A VARIABLE INPUT BOUNDARY VALUE PROBLEM IN CONTAMINANT TRANSPORT

G. C. Warner

A Thesis Submitted for the Degree of MPhil
at the
University of St Andrews

1997

Full metadata for this item is available in St Andrews Research Repository at:
http://research-repository.st-andrews.ac.uk/

Please use this identifier to cite or link to this item:
http://hdl.handle.net/10023/13801

This item is protected by original copyright
A Variable Input Boundary Value Problem in Contaminant Transport

M. Phil. Thesis

Submitted 6 January 1997

G.C. Warner
The C213
Abstract

This thesis considers the large-time behaviour of the equation

$$\frac{\partial (u + w^r)}{\partial t} + Q(t) \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}, \quad p > 0, \quad r \geq -1$$

with $0 \leq x < \infty$, $t \geq 0$ and $Q(t) \sim t^r$, $t \to \infty$. This equation models, after suitable scalings are introduced, the one-dimensional flow of a solute through a porous medium with the solute undergoing adsorption by the solid matrix. We consider two models for the contaminant input at $x = 0$, the first being continuous input and the second being an initial pulse of contaminant which terminates after a finite time. Thus the total mass of the solute both adsorbed and in solution is considered to be dependent on time. It is found that the asymptotic solution depends crucially on both $p$ and $r$. In finding the asymptotic solution, a similarity variable is introduced which for $p \geq 1$ may involve spatial translation. We also have that when $p < 1$ interfaces appear and hence we have bounded support, whilst for $p \geq 1$ we do not. The principal role of $r$ is to determine the balance between diffusion and convection effects. In the continuous input case this balance is independent of $p$, whilst in the pulse problem $p$ is also involved in determining the balance.
I, G.C. Warner, hereby certify that this thesis, which is approximately 20,000 words in length, has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

I was admitted as a research student in January 1996 and as a candidate for the degree of M. Phil. in January 1996; the higher study for which this is a record was carried out in the University of St. Andrews during 1997.

I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate for the degree of M. Phil. in the University of St. Andrews and that the candidate is qualified to submit this thesis in application for that degree.
## Contents

1 Introduction ................................................. 3

2 Model and Historical Background ................. 5
   2.1 The Model ............................................. 5
   2.2 Historical Background ......................... 11

3 Formulation of the Problem ......................... 15
   3.1 Analytical Remarks on the Mass Condition ... 15
   3.2 The Solution for $p = 1$: Continuous Input ... 16
   3.3 The Solution for $p = 1$: The Pulse ........... 29
   3.4 The General Formulation of the Equation when $p \neq 1$ ... 32

4 The Continuous Input Solution ..................... 36
   4.1 Case I) $-1 < r < -1/2$ ....................... 36
   4.2 Case II) $r > -1/2$ .............................. 40
   4.3 Case III) $r = -1/2$ ............................ 46
   4.4 Case IV) $r = -1$ ................................. 51

5 The Pulse Solution ........................................ 58
   5.1 Case A) $-1 \leq r < \frac{p}{2} - 1, p > 1$ ........ 58
   5.2 Case B) $r > \frac{p}{2} - 1, p > 1$ ............... 59
   5.3 Case C) $r = \frac{p}{2} - 1, p > 1$ .......... 63
   5.4 Case D) $-1 \leq r < -\frac{p}{p+1}, p < 1$ ........ 65
   5.5 Case E) $r > -\frac{p}{p+1}, p < 1$ ............ 66
   5.6 Case F) $r = -\frac{p}{p+1}, p < 1$ .......... 70

6 The Algorithm ............................................. 72
   6.1 The Algorithm when $p < 1$ ..................... 72
7 Summary 76

A Mathematical Functions 84
A.1 The Kummer Function ........................................... 84
A.2 Parabolic Cylinder Functions ................................. 85
Section 1

Introduction

The aim of this thesis is to look at a non-linear convection-diffusion problem which is derived from the transport of contaminant through a porous media by a solute. An example of the physical situation might be some form of chemical spillage being carried through the soil by ground-water. Before we actually look at how to approach solving this problem, we will give a review of the development of this problem from some early non-linear diffusion equations to the work of Grundy et.al. [1]. This will show the context in which the aspect of the problem this thesis studies, lies. The primary difference between this thesis and the work that has been carried out previously is that the specific discharge is now a function of time.

Before presenting the review of the other work in this area, the model this thesis is interested in is introduced. This is because the equation the model is based on, namely

$$\frac{\partial}{\partial t} (\theta c + \rho s) = \nabla \cdot (D \nabla (c) - qc)$$  \hspace{1cm} (1.1)

where $c$ is the concentration in the solute and $s$ is the adsorbed concentration, will appear in some of the papers we will review (and hence we will be able to draw comparisons between these papers and the problem this thesis looks at) and it is also useful to keep in mind whilst reviewing the other papers.

On introducing the problem it will become quickly apparent that the problem, and hence the solution is dependent on two parameters:

- $p$ which comes from the model we choose for the adsorbed concentration $s$.
- $r$ which comes from the model we choose for the specific discharge $q$.

The solution to the problem will be shown to take different forms based on the different regions of the $r-p$ parameter space. We in fact look at two slightly different physical
situations, i.e. continuous input of contaminant into the system (note that the different balances between \( p \) and \( r \) will be referred to as Case 1, Case 2, etc.) and an initial pulse only, of contaminant into the system (this time the balances will be referred to as Case A, Case B, etc.). In this thesis the approach taken to solve these different regions is by means of a similarity variable and a first order expansion of the dependent variable. The exact choice of similarity variable will differ between two different forms depending again on \( r \) and \( p \).

It must be noted that there is unfortunately not a lot of corroborating evidence available for this thesis. If more time was available, the problem for finite time would be looked at, and a numerical method of solving it derived. This would corroborate this thesis by demonstrating how as time increased, this solution tends to the infinite time solution. It would also establish how large time needs to be, for the behaviour of the problem to be similar to that as time tends to infinity.
Section 2

Model and Historical Background

2.1 The Model

In many physical situations that involve some form of contaminant transport, the contaminant, possibly a hazardous chemical, is introduced into the subsurface where it will be carried downwards by the ground-water, and may eventually reach drinking water supplies. For this reason it is important to be able to examine how the contaminant is adsorbed by the soil, and how deep into the soil the contaminant reaches after a large time. We can hence define our model so as to have the contaminant introduced at time $t = 0$ and at a depth of $x = 0$ ie. we are only considering solutions in the space

$$\Omega = \{(x, t) : 0 \leq x < \infty, \ t > 0\}.$$  (2.1.1)

Note that we are not considering gravity in this model. The co-ordinates have been chosen in these directions so as to simplify the physical explanation of the model. For an examination of the case when the range $-\infty < x < \infty$ is considered, see Grundy, et. al.[1] which is reviewed later in this section. To keep the situation relatively simple, we assume that the contaminant is of a single species and does not decay in time. A similar situation for 2 and 3 dimensions is examined in Van Duijn and Knaber[2].

If the contaminated water flow was passing through a totally non-absorbent material then the problem would be characterized by the conservation equation

$$\frac{\partial \theta}{\partial t} = -\frac{\partial q}{\partial x}$$  (2.1.2)

where $q[\text{m/s}]$ is the water flux vector and $\theta[\text{m}^3/\text{m}^3]$ is the water content. See Bear[3].
Section 2 Model and Historical Background

Allowing for the soil adsorbing some of the contaminant, we now have the more general equation

\[
\frac{\partial (\theta c)}{\partial t} + \frac{\partial (qc)}{\partial x} = \frac{\partial}{\partial x} \left( D \frac{\partial c}{\partial x} \right) - \frac{\partial (\rho s)}{\partial t}, \quad (x, t) \in \Omega
\]  

(2.1.3)

where \( c \text{[mol/m}^3\text{]} \) is the concentration of the dissolved contaminant, \( s \text{[mol/kg]} \) is the concentration of adsorbed contaminant, \( \rho \text{[kg/m}^3\text{]} \) is the density of the soil and \( D \text{[m}^2\text{/s]} \) is the sum of molecular diffusion and mechanical dispersion. The model is illustrated in figure 2.1

Figure 2.1: The model of contaminant entering the soil

At \( x = 0 \) we have a boundary condition representing the convective flux of the contaminant in the water. So as to model the variable input we consider \( q = q(t) \) (note that we have arbitrarily chosen \( q \) to have no dependence on \( x \)) in this thesis although this could also be done by allowing \( c_0 = c_0(t) \). If we assume continuous input, we get by considering the amount of contaminant as \( x \to 0 \) and as \( x \to 0_+ \) to be equal

\[-D \theta \frac{\partial c}{\partial x} + q(t)c = c_0 q(t), \quad t > 0, \quad x = 0\]

(2.1.4)

where \( c_0 \) is physically found from the initial condition

\[
c(x, 0) = \begin{cases} 
c_0 & x = 0 \\
0 & x > 0 \end{cases}
\]  

(2.1.5)

Note that the boundary condition for the pulse is identical for \( t < t_0 \) where \( t_0 \) is defined by the pulse entering the porous matrix between time \( t = 0 \) and \( t = t_0 \). For \( t > t_0 \), since no contaminant is entering or leaving the system then \( c_0 = 0 \) and the boundary condition is

\[-D \theta \frac{\partial c}{\partial x} + q(t)c = 0, \quad x = 0, \quad t > t_0 \]

(2.1.6)
Section 2 Model and Historical Background

At all stages in this thesis where continuous input and the pulse are dealt with together then the boundary condition for continuous input only will be given since the boundary condition for the pulse can always be found by substituting in $c_0 = 0$ for $t > t_0$ (or as will soon be shown $u_0 = 0$ for $t > t_0$).

This is supplemented by the boundary condition

$$c \to 0, \quad \frac{\partial c}{\partial x} \to 0, \quad x \to \infty, \quad t \geq 0.$$  \hspace{1cm} (2.1.7)

We now turn to look at the relationship between $c$ and $s$, which we will call the adsorption relationship. The most general approach to this is by considering that at each adsorption site, $\lambda$, the adsorption relationship may be different. We can then group these sites into members of a set $\Lambda$, where each member of $\Lambda$ represents each of the different adsorption relationships.

We are now able to consider $s$ as

$$s = \int_{\Lambda} s_\lambda \, d\lambda$$  \hspace{1cm} (2.1.8)

where $\int_{\Lambda} f \, d\lambda = \int f \, d\lambda / |\Lambda|$. We must also restrict ourselves to looking only at relationships which can be written in the form

$$\frac{\partial s_\lambda}{\partial t} = k_\lambda F_\lambda(c, s_\lambda), \quad \lambda \in \Lambda.$$  \hspace{1cm} (2.1.9)

We can now consider each $\lambda$ as either representing

- equilibrium adsorption, in which case the reaction rate is very fast. We have

$$k_\lambda \to \infty \Rightarrow F_\lambda(c, s_\lambda) = 0 \Rightarrow s_\lambda = \Psi_\lambda(c)$$  \hspace{1cm} (2.1.10)

and we will say that all members of this set are in $\Lambda_1$ (which is clearly a subset of $\Lambda$).

- non-equilibrium adsorption, ie. all of these not in the $\Lambda_1$. Hence we must have

$$k_\lambda < \infty$$  \hspace{1cm} (2.1.11)

and we will call this set $\Lambda_2$.

---

1An adsorption site being defined as an area of the porous matrix which has identical adsorption. A porous matrix could have multiple adsorption sites if, say, the porous matrix was made of multiple materials.
If we now define \( \lambda_i = |\Lambda_i|/|\Lambda| \) \( i = 1, 2 \) then (2.1.3) becomes

\[
\frac{\partial \theta c}{\partial t} + \frac{\partial qc}{\partial x} = \frac{\partial}{\partial x} \left( D \theta \frac{\partial c}{\partial x} \right) - \frac{\partial}{\partial t} \left\{ \rho \left( \lambda_1 \int_{\Lambda_1} \Psi(\lambda(c)) d\lambda + \lambda_2 \int_{\Lambda_2} s(\lambda) d\lambda \right) \right\}
\]

and if we consider that all sites in \( \Lambda_1 \) are homogeneous then we can say

\[
\Psi(c) = \int_{\Lambda_1} \Psi(\lambda(c)) d\lambda
\]

where \( \Psi(c) \) is the adsorption isoline.

For the case this thesis looks at, we consider that all sites are in \( \Lambda_1 \) and hence \( \lambda_1 = 1 \), \( \lambda_2 = 0 \) and hence we are interested in

\[
\frac{\partial \theta c}{\partial t} + \frac{\partial qc}{\partial x} = \frac{\partial}{\partial x} \left( D \theta \frac{\partial c}{\partial x} \right) - \frac{\partial \rho \Psi(c)}{\partial t}
\]

We must now look more closely at \( \Psi(c) \). \( \Psi(c) \) can be classified into different types, by examining its behaviour near \( c = 0 \), i.e.

- \( \Psi \) strictly concave and \( \Psi'(0+) < \infty \) - Langmuir (L) type, e.g.
  \[
  \Psi(c) = \frac{k_1c}{1 + k_2c}, \quad k_1, k_2 > 0
  \]

- \( \Psi \) strictly concave and \( \Psi'(0+) = \infty \) - Freundlich (F) type, e.g.
  \[
  \Psi(c) = k_1c^{k_2}, \quad k_1 > 0, \quad 0 < k_2 < 1
  \]

- \( \Psi \) strictly convex - Convex (S) type, e.g.
  \[
  \Psi(c) = k_1c^{k_2}, \quad k_1 > 0, \quad k_2 > 1
  \]

Other examples are given in Van Genuchten and Cleary [4]. For the purposes of this thesis only the Freundlich and S types will be considered. Hence we will take

\[
s = Kc^p, \quad p > 0
\]

which has also been shown to be consistent with many organic chemicals being our contaminant (see for example Davidson and Change[5]). Note that, from this point on, when talking about \( p \), we will not repeat \( p > 0 \) although it is always true.
Section 2 Model and Historical Background

We must now look at $q(t)$. If we take

$$q(t) = q_\infty Q(t)$$  \hspace{1cm} (2.1.19)

where

$$Q(t) \sim t^r, \quad t \to \infty$$  \hspace{1cm} (2.1.20)

and $q_\infty$ and $r$ are known, then we are able to eliminate the physical constants from the problem. These scalings in the limit $t \to \infty$ are, for $p \neq 1$

$$c = \left(\frac{\theta}{K \rho}\right)^{\frac{1}{r-1}} u, \quad x = \frac{q_\infty}{\theta} \left(\frac{D\theta^2}{q_\infty^2}\right)^{\frac{1}{r+1}} x_1, \quad t = \left(\frac{D\theta^2}{q_\infty^2}\right)^{\frac{1}{r+1}} t_1$$  \hspace{1cm} (2.1.21)

and for $p = 1$

$$c = u, \quad x = \left(\frac{D\theta}{q_\infty} \left(\frac{D\theta}{\theta + \rho K}\right)^{r}\right)^{\frac{1}{r+1}} x_1, \quad t = \left(\theta + \rho K\right)^{\frac{1}{r+1}} t_1$$  \hspace{1cm} (2.1.22)

Note that these scalings are not valid when $r = -\frac{1}{2}$. In fact when $r = -\frac{1}{2}$ we cannot totally scale out the physical constants, but we can reduce them to appear as one constant only in the problem. To this end we introduce the $r = -\frac{1}{2}$ scalings of, when $p \neq 1$

$$c = \frac{\theta}{\rho K} u, \quad x = \frac{q_\infty}{\theta} x_1, \quad t = t_1$$  \hspace{1cm} (2.1.23)

and when $p = 1$

$$c = u, \quad x = \frac{q_\infty}{\theta + \rho K} x_1, \quad t = t_1$$  \hspace{1cm} (2.1.24)

To allow for this difference of scalings in the model it is helpful to introduce $\lambda$ and $\lambda_1$ which are defined as follows

$$\lambda = \begin{cases} 
1, & r \geq -1, \quad r \neq -\frac{1}{2}, \quad p \neq 1 \\
\frac{D\theta^2}{q_\infty^2}, & r = -\frac{1}{2}
\end{cases}$$  \hspace{1cm} (2.1.25)

$$\lambda_1 = \begin{cases} 
1, & r \geq -1, \quad r \neq -\frac{1}{2}, \quad p = 1 \\
\frac{D\theta(\theta + \rho K)}{q_\infty^2}, & r = -\frac{1}{2}
\end{cases}$$  \hspace{1cm} (2.1.26)

The scalings for finite time are not of interest to this thesis, since we are only interested in the large-time solutions and hence we do not need to know the finite-time function of $q(t)$. By introducing these scalings (2.1.3), and its boundary conditions (2.1.6) and (2.1.7) now
become for $p \neq 1$
\[
\frac{\partial (u + u^p)}{\partial t} + Q(t) \frac{\partial u}{\partial x} = \lambda \frac{\partial^2 u}{\partial x^2}
\]  \hspace{1cm} (2.1.27)

and for $p = 1$
\[
\frac{\partial u}{\partial t} + Q(t) \frac{\partial u}{\partial x} = \lambda \frac{\partial^2 u}{\partial x^2}
\]  \hspace{1cm} (2.1.28)

which must be solved subject to, for $p \neq 1$
\[
-\lambda \frac{\partial u}{\partial x} + Q(t)u = Q(t)u_0, \quad x = 0, \quad t > 0
\]  \hspace{1cm} (2.1.29)

and for $p = 1$
\[
-\lambda \frac{\partial u}{\partial x} + Q(t)u = Q(t)u_0, \quad x = 0, \quad t > 0
\]  \hspace{1cm} (2.1.30)

together with
\[
\frac{\partial u}{\partial x} \to 0, \quad u \to 0, \quad x \to \infty, \quad t \geq 0
\]  \hspace{1cm} (2.1.31)

and
\[
u = 0, \quad t = 0, \quad 0 < x < \infty
\]  \hspace{1cm} (2.1.32)

where $u_0$ is defined by
\[
u_0 = \left(\frac{Kp}{\theta}\right)^{1/(p-1)} u_0.
\]

We can further improve this definition of our model by looking at (2.1.31). We initially look at the special case of (2.1.27)-(2.1.31) when $r = 0$. If we set
\[
\beta(u) = \begin{cases} 
  u + u^p & p \neq 1 \\
  u & p = 1
\end{cases}
\]  \hspace{1cm} (2.1.33)

and then introduce
\[
y = \beta(u), \quad u = \phi(y)
\]  \hspace{1cm} (2.1.34)

where $\phi$ is the inverse of the function $\beta$, we can then write (2.1.38) and (2.1.39) as
\[
\frac{\partial y}{\partial t} + \frac{\partial \phi(y)}{\partial x} - \frac{\partial^2 \phi(y)}{\partial x^2} = 0
\]  \hspace{1cm} (2.1.35)

which is subject to
\[
y(x, 0) = y_0(x) = \begin{cases} 
  u_0 + u_0^p & p \neq 1 \\
  u_0 & p = 1
\end{cases}
\]  \hspace{1cm} (2.1.36)

The problem (2.1.35), (2.1.36) has been studied by Gilding[6], who concluded that this
problem had bounded support for \( p < 1 \) and unbounded support for \( p > 1 \). The implication of this is that where interfaces appear we adopt the zero flux boundary condition, i.e.

\[
\frac{\partial u}{\partial x} = 0, \quad u = 0 .
\]  

(2.1.37)

It can be shown that this is also true for the more general case of \( r > -1 \). We are now able to refine our model to

\[
\frac{\partial (u + w^p)}{\partial t} + Q(t) \frac{\partial u}{\partial x} = \lambda \frac{\partial^2 u}{\partial x^2}
\]  

(2.1.38)

and for \( p = 1 \)

\[
\frac{\partial u}{\partial t} + Q(t) \frac{\partial u}{\partial x} = \lambda \frac{\partial^2 u}{\partial x^2}
\]  

(2.1.39)

subject to

\[
-x_1(t), \quad t > 0
\]

\[
\frac{\partial u}{\partial x} = 0, \quad u = 0, \quad \left\{ \begin{array}{l}
x \to \infty, \quad t \geq 0 \quad p \geq 1 \\
x = x_1(t), \quad t \geq 0 \quad p < 1
\end{array} \right.
\]  

(2.1.41)

where \( x_1(t) \) represents the interface moving with time, and

\[
u = 0, \quad t = 0, \quad 0 < x < \infty .
\]  

(2.1.42)

2.2 Historical Background

In looking at the background to this problem we have to examine the version of our model where in fact \( Q(t) \) is taken as constant instead of having the time dependency of our model. This model was proposed by Van Genuchten and Cleary[4] who looked at the case \( p = 1 \) but gave a solution for finite time. The problem then was to generalize this model so as to extend to all \( p > 0 \). Whilst finite-time analytical solutions have not been found, Grundy[7] gave a solution for large time for the range \( 0 < p < 1 \) as part of a more general model, and then together with others, Grundy et. al.[1] was able to solve, again for large times, the case \( p > 0 \). Although not so directly related, we must make mention of the porous-medium equation since the model when \( p < 1 \) can be considered as a generalization of it, and also it is here that the ideas of a similarity variable are introduced into these types of problems. The paper by Grundy et. al.[1] is of special interest to this thesis due to its similarity with our own model, in particular with our pulse case. The immediate effect of having \( Q(t) = \text{constant} \) is that we can effectively write \( Q(t) = 1 \) in the model.
We first examine the paper by Van Genuchten and R. W. Cleary[4]. Although we look at this paper first because it was the first paper to investigate our model, it has to be treated separately to the other papers we will examine, mainly because it deals with a finite-time solution to a special case of the model without the scalings in it. The relevance of this paper to this thesis is that by taking an asymptotic expansion of this result as \( t \to \infty \) we can corroborate our own solution in this special case and hence we present that result here.

The special case considered by Van Genuchten and Cleary is our \( p = 1 \) problem. This choice of value of \( p \) removes the non-linear nature of the problem and hence an analytic solution for finite time can be found. This paper also considers the possibility of a decay term which we have not considered in our model and hence again it is only a special case of this paper that is of interest. The model they consider is

\[
\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} - \nu \frac{\partial c}{\partial x}
\]

subject to the same boundary and initial conditions as our model in the pulse case. Here we have that

\[
\nu = \frac{q}{\theta}.
\]

The solution given by Van Genuchten and Cleary becomes comparable with our continuous input model by considering \( t_0 \to \infty \). The solution they then give is as follows

\[
c = \begin{cases} 
  c_1(x, t) & 0 < t \leq t_0 \\
  c_1(x, t) - c_1(x, t - t_0) & t > t_0 
\end{cases}
\]

\[
c_1(x, t) = \frac{1}{2} \text{erfc} \left( \frac{R(t) - \nu t}{(4DR_f t)\frac{1}{2}} \right) + \left( \frac{\nu^2 t}{\pi DR_f} \right) \exp \left( -\frac{(R(t) - \nu t)^2}{4DR_f t} \right) \\
-\frac{1}{2} \exp \left( \frac{\nu x}{D} \right) \left( 1 + \frac{\nu x}{D} + \frac{\nu^2 t}{DR_f} \right) \text{erfc} \left( \frac{r(t) + \nu t}{(4DR_f t)\frac{1}{2}} \right).
\]

We now turn to try and establish the nature of our model when \( p < 1 \). This was the next stage in the development of our model and by transforming our equation (2.1.38) into the form that was looked at by Grundy[7] we can determine the essential nature of the problem when \( p < 1 \). If we first make the assumption that for large time \( u \) is small we can then state

\[
u + u^p \sim u^p.
\]
Now if $y = u^p$ we can write the special case of our model as

$$\frac{\partial y}{\partial t} + Q(t) \frac{\partial y^m}{\partial x} = \frac{\partial^2 y^m}{\partial x^2}, \quad m = \frac{1}{p}.$$  \hspace{1cm} (2.2.6)

We can now see that when $p < 1$ our model has non-linear diffusion and non-linear convection. This form of equation when $Q(t) =$ constant is a special case of the equation studied by Grundy[7] amongst others. The method that Grundy uses to solve the problem has similarities with our own problem in that it also uses a method of balancing dominant terms and matched asymptotics. If instead of just considering $Q(t) =$ constant, we had considered the case $Q(t) = 0$ then our model reduces to the porous medium-equation. Hence we can say that for $p < 1$ that our model is just the porous-medium equation with a non-linear convection term added on. The porous-medium equation was independently solved by Pattle[8] and Barenblatt[9] who solved our initial-value problem (with $Q(t) = 0$) for arbitrary initial data using an exact similarity solution. In solving the problem in this way, Pattle and Barenblatt established the importance of similarity variables in the theory of problems of this type (ie. parabolic equations).

We finally turn to look at the paper by Grundy, et al.[1], which is probably the most relevant to this thesis. The first point to note about this paper is that it presents the solutions over an infinite domain with compactly supported initial data, hence modelling the spread of initially confined contaminant. The effect of this is that the boundary condition at $x = 0$ in our problem is replaced by a boundary condition at $x = -\infty$. A result of having $Q(t)$ constant is that the mass of contaminant in the system must also be constant and finite, hence the new boundary condition must be

$$u = \frac{\partial u}{\partial x} = 0, \quad x \to \infty.$$  \hspace{1cm} (2.2.7)

In solving the model Grundy et al. again use methods of balancing dominant terms and asymptotic balancing. The ideas introduced in this paper allow us to establish the nature of our problem when $p < 1$. To do this we must look at the introduction into the problem of a travelling wave, as shown by Grundy, et al.. This is relevant to the problem this thesis is interested in, because the argument used equally applies to our model. In introducing the travelling wave into the problem, Grundy et al. have made use of the work by Van Duijn and Knabner[10] who looked at this problem and Van Duijn and De Graaf[11] who looked at rarefraction wave solutions.

Grundy, et. al show that the translational properties of the solution are controlled by
convection and hence it is useful to examine the equation

\[ \frac{\partial (u + u^p)}{\partial t} + \frac{\partial u}{\partial x} = 0 \]  

(2.2.8)

which may be written as

\[ \frac{\partial u}{\partial t} + \frac{1}{1 + pw^{p-1}} \frac{\partial u}{\partial x} = 0 \]  

(2.2.9)

It is easy to see from this that since \( u \) is very small at large times then for \( p \geq 1 \) the wave has a non-zero translational speed whilst for \( p < 1 \) the translational speed is zero. It is easy to see from this that provided \( Q(t) \) is non-zero then \( Q(t) \) can only at most effect the constant nature of the translational speed for \( p > 1 \). This travelling wave can be incorporated into the model by introducing the change of variable when \( p \geq 1 \),

\[ x = X - \int Q(t) \, dt \]  

(2.2.10)

thus transforming (2.1.38) into

\[ \frac{\partial (u + u^p)}{\partial t} + Q(t) \frac{\partial u^p}{\partial x} = \frac{\partial^2 u}{\partial x^2} \]  

(2.2.11)

The transformation of (2.1.39) is analogous to this. The effect of this is that when \( p \geq 1 \) we are actually looking at a problem with non-linear convection but linear diffusion. This is incorporated into the problem by the choice of our similarity variable.

There is one other important feature of our solutions that can be established from the paper of Grundy, et al. In the solution of their model, Grundy, et al. use a first-order expansion of the dependent variable for large time. This has the effect of making some of the solutions (multiple solutions appear due to different values of \( p \) ) non-uniformly valid in space and hence boundary-layer solutions must be introduced into the problem to compensate for this.

To summarize then, the historical background to our problems shows us that what initially looks like a linear diffusion and linear convection problem, because of the \( \frac{\partial u^p}{\partial t} \) term, is actually for \( p < 1 \) a non-linear diffusion and non-linear convection problem. The case for \( p > 1 \) can be seen, after the introduction of a travelling wave, to be a linear convection and non-linear diffusion problem. In both cases we can expect the solution at some points to involve boundary layers.
Section 3

Formulation of the Problem

3.1 Analytical Remarks on the Mass Condition

3.1.1 Continuous Input

An important quantity is the mass $M(t)$ of contaminant in the soil at any one moment. The mass at any point $m(x,t)$ is clearly the sum of the mass of the adsorbed contaminant and the contaminant still dissolved in the water, i.e.

$$m(x,t) = \theta c + \rho s$$  \hspace{1cm} (3.1.1)

For $p \neq 1$ we use the following analysis. By changing (3.1.1) to the new variables introduced in §2.1 and summing all of these masses,

$$M(t) = \int_0^\infty (u + u^p) \, dx.$$  \hspace{1cm} (3.1.2)

Differentiating (3.1.2) we have

$$\frac{\partial M}{\partial t} = \int_0^\infty \frac{\partial (u + u^p)}{\partial t} \, dx = \int_0^\infty \left\{ \frac{\partial^2 u}{\partial x^2} - t^r \frac{\partial u}{\partial x} \right\} \, dx$$  \hspace{1cm} (3.1.3)

$$\frac{\partial M}{\partial t} = \left[ \frac{\partial u}{\partial x} - t^r u \right]_0^\infty = \left[ -\frac{\partial u}{\partial x} + t^r u \right]_{x=0} = u_0 t^r.$$  \hspace{1cm} (3.1.4)

Hence we can define our mass condition as for $p \neq 1$

$$\int_0^\infty (u + u^p) \, dx = \begin{cases} \frac{u_0}{-1 + r + 1} & t \to \infty \quad r > -1 \\ u_0 \log t & t \to \infty \quad r = -1 \end{cases}$$  \hspace{1cm} (3.1.5)
Section 3 Formulation of the Problem

For $p = 1$ by following the same style of derivation we get the mass condition as

$$\int_0^\infty u \, dx = \begin{cases} \frac{u_0}{r+1} t^{r+1} & t \to \infty \quad r > -1 \\ u_0 \log t & t \to \infty \quad r = -1 \end{cases} \quad (3.1.6)$$

3.1.2 The Pulse

In the case of the pulse the statement of the mass condition is much simpler. We again have

$$M(t) = \int_0^\infty (u + u^p) \, dx \quad (3.1.7)$$

but since no mass enters the porous medium after time $t_0$ then if the mass at time $t_0$ is $M_0$ then the mass condition is simply

$$M_0 = \int_0^\infty (u + u^p) \, dx \quad (3.1.8)$$

An interesting point to note is that this does not involve logarithmic terms for $r = -1$ and hence the $r = -1$ case will not necessarily differ from its immediately adjacent regions in the $r$-$p$ plane (except that is, for the initial change of variable from $x$ to $X$ as will be shown).

3.2 The Solution for $p = 1$: Continuous Input

For $p = 1$ the problem is linear and is thus dealt with separately from the significantly more complicated non-linear problem. So saying, the solution exhibits similar characteristics to the non-linear problem in that the solution changes its behaviour over the range of $r$ at the same values. Hence looking at $p = 1$ first allows us to introduce the principles the rest of the problem relies on.

We first look at $r > -1$. The problem in this case is

$$\frac{\partial u}{\partial t} + t^r \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \quad (3.2.1)$$

subject to the conditions

$$-\frac{\partial u}{\partial x} + t^r u = u_0 t^r, \quad x = 0 \quad (3.2.2)$$

$$u, \frac{\partial u}{\partial x} \to 0, \quad x \to \infty \quad (3.2.3)$$
Section 3 Formulation of the Problem

If we introduce the change of variable

\[ X = x - \frac{t^{r+1}}{r+1} \]  

(3.2.4)

the problem takes the form of a heat equation

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial X^2} \]  

(3.2.5)

\[ -\frac{\partial u}{\partial X} + t^r u = u_0 t^r, \quad x = 0 \]  

(3.2.6)

\[ u, \frac{\partial u}{\partial X} \to 0, \quad X \to \infty . \]  

(3.2.7)

A closed-form solution to (3.2.5) that satisfies (3.2.6) and (3.2.7) has not been found, hence we must introduce the similarity variable

\[ \eta = \frac{X}{t^\delta}, \quad \delta > 0 \]  

(3.2.8)

together with the change of dependent variable

\[ u(x, t) = t^\alpha v(\eta, t), \quad \alpha \leq 0 \]  

(3.2.9)

where the \( \alpha \leq 0 \) represents temporal decay.

For the purposes of this paper, where we are looking at the large-time behaviour of the problem, we will in fact only look at a first-order expansion of \( v(\eta, t) \). We therefore replace (3.2.9) with

\[ u(x, t) = t^\alpha (v_0(\eta) + o(1)) \]  

(3.2.10)

Before finally stating the problem, note that the position of the boundary condition corresponding to \( x = 0 \) becomes in our new co-ordinate system as \( t \to \infty \)

\[ \eta = \eta_{BC} = -\frac{t^{r+\delta}}{r+1} = \begin{cases} 
0 & r+1-\delta < 0 \\
-\frac{1}{r+1} & r+1-\delta = 0 \\
-\infty & r+1-\delta > 0 
\end{cases} \]  

(3.2.11)

Hence if \( r+1-\delta \leq 0 \) then our change of variables is at best only a translation (as \( t \to \infty \)) and we must therefore go back to the initial problem (3.2.1) - (3.2.3) and instead introduce immediately the similarity variable
Section 3 Formulation of the Problem

\[ \eta = \frac{x}{\varepsilon}, \quad \delta > 0 \quad (3.2.12) \]

If we introduce these change of variables into (3.1.6) for \( r > -1 \) we get

\[ \int_0^\infty v_0 \, d\eta = \frac{u_0}{r + 1} t^{r+1-\alpha-\delta}. \quad (3.2.13) \]

Clearly to remove the \( t \) from the right-hand side of this we must choose \( \alpha \) and \( \delta \) to satisfy

\[ r + 1 = \alpha + \delta. \quad (3.2.14) \]

We now have that the right-hand side of (3.2.13) is finite and hence for the left-hand side to also be finite, we require

\[ v_0, \frac{dv_0}{d\eta} \rightarrow 0, \quad \eta \rightarrow \infty. \quad (3.2.15) \]

Our problem is then finally described by

\[ \alpha v_0 - \delta \eta \frac{dv_0}{d\eta} = t^{1-2\delta} \frac{d^2v_0}{d\eta^2}, \quad (3.2.16) \]

\[ -t^{\alpha-\delta} \frac{dv_0}{d\eta} + t^{\alpha+\delta} v_0 = u_0 t^r, \quad \eta = \eta_{BC} \quad (3.2.17) \]

\[ v_0 = \frac{dv_0}{d\eta} = 0, \quad \eta \rightarrow \infty. \quad (3.2.18) \]

We can also use (3.2.14) to rewrite (3.2.17) as

\[ -t^{1-2\delta} \frac{dv_0}{d\eta} + v_0 t^{r+1-\delta} = u_0, \quad \eta = \eta_{BC}. \quad (3.2.19) \]

### 3.2.1 \(-1 < r < -\frac{1}{2}\)

This range of \( r \) can be obtained by setting \( \delta = \frac{1}{2} \) and \( \alpha = r + \frac{1}{2} \). Before stating the problem we must look at the boundary condition at \( \eta = \eta_{BC} \). In this case it is obvious that \( \eta_{BC} = 0 \) and we have

\[ -\frac{dv_0}{d\eta} + v_0 t^{r+\frac{1}{2}} = u_0, \quad \eta = 0 \quad (3.2.20) \]

but since \( r < -\frac{1}{2} \) the second term on the left-hand side can be neglected and we can state the problem as

\[ \left( r + \frac{1}{2} \right) v_0 - \frac{\eta}{2} \frac{dv_0}{d\eta} = \frac{d^2v_0}{d\eta^2} \quad (3.2.21) \]
Section 3 Formulation of the Problem

\[-d\varphi_0 d\eta = u_0, \quad \eta = 0 \]

\[v_0, \frac{dv_0}{d\eta} = \rightarrow 0, \quad \eta \rightarrow \infty. \]

Result (3.2.21) can be reduced to the parabolic cylinder equation by the following procedure. The solutions of the parabolic cylinder equation are well known, and hence we may find our solution through them.

We let

\[v_0 = y(\eta)w(\eta) \]

where \(w(\eta)\) is arbitrary we can then choose \(w\) so as to eliminate those terms in our modified (4.1.7) that contain \(\frac{dy}{d\eta}\). We in fact take

\[w = e^{-\frac{r^2}{4}} \]

resulting in our problem becoming

\[\frac{\partial^2 y}{\partial \eta^2} - y \left( \frac{\eta^2}{16} + \alpha + \frac{1}{4} \right) = 0. \]

The boundary conditions are no longer being shown, since once a solution has been found in terms of arbitrary constants, we will immediately change our dependent variable back to \(v_0\).

Now let \(\eta = A \xi\), giving

\[\frac{1}{A^2} \frac{\partial^2 y}{\partial \xi^2} - y \left( \frac{A^2 \xi^2}{16} + \alpha + \frac{1}{4} \right) = 0. \]

Choose \(A\) such that

\[\frac{A^4}{16} = \frac{1}{4} \Rightarrow A = \sqrt{2} \]

ie. \(v_0 = e^{-\frac{r^2}{4}} y\) and hence we have the parabolic cylinder equation

\[\frac{d^2 y}{d\xi^2} - y \left( \frac{\xi^2}{4} + a \right) = 0 \]

where

\[a = 2 \left( \alpha + \frac{1}{4} \right) = 2 \left( r + \frac{3}{4} \right) \Rightarrow |a| < \frac{1}{2}. \]

Since the solutions of (3.2.29) (see Appendix A.2) are well known we can now immediately
state

\[ v_0 = e^{-\frac{\xi^2}{4}} (C_1 U + C_2 V) \]  

(3.2.31)

where \( U \) and \( V \) are the solutions of the parabolic-cylinder equation. If we now examine the asymptotic expansions of \( U \) and \( V \) as \( \xi \to \infty \),

\[
U \sim e^{-\frac{\xi^2}{4}} \xi^{-\frac{1}{2}} \left\{ 1 - \frac{(a + \frac{1}{2})(a + \frac{3}{2})}{2\xi^2} + \frac{(a + \frac{1}{2})(a + \frac{3}{2})(a + \frac{5}{2})(a + \frac{7}{2})}{2 \cdot 4\xi^4} - \ldots \right\}
\]

(3.2.32)

\[
V \sim \sqrt{\frac{2}{\pi}} e^{-\frac{\xi^2}{4}} \xi^{a-\frac{1}{2}} \left\{ 1 + \frac{(a - \frac{1}{2})(a - \frac{3}{2})}{2\xi^2} - \frac{(a - \frac{1}{2})(a - \frac{3}{2})(a - \frac{5}{2})(a - \frac{7}{2})}{2 \cdot 4\xi^4} + \ldots \right\}.
\]

(3.2.33)

The mass condition (3.1.5) states that in this case the integral of \( v_0 \) must be finite when evaluated between 0 and \( \infty \), subsequently the following conditions, involving indefinite integrals, must hold true

\[
\lim_{n \to \infty} \int_0^n e^{-\frac{\xi^2}{4}} U_\alpha d\xi = B_1, \quad \lim_{n \to \infty} \int_n^\infty e^{-\frac{\xi^2}{4}} V_\alpha d\xi = B_2
\]

(3.2.34)

for \( B_1, B_2 \) finite constants.

Clearly \( B_1 = 0 \), since we are integrating an expression involving a negative exponential, so we will just examine the first term of \( e^{-\frac{\xi^2}{4}} V_\alpha \), i.e.

\[
\lim_{n \to \infty} \int_0^n e^{-\frac{\xi^2}{4}} V_\alpha d\xi \sim \lim_{n \to \infty} \int_0^n \sqrt{\frac{2}{\pi}} \xi^{a-\frac{1}{2}} d\xi = \lim_{n \to \infty} \left( \sqrt{\frac{2}{\pi}} \frac{\xi^{1+a-\frac{1}{2}}}{1+a-\frac{3}{2}} \right) = \infty
\]

(3.2.35)

since \( 0 < 1 + a - \frac{1}{2} < 1 \). Clearly (3.2.35) contradicts (3.2.34) and hence the only way out of this contradiction is if

\[ C_2 = 0. \]

(3.2.36)

We must now solve

\[ v_0 = C_1 e^{-\frac{\xi^2}{4}} U \]

(3.2.37)

subject to

\[
\frac{dv_0}{d\xi} = -\sqrt{2} v_0, \quad \xi = 0
\]

(3.2.38)
Figure 3.1: The solution when \( u_0 = 1, r = \frac{3}{4} \)

\[
v_0, \frac{dv_0}{d\xi} \to 0, \quad \xi \to \infty . \tag{3.2.39}
\]

Note that we do not need to use the definition of \( V \) at all, but to evaluate \( C_i \) we must use the definition of \( U \) at \( \xi = 0 \) as given in Appendix A.2. If we use

\[
\frac{dv_0}{d\xi} = C_1 \left( e^{-\frac{\xi^2}{4}} \frac{dU}{d\xi} - \frac{\xi}{2} e^{-\frac{\xi^2}{4}} U \right) \tag{3.2.40}
\]

then we get

\[
C_1 = \frac{2^{r+1} \Gamma(r+1) u_0}{\sqrt{\pi}} . \tag{3.2.41}
\]

In summary the solution is

\[
v_0(\xi) = \frac{2^{r+1} \Gamma(r+1) u_0}{\sqrt{\pi}} U(a, \xi) . \tag{3.2.42}
\]

See Figure 3.1 for the solution when \( u_0 = 1, r = \frac{3}{4} \), which has been computed using

\[1\text{Because }|a| < \frac{1}{2} \text{ then the } \Gamma \text{ function never has a zero argument and our solution stays finite.} \]
numerical means, due to the difficulties imposed by $U(a, \xi)$ being an infinite series. The numerical method just used a Runge-Kutta method together with the initial values at $\xi = 0$ of $v_0$ and $\frac{dv_0}{d\xi}$ which are already known.

### 3.2.2 $r > -\frac{1}{2}$

For the case $r > -\frac{1}{2}$ it is found that we require $\delta > \frac{1}{2}$ and $\alpha = 0$. Since we now have $\delta = r + 1$, we can now redefine $\eta$ as

$$\eta = \frac{x}{t^{r+1}}$$

(3.2.43)

giving the differential equation to be solved as

$$\frac{dv_0}{d\eta} = 0, \quad \eta > 0$$

(3.2.44)

The first step again is to look at the boundary condition which in this case is

$$-t^{-1-2r} \frac{dv_0}{d\eta} + v_0 = u_0, \quad \eta = 0$$

(3.2.45)

and since $r > -\frac{1}{2}$ the first term on the left-hand side disappears and we can then state the problem as

$$\frac{dv_0}{d\eta} = 0 \text{ for } \eta > 0, \quad v_0 = u_0 \text{ at } \eta = 0, \quad v_0 \to 0, \text{ as } \eta \to \infty.$$  

(3.2.46)

Clearly this requires the introduction of a boundary layer into our problem. We will take the internal layer to be around $\eta = \eta_0$. We can find $\eta_0$ by considering the mass condition (3.1.6) for this case, i.e.

$$\frac{1}{r + 1} = \int_0^{\eta_0} d\eta$$

(3.2.47)

i.e.

$$\eta_0 = \frac{1}{r + 1}.$$  

(3.2.48)

If we now let

$$\xi = \frac{\eta - \eta_0}{t^{-(r + \frac{1}{2})}}, \quad u(x, t) = w_0(\xi) + o(1)$$

(3.2.49)

the problem becomes

$$-\frac{\xi}{2} \frac{dw_0}{d\xi} = \frac{d^2 w_0}{d\xi^2}$$

(3.2.50)

$$w_0 = u_0, \quad \xi \to -\infty$$

(3.2.51)
Section 3 Formulation of the Problem

Figure 3.2: The boundary layer when \( p = 1 \) and \( u_0 = 1 \)

\[ w_0 = 0, \quad \xi \to \infty \]

which has the solution

\[ w_0 = \frac{u_0}{2} \left( 1 - \text{erf} \left( \frac{\xi}{2} \right) \right). \quad (3.2.53) \]

Figure 3.2 shows this solution for \( u_0 = 1 \).

We can verify that our boundary condition at \( \eta = 0 \) holds by looking at the work of Genuchten and Cleary\[^4\] for the special case \( r = 0 \). They give the solution to (3.2.1)–(3.2.3) for \( r = 0 \) as, after introducing scalings to remove the physical constants (which is equivalent to setting each physical constant to 1 hence allowing us to still write the solution in terms of \( x \) and \( t \) although we do use the fact that \( c/c_0 = u/u_0 \)),

\[ \frac{u}{u_0} = \frac{1}{2} \text{erfc} \left( \frac{x - t^{\frac{1}{2}}}{2 t^{\frac{1}{2}}} \right) + \left( \frac{t}{\pi} \right)^{\frac{1}{4}} \exp \left( -\frac{(x - t)^2}{4t} \right) - \frac{1}{2} \exp(x) (1 + x + t) \text{erfc} \left( \frac{x + t^{\frac{1}{2}}}{2 t^{\frac{1}{2}}} \right). \quad (3.2.54) \]
which must satisfy the boundary condition

\[ \frac{\partial u}{\partial x} + u = 1 \]  

(Note that we have now also scaled out \( u_0 \).) The behaviour of this solution can be seen by temporarily introducing the change of variable

\[ x = \xi t \]  

where \( \xi = o(1) \) which gives

\[ u = \frac{1}{2} \text{erfc} \left( \frac{t^{\frac{1}{2}}}{2} (\xi - 1) \right) + \left( \frac{t}{\pi} \right)^{\frac{1}{2}} \exp \left( -\frac{t}{4} (\xi - 1)^2 \right) - \frac{1}{2} \exp(\xi t) \left( 1 + t(\xi - 1) \right) \text{erfc} \left( \frac{t^{\frac{1}{2}}}{2} (\xi + 1) \right) \]  

It is easy to establish from (3.2.57) that as \( t \to \infty \),

- if \( \xi < 1 \) then \( u \to 1 \)
- if \( \xi > 1 \) then \( u \to 0 \).

This verifies that we do indeed have a stepwise solution which requires a boundary layer.

If we instead introduce the change of variables

\[ \eta = \frac{x - t}{t^{\frac{1}{2}}} \]  

then (3.2.54) subsequently becomes

\[ u = \frac{1}{2} \text{erfc} \left( \frac{\eta}{2} \right) + \left( \frac{t}{\pi} \right)^{\frac{1}{2}} \exp \left( -\left( \frac{\eta^2}{4} \right) \right) - \frac{1}{2} \exp \left( t + \eta t^{\frac{1}{2}} \right) \left( 1 + 2t + \eta t^{\frac{1}{2}} \right) \text{erfc} \left( t^{\frac{1}{2}} + \frac{\eta}{2} \right) \]  

Now looking at the case \( t \to \infty \) and \( \eta = O(1) \) we can make use of the asymptotic expansion of \( \text{erfc}(z) \) as \( z \to \infty \), i.e.

\[ \text{erfc}(z) = \frac{1}{\sqrt{\pi z}} e^{-z^2} \left( 1 + O\left(z^{-2}\right) \right), \quad z \to \infty \]  

24
\[ \text{erfc} \left( t^{\frac{1}{2}} + \frac{\eta}{2} \right) \sim \frac{1}{\sqrt{\pi} \left( t^{\frac{1}{2}} + \frac{\eta}{2} \right)} \exp \left( \left( t^{\frac{1}{2}} + \frac{\eta}{2} \right)^2 \right) \left( 1 + O \left( \frac{1}{t} \right) \right) + \ldots \]  
\tag{3.2.61}

\[ = \frac{\exp \left( -t - \eta t^{\frac{1}{2}} - \frac{\eta^2}{4} \right)}{\sqrt{\pi} t^{\frac{3}{2}}} \left( 1 - \frac{\eta}{2t^{\frac{1}{2}}} + O \left( \frac{1}{t} \right) \right) \left( 1 + O \left( \frac{1}{t} \right) \right) \]  
\tag{3.2.62}

Consequently putting this into (3.2.59) after tidying up we get
\[ u = \frac{1}{2} \text{erfc} \left( \frac{\eta}{2} \right) \]  
\tag{3.2.63}

Now to finally satisfy the boundary condition
\[ \left. \left| \frac{\partial u}{\partial x} + u \right| \right|_{x=0} = \left. -t^{-\frac{1}{2}} \frac{\exp \left( -\frac{\eta^2}{4} \right)}{\sqrt{\pi}} + \frac{1}{2} \text{erfc} \left( \frac{\eta}{2} \right) \right|_{\eta \to -\infty, t \to \infty} = 1 \]  
\tag{3.2.64}

This also helps to establish the choice of change of variables \( \eta = \frac{x - t}{t^{\frac{1}{2}}} \).
3.2.3 \( r = \frac{-1}{2} \)

This time, since \( r = \frac{-1}{2} \) we must take

\[
\alpha = 0, \quad \delta = \frac{1}{2}
\]

and the problem this time is

\[
-\lambda_1 \frac{\eta d^2 v_0}{d\eta^2} = \lambda_1 \frac{d^2 v_0}{d\eta^2} - \frac{dv_0}{d\eta}
\]

(3.2.66)

\[
- \frac{dv_0}{d\eta} + v_0 = u_0, \quad \eta = 0
\]

(3.2.67)

\[
v_0, \frac{dv_0}{d\eta} \rightarrow 0, \quad \eta \rightarrow \infty.
\]

(3.2.68)

Solving (3.2.66) gives

\[
v_0 = c_1 + c_2 \text{erf}\left( \frac{1}{\sqrt{\lambda_1}} \left( \frac{\eta}{2} - 1 \right) \right)
\]

(3.2.69)

and using (3.2.67) and (3.2.68) we get

\[
v_0 = \frac{u_0}{1 + \text{erf}\left( \frac{1}{\sqrt{\lambda_1}} \left( \frac{\eta}{2} - 1 \right) \right)}
\]

(3.2.70)

The solution when \( u_0 = 1 \) and we have choosen \( \lambda_1 = 1 \) is shown in Figure 3.3.

3.2.4 \( r = -1 \)

The case \( r = -1 \) has to be dealt with separately since the change of variable involves the logarithmic function. The problem is

\[
\frac{\partial u}{\partial t} + t^{-1} \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}
\]

(3.2.71)

\[
- \frac{\partial u}{\partial x} - t^{-1} u = u_0 t^{-1}, \quad x = 0
\]

(3.2.72)

\[
u = \frac{\partial u}{\partial x} = 0, \quad x \rightarrow \infty.
\]

(3.2.73)

This time we introduce the change of variable

\[
X = x - \log t
\]

(3.2.74)
to again make the problem take the form of a heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial X^2}$$

(3.2.75)

$$-\frac{\partial u}{\partial X} + t^{-1}u = u_0t^{-1}, \quad x = 0$$

(3.2.76)

$$u, \frac{\partial u}{\partial X} \to 0, \quad X \to \infty.$$  

(3.2.77)

Yet again the next change of variables is similar to the cases \( r > -1 \), but involves the logarithmic function. We introduce

$$\eta = \frac{X}{t^\delta (\log t)^\gamma}, \quad \delta, \gamma > 0$$

(3.2.78)

and the change of dependent variable

$$u = t^\alpha (\log t)^\beta (v_0(\eta) + o(1)), \quad \alpha, \beta < 0$$

(3.2.79)

and our problem becomes, after discarding terms that are always dominated by others no matter what values for the parameters are taken,

$$\alpha v_0 - \approx \eta \frac{dv_0}{d\eta} = t^{1-2\delta} (\log t)^{-2\gamma} \frac{d^2 v_0}{d\eta^2}$$

(3.2.80)

$$-t^{\alpha-\delta} (\log t)^{\gamma - \delta} \frac{dv_0}{d\eta} + t^{\alpha-1} (\log t)^{\beta} = u_0t^{-1}, \quad x = 0$$

(3.2.81)

$$v_0 = \frac{dv_0}{d\eta} = 0, \quad x \to \infty.$$  

(3.2.82)

It can be shown (the method used is gone into in more detail in §4.4) that the only solution for (3.2.80) that is consistent with the boundary conditions (3.2.81) and (3.2.82) is when

$$\delta = \frac{1}{2} \implies \alpha = -\frac{1}{2}$$

(3.2.83)

$$\gamma = 0 \implies \beta = 1.$$  

(3.2.84)

If we now look at the boundary condition corresponding to \( x = 0 \) we have

$$\eta = \eta_{BC} = -t^{-\frac{1}{2}} (\log t) \to 0, \quad t \to \infty$$

(3.2.85)

so we have to return to the original problem (3.2.71) - (3.2.73) and instead introduce the
change of variable

\[ \eta = \frac{x}{t^{\frac{1}{2}}} \quad . \quad (3.2.86) \]

Since the convection term is now dominated by the diffusion term we again get problem (3.2.80) - (3.2.80) but with this different definition for \( \eta \). Hence we finally have the problem that needs to be solved as

\[ -\frac{1}{2} \left( \frac{d^2 v_0}{d\eta^2} + \frac{dv_0}{d\eta} \right) = \frac{d^2 v_0}{d\eta^2} \quad (3.2.87) \]

\[ \frac{dv_0}{d\eta} = 0, \quad \eta = 0 \quad (3.2.88) \]

\[ v_0, \frac{dv_0}{d\eta} \to 0, \quad \eta \to \infty \quad (3.2.89) \]

and the solution to this is

\[ v_0 = \frac{u_0}{2\sqrt{\pi}} e^{-\frac{x^2}{\xi}} \quad . \quad (3.2.90) \]

We show the solution in Figure 3.4 when \( u_0 = 1 \).
3.3 The Solution for $p = 1$: The Pulse

As in the continuous-input case, the linearization caused by setting $p = 1$ significantly simplifies the problem. Our problem here is

$$\frac{\partial u}{\partial t} + t^r \frac{\partial u}{\partial x} = \lambda_1 \frac{\partial^2 u}{\partial x^2}$$

subject to

$$-\lambda_1 \frac{\partial u}{\partial x} + t^r u = \begin{cases} u_0 t^r & x = 0, \quad 0 < t < t_0 \\ 0 & x = 0, \quad t > t_0 \end{cases}$$

$$u_0 \frac{\partial u}{\partial x} \to 0, \quad x \to \infty.$$  (3.3.3)

As we are only interested in the large time solution we will from now on in all of the pulse cases, neglect the boundary condition for $0 < t < t_0$ at $x = 0$.

If we introduce the change of variable

$$X = \begin{cases} x - \frac{r+1}{r+1}, & r > -1 \\ x - \log t, & r = -1 \end{cases}$$

we are then able to follow the method used in the previous section (§3.2). It must be noted though, that due to our different mass condition, we have

$$\delta = -\alpha.$$  (3.3.5)

It is easy to establish that the only value for $\delta$ that will allow the problem to satisfy the boundary conditions is in fact

$$\delta = \frac{1}{2}.$$  (3.3.6)

(any other value will leave the diffusion term either dominated by the other terms or dominant on its own) and hence we may state the problem to be solved as

$$\frac{1}{2} \frac{d(\eta v_0)}{d\eta} = \lambda_1 \frac{d^2 v_0}{d\eta^2}$$

$$-\lambda_1 \eta^{-r-\delta} \frac{dv_0}{d\eta} + v_0 = 0, \quad \eta = \eta_{bc}.$$  (3.3.8)
Section 3 Formulation of the Problem

The solution of (3.3.7) with (3.3.9) is

\[ v_0 = C_1 e^{-\frac{\eta^2}{4\lambda_1}} \]  

(3.3.10)

where \( C_1 \) is constant. Due to the differing values of \( \eta_{BC} \) we then finally get the solution as

\[ v_0 = \begin{cases} 
\frac{M_0}{\sqrt{\pi}} e^{-\frac{\eta^2}{4}}, & 0 < \eta < \infty, \quad -1 \leq r < -\frac{1}{2} \\
\frac{M_0}{\sqrt{\pi\lambda_1}(1+erf(\frac{1}{\sqrt{\lambda_1}}))} e^{-\frac{\eta^2}{4\lambda_1}}, & -2 < \eta < \infty, \quad r = -\frac{1}{2} \\
\frac{M_0}{2\sqrt{\pi}} e^{-\frac{\eta^2}{4}}, & -\infty < \eta < \infty, \quad r > -\frac{1}{2} 
\end{cases} \]

(3.3.11)

The plots of the solution when \( M_0 = 1 \) (when \( r = -\frac{1}{2} \) we have chosen \( \lambda_1 = 1 \)), are shown in Figure 3.5.

We can verify this result again for the case \( r = 0 \) by looking again at the work of
Genuchten and Cleary[4]. If we introduce our scalings to remove the physical constants and let

\[
f(x, t) = \frac{1}{2} \text{erfc}\left(\frac{x - t}{2t^{\frac{1}{4}}}\right) + \left(\frac{t}{\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{(x - t)^2}{4t}\right) - \frac{1}{2} \exp(x)(1 + x + t)\text{erfc}\left(\frac{x + t}{2t^{\frac{1}{4}}}\right)
\]

then

\[
u = f(x, t) - f(x, t - t_0), \quad t > t_0.
\]

(3.3.12)

We can firstly verify our boundary condition by setting \(x = 0\) in (3.3.13) and then taking a series expansion about \(t \to \infty\). This yields the result that

\[
u \sim O\left(t^{-\frac{3}{2}} e^{-\frac{x^2}{t}}\right), \quad t \to \infty
\]

which is consistent with the boundary condition

\[
v_0 = 0, \quad \eta = 0, \quad t \to \infty.
\]

(3.3.15)

To verify our solution is correct we make a change of variable in (3.3.13) from \(x\) to \(\eta\), but must remember that \(\eta\) also involves \(t\), hence (3.3.13) becomes

\[
u = g(\eta, t) - g\left(\frac{\eta t^\frac{1}{2} + t_0}{(t - t_0)^{\frac{1}{2}}}, t - t_0\right)
\]

(3.3.16)

where

\[
g(\eta, t) = \frac{1}{2} \text{erfc}\left(\frac{\eta}{2}\right) + \left(\frac{t}{\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{\eta^2}{4}\right)
\]

\[-\frac{1}{2} \exp\left(t + \eta t^\frac{1}{2}\right)\left(1 + 2t + \eta t^\frac{1}{2}\right)\text{erfc}\left(t^\frac{1}{2} + \frac{\eta}{2}\right).
\]

(3.3.17)

Again by taking a series expansion of (3.3.16) about \(t \to \infty\) we get

\[
u \sim t^{-\frac{1}{2}} \frac{t_0}{2\sqrt{\pi}} e^{-\frac{x^2}{t}} + O\left(\frac{1}{t}\right)
\]

(3.3.18)

which verifies our solution.
3.4 The General Formulation of the Equation when $p \neq 1$

Before looking at the general formulation of the problem it is important to point out that this formulation does not include the continuous input case of $r = -1$ for reasons that will become clear although the formulation for this case is similar. The approach taken is to postulate an analytic form of the solution and then to look for solutions that fit this form. The choice of this analytic form is initially based on whether bounded support ($p < 1$) or unbounded support ($p > 1$) applies. Depending on how the obtained results behave, we then review whether the other analytic form would be more sensible to use. In each case the analytic form involves two parameters $\alpha$ and $\delta$, and it is these parameters that cause our solution to be split into four different cases ($r = -1, -1 < r < -\frac{1}{2}, r = -\frac{1}{2}$ and $r > -\frac{1}{2}$). These forms of solution have been previously used to good effect, particularly in the work of Grundy[1].

3.4.1 $p > 1$

If we take the initial problem (2.1.38) - (2.1.41) and introduce the following change of variable, so as to follow the wave of contaminant as it moves along (remember this is the unbounded case),

$$X = x - \frac{p+1}{r+1}$$

and let $u = u(X, t)$. The problem becomes

$$\frac{\partial (u + u^p)}{\partial t} - t^r \frac{\partial u^p}{\partial X} = \lambda \frac{\partial^2 u}{\partial X^2}, \ t \to \infty$$

$$-\lambda \frac{\partial u}{\partial X} + t^r u = u_0 t^r, \ x = 0$$

$$u, \frac{\partial u}{\partial X} \rightarrow 0, \ x \rightarrow \infty.$$  \hspace{1cm} (3.4.3)

Now if we introduce the similarity variable

$$\eta = \frac{X}{t^\delta}, \ \delta > 0$$

and a change of dependent variable to

$$u = t^\alpha v(\eta, t), \ \alpha \leq 0$$

$$\eta = \frac{X}{t^\delta}, \ \delta > 0$$
where the $\alpha < 0$ represents temporal decay. As in the $p = 1$ case, since we are only looking at large time solutions, we will confine our interest to first order terms and hence we will instead introduce the change of dependent variable to

$$u = t^{\alpha} (v_0(\eta) + o(1)), \alpha \leq 0 \ . \quad (3.4.7)$$

Hence our the problem (3.4.2) - (3.4.4) has become

$$\left( \alpha v_0 - \delta \eta \frac{dv_0}{d\eta} \right) + t^{\alpha(p-1)} \left( \alpha p v_0^{p-1} - \delta \eta \frac{dv_0^p}{d\eta} \right) = t^{1-2\delta} \lambda \frac{d^2v_0}{d\eta^2} + t^{r+\alpha(p-1)+1-\frac{1}{r}} \frac{dv_0^p}{d\eta} \quad (3.4.8)$$

which we can write as

$$A_{1+} + B_{1+} = C_{1+} + D_{1+}$$

subject to

$$-\lambda t^{-\delta - r} \frac{dv_0}{d\eta} + v_0 = u_0 t^{-\alpha}, x = 0 \quad (3.4.9)$$

$$v_0 \frac{dv_0}{d\eta} \to 0, \quad x \to \infty \ . \quad (3.4.10)$$

Note that the location of (3.4.9) and (3.4.10) are in terms of $x$ and not $\eta$ since the value of our parameters will determine these corresponding values of $\eta$.

At this point it is useful to note that our mass condition for continuous input with $r > -1$ gives us

$$r + 1 = \alpha + \delta \quad (3.4.11)$$

as in the $p = 1$ case. This allows us to use the same boundary conditions for the different cases of $r$ as in the $p = 1$ case. The mass condition for the pulse gives a different relationship, ie.

$$\alpha = -\delta \quad (3.4.12)$$

again as in the $p = 1$ case.

Whilst for the continuous input case we gain very little by replacing $\alpha$ in terms of $r$ and $\delta$, it is worth our while replacing $\alpha$ with $-\delta$ in the pulse case. An implication of $\alpha = -\delta$ is that we can now say $\alpha < 0$ and hence we can neglect the $B_{1+}$ term in (3.4.8) since it is always dominated by $A_{1+}$. Hence redefining the problem to be solved for $p > 1$ we get

$$-\delta \frac{d}{d\eta} (\eta v_0) = t^{1-2\delta} \lambda \frac{d^2v_0}{d\eta^2} + t^{r-\delta p + 1-\frac{1}{r}} \frac{dv_0^p}{d\eta} \ , \quad (3.4.13)$$
which we can write as
\[ A_{1+} = C_{1+} + D_{1+} \] subject to
\[ -\lambda t^{-\gamma} \frac{dv_0}{d\eta} + v_0 = 0, \quad x = 0, \quad (3.4.14) \]
\[ v_0, \frac{dv_0}{d\eta} \to 0, \quad \eta \to \infty. \quad (3.4.15) \]

### 3.4.2 \( p < 1 \)

If we instead introduce the similarity variable straight away, since here we have bounded support and hence do not need to attempt to follow the wave, i.e.
\[ \eta = \frac{x}{t^\delta}, \quad \delta > 0 \quad (3.4.16) \]

and
\[ u = t^\alpha (v_0(\eta) + o(1)), \quad \alpha \leq 0 \quad (3.4.17) \]
in the same way as in §3.4.1 then the problem (2.1.38)-(2.1.41) becomes
\[ \left( \alpha v_0 - \delta \eta \frac{dv_0}{d\eta} \right) + t^\alpha (p-1) \left( \alpha \eta v_0^p - \delta \eta \frac{dv_0^p}{d\eta} \right) = t^{1-2\delta} \lambda \frac{d^2v_0}{d\eta^2} - t^{p+1-\delta} \frac{dv_0}{d\eta} \quad (3.4.18) \]

which we can write as
\[ A_{1-} + B_{1-} = C_{1-} + D_{1-} \] subject to
\[ -\lambda t^{-\gamma} \frac{dv_0}{d\eta} + v_0 = u_0 t^{-\alpha}, \quad x = 0 \quad (3.4.19) \]
\[ v_0, \frac{dv_0}{d\eta} \to 0, \quad x \to \infty. \quad (3.4.20) \]

In fact, due to the bounded support of \( u \) we can replace (3.4.20) with
\[ v_0 = \frac{dv_0}{d\eta} = 0, \quad \eta = \eta_0 \quad (3.4.21) \]
for some \( \eta_0 \).

As the different balances of \( C_{1-} \) and \( D_{1-} \) are considered, in the next section, the balances of (3.4.19) will also have to be considered. This consideration follows the same method as in the \( p = 1 \) case except that the mass condition for \( r > -1 \) this time gives for
continuous input

\[ r + 1 = \alpha p + \delta \] (3.4.22)

and for the pulse

\[ \alpha p = -\delta \] (3.4.23)

Again it is appropriate for the pulse case to substitute \( \alpha \) with \(-\frac{\delta}{p}\). This time though we have that \( B_{1-} \) dominates \( A_{1-} \). We can therefore state the problem as

\[ -\delta t^{-\frac{\delta}{p}(p-1)} \frac{d}{d\eta} (\eta v_0^p) = t^{1-2\delta} \lambda \frac{d^2 v_0}{d\eta^2} - t^{\gamma+1-\delta} \frac{dv_0^p}{d\eta} \] (3.4.24)

which we can write as

\[ B_{1-} = C_{1-} + D_{1-} \]

subject to

\[ -\lambda t^{-\gamma-\delta} \frac{dv_0}{d\eta} + v_0 = 0, \quad \eta = 0 \] (3.4.25)

\[ \frac{dv_0}{d\eta}, v_0 \to 0, \quad x \to \infty. \] (3.4.26)

In the same way as above we can replace (3.4.26) with

\[ v_0 = \frac{dv_0}{d\eta} = 0, \quad \eta = \eta_0 \] (3.4.27)

for some \( \eta_0 \).
Section 4

The Continuous Input Solution

4.1 Case I) $-1 < r < -1/2$

4.1.1 $p > 1$

Here we are looking at the balance $A_{1+} \sim C_{1+}$ (see §3.4.1). The choice of our arbitrary parameters depends on the exponents of $t$ in (3.4.8). For $A_{1+}$ and $C_{1+}$ to balance we require

$$1 - 2\delta = 0, \quad \Rightarrow \quad \delta = \frac{1}{2}$$

(4.1.1)

and for $D_{1+}$ to be dominated we require that it tends to zero, and hence

$$r + \alpha(p - 1) + 1 - \delta < 0$$

(4.1.2)

but from the mass condition (3.1.5) we have that

$$r + 1 - \delta = \alpha$$

(4.1.3)

consequently

$$\alpha < 0$$

(4.1.4)

If we now remember that our similarity variable is

$$\eta = \frac{x - \frac{t^{r+1}}{r+1}}{t^{\delta}} = \frac{x}{t^{\delta}} - \frac{t^{r+1-\delta}}{r+1}$$

(4.1.5)

It is immediately obvious that, since we are considering $t \to \infty$, it is more sensible to instead introduce the similarity variable
Instead. Note that this only changes $D_{1+}$ and even then does not change the exponent of $t$ and hence the problem we need to solve is unaffected by this alternate similarity variable.

The problem we must now solve is hence

\[
\frac{d^2v_0}{d\eta^2} = \alpha v_0 - \frac{\eta}{2} \frac{dv_0}{d\eta} \tag{4.1.7}
\]

\[
\frac{dv_0}{d\eta} = -u_0, \quad \eta = 0 \tag{4.1.8}
\]

\[
v_0, \frac{dv_0}{d\eta} \to 0, \quad \eta \to \infty \tag{4.1.9}
\]

which is identical to the problem in §3.2.1, and hence we can just state the solution as

\[
v_0(\xi) = \frac{2^{r+1} \Gamma(r + 1) u_0}{\sqrt{\pi}} U(a, \xi) \tag{4.1.10}
\]

where $\xi = \frac{\eta}{\sqrt{2}}$ and $a = 2 \left( r + \frac{3}{4} \right)$.

### 4.1.2 $p < 1$

Here we are interested in $B_{1-} \sim C_{1-}$. So here we require in the same way as in the previous case ($p > 1$) that the exponents of $t$ in $B_{1-}$ and $C_{1-}$ are equal and that they are larger than the exponent of $t$ in $D_{1-}$. In other words we have the conditions

\[
1 - 2\delta = \alpha(p - 1) \tag{4.1.11}
\]

and

\[
r + 1 - \delta < 1 - 2\delta \Rightarrow r < -\delta . \tag{4.1.12}
\]

If we now use (4.1.11) along with the mass condition we get

\[
\alpha = \frac{2r + 1}{1 + p}, \quad \delta = \frac{1 + r(1 - p)}{1 + p} . \tag{4.1.13}
\]

To re-cap, our boundary condition is

\[
-t^{-\delta-r} \frac{dv_0}{d\eta} + v_0 = u_0 t^{-\alpha} . \tag{4.1.14}
\]
If we use (4.1.11) and the mass condition

$$1 + r = \alpha p + \delta$$  \hspace{1cm} (4.1.15)

then we get

$$-\delta - r = 1 - \alpha p - 2\delta = 1 - \alpha p + \alpha(p - 1) - 1 = -\alpha$$  \hspace{1cm} (4.1.16)

and hence the problem becomes

$$\frac{2r + 1}{1 + p} Dw_a^p - \frac{1 + r(1 - p)}{1 + p} \eta \frac{dv_0}{d\eta} = \frac{d^2v_0}{d\eta^2}$$ \hspace{1cm} (4.1.17)

$$\frac{dv_0}{d\eta} = -u_0, \quad \eta = 0$$  \hspace{1cm} (4.1.18)

$$v_0 = \frac{dv_0}{d\eta} = 0, \quad \eta \to \infty$$ \hspace{1cm} (4.1.19)

We are unable to find an exact solution so a numerical method is used. The approach taken revolves around re-writing (4.1.17) as

$$\frac{dy_1}{d\eta} = y_2$$ \hspace{1cm} (4.1.20)

$$\frac{dy_2}{d\eta} = \alpha p y_1^p - \delta \eta y_1^{p-1} y_2$$ \hspace{1cm} (4.1.21)

with $y_1 = v_0, y_2 = \frac{dv_0}{d\eta}$.

Then by using the algorithms outlined in §6, and making the arbitrary choice for $p$ and $r$ of

$$p = \frac{1}{2}, \quad r = -\frac{3}{4}$$ \hspace{1cm} (4.1.22)

we can then plot a profile of $v_0$ against $\eta$ when $u_0 = 1$, see Figure 4.1 and a graph showing the relationship between $u_0$ and $\eta_0$, see Figure 4.2. Note that an error can be seen in Figure 4.1 where $v_0 = 0$. This is due to having taken an approximation to the solution at this point to avoid problems associated with $\frac{dv_0}{d\eta} \big|_{\eta = \eta_0} = 0$.

In attempting to solve (4.1.17) - (4.1.19) analytically, it is possible to reduce the problem to a first order nonlinear differential equation by using Euler's transformation, i.e. let

$$\eta = e^\beta y, \quad v_0(\eta) = e^y u(y)$$ \hspace{1cm} (4.1.23)
Section 4 The Continuous Input Solution

Figure 4.1: The solution when $u_0 = 1, r = -\frac{3}{4}, p = \frac{1}{2}$

where $\beta$ is an arbitrary constant. We now have (4.1.17) as

$$
\frac{e^{\nu(1-2\beta)}}{\beta^2} \left( \frac{d^2 u}{dy^2} + 2 \frac{du}{dy} + (1 - \beta) u \right) = \alpha p e^{\nu y} u^p - \frac{\delta e^{\nu y} u^{p-1}}{\alpha} \left( \frac{du}{dy} + u \right). \quad (4.1.24)
$$

So as to eliminate all of the exponential terms and hence leave no terms explicitly involving $y$ we can choose

$$
\beta = \frac{1 - p}{2}. \quad (4.1.25)
$$

If we now introduce

$$
z = \frac{du}{dy} \Rightarrow \frac{d^2 u}{dy^2} = \frac{dz}{du} \quad (4.1.26)
$$

then (4.1.24) becomes

$$
\frac{2}{p - 1} \left( 2z \frac{dz}{du} + 4z + (p - 1) u \right) = u^p \left( \alpha p^2 - \alpha p + 2 \delta \right) + 2u^{p-1} \delta z. \quad (4.1.27)
$$
Section 4 The Continuous Input Solution

Figure 4.2: The relationship between \( u_0 \) and \( \eta_0 \) when \( p = \frac{1}{2}, r = -\frac{3}{4} \)

Now introducing one final change of variable

\[
    w = \frac{1}{z}
\]

we get that (4.1.27) is equivalent to

\[
    \frac{dw}{du} = w^q \left\{ \frac{u(p + 1)}{2} - \frac{u^p(p - 1)}{4} \left( \alpha p^2 - \alpha p + 2\delta \right) \right\} + w^2 \left\{ 2 - \frac{(p - 1)\delta u^{p-1}}{2} \right\}
\]

which is a type of Abel's equation and the solution of this is not known (see Kamke[12]).

4.2 Case II) \( r > -1/2 \)

This time we are interested in either \( A_{1+} \) balancing with \( D_{1+} \) or \( A_{1-} \) balancing with \( D_{1-} \).

The balancing of the exponents of \( t \), and considerations of dominance, in the same way as
in §4.1 this time give us that, for $p > 1$

\[
\begin{align*}
1 - 2\delta &< 0 \\
r + 1 - \delta &= 0
\end{align*}
\]  
(4.2.1)

and for $p < 1$

\[
\begin{align*}
1 - 2\delta &< \alpha(p - 1) \\
r + 1 - \delta &= \alpha(p - 1)
\end{align*}
\]  
(4.2.2)

We can easily see that we have

\[
\delta > \frac{1}{2}
\]  
(4.2.3)

For $p > 1$ clearly $\alpha = 0$ while for $p < 1$, since $r + 1 - \delta = \alpha p$, we have $\alpha p = \alpha(p - 1)$ and again $\alpha = 0$. It is then clear that the range of $r$ here is $r > -\frac{1}{2}$.

If we now look at our similarity variable

\[
\eta = \frac{x}{t^\delta} - \frac{1}{r + 1} = \frac{x}{t^\delta} - \frac{1}{r + 1}
\]  
(4.2.4)

it is easy to see that we would in fact be better to introduce the similarity variable

\[
\eta = \frac{x}{t^\delta}
\]  
(4.2.5)

Our problem now becomes

\[
\delta \frac{d}{d\eta} (v_0 + v_0^\delta) = \frac{dv_0}{d\eta}
\]  
(4.2.6)

\[
v_0 = u_0, \quad \eta = 0
\]  
(4.2.7)

\[
\begin{align*}
v_0, \frac{dv_0}{d\eta} &\to 0, \quad \eta \to \infty, \quad p > 1 \\
v_0 = \frac{dv_0}{d\eta} &\to 0, \quad \eta = \eta_0 \quad p < 1
\end{align*}
\]  
(4.2.8)

Equation (4.2.6) on its own (i.e. without (4.2.7) and (4.2.8)) has two solutions

\[
v_0 = \text{constant}
\]  
(4.2.9)

\[
\delta \eta \left(1 + pv_0^\delta - 1\right) = 1, \quad \Rightarrow \quad v_0 = \left\{ \left(\frac{1}{\eta} - 1\right) \frac{1}{p} \right\}^{\frac{1}{p-1}}, \quad p \neq 1
\]  
(4.2.10)

So the solution can take two forms, either
Section 4 The Continuous Input Solution

1. Ignore solution (4.2.10) completely and use a boundary layer to get from \( v_0 = u_0 \) to \( v_0 = 0 \) centered on \( \eta = \eta_0 \), see 4.3(a).

2. Use the solution \( v_0 = u_0 \) for \( 0 < \eta < \eta_1 \) then switch to the solution of (4.2.10) to get from \( v_0(\eta_1) = u_0 \) to \( v_0(\eta_2) = 0 \) and finally switch to \( v_0 = 0 \) for \( \eta_2 < \eta < \infty \). See 4.3(b) where the solution of (4.2.10) is represented by a straight line so as to exaggerate the difference between the two forms of solution, although this is not actually the case.

It can be established by noting that any other possibilities give invalid solutions, that we must use a boundary layer for \( p < 1 \) and a two part solution for \( p > 1 \) (any other possibilities result in \( v_0 \) being infinite at some point).

4.2.1 \( p > 1 \)

If we assume that solution (4.2.9) and solution (4.2.10) intersect at \( \eta_1 \) and \( \eta_2 \) where \( 0 < \eta_1 < \eta_2 \) then the solution is

\[
v_0 = \begin{cases} 
  u_0 & 0 < \eta < \eta_1 \\
  \left( \frac{1}{\delta} - 1 \right) \frac{1}{p} \frac{1}{\eta^{p-1}} & \eta_1 < \eta < \eta_2 \\
  0 & \eta_2 < \eta 
\end{cases}
\]

(4.2.11)

It is then a trivial matter to establish \( \eta_1 \) and \( \eta_2 \) as shown below:
Section 4 The Continuous Input Solution

\[ \begin{align*}
\eta_1) & \quad \left( \frac{1}{(\delta \eta_1 - 1)} \right)^{\frac{1}{p}} = u_0 \quad \Rightarrow \quad \eta_1 = \frac{1}{(r+1)} \left( pu_0^{p-1} + 1 \right) \\
\eta_2) & \quad \left( \frac{1}{(\delta \eta_2 - 1)} \right)^{\frac{1}{p}} = 0, \quad \Rightarrow \quad \eta_2 = \frac{1}{r+1}.
\end{align*} \]

Figure 4.4: The solution when \( u_0 = 1, r = 0, p = \frac{3}{2}, 2, \frac{5}{2} \)

Hence the solution here is

\[ v_0 = \begin{cases} 
    u_0 & \eta < \frac{1}{(r+1)(pu_0^{p-1} + 1)} \\
    \left( \frac{1}{\delta \eta - 1} \right)^{\frac{1}{p}} & \frac{1}{(r+1)(pu_0^{p-1} + 1)} < \eta < \frac{1}{r+1} \\
    0 & \frac{1}{r+1} < \eta
\end{cases} \]  

(4.2.14)

Note that if \( p > 2 \) then \( \frac{1}{p-1} < 1 \) and so we show in figure 4.4 the solution for \( p = \frac{3}{2}, p = 2 \) and \( p = \frac{5}{2} \) when \( u_0 = 1 \) and \( r = 0 \).
4.2.2 $p < 1$

If we consider solution (4.2.9) as an outer expansion then we can introduce a boundary layer about $\eta = \eta_0$

\[
\eta = \eta_0 + \frac{\xi}{\xi^{2\delta-1}}, \quad v_0(\eta) = w_0(\xi) ,
\]

subject to

\[
w_0 \to u_0, \quad \frac{dw_0}{d\eta} \to 0, \quad \xi \to -\infty \tag{4.2.16}
\]

\[
w_0 \to 0, \quad \frac{dw_0}{d\eta} \to 0, \quad \xi \to \infty \tag{4.2.17}
\]

where the exponent of $t$ has been chosen so as to ensure that $C_{1-}$ has equal dominance with $D_{1-}$. With this change of variable (4.2.6) becomes

\[
-\delta \eta_0 \frac{\partial}{\partial \xi} (w_0 + w_0^\delta) = \frac{\partial^2 w_0}{\partial \xi^2} - \frac{\partial w_0}{\partial \xi} \tag{4.2.18}
\]

i.e.

\[
\frac{\partial w_0}{\partial \xi} = (1 - \delta \eta_0) w_0 - \delta \eta_0 w_0^\delta + C_1, \quad C_1 \text{ constant.} \tag{4.2.19}
\]

So as to satisfy (4.2.17) we must choose $C_1 = 0$, from whence we subsequently have

\[
w_0 = \left( e^{(\xi+C_2)(1-\delta \eta_0)(1-p)} - \delta \eta_0 \right)^{\frac{1}{1-p}}, \quad C_2 \text{ constant.} \tag{4.2.20}
\]

This solution must also satisfy (4.2.16). We can establish that the solution does in fact satisfy this requirement by examining the mass condition when $\alpha = 0$, where we have

\[
\frac{u_0}{r + 1} = \frac{u_0}{\delta} = \int_0^{\eta_0} u_0 + u_0^\delta d\eta \tag{4.2.21}
\]

Solving this gives that

\[
\delta \eta_0 = \frac{u_0^{1-p}}{1 + u_0^{1-p}} \tag{4.2.22}
\]

which when put into (4.2.20) as $\xi \to -\infty$ gives $w_0 = u_0$ as required.

The evaluation of $C_2$ is not looked at here, but will come from an examination of the second order or higher terms in the expansion of $v(\eta, t)$.

The graph of $u_0 = 1$ is given in Figure 4.5 where we have taken an arbitrary value of $C_2 (= -6)$.
Figure 4.5: The solution when $u_0 = 1, \ p = \frac{1}{2}$
4.3 Case III) $r = -1/2$

Here we find that $r = -\frac{1}{2}$ requires the same choices of $\alpha$ and $\delta$ for both $p < 1$ and $p > 1$. This is shown to be true by first noting that we are interested in $A_{1+} \sim C_{1+} + D_{1+}$ and $B_{1-} \sim C_{1-} + D_{1-}$ for $p > 1$ and $p < 1$ respectively. Balancing of the exponents of $t$ for $p > 1$ between $A_{1+}$ and $C_{1+}$, gives us

$$1 - 2\delta = 0 \quad \Rightarrow \quad \delta = \frac{1}{2} \quad (4.3.1)$$

and the balance between $A_{1+}$ and $D_{1+}$ similarly gives

$$r + \alpha(p - 1) + 1 - \delta = 0 \quad (4.3.2)$$

but from the mass condition (3.1.5) we have that $r + 1 - \delta = \alpha$ which means that

$$\alpha = 0 \quad (4.3.3)$$

since $p \neq 0$. Putting these values of $\delta$ and $\alpha$ into (4.3.2) gives

$$r = \frac{1}{2} \quad (4.3.4)$$

In a similar way for $p < 1$ we get

$$1 - 2\delta = \alpha(p - 1) \quad (4.3.5)$$

and

$$r + 1 - \delta = \alpha(p - 1) \quad (4.3.6)$$

but this time the mass condition (3.1.5) gives $r + 1 - \delta = \alpha p$, so once again we get

$$\alpha = 0, \quad \delta = \frac{1}{2}, \quad r = \frac{1}{2} \quad (4.3.7)$$

Because of $\alpha$ being zero we may once again (like in §4.2) consider $p > 1$ and $p < 1$ together, and since we again have $r + 1 - \delta = 0$, we can use the change of variable

$$\eta = \frac{x}{\delta^2}, \quad p > 0, \quad p \neq 1 \quad (4.3.8)$$

Yet again, we must at this point look at the balances of the boundary condition, but this
time we simply have
\[ -\delta - \tau = 0 \]  
(4.3.9)
and again the boundary condition is the same as for \( p = 1 \).

We now only have one problem to consider,
\[
\lambda \frac{d^2 v_0}{d\eta^2} = \left(1 - \frac{\eta}{2} - \frac{\eta}{2} p v_0^{p-1} \right) \frac{dv_0}{d\eta} 
\]  
(4.3.10)
\[
v_0 - \lambda \frac{dv_0}{d\eta} = u_0, \quad \eta = 0 
\]  
(4.3.11)
\[
v_0, \frac{dv_0}{d\eta} \to 0, \quad \eta \to \infty \quad p > 1 \text{ } 
\]  
(4.3.12)
\[
v_0 = \frac{dv_0}{d\eta} = 0, \quad \eta = \eta_0 \eta_0 \text{ finite } \quad p < 1 \}
\]

Clearly (4.3.10) supports two possibilities, that essentially come down to whether
\[
\frac{dv_0}{d\eta} = 0
\]  
(4.3.13)
or
\[
\frac{dv_0}{d\eta} \neq 0 
\]  
(4.3.14)

Examining our boundary conditions (4.3.11) and (4.3.12) shows us that the solution can take two different forms:

1. We take (4.3.13) over \( 0 < \eta < \eta_1 \), (4.3.14) over \( \eta_1 < \eta < \eta_2 \) and (4.3.13) again over \( \eta_2 < \eta \) where \( 0 < \eta_1 < \eta_2 \),

2. We use only (4.3.14).

An analytic solution for (4.3.14) cannot be found and hence numerical methods must be employed. The difficulty then arises as to whether the solution takes form 1 or form 2. To date numerical solutions have only been found for form 2. As a result of having to take numerical solutions we must also arbitrarily fix \( \lambda \). We will take \( \lambda = 1 \) for both of the following cases.

This time we will consider the \( p < 1 \) case first.

\textbf{4.3.1 } \textit{p < 1}

Here we take the arbitrary choice of
\[
p = \frac{1}{2} 
\]  
(4.3.15)
Section 4 The Continuous Input Solution

If we define $y_1 = v_0$ and $y_2 = \frac{dv_0}{d\eta}$, we get the system of equations

\[
\frac{dy_1}{d\eta} = y_2 \\
\frac{dy_2}{d\eta} = \left(1 - \frac{\eta}{2} - \frac{\eta}{4}y_1^{-\frac{1}{2}}\right)y_2.
\]  

We can then plot the relationship between $u_0$ and $\eta_0$, see Figure 4.6, and if we take $u_0 = 1$ the plot of $v_0$ against $\eta$ as in §4.1, see Figure 4.7.

4.3.2 $p > 1$

Since in this case we have unbounded support, then there is no finite point $\eta_0$ where $v_0 = 0$ and hence we are only able to plot the solution for an arbitrary choice of $u_0$, in this case $u_0 = 2$. We have here also chosen $p = 2$, see Figure 4.8.
Figure 4.7: The Solution when $u_0 = 1, r = \frac{1}{2}, p = \frac{1}{2}$
Figure 4.8: The numerical solution when $u_0 = 2, r = \frac{1}{2}, p = 2$
4.4 Case IV) $r = -1$

4.4.1 $p > 1$

Since in this case we have $Q(t) \sim \frac{1}{t}$ as $t \to \infty$, to have the same effect as the previous choice (for $r > -1, p > 1$) of change of variable, so as to follow the wave, we must choose

$$X = x + \log t$$

(4.4.1)

giving us our problem as

$$\frac{\partial (u + u^p)}{\partial t} - \frac{1}{i} \frac{\partial u^p}{\partial X} = \frac{\partial^2 u}{\partial X^2}$$

(4.4.2)

$$-\frac{\partial u}{\partial X} + \frac{1}{t} u = u_0 \frac{1}{t}, \quad x = 0$$

(4.4.3)

$$u = \frac{\partial u}{\partial X} = 0, \quad x \to \infty .$$

(4.4.4)

We now introduce our similarity variable

$$\eta = \frac{X}{t^\delta}$$

(4.4.5)

together with the change of dependent variable

$$u = t^\delta \log t (v_0(\eta) + o(1)) .$$

(4.4.6)

It can be shown easily (by the same method as will be used when $p < 1$) that if we had introduced $(\log t)^\gamma$ in the similarity variable and $(\log t)^\alpha$ instead of $\log t$ in the change of dependent variable then the only possibility is $\gamma = 0, \alpha = 1$. The problem now becomes

$$(\beta v_0 + v_0(\log t)^{-1} - \delta \eta \frac{dv_0}{d\eta}) + (t^\delta \log t)^{p-1} \left( \beta p v_0^p + p v_0^p (\log t)^{-1} - \delta \eta \frac{dv_0^p}{d\eta} \right)$$

$$= t^{1-2\delta} \frac{d^2 v_0}{d\eta^2} + (t^\delta \log t)^{p-1} t^{-\delta} \frac{dv_0^p}{d\eta}$$

(4.4.7)

$$-t^{\delta + \beta} \log t \frac{dv_0}{d\eta} + t^{r+\beta} \log t v_0 = u_0 t^r, \quad \eta = 0$$

(4.4.8)

$$v_0, \frac{dv_0}{d\eta} \to 0, \quad \eta \to \infty .$$

(4.4.9)

Now considering which terms dominate and balance with each other as $t \to \infty$, since
Section 4 The Continuous Input Solution

$p > 1$, then (4.4.7) - (4.4.9) become

\[ \beta v_0 - \delta \eta \frac{dv_0}{d\eta} = t^{1-2\delta} \frac{d^2 v_0}{d\eta^2} \]  
(4.4.10)

\[ \frac{\partial v_0}{\partial \eta} = 0, \quad \eta = 0 \]  
(4.4.11)

\[ v_0 = \frac{dv_0}{d\eta} = 0, \quad \eta \to \infty . \]  
(4.4.12)

If we now turn our attention to the mass condition which in these new variables has become

\[ u_0 \log t = \int_0^\infty t^{\delta + \beta} \log t \, v_0 \, d\eta . \]  
(4.4.13)

We must choose $\delta = -\beta$ and consequently (4.4.10) becomes

\[ \beta v_0 + \beta \eta \frac{dv_0}{d\eta} = t^{1-2\delta} \frac{d^2 v_0}{d\eta^2} . \]  
(4.4.14)

It can be shown (again by the same method that is used when we turn to look at the $p < 1$ case, although the $p < 1$ is more complicated) that, in fact, the only solution possible is when

\[ 1 + 2\beta = 0 \quad \Rightarrow \quad \beta = -\frac{1}{2} \quad \Rightarrow \quad \delta = \frac{1}{2} . \]  
(4.4.15)

This finally gives us (4.4.10) as

\[ \frac{d^2 v_0}{d\eta^2} + \frac{\eta}{2} \frac{dv_0}{d\eta} + \frac{v_0}{2} = 0 \]  
(4.4.16)

which is identical to the problem in §4.1 except that this time by writing it as

\[ \frac{d^2 v_0}{d\eta^2} + \frac{d}{d\eta} \left( \frac{\eta}{2} v_0 \right) = 0 , \]  
(4.4.17)

we can immediately see that since if we have to satisfy (4.4.12) then the solution is

\[ v_0 = C_1 e^{-\frac{x^2}{4}} . \]  
(4.4.18)

Clearly (4.4.18) satisfies our boundary condition (4.4.11). Consequently we must use our mass condition to evaluate $C_1$, ie.

\[ u_0 = C_1 \int_0^\infty e^{-\frac{x^2}{4}} \, dx = C_1 \sqrt{\pi} . \]  
(4.4.19)
Section 4 The Continuous Input Solution

Figure 4.9: The solution when \( u_0 = 1, r = -1, p > 1 \)

Hence

\[
v_0 = \frac{u_0}{\sqrt{\pi}} e^{-\frac{\eta^2}{4}}.
\]  

(4.4.20)

For \( u_0 = 1 \) we get Figure 4.9.

4.4.2 \( p < 1 \)

This case can be considered as taking a slightly more generalized, but similar, approach to the previous section. This time we introduce the similarity variable

\[
\eta = \frac{x}{t^\delta (\log t)^\gamma}, \quad \delta, \gamma > 0
\]  

(4.4.21)

and the change of dependent variable in a similar way to all the previous cases

\[
u = t^\alpha (\log t)^\beta (v_0(\eta) + o(1))
\]  

(4.4.22)

where \( \alpha < 0 \) (giving \( t^\alpha (\log t)^\beta \to 0 \) for any \( \beta \)) as \( t \to \infty \). Introducing the following approximations
Section 4 The Continuous Input Solution

\[ \frac{\partial \left( t^\alpha (\log t)^\beta v_0 \right)}{\partial t} = \alpha t^{\alpha - 1} (\log t)^\beta v_0 + \beta t^{\alpha - 1} (\log t)^{\beta - 1} v_0 \sim \alpha t^{\alpha - 1} (\log t)^\beta v_0 \quad \text{as} \quad t \to \infty \]  
(4.4.23)

\[ \frac{\partial \left( t^\alpha (\log t)^\beta v_0 \right)}{\partial t} = p t^\alpha (p-1) (\log t)^{\beta (p-1)} v_0^p \left( \alpha t^{\alpha - 1} (\log t)^\beta + \beta t^{\alpha - 1} (\log t)^{\beta - 1} \right) \sim \alpha p t^{\alpha - 1} (\log t)^\beta \left( t^{\alpha(p-1)} (\log t)^{\beta (p-1)} \right) v_0^p \quad \text{as} \quad t \to \infty \]  
(4.4.24)

allows us to state our problem as

\[ \alpha v_0 - \delta \eta \frac{dv_0}{d\eta} + t^\alpha (\log t)^{\beta (p-1)} \left( \alpha p v_0^p - \delta \eta \frac{dv_0^p}{d\eta} \right) = t^{1-2\delta} (\log t)^{-2\gamma \eta + 2} - t^{-\delta} (\log t)^{1-\gamma} \frac{dv_0}{d\eta} \]  
(4.4.25)

\[ t^{1+\alpha - \delta} (\log t)^{\beta - \gamma} \frac{dv_0}{d\eta} + t^\alpha (\log t)^\beta v_0 = u_0, \quad \eta = 0 \]  
(4.4.26)

\[ v_0 = \frac{dv_0}{d\eta} = 0, \quad \eta = \eta_0 . \]  
(4.4.27)

If we now write (4.4.25) as

\[ A_{-1} + B_{-1} = C_{-1} + D_{-1} \]  
(4.4.28)

we can easily see that since \( \alpha < 0, p < 1, B_{-1} \) dominates \( A_{-1} \).

At this point it is helpful to consider the mass condition (3.1.5) in the new co-ordinate system ie.

\[ u_0 \log t = \int_0^\infty t^{\alpha p + \delta} (\log t)^{\beta p + \gamma} v_0^p d\eta \]  
(4.4.29)

from which we get

\[ \alpha p = -\delta, \quad \beta p + \gamma = 1 . \]  
(4.4.30)

By examining (4.4.25) it is clear that \( C_{-1} \) cannot be the only dominant term in the whole equation. This leaves us with 4 possibilities:

1. \( D_{-1} \) dominates everything

\[ -\delta > \alpha(p - 1) \Rightarrow \alpha p > \alpha p - \alpha \Rightarrow \alpha > 0 \]  
(4.4.31)

but \( \alpha \) is defined as \( \alpha < 0 \) - hence there is a contradiction.

2. \( D_{-1} \sim B_{-1} \)

\[ -\delta = \alpha(p - 1) \Rightarrow \alpha = 0, \quad \delta = 0 \]  
(4.4.32)
Section 4 The Continuous Input Solution

\[
\beta p = 1 - \gamma = \beta(p - 1) \quad \Rightarrow \quad \beta = 0 \quad \gamma = 1 \tag{4.4.33}
\]

but \( C_{-1} \) now becomes \( \frac{t}{(\log t)^2} \frac{d^2v_0}{dn^2} \) and \( \frac{t}{(\log t)^2} \to \infty \) as \( t \to \infty \), which clearly dominates \( D_{-1} \) hence there is a contradiction.

3. \( C_{-1} + D_{-1} \sim B_{-1} \)

\[
1 - 2\delta = -\delta \quad \Rightarrow \quad \delta = 1 \quad \Rightarrow \quad \alpha = -\frac{1}{p} \tag{4.4.34}
\]

\[
\alpha(p - 1) = -\delta \quad \Rightarrow \quad \frac{1}{p}(p - 1) = -1 \quad \Rightarrow \quad p = \frac{1}{2}
\]

\[
1 - \gamma = -2\gamma \quad \Rightarrow \quad \gamma = -1 < 0 \tag{4.4.35}
\]

but \( \gamma \) is defined as \( \gamma > 0 \) - so there is a contradiction.

4. \( C_{-1} \sim B_{-1} \)

\[
1 - 2\delta = \alpha(p - 1) \tag{4.4.36}
\]

but \( -\delta = \alpha p \), hence

\[
\alpha = \frac{-1}{p + 1} \ (\ < 0) , \quad \delta = \frac{p}{p + 1} \ (> 0) \tag{4.4.37}
\]

similarly

\[
\beta(p - 1) = -2\gamma \tag{4.4.38}
\]

but \( \gamma = 1 - \beta p \), hence

\[
\beta = \frac{2}{p + 1}, \quad \gamma = 1 - \frac{2p}{p + 1} = \frac{1 - p}{1 + p} \ (> 0) . \tag{4.4.39}
\]

Note that these values of \( \alpha \) and \( \beta \) satisfy \( t^\alpha (\log t)^\beta \to 0 \) as \( t \to \infty \).

We are left with only one possibility. Before finally stating the problem it is interesting to note that in (4.4.26) we have

\[
1 + \alpha - \delta = 1 - \frac{1}{p+1} - \frac{p}{p+1} = 0 \tag{4.4.40}
\]

\[
\beta - \gamma = \frac{2}{p+1} - 1 + \frac{2p}{p+1} = 1
\]

which means that our boundary condition is not dependent on \( p \) at all. So finally we have the problem

\[
\frac{d^2v_0}{dn^2} = -\frac{p}{p + 1} \left( v_0^p + \eta \frac{dv_0^p}{dn} \right) \tag{4.4.41}
\]
Section 4 The Continuous Input Solution

Figure 4.10: The solution when \( \nu_0 = 1, r = -1 \) and \( p = \frac{1}{2} \)

\[
\frac{dv_0}{d\eta} = 0, \quad \eta = 0 \quad (4.4.42)
\]

\[v_0 = \frac{dv_0}{d\eta} = 0, \quad \eta \rightarrow \eta_0 \quad (4.4.43)\]

for which a solution is easily found as

\[v_0 = \left( \frac{p(1-p)(\eta_0^2 - \eta^2)}{2(p+1)} \right)^{-\frac{1}{p+1}} \text{ for } \eta \leq \eta_0 \quad (4.4.44)\]

and

\[v_0 = 0 \text{ for } \eta > \eta_0 \quad (4.4.45)\]

where \( \eta_0 \) is defined by \( v_0(\eta_0) = 0 \). (The other possibility is \( v_0 = 0, \forall \eta \) where the solution must come from a consideration of second order or higher terms and hence we do not look at this solution here.) Since this satisfies both (4.4.42) and (4.4.43) then the only possibility to determine \( \eta_0 \) is by the use of the mass condition (4.4.29). If we scale

\[\eta = \eta_0 s \quad (4.4.46)\]
then the integral condition to be solved is

\[ u_0 = \left( \frac{p(1-p)}{2(p+1)} \right)^{\frac{p}{p+1}} \eta_0^{\frac{1+p}{p}} \int_0^1 \theta (1-s)^{\frac{1-p}{1-p}} (1+s)^{\frac{p}{1-p}} \, ds \quad . \tag{4.4.47} \]

By making the change of variable

\[ s = 2t - 1 \tag{4.4.48} \]

the problem becomes

\[ u_0 = \left( \frac{p(1-p)}{2(p+1)} \right)^{\frac{p}{p+1}} (2\eta_0)^{\frac{1+p}{p}} \int_0^1 t^{\frac{1}{1-p}} (1-t)^{\frac{p}{1-p}} \, dt \tag{4.4.49} \]

i.e.

\[ u_0 = \frac{1}{2} \left( \frac{p(1-p)}{2(p+1)} \right)^{\frac{p}{p+1}} (2\eta_0)^{\frac{1+p}{p}} \left( B \left( \frac{1}{1-p}, \frac{1}{1-p} \right) - B_{\frac{1}{2}} \left( \frac{1}{1-p}, \frac{1}{1-p} \right) \right) \tag{4.4.50} \]

where this is in terms of the Incomplete Beta Function denoted by \( B_n(x, y) \) for some constant \( n \) and variables \( x \) and \( y \) (if \( n = 1 \) then it is just the Beta Function \( B(x, y) \)).

\[ \eta_0 = \frac{1}{2} \left( \frac{p(1-p)}{2(p+1)} \right)^{\frac{p}{p+1}} \left( \frac{u_0}{B \left( \frac{1}{1-p}, \frac{1}{1-p} \right)} - B_{\frac{1}{2}} \left( \frac{1}{1-p}, \frac{1}{1-p} \right) \right)^{\frac{1+p}{p}} \tag{4.4.51} \]

Figure 4.10 shows the solution when \( p = \frac{1}{2} \) and \( u_0 = 1 \).
Section 5

The Pulse Solution

5.1 Case A) \(-1 \leq r < \frac{p}{2} - 1, p > 1\)

Here we find that the only possibility is to take

\[ \delta = \frac{1}{2} \]  

(5.1.1)

and hence we are looking at the balance \( A_{1+} \sim C_{1+} \). It is simple to see from this that the problem we now must solve is

\[ \frac{1}{2} \frac{d(\eta v_0)}{d\eta} = \lambda \frac{d^2 v_0}{d\eta^2} \]  

(5.1.2)

\[ -\lambda \frac{d}{d\eta} \frac{1}{r} \frac{dv_0}{d\eta} + v_0 = 0, \quad x = 0 \]  

(5.1.3)

\[ v_0, \left. \frac{dv_0}{d\eta} \right|_{\eta = 0} = 0, \quad \eta \to \infty . \]  

(5.1.4)

which is identical to the problem when \( p = 1 \) and hence we can just state the solution as

\[ v_0 = \begin{cases} 
\frac{M_0 e^{-\frac{\eta^2}{4}}}{\sqrt{\pi}}, & 0 < \eta < \infty, \quad -1 \leq r < -\frac{1}{2} \\
\frac{M_0}{\sqrt{\pi \lambda_1} \left( 1 + \text{erf} \left( \frac{\eta}{\sqrt{\lambda_1}} \right) \right)} e^{-\frac{\eta^2}{4\lambda_1}}, & -2 < \eta < \infty, \quad r = -\frac{1}{2} \\
\frac{M_0}{2 \sqrt{\pi}} e^{-\frac{\eta^2}{4}} & -\infty < \eta < \infty, \quad -\frac{1}{2} < r < \frac{p}{2} - 1
\end{cases} \]  

(5.1.5)

which is plotted in Figure 3.5 when \( M_0 = 1 \) with the one difference being that we must relabel the line \( r > -\frac{1}{2} \) as \( -\frac{1}{2} < r < \frac{p}{2} - 1 \).
5.2 Case B) \( r > \frac{p}{2} - 1, \ p > 1 \)

We are now looking at the balance \( A_{1+} \sim D_{1+} \) which requires that

\[
r - \delta p + 1 = 0
\]

and since

\[
r > \frac{p}{2} - 1
\]

we can see that, we must have

\[
\delta > \frac{1}{2}
\]

If we now note that

\[
-\delta - r = -\delta(p + 1) + 1 < 0
\]

then it easy to see that the problem to be solved here is

\[
\frac{r + 1}{p} \frac{d}{d\eta} (\eta v_0) = \frac{dv_0^p}{d\eta}
\]

\[
v_0 = 0, \ \eta \to -\infty
\]

\[
v_0, \ \frac{dv_0}{d\eta} \to 0, \ \eta \to \infty
\]

The solution to (5.2.5) - (5.2.7) clearly requires the introduction of boundary layers. Hence the outer solution is

\[
v_0 = \begin{cases} 
0 & \eta < \eta_0 \\
\left(\frac{r+1}{r}\right)^{-1} \eta^{\frac{1}{p-1}} & \eta_0 < \eta < 0 \\
0 & \eta > 0 
\end{cases}
\]

Where the boundary layers are at \( \eta = \eta_0 \) and \( \eta = 0 \). The value of \( \eta_0 \) can be found by using the mass condition to be

\[
\eta_0 = -\frac{p}{r+1} \left(\frac{M_0(r+1)}{p-1}\right)^{\frac{p-1}{p}}
\]
(Note that this is always negative.) An important point to also note is that the shape of the outer solution significantly changes as \( p \) varies. The effect of this change can be shown by looking at the behaviour of the outer solution as it approaches \( 0_- \). If we observe that

\[
\frac{dv_0}{d\eta} = \frac{1}{p-1} \left( \frac{r+1}{p} \right)^{\frac{1}{p-1}} (-\eta)^\frac{2-p}{p-1}
\]  

(5.2.10)

then it is easy to see that if:

- \( p > 2 \) then as \( \eta \to 0_- \), \( \frac{dv_0}{d\eta} \to \infty \)

- \( p = 2 \) then when \( \eta = 0 \), \( \frac{dv_0}{d\eta} = \frac{1}{p-1} \left( \frac{r+1}{p} \right)^{\frac{1}{p}} \)

- \( p < 2 \) then when \( \eta = 0 \), \( \frac{dv_0}{d\eta} = 0 \).

Figure 5.1 shows the outer solution when \( M_0 = 1, p = \frac{3}{2} \) and \( r = 1 \).
Section 5 The Pulse Solution

The boundary layer at \( \eta = \eta_0 \) can be found by introducing the change of variable

\[
\xi = \frac{\eta - \eta_0}{t^{1-2\frac{r+1}{p}}} \quad u(\eta, t) = w_0(\xi) + o(1) \quad (5.2.11)
\]

We have this time described \( u \) as a function of \( \eta \) and \( t \) for convenience, since \( \eta \) is a function of \( x \) and \( t \). We then have to solve the problem

\[
-\frac{r+1}{p} \eta_0 \frac{dw_0}{d\xi} = \frac{d^2w_0}{d\xi^2} + \frac{dw_0^p}{d\eta} \quad (5.2.12)
\]

subject to

\[
w_0 \to 0, \quad \xi \to -\infty \quad (5.2.13)
\]

\[
w_0 = \left( -\frac{r+1}{p} \eta_0 \right)^{\frac{1}{r-1}}, \quad \xi = \xi_0 \quad (5.2.14)
\]

where \( \xi_0 \) may be found by examining higher-order terms in the expansion of \( u(\eta, t) \).

The solution to (5.2.12) and (5.2.13) can be easily found as
where $\eta_0$ is given by (5.2.9).

Note that although we could write $C_1$ in terms of $\xi_0$ nothing is gained, since we do not know the value of $\xi_0$. The constant $\xi_0$ is introduced so as to take account of the singularity that exists in (5.2.15) when $\xi = C_1$. Figure 5.2 shows the boundary layer for the same values as in Figure 5.1, where $C_1$ has arbitrarily been chosen to be 2.

We must now also look at the boundary layer about $\eta = 0$. As was noted earlier the outer solution has $\frac{\partial w_0}{\partial \eta} \rightarrow 0$ as $\eta \rightarrow 0_-$ only when $1 < p < 2$ and hence this is the only region where we can look at finding the boundary layer solution. The boundary layer solution gets into difficulties straight away since we are considering a boundary layer about $\eta = 0$, which means that we are also going to have to consider a scaling of $\eta_0$. The problem with this, is that to introduce this scaling means that derivatives with respect to $\eta$ must again be considered, hence we must consider (3.4.8) with the time derivatives included and in terms of $v$ not $v_0$. We then find that

$$I \frac{\partial v}{\partial t} + A_{1+} \gg I \frac{\partial v^p}{\partial t} + B_{1+}$$

(5.2.16)

giving us (3.4.8) as

$$I \frac{\partial v}{\partial t} - \frac{r + 1}{p} \frac{\partial (\eta v)}{\partial \eta} = t^{1-2r+1} \frac{\partial^2 v^p}{\partial \eta^2} + \frac{\partial v^p}{\partial \eta}$$

(5.2.17)

If we now introduce the change of variables

$$\xi = \frac{\eta}{t-\epsilon}, \quad v(\eta, t) = t^\epsilon (w_0(\xi) + 0(1))$$

(5.2.18)

we can write (5.2.16) as

$$\phi w_0 + \epsilon \xi w_0 - \frac{r + 1}{p} \left( \frac{d w_0}{d \xi} + w_0 \right) = t^{1-2r+1+2\epsilon} \frac{d^2 w_0}{d \xi^2} + t^{\epsilon + \phi(p-1)} \frac{d w_0}{d \xi}$$

(5.2.19)

For the right-hand side of (5.2.19) to balance with the left-hand side we clearly require that

$$1 - 2r + 1 \frac{1}{p} + 2\epsilon = 0, \quad \Rightarrow \quad \epsilon = \frac{2(r + 1) - p}{2p}$$

(5.2.20)
Section 5 The Pulse Solution

for the continuous input. This time the problem can be solved because the left-hand side of
(5.3.2) can be simplified down to just one term.

and
\[ \epsilon + \phi(p-1) = 0, \quad \Rightarrow \quad \phi = -\frac{\epsilon}{p-1} = \frac{p - 2(r+1)}{2p(p-1)} \]  

(5.2.21)

We are now able to state the boundary layer problem that needs solving as

\[ \frac{d^2w_0}{d\xi^2} + \frac{dw_0}{d\xi} + \frac{\xi}{2} \frac{dw_0}{d\xi} + \frac{1+2r}{2(p-1)}w_0 = 0 \]  

(5.2.22)

subject to

\[ w_0 \sim \left( -\frac{r+1}{p} \xi \right)^{\frac{1}{p-1}}, \quad \xi \to -\infty \]  

(5.2.23)

\[ w_0 \to 0, \quad \xi \to \infty \]  

(5.2.24)

The solution to (5.2.21)-(5.2.23) can not be so far solved analytically. The case \( r = 0 \) has been looked at numerically in a paper by Cazenave and Escobedo [13]. We do not attempt in this thesis to find a numerical solution to the more general case \( r > \frac{p}{2} - 1 \), mainly because of the complexity in solving it for the case \( r = 0 \) and time constraints on this thesis.

5.3 Case C) \( r = \frac{p}{2} - 1, \ p > 1 \)

The final balance that needs to be considered for \( p > 1 \) is \( A_{1+} \sim C_{1+} + D_{1+} \) from which it is easy to establish that given \( r = \frac{p}{2} - 1 \)

\[ \delta = \frac{1}{2} \]  

(5.3.1)

and consequently the problem we are interested in this time is

\[ -\delta \frac{d(\eta v_0)}{d\eta} = \frac{d^2v_0}{d\eta^2} + \frac{dv_0}{d\eta} \]  

(5.3.2)

\[ v_0 \to 0, \quad \eta \to -\infty \]  

(5.3.3)

\[ \frac{dv_0}{d\eta}, v_0 \to 0, \quad \eta \to \infty . \]  

(5.3.4)

It is interesting to compare this problem with the problem faced in Case III \( p > 1 \) (see §4.3) for the continuous input. This time the problem can be solved because the left-hand side of (5.3.2) can be simplified down to just one term.
Clearly (5.3.4) allows us to simplify (5.3.2) to

\[ \frac{dv_0}{d\eta} = -\frac{\eta}{2} v_0 - v_0^p. \]  

(5.3.5)

If we introduce the change of dependent variable

\[ v_0 = y^{1-p} \]

(5.3.6)

then (5.3.5) becomes

\[ \frac{dy}{d\eta} = \frac{p - 1}{2} (\eta y + 2) \]

(5.3.7)

which has the solution

\[ y = \frac{e^{\frac{p-1}{4} \eta^2}}{\sqrt{p-1}} \left( \sqrt{\pi (p-1)} \text{erf} \left( \frac{1}{2} \eta \sqrt{p-1} \right) + C_1 \sqrt{p-1} \right) \]

(5.3.8)

from which we can state

\[ v_0 = e^{-\frac{y^2}{4} (p-1) \frac{1}{(p-1)}} \left( \sqrt{\pi (p-1)} \text{erf} \left( \frac{1}{2} \eta \sqrt{p-1} \right) + C_1 \sqrt{p-1} \right)^{1-p}. \]

(5.3.9)
The evaluation of \( C_1 \) comes from the mass condition, but since so far this integral has not been analytically solved, it has been necessary to evaluate \( C_1 \) numerically. The method used was to choose a value for \( M_0 \) and then use simpson's rule to evaluate numerically the integral of \( v_0 \) with respect to \( \eta \), where \( C_1 \) was arbitrarily chosen. The value of \( C_1 \) was then improved on by a method of trial and error until a reasonable accuracy integral evaluation to our arbitrary choice of \( M_0 \). This method clearly does not account for the possibility of there being multiple values of \( C_1 \) that satisfy this arbitrary value of \( M_0 \), and due to time constraints this thesis will not attempt to establish whether or not multiple values do exist, but it is a reasonable intuitive assumption to make, based on all of the other solutions presented in this thesis, that there is in fact only one such value.

Figure 5.3 shows the solution when \( M_0 = 1 \) and \( p = 3 \). By using the above described numerical method, \( C_1 \) has been evaluated as

\[
C_1 \approx 12.7395 .
\]

### 5.4 Case D) \(-1 \leq r < -\frac{p}{p+1}, p < 1\)

We now turn to look at the different balances when \( p < 1 \). The first balance we look at is \( B_{1-} \sim C_{1-} \) which clearly requires that

\[
-\frac{\delta}{p} (p - 1) = 1 - 2\delta ,
\]

hence we have

\[
\delta = \frac{p}{p + 1} .
\]

The problem we are now interested is

\[
-\frac{p}{p + 1} \frac{d}{d\eta} (\eta \eta_0^p) = \lambda \frac{d^2 v_0}{d\eta^2}
\]

subject to

\[
\frac{dv_0}{d\eta} = 0, \quad \eta = 0 \quad \text{(5.4.4)}
\]

\[
v_0 = \frac{dv_0}{d\eta} = 0, \quad \eta = \eta_0 \quad \text{(5.4.5)}
\]
Note the introduction of $\eta_0$ in (5.4.5). We know that $\eta_0$ is finite since for $p < 1$ the support of $u$ is bounded. We can now find the solution of (5.4.3)-(5.4.5) as, in terms of $\eta_0$,

$$v_0 = \left( \frac{p(p-1)}{2\lambda(p+1)} \right)^{\frac{1}{1-p}}.$$  

(5.4.6)

All that remains to be done is to evaluate $\eta_0$, and this can be done by using the mass condition, i.e.

$$M_0 = \int_0^{\eta_0} \left( \frac{p(p-1)}{2\lambda(p+1)} \right)^{\frac{1}{1-p}} \frac{\eta}{\eta_0} \eta_0^{\frac{1}{1-p}} \eta^{\frac{1}{2}} \eta^{\frac{1}{2}} \eta^{\frac{1}{2}} \eta^{\frac{1}{2}} d\eta.$$  

(5.4.7)

If we now scale $\eta_0$ out of the integral by using $\eta = s\eta_0$ we consequently get

$$M_0 = \left( \frac{p(1-p)}{2\lambda(p+1)} \right)^{\frac{1}{1-p}} \eta_0^{\frac{1}{1-p}} \int_0^1 (1-s^2)^{\frac{1}{2}} ds.$$  

(5.4.8)

To solve this integral we introduce the change of variable $s = 2w - 1$ which transforms the problem to a known problem with a known answer, again in terms of the Incomplete Beta Function and the Beta Function (compare with (4.4.50) and (4.4.51)). We can now say

$$M_0 = \left( \frac{p(1-p)}{2\lambda(p+1)} \right)^{\frac{1}{1-p}} \eta_0^{\frac{1}{1-p}} \left( B \left( \frac{p}{1-p}, \frac{p}{1-p} \right) - B_\frac{1}{2} \left( \frac{p}{1-p}, \frac{p}{1-p} \right) \right)$$  

(5.4.9)

or

$$\eta_0 = \left( \frac{2\lambda(p+1)}{p(1-p)} \right)^{\frac{1}{2}} \frac{M_0}{B \left( \frac{p}{1-p}, \frac{p}{1-p} \right) - B_\frac{1}{2} \left( \frac{p}{1-p}, \frac{p}{1-p} \right)}$$  

(5.4.10)

It is worth pointing out the similarity of this equation with (4.4.51).

Figure 5.4 shows the solution when $p = \frac{1}{2}$ and $M_0 = 1$ which give a value of $\eta_0 \sim 2.6211$. If $r = -\frac{1}{2}$ then we choose $\lambda = 1$ for consistency with other values of $r$.

### 5.5 Case E) $r > -\frac{p}{p+1}, p < 1$

We now turn to look at the balance $B_{1-} \sim D_{1-}$, which in the same way as in all of the previous cases gives

$$-\delta(p-1) = r + 1 - \delta$$  

(5.5.1)

and hence

$$\delta = (r + 1)p.$$  

(5.5.2)
Section 5 The Pulse Solution

Figure 5.4: The solution when \( M_0 = 1 \) and \( p = \frac{1}{2} \) and \( r < -\frac{1}{3} \)

The problem to be solved in this case is now

\[-(r+1)p \frac{d}{d\eta} (\eta v_0^p) = -\frac{dv_0}{d\eta}\]  (5.5.3)

\[v_0 = 0, \quad \eta = 0\]  (5.5.4)

\[v_0 = \frac{dv_0}{d\eta} = 0, \quad \eta = \eta_0\]  (5.5.5)

Investigation of the problem (5.5.3)–(5.5.5) shows us that we have to introduce a boundary layer at \( \eta = \eta_0 \). (It is worth while noting that the problem here is very similar to the problem investigated in Case B §5.2 with the major difference being that because the problem is bounded and there has been no introduction of a travelling wave, we only need to introduce one boundary layer.) The outer solution can be easily found to be

\[v_0 = \begin{cases} ((r+1)p\eta)^{\frac{1}{1-p}} & 0 < \eta < \eta_0 \\ 0 & \eta > \eta_0 \end{cases}\]  (5.5.6)
Section 5 The Pulse Solution

We can as usual establish $\eta_0$ from the mass condition

\[ M_0 = \int_0^{\eta_0} ((r+1)p\eta)^{\frac{p}{1-p}} \, d\eta \]  \hfill (5.5.7)

Consequently we find

\[ \eta_0 = \left( \frac{M_0}{1-p} \right)^{1-p} \left( \frac{1}{(r+1)p} \right)^p . \]  \hfill (5.5.8)

The outer solution when $p = \frac{1}{2}$, $r = 0$ and $M_0 = 1$ which give $\eta_0 = 2$ is shown in Figure 5.5. The boundary layer can be found by introducing the change of variables in a similar way to Case B (see §5.2),

\[ \xi = \frac{\eta - \eta_0}{t^{\frac{1}{2}}(r+1)p}, \quad u(\eta, t) = w_0(\xi) + O(1) \]  \hfill (5.5.9)

We now have the problem for the boundary layer as

\[-(r+1)p\eta_0 \frac{d^2 w_0}{d\xi^2} = \frac{d^2 w_0}{d\xi^2} - \frac{dw_0}{d\xi} \]  \hfill (5.5.10)

subject to

\[ w_0 = \frac{dw_0}{d\xi} = 0, \quad \xi = \xi_1 \]  \hfill (5.5.11)
The introduction of $\xi_1$ is because of the bounded support for all the $p < 1$ cases. The solution for the boundary layer can then be found to be

$$w_0 = ((r + 1)p\eta_0)^{1/r} \cdot \xi \to -\infty \quad . \quad (5.5.12)$$

where $C_1$ can be written in terms of $\xi_0$ and is a constant dependent on higher order terms in the expansion of $u$. Figure 5.6 shows the boundary layer when $p = \frac{1}{2}, \ r = 0$ and $M_0 = 1$. The boundary layer is thus

$$w_0 = \left(1 - e^{\frac{1}{2}(\xi - C_1)}\right)^2 \quad (5.5.14)$$

and we can then see that in this example $\xi_1 = C_1$. We have arbitrarily chosen $C_1 = 4$. 

Figure 5.6: The Boundary layer when $p = \frac{1}{2}, \ r = 0$ and $M_0 = 1$
5.6 Case F) \( r = -\frac{p}{p+1}, \ p < 1 \)

We finally turn to look at the last remaining balance, i.e. \( B_{1-} \sim C_{1-} + D_{1-} \) which obviously requires that

\[
\frac{\delta}{p} (p - 1) = 1 - 2\delta, \quad 1 - 2\delta = r + 1 - \delta
\]  

(5.6.1)

from which we can state

\[
\delta = \frac{p}{p+1}
\]  

(5.6.2)

and the problem to be solved for this final case now becomes

\[
-\frac{p}{p+1} \frac{d}{d\eta} (\eta \nu_0^p) = \frac{d^2 \nu_0}{d\eta^2} - \frac{d\nu_0}{d\eta}
\]  

(5.6.3)

\[
-\frac{d\nu_0}{d\eta} + \nu_0 = 0, \quad \eta = 0
\]  

(5.6.4)

\[
v_0 = \frac{d\nu_0}{d\eta} = 0, \quad \eta = \eta_0
\]  

(5.6.5)

The solution to (5.6.3)–(5.6.5) can be easily found, since (5.6.3) has the same form as the Bernoulli equation, to be

\[
v_0 = \left( p((p-1)\eta - 1) + C_1 (p^2 - 1) e^{-(p-1)\eta} \right)^{\frac{1}{1-p}}
\]  

(5.6.6)

where

\[
C_1 = \frac{p (1 - (p-1)\eta_0) e^{(p-1)\eta_0}}{p^2 - 1}
\]  

(5.6.7)

The evaluation of \( \eta_0 \) again comes from the mass condition. We have again been unable so far to analytically solve this integral. The value of \( \eta_0 \) can however be easily found numerically by first noting that if we define a function

\[
f(y) = \int_0^y \nu_0(s)^p \ ds
\]  

(5.6.8)

then we are looking for the solution of

\[
f(y) = M_0
\]  

(5.6.9)

The numerical evidence is that \( f \) is a monotonic function and hence a simple bisection method will yield the root of (5.6.9). Note though that this is the same assumption as was outlined in Case C (see §5.3) and hence subject to the same difficulty.
Figure 5.7: The solution when \( p = \frac{1}{2}, \, r = -\frac{1}{3} \) and \( M_0 = 1 \)

Figure 5.7 shows the solution when \( p = \frac{1}{2} \) and \( M_0 = 1 \) and the numerical method has yielded \( \eta_0 \sim 3.1125 \).
Section 6

The Algorithm

The approach taken in solving the problem numerically is, as would be expected, different for \( p < 1 \) and \( p > 1 \) due to the fact that the \( p < 1 \) case has bounded support.

6.1 The Algorithm when \( p < 1 \)

Before describing the numerical method used to solve the problem numerically, it is worth describing a feature of the problem which appeared whilst attempting to ascertain a numerical solution. The nature of this problem is that the numerical computer package used to solve the problem fails at the point \( \eta_0 \) where \( \eta_0 \) is defined by

\[
v_0(\eta_0) = \frac{dv_0}{d\eta} \bigg|_{\eta=\eta_0} = 0 .
\]  

(6.1.1)

The nature of this failure is illustrated in Figure 6.1. In Figure 6.1 we have defined \( \epsilon = v_0(0) \) and have chosen \( r = -\frac{1}{2}, p = \frac{1}{2} \) and \( u_0 = 1 \). As we increased \( \epsilon \) towards its correct value of around 0.75, just before reaching that point the numerical value of \( \eta_0 \) started to increase rapidly. A result of this behaviour is that the approach taken was to start from \( \eta_0 \) and work back to \( \eta = 0 \). The method used differs slightly between the cases \( r = -\frac{1}{2} \) and \(-1 < r < -\frac{1}{2} \). We will first look at the \( r = -\frac{1}{2} \) case.

6.1.1 The \( r = -\frac{1}{2}, p < 1 \) Algorithm

To re-cap, the equation to be solved is

\[
\frac{d^2v_0}{d\eta^2} = \left(1 - \frac{\eta}{2} - \frac{\eta}{2} p_{n_0}^{r-1}\right) \frac{dv_0}{d\eta} .
\]  

(6.1.2)
If we approximate (6.1.2) for $v_0$ small by

$$
\frac{d^2v_0}{d\eta^2} \approx -\frac{\eta}{2}pv_0^{p-1} \frac{dv_0}{d\eta}
$$

(6.1.3)

and then consider the first term of the expansion of $v_0$ about $\eta_0$ (since $\eta \to \eta_0$ here all subsequent terms in the expansion are dominated by the first, and we can consider $\eta$ in (6.1.2) as $\eta_0$), in other words let

$$
v_0 \sim A (\eta_0 - \eta)^a
$$

(6.1.4)

where $A$ and $a$ are constants. If we choose $a$ such that

$$
a = \frac{1}{1-p}
$$

(6.1.5)

we can solve (6.1.3) for $A$, i.e.

$$
A = \left( \frac{2}{\eta_0(1-p)} \right)^{\frac{1}{2}}
$$

(6.1.6)
and we subsequently have

\[ v_0 = \left( \frac{2}{\eta_0 (1 - p) (\eta_0 - \eta)} \right)^{\frac{1}{r-1}} . \]  \hspace{1cm} (6.1.7)

If we now make an arbitrary choice of \( p \) and define \( y_1 = v_0 \) and \( y_2 = \frac{dv_0}{d\eta} \). We can then rewrite (6.1.2) as the system of equations

\[ \frac{dy_1}{d\eta} = y_2 \] \hspace{1cm} (6.1.8)

\[ \frac{dy_2}{d\eta} = \left( 1 - \frac{\eta}{2} - \frac{\eta}{4} \frac{1}{y_1^{\frac{1}{2}}} \right) y_2 . \] \hspace{1cm} (6.1.9)

We must also remember that these must be solved subject to

\[ -\frac{dv_0}{d\eta} - v_0 = u_0, \quad \eta = 0 \] \hspace{1cm} (6.1.10)

\[ \frac{dv_0}{d\eta} = v_0 = 0, \quad \eta = \eta_0 . \] \hspace{1cm} (6.1.11)

The approach taken now is to choose an arbitrary \( \eta_0 \) and use approximation (6.1.7) to establish \( v_0 \) and \( \frac{dv_0}{d\eta} \) at a point just short of \( \eta_0 \), hence avoiding the problems associated with the point \( \eta_0 \). It is then a simple matter to use a computational package to numerically work back and find \( v_0(0) \) and \( \frac{dv_0}{d\eta} \bigg|_{\eta=0} \). To finally establish the corresponding \( u_0 \) we then use the boundary condition (6.1.10). By repeating this process over a range of different values of \( \eta_0 \) it is then possible to graphically represent the relationship between \( \eta_0 \) and \( u_0 \). It is worth noting that in plotting the graph of \( v_0 \) against \( \eta \) for a specific value of \( u_0 \) we must stop just short of \( \eta_0 \) for the same reasons. Note that also this algorithm could be refined by taking more terms in the expansion of \( v_0 \) about \( \eta = \eta_0 \).

### 6.1.2 The \( -1 < r < -\frac{1}{2}, p < 1 \) Algorithm

This time we use a slightly better method than in the previous section to establish the relationship between \( \eta_0 \) and \( u_0 \). The main difference is that this time it is possible to scale out \( \eta_0 \). To re-cap, the problem to be solved is

\[ \frac{d^2 v_0}{d\eta^2} = \frac{2r + 1}{1 + p} \nu_0^p - \frac{1 + r(1 - p)}{1 + p} \eta \frac{dv_0}{d\eta} . \] \hspace{1cm} (6.1.12)
subject to

\[ \frac{d\nu_0}{d\eta} = -u_0, \quad \eta = 0 \]  \hspace{1cm} (6.1.13)

\[ \frac{d\eta_0}{d\eta} = u_0 = 0, \quad \eta = \eta_0 \]  \hspace{1cm} (6.1.14)

By introducing the change of variables

\[ \eta = \xi \eta_0, \quad \nu_0 = w_0 \eta_0^{2-p} \]  \hspace{1cm} (6.1.15)

we effectively scale \( \eta_0 \) out of (6.1.14) and the problem becomes

\[ \frac{d^2w_0}{d\xi^2} = \frac{2r+1}{1+p} \nu w_0^p - \frac{1+r(1-p)}{1+p} \frac{dw_0}{d\xi} \]  \hspace{1cm} (6.1.16)

\[ \frac{dw_0}{d\xi} = -u_0 \eta_0^{-1+p} \eta_0^{-1-p}, \quad \xi = 0 \]  \hspace{1cm} (6.1.17)

\[ \frac{dw_0}{d\xi} = w_0 = 0, \quad \xi = 1 \]  \hspace{1cm} (6.1.18)

The numerical approach now taken is the same as in the previous section, with the major exception being that we now know the \( \eta_0 \) equivalent (\( \xi = 1 \)) and do not need to use a range of values for this point. To construct the relationship between \( u_0 \) and \( \eta_0 \) we use the fact that we know now the value of \( \frac{dw_0}{d\xi} \bigg|_{\xi=0} = B \) say, and may hence rearrange (6.1.17) to give

\[ u_0 = -B \eta_0^{\frac{1+p}{1-p}} \]  \hspace{1cm} (6.1.19)
Section 7

Summary

The results for continuous input have been shown to be split into 3 different divisions over $p$ and 4 different divisions over $r$. Figure 7.1 summarizes how these regions fit together.

![Diagram](image)

**Figure 7.1:** The different regions of solutions for continuous input

The results for these different regions are now summarized in the following table.
### Section 7 Summary

<table>
<thead>
<tr>
<th>$p$</th>
<th>$r$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; 1$</td>
<td>$= -1$</td>
<td>$u \sim t^{-\frac{1}{p+1}} \left( \log t \right)^{\frac{2}{p+1}} \left( \frac{p(1-p)}{2(p+1)} \left( \eta_0^2 - \frac{x^2}{t^{p+1}} \left( \log t \right)^{\frac{2(1-p)}{1+p}} \right) \right)^{-\frac{1}{p+1}}$ for $0 &lt; x \leq \eta_0 t^{\frac{p}{p+1}} \left( \log t \right)^{\frac{1-p}{1+p}}$ and $u \sim 0$ for $x &gt; \eta_0 t^{\frac{p}{p+1}} \left( \log t \right)^{\frac{1-p}{1+p}}$ where $\eta_0 = \frac{1}{2} \left( \frac{t^{\frac{p-1}{2(p+1)}}}{2(p+1)} \right)^{\frac{p-1}{p}} \left( B \left( \frac{1}{1-p}, \frac{1}{1-p} \right) - B_\frac{1}{2} \left( \frac{1}{1-p}, \frac{1}{1-p} \right) \right)$</td>
</tr>
<tr>
<td>$&lt; 1$</td>
<td>$-1 &lt; r &lt; -\frac{1}{2}$</td>
<td>$u \sim t^{-\frac{2r+1}{1+r(p+1)}} \left( \frac{x}{t^{\frac{1}{1+r(p+1)}}} \right)$ where $v_0$ is obtained numerically.</td>
</tr>
<tr>
<td>$&lt; 1$</td>
<td>$= -\frac{1}{2}$</td>
<td>$u \sim v_0 \left( \frac{x}{t^{\frac{1}{2}}} \right)$ where $v_0$ is obtained numerically.</td>
</tr>
<tr>
<td>$&lt; 1$</td>
<td>$&gt; -\frac{1}{2}$</td>
<td>$u \sim \begin{cases} u_0, &amp; 0 &lt; x &lt; u_0^{1-p} t^{r+1} / (1 + u_0^{1-p}) \ 0, &amp; x &gt; u_0^{1-p} t^{r+1} / (1 + u_0^{1-p}) \end{cases}$ with a boundary layer about $u_0^{1-p} t^{r+1} / (1 + u_0^{1-p})$ defined by $u \sim \left( \frac{e^{t(\xi+C_2)(1-u_0^{1-p}/(1+u_0^{1-p}))(1-p)} - \delta \eta_0}{u_0^{1-p}/(1 + u_0^{1-p}) - 1} \right)^{\frac{1}{1-p}}$, $C_2$ constant where $\xi = \frac{\eta - u_0^{1-p}/(1 + u_0^{1-p})}{t^{-(2r+1)}}$ where $\eta = 4 t^{r+1}$</td>
</tr>
<tr>
<td>$= 1$</td>
<td>$= -1$</td>
<td>$u \sim t^{-\frac{1}{2}} \log(t) \frac{u_0 e^{-\frac{x^2}{4t}}}{\sqrt{\pi}}$</td>
</tr>
<tr>
<td>$= 1$</td>
<td>$-1 &lt; r &lt; -\frac{1}{2}$</td>
<td>$u \sim t^{r+\frac{1}{2}} \frac{2^{r+1}}{\sqrt{\pi}} \Gamma(r+1) u_0 U \left( 2r + \frac{3}{2}, \sqrt{\frac{x}{2t}} \right)$</td>
</tr>
<tr>
<td>$= 1$</td>
<td>$= -\frac{1}{2}$</td>
<td>$u \sim \frac{u_0}{1 + \text{erf} \left( \frac{1}{\sqrt{\lambda_1}} \right) + \sqrt{\frac{2\lambda_1}{\pi}} e^{-\frac{x^2}{4\lambda_1}} \left( 1 - \text{erf} \left( \frac{1}{\sqrt{\lambda_1}} \frac{x}{2\sqrt{t}} - 1 \right) \right) \right)$</td>
</tr>
</tbody>
</table>
## Section 7 Summary

### Summary of our conclusions for Continuous Input

<table>
<thead>
<tr>
<th>$p$</th>
<th>$r$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$= 1$</td>
<td>$&gt; -\frac{1}{2}$</td>
<td>$u \sim \begin{cases} u_0, &amp; 0 &lt; x &lt; t^{r+1}/2(r+1) \ 0, &amp; x &gt; t^{r+1}/2(r+1) \end{cases}$ with a boundary layer about $x = t^{r+1}/2(r+1)$ defined by $u \sim \frac{u_0}{2} \left( 1 - \text{erf} \left( \frac{r + 1}{2} \xi \right) \right)$ where $\xi = \frac{\eta - 1/2(r+1)}{t^{r+1/2}} = O(1)$ where $\eta = \frac{x}{r^{r+1}}$</td>
</tr>
<tr>
<td>$&gt; 1$</td>
<td>$= -1$</td>
<td>$u \sim t^{-1/2} \left( \log t \right) \frac{u_0}{\sqrt{\pi}} e^{-\frac{x^2}{4t}}$</td>
</tr>
<tr>
<td>$&gt; 1$</td>
<td>$-1 &lt; r &lt; -\frac{1}{2}$</td>
<td>$u \sim t^{r+1/2} 2^{r+1} \sqrt{\frac{\Gamma(r+1)}{\pi}} u_0 U \left( 2r + 3, \frac{x - \frac{r+1}{r+1}}{\sqrt{2t}}, \frac{1+1-p}{1+p} \right)$</td>
</tr>
<tr>
<td>$&gt; 1$</td>
<td>$= -\frac{1}{2}$</td>
<td>$u \sim u_0 \left( \frac{x}{t^{1/2}} \right)$ where $u_0$ is obtained numerically.</td>
</tr>
<tr>
<td>$&gt; 1$</td>
<td>$&gt; -\frac{1}{2}$</td>
<td>$u \sim \begin{cases} u_0, &amp; 0 &lt; x &lt; \frac{r+1}{\left( (r+1)\left( p_0^{r+1} - 1 \right) \right)^{1/2}} \ \left( \left( \frac{r+1}{(r+1)x} - 1 \right) \frac{1}{p} \right)^{r-1}, &amp; \frac{r+1}{\left( (r+1)\left( p_0^{r+1} - 1 \right) \right)^{1/2}} &lt; x &lt; \frac{r+1}{r+1} \ 0, &amp; x &gt; \frac{r+1}{r+1} \end{cases}$</td>
</tr>
</tbody>
</table>

The situation for the pulse can not be described so easily as the continuous-input situation. Like the continuous-input problem, the pulse problem splits into 3 regions over $p$, but unlike the continuous-input problem, it splits into 3 different regions over $r$. The main difficulty that arose was that the dividing line was defined differently for $p > 1$ and $p < 1$ although these two lines do meet. This situation can be seen in Figure 7.2.
A Summary of the results for the pulse are shown in the following table.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$r$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; 1$</td>
<td>$-1 \leq r &lt; -\frac{p}{1-p}$</td>
<td>$u \sim t^{-\frac{1}{r+1}} \left( \frac{p(p-1)}{2\lambda(p+1)} \left( x^2 / t^{\frac{2}{r+1}} - \eta_0^2 \right) \right)^{\frac{p}{1-p}}$ for $0 &lt; x \leq \eta_0 t^{\frac{p}{r+1}}$ and $u \sim 0$ for $x &gt; \eta_0 t^{\frac{p}{r+1}}$ where $\eta_0 = \left( \frac{2\lambda(p+1)}{p(1-p)} \right)^{\frac{p}{1-p}} M_0$ $- B \left( \frac{p}{1-p}, \frac{p}{1-p} \right) - B_\frac{1}{2} \left( \frac{p}{1-p}, \frac{p}{1-p} \right)$ $\frac{1}{1-p}$</td>
</tr>
<tr>
<td>$&lt; 1$</td>
<td>$= -\frac{p}{p+1}$</td>
<td>$u \sim t^{-\frac{1}{r+1}} \left( \frac{p \left( \frac{(p-1)z}{t^{\frac{p}{p+1}}} - 1 \right) + C_1 (p^2 - 1) e^{-\frac{(p-1)}{\eta_0}}}{p^2 - 1} \right)$ where $C_1 = \frac{p(1-(p-1)\eta_0)e^{(p-1)\eta_0}}{p^2 - 1}$ and $\eta_0$ has to be found numerically.</td>
</tr>
</tbody>
</table>
### Section 7 Summary

#### Summary of our conclusions for the Pulse Solution

<table>
<thead>
<tr>
<th>$p$</th>
<th>$r$</th>
<th>Solution</th>
</tr>
</thead>
</table>
| $< 1$   | $r > -\frac{p}{p+1}$ | $u \sim \begin{cases} t^{-(r+1)} \left( (r+1)p \frac{\pi}{2(r+1)p} \right)^{1/p} & x < \eta_0 \\ 0 & x > \eta_0 \end{cases}$  \\
|         |           | where $\eta_0 = \left( \frac{M_0}{1-p} \right)^{1-p} \left( \frac{1}{(r+1)p} \right)^p$  \\
|         |           | together with a boundary layer about $x = \eta_0$ defined by  \\
|         |           | $u \sim t^{-(r+1)} \left( (r+1)p \eta_0 - e^{(1-p)(\xi-C_1)} \right)^{1/p}$  \\
|         |           | where $\xi = \frac{x - \eta_0 t^{(r+1)p}}{t^{3(r+1)p-1}}$ and $C_1$ is unknown |
| $\geq 1$| $= -1$    | $u \sim t^{-\frac{1}{2}} \frac{M_0}{\pi^\frac{1}{2}} \exp \left( \frac{(x - \log t)^2}{4t} \right)$ |
| $\geq 1$| $-1 < r < -\frac{1}{2}$ | $u \sim t^{-\frac{1}{2}} \frac{M_0}{\pi^\frac{1}{2}} \exp \left( \frac{(x - r t^{\frac{1}{2}})^2}{4t} \right)$ |
| $= 1$   | $= -\frac{1}{2}$ | $u \sim t^{-\frac{1}{2}} \frac{M_0}{\pi^\frac{1}{2}} \exp \left( \frac{(x - 2t^{-\frac{1}{2}})^2}{4\lambda t} \right)$ |
| $> 1$   | $= -\frac{1}{2}$ | $u \sim t^{-\frac{1}{2}} \frac{M_0}{\pi^\frac{1}{2}} \exp \left( \frac{(x - r t^{\frac{1}{2}})^2}{4\lambda t} \right)$ |
| $= 1$   | $r > -\frac{1}{2}$ | $u \sim t^{-\frac{1}{2}} \frac{M_0}{2\pi^\frac{1}{2}} \exp \left( \frac{(x - r t^{\frac{1}{2}})^2}{4t} \right)$ |
| $> 1$   | $-\frac{1}{2} < r < \frac{p}{2} - 1$ | $u \sim t^{-\frac{1}{2}} \frac{M_0}{2\pi^\frac{1}{2}} \exp \left( \frac{(x - r t^{\frac{1}{2}})^2}{4t} \right)$ |
| $> 1$   | $= \frac{p}{2} - 1$ | $u \sim t^{-\frac{1}{2}} \exp \left( \frac{(x - \frac{r \frac{1}{2}}{2t^\frac{1}{2}})^2}{4t} \right) \left( p - 1 \right)^{-\frac{1}{2(p-1)}} \ast y$  \\
|         |           | where $y = \left( \frac{\pi^\frac{3}{2}(p-1)\text{erf}\left( \frac{x - \frac{r \frac{1}{2}}{2t^\frac{1}{2}}} \right)}{2t^\frac{1}{2}} \left( p - 1 \right)^\frac{1}{2} + C_1 \left( p - 1 \right)^\frac{1}{2} \right)^{1/p}$  \\
<p>|         |           | where $C_1$ has not been analytically found |</p>
<table>
<thead>
<tr>
<th>$p$</th>
<th>$r$</th>
<th>Solution</th>
</tr>
</thead>
</table>
| $> 1$ | $r > \frac{p}{2} - 1$ | $u \sim \begin{cases} 
0 & \text{where } \eta_0 = - \frac{p}{r+1} \left( \frac{M_0(r+1)}{p-1} \right)^{\frac{p-1}{r}} \\
- \frac{r+1}{p} \left( \frac{r+1}{p} \frac{x - \eta t^{\frac{1}{r+1}}}{t^{\frac{1}{r+1}}} \right)^{\frac{1}{p-1}} & x < 0 \\
0 & x > 0 
\end{cases}$ |

The boundary layer about $x = \eta_0 t^{\frac{1}{r+1}} + \frac{r+1}{r+1} t^{\frac{1}{r+1}}$ and $x = 0$. The boundary layer about $x = 0$ is not solved here. The boundary layer about $x = \eta_0 t^{\frac{1}{r+1}} + \frac{r+1}{r+1} t^{\frac{1}{r+1}}$ is described by

$$u \sim \left( \frac{p \exp \left( \frac{r+1}{p} \eta_0 (p-1)(\xi + C_1) \right)}{(r+1) \eta_0} \right)^{\frac{1}{r-1}}$$

where $\xi = \frac{x - \eta t^{\frac{1}{r+1}} \left( \frac{1}{r+1} - \eta t^{\frac{1}{r+1}} \right)}{t^{\frac{1}{r+1}} - \eta t^{\frac{1}{r+1}}}$ and $C_1$ is unknown.
Bibliography


Appendix A

Mathematical Functions

A.1 The Kummer Function

The Kummer Function is defined as the solution of

\[ \frac{z}{d^2} \frac{d^2 w}{dz^2} + (b-z) \frac{dw}{dz} - \alpha w = 0 \]  

(A.1.1)

and hence Kummer's function is

\[ M(a, b, z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \ldots + \frac{(a)_n z^n}{(b)_n n!} + \ldots \]  

(A.1.2)

where

\( (a)_n = a(a+1)(a+2)\ldots(a+n-1), \quad (a)_0 = 1 \)  

(A.1.3)

and similar for \((b)_n\).

Properties of the Kummer function that, whilst simple, are useful, due to the fact that truncations of the Kummer function need not be used, are

- \[ \frac{d}{dz} M(a, b, z) = \frac{a}{b} M(a+1, b+1, z) \]  

(A.1.4)

which gives us, the result we are more interested in, when \( z = \frac{x^2}{2} \), that

\[ \frac{\partial}{\partial x} M \left( a, b, \frac{x^2}{2} \right) = \frac{Ax}{B} M \left( a+1, b+1, \frac{x^2}{2} \right) \]  

(A.1.5)

- \[ M(a, b, 0) = 1 \]  

(A.1.6)
A.2 Parabolic Cylinder Functions

These are defined as the solutions of

$$\frac{d^2y}{dx^2} - y \left( \frac{x^2}{4} + a \right) = 0$$  \hspace{1cm} (A.2.1)

and are

$$U(a, x) = \cos \left( \pi \left( \frac{1}{4} + \frac{1}{2}a \right) \right) Y_1 - \sin \left( \pi \left( \frac{1}{4} + \frac{1}{2}a \right) \right) Y_2$$  \hspace{1cm} (A.2.2)

$$V(a, x) = \frac{1}{\Gamma \left( \frac{1}{2} - a \right)} \left( \sin \left( \pi \left( \frac{1}{4} + \frac{a}{2} \right) \right) Y_1 + \cos \left( \pi \left( \frac{1}{4} + \frac{a}{2} \right) \right) Y_2 \right)$$  \hspace{1cm} (A.2.3)

where

$$Y_1 = \frac{1}{\sqrt{\pi}} \frac{\Gamma \left( \frac{1}{4} - \frac{1}{2}a \right)}{2^{\frac{3}{2}a+\frac{1}{4}}}$$  \hspace{1cm} (A.2.4)

$$Y_2 = \frac{1}{\sqrt{\pi}} \frac{\Gamma \left( \frac{3}{4} - \frac{1}{2}a \right)}{2^{\frac{3}{2}a-\frac{1}{4}}} y_2$$  \hspace{1cm} (A.2.5)

where

$$y_1 = e^{-\xi^2} M \left( \frac{1}{2} a + \frac{1}{4}, \frac{1}{2}, \frac{\xi^2}{2} \right)$$  \hspace{1cm} (A.2.6)

$$y_2 = \xi e^{-\xi^2} M \left( \frac{1}{2} a + \frac{3}{4}, \frac{3}{2}, \frac{\xi^2}{2} \right)$$  \hspace{1cm} (A.2.7)

Useful facts are

$$U(a, 0) = \frac{\sqrt{\pi}}{2^{\frac{3}{2}a+\frac{1}{4}} \Gamma \left( \frac{3}{4} + \frac{1}{2}a \right)}, \quad U'(a, 0) = -\frac{\sqrt{\pi}}{2^{\frac{3}{2}a-\frac{1}{4}} \Gamma \left( \frac{1}{4} + \frac{1}{2}a \right)}$$  \hspace{1cm} (A.2.8)

$$V(a, 0) = \frac{2^{\frac{3}{2}a+\frac{1}{4}} \sin \left( \pi \left( \frac{3}{4} - \frac{1}{2}a \right) \right)}{\Gamma \left( \frac{3}{4} - \frac{1}{2}a \right)}, \quad V'(a, 0) = \frac{2^{\frac{3}{2}a+\frac{1}{4}} \sin \left( \pi \left( \frac{1}{4} - \frac{1}{2}a \right) \right)}{\Gamma \left( \frac{1}{4} - \frac{1}{2}a \right)}$$  \hspace{1cm} (A.2.9)

Note that the information presented in this appendix was obtained from the book by Abramowitz and Stegun[14].