / 518400

```
•S07:=int((y-x)*(y-x-1)*(y-x-2)*(y-x-3)*(y-x-4)*(y-x-5)*(y-x-6) /7!,y=x..7):
```

I07 := 8183 / 64800

```
•S17:=int(x*(y-x)*(y-x-1)*(y-x-2)*(y-x-3)*(y-x-4)*(y-x-5) /6!,y=x..7):
```

I17 := 146461 / 518400

```
•S27:=int(x*(x-1)*(y-x)*(y-x-1)*(y-x-2)*(y-x-3)*(y-x-4)/ (2!*5!),y=x..7):
```

I27 := 175273 / 518400

```
•S37:=int(x*(x-1)*(x-2)*(y-x)*(y-x-1)*(y-x-2)*(y-x-3)/ (3!*4!),y=x..7):
```

I37 := 26411 / 103680

```
•S47:=int(x*(x-1)*(x-2)*(x-3)*(y-x)*(y-x-1)*(y-x-2)/ (3!*4!),y=x..7):
```

I47 := 26411 / 103680

```
• S57:= int(x*(x-1)*(x-2)*(x-3)*(x-4)*(y-x)*(y-x-1)/(2!*5!),y=x..7):
```

I57 := 175273 / 518400

```
• S67:= int(x*(x-1)*(x-2)*(x-3)*(x-4)*(x-5)*(y-x)/6!,y=x..7):
```

I67 := 146461 / 518400

```
• S77:= int(x*(x-1)*(x-2)*(x-3)*(x-4)*(x-5)*(x-6)/7!,y=x..7):
```

I77 := 8183 / 64800

***

Appendix 6A Evaluation of the Lebesgue function $\lambda_n(x, y; q)$

```
Program Ldim; {To find data point for λn(x, y; q)}

var
f: text;
n: integer;
r, s, nomq, q, d: real;


function QI(k: integer): real;  {QI(k) = [k], the q-integer}
begin
if q = 1 then QI:= k
  else QI:= (1 – exp(k*ln(q)))/(1 – q);
end;


function QF(i: integer): real;  {QF(i) = [i]!}
var
k: integer;
qfac: real;
begin
qfac:= 1;
if i > 0 then for k:= 1 to i do qfac:= qfac*QI(k);
QF:= qfac;
end;


function LN(x, y: real): real;  {LN(x, y) = λn(x, y; q)}
var
v, u, w, i, j: integer;
vluex, vluey, vluez, prdct, qsum: real;
begin
qsum:= 0;
for j:=0 to n do
  for i:= 0 to j do
    begin
    vluex:= 1;
    if i – 1 >= 0 then for v:= 0 to i – 1 do vluex := vluex * (x – QI(v));
    vluey:= 1;
    if j + 1 <= n then for u:= j + 1 to n do vluey := vluey * (QI(u) – y);
    vluez:= 1;
    if j – i – 1 >= 0 then for w:= 0 to j – i – 1 do
    vluez := vluez * (y – exp(w * ln(q)) * x – QI(w));
   prdct:= exp((– (2 * n – j – 1) * j/2) * ln(q)) * vluex * vluey * vluez/
    (QF(i) * QF(n – j) * QF(j – i));
   qsum:= qsum + abs(prdct);
   end;
LN:= qsum;
end;


begin  {main program}
rewrite(f, 'Ldim.out');
writeln('Enter degree n, value q and spacing d');
readln(n, q, d);
writeln(f, 'LBfunction n, q, d', n, q: 8: 4, d: 8: 4);
writeln('        value of x          value of y          LN(x, y)');
writeln(f, '     gridx               gridy               LN(x, y)');
```

```
s:= 0;  while s<= QI(n) + 0.00001 do
   begin
   r:= 0;  while r<= s + 0.00001 do
      begin
      nomq:= LN(r, s);
      writeln(r, '         ', s, '         ', nomq);
      writeln(f, r: 8: 4, s: 8: 4, nomq: 10: 6);
      r:= r + d
      end;
   s:= s + d
   end;
end.
```

Appendix 6B   Evaluation of the Lagrange coefficient $L^n_{i,j}(x, y; q)$

```
Program LC;  {Evaluation of the Lagrange coefficient}

var
f: text;
i, j, n: integer;
r, s, nomq, q, d: real;

function QI(k: integer): real;   {QI(k) = [k], the q-integer}

function QF(i: integer): real;   {QF(i) = [i]!}

function LNC(i, j: integer;  x, y: real): real;   {LNC(x, y) = L^n_i,j(x, y; q)}
var
v, u, w: integer;             vluex, vluey, vluez: real;
begin
vluex:= 1;
if i - 1 >= 0 then for v:= 0 to i - 1 do vluex := vluex * (x - QI(v));
vluey:= 1;
if j + 1 <= n then for u:= j + 1 to n do vluey := vluey * (QI(u) - y);
vluez:= 1;
if j - i - 1 >= 0 then for w:= 0 to j - i - 1 do
         vluez := vluez * (y - exp(w * ln(q)) * x - QI(w));
LNC:= exp((- (2 * n - j - 1) * j/2) * ln(q)) * vluex * vluey * vluez/
         (QF(i) * QF(n - j) * QF(j - i));
end;

begin   {main program}
rewrite(f, 'LC.out');
writeln('Enter degree n, value q and spacing d');         readln(n, q, d);
writeln(f, 'LBfunction n, q, d =', n, q: 8: 4, d: 8: 4);
writeln('Enter i, j for Lagrange coefficient LNij(x, y)');
readln(i, j);
writeln(f, 'Consider the Lagrange coefficient LN', i, j, '(x, y)');
writeln( '        value of x                  value of y                   LNij(x, y)');
writeln(f, '        value of x                  value of y                   LNij(x, y)');
s:= 0;  while s<= QI(n) + 0.00001 do
   begin
   r:= 0;  while r<= s + 0.00001 do
      begin      nomq:= LNC(i, j, r, s);       writeln(r, '        ', s, '         ', nomq);
      writeln(f, r: 8: 4, s: 8: 4, nomq: 10: 6);
      r:= r + d
      end;
   s:= s + d
   end
end.
```

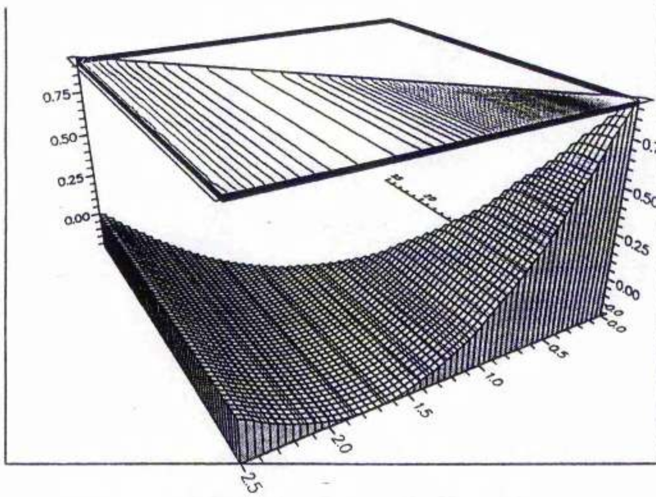


Figure 6.17 The surface of $L^2_{0,0}(x, y)$.

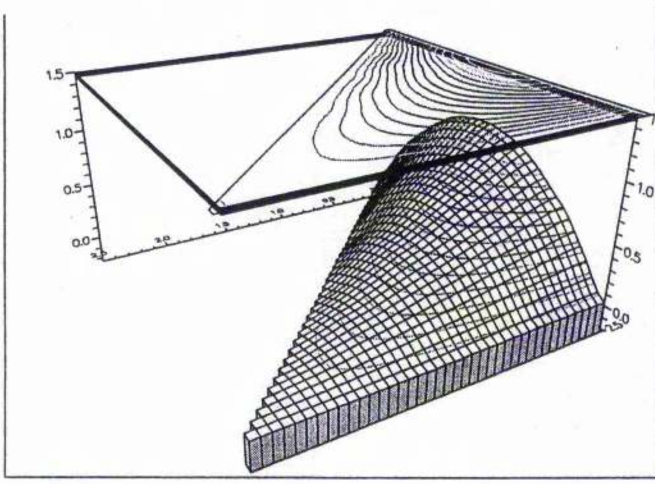


Figure 6.18 The surface of $L^2_{0,1}(x, y)$.

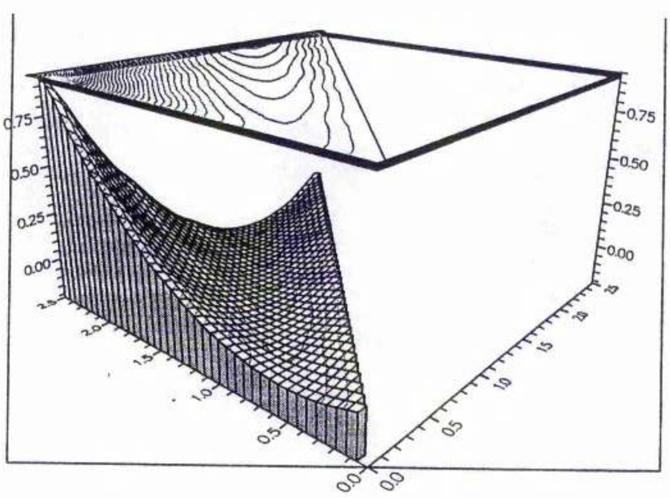


Figure 6.19 The surface of $L^2_{0,2}(x, y)$.

Appendix 6C The Minimum of $\Lambda_2(q)$ and $\Lambda_3(q)$

Select 20 regular values for q near 1, ranging from 0.957 to 1.052. For each q we subdivide each triangle S_2 and S_3, using a grid with regular spacing d. We took five choices of d, ranging from 0.160 to 0.004. For each spacing d, we then estimate $\Lambda_2(q)$ and $\Lambda_3(q)$ up to 6 decimal places. The results of the calculations are given in Tables 6.6 and 6.7 . The Pascal program used for these calculations is as follows.

```
Program Lcnst; {To find the smallest Lebesgue constant}

var
f: text;
n, meshq, testd: integer;
r, s, nomq, q, delq, d1, d, deld, Lmdc, xmax, ymax: real;

function QI(k: integer): real;  {QI(k) = [k], the q-integer}

function QF(i: integer): real;  {QF(i) = [i]!}

function LN(x, y: real): real;  {LN(x, y) = λn(x, y; q)}

begin
rewrite(f, 'Lcnst.out');
writeln('Enter the degree of polynomial: n');          readln(n);
writeln('Enter initial value for q and increment delq');          readln(q, delq);
writeln('Enter initial grid interval d1 and increment deld');          readln(d1, deld);
writeln(f, 'Lebesgue constant  of order:  ', n);
writeln(f, 'Initial value for q:   ', q: 10: 6,    'increment delta q:   ', delq: 10: 6);
writeln(f, 'Initial value for d: ', d1: 10: 6,  'increment delta d: ', deld: 10: 6);

for meshq:= 1 to 20 do
begin
writeln('  Meshq     LBconst     Grid      Maximum point');
writeln(f, '  Meshq     LBconst     Grid      Maximum point');
d:= d1;
for testd:= 1 to 5 do
 begin
 Lmdc:= 0
 s:= 0;  while s<= QI(n) + 0.00001 do
  begin
  r:= 0;  while r<= s + 0.00001 do
   begin    nomq:= LN(r, s);
   if nomq > Lmdc then begin Lmdc:= nomq;  xmax:= r;  ymax:= s  end;
   r:= r + d
   end;
  s:= s + d
  end;
 writeln(q: 10: 6, '    ', Lmdc: 12: 6,'    ', d: 10: 6, '    ', xmax: 10: 6, ymax: 10: 6);
 writeln(f, q: 10: 6, '    ', Lmdc: 12: 6,'    ', d: 10: 6, '    ', xmax: 10: 6, ymax: 10: 6);
 d:= d – deld;
 end;
q:= q + delq;
end;
end.
```

Lebesgue constant of order: 2.

Initial value for q:	0.957000	Initial value for d:	0.016000
increment delta q:	0.005000	increment delta d:	0.003000

Meshq	LBconst	Grid	Maximum point
0.957000	1.667954	0.004000	(0.652000, 1.304000)
0.962000	1.667659	0.004000	(0.656000, 1.308000)
0.967000	1.667417	0.004000	(0.656000, 1.312000)
0.972000	1.667202	0.007000	(0.658000, 1.316000)
0.977000	1.667021	0.010000	(0.660000, 1.320000)
0.982000	1.666884	0.010000	(0.660000, 1.320000)
0.987000	1.666778	0.013000	(0.663000, 1.326000)
0.992000	1.666710	0.004000	(0.664000, 1.328000)
0.997000	1.666670	0.007000	(0.665000, 1.330000)
1.002000	1.666667	0.004000	(0.668000, 1.336000)
1.007000	1.666693	0.010000	(0.670000, 1.340000)
1.012000	1.666759	0.010000	(0.670000, 1.340000)
1.017000	1.666855	0.016000	(0.672000, 1.344000)
1.022000	1.666975	0.004000	(0.672000, 1.348000)
1.027000	1.667139	0.013000	(0.676000, 1.352000)
1.032000	1.667318	0.013000	(0.676000, 1.352000)
1.037000	1.667547	0.007000	(0.679000, 1.358000)
1.042000	1.667793	0.010000	(0.680000, 1.360000)
1.047000	1.668065	0.004000	(0.680000, 1.364000)
1.052000	1.668380	0.004000	(0.684000, 1.368000)

Table 6.6

Lebesgue constant of order: 3.

Initial value for q:	0.957000	Initial value for d:	0.020000
increment delta q:	0.005000	increment delta d:	0.003000

Meshq	LBconst	Grid	Maximum point
0.957000	2.374327	0.020000	(0.440000, 0.880000)
0.962000	2.360392	0.017000	(0.442000, 0.884000)
0.967000	2.346697	0.017000	(0.442000, 0.884000)
0.972000	2.333230	0.017000	(0.442000, 0.884000)
0.977000	2.320029	0.014000	(0.448000, 0.896000)
0.982000	2.307085	0.014000	(0.448000, 0.896000)
0.987000	2.296035	0.017000	(0.544000, 1.088000)
0.992000	2.285757	0.017000	(0.544000, 1.088000)
0.997000	2.275702	0.014000	(0.546000, 1.092000)
1.002000	2.273693	0.008000	(1.904000, 2.456000)
1.007000	2.283584	0.008000	(1.912000, 2.464000)
1.012000	2.293647	0.020000	(1.920000, 2.480000)
1.017000	2.303830	0.014000	(1.932000, 2.492000)
1.022000	2.316199	0.008000	(2.136000, 2.600000)
1.027000	2.328688	0.008000	(2.144000, 2.616000)
1.032000	2.341315	0.008000	(2.152000, 2.624000)
1.037000	2.353999	0.017000	(2.159000, 2.635000)
1.042000	2.366790	0.008000	(2.168000, 2.648000)
1.047000	2.379669	0.011000	(2.178000, 2.662000)
1.052000	2.392627	0.008000	(2.184000, 2.672000)

Table 6.7

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