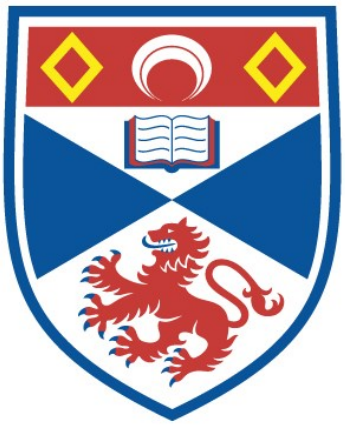
INNER PRODUCT QUADRATURE FORMULAS

Julian de Gruchy Gribble

A Thesis Submitted for the Degree of PhD
at the
University of St Andrews



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A thesis presented for the degree of Doctor of Philosophy in the Faculty of Science of the University of St. Andrews.

by

Julian de Gruchy Gribble B.Sc. (HON)

1979

St. Andrews

ProQuest LLC.
789 East Eisenhower Parkway
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DECLARATION

I declare that this thesis is of my own composition and that the work of which it is a record has been carried out by myself. It has not been submitted in any previous application for a higher degree.

This thesis describes the results of research done in the Department of Applied Mathematics, University of St. Andrews, where I was admitted as a research student under the supervision of Dr. George M. Phillips in October 1976.

Julian de Gruchy Gribble

CERTIFICATE

I hereby certify that J. de Gruchy Gribble has fulfilled the conditions of Ordinance No. 12 and Resolution of the University Court No. 1 (St. Andrews) and is qualified to submit the accompanying thesis in application for the degree of Doctor of Philosophy.

G.M. Phillips
(Supervisor)

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CONTENTS

0. SUMMARY

We investigate an approach to approximating the integral

$$\int_R w(x)f(x)g(x)dx \equiv I(f;g), \tag{0.1}$$

where R is a region in one-dimensional Euclidean space, and w a weight function. Since (0.1) may be regarded as a continuous bi-linear functional in f and g we approximate it by a discrete bi-linear functional, which we term an Inner Product Quadrature Formula (I.P.Q.F.):

$$Q(f;g) \equiv \underline{f}^T A \underline{g} , \tag{0.2}$$

where

$$\underline{f}^T = (S_o(f),\ldots,S_m(f))^T ,$$

$$\underline{g}^T = (T_o(g),\ldots,T_n(g))^T ,$$

$$A = (a_{ij})_{i=0,j=0}^{m,n},$$

and a_{ij} are real numbers, $\sum_{i=0}^{m} \sum_{j=0}^{n} |a_{ij}| > 0$.

The so-called elementary functionals $\{S_i\}_{i=0}^{m}$ and $\{T_j\}_{j=0}^{n}$ are two sets of linearly independent linear functionals, acting on f and g respectively, defined over a certain subspace of functions to which f and g belong. The simplest example of these functionals is function evaluation at a given point.

The matrix A is determined by requiring (0.2) to be exact for certain classes of functions f and g, say

$$(0.3) \qquad \begin{aligned} f \varepsilon \Phi^{\gamma} &\equiv \{\phi_0, \ldots, \phi_{\gamma}\}, \ \gamma \geq 0, \\ g \varepsilon \Psi^{\delta} &\equiv \{\psi_0, \ldots, \psi_{\delta}\}, \ \delta \geq 0. \end{aligned}$$

In Chapter 1 we introduce the concept of I.P.Q.F. in more detail and make some general comments about approaches available when examining numerical integration. After explaining in some detail why we feel I.P.Q.F. are a useful tool in §2.1, we proceed in the remainder of Chapter 2 to investigate various conditions which may be placed on $\Phi^{\gamma}, \Psi^{\delta}$ $\{S_i\}_{i=0}^{m}$, and $\{T_j\}_{j=0}^{n}$, in order to guarantee the existence of I.P.Q.F. exact when $f \varepsilon \Phi^{\gamma}$ and $g \varepsilon \Psi^{\delta}$.

In particular we investigate the question of maximising $\gamma+\delta$. In the case where ϕ_i and ψ_j are the standard monomials of degree i and j respectively, some results have already been published in B.I.T. (1977) p. 392-408. We investigate various choices of ϕ_i and ψ_j:

(a) $\{\phi_i\}_{i=0}^{m+1}$ (i.e. $\gamma=m+1$) and $\{\psi_j\}_{j=0}^{m}$ (i.e. $\delta=m$) being Tchebychev sets (§2.7),

(b) $\{\phi_i\}_{i=0}^{2m+1}$ (i.e. $\gamma=2m+1$) being a Tchebychev set and Ψ^{δ} contains only one function (i.e. $\delta=0$) (§2.6),

(c) $\phi_i \equiv (\phi_1)^i$, $i=0,1,\ldots$ and $\phi_i \equiv \psi_i$, $i=0,1,\ldots$ (§2.8).

In Chapter 3 we consider the question of compounding I.P.Q.F. both in the classical sense, and, briefly, by examining spline functions, regarding them as providing a link between an I.P.Q.F on one hand and a compounded I.P.Q.F. on the

other. Various methods of theoretically estimating the errors involved are considered in Chapter 4. In the fifth Chapter we examine various ways in which the concept of I.P.Q.F. might (or might not) be extended. Finally, we make some brief comments about the possible applications of I.P.Q.F., and give a few examples.

CHAPTER 1: INTRODUCTION

§1.1 QUADRATURE FORMULAE

The subject of Quadrature, or Numerical Integration, in its general setting may be considered as the problem of approximating the linear functional I, the integration operator, which for some reason we are unable to evaluate directly, by another functional Q, which we are able to evaluate. Let

$$\int_R w(x)f(x)dx \equiv I(f) \tag{1.1.1}$$

where R is a region in d-dimensional Euclidean space R^d, $d \geq 1$. The weight function w is assumed to be non-trivial in R and such that I(f) exists in a Riemann-Stieltjes sense for $f \in \Gamma$, $f: R^d \to R^1 \equiv R$, where Γ is some class of functions under consideration. Unless stated otherwise, we shall assume that w is non-negative throughout R.

The functional Q, although ultimately acting on f, may be regarded as operating on a real (m+1) vector, each of whose elements is itself the result of the operation of a linear functional acting on f. Throughout, m may be regarded as an arbitrarily fixed positive integer. These latter linear functionals we shall call 'elementary'. For example, it is often the case that the elementary functionals are function evaluations at specified points (usually with R). We denote these elementary functionals by $S_0, \ldots, S_m$ and, more conveniently, by $\{S_i\}_{i=0}^m$. We now have

$$(1.1.2) \qquad Q^{(m)}(f) \equiv Q(S_o(f),\ldots,S_m(f)).$$

When there is no ambiguity, we will delete the superscript (m) from $Q^{(m)}$.

The choice of the elementary functions $\{S_i\}_{i=0}^{m}$ is affected by many factors; for example, the availability of information about the function f (i.e. ensuring that we are actually able to evaluate $S_i(f)$), ease of computation, properties that we might desire Q to possess - over and above the necessary one that Q exists for all $f\varepsilon\Gamma$.

We shall say that a Quadrature Formula (Q.F.) is exact for a function f if the error functional

$$(1.1.3) \qquad E(f) \equiv I(f)-Q(f)$$

is zero. If

$$(1.1.4) \qquad E(f) = 0 \ , \ \forall f\varepsilon\Phi,$$

we say that the Q.F. is exact on the class Φ.

The usual approach when obtaining a Q.F. is to choose m, the elementary functionals $\{S_i\}_{i=0}^{m}$, and then require Q to be exact on some specified class of functions $\Phi^{\gamma}c\Gamma$. We shall assume that we may write

$$(1.1.5) \qquad \Phi^{\gamma} \equiv \{\phi_o,\ldots,\phi_\gamma\}, \ \gamma\geq 0,$$

and that the basis functions $\phi_0,\ldots,\phi_\gamma$ are linearly independent. If we wish to discuss a function ϕ which is a linear combination of the functions ϕ_i i.e. ϕ is in the function space spanned by Φ^γ, we write

$$\phi\varepsilon sp\Phi^\gamma, \tag{1.1.6}$$

or, if no ambiguity arises, $\phi\varepsilon\Phi^\gamma$. Obviously if a Q.F. is exact on Φ^γ it will also be exact on $sp\Phi^\gamma$ when Q is a linear functional. Clearly the values the number γ may take depend upon m and possibly the choice of functionals $\{S_i\}_{i=0}^m$. If anything were known of the behaviour of functions in Γ it would be natural to try to choose the functions Φ^γ so that they reflect this behaviour. Having chosen m and Φ^γ, we may sometimes investigate which choice of functionals $\{S_i\}_{i=0}^m$ leads to a Q.F. which is best in some sense (see §1.2).

Since Q is approximating the linear functional I it is natural that we require Q to be linear also. The general form of Q is thus (see, for example, [6])

$$Q^{(m)}(f) = \sum_{i=0}^m a_i S_i(f),\ a_i\varepsilon\mathbb{R},\ i=0,\ldots,m. \tag{1.1.7}$$

Thus, in determining Q we have (m+1) linear parameters, a_i $i=0,\ldots,m$, which in principle are found by solving the so called moment equations. The moment equations are a system of simultaneous linear equations derived by replacing f in (1.1.7) by each of $\phi_0,\ldots,\phi_r$, $r=\min(m,\gamma)$. It is possible that in deter-

mining our choice of elementary functions $\{S_i\}_{i=0}^{m}$ further parameters, usually nonlinear, become available in Q.

Let

$$(1.1.8) \qquad B = \begin{bmatrix} S_o(\phi_o),\ldots,S_m(\phi_o) \\ S_o(\phi_1),\ldots,S_m(\phi_1) \\ \cdot \\ \cdot \\ \cdot \\ S_o(\phi_m),\ldots,S_m(\phi_m) \end{bmatrix} .$$

The following is immediate (see, for example, [2]).

THEOREM 1.1.1 If Φ^{γ} has dimension at least m+1, and $\{S_i\}_{i=0}^{m}$ are linearly independent elements of $(\Phi^m)^*$, then $\det(B)\neq 0$. ///

As usual $(\Phi^m)^*$ denotes the dual space of Φ^m.

We generalize the usual idea of a polynomial in a standard manner (see, for example, [9,97]).

DEFINITION 1.1.1 A ϕ-polynomial of degree r is a function of the form

$$\sum_{i=0}^{r} \alpha_i \phi_i ,$$

where $\alpha_i \varepsilon \mathbb{R}$, $\phi_i \varepsilon \Phi^r$, $i=0,\ldots,r$. ///

If no ambiguity results we simply call a ϕ-polynomial a polynomial. If we refer to polynomials, with no obvious reference to some specific set ϕ^{γ}, we mean polynomials in the usual sense.

DEFINITION 1.1.2 An interpolatory ϕ-polynomial of degree r to a function $f \in \Gamma$ at $\{S_i\}_{i=0}^{s}$ $r \geq s$, is a ϕ-polynomial ξ of degree at most r, such that

$$S_i(f) = S_i(\xi), \quad i=0,\ldots,s \qquad ///$$

Usually we choose r=s in the above definition.

DEFINITION 1.1.3 The Q.F. (1.1.7) is called interpolatory if it may be derived by integrating the interpolatory ϕ-polynomial of degree m to f at $\{S_i\}_{i=0}^{m}$. ///

When the matrix B has full rank it is clear that we may derive a Q.F. in one of two ways which are equivalent in the sense that, although they have different representations, they will always give the same approximation to $I(f)$.
These methods are

I. Derive the interpolatory Q.F. by solving

(1.1.9) $$B^T \underline{b} = \underline{f}$$

where $$\underline{b}^T = (b_o,\ldots,b_m),$$
$$\underline{f}^T = (S_o(f),\ldots,S_m(f)),$$

thus the interpolating ϕ-polynomial, p, to f at $\{S_i\}_{i=0}^m$ is

$$p = \sum_{i=0}^m b_i \phi_i \ .$$

Then integrate p.

II. Solve the moment equations

(1.1.10) $$B\underline{a} = \underline{\Lambda}$$

where $$\underline{a}^T = (a_o,\ldots,a_m),$$
$$\underline{\Lambda}^T = (I(\phi_o),\ldots,I(\phi_m)),$$

thus deriving the Q.F. (1.1.7) directly.

However, if the matrix B does not have full rank and thus a solution to (1.1.9) may not exist, it is still possible that a solution to (1.1.10) does exist. See [8].

Thus, we have

THEOREM 1.1.2 If the matrix B has full rank and Q is a Q.F. exact on Φ^γ, $\gamma \geq m$, then Q is interpolatory. ///

Without any real loss in generality, unless it is stated otherwise, we shall assume that the elementary functionals $\{S_i\}_{i=0}^m$ are linearly independent in $(\Phi^m)^*$.

When we seek Q.F. exact on Φ^γ, $\gamma > m$, we almost invariably become involved in the solution of non-linear equations including the parameters determining the choice of $\{S_i\}_{i=0}^m$. The deriviation of such Q.F. is a more difficult problem, and in

order to guarantee the existence of such Q.F., we may be forced to impose stronger conditions on either $\{S_i\}_{i=0}^{m}$ or Φ^γ (or both). An obvious example of this is to require the functions in Φ^γ to form a Tchebychev set on R (see §2.2).

Our approach to deriving Q.F.'s, namely requiring exactness on a certain class of functions, is not the only approach that can be utilized. There is a large body of literature concerning Monte Carlo and other number-theoretic Q.F.'s, see for example [10,11,79]. These methods appear to be more effective for higher dimensional regions. There are other Q.F. which, while not integrating a large class of functions exactly, attempt to give good approximations over many integrals. See [7]. For other approaches see [80,81], and for other work describing some unifying theories of Quadrature, see [73,82].

§1.2. CHOICE OF QUADRATURE FORMULA

Having obtained a Q.F. it still remains to investigate the properties that it possesses so that we may compare its effectiveness when compared with other Q.F.'s. Usually, by setting $m=0,1,2,\ldots$, we are able to obtain a family of Q.F.'s exact on larger and larger classes of functions - γ increasing as a function of m. Thus we can investigate asymptotic properties like convergence. It is possible that we might need to impose conditions on the set Φ^γ and/or the elementary functionals S_i to ensure that we at least obtain w^* convergence of the sequence of linear operators $\{Q^{(m)}\}_{m=0}^{\infty} \to I$.

It is important to be able to estimate the error E(f).

Often the theoretical error estimates available cannot be used in practical situations - for example when they require the evaluation of high order derivatives at unknown points. The problem of estimating errors in a manner which is easy and practical and yet reliable is by no means trivial. The behaviour of both E and $Q^{(m)}$ will be affected by the choice of Φ^{γ}. For instance, the width of the subspace $sp\Phi^{\gamma}$ in Γ, see for example [1], or the choice of points at which we evaluate f when $S_i(f)\equiv f(x_i)$, $i=0,\ldots,m$, bearing in mind the fact that an increase in the number of interpolation points does not necessarily give a better uniform approximation to a function, could be important considerations.

Distinct from errors arising from a theoretical approximation of I, in practice, since we are almost always using computers, we must also consider roundoff error. Roundoff error is not as serious as it has been in the past as we may always use double precision arithmetic if necessary. Clearly however, it remains desirable, and certainly aesthetically pleasing, that the coefficients a_i in (1.1.7) are all positive. Also, if some a_i are negative, it is important that the sum of the modulus of all the a_i's is reasonably small, say of the same order of magnitude as their algebraic sum. Depending upon the weight function w it may also be desirable that the numbers $|a_i|$ are themselves of the same order of magnitude.

When we design or choose a Q.F. there are various things that we should take into consideration:

(1.2.1) The amount of time and money one has available. If

we are only going to evaluate a given integral once we may be happy using an inefficient but simple or easily available Q.F.. On the other hand, if we are going to be evaluating a large number of similar integrals it becomes important to spend time and effort in discovering efficient (though perhaps not so simple) Q.F. so that a computer may do its job quickly and cheaply.

(1.2.2) The information available about the integrand. This will range between the extremes of being given, a priori, certain data (i.e. the elementary functions $\{S_i\}_{i=0}^{m}$ are predetermined, say, in some physical experiment) to having the integral available in a form that allows us a free rein in our choice of $\{S_i\}_{i=0}^{m}$. In this latter case the ease and accuracy with which we may extract the desired information becomes an important consideration.

(1.2.3) The way in which we choose the set Φ^{γ}. Depending upon the role in which we want to use the Q.F. how do we choose the functions in Φ^{γ}: generally, or aimed at the solution of one or a few specific problems? For example, the integrand may be oscillatory or singular. There is a school of thought which favours the latter alternative if we are actually going to make serious practical use of any Q.F. See [46].

(1.2.4) If the Q.F. is going to be used in some compound manner (either adaptive or not), do we wish to be able to use

information obtained more than once? This is a more important consideration in the one-dimensional case (d=1) than in the higher dimensional cases since the amount of new information required at each stage in the compounding procedure is of the order of the amount required at the previous stage to the power d.

(1.2.5) When our region of integration is multi-dimensional, (d>1), there are an infinite number of regions which may not be transformed into one another with affine transformations, for example, hypercubes, spheres, simplices and so on. Consequently, different Q.F. are required for different regions. It is often possible to simplify matters considerably by taking into account the symmetry properties of the region under consideration. See, for example, [87,88,89]. The functions ϕ_i are almost invariably chosen to be the standard monomials. Because of the large number of points required in multidimensional Q.F., exact even for polynomials of moderate degree, much effort has been expended in the investigation of minimal point Q.F. Compared to the one dimensional case, there remains much work to be done in this area. See, for some recent developments, [90,...,93].

Our choice of Q.F. above is dominated by a "trade-off" between ease (the Q.F. is familiar, easy to programme, "nice" easily handled parameters perhaps already available as a subroutine on the computer, and so on) and efficiency in some

sense (the amount of computer and/or programmer time used, optimal characteristics of some type, and so on).

Apart from the above considerations, we are often interested in obtaining "best" Q.F.'s in same sense usually related to a method of estimating errors. For example:

(1.2.6) We may wish to maximise γ. By far the most common occurrence of this is an investigation where the elementary functionals are point evaluations of the function, namely traditional Gaussian Q.F.'s. See, for example, [38,59].

All elementary books on numerical analysis deal with Gaussian Q.F. in some detail for the standard polynomial case. The derivation depends upon the theory of orthogonal polynomials. For the one-dimensional case tables of knots and weights are readily available, e.g. [38,60].

(1.2.7) We may seek Q.F. that are optimal in the sense of Sard, minimizing the Peano kernel of the Q.F. See, for example, [61,...,67].

(1.2.8) We may seek Q.F. that give a "best" performance over a function space or collection of test functions. Here "best" is usually measured in some sort of statistical manner. We may choose to use the Sarma-Eberlein estimate of goodness, see Ch. 5 of [37], [39,40]. Alternatively we may seek some type of "performance profile" over a set of carefully chosen test functions. See [41].

(1.2.9) We may examine Q.F. which minimise $\sum_{i=0}^{m} |a_i|^2$ (cf. (1.1.7)), usually called Tchebychev Q.F. . See, for example, [68,69,70].

Once we choose a method of estimating errors, or examine a general theory of Q.F., we obtain a theoretical means of comparison of Q.F.. However, most of the work done in the area of comparison of Q.F. has been via the practical expedient of examining the performance of various Q.F. on carefully chosen sets of test functions. See [41,...,45].

Other approaches to the problem of estimating errors include (not necessarily mutually exclusively):

(1.2.10) The commonest approach, in the polynomial case, of using Taylor series, and obtaining error bonds in terms of high order derivatives. Other truncated series have also been used, in particular Tchebychev series. See [78] and the consequent literature.

(1.2.11) The theory of analytic functions, in which case techniques dependent on Cauchy's theorem and contour integration are used. See, for example, [71,72,73,98].

(1.2.12) The use of functional analysis to minimize the norm of E in some sense. See, for example, [39,67,74].

(1.2.13) The use of differences or other means to avoid the need to estimate high order derivatives. See, for example, [76,77,83].

(1.2.14) Special techniques for error estimates for functions with special characteristics, say low orders of continuity. See, for example, [75,99].

When we use a Q.F. we must remember that we only have a finite number of pieces of information to use, namely $\{S_i\}_{i=0}^{m}$ and consequently it is impossible to obtain infallible error estimates based only on this information. The common technique of simply doubling m and then comparing the results obtained using $Q^{(m)}$ and $Q^{(2m)}$ can be easily fooled. See [5,122]. Given a little more information about the general behaviour of the integrand it is possible, using so called Adaptive Q.F., to obtain fairly accurate results at a cost that is not exorbitant. See, for example, [123,...,126]. Other approaches to the problem of practical error estimation and obtaining a desired degree of accuracy include,

(1.2.14) Approximation, or more exactly the obtaining of upper bounds for theoretical estimates. This tends to be a very conservative, and thus very expensive, approach.

(1.2.15) The use of a sequence of Q.F. and some type of extrapolation or acceleration procedure. See, for example, [57,58], and other literature on Romberg Q.F. of various types.

Analogous to the comment we made concerning the choice of Q.F. when estimating errors, we always have to consider the "trade-off" between price and reliability.

Further comments concerning the criteria for deriving, choosing, evaluating, and estimating errors for various choices of Q.F., can of course be found in any of the authoritative books written on Numerical Integration, in particular [5], [37]. See also [4], [84,85,86]. There is also an extensive Russian literature on the subject of Numerical Integration.

§1.3 INNER PRODUCT QUADRATURE FORMULAE

An Inner Product Quadrature Formula (I.P.Q.F.) is a Q.F. designed especially for approximating the integral

$$\int_R w(x)f(x)g(x)dx \equiv I(f;g) \tag{1.3.1}$$

where R and w are as in §1.1. The functions f and g are assumed to be members of classes of functions Γ and Δ respecitvely. In some cases it is advantageous to have $\Gamma \equiv \Delta$.

For ease of reference we list the notation that is used throughout in Appendix 1, and the assumptions that are made during the discussion of I.P.Q.F. are listed in Appendix II. So far we have

(A1) $f \in \Gamma$ and $g \in \Gamma$ are real valued functions acting on points in R, $R \subset R^d$.

(A2) The integral (1.3.1) exists (perhaps improperly) and is finite in a Riemann-Stieltjes sense.

(A3) The weight function w is non-negative throughout R,

For the present, we also have

(A4) d=1. That is, R is a one dimensional finite interval, whose interior we may take, without loss of generality, to be (-1,1).

The integration operator I in (1.3.1) is a bi-linear functional. Thus it is natural, analogous to the derivation of Q in §1.1, that we should seek a bi-linear functional Q(f;g) with

which to approximate I. Suppose we have elementary linear functionals $S_o,\ldots,S_m$ and $T_o,\ldots,T_n$, denoted $\{S_i\}_{i=0}^m$ and $\{T_j\}_{j=0}^n$, giving us information about f and g respectively. We also have the assumption

(A5) Each of the linear functionals S_i and T_j is bounded and therefore continuous on Γ and Δ respectively.

With a customary abuse of notation the bi-linear functional $Q(f;g)$, written $Q^{(m,n)}(f;g)$ where necessary, may be regarded as a mapping

$$Q^{(m,n)} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}, \tag{1.3.2}$$

and has the general form

$$Q(f;g) \equiv \sum_{i=0}^m \sum_{j=0}^n S_i(f) a_{ij} T_j(g) \equiv \underline{f}^T A \underline{g} \tag{1.3.3}$$

where $$\underline{f}^T = (S_o(f),\ldots,S_m(f)),$$

$$\underline{g}^T = (T_o(g),\ldots,T_n(g)),$$

and $A=(a_{ij})_{i=0,j=0}^{m,n}$ is a real $(m+1) \times (n+1)$ matrix called the coefficient matrix. The numbers a_{ij} are called the coefficients (or sometimes weights), just as in (1.1.7) the numbers a_i are known as the coefficients (or weights) of the Q.F. $Q^{(m)}$.

We call $Q(f;g)$ in (1.3.3) an I.P.Q.F. To distinguish them, we call Q.F. of the type (1.1.7) Regular (R.Q.F.). We have used the name I.P.Q.F. instead of Product-type Q.F., see

[12,...,21,23], to avoid confusion with other more established meanings of the term, in particular in the context of integration in multi-dimensional regions (i.e. d>1). See also §1.4.

We note that it can be shown by example that setting m=n, and $S_i \equiv T_i$, i=0,...,m, does not necessarily reduce an I.P.Q.F. to a R.Q.F..

We proceed as in §1.1. We shall say that an I.P.Q.F. is exact for an ordered pair (f;g), $f \in \Gamma$, $g \in \Delta$, if the error functional

$$E(f;g) \equiv I(f;g) - Q(f;g) \tag{1.3.4}$$

is zero. Hence, say that an I.P.Q.F. is exact on the product space $\Phi \times \Psi$ if, for each choice of $f \in \Phi$ and $g \in \Psi$ we have $E(f;g)=0$.

After predetermining m and n, and choosing $\{S_i\}_{i=0}^{m}$ and $\{T_j\}_{j=0}^{n}$, we seek an I.P.Q.F. exact on $\Phi^{\gamma} \times \Psi^{\delta}$, where

$$\begin{cases} \Phi^{\gamma} \subset \Gamma \\ \Psi^{\delta} \subset \Delta \end{cases} . \tag{1.3.5}$$

Clearly the integers γ and δ are functions of m and n respectively, and may also depend on the choice of any (usually non-linear) parameters we may make in the selection of $\{S_i\}_{i=0}^{m}$ and $\{T_j\}_{j=0}^{n}$. We make the following assumption:

(A6) The sets Φ^{γ} and Ψ^{δ} are each made up of functions which are linearly independent over R.

Thus, analogous to (1.1.5), we will always be able to write

$$(1.3.6)\qquad \begin{cases} \Phi^{\gamma} \equiv \{\phi_0, \ldots, \phi_{\gamma}\}, \ \gamma \geq 0, \\ \Psi^{\delta} \equiv \{\psi_0, \ldots, \psi_{\delta}\}, \ \delta \geq 0. \end{cases}$$

Obviously if an I.P.Q.F. is exact on $\Phi^{\gamma} \times \Psi^{\delta}$, due to bi-linearity, it will be exact on $(sp\Phi^{\gamma}) \times (sp\Psi^{\delta})$.

In determining $Q^{(m,n)}$ we see that we have $(m+1)(n+1)$ linear parameters a_{ij} and possibly some non-linear parameters at our disposal. Once the non-linear parameters have been assigned, we may find the linear parameters as follows:

Let

$$(1.3.7)\quad B = (S_i(\phi_\ell))_{\ell=0,i=0}^{r \quad m} \equiv (b_{\ell i})_{\ell=0,i=0}^{r \quad ,m}, \ (r=\min(m,\gamma),$$

$$= \begin{bmatrix} S_0(\phi_0), & \ldots, & S_m(\phi_0) \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ S_0(\phi_r), & \ldots, & S_m(\phi_r) \end{bmatrix}$$

$$(1.3.8)\quad C = (T_j(\psi_\ell))_{\ell=0,j=0}^{s \quad ,m} \equiv (c_{\ell j})_{\ell=0,j=0}^{s \quad ,n}, \ s=\min(n,\delta),$$

$$= \begin{bmatrix} T_0(\psi_0), & \ldots, & T_n(\psi_0) \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ T_0(\psi_s), & \ldots, & T_n(\psi_s) \end{bmatrix}$$

and

$$D = C \otimes B \ . \tag{1.3.9}$$

That is, D is the Kronecker product of the two matrices C and B, having the block matrix structure

$$\begin{bmatrix} c_{oo}B, \ . \ . \ . \ , c_{on}B \\ . \\ . \\ . \\ c_{so}B, \ . \ . \ . \ , c_{sn}B \end{bmatrix} \tag{1.3.10}$$

We now have, analogous to Theorem 1.1.1,

THEOREM 1.3.1 Assume that Φ^{γ} and Ψ^{δ} have dimensions at least $m+1$ and $n+1$ respectively, i.e. $\gamma \geq m$, $\delta \geq n$. Assume that $\{S_i\}_{i=0}^{m}$ and $\{T_j\}_{j=0}^{n}$ are linearly independent in $(\Phi^m)^*$ and $(\Psi^n)^*$ respectively. Then det(D) is non zero.

PROOF: By block row operations D may be reduced to a block matrix in which all blocks below the main diagonal are zero. The matrix C is non singular by Theorem 1.1.1. The result follows by applying Theorem 1.1.1 to each of the block matrices on the diagonal. ///

We define ψ-polynomials and interpolating ψ-polynomials

in a way completely analogous to the definition of ϕ-polynomials.

Now we have

DEFINITION 1.3.1 An I.P.Q.F. will be called interpolatory if it may be obtained by integrating pq, where $(p;q)\varepsilon\Phi^m\times\Psi^n$, and p(q) is an interpolating ϕ-(ψ-)polynomial of degree at most m(n) to f(g) at $\{S_i\}_{i=0}^m$ ($\{T_j\}_{j=0}^n$). ///

As before there are at least two approaches to determining an I.P.Q.F. exact on $\Phi^m\times\Psi^n$ which, although they may have different representations, will give the same approximation to I. I. Derive the interpolatory I.P.Q.F. Solve

$$\text{(1.3.11)} \qquad \begin{cases} B^T\underline{b} = \underline{f} \\ C^T\underline{c} = \underline{g} \ , \end{cases}$$

where $\underline{b}^T = (b_o,\ldots,b_m)$, $\underline{c}^T = (c_o,\ldots,c_n)$,
$\underline{f}^T = (S_o(f),\ldots,S_m(f))$,
$\underline{g}^T = (T_o(g),\ldots,T_n(g))$,

and thus obtain the interpolating ϕ-(ψ-)polynomials

$$\text{(1.3.12)} \qquad \begin{cases} p = \sum_{i=0}^m b_i\phi_i \\ q = \sum_{j=0}^n c_j\psi_j \ . \end{cases}$$

Finally, integrate the product pq.

II. Solve the moment equations

$$D\underline{a} = \underline{\Lambda} \,, \tag{1.3.13}$$

$$\begin{cases} \underline{a}^T = (\underline{a}_o^T, \ldots, \underline{a}_n^T) \\ \underline{a}_i^T = (a_{oi}, \ldots, a_{mi}), \; i=0, \ldots, n, \end{cases} \tag{1.3.14}$$

$$\begin{cases} \underline{\Lambda}^T = (\underline{\Lambda}_o^T, \ldots, \underline{\Lambda}_n^T) \\ \underline{\Lambda}_i^T = (I(\phi_o;\psi_i), \ldots, I(\phi_m;\psi_i)), \\ \qquad\qquad\qquad i=0, \ldots, n. \end{cases} \tag{1.3.15}$$

As before, it is still possible that when either B or C does not have full rank, that there will be a solution to (1.3.13) but not to (1.3.12). Again see [8] for further details. However we shall not consider Q.F. of this type, sometimes rather aptly referred to as "fortuitous" Q.F. See [88]. Hence we make the following assumption:

(A7) The elementary linear functionals $\{S_i\}_{i=0}^m$ and $\{T_j\}_{j=0}^n$ are linearly independent in $(\Phi^m)^*$ and $(\Psi^n)^*$ respectively.

The comment following Th. 2.1.2 explains why, if $\gamma > m$, this might be an important assumption. Analogous to Theorem 1.1.2

we have

THEOREM 1.3.2 Assume (A7). Assume an I.P.Q.F. is exact on $\Phi^{\gamma}\times\Psi^{\delta}$, $\gamma\geq m$, $\delta\geq n$. Then the I.P.Q.F. is interpolatory.

PROOF: (A7) tells us via Theorem 1.3.1 that B and C have full rank. Hence a unique solution to (1.3.12) exists. The results obtained using methods I and II must be identical since the I.P.Q.F. is exact on $\Phi^{\gamma}\times\Psi^{\delta}$. ///

The following results are also obvious:

THEOREM 1.3.3 If (A1), (A2), (A5), (A6) and (A7) hold then there exists an interpolatory I.P.Q.F. exact on $\Phi^{m}\times\Psi^{n}$. ///

Note that we do not necessarily require (A3), or even that w is continuous in (-1,1).

THEOREM 1.3.4 An interpolatory I.P.Q.F. is uniquely determined by

(1) w in (1.3.1)

(2) the choice of function subspaces Φ^{m} and Ψ^{n}

(3) the choice of elementary functionals $\{S_i\}_{i=0}^{m}$ and $\{T_j\}_{j=0}^{n}$. ///

The investigation of the various properties of I.P.Q.F.'s, convergence, error estimations, effect of choice of functions and

elementary functionals and so on, is the objective of the rest of this thesis.

Before continuing we make the obvious comment that a R.Q.F., when applied to an integrand which is the product of two functions, or even one function ξ, which may always be written $\xi.1$, may always be written as an I.P.Q.F. on setting $m=n$, $S_i \equiv T_i$, $i=0,\ldots,m$, and

$$(1.3.16) \qquad a_{ij} = \begin{cases} 0, & i \neq j \\ a_i, & i=j \end{cases} \qquad \begin{matrix} i=0,\ldots,m,\ j=0,\ldots,n. \\ \text{(see (1.1.7))} \end{matrix}$$

From now on, unless it is specifically stated to the contrary, we assume that (A1-A7) hold.

§1.4 OTHER WORK ON INTEGRATING THE PRODUCT OF SEVERAL FUNCTIONS

Before continuing, we would like to comment on the use of the name I.P.Q.F. again. There are other approaches to the problem of integrating the product of several functions than the one investigated here. Some, not unnaturally, are termed "product integration" of one sort or another. It is to avoid confusion with other approaches that we use the somewhat more descriptive term Inner Product Quadrature Formula . For example, in [47] the term product integral is used: however, the subject under investigation is that of approximating a matrix valued Riemann product integral.

The investigation of the numerical approximation of the integral of the product of two (or more) functions is not a new or exhausted topic, as an examination of [47,...,56] shows. It seems that the earlier impetus for such investigations comes from practical questions arising in actuarial work, and much of the later impetus, with the advent of the computer, from the area of numerically solving integral equations.

In fact, possibly the first example of an I.P.Q.F., see [48, 49], is derived via method I of the previous section. We set

$$\phi_i = \psi_i = \tau_i, \quad i=0,\ldots,m, \tag{1.4.1}$$

where τ_i is the i-th polynomial orthogonal with respect to w on R, and

$$S_i(\xi) \equiv T_i(\xi) = \frac{I(\xi;\tau_i)}{I(\tau_i;\tau_i)}, \quad i=0,\ldots,m, \tag{1.4.2}$$

namely the i-th generalized Fourier coefficient of the function ξ. We then find that A is a diagonal matrix with elements

$$(1.4.3) \qquad a_{ij} = \begin{cases} 0 , & i \neq j \\ I(\tau_i;\tau_i), & i=j \end{cases} \qquad i,j=0,\ldots,m.$$

This I.P.Q.F. is interpolatory and exact on $\Phi^m \times \Psi^m$.

Of course, in practice, the evaluation of S_i and T_i may not be a straightforward procedure.

In the oft quoted paper by Young [51], and in [50], we discover the source of both the term and the technique of product integration in its usual sense. We consider the integral

$$(1.4.4) \qquad \int_R f(x)\psi(x)\,dx .$$

The function ψ essentially plays the role of a weight function. For the first time we encounter a more general formulation of what is, at present, essentially the problem of finding a R.Q.F.: matrix notation is used and the functions ϕ_i (of §1.1) are not immediately assumed to be polynomials. This R.Q.F. is interpolatory and, in our notation, is derived as follows (cf §1.1.):

$$(1.4.5) \qquad S_i(\xi) = \xi(x_i), \quad i=0,\ldots,m,$$

and the coefficients a_i in (1.1.7) are chosen (by solving the linear system analogous to (1.1.9)) so the R.Q.F. is exact on Φ^m.

When we wish to attain greater accuracy than we can with

$Q^{(m)}$ as above, the usual procedure is, in principle, to compound $Q^{(m)}$, as opposed to attempting to increase m. As m increases, apart from the possible difficulty in calculating the moments of the functions ϕ_i with respect to ψ, namely the numbers

$$\int_R \phi_i(x)\psi(x)dx, \tag{1.4.6}$$

convergence of the R.Q.F. is not as rapid as might be desired. Usually, see [53], [56] as examples of this well known procedure, we approximate f in (1.4.4) by sequences of low order interpolating piecewise ϕ-polynomials, taking $\phi_i=x^i$.

The term product-integration has also been used by Elliot and Paget, [54,55], in their investigation of integrals of the form

$$\int_R w(x)f(x)K(\lambda;x)dx = I(f;K) \tag{1.4.7}$$

where w(x) and f(x) are as in §1.1 and K is a function, depending on the parameter λ, which is assumed to be such that the application of a "standard" (i.e. regular, of some sort) Q.F. to (1.4.7) would require a large number of knots in order to attain a modest degree of accuracy. This may also be regarded as an example of an I.P.Q.F.

Set $\phi_i=x^i$, i=0,...,m. Then f is approximated by an interpolating polynomial at the knots x_i, i=0,...,m, which are the zeros of the polynomial, τ_m, of degree m+1 orthogonal with respect to w over R. Thus

$$S_i(\xi) = \xi(x_i), \quad i=0,\ldots,m. \tag{1.4.8}$$

K is approximated by its truncated (generalized) Fourier Series, and Ψ^{δ} has $\delta=0$, $\psi_o \equiv K$. Thus

$$T_j(K) = \frac{I(\tau_j;K)}{I(\tau_j;\tau_j)}, \quad j=0,\ldots,m, \tag{1.4.9}$$

where τ_j is the orthogonal polynomial of degree j with respect to w over R.

We end up with an I.P.Q.F., Q(f;K), which we may write as

$$Q^{(m,n)}(f;K)(=Q(f;\lambda)) = \underline{f}^T A \underline{K}, \tag{1.4.10}$$

where $$a_{ij} = \begin{cases} 0, & i \neq j \\ I(\tau_j;\tau_j), & i=j \end{cases} \qquad i,j=0,\ldots,m.$$

This Q.F. is not exact on $\Phi^m \times \Psi^o$ unless K can be written as a polynomial of degree m. As remarked earlier, the determination of $T_j(K)$ may not be easy.

CHAPTER 2: INNER PRODUCT QUADRATURE FORMULAS

§2.1 MOTIVATION

Most, though not all, of the work published dealing with I.P.Q.F. deals with the standard polynomial case, i.e. $\phi_i=\psi_i=x^i$. We include a list of, to the best of our knowledge, all previously published work concerning I.P.Q.F., i.e. [12,...,28, 118, 119].

The main motivation for investigating I.P.Q.F. is that when dealing with integrals of the type (1.3.1) it is often the case that the two functions f and g have different characteristics. One may be smooth and "well behaved"; the other not, perhaps being oscillatory, having singularities in or near R, or having only a low order of continuity. Thus, if we choose Φ^γ and Ψ^δ to reflect the properties that f and g possess, we might expect more accurate approximations of (1.3.1). It is also possible that we might have different information available for each of the two functions, thus making it desirable to choose elementary functionals, giving us information about one function, which we may not apply to the other function. Thus, assuming we know enough about the integrand to be able to implement an I.P.Q.F., the use of I.P.Q.F. offers two advantages, namely the divorcing of information used and the spaces approximated over when dealing with f and g.

Since I.P.Q.F. has more linear (and non-linear) parameters in its construction than a R.Q.F., (m+1)(n+1) as opposed to m+1, it is reasonable to regard I.P.Q.F. as generalizations of R.Q.F. Clearly, we may always regard R.Q.F. as special cases of

I.P.Q.F. exact on $\Phi^{\delta} \times \Psi^{o}$, where ψ_o is almost always chosen to be unity. See also comments following Th. 1.3.4. As always, we have to pay a price for a gain, and to use an I.P.Q.F. we must be able to distinguish between the functions f and g. As will become obvious later, the way in which the integrand in (1.3.1) is broken up into the product of two functions can be important.

With (m+1)(n+1) linear parameters available, we might expect to be able to integrate exactly at least (m+1)(n+1) distinct linear independent functions when using an I.P.Q.F. (analogous in some way to an interpolatory R.Q.F.). However, using an R.Q.F. with essentially the same information available (i.e. forcing $S_i \equiv T_i$, m=n), we might only expect to be able to integrate m+1 linearly independent functions exactly, treating the product fg as one function. This, in essence, turns out to be the case, that is, we can always integrate exactly the potentially linearly independent functions

$$\phi_i \psi_j, \quad i=0,\ldots,m, \quad j=0,\ldots,n. \tag{2.1.1}$$

However, an investigation of the standard polynomial case manages to obscure completely this important fact. Set

$$\phi_o \equiv 1, \tag{2.1.2}$$

$$\phi_i = (\phi_1)^i, \quad i=1,2,\ldots, \tag{2.1.3}$$

$$\psi_o \equiv 1, \tag{2.1.4}$$

$$\psi_j = (\psi_1)^j, \quad j=1,2,\ldots \tag{2.1.5}$$

Of course we require $\phi_1 \neq 1$ and $\psi_1 \neq 1$. If, in addition, we have

$$\phi_1 = \psi_1, \tag{2.1.6}$$

we find that

$$\phi_i \psi_j \equiv \phi_k \psi_\ell, \quad i+j=k+1 = \text{constant}. \tag{2.1.7}$$

Equation (2.1.7) means that the (m+1)(n+1) potentially linearly independent functions (2.1.1) degenerate to only m+n+1 linearly independent functions. This of course is the situation when we choose, in particular,

$$\phi_1 = \psi_1 = x. \tag{2.1.8}$$

We shall use the following:

DEFINITION 2.1.1 - If the action of an elementary functional L on a function, ξ, is the evaluation of the function at a given point t, we call t a knot, and say that L is of class χ_1, writing $L \varepsilon \chi_1$, and denote this functional by $L(t;.)(\equiv \xi(t))$. ///

Obviously, we always assume that knots are in the domain of definition of functions we are likely to evaluate, however, this does not necessarily (though it is preferable) mean that knots must be within the interval of integration R.

We have, see [24],

THEOREM 2.1.1 Let $S_i \in \chi_1$, $T_j \in \chi_1$, $i=0,\ldots,m$, $j=0,\ldots,n$. Define Φ^γ and Ψ^δ by (2.1.2) - (2.1.6). Then, no matter how we choose the knots x_i, $i=0,\ldots,m$, and y_j, $j=0,\ldots,n$, it is not possible to find an I.P.Q.F. exact on

$$\Phi^\gamma \times \Psi^\delta$$

if $\gamma>m$ and $\delta>n$, or if $\gamma+\delta \geq 2m+2$ with $\gamma>m$ and $\delta \leq n$, or if $\gamma+\delta \geq 2n+2$ with $\gamma \leq m$ and $\delta>n$.

PROOF: See Lemma's 3.1 and 3.2 of [24]. ///

Recall that, when dealing with R.Q.F., and choosing $S_i \in \chi_1$, $i=0,\ldots,m$, we cannot find a R.Q.F. exact on Φ^γ $\phi_i = x^i$, $i=0,..,\gamma$, if $\gamma>2m+1$. See, for example, [4] p. 102. The comments made earlier explain the above Theorem, which is initially perhaps somewhat surprising.

In fact, when dealing with the polynomial case, the only advantage an I.P.Q.F. has over a R.Q.F. is that, even when $\gamma+\delta$ is maximal, we do obtain some degrees of freedom in the choice of knots. One way of employing these degrees of freedom is in making the I.P.Q.F. exact on $\Phi^\gamma \times \Psi^\delta$ for various choices of γ and δ. Again see [24].

Whilst the choice of the elementary functionals is, in theory, only restricted by the requirement

$$(2.1.9) \qquad \begin{cases} S_i \in \Gamma^* , & i=0,\ldots,m \\ T_j \in \Delta^* , & j=0,\ldots,n, \end{cases}$$

in practice the choice is usually more limited: by the information available and the ease of access of the information. By far the most common choice is a χ_1 functional. Other choices of functional include function derivative evaluation at a knot (see, for example, [94, 95, 96]) and an integral over a small interval centred on a given point, see [30, 31]. We shall define further classes of elementary functional, analogous to χ_1, if and when we need them.

For future reference, we include some basic facts concerning the inter-relations between linear function spaces.

THEOREM 2.1.2 Let F and G be two linear function spaces, such that $F \subset G$. Then $G^* \subset F^*$.

PROOF: See [2]. ///

We note, however, that if $\{L_i\}_{i=0}^r$ are linearly independent in F^*, and $L_i \in G^*$, $i=0,\ldots,r$, we are not assured that $\{L_i\}_{i=0}^r$ are linearly independent in G^*.

Example 2.1.1 Let $L_i \in \chi_1$, $i=0,\ldots,r$,

$$L_o \equiv L_o(0;\cdot),$$
$$F = \{x^i\}_{i=0}^r,$$
$$G = \{x^i\}_{i=0}^{r+1}, \quad \text{and} \quad \det\left(L_i(x^{j+1})\right)_{i,j=0}^r = 0.$$

Thus, when talking of linear functionals, it is essential that we say with respect to what function space they are defined.

DEFINITION 2.1.2 A set of functions $F \subset G$ is called dense in the normed linear space G if an arbitrary element $\xi \in G$ can be approximated arbitrarily closely by a finite linear combination of elements $\xi_i \in F$. That is, given $\varepsilon > 0$ there exist s, $\alpha_0, \ldots, \alpha_s$, such that

$$\| \xi - \sum_{i=0}^{s} \alpha_i \xi_i \| < \varepsilon$$

where $\| \cdot \|$ is the norm on the space G. ///

As an example we have (see, for example, [169])

LEMMA 2.1.3 The set of all polynomials is dense in the space of continuous real valued functions defined on [-1,1]. ///

We recall that the hierarchy of commonly used function spaces is (see, for example, [2,169]),

$$(2.1.10) \quad \begin{cases} E \subset A \subset C^{\infty}, .., \subset C^{n+1} \subset D^{n} \subset C^{n}, \ldots, \subset D^{0} \subset Lip_{\beta} \subset Lip_{\alpha} \\ \qquad\qquad \subset C^{0} \subset B \subset L_q \subset L_p, \\ 0 < p < q \leq \infty, \ \alpha < \beta, \ n \text{ a positive integer.} \end{cases}$$

and the spaces are defined as follows:

$$(2.1.11) \qquad L_p \equiv L_p(R) \equiv \{f: \int_R |f(x)|^p dx < \infty\},$$

$$(2.1.12) \qquad B \equiv B(R) \equiv \{f: \exists M < \infty, |f(x)| < M, \ \forall x \in R\},$$

$$(2.1.13) \qquad C^r(R) \equiv \{f: f \text{ has a continuous } r^{th} \text{ derivative } \forall x \in R\},$$

(2.1.14) $D^r(R) \equiv \{f: f$ has a continuous $(r-1)$st derivative, differentiable almost everywhere in $R\}$,

(2.1.15) $Lip_M\alpha \equiv Lip\alpha \equiv \{f: |f(x_1)-f(x_2)| \leq M|x_1-x_2|^{\alpha}, \ \forall x_1, x_2 \varepsilon R\}$,

(2.1.16) $A(R) \equiv \{f: f(x) = \sum_{n=0}^{\infty} \alpha_n (x-x_o)^n$, $(x-x_o)\varepsilon h(x_o)$ a neighbourhood of x_o, $\forall x_o \varepsilon R\}$,

(2.1.17) $E \equiv \{f: f\varepsilon A, \ \forall x, \ |x|<\infty\}$.

As functions become more specialized we find that more can be said about the structure of the function spaces containing these functions. We also have

THEOREM 2.1.4 Let $R = [-1,1]$, and assume $F \subset G \subset C^o(R)$ are two of the above mentioned spaces. Then F is dense in G.

PROOF: From Lemma 2.1.3 since the polynomials will be in the space F. ///

Thus, given a set of functions dense in E, this set of functions will also be dense in any of the above spaces contained in $C(R), R=[-1,1]$. This, of course, is the justification for constructing Q.F. of any type, exact on well behaved sets of functions. The classic examples being the polynomials, and to a lesser extent the trigonometric functions. The error of a Q.F. exact on certain functions (often a basis for a subset of a set of functions dense in E), for a particular integrand depends on the integrand and the functions the Q.F. is exact on.

nomials when employed on integrands that are oscillatory.

Recalling (A1), it is natural to have

(A8) $\Gamma \subset C^0(R), \quad \Delta \subset C^0(R).$

§2.2 TCHEBYCHEV SETS

We begin by quoting some definitions and summarizing some results concerning Tchebychev sets, or as we shall abbreviate them, T-sets. See also [9].

Let * be a total order relation on a set K. If we use the symbol < we mean the usual ordering used in the real number system. Thus, if $K \subset R$, we may always replace * by < in what follows (although the converse may not be valid). By $\Xi_r = \{\xi_i\}_{i=0}^r$ we mean a set of linearly independent real valued functions defined on K. To avoid trivialities, whenever we define sets of functions (e.g. $\textcircled{H}_r = \{\theta_i\}_{i=0}^r$ etc.), we will always assume their linear independence.

DEFINITION 2.2.1 Let K and Ξ_r defined on K be given. Then:

(a) Ξ_r is a T-set on K if and only if every $\xi \in sp\Xi_r$, $\xi \neq 0$, has at most r distinct points in K at which it is zero.

(b) Ξ_r is a complete T-set (CT-set) on K if $\Xi_s, s=0,\ldots,r$, are all T-sets on K.

(c) Ξ_r is a weak T-set (WT-set) on K if every $\xi \in sp\Xi_r$, $\xi \neq 0$, has at most r sign changes in K. ///

We note that if K is a point set, Ξ_r being a T-set is

equivalent to Ξ_r, being a unisolvent set (in the sense of unisolvence as defined in [2]; this is a somewhat restricted definition, not least in that it depends upon linear algebraic arguments, see [117]). However, we are primarily concerned (cf. (A3), (A8)) with the case where K is a connected set, particularly an interval of the real line, and ξ_i are all continuous functions on K. In this context we have an alternative definition of T-sets available (see [9]). Let

$$(2.2.1) \qquad U_r \equiv U(t_0,\ldots,t_r) = \det \begin{bmatrix} \xi_0(t_0),\ldots,\xi_0(t_r) \\ . \\ . \\ . \\ \xi_r(t_0),\ldots,\xi_r(t_r) \end{bmatrix},$$

$$t_i \varepsilon K, \quad i=0,\ldots,r.$$

THEOREM 2.2.1 Let K and Ξ_r defined on K be given. Assume K is an interval of the real line, and $\xi_i \varepsilon C^0(K)$, $i=0,\ldots,r$. If, for all choices of distinct points

$$t_0<t_1<\ldots<t_r, \qquad t_i \varepsilon K,$$

(a) U_r is non zero, then Ξ_r is a T-set on K.

(b) U_r is of one sign, then Ξ_r is a WT-set on K.

(c) U_s, $s=0,\ldots,r$, is non zero, then Ξ_r is a CT-set on K.

The converse also holds. ///

In a more general setting the extension of Theorem 2.2.1 (a)

leads to the concept of an oriented T-set (OT-set) which is more restrictive than Definition 2.2.1 (a). For completeness we include the definition of an OT-set, see [104]:

DEFINITION 2.2.1 (d) Ξ_r is an OT-set provided U_r, as defined by (2.2.1) is of one sign and non-zero for all choices of $t_0 * t_1 * , \ldots , * t_r$, $t_i \epsilon K$, $i=0,\ldots,r$. (As in the rest of Definition 2.2.1, Ξ_r is defined on K_1 which has the total ordering relation * defined on it).

In the setting of Theorem 2.2.1, OT-sets and T-sets are equivalent. Parts (b) and (c) of Theorem 2.2.1 may also be extended to more general settings, see [35].

Observe that when considering a CT-set the ordering of the functions ξ_i may well be important. For example, the set $\{x^0, x^2, x^4, x^1, x^3\}$ defined on, say (-1,1), is not a CT-set, whereas $\{x^i\}_{i=0}^4$ is.

We now present some results concerning the relationship between T-sets and sets of linearly independent functions. We find from [32]

LEMMA 2.2.2 Let K and Ξ_r defined on K be given. Then there exist distinct points $t_0,\ldots,t_r$, in K, such that

$$U_r \neq 0.$$

PROOF: We proceed by induction on r.

The case r=0 is trivial.

Fix $t_0,\ldots,t_{r-1}$ and let t_r (=t say) vary. Now

$U_r = M_0(t_0,\ldots,t_{r-1})\xi_0(t)+\ldots+M_r(t_0,\ldots,t_{r-1})\xi_r(t)$

where M_i is the signed minor associated with $\xi_i(t)$ in the expansion of the determinant U_r.

If $U_r \equiv 0$, this implies that $\{\xi_i\}_{i=0}^{r}$ are linearly dependent unless, in particular, $M_r(t_0,\ldots,t_{r-1})=0$ for all choices of distinct points $t_0,\ldots,t_{r-1}$ in K.

This however, contradicts the inductive hypothesis. ///

We note that, if we so desire, we may immediately assume

$$t_0 * t_1 * \ldots t_r$$

in Lemma 2.2.2.

LEMMA 2.2.3 Let K and Ξ_r defined on K be given. Let $t_0,\ldots,t_{r-1}$ be distinct points such that $U_{r-1}\neq 0$. Then there exists $t_r \varepsilon K$ distinct from $t_0,\ldots,t_{r-1}$ such that $U_r \neq 0$.

PROOF: Assume the converse. Arbitrarily fix $t_r \varepsilon K$. M_r in Lemma 2.2.2 is assumed non-zero. Thus there exist $\alpha_0,\ldots,\alpha_r$, not all zero, such that

$$\xi_r(t_i) = \sum_{j=0}^{r-1} \alpha_j \xi_j(t_i),\ i=0,\ldots,r-1.$$

If, for arbitrary t_r, $\alpha_0,\ldots,\alpha_{r-1}$ are independent of t_r, we contradict the assumption of linear independence. If α_i depend on t_r then, since $U_r \neq 0$, the result is demonstrated. ///

The above Lemma has a useful constructive aspect.

We make the following comments about the above results. First, we have an alternative means of characterizing a set, Ξ_r, of linearly independent functions, i.e. the requirement that there exist $t_0 * t_i * \ldots * t_r$, $t_i \varepsilon K$, such that $U_r \neq 0$. Second, the above results include, as a particular case, multi-dimensional regions obtained by taking a Cartesian product of spaces. Thus we have a useful supplementary result to the well-

known theorem of Tchakaloff, see [100,101], stating that we can always find an interpolatory Q.F. with positive coefficients on a given region. Third, the requirement that Ξ_r is a T-set means that for arbitrary (distinct) $t_i \varepsilon K$ we have $U_r \neq 0$, whereas if Ξ_r are merely linearly independent all we are guaranteed is the existence of at least one set of points $t_o,\ldots,t_r$ such that $U_r \neq 0$.

We would expect that only under exceptional circumstances would the points $t_o,\ldots,t_r$ be unique.

We now give some results investigating the relationship between various types of T-set. Recalling that K is a totally ordered set, we may represent it when it is an open set by an "interval",

$$(2.2.2) \qquad (\lambda,\mu) \equiv \{t:\lambda * t \text{ and } t * \mu, t \neq \lambda, t \neq \mu\}$$

and when it is a closed set by an "interval"

$$(2.2.3) \qquad [\lambda,\mu] \equiv (\lambda,\mu) \cup (\inf K) \cup (\sup K).$$

The interior of a set K is the "interval" (λ,μ) and the closure of K, denoted $\overline{K}$, is given by (2.2.3).

THEOREM 2.2.4 Let Ξ_r be a WT-set of continuous functions defined on a closed real interval (i.e. let $K=[\lambda,\mu]$). Assume $1 \varepsilon \Xi_r$. Then one and only one of the following may occur:
(a) Ξ_r is a T-set.

(b) there exists a non-trivial $\xi \varepsilon sp\Xi_r$ with $\xi \equiv 0$ on a sub-interval of K.

PROOF: See [102, 109]. ///

Following [106] we have

DEFINITION 2.2.2 Let Ξ_r be a T-set defined on a set K. A function ξ is called adjoint to Ξ_r if the set of functions $\Xi_r \cup \{\xi\}$ is also a T-set on K. The set of functions $\Xi_r \cup \{\xi\}$ is called an adjoined T-set. ///

DEFINITION 2.2.3 If we replace "T-set" with "OT-set" in Def. 2.2.2 we prefix adjoint with "strongly". If we replace "T-set" with "WT-set" in Def. 2.2.2, we prefix adjoint with "weakly". ///

Before continuing we recall the fact that if K is a real interval and the functions ξ_i, ξ are continuous, then an OT set is equivalent to a T-set. We also have the following inclusion relations:

$$(2.2.4) \qquad \begin{cases} \text{OT-set} \subset \text{T-set} \subset \text{WT-set}, \\ \text{CT-set} \subset \text{T-set}. \end{cases}$$

DEFINITION 2.2.4 Let Ξ_r be a WT-set on K. The set of functions weakly adjoint to Ξ_r is called the convex cone of Ξ_r and

is denoted by $C\{\Xi_r\}$.

LEMMA 2.2.5 Let Ξ_{r-1} be a T-set of continuous functions defined on K. Let the interior of K be an open interval. Let $\xi_r \varepsilon C\{\Xi_{r-1}\}$. If $U_r=0$ for some choices of points $t_o * t_1 * \ldots * t_r$, $t_i \varepsilon K$, then there exists a ξ-polynomial of degree r, which is equivalently zero on $[t_o,t_r]$.

PROOF: See [103]. ///

This result complements, though under more restrictive conditions, Theorem 2.2.4.

THEOREM 2.2.6 Let Ξ_r be a finite dimensional WT-set of continuous functions defined on a real interval $K=[\lambda,\mu]$. Assume $1 \varepsilon \Xi_r$. Assume there exists a $\alpha>0$ such that if $\xi \varepsilon \Xi_r$, and $\xi \not\equiv 0$ on $[\lambda_1,\mu_1] \subset [\lambda,\mu]$, then $\mu_1-\lambda_1 \geq \alpha$. Then there exist knots, $\lambda=t_o<t_1<\ldots<t_s=\mu$, such that $\Xi_r|[t_j,t_{j+1}]$ is a T-set $j=0,\ldots,s-1$.

PROOF: See [102]. ///

As a partial converse, generalizing the result that polynomial splines are WT-sets, we have:

THEOREM 2.2.7 Let $\lambda=t_o<t_1<\ldots<t_s=\mu$ be points in the real interval $[\lambda,\mu]$. For $j=1,\ldots,s$, let Ξ_{n_j} be T-sets of dimension n_j+1 on the intervals $[t_{j-1},t_j]$. Assume $1 \varepsilon \Xi_{n_j}, j=1,\ldots,s$. Let

$$\Xi = \{\xi \in C^{o}[\lambda,\mu]:\xi|[t_{j-1},t_j] \in \Xi_{n_j}, j=1,\ldots,s\}.$$

Then Ξ is a WT-set on $[\lambda,\mu]$ with dimension $(\sum_{j=1}^{s} n_j)+1$.

PROOF: See [102]. ///

Turning our attention to CT-sets we have the very important results:

THEOREM 2.2.8 Let K be an open set. Then the span of every OT-set on K contains a CT-set thereon.

PROOF: See [35, 104]. ///

THEOREM 2.2.9 Let Ξ_r be a set of continuous functions defined on the real interval $\overline{K}=[\lambda,\mu]$. Assume Ξ_r is a CT-set on (λ,μ). Then Ξ_r is a CT-set on $\overline{K}$ if and only if $\xi_o>0$ on $\overline{K}$.

PROOF: See [103]. ///

Theorem 2.2.9 does not immediately extend to the more general situation discussed above (see [103]). Theorem 2.2.8 is not constructive in nature, being concerned with existence, and it appears that an algorithm for constructing a CT-set from the span of a T-set remains to be found. It is clear that we should not necessarily expect any CT-set so constructed to be unique even up to multiplicative constants since, for example, both the following are CT-sets on (0,1):

(2.2.5) $$\xi_o = x^3, \quad \xi_1 = x^2, \quad \xi_2 = x .$$

(2.2.6) $$\xi_o = x , \quad \xi_1 = x^2, \quad \xi_2 = x^3.$$

We may combine the above two theorems as follows:

LEMMA 2.2.10 Let Ξ_r be a set of continuous functions defined on the real interval $[\lambda,\mu]$. Assume Ξ_r is a T-set on (λ,μ). There exists a CT-set, Ξ_r^*, within $sp\Xi_r$, such that

$$\lim_{t\to\lambda^+} \xi_o^* \neq 0 \quad \text{and} \quad \lim_{t\to\mu^-} \xi_o^* \neq 0$$

if and only if Ξ_r^* is a CT-set on $[\lambda,\mu]$. ///

See also [106].

The condition in Lemma 2.2.10 is only trivial if $r<2$. This is demonstrated by various examples of T-sets on closed intervals that do not contain CT-sets in their span. See [104, 107].

The question of whether to include the end parts of a real interval when we are considering I.P.Q.F. using χ_1 functionals, depends upon whether we choose to allow these end points to be used as knots (i.e. whether we restrict ourselves to so-called "open" Q.F. or not).

The requirement that Ξ_r be a T-set on a set K is quite restrictive as the following theorem shows:

THEOREM 2.2.11 Let K be a compact Hausdorff-space and Ξ_r an

(r+1) dimensional T-set of continuous functions defined on K, r≥1. Then K is homeomorphic to a topological sub-space of the unit circle.

PROOF: See [104, 108]. ///

We note that, in particular, the closure of every bounded subspace in IR^d, d≥1, is compact, and that IR^d is a Hausdorff space. See [105].

Nevertheless, on sets on which there exist non-trivial T-sets (i.e. having dimension greater than 1) we have

THEOREM 2.2.12 Let K be an open set such that if x,y∈K, there exists z, x*z*y. Let Ξ_r be an (r+1) dimensional OT-set on K, r≥1. There is a function ξ strongly adjoint to Ξ_r.

PROOF: See [106]. ///

Note that the above result does not require that the set K be a connected set. Of more immediate interest we have:

THEOREM 2.2.13 Let K be a real interval. Let Ξ_r be an (r+1) dimensional T-set of continuous functions defined on Ξ_r. Then there exists a continuous function ξ adjoint to Ξ on K.

PROOF: See [106, 110]. ///

The function ξ in Theorem 2.2.13 is clearly not unique and if some mechanism for obtaining the function(s) were available, we observe that we could use the above theorem recursively to obtain infinite dimensional bases for the function space $C^o(K)$.

From what we have said about T-sets of various types it is apparent that, when K is a real open interval there is no restriction in assuming a T-set is a CT-set. Also, if K is not an open interval (i.e. we consider "closed" Q.F.), it is likely, though not certain, that Theorem 2.2.9 or Lemma 2.2.10 will be applicable. Note the importance of endpoints of intervals when considering closed intervals.

This last comment leads us naturally to the question of whether, when given a T-set Ξ_r on an interval K, there are sets of points κ such that Ξ_r is a T-set on $K \cup \kappa$. This question may be rephrased in another way: namely, given a set of linearly independent functions on the interval K, under what conditions can we find a set κ, $\kappa \subset K$, such that Ξ_r is a T-set on κ. Thus we are led to an investigation of what might loosely be termed "almost" T-sets.

DEFINITION 2.2.5 Let Ξ_r be defined on a set K. Let $\textcircled{H}_r = \{\theta_i\}_{i=0}^{r}$ be defined on $K \cup \kappa$, $\kappa \neq \phi$, $\kappa \cap K = \phi$. Assume that $\theta_i | K \equiv \xi_i$. We say that $\textcircled{H}_r$ is an extension of Ξ_r on $K \cup \kappa$. Conversely we say that Ξ_r is a restriction of $\textcircled{H}_r$ on K.

If Ξ_r is a T-set, we call it a T-restriction of $\textcircled{H}_r$. If $\textcircled{H}_r$ is also a T-set, we call it a T-extension of Ξ_r. ///

Taking a lead from [111] we have

DEFINITION 2.2.6 We call κ in the above definition an expansion set of K. If κ is an expansion set and $\theta_i = 0$, $i = 0, \ldots, r$ on

κ we call κ a null set. Conversely, we call K a contraction set of $K \cup \kappa$. ///

Note that Theorem 2.2.9 and Lemma 2.2.10 address themselves to the question of existence of T-extensions of Ξ_r. We have the following interesting result:

THEOREM 2.2.14 Let Ξ_r be an (r+1) dimensional T-set on a set K. Assume $r \geq 1$. Then the following statements are equivalent:

(1) Ξ_r contains an r dimensional T-set on K.

(2) There is a point $t \notin K$ and a T-extension of Ξ_r on $K \cup \{t\}$.

PROOF: See [104, 107]. ///

For some further results concerning T-extensions, including the fact that they may not exist, see [107]. For some further results concerning particular choices of adjoint functions and T-sets on intervals whose endpoints may be a null set, see [111].

We can easily derive some further results, based on the fact that (see [9])

$$\det \left(\exp \left(x_i \, y_j \right)_{i,j=0}^{n} \right) \tag{2.2.7}$$

is strictly positive whenever $-\infty < x_0 < \ldots < x_n < \infty$ and $-\infty < y_0 < \ldots < y_n < \infty$.

LEMMA 2.2.15 Let ξ be a positive strictly increasing real valued function defined on a set K. Let $\alpha_i \in R$ $i=0,\ldots,r \geq 0$, be pairwise distinct. Then Ξ_r, with

$$\xi_i \equiv (\xi)^{\alpha_i} ,$$

is a T-set on K. ///

COROLLARY 2.2.16 If one of $\alpha_o,\ldots,\alpha_r$ is zero we may replace "positive" by "non-negative" in Lemma 2.2.15. ///

We now introduce the following:

DEFINITION 2.2.7 Let ξ be a real function defined on a set K. Let t_1 and t_2 be two distinct points in K. If

$$\xi(t_1) \neq \xi(t_2)$$

we say that t_1 and t_2 are ξ-distinct points, otherwise we say t_1 and t_2 are ξ-equivalent points. ///

Thus, we obtain:

COROLLARY 2.2.17 Let ξ be a real function defined on a set K. Let κ be a subset of K, containing at least r+1 points on which ξ is strictly positive and any two points in κ are ξ-distinct. Let $\alpha_o,\ldots,\alpha_r$ be real, $r\geq 0$. Then Ξ_r, with

$$\xi_i \equiv (\xi)^{\alpha_i} ,$$

is a T-set on κ. ///

We observe that Corollary 2.2.16 has an analogue following

Corollary 2.2.17. Further, it is clear that if we denote by S_t the set of points in an interval K which are ξ-equivalent to t, that if $t \varepsilon K$, we may replace t with any other point in S_t. Thus it is unlikely (impossible if ξ is continuous but not positive and strictly increasing) that κ in Corollary 2.2.17 will be unique. Clearly, in the above results any set κ on which ξ is a monotonic function is a T-restriction of K (see also §2.8).

If we impose restrictions on the choice of the exponents $\alpha_0,\ldots,\alpha_r$ in the above three results it is fairly obvious that we can derive other T-sets on certain subsets κ of K.

Thus, for certain sets of functions Ξ_r, we are able to obtain information about some of their T-restrictions.

An investigation of what we have termed T-restrictions would help us obtain information about sets of functions which are "intermediate"; not being T-sets, but not "nearly" linearly dependent in the sense that sets of points with $U_r \neq 0$ are a rare occurrence. In this way we might be able to obtain a spectrum of results ranging from, for example, Lemma 2.2.2, through to results dealing with T-sets on the set K.

If we pause to consider the definition of T-sets (either Definition 2.2.1 or Theorem 2.2.1) we observe that we make crucial use of a linear functional - namely point evaluation of a function. It seems obvious to ask the question of whether, by analogy, we may use other linear functionals instead. Another generalization of T-sets (essentially non-linear) is given in [117], with the so-called "Property Z". See also [112].

Often, once the form of a linear functional has been given it will depend on one, or perhaps more, parameters. For example, a linear functional $L\varepsilon\chi_1$, depends on one parameter, namely the point, say t, at which we evaluate a function ξ, and it could be written as

$$L(\xi) \equiv L(t;\xi). \tag{2.2.8}$$

More generally, if a linear functional L depends upon several parameters, $\rho_1,\ldots,\rho_s$, and acts on a function ξ we could write

$$L(\xi) \equiv L(\rho_1,\ldots,\rho_s;\xi) \equiv L(\underline{\rho};\xi). \tag{2.2.9}$$

For example, if L were an integration operator, acting on an integrable function over an interval (λ,μ) we could write it as $L(\lambda,\mu;\xi)$. We would expect the parameters $\rho_1,\ldots,\rho_s$ to depend upon either the function ξ (in some general way) or the set on which ξ is defined, and so it is not necessary that $\rho_1,\ldots,\rho_s$ are real. We shall, however, assume (cf. (A1)).

(A9) All linear functionals are real valued and, if P is a connected set, the linear functionals are continuous functions of $\underline{\rho}\varepsilon P$.

We might proceed by using one of the following:

DEFINITION 2.2.8 Let Ξ_r be defined on a set K. Let $L(\underline{\rho};\cdot)\varepsilon(\Xi_r)^*$

for all choices $\underline{\rho}\varepsilon P$, P being some set containing at least r distinct elements.

Then we say that Ξ_r is a generalized T-set, GT-set, with respect to L (and, implicitly, P) if for every $\xi\varepsilon sp\Xi_r$ there are at most r distinct choices $\underline{\rho}\varepsilon P$ such that

$$L(\underline{\rho};\xi) = 0. \qquad ///$$

DEFINITION 2.2.9 Let Ξ_r be a set of functions defined on a set K. Let $L_i(\underline{\rho}_i;\cdot)\varepsilon(\Xi_r)^*$, $\underline{\rho}_i\varepsilon P_i$, $i=0,\ldots,r$. Define

$$(2.2.10) \qquad V_r \equiv V(\underline{\rho}_o,\ldots,\underline{\rho}_r) \equiv \det \begin{bmatrix} L_o(\underline{\rho}_o;\xi_o),\ldots,L_r(\underline{\rho}_r;\xi_o) \\ \cdot \\ \cdot \\ \cdot \\ L_o(\underline{\rho}_o;\xi_r),\ldots,L_r(\underline{\rho}_r;\xi_r) \end{bmatrix}.$$

Then, if for all choices $\underline{\rho}_i\varepsilon P_i$, $i=0,\ldots,r$, we have $V_r\neq 0$ we call Ξ_r a generalized T-set, GT-set, with respect to $\{L_i\}_{i=0}^r$ (and implicitly $\{P_i\}_{i=0}^r$). ///

From now on when we talk of a collection of functionals $\{L_i\}_{i=0}^r$ we shall always assume that the functionals are distinct.

Clearly we may define complete GT-sets, (WT-sets) denoted CGT-sets (WGT-sets), analogously to the way CT-sets (WT-sets) are defined.

An alternative way of looking at Definition 2.2.9 is (cf. Theorem 1.1.1),

THEOREM 2.2.18 A set of functions Ξ_r is a GT-set w.r.t $\{L_i\}_{i=0}^{r}$ if and only if the linear functionals $L_o,\ldots L_r$ are linearly independent for all possible choices of $\underline{\rho}_i \varepsilon P_i$, $i=0,\ldots,r$. ///

It is clear that Definition 2.2.9 includes Definition 2.2.8 as a special case (i.e. $L=L_i$, $i=0,\ldots,r$). Thus there is no inconsistency between the two definitions. If the sets P_i in Definition 2.2.9 all consist of single points (P in Definition 2.2.8 of r+1 distinct points) we say that the GT-set is degenerate, since we are merely investigating the linear independence of a given set of r+1 linear functionals over a given r+1 dimensional function space. We note that we have not required the sets P_i to be independent of each other, since we recall the situation when we define T-sets, where the definition of P_i may be viewed as depending on $P_{i-1},\ldots,P_o$. As was the case with CT-sets we note that the ordering of the functions ξ_i may be important when considering CGT-sets. In addition, we observe that it may also come about that the ordering of the linear functionals L_i is important (if they are not the same). It is also clear that the above definitions are vacuous unless Ξ_r spans a function space of dimension r+1.

We give a simple example of a GT-set (see [115]);

$$(2.2.11)\begin{cases} \xi_i = x^i, \quad i=0,\ldots,r, \quad \text{on } [0,1], \\ L_i(\xi) = \left.\dfrac{d^{(i)}\xi}{dx^i}\right|_{x=t}, \quad t\varepsilon P_i = [0,1], \quad i=0,\ldots,r. \end{cases}$$

The following are immediate generalizations of the properties of T-sets.

LEMMA 2.2.19 Let the elements of Ξ_r be continuous functions defined on an interval, which are a GT-set with respect to $\{L_i\}_{i=0}^r$. Then V_r is of one sign for all possible choices of $\underline{\rho}_i$, $i=0,\ldots,r$.

PROOF: Immediate consequence of continuity of ξ_i, $i=0,\ldots,r$. ///

Generalizing Lemma 2.2.2 we have

LEMMA 2.2.20 Let Ξ_r be defined on a set K. Let $\{L_i\}_{i=0}^r \varepsilon(\Xi_r)^*$. Assume that $L_i(\underline{\rho}_i;\xi)\neq 0$, for some $\underline{\rho}_i \varepsilon P_i$, for every non-trivial $\xi \varepsilon sp\Xi_r$, $i=0,\ldots,r$. Then there exist $\underline{\rho}_o,\ldots\underline{\rho}_r$ such that $V_r \neq 0$.

PROOF: By induction on r (cf. Lemma 2.2.2). The case r=0 is obvious.

Fix $\underline{\rho}_o,\ldots\underline{\rho}_{r-1}$ and let $\underline{\rho}_r=\underline{\rho}$ vary. Now

$V_r = M_o(\rho_o,\ldots,\rho_{r-1})\ L_o(\underline{\rho};\xi_o)+\ldots+M_r(\underline{\rho}_o\ldots,\rho_{r-1})\ L_r(\underline{\rho};\xi)$,

where M_i is the signed minor associated with $L_r(\underline{\rho};\xi)$ in the expansion of the determinant V_r. Thus,

$$V_r = L_r(\underline{\rho};\textstyle\sum_{j=0}^r M_j(\underline{\rho}_o,\ldots,\underline{\rho}_{r-1})\xi_j).$$

If $V_r \equiv 0$, it follows that, in particular,

$$M_r(\rho_o,\ldots,\rho_{r-1}) = 0$$

for all possible choices of $\rho_o,\ldots,\rho_{r-1}$. This contradicts the inductive hypothesis. ///

Note that the ordering of the L_i's in Lemma 2.2.20 may be important. The next result is immediate from definitions:

LEMMA 2.2.21 Let Ξ_r be a GT-set with respect to $\{L_i\}_{i=0}^{r}$. Then any ξ-polynomial, ξ, is specified uniquely by giving the numbers

$$L_i(\rho_i;\xi)$$

for given $\rho_i \varepsilon P_i$, $i=0,\ldots,r$. In particular, any non-trivial ξ-polynomial cannot have

$$L_i(\rho_i;\xi) = 0, \quad i=0,\ldots,r$$

for any choice of $\underline{\rho}_i \varepsilon P_i$, $i=0,\ldots,r$.

Conversely, if a set of functions Ξ_r is such that for any $\xi \varepsilon sp\Xi_r, \xi$ has at most r isolated zeros with respect to L_i, $i=0,\ldots,r$, then it is a GT-set with respect to $\{L_i\}_{i=0}^{r}$.

///

DEFINITION 2.2.10 Let ξ be a function and L defined in $(\xi)^*$. We say the ξ has a zero $\underline{\rho}$, with respect to L, if

$$L(\underline{\rho};\xi) = 0.$$

If P is a connected set in a space with metric d(.,.), we say $\underline{\rho} \varepsilon P$ is an isolated zero of ξ if there exists $\Omega>0$ such that

$$L(\underline{\rho}+\underline{\nu};\xi) \neq 0,$$

for all $\underline{\nu}$, $0<d(0,\underline{\nu})<\Omega$.

In addition, we say that an isolated zero $\underline{\rho}$ is non-nodal, if for all $\underline{\nu}_1, \underline{\nu}_2$, such that $0<d(0,\underline{\nu}_1), d(0,\underline{\nu}_2)<\Omega$,

$$L(\underline{\rho}+\underline{\nu}_1;\xi)\ L(\underline{\rho}+\underline{\nu}_2;\xi)>0\ .$$

Otherwise we call an isolated zero $\underline{\rho}$ nodal. ///

When no ambiguity results, we do not use the phrase "with respect to L" in the above definition. The following is immediate from the assumed continuity of linear functionals, in $(\Xi_r)^*$, with respect to both parameters and functions operated on.

LEMMA 2.2.22 Let $\{L_i\}_{i=0}^{r}$ be elements of $(\Xi_r)^*$, $r\geq 0$. Assume that P_i are connected non-trivial sets, and that there is a non-trivial $\theta_i \varepsilon sp\Xi_r$ such that there is $\underline{\rho}_i \varepsilon P_i$ and $\underline{\rho}_i$ is a nodal zero of θ_i, $i=0,\ldots,r$. Then all components of the vector $\underline{\rho}_i$ depend on only one independent parameter, $i=0,\ldots,r$. ///

The above Lemma is not necessarily valid in the case where P_i is disconnected; for example, a collection of discrete points. The importance of the zero properties of sets of elementary functionals is seen in the remainder of this chapter.

As a consequence of the above Lemma, when we talk of linear functionals depending on parameters ρ_i we have:

(A10) Elementary functionals S_i and T_j will each only depend on one parameter.

Given (A10) we note that if, in definition 2.2.10, ρ is an isolated zero on the boundary of P, then it is automatically a non-nodal zero.

Generalizing results in [9] slightly, we obtain

THEOREM 2.2.23 Let Ξ_r be a GT-set of continuous functions, defined on a connected interval, with respect to $\{L_i\}_{i=0}^{r}$. Assume P_i, $i=0,\ldots,r$, is connected. Then, counting every nodal zero once and every non-nodal zero twice, (unless it is on the boundary of P_i, when it counts only once), no non-trivial $\xi \varepsilon sp \Xi_r$ may have more than r zeros.

PROOF: Analogous to the proof of Theorem I.4.2 of [9]. ///

THEOREM 2.2.24 Let Ξ_r be a GT-set of continuous functions defined on a connected interval K with respect to $\{L_i\}_{i=0}^{r}$. Assume the interior of P_i is a connected interval, $i=0,\ldots,r$. Assume we want non-nodal zeros at $\rho_o,\ldots,\rho_k$ and nodal zeros at $\rho_{2(k+1)},\ldots,\rho_{2(k+1)+\ell}$, $s=2(k+1)+\ell<r$. Assume that

(a) $L_i \equiv L_{k+1+i}$ $\quad i=0,\ldots,k$,

(b) $L_{r-1} \equiv L_{r-2},\ldots,L_{r-s+2} \equiv L_{r-s+1}$, $r-s-1$ even.

Then there is a ξ-polynomial with the desired zeros and no others.

PROOF: Analogous to Theorem I.5.1 of [9] on considering the ξ-polynomial given in (2.2.12) and Lemma 2.2.25.

$$(2.2.12)\quad \det \begin{bmatrix} L_o(\rho_o;\xi_o), \ldots, L_o(\rho_o;\xi_r) \\ L_o(\rho_o+\varepsilon_o;\xi_o), \ldots, L_o(\rho_o+\varepsilon_o;\xi_r) \\ \cdot \\ \cdot \\ L_k(\rho_k;\xi_o), \ldots, L_k(\rho_k;\xi_r) \\ L_k(\rho_k+\varepsilon_k;\xi_o), \ldots, L_k(\rho_k+\varepsilon_k;\xi_r) \\ L_{2k+3}(\rho_{2k+3};\xi_o), \ldots, L_{2k+3}(\rho_{2k+3};\xi_r) \\ \cdot \\ \cdot \\ L_s(\rho_s;\xi_o), \ldots, L_s(\rho_s;\xi_r) \\ L_{s+1}(\rho_{s+1};\xi_o), \ldots, L_{s+1}(\rho_{s+1};\xi_r) \\ L_{s+1}(\rho_{s+1}+\varepsilon_{s+1};\xi_o), \ldots, L_{s+1}(\rho_{s+1}+\varepsilon_{s+1};\xi_r) \\ \cdot \\ \cdot \\ \cdot \\ L_{r-2}(\rho_{r-2};\xi_o), \ldots, L_{r-2}(\rho_{r-2};\xi_r) \\ L_{r-2}(\rho_{r-2}+\varepsilon_{r-2};\xi_o), \ldots, L_{r-2}(\rho_{r-2}+\varepsilon_{r-2};\xi_r) \\ \xi_o(t), \ldots, \xi_r(t) \end{bmatrix} \equiv \xi(t) \quad ///$$

LEMMA 2.2.25 Let Ξ_r and K be as in Theorem 2.2.24. Then $L_r(\rho_r;\xi)\neq 0$, $\forall \rho_r \varepsilon P_r$, where ξ defined by (2.2.13). Hence ξ is non-trivial.

PROOF: We consider

$$(2.2.13)\quad \det \begin{bmatrix} L_o(\rho_o;\xi_o), \ldots, L_o(\rho_o;\xi_r) \\ \cdot \\ \cdot \\ \cdot \\ L_{r-1}(\rho_{r-1};\xi_o), \ldots, L_{r-1}(\rho_{r-1};\xi_r) \\ \xi_o(t), \ldots, \xi_r(t) \end{bmatrix} \equiv \xi(t).$$

Consider $L_r(\rho_r;\xi)$ for some $\rho_r \varepsilon P_r$. This can be evaluated by replacing the last row of the determinant in (2.2.13) by $(L_r(\rho_r;\xi_o),\ldots,L_r(\rho_r;\xi_r))$. This must always be non-zero or we contradict Ξ_r being a GT-set with respect to $\{L_i\}_{i=0}^{r}$. ///

Theorem 2.2.24 is not as restrictive as it first appears since we can always renumber the functionals L_i if we wish. If we are unable to fulfil requirement (b) of Theorem (2.2.24) for those functionals not occurring in pairs it is possible that they will each provide ξ with an extra zero (clearly we may choose this zero, if we so desire, to be "placed" somewhere convenient).

It is also apparent that the definitions of extensions, adjoints, expansion sets, and equivalence points can be extended, by analogy, to GT-sets.

Much more is known about the properties and interrelationships of the various types of T-set than we have quoted here. We shall quote further results if and when the need for them arises.

§2.3 IMMEDIATE RESULTS

In this section we return to a direct investigation of I.P.Q.F. We already have some results concerning existence and non-existence of I.P.Q.F., i.e. Theorems 1.3.1 - 1.3.4 and 2.1.1 With the aid of information in the previous section, we can obtain some further results.

We note (cf. Lemma 2.2.20) that if $\{S_i\}_{i=0}^{m}$ obey (A7), as earlier assumed, then if S_i depends on a parameter set P_i, there must exist $\rho_i \varepsilon P_i$ such that $S_i(\rho_i;\xi)\neq 0$ for every $\xi \varepsilon sp\Phi^m$ (analogously for $\{T_j\}_{j=0}^{n}$).

THEOREM 2.3.1 Assume that $\{S_i\}_{i=0}^{m}$, $\{T_j\}_{j=0}^{n}$ fulfil (A7) on Φ^m and Ψ^n respectively. Then there exists at least one choice of parameters such that an interpolatory I.P.Q.F., exact on $\Phi^m \times \Psi^n$, exists.

PROOF: Follows from Lemma 2.2.20. ///

The problem of actually finding the parameters mentioned above does not appear to be a trivial one. The next result demonstrates the potency of GT-sets in the context of I.P.Q.F.:

THEOREM 2.3.2 Let Φ^m be a GT-set with respect to $\{S_i\}_{i=0}^{m}$. Let $\{T_j\}_{j=0}^{n}$ be such that there is a choice of parameters for which $\det(C)\neq 0$ (recall Lemma 2.2.20 and see (1.3.8)). Then there is an interpolatory I.P.Q.F. exact on $\Phi^m \times \Psi^n$ for arbitrary choice of parameters of S_i.

If, in addition, Ψ^n is a GT-set with respect to $\{T_j\}_{j=0}^{n}$ then the choice of parameters of $\{T_j\}_{j=0}^{n}$ is arbitrary

as well. ///

We can now extend Theorem 2.1.1 as follows:

THEOREM 2.3.3 Let Φ^{γ}, $\gamma>m$, and Ψ^{δ}, $\delta\geq 0$, be T-sets on R. There is no choice of χ_1 functionals (knots) such that there is an I.P.Q.F. exact on $\Phi^{\gamma}\times\Psi^{\delta}$ when $\gamma+\delta>2m+1$.

PROOF: Analogous to Theorem 2.1.1, using Theorem 2.2.24 in the case where all the linear functionals $L\varepsilon\chi_1$. ///

This is clearly a symmetric result in Φ^{γ} and Ψ^{δ}, and we consequently have:

COROLLARY 2.3.4 Let Φ^{γ}, $\gamma>m$, and Ψ^{δ}, $\delta>n$, be T-sets on R. There exists no I.P.Q.F. exact on $\Phi^{\gamma}\times\Psi^{\delta}$.

PROOF: Without loss of generality assume $m>n$ (if $m=n$, see Theorem 2.3.3). Apply the symmetric analogue of Theorem 2.3.3. ///

These results have what might be termed a "Gaussian" flavour in that, analogous to R.Q.F. exact on polynomials, even setting $\delta=0$ we are unable to find an I.P.Q.F. exact on Φ^{2m+2} (this latter result is also in [9], see §2.6).

We note that in view of what was said in the previous section concerning T-restrictions and so on it is possible that the above two results hold even if Φ^{γ} and Ψ^{δ} are T-sets only on a subset κ of R.

Extensions of Theorem 2.3.3 to other types of linear functional are not obvious since a knowledge of the zeros (and hence

sign changes) of a continuous function with respect to some linear functional L does not immediately give us information about when, and if, the function itself changes sign.

In view of Theorem 2.3.3, there is no loss in making the following assumption.

(A11) $$m \geq n.$$

§2.4 GENERAL SOLUTION

In view of Corollary 2.3.4 and what we already know of interpolatory I.P.Q.F., we shall be primarily interested in investigating I.P.Q.F. exact on $\Phi^{\gamma} \times \Psi^{\delta}$ under the assumption

(A12) $$\gamma > m, \quad \delta \leq n.$$

We recall that, (see §1.3) if the required I.P.Q.F. exists, the system of linear equations given below must have a solution:

(2.4.1) $$D\,\underline{a} = \underline{A}\ ,$$

where

(2.4.2) $$D = C \otimes B,$$

(2.4.3) $$C = \begin{bmatrix} T_0(\psi_0), \ldots, T_n(\psi_0) \\ . \\ . \\ . \\ T_0(\psi_\delta), \ldots, T_n(\psi_\delta) \end{bmatrix}\ ,$$

$$(2.4.4) \qquad B = \begin{bmatrix} S_o(\phi_o),\ldots,S_m(\phi_o) \\ \cdot \\ \cdot \\ \cdot \\ S_o(\phi_\gamma),\ldots,S_m(\phi_\gamma) \end{bmatrix},$$

and

$$(2.4.5) \qquad \underline{\Lambda}^T = (\underline{\Lambda}_o^T,\ldots,\underline{\Lambda}_\delta^T),$$

$$(2.4.6) \qquad \underline{\Lambda}_j^T = (I(\phi_o;\psi_j),\ldots,I(\phi_\gamma;\psi_j),\ j=0,\ldots,\delta.$$

$\underline{a}$ is given by (1.3.14).

To guarantee that a solution to (2.4.1) exists, the augmented matrix $(D|\underline{\Lambda})$ must have the same rank as D. From (A6), (A7) and (A10), analogous to Theorem 1.3.1, the following is obvious.

LEMMA 2.4.1 The rank of D in (2.4.1) is the product of the ranks of B and C, namely $(m+1)\times(\delta+1)$. ///

The assumptions (A6) and (A7) allow us to assume that the first (m+1) rows of B are linearly independent, and thus there must exist constants $\alpha_{or},\ldots,\alpha_{mr}$, $r=1,\ldots,\gamma-m=k$, not all zero, such that

$$(2.4.7) \qquad \alpha_{or}S_i(\phi_o)+\ldots+\alpha_{mr}S_i(\phi_m) = S_i(\phi_{m+r})$$

$$r = 1,\ldots,k, \quad i=0,\ldots,m.$$

For the rank of $(D|\underline{\Lambda})$ to be the same as that of D, we must have

$$(2.4.8) \qquad I(\alpha_{or}\phi_o+\ldots+\alpha_{mr}\phi_m;\psi_j) = I(\phi_{m+r};\psi_j)$$

$$r = 1,\ldots,k, \quad j=0,\ldots,\delta,$$

or, on rewriting,

$$(2.4.9) \qquad I(h_r;\psi_j) = 0$$

$$r = 1,\ldots,k, \quad j=0,\ldots,\delta,$$

where

$$(2.4.10) \qquad h_r = \alpha_{or}\phi_o+\ldots+\alpha_{mr}\phi_m-\phi_{m+r}$$

$$r = 1,\ldots,k.$$

The requirements (2.4.7) and (2.4.9) impose more conditions than there are parameters (α_{ir}) available in general. Whether these conditions can be satisfied depends on the choice of Φ^γ,Ψ^δ and the elementary functionals S_i.

We note that the choice of the linear functionals $\{T_j\}_{j=0}^n$ plays no role in (2.4.7) - (2.4.10) and thus is unable to affect the question of existence of I.P.Q.F. exact on $\Phi^\gamma\times\Psi^\delta$, $\gamma>m, \delta\leq n$. Recalling Lemma 2.2.22 we see that we have at most (m+1) non-linear parameters to choose, namely those upon which the functionals S_i depend. Thus, since it may be that $m+1<k(\delta+1)$ we might expect it to be an impossible task satisfying (2.4.9) in all possible cases. We shall consider three special cases;

I. $\gamma = 2m+1$, i.e. $k=m+1$; and $\delta=0$.

II. $\gamma = m+1$, i.e. $k=1$; and $\delta=n$.

III. $\{\phi_i\}_{i=0}^{\gamma}$ and $\{\psi_j\}_{j=0}^{\delta}$ having properties akin to (2.1.2)-(2.1.6).

In cases I and II we have as many parameters available as there are conditions to be satisfied, and we do not have to place strong conditions on ϕ_i and ψ_j, in direct contrast with what occurs in case III. Also, we note that we may satisfy $(m+1)\geq k(\delta+1)$ be either fixing n and k and choosing m sufficiently large, or fixing m and k and choosing n (thus δ) sufficiently small.

Before continuing we recall that once we have chosen parameters to satisfy (2.4.9) we may find the matrix A (the vector $\underline{a}$) by solving the linear equations requiring the I.P.Q.F. to be exact on $\Phi^m \times \Psi^n$. That is, we are able to separate, and consider independently, the linear and non-linear aspects of the problem. We consider the solution of this set of linear equations in §2.9.

We introduce the following definitions:

<u>DEFINITION 2.4.1</u> Let Ξ_r, $r\geq s\geq 1$, and $\textcircled{H}_s=\{\theta_i\}_{i=0}^{s}$ be two sets of functions defined on a set K. We say that Ξ_r is weakly orthogonal (W-orthogonal) with respect to a bi-linear functional L, to $\textcircled{H}_s$, if for each j=1,...,s, we have

$$L(\xi_j;\theta_i) = 0, \quad i=0,\ldots,j-1. \tag{2.4.11}$$

If, in addition we have

(2.4.12) $$L(\xi_j;\theta_j) = \gamma_j \neq 0 \qquad j=0,\ldots,s,$$

We say that Ξ_r is orthogonal, with respect to L, to $\textcircled{H}_s$. If $\gamma_j=1$, $j=0,\ldots,s$, we say orthonormal instead of orthogonal. ///

Obviously, if $\xi_i \equiv \theta_i$, $i=0,\ldots,s=r$, then the above definition reduces to the usual one of orthogonality when L is interpreted as the integration operator. We note that the ordering of the functions ξ_j and θ_i may be important, and that the concepts in Definition 2.4.1 are related to that of biorthogonality.

DEFINITION 2.4.2 A function ξ is said to be W-orthogonal to a set of real valued linearly independent functions $\textcircled{H}_s$, defined on a set K, with respect to a bi-linear functional L if

(2.4.13) $$L(\xi;\theta_i) = 0, \quad i=0,\ldots,s. \qquad ///$$

The phrase "with respect to a linear functional L" is deleted unless required in the interests of clarity.

We can summarize as follows:

THEOREM 2.4.2 Given $\{S_i\}_{i=0}^{m}$, a necessary and sufficient condition that there exists an I.P.Q.F. exact on $\Phi^{\gamma}\times\Psi^{\delta}$, $\gamma>m, \delta\leq n$, is that the ϕ-polynomials h_r, $r=1,\ldots\gamma-m$, in (2.4.10) are each W-orthogonal to Ψ^{δ}. ///

This Theorem allows us to test, when given a specific set

of functionals $\{S_i\}_{i=0}^{m}$, whether or not the required I.P.Q.F. exists. However, the questions of more interest are those of discovering for what choice of parameters (if any) $\rho_i \varepsilon P_i$ from non-degenerate intervals P_i, and what choices of elementary functionals S_i, the desired I.P.Q.F. do exist. Once we allow the parameters ρ_i to be chosen from non-degenerate intervals the importance of GT-sets in guaranteeing that $\{S_i\}_{i=0}^{m}$ remain linearly independent in $(\Phi^m)^*$ is clear.

THEOREM 2.4.3 Let S_i depend upon $\rho_i \varepsilon P_i$, $i=0,\ldots,m$. For an I.P.Q.F. exact on $\Phi^\gamma \times \Psi^\delta$, $\gamma > m$, $\delta \leq n$, to exist it is sufficient that the following conditions are fulfilled:

Let h_r be defined similarly to (2.4.10).

(2.4.14) There exist α_{ir}, $i=0,\ldots,m$, $r=1,\ldots,\gamma-m$, such that h_r is W-orthogonal to Ψ^δ with respect to I.

(2.4.15) There exist $\mathring{\rho}_i \varepsilon P_i$ such that for each $i=0,\ldots,m$

$$S_i(\mathring{\rho}_i; h_r) = 0, \quad r=1,\ldots,\gamma-m.$$

(2.4.16) $(S_i(\mathring{\rho}_i; \ ))_{i=0}^{m}$ are linearly independent in $(\Phi^m)^*$.

(2.4.17) The matrix C has maximal rank. ///

We observe that (2.4.14) in the above Theorem depends upon I and the choice of the set of functions Ψ^δ, whereas (2.4.15) and (2.4.16) depend upon the choice of elementary functionals $\{S_i\}_{i=0}^{m}$. The choice of the elementary functionals $\{T_j\}_{j=0}^{n}$ plays no role. The importance of GT-sets is obvious from (2.4.16).

It is also important to note that the above Theorem does not require all the assumptions (A1) - (A11).

Before moving on to the next section we note that, while the above results were obtained under the assumption that the interior of R is (-1,1), provided the weight function w is such that all the necessary moments exist and are finite, there is no difficulty in extending these results to semi-infinite or infinite intervals of IR. This comment remains equally applicable to the results of future sections, although we shall not explicitly state it at each point where it might be applicable. See also [23].

§2.5 ORTHOGONALITY

Prompted by Theorem 2.4.3 we investigate when, given two sets of linearly independent functions Ξ_r and $\textcircled{H}_r$, they can be made W-orthogonal or orthogonal. There is no real point in considering $\textcircled{H}_s$ with $s<r$, since this would immediately restrict us to Ξ_s as well. All the results obtained in this section are derived using elementary linear algebra.

DEFINITION 2.5.1 If Ξ_r is W-orthogonal (orthogonal) to $\textcircled{H}_r$, with respect to a bi-linear functional L, and $\textcircled{H}_r$ is W-orthogonal (orthogonal) to Ξ_r with respect to L, we say that Ξ_r and $\textcircled{H}_r$ are mutually W-orthogonal (orthogonal) with respect to L. ///

Set

$$m_{ij} = L(\xi_i;\theta_j) \tag{2.5.1}$$

and then define the so-called moment matrices, with respect to L, by

$$M_{s,t} = \{m_{ij}\}_{i=0,\ j=0}^{s\ \ t} = \begin{bmatrix} m_{oo},\ldots,m_{ot} \\ \cdot \\ \cdot \\ m_{so},\ldots,m_{st} \end{bmatrix}, \tag{2.5.2}$$

$$M_s \equiv M_{s,s}, \tag{2.5.3}$$

for any choice $s,t \leq r$.

Throughout this section we shall employ the following

(A13) The moment matrix M_r has no row or column which is entirely composed of zeros.

If a moment matrix does not obey A13 we shall say that it is degenerate, and the following results will not necessarily be applicable. Perhaps surprisingly, degenerate moment matrices may occur, even when one of Ξ_r and $\textcircled{H}_r$ are T-sets.

Example 2.5.1. Let $\xi_i = t^i$, $i=0,1,\ldots,$

$$\theta_j = \sin(j\pi(t+1)),\ j=1,2,\ldots\ .$$

Since θ_j is an odd function for any value of j, it is clear that as int(R)=(-1,1), the (2k+1)st row k=0,1,... of the moment matrix will be equivalently zero, when the functional L is the integration operator with w≡1. ///

We seek $\hat{\Xi}_r = \{\hat{\xi}_i\}_{i=0}^r$, such that $\hat{\Xi}_r$ spans the same space as Ξ_r, and $\hat{\Xi}_r$ is W-orthogonal (orthogonal) to $\textcircled{H}_r$. $\hat{\textcircled{H}}_r \equiv \{\hat{\theta}_i\}_{i=0}^r$ is defined analogously. Initially we investigate W-orthogonality, and then continue and examine orthogonality. Let

$$\hat{\xi}_i = \sum_{\ell=0}^{i} \alpha_{i,\ell} \xi_\ell, \quad i=1,\ldots,r. \tag{2.5.4}$$

If, for arbitrary but fixed i, $\hat{\xi}_i$ is W-orthogonal to $\{\theta_j\}_{j=0}^{i-1}$, the following linear equations are satisfied.

$$M_{i,i-1}^T \underline{\alpha}_i = \underline{0} \tag{2.5.5}$$

where $\underline{0}$ is an i-vector of zeros and

$$\underline{\alpha}_i^T = (\alpha_{i,0},\ldots,\alpha_{ii}).$$

Clearly, there are

$$i - \text{rank}\,(M_{i,i-1}) \geq 1 \tag{2.5.6}$$

degrees of freedom in the vector $\underline{\alpha}_i$. To guarantee that $\hat{\Xi}_r$ spans the same space as Ξ_r, it is sufficient to choose $\alpha_{ii}=1$. This means we require

(2.5.7) $\qquad \text{rank } (M_{i,i-1}) = \text{rank } (M_{i-1}).$

However, by considering first $M_{r,r-1}$, then $M_{r-1,r-2}$ and so on, it is apparent that reordering the functions $\{\xi_i\}_{i=0}^{r}$ only, condition (2.5.7) can be guaranteed. Thus we may state the following result:

LEMMA 2.5.1 Given Ξ_r and $\textcircled{H}_r$, possibly after reordering the set Ξ_r, there exist functions $\hat{\Xi}_r$, each with leading coefficient 1, such that $\hat{\Xi}_r$ is W-orthogonal to $\textcircled{H}_r$.

In addition, $sp\Xi_r = sp\hat{\Xi}_r$. ///

COROLLARY 2.5.2 If rank $(M_{r,r-1})$ = rank (M_{r-1}) then we may write

$$\hat{\xi}_r = \xi_r + \sum_{\ell=0}^{r-1} \alpha_{r,\ell}\xi_\ell \ . \qquad ///$$

The above argument may not then be used to obtain $\hat{\textcircled{H}}_r$ mutually orthogonal to $\hat{\Xi}_r$, since in the second reordering in M_r (of columns instead of rows) may introduce the vector $(m_{or},\ldots,m_{rr})$ into the new $M_{r,r-1}$ and thus possibly refute the above analysis. However, we may still obtain the result:

THEOREM 2.5.3 Given Ξ_r and $\textcircled{H}_r$, possibly after reordering the sets Ξ_r and $\textcircled{H}_r$, there exist functions $\hat{\Xi}_r$ and $\hat{\textcircled{H}}_r$, each function with leading coefficient 1, such that $\hat{\Xi}_r$ and $\hat{\textcircled{H}}_r$ are mutually W-orthogonal with respect to L.

In addition $sp\Xi_r = sp\hat{\Xi}_r$ and $sp\textcircled{H}_r = sp\hat{\textcircled{H}}_r$.

PROOF: Given the matrix M_r it is clearly possible using only row operations to reduce M_r to an upper triangular matrix.

Now using column operations M_r may be reduced to a matrix $\tilde{M}_r=(\tilde{m}_{ij})_{i,j=0}^{r}$ such that for some $k\in\{1,\ldots,r\}$ we have

$$\tilde{m}_{ii}\neq 0, \quad i=0,\ldots,k-1,$$

$$\tilde{m}_{ij}=0, \quad i=k,\ldots,r, \quad j=i,\ldots,r.$$

Reordering ξ_i (θ_i) according to the row (columns) interchanges made (i.e. if row i in M_r ends up being row j in $\tilde{M}_r$ the reordered ξ_i becomes $\tilde{\xi}_j$) ensures that, for the new sets $\tilde{\Xi}_r$ and $\tilde{(H)}_r$, the moment matrices $\tilde{M}_{ij}$ obey

$$\text{rank}\,(\tilde{M}_{i,i-1}) = \text{rank}\,(\tilde{M}_{i-1}) = \text{rank}\,(\tilde{M}_{i-1,i}). \qquad ///$$

COROLLARY 2.5.4 If $\text{rank}\,(M_{r,r-1}) = \text{rank}\,(M_{r-1}) = \text{rank}\,(M_{r-1,r})$ then we can write

$$\hat{\xi}_r = \xi_r + \sum_{\ell=0}^{r-1}\alpha_{r,\ell}\xi_\ell ,$$

$$\hat{\theta}_r = \theta_r + \sum_{\ell=0}^{r-1}\beta_{r,\ell}\theta_\ell . \qquad ///$$

COROLLARY 2.5.5 The functions $\hat{\Xi}_r$ are unique, up to a multiplicative factor, if and only if M_{r-1} has full rank. ///

We now consider orthogonality. This means that the requirement (2.5.7) is replaced by the requirement

(2.5.8) $$\text{rank}\,(M_i) = \text{rank}\,(M_i|\underline{e}_i) \quad i=0,\ldots,r$$

where $\underline{e}_i$ is an (i+1) - vector in which all elements are zero

except the (i+1)-st, which is unity. We have

THEOREM 2.5.6 If the matrices M_i, $i=0,\ldots,r$, all have full rank then mutually orthogonal polynomials $\hat{\Xi}_r$ and $\hat{(H)}_r$ exist, are unique, and span the same spaces as Ξ_r and $(H)_r$ respectively.

PROOF: We exhibit the polynomials $\hat{\xi}_i$ and $\hat{\theta}_j$.

(2.5.9) Set $\xi_o^* = \xi_o$, and then $\hat{\xi}_o = \xi_o^* | (L(\xi_o^*;\theta_o^*))^{\frac{1}{2}}$,

$\theta_o^* = \theta_o$, and then $\hat{\theta}_o = \theta_o^* | (L(\xi_o^*;\theta_o^*))^{\frac{1}{2}}$.

(2.5.10) Set $\xi_1^* = \xi_1 - L(\xi_1;\hat{\theta}_o)\hat{\xi}_o$, then $\hat{\xi}_1 = \xi_1^* | (L(\xi_1^*;\theta_1^*))^{\frac{1}{2}}$,

$\theta_1^* = \theta_1 - L(\hat{\xi}_o;\theta_1)\hat{\theta}_o$, then $\hat{\theta}_1 = \theta_1^* | (L(\xi_1^*;\theta_1^*))^{\frac{1}{2}}$,

and thus, for each $\ell=1,\ldots,r$;

(2.5.11) Set $\xi_\ell^* = \xi_\ell - \sum_{k=0}^{\ell-1} L(\xi_\ell;\hat{\theta}_k)\hat{\xi}_k$, then $\hat{\xi}_\ell = \xi_\ell^* | (L(\xi_\ell^*;\theta_\ell^*))^{\frac{1}{2}}$,

$\theta_\ell^* = \theta_\ell - \sum_{k=0}^{\ell-1} L(\hat{\xi}_k;\theta_\ell)\hat{\theta}_k$, then $\hat{\theta}_\ell = \theta_\ell^* | (L(\xi_\ell^*;\theta_\ell^*))^{\frac{1}{2}}$.

By construction we have

(2.5.12) $$L(\hat{\xi}_\ell;\hat{\theta}_k) = \delta_{k\ell},$$

where $\delta_{k\ell}$ is the usual Kronecker delta function.

We observe that it is not possible for $L(\xi_\ell^*;\theta_\ell^*)$ to be zero, since if it were zero, we have found a non-trivial ξ-polynomial, namely ξ_ℓ^*, such that $L(\xi_\ell^*;\theta_k)$ is zero,

$k=0,\ldots,\ell$, and this contradicts the assumption that M_ℓ has full rank. Clearly $\hat{\Xi}_r$ and $\hat{\textcircled{H}}_r$ span the same spaces as Ξ_r and $\textcircled{H}_r$ as in $\hat{\xi}_\ell(\hat{\theta}_\ell)$ we have $\xi_\ell(\theta_\ell)$ with coefficient unity. Last, we have to show uniqueness (up to multiplicative constants at least). Assume that $\hat{\Xi}_r$ and $\hat{\Xi}_r^{(1)}$ are two sets of functions orthogonal to $\hat{\textcircled{H}}_r$. Let

$$\hat{\xi}_i^{(2)} = \hat{\xi}_i - \alpha_i \hat{\xi}_i^{(1)}, \quad \alpha_i = \frac{L(\hat{\xi}_i;\theta_i)}{L(\hat{\xi}_i^{(1)};\theta_i)}, \quad i=1,\ldots,r.$$

Clearly

$$L(\hat{\xi}_i^{(2)};\hat{\theta}_j) = 0, \quad j=0,\ldots,i.$$

If $\hat{\xi}_i^{(2)}$ is non-trivial, this is a contradiction as pointed out above. Hence $\hat{\Xi}_r$ is unique. Similarly $\hat{\textcircled{H}}_r$. ///

The procedure (2.5.9) - (2.5.12) is a generalization of the standard Gram-Schmidt orthogonalization process, and is clearly constructive in nature.

It has been noticed before that the ordering of the functions ξ_i and θ_j plays an important role in the determination of W-orthogonal and orthogonal sets of functions. The following is obvious:

LEMMA 2.5.7 Given a matrix of full rank, possibly after reordering rows (or columns) we may assume that all the leading minors have full rank. ///

We now note

LEMMA 2.5.8 If M_r does not have full rank, then no matter how we reorder ξ_i and θ_j it is not possible to find $\hat{\Xi}_r$ orthogonal to $\textcircled{H}_r$ with respect to L.

PROOF: The requirement for orthogonality is

$$\text{rank } (M_i) = \text{rank } (M_i | \underline{e}_i), \quad i=0,\ldots,r.$$

Thus the (i+1)st row of M_i is linearly independent of the the other i rows.

Letting i=r, then r-1, and so on, this implies that the rank of M_r must maximal. ///

This we can summarize with the next theorem:

THEOREM 2.5.9 Given Ξ_r and $\textcircled{H}_r$ it is possible to find $\hat{\Xi}_r$ orthogonal to $\textcircled{H}_r$ (mutually orthogonal to $\hat{\textcircled{H}}_r$) with respect to a bilinear functional L if and only if the matrix M_r has full rank.

In addition, possibly after reordering the functions ξ_r and $\textcircled{H}_r$, we find that $\hat{\xi}_i$ and $\hat{\theta}_j$ are unique up to a multiplicative constant, and

$$\hat{\xi}_i = \xi_i + \sum_{\ell=0}^{i-1} \alpha_{i,\ell} \xi_\ell, \quad i=0,\ldots,r, \tag{2.5.13}$$

$$\hat{\theta}_j = \theta_j + \sum_{\ell=0}^{j-1} \beta_{j,\ell} \theta_\ell, \quad j=0,\ldots,r.$$

thus $\hat{\Xi}_r$ and $\hat{\textcircled{H}}_r$ span the same spaces as Ξ_r and $\textcircled{H}_r$ respectively. ///

There are several comments which seem to be worth making at this stage. First, if M_r does not have full rank then, possibly after reordering, some submatrix $M_\ell, \ell \leq m$, will have, and despite the fact that we cannot find $\hat{\Xi}_r$ mutually orthogonal to $\hat{\textcircled{H}}_r$ we can find $\hat{\Xi}_r$ mutually W-orthogonal to $\hat{\textcircled{H}}_r$, with the additional property that $\hat{\Xi}_\ell$ is mutually orthogonal to $\hat{\textcircled{H}}_\ell$. Second, we cannot use the generalized Gram-Schmidt process (2.5.9) - (2.5.11) to obtain W-orthogonal polynomials, since we cannot guarantee $L(\xi_\ell^*;\theta_\ell^*) \neq 0$, $\ell = 0,\ldots,r$. Third, it is not immediately clear what is the best way to assign any arbitrary constants when obtaining W-orthogonal polynomials. Fourth, when considering two sets of functions, Ξ_r and $\textcircled{H}_r$ the re-ordering processes which may be required are of no real consequence in the context of I.P.Q.F. since the sets of functions still span the same spaces. Fifth, if we consider Ξ_s and $\textcircled{H}_r, s>r$, there may be several reorderings of Ξ_s possible which enable us to derive W-orthogonal or orthogonal polynomials $\hat{\Xi}_r$. Sixth, the results of this section so far depend merely on the rank of the matrix M_r, and not on the structure of the bi-linear operator L. In particular, when dealing with the integration operator I, we do not necessarily require (A3). Finally, given a functional L, and examining orthogonality the question of what conditions we need to impose on Ξ_r and $\textcircled{H}_r$ in order to guarantee that M_r has maximal rank is an interesting one, which would appear to be closely connected with Moment Theory.

Using a slightly different approach it is possible, see, see [9,27], to obtain the following result, now however requiring

(A3);

LEMMA 2.5.10 Let Ξ_r and $\textcircled{H}_r$ be CT-sets on R. Then there exist $\hat{\Xi}_r$ and $\hat{\textcircled{H}}_r$ mutually orthogonal with respect to I, on R.

PROOF: We prove the result by exhibiting $\hat{\xi}_i, \hat{\theta}_i$, $i=0,\ldots,r$. Let M_ℓ be the ℓ-th moment matrix with respect to I. For each choice of $\ell=1,\ldots,r$, define ξ^*_ℓ and θ^*_ℓ by the following determinental equations;

$$(2.5.14) \qquad \xi^*_\ell = \det \begin{bmatrix} & & \xi_0(x) \\ & & \cdot \\ & M_{\ell-1} & \cdot \\ & & \cdot \\ & & \xi_{\ell-1}(x) \\ m_{\ell,0},\ldots,m_{\ell,\ell-1} & & \xi_\ell(x) \end{bmatrix}$$

$$(2.5.15) \qquad \theta^*_\ell = \det \begin{bmatrix} & & m_{0,\ell} \\ & & \cdot \\ & M_{\ell-1} & \cdot \\ & & \cdot \\ & & m_{\ell-1,\ell} \\ \theta_0(x),\ldots,\theta_{\ell-1}(x) & & \theta_\ell(x) \end{bmatrix}$$

and then set

$$(2.5.16) \qquad \hat{\xi}_\ell = \xi^*_\ell \Big| \det M_{\ell-1};\ \hat{\theta}_\ell = \theta^*_\ell \Big| \det M_{\ell-1}.$$

(take $\hat{\xi}_0 = \xi_0$, $\hat{\theta}_0 = \theta_0$).

It remains to show that det $M_{\ell-1} \neq 0$, $\ell=0,\ldots,r$, and we prove this in the following result. ///

LEMMA 2.5.11 Let Ξ_r and $\textcircled{H}_r$ be T-sets on R. Then the moment matrix M_r, with respect to I, has non-zero determinant.

PROOF: Assume the contrary.

Thus there exist $\alpha_0,\ldots,\alpha_r$, $\sum_{i=0}^{r}|\alpha_i|>0$, such that

$$\sum_{i=0}^{r}\alpha_i m_{ij} = 0, \quad j=0,\ldots,r.$$

This implies

$$I(\sum_{i=0}^{r}\alpha_i\xi_i;\theta_j) = 0, \quad j=0,\ldots,r.$$

However, using Theorem 2.2.24, we know it is possible to construct a non-trivial θ^* which has zero at the same points as $\sum_{i=0}^{r}\alpha_i\xi_i$ in (-1,1). This leads to the contradiction, since it is implied that

$$I(\sum_{i=0}^{r}\alpha_i\xi_i;\theta^*) = 0. \qquad ///$$

COROLLARY 2.5.12 The functions $\hat{\Xi}_r$ and $\hat{\textcircled{H}}_r$ of Lemma 2.5.10 are unique (up to multiplicative constants) and span the same spaces as Ξ_r and $\textcircled{H}_r$.

PROOF: By construction and Lemma 2.5.11 coupled with Theorem 2.5.9. ///

It is clear that the process used in the proof of Lemma 2.5.10 can be used for arbitrary Ξ_r, $\textcircled{H}_r$ and L provided we can

guarantee the matrices M_i $i=1,\ldots,r$ are non-singular. Obviously Corollary 2.5.12 remains valid. Again summarizing, we have

THEOREM 2.5.13 Let Ξ_r and $\textcircled{H}_r$ be T-sets on R. Then there exist $\hat{\Xi}_r$ and $\hat{\textcircled{H}}_r$, mutually orthogonal with respect to I, on R. ///

Any extension of these results depends upon the relation of linear functionals we might use in defining GT-sets Ξ_r and $\textcircled{H}_r$, and the bi-linear functional L with respect to which we are considering orthogonality.

THEOREM 2.5.14 Let Ξ_r, $\textcircled{H}_r$ be given. Assume (A3). Let $\xi_i \equiv \theta_i$, $i=0,\ldots,r$. Then M_r has full rank.

PROOF: Assume the contrary. Then there exist $\alpha_0,\ldots,\alpha_r$, $\sum_{i=0}^{r}|\alpha_i|>0$, such that

$$I(\xi_j;\textstyle\sum_{i=0}^{r}\alpha_i\xi_i) = 0, \quad j=0,\ldots,r.$$

However, this implies that

$$I(\textstyle\sum_{i=0}^{r}\alpha_i\xi_i;\sum_{i=0}^{r}\alpha_i\xi_i) = 0 ,$$

and this is a contradiction. ///

We now return to our original motivation for the section, Theorem 2.4.3, and direct consideration of I.P.Q.F. Noting (A11) in particular, we have,

THEOREM 2.5.15 Let $\Xi_{\delta+1}^{(r)}$ consist of $\delta+1$ of the functions $\{\phi_i\}_{i=0}^{m}$, and set

$$\xi_{\delta+1}^{(r)} = \phi_{m+r}, \quad r=1,\ldots,\gamma-m.$$

Then condition (2.4.14) is satisfied provided,

(2.5.13) $$\text{rank } (M_{\delta+1,\delta}^{(r)}) = \text{rank } (M_{\delta}^{(r)}), \quad r=1,\ldots,\gamma-m,$$

where $M_{\delta+1}^{(r)}$ are the moment matrices with respect to I using the sets of functions $\Xi_{\delta+1}^{(r)}$ and $\{\psi_j\}_{j=0}^{\delta}$. ///

The condition (2.5.13) does not appear to be a very restrictive one, and can of course always be guaranteed if the rank of $M_{\delta}^{(r)}$ is maximal.

§2.6 I.P.Q.F. AND T-SETS: CASE I

Having investigated orthogonality and thus the first condition of Theorem 2.4.3 we now begin our investigation of the remaining conditions of the aforementioned theorem with some results concerning case I. We are not, as before, able to reduce the problem to being essentially linear. The following results come basically from [9], Chapter 2. Let

(2.6.1) $$M_r \equiv \{\underline{c}=(c_0,\ldots,c_r)\in R^{r+1}, c_i=\int_R \xi_i(t)d\sigma(t), i=0,\ldots,r\}$$

where $\{\xi_i\}_{i=0}^{r}$ is a T-set on $[-1,1]$ and σ traverses the set of all non-decreasing right continuous functions of bounded variation. It can be shown that M_r is a closed convex cone.

We have the following result:

THEOREM 2.6.1 Assume we are given a measure σ° and a T-set $\{\xi_i\}_{i=0}^{r}$. Let $\underline{c}^\circ$ be the vector determined by the measure σ°.

(a) If $r=2s+1$ and $\underline{c}^\circ \varepsilon \mathrm{Int} M_r$ then there is exactly one Q.F., using $s+1$ distinct knots, $x_i \varepsilon (-1,1)$, $i=0,\ldots,s$, such that

$$I(\xi) - \sum_{i=0}^{s} a_i \xi(x_i) = 0, \qquad \forall \xi \varepsilon \Xi_r.$$

In addition, $a_i > 0$, $i=0,\ldots,s$.

(b) If $r=2s$ and $\underline{c}^\circ \varepsilon \mathrm{Int} M_r$ then there is exactly one Q.F. exact on Ξ_r, using $s+1$ distinct knots x_i, $i=0,\ldots,s$, $x_o=-1$, and $x_i \varepsilon (-1,1)$, $i=1,\ldots,s$. In addition, the coefficients a_i are all positive.

The analogous result holds setting $x_s=1$.

(c) If $\underline{c}^\circ$ is on the boundary of M_r then there is a unique Q.F., with positive coefficients, using at most $s+1$ distinct knots in $[-1,1]$, whether $r=2s+1$ or $2s$. ///

Parts (a) and (b) of the above Theorem are, perhaps not unexpectedly, analogous to the well known Gaussian and Radau Q.F., which deal with the case $\xi_i = x^i$, $i=0,\ldots,r$. Part (a) is, in view of Theorem 2.3.3, the best result we could obtain. If the Q.F. in part (c) above uses less than $s+1$ knots, it would seem that some type of non-linear degeneracy, dependent on the choice of functions ξ_i and the measure σ°, occurs (the degeneracy

cannot be linear since $\{\xi_i\}_{i=0}^{r}$ is a T-set). However, we have

LEMMA 2.6.2 Assume σ° of Theorem 2.6.1 is differentiable, i.e. we have $d\sigma^\circ(t) = w(t)\,dt$. Then the Q.F. of part (c) in the above Theorem uses exactly s+1 knots.

PROOF: Else Theorem 2.3.3 is contradicted. ///

The situation in the above Lemma is that in which we are interested. As commented earlier, we may regard the Q.F. obtained when σ is differentiable either as a weighted R.Q.F., or an I.P.Q.F. exact on $\Xi^r \times \Psi^\delta$, $\delta=0$, $\psi_o = w(t)$.

We recall that when dealing with R.Q.F. exact on polynomials, it is sometimes possible to prescribe $t \le s+1$ knots x_i, $i=0,\ldots,t$, and obtain R.Q.F. exact on $\{x^i\}_{i=0}^{i=2s-t}$ (namely when the moment matrix with respect to $\prod_{i=0}^{t}(x-x_i)\,w(t)$ is non-singular. See also Theorem 2.8.8). When dealing with arbitrary T-sets we can also obtain some results of this type.

THEOREM 2.6.3 Assume σ°, $\underline{c}^\circ$ and $\{\xi_i\}_{i=0}^{r}$ are as in Theorem 2.6.1. Choose $\hat{t} \in [-1,1]$.

(a) Let $r=2s+1$, and $\underline{c}^\circ \in \mathrm{Int} M_r$. Assume $\hat{t}$ is not one of the knots in Theorem 2.6.1 (a).

(i) If $\hat{t} \in (-1,1)$ there is a unique R.Q.F., of the form

$$\sum_{i=0}^{s+1} a_i \xi(x_i),$$

exact for $\xi \in \Xi^r$. One of these knots x_i is $\hat{t}$ and one, but not both, endpoints (-1 or 1) occur as

a knot. In addition, $a_i>0$, $x_i\varepsilon[-1,1]$, $i=0,\ldots,s+1$.

(ii) If $\hat{t}=1$ (or, analogously, -1) there is a unique $s+2$ point R.Q.F. also using -1 (1) as a knot exact on Ξ^r. In addition, $a_i>0$, $x_i\varepsilon[-1,1]$, $i=0,\ldots,s+1$, and $x_o=-1$, $x_{s+1}=1$.

(b) Let $r=2s$ and $\underline{c}^o\varepsilon \mathrm{Int}M_r$. Assume that $\hat{t}$ is not one of the knots of either R.Q.F. in Theorem 2.6.1 (b). (Thus $\hat{t}\varepsilon(-1,1)$). Then there exists a unique R.Q.F.,

$$\sum_{i=0}^{s} a_i \xi(x_i),$$

exact for $\xi\varepsilon\Xi^r$. One of the knots x_i is $\hat{t}$, and neither endpoint, -1 or 1, is a knot. In addition, $a_i>0$, $x_i\varepsilon(-1,1)$, $i=0,\ldots,s$. ///

We note that the R.Q.F. of part (a) (ii) above is the analogue of the well-known Lobotto Q.F. From Theorem 2.1 of Chapter 2 of [9] we have a criteria for determining whether $\underline{c}^o\varepsilon \mathrm{Int}M_r$ (see also Ch. 4, Cor. 1.1).

The results above do not have constructive proofs in [9] and thus are best regarded as existence (and uniqueness) theorems. Clearly, if we are able to determine the knots of the R.Q.F. in some way, the coefficients a_i may be determined by solving a set of linear moment equations involving s+1 or s+2 functions

chosen from $\{\xi_i\}_{i=0}^{r}$, such that the matrix B has full rank. These functions need not be $\{\phi_i\}_{i=0}^{t}$, t=s+1 or s+2, as, for the given knots $\{\xi_i\}_{i=0}^{t}$ need not be linearly independent on the given set of knots, however, since $\{\xi_i\}_{i=0}^{r}$ is a T-set, there must exist a set of t functions for which B is non-singular. In this general case, we can obtain a partial characterization of the required knots, as non-nodal zeros of polynomials constructed from $\{\xi_i\}_{i=0}^{r+1}$ (an adjoined T-set) in the form of Lemma 2.1 in Chapter 4, [9]. Of course, knowing a solution exists, we may, in the absence of more sophisticated methods, always solve the non-linear system of equations (cf. (1.3.7)),

$$(2.6.2)\qquad \begin{bmatrix} \xi_o(x_o),\ldots,\xi_o(x_t) \\ \cdot \\ \cdot \\ \cdot \\ \xi_r(x_o),\ldots,\xi_r(x_t) \end{bmatrix} \begin{bmatrix} a_o \\ \cdot \\ \cdot \\ a_t \end{bmatrix} = \begin{bmatrix} I\ (\xi_o) \\ \cdot \\ \cdot \\ \cdot \\ I\ (\xi_r) \end{bmatrix} .$$

We also note that extensions of Theorem 2.6.3 to the case of predetermining two or more knots without being able to arbitrarily choose their coefficients are not, as yet, available.

§2.7 I.P.Q.F. AND T-SETS: CASE II

As in the previous section the case II problem is non-linear. We recall that the following conditions are in force:

(2.7.1) $\gamma = m+1, \quad$ i.e. $k=1$

(2.7.2) $\delta = n$

With regard to (2.7.2) we note that there is little loss in generality in this requirement as opposed to $\delta \leq n$. If $\delta < n$ we find, see [24], that we can preassign $n-\delta$ elements in each row of the matrix A. There does not appear to be any gain in setting these elements to be other than zero, and consequently we may as well have (2.7.2).

We begin by imposing the additional conditions:

(2.7.3) $m = n,$

(2.7.4) $S_i \varepsilon \chi_1, \quad i=0,\ldots,m.$

Thus given Φ^{m+1} and Ψ^m we are faced with the problem of discovering whether knots $x_0,\ldots,x_m$ exist such that, as required by Theorem 2.4.3,

(2.7.5) $S_i(x_i;h_1) = h_1(x_i) = 0, \quad i=0,\ldots m.$

Then we must ensure that the matrix B has full rank. We begin an investigation of the properties of the zeros of h_1 with a generalization of a well-known result dealing with the zeros of orthogonal polynomials.

THEOREM 2.7.1 Let Ξ^{m+1} be defined on $[-1,1]$. Assume Ξ^{m+1} is weakly orthogonal to $\textcircled{H}^m$ with respect to I.

(a) If $\textcircled{H}^m$ is a CT-set on (-1,1) then ξ_i, i=1,...,m+1 has at least i distinct nodal zeros in (-1,1). If, in addition, Ξ^{m+1} is a CT-set in (-1,1) ([-1,1]) then ξ_i has exactly i distinct zeros, all nodal, in (-1,1) ([-1,1]).

(b) If $\textcircled{H}^m$ is a CT-set on [-1,1] then ξ_i, i=1,...,m+1 has at least i distinct nodal zeros in [-1,1]. (Here we count a zero at -1 or 1 as nodal).

If, in addition, Ξ^{m+1} is a CT-set in (-1,1) ([-1,1]) then ξ_i has at most (exactly) i distinct zeros, all nodal, in (-1,1) ([-1,1]).

PROOF: (a) ((b) is analogous). Assume the contrary. Arbitrarily fix $i\varepsilon\{1,2,\ldots,m+1\}$. Let $t_1,\ldots,t_r$ $r<i$ be all the nodal zeros of ξ_i in (-1,1). From Lemma 2.2.21 there exists a non-trivial θ-polynomial, θ, of degree r, with nodal zeros at $t_1,\ldots,t_r$ and no nodal zeros elsewhere in (-1,1). However, we now have

$$I(\xi_i;\theta)\neq 0,$$

contradicting the assumption of weak orthogonality. The results when Ξ^{m+1} is a CT-set follow immediately from the definition of CT-set. ///

Note the explicit dependence on (A3) in the above proof. We also have

COROLLARY 2.7.2 Replacing the phrase "CT-set" by "T-set" in Theorem 2.7.1, and merely requiring that ξ_{m+1} is W-orthogonal to $\textcircled{H}^m$, the results remain valid in the case i=m+1.

PROOF: As above, except Theorem 2.2.24 may be needed in place of Lemma 2.2.22 (if r≠m). ///

Recall that, in view of Theorem 2.2.8, the restriction of $\textcircled{H}^m$ or Ξ^{m+1} to being CT-sets on open intervals is not very restrictive. Also observe that if we only wish to use Corollary 2.7.2 we may reorder the function θ_j and ξ_i at will, since $\textcircled{H}^m$ and Ξ^{m+1} will remain T-sets regardless of the ordering of their constituent functions.

If the conditions on $\textcircled{H}^m$ in the above results are not met it is still possible that results remain valid, however each individual case would have to be investigated. The importance of $\textcircled{H}^m$ having the stated properties is seen from the following examples:

(2.7.6) Let $\xi_o = \theta_o = 1$, $\xi_1 = \theta_1 = x^2$, and $\xi_2 = x^3$.

Then $\hat{\xi}_o(x) = 1$,

$\hat{\xi}_1(x) = x^2 - 2/3$,

$\hat{\xi}_2(x) = x^3$

is weakly orthogonal to $\{\theta_j\}_{j=0}^{1}$ over R. However $\hat{\xi}_2$ has only one zero on the real line (and in R) (cf (A4)).

(2.7.7) Let $\xi_i = x^i$, $i=0,1,2$, $\theta_o(x)=1$ and $\theta_1(x)=-x^2$

Then $\hat{\xi}_o(x) = 1$,

$\hat{\xi}_1(x) = x^2-2/3$,

$\hat{\xi}_2(x) = x$,

is weakly orthogonal to $\{\theta_j\}_{j=0}^{1}$, over R.

However, again ξ_2 does not have the required number of zeros in $\mathbb{R}$, let alone in R.

Note that $\textcircled{H}_1$ is not a T-set in either (2.7.6) or (2.7.7) but in (2.7.6) Ξ_2 is a T-set, and in (2.7.7) Ξ_2 is a CT-set.

We now examine the conditions of Theorem 2.4.3 in the current context, and obtain:

THEOREM 2.7.3 Assume we are given Φ^{m+1} and Ψ^m. Assume also that Ψ^m is a T-set on R, and that Φ^{m+1} is W-orthogonal to Ψ^m. Then

(a) The knots x_i, $i=0,\ldots,m$, of an I.P.Q.F. exact on $\Phi^{m+1}\times\Psi^m$ may be any set of m+1 zeros of ϕ_{m+1} such that $\det(B)\neq 0$.

(b) If Φ^{m+1} is a T-set there is a unique choice of knots x_i, $i=0,\ldots,m$, (zeros of ϕ_{m+1}) available for an I.P.Q.F. exact on $\Phi^{m+1}\times\Psi^m$.

In both cases above, the choice of the knots $y_o,\ldots,y_n$ is arbitrary, so long as they are distinct and lie in R.

PROOF: That ϕ_{m+1} has sufficient zeros has been proved in Corollary 2.7.2, thus (2.4.15) is satisfied. Conditions (2.4.14) and (2.4.16) are implicit in the statement of the Theorem. ///

There are several things to note. First, the requirement that Φ^{m+1} is W-orthogonal to Ψ^m is trivial in light of Lemma 2.5.1. Second, in part (a) apart from there being possibly more than one set of roots of ϕ_{m+1} obeying the condition, it is possible that there may be more than one choice of ϕ_{m+1}. This occurs if the rank of M_m is not maximal or there is more than one choice of m+1 functions in Φ^{m+1} with rank M_m maximal. Third, if there is a choice of zeros of ϕ_{m+1} (or parameters to assign if rank M_m is not maximal) what is the best choice? Fourth, the importance of the role of the functions Ψ^m being a T-set (although bearing in mind our earlier comments) although we are interested in zeros of ϕ_{m+1} and the knots y_j are arbitrary. Fifth, this result is the best we could hope for in view of Theorem 2.3.3. Last, if we do not require Ψ^m to be a T-set it is possible that I.P.Q.F. exact on $\Phi^{m+1}\times\Psi^m$ still exist however we must show that ϕ_{m+1} has at least m+1 zeros, and then that $\det(B)\neq 0$. (There always exists at least one choice of knots y_j such that $\det(C)\neq 0$ by Lemma 2.2.2.)

We are able to "use" the freedom in the choice of the knots y_j as follows:

THEOREM 2.7.4 Assume we are given Φ^{m+1} and Ψ^{m+1}. Assume Φ^m and Ψ^m are T-sets on R. Assume that Φ^{m+1} and Ψ^{m+1} are mutually W-orthogonal.

Then there are I.P.Q.F. (unique if both Φ^{m+1} and Ψ^{m+1} are T-sets) exact on

$$\Phi^{m+1}\times\Psi^{m} \quad \text{and} \quad \Phi^{m}\times\Psi^{m+1}.$$

The knots x_i (y_j) are m+1 zeros of ϕ_{m+1} (ψ_{m+1}).

PROOF: Immediate from Theorems 2.5.9 and 2.7.3 ///

We now consider what happens if we drop (2.7.3), and allow in m>n, while still seeking I.P.Q.F. exact on $\Phi^{m+1}\times\Psi^{n}$. Proceeding as before, using the analogue of Corollary 2.7.2, clearly we can guarantee that ϕ_{m+1} has at least n+1 distinct nodal zeros. However, if we allow ϕ_{m+1} to be a linear combination of all the basis functions in Φ^{m+1}, there are m-n degrees of freedom (at least) in the determination of ϕ_{m+1}. Obviously we want to employ this freedom in ensuring that ϕ_{m+1} has at least m+1 zeros. If Ψ^{n} can have functions $\psi_{n+1},\ldots,\psi_{m}$ adjoined so that the set Ψ^{m} is a T-set (see § 2.2, Theorem 2.2.12 etc.) then we know that I.P.Q.F. exact on $\Phi^{m+1}\times\Psi^{n}$ exist, using the above results, although we now need to use m+1 knots y_j instead of n+1 as we might have hoped, since the I.P.Q.F. is in fact exact on $\Phi^{m+1}\times\Psi^{m}$ (m>n). However, we can also state the fairly obvious result.

THEOREM 2.7.5 Assume we are given Φ^{m+1} and Ψ^{n}, m>n. Assume that Ψ^{n} is a T-set on R.

Let the linear system

$$(2.7.8)\quad \begin{bmatrix} I(\phi_o;\psi_o), \ldots, I(\phi_{m+1};\psi_o) \\ I(\phi_o;\psi_1), \ldots, I(\phi_{m+1};\psi_1) \\ \cdot \\ \cdot \\ I(\phi_o;\psi_n), \ldots, I(\phi_{m+1};\psi_n) \\ \phi_o(x_o), \ldots, \phi_{m+1}(x_o) \\ \cdot \\ \cdot \\ \phi_o(x_{m-n-1}), \ldots \phi_{m+1}(x_{m-n-1}) \end{bmatrix} \begin{bmatrix} b_o \\ \cdot \\ \cdot \\ \cdot \\ b_{m+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

have a non-trivial solution, $\hat{\phi}_{m+1} = \sum_{i=0}^{m+1} b_i \phi_i$, then there exist I.P.Q.F. exact on $\Phi^{m+1} \times \Psi^n$, with prescribed knots $x_o, \ldots, x_{m-n-1}$, and n+1 knots y_j, if

(a) $\hat{\phi}_{m+1}$ has at least m+1 distinct zeros.

(b) If Φ^{m+1} is not a T-set, there are m+1 zeros of $\hat{\phi}_{m+1}$ such that detB≠0 (If Φ^{m+1} is a T-set there are at most m+1 zeros of $\hat{\phi}_{m+1}$, and if there are m+1, detB is guaranteed to be non zero).

The choice of knots y_j, j=0,...,n, is arbitrary so long as they are distinct and lie in R.

PROOF: Immediate from Theorem 2.4.3 (analogue of Theorem 2.7.3).

///

Note that the matrix in (2.7.8) is an (m+1) × (m+2) matrix and so there must always exist a solution to the system (2.7.8). As before, the most difficult condition is (a) above. It is perhaps surprising to discover that the I.P.Q.F. desired in

Theorem 2.7.5 may not exist, although an I.P.Q.F. exact on $\Phi^{m+1} \times \Psi^m$, where Ψ^m is the adjoined set mentioned in the preceding paragraph may exist. It would seem that this is due to our choice of preassigned knots $x_o,\ldots,x_{m-n-1}$ in the above Theorem (see §2.8). This is particularly striking when we require that Φ^m is a T-set. Comments analogous to those following Theorem 2.7.3 again apply.

Finally, we note that the analysis of Theorem 2.7.5 can be easily adapted to situations where, dealing with $\Phi^{m+1}{}_{\times}\Psi^m$ again, the moment matrix M_m does not have full rank, although perhaps with more hope, since when Ψ^m is a T-set, condition (a) would automatically be satisfied. Recall that if Φ^m is also a T-set then the rank of M_m is guaranteed to be m (Lemma 2.5.11).

We have

LEMMA 2.7.6 Assume we are given Φ^{m+1} and Ψ^m, with Ψ^m being a T-set on R, and M_m has rank $r \le m$. Let $\hat{\Psi}^r = \{\hat{\psi}_j\}_{j=0}^r$ be such that the moment matrix $\hat{M}_{m+1,r}$ (i.e. using the functions ϕ_i and $\hat{\psi}_j$) has full rank. Predetermine knots $x_o,\ldots,x_{m-r-1}$. Let $\hat{\phi}_{m+1} = \sum_{i=0}^{m+1} b_i \phi_i$ be a non-trivial solution of

$$(2.7.9)\quad \begin{bmatrix} & (\hat{M}_{m+1,r})^T & \\ \phi_o(x_o), & \ldots & \phi_{m+1}(x_o) \\ \cdot & & \\ \cdot & & \\ \phi_o(x_{m-r-1}) & \ldots & \phi_{m+1}(x_{m-r-1}) \end{bmatrix} \begin{bmatrix} b_o \\ \cdot \\ \cdot \\ b_{m+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}.$$

Then if there are m+1 zeros of $\hat{\phi}_{m+1}$ such that detB≠0, there is an I.P.Q.F. exact on $\Phi^{m+1}\times\Psi^{m}$ with the knots x_i being these zeros, and the knots y_j, j=0,...,m, being arbitrary, distinct and within R. ///

Some closely related but independently obtained results are given in [26,27]. In [27] results similar to Theorem 2.7.3 and 2.7.4 are given in the case where Φ^{m+1} and Ψ^{m} are both CT-sets. In [26] the special case

$$\phi_i = x_i, \quad i=0,1,\ldots,m+1, \tag{2.7.10}$$

$$\psi_j = (\xi)^j, \quad j=0,1,\ldots,m+1, \tag{2.7.11}$$

where ξ is a strictly monotonic function defined on R is investigated. Both the sets Φ^{m+1} and Ψ^{m+1} are CT-sets (see Lemma 2.2.15). Clearly we could replace (2.7.10) by

$$\phi_i = (\theta)^i, \quad i=0,1,\ldots,m+1, \tag{2.7.12}$$

where θ is a strictly monotonic function, with analogous results. In this case I.P.Q.F. would be very useful when I has the special form

$$I(f;g) = \int_R w(x)f(\theta(x))g(\xi(x))dx \quad . \tag{2.7.13}$$

In the next section we assume θ=ξ and are thus able to drop the requirement that ξ is strictly monotonic.

§2.8 I.P.Q.F. AND T-SETS: CASE III

We now consider the third of the special cases mentioned in §2.4. In §2.1 we noted the properties of polynomials which obscure the power of the concept of I.P.Q.F. For convenience we restate them here before examining case III, which can be regarded as a generalization of conventional polynomials. They are:

$$\phi_i = (\phi_1)^i \quad i=0,1,2,\ldots \tag{2.8.1}$$

where ϕ_1 is not the constant function, but is at least a piecewise continuous function, which is not a step function, on R.

$$\phi_i \equiv \psi_i \quad i=0,1,2,\ldots \tag{2.8.2}$$

LEMMA 2.8.1 The functions Φ^γ, $\gamma \geq 0$, ϕ_i defined by (2.8.1), are linearly independent on R.

PROOF: Immediate from Corollary 2.2.17, as there must exist a non-trivial interval within R on which ϕ_1 is continuous and monotonically increasing or decreasing. ///

Since we are dealing with T-sets, we naturally assume throughout this section

$$S_i \epsilon \chi_1, \quad i=0,\ldots,m, \tag{2.8.3}$$

$$T_j \epsilon \chi_1, \quad j=0,\ldots,n. \tag{2.8.4}$$

We may now obtain the property of polynomials which is not available when dealing with T- or GT-sets in general, that is the ability to factor out a zero. It is this property, in addition to the T-set properties, that allows us to obtain results for I.P.Q.F. exact $\Phi^{\gamma}\times\Psi^{\delta}$, $\gamma+\delta = 2m+1$, $\gamma>m\geq n$, when (2.8.1) and (2.8.2) hold.

The following is fairly obvious:

THEOREM 2.8.1 Assuming (2.8.1) and (2.8.2), if ξ_r is a ϕ-polynomial of degree r, with a zero at t, there is a unique ϕ-polynomial, θ_{r-1} of degree r-1 such that

$$\xi_r = (\phi_1-\phi_1(t))\theta_{r-1}. \tag{2.8.5}$$

PROOF: Immediate by analogy with the case of standard monomials (i.e. $\phi_1\equiv x$). ///

We now proceed as we did in §2.4, and analogous to (2.4.9) and (2.4.10), obtaining the following necessary conditions,

$$I(\phi_j;h_r) = 0,\ r=1,\ldots,\gamma-m=k,\ \ j=0,\ldots,\delta, \tag{2.8.6}$$

where

$$\alpha_{or}\phi_o + \ldots + \alpha_{mr}\phi_m - \phi_{m+r} = h_r,\ \ r=1,\ldots,k. \tag{2.8.7}$$

We now use the same argument as [24] where we examined the case $\phi_1\equiv x$. If the required knots $x_o,\ldots,x_m$ exist, then they must be ϕ_1-distinct zeros of h_r, r=1,...,k, (thus $\det(B)\neq 0$). If this is the case, we can write, due to Lemma 2.8.1,

(2.8.8) $$h_r = (\phi_1-\phi_1(x_o)) \ldots (\phi_1-\phi_1(x_m))\beta_r,$$

$r=1,\ldots,k$, where β_r is a ϕ-polynomial of degree $r-1$. That is

(2.8.9) $$h_r = h_1\beta_r, \quad r=1,\ldots,k-1$$

Thus the requirement (2.8.6) can now be written

(2.8.10) $$I(\phi_j;h_1) = 0, \quad j=0,\ldots,\delta+k.$$

Clearly, this is a W-orthogonality condition. From Lemma 2.5.1 we know that h_1 exists, (Theorem 2.3.3 forces $\delta+k\leq m$) so it remains to show that h_1 has at least $m+1$ ϕ_1-distinct zeros, in the region R. We have

THEOREM 2.8.2 Let ϕ_i be defined by (2.8.1) and (2.8.2). Then the ϕ-polynomial

$$\tau_r = \sum_{i=0}^{r+1}\alpha_i^{(r)}\phi_i, \quad \alpha_{r+1}^{(r)}\neq 0, \qquad \text{orthogonal to}$$

to ϕ^{r+1} exists, and is made unique by requiring $\alpha_{r+1}^{(r)}=1$.

Further τ_r has exactly $r+1$ ϕ_1-distinct zeros in R, $x_o^{(r)},\ldots,x_r^{(r)}$. Thus we can write

(2.8.11) $$\tau_r = \prod_{i=0}^{r+1}(\phi_1-\phi_1(x_i^{(r)})).$$

PROOF: Existence and uniqueness (up to multiplicative constant) of τ_r follow immediately from the usual Gram-Schmidt orthogonalization process (see, for example, [2]). If there are

less than $r+1$ ϕ_1-distinct zeros, (2.8.10) is contradicted, but we know τ_r exists from Lemma 2.5.1. Clearly, after repeated factorization, it is not possible for τ_r to have more than $r+1$ ϕ_1-distinct zeros. ///

Analogous to Theorem 2.3.3 we have, despite the fact that Φ^r, $r\geq 0$, is not a T-set unless ϕ_1 is a monotonic function on R (see Lemma 2.2.15 and Corollary 2.2.16),

LEMMA 2.8.3 We cannot have $\delta+k>m$ in the above analysis. That is, under conditions (2.8.1) - (2.8.4) there cannot exist an I.P.Q.F. exact on $\Phi^\gamma \times \Phi^\delta$ if $\gamma+\delta>2m+1$.

PROOF: If $\delta+k>m$ then, via (2.8.10), we require h_1 to be W-orthogonal to itself. If h_1 is non-trivial this is impossible. ///

We state the results concerning I.P.Q.F. explicitly:

THEOREM 2.8.4 Assume $\gamma+\delta=2m+1$, and (2.8.1) - (2.8.4) hold. Then there exist I.P.Q.F. exact on $\Phi^{2m+1-n} \times \Phi^n$, where the knots $x_0,\ldots,x_m$ are ϕ_1-distinct zeros of τ_m (as given in Theorem 2.8.2) and $y_0,..,y_n$ are arbitrary ϕ_1-distinct points. ///

COROLLARY 2.8.5 Let $x_0,\ldots,x_m$ be as in Theorem 2.8.4 and assume (2.8.1) - (2.8.4). Given an I.P.Q.F. exact on $\Phi^{2m+1-n} \times \Phi^n$, then it is also exact on

$$\Phi^{2m+1-n+k}\times\Phi^{n-k}, \quad k=0,\ldots,n.$$

PROOF: The conditions on the knots $x_0,\ldots,x_m$ are identical in all cases, and once satisfied, if we obtain the coefficients a_{ij}, $i=0,\ldots,m$, $j=0,\ldots,n$, so that the I.P.Q.F. is exact on $\Phi^m\times\Phi^n$, then it is clearly exact on $\Phi^m\times\Phi^{n-k}$ also. ///

It is not necessarily the case that an I.P.Q.F. exact on $\Phi^\gamma\times\Phi^\delta$, $\gamma+\delta=2m+1$, $\delta<n$, will be exact on $\Phi^{\gamma-1}\times\Phi^{\delta+1}$ (see Theorem 2.10.8). Similarly we have:

COROLLARY 2.8.6 Let $y_0,\ldots,y_n$, be ϕ_1-distinct zeros of τ_n, and (2.8.1) - (2.8.4) hold. Given an I.P.Q.F. exact on $\Phi^n\times\Phi^{n+1}$, then it is also exact on

$$\Phi^{n-k}\times\Phi^{n+1+k}, \quad k=0,\ldots,n. \qquad ///$$

Analogous to Theorem 2.7.4,

COROLLARY 2.8.7 If, under the assumptions of Theorem 2.8.4, the knots x_i are $m+1$ ϕ_1-distinct zeros of τ_m, and the knots y_j are $n+1$ ϕ_1-distinct zeros of τ_n, then the I.P.Q.F. of Theorem 2.8.4 is exact on both

$$\Phi^{m+1}\times\Phi^n \quad \text{and} \quad \Phi^m\times\Phi^{n+1}. \qquad ///$$

Of course, Corollaries 2.8.5 and 2.8.6 still apply.

The I.P.Q.F. of Theorem 2.8.4 are by no means unique since

we have freedom in the choice of knots y_j, and x_i if ϕ_1 is not a monotonic function. Also, if rank M_m is $r<m$, there, will be $m-r$ degrees of freedom in τ_m. We observe that the matrix A remains unchanged no matter which point in any set of ϕ_1-equivalent points we may use. Thus there remains the question of which (if any) point in a set of ϕ_1-equivalent points is the best (in some sense) to choose.

Having considered the case $\gamma+\delta=2m+1$ we now examine $\gamma+\delta<2m+1$ and ask whether it is possible to use any extra degrees of freedom in preassigning knots. Analogous to Theorem 4.4 of [24] (in its corrected form) we obtain:

THEOREM 2.8.8 Assume that (2.8.1) - (2.8.4) hold, that $2m+1-\gamma-n = s>0$, $\gamma\geq m$, and that ϕ_1-distinct knots $x_0,\ldots,x_{s-1}$ have been preassigned. Then, if $\tau_{m-s}^{(s)}$ W-orthogonal to Φ^{m-s}, with respect to I with weight function $w\prod_{i=0}^{s-1}(\phi-\phi(x_i))$, has $m-s+1$ ϕ_1-distinct zeros, $x_s,\ldots,x_m$ such that $x_0,\ldots,x_m$ are ϕ_1-distinct, there are I.P.Q.F. exact on $\Phi^\gamma\times\Phi^n$, with knots $x_0,\ldots,x_m$ (or their ϕ_1-equivalents) and knots y_j being arbitrary but ϕ_1-distinct. ///

Clearly, analogues of Corollaries (2.8.5), (2.8.6) and (2.8.7) hold. We note (§2.5) that $\tau_{m-s}^{(s)}$ of Theorem 2.8.8 always exists, but with the weight function changing sign, it is not certain that the ϕ_1-distinct roots $x_s,\ldots,x_m$, in R, exist. However, we may obtain (cf. Theorem 4.3 of [12]).

THEOREM 2.8.9 Assume (2.8.1) - (2.8.4) are valid. Assume that the weight function w of I changes sign $s<m$ times in R, and that $M_m^{(s)}$ and $M_{m+1}^{(s)}$ are non-singular.

(a) Then there exists a unique (up to multiplicative constant) ϕ-polynomial $\tau_m^{(s)}$ of degree m+1, orthogonal to Φ^{m+1}, with at least m+1-s ϕ_1-distinct zeros in R.

(b) Denoting the ϕ_1-distinct zeros of $\tau_m^{(s)}$ by $x_o,\ldots,x_r$ $r>m-s$ then:

(i) if t is ϕ_1-distinct from x_i, $i=0,\ldots,r$, there exists a unique (up to multiplicative constants) ϕ-polynomial of degree m+1, $\tau_m^{(s+1)}$ orthogonal to Φ^{m+1} with respect to $I^{(s+1)}$ (I with weight function $w(\phi_1-\phi_1(t))$). Further $\tau_m^{(s+1)}$ has at least m-s ϕ_1-distinct zeros in R.

(ii) If t is ϕ_1-equivalent to x_i, for some i, there exists no $\tau_m^{(s+1)}$ W-orthogonal to Φ^m with respect to $I^{(s+1)}$.

Clearly

$$\theta_{m-1} = \frac{\tau_m^{(s)}}{(\phi_1-\phi_1(x_i))}$$

where t_i are m-s of the knots $x_o,\ldots,x_r$, and θ_{m-1} has degree m, is W-orthogonal to Φ^m with respect to $I^{(s+1)}$.

PROOF: (a) $\tau_m^{(s)}$ exists since $M_m^{(s)}$ has full rank, and it has at least m-s+1 zeros by an argument analogous to the proof of Theorem 2.8.2.

(b) ii) If such a $\tau_m^{(s+1)}$ exists, clearly

$$I(\tau_m^{(s)};\tau_m^{(s+1)}) = 0.$$

If $\tau_m^{(s+1)}$ is of degree m+1, we can write

$$\tau_m^{(s+1)} = \alpha\tau_m^{(s)} + \theta \ , \ \theta \text{ of degree } m,$$

but this implies $I(\tau_m^{(s)}; \tau_m^{(s)}) = 0$,

which implies $I(\tau_m^{(s)};\phi_i) = 0$, $i=0,\ldots,m+1$,

and this contradicts the rank of M_{m+1} being full.

(b) (i) Clearly, if $\tau_m^{(s+1)}$ exists it must have the required number of zeros (cf. (a)).

Only requiring W-orthogonality, $\tau_m^{(s+1)}$ exists by the results of §2.5, and since $M_m^{(s)}$ has full rank it follows that $\tau_m^{(s+1)}$ is unique up to a constant and the coefficient of ϕ_{m+1} must be non-trivial. $M_{m+1}^{(s)}$ being of full rank gives $I(\tau_m^{(s+1)}; \tau_m^{(s+1)})\neq 0$. ///

COROLLARY 2.8.10 Assume (2.8.1) - (2.8.4). Let w be as in (A3), i.e. of one sign on R. Then the conclusion of Theorem 2.8.9 is valid.

PROOF: The structure of ϕ_i ensures that $M_m(\equiv M_m^{(0)})$ and M_{m+1} $(\equiv M_{m+1}^{(0)})$ have full rank in this case. ///

COROLLARY 2.8.11 Assume (2.8.1) - (2.8.4). Let w and other notation be as in Theorem 2.8.9. Then, if

(a) $t\neq x_i$, for same i, the moment matrices $M_m^{(s+1)}$, $M_{m+1}^{(s+1)}$ with respect to $I^{(s+1)}$ have full rank.

(b) If $t=x_i$, for same i, $M_m^{(s+1)}$ must be singular.

PROOF: (a) follows from uniqueness of $\tau_m^{(s+1)}$ in (b) (ii) of Theorem (2.8.9).

(b) follows from existence of a non-trivial solution of a homogeneous system of m+1 linear equations in m+1 unknowns. ///

We observe that if we choose points $t_o,\ldots,t_s$, s<m, and seek ϕ-polynomial's $\tau_m^{(r)}$ of degree m+1, orthogonal to ϕ^{m+1} with respect to $I^{(r)}$ (I with weight function $w\prod_{j=0}^{r-1}(\phi_1-\phi_1(t_j))$, w obeying (A3)) we may proceed as follows:

(a) Let $x_o,\ldots,x_m$ be the zeros of $\tau_m(\equiv\tau_m^{(0)})$.

(b) If t_j are ϕ_1-distinct from $x_i(\equiv x_i^{(0)})$, i=0,...,m, j=0,...,s, calculate $\tau_m^{(1)}$. Let the zeros of $\tau_m^{(1)}$ be

$$x_o^{(1)},\ldots,x_{r_1}^{(1)},\ r_1\geq m-1.$$

(c) If t_j are ϕ_1-distinct from all x_i and $x_i^{(1)}$, j=1,...,s, we can calculate $\tau_m^{(2)}$. Let the zeros of $\tau_m^{(2)}$ be

$$x_o^{(2)},\ldots,x_{r_2}^{(2)},\quad r_2\geq m-2.$$

(d) For k=2,...,s:

If t_j are ϕ_1-distinct from all x_i, $x_i^{(1)},\ldots,x_i^{(k)}$, j=k+1, ...,s, we calculate $\tau_m^{(k+1)}$ with zeros

$$x_o^{(k+1)},\ldots,x_{r_{k+1}}^{(k+1)},\ r_{k+1}\geq m-(k+1).$$

Noting that since the matrices M_r, r=0,1,..., are non-singular when (2.8.1) - (2.8.2) hold, the above process can

obviously be modified to seek $\tau_{m-r}^{(r)}$, of degree m-r+1, orthogonal to Φ^{m-r} with respect to $I^{(r)}$, r=0,...,s+1.

Thus, although we cannot obtain any further insight into the zero distribution of $\tau_{m-s}^{(s)}$, we have gained some knowledge concerning the existence of the polynomials $\tau_{m-s}^{(s)}$ and $\tau_{m}^{(s)}$. Returning to Theorem 2.8.8 we have:

COROLLARY 2.8.12 Assume (2.8.1) - (2.8.4). Let $2m+1-\gamma-n=1$, $\gamma>m$. Preassign a knot x_o. If the range of ϕ_1(domain not restricted to R) is $\mathbb{R}$, then $\tau_{m-1}^{(1)}$ is guaranteed m zeros, and so the I.P.Q.F. in Theorem 2.8.8 (s=1) exists if all these zeros are ϕ_1-distinct.

PROOF: Clearly we may write

$$\tau_{m-1}^{1} = \prod_{i=1}^{m-1} (\phi_1-\phi_1(x_i))(\phi_1-s)$$

where $x_1,\ldots,x_{m-1}$ are ϕ_1-distinct points in R. The range of ϕ_1 being $\mathbb{R}$ guarantees the existence of x_m such that $\phi_1(x_m)=s$. ///

In particular if $\phi_1\equiv x$, since x is a strictly monotonic function defined on $\mathbb{R}$ with range R, Theorem 2.8.8 is always valid with s=1.

We note that (cf. Lemma 4.5 of [24]), if we are given a set of knots $x_o,\ldots,x_m$, by using (2.8.10) we may decide whether an I.P.Q.F. exact on $\Phi^{\gamma}\times\Phi^{\delta}$, $\gamma>m$, exists. Also the above analysis is symmetric in m and n.

It is clear that (2.8.2) is a central tenet of the arguments

presented in this section. Finally, we comment that although our motivation for this section we originally the investigation of the T-set $\{x^i\}_{i=0}^m$, the results we have obtained depend much more on the fact that, given (2.8.1) and (2.8.2), we are able to factorize at zeros than on T-set properties, which we only need to the extent of assuming that ϕ_1 has at least $m+1$ ϕ_1-distinct points, or if we are interested in uniqueness of knots in some I.P.Q.F.

Recalling the comments made at the end of §2.7, in view of the analysis of this section, it is clear that given (2.7.11) and/or (2.7.12) the requirement that the ξ and θ mentioned there need not be strictly monotonic when we consider the case of I.P.Q.F. exact on $\Phi^{m+1}\times\Psi^m$ and/or $\Phi^m\times\Psi^{m+1}$. The only difference is that the knots x_i and/or y_j will be ϕ_1-distinct (ψ_1-distinct) and may not now be unique, though the matrix A will be unchanged when any point ϕ_1-equivalent to a knot is used in place of that knot.

Let us return and consider (2.8.1) - (2.8.2) for a moment. We recall, see for example [157], that standard orthogonal polynomials can be expressed by a three-term recurrence relationship

$$\xi_{i+1} = (t-\mu_i)\,\xi_i - \nu_i\xi_{i-1}, \tag{2.8.12}$$

with the initial conditions

(2.8.13)
$$\begin{cases} \xi_{-1} \equiv 0, \\ \xi_0 \quad \text{usually being 1, the constant function,} \\ \mu_i \text{ and } \nu_i \text{ being given constants, } i=0,1,\ldots \end{cases}$$

It is clearly possible to generalize (2.8.12) and derive a set of basis functions ϕ_i using generalized recurrence relation. For example, we may use, see [166],

(2.8.14)
$$\phi_{i+1} = (\xi-\alpha_i)\,\phi_i-\beta_i\phi_{i-1}$$

with the initial conditions

(2.8.15)
$$\begin{cases} \phi_{-1} \equiv 0, \\ \phi_0 \text{ and } \xi \text{ are given functions,} \\ \alpha_i \text{ and } \beta_i \text{ are given constants, } i=0,1,\ldots \end{cases}$$

Clearly (2.8.14) is a generalization of (2.8.1) since we obtain (2.8.1) on choosing $\alpha_i=\beta_i=0$, $i=0,1,\ldots$, and $\xi\equiv\phi_1$ (as defined in (2.8.1)) and ϕ_0 (from (2.8.15)) to be unity. However, it is clear that the function ϕ_{i+1} in (2.8.14) can be regarded as a polynomial of degree (i+1) (in the integer powers of ξ) multiplied by ϕ_0. Consequently, we observe that all the results of this section now hold in a more general context; namely that obtained by defining the basic functions ϕ_i and ψ_j as follows:

(2.8.16)
$$\begin{cases} \phi_{i+1} = (\xi-\alpha_{i1})\phi_i-\beta_{i1}\phi_{i-1} \\ \psi_{j+1} = (\xi-\alpha_{j2})\psi_j-\beta_{j2}\psi_{j-1} \end{cases}$$

under the initial conditions

(2.8.17)
$$\begin{cases} \phi_{-1} \equiv \psi_{-1} \equiv 0, \\ \phi_o,\psi_o \text{ and } \xi \text{ are given functions,} \\ \alpha_{i1},\beta_{i1},\alpha_{j2},\beta_{j2} \text{ are given scalars.} \end{cases}$$

Now, provided ϕ_o and ψ_o are of one sign on R it is clear the above results hold (since we merely regard the function $w\phi_o\psi_o$ as the weight function in I in place of w).

§2.9 OBTAINING AN I.P.Q.F.

In cases II and III, we have seen that the knots, $x_o,\ldots,x_m$, can be characterized as the zeros of a single ϕ-polynomial (even if that polynomial is not unique). In case I we know that the required knots exist, and in the case $\phi_i \equiv x^i$, i=0, ...,2m+1, the knots may also be characterized as zeros of certain polynomials, see [9], Ch. 4. We also have several methods for obtaining the W-orthogonal polynomial which has $x_o,\ldots,x_m$ as zeros, given in §2.5:

(2.9.1) The solution of linear system of equations.

(2.9.2) When the moment matrix M_m has full rank we may use the generalized Gram-Schmidt process, (2.5.9) - (2.5.12), and this is available if both Φ^{m+1} and Ψ^m are T-sets

(Theorem 2.5.13). In fact, we observe that we only need Φ^m and Ψ^m being T-sets, since there is no need to normalize $\hat{\xi}_{m+1}$ (see 2.5.11).

(2.9.3) We may define ϕ_{m+1}, W-orthogonal to Ψ^m by (2.5.14) ($\ell=m+1$, and replacing ξ_i by ϕ_i, $i=0,\ldots,m+1$). This requires that rank $M_{m+1,\,m}$ has to be m (else $\phi_{m+1}\equiv 0$) and a sufficient though not necessary condition for this is that Φ^m and Ψ^m are T-sets.

In the context of T-sets (and thus χ_1 elementary functionals), we can use (2.9.3) to amplify the comments made in §2.4 concerning the unlikeliness of there existing I.P.Q.F. exact on $\Phi^\gamma \times \Psi^\delta$, $\gamma+\delta=2m+1$, except in cases I, II or III, in which we eventually have no more conditions (i.e. at most m+1) to satisfy than there are parameters (the knots x_i) available. Recalling condition (2.4.14) of Theorem 2.4.3, obvious candidates for the functions h_r are

$$(2.9.4)\quad h_r=\det\begin{bmatrix} I(\phi_o;\psi_o), & \ldots, & I(\phi_o;\psi_m), & \phi_o \\ \vdots & & \vdots & \vdots \\ I(\phi_m;\psi_o), & & I(\phi_m;\psi_m), & \phi_m \\ I(\phi_{m+r};\psi_o), & \ldots, & I(\phi_{m+r};\psi_m), & \phi_{m+r} \end{bmatrix},\quad r=1,\ldots,\gamma-m$$

However, condition (2.4.15) of Theorem 2.4.3 in this context requires that h_r, $r=1,\ldots,\gamma-m$, all have m+1 common zeros,

$x_o,\ldots,x_m$, and once we have determined $x_o,\ldots,x_m$ from examining, say h_1, it seems improbable that, without imposing stringent conditions, even h_2 will have $x_o,\ldots,x_m$ as zeros, let alone $h_3,\ldots,h_{r-m}$. Replacing $I(\phi_i;\psi_j)$ by $\phi_i(x_j)$, $i=0,\ldots,m,m+r$, $j=0,\ldots,m$, in (2.9.4) guarantees the common zeros, but in general we would not expect the h_r so defined, $r=1,\ldots,\gamma-m$, all to be W-orthogonal to Ψ^m.

We now consider how to obtain the matrix A for an I.P.Q.F. exact on $\Phi^m\times\Psi^n$. We assume that the elementary functions $\{S_i(\zeta_i;.)\}_{i=0}^m$, $\{T_j(\eta_j;.)\}_{j=0}^n$ have been determined, and that the matrices B and C are non-singular. As pointed out in §1.3, there are two methods available; the use of interpolating polynomials and the direct solution of the moment equations.

The use of interpolating polynomials is straightforward. Since the matrices B and C are non-singular (cf. Lemma 2.2.21) it is immediate that there exist unique ϕ- and ψ-polynomials, p_i and q_j, of degrees m and n respectively such that

$$(2.9.5)\qquad \begin{cases} S_i(\zeta_i;p_s) = \delta_{is}, & i,s=0,\ldots,m, \\ T_j(\eta_j;q_t) = \delta_{jt}, & j,t=0,\ldots,n. \end{cases}$$

Clearly, any ϕ-polynomial, ϕ, and ψ-polynomial, ψ, can be written

$$(2.9.6)\qquad \begin{cases} \phi = \sum_{i=0}^m p_i S_i(\zeta_i;\phi)\ , \\ \psi = \sum_{j=0}^n q_j T_j(\eta_j;\phi)\ . \end{cases}$$

Thus it follows that we may find a_{ij} by

$$a_{ij} = I(p_i;q_j), \quad i=0,\ldots,m, \quad j=0,\ldots,n. \tag{2.9.7}$$

The polynomials p_i may be found either by solving the relevant linear equations or as (see (2.2.10))

$$p_i = V_{m,i}/V_m, \quad i=0,\ldots,m, \tag{2.9.8}$$

where $V_{m,i}$ is V_m with the (i+1)-st column replaced with the vector $(\phi_0,\ldots,\phi_m)^T$. We may obtain q_j analogously. Hence

$$a_{ij} = I(V_{m,i};V_{n,j})/V_m V_n. \tag{2.9.9}$$

If (2.8.1) and (2.8.2) are valid, then we may obtain p_i and q_j in a format analogous to the standard Lagrange interpolation formula for polynomials.

Now, let us consider solving the moment equations (1.3.13) directly. For quite small values of m and n, the matrix can become quite large, however it is possible, see [23], and in a general context, [31], to obtain the solution to (1.3.13) by examining only (m+1) x (m+1) and (n+1) x (n+1) systems of linear equations.

For each i=0,...,m solve the equations:

$$C \begin{bmatrix} \alpha_{0i} \\ \cdot \\ \cdot \\ \alpha_{ni} \end{bmatrix} = \begin{bmatrix} I(\phi_i;\psi_0) \\ \cdot \\ \cdot \\ I(\phi_i;\psi_n) \end{bmatrix}, \tag{2.9.10}$$

where

$$\alpha_{ji} = \sum_{k=0}^{m} a_{kj} S_k(\phi_i) \ . \tag{2.9.11}$$

Then we obtain the elements of A as solutions of

$$B \begin{bmatrix} a_{oj} \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix} = \begin{bmatrix} \alpha_{jo} \\ \cdot \\ \cdot \\ \alpha_{jm} \end{bmatrix} \quad , \quad j=0,\ldots,n. \tag{2.9.12}$$

Observe that in the process (2.9.10) - (2.9.12) we only need to decompose the matrices B and C once, and so the derivation of the matrix A can be done much more easily and efficiently than by attempting to solve (1.3.13) directly.

Of course, no matter how we obtain the matrix A, we know that it is unique (Theorem 1.3.4).

2.10 PROPERTIES OF THE MATRIX A

We begin by noticing some fairly straightforward properties of A. Initially we examine properties which A has once an I.P.Q.F. has been determined, and later we investigate the effect on I.P.Q.F. of imposing certain conditions on A.

LEMMA 2.10.1 Let A be an (m+1)x(n+1) matrix. If we desire an I.P.Q.F. to be exact on $\Phi^{\gamma} \times \Psi^{\delta}$, $n-\delta=\ell>0$, then we may arbitrarily preassign ℓ elements in each row of the matrix

A, provided the rank of the matrix C (see (1.3.8)) is not decreased after the relevant columns have been deleted (i.e. in a given row of A, if we preassign the r^{th} element, we delete the r^{th} column of C).

An analogous result holds if $m-\gamma=k>0$, when dealing with the columns of A.

PROOF: The rank of D (1.3.9) - (1.3.10) remains unchanged. See [24]. ///

COROLLARY 2.10.2 If Ψ^{δ} of Lemma 2.10.1 is a GT-set with respect to a bilinear functional L (so $T_j \equiv L, j=0,\ldots,n$) the rank of C in Lemma 2.10.1 remains unchanged. ///

COROLLARY 2.10.3 If I.P.Q.F. as described in Lemma 2.10.1 exist exact on $\Phi^{\gamma}\times\Psi^{n-\ell}$, $\ell>0$, $\gamma\geq m$ then, once the ℓ elements in each row of A have been preassigned, the matrix A is unique. ///

Consequently, we have the following result, which is complimentary to Corollary 2.8.5.

COROLLARY 2.10.4 If we seek an I.P.Q.F. exact on $\Phi^{\gamma}\times\Psi^{n-\ell}$, $\ell>0$, we may not arbitrarily preassign more than ℓ elements in each row of A. ///

We are thus able to conclude that, when seeking an I.P.Q.F. exact on $\Phi^{\gamma}\times\Psi^{\delta}$, there is, in general, no real gain in considering $n>\delta$, and the usual choice would be $n=\delta$ (cf. (2.7.2)).

Considering the rank of A, we have (see also [13,24]).

LEMMA 2.10.5 Assume we are given an I.P.Q.F., exact on $\Phi^{\gamma}\times\Psi^{n}$, and that the rank of $M_{\gamma,n}$ is n+1. Then the rank of A is maximal, i.e. n+1.

PROOF: Assume the contrary, i.e. rank of A is less than n+1. Then there exist constants α_j, with $\sum_{j=0}^{n}|\alpha_j|>0$, such that

$$\sum_{j=0}^{n} \alpha_j \underline{a}_j = \underline{0} ,$$

where $\underline{a}_j$ is (see (1.3.14)) the (j+1)-st column of A. However recalling that C (see (1.3.8) has maximal rank, there exists a non-trivial ψ-polynomial ψ such that

$$T_j(\psi) = \alpha_j, \quad j=0,\ldots,n.$$

Hence $Q(\phi_i;\psi) = 0$, $i=0,\ldots,\gamma$, and since the I.P.Q.F. is exact for $\Phi^{\gamma}\times\Psi^{n}$, it follows that

$$I(\phi_i;\psi) = 0, \quad i=0,\ldots,\gamma.$$

Thus contradicts the assumption on the rank of $M_{\gamma,n}$. ///

We also have:

LEMMA 2.10.6 Assume we are given an I.P.Q.F. exact on $\Phi^{\gamma}\times\Psi^{n}$. Assume $\phi_i\equiv\psi_i$, $i=0,\ldots,n$. Then the rank of A is n+1

PROOF: Again, we assume the contrary, namely that rank of A is at most n. We show that the rank cannot be n (the

argument is analogous for n-1,...). Assume the first n columns of A are linearly independent. We now show that the remaining column cannot be equivalently zero. Since if is equally zero, we construct a non-trivial ϕ-polynomial, ϕ, of degree n, such that

$$T_j(\phi) = 0, \quad j=0,\ldots,n-1,$$

and $T_n(\phi)=1$. Such a polynomial exists since C has maximal rank. However, we now find that

$$Q(\phi;\phi) = 0,$$

while

$$I(\phi;\phi)>0,$$

and thus have a contradiction. We now derive the polynomial ψ as done in the proof of Lemma 2.10.5, and we have a contradiction since

$$Q(\psi;\psi) = 0,$$

while

$$I(\psi;\psi) > 0. \qquad ///$$

LEMMA 2.10.7 Assume we are given an I.P.Q.F. exact on $\Phi^{\gamma}\times\Psi^{\delta}$, $\delta\leq n$, $\gamma+\delta\geq 2m$, $S_i\varepsilon\chi_1$, $i=0,\ldots,m$, and that (2.8.1), (2.8.2) are valid. Then the rank of A is at least $\delta+1$.

PROOF: We proceed in exactly the same way as the proof of Lemma 2.10.6, examining rows and elementary functionals S_i and on replacing n by δ it follows that at least one of the δ-th,...,m-th rows of A cannot be identically zero. Assume the δ-th row is not identically zero. Consequently there exist constant $\alpha_o,\ldots,\alpha_\delta$, not all zero, such that

$$(\alpha_o,\ldots,\alpha_\delta,0,\ldots,0)\ A = \underline{0}.$$

Set

$$\xi = \prod_{i=\delta+1}^{m} (\phi_1-\phi_1(x_i))^2$$

If $\xi(x_i)=0$, $i=0,\ldots,\delta$, we are finished since $I(\xi;1)>0$ but $Q(\xi;1)=0$. Assume $\xi(x_i)\neq 0$ for same $i\varepsilon\{0,\ldots,\delta\}$. We now obtain θ, of degree at most δ, such that if $\xi(x_i)\neq 0$, $i=0,\ldots,\delta$,

$$\theta(x_i) = \frac{\alpha_i}{\xi(x_i)} \quad .$$

The existence of such a θ is guaranteed by the ϕ-distinctions of the knots $x_o,\ldots,x_m$. However $Q(\xi\theta;\theta)=0$ by construction, while

$$I(\xi\theta;\theta) > 0. \qquad ///$$

As usual, the analogous results hold an interchanging m and n.

We can obtain some results, concerning the symmetry of A,

which will be of particular use when actually calculating the elements a_{ij}.

DEFINITION 2.10.1 We say that a function ξ is even with respect to a set of elementary functionals $\{L_i\}_{i=0}^{r}$ if

(2.10.1) $$L_i(\xi) = L_{r-i}(\xi), \quad i=0,\ldots,[\tfrac{r}{2}].$$

Similarly, ξ is odd with respect to $\{L_i\}_{i=0}^{r}$ if

(2.10.2) $$L_i(\xi) = -L_{r-i}(\xi), \quad i=0,\ldots,[\tfrac{r+1}{2}].$$ ///

As usual [s] denotes the integer part of s, i.e. the greatest integer less than or equal to s.

THEOREM 2.10.8 Assume we are given an I.P.Q.F., exact on $\Phi^m \times \Psi^n$, and the following conditions are obeyed,

(a) $[\frac{m+1}{2}]$ of the functions Φ^m are even with respect to $\{S_i\}_{i=0}^{m}$, the other $m+1-[\frac{m+1}{2}]$ being odd,

(b) $[\frac{n+1}{2}]$ of the functions Ψ^n are odd with respect to $\{T_j\}_{j=0}^{n}$, the other $n+1-[\frac{n+1}{2}]$ being odd,

(c) If $\phi_i\psi_j$ is the product of an even and an odd function, $I(\phi_i;\psi_j) = 0$.

Then the matrix A has rotational symmetry of order 2.

PROOF: The requirement that A has rotational symmetry of order 2 can be written as

$$(2.10.3) \qquad \begin{bmatrix} 1,0,\ldots,0,0,\ldots,0,-1 \\ 0,1,\ldots,0,0,\ldots,-1,0 \\ \cdot \\ \cdot \\ \cdot \\ 0,0,\ldots,1,-1,\ldots,0,0 \end{bmatrix} \underline{a} \equiv U\underline{a} = \underline{0}$$

where $\underline{a}$ is given by (1.3,14) (we insert a column of zeros between the $\frac{(m+1)(n+1)-1}{2}$'th and following column of U if (m+1)(n+1) is odd). If an I.P.Q.F. with the desired property exists then rank $\left(\frac{D}{U}\right)$ is the same as rank $\left(\frac{D|\underline{\underline{A}}}{U|\underline{\underline{0}}}\right)$. An examination of $\left(\frac{D}{U}\right)$ shows that U and the matrix formed by taking all rows of D corresponding to the pairs of functions in condition (c) are equivalent, and consequently only one of them is required in the determination of A. ///

COROLLARY 2.10.9 Unless both m and n are even, the words "even" and "odd" may be interchanged in either or both of conditions (a) and (b) of Theorem 2.10.8. If both m and n are even, even and odd must be interchanged in both (a) and (b). ///

COROLLARY 2.10.10 Condition (c) of Theorem 2.10.8 may require that both ϕ_i and ψ_j are either odd or even, and then A has antisymmetry of order 2. ///

In particular, we have the following special cases:

COROLLARY 2.10.11 Let $\phi_i(\psi_j)$ be even if i(j) is even, and odd otherwise. Let $S_i \epsilon \chi_1$, $T_j \epsilon \chi_1$, and assume the knots $x_i(y_j)$ are symmetrically distributed about the origin. Then

(a) if w is even, Theorem 2.10.8 is valid.

(b) if w is odd, Corollary 2.10.10 holds. ///

Again extending slightly some results in [24], we can make the following observation. If a function is even or odd, its zeros will be placed symmetrically about the origin. Consequently the following is immediate:

COROLLARY 2.10.12 Let ϕ_i and ψ_j be as in Corollary 2.10.11 i,j=0,1,... Assume $S_i \epsilon \chi_1$, $T_j \epsilon \chi_1$. Then, if w is also an even function, and the I.P.Q.F. is exact on $\Phi^{\alpha} \times \Psi^{\beta}$ where α and β are both even or both odd, the I.P.Q.F. will also be exact on

$$\Phi^{\alpha+1} \times \Psi^{\beta} \quad \text{and} \quad \Phi^{\alpha} \times \Psi^{\beta+1}.$$

PROOF: Immediate from the structure of A determined in Theorem 2.10.8. ///

Clearly there is an analogous Corollary in the case when ω is an odd function.

We now have, following [12]:

DEFINITION 2.10.2 An I.P.Q.F. will be called symmetric if m=n, $S_i \equiv T_i$, i=0,...,m, and $A=A^T$.

The following results are obvious and extend results in

[12,13].

LEMMA 2.10.13 An I.P.Q.F. exact on $\Phi^{m+1}\times\Phi^{m}$ (and $m\geq n$) with $S_i\equiv T_i$, $i=0,\ldots,m$, is symmetric, and the matrix A is positive definite.

PROOF: $m=n$ follows from exactness on the product space (see Corollary 2.3.4). The fact $a_{ij}=a_{ji}$ is now immediate on calculating the elements of A using interpolating polynomials ((2.9.5) = (2.9.7)). Positive definiteness of A follows since $\det B\neq 0$, and thus for an arbitrary (m+1) vector $(\alpha_0,..,\alpha_m)$, there is an unique non-trivial ϕ-polynomial ϕ such that $S_i(\phi)=\alpha_i$, $i=0,\ldots,m$, and

$$Q(\phi;\phi) = I(\phi;\phi) > 0. \qquad ///$$

LEMMA 2.10.14 Assume (2.8.1), (2.8.2) hold, $S_i\equiv T_i$, $i=0,\ldots,m=n$, and that

$$S_i(\phi_j)S_i(\phi_k) = S_i(\phi_j\phi_k),\ i,j,k=0,\ldots,m.$$

Then any R.Q.F. exact on Φ^{γ}, $\gamma\leq 2m+1$, can be written as a symmetric I.P.Q.F. with A diagonal. Conversely any symmetric I.P.Q.F. with A diagonal exact on $\Phi^{\gamma}\times\Phi^{\delta}$ is a R.Q.F. exact on $\Phi^{\gamma+\delta}$. ///

LEMMA 2.10.15 Assume (2.8.1), (2.8.2) hold, $S_i\equiv T_i\varepsilon\chi_1$, $i=0,\ldots,m=n$, and we are given an I.P.Q.F. exact on $\Phi^{m+1}\times\Phi^{m}$. Then the matrix A is diagonal.

PROOF: The knots, $x_0,\ldots,x_m$, $(y_0,\ldots,y_m)$ are, up to ϕ_1-dis-

tinctness, unique (see §2.8). The matrix A is unique (Theorem 1.3.1), and may be calculated using interpolatery polynomials. Observe that in this case we may write

$$p_i = \alpha_i \Big(\prod_{\substack{j=0 \\ j\neq i}}^{m} (\phi_1 - \phi_1(x_j)) \Big), i=0,\ldots,m, \tag{2.10.4}$$

where α_i is some non zero constant. Observe that

$$a_{ii} = I(p_i;p_i) > 0, \quad i=0,\ldots,m,$$

and

$$a_{ij} = I(p_i;p_j) = I(\tau_m;\theta) = 0,$$

where τ_m is given by (2.8.10) and θ is a ϕ-polynomial of degree m-1. ///

COROLLARY 2.10.16 Assume (2.8.1) - (2.8.3) and that we are given an I.P.Q.F., exact on $\Phi^{m+1} \times \Phi^m$, in which the matrix A is diagonal. Then the knots y_j must be ϕ_1-distinct but, as a set, ϕ_1-equivalent to $x_0,\ldots,x_m$. Thus A is unique.

PROOF: Immediate from the proof of Lemma 2.10.15 and Lemma 2.5.15, which guarantees that τ_m is unique (up to a multiplicative constant). ///

Having seen that in certain circumstances A may be a diagonal matrix, we now reverse our stance and examine what happens if, from the outset, we require A to be diagonal. If m>n, by A being diagonal we shall mean that (after reordering

rows and columns of A if necessary) we can write

$$a_{ij} = 0,\ i\neq j,\ i,j = 0,\ldots,n. \tag{2.10.5}$$

We shall only consider I.P.Q.F. exact on $\Phi^{\gamma}\times\Psi^{n}$, $\gamma\geq m$. The requirement that A be diagonal may be expressed as

$$I(p_i;q_j) = 0,\ i\neq j,\ i,j=0,\ldots,n, \tag{2.10.6}$$

where p_i and q_j are defined as in §2.9. This gives us $n(n+1)$ conditions and we have $m+n+2$ parameters in the choice of elementary functions S_i and T_j available. Thus without the aid of some conditions on the choice of ϕ_i, ψ_j (e.g., (2.8.1) - (2.8.2), $\phi_i\equiv\psi_i$) and perhaps S_i and T_j, we might only expect it possible to force A to be diagonal when, seeking I.P.Q.F. exact on $\Phi^{m}\times\Psi^{n}$,

$$n(n+1) \leq m+n+2. \tag{2.10.7}$$

Clearly, given n, there is always an m_o such that (2.10.7) is satisfied for $m>m_o$. Analogous comments apply if we seek an I.P.Q.F. exact on $\Phi^{\gamma}\times\Psi^{n}$ (though there are more conditions to be satisfied now), when (2.10.7) is replaced by (see §2.4).

$$(n+\gamma-m)(n+1) \leq m+n+2 \tag{2.10.8}$$

In the most structured case we have examined, i.e. case III of §2.4 (§2.8), and using arguments similar to those in the proof of Lemma 2.10.15 (see also [12,13]) we immediately find

THEOREM 2.10.17 Assume (2.8.1) - (2.8.4). Let $x_0,\ldots,x_m$ be ϕ_1-distinct zeros of τ_m, orthogonal to Φ^{m+1}. If, y_j, $j=0,\ldots,n$, are ϕ_1-distinct and such that $y_j\varepsilon\{x_0,\ldots,x_m\}$ (up to ϕ_1-equivalence), then the I.P.Q.F. using these knots is exact on $\Phi^\gamma\times\Phi^n$, $\gamma+n=2m+1$, and A is diagonal. The converse is also true.

PROOF: We only need comment on the converse. Assume $a_{ij}=0$, $i\neq j,i,j=0,\ldots,n$ (by renumbering knots if necessary). Clearly, if the y_j's are notϕ_1-equivalent to x_i's, we have two distinct sets of functions orthogonal to each other, with respect to the weight function $w\prod_{i=n+1}^{m}(\phi_1-\phi_1(x_i))$. Thus, the moment matrix with respect to this new weight function has full rank (Theorem 2.5.9). This contradicts Corollary 2.8.11. ///

COROLLARY 2.10.18 Assume an I.P.Q.F. as postulated in Theorem 2.8.8 exists. Then, if the knots are chosen as in Theorem 2.10.17, the matrix A will be diagonal. The converse is also true. ///

The analogue of Corollary 2.8.10 obviously holds also. Having examined this case in some detail, we can also obtain some information for a more general situation, see [25,26,27], when $S_i,T_j\varepsilon\chi_1,i,j=0,1=m=n$.

THEOREM 2.10.19 Assume Φ^2 and Ψ^1 are CT sets on R. There is exactly one I.P.Q.F. exact on $\Phi^2\times\Psi^1$ with A being a diagonal

matrix. The diagonal elements of A are positive.

PROOF: m=1 and exactness on $\Phi^2 \times \Psi^1$ force x_o and x_1 to be the zeros of τ_1. Diagonality requires that $I(p_o;q_1) = I(p_1;q_o)=0$. If y_o, y_1 exist such that the required I.P.Q.F. exists, y_o is a zero of q_1 and y_1 a zero of q_o. Let λ be the ϕ-polynomial of degree 2 such that

$$\lambda(x_o) = 1$$

and $$\lambda(x_1) = 0 \qquad \text{(a nonnodal zero)}.$$

Thus $\lambda - p_o$ is zero at x_o and x_1 and so $\lambda - p_o$ is proportional to τ_1. Thus $I(\lambda - p_o;q) = 0$ for any $q \varepsilon sp\Psi^1$,

$$\Rightarrow I(\lambda;q) = I(p_o;q), \quad \forall q \varepsilon sp\Psi^1 .$$

Thus we need q, non trivial, with a zero which we will choose to be y_o, such that

$$I(\lambda;q_1) = 0 \quad (\Rightarrow I(p_o;q_1) = 0).$$

Let $\hat{q}$ be a ψ-polynormal of degree 1, with a zero in R. If $I(\lambda;\hat{q}) = 0$ set $q_1 = \hat{q}$. Else, since ψ_o is of one sign, say positive (Ψ^1 a CT-set), thus $I(\lambda;\psi_o)$ is non-zero, choose $q_1 = \hat{q} - \alpha\psi_o$ where

$$I(\lambda;\hat{q}) = \alpha I(\lambda;\psi_o).$$

$\hat{q}$ and q_1 contain ψ_1, with non zero coefficient (by their construction), thus since $I(\lambda;q_1)=0$, q_1 must have a zero in R. We obtain y_1 analogously (q_o and q_1 are distinct else we contradict orthogonality). We now show that $a_{oo}>0$

(a_{11} analogously).

$$a_{oo} = I(p_o;q_o) = I(\lambda;q_o).$$

But $q_o=\alpha q_1+\beta\psi_o$ ($\alpha\neq 0$ since q_o has a zero in R, and $\beta\neq 0$ since $q_o\neq q_1$). Thus $a_{oo} = \beta I(\lambda_1;\psi_o)>0$. ///

We observe that Theorem 2.10.18 is symmetric in m and n, and that it is valid on preassigning x_o and x_1 (distinct) arbitrarily, although we then have exactness on $\Phi^1\times\Psi^1$ only. Further, we note that the proof does not extend to m and/or n larger than 1.

§2.11 I.P.Q.F. AND GT-SETS

We begin with some straightforward comments based on the properties of linear functionals. It is elementary that, given two r x r non-singular distinct matrices F and G that there are unique, distinct non-singular transformation matrices W_1 W_2 such that

(2.11.1) $$F = W_1G,$$

(2.11.2) $$F = GW_2,$$

(2.113) $$W_3F = G \quad (W_3=W_1^{-1}),$$

(2.11.4) $$FW_4 = G \quad (W_4=W_2^{-1}).$$

Now let F,G be r×s (r<s) and of full rank. It follows that W_2 (similarly W_3), now s×s, always exists, has rank at least r, and that s-r elements in each column of W_2 may be preassigned. By carefully preassigning elements we may guarantee that W_2 has rank s (or any number between r and s inclusive). W_2 is of course not unique, thus $W_4=W_2^{-1}$ need not hold. On the other hand W_1 if it exists will be unique, and it will exist if and only if the following conditions are fulfilled:

(2.11.5) Some given r columns in both F and G are linearly independent.

(2.11.6) In both F and G all other columns are the same linear combination of the given s linearly independent columns.

Clearly, if W_1 exists, then $W_3=W_1^{-1}$. If r>s then by interchanging W_1 and W_2, r and s, and rows for columns, the above comments remain valid.

Now we examine the question of replacing one set of elementary functionals, say $\{S_i\}_{i=0}^{m}$, by another set, $\{\hat{S}_i\}_{i=0}^{m}$. Naturally, we assume that any replacement set of elementary functionals obeys (A7) if they are not a GT-set. The comments when considering $\{T_j\}_{j=0}^{n}$ (and we consider the case r>s above) are analogous. From our initial remarks it is easy to obtain:

THEOREM 2.11.1 Assume we are given an I.P.Q.F. exact on

$\phi^{\gamma}\times\psi^{\delta}$ $\gamma\leq m$. Then we may replace $\{S_i\}_{i=0}^{m}$ by $\{\hat{S}_i\}_{i=0}^{m}$, if we replace the coefficient matrix A by WA where W is a transformation matrix such that $B=\hat{B}W$, where B is given by (1.3.7), and $\hat{B}$ on replacing S_i with $\hat{S}_i$ in (1.3.7). If $\gamma=m$ the matrix W is of full rank and unique. ///

THEOREM 2.11.2 If, in Theorem 2.11.1 $\gamma>m$, the conclusion is valid if and only if (2.11.5), (2.11.6) hold. If the matrix W exists it is unique and of full rank.
(Here we have $F = \{S_j(\phi_i)\}_{i=0,j=0}^{\gamma,\ m} = B_{\gamma,m}$ and
$G = \{\hat{S}_j(\phi_i)\}_{i=0,j=0}^{\gamma,\ m} = \hat{B}_{\gamma,m}$. (cf (2.11.2))) ///

From §2.10 it is clear that the rank of the matrix WA will be at least $\delta+1$ in the above results (we may be able to say more, depending on the nature of $\{S_i\}_{i=0}^{m}$ and $\{\hat{S}_i\}_{i=0}^{m}$).

We see that the conditions (2.11.5) and (2.11.6) in Theorem 2.11.2 are in practice conditions on the choice of parameters of $\{\hat{S}_i\}_{i=0}^{m}$. Consequently we end up examining conditions similar to (2.4.7), and the coefficients α_{ir} mentioned are determined by linear combinations of rows in $B_{\gamma,m}$.

Returning to Theorem 2.4.3 we recall that the way in which we arranged for conditions (2.4.14) - (2.4.16) in §2.6 - §2.8 to be satisfied was to obtain the orthogonal (or W-orthogonal) polynomials h_r, $r=1,\ldots,\gamma-m$, and then to see if we could guarantee that (2.4.15) and (2.4.16) were satisfied. This means that the coefficients α_{ir} of h_r in (2.4.10) are fixed by orthogonality conditions which are independent of our choice

of elementary functionals S_i and T_j. In turn, this means that the coefficients α_{ir} in (2.4.7) are determined independent of our choice of functionals S_i. Thus it is apparent that the coefficients α_{ir} are independent of the method used (i.e. trying to replace one set $\{S_i\}_{i=0}^m$ of elementary functionals by another $\{S_i\}_{i=0}^m$ or examining the moment equations (using $\{S_i\}_{i=0}^m$) directly) in obtaining them. Hence, emphasizing (2.4.15), we see that to determine whether an I.P.Q.F. exact on $\Phi^\gamma \times \Psi^n$ exists we need only consider the action of $\{S_i\}_{i=0}^m$ on same special ϕ-polynomials, which are W-orthogonal to Ψ^n. This means we have two ways of obtaining an I.P.Q.F. using $\{\hat{S}_i\}_{i=0}^m$, once we have parameters such that $\hat{S}_i$ acting on the (W-) orthogonal polynomials is zero, i=0,...,m, then we may either determine W as in Theorem 2.11.2 if another I.P.Q.F. exact on $\Phi^\gamma \times \Psi^n$ using functionals $\{S_i\}_{i=0}^m$ is already known, or we are able to obtain the matrix A as in §2.9. Clearly the existence of an I.P.Q.F. exact on $\Phi^\gamma \times \Psi^n$, $\gamma > m$, using $\{S_i\}_{i=0}^m$, does not imply the existence of an I.P.Q.F. exact on $\Phi^\gamma \times \Psi^n$ using functionals $\{\hat{S}_i\}_{i=0}^m$.

The transformation matrix W can be calculated directly using discrete vector scalar products, from the square matrices B $(B_{m,m})$ and $\hat{B}$, as follows:

(a) Find an upper triangular matrix, U, such that

$$\hat{B}^* = \hat{B}U, \tag{2.11.7}$$

where $\hat{B}^*$ has columns which are orthonormal to each other, and span the same vector space as the columns of $\hat{B}$. This may be done by using the discrete version

of the Gram-Schmidt orthonormalization process (see, e.g., [116]).

(b) Obtain the matrix W* such that

$$B = \hat{B}^* W^*. \tag{2.11.8}$$

This is straightforward now that the columns of $\hat{B}^*$ are orthonormal, i.e. $(\hat{B}^*)^T(\hat{B}^*) = I_m$ (I_m is the $(m+1)\times(m+1)$ identity matrix).

(c) The matrix W is now given by

$$W = U(\hat{B}^*)^T B = UW^* \tag{2.11.9}$$

A great deal of analysis in this chapter has dealt with T-sets, and thus with χ_1-functionals. There are two reasons for this. First, the most common type of elementary functional to be used is χ_1. Second, using point evaluation, and thus being interested in the zeros of functions, we are able to make use of some known properties of integrals of functions with certain zero properties, in particular the fact that the integral of a non-trivial function of one sign is non-zero. It is hard to obtain similar properties when using other types of elementary functional. However, from the comments made earlier in this section it is clear that much of the analysis depends upon orthogonality which is independent of the choice of the elementary functionals, and some results that have been proved using properties of functionals can be proved for more general cases using orthogonality properties. In particular, we can

extend Theorem 2.3.3 (and Corollary 2.3.4). Given Φ^{γ}, $\gamma>m$, satisfying (2.8.1), and Ψ^{δ}, we define the moment matrices $M^{r}_{m+1,\ell}$ $r=1,\ldots,\gamma-m$, $\ell=(\gamma-m)(\delta+1)$, (cf (2.5.1), (2.5.2)) as the moment matrices, with respect to I, using the set of functions

$$\Phi^{r}_{m} = \{\phi_{o},\ldots,\phi_{m},\phi_{m+r}\}, \tag{2.11.10}$$

$$\textcircled{H}_{\ell} = \{\phi_{o}\psi_{o},\ldots,\phi_{o}\psi_{\delta};\ldots;\phi_{\gamma-m}\psi_{o},\ldots,\phi_{\gamma-m}\psi_{\delta}\}. \tag{2.11.11}$$

THEOREM 2.11.3 Let Φ^{γ}, $\gamma>m$, satisfy (2.8.1) and Ψ^{δ}, $\delta>0$, $\ell=(\gamma-m)(\delta+1)>m+1$, be such that the rank of one of the moment matrices $M^{r}_{m+1,\ell}$, $r=1,\ldots,\gamma-m$, is $m+2$. Then there cannot exist an I.P.Q.F. exact on $\Phi^{\gamma}\times\Psi^{\delta}$ when $S_i\varepsilon\chi_1$.

PROOF: Choose r such that rank $M^{r}_{m+1,\ell}$ is $m+2$. In calculating h_r (see (2.4.10)), W-orthogonal to at least $m+2$ functions (cf (2.8.8)-(2.8.9)) we are attempting to find a non-trivial solution to a homogeneous system of equations, in which the matrix has full rank. This is impossible. ///

We conclude this section with a Theorem, the proof of which emphasizes the role orthogonality plays in our considerations.

THEOREM 2.11.4 Let Φ^{m+1} satisfy (2.8.1). Let Ψ^{m} be a T-set. Assume $\phi_1,\psi_j\varepsilon C^1(R)$, $j=0,\ldots,m$ and ϕ_1' has no zeros. Let $x_o,\ldots,x_m$ be m ϕ_1-distinct zeros of τ_m W-orthogonal to Ψ^{m}. Assume $S_i\varepsilon\chi_1$, $S_i\equiv S_i(x_i;.)$, $i=1,\ldots,m$. Let $z_o,\ldots,z_r$ be the zeros of $(\tau_m)'$. Then for at least one $z\varepsilon\{z_o,\ldots,z_r\}$ defining S_o to be derivative evaluation at z, there is an

I.P.Q.F. exact on $\Phi^{m+1}\times\Psi^{m}$ (m=n).

PROOF: Examine the matrix

$$(2.11.12)\qquad B = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ \phi_1'(z) & \phi_1(x_1) & \cdots & \phi_1(x_m) \\ 2\phi_1(z)\phi_1'(z) & \phi_1^2(x_1) & \cdots & \phi_1^2(x_m) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ m\phi_1^{m-1}\phi_1'(z) & \phi_1^m(x_1) & \cdots & \phi_1^m(x_m) \end{bmatrix} .$$

The matrix obtained by deleting the first column and last row is nonsingular (Corollary 2.2.17). Let

$$(2.11.13)\qquad \xi = \prod_{i=1}^{m}(\phi_1-\phi_1(x_i)) = \sum_{j=0}^{m-1}\alpha_j\phi_1^j+\phi_1^m.$$

B will be singular if z is a zero of

$$(2.11.14)\qquad \theta = \phi_1'(m\phi_1^{m-1}+\sum_{j=0}^{m-1} j\alpha_j\phi_1^{j-1}) .$$

θ has at most m-1 ϕ_1-distinct zeros. On the other hand $(\tau_m)'$, since τ_m has at least m+1 nodal ϕ_1-distinct zeros, has at least $m\phi_1$-distinct zeros. We can now choose z to be any zero of $(\tau_m)'$ which is not also a zero of θ, thus ensuring that B is nonsingular. Also, assuming that

$$\tau_m = \phi_1^{m+1} + \sum_{i=0}^{m}\beta_i\phi_1^i ,$$

as required for there to be an I.P.Q.F. exact on $\Phi^{m+1}\times\Psi^{m}$

we clearly have

$$((m+2)\text{nd row of } B_{m+1,m})+\sum_{i=0}^{m}\beta_i((i+1)\text{st row of } B_{m+1,m})\equiv 0.$$

///

COROLLARY 2.11.5 Let everything be as in Theorem 2.11.4 except that ϕ_1' may now have zeros. The conclusion of Theorem 2.11.4 remains valid provided there is a zero of τ_m' which is not a zero of θ given by (2.11.14). ///

COROLLARY 2.11.6 If in addition to the conditions of Theorem 2.11.4, (2.8.2) holds, the conclusion of Theorem 2.11.4 is valid for $\Phi^{\gamma}\times\Psi^{n}$, $\gamma+n=2m+1$. ///

Clearly the I.P.Q.F. of Theorem 2.11.4 and its Corollaries are by no means unique, since, apart from the possible choices of the point z, there are $\binom{s}{m}$ ways of choosing the knots $x_1,\ldots,x_m$, where $s>m$ is the number of ϕ_1-distinct zeros that τ_m possesses. We may extend Theorem 2.11.4 to deal with interpolatory I.P.Q.F.:

COROLLARY 2.11.7 Let Φ^m satisfy (2.8.1). Assume $\phi_1,\psi_j\in C'(R)$, $j=0,\ldots,m$. Assume $S_i\in\chi_1, x_i$ fixed and ϕ_1-distinct, $i=1,\ldots,m$. Let $\xi=\prod_{i=1}^{m}(\phi_1-\phi_1(x_i))$.

Let S_o be derivative evaluation at z. If $\xi'(z)\neq 0$, then $\det(B)\neq 0$, B given by (2.11.12):

PROOF: See Theorem 2.11.4. ///

For S_i as in Corollary 2.11.7 we see T_j are still arbitrary (so long as $\det(C)\neq 0$) and S_o has only a weak condition applied to it, so the elementary functionals S_i retain many degrees of freedom in their choice. Corollary 2.11.7 may be extended further.

COROLLARY 2.11.8 Assume the conditions of Corollary 2.11.7, except assume $S_i \varepsilon \chi_1, x_i$ fixed and ϕ_1-distinct, $i=r,\ldots,m$. Let ξ_j be the ϕ-polynomial determined by

$$\alpha_{jr}\phi_1^o+\ldots+\alpha_{jm}\phi_1^{m-r}-\phi_1^{m-r+j}\Big|_{x_i}=0,\ j=1,\ldots,r,\ i=r,\ldots,m$$

Let $z_o,\ldots,z_{r-1}$ be such that

(1) $\phi_1'(z_i)\neq 0 \quad i=0,\ldots,r-1$

(2) z_i are ϕ_1-distinct $\quad i=0,\ldots,r-1$

(3) $\det(\xi_j'(z_i))_{j=1,i=0}^{r,r-1}\neq 0$

Then $\det B\neq 0$, B given by (2.11.15).

PROOF: Analogous to Theorem 2.11.4 on considering

$$(2.11.15)\qquad B=\begin{bmatrix} 0 & \ldots, & 0, & 1 & \ldots, & 1 \\ \phi_1'(z_o) & \ldots,\phi_1'(z_{r-1}) & & \phi_1(x_r)\ldots,\phi_1(x_m) & & \\ 2\phi_1\phi_1'(z_o) & \ldots,2\phi_1\phi_1'(z_{r-1}) & & \phi_1^2(x_r)\ldots,\phi_1^2(x_m) & & \\ \vdots & & & & & \\ m\phi_1^{m-1}\phi_1'(z_o) & \ldots,m\phi_1^{m-1}\phi_1'(z_{r-1}) & & \phi_1^m(x_r)\ldots\phi_1^m(x_m) & & \end{bmatrix}$$

///

CHAPTER 3: COMPOUND I.P.Q.F.

§3.1 USUAL COMPOUNDING

We remember that a prime reason for investigating I.P.Q.F. is the desire to make full use of the differing characteristics of the functions f and g in the bilinear functional I (see (1.3.1)). We should not forget this when we use an I.P.Q.F. in a compound manner. When a R.Q.F. is compounded, the usual procedure is to split the interval of integration into a number of subintervals (usually of equal length, for simplicity), apply the R.Q.F. to each subinterval, and sum the results. It is likely when using an I.P.Q.F. that we will want to use different selections of subintervals when considering the functions f and g. Some comments in connection with the standard polynomial case are made in [12], and a more detailed investigation is carried out in [20], based on evaluating the integral (2.9.7) exactly, when it has been modified to deal with the compound I.P.Q.F. We employ a different approach.

In order to discuss compound I.P.Q.F., we must initially establish the criteria under which we may change the interval of integration and yet retain exactness. Assume that we have an interpolatory I.P.Q.F. exact on $\Phi^{\gamma}\times\Psi^{\delta}$, with $\gamma\geq m$, $\delta\geq n$. For generality we denote the interior of R by (a,b) (not (A4)), and the I.P.Q.F. by Q_R. We seek a shifted I.P.Q.F., Q_K, exact on an interval K with interior, say, (c,d). We want Q_K to be exact on ${}^{(K)}\Phi^{\gamma}\times{}^{(K)}\Psi^{\delta}$ (i.e. $\{{}^{(K)}\phi_i\}_{i=0}^{\gamma}\times\{{}^{(K)}\psi_j\}_{j=0}^{\delta}$), where

$$(3.1.1)\qquad \begin{cases} {}^{(K)}\phi_i\left(\dfrac{(d-c)t+bc-ad}{b-a}\right) \equiv \phi_i(t), & i=0,\ldots,\gamma, \\ {}^{(K)}\psi_j\left(\dfrac{(d-c)t+bc-ad}{b-a}\right) \equiv \psi_j(t), & j=0,\ldots,\delta, \\ t\varepsilon R. \end{cases}$$

The elementary functions used in Q_K, $\{{}^{(K)}S_i\}_{i=0}^{m}$ and $\{{}^{(K)}T_j\}_{j=0}^{n}$, will obviously be of the same form as $\{S_i\}_{i=0}^{m}$ and $\{T_j\}_{j=0}^{n}$ but probably with a different choice of parameter. Naturally, we assume

$$(3.1.2)\qquad \begin{cases} S_i(\phi_\mu) = {}^{(K)}S_i({}^{(K)}\phi_\mu), & i=0,\ldots,m,\ \mu=0,\ldots,\gamma, \\ T_j(\psi_\nu) = {}^{(K)}T_j({}^{(K)}\phi_\nu), & j=0,\ldots,n,\ \nu=0,\ldots,\delta. \end{cases}$$

Now Q_K approximates I_K, the bilinear integration functional over K with respect to weight function w_K, where

$$(3.1.3)\qquad w_K\left(\frac{(d-c)t+bc-ad}{b-a}\right) \equiv w(t),\ t\varepsilon R.$$

Define Q_K by

$$(3.1.4)\qquad Q_K(f;g) \equiv \left(\frac{d-c}{b-a}\right){}^{(K)}\underline{f}^T A\,{}^{(K)}\underline{g},$$

where A is given by Q

$${}^{(K)}\underline{f}^T = ({}^{(K)}S_o(f),\ldots,{}^{(K)}S_m(f)),$$

$${}^{(K)}\underline{g}^T = ({}^{(K)}T_o(g),\ldots,{}^{(K)}T_n(g)).$$

The following Lemma is immediate (see [12]):

LEMMA 3.1.1 Let Q be an I.P.Q.F. exact on $\Phi^{\gamma}\times\Psi^{\delta}$. The shifted I.P.Q.F. Q_K, defined by (3.1.4) is exact on ${}^{(K)}\Phi^{\gamma}\times{}^{(K)}\Psi^{\delta}$, defined in (3.1.1). ///

We note that if, assuming ϕ_i and ψ_j are defined on K, we replace (3.1.1) by

$$(3.1.5)\qquad \begin{cases} {}^{(K)}\phi_i(s) = \phi_i(s) & i=0,\ldots,\gamma, \\ {}^{(K)}\psi_j(s) = \psi_j(s) & j=0,\ldots,\delta, \\ s\varepsilon K. \end{cases}$$

the conclusion of Lemma 3.1.3 is not valid in general. However, if as in the case of standard polynomials, i.e. (2.8.1) - (2.8.2), and $\phi_1\equiv t$, $t\varepsilon R$, hold, the conclusion of Lemma 3.1.1 remains true on replacing (3.1.1) by (3.1.5).

Noting the condition (3.1.3) we see that, except in very special circumstances, where w is periodic, there is no point in considering compound I.P.Q.F. unless, as we shall assume for the rest of this section, $w\equiv 1$. We now need some notation:

DEFINITION 3.1.1 An r-partition of the interval R is a partition of R into r non-trivial, non-intersecting (but not necessarily equal length) subintervals,

$$(-1,W_1),(W_1,W_2),\ldots,(W_{r-2},W_{r-1}),(W_{r-1},1)$$

(we sometimes will use $-1\equiv W_o$, $1\equiv W_r$). We denote this partition by

(3.1.6) $\{-1,W_1,\ldots,W_{r-1},1\}$. ///

DEFINITION 3.1.2 Let M(N) be a u(v) partition of R. By ${}^{M}\Phi^{\gamma}$ (${}^{N}\Psi^{\delta}$) we mean the set of functions which are ${}^{\mu}\phi$-(${}^{\nu}\psi$)-polynomials of degree at most $\gamma(\delta)$ on the μ-th (ν-th) subinterval of the partition $M\equiv\{-1,X_1,\ldots,X_{u-1},1\}$ ($N\equiv\{-1,Y_1,\ldots,Y_{v-1},1\}$), $\mu=1,\ldots,u$ ($\nu=1,\ldots,v$). ///

In the above definition, a ${}^{\mu}\phi$-polynomial is a polynomial in ${}^{(X_{\mu-1},X_{\mu})}\Phi^{\gamma}$ defined by (3.1.1) (on replacing K with $(X_{\mu-1},X_{\mu})$). We define a ${}^{\nu}\psi$-polynomial analogously.

DEFINITION 3.1.3 An (M,N) copy of an I.P.Q.F. Q is an I.P.Q.F., denoted ${}^{(M,N)}Q$, with the properties

(a) If Q is exact on $\Phi^{\gamma}\times\Psi^{\delta}$, then ${}^{(M,N)}Q$ is exact on ${}^{M}\Phi^{\gamma}\times{}^{N}\Psi^{\delta}$, where M and N denote partitions of R.

(b) The elementary functionals used by ${}^{(M,N)}Q$, denoted by $\{{}^{\mu}S_i\}_{i=0}^{m}$, $\{{}^{\nu}T_j\}_{j=0}^{n}$, are such that (3.1.2) holds (the interval K is now $(X_{\mu-1},X_{\mu})$ ($(Y_{\nu-1},Y_{\nu})$)). $\mu=1,\ldots,u(\nu=1,\ldots,v)$. ///

The I.P.Q.F. ${}^{(M,N)}Q$ can be written

(3.1.7) $${}^{(M,N)}Q = ({}^{M}\underline{f})^{T}\,{}^{(M,N)}A\,({}^{N}\underline{g}) ,$$

where ${}^{(M,N)}A$ is a $u(m+1)\times v(n+1)$ real matrix, and

$$(3.1.8)\qquad \begin{cases} ({}^{M}\underline{f})^{T} = ({}^{1}\underline{f}^{T},\ldots,{}^{u}\underline{f}^{T}), \\ {}^{\mu}\underline{f}^{T} = ({}^{\mu}S_{o}(f),\ldots,{}^{\mu}S_{m}(f)),\ \mu=1,\ldots,u, \end{cases}$$

$$(3.1.9)\qquad \begin{cases} ({}^{N}\underline{g})^{T} = ({}^{1}\underline{g}^{T},\ldots,{}^{v}\underline{g}^{T}), \\ {}^{\nu}\underline{g}^{T} = ({}^{\nu}T_{o}(g),\ldots,{}^{\nu}T_{n}(g)),\ \nu=1,\ldots,v. \end{cases}$$

In the most common case, in which $S_i(T_j)$ is function evaluation at $x_i(y_j)$, it follows that

$$(3.1.10)\qquad {}^{\mu}S_i(f) \equiv f(x_{\mu i}),\ x_{\mu i} = X_{\mu}+\frac{(x_i+1)}{2}(X_{\mu+1}-X_{\mu}),$$

$$i=0,\ldots,m,\ \mu=1,\ldots,u,$$

$$(3.1.11)\qquad {}^{\nu}T_j(g) \equiv g(y_{\nu j}),\ y_{\nu j}=Y_{\nu}+\frac{(y_j+1)}{2}(Y_{\nu+1}-Y_{\nu}),$$

$$j=0,\ldots,n,\ \nu=1,\ldots,v.$$

In [20] a third desirable property for a compound I.P.Q.F. is given namely,

(3.1.12) When f is set equivalent to 1, both Q and ${}^{(M,N)}Q$ reduce to R.Q.F., say $Q^{(n)}$ and $Q^{(vn)}$, where $Q^{(vn)}$ is the N-copy of $Q^{(n)}$. An analogous result holds on setting g≡1.

This property patently requires that $1\varepsilon sp\Phi^{\gamma}$ and $1\varepsilon sp\Psi^{\delta}$, and this may not always be the case.

Assuming that all the elementary functions ${}^{\mu}S_i$ and ${}^{\nu}T_j$ exist, we need to calculate the matrix ${}^{(M,N)}A$, preferably from the matrix A. In the process of our calculations we note that

in each subinterval of a partition of R we may use any set of $\gamma+1$ ($\delta+1$) functions which span ${}^{\mu}\Phi^{\gamma}({}^{\nu}\Psi^{\delta})$, including, of course, $\{{}^{\mu}\phi_i\}_{i=0}^{\gamma}$ ($\{{}^{\nu}\psi_j\}_{j=0}^{\delta}$). This is of particular interest when the functions ϕ_i,ψ_j are monomials since then we can (if we wish) replace ${}^{\mu}\phi_i({}^{\nu}\psi_j)$ by $\phi_i(\psi_j)$, $\mu=1,\ldots,u$ ($\nu=1,\ldots,v$).

It is not immediately clear that compound I.P.Q.F. even exist, let alone how they may be derived. Using the same approach as in §2.4 we write down the system of linear equations which the elements of the matrix ${}^{(M,N)}A$ must satisfy. We can do this by assuming that f(g) is equivalently zero on all subintervals of the partition M(N) except the μ-th (ν-th), on which we set $f\equiv{}^{\mu}\phi_i$ ($g\equiv{}^{\nu}\psi_j$), $i=0,\ldots,m$ ($j=0,\ldots,n$), for each $\mu=1,\ldots,u$ ($\nu=1,\ldots v$). For the present we assume $\gamma=m$ and $\delta=n$. We comment on what happens if we require $\gamma>m$ (or $\delta>n$) later.

From (3.1.1) and (3.1.2) it is clear that

$$(3.1.10)\qquad {}^{\mu}B \equiv ({}^{\mu}S_i({}^{\mu}\phi_r))_{i=0,r=0}^{m,m} = B, \quad \mu=1,\ldots,u,$$

where B is given by (1.3.7); and

$$(3.1.11)\qquad {}^{\nu}C = ({}^{\nu}T_j({}^{\nu}\psi_r))_{j=0,r=0}^{n,n} = C = (c_{\ell k})_{\ell,k=0}^{n},\nu=1,\ldots,v,$$

where C is given by (1.3.8). It follows that the system of linear equations we seek is

$$(3.1.12)\qquad {}^{(M,N)}D\,{}^{(M,N)}\underline{b} = {}^{(M,N)}\underline{\Lambda}\ .$$

Some notation helps us in defining ${}^{(M,N)}D$.

DEFINITION 3.1.4 Given an (r+1) × (r+1) matrix $F = (f_{ij})_{i=0,j=0}^{r\ \ r}$, by $F[I]_t$ we mean the block matrix

$$(3.1.13) \qquad \begin{bmatrix} f_{oo}I_t,\ldots,f_{or}I_t \\ \cdot \\ \cdot \\ \cdot \\ f_{ro}I_t,\ldots,f_{rr}I_t \end{bmatrix} = F \otimes I_t,$$

where I_t is the (t+1)×(t+1) identity matrix. By $\mathrm{diag}_r(F_o,\ldots,F_r)$, or $\mathrm{diag}_r(F)$ when $F_i \equiv F_j$, i,j=0,...,r, we mean the block matrix

$$(3.1.14) \qquad \begin{bmatrix} F_o,0,\ldots,0 \\ 0,F_1,\ldots,0 \\ \cdot \\ \cdot \\ 0,\ldots,0,F_r \end{bmatrix},$$

where 0 denotes appropriately sized zero matrices. ///

We find

$$(3.1.15) \qquad {}^{(M,N)}D = \mathrm{diag}_{v-1}(C \otimes (\mathrm{diag}_{u-1}B)).$$

We now define

$$(3.1.16) \qquad {}^{\mu\nu}\underline{\Lambda}^r = (I({}^{\mu}\phi_o;{}^{\nu}\psi_r),\ldots,I({}^{\mu}\phi_m;{}^{\nu}\psi_r))^T,$$

$$r = 0,\ldots,n,$$

$$(3.1.17)\qquad {}^{\nu}\underline{\Lambda}^{r} = (({}^{1}\underline{\Lambda}^{r})^{T},\ldots,({}^{u\nu}\underline{\Lambda}^{r})^{T})^{T},$$

$$(3.1.18)\qquad {}^{\nu}\underline{\Lambda} = (({}^{\nu}\underline{\Lambda}^{0})^{T},\ldots,({}^{\nu}\underline{\Lambda}^{n})^{T})^{T},\quad \nu=1,\ldots,v,$$

and so derive

$$(3.1.19)\qquad {}^{(M,N)}\underline{\Lambda} = ({}^{1}\underline{\Lambda}^{T},\ldots,{}^{v}\underline{\Lambda}^{T})^{T}.$$

Let the matrix ${}^{(M,N)}A$ have elements b_{ij}, $i=0,\ldots,(m+1)u$, $j=0,..,(n+1)v$ (to avoid confusion with the elements of the matrix A). It is possible that for certain choices of elementary functionals (for example χ_1-functionals at the endpoints of intervals), some rows and/or columns of the matrix ${}^{(M,N)}A$ may be coalesced if the functions I operates on are sufficiently smooth. Set

$$(3.1.20)\qquad {}^{\mu j}\underline{b}^{T} = (b_{(m+1)(\mu-1),j-1},\ldots,b_{\mu(m+1)-1,j-1}),$$

$$\mu=1,\ldots,v,\ j=0,\ldots,v(n+1),$$

and then

$$(3.1.21)\qquad {}^{(M,N)}\underline{b}^{T} = ({}^{(1}\underline{b})^{T},\ldots({}^{u}{}_{1}b)^{T};\ldots;({}^{1,v(n+1)}\underline{b})^{T},\ldots,$$

$$({}^{u,v(n+1)}\underline{b})^{T})\ .$$

The large system of equations (3.1.12) may be broken down into many smaller systems. Let ${}^{(M,N)}A$ be made up of uv submatrices ${}^{\mu\nu}A$, $\mu=1,\ldots,u$, $\nu=1,\ldots,v$, each of which is an $(m+1)\times(n+1)$ matrix. We interpret ${}^{\mu\nu}A$ as determining the contribution made to I by the function f defined on $(X_{\mu-1},X_{\mu})$

and g defined on $(Y_{\nu-1},Y_\nu)$. An examination of ${}^{(M,N)}D$ shows that we can write

$$(3.1.22) \qquad D^{\mu\nu}\underline{a} = {}^{\mu\nu}\underline{A},\ \mu=1,\ldots,u,\ \nu=1,\ldots,v,$$

where D is given by (1.3.4), and

$$(3.1.23) \qquad {}^{\mu\nu}\underline{A} = (({}^{\mu\nu}\underline{A}^{o})^T,\ldots,({}^{\mu\nu}\underline{A}^{n})^T)^T\ ,$$

$$(3.1.24) \qquad {}^{\mu\nu}\underline{a} = ({}^{\mu,(\nu-1)(n+1)+1}\underline{b}^T,\ldots,{}^{\mu,\nu(n+1)}\underline{b}^T)^T,$$

$$(3.1.25) \qquad {}^{\mu\nu}A = ({}^{\mu,(\nu-1)(n+1)+1}\underline{b},\ldots,{}^{\mu,\nu(n+1)}\underline{b}).$$

In dealing with (3.1.22) we may utilise results obtained in the previous chapter, for instance we already know that the rank of D is maximal (Lemma 2.4.1). Thus the following are immediate:

LEMMA 3.1.2 If $I({}^{\mu}\phi_i;{}^{\nu}\psi_j) = 0$, $i=0,\ldots,m$, $j=0,\ldots,n$, then ${}^{\mu\nu}A$ is a matrix with all elements zero. This occurs when $(X_{\mu-1},X_\mu) \cap (Y_{\nu-1},Y_\nu)$ is the empty set. ///

Thus we will not have to solve (3.1.22) for all values of μ and ν.

THEOREM 3.1.3 Let Q be an I.P.Q.F. exact on $\Phi^m \times \Psi^n$. Let M and N be two partitions of R. Then the (M,N) copy of Q always exists and is unique. ///

THEOREM 3.1.4 Let Q be an I.P.Q.F. exact on $\Phi^{\gamma}\times\Psi^{\delta}$, $\gamma>m$. Let M be a partition of R. Then the (M,M) copy of Q always exists and is unique, and can be written as in (3.1.26).

PROOF: Let $M = \{-1,X_1,\ldots,X_{u-1},1\}$.

Then it is clear that

$$\mu\nu\underline{\Lambda} = \begin{cases} \frac{(X_\mu - X_{\mu-1})}{2}\,\underline{\Lambda}\,, & \mu=\nu \\ 0\,, & \mu\neq\nu\,. \end{cases}$$

Where $\underline{\Lambda}$ is given by (1.3.14). Hence we obtain

$$(3.1.26)\qquad {}^{(M,M)}A = \tfrac{1}{2}\begin{bmatrix} (X_1-X_0)A,0,\ldots,0 \\ 0,(X_2-X_1)A,\ldots,0 \\ \cdot \\ \cdot \\ \cdot \\ 0,\ldots,0,(X_\mu-X_{\mu-1})A \end{bmatrix}. \qquad ///$$

This Theorem means that in determining an (M,N) copy of an I.P.Q.F. we may as well assume that in the partitions M and N we find

$$(3.1.27)\qquad X_\mu \neq Y_\nu,\ \mu=1,\ldots,u-1,\ \nu=1,\ldots,v-1.$$

Returning to (3.1.22) we see that, on recalling (1.3.13), and defining a square matrix F such that

$$(3.1.28)\qquad \mu\nu\underline{\Lambda} = F\underline{\Lambda},$$

it follows that

(3.1.29) $$ {}^{\mu\nu}\underline{a} = D^{-1}\ {}^{\mu\nu}\underline{\Lambda}, $$

$$ = D^{-1}\ F\underline{\Lambda}, $$

(3.1.30) $$ {}^{\mu\nu}\underline{a} = D^{-1}\ FD\underline{a}. $$

It is clear that there are many matrices F which satisfy (3.1.28). If ${}^{\mu\nu}\underline{\Lambda}$ can be calculated from $\underline{\Lambda}$, given the parameters of the subintervals (i.e. $X_{\mu}, X_{\mu-1}, Y_{\nu}$, and $Y_{\nu-1}$) in the form (3.1.28) we choose F to be the required transformation matrix. However, if we are merely given ${}^{\mu\nu}\underline{\Lambda}$ and $\underline{\Lambda}$ (i.e. we are forced to evaluate them independently in some way), we assume that F takes some convenient form, probably diagonal or near diagonal. Instead of using (3.1.30) it is possible that it will be more convenient to use (3.1.29), particularly if D^{-1} is easy to obtain and ${}^{\mu\nu}\underline{\Lambda}$ cannot be obtained from $\underline{\Lambda}$ by a transformation of the type (3.1.28). Note that solving (3.1.29) by determining D^{-1} also specifies a third method of obtaining the matrix A (see §2.9).

We now explain one method of determining D^{-1}. Analogous to the procedure mentioned earlier (§2.11), given a $(m+1)\times(m+1)$ matrix G of full rank it is possible, in a recursive manner, to find a lower triangular matrix T such that TG has rows which are orthonormal to each other. In particular let U and V be lower triangular matrices such that

(3.1.31) $$ (B^{T}U^{T})UB = I_{m}, $$

$$(C^T V^T)VC = I_n . \tag{3.1.32}$$

In passing we note that the ϕ-polynomials defined by

$$U \begin{bmatrix} \phi_o \\ . \\ . \\ . \\ \phi_m \end{bmatrix} = \begin{bmatrix} \xi_o \\ . \\ . \\ . \\ \xi_m \end{bmatrix} , \tag{3.1.33}$$

are orthonormal with respect to summation over $\{S_i\}_{i=0}^m$, that is,

$$\begin{cases} \sum_{i=0}^m S_i(\xi_k)S_i(\xi_\ell) = \delta_{k\ell} , \\ k,\ell = 0,\ldots,m;\ \xi_k,\xi_\ell \text{ in (3.1.33)}. \end{cases} \tag{3.1.34}$$

An analogous result holds for ψ-polynomials on replacing U by V. This generalizes the results given in [120] and it is of interest to note it seems possible that the elements of the matrix U may be calculated more efficiently by using a method of partitioning given in [121].

We now find that

$$(\mathrm{diag}_n(B^T U^T U))D = C[I]_n , \tag{3.1.35}$$

and thus

$$(C^T V^T V)[I_n](\mathrm{diag}_n(B^T U^T U)) = D^{-1} . \tag{3.1.36}$$

Consequently the cost of obtaining D^{-1} is that of obtaining the matrices U and V, and then evaluating the expression on

the left hand side of (3.1.36). Apart from calculating the matrices U and V recursively as above, the following results are sometimes useful:

THEOREM 3.1.5 Assume $S_i=T_i\varepsilon\chi_1$, $i=0,\ldots,m$, and (2.8.1) - (2.8.2) hold. Let Q be an I.P.Q.F. exact on $\Phi^{m+1}\times\Phi^m$. Assume that Φ^m is orthonormal to itself with respect to I. Then Φ^m is an orthonormal set with respect to the discrete inner product.

$$(3.1.37) \qquad \sum_{i=0}^{m} a_i \xi(x_i)\theta(x_i), \quad \xi,\theta\varepsilon\Phi^m,$$

where the coefficient matrix $Q=\mathrm{diag}(a_0,\ldots,a_m)$ (x_i are zeros of ϕ_{m+1} orthogonal to Φ^{m+1}).

PROOF: That the coefficient matrix of Q is diagonal follows from Lemma 2.10.14. The rest of the theorem follows from Q being exact on $\Phi^m\times\Phi^m$ (see [3] also). ///

Thus in some circumstances the rows of U and V depend essentially on the coefficient of orthogonal ϕ- and ψ-polynomials written in terms of Φ^m and Ψ^n.

More generally, we have:

THEOREM 3.1.6 Assume that $S_i\equiv T_i$, $i=0,\ldots,m$, and that Φ^m is orthonormal to Φ^m. Let Q be exact on $\Phi^m\times\Phi^m$. Then Φ^m is orthonormal with respect to the discrete inner product defined by the I.P.Q.F. $Q^{(m,m)}$. ///

This means that we can obtain D^{-1} as follows:

(3.1.38) Let $\tilde{U}$ ($\tilde{V}$) be a lower triangular matrix in which the elements of the i-th (j-th) row are the coefficients of the ϕ-(ψ-) polynomial of degree i(j), in terms of $\Phi^i(\Psi^j)$, which is W-orthogonal to $\Phi^{i-1}(\Psi^{j-1})$, whose square is integrated to unity. Clearly, we choose $\tilde{u}_{oo}=I(\phi_o;\phi_o)^{-1}$ ($v_{oo}=I(\psi_o;\psi_o,)^{-1}$). From §2.5 we know that the ϕ-(ψ-) polynomials defined by $\tilde{U}(\tilde{V})$ are unique.

(3.1.39) Let $Q_S(Q_T)$ be the I.P.Q.F. exact on $\Phi^m\times\Phi^m(\Psi^n\times\Psi^n)$ in which $T_i\equiv S_i(T_j\equiv S_j)$, i=0,...,m, (j=0,...,n) where $S_i(T_j)$ are given. Denote its coefficient matrix by $A_S(A_T)$.

(3.1.40) We now find that

(3.1.41) $$(C^T\tilde{V}^TA_T\tilde{V})\ [I]_n\ (\mathrm{diag}_n(B^T\tilde{U}^TA_S\tilde{U})) = D^{-1}.$$

The process (3.1.38) - (3.1.40) will be useful if the I.P.Q.F. $Q_S(Q_T)$ is already known from elsewhere, since the orthonormalization process mentioned in (3.1.37), i.e. (2.5.9) - (2.5.12), is straightforward (see also Lemma 2.5.14).

Having obtained D^{-1} we use either (3.1.29) or (3.1.30) depending on how convenient it is to obtain the matrix F.

We find

$$(3.1.42)\qquad {}^{\mu\nu}\underset{i}{\Lambda}^{j} = {}^{\mu\nu}I\left(\phi_i\left(\frac{2t-(X_\mu+X_{\mu-1})}{X_\mu-X_{\mu-1}}\right);\ \psi_j\left(\frac{2t-(Y_\nu+Y_{\nu-1})}{Y_\nu-Y_{\nu-1}}\right)\right)$$

$$= {}^{\mu\nu}I({}^{\mu}\phi_i;{}^{\nu}\psi_j),\ t\in R,$$

where ${}^{\mu\nu}I$ is I restricted to the subinterval $(X_{\mu-1},X_\mu)\cap(Y_{\nu-1},Y_\nu)$. From (3.1.42) we see that if (2.8.1) - (2.8.2) hold with $\phi_1\equiv x$ (i.e. the standard polynomial of degree 1) we have

$$(3.1.43)\qquad {}^{\mu}\phi_i(t) = \sum_{r=0}^{i}\binom{i}{r}\left(\frac{X_\mu+X_{\mu-1}}{X_{\mu-1}-X_\mu}\right)^{i-r}\left(\frac{2}{X_\mu-X_{\mu-1}}\right)^{r}\phi_r(t).$$

An analogous relationship between ${}^{\nu}\psi_j$ and ψ_j holds. Let

$$(3.1.44)\qquad \begin{cases} \dfrac{X_\mu+X_{\mu-1}}{X_{\mu-1}-X_\mu} = \eta_\mu,\ \dfrac{2}{X_\mu-X_{\mu-1}} = \zeta_\mu,\ \mu=1,\ldots,u, \\ \dfrac{Y_\nu+Y_{\nu-1}}{Y_{\nu-1}-Y_\nu} = \eta_\nu,\ \dfrac{2}{Y_\nu-Y_{\nu-1}} = \zeta_\nu,\ \nu=1,\ldots,v, \end{cases}$$

$$(3.1.45)\qquad H_\mu = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \eta_\mu & \zeta_\mu & \cdots & 0 \\ \eta_\mu^2 & \binom{2}{1}\eta_\mu\zeta_\mu & \cdots & 0 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \eta_\mu^m & \binom{m}{1}\eta_\mu^{m-1}\zeta_\mu & \cdots & \binom{m}{m}\eta_\mu^{0}\zeta_\mu^{m} \end{bmatrix}.$$

H_ν is defined as H_μ, on replacing η_μ,ζ_μ, and m, by η_ν,ζ_ν, and n, respectively. We now find

$$(3.1.46)\qquad {}^{\mu\nu}\underline{\Lambda} = (\mathrm{diag}_n(H_\mu))H_\nu[I]_m\underline{\Lambda}$$
$$= (H_\nu[I]_m)(\mathrm{diag}_n(H_\mu))\underline{\Lambda}.$$

Note that (3.1.46) does not hold unless we assume (2.8.1) - (2.8.2) and $\phi_1 \equiv x$, and in this case (3.1.30) becomes

$$(3.1.47)\qquad {}^{\mu\nu}\underline{a} = D^{-1}(H_\nu[I]_m)(\mathrm{diag}_n(H_\mu))D\underline{a}.$$

This completes our treatment of compound I.P.Q.F. in the case $\gamma=m$, $\delta=n$. We now consider the case $\gamma>m$. Observe that there are three distinct ways the subintervals $(X_{\mu-1},X_\mu)$ and $(Y_{\nu-1},Y_\nu)$ may intersect:

$$(3.1.48)\qquad (X_{\mu-1},X_\mu) \subset (Y_{\nu-1},Y_\nu),$$

$$(3.1.49)\qquad (Y_{\nu-1},Y_\nu) \subset (X_{\mu-1},X_\mu),$$

(3.1.50) $(X_{\mu-1},X_\mu) \cap (Y_{\nu-1},Y_\nu)\neq\phi$, but neither (3.1.48) nor (3.1.49) hold.

Recall from Theorem 2.4.3 that if an I.P.Q.F. is exact on $\Phi^\gamma\times\Psi^n$, $\gamma>m$, this imposes conditions which are quite restrictive on the choices of parameters for $\{S_i\}_{i=0}^m$. If cases (3.1.49) or (3.1.50) occur we find we are imposing at least twice as many conditions on $\{S_i\}_{i=0}^m$. In general, we would expect that this leads to the non-existence of a solution. As an example, we note that if $S_i\varepsilon X_1$, and the I.P.Q.F. Q is exact on $\Phi^{m+1}\times\Psi^m$ (or $\Phi^\gamma\times\Psi^n$, $\gamma+n=2m+1$, and (2.8.1) - (2.8.2) hold) then all the roots of h_1 must be in the interval of integration.

In cases (3.1.49), (3.1.50), this is not possible (as we also have to consider $(Y_{\nu-2},Y_{\nu-1}) \cap (X_{\mu-1},X_{\mu})$ and $(Y_{\nu},Y_{\nu+1}) \cap (X_{\mu-1},X_{\mu})$). Consequently, unless the elementary functionals have some very unusual properties, we would not expect to obtain more than

LEMMA 3.1.7 Given an I.P.Q.F. exact on $\Phi^{\gamma}\times\Psi^{n}$, and two partitions, M and N, of R. If (3.1.48) is true $\mu=1,\ldots,u$, $\nu=1,\ldots,v$, then an (M,N) copy of the I.P.Q.F. exists and is unique. ///

We conclude this section with some further comments about compound I.P.Q.F.

(3.1.51) It is obvious that if the partitions M and N of R are given, and Q is exact on $\Phi^{\gamma_1}\times\Psi^{\delta_1}$ and $\Phi^{\gamma_2}\times\Psi^{\delta_2}$, $\max(\gamma_1,\gamma_2)>m$, then the (M,N) copy of Q will be exact on ${}^{M}\Phi^{\gamma_1}\times{}^{N}\Psi^{\delta_1}$ and ${}^{M}\Phi^{\gamma_2}\times{}^{N}\Psi^{\delta_2}$. Also we observe that we will have to require M and N are the same partition if $\max(\gamma_1,\gamma_2)>m$ and $\max(\delta_1,\delta_2)>n$.

(3.1.52) If we define ${}^{\mu}\Phi^{\gamma}$ and ${}^{\gamma}\Psi^{\delta}$ analogous to (3.1.5), instead of (3.1.1), then in the general case we are led to the consideration of more general "compound" I.P.Q.F. than we have considered above. In this situation we would be considering the sets of functions ${}^{\mu}\Phi^{\gamma}$ and ${}^{\nu}\Psi^{\delta}$ as depending on μ and ν respectively and possibly finding that, for example, ${}^{\mu_1}\Phi^{\gamma}$ and ${}^{\mu_2}\Phi^{\gamma}$, $\mu_1\neq\mu_2$

are not related to each other in any way. Of course, we could then make $\gamma(\delta)$ depend upon $\mu(\nu)$ as well, and we may consider not requiring (3.1.2) to hold. This situation could be analysed in much the same way as above: we would however find that ${}^{\mu}B$ and ${}^{\nu}C$ depended on μ and ν and thus the matrix D in (3.1.22) would depend upon μ and ν. However the method of solving (3.1.22) would still remain, in principle, the same although now somewhat more tedious.

(3.1.53) Considering (3.1.12) we have

LEMMA 3.1.8 If an I.P.Q.F. is interpolatory and $l \varepsilon sp\Phi^{\gamma}$, $l \varepsilon sp\Psi^{\delta}$, then its (M,N) copy has property (3.1.12).

PROOF: This follows immediately from the uniqueness of interpolatory I.P.Q.F. ///

We note that if the I.P.Q.F. is not interpolatory (for example if $\delta<n$) then Lemma 3.1.8 need not necessarily be valid.

(3.1.54) Bearing (3.1.52) in mind we note that a compound I.P.Q.F. as we have derived it in this section is exact on a larger class of functions than $M_{\Phi^{\gamma}} \times N_{\Psi^{\delta}}$. Let

(3.1.55) $$Z = \{-1, Z_1, \ldots, Z_{s-1}, 1\}, \quad s \leq u+v-1,$$

be the partition of R obtained on applying both the

partition M and N to R. Let

$$(3.1.56)\quad \begin{cases} {}^{r}\phi_i \equiv {}^{\mu}\phi_i, \text{ where } (Z_{r-1},Z_r)\subset(X_{\mu-1},X_\mu),\ \mu\in\{1,\ldots,u\}, \\ {}^{r}\psi_j \equiv {}^{\nu}\psi_j, \text{ where } (Z_{r-1},Z_r)\subset(Y_{\nu-1},Y_\nu),\ \nu\in\{1,\ldots,v\}, \\ r=1,\ldots,s. \end{cases}$$

From the construction of our compound I.P.Q.F. it is clear that ${}^{(M,N)}Q$ is in fact exact on

$$(3.1.57)\qquad Z_\Phi{}^{\gamma}\times Z_\Psi{}^{\delta}.$$

This result combines the approach we used in this section and, in a simplified form, the approach suggested in (3.1.52) which is also implicitly followed.

§3.2 SPLINES AND I.P.Q.F.

In deriving compound I.P.Q.F. in the previous section we did not impose any continuity restrictions on the integrands as they pass from one subinterval to the next. We generalize the usual definition of a polynomial spline as follows:

DEFINITION 3.2.1 Let $\{-1,W_1,\ldots,W_{s-1},1\}$ be an s partition of R. Let $\Xi^{\sigma_r}\equiv\{{}^{r}\xi_i\}_{i=0}^{\sigma_r}$ be defined on (W_{r-1},W_r), $r=0,\ldots,s+1$, $(W_{-1}=-\infty, W_{s+1}=\infty)$ and let ${}^{r}\xi_i$ be t times continuously differentiable in (W_{r-1},W_r), $r=1,\ldots,s$, $i=0,\ldots,\sigma_r$, $\sigma_r\geq t$. We say that a spline ξ, with deficiency $\underline{k}=(k_0,\ldots,k_s;t)$

of order $(\sigma_0,\ldots,\sigma_{s+1})$ with respect to $\Xi^{\sigma_0},\ldots,\Xi^{\sigma_{s+1}}$, is a function such that

(a) On (W_{r-1},W_r), $\xi\varepsilon sp\Xi^{\sigma_r}$ (a $^r\xi$-polynomial), $r=0,\ldots,s+1$,

(b) $\xi^{(\ell)}(W_r-0)=\xi^{(\ell)}(W_r+0)$, $\ell=0,\ldots,t-k_r$, $r=0,\ldots,s$.

We denote the class of all such splines by

$$\Omega\frac{W}{k}(\Xi^{\sigma_0},\ldots,\Xi^{\sigma_{s+1}})\equiv\Omega\frac{W}{k}(\underline{\Xi}), \tag{3.2.1}$$

where $\underline{W} = \{W_0,W_1,\ldots,W_s\}$,

$\underline{k} = (k_0,\ldots,k_s;t)$. ///

Clearly this definition can be extended in the case where R is an arbitrary interval. If their meaning is clear from the context we may delete or abbreviate $\underline{w}$, $\underline{k}$ or $(\underline{\Xi})$ in (3.2.1.) In particular if $k=k_i$, $i=0,\ldots,s$ we replace $\underline{k}$ by k,t, and if $^r\xi_i$ is defined by (3.1.1) for all r (K replaced by (W_{r-1},W_r)) we replace $(\Xi^{\sigma_0},\ldots,\Xi^{\sigma_{s+1}})$ with (Ξ). Again note that defining $^r\xi_i$ by (3.1.5), except in the case where (2.8.1) (on replacing ϕ by ξ) and $\xi_1\equiv t, t\varepsilon R$ hold, leads to the spaces of functions Ξ^{σ_i} being different from those mentioned in the above sentence.

We now observe that the compound I.P.Q.F. derived in the previous section are I.P.Q.F. exact on

$$\Omega^{(M)}_{\alpha+1,\alpha}(\Phi^\gamma)\times\Omega^{(M)}_{\beta+1,\beta}(\Psi^\delta), \tag{3.2.2}$$

where α and β depend on how many derivatives $\phi\varepsilon sp\Phi^\gamma$ and $\psi\varepsilon sp\Psi^\delta$ have. By setting $k=\alpha+1$, $\beta+1$, we mean that we do not even require continuity at M_i or N_j, $i=0,\ldots,u$, $j=0,\ldots,v$.

In fact, from (3.1.54) we know that we may replace M and N in (3.2.2) by Z ((3.1.55)), provided $\Xi^{s}{}_{r}$ in (3.2.1) are defined by (3.1.55), and this new set of splines includes both those mentioned in (3.2.2). Clearly the compound I.P.Q.F. will remain exact if the classes of splines in (3.2.2) have some continuity conditions imposed on them.

It is easy to see that I.P.Q.F. exact on various product spaces of splines may be regarded as completing a "spectrum" of results between I.P.Q.F. exact product spaces of "smooth" functions on the one hand and compound I.P.Q.F. on the other.

In addition, from §2.5, we note that, provided $l \in sp\Xi^{\sigma}{}_{r}$, $r=0,\ldots,s+1$, and $\Xi^{\sigma}{}_{r}$ are T-sets, then $\Omega^{W}_{\underline{k}}(\underline{\Xi})$ of (3.2.1) may be regarded as a WT-set.

In the literature there are, as far as we can discover, no references to splines as we defined them in Definition 3.2.1, and the majority of research appears to deal with polynomial splines (i.e. ${}^{r}\xi_{i}=t^{i}, t \in R$). However in [9] there is a more general definition of splines defined by means of differential operators. Further, there appear to be few results which deal directly with R.Q.F. exact on function spaces of splines and those which are available depend upon the integration of interpolatory splines (usually cubic polynomial splines). However, see [127,128,129]. This is perhaps a little surprising in view of the wealth of results available dealing with "best" R.Q.F. in the sense of Sard and the consequent investigation of the splines which constitute the Peano kernel of the Q.F. (see §4.2). For some summaries of this type of investigation see, for

example, [130,...,135].

In restricting ourselves to considering only χ_1 elementary functionals (as in almost all the literature) we notice that when we came to consider either compound Q.F., or even Q.F. exact on some class of splines, it is possible that a given elementary functional may contribute to evaluating a Q.F. on more than one subinterval of a partition, thus significantly reducing the total numbers of function evaluations required by the Q.F. Some work along these lines has been done, see [93].

CHAPTER 4: ERROR ANALYSIS

§4.1 IMMEDIATE RESULTS

Many well known error estimates used for R.Q.F. can be adapted to deal with I.P.Q.F. Some results of this type have already been obtained, particularly when dealing with I.P.Q.F. exact on polynomials, and having S_i, $T_j \varepsilon \chi_1$. See [12,14] and [28] in particular. Following [12], if an I.P.Q.F. is exact on $\Phi^{\gamma} \times \Psi^{\delta}$, the following relationships are an obvious consequence of the bi-linearity of both I and Q:

$$(4.1.1) \qquad E(f;g) = E(f-\phi;\psi) + E(f;g-\psi)$$

$$(4.1.2) \qquad E(f;g) = E(f-\phi;g) + E(f;g-\psi) - E(f-\phi;g-\psi)$$

$$(4.1.3) \qquad E(f;g) = E(f-\phi;g-\psi) + E(\phi;g-\psi) + E(f-\phi;\psi)$$

where $f \varepsilon \Gamma$, $g \varepsilon \Delta$, $\phi \varepsilon sp\Phi^{\gamma}$, $\psi \varepsilon sp\Psi^{\delta}$ and $E(f;g)$ is given by (1.3.4).

Let $\|\xi\|_{p,}$ denote the weighted L_p norm (see (2.1.11)), i.e.

$$(4.1.4) \qquad \|\xi\|_{p,w} = \left(\int_R w(x)|\xi(x)|^p dx\right)^{\frac{1}{p}}, \quad \xi \varepsilon L_{p,w}(R) ,$$

where $L_{p,w}(R)$ is the set of functions for which $\|\cdot\|_{p,w} < \infty$, and w is of one sign in R (see (A3)), $1 \le p \le \infty$. The discrete analogue of (4.1.4) is dealing with $\underline{\xi} = (\xi_0, \ldots, \xi_r)^T$,

$$(4.1.5) \qquad \|\xi\|_{p,w} = \left(\sum_{i=0}^{r} w_i |\xi_i|^p\right)^{\frac{1}{p}}, \quad w_i > 0, \ i = 0, \ldots, r.$$

Recall the generalized Cauchy-Schwarz Inequality;

$$(4.1.6) \qquad \int_R w(x)|\xi(x)\theta(x)|dx \le \|\xi\|_{p,w}\|\theta\|_{q,w}, \quad q=\frac{p}{p-1} .$$

(The discrete version is analogous.)

Since we have assumed $\Gamma\subset C^o(R)$, $\Delta\subset C^o(R)$ in (A8), we know that $f,g\in L_p(R)$ for all p (see 2.1.10).

We now examine (4.1.2) in more detail, obtaining

$$(4.1.7) \qquad E(f;g) = I(f-\phi;g) + I(f;g-\psi) - I(f-\phi;g-\psi) \\ -Q(f-\phi;g) - Q(f;g-\psi) + Q(f-\phi;g-\psi),$$

which implies

$$(4.1.8) \qquad |E(f;g)| \le |I(f-\phi;g) + I(f;g-\psi) + I(f-\phi;g-\psi)| \\ + |Q(f-\phi;g-\psi) - Q(f-\phi;g) - Q(f;g-\psi)|,$$

and, applying (4.1.6)

$$(4.1.9) \qquad |E(f;g)| \le \|f-\phi\|_{p_1,w}\|g\|_{q_1,w} + \|f\|_{p_2,w}\|g-\psi\|_{q_2 w} + \\ + \|f-\phi\|_{p_3,w}\|q-\psi\|_{q_3,w} + E_Q,$$

where $E_Q = |Q(f-\phi;g-\psi) - Q(f-\phi;g) - Q(f;g-\psi)|$.

$$q_i = \frac{p_i}{p_i-1} \qquad i=1,2,3.$$

Before pursuing (4.1.9) further, we make several comments. With regard to E_Q note that we already know $\underline{f}$ and $\underline{g}$ (as we approximate $I(f,g)$ by $Q(f,g)$). Consequently, if $S_i(\phi)$ and $T_j(\psi)$ are readily available then it is easy to evaluate the contribution to the right hand side of (4.1.9) due to E_Q exactly,

thus reducing our estimate of $E(f;g)$. If $S_i(\phi)$ and $T_j(\psi)$ are not immediately available and (as in most cases of interest) Q is exact on $\Phi^m \times \Psi^n$ then we may require ϕ and ψ to be interpolatory polynomials to f and g at $\{S_i\}_{i=0}^m$, and $\{T_j\}_{j=0}^n$, respectively (and thus $E_Q=0$). In this case we would hope that we had some means of approximating the various norms in (4.1.9) easily available. Clearly, the way in which we deal with (4.1.9) depends upon the knowledge and tools we have available. So far we have implied that it may be useful to actually obtain ϕ and ψ. In the situation where we know a lot about f and g it may well be possible to avoid determining ϕ and ψ and still obtain estimates for the quantities appearing on the right in (4.1.9), although these estimates will be more conservative than those obtainable if we determine ϕ and ψ. However, if we knew little about f and g it is possible that the best approach is that of examining the analogue of (4.1.9) derived from (4.1.3), i.e.

$$(4.1.10)\quad |E(f;g)| \le \|f-\phi\|_{p_1,w}\|\psi\|_{q_1,w} + \|\phi\|_{p_2,w}\|g-\psi\|_{q_2,w} + \|f-\phi\|_{p_3,w}\|g-\psi\|_{q_3,w} + E_Q ,$$

$$\text{where } E_Q = |Q(f-\phi;\psi)+Q(\phi;g-\psi)+Q(f-\phi;g-\psi)|$$

and investigating ϕ and ψ in more detail.

It is immediate that it is computationally advantageous to choose $p_1=p_2=p_3$. Note that we also have, for $1\le p\le\infty$,

$$(4.1.11)\quad \|\xi\|_{p,w} \le I(1;1)\max_{t\in R}|\xi(t)| \equiv I(1;1)\|\xi\|_{\infty,1}$$

The relation (4.1.11) may also be employed in approximating (4.1.10) or (4.1.9).

If we have to approximate $|Q(\xi;\theta)|$ we proceed in an analogous manner. From the discrete Cauchy-Schwarz Inequality (with w taken to be unity) we have

$$(4.1.12) \quad |Q(\xi;\theta)| \le \|\underline{\xi}\|_{p,1} \|A\underline{\theta}\|_{q,1} \equiv \|\underline{\xi}\|_p \|A\underline{\theta}\|_q \quad q=\frac{p}{p-1} .$$

Denoting the matrix norm compatible with $\|.\|_q$ by $\|.\|_q^*$ we obtain

$$(4.1.13) \qquad |Q(\xi;\theta)| \le \|\underline{\xi}\|_p \|A\|_q^* \|\underline{\theta}\|_q,$$

and

$$(4.1.14) \qquad |Q(\xi;\theta)| \le \|\xi\|_p \|A\|_p^* \|\theta\|_q .$$

Also, if $a_{ij}>0$, $i=0,\ldots,m$, $j=0,\ldots,n$, we obtain the analogue of (4.1.11), namely,

$$(4.1.15) \qquad \|A\theta\|_q \le Q(1;1)\max_j|\theta_j| \equiv Q(1;1)\|\underline{\theta}\|_\infty, \quad 1\le q\le\infty .$$

If some $a_{ij}<0$, we may replace $Q(1;1)$ in (4.1.15) by $\sum_{i=0}^{M}\sum_{j=0}^{n}|a_{ij}|$. As before, when approximating the contribution to $|E(f;g)|$ due to E_Q in (4.1.9) or (4.1.10), we may use different norms when dealing with different components in the summation. In addition, we note that before approximation, we may recombine some of those components due to the bi-linearity of Q, obtaining (in both cases)

$$(4.1.16) \qquad E_Q = |Q(f-\phi;\psi) + Q(f;g-\psi)|,$$

(4.1.17) $$E_Q = |Q(f-\phi;g) + Q(\phi;g-\psi)|.$$

Thus there are a large number of error estimates available and which one we use depends upon what information we have to hand. In particular, we note that if we use the uniform (i.e. L_∞) norm we may always seek to approximate the norm of a function in R by the norm of the function over a point set in R.

We summarise our comments so far as a Lemma (cf. [12], [28]).

LEMMA 4.1.1 Let Q be an I.P.Q.F. exact on $\Phi^\gamma \times \Psi^\delta$, and $\phi \varepsilon sp\Phi^\gamma$, $\psi \varepsilon sp\Psi^\delta$, be such that

(4.1.18) $$\begin{cases} \|f-\phi\|_{p,w} \leq \varepsilon_1 \\ \|g-\psi\|_{p,w} \leq \varepsilon_2, \ 1 \leq p \leq \infty, \ (q = \frac{p}{p-1}) \ . \end{cases}$$

Then

(4.1.19) $$|E(f;g)| \leq \varepsilon_1 \|g\|_{q,w} + \varepsilon_2 \|f\|_{q,w} + \varepsilon_1\varepsilon_2 + E_Q,$$

where E_Q obtained from (4.1.16) or (4.1.17). If $p=\infty$ we obtain (see [12])

(4.1.20) $$|E(f;g)| \leq \varepsilon_1 \|g\|_{1,w} + \varepsilon_2 \|f\|_{1,w} + \varepsilon_1\varepsilon_2 + \varepsilon_1 \|A\|^* \|\underline{\psi}\|_1 + \varepsilon_2 \|A\|^* \|\underline{f}\|_1 \ . \qquad ///$$

In (4.1.19) we may replace $\|g\|$ ($\|f\|$) by $\|\psi\|$ ($\|\phi\|$), and the expression on the right hand side of (4.1.20) depends upon

which norm we use for $\|A\|$ (cf (4.1.13), (4.1.14)) and which of (4.1.16) and (4.1.17) we use.

The derivation of the numbers ε_1 and ε_2 and the polynomials ϕ and ψ in Lemma 4.1.1 is not a trivial matter. If we want ϕ and ψ to be interpolatery, assuming that our I.P.Q.F. is exact on at least $\phi^m \times \psi^n$, we may obtain them either by using the polynomials p_i and q_j of §2.9, or on slightly generalizing the Newton interpolation scheme given in [136].

When the functions ϕ_i and ψ_j are standard polynomials, we can obtain error estimates for interpolatory polynomials involving derivatives of f and g (assuming f and g have sufficient derivatives). See [12,13]. If an estimate of the required derivative of f is available, say $|f^{(m+1)}(x)| \leq M, x \in R$ we obtain (see also [23])

$$|f-\phi| \leq \frac{M}{(m+1)!} \int_R w(t) \prod_{i=0}^{m} (t-x_i)\,dt \tag{4.1.20}$$

when $S_i \in \chi_1$, $i=0,\ldots,m$. Clearly on investigation of what choice of knots leads to a minimal norm interpolatory polynomial would be of interest. The expression (4.1.20) can be generalized to deal with other choices of basic functions than polynomials. See §4.2. Under certain circumstances derivatives may be replaced by divided differences. See [13,137]. When dealing with derivatives of higher order we face the additional problems that, except for certain entire functions, ultimately derivatives of a function will increase without bound, and further, even if higher derivatives are available they are difficult to

approximate. This is a major reason why, in practice, high-order (i.e. γ large) R.Q.F. tend not to be used, and Compound R.Q.F. are used instead. Further, it should be noted that as we increase the degree of an interpolating polynomial to a given continuous function, we do not necessarily obtain a better pointwise approximation to the function, and we do not always obtain convergence of interpolatory R.Q.F. to I. See [138]. If we assume more smoothness properties than continuity, it is possible to guarantee both convergence of a R.Q.F. to I, and of interpolatory polynomials to the interpolated function in various norms. See [139,...,142]. Some work on interpolation with basis functions other than the polynomials has also been done. See [143,144].

Error bounds of the type mentioned in this § may be easily modified, in the standard manner, to deal with compound I.P.Q.F.

4.2 PEANO'S THEOREM

Let us initially assume

$$\phi_i \equiv t^i, \quad i = 0,1,... \tag{4.2.1}$$

$$\psi_j \equiv t^j, \quad j = 0,1,... \tag{4.2.2}$$

We can now derive the analogue of Peano's Theorem for the bi-linear functional E. As usual we assume our I.P.Q.F. is exact on $\Phi^\gamma \times \Psi^\delta$. Assume f and g have sufficient derivatives we expand

them in Taylor Series about -1, obtaining

$$(4.2.3)\qquad f(t) = \sum_{i=0}^{k} \frac{(t+1)^i}{i!} f^{(i)}(-1) + \frac{1}{k!}\int_R f^{(k+1)}(r)(t-r)_+^k dr,$$

$$(4.2.4)\qquad g(t) = \sum_{j=0}^{\ell} \frac{(t+1)^j}{j!} g^{(j)}(-1) + \frac{1}{\ell!}\int_R g^{(\ell+1)}(r)(t-r)_+^\ell dr,$$

where $$x_+^m = \begin{cases} x^m, & x>0 \\ 0, & x\le 0 \end{cases}, \quad f\in C^{(k+1)}(R),\ g\in C^{(\ell+1)}(R).$$

Thus we find that, if $k\le\gamma$ and $\ell\le\delta$,

$$\begin{aligned}(4.2.5)\quad E(f;g) &= E_t\Big(\frac{1}{k!}\int_R f^{(k+1)}(r)(t-r)_+^k dr;\ \frac{1}{\ell!}\int_R g^{(\ell+1)}(s)(t-s)_+^\ell ds\Big)\\ &+ E_t\Big(\frac{1}{k!}\int_R f^{(k+1)}(r)(t-r)_+^k dr;\ \sum_{j=0}^{\ell}\frac{(t+1)^j}{j!}g^{(j)}(-1)\Big)\\ &+ E_t\Big(\sum_{i=0}^{k}\frac{(t+1)^i}{i!}f^{(i)}(-1);\ \frac{1}{\ell!}\int_R g^{(\ell+1)}(s)(t-s)_+^\ell ds\Big)\\ &+ E_t\Big(\sum_{i=0}^{k}\frac{(t+1)^i}{i!}f^{(i)}(-1);\ \sum_{j=0}^{\ell}\frac{(t+1)^j}{j!}g^{(j)}(-1)\Big),\end{aligned}$$

where E_t means that the parameter t is the one which is regarded as a variable. Consequently

$$\begin{aligned}(4.2.6)\quad E(f;g) &= \int_R\int_R \Big\{E_t\Big(\frac{(t-r)_+^k}{k!},\ \frac{(t-s)_+^\ell}{\ell!}\Big)\Big\} f^{(k+1)}(r)g^{(\ell+1)}(s)\,dr\,ds\\ &+ \int_R \Big\{E_t\Big(\frac{(t-r)_+^k}{k!};\ \sum_{j=0}^{\ell}\frac{(t+1)^j}{j!}g^{(j)}(-1)\Big\} f^{(k+1)}(r)\,dr\\ &+ \int_R \Big\{E_t\Big(\sum_{i=0}^{k}\frac{(t+1)^i}{i!}f^{(i)}(-1);\frac{(t-s)_+^\ell}{\ell!}\Big)\Big\} g^{(\ell+1)}(s)\,ds\\ &+ E_t\Big(\sum_{i=0}^{k}\frac{(t+1)^i}{i!}f^{(i)}(-1);\sum_{j=0}^{\ell}\frac{(t+1)^j}{j!}f^{(j)}(-1)\Big).\end{aligned}$$

and using the multi-dimensional Cauchy Schwarz Inequality, we obtain

$$(4.2.7)\quad |E(f;g)| \le \|E_t(\frac{(t-r)_+^k}{k!};\ \frac{(t-s)_+^\ell}{\ell!})\|_{p_1} \|f^{(k+1)}(r)g^{(\ell+1)}(s)\|_{q_1}$$
$$+ \|E_t(\frac{(t-r)_+^k}{k!};\ \psi_g(t))\|_{p_2} \|f^{(k+1)}\|_{q_2}$$
$$+ \|E_t(\phi_f(t);\frac{(t-s)_+^\ell}{\ell!})\|_{p_3} \|g^{(\ell+1)}\|_{q_3}$$
$$+ E_t(\phi_f(t);\ \psi_g(t)),$$

where $\frac{1}{q_i} + \frac{1}{q_i} = 1$, $i=1,2,3$, and

$$(4.2.8)\quad \begin{cases} \phi_f(t) = \sum_{i=0}^{k} \frac{(t+1)^i}{i!} f^{(i)}(-1), \\ \psi_g(t) = \sum_{j=0}^{\ell} \frac{(t+1)^j}{j!} g^{(j)}(-1). \end{cases}$$

We also note that

$$(4.2.9)\quad \|f^{(k+1)}(r)g^{(\ell+1)}(s)\|_{q_1} = \|f^{(k+1)}(r)\|_{q_1} \|g^{(\ell+1)}(s)\|_{q_1},$$

where the norms on the right are with respect to one variable and that on the left is with respect to two.

Since the I.P.Q.F. integrates the monomials in $\phi^\gamma \times \psi^\delta$ exactly it is clear that the last term in the above error estimates is zero. We also note that the choice of the elementary functionals S_i and T_j does not affect the error estimates, provided we know the degree of exactness of the I.P.Q.F.

In the case of R.Q.F., the Peano error estimate is

$$(4.2.10)\qquad E(f) = \int_R E_t\left(\frac{(t-r)_+^k}{k!}\right) f^{(k+1)}(r)\,dr$$

$$\leq \left\| E_t\left(\frac{(t-r)_+^k}{k!}\right)\right\|_{p_1} \left\| f^{(k+1)}\right\|_{q_1}.$$

Clearly, if f(g) is equivalent to unity and $l \varepsilon sp\Phi^{\gamma}$, $(sp\Psi^{\delta})$, (4.2.7) reduces to (4.2.10). As mentioned in §3.2, best R.Q.F. in the sense of Sard are obtained by choosing the functionals S_i (usually χ_1 functionals) so that $\left\| E_t\left(\frac{(t-r)_+^k}{k!}\right)\right\|$ is minimized. This leads to the examination of minimal norm polynomial monosplines. for some references to the large body of literature devoted to this topic see §3.2. This approach requires that we can separate the contributions made to E by the R.Q.F. (i.e., $\|E_t\|_{p_1}$) and the function (i.e. $f^{(k+1)}$). In the case of I.P.Q.F., due to the second and third terms in (4.2.7) it is not so easy to separate the contribution to E due to f and g from that inherently due to the I.P.Q.F. Examining the second term in (4.2.7) we see that

$$(4.2.11)\qquad E_t\left(\frac{(t-r)_+^k}{k!};\psi_g(t)\right) = \int_R w(t)\frac{(t-r)_+^k}{k!}\psi_g(t)\,dt - \underline{\xi}^T(r)A\underline{\psi}_g,$$

where $$\underline{\xi}^T(r) = \left(S_o\left(\frac{(t-r)_+^k}{k!}\right),\ldots,S_m\left(\frac{(t-r)_+^k}{k!}\right)\right),$$

$$\underline{\psi}_g^T = (T_o(\psi_g),\ldots,T_n(\psi_g)).$$

We can approximate (4.2.11) by

$$(4.2.12)\qquad \left|E_t\left(\frac{(t-r)_+^k}{k!};\psi_g(t)\right)\right| \leq \left\|\frac{(t-r)_+^k}{k!}\right\|_{p_4,w} \left\|\psi_g(t)\right\|_{q_4,w} +$$

$$+ \left\|\underline{\xi}(r)\right\|_{p_5} \|A\|^* \left\|\underline{\psi}_g\right\|_{q_5}$$

$$\frac{1}{p} + \frac{1}{q} = 1, \quad i=4,5$$

and the norm of A is discussed in §4.1 ((4.1.13), (4.1.14)). An expression analogous to (4.2.12) can be obtained for $|E_t(\phi_f(t);\frac{(t-s)_+^\ell}{\ell!})|$, in which $\underline{\xi(r)}$ and $\underline{\psi}_g$ are replaced by $\underline{\theta(s)}$ and $\underline{\phi}_f$,

$$\underline{\theta}^T(s) = (T_o(\frac{(t-s)_+^\ell}{\ell!}),\ldots,T_n(\frac{(t-s)_+^\ell}{\ell!})),$$

and we use norms, as determined by subscripts 6 and 7 (replacing 4 and 5).

Substituting (4.1.12) and its analogue in (4.2.7) we obtain

$$\begin{aligned}(4.2.13)\quad |E(f;g)| \le{}& \|E_t(\frac{(t-r)_+^k}{k!};\frac{(t-s)_+^\ell}{\ell!})\|_{p_1}\|f^{(k+1)}\|_{q_1}\|g^{(\ell+1)}\|_{q_1}\\ &+\|\psi_g\|_{q_4,w}\|\frac{(t-r)_+^k}{k!}\|_{p_4,w}{}_{p_2}+\|\|\underline{\xi(r)}\|_{p_5}\|_{p_2}\|A\|^*\\ &\times\|\psi_g\|_{q_5}+\|\phi_f\|_{q_6,w}\|\frac{(t-s)_+^\ell}{\ell!}\|_{p_6,w}{}_{p_3}+\\ &+\|\phi_f\|_{p_7}\|A\|^*\|\|\underline{\theta(s)}\|_{q_7}\|_{q_3}+E_t(\phi_f;\psi_g).\end{aligned}$$

A slightly different result, based on the use of Taylor's formula, approximating $\|\phi_f\|$ by

$$\sum_{i=0}^k \max_{t\in R}|f^{(i)}(t)|(i!)^{-1}\int_R |E(\frac{(t-r)_+^i}{i!};\frac{(t-s)_+^\ell}{\ell!})|dt$$

(and $\|\psi_g\|$ analogously) is given in [23].

So an I.P.Q.F. could be said to be best in the sense of Sard if some combination of

$$\| E_t(\frac{(t-r)_+^k}{k!};\frac{(t-s)_+^\ell}{\ell!})\|_{p_1}, \| \|\underline{\xi(r)}\|_{p_5}\|_{p_2} \text{ and } \| \|\underline{\xi(s)}\|_{q_7}\|_{q_3}$$

is minimized.

There are many different errors estimates included in (4.2.13) depending upon the choices of $k \leq \gamma$, $\ell \leq \delta$, $p_1,\ldots,p_7$. There is some advantage, from the computational point of view, in choosing $p_1=p_2=p_3$ and we would usually expect that $p_i \in \{1,2,\infty\}$, $i=1,\ldots,7$. Other approximations of $|E(f;g)|$ can be obtained. For example, we may apply an analysis similar to (4.2.11) - (4.2.12) to $|E_t(\frac{(t-r)_+^k}{k!};\frac{(t-s)_+^\ell}{\ell!})|$ or we may, for certain choices of the function g, be able to approximate (4.2.11) by

$$(4.2.14) \quad |\|\frac{(t-r)_+^k}{k!}\|_{p_4,w}\|\psi_g(t)\|_{q_4,w}\|\underline{\xi(r)}\|_{p_5}\|A\|^*\|\underline{\psi}_g\|_{q_5}|,$$

in which case a careful choice of $\|\underline{\xi(r)}\|_{p_5}$ might substantially reduce our approximation of $|E(f;g)|$.

It is also observed that, even if (4.2.1) - (4.2.2) do not hold, we may still apply the above analysis, except that now the term $E_t(\phi_f;\psi_g)$ is non zero.

This leads us to ask the question of whether we may make an analysis similar to that above when we deal with other choices of function (i.e., we do not require (4.2.1) - (4.2.2)) and we use other approximations to the functions f and g (i.e., we do not use (4.2.3) - (4.2.4)). We see that (4.2.3) may be regarded as deriving an interpolating ϕ-polynomial of degree k to f at $\{\hat{S}_i\}_{i=0}^k$, where

$$(4.2.15) \qquad \hat{S}_i \equiv \frac{d^i f}{dt^i}\Big|_{t=-1}, \quad i=0,\ldots,k.$$

We return, momentarily, to the case of R.Q.F. In [150] we find that it is possible to expand a function, possessing sufficient derivatives, as a power series of a function ξ, which plays the same role in the Bürmann series as x does in the Taylor series. However, following [149], we can glean a more general approach.

Given a set of functions Ξ_r, we define its Wronskian, W_r, by

$$(4.2.10) \qquad W_r = \det \begin{bmatrix} \xi_o & , \ldots , & \xi_r \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \xi_o^{(r)} & , \ldots , & \xi_r^{(r)} \end{bmatrix} .$$

If W_r has no zeros in R, we can define the linear operator L_{r+1} by

$$(4.2.17) \qquad L_{r+1}(f) = W_r^{-1} \det \begin{bmatrix} \xi_o & , \ldots , & \xi_r, f \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \xi_o^{(r)} & , \ldots , & \xi_o^{(r)}, f^{(r)} \\ \xi_o^{(r+1)} & , \ldots , & \xi_r^{(r+1)}, f^{(r+1)} \end{bmatrix} .$$

It is clear that every solution of $L_{r+1}(f)=0$ is a ξ-polynomial of degree at most r. Now define

$$(4.2.18)\quad \kappa(t,s) = W_r^{-1}(s)\det\begin{bmatrix} \xi_0(s) & , \ldots , & \xi_r(s) \\ \cdot & & \\ \cdot & & \\ \xi_0^{(r-1)}(s) & , \ldots , & \xi_r^{(r-1)}(s) \\ \xi_0(t) & , \ldots , & \xi_r(t) \end{bmatrix}$$

$$= \sum_{i=0}^{r}\kappa_i(s)\xi_i(t)\ ,$$

and it follows that

$$(4.2.19)\qquad \mu = \sum_{i=0}^{r}\alpha_i\xi_i + \int_{-1}^{t}\kappa(t,s)\zeta(s)ds$$

satisfies

$$(4.2.20)\qquad L_{r+1}(\mu) = \zeta$$

for arbitrary $\alpha_0,\ldots,\alpha_r$. Conversely, given f, there exist constants α_i such that (4.2.19) is valid (with μ replaced by f). These constants, which we denote $\hat{\alpha}_i$ can be found by solving a set of linear equations determined by knowing the value of f at r+1 distinct points.

Assume that we are given ϕ-polynomial of degree m, which interpolates f at the elementary functionals $\{S_i\}_{i=0}^{m}$,

$$(4.2.21)\qquad f = \sum_{i=0}^{m}\alpha_i^*\phi_i + R_m(t) = \phi_f + R_m(t).$$

Combining (4.2.19) and (4.2.21) we see that we can write

$$(4.2.22)\quad R_m(t) = \sum_{i=0}^{r}\hat{\alpha}_i\xi_i - \sum_{i=0}^{m}\alpha_i^*\phi_i + \int_{-1}^{t}\kappa(t,s)L_{r+1}(f)ds.$$

On observing that we may replace -1 by 1 as a limit of integration in (4.2.19), and denoting the coefficients α_i determined by f in this alternative expression for f by $\tilde{\tilde{\alpha}}_i$, we obtain

$$(4.2.23)\quad R_m(t) = \sum_{i=0}^{r}(\beta\hat{\alpha}_i+(1-\beta)\tilde{\alpha}_i)\xi_i-\sum_{i=0}^{m}\alpha_i^{*}\phi_i + \int_R K_\xi(t,s)L_{r+1}(f)ds$$

where $$K_\xi(t,s) = \begin{cases} \beta\kappa(t,s), & s<t \\ (\beta-1)\kappa(t,s), & s>t, \quad \beta\in[0,1]. \end{cases}$$

The function $\kappa(t,s)\in C^{(r-1)}(R)$, with respect to t, and satisfies

$$(4.2.24)\quad \frac{\partial^i}{\partial t^i}(\kappa(t,s))\Big|_{t=s} = \begin{cases} 0, & i=0,\ldots,r-1 \\ 1, & i=r \end{cases},$$

hence $K_\xi\in C^{(r-1)}(R)$, with respect to t.

We now apply a R.Q.F., exact on Φ^γ, $\gamma>m$, to f. The expression for the error is thus

$$(4.2.25)\quad E(f) = I(R_m(f)) - Q(R_m(f))$$
$$= I(\sum_{i=0}^{r}(\beta\hat{\alpha}_i+(1-\beta)\tilde{\alpha}_i)\xi_i)-Q(\sum_{i=0}^{r}(\beta\hat{\alpha}_i+(1-\beta)\tilde{\alpha}_i)\xi_i)$$
$$+ \int_R E_t(K_\xi(t,s))L_{r+1}(f)ds.$$

Obviously, if $r\le\gamma$ and $\phi_i\equiv\xi_i$, (4.2.25) reduces to

$$(4.2.26)\quad E(f) = \int_R E_t(K_\phi(t,s))L_{r+1}(f)ds.$$

DEFINITION 4.2.1 We say that an R.Q.F., Q, exact on Φ^γ, is r-p-Ξ minimal if

$$\|E_t(K_\xi(t,s))\|_p$$

is minimal. If $\xi_i=\phi_i$ we say that Q is r-p-minimal. If, in addition, $r=\gamma$, we say Q is p-minimal. ///

There is some freedom in the definition of K_ξ in the choice of the numbers β and r, and it is not immediately clear what their optimal choices are. In (4.2.26) the larger r is the more functions f have E(f)=0, however it may be, if E(f)≠0 for all possible choices of r, that choosing r small will give a better estimate of E(f) in the sense that $\|E_t(K_\xi(t,s))\|_p$ in Definition 4.2.1 may reduce. An additional advantage of using a small value of r may be that this would facilitate approximation of $L_{r+1}(f)$ when we are seeking an error estimate for a particular function f.

The function $\kappa(t,s)$ remains unchanged if we replace Ξ_r by any other set of functions spanning Ξ_r. This can be utilized in showing that for certain choices of elementary functionals $\{S_i\}_{i=0}^m$, if $\xi_i=\phi_i$, the coefficients $\hat{\alpha}_i$ and $-\alpha_i^*$ are the same, and thus we obtain an explicit expression for $R_m(f)$ from (4.2.22). If either

$$S_i \varepsilon \chi_1, \quad i=0,\ldots,m, \tag{4.2.27}$$

or

$$(4.2.28) \qquad S_i \equiv \frac{d^i}{dt^i}\Big|_{t=\alpha}, \quad \alpha \in R, \ i=0,\ldots,m,$$

is valid, then this is the case. The conditions (4.2.28) include those prescribing a Taylor series expansion of f as a special case.

The extension of error estimates of the type (4.2.26), and even (4.2.25) to deal with I.P.Q.F. is entirely analogous to that done earlier in this section when we considered the extension of Peano's theorem, on replacing (4.2.3) by (4.2.21) and (4.2.4) by the analogue of (4.2.21). We note that if $\|L_{r+1}\|$ is available we may be able to replace $\|L_{r+1}(f)\|$ by $\|L_{r+1}\|\|f\|$ in our estimations (cf. 4.2.9)) where $\|L_{r+1}\|$ is some suitable norm of the linear functional L_{r+1}.

§4.3 SARMA-EBERLEIN ESTIMATES

In the previous sections of this chapter we have detailed various methods of estimating the errors involved in approximating I by a given I.P.Q.F. Of course, once we have more then one I.P.Q.F. available it is natural to want to compare their performances. This is not always a straightforward procedure as we have yet to decide on criteria for comparison. It appears to be a common practice in the literature to compare Q.F. of any sort by examining their performance on sets of judiciously chosen test functions (cf §1.2). If these test functions are chosen carefully, manifesting a wide range of behavioural characteristics, we will be able to obtain a "performance profile" of a Q.F. which should give us quite a strong intuition into

how a given Q.F. will behave in various given situations. This will be very useful if we know, in advance, quite a lot about the integrand. However, the above procedure is somewhat reminiscent of attempting to prove a theorem by examining some carefully chosen examples - a practice not generally condoned. It would be very useful, particularly when we are dealing with integrands about which we know little, if some more general and rigorous "measure" of the performance of a Q.F. over a specified class of functions were available. One approach, pursued in the previous section is to minimize the error estimate usedffor the Q.F. in some way, using methods obtained from Functional Analysis. Another approach, which we examine in this section, is to obtain a measure of "goodness", the Sarma-Eberlein estimate of goodness, S_E, of a Q.F. based on the performance of the Q.F. on each function in a given space of functions. Other approaches might, for example, include some sort of statistical estimate of measures of goodness. Before progressing any further, we note that the derivation of such measures of goodness is not a straightforward matter, and their application in practice is time consuming and not without difficulties, which to a large extent explains their lack of popularity and/or acceptance. In fact, the only references we know of dealing with S_E are [37], [39], [40] and [148]. In any event, we will use a different approach. Although we only consider real valued functions of one variable, the following analysis may clearly be generalized to higher dimensions and other linear functionals.

Let $\{\xi_i\}_{i=0}^{\infty}$ be a linearly independent set of functions which

is dense in C(R). Thus, if $f \in C(R)$, we can write

$$(4.3.1) \qquad f = \sum_{i=0}^{\infty} \alpha_i \xi_i, \quad \forall t \in R,$$

where the numbers α_i depend upon the function f, but not on the point $t \in R$. Now let Q be an R.Q.F. Clearly

$$(4.3.2) \qquad E(f) = I(f) - Q(f) = \sum_{i=0}^{\infty} \alpha_i E(\xi_i).$$

We observe that the contribution dependent on the function f in the error expression (4.3.2) is contained in the numbers α_i, and the numbers $E(\xi_i)$ are independent of f and may be regarded as being due to the R.Q.F. There are two directions in which we may progress from here.

(4.3.3) If the numbers $E(\xi_i)$ are known, and we are dealing with a given function f, we investigate the properties of the numbers α_i. Knowing α_i, or an approximation of α_i, we are then able to estimate (4.3.2) by a truncated summation. In this situation it is usual (and sensible) to require that the functions ξ_i are orthogonal, then α_i is the generalized Fourier coefficient.

$$(4.3.4) \qquad \alpha_i \equiv \frac{I(\xi_i f)}{I(\xi_i^2)} \qquad i=0,1,2,\ldots$$

In some special circumstances it is possible to calculate the first few of these coefficients using summations over the zeros of orthogonal polynormals, see [3]. This is the basis of the well known Clenshaw-Curtis Q.F. (see [145,146] and the con-

sequent literature, eg [147]). Note that for a given R.Q.F. the numbers $E(\xi_i)$ only ever need to be calculated once.

The type of error estimate obtained in this way, namely of predicting how an infinite sequence behaves on the basis of a few initial terms, will not lead to foolproof estimates. However, these estimates do have the great advantage of being able to be used in practice, and unless we are dealing with pathological integrands can probably be expected to give useful error estimates.

(4.3.5) If we wish to compare two R.Q.F., instead of examining their performance over certain test functions we may examine the numbers $E(\xi_i)$ due to each of the R.Q.F. Following [39] we do this by examining the weighted sum,

$$\sum_{i=0}^{\infty} \beta_i E(\xi_i)^2, \tag{4.3.6}$$

where the numbers β_i are chosen in order to guarantee convergence of the series and to accentuate desired aspects of the behaviour of the R.Q.F. If we wish to emphasize the behaviour of the Q.F. on certain subsets of $\{\xi_i\}_{i=0}^{\infty}$, we merely increase the corresponding β_i's or decrease the residual β_i's. Thus we may compare Q.F.'s performance over various subspaces of C(R).

The Sarma-Eberlein estimate of goodness referred to above has

$$\begin{cases} \xi_i = x^i \\ \beta_i = 3^{-(i+1)} \end{cases} \qquad i=0,1,2,\ldots \tag{4.3.7}$$

and can be derived by defining an integral over the function space C(R).

Obviously, having chosen the numbers β_i, we may attempt to obtain "optimal" Q.F. with respect to the error measure characterized by β_i, by minimizing (4.3.4) with respect to the parameters available in the R.Q.F.

We now examine I.P.Q.F. Assume $\{\phi_i\}_{i=0}^{\infty}$ and $\{\psi_j\}_{j=0}^{\infty}$ are sets of functions dense in C(R). Clearly we may write

$$(4.3.8) \qquad \begin{cases} f = \sum_{i=0}^{\infty} \mu_i \phi_i \\ g = \sum_{i=0}^{\infty} \nu_j \psi_j \ , \end{cases}$$

and we then have

$$(4.3.9) \qquad E(f;g) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mu_i \nu_j E(\phi_i;\psi_j).$$

Obviously, comments entirely analogous to (4.3.3) and (4.3.5) can be made with regard to (4.3.9), and we investigate the weighted sum

$$(4.3.10) \qquad \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \beta_{ij} E(\phi_i;\psi_j)^2 \ ,$$

for various choices of β_{ij}.

If the I.P.Q.F. we are considering is exact on $\Phi^{\gamma} \times \Psi^{\delta}$ then the summations (4.3.9) and (4.3.10) will start from $i=\gamma+1$ and $j=\delta+1$. Provided the numbers $E(\phi_i;\psi_j)$ are bounded then (4.3.10), which can be regarded as a generalization of (4.3.9), will be

convergent provided $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \beta_{ij}$ is. This is not a severe restriction. However, if the series $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \beta_{ij}$ converges too slowly there may be practical problems in evaluating (4.3.10).

Some numerical results are available in the case of the Sarma-Eberlein estimates. See [37], [148]. In [148] it is reported that attempts to calculate optimal R.Q.F. (i.e. minimizing (4.3.6)) were made. In all but the simplest cases (using either two or three χ_1 elementary functionals) these attempts were unsuccessful due to computational difficulties.

In [148] it was pointed out that the Sarma-Eberlein estimate can be regarded as the variance of the error functional. Thus, using Chebychev's inequality we can obtain an estimate of the probability that, choosing an integrand at random, the error is less than a given tolerance. Clearly, randomness is with respect to the distribution function due to the Eberlein measure used to obtain S_E in [39,40]. It would be of interest to know whether these results can be parallelled for other choices of distribution function, thus allowing us to obtain useful probabilistic error estimates when using Q.F. in various circumstances.

We also observe that extending the idea of a Sarma-Eberlein estimate of goodness to I.P.Q.F. is very straightforward in comparison to the other techniques of estimating errors considered earlier in this Chapter.

§4.4 CONVERGENCE

There are essentially two ways in which we may examine convergence of a Q.F. to the integral it is approximating. We may use compound Q.F. applied to smaller and smaller subregions of R or we may increase the degree of exactness of the Q.F. we are using. The first approach is clearly analogous to standard Riemann integration, and the second leads us to investigate the properties of families of Q.F. Although the properties of families of Q.F. are important for theoretical reasons, due to practical difficulties, it is unusual to find these properties being exploited in the course of practical computation.

There is a well known result which states that, provided it is exact for the function 1, and given a tolerance ε, there is an N such that a R.Q.F., applied to N (equal) subintervals of R will guarantee $E(f)\leq\varepsilon$. Naturally, N is dependent on the choice of function f. Recalling (A2) in particular the following result is immediate. See also [12].

THEOREM 4.4.1 Let Q be an I.P.Q.F. exact on 1×1 (i.e. exact when $\phi\equiv\psi\equiv 1$) at least. Let ${}^{(M,N)}Q$ be an (M,N) copy of Q. Assume $\max_{1\leq\mu\leq M}(X_\mu - X_{\mu-1})\to 0$ as $M\to\infty$ and $\max_{1\leq\nu\leq N}(Y_\nu - Y_{\nu-1})\to 0$ as $N\to\infty$. Then

$$\lim_{\substack{M\to\infty\\N\to\infty}} {}^{(M,N)}Q(f;g) = I(f;g). \qquad ///$$

Note that the statement of Theorem 4.4.1 is independent of the choice of elementary functionals S_i and T_j.

We now consider the second approach to convergence mentioned above. We have the following theorem due to Banach (see [4]):

THEOREM 4.4.2 In order that the sequence of linear operators $\{L_i\}_{i=0}^{\infty}$ be convergent it is necessary and sufficient that they satisfy the two conditions:

(a) The norms of the operators $\|L_i\|$ have a common bound,

(b) $L_i\phi$ is convergent for each ϕ in a set Φ, which is everywhere dense in $C^o(R)$. ///

Clearly, with slight modifications to condition (b) the above Theorem is applicable to bi-linear operators.

At first sight the above Theorem seems to clash with the comments made in §4.1 concerning the divergence of interpolating polynomials from the functions they are approximating. This difficulty is removed when we recall that in dealing with the convergence of Q.F. to an integral we are primarily concerned with the weak* convergence of the (bi-)linear operators $Q^{(m,n)}$ to I. This is a weaker condition than requiring strong convergence (for example, requiring that the interpolating polynomials also approximate the function under consideration). See also [2,14]. Thus it is immediate that we obtain, analogous to the celebrated result for R.Q.F.

THEOREM 4.4.3 Let $Q^{(m,n)}$ be exact on $\Phi^{\gamma(m)}\times\Psi^{\delta(n)}$. Assume that $\{\phi_i\}_{i=0}^{\infty}$ and $\{\psi_j\}_{j=0}^{\infty}$ are dense in $C^o(R)$. Then, provided the following are satisfied:

(a) $\|Q^{(m,n)}\| \leq \Omega$, $m=0,1,\ldots,n=0,1,\ldots$

(b) $\gamma(m)\to\infty$ as $m\to\infty$, and $\delta(n)\to\infty$ as $n\to\infty$, we find

$$\lim_{m,n\to\infty} Q^{(m,n)}(f;g) = I(f;g)$$

for all $f,g\in C^{o}(R)$.

PROOF: See [4,14]. ///

We note two things about Theorem 4.4.3; first, it is independent of the choice of elementary functionals S_i and T_j, and second we are free to choose the norm used in part (a). The usual, though not the only, choices would be either the least square or absolute norm, i.e.

$$\|Q^{(m,n)}\|_2 = \left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|a_{ij}^{(m,n)}|^2\right)^{\frac{1}{2}}, \tag{4.4.1}$$

and

$$\|Q^{(m,n)}\|_1 = \sum_{i=0}^{m}\sum_{j=0}^{n}|a_{ij}|, \tag{4.4.2}$$

respectively.

If we place more conditions on the functions f and g, for example, requiring them to be holomorphic in some region containing R, we can obtain stronger results.

However, we must remember that it is no easy matter to guarantee that the required conditions for convergence of a family of I.P.Q.F., or R.Q.F. for that matter, can be satisfied. Some simplifying results are available in [14].

When using a Q.F. it is natural to desire that the conver-

gence of the error-functional to zero is as rapid as possible. This leads us to consider the question of best choice of elementary functionals and the functions ϕ_i and ψ_j. Often, due to circumstances beyond our control, our choice of elementary functionals in particular, will be limited. Our choice of functions ϕ_i and ψ_j depends upon our circumstance also. If the I.P.Q.F. is to be used for a specific problem about which we have information available, this will be a major factor in our choice of ϕ_i and ψ_j. If, on the other hand, we are to use the I.P.Q.F. for a variety of problems it would seem advantageous to choose the functions of ϕ_i and ψ_j from dense subspaces of $C^o(R)$ whose n-widths are maximal or near maximal. See [1,160, ...,163]. Also, much work has been done concerning best approximation of linear functionals, see for example [164, 165], and it seems reasonable to expect some of these results to carry over to approximation of bi-linear functionals.

As a final remark, we note the important role of the function 1 in Theorem 4.4.1 and recalling the Stone-Weierstraus Theorem (see [3]) we see that, unless circumstances are exceptional, an excellent choice for ϕ_o and ψ_o would be the function 1.

CHAPTER 5: EXTENSIONS AND APPLICATIONS

§5.1 EXTENSIONS - HIGHER DIMENSIONS

There are several ways in which we might want to extend the concept of I.P.Q.F. Some work has already been done, see [22, 119,141,152].

An obvious question to ask is whether we can develop I.P.Q.F. in multidimensional spaces, i.e. $R \subset \mathbb{R}^d$, $d>1$, and f and g are functions mapping $\mathbb{R}^d$ to $\mathbb{R}$. Some initial work on this question has been done in [22], under the assumptions that all the elementary functionals are of type χ_1, and the functions ϕ_i and ψ_j are monomials. In deriving interpolatory I.P.Q.F. the procedure is analogous to the one dimensional case. However not all results remain valid. For example the analogue of Theorem 1.3.2, stating that if an I.P.Q.F. is exact on $\phi^m \times \psi^n$ it must be interpolatory, is false. It is clear that the second approach to deriving I.P.Q.F. in §1.3 is much more general when $d>1$. The only difficulty lies in finding interpolating sets of points for the monomials - that is, sets of points such that $\det B \neq 0$ and $\det C \neq 0$. The problem of obtaining I.P.Q.F. exact on $\phi^\gamma \times \psi^\delta$ where, say $\gamma > m$, will depend upon the properties of orthogonal polynomials in more than one dimension. The study of these properties is not as advanced as in the one dimensional case. See [37].

Nevertheless, we can make some comments. We shall only consider the case $d=2$. As is almost universal, we shall also only consider the standard monomials, i.e. we assume that ϕ_i

and ψ_j are monomials. However we note that in principle the ensuing comments apply to higher dimensions. First, we note that when d=2 there is an (r+1)-dimensional subspace of polynomials of degree exactly r which is weakly orthogonal to the subspace of all polynomials of degree less than r. This space is spanned by basis functions of the form

$$x^{r-i}y^{i} + Q_{r-1}^{(i)}(x,y), \qquad i=0,\ldots,r, \tag{5.1.1}$$

where $Q_{r-1}^{(i)}(x,y)$ is a polynomial of degree at most r-1 in the two independent variables x and y. This situation is intrinsically different from that when d=1, when there is only a one dimensional subspace of polynomials orthogonal to polynomials of a lower degree. Second, recalling Theorem 2.2.11, we note that an arbitrary set of given knots need not be an interpolating set, although it is known, see [100], that such sets are dense in regions of $\mathbb{R}^2$. However, the analysis of §2.4 remains entirely applicable, and Theorem 2.4.3 (with slight modifications) remains our starting point. Even in the situation we are discussing, conditions (2.4.15) and (2.4.16) become difficult to satisfy.

When we consider Numerical Quadrature in more than one dimension (sometimes known as Numerical Cubature) it is traditional to seek Q.F. exact for all polynomials up to a given degree. This is despite the fact that in progressing from degree r to degree r+1, in d dimensions, we have to consider $\binom{r+d}{d-1}$ further monomials.

However, instead of integrating exactly all polynomials up to a given degree, it may be preferable to consider only sets of polynomials drawn from all polynomials up to a given degree. It may be possible to obtain results, for example, which integrate exactly all monomials up to degree r and some selected monomials of degree higher than r, probably r+1. Finally, since the total number of monomials, $\binom{r+d}{d}$, which we have to consider increases very rapidly with both r and d it is desirable to reduce the number we have to consider by, for example, appealing to properties due to symmetries, see example, [92,93,153]. However, if we seek I.P.Q.F. exact on $\Phi^{\gamma}\times\Psi^{\delta}$, $\gamma=\binom{r+2}{2}$, $\delta=\binom{s+2}{2}$, r,s>0, this approach is invalid. This can be demonstrated as follows:

Consider first a R.Q.F. Assume that the region of integration is R×R and

$$w(x,y) = w(-x,y) = w(x,-y) = w(-x,-y) \tag{5.1.2}$$

and so we naturally use a symmetric Q.F. which can be written in the form (cf [93,153])

$$\begin{aligned} Q(f) = {} & \sum_{i=0}^{t_1} a_i (f(x_i,y_i)+f(x_i,-y_i)+f(-x_i,y_i)+f(-x_i;-y_i)) \\ & + \sum_{i=t_1+1}^{t_2} a_i (f(x_i,0)+f(-x_i,0)) \\ & + \sum_{i=t_2+1}^{t_3} a_i (f(0,y_i)+f(0,-y_i)). \end{aligned} \tag{5.1.3}$$

It is now immediate that, letting $f\equiv x^{\alpha} y^{\beta}$, α and β being non-negative integers, that

(5.1.4) $$I(x^{\alpha}y^{\beta}) = Q(x^{\alpha}y^{\beta}) = 0,$$

if either α or β is odd. On choosing the numbers x_i, y_i, t_1, t_2 and t_3 with some care it is possible to obtain many different Q.F.'s, considering only the moment equations due to $x^{\alpha}y^{\beta}$, where both α and β are even. This reduces the number of equations we have to deal with to the order of one quarter of the possible total.

If we attempt to obtain I.P.Q.F. using a similar approach (i.e., using configurations of knots for both f and g as suggested by (5.1.3) we see that $I(x^{\alpha_1}y^{\beta_1}; x^{\alpha_2}y^{\beta_2})$ is non-zero if $\alpha_1+\alpha_2$ and $\beta_1+\beta_2$ are even, whereas $Q(x^{\alpha_1}y^{\beta_1}; x^{\alpha_2}y^{\beta_2})$ will be zero if α_1, α_2 and/or β_1, β_2 are odd. Note that if we restrict Ψ^{δ} so that both α_2 and β_2 are even then the required I.P.Q.F. may exist.

We might also attempt to derive I.P.Q.F. which use other predetermined combinations of knots, in particular we might consider using

(5.1.5) $$f(x,y) + f(x,-y),$$

or

(5.1.6) $$f(x,y) + f(-x,-y) .$$

However this will require us to deal with larger systems of equations than mentioned above. Finally, we may consider the largest system of equations that derived by allowing each knot to have its own coefficient.

In all the above cases if $\gamma > m$, we have to investigate the properties of common zeros of orthogonal polynomials, properties which are not well understood. For example, it is not known if moment matrices derived on using these zeros as knots are singular or not.

There is one special case where I.P.Q.F. can be easily applied to functions of two variables, that is when the function of two variables is known to be a product of two functions of one variable. In this case, since $I(f;g)$ can be split into the product of two integrals, on applying the analysis of §2.4 and §2.7 we see that there are I.P.Q.F. exact on $\Phi^{2m-1} \times \Psi^{2n-1}$. However, since we are able to divorce the contributions of f and g in $Q(f;g)$ we see that the I.P.Q.F. obtained, which is unique, is the product of two R.Q.F. Let

$$Q_1^{(m)}(f) = \sum_{i=0}^{m} a_i S_i(f), \tag{5.1.7}$$

and

$$Q_2^{(n)}(g) = \sum_{j=0}^{n} b_j T_j(g), \tag{5.1.8}$$

be exact on Φ^{2m-1} and Ψ^{2n-1} respectively. Then

$$\begin{aligned} Q^{(m,n)}(f;g) &= \underline{f}^T \mathrm{diag}(a_i) \underline{1}_m \underline{1}_n^T \mathrm{diag}(b_j) \underline{g} \\ &= \underline{f}^T A \underline{g} \ , \end{aligned} \tag{5.1.9}$$

where $\underline{1}_m (\underline{1}_n)$ is a $(m+1)((n+1))$ vector of 1's,

$\underline{f}, \underline{g}$ are as in (1.3.3),

and

$$(5.1.10) \qquad a_{ij} = a_i b_j, \quad i=0,\ldots,m, \quad j=0,\ldots,n.$$

Obviously this process is valid for d>2 as well. There does not seem to be anything to recommend further study of I.P.Q.F. in this situation as we do not gain any advantage over the use of two R.Q.F. and multiplying the results together. Similarly, replacing the R.Q.F. in (5.1.7) and/or (4.1.5) with I.P.Q.F. does not seem to offer any advantage except possibly in the calculation of one dimensional integrals.

Another possible use of I.P.Q.F. in higher dimensional spaces is in the evaluation of line integrals. See [151] and the references contained therein. We consider the following special case of an I.P.Q.F.:

$$(5.1.11) \qquad I(f;\Omega) \equiv \int_{\Omega} f(s) \left(\frac{d\Omega}{ds}\right) ds,$$

where s is the distance along a rectifiable curve Ω in $\mathbb{R}^2$, and f is a given function. The elementary functional S_i are interpreted as information concerning the function f, and the functionals T_j as information concerning the curve Ω. Obviously the most likely form of S_i and T_j will be that of the values of f and Ω for given (distinct) values of s, so requiring m=n. We note that I.P.Q.F. derived in other contexts cannot be immediately employed in approximating (5.1.11), since (5.1.11) involves the derivative of Ω as an integrand.

If the curve Ω is given in parametric form, i.e. considering R^2 as the xy-plane and Ω is the set of points

$$\Omega(t) = \{(\xi(t),\theta(t)), \quad a \le t \le b\} ,$$

we find that, integrating in the positive direction, (5.1.11) becomes

$$\int_a^b f(\xi(t),\theta(t))((\xi'(t))^2 + (\theta'(t))^2)^{\frac{1}{2}}dt. \tag{5.1.12}$$

We observe that unless ξ and θ are given and $f(\xi,\theta)$ is regarded as the product of a ϕ-polynomial and a ψ-polynomial, $\xi \equiv \phi$, $\theta \equiv \psi$, and (2.8.1) being valid for both Φ^γ and Ψ^δ, (5.1.12) will become a nonlinear functional of ξ and θ, and so I.P.Q.F. as developed here will be inapplicable.

§5.2 EXTENSIONS - MORE THAN TWO FUNCTIONS

When dealing with real variables, the extensions of the concept of I.P.Q.F. dealing with the product of two functions in more than one dimension and, alternatively, the product of more than two functions in one dimension are the most natural to investigate. We examine the second of these in this section. See also [119].

First, we examine the interpolatory case, which is straightforward. We consider the case of the product of three functions. Extension to the product of four or more functions are obvious by analogy. We are given

$$(5.2.1) \qquad I(f;g;h) \equiv \int_R w(x)f(x)g(x)h(x)dx$$

and we approximate this by

$$(5.2.2) \quad Q(f;g;h) \equiv \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{p} a_{ijk}S_i(f)T_j(g)U_k(h),$$

where $S_i \varepsilon (\Phi^m)^*$, $T_j \varepsilon (\Psi^n)^*$, and $U_k \varepsilon (\Xi^p)^*$ are sets of linearly independent elementary linear functionals defined on their respective function spaces. We may also regard $A \equiv (a_{ijk})_{i=0,j=0,k=0}^{m,n,p}$ as a "three-dimensional" matrix, abbreviated as a 3-matrix, made up of real numbers a_{ijk}. In this context the matrix A in 1.3.3 would be called a 2-matrix. Naturally, we seek a_{ijk} such that Q, which we will denote $Q^{(m,n,p)}$ when necessary, is exact on $\Phi^m \times \Psi^n \times \Xi^p$ that is,

$$(5.2.3) \qquad E(f;g;h) \equiv I(f;g;h) - Q^{(m,n,p)}(f;g;h)$$

is zero when $f \varepsilon \Phi^m$, $g \varepsilon \Psi^n$, and $h \varepsilon \Xi^p$. We now proceed as in §1.3: Let

$$(5.2.4) \qquad F = (U_k(\xi_\ell))_{k=0,\ell=0}^{p,p} \equiv (f_{k\ell})_{k=0,\ell=0}^{p,p}$$

$$= \begin{bmatrix} U_o(\xi_o), & \ldots, & U_p(\xi_o) \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ U_o(\xi_p), & \ldots, & U_p(\xi_p) \end{bmatrix}.$$

We now define

(5.2.5) $$D = F \otimes (C \otimes B),$$

and seek to solve the system of linear moment equations

(5.2.6) $$D\underline{a} = \underline{\Lambda},$$

where

(5.2.7)
$$\begin{cases} \underline{a}^T = (\underline{a}_o^T , \ldots , \underline{a}_p^T) , \\ \underline{a}_k^T = (\underline{a}_{ok}^T , \ldots , \underline{a}_{nk}^T) , \quad k=0,\ldots,p, \\ \underline{a}_{jk}^T = (a_{ojk}, \ldots , a_{mjk}), \quad j=0,\ldots,n. \end{cases}$$

and

(5.2.8)
$$\underline{\Lambda}^T = (\underline{\Lambda}_o^T , \ldots , \underline{\Lambda}_p^T) ,$$

$$\underline{\Lambda}_k^T = (\underline{\Lambda}_{ok}^T, \ldots , \underline{\Lambda}_{nk}^T), \quad k=0,\ldots,p,$$

$$\underline{\Lambda}_{jk}^T = (I(\phi_o;\psi_j;\xi_k),\ldots,I(\phi_m;\psi_j;\xi_k)), \quad j=0,\ldots,n.$$

Clearly there are direct analogues of all the Theorems and Definitions of §1.3. The matrix A may be found by analogues of the methods suggested in §2.9. In particular note that the analogue of (2.9.10)-(2.9.12) requires the examination of only one $(m+1)\times(m+1)$, one $(n+1)\times(n+1)$ and one $(p+1)\times(p+1)$ matrix.

We now consider the question of exactness on, say, $\Phi^\gamma \times \Psi^\delta \times \Xi^\iota$. As before (see §2.7), there is no real gain in considering $\gamma<m$, $\delta<n$ or $\iota<p$. We examine the case (say) $\gamma>m$, $\delta=n$, and $\iota=p$. Following the analysis of §2.4, it is clear that,

defining h_r as in (2.4.10), we have

THEOREM 5.2.1 Let S_i depend upon $p_i \varepsilon P_i$, $i=0,\ldots,m$. For an I.P.Q.F. exact on $\Phi^\gamma \times \Psi^n \times \Xi^p$, $\gamma > m$, to exist it is necessary and sufficient that the following conditions are fulfilled:

(5.2.9) There exist h_r, W-orthogonal to $\Psi^n \times \Xi^p$, with respect to I, $r=1,\ldots,\gamma-m$. That is,

$$I(h_r;\psi_j;\xi_k) = 0, \quad j=0,\ldots,n, \quad k=0,\ldots,p.$$

(5.2.10) There exist $\tilde{p}_i \varepsilon P_i$, $i=0,\ldots,m$, such that

$$S_i(\tilde{p}_i;h_r) = 0, \quad r=1,\ldots,\gamma-m.$$

(5.2.11) The elementary functionsl $(S_i(\tilde{p}_i;\ ))_{i=0}^m$ are linearly independent on $(\Phi^m)^*$. ///

We have already assumed independence of T_j on $(\Psi^n)^*$ and U_k on $(\Xi^p)^*$ and so (5.2.11) guarantees that the matrix D in (5.2.5) has full rank. Analogous comments to those following Theorem 2.4.3 obviously apply here.

Clearly the above result is symmetric in γ,δ and ι. In general, we would expect (5.2.9) not to be satisfied if $(n+1)(p+1)>m+1$. However, in certain special circumstances, we might expect more advantageous results. For example, if we find

(5.2.11)
$$\begin{cases} \psi_j \equiv \xi_j & j=0,1,\ldots \\ \psi_j \equiv (\psi_1)^j & j=0,1,\ldots, \end{cases}$$

we obtain (cf §2.7),

THEOREM 5.2.2 Assume $S_i \varepsilon \chi_1$, i=0,...,m, (5.2.11) holds and n+p=m. If $\{\psi_j\}_{j=0}^{m}$ is a T-set on R, then there exists an I.P.Q.F. exact on

$$\Phi^{m+1} \times \Psi^n \times \Xi^p ,$$

provided that there exists a set of (m+1) zeros of h_1 such that detB≠0. The functionals T_j and U_k may be chosen arbitrarily provided detC≠0 and detF≠0. The choice of the functionals T_j and U_k uniquely determines the I.P.Q.F. ///

Clearly, if Φ^{m+1} is a T-set on R, the knots x_i are unique. We also note that the requirement that $\{\psi_j\}_{j=0}^{m}$ be a T-set in the above result in effect requires that ψ_1 is a monotonically increasing or decreasing function. Further, analogous with results in §2.8, we have

LEMMA 5.2.3 Let ϕ_i obey (2.8.1) with $\phi_1 \equiv \psi_1$. Let (5.2.11) hold. Assume ψ_1 is a monotonic function on R. Then there exist I.P.Q.F. exact on

$$\Phi^{\gamma} \times \Psi^n \times \Xi^p$$

$\gamma+n+p=2m+1$, $\gamma=2m+1-n-p>m$. The functionals, chosen as in Theorem 5.2.2, uniquely determining the I.P.Q.F.

PROOF: Since ψ_1 is monotonic, all the sets of functions under consideration are T-sets. The Lemma is now immediate. ///

We see that these results, which are immediate generalizations of results concerning I.P.Q.F. which deal with the product of two functions, suffer the restriction $\gamma+n+p=2m+1$ if we seek $\gamma>m$, despite the increased number of parameters (in A) we have available. We have the following:

LEMMA 5.2.4 Let Φ^m, $m>n$ and Ψ^δ, $\delta\geq n$ be T-sets on R. If there exists a function $\xi\epsilon sp\Xi^p$ which is of one sign throughout R, there exists no choice of χ_1 functionals such that there is an I.P.Q.F. exact on

$$\Phi^m\times\Psi^\delta\times\Xi^p.$$

PROOF: Analogous to Theorem 2.1.1. ///

COROLLARY 5.2.5 Let Φ^γ, $\gamma\geq m$ and Ψ^δ, $\delta\geq n$ be T-sets on R. Let $\gamma+\delta>\min(2n+1,2m+1)$. If there exists a function ξ as in Lemma 5.2.4, there exists no choice of χ_1 functionals such that there is an I.P.Q.F. exact on

$$\Phi^\gamma\times\Psi^\delta\times\Xi^p\ .$$ ///

In this context, concerning functions of one sign, it is useful to note the following result:

THEOREM 5.2.6 Let R be a closed real interval, and Ξ^p a T-set on R. Then there is $\theta\epsilon sp\Xi^p$ such that $\theta>0$ on R.

PROOF: See [154,155]. ///

If we remove the restriction that the above sets of functions are T-sets, it is clear from §2.7 that we also lose our guarantee that the relevant orthogonal polynomials have sufficient zeros to act as knots of an I.P.Q.F. Of course, all the above results are symmetric in ϕ,ψ and ξ.

Thus we see that seeking an I.P.Q.F. exact on $\Phi^{\gamma}\times\Psi^{n}\times\Xi^{p}$ is a strong restriction on our choice of m, n and p, since we can only obtain results when $\gamma+n+p\geq 2m+1$. This is to be contrasted with the interpolatory case where there is no such restriction on m, n and p. Thus we conclude that when integrating the product of two or more functions we should use interpolatory I.P.Q.F., perhaps assigning any spare (non-linear) parameters in a way which improves the behaviour of the error functional in some way.

The results concerning errors for I.P.Q.F. in Chapter 4 can clearly be generalized to deal with the product of more than two functions.

§5.3 EXTENSIONS - COMPLEX VARIABLES

Having dealt with the above forms of extension of I.P.Q.F. when dealing with real variables and real valued functions, we now propose to briefly examine the effects of allowing the variables to become complex. We note that there appears to be dearth of results in the literature dealing exclusively with approximate integration in the complex domain, although there are some results investigating the connection between analytic

functions and integration over real intervals. In fact consideration of analytic functions allows us to derive yet another class of error estimates, both for R.Q.F. and I.P.Q.F., see [5,12,71,72]. Thus we obtain yet another way of defining a "best" Q.F., since, with the aid of some functional analysis we are able to separate the contributions due to the intergrand and the Q.F. in the error estimate.

From [2] we have the following well known result:

THEOREM 5.3.1 Given n+1 distinct (real or complex) points, $z_0,\ldots z_n$, and n+1 (real or complex) values, $w_0,\ldots,w_n$, there exists a unique polynomial, p, of degree at most n, such that

$$p(z_i) = w_i \qquad i=0,\ldots,n. \qquad ///$$

This result is immediate since the so-called Vandermonde determinant is non-zero. If fact, see [156], this result can be extended to ϕ-polynomials, where ϕ_i obey (2.8.1) and the points z_i are all ϕ-distinct.

Theorem 5.3.1 means that we can construct interpolatory I.P.Q.F. which will integrate ϕ- and ψ-polynomials exactly, (ϕ_i and ψ_j satisfying (2.8.1)), over arbitrary regions in the complex plane, using χ_1 functionals. The analogue of Theorem 2.4.3 also clearly holds. However, still examining χ_1 functionals, we find, see [2,157], that orthogonal polynomials on regions in the complex plane may not possess distinct zeros, and these zeros may not even be in the region itself. Consequently,

except in special cases, we would not expect to be able to obtain Q.F. which might be regarded as analogous to Gaussian QF in the complex plane.

However, we note that the Vandermande determinant remains non zero if, on allowing, say, $z_i=z_{i+1}$, we replace the $(i+2)^{nd}$ row, i.e. $(1, z_{i+1}, z_{i+1}^2,\ldots,z_{i+1}^n)$, with the row $(0,1,z_i,z_i^2,\ldots, z_i^{n-1})$. This procedure can be generalized in the obvious way to deal with both higher derivatives and several groups of numbers z_i coalescing. Thus, provided the integrands are defined at the zeros of the orthogonal polynomials and the relevant derivatives may be approximated, we can obtain I.P.Q.F. (and R.Q.F.) in which $\gamma>m$ by choosing the elementary functionals S_i to be function and derivative evaluation at the zeros of the orthogonal polynomial (if a zero z^* has multiplicity r we use the elementary functionals $f(z^*),\ldots,f^{(r-1)}(z^*)$).

§5.4 APPLICATIONS

In a sense, all applications of I.P.Q.F. are obvious, that is, we would consider using them in any circumstances where we integrate the product of two (or possibly more) functions, each of which can be distinguished. Some work has already been done. We list various applications which come readily to mind:

(5.4.1) Solutions of Integral Equations. See [15,18].

(5.4.2) Solution of various transforms and convolution integrals.

(5.4.3) The evaluation of (generalized) Fourier coefficients of functions.

(5.4.4) Evaluation of the integral of the product of a large number of relatively simply functions. See [118].

With regard to (5.4.3) we note that there are some very good algorithms already available for the evaluation of the Fourier sine and cosine coefficients of a function. See, for example, [158,159]. However, we note that these methods rely on (standard) polynomial approximations, and as remarked at the beginning of Chapter 2, we feel that the intrinsic strength of the concept of I.P.Q.F. lies in dealing with the situation in which either ϕ_i or ψ_j are not standard polynomials, and we can benefit fully from not having to discover an unnecessary amount of information about both integrands. For example, as always, in the obvious situation where one of the functions is badly behaved and the other is not.

§5.5 EXAMPLES

Here we propose to give a few simple examples of I.P.Q.F. There are, of course, some examples of I.P.Q.F. in the literature, see [12,15,16,17,18,23,24,25,118]. However all these examples deal with the special case $\phi_i \equiv \psi_i \equiv t^i$, $i=0,1,2,\ldots$, and although it is reported that the use of these I.P.Q.F. is more accurate that R.Q.F., ([23]) that is, the same accuracy is achieved for

less function evaluations, as commented in §2.1, we feel that the main advantage in using I.P.Q.F. becomes apparent when at least one of the sets of functions ϕ_i or ψ_j are not the standard polynomials. As pointed out earlier, it is under these circumstances that we may claim to make "full use" of all the linear parameters (a_{ij}) available in obtaining interpolatory I.P.Q.F.'s which can integrate exactly (m+1)(n+1) distinct functions as opposed to the at most 2m+1 a R.Q.F. can cope with.

Examples demonstrating some of the symmetry properties mentioned in §2.10 are given in [24]. We now proceed to give some examples of I.P.Q.F. We shall examine the following sets of functions:

(5.5.1) $\qquad (t+1)^{\alpha i}, \quad i=0,1,\ldots, \ \alpha \neq 0,$

(5.5.2) $\qquad e^{i\beta t}, \quad i=0,1,\ldots, \ \beta \neq 0,$

(5.5.3) $\qquad \cos(i\pi(t+1)/2) \quad i=0,1,\ldots,$

(5.5.4) $\qquad \sin(i\pi(t+1)/2) \quad i=1,\ldots$

Throughout, we recall that the interior of R is (-1,1). In the interests of simplicity we assume $w \equiv 1$. Also, we shall only consider using χ_1 functionals. Once the knots have been determined we obtain the matrix A using (2.9.10) - (2.9.12). The zeros of the relevant orthogonal polynomials are obtained using the algorithm given in [167] for obtaining simple real zeros of polynomials and modifying the results obtained in the obvious

manner. That is, given the coefficients of the orthogonal polynomial, find the roots using the algorithm, obtaining for example $\alpha_i(=\phi(x_i))$ $i=1,\ldots,r$, r being the degree of the polynomial, and then solving $\alpha_i=\phi(x_i)$, $i=1,\ldots,r$. We know that the sets of functions (5.5.1) - (5.5.3) are T-sets on $[-1,1]$ from, for example, [111].

Example 5.5.1 Let ϕ_i be given by ξ_i of (5.5.2), $\beta=1$, and ψ_j be given by ξ_j of (5.5.1), $\alpha=1$. Let $m=n=3$. We then obtain the following I.P.Q.F. exact on $\Phi^4\times\Psi^3$ and $\Phi^3\times\Psi^4$.

$$(5.5.5)\quad \underline{f}^T \begin{bmatrix} 0.4697446460 & 0.0505350026 & -0.0259709689 & 0.0082369280 \\ -0.0206882436 & 0.6886947864 & 0.0706153851 & -0.0179163084 \\ 0.0045573657 & -0.0242395656 & 0.5361207742 & 0.0219676153 \\ -0.0007807973 & 0.0034976556 & -0.0103330767 & 0.2459588018 \end{bmatrix} \underline{g},$$

where $\underline{f}^T \equiv (f(-0.7913959534), f(-0.1331193345), f(0.5153672678), f(0.9069848137))$,

$\underline{g}^T = (g(-0.8154575387), g(-0.1896084753), g(0.4791128782), g(0.8991238864))$.

We note that elements of the moment matrices required in order to calculate the parameters in (5.5.5) can be obtained in closed form only when α above is an integer. ///

Example 5.5.2 Let ϕ_i be given by ξ_i of (5.5.3), and ψ_j be given by ξ_j of (5.5.1), $\alpha=1$. Let $m=n=3$. We obtain the following I.P.Q.F. exact on $\Phi^4\times\Psi^3$ and $\Phi^3\times\Psi^4$.

$$(5.5.6)\quad \underline{f}^T \begin{bmatrix} 0.4180077850 & 0.0467218887 & -0.0095186877 & 0.0007664687 \\ -0.0184376039 & 0.5580034932 & 0.0014445871 & 0.0030120689 \\ 0.0030120689 & 0.0014445871 & 0.5580034932 & -0.0184376039 \\ 0.0007664687 & -0.0095186877 & 0.0467218887 & 0.4180077850 \end{bmatrix} \underline{g}$$

where $\underline{f}^T = (f(-0.7809256111), f(-0.2758715666), f(0.2758715666)$
$f(0.7809256111))$,

$\underline{g}^T = (g(-0.8342341385), g(-0.2969834986), g(0.2969834986)$
$g(0.8342341385))$.

We note that the above mentioned algorithm employed for finding zeros of polynomials can be used when dealing with ϕ_i on noting that we can write

$$\cos ix = \sum_{j=0}^{i} \alpha_{ij} (\cos x)^j,$$

for specified α_{ij} (see, for example, [168]).

We can also observe, as predicted in Theorem 2.10.8, that the matrix A above has rotational symmetry of order 2.

From Corollary 2.10.12 it is clear that the above I.P.Q.F. will also be exact on

$$\Phi^5 \times \Psi^2 \text{ and } \Phi^2 \times \Psi^5. \qquad ///$$

Example 5.5.3 We now demonstrate the fact that we do not require m=n.

Let ϕ_i be given by (5.5.2), $\beta=1$, and ψ_j by (5.5.3). Let m=2, and n=3. We then obtain, for example, the following I.P.Q.F. exact on $\Phi^2 \times \Psi^3$.

$$(5.5.7)\quad \underline{f}^T \begin{bmatrix} 0.5986705423 & 0.2234366014 & -0.1020851981 & 0.0209795908 \\ -0.0628180635 & 0.3764349217 & 0.5484907508 & -0.0492236995 \\ 0.0094644135 & -0.0325905421 & 0.0715457235 & 0.3976949592 \end{bmatrix} \underline{g}$$

where $\underline{f}^T = (f(-0.6633426243), f(0.1587977139), f(0.8229651623))$,

$\underline{g}^T = (g(-0.7298108309), g(-0.1709472124), g(0.3801006302),$
$g(0.8292503279))$.

The knots x_i and y_j were chosen to be zeros of ϕ_3 and ψ_4 respectively, and consequently the I.P.Q.F. will also be exact on

$$\Phi^3 \times \Psi^2 \text{ and } \Phi^2 \times \Psi^4.$$

We note that while the knots x_i are uniquely determined, it is possible, since there are degrees of freedom in the choice of ψ_4 (we made the natural choice of requiring ψ_4 orthonormal to Φ_4, but this is not necessary), that there might be some element of choice in the determination of the knots y_j, while still retaining exactness on all the above mentioned product spaces of function.

We observe that the asymmetry of the knots reflects the asymmetry of the exponential functions ϕ_i. ///

In Examples (5.5.1) and (5.5.2) we see that the coefficient matrices are diagonally dominant, those elements which are negative have relatively small modules, and the diagonal elements are of similar size. From the point of view of estimating errors, via some norm of the coefficient matrix, and roundoff error, the above

properties of the coefficient matrices are advantageous. Even the coefficient matrix in Example (5.5.3) is almost "diagonally dominant" in the sense that a_{11}, a_{22}, a_{23} and a_{34} are its largest elements. These properties are not necessarily general, particularly in the case where $m \neq n$. It is also possible that coefficient matrices with smaller norms can be obtained when the knots x_i and/or y_j are not zeros of the relevant orthogonal polynomials.

It is not yet clear under what circumstances we may obtain a theoretical justification for these clearly desirable properties.

We now proceed to examine the following:

$$(5.5.8) \qquad \int_{-1}^{1} \ln(\pi(1-x)) \cos(j\pi(x+1))dx = -\frac{1}{\pi j} S_i(j2\pi),$$

and

$$(5.5.9) \qquad \int_{-1}^{1} \ln(\pi(1-x))\sin(j\pi(x+1))dx = \frac{1}{j\pi}(\gamma + \ln(2j\pi) - ci(2j\pi)),$$

where γ is Euler's constant, and Si and ci are the Sin and Cosine integrals (see [168]). Some numerical results concerning (5.5.9) are given in [5], p. 66, and it is apparent, even using some R.Q.F. specifically designed to deal with oscillatory functions, that (5.5.9) (and analogously (5.5.8)) is difficult to approximate using a Q.F. This is for two reasons, first, the oscillatory behaviour of the trigonometric functions and second, particularly in (5.5.8), the fact that $\lim_{x \to 1} \ln\pi(1-x) = -\infty$.

We list some results in Tables 5.5.1 and 5.5.2. The results were calculated to 18 decimal places, then rounded to 6.

TABLE 5.5.1

Numerical Results for (5.5.8)

	j=2	j=3	j=4	j=5	j=10	j=15
exact[1]	-0.237485	-0.161068	-0.121843	-0.097977	-0.0494937	-0.0335511
RGS4 x1	-0.547546	1.49456	-0.658372	-1.01690	-0.549075	1.65892
x4	-0.221867	-0.145626	-0.103818	-0.0810572	-0.123038	0.0839035
x10	-0.231199	-0.154792	-0.115580	-0.0917305	-0.0411894	-0.0243722
R31B x1	-0.0915452	0.781835	-1.32892	-0.0692672	-0.432809	-0.159776
x4	-0.217257	-0.140748	-0.0913534	-0.0745310	-0.0120172	0.234376
x10	-0.229546	-0.153073	-0.107225	-0.0898865	-0.0409699	-0.0167606
R23A x1	-0.110081	-0.433241	-1.15091	0.174733	0.0260176	0.918274
x4	-0.217289	-0.135790	-0.103220	-0.0797689	0.00363	-0.485379
x10	-0.230510	-0.153887	-0.114300	-0.0898969	-0.0428029	-0.0195204
R23F x1	-0.267987	-0.414278	-1.11524	0.100070	-0.0587018	0.946139
x4	-0.244938	-0.153978	-0.109160	-0.0997442	-0.0264598	-0.467996
x10	-0.235470	-0.161415	-0.124129	-0.100905	-0.0454539	-0.0422994
R24A x1	-0.184586	-0.140858	-1.40276	0.747689	-0.597365	0.747689
x4	-0.209730	-0.122653	-0.168245	-0.0647461	-0.0593925	-0.104526
x10	-0.221990	-0.150150	-0.113348	-0.0904064	-0.114602	-0.0353209
R211Ax1	-0.297506	-0.168364	0.0405767	0.312668	0.272750	0.312668
x4	-0.243312	-0.156961	-0.111315	-0.0995339	-0.0628976	-0.0840902
x10	-0.235687	-0.161297	-0.123673	-0.100343	-0.0463140	-0.0442142

[1]Those values (of Sin and Cosine Integrals), suitably modified, come from [5,60,170].

[2]From [5].

TABLE 5.5.2

Numerical Results for (5.5.9)

	j=2	j=3	j=4	j=5	j=10	j=20
exact[1]	0.495672	0.373110	0.302617	0.256274	0.150179	0.086103
RGS32x1[2]	0.495606		0.302615		0.150289	-0.0393777
RG54 x1	0.301574	0.843106	1.30699	0.366074	-0.409023	0.309358
x4	0.496665	0.374548	0.305987	0.262631	-0.112950	0.115196
x10	0.495828	0.373353	0.302952	0.256673	0.151751	0.0402575
R31B x1	0.329849	1.00763	0.992598	0.00899808	-0.799542	-0.639485
x4	0.496973	0.375830	0.302753	0.239458	0.0597910	-0.0850850
x10	0.495871	0.373406	0.303031	0.256810	0.150181	0.0532306
R23A x1	0.689656	1.65519	-0.987605	-0.283092	0.956108	-0.694964
x3	0.507802	0.390604	0.364925	0.289665	0.198122	-0.257117
x10	0.497902	0.376598	0.307346	0.262190	0.193077	0.171601
R23F x1	0.978836	1.77403	-0.927977	-0.301661	0.819257	-1.01980
x4	0.506996	0.402527	0.360629	0.268556	0.233597	-0.260382
x10	0.494209	0.372619	0.304643	0.261885	0.190145	0.202251
R24A x1	0.568128	0.831934	-1.01917	-0.391046	0	0
x4	0.494187	0.401061	0.297187	0.217148	0.211661	-0.141538
x10	0.474224	0.359886	0.295431	0.255685	0.147846	0.0916444
R211Ax1	0.590286	0.470743	0.479063	0.684289	0	0
x4	0.496192	0.381181	0.296655	0.233954	0.162262	0.217225
x10	0.492742	0.370171	0.300903	0.256493	0.147633	0.0938493
R24B x1	0.510640	0.875050	-0.995828	-0.374886	0	0
x4	0.496153	0.398672	0.301930	0.233098	0.219969	-0.138579
x10	0.475997	0.361930	0.297581	0.256643	0.149950	0.0862617

Calculations were carried out, using APL language, on the DEC-10 computers at the Universities of Toronto, Canada, and Aberdeen, Scotland.

The notation, x1,x4, and x10 indicates how many equal subintervals R was divided into when the rules were used in a compound manner. The rules RGS32 and RGS4 are the Gaussian 32 and 4 point rules respectively. The rules R31B and R23A are given by (5.5.6) and (5.5.7). The remaining rules can be described as follows:

R23F : $m=\gamma=3$, $n=3$, $\delta=4$. ϕ_i defined by (5.5.2), $\beta=1$; ψ_j defined by (5.5.3). The knots x_i, $i=0,\ldots,3$, and y_j, $j=0,\ldots,3$, are $\bar{0}.5,0.2,0.75,0.97$, and $\bar{0}.7298108039$, $\bar{0}.1709472124$, 0.3801006302, 0.8292503279 respectively.

R24A : $m=\gamma=n=\delta=3$. ϕ_i defined by (5.5.2), $\beta=1$; ψ_j defined by (5.5.3). The knots x_i, $i=0,\ldots,3$, and y_j, $j=0,\ldots,3$, are $-0.5,0.2,0.75,0.97$ and $-0.7,-0.2,0.3,0.8$ respectively.

R24B : As R24A except that the knots x_i, $i=0,\ldots,3$, are now $-0.6,0.1,0.6,0.9$.

R211A : $m=\gamma=3$, $n=\delta=6$. ϕ_i defined by (5.5.2), $\beta=1$; ψ_j defined as

(5.5.10)
$$\begin{cases} \psi_o \equiv 1 , \\ \psi_{2k+1} = \cos(\pi(k+1)(x+1)), & k=0,1,.., \\ \psi_{2k} = \sin(\pi k(\lambda+1)), & k=1,2,... \end{cases}$$

This is a T-set on R. The knots x_i, $i=0,\ldots,3$, and

$y_j, j=0,\ldots,6$, are $-0.5, 0.2, 0.75, 0.98$ and $-0.9, -0.6, -0.3, 0, 0.3, 0.6, 0.9$ respectively.

The choice of the knots x_i and y_j is heuristic and has not been the subject of a detailed investigation. The asymmetry (with respect to the origin) of the knots x_i reflects the asymmetry of the exponential function. In all rules except R211A, the asymmetry of the knots y_j reflects the importance of the behaviour of the integrand near 1. We observe, in particular, how as $x_3 \to 1$ when dealing with (5.5.8), and $x_3 \to 0.9$ when dealing with (5.5.9) (thus reflecting to an extent the positioning of minima of the integrands) the I.P.Q.F. give better approximations of I. We see that in approximating (5.5.8) both R23F and R211A slightly outperform RGS4. Also, as j increases in value, we see that all the I.P.Q.F. used have improved performances - particularly R24A. As might be expected since $x^{-i}e^{ix}$ is an increasing function ($x>0$), R31B is outperformed by R211A, R23F and R23A. When we consider (5.5.9), as might be expected, we find R211A and R24B performing better than the other I.P.Q.F., and they perform better than RGS7 for $j \geq 5$. Finally, we observe that while R24A performs reasonably well on (5.5.8), R23F does not perform so well on (5.5.9). This, and the deteriorating results of R24A and R24B for small values of j are probably explained by the fact that $\lim_{x \to 1} \ln(\pi(1-x)) \sin j\pi(x+1) = 0$ and so when these rules are compounded the interpolatery approximation of the integrand is forced toward zero in places where this should not occur.

We also note that, from the point of view of function evaluations, counting f and g as separate functions, that all rules mentioned, except R211A and RGS32, use the same number of function evaluation; 8,32 or 80, whereas R211A uses 11,44 or 110 and RGS32 64. In general, of course, an I.P.Q.F. uses

(5.5.11) (m+n+2) Elementary functional evaluations and (m+2)(n+1) multiplications during the course of its evaluation,

whereas a R.Q.F. uses, when applied to a product of two functions.

(5.5.12) (2m+2) Elementary functional evaluation and 2m+2 multiplications during the course of its evaluation.

In (5.5.12) we have in effect, written the R.Q.F. as an I.P.Q.F., chosen m=n and then taken into account the special structure of the coefficient matrix.

It is clear, particularly if multiplications are cheap and elementary functional evaluation is expensive, that the use of an I.P.Q.F. may well be cheaper than the use of an R.Q.F. if m is noticably bigger than n.

Consequently, in the above examples, the improved performance of I.P.Q.F. over RGS7 cannot be attributed to the fact that we obtain more information about the integrand using an I.P.Q.F. than the R.Q.F., though perhaps it can be attributed to the fact that we use what information we have "more effectively".

The question of how we may realistically compare two (or more) Q.F. with one another is not easily resolved. It is clear that simply comparing results when the Q.F. are applied to a few carefully chosen "test-functions" is not really an adequate procedure. Some work on so-called "performance profiles" of Q.F. has been done, see [41, 42, 43]. In dealing with I.P.Q.F., which are designed as special purpose Q.F. we propose a simple method of comparison.

Let the functions f and g depend upon parameters α and β respectively, thus obtaining a two parameter family of integrands. Approximate these integrals using two Q.F., Q_1 and Q_2 and we may compare the performance of the Q.F., on this family of functions in the following ways:

(5.5.13) Maximum modulus relative error incurred by Q_1 and Q_2

(5.5.14) Number of times the relative error for one Q.F. is smaller than the relative error for the other Q.F.

We notice that (5.5.14) is a very crude comparison and augment it by calculating the following ratio for Q_1 (and Q_2 analogously);

$$(5.5.15)\quad \left(\sum_{\substack{\alpha,\beta \\ |I_{\alpha\beta}-Q_{1\alpha\beta}|<|I_{\alpha\beta}-Q_{2\alpha\beta}|}} \frac{|I_{\alpha\beta}-Q_{1\alpha\beta}|}{|I_{\alpha\beta}-Q_{2\alpha\beta}|} \right) \div \left(\begin{array}{c} \text{number of times} \\ |I_{\alpha\beta}-Q_{1\alpha\beta}|<|I-Q_{2\alpha\beta}| \end{array} \right) = R_1$$

where $I_{\alpha\beta}$, $Q_{1\alpha\beta}$, $Q_{2\alpha\beta}$ stand for I or Q_1 or Q_2 acting on f and g with parameters α and β. The ratio R_2 (for Q_2) is calculated analogously. It is clear that if R_1 is near 1 then, in the cases where Q_1 has smaller error than Q_2, the inter-

pretation is that Q_1 and Q_2 approximate I with similar accuracy, whereas if R_1 is near 0 , Q_1 gives a noticeably better approximation than Q_2 (on average). Thus we can obtain some sort of measure of how much "better" Q_1 is when compared to Q_2 in the cases where Q_1 gives a better approximation to I than Q_2 .

In order to be able to obtain exact solutions easily we examine some integrals of the form

$$(5.5.16) \qquad I(f';g) + I(f;g') = fg\Big|_{-1}^{1} .$$

Example (5.5.17) Let $f(x) = e^{\alpha x}$, $\alpha = 0,1,\ldots,4$

$g(x) = (1+x)^{\beta}$, $\beta = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$.

We approximate (5.5.16) by using

RGS4 (as defined earlier in this section),

R21A (as given in Example (5.5.1)),

and R21B,

where R21B is determined by letting ϕ_i be given by (5.5.2) (with $\beta = 1$), and ψ_j by (5.5.1) (with $\alpha = 1$), and $m = 2$, $n = 4$. The I.P.Q.F. is exact on $\Phi^3\times\Psi^2$ and $\Phi^2\times\Psi^5$, with

$$A = \begin{bmatrix} 0.3808777266 & 0.4048531936 & 0.01637874803 & -0.08176517185 \\ & & & 0.03664060927 \\ -0.07054513554 & 0.1900368667 & 0.5439376461 & 0.2772342454 \\ & & & -0.09432447568 \\ 0.01188582114 & -0.02546960111 & -0.006836955378 & 0.1901147600 \\ & & & 0.2269817228 \end{bmatrix}$$

$$\underline{f}^T = \big(f(-0.6740553004),\ f(0.2073300597),\ f(0.8448222766)\big),$$
$$\underline{g}^T = \big(g(-0.8702937525),\ g(-0.4019057276),\ g(0.1781971940),\ g(0.6549588641),\ g(0.9339042206)\big).$$

We obtain the following results:

TABLE 5.5.4

I.P.Q.F.	MAX. MOD. REL. ERROR	COMPARED WITH R21A				COMPARED WITH R21B			
		SMALLER ERROR		LARGER ERROR		SMALLER ERROR		LARGER ERROR	
		NO. TIMES	RATIO	NO. TIMES	RATIO	NO. TIMES	RATIO	NO. TIMES	RATIO
RGS4	0.0968	9	0.595	16	0.0729	3	0.960	22	0.505
R21A	0.113					9	0.0112	16	0.459
R21B	0.0941								

We see that on the family of test functions chosen in this example, both I.P.Q.F. may be claimed to perform better than RGS4. Note the use of exponential basis functions in both R21A and R21B, and the fact that all three Q.F. use 8 function evaluations.

Example 5.5.18. We repeat Example (5.5.17) with $g(x)$ now given by

$$g(x) = \big(1+(1-x)\big)^{-\beta} \qquad \beta = 1,2,3,4\ ,$$

and obtain the following results:

TABLE 5.5.5

I.P.Q.F.	REL. ERROR	COMPARED WITH R22A				COMPARED WITH R22B				COMPARED WITH R21B			
		SMALLER ERROR		LARGER ERROR		SMALLER ERROR		LARGER ERROR		SMALLER ERROR		LARGER ERROR	
		NO. TIMES	RATIO	NO. TIMES	RATIO	NO. TIMES	RATIO	NO. TIMES	RATIO	NO. TIMES	RATIO	NO. TIMES	RATIO
RGS4	0.0161	3	0.326	17	0.233	4	0.370	16	0.107	6	0.375	14	0.216
R22A	0.0337					6	0.173	14	0.213	10	0.237	10	0.313
R22B	0.0129									19	0.381	1	0.760
R21B	0.0145												

The I.P.Q.F. RGS4 and R21B are as before. The I.P.Q.F. R22A is determined by letting $\phi_i = \psi_i$ be given by (5.5.2) (with $\beta = 1$) and setting $m = n = 3$. Thus R22A is exact on $\Phi^{7-r} \times \Phi^r$, $r = 0,1,\ldots,7$ (see section 2.8), with

$A = \text{diag}(0.1847174964 , 0.6517223295 , 0.4487931752 , 0.7147669989)$,

$\underline{f}^T = \left(f(-0.7148808021),\ f(0.02752871693),\ f(0.6141299470),\ f(0.9284405505)\right)$

and $\underline{g}^T$ is as $\underline{f}^T$ (with g replacing f).

The I.P.Q.F. R22B has the same basis functions as R22A, but we now choose $m = 2$, $n = 4$. Thus R22B is exact on $\Phi^3 \times \Phi^2$ and $\Phi^2 \times \Phi^5$, with

$$A = \begin{pmatrix} 0.5802520778 & 0.3611532159 & -0.01666076724 & -0.03874154766 \\ & & & 0.01802973033 \\ -0.1160017994 & 0.3183096250 & 0.4774214710 & 0.1520157346 \\ & & & -0.05575714321 \\ 0.02524390504 & -0.05149889088 & 0.01285685095 & 0.1755889376 \\ & & & 0.1577886002 \end{pmatrix}$$

$\underline{f}^T = \left(f(-0.5873716418 , f(0.32999398216) , f(0.8763713498)\right)$,

$\underline{g}^T = \left(g(-0.7916367808) , g(-0.1891407712) , g(0.3699854772) , g(0.7502862223) , g(0.9534091960)\right)$.

For the family of test functions used in this example, we see that the effectiveness of both a suitable choice of nonpolynomial basis functions (R22A as opposed to RGS4) and the use of I.P.Q.F. (R21B and R22B) is apparent.

We can also note that, not surprisingly, the way in which we subdivide the interval of integration as well as the choice of basis functions can affect the accuracy of a compound I.P.Q.F. We can illustrate this with the following results obtained (rounded to 6 figures) when examining (5.5.8) with n=2.

Table 5.5.6 (exact result is -0.237485)

I.P.Q.F.	RATIO OF SUBINTERVAL LENGTHS	RESULT
R11A	1:1:1:1:1:1:1:1:1	-0.228065
	6:6:6:3:2:1	-0.231166
	10:10:10:4:3:2:1	-0.232643
	21:21:21:6:5:4:3:2:1	-0.233860
R31B	21:21:21:6:5:4:3:2:1	-0.236466
R23C	10:10:10:4:3:2:1	-0.236237
	15:15:15:5:4:3:2:1	-0.236687
	21:21:21:5:4:3:2:1	-0.236943

As an example, to clarify any ambiguity about the subintervals used above, we observe that if (-1,1) is subdivided with ratios 3:2:1, then we use the subintervals $(-1,0)$, $\left(0,\frac{2}{3}\right)$, and $\left(\frac{2}{3},1\right)$.

The I.P.Q.F. used in the above table are describes as follows.

R11A : m=3, n=2. Exact on $\Phi^4\times\Phi^2$ and $\Phi^3\times\Phi^3$, where the functions ϕ_i are given by (5.5.1) with $\alpha=1$. The knots x_i and y_j are -0.7946544723, -0.1875924741, 0.1875924741, 0.7946544723 and -0.7071067812, 0, 0.7071067812 respectively.

R23C : $m=n=3$, ϕ_i are defined by (5.5.2), with $\beta=1$ and ψ_j by (5.5.3). The I.P.Q.F. is exact on $\Phi^4\times\Psi^3$ and $\Phi^3\times\Psi^4$. The knots x_i and y_j are -0.7657840322, -0.1436874983, 0.4381215835, 0.8837483052 and -0.7298108309, -0.1709472124, 0.3801006302, 0.8292503279 respectively.

R31B : This is the I.P.Q.F. in Example (5.5.2).

From the above results we may make a few tentative conclusions:

(5.5.19) Using I.P.Q.F. it is possible in some circumstances to obtain improved approximations to I (as opposed to using compound low order R.Q.F. usually based on being exact for standard polynomials).

(5.5.20) Not only the choice of basis functions, but the choice of the parameters of the elementary functionals plays a crucial role in the effectiveness of I.P.Q.F.

(5.5.21) The number of times we compound our I.P.Q.F. can be critical, and is determined by matching the behaviour of the integrand and known properties of the basis functions ϕ_i and ψ_j.

The best example of this last part is seen by examining the results using R24B in Table 5.5.2. As always if we compound a Q.F. too many times, roundoff error will affect our results, however this is a separate issue to that raised in (5.5.21).

We also give an example in which we use elementary functionals other than χ_i functionals. (See §2.11).

Example (5.5.22). m=4, n=3, and the functions ϕ_i and ψ_j are defined by (5.5.1). The I.P.Q.F. is exact on $\Phi^5 \times \Phi^3$ and $\Phi^4 \times \Phi^4$.

$$A = \begin{bmatrix} -0.1286093407, & 0.2435260610, & -0.004436121970, & -0.003852499465 \\ -0.05072803185, & 0.5648462566, & -0.1144333710, & 0.03608058546 \\ 0.02423649750, & 0.2775072614, & 0.3494911888, & -0.08764953983 \\ -0.008119534890, & -0.05436317277, & 0.3835521890, & 0.4097311628 \\ 0.001068461289, & 0.005696332245, & -0.01775443549, & 0.08930149472 \end{bmatrix}$$

and

$$\underline{f}^T = (f(-0.9061798459),\ f(-0.5384631011),\ f(0),\ f(0.5384631011)\ f'(0.7650553239))$$

$$\underline{g}^T = (g'(-0.6546536707),\ g(-0.3399810436),\ g(0.3399810436)\ g(0.8611363116)).$$

It is interesting to note that the sume of the modulus of the elements in the matrix A in Example (5.5.22) is less than that for the corresponding I.P.Q.F. with the last element of $\underline{f}^T$ replaced with f(0.9061798459) and the first element of $\underline{g}^T$ replaced with g(-0.8611363116).

§5.6 COMPARISON OF I.P.Q.F. AND R.Q.F.: A SUMMARY

It is a little difficult to be able to compare I.P.Q.F. and R.Q.F. directly. When we talk of the degree of precision or exactness, once the functions ϕ_i have been announced, this has a fairly unambiguous meaning in the context of R.Q.F. This is not the case with I.P.Q.F. since first we have to announce both ϕ_i and ψ_j and then γ and δ, and in general these may not be interchanged. Thus more care is required when implementing on I.P.Q.F.

There appear to be several criteria which we might consider:

(5.6.1) Number of elementary functional evaluations required.

(5.6.2) Total number of linearly independent functions whose integral the Q.F. is able to evaluate exactly.

(5.6.3) Number of operations required in order to evaluate the Q.F.

The criteria (5.6.1) and, to an extent, (5.6.2) are often used to allow us to group R.Q.F. Once this has been done, we may seek "best" R.Q.F. in some sense, for example, by minimization of an error estimate of some sort, or obtaining maximal degree of exactness.

We recall that I.P.Q.F. are special purpose Q.F. in that they are designed to integrate the product of two functions. Thus we should only try to compare them with R.Q.F. under this restriction. This makes the natural tendency to compare the sum $\gamma+\delta$ from the I.P.Q.F. with the number γ from the R.Q.F. a

little dubious when, for example, we consider (5.6.1). We can make some comments.

The use of an I.P.Q.F. would appear to offer several advantages:

(5.6.4) The ability to reflect known characteristics of the behaviour of the integrand both in the choice of m and n, the choice of the functions ϕ_i and ψ_j, and perhaps in the choice of the elementary functionals S_i and T_j.

(5.6.5) Consider an I.P.Q.F. exact on $\Phi^m \times \Psi^m$. This requires no more elementary functional evaluations than the use of a R.Q.F. exact on Φ^m. However, in general, the I.P.Q.F. will integrate many more functions (regarding fg as a single function temporarily) exactly than will the R.Q.F. If we try and increase the degree of exactness of the R.Q.F. ($\gamma > m$) we may only do so at the expense of using certain special elementary functionals, while an interpolatory I.P.Q.F. retains the advantage of using essentially arbitrary elementary functionals (provided B and C retain full rank).

(5.6.6) An interpolatory I.P.Q.F. exact on $\Phi^m \times \Psi^m$ is equally easily derived, once the moments $\underline{\Lambda}$ are known, whether ϕ_i (ψ_j) are standard monomials or not. If we seek a R.Q.F. exact on Φ^γ, $\gamma > m$, (cf. (5.6.5)) this is not the case.

These gains are obtained at a certain cost however:

(5.6.7) An I.P.Q.F. is more costly to evaluate than a R.Q.F. (see comments in previous §).

(5.6.8) An I.P.Q.F. is very sensitive to the way in which the integrand is split into the appropriate product of two functions. This is not the case when using a R.Q.F.

Lastly we note that if we use an I.P.Q.F. exact on $\Phi^{\gamma}\times\Psi^{n}$, $\gamma>m$, although we increase the dimension of the product space of functions integrated exactly, this is only achieved at a marked drop in the number of elementary functionals we may choose at will. In particular consider the case when $m=n$ and $\gamma=m+1$, in which case all the elementary functionals S_i are prescribed.

APPENDIX I INDEX OF NOTATION

	Equation
$I(f;g)$ $(I(f))$	(1.3.1) (1.1.1)
$f\varepsilon\Gamma$	(1.3.1) (1.1.1)
$g\varepsilon\Delta$	(1.3.1)
w	(1.3.1) (1.1.1)
$R\subset\mathbb{R}^d$, $d\geq 1$, (see (A4)); $\mathbb{R}$	(1.3.1) (1.1.1)
S_i, $\{S_i\}_{i=0}^m$; T_j, $\{T_j\}_{j=0}^m$	(1.3.3) (1.1.2)
$Q^{(m,n)}(f;g)$; $Q(f;g)$ $(Q^{(m)}(f);\ Q(f))$	(1.3.3) (1.1.2)
$\underline{f};\underline{g}$	(1.3.3)
$A=(a_{ij})_{i=0,j=0}^{m,n}$ (a_i)	(1.3.3) (1.1.7)
$E(f;g)$ $(E(f))$	(1.3.4) (1.1.3)
$\Phi^\gamma=\{\phi_0,\ldots,\phi_\gamma\}$; $\Psi^\delta=\{\psi_0,\ldots,\psi_\delta\}$	(1.3.6) (1.1.5)
$\phi\varepsilon sp\Phi^\gamma$; $\psi\varepsilon sp\Psi^\delta$	(1.3.6) (1.1.6)
$B=(S_i(\phi_\ell))_{\ell=0,j=0}^{r,m}$, $r=\min(m,\gamma)$	(1.3.7) (1.1.8)
$C=(T_j(\psi_\ell))_{\ell=0,j=0}^{s,n}$, $s=\min(n,\delta)$	(1.3.8)
$D=C\otimes B$	(1.3.9)
$(\Phi^m)^*$; $(\Psi^n)^*$	Th. 1.3.1 (Th. 1.1.1)
$p;q$	Def. 1.3.1 (Def. 1.1.2)
$\underline{b};\underline{c}$	(1.3.12) (1.1.9)
$\underline{a};\underline{a}_i$	(1.3.13) (1.1.10)
$\underline{\Lambda},\underline{\Lambda}_i$	(1.3.14) (1.1.10)
$K(\lambda;x)$	(1.4.7)
$S_i(T_j)\varepsilon\chi_1$; $S_i\equiv S_i(x_i;\ )$	Def. 2.1.1
$x_0,\ldots,x_m$; $y_0,\ldots,y_n$	Th. 2.1.1
$E,A,C^r,D^r,Lip\alpha,B,L_p$	(2.1.10)

	Equation
$U_r \equiv U(t_o,\ldots,t_r)$	(2.2.1)
$V_r \equiv V(\underline{\rho}_o,\ldots,\underline{\rho}_r)$ $\rho_i \varepsilon P_i$	(2.2.9)
h_r, r=1,..., k=γ-m	(2.4.10)
$\underset{\sim}{\rho}_i$	(2.4.15)
$M_{s,t}=(m_{ij})_{i=0}^{s},_{j=0}^{t},M_s$	(2.5.1) - (2.5.3)
$\underline{e}_i$	(2.5.8)
M_m, $\underline{c}$, $\underline{c}^o$, σ, σ^o	(2.6.1)
τ_r	(2.8.10)
$\tau_r^{(s)}$	Th. 2.8.8
$S_i(\zeta_i;\ )$; $T_j(\eta_j;\ )$	(2.9.5)
P_i; q_j	(2.9.7)
$V_{m,i}$, $(V_{n,j})$	(2.9.8)
W, $W_1,\ldots,W_4$	(2.11.1) - (2.11.4)
$\{\tilde{S}_i\}_{i=0}^{m}$	Th. 2.11.1
$\hat{B}$	Th. 2.11.1
Φ_m^r	(2.11.10)
$(K)_{\Phi^\gamma}$; $(K)_{\Psi^\delta}(\{(K)_{\phi_i}\}_{i=0}^{\gamma};\{(K)_{\psi_j}\}_{j=0}^{\delta})$	(3.1.1)
$(K)_{S_i}$; $(K)_{T_j}$	(3.1.2)
Q_K, $(K)_{\underline{f}}$; $(K)_{\underline{g}}$	(3.1.4)
$\{W_o,W_1,\ldots,W_{r-1},W_r\}$	(3.1.6)
M_{Φ^γ}; N_{Ψ^δ}; $\mu_{\phi\text{-pl}}{}^m$; $\nu_{\psi\text{-pl}}{}^m$	Def. 3.1.2
$(M,N)_Q$ $((M,N)_Q(m,n))$	Def. 3.1.3
$\{{}^\mu S_i\}_{i=0}^{m};\{{}^\nu T_j\}_{j=0}^{n},\{{}^\mu\phi_i\}_{i=0}^{m};\{{}^\nu\psi_j\}_{j=0}^{n}$	Def. 3.1.3
${}^\mu B$; ${}^\nu C$	(3.1.10),(3.1.11)
$(M,N)_D$	(3.1.15)
$(M,N)_{\underline{A}}$	(3.1.16) - (3.1.19)
$(M,N)_{\underline{b}}$	(3.1.20) - (3.1.21)

	Equation
$\mu\nu_{\underline{\Lambda}}$, $\mu\nu_{\underline{a}}$, $\mu\nu_{A}$	(3.1.23) - (3.1.25)
U, V	(3.1.31) - (3.1.32)
$\hat{U}, \hat{V}$	(3.1.38)
Z	(3.1.55)
$\Omega_{\underline{k}}^{W} \equiv \Omega_{\underline{k}}^{W}(\underline{\Xi})$	(3.2.1)
$\|\cdot\|_{p,w}$	(4.1.4),(4.1.5)
P_i, q_i	(4.1.9)
E_Q	(4.1.10)
x_t^m	(4.2.4)
E_t	(4.2.5)
$\phi_f; \psi_g$	(4.2.8),(4.2.21)
W_r	(4.2.16)
$L_{r+1}(.)$	(4.2.17)
$\kappa(t,s)$	(4.2.18)
$K_\xi(t,s)$	(4.2.23)
S_E	(4.3.6) - (4.3.7)
$\{U_k\}_{k=0}^{p}$	(5.2.2)
$D, \underline{a}, \underline{\Lambda}$	(5.2.5) - (5.2.8)

GENERAL NOTATION

arbitrary indices r,s,t,i,j,l,k,μ,ν

arbitrary functions $\theta,\hat{\theta},\hat{\xi},\xi$

arbitrary linear functionals $L,L_i,L_i(\underline{\rho}_i)$ $\underline{\rho}_i \epsilon P_i$, $\underline{\rho}=(\rho_1,\ldots,\rho_r)$ $(\hat{\underline{\rho}}_{(i)})$

arbitrary points $t,t_i,s,s_i,\tilde{t}$

arbitrary coefficients $\alpha,\alpha_i,\alpha_{ij},\beta_i,\beta_{ij}$

arbitrary polynomials ξ,θ

arbitrary sets of functions $\Xi_r=\{\xi_i\}_{i=0}^r$, $\textcircled{H}_r=\{\theta_i\}_{i=0}^r$, $\hat{\Xi}_r=\{\hat{\xi}_i\}_{i=0}^r$, $\hat{\textcircled{H}}_r=\{\hat{\theta}_i\}_{i=0}^r$

arbitrary regions $K,K[\kappa,\lambda]$, $K[\kappa,\lambda)$, $K(\kappa,\lambda)$, $K[\lambda,\mu]$, $K[\lambda,\mu)$ $K(\lambda,\mu)$

arbitrary function spaces F,G

arbitrary matrices F, G

APPENDIX II INDEX OF ASSUMPTIONS

(A1)	$f\in\Gamma, g\in\Delta$ real valued functions acting on $R\subset\mathbb{R}^d$.	§1.3
(A2)	(1.3.1) exists and is finite in Riemann-Stieltjes sense.	§1.3
(A3)	$w(x)\geq 0$ on R.	§1.3
(A4)	d=1; R a finite real interval, with interior (-1,1).	§1.3
(A5)	S_i, T_j bounded and (therefore) continuous	§1.3
(A6)	$\{\phi_i\}_{i=0}^{\gamma}$, $\{\psi_j\}_{j=0}^{\delta}$ linearly independent over R.	§1.3
(A7)	$\{S_i\}_{i=0}^{m}$, $\{T_j\}_{j=0}^{n}$ linearly independent in $(\Phi^m)^*$ and $(\Psi^n)^*$.	§1.3
(A8)	$\Gamma\subset C^o$, $\Delta\subset C^o$.	§2.1
(A9)	All linear functions are real valued.	§2.2
(A10)	$\underline{\rho}_i$ depends on only one independent parameter.	§2.2
(A11)	$m\geq n$.	§2.3
(A12)	$\gamma>m$, $\delta\leq n$.	§2.4
(A13)	M_r has no row or column which is entirely composed of zeros.	§2.5

APPENDIX III POSSIBLE TOPICS FOR FURTHER INVESTIGATION

§2.2 T-extensions, T-restrictions ("almost" T-sets).

§2.2 GT-sets.

§2.2 How to obtain CT-sets from T-sets.

§2.2 Conjecture (after Th. 2.2.14): "Closed real interval, (T sets $\not\Rightarrow$ CT-set) $\Rightarrow$ (no proper extensions)."

§2.2 Further criteria for Ξ_r being a T-set.

§2.2 Conditions such that matrix M has full rank (how to choose the knots).

§2.3 Th. 2.3.1: Finding the required parameters (related to finding sets of points in $\mathbb{R}^d$, $d \geq 2$, such that the determinant using monomials is non-zero (and extensions)).

§2.4 Extension of results to semi-infinite and infinite intervals of $\mathbb{R}$ (appears to be straightforward).

§2.5 Conditions under which M_r has full rank.

§2.6 Prescribing more than one knot.

§2.6 Use of GT sets instead of T-sets.

§2.8 Application of these results to integration of periodic functions.

§2.10 Further investigation of conditions under which A can be a diagonal matrix.

§2.11 More examples dealing with GT-sets.

§3.1 Extension of (3.1.51)-particularly bearing in mind to comments made in §5.6 of [20].

§3.2 (I.P.) Q.F. for spaces of splines ($S_i, T_j \varepsilon \chi_1$, Super-functions, B-splines, WT-sets).

§3.3 Relationship between I.P.Q.F. and "best" Q.F. (if any) in the sense of Sard (in minimising norms).

§4.2 Connection between minimal interpolation results and minimal norm QF.

§4.2 Investigation of splines and monosplines (not necessairly polynomial) in 1-D - either directly or as a combination of 1-D results. Application of these results to minimal norm QF.

§4.2 Connection between minimal norm one and higher dimensional results (if any) (i.e. viewing I.P.Q.F. as a two dimensional functionals).

§4.3 Pursue investigation of various measures of function space(s) to practice both alternative estimates analogous to S_E and probabilistic estimates of errors (with chosen distributions to show desired emphasis on functions) using various QF.

§5.1 Investigation of various types of symmetry we may impose on both Q and I in order to reduce the number of moment equations we need to consider.

§5.1 (cf last paragraph). Other ways of expressing f as a function of ξ and θ. (linear or nonlinear).

PUBLICATIONS

1. "Further Properties of Inner Product Quadrature Formulas" B.I.T. 17, No. 4, (1977), pp. 392-408.
2. "Stencil Quadrature Formulas", jointly with Dr. G.M. Phillips and Prof. P. Keast, submitted J.I.M.A., 1979.

Proposed Papers (ex Thesis)

1. Inner Product Quadrature Formulas for Generalized Polynomials.
2. A Note Concerning Generalized Polynomials.
3. An Extension of the Idea of Orthogonality.
4. Inner Product Quadrature Formulas Exact on Maximal Product Spaces of Functions.
5. A Note on Non Singular Vandermonde like Matrices.
6. Compand Inner Product Quadrature Formulas.

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