A STUDY OF THE WORK AND METHODS OF HENRY BRIGGS, WITH SPECIAL REFERENCE TO THE EARLY HISTORY OF INTERPOLATION

Andrew Waterson

**A Thesis Submitted for the Degree of PhD
at the
University of St Andrews**

1941

"A STUDY OF THE WORK AND METHODS OF HENRY BRIGGS, WITH SPECIAL REFERENCE TO THE EARLY HISTORY OF INTERPOLATION."

being a Thesis presented

by

ANDREW WATERSON, M.A.

to the University of St. Andrews in application for the degree of Ph.D.

Career.

I matriculated in the University of St. Andrews in October 1924, and followed a course leading to graduation in Arts (Honours Mathematical and Physical Science) until June 1928.

On 1st October 1937 I commenced the research on the Work and Methods of Henry Briggs, which is now being submitted as a Ph.D. Thesis.

Declaration.

31. 1. 41

I hereby declare that the following Thesis is based on the results of research work carried out by me, that the Thesis is my own composition, and that it has not previously been presented for a Higher Degree.

Certificate.

I certify that Andrew Waterson

has spent nine terms at Research Work in the Study of the Mathematical Work of Henry Briggs and the History of Interpolation and

that he has fulfilled the conditions of Ordinance No. 16 (St. Andrews) and that he is qualified to submit the accompanying Thesis in application for the degree of Ph.D.

List of Contents.

List of Contents. contd.

List of Contents. contd.

BIBLIOGRAPHY

Author.	Title of Work.	Brief Title.
ANDERSON, Alex.	Ad Angulares Sectiones, contained in Vieta's Opera Machematica.	Anderson.
AUBREY, John.	Lives of Eminent Men of the XVIIth Century. 2 Vols.	Aubrey.
BALL, W. R. R.	A short Account of the History of Mathematics.	Ball.
BALL, W. R. R.	History of Study of Mathematics at Cambridge.	Ball. (Camb).
BRIGGS, Henry.	Arithmetica Logarithmica. (London 1624).	Arith. Log.
BRIGGS, Henry.	Trigonometria Britannica. (Goudae 1633).	Trig. Brit.
BRIGGS, Henry.	Mathematica ab Antiquis minus cognita.	
CAJORI, Florian.	History of Mathematics. (London 1894).	Cajori.
	Dictionary of National Biography.	D. N. B.
	Encyclopaedia Britannica, XIth edition.	Encyc. Brit.
HOBSON, E. W.	Squaring the Circle. (Cam. 1913).	Hobson.
HUTTON, Charles.	Mathematical Dictionary. (1815).	Math. Dict.
HUTTON, Charles.	Introduction of Mathematical Tables.	Math. Intro.
	Library of Useful Knowledge. (Biography).	L. U. K.
MAURICE. F.	Journal of Institute of Actuaries. Vol. 14. (1869).	J. I. A.

BIBLIOGRAPHY

Author.	Title of Work.	Brief Title.
NAPIER, John.	Mirifici Logarithmorum Canonis Constructio. (Translation by W. R. Macdonald. Edin. 1889).	Constructio.
	Napier Tercentenary Memorial Volume. (London 1915).	Tercen. Vol.
NAPIER, Mark.	Memoirs of John Napier.	Mark Napier.
	Philosophical Transactions. Vol. I.	Phil. Trans.
RIGAUD, S.	Correspondence of Scientific Men of the XVIIth Century. (Oxford 1841).	Rigaud.
SCOTT, J. F.	The Mathematical Work of John Wallis. (London 1938).	Scott.
SMITH, D. E.	History of Mathematics. (1923) 2 Vols.	Smith.
TURNBULL, H. W.	James Gregory: A Study in the Early History of Interpolation.	Turnbull.
VIETA, Franciscus.	Opera Mathematica, edited by F. van Schooten. (Leyden 1646).	Vieta.
WARD, J.	Lives of the Professors of Gresham College. (London 1740).	Ward.
WHITTAKER & ROBINSON.	The Calculus of Observations. (Glasgow 1926).	Whittaker.
FRASER, D.C.	NEWTON'S INTERPOLATION FORMULAS.	FRASER.

CHAPTER I.

A Short Biography.

Henry Briggs was born at Warley Wood near Halifax, Yorkshire in February 1560/61. There is some doubt as to the actual year 1556 being the one usually given, but it is pointed out by D. E. Smith+ that the parish register shows 1560/61 and for that reason that year must be taken. He attended a grammar school in that county and in 1579 he entered St. Johns College, Cambridge where he became intensely interested in Mathematics. He graduated B.A. in 1581 and M.A. in 1585, and three years later he was elected fellow of his college. He applied himself chiefly to the study of Mathematics, in which he so greatly excelled that in 1592 he was appointed examiner and lecturer in Mathematics at St. John's, and soon after reader of the physical lectures founded by Dr. Linacre.

It seems clear that Briggs began to acquire a reputation as a geometrician for on the foundation of Gresham House (afterwards College) London, he was chosen first professor of Geometry, and thus has the distinction of being the first occupant of the first Chair of Mathematics in England. He settled down to a life of close and severe studies carrying out the duties of his office with that thoroughness which became so characteristic of him, and devoting all his leisure to mathematical pursuits.

Navigational and Astronomical problems seem to have claimed his attention first, for in 1602 the first result of his efforts, "A table to find the Height of the Pole, the Magnetic Declination being given", was published in Mr. Thomas Blunderville's "Theoriques of the Seven Planets". Meanwhile he had started to communicate with others, for there are extant six letters to him from Sir Cristopher Heydon, which are chiefly on the subject of Comets/

+ Smith. vol. 1. P. 391.

Comets, and the second of which is dated 1st November 1603.

Briggs, however, had begun to think about something which no English mathematician before him had attempted, namely the compilation of a canon of sines. In view of the prominent part played by English mathematicians at a later date in the calculation of the canons of the logarithms of the trigonometrical ratios, it is rather remarkable that they had no part in the calculation of the canons of natural ratios and had made few contributions of note in the field of trigonometry since the time of Thomas Bradwardine (c. 1290 - 1349), who along with Richard of Wallingford, John Maudith and Simon Bredon of Winchecombe gave to England the honour of having produced the earliest European writers on Trigonometry.+ Some time round about 1603 Briggs was engaged upon the study of angular sections, which form the basis of such a canon, and this work led to most of the important discoveries upon which rest Briggs's claims to fame, and which will be treated of later. This work seems to have been entirely original, and this, among other studies, must have kept him busy for the next few years, for he published nothing more until 1610 when there appeared "Tables for the improvement of navigation" printed in the second edition of Edward Wright's treatise entitled "Certain errors in Navigation detected and corrected".

In 1609 he formed a friendship with the learned James Ussher, afterwards Archbishop of Armagh,++ which continued over a period of years as is evident from two letters from Briggs, published in the collection of Ussher's letters. In the first of these, dated August 1610, he writes that he was engaged upon the subject of eclipses, which was also the substance of the fourth letter from Sir Cristopher Heydon to Briggs dated 14th December 1609.

Although we know that Briggs's own private researches had already led him to discoveries of great mathematical importance, these discoveries remained unpublished, and it must be recorded that/

+ Cajori. ++ Ward.

that up to 1614 he had published nothing of note in the field of mathematics. I have taken 1614 as a milestone in his life for in that year, like a bolt from the blue, came the publication of Napier's "Descriptio" or to give it its fuller title "Mirifici Logarithmorum Canonis descriptio", giving logarithms to the world for the first time. This publication at once attracted the attention of two of England's most eminent mathematicians of that period, Edward Wright, well known for his works on navigation and Henry Briggs, then professor of Geometry at Gresham College. The former undertook its translation into English, but died in 1615 before its publication could be accomplished, the duty of so doing falling upon Samuel Wright, the translator's son, and Henry Briggs. Briggs himself was greatly excited by Napier's invention, and seems to have abandoned everything else in order to devote himself to it. In a letter dated 10th March 1615 to James Ussher he writes "Naper Lord of Markinston hath set my head and hands at work with his new and admirable logarithms. I hope to see him this summer if it please God; for I never saw a book which pleased me better and made me more wonder. I purpose to discourse with him concerning eclipses, for what is there which we may not hope for at his hands," and further speaks of himself as being "wholly taken up and employed about the noble invention of logarithms but lately discovered."

He lectured on them immediately, thought of improvements, communicated with Napier, began to calculate them, and duly visited Napier in 1615, staying with him a whole month. The extent of his regard and respect for Napier may be deduced from the account of their meeting, recorded by William Lilly, who writes "When the two great mathematicians met, almost one quarter of an hour was spent, each beholding the other, almost with admiration before one spoke a word. At last Mr. Briggs began 'My Lord I have undertaken this journey purposely to see your person and to know by what engine of wit or ingenuity you came first to think of this most excellent help unto astronomy viz. the logarithms: but, my Lord, being by you/

you found out I wonder no body else found it out before when now known it is so easy'."

Briggs pointed out to Napier the advantages that would accrue from having 10 as a base, an improvement he had already explained in his public lectures at Gresham College. Napier, however, seems to have thought of this also, and suggested a further improvement to Briggs, who on his return to London set about calculating this new kind of logarithms. He visited Napier again in 1616 and writes that he would gladly have paid him a third visit in 1617 had Napier's life been spared. Briggs set to work with a will and must have been an exceedingly busy man, for in 1616 he published "A Description of an Instrumental Table to find the part proportional, devised by Mr. Edward Wright", and then in 1617 "Logarithmorum chilias prima" or the logarithms of the first thousand natural numbers. This work is the first published on decimal or common logarithms, and although it bears no author's name, place, or date, the author was undoubtedly Briggs. The date of publication is fixed by a letter from Sir Henry Bourchier to Ussher dated 6th December 1617 in which he writes "Our kind friend Mr. Briggs hath lately published a supplement to the most excellent tables of logarithms, which I presume he has sent to you." Again, in 1619 he assisted Robert Napier to publish his father's "Mirifici Logarithmorum Canonis Constructio", which gave the method by which Napier calculated his logarithms, and to it Briggs himself contributed Lucubrationes et Annotationes in Opera posthuma J. Naperi.

In 1619 Briggs was appointed Savilian Professor of Geometry at Oxford on the foundation of the Chair in that year, and thus had the singular distinction of holding in succession the two earliest chairs of Mathematics that were founded in England, surely a fitting tribute to his abilities and an indication of the esteem in which he was held by his contemporaries. Sir Henry Savile, the founder of the chair must have had very definite ideas on the subject of geometry for it is recorded by John Aubrey that Bishop Seth Wood of/

of Sarum, told him that "Savile first sent for Mr. Edmund Gunter, from London, (being of Oxford University) to have been his Professor of Geometrie, so he came and brought with him his sector and quadrant and fell to resolving of triangles and doeing a great many fine things.

"Said the grave Knight 'Doe you call this reading of Geometrie? This is shewing of tricks, man!' and so dismisst him with scorne and sent for Henry Briggs from Cambridge." +

Briggs entered on his new duties on January 8th 1620, beginning his lectures with the ninth proposition of Euclid, but he continued to hold his professorship at Gresham College till 25th July 1620, when he resigned. In this same year he published in folio "Euclidis Elementorum VI Libri priores".

At Oxford he settled down at Merton College, where he continued his most laborious and studious life, employed partly in the duties of his office and partly in the computation of his logarithms and other useful works. In 1622 he published "A Treatise of the North West Passage to the South Seas through the Continent of Virginia and by Hudson Bay", the reason leading him to this being probably that he was then a member of the company trading to Virginia, the first English Colony in America. ++ Round about this time Briggs seems to have had quite an interest in commercial undertakings, for he became the instigator and the advocate of a canal between the Thames and the Avon, a scheme which afterwards attracted among other men John Collins. John Aubrey writes +++ at some length on it and says "Looking one time on the mappe of England he (Briggs) observed that the two rivers, the Thames and that Avon which runnes to Bathe and so to Bristowe, were not far distant, scilicet, about 3 miles - vide the mappe. He sees 'twas but about 25 miles from Oxford: getts a horse and viewes it and found it to be a levell ground and easie to be digged. Then he considered the chardge of cutting between those rivers which would be of great consequence for/

+ Aubrey. vol II. ++ Ward. p.125. +++ Aubrey. vol I. p 124 for cheape and safe carrying of goods between London and Bristowe, and though the boates goe slowly and with meanders, yet considering they goe day and night they would be at their journey's end almost as soon as the waggons, which often are overthrowne and liquors spilt and other goods broken. Not long after this he dyed and the civill warres brake-out. It happened by good luck that one Mr. Matthewes of Dorset had some acquaintance with this Mr. Briggs, and had heard him discourse of it. He was an honest simple man, and had runne out of his estate, and this project did much run in his head. He would revive it (or els it had been lost and forgott) and went into the country to make an ill survey of it (which he printed) about anno....., but with no great encouragement of the countrey or others. Upon the restauration of King Charles II he renewed his designe and applyed himself to the King and Counsell. His majestic espoused it more (he told me) than any one els. In short, for want of management and his non-ability it came to nothing, and he is now dead of old age."

His next publication was the result of his assiduous labour at the computation of logarithms, and in 1624 appeared his great and elaborate work, the Arithmetica Logarithmica. The more one studies this work and thinks of the labour and the genius it involves the more one realises how stupendous a work it is for so short a time. This is one of the two works, for which, it may confidently be stated, Briggs merits a place in Mathematical history.

The Arithmetica Logarithmica contains the logarithms of numbers from 1 - 20,000 and from 90,000 to 100,000 (in some 101,000), and it had been Briggs's intention to fill in the gap. In his book he calls for volunteers to help him in his calculation, the plans for which he had already made. That he was baulked in the publication of the intervening chiliads without being saved the labour of their calculation is well known, and such a happening would have been a severe blow to a much younger man than Briggs, who/

who by this time was getting well on in years. He writes to Pell [+] in a letter dated October 25th 1628 -

"My desire was to have those chiliads that are wantinge betwixt 20 and 90 calculated and printed, and I had done them all almost by my selfe, and by some frendes whom my rules had sufficiently informed, and by agreement the business was conveniently parted amongst us: but I am eased of that charge and care by one Adrian Vlacque, an Hollander, who hath done all the whole hundred chiliads and printed them in Latin, Dutche and Frenche, 1000 bookes in these three languages and hath solde them almost all. But he hath cutt off 4 of my figures throughout and hath left out my dedication, and to the reader, and two chapters the 12th and 13th, in the rest he hath not varied from me at all."

That Briggs was made of stern stuff, however, is evident from his subsequent conduct, for in spite of his advancing years and his disappointment, he set about a task nearly as laborious as his former one, namely the calculation of canons of the logarithms of the trigonometrical ratios. It may be recalled that about 1603 he had already worked out the theoretical basis of a canon of natural sines, and had subsequently compiled the tables without ever publishing them. He now meditated their publication together with an account of their construction and use, and intended to add as well the logarithmic canons of the trigonometrical ratios. Briggs applied himself to the job in hand, but did not live to see the completion of his project, for when he had completed the account of the construction of his canons, he died on the 26th January 1630. The duty of giving an account of their use fell upon Briggs's friend Henry Gellibrand, who not only discharged it faithfully, but in addition saw to it that the original project was carried through as intended. He edited the work and allowed Adrian Vlacq to publish it, which he did at his own expense at Gouda in 1633 under the title of Trigonometria Britannica.

"Thus" Ward writes [++] "lived and died this celebrated mathematician/

+ Encyc. Brit. vol 16 p. 870. ++ Ward.

mathematician, inferior to none whom he left behind him."

He finished his life on January 26th 1630 in Merton College and was buried in the choir of the chapel there under the honorary moment of Sir Henry Savil, a plain stone being laid over him, with his name only inscribed upon it, which has been since removed, upon the new paving of the choir.

The following account of him stands yet in the College register:-

> Jan. 26, 1630. Obiit apud nos commensalis, magister Henricus Brigges, vir quidem moribus ac vita intigerrimus: quem in rebus geometricis, quarum studiis primum Cantabrigiae in societate collegii S. Johannis sese a juventute addiperat, dein publicus praelector Londini in collegio Greshamensi multos per annos sustinuerat, omnium sui temporis eruditissimum D. Henricus Savilius ut primo ex fundatione sua geometriae professoris munere fungeretur, Oxonium evocavit; cuius exequias 29 die proxime sequente, concione habita a magistro Sellar, et oratione funebri a magistro Cressy, una cum primoribus academiae celebravimus."

Oughtred called him the English Archimedes,[+] and "the mirrour of the age for excellent skill in Geometry." [++] A Greek epitaph was written on him by one Henry Jacob, [+++] one of the fellows of Merton, which ends by saying that "his soul still astronomises and his body geometrises."

It is obvious from his achievements that he was a man of tremendous industry, having that capacity for taking pains which we are told is the essence of genius. He is characterised as a man of great probity, a contemner of riches, a man contented with his own station, preferring a studious retirement to all the splendid circumstances of life. But though his life was spent in close and severe studies, more for the benefit of mankind than his own private interest, yet they do not seem to have affected, or had any bad influence upon his temper, for his letters not only discover an ease and sedateness of mind, but likewise an agreeable cheerfulness/

[+] Aubrey. vol 1. p 124. [++] Ward. [+++] D.N.B.

cheerfulness and pleasantry. In an age when scholars like Kepler and Napier, though in the forefront of scientific progress and inquiry, still clung to the scholastic ideas of the preceding ages as witnessed by their astrological beliefs, it is interesting to note that Briggs was entirely free from them. Mr. Thomas Gataker, one of his students with whom he had kept up a friendship, says that desiring Briggs once to give him his judgment concerning judicial astrology, his answer was that - "he conceived it to be a mere system of groundless conceits".+

I can think of no more fitting conclusion to this short history of Briggs than to quote the tribute to him by one of his successors at Gresham College, Dr. Isaac Barrow, who in his oration on his admission there says:- ++

> "Most distinguished Briggs, I bear witness to your name, which heads the list in our records, a name which is beyond all praise, and is too widely known for any eulogy in erudition, skill and experience: you who have devised, not indeed by any good fortune of yours, that most famous system of logarithms, but, and this is equally deserving of praise, you have completed it by your diligence, and freed us from (the servitude of) all numbers. This work would perhaps have remained useless and incomplete yet, buried in the rubbish of its own foundation, had you not applied to it the polish of a most acute intellect and the carefulness of an indefatigable hand. You, who gather together these hosts of numbers until you have drawn them up regularly as if in battle array, and have prepared endless models for us, have procured our leisure hours by so spending your own: by your toil you have relieved our feelings of irksomeness, and to spare our sleep you have worn yourself out with wretched wakefulness: for these reasons you would be deserving of obtaining from us incalculable thanks and gratitude that cannot be reckoned, even by your logarithms."

CHAPTER II.

The Trigonometria Britannica.

Briggs's memory is so largely logarithmic that the casual intruder into the field of mathematical history might well be excused/

+ Ward. ++ Ward.

excused for thinking that he had no other claims to be remembered. This estimate of him is strengthened rather than otherwise by a perfunctory perusal of his works, most of which seem to deal with logarithms, and since historians seem to have been quite content with merely scratching the surface as it were, this is the estimate of him which has come down to us. How far wide of the mark it is I hope to show, for in my opinion Briggs had already staked his claim to fame long before Napier's "Descriptio" was given to the world. The only historian who seems to have championed Briggs was Charles Hutton, who recognised in him the mathematical genius he was. Whether, recognising his true worth, he was stirred by the almost criminal neglect of historians before him and wished to make reparation, I do not know, but the fact remains that Hutton seems over zealous on occasion, and leaves himself open to the suspicion that his judgment is not entirely without bias in matters concerning Briggs. This is especially true with regard to the invention of decimal logarithms for he clearly misinterprets all Briggs's statements in this connection, reading them in a manner contrary to their true meaning, and it is most unfortunate, for Briggs was great enough to require only an unbiassed and impartial appreciation of his original work to have his merit made manifest. His writings were more important than numerous, and by far his two most important were his "Arithmetica Logarithmica" published in 1624 and his "Trigonometria Britannica", for in those two works he gives the results and discoveries of his private researches, and, as Hutton writes + - "in their construction Briggs, besides extreme labour and application, manifests the highest power of genius and invention, as we here for the first time meet with several of the most important discoveries in mathematics, and what have been considered as of much later invention." These inventions include the Differential Method and Construction of Tables by Differences, Interpolation by Differences, Angular Sections, the Binomial Theorem, and our Decimal Notation.

I/

+ Math. Dict.

I propose to examine carefully each of these works and to investigate how far his writings bear out the claims which have been made for him. I begin with the Trigonometria Britannica, for although it was published in 1633, three years after Briggs's death, it contains the results of his researches made in the first decade of the seventeenth century or even earlier. This is borne out by Henry Gellibrand's statement in his "address to the reader" at the beginning of the Trigonometria Britannica, and is further confirmed by Briggs himself in a similar address in the Arithmetica Logarithmica. Thus although of later publication it embodies his earliest researches.

Theory of Angular Sections.

One does not require to wait long for evidence of Briggs's originality of thought, for in the first chapter, he points out that although all previous workers in this field had divided the degree sexagesimally into minutes and seconds, he, influenced by the Calendarii Gregoriani of Vieta, and at the exhortation of others, proposes to break with tradition and to follow his own inclination by dividing the degree decimally into 100 primary parts or centesms and each of these into ten parts or millesms. These parts render calculation much easier and no less sure. It is probable that computation by this decimal division would have come into general use, had it not been for the publication of Vlacq's tables which retained the sexagesimal division previously in vogue. In many ways Briggs displayed remarkable foresight in this, for it must be remembered that at this time even decimal fractions were neither commonly used nor understood, yet he saw the many advantages of decimal division and so anticipated a trend which exists even at present, when the advantages of a decimal division in all kinds of measures are freely admitted, its adoption seen to be desirable, without its being adopted.

Briggs then proceeds to define the sine, chord, tangent, and/

and secant as lines in and about a circle, following the definitions with a brief description of the methods by which the Ancients found chords, naming theorems, which in modern notation are the Theorem of Pythagoras, viz. $\sin^2\theta + \sin^2(90^\circ-\theta) = 1$ and Ptolemy's Theorem on the cyclic quadrilateral, showing its use in finding the chord of the sum and of the difference of two arcs. He next goes on to what had been discovered in the years just prior to his time of writing, and by which the whole canon of chords, and therefore of sines, might be calculated very easily and very accurately. The subsequent five chapters are devoted to a geometrical treatment, from first principles, of Angular Sections, the relationship between the chords of multiple and submultiple arcs being fully investigated. He treats in detail the finding of the chord of a triple arc in terms of that of the simple arc (Triplation), and uses the equation thus found to derive the chord of one third of the arc from that of the given arc (Trisection); similarly with quintuplation and quinquesection, also with septisection. In each case of the multiple arc he finds the equation connecting the chord of the multiple arc with that of the simple arc, and then for the converse process the same equation connects the given chord with that of the submultiple arc, so that solving this equation gives the chord of the submultiple arc. The method given is perfectly general, quintuplation being developed from Triplation and septulation from quintuplation. For septisection he says the method of working scarcely differs from that recorded for the chord of one-third and one-fifth parts, but the working for the multiplication of the terms is much more laborious.

Similar equations may be found and demonstrated if any arc be divided into any odd number of equal parts, but not with the same ease as those above mentioned, for the greater the number of equal segments in the arc the greater the number of terms in the equation, and the greater the number added to any particular term in the solution of the equation in the converse process, so that in all these/

ABACUS PANCHRESTUS

M	L	K	I	H	G	F	E	D	C	B	A
− 12	− 11	+ 10	+ 9	− 8	− 7	+ 6	+ 5	− 4	− 3	+ 2	+ 1
1	1	1	1	1	1	1	1	1	1	1	1
13	12	11	10	9	8	7	6	5	4	3	2
91	78	66	55	45	36	28	21	15	10	6	3
455	364	286	220	165	120	84	56	35	20	10	4
1820	1365	1001	715	495	330	210	126	70	35	15	5
6188	4368	3003	2002	1287	792	462	252	126	56	21	6
18564	12376	8008	5005	3003	1716	924	462	210	84	28	7
50388	31824	19448	11440	6435	3432	1716	792	330	120	36	8
125970	75582	43758	24310	12870	6435	3003	1287	495	165	45	9
293930	167960	92378	48620	24310	11440	5005	2002	715	220	55	10
646646	352716	184756	92378	43758	19448	8008	3003	1001	286	66	11
									364	78	12
									455	91	13
									560	105	14
									680	120	15
17383860									816	136	16
30421755	13037895	5311735							969	153	17
51895935	21474180	8436285	3124550						1140	171	18
			4686825	1562275	480700				1330	190	19
				2220075	657800	177100	42504			210	20
						230230	53130	10626	1771	231	21
								12650	2024	253	22
										276	23
										300	24
										325	25
										351	26
										378	27
										406	28
										435	29
											30

											1
											2
91											3
455	364	286									4
		1001	715	495							5
				1287	792	462					6
						924	462	210			7.
								330	120	36	8
										45	9
17383860											
30421755	13037895										
	21474180	8436285	3124550								18
			4686825	1562275	480700						19
					657800	177100	42504				20
	13037895										
	21474180	8436285	3124550				53130	10626	1771		21
									2024	253	22
			4686825	1562275	480700						23.

these cases for one reason or another, the working may be more difficult. However, in Chapter VIII, a very interesting and a very important chapter, Briggs gives a table from which the equation for any section can be picked out, and so we have the first instance of a characteristic of all Briggs's work, viz. a penchant to generalise wherever possible.

The Abacus Panchrestus.

This table is as follows and is best described in the words of Briggs himself, who on account of its numerous and uncommon uses was wont to call it the Abacus Panchrestus. (The abacus good for all work). I give it on the opposite page, without comment at present.

"The table is separated by perpendicular lines into columns, marked off at the top by the letters A, B, C, D, E, etc. and by the characters of the figurate numbers or of powers. Among these numbers those are called Diagonal which are next to each other and not in the same column. All these numbers are made by the addition of Diagonal numbers the sum of which is always placed in the next lower place of the same column as the diagonal number farther away from the right hand margin..........."

Another use no less important and valuable is the finding of the equations by the help of which may be obtained the chords for all sorts of division of the circumference and especially for any section in which the equal parts are given odd in number, in which case the chords themselves may be found by a single operation. Also from the same table may be had the equations of the equal parts of the arc given even in number. True these equations do not give by a single operation the chords themselves, but the squares of the chords from which then the chords themselves are got. The powers to be placed in the separate equations are to be had from the top of each column together with the signs of Addition or Subtraction.

The/

M	L	K	I	H	G	F	E	D	C	B	A
− 12	− 11	+ 10	+ 9	− 8	− 7	+ 6	+ 5	− 4	− 3	+ 2	+ 1
1	1	1	1	1	1	1	1	1	1	1	1
14	13	12	11	10	9	8	7	6	5	4	3
104	90	77	65	54	44	35	27	20	14	9	5
546	442	352	275	210	156	112	77	50	30	16	7
2275	1729	1287	935	660	450	294	182	105	55	25	9
8008	5733	4004	2717	1782	1122	672	378	196	91	36	11
24752	16744	11011	7007	4290	2508	1386	714	336	140	49	13
68952	44200	27456	16445	9438	5148	2640	1254	540	204	64	15
176358	107406	63206	35750	19305	9867	4719	2079	825	285	81	17
419900	243542	136136	72930	37180	17875	8008	3289	1210	385	100	19
940576	520676	277134	140998	68068	30888	13013	5005	1716	506	121	21
					51272	20384	7371	2366	650	144	23
						30940	10556	3185	819	169	25
							14756	4200	1015	196	27
								5440	1240	225	29
									1496	256	31
										289	33
	34512075									324	35
			7811375							361	37
					1138500					400	39
							95634			441	41
									3795	484	43
										529	45

The coefficients of the powers may be had from the addition of two successive numbers in the same column.

The first number of each equation for the chords is the number of the section itself in column A, the second in C, the third in E, and the others in the same way alternate columns being omitted. These numbers all ascend obliquely to the left. Thus when the number of parts is odd: for trisection 3① - 1③ for septisection 7① - 14③ + 7⑤ - 1⑦ , which have been previously shown. For the remaining odd sections it will be the same, both the method of demonstration used before and the finding from the table.

But in the equations for the squares of the chords, the first number is the square of the number of the section, to be sought in column B, the second in column D, the third in F, etc. Thus for Bisection 4② - 1④ , for quadrisection 16② - 20④ + 8⑥ - 1⑧ . Nor is this true only for even sections but also for odd, e.g. trisection, 9② - 6④ + 1⑥ .

Special Abacus for Angular Sections.

Briggs then subjoins another table in which the addition of successive numbers in the above is already performed. It is constructed in the same way as the first, by the addition of Diagonal numbers and is given on the opposite page. This table is expressly for the purpose of Angular Sections, and from it all the equations both for the chords and the squares can be got with less trouble than before. Briggs carefully points out that it is much more difficult in this table to write down a bit of it without the unbroken series of numbers as he could easily do in the first, and that the first has many uses which the second has not, for which reason he values the first much more than the second. The positions of the numbers for any equation is the same in either table, but what was previously expressed by two numbers is here recorded by a single number. Briggs then gives a geometrical proof/

proof of the equation of the square of the chord in the case of bisection and the even sections got by repeated bisection.

These tables of Briggs are very remarkable indeed. His lack of a suitable algebraic notation such as Newton later possessed, has in all probability contributed to the comparative obscurity of his results, which had perforce to be expressed arithmetically in the form of a table, the only means of generalising at his disposal.

This second table expressed in a modern notation, and using sines instead of chords, gives the following well known expansions:-

(1) For odd Sections,

$$2 \sin (2n+1).\theta = \sum_{r=1}^{r=n+1} (-)^{r+1} \frac{2n+1}{n+r} \cdot \binom{n+r}{2r-1} \cdot (2 \sin \theta)^{2r-1}$$

(2) For any section, odd or even,

$$(2 \sin n\theta)^2 = \sum_{r=1}^{r=n} (-)^{r+1} \frac{2n}{n+r} \cdot \binom{n+r}{2r} \cdot (2 \sin \theta)^{2r}$$

where $\binom{x}{r} = C^x_r = \frac{\lfloor x}{\lfloor r \,.\, \lfloor x-r}$, a general binomial coefficient.

In Chapter IX he points out that from the appropriate equation the chord of the multiple arc can be found by the mere substitution of the given chord, but if the chord of the multiple arc be given that of the fractional arc is found by solving the equation for the chord itself, if the section be odd, and for the square of the chord if the section be even. Then in order to avoid the taking of square roots he gives three theorems whereby the squares of the chords may give other chords without the necessity of finding the square roots.

These theorems may be stated as follows:-

$$2R + \text{chord}\ \theta = \text{chord}^2\ (90^\circ + \tfrac{\theta}{2})$$
$$2R - \text{chord}\ \theta = \text{chord}^2\ (90^\circ - \tfrac{\theta}{2})$$
$$2R - \text{chord}^2\theta = \text{chord}\ (180^\circ \sim 2\theta).$$

This really ends Briggs's treatise on Angular Sections, but in Chapter/

Chapter XI he unexpectedly gives an alternative method of expressing the chord of any multiple arc in terms of that of the simple arc

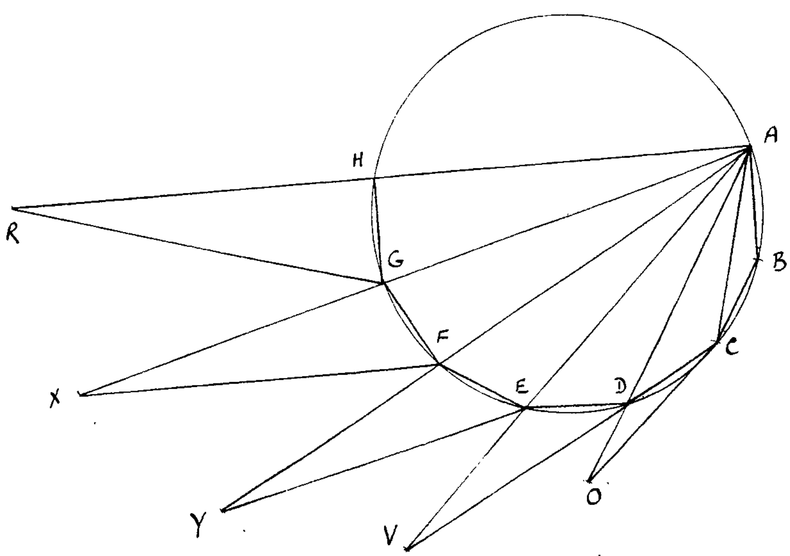
Alternative method of treating Angular Sections.

This method differs entirely from his previous one, and it will be necessary to quote it in detail partly because of its many points of interest and partly because of the use I wish to make of it.

The fundamental proposition, which Briggs proves is as follows:-

"If AC bisects angle BAD at the circumference and if CE is made equal to CA, E being on BD produced, then DE equals BA".
He then proceeds -

"Let there be drawn within the circle the equal chords AB, BC, CD, DE, EF, etc. and let there be made CO = AC, DV = AD EY = AE, FX = AF and AG = GR. Then by the preceding proposition DO = AB, EV = AC, FY = AD, etc.



Further let there be a series of continued proportionals of which AB is the first (denoted by 1①) AC the second (denoted by 1②), and AO the third (1③) since $\frac{AB}{AC} = \frac{AC}{AO}$.

Then $\underline{AD} = AO - OD = AO - AB = 1③ - 1①$

$\therefore AV = 1④ - 1②$ [since $\frac{AV}{AD} = \frac{AC}{AB}$ = common ratio of proportionals]

$\therefore \underline{AE} = AV - VE = AV - AC = 1④ - 1② - 1② = 1④ - 2②.$

and so on.

Briggs/

FIRST							
SECOND	—	+	—	+	—	+	—
THIRD	1 1						
FOURTH	2 2						
FIFTH	3 3	1 1					
SIXTH	4 4	3 2					
SEVENTH	5 5	6 3	1 1				
EIGHTH	6 6	10 4	4 2				
NINTH	7 7	15 5	10 3	1 1			
TENTH	8 8	21 6	20 4	5 2			
ELEVENTH	9 9	28 7	35 5	15 3	1 1		
TWELFTH	10 10	36 8	56 6	35 4	6 2		
THIRTEENTH	11 11	45 9	84 7	70 5	21 3	1 1	
FOURTEENTH	12 12	55 10	120 8	126 6	56 4	7 2	
FIFTEENTH	13 13	66 11	169 9	210 7	126 5	28 3	1 1
SIXTEENTH	14 14	78 12	220 10	330 8	252 6	84 4	8 2
A	B	C	D	E	F	G	H

Briggs points out that in all these isosceles triangles the ratio of a side to its base is the same, and thus when the side is known the base is found.

He tabulates his results in the table given on the opposite page which can be easily extended if required.

Column A gives the lines in continued proportion: columns B, D, F. H. etc. are subtracted from and C, E, G added to the proportionals E.G. the chord of ten times the arc is the tenth proportional plus 21 times the sixth and 5 times the second from which must be subtracted 8 times the eighth and 20 times the fourth.

In our modern notation the proportionals have first term $(2 \sin\theta)$, and second term $2 \sin 2\theta$ i.e. a common ratio $(2 \cos\theta)$, so that the proportionals are

$(2 \sin\theta)$, $(2 \sin\theta . 2 \cos\theta)$, $2 \sin\theta\,(2 \cos\theta)^2$, $2 \sin\theta\,(2 \cos\theta)^3$, ... and the table for expressing the above is equivalent to the present analytic formula for expressing $\sin n\theta$ in terms of $2 \cos\theta$.

namely, $$\frac{\sin n\theta}{\sin\theta} = \sum_{r=1}^{r=\frac{n+1}{2}} (-)^{r-1} \binom{n-r}{r-1} . (2\cos\theta)^{n+1-2r}$$

or $$\sin n\theta = \sin\theta\left[(2\cos\theta)^{n-1} - \binom{n-2}{1} . (2\cos\theta)^{n-3} + \binom{n-3}{2} . (2\cos\theta)^{n-5} - \binom{n-4}{3}(2\cos\theta)^{n-7} + \cdots\cdots\right]$$

CHAPTER III.

The Angular Sections of Franciscus Vieta.

The question which now arises is: Is this work of Briggs original? If not entirely, how much did he get from other sources? There were no English sources from which he could borrow, and of the Continental writers the only one to whom he could have been indebted was Franciscus Vieta (1540 - 1603) one of the most capable mathematicians of the 16th century or indeed of any age, and the first/

first man in the West to give a systematic elaboration of the methods of computing plane and spherical triangles by the aid of the six trigonometric functions. He is probably best known as the founder of analytical methods, introducing into Algebra a notation and symbolism which transformed it into a powerful tool in his hands, and so laid the foundation of modern Analysis.

The Canon Mathematicus, (1579)

Among his writings are many pieces on trigonometry which appear in the collection of his works published at Leyden in 1646 by Frans van Schooten, but which first appeared in 1579 in a volume of trigonometrical tables with their construction and use. The first part is entitled - "Canon Mathematicus seu ad Triangula cum Appendicibus" and contains a great variety of tables beautifully printed, but inaccurate as he himself testifies in a later work. The second part, "Universalium Inspectionum ad Canonem Mathematicum Liber Singularis", is theoretical and contains the construction of the tables among other things. It is in this part that we are interested for it is quite possible for Briggs to have been acquainted with it. Vieta was a wealthy man, who frequently printed his works at his own expense, sending copies to other mathematicians, with the result that his writings were rapidly circulated abroad. + In spite of this I have not been able to trace any mention of this work of 1579 being known in England before the time of Briggs's own researches in the same field, and it would not be the least surprising if no copy did find its way thither, for England at this time had no mathematician of note to whom a copy was likely to be sent, and the diffusion of new knowledge by ordinary channels was a remarkably slow process. I myself do not think there can be the slightest doubt that the whole theory and the proofs given by Briggs are entirely original and his own work, and am of the opinion that he is deserving of the greatest credit for it. Gellibrand ++ definitely states that the canon of sines was derived from first principles/

+ Encyc. Brit. vol 28 p. 58. ++ Trig. Brit.

VIETA'S TABLE.

1st									
—									
2	2nd								
3	+								
4	2	3rd							
5	5	—							
6	9	2	4th.						
7	14	7	+						
8	20	16	2	5th.					
9	27	30	9	—					
10	35	50	25	2	6th				
11	44	77	55	11	+				
12	54	112	105	36	2	7th			
13	65	156	182	91	13	—			
14	77	210	294	196	49	2	8th		
15	90	275	450	378	140	15	+		
16	104	352	660	672	336	64	2	9th.	
17	119	442	935	1122	714	204	17	—	
18	135	546	1287	1782	1386	540	81	2	10th.
19	152	665	1729	2717	2508	1254	285	19	+
20	170	800	2275	4604	4290	2640	825	100	2
21	189	952.	2940.	5733	7007	5148	2079	385	21.

principles, and I hope to show that Vieta's work does differ from that of Briggs. The latter, it may be remembered, defines his chords, sines, tangents and secants as lines which are drawn in and about a circle whereas Vieta, after the manner of Rhaeticus (1514 - 1567), treats them as lines representing the sides of right angled triangles, the different functions being found by taking each side in turn to be constantly 100,000, and calculating the other two sides, so that strictly speaking Vieta's Tables are tables of sides of right angled triangles. He establishes geometrically the equivalent of sin $(A \pm B)$ and cos $(A \pm B)$, and then finds the sides for the multiple angle 2A by considering it as A + A. He then states the rule giving the sides of the right angled triangle containing any multiple angle in terms of the sides of that containing the simple angle. This is equivalent to expressing $\sin n\theta$ and $\cos n\theta$ in terms of both $\sin\theta$ and $\cos\theta$, and there is no mention of this in Briggs at all.

Vieta's Table: comparison with Briggs's.

Vieta further treats Angular Sections as follows:-

"If from the end of a diameter any number of equal arcs be taken, and from the other extremity be drawn lines to the ends of the equal arcs taken, these straight lines become the bases of triangles in which the common hypotenuse is the diameter, and that particular base nearest to the diameter is understood to be the base for the single angle, the next of the double angle, and so on in order. Moreover let there be set up a series of lines in continued proportion in which the radius is the first term and the base of the single angle the second etc. The following will give the bases of the others taken in regular succession" and he proceeds to give the table shown on the opposite page, with its mode of construction but without proof of the rule.

True this table achieves the same object as Briggs's table for the/

the chord of the multiple arc, but its mode of construction is entirely different, Vieta's agreeing more with that given by Briggs in his alternative treatment in Chapter XI. Charles Hutton in his review of the two works [+] points out that while they achieve the same purpose they are quite different, but does not seem to have appreciated wherein the difference lay. Vieta's table expresses that

$$2 \cos n\theta = \sum_{r=0}^{r \leq \frac{n}{2}} (-)^r \frac{n}{n-r} \cdot \binom{n-r}{r} \cdot (2\cos\theta)^{n-2r}$$

i.e. $\cos n\theta$ in terms of $\cos\theta$ whereas Briggs's tables give $\sin n\theta$ in terms of $\sin\theta$ in Chapter VIII and in terms of $\cos\theta$ in Chapter XI. Vieta himself considered his methods original for he adds "Thus the analysis of angular sections involves geometric and arithmetic secrets which hitherto have been penetrated by no one."

<u>Arguments in favour of originality of Briggs's work</u>.

The 1579 edition therefore contained the rules without any proofs, and if Briggs ever saw it he certainly made no use of it. Furthermore there was no other edition until 1615 by which time Briggs had already completed his work on Angular Sections. It was this 1615 edition which was later incorporated in the 1646 edition of his collected Works. It was published by Alexander Anderson, a Scotsman who was Professor of Mathematics at Paris, and was no mean mathematician himself. He applied himself to editing Vieta's work on Trigonometry, adding proofs of the rules which Vieta had merely stated, and not content with this he added some theorems of his own to make the treatise more complete. We have this on his own authority, [++] in his denial of the charge made by some persons who accused him of passing off Vieta's work as his own, and furthermore he plainly states which of the theorems are his. These theorems are of the greatest interest for one contains the figure for the fundamental theorem of Briggs's alternative treatment, and another gives the results which he stated there. Briggs's treatment is the more straightforward and elegant of the two, the fundamental theorem being developed geometrically/

[+] Maths. Intro. p. 77. [++] Anderson.

geometrically in his case but analytically in Anderson's, who was following the methods of Vieta. Since Anderson's results were published before those of Briggs, it is quite possible that the latter may have seen them, and in this way he may have found a hint in the new theorem which Anderson gives. I scarcely think this to be the true account, but the possibility must be admitted. Even if Briggs did get his first hint from Anderson however, I do not think it detracts in the slightest from the credit due to Briggs. I think rather that the true course of events was that early in the first decade of the seventeenth century Briggs worked out his theory of Angular Sections, the work being entirely original in his hands, and compiled from it his canon of natural sines, which was certainly under way about 1604. In spite of this Briggs was only contemplating its publication at the time of his death, and it seems highly probable to me that, having in the interval come to know of this alternative way of finding chords either as an original discovery or from Anderson's theorem, he merely adds it to his own work as a matter of interest, for the fact remains that although he put it in his book he never actually used it. The evidence for this is found in Chapter XIII where he describes how he calculated the primary sines of the equidifferent arcs, which he used in the construction of his canon, and these are derived from his own methods stated in the early chapters of his book. Indeed his alternative treatment could have been omitted without loss of unity of the volume. There is still further, though indirect, evidence that Briggs considered his Angular Sections to be original work, for in his "Mathematica ab Antiquis minus Cognita" he quotes the expression of the chord of the multiple arc in terms of that of the simple arc as one of the problems whose solution was unknown to the Ancients, and whereas in most of the others he gives the inventor or discoverer of the new facts, in this case he does not, which would indicate that he knew of no other before himself. This conclusion is further strengthened by the fact that he differentiates between odd and even sections showing it was his own work he had in mind, for if he had been familiar/

familiar with either Vieta or Anderson at this time he would have known that their treatment makes no such differentiation necessary. This work of Briggs mentions the invention of Logarithms by Napier and is therefore later than 1614, so that at this late date he seems to have had no knowledge of Vieta, while we know his own work to be of a much earlier date. I can only conclude therefore that the treatise on Angular Sections was entirely original in the hands of Briggs, who must therefore have been the first Englishman to contribute anything of real note in Trigonometry.

CHAPTER IV.

The Method of Differences. - QUINQUESECTION.

Having shown considerable originality in his treatise on Angular Sections, Briggs displays nothing short of genius when he comes to the actual construction of his tables. He must have been a man of singularly penetrating mathematical insight, for there are few things more remarkable than the way in which he divined the method of subtabulation by central differences which he uses in the construction of his canon of Sines and in his later Tables. In Chapter XII he fully describes and applies the working process, which could scarcely be bettered even today without mechanical aid, and so laid the foundation of all future work on central differences. For this work alone he has every right to be hailed as the inventor of the Differential Method, and of Interpolation, the credit for the latter usually being assigned to James Gregory, who was Briggs's junior by about 80 years, because his results were expressed by series in analytic notation, a thing which Briggs unfortunately lacked. That Briggs's claim is no empty one the reader can judge for himself.

<u>Briggs's title to invention of Method</u>.

Briggs/

Briggs's title to invention of Method.

Briggs gives the rules but not the slightest clue as to the method by which he reached them. That it was one of his early discoveries, however, is certain for he himself + and later Gellibrand ++ fixes the date of its use round about 1600, which is 70 years earlier than the date of Gregory's discovery. Although invented at the time he was doing Angular Sections, the first publication of the method is when he proposes to use it for filling in the gaps in the chiliads in his tables of Logarithms, and for that reason he describes it very fully in the XIIIth Chapter of the Arithmetica Logarithmica, where he shows himself to be fully conversant with the behaviour of a difference table for equidistant arguments where all the differences are either rational or irrational That he was excited by his discovery can be inferred from the Arithmetica Logarithmica, for he confesses +++ to letting his enthusiasm carry him away, and that at a time when every moment was precious to him.

The whole idea of the difference table seems to be due to Briggs, for I have carefully probed and investigated the works of those who came before him to discover if they made use of differences at all. Neither Vieta nor Napier seem aware of differences and histories of Mathematics make no mention of their first use. Hutton ++++ speaks of Briggs as the first he knows of to use a Differential method, and I have established +++++ that in 1673 Leibnitz wrote to Oldenburg concerning the scheme of treating the series of cubes by a difference table finishing with zero differences, saying that John Pell attributed the discovery to Gabriel Mouton of Lyons (1618-1694) This latter statement cannot be correct for this kind of treatment certainly goes back to Briggs who actually treated the fourth powers in this way as an example demonstrating Trisection in the Arithmetica Logarithmica, and the sixth powers to demonstrate Quinquesection in his Trigonometria Britannica. The only other reference to/

+ Arith. Log. ++ Trig. Brit.

+++ Arith. Log. ch. 13. ++++ Math, Intro. p. 65.

+++++ Smith vol. II.

to the early use of differences that I can find points out that writers previous to Briggs seem <u>not</u> to have been possessed of the method of differences, so profitably used since, and first of all by Briggs in computing his trigonometrical and his logarithmic canons. They took, however, the successive differences of numbers after they were computed to verify or prove the truth of them; and if erroneous, by any irregularity in the last differences, from thence they had a method of correcting the original numbers themselves. At least this method is used by Pitiscus in his Trigonometry Book 2 published at Frankfort in 1599, where the differences are extended to the third order.

The Process of Quinquesection: Briggs's Table.

Now to the method itself: it seems to me that it can best be described from the Arithmetica Logarithmica in which it was first published, and most fully given, while on the other hand it is to the Trigonometria Britannica we must go in our search for any clue to a possible solution of it, since there we find it in the very setting in which it was discovered. Briggs explains fully the process of Quinquesection, that is the interpolation of four intermediate entries between each pair of the given ones, thus making five intervals where previously there was only one. He writes -

"Let there be taken the first, second, third and fourth etc. differences of the given entries, and let the first be divided by 5, the second by 25, the third by 125 etc., the divisors increasing with the common ratio 5. Let the quotients be called the first, second, third etc. mean differences, or better, in place of division let the first differences be multiplied by 2, the second by 4, the third by 8 etc., cutting off one digit of the product of the first, two in the next, three in the third etc. The latter products which are equal to the former quotients will be the first, second, third etc. mean differences."

To/

BRIGGS'S TABLE FOR QUINQUESECTION.

20								
19								
18	18 (20)							
17	17 (19)							
16	16 (18)	$123^{\underline{2}}$ (20)						
15	15 (17)	$108^{\underline{0}}$ (19)						
14	14 (16)	$93^{\underline{8}}$ (18)	$400^{\underline{4}}$ (20)					
13	13 (15)	$80^{\underline{6}}$ (17)	$317^{\underline{2}}$ (19)					
12	12 (14)	$68^{\underline{4}}$ (16)	$246^{\underline{4}}$ (18)	$629^{\underline{64}}$ (20)				
11	11 (13)	$57^{\underline{2}}$ (15)	$187^{\underline{0}}$ (17)	$431^{\underline{20}}$ (19).				
10	10 (12)	$47^{\underline{0}}$ (14)	$138^{\underline{0}}$ (16)	$283^{\underline{80}}$ (18)	$434^{\underline{40}}$ (20)			
9	9 (11)	$37^{\underline{8}}$ (13)	$98^{\underline{4}}$ (15)	$177^{\underline{84}}$ (17)	$236^{\underline{88}}$ (19).			
8	8 (10)	$29^{\underline{6}}$ (12)	$67^{\underline{2}}$ (14)	$104^{\underline{72}}$ (16)	$118^{\underline{72}}$ (18)	$111^{\underline{248}}$ (20)		
7	7 (9)	$22^{\underline{4}}$ (11)	$43^{\underline{4}}$ (13)	$56^{\underline{84}}$ (15)	$53^{\underline{20}}$ (17)	$36^{\underline{680}}$ (19)		
6	6 (8)	$16^{\underline{2}}$ (10)	$26^{\underline{0}}$ (12)	$27^{\underline{60}}$ (14)	$20^{\underline{40}}$ (16)	$10^{\underline{760}}$ (18)	$4^{\underline{080}}$ (20)	
5	5 (7)	$11^{\underline{0}}$ (9)	$14^{\underline{0}}$ (11)	$11^{\underline{40}}$ (13)	$6^{\underline{20}}$ (15)	$2^{\underline{280}}$ (17)	$\underline{560}$ (19)	
4	4 (6)	$6^{\underline{8}}$ (8)	$6^{\underline{4}}$ (10)	$3^{\underline{64}}$ (12)	$1^{\underline{28}}$ (14)	$\underline{272}$ (16)	$\underline{032}$ (18)	$\underline{0016}$ (20)
3	3 (5)	$3^{\underline{6}}$ (7)	$2^{\underline{2}}$ (9)	$\underline{72}$ (11)	$\underline{12}$ (13)	$\underline{008}$ (15).		
2	2 (4)	$1^{\underline{4}}$ (6)	$\underline{4}$ (8)	$\underline{04}$ (10)				
1	1 (3)	$\underline{2}$ (5).						
A	B.	C	D	E	F	G	H	I.

To illustrate this he then gives an example in which the fifth differences are constant, and having found the respective differences he forms the mean differences in the manner above stated. He then continues - "These mean differences should next be corrected in this way. The two most remote, namely the fourth and fifth, do not require to be corrected (since the sixth and seventh are zero: and furthermore every correction of differences is made by the subtraction of every second difference more remote and corrected, as the subtraction of the seventh corrects the fifth, of the sixth the fourth etc.): therefore the fourth and fifth mean differences are taken as the fourth and fifth correct differences. The third mean differences are corrected if from them are subtracted three times the correct fifth differences. From the second mean differences should be subtracted twice the correct fourth one and in addition should be taken away $1\frac{2}{5}$ the sixth, if there is any sixth within the limits.. ..... In this way all the mean differences are corrected and prepared for the carrying out of their function."

Having explained the process as far as it affects the practical problem in hand, Briggs then extends it and generalises it, a trait of his to which attention has already been drawn. Of course, he lacked the modern convenient algebraic notation whereby a generalised result can be very easily and compactly expressed by means of a general symbol like 'n', but he sums up his results in the form of a table which gives the particular cases in turn. "If there are many orders of differences" he writes, "we should use them in the same way, by beginning with the least and most remote. Furthermore the table placed on the opposite page indicates how many should be subtracted in the case of any one order of differences.

The numbers placed in column A signify the mean differences first, second, third ...... up to 20. The numbers in columns B, C, D etc. show how many and what order of correct differences require to be subtracted from these mean differences which are situated/

situated in column A in the same straight line with them e.g. from the sixth mean difference should be subtracted 6 times the correct eighth difference, $16\frac{2}{10}$ times the tenth, 26 times the twelfth etc. In the same way from the first mean difference should be taken 1 times the correct third difference and $\frac{1}{5}$ of the fifth.

After these correct differences have been found, each one must next be suitably placed in its right position, so that in such a complicated affair, every one can be done as quickly as possible and confusion avoided. Moreover we shall attain this more easily if we have squared paper, distinctly marked off for this method by straight lines and if the first, third, fifth and seventh are written down in a colour different from the others. The given entries marked A (in the example given) occupy every fifth place and the (corrected) second differences C, the fourth E, the sixth G etc. are placed in the same line as the entries and to the left of them. The first difference B, the third D, the fifth F, seventh etc. occupy the bottom parts, midway between each space. Finally the empty spaces should be filled up beginning from the left. By the addition of the fourth (supposing this to be the most remote) the third are formed, by the addition of the third the second and so on. Briggs suggests also retaining one or two places more than the number wanted in order to ensure very good approximations, when the differences are not rational but correct only to a fixed degree of accuracy."

Such then are the rules and practice of subtabulation laid down by Briggs himself, but while the actual mode of working is of some interest, it is the theory behind the practice, as shown by the table which must claim our attention. This work shows that Briggs possessed all that Cotes afterwards delivered in his Canontechnia Sive Constructio Tabularum per Differentias as their general rules exactly agree, although the notations differ, Briggs's mean and correct differences being by Cotes called round and quadrat differences, because he expresses them by the numbers 1, 2, 3 etc. written/

written respectively within a small circle and square.

History of Attempts to explain the process.

Considering the importance of this work of Briggs in the construction of tables and interpolation by differences it is astonishing to find how neglected his works were by later mathematicians. That he was well thought of by his contemporaries and immediate successors we know from the offices he held, and from the tributes paid him by these men, but his works seem to have been soon forgotten It may be due to his arithmetical notation being unsuited to an age to which algebraic notation had become a commonplace, but whatever the reason it has denied him the place he should rightfully have occupied in mathematical history. From time to time, however, there have arisen mathematicians who have appreciated his worth, and have deplored his comparative obscurity. Charles Hutton's appreciation has already been commented upon, but though drawing attention to the work he did not or could not explain it. Legendre was another such one, and he made an attempt + to explain the theory of the process of quinquesection just given, but had to make use of results and notation known in his day but not in Briggs's. He has virtually pointed out that this table of Briggs is equivalent to the expansion of $\left[\frac{(x^5-1)}{x-1}\right]^n x^{-2n}$ in ascending powers of Z where $Z = x + x^{-1} - 2$, ++ which does give the correct coefficients for quinquesection, but gives no indication of how to treat trisection and septisection which were perfectly well known to Briggs. This appears in Legendre's "Connaissance des Termes" which was quoted in the Journal of the Institute of Actuaries Vol. 14 in which also appears an attempt by one Maurice to explain the same, and to compare it with the work of Mouton. It is my opinion that Maurice has entirely misunderstood Briggs's work, and his remarks on the simplicity of Mouton's work are far more applicable to Briggs. Legendre too, though recognising the elegance and beauty of Briggs's work has failed to grasp the simplicity of it. Their failure and/

+ J.I.A. vol. 14. ++ Turnbull. p. 165.

and that of others who have tried, is due, I think, to their taking only the 12th and 13th Chapters of the Arithmetica Logarithmica, paying little attention to the rest of his work and consequently failing to grasp his methods.

My Own Theory of probable method of Discovery.

But to the table: Briggs gives his results but no indication as to how he achieved these results, and so far as I have been able to find out no one has ever successfully explained how he might have done so. It is now my object to advance a theory to account for these results, a theory which I feel convinced explains fairly accurately the way Briggs set about the problem, and one which shows the formation of the table by just as simple a process as that employed in those tables where he gives the mode of construction.

In actual application the first correct difference is found last, since he starts with the most remote differences as explained and works towards the first, but I think it highly probable that in the theory the process was reversed and that it was the correct first differences which were first found, and from these the others were easily derived.

That at least is my theory, and in its support I shall show
(1) that having found the first correct difference all the other correct differences could be simply found to any degree of accuracy demanded, and
(2) that these first correct differences would present no difficulty for they were known to Briggs for any odd section.
Thus,

Let Δ denote the "mean" difference found from the given difference table for the larger intervals of argument and δ denote the corresponding correct difference in the difference table for the subtabulated entries.

Then in the case of Quinquesection, from the table we have

$\delta = \Delta - \delta^3 - \frac{1}{5}\delta^5$ where the index shows order of difference.

$\therefore \Delta = \delta + \delta^3 + \frac{1}{5}\delta^5$

$\therefore \Delta^n = (\delta + \delta^3 + \frac{1}{5}\delta^5)^n$ which is the symbolic formula for Briggs's table of differences in Quinquesection!

Thus putting n = 1, 2, 3, ... 20 (where Briggs stops) and expanding as far as the 20th order, which was as far as Briggs went in his table, we get

$$\Delta = \delta + \delta^3 + \cdot 2\,\delta^5$$

$$\Delta^2 = \delta^2 + 2\,\delta^4 + 1{\cdot}4\,\delta^6 + \cdot 4\,\delta^8 + \cdot 04\,\delta^{10}$$

$$\Delta^3 = \delta^3 + 3\,\delta^5 + 3{\cdot}6\,\delta^7 + 2{\cdot}2\,\delta^9 + \cdot 72\,\delta^{11} + \cdot 12\,\delta^{13} + \cdot 008\,\delta^{15}$$

$$\Delta^4 = \delta^4 + 4\,\delta^6 + 6{\cdot}8\,\delta^8 + 6{\cdot}4\,\delta^{10} + 3{\cdot}64\,\delta^{12} + 1{\cdot}28\,\delta^{14} + \cdot 272\,\delta^{16} + \cdot 032\,\delta^{18} + \cdot 0016\,\delta^{20}$$

$$\Delta^5 = \delta^5 + 5\,\delta^7 + 11{\cdot}0\,\delta^9 + 14{\cdot}0\,\delta^{11} + 11{\cdot}40\,\delta^{13} + 6{\cdot}20\,\delta^{15} + 2{\cdot}280\,\delta^{17} + \cdot 560\,\delta^{19}$$

$$\Delta^6 = \delta^6 + 6\,\delta^8 + 16{\cdot}2\,\delta^{10} + 26{\cdot}0\,\delta^{12} + 27{\cdot}60\,\delta^{14} + 20{\cdot}40\,\delta^{16} + 10{\cdot}760\,\delta^{18} + 4{\cdot}080\,\delta^{20}$$

$$\Delta^7 = \delta^7 + 7\,\delta^9 + 22{\cdot}4\,\delta^{11} + 43{\cdot}4\,\delta^{13} + 56{\cdot}84\,\delta^{15} + 53{\cdot}20\,\delta^{17} + 36{\cdot}680\,\delta^{19}$$

$$\Delta^8 = \delta^8 + 8\,\delta^{10} + 29{\cdot}6\,\delta^{12} + 67{\cdot}2\,\delta^{14} + 104{\cdot}72\,\delta^{16} + 118{\cdot}72\,\delta^{18} + 101{\cdot}248\,\delta^{20}$$

$$\Delta^9 = \delta^9 + 9\,\delta^{11} + 37{\cdot}8\,\delta^{13} + 98{\cdot}4\,\delta^{15} + 177{\cdot}84\,\delta^{17} + 236{\cdot}88\,\delta^{19}$$

$$\Delta^{10} = \delta^{10} + 10\,\delta^{12} + 47{\cdot}0\,\delta^{14} + 138{\cdot}0\,\delta^{16} + 283{\cdot}80\,\delta^{18} + 434{\cdot}40\,\delta^{20}$$

$$\Delta^{11} = \delta^{11} + 11\,\delta^{13} + 57{\cdot}2\,\delta^{15} + 187{\cdot}0\,\delta^{17} + 431{\cdot}20\,\delta^{19}$$

$$\Delta^{12} = \delta^{12} + 12\,\delta^{14} + 68{\cdot}4\,\delta^{16} + 246{\cdot}4\,\delta^{18} + 629{\cdot}64\,\delta^{20}$$

$$\Delta^{13} = \delta^{13} + 13\,\delta^{15} + 80{\cdot}6\,\delta^{17} + 317{\cdot}2\,\delta^{19}$$

$$\Delta^{14} = \delta^{14} + 14\,\delta^{16} + 93{\cdot}8\,\delta^{18} + 400{\cdot}4\,\delta^{20}$$

$$\Delta^{15} = \delta^{15} + 15\,\delta^{17} + 108{\cdot}0\,\delta^{19}$$

$$\Delta^{16} = \delta^{16} + 16\,\delta^{18} + 123{\cdot}2\,\delta^{20}$$

$$\Delta^{17} = \delta^{17} + 17\,\delta^{19}$$

$$\Delta^{18} = \delta^{18} + 18\,\delta^{20}$$

$$\Delta^{19} = \delta^{19}$$

$$\Delta^{20} = \delta^{20}$$

and from these we derive

$$\delta^{20} = \Delta^{20}$$

$$\delta^{19} = \Delta^{19}$$

$$\delta^{18} = \Delta^{18} - 18\,\delta^{20}$$

$$\delta^{17} = \Delta^{17} - 17\,\delta^{19}$$

$$\delta^{16} = \Delta^{16} - 16\,\delta^{18} - 123.2\,\delta^{20}$$

$$\delta^{15} = \Delta^{15} - 15\,\delta^{17} - 108.0\,\delta^{19}$$

$$\delta^{14} = \Delta^{14} - 14\,\delta^{16} - 93.8\,\delta^{18} - 400.4\,\delta^{20}$$

$$\delta^{13} = \Delta^{13} - 13\,\delta^{15} - 80.6\,\delta^{17} - 317.2\,\delta^{19}$$

$$\delta^{12} = \Delta^{12} - 12\,\delta^{14} - 68.4\,\delta^{16} - 246.4\,\delta^{18} - 629.64\,\delta^{20}$$

$$\delta^{11} = \Delta^{11} - 11\,\delta^{13} - 57.2\,\delta^{15} - 187.0\,\delta^{17} - 431.20\,\delta^{19}$$

$$\delta^{10} = \Delta^{10} - 10\,\delta^{12} - 47.0\,\delta^{14} - 138.0\,\delta^{16} - 283.80\,\delta^{18} - 434.40\,\delta^{20}$$

$$\delta^{9} = \Delta^{9} - 9\,\delta^{11} - 37.8\,\delta^{13} - 98.4\,\delta^{15} - 147.84\,\delta^{17} - 236.88\,\delta^{19}$$

$$\delta^{8} = \Delta^{8} - 8\,\delta^{10} - 29.6\,\delta^{12} - 67.2\,\delta^{14} - 104.72\,\delta^{16} - 118.72\,\delta^{18} - 101.248\,\delta^{20}$$

$$\delta^{7} = \Delta^{7} - 7\,\delta^{9} - 22.4\,\delta^{11} - 43.4\,\delta^{13} - 56.84\,\delta^{15} - 53.20\,\delta^{17} - 36.680\,\delta^{19}$$

$$\delta^{6} = \Delta^{6} - 6\,\delta^{8} - 16.2\,\delta^{10} - 26.0\,\delta^{12} - 27.60\,\delta^{14} - 20.40\,\delta^{16} - 10.760\,\delta^{18} - 4.080\,\delta^{20}$$

$$\delta^{5} = \Delta^{5} - 5\,\delta^{7} - 11.0\,\delta^{9} - 14.0\,\delta^{11} - 11.40\,\delta^{13} - 6.20\,\delta^{15} - 2.280\,\delta^{17} - .560\,\delta^{19}$$

$$\delta^{4} = \Delta^{4} - 4\,\delta^{6} - 6.8\,\delta^{8} - 6.4\,\delta^{10} - 3.64\,\delta^{12} - 1.28\,\delta^{14} - .272\,\delta^{16} - .032\,\delta^{18} - .0016\,\delta^{20}$$

$$\delta^{3} = \Delta^{3} - 3\,\delta^{5} - 3.6\,\delta^{7} - 2.2\,\delta^{9} - .72\,\delta^{11} - .12\,\delta^{13} - .008\,\delta^{15}$$

$$\delta^{2} = \Delta^{2} - 2\,\delta^{4} - 1.4\,\delta^{6} - .4\,\delta^{8} - .04\,\delta^{10}$$

$$\delta = \Delta - 1\,\delta^{3} - .2\,\delta^{5}$$

and these are exactly the equations given by Briggs's table!

It should further be noted how easily this table for Δ^n is formed and how it can be extended indefinitely with no labour at all. The whole table is made up from $\Delta = \delta + \delta^3 + \frac{1}{5}\delta^5$ in this very simple way: any coefficient is the sum of that immediately above it together with that to the left of this first and $\frac{1}{5}$ (or $\cdot 2$) of that to the left of this second, thus,

in Δ^4, the coefficient of δ^{10} is $2\cdot 2 + 3\cdot 6 + \cdot 2(3) = 6\cdot 4$

the coefficient of δ^{18} is $\cdot 008 + \cdot 12 + \cdot 2(\cdot 42) = \cdot 242$.

as was stated above Briggs did not go beyond δ^{20}, but of course, there is no necessity to stop there, and in practice one would proceed just as far as the differences remained within the limits of accuracy required. For instance, in the example given by way of illustration, the fifth differences are constant within the degree of accuracy required so that in this case δ^6 and all higher orders are zero and $\delta^5 = \Delta^5$ and $\delta^4 = \Delta^4$.

In exactly the same way it may be shown that the different orders of correct differences for Trisection may be generated from $\Delta^n = (\delta + \frac{1}{3}\delta^3)^n$, which is the symbolic expression for the table which Briggs gives.

The calculation of the coefficients entails even less work in this case than for quinquesection, for any coefficient is the sum of the one above it together with $\frac{1}{3}$ of that to the left of this first one. Thus, going as far as δ^{12}

$$\Delta = \delta + \tfrac{1}{3}\delta^3 \qquad \therefore\ \delta = \Delta - \tfrac{1}{3}\delta^3$$

$$\Delta^2 = \delta^2 + \tfrac{2}{3}\delta^4 + \tfrac{1}{9}\delta^6 \qquad \delta^2 = \Delta^2 - \tfrac{2}{3}\delta^4 - \tfrac{1}{9}\delta^6$$

$$\Delta^3 = \delta^3 + 1\,\delta^5 + \tfrac{3}{9}\delta^7 + \tfrac{1}{27}\delta^9 \qquad \delta^3 = \Delta^3 - 1\,\delta^5 - \tfrac{3}{9}\delta^7 - \tfrac{1}{27}\delta^9$$

$$\Delta^4 = \delta^4 + 1\tfrac{1}{3}\delta^6 + \tfrac{6}{9}\delta^8 + \tfrac{4}{27}\delta^{10} + \tfrac{1}{81}\delta^{12} \qquad \delta^4 = \Delta^4 - 1\tfrac{1}{3}\delta^6 - \tfrac{6}{9}\delta^8 - \tfrac{4}{27}\delta^{10} - \tfrac{1}{81}\delta^{12}$$

$$\Delta^5 = \delta^5 + 1\tfrac{2}{3}\delta^7 + 1\tfrac{1}{9}\delta^9 + \tfrac{10}{27}\delta^{11} \qquad \delta^5 = \Delta^5 - 1\tfrac{2}{3}\delta^7 - 1\tfrac{1}{9}\delta^9 - \tfrac{10}{27}\delta^{11}$$

etc. etc.

Thus knowing the first mean difference in terms of the corrected finer/

finer differences, he is able to find all the others with scarcely any labour at all. The real problem therefore is just this: Could Briggs get this first relationship? and the answer is yes. I shall endeavour to show that not only could he get it but that he had <u>all</u> such relationships ready at hand, so that he was in a position to give such a table for any such odd section.

It should be recalled that the discovery of this method of differences is contemporaneous with his researches on Angular Sections, or came shortly after them, and I would advance the theory that it was the latter which made the former possible. It may be illuminating to try to follow the course of Briggs's work round about the end of the sixteenth and the beginning of the seventeenth century, and my reconstruction of it would be as follows:

Briggs, the geometer, is engaged upon original work, in the course of which he successfully builds up the Theory of Angular Sections, the treatment being purely geometrical as was to be expected from a geometer. He discovers, and illustrates geometrically that the differences of the sines of equidifferent arcs are proportional to the sines of the complements of the mean arcs. These sines in turn are sines of equidifferent arcs, and therefore, by the above, their differences or the second differences of the original sines will be proportional to the sines of the complements of the mean arcs in this second case, and these complements are merely the given arcs. Thus by repeating the process starting from these second differences, we arrive at similar results for the third, fourth and remaining differences. In fact the second, fourth, sixth and all even differences are proportional to the given sines themselves, and the first, third and all odd differences are proportional to themselves and to the sines of the complements of the mean arcs. This is not conjecture, but what Briggs himself states in the XIIth Chapter of the Trigonometria Britannica. I would not push it too far, but I am prepared to see in this the possible origin of the difference table itself, which we noted before/

before seems to have started with Briggs, Something must in that case have suggested repeated differencing, and what more likely than this property of sines, for this repeated proportionality simply invites continued differencing, and once having started with Sines, the prying mind of Briggs would want to try other things. He certainly did try other things as witnessed by his examples.

Be that as it may, Briggs certainly discovered and states in continuing the above that all those sines and differences which lie in the same straight line will be continued proportionals, and this fact may be used to investigate the last and smallest differences from the early ones to a high degree of accuracy.

The real importance to us of these two properties of the differences is that we are able to place any difference of any entry in its proper place without reference to the others. The real problem confronting Briggs will be evident by a comparison of the difference table for the coarser differences with that for the finer differences, and we now proceed to these tables.

Let the difference of argument in the one case be 5θ so that the interval may easily be divided to give difference of argument θ in the table of finer differences

ENTRY	Δ	Δ^2	Δ^3	Δ^4	Δ^5
$\sin(\alpha-10\theta)$		$-(2\sin\frac{5\theta}{2})^2\sin(\alpha-10\theta)$		$+(2\sin\frac{5\theta}{2})^4\sin(\alpha-10\theta)$	
	$2\sin\frac{5\theta}{2}\cos(\alpha-7\frac{1}{2}\theta)$		$-(2\sin\frac{5\theta}{2})^3\cos(\alpha-7\frac{1}{2}\theta)$		$+(2\sin\frac{5\theta}{2})^5\cos(\alpha-7\frac{1}{2}\theta)$
$\sin(\alpha-5\theta)$		$-(2\sin\frac{5\theta}{2})^2\sin(\alpha-5\theta)$		$+(2\sin\frac{5\theta}{2})^4\sin(\alpha-5\theta)$	
	$2\sin\frac{5\theta}{2}\cos(\alpha-2\frac{1}{2}\theta)$		$-(2\sin\frac{5\theta}{2})^3\cos(\alpha-2\frac{1}{2}\theta)$		$+(2\sin\frac{5\theta}{2})^5\cos(\alpha-2\frac{1}{2}\theta)$
$\sin\alpha$		$-(2\sin\frac{5\theta}{2})^2\sin\alpha$		$+(2\sin\frac{5\theta}{2})^4\sin\alpha$	
	$2\sin\frac{5\theta}{2}\cos(\alpha+2\frac{1}{2}\theta)$.		$-(2\sin\frac{5\theta}{2})^3\cos(\alpha+2\frac{1}{2}\theta)$		$+(2\sin\frac{5\theta}{2})^5\cos(\alpha+2\frac{1}{2}\theta)$
$\sin(\alpha+5\theta)$		$-(2\sin\frac{5\theta}{2})^2\sin(\alpha+5\theta)$		$+(2\sin\frac{5\theta}{2})^4\sin(\alpha+5\theta)$	
	$2\sin\frac{5\theta}{2}\cos(\alpha+7\frac{1}{2}\theta)$		$-(2\sin\frac{5\theta}{2})^3\cos(\alpha+7\frac{1}{2}\theta)$		$+(2\sin\frac{5\theta}{2})^5\cos(\alpha+7\frac{1}{2}\theta)$
$\sin(\alpha+10\theta)$.		$-(2\sin\frac{5\theta}{2})^2\sin(\alpha+10\theta)$		$+(2\sin\frac{5\theta}{2})^4\sin(\alpha+10\theta)$.	

This/

This table of coarser differences is the one known and the problem is how from this table can be found the differences in the table of finer differences corresponding to the same entry. Let us form the table of finer differences and compare the entries in it, with those above.

ENTRY	Δ	Δ^2	Δ^3	Δ^4	Δ^5
$\sin(\alpha-5\theta)$		$-(2\sin\frac{\theta}{2})^2\sin(\alpha-5\theta)$		$(2\sin\frac{\theta}{2})^4\sin(\alpha-5\theta)$	
	$(2\sin\frac{\theta}{2})\cos(\alpha-4\frac{1}{2}\theta)$		$-(2\sin\frac{\theta}{2})^3\cos(\alpha-4\frac{1}{2}\theta)$		$(2\sin\frac{\theta}{2})^5\cos(\alpha-4\frac{1}{2}\theta)$
$\sin(\alpha-4\theta)$		$-(2\sin\frac{\theta}{2})^2\sin(\alpha-4\theta)$		$(2\sin\frac{\theta}{2})^4\sin(\alpha-4\theta)$	
	$(2\sin\frac{\theta}{2})\cos(\alpha-3\frac{1}{2}\theta)$		$-(2\sin\frac{\theta}{2})^3\cos(\alpha-3\frac{1}{2}\theta)$		$(2\sin\frac{\theta}{2})^5\cos(\alpha-3\frac{1}{2}\theta$
$\sin(\alpha-3\theta)$		$-(2\sin\frac{\theta}{2})^2\sin(\alpha-3\theta)$		$(2\sin\frac{\theta}{2})^4\sin(\alpha-3\theta)$	
	$(2\sin\frac{\theta}{2})\cos(\alpha-2\frac{1}{2}\theta)$		$-(2\sin\frac{\theta}{2})^3\cos(\alpha-2\frac{1}{2}\theta)$		$(2\sin\frac{\theta}{2})^5\cos(\alpha-2\frac{1}{2}\theta)$
$\sin(\alpha-2\theta)$		$-(2\sin\frac{\theta}{2})^2\sin(\alpha-2\theta)$		$(2\sin\frac{\theta}{2})^4\sin(\alpha-2\theta)$	
	$(2\sin\frac{\theta}{2})\cos(\alpha-1\frac{1}{2}\theta)$		$-(2\sin\frac{\theta}{2})^3\cos(\alpha-1\frac{1}{2}\theta)$		$(2\sin\frac{\theta}{2})^5\cos(\alpha-1\frac{1}{2}\theta)$
$\sin(\alpha-\theta)$		$-(2\sin\frac{\theta}{2})^2\sin(\alpha-\theta)$		$(2\sin\frac{\theta}{2})^4\sin(\alpha-\theta)$	
	$(2\sin\frac{\theta}{2})\cos(\alpha-\frac{1}{2}\theta)$		$-(2\sin\frac{\theta}{2})^3\cos(\alpha-\frac{1}{2}\theta)$		$(2\sin\frac{\theta}{2})^5\cos(\alpha-\frac{1}{2}\theta)$
$\sin\alpha$		$-(2\sin\frac{\theta}{2})^2\sin\alpha$		$(2\sin\frac{\theta}{2})^4\sin\alpha$	
	$(2\sin\frac{\theta}{2})\cos(\alpha+\frac{1}{2}\theta)$		$-(2\sin\frac{\theta}{2})^3\cos(\alpha+\frac{1}{2}\theta)$		$(2\sin\frac{\theta}{2})^5\cos(\alpha+\frac{1}{2}\theta)$
$\sin(\alpha+\theta)$		$-(2\sin\frac{\theta}{2})^2\sin(\alpha+\theta)$		$(2\sin\frac{\theta}{2})^4\sin(\alpha+\theta)$	
	$(2\sin\frac{\theta}{2})\cos(\alpha+1\frac{1}{2}\theta)$		$-(2\sin\frac{\theta}{2})^3\cos(\alpha+1\frac{1}{2}\theta)$		$(2\sin\frac{\theta}{2})^5\cos(\alpha+1\frac{1}{2}\theta$
$\sin(\alpha+2\theta)$		$-(2\sin\frac{\theta}{2})^2\sin(\alpha+2\theta)$		$(2\sin\frac{\theta}{2})^4\sin(\alpha+2\theta)$	
	$(2\sin\frac{\theta}{2})\cos(\alpha+2\frac{1}{2}\theta)$		$-(2\sin\frac{\theta}{2})^3\cos(\alpha+2\frac{1}{2}\theta)$		$(2\sin\frac{\theta}{2})^5\cos(\alpha+2\frac{1}{2}\theta$
$\sin(\alpha+3\theta)$		$-(2\sin\frac{\theta}{2})^2\sin(\alpha+3\theta)$		$(2\sin\frac{\theta}{2})^4\sin(\alpha+3\theta)$	
	$(2\sin\frac{\theta}{2})\cos(\alpha+3\frac{1}{2}\theta)$		$-(2\sin\frac{\theta}{2})^3\cos(\alpha+3\frac{1}{2}\theta)$		$(2\sin\frac{\theta}{2})^5\cos(\alpha+3\frac{1}{2}\theta$
$\sin(\alpha+4\theta)$		$-(2\sin\frac{\theta}{2})^2\sin(\alpha+4\theta)$		$(2\sin\frac{\theta}{2})^4\sin(\alpha+4\theta)$	
	$(2\sin\frac{\theta}{2})\cos(\alpha+4\frac{1}{2}\theta)$		$-(2\sin\frac{\theta}{2})^3\cos(\alpha+4\frac{1}{2}\theta)$		$(2\sin\frac{\theta}{2})^5\cos(\alpha+4\frac{1}{2}\theta$
$\sin(\alpha+5\theta)$.		$-(2\sin\frac{\theta}{2})^2\sin(\alpha+5\theta)$		$(2\sin\frac{\theta}{2})^4\sin(\alpha+5\theta)$	

Thus sin $(\alpha \pm 5n\theta)$ for integral values of n will be entries in both tables. Take sinα for instance. In the table of coarser differences the first difference of sinα is sin $(\alpha + 5\theta)$ - sinα = 2 sin$\frac{5\theta}{2}$ cos $(\alpha + 2\frac{1}{2}\theta)$ placed mid way between sinα and sin $(\alpha + 5\theta)$ i.e. in row $2\frac{1}{2}$ as it were. In the table of finer differences, the first difference occupying this very position would be, not the difference between sinα and sin $(\alpha + 5\theta)$, but the difference between sin $(\alpha + 2\theta)$ and sin $(\alpha + 3\theta)$ i.e. $2\sin\frac{\theta}{2}$ cos $(\alpha + 2\frac{1}{2}\theta)$ and all the differences in this row would be proportional to cos $(\alpha + 2\frac{1}{2}\theta)$, and would also be continued proportionals, Thus the third difference would be $-\left(2\sin\frac{\theta}{2}\right)^3$ cos $(\alpha + 2\frac{1}{2}\theta)$, the fifth $\left(2\sin\frac{\theta}{2}\right)^5$ cos $(\alpha + 2\frac{1}{2}\theta)$ etc. the index of $\left(2\sin\frac{\theta}{2}\right)$ being exactly the same as the order of the difference concerned.

Using modern central difference notation, Δ for the coarser and δ for the finer, we have

$$\Delta \sin(\alpha + 2\tfrac{1}{2}\theta) = 2\sin\frac{5\theta}{2}\cos(\alpha + 2\tfrac{1}{2}\theta) \quad \text{and} \quad \Delta^n \sin(\alpha + 2\tfrac{1}{2}\theta) = \left(2\sin\frac{5\theta}{2}\right)^n \cos(\alpha + 2\tfrac{1}{2}\theta)$$

$$\delta \sin(\alpha + 2\tfrac{1}{2}\theta) = 2\sin\frac{\theta}{2}\cos(\alpha + 2\tfrac{1}{2}\theta) \quad \text{and} \quad \delta^n \sin(\alpha + 2\tfrac{1}{2}\theta) = \left(2\sin\frac{\theta}{2}\right)^n \cos(\alpha + 2\tfrac{1}{2}\theta)$$

where no attention has been paid to the sign, of which more later. We see therefore that operating by Δ corresponds to a multiplication by 2 sin $\frac{5\theta}{2}$, and operating by δ corresponds to a multiplication by 2 sin $\frac{\theta}{2}$. Thus the problem of expressing Δ in terms of δ is just that of expressing 2 sin $\frac{5\theta}{2}$ in terms of 2 sin$\frac{\theta}{2}$ which is a problem in Angular Sections, the very work Briggs was or had been engaged upon at the time he was dealing with differences. He already had a table for expressing the chord (i.e. the double sine) of any odd multiple arc in terms of that of the simple arc. We see therefore that he was in possession of all the knowledge necessary for any odd section and not only that of quinquesection, a statement which Briggs himself makes for trisection and septisection, but which nobody+ seems to have tried to substantiate.

The equation of quinquesection given by Briggs is

$$\text{chord } 5\alpha = 5 \text{ chord } \alpha - 5 \text{ chord}^3\alpha + \text{chord}^5\alpha$$

$$\text{or} \quad 2\sin\frac{5\theta}{2} = 5\left(2\sin\frac{\theta}{2}\right) - 5\left(2\sin\frac{\theta}{2}\right)^3 + \left(2\sin\frac{\theta}{2}\right)^5$$

for/

+ Arith. Log. ch. 13.

for the case in point.

It was pointed out above that no attention had been paid to the signs, a matter which must now be brought up.

It may be recalled that Briggs states that in the sine difference table those entries which are in the same straight line are continued proportionals, but these it should be remembered give only alternate differences, either the even or the odd.

If we take the two rows which give <u>all</u> the orders of differences say of $\sin\alpha$, the coefficients of $\cos(\alpha + \frac{1}{2}\theta)$ in the odd differences and those of $\sin\alpha$ in the even differences, give the complete series of continued proportionals apart from sign. Briggs himself did not put in the actual signs, but rather in his tables of coefficients gave instructions in words as to which columns should be added and which subtracted. Now Briggs was perfectly aware of the fact, and is at some pains to bring to the reader's notice +that the case of sines is peculiarly different from all other cases such as logarithms, tangents, secants and all homogeneous powers of equidistant numbers, for which all the differences placed in columns B, C, D, E, etc of his tables are to be subtracted from the mean differences in column A, while for sines those in column B, D, F, etc are not to be subtracted but added. This apparent difference for the case of the sine is easily explained, and is due entirely to the peculiar property of the sine difference table.

The series of continued proportionals giving the coefficients is not 1, $(2\sin\frac{\theta}{2})$, $(2\sin\frac{\theta}{2})^2$, $(2\sin\frac{\theta}{2})^3$, $(2\sin\frac{\theta}{2})^4$, etc. which would keep step exactly with entry, δ, δ^2, δ^3, δ^4, in size and sign, but is rather

1, $(2\sin\frac{\theta}{2})$, $-(2\sin\frac{\theta}{2})^2$ $-(2\sin\theta/2)^3$, $(2\sin\frac{\theta}{2})^4$ etc.

where the successive pairs are alternately + and − , so that it is not $+(2\sin\frac{\theta}{2})^{2r}$ and $+(2\sin\theta/2)^{2r+1}$ (where r is odd) but $-(2\sin\frac{\theta}{2})^{2r}$ and $-(2\sin\theta/2)^{2r+1}$ which correspond to the operation δ^{2r} and δ^{2r+1}.

Therefore/

+ Arith. Log. ch. 13. Trig. Brit. ch. 12.

Therefore, keeping this in mind when changing from the equation $2 \sin \frac{5\theta}{2} = 5\left(2 \sin \frac{\theta}{2}\right) - 5\left(2 \sin \theta/2\right)^3 + \left(2 \sin \frac{\theta}{2}\right)^5$ to that giving $\boldsymbol{\Delta}$ in terms of δ, we have for quinquesection

$$\boldsymbol{\Delta} = 5\delta + 5\delta^3 + \delta^5 \qquad \left[\delta^3 = -(2 \sin \theta/2)^3\right]$$

and since the mean difference $\Delta = \frac{\boldsymbol{\Delta}}{5}$

$$\therefore \quad \Delta = \delta + \delta^3 + \tfrac{1}{5}\delta^5$$

$$\therefore \quad \Delta^n = \left(\delta + \delta^3 + \tfrac{1}{5}\delta^5\right)^n$$

from which we derive δ^n in terms of Δ^n and higher orders of δ as previously shown.

Similarly we get for

<u>TRISECTION</u> $\qquad \boldsymbol{\Delta} = 3\delta + \delta^3$

$$\therefore \quad \Delta = \delta + \tfrac{1}{3}\delta^3$$

$$\therefore \quad \Delta^n = \left(\delta + \tfrac{1}{3}\delta^3\right)^n$$

which gives Brigg's table for trisection, when δ^n is expressed as Δ^n - orders of δ greater than n.

<u>SEPTISECTION</u> $\qquad \boldsymbol{\Delta} = 7\delta + 14\delta^3 + 7\delta^5 + \delta^7$

$$\therefore \quad \Delta = \delta + 2\delta^3 + \delta^5 + \tfrac{1}{7}\delta^7$$

$$\therefore \quad \Delta^n = \left(\delta + 2\delta^3 + \delta^5 + \tfrac{1}{7}\delta^7\right)^n$$

which would lead to a table such as Briggs gives for trisection and quinquesection.

<u>Generalisation of Briggs's results</u>.

Briggs does not actually give a table for septisection, but this method which I have given, and which I think in all probability Briggs used (without the symbolism), leads to the above result. Indeed the whole can be summed up as follows:-

If $\boldsymbol{\Delta}$ denote a coarser difference, and Δ the mean difference derived from it, and if each interval be divided into $(2p+1)$ intervals, δ denoting a correct finer difference, then since

$$2 \sin (2p+1)\,\theta/2 = \sum_{r=1}^{r=p+1} (-)^{r+1} \frac{2p+1}{p+r} \cdot \binom{p+r}{2r-1} \cdot \left(2 \sin \theta/2\right)^{2r-1}$$

we get as above

$$\boldsymbol{\Delta} = \sum_{r=1}^{r=p+1} \frac{2p+1}{p+r} \cdot \binom{p+r}{2r-1} \cdot \delta^{2r-1} \qquad \therefore \quad \Delta = \frac{\boldsymbol{\Delta}}{2p+1} = \sum_{r=1}^{r=p+1} \frac{1}{p+r} \cdot \binom{p+r}{2r-1} \cdot \delta^{2r-1}$$

$$\therefore \quad \Delta^n = \left[\sum_{r=1}^{r=p+1} \frac{1}{2r-1} \cdot \binom{p+r-1}{2r-2} \delta^{2r-1}\right]^n$$

which is the fundamental equation giving the correct differences in/

in the table of finer differences of a $(2p+1)^{th}$ section.

It follows from the above, that when the k^{th} finer differences are constant, i.e. δ^{k+1} and all higher orders are zero then $\Delta^k = \left[\sum_{r=1}^{r=p+1} \frac{1}{2r-1} \cdot \binom{p+r-1}{2r-2} \delta^{2r-1}\right]^k$, and therefore all terms above r = 1 lead to orders greater than the k^{th} and so vanish, in which case only one term survives and we get $\Delta^k = \left[\frac{1}{2.1-1} \cdot \binom{p}{0} \delta^{2.1-1}\right]^k = \delta^k$ since $\binom{p}{0} = 1$

Thus $\delta^k = \Delta^k = \dfrac{\triangle^k}{(2p+1)^k}$ .... (1).

and $\Delta^{k+1} = \delta^{k+1} + \text{higher orders} = 0.$

so that the coarser differences and finer differences end at the same order. Furthermore, result (1) shows that if the k^{th} coarser differences are constant, the k^{th} finer differences are constant and equal to $\dfrac{\text{coarser const. difference}}{(2p+1)^k}$, a result whose discovery is attributed + to Gabriel Mouton about the year 1670, but which, as can be seen, is a necessary corollary to Briggs's work.

<u>Arguments in favour of above Method</u>.

Whether Briggs used the above method or not, it certainly produces his results and the question naturally arises, Is there any evidence to show that Briggs did use it?

It should be reiterated that Briggs's method of differences was invented at the same time or immediately after he had discovered his results in Angular Sections, and the method is consequently a very likely one for him to have taken because of its basic dependence on Angular Sections. Furthermore there is his own evidence from which I would conclude that he associated the two problems together in his own mind, for in dealing with the problem of even sections he says (Arithmetica Logarithmica Chap. XIII) -

"The remaining sections to which the names are assigned by even numbers, such as Bisection and Quadrisection, are more difficult. This we experience also in the finding of chords in a circle, since the sections named after the odd numbers produce the required/

+ Whittaker.

required chords themselves by a single operation, but the others named by even sections produce not the chords themselves but only the squares of the chords."

If the two problems are not connected, it seems to me remarkable that while Briggs could do subtabulation for any odd section (which in Angular Sections produces an equation for the multiple chord in terms of the simple chord) he certainly could not do it for the even sections, which he admits are more difficult, and of which he gives no example. By the theory I have propounded one would not expect him to, since in his Angular Sections, from which I have pointed out he gets his basic relations for subtabulation, the even sections do not produce an equation in terms of the chord but in terms of the square of the chord. This absence of an even section then fits in with my theory and would indicate that the equations of Angular Sections are the true source of the basic relation between the coarser and finer differences.

But, it may be objected, all this has been derived from a table of differences of sines, and what justification is there for applying the same rules to other functions? The objection is sustained. Let it be remembered however that most mathematical results of importance have come well in advance of any rigid mathematical proof of these results. Mathematical truths have been formulated and stated before they could be demonstrated to be truths, being first half-guessed at, and enunciated when the truth of, say a few particular cases, has been established. Newton did not prove the Binomial Theorem for fractional indices before he gave it to the world, and Brook Taylor's proof of his theorem is no longer considered valid, but in spite of these facts, the truth of the Theorems remain. At this period of mathematical discovery, many results came from a process akin to trial and error, and the fact that a theory worked in all known cases was considered a good enough justification to be going on with. I can imagine Briggs reasoning something like this:-

Sines/

Sines are "continuous" functions in that there are no sudden breaks or jumps in the values, so that in this respect they behave like most other mathematical functions which can be expressed rationally. Therefore if there are any general laws in difference tables, these laws should be present also in a sine difference table, which happens to be perhaps the very simplest of such tables, due to the proportionality properties already noted. This being so, the laws, if any, should be most easily found from such a table. By careful inspection and investigation he would discover the relationships which I have pointed out, so that all that remained was to discover whether these laws were general or merely peculiar to the sine difference table. Trial and error in applying these laws to other functions where the results obtained could be easily verified, would convince him that they were general laws, and as such he stated them. They worked in known cases and that would be good enough for him. Colour is lent to this by the fact that he discovered another important property of the sine difference table [+], but one which is peculiar to that table, and of which after the first mention we hear no more.

Briggs's Knowledge of Difference Table properties.

Briggs has been referring all along to sines and logarithms and tells how to ensure getting the most accurate results, since the differences in these cases are not exact, but "irrational" and only approximations within given limits the which is also true of tangents and secants. To appreciate just how thoroughly Briggs had mastered the difference table and its properties we must have recourse to what he has to say in connection with rational differences. He writes [++] - "But in the case of powers of equidistant numbers, where all the differences and the given numbers themselves are rational, the whole can be confined to fixed limits, for the number of separate differences is definite, a number beyond which they cannot go should the arguments be equidifferent. For instance in/

[+] Trig. Brit. Ch. 12. [++] Arith. Log. Ch. 13.

in quadratics there are two orders of differences, in cubes three, in fourth powers four etc, and the most remote differences are always equal to each other, and to the product of the same power of the difference in argument multiplied by the continued product of the indices of that power and all those below it. Thus if the difference in argument is 1, the final differences will be for the square 2, the cube 6, the fourth power 24, the fifth 120, the sixth 720, the seventh 5040 etc. namely the continued products of 1.2 = 2, 1.2.3 = 6, 1.2.3.4 = 24 etc." What he says amounts to this, that if the entry is of the n^{th} degree and the difference in argument is h, then there will be n orders of differences, the n^{th} order being constant and equal to $\underline{|n} \, h^n$.

Briggs further points out that in the case of sines, it is enough to know three sines of equidifferent arcs in order to find all the differences, since these can be found from the continued proportion properties of the sine difference table, but that in all other cases such as powers, logarithms, tangents and secants, a few more entries must be taken without which we cannot obtain the last differences.

All this it should be marked was given by Briggs at least 50 years previous to the time when Interpolation was supposed to have been invented. In view of the evidence which he himself has supplied as to his absolute mastery of the difference table, and his invention of subtabulation by central differences, can any one deny that he is fully entitled to be hailed as the inventor of Interpolation?

CHAPTER V.

The Method of Differences - Decisection.

This, however, does not nearly represent all Briggs's work

in/

in this field, and his claims are further strengthened when we consider the application of his differential method to logarithms. Although Briggs described his methods of quinquesection and trisection so fully in his Arithmetica Logarithmica he did not use it so very extensively in that volume, Its chief use was to be in the filling in of the chiliads that were missing, and for this purpose he holds it to be the best of all the methods available. As was pointed out before, he did use it for that purpose, but the missing chiliads which he did calculate were never published due to his being forestalled in this by Vlacq. Whether it was because of the large number of logarithms to be calculated, or because of the nature of logarithms, Briggs found decisection to be very convenient. This may have been partly because decisection gives divisors which are powers of 10 and therefore very easy to work with, but chiefly because of the decimal nature of our number system, which fact being true would be but one other instance of Briggs's fondness for decimal division, which we noted previously and on which we shall comment later. The convenience of decisection will manifest itself if we consider the following example -

Suppose Briggs had already found the logarithms for numbers of the tenth chiliad, say the logarithms of the equidistant numbers 9010, 9011, 9012 etc., he could write down the logarithms of 90100, 90110, 90120 etc. simply by changing the characteristic, and therefore these latter would have exactly the same differences as the former, for which they are known.

The problem is to interpolate the logarithms of 90101, ...2, ...3 etc. and obviously it will be much easier if all the intermediate numbers required between each two entries can be found at the same working i.e. decisection is required. It will be noted however, that this is an even section and up to his calculation of logarithms Briggs did not know how to deal with such a section. His difficulty however could not be avoided by quinquesection, since/

since to complete the whole chiliad it would have been necessary to apply Bisection next, which is still an even section

Briggs's Account of Decisection.

Briggs must therefore have applied himself to this problem and he sets down his rules without any indication whatever as to the method by which they were derived. For reasons previously stated he could not have had recourse to Angular Sections, which explained his previous method, and so he had to find some other way. This other way has given rise to much discussion, since he treats only of the case where second differences are equal or so nearly equal that third differences are negligible, and consequently there is not much to go on, so that his method is very much a matter of conjecture, conclusions having to be based upon the very scantiest of evidence. Before we proceed farther, however, let us see what Briggs himself has to say about it. In Chapter XII of his Arithmetica Logarithmica he writes:-

"Given two consecutive numbers with their logarithms, to insert between them nine other equidistant numbers and to find their logarithms," and continues

"If the second differences of the given logarithms are nearly equal it will not be difficult to carry this out, but otherwise, this method, should third differences require to be called in, has something wanting. (alias hic modus si tertae differentiae adhibendae <u>non nihil deficiet</u>).

Let there be taken two consecutive numbers A and their logarithms B together with their first differences C and second differences D. If the second differences are equal let one of them be multiplied by the numbers in Table E placed opposite the first 10 numbers : moreover the products F,G,H,I,K the last three digits having been cut off, are added in the case of the first five and subtracted for the other five, from the tenth part of the intervening first difference C ; the sums and differences are the (first) differences of/

of the logarithms required, which, added continuously in turn to the smaller given logarithm, will give the required logarithms as here shown.

If the second differences are unequal, let the two near values be added, and let half the sum be taken in place of the second difference, and let it be multiplied as before.

		47602 0016 C.	
91235 A	4 96016 14763 8639 B.		5217 D.
		47601 4799 C.	
91236 A	4 96016 62365 3438 B.		5217 D.
		47600 9582 C.	

	E	
1	45	Products to be added.
2	35	
3	25	
4	15	
5	5	
6	5	Products to be subtracted
7	15	
8	25	
9	.35	
10	45	

912350	4 96016 14763 8639	
	4760 1715	C + F
912351	4 96016 19524 0354	
	4760 1662	C + G
912352	4 96016 24284 2016	
	4760 1610	C + H
912353	4 96016 29044 3626	
	4760 1558	C + I
912354	4 96016 33804 5184	
	4760 1506	C + K
912355	4 96016 38564 6690	
	4760 1454	C − K
912356	4 96016 43324 8144	
	4760 1402	C − I
912357	4 96016 48084 9546	
	4760 1350	C − H
912358	4 96016 52845 0896	
	4760 1297	C − G
912359	4 96016 57605 2193	
	4760 1245	C − F
912360	4 96016 62365 3438.	

PRODUCT		5217 ×
F	234 765	45
G	182 595	35
H	130 425	25
I	78 255	15
K	26 085	

$\left.\begin{matrix}F\\G\\H\\I\\K\end{matrix}\right\}$ REMAINING FACTORS.

47601479	9	$\frac{1}{10}$C
47601714	7	C + F
47601662	5	C + G
47601610	3	C + H
47601558	2	C + I
47601506	0	C + K
47601453	8	C − K
47601401	6	C − I
47601349	5	C − H
47601297	3	C − G
47601245	1.	C − F.

Briggs then gives a second example where the second differences are only nearly equal, so he takes as the second difference the mean of the two.

		4	51707	8187	C	
9615	A	3 98294	92885	7450	B	469771
		4	51660	8416	C	
9616	A	3 98299	44546	5866	B	469672
		4	51613	8744	C	939443 ...... sum
						469721 ...... ½ sum.

But if you wish to find any one of them, omitting the rest, the number less than ten adjoining the given number A multiplies the given difference C, and the number opposite the same number in table E' multiplies the second difference, and having cut off three digits in the latter and a single digit in the former, the products are added, and the sum added to the given logarithm will give the logarithm sought. For instance, if I want to know the logarithm of the number 96157, the given difference 45516608416 is multiplied by 7, the product being 31616258912 : then 105 placed opposite the number 7 in table E' multiplies the second difference 469721, and the product 49320/705, after the last three digits have been cut off, is added to the former product, which with a single digit cut off, will be 3161625891, and the total 3161675212 is added to the given logarithm. The sum 398298090532662 will be the logarithm of 96517.

	E'
1	45
2	80
3	105
4	120
5	125
6	120
7	105
8	80
9	45

these products are added.

If you wish to know the difference between the logarithm of this number and the next greater, the number in Table E opposite the number 8, which exceeds the given one 7 by unity, multiplies the second difference 469721. The product 11743/025 after the removal of three digits is taken from the tenth part of the first difference/

difference and the remainder 451649099 will be the required difference."

That is the whole of what Briggs gives for this method. How, therefore, are we to explain it?

Points raised by Briggs's limited Process.

Before we attempt an explanation there are one or two points which must be noted because of their bearing on the problem.

Firstly, the whole treatment would be expected to be entirely different from the theory of quinquesection, since their sources are very different.

Secondly, much stress has been laid upon the fact that it only goes to second differences, and that when they are constant or very nearly so. This fact seems to have led to the value of the work being overlooked, and wrongly so. It is my considered opinion from the very scanty evidence there is, that Briggs was in possession of a much fuller treatment, if not in possession of the full facts which were later expressed analytically by series.

Arguments in favour of a fuller Knowledge on part of Briggs.

In the first place it is quite contrary to all of Briggs's previous work for him to be content with only a very limited part of a theory, especially when he was going to work extensively with it : it is all the other way. My reading of Briggs's character is that his nature would not have allowed him to rest satisfied with the little he has actually given, but rather he would have applied himself with unsparing effort to master the problem in hand. Recall the tributes to his industry, the thoroughness of his Angular Sections and his Subtabulation with their results generalised far in excess of his practical needs, and ask yourself if/

if this was a man who would shirk the challenge of a problem unsolved. Further, he himself was aware that third differences might on occasion be required, but since the work was entirely new even to the mathematical world, a reader might be excused for thinking that the treatment given was all that there was to give. Is it not safe to assume that it was the recognition of this very danger which made Briggs go to the trouble of pointing out so emphatically that there was something wanting (non nihil deficiet) should the third differences not be negligible? Had he not known the fuller treatment, it would appear to me that he would rather have refrained from drawing attention to its incompleteness, and taken out of it what credit there was in having stated a special case.

But it may be asked, why did he not state the general theory if he were in possession of the facts as he did with Angular Sections and Subtabulation. In answering this it should be remembered that the circumstances were entirely different. In the latter he had years to think out and develop his theory, and we know that he did wait until he had brought them to perfection before deciding to publish them, while in the case in hand he was in the midst of a large sized computation, in which he had not only to do the actual calculations of some 40,000 logarithms, but in some cases he had to invent the methods for doing so. The job in hand was the computation and in the course of this he did not require to go beyond second differences, so that he gave only as much of his theory as was necessary to explain the computation of his logarithms. He probably knew the whole thing, but being fully engaged on his great task, he had not the time to bother about putting his ideas into general form. Even after the publication of the Arithmetica Logarithmica Briggs was kept fully occupied first with further logarithms and then with the Trigonometria Britannica, upon which he was engaged when he died. Had he been spared, and had had the time, I feel sure that we would have heard more about this process of interpolation, of an isolated entry.

Briggs's work in Algebraic Form.

Let us look farther into Briggs's rules, and give his work in algebraic form. Having found his first differences C and second difference D , he multiplies D in turn by 45, 35, 25, 15, 5 and cuts off the last three digits i.e. divides by 1000. That is, he multiplies in turn by $\frac{45}{1000}$, $\frac{35}{1000}$, $\frac{25}{1000}$, $\frac{15}{1000}$, $\frac{5}{1000}$,

i.e. by $\frac{9}{200}$, $\frac{7}{200}$, $\frac{5}{200}$, $\frac{3}{200}$, $\frac{1}{200}$ to get F,G,H,I,K.

He says that the first differences, and therefore by addition the interpolated terms are as given in the following table, though of course he does not give his results in this tabular form.

THE DIFFERENCES	SERIES OF TERMS
	A
$\frac{1}{10}C + \frac{45}{1000}D = \frac{1}{10}C + \frac{9}{200}D$	$A + \frac{1}{10}C + \frac{9}{200}D.$
$\frac{1}{10}C + \frac{35}{1000}D = \frac{1}{10}C + \frac{7}{200}D$	$A + \frac{2}{10}C + \frac{16}{200}D.$
$\frac{1}{10}C + \frac{25}{1000}D = \frac{1}{10}C + \frac{5}{200}D$	$A + \frac{3}{10}C + \frac{21}{200}D.$
$\frac{1}{10}C + \frac{15}{1000}D = \frac{1}{10}C + \frac{3}{200}D$	$A + \frac{4}{10}C + \frac{24}{200}D.$
$\frac{1}{10}C + \frac{5}{1000}D = \frac{1}{10}C + \frac{1}{200}D$	$A + \frac{5}{10}C + \frac{25}{200}D.$
$\frac{1}{10}C - \frac{5}{1000}D = \frac{1}{10}C - \frac{1}{200}D$	$A + \frac{6}{10}C + \frac{24}{200}D.$
$\frac{1}{10}C - \frac{15}{1000}D = \frac{1}{10}C - \frac{3}{200}D$	$A + \frac{7}{10}C + \frac{21}{200}D.$
$\frac{1}{10}C - \frac{25}{1000}D = \frac{1}{10}C - \frac{5}{200}D$	$A + \frac{8}{10}C + \frac{16}{200}D.$
$\frac{1}{10}C - \frac{35}{1000}D = \frac{1}{10}C - \frac{7}{200}D$	$A + \frac{9}{10}C + \frac{9}{200}D.$
$\frac{1}{10}C - \frac{45}{1000}D = \frac{1}{10}C - \frac{9}{200}D.$	$A + C$

It may be easily verified that the series of terms given in this table agree with the formulae

$$u_x = u_0 + x\,\Delta u_0 + \frac{x(x-1)}{\lfloor\underline{2}}\,\Delta^2 u_0$$

where Δ has its modern meaning, or in the central difference notation

$$u_x = u_0 + x\,\delta u_{1/2} + \frac{x(x-1)}{\lfloor\underline{2}}\cdot\delta^2 u_0$$

where $u_0 = A$, $\Delta u_0 = \delta u_{1/2} = C$ and $\Delta^2 u_0 = \delta^2 u_0 = -D.$

and x is taken in turn $= \frac{0}{10}, \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \ldots\ldots\ldots\ldots, \frac{9}{10}, \frac{10}{10}.$

/

$\delta^2 u_0$ in general is the central difference notation for $\Delta^2 u_{-1}$ but since second differences are equal $\Delta^2 u_{-1} = \Delta^2 u_0$.

Did Briggs know the Gregory or the Gauss Interpolation series?

This first result is the modern Gregory Formula to second differences, while the latter is the modern Gauss Formula to second differences, and the question arises whether he knew either of them, for they are both supposed to be of much later origin, or did he, contrary to my previous conclusion, really know the theory only for constant second differences."

The real difficulty in deciding is that the discovery of both lay well within his power, and I could show how to discover both the Gregory and the Gauss Formulae by methods which Briggs frequently used.

Take the Gregory Formula. Hutton + points out the agreement of Briggs's results with the formula, and thereupon gives him the credit of its discovery though failing to show how Briggs did or could have discovered it. I myself held his opinion until assailed by doubts, but did so partly because I could see how Briggs could easily have arrived at it, and partly because I could see that Briggs had reached a stage where it was necessary or at least highly advantageous for him to be able to write down by itself the entry corresponding to some intermediate value of the argument without writing down all the others as well.

Two methods proposed whereby Briggs might have arrived at the Gregory and Gauss Series.

Let me marshal the facts, and recall that, in my opinion, Briggs derived the relation between the coarser differences and the finer differences in an odd section from his Theory of Angular Sections, and from the fact that in the sine difference table, the entries in/

+ Math. Intro. p. 70.

in any row are in continued proportion. Now one has only to peruse Briggs's work to be struck with the fact that he invariably works in terms of continued proportionals. One just simply can't get away from them, and indeed he composed, though did not publish, a treatise on them. +

Again, the theory which supplied him with his facts for an odd section, breaks down when he is confronted with an even one, so that he must try something else, What more natural then, than that, since he solved his problem of subtabulation by considering continued proportionals in a horizontal line, he should now try the effect of continued proportionals in a diagonal line?

Thus he would have for his first entry and leading differences a, ar, ar^2, ar^3, ar^4, ar^5 etc., and his difference table would be as follows:-

ENTRY	Δ	Δ^2	Δ^3	Δ^4	Δ^5	
a						
	ar					
$a(1+r)$		ar^2				
	$ar(1+r)$		ar^3			
$a(1+r)^2$		$ar^2(1+r)$		ar^4		
	$ar(1+r)^2$		$ar^3(1+r)$		ar^5	
$a(1+r)^3$		$ar^2(1+r)^2$		$ar^4(1+r)$		etc.
	$ar(1+r)^3$		$ar^3(1+r)^2$		$ar^5(1+r)$	
$a(1+r)^4$		$ar^2(1+r)^3$		$ar^4(1+r)^2$		
	$ar(1+r)^4$		$ar^3(1+r)^3$		$ar^5(1+r)^2$.	
$a(1+r)^5$.		$ar^2(1+r)^4$		$ar^4(1+r)^3$		

etc.

where starting from the leading differences, by simple addition he can fill up the rest of the table as above, and arrive at the entries. He would be aware therefore of the fact that the orders of/

+ Ward.

of differences keep step with the indices of r , and further that this table leads to $a(1+r)^x$ being the entry corresponding to the xth interval of the argument. Now by expanding $a(1+r)^x$, he would get its value in terms of the continued proportionals of the leading differences, which would give the connection between the entry and the several orders of differences, a connection which might be peculiar to this particular table, but on the other hand might be the general laws of the difference table as was argued in the previous case of subtabulation, and further experiment with functions where the results could easily be verified, would show him whether the relationship were of particular or general application. In this case it would be of general application.

Furthermore the expansion would cause no difficulty to Briggs for his Abacus Panchrestus, discovered long before this, allowed him to expand any power of a Binomial. He would thus get

$$a(1+r)^x = a + \binom{x}{1} ar + \binom{x}{2} ar^2 + \binom{x}{3} ar^3 + \cdots\cdots + \binom{x}{k} ar^k + \cdots\cdots + \binom{x}{x} ar^x$$

which translated into terms of entries and differences gives

$$u_x = u_0 + \binom{x}{1}\Delta u_0 + \binom{x}{2}\Delta^2 u_0 + \binom{x}{3}\Delta^3 u_0 + \cdots\cdots + \binom{x}{k}\Delta^k u_0 + \cdots\cdots + \binom{x}{x}\Delta^x u_0$$

which is Gregory's Formula!

It will be seen that this method corresponds almost exactly with that adopted when the entries in the same row were continued proportionals, the use of which is in accordance with Briggs's usual procedure, and further it contains nothing which would have caused him any difficulty. It must therefore be admitted as at least a probable method of attaining his results, his stopping at the second differences in practice being due merely to the fact that he always kept his work on logarithms within the limits of constant second differences. The only objection I see to this theory is that in practice Briggs does not go beyond second differences, and therefore there is no means of knowing whether he was dealing with central differences as he does in all his other work or with differences along a diagonal as this theory demands, for in constant differences $\Delta^2 u_0 = \delta^2 u_0$.

All/

All this however, is mere conjecture, no matter how attractive the speculation may be. If we stick to the facts as we have them it must be granted that Briggs could have derived all that he gives without knowing any general result, by a very simple knowledge of the difference table, though an extension of the method I am about to propose leads both to the Gregory result and to the Gauss result.

Briggs must have known that in a difference table, the sum of the entries in any column is equal to the difference between the first and last entries of the preceeding column, and further that before the entries in any one column could be the same those in the preceeding column must be in Arithmetical progression. With only this knowledge Briggs could have worked out all that is stated above. Thus,

Let the two given numbers A and A + C be denoted by u_0 and u_{10} so that the interpolated terms may be denoted by $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9$.

Since the second differences are constant

$\delta u_{1/2}, \delta u_{1\frac{1}{2}}, \delta u_{2\frac{1}{2}}, \delta u_{3\frac{1}{2}}, \ldots\ldots\ldots\ldots \delta u_{9\frac{1}{2}}$ are in Arithmetic Progression. Further since the second coarser differences are constant and equal to −D, Briggs knew from his previous work, or could easily have found from first principles that the finer second differences would be constant and equal to $-\frac{D}{10^2}$ i.e. $\frac{-D}{100}$.

Now even to this very day the simplest way of writing 10 numbers in Arithmetical Progression in descending order of magnitude is

$a+9d,\ a+7d,\ a+5d,\ a+3d,\ a+d,\ a-d,\ a-3d,\ a-5d,\ a-7d,\ a-9d.$

where −2d is the common difference, so that the above first differences are of this form.

$$\therefore\ -2d = \frac{-D}{100} \qquad \therefore\ d = \frac{D}{200}.$$

and since

$$\delta u_{\frac{1}{2}} + \delta u_{1\frac{1}{2}} + \delta u_{2\frac{1}{2}} + \delta u_{3\frac{1}{2}} + \ldots\ldots\ldots + \delta u_{9\frac{1}{2}} = u_{10} - u_0$$

$$\therefore\ (a+9d)+(a+7d)+(a+5d)+(a+3d)+(a+d)+(a-d)+(a-3d)+(a-5d)+(a-7d)+(a-9d) = C$$

$$\therefore\ 10a = C \qquad \text{i.e.}\ a = \tfrac{1}{10}C.$$

From this simple treatment

$\delta u_{1/2} = \frac{1}{10}C + \frac{9}{200}D$ $\qquad \delta u_{1\frac{1}{2}} = \frac{1}{10}C + \frac{7}{200}D$ $\qquad \delta u_{2\frac{1}{2}} = \frac{1}{10}C + \frac{5}{200}D$.

$\delta u_{3\frac{1}{2}} = \frac{1}{10}C + \frac{3}{200}D$ $\qquad \delta u_{4\frac{1}{2}} = \frac{1}{10}C + \frac{1}{200}D$ $\qquad \delta u_{5\frac{1}{2}} = \frac{1}{10}C - \frac{1}{200}D$ etc.

exactly as Briggs states.

This treatment would explain much of what appears in the Arithmetica Logarithmica. It would account for his notation of $C+F$, $C+G$, $C+H$, $C+I$, $C+K$, $C-K$, $C-I$, $C-H$, $C-G$, $C-F$ for the first differences, account for why products F,G,H,I,K are added for the first five differences and subtracted for the second five, and where exactly Briggs obtained his table E with its values

To be added					To be subtracted.				
45	35	25	15	5	5	15	25	35	45
1	2	3	4	5	6	7	8	9	10

since instead of multiplying by $\frac{9}{200}$, $\frac{7}{200}$, $\frac{5}{200}$, $\frac{3}{200}$, $\frac{1}{200}$ he prefers the slightly easier multiplication by $\frac{45}{1000}$, $\frac{35}{1000}$, $\frac{25}{1000}$, $\frac{15}{1000}$, $\frac{5}{1000}$.

It is then easy to see how he can get his table for finding a single intermediate entry without the necessity for finding all the others. For, referring back to the table of differences and series of terms, should he wish to find say u_7, which is the actual example he quotes, he knows that $u_7 - u_0 =$ sum of the first differences between these, and this right hand side can be summed at sight.

Thus:-

$$u_1 = u_0 + \tfrac{1}{10}C + \tfrac{45}{1000}D = u_0 + \tfrac{1}{10}C + \tfrac{45}{1000}D.$$

$$u_2 = u_0 + \left(\tfrac{1}{10}+\tfrac{1}{10}\right)C + \left(\tfrac{45}{1000}+\tfrac{35}{1000}\right)D = u_0 + \tfrac{2}{10}C + \tfrac{80}{1000}D.$$

$$u_3 = u_0 + \left(\tfrac{1}{10}+\tfrac{1}{10}+\tfrac{1}{10}\right)C + \left(\tfrac{45}{1000}+\tfrac{35}{1000}+\tfrac{25}{1000}\right)D = u_0 + \tfrac{3}{10}C + \tfrac{105}{1000}D.$$

- -

$$u_x = u_0 + \left(\tfrac{1}{10}+\tfrac{1}{10}+\tfrac{1}{10}+\cdots \text{ to } x \text{ terms}\right).C + \tfrac{D}{1000}\left(45+35+25+\cdots \text{ to } x \text{ terms}\right).$$

Therefore the entries in the table E' are merely got by taking the algebraic sums of those in table E. Thus if

$$S_{10} = 45+35+25+15+5-5-15-25-35-45$$

the entries in E' opposite 1, 2, 3, 4, 5, 6, 7, 8, 9 are S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9

In view of the way in which Briggs writes down his results it seems almost impossible to escape the conclusion that it was this line of reasoning that he took. Even should this be the case, however, I still feel sure that he knew the fuller facts beyond second differences, for this same method need only be extended in order/

order to arrive at first the Gregory Series and then the Gauss Series, and as I have already emphasised Briggs's nature was such that he would not have desisted until the whole truth had been found. Thus forming a difference table for entries u_0, u_1, u_2, u_3, etc. we have, taking an entry u_x,

$$
\begin{array}{rllllll}
u_x - u_0 = & \delta u_{\frac{1}{2}} & + \delta u_{1\frac{1}{2}} & + \delta u_{2\frac{1}{2}} & + \delta u_{3\frac{1}{2}} + \cdots & \cdots\cdots & + \delta u_{x-\frac{1}{2}} \\
= & \delta u_{\frac{1}{2}} & + \delta u_{\frac{1}{2}} & + \delta u_{\frac{1}{2}} & + \delta u_{\frac{1}{2}} + & & + \delta u_{\frac{1}{2}} \\
 & & + \delta^2 u_1 & + \delta^2 u_1 & + \delta^2 u_1 & & + \delta^2 u_1 \\
 & & & + \delta^2 u_2 & + \delta^2 u_2 & & + \delta^2 u_2 \\
 & & & & + \delta^2 u_3 & & + \delta^2 u_3 \\
 & & & & & & \vdots \\
 & & & & & & + \delta^2 u_{x-1}
\end{array}
$$

$$= x\,\delta u_{\frac{1}{2}} + (x-1)\,\delta^2 u_1 + (x-2)\,\delta^2 u_2 + (x-3)\,\delta^2 u_3 + \cdots\cdots + \delta^2 u_{x-1}$$

$$= x\,\delta u_{\frac{1}{2}} + (x-1)\,\delta^2 u_1 + (x-2)\begin{bmatrix} \delta^2 u_1 \\ +\delta^3 u_{1\frac{1}{2}} \end{bmatrix} + (x-3)\begin{bmatrix} \delta^2 u_1 \\ +\delta^3 u_{1\frac{1}{2}} \\ +\delta^3 u_{2\frac{1}{2}} \end{bmatrix} + \cdots\cdots + \begin{bmatrix} \delta^2 u_1 \\ +\delta^3 u_{1\frac{1}{2}} \\ +\delta^3 u_{2\frac{1}{2}} \\ \vdots \\ +\delta^3 u_{x-1\frac{1}{2}} \end{bmatrix}$$

$$= x\,\delta u_{\frac{1}{2}} + \sum_{t=1}^{t=x-1} t.\,\delta^2 u_1 + \sum_{t=1}^{t=x-2} t.\,\delta^3 u_{1\frac{1}{2}} + \sum_{t=1}^{t=x-3} t.\,\delta^3 u_{2\frac{1}{2}} + \cdots\cdots + \sum_{t=1}^{t=1} t.\,\delta^3 u_{x-1\frac{1}{2}}$$

$$= \binom{x}{1}\delta u_{\frac{1}{2}} + \binom{x}{2}\delta^2 u_1 + \binom{x-1}{2}\delta^3 u_{1\frac{1}{2}} + \binom{x-2}{2}\delta^3 u_{2\frac{1}{2}} + \cdots\cdots + \delta^3 u_{x-1\frac{1}{2}}.$$

$$= \binom{x}{1}\delta u_{\frac{1}{2}} + \binom{x}{2}\delta^2 u_1 + \sum_{t=2}^{t=x-1}\binom{t}{2}\delta^3 u_{1\frac{1}{2}} + \sum_{t=2}^{t=x-2}\binom{t}{2}\delta^4 u_2 + \cdots\cdots + \delta^4 u_{x-2}.$$

$$= \binom{x}{1}\delta u_{\frac{1}{2}} + \binom{x}{2}\delta^2 u_1 + \binom{x}{3}\delta^3 u_{1\frac{1}{2}} + \binom{x-1}{3}\delta^4 u_2 + \binom{x-2}{3}\delta^4 u_3 + \cdots + \delta^4 u_{x-2}$$

which by repeated application generates into

$$u_x = u_0 + \binom{x}{1}\delta u_{\frac{1}{2}} + \binom{x}{2}\delta^2 u_1 + \binom{x}{3}\delta^3 u_{1\frac{1}{2}} + \binom{x}{4}\delta^4 u_2 + \binom{x}{5}\delta^5 u_{2\frac{1}{2}} + \cdots + \binom{x}{t}\delta^t u_{t/2} + \cdots + \delta^x u_{x/2}$$

which is Gregory's Series using differences along a diagonal. The same method leads to the series for differences in the same line or as near that as possible, Thus from above

$$\begin{aligned} u_x &= u_0 + \binom{x}{1}\delta u_{\frac{1}{2}} + \binom{x}{2}\delta^2 u_1 + \binom{x}{3}\delta^3 u_{1\frac{1}{2}} + \binom{x}{4}\delta^4 u_2 + \binom{x}{5}\delta^5 u_{2\frac{1}{2}} + \cdots\cdots + \delta^x u_{x/2} \\ &= u_0 + \binom{x}{1}\delta u_{\frac{1}{2}} + \binom{x}{2}\delta^2 u_0 + \binom{x}{3}\delta^3 u_{\frac{1}{2}} + \binom{x}{4}\delta^4 u_1 + \binom{x}{5}\delta^5 u_{1\frac{1}{2}} + \cdots\cdots \\ &\qquad + \binom{x}{2}\delta^3 u_{\frac{1}{2}} + \binom{x}{3}\delta^4 u_1 + \binom{x}{4}\delta^5 u_{1\frac{1}{2}} + \binom{x}{5}\delta^6 u_2 \\ &= u_0 + \binom{x}{1}\delta u_{1/2} + \binom{x}{2}\delta^2 u_0 + \binom{x+1}{3}\delta^3 u_{\frac{1}{2}} + \binom{x+1}{4}\delta^4 u_1 + \binom{x+1}{5}\delta^5 u_{1\frac{1}{2}} + \cdots\cdots \end{aligned}$$

which by the same process repeated gives

$$u_x = u_0 + \binom{x}{1}\delta u_{\frac{1}{2}} + \binom{x}{2}\delta^2 u_0 + \binom{x+1}{3}\delta^3 u_{\frac{1}{2}} + \binom{x+1}{4}\delta^4 u_0 + \binom{x+2}{5}\delta^5 u_{\frac{1}{2}} + \cdots\cdots$$

which is Gauss's Series.

All the coefficients are binomial coefficients and well known to Briggs, and although the repeated working makes it look lengthy the principles are very easy to see, and Briggs once on the track, and he decidedly was, is almost bound to have seen them. Getting on the proper track would be the main difficulty, not the carrying on once that had been achieved.

Let it be repeated again therefore, that although it cannot be definitely established, it is highly probable that Briggs was in possession of a far fuller treatment than he gives, and while it would be going too far to credit him with the discovery of either the Gregory or the Gauss Series, there is a very strong suspicion that he was in possession of the facts of the one or the other or both, some considerable time before they were discovered by the men by whose names they are now known.

It has been stated earlier that in dealing with logarithms Briggs required a method which would give him the logarithm corres-p onding to a given entry without finding those of the unbroken series. This was due to the fact that he formed his tables of logarithms chiefly by calculating the logarithms of the smaller prime numbers and then formed by addition and subtraction the logarithms of all numbers which could be formed from the multiplication or division of any combination of these primes.

Briggs/

Briggs in Chapter IX starts, "After finding the logarithms of the numbers 1, 2, 3, 5, 10 and by axioms 1 & 2 Chapter 2 of all those which come from the multiplication or division of these among themselves, it remains for us to seek the logarithms of the remaining primes" and proceeds to show an easy method of getting the logarithms of all the primes up to 100. The only logarithms therefore which would cause him any difficulty would be those of prime numbers above 100, and it is in the calculation of these isolated values that a method of interpolation would be invaluable, and it was for this purpose that Briggs used his rules laid down in dealing with decisection, where the problem of the isolated value was discussed. There can be scarcely any doubt whatever that this was what he did, for it is very significant that the number (96157) whose logarithm he found to illustrate his method for the interpolation of an isolated entry is a prime number!

CHAPTER VI.

The Trigonometria Britannica. (contd)

Complaint against Vlacq.

But to return to the Trigonometria Britannica which we left at Chapter XII where Briggs describes his process of quinquesection. The rest of the Chapter is taken up with a large specimen in full detail of the application of quinquesection, but we cannot leave it without commenting upon one passage appearing in it in the form of a marginal note, which is of more than passing interest. Vlacq, it may be remembered, forestalled Briggs in the publication of the chiliads between 20,000 and 90,000 and while calling his work a second edition of Briggs's Arithmetica Logarithmica he makes certain omissions which Briggs resents, and in this marginal note he deals with his reasons for repeating his rules of quinquesection which he/

he had already given so fully in his Arithmetica Logarithmica of 1624. He writes - "The method of correction has been recorded by me in the London Edition Chapter 13 of the Arithmetica Logarithmica. That chapter, however, and the preceding one have been omitted from the Batavian Edition, unknown to me and without my being consulted. Nor does the author of that edition, a man industrious and in other respects not unlearned, seem to have understood my mind in all things. On that account, lest anything should be missing for any-one who may wish to complete the whole canon, I have considered that certain of the most essential facts required to be transferred from there to here."

It appears to me from his complaint in this passage thatBriggs felt himself entitled to the credit of the invention of interpolation by successive differences, and was afraid of an intention to de-prive him of that honour.

Method of Construction of Canon of Sines.

He then proceeds to show how his process of quinquesection is to be applied to the construction of the whole canon of sines both to 100ths and 1000 ths of a degree i.e. to centesms and millesms. For centesms the quadrant is first divided into 72 equal parts and their sines calculated by the primary methods stated. These are the basic sines, for by quinquesection he gets 360 parts, a second quinquesection giving 1800 parts and a third 9000 parts, or centesms of a degree. For millesms he starts with the quadrant divided into 144 equal parts, which by four quinquesections gives 90,000 parts or millesms.

Note on Decimal Division.

It was remarked upon that Briggs was very keen to adopt the decimal division of the degree and that he adopted it in his Tables, so that it is a matter of interest to realise that there were others who wished to be even more revolutionary. They wanted not only/

only the degree but the whole circumference to be divided thus, and also to adjust to these divisions all canons of sines, tangents, secants, and their logarithms, together with tables of mean motions and Prosthaphaereses, and for their sakes he adds a small table of sines for the quadrant divided into 40 equal parts, whose number may be increased by quinquesection to 200, 1000, 5000 and lastly to 25,000.

He then goes on to the canons of tangents and secants, to be constructed in the same way as for the sines namely by interpolation from a few primary ones, which would be calculated from the known relationships between sines, tangents and secants.

Construction of Logarithmic Canons.

The other two chapters of the first Book, i.e. the one due to Briggs himself are concerned with the logarithms of the sines, tangents and secants, and these two chapters of course must be of a later date than the others. The most important of the three canons is that for the sines for once it has been formed, those for the tangents and secants are derived from it, using the relationships that $\tan\theta = \frac{\sin\theta}{\sin(90°-\theta)}$ and $\sec\theta = \frac{1}{\sin(90°-\theta)}$. The method of procedure is very simple whether it be required for intervals of one centesm or one millesm, for it follows exactly that of constructing the canons of sines for these divisions. It may be remembered that the quadrant was divided into 72 equal parts for the one case and 144 for the other, for which primary parts the sines were calculated by primary methods, and then the whole canon constructed by repeated quinquesections. In the case of the logarithmic canons, the logarithms of the above primary parts are calculated by the Radix Method, to be described later in dealing with the Arithmetica Logarithmica, and then the whole canon completed by repeated quinquesections as above. Briggs also adds that since the inequality of the differences of the logarithms of the sines at the beginning of the quadrant may be unusually irregular,/

irregular, the method of quinquesection could scarcely produce accurate logarithms for the first degree. This defect therefore should be compensated for in another way. He further remarks that the logarithmic sines of only half the quadrant need be found in this manner as the other half may be found by mere addition or subtraction, by means of the relationship which, given in modern notation is $\sin\theta = \dfrac{\sin 2\theta}{2\sin(90^\circ-\theta)}$.

CHAPTER VII.

Logarithms.

All that has been w ritten so far is quite independent of logarithms, and it shows such originality of thought and treatment that had Briggs done nothing else, he had yet accomplished enough to merit a high place in the history of mathematics. He is, however, remembered not so much for the above as he deserved to be, but rather for the association of his name with that system of logarithms which is now called common or Briggian logarithms. Important as is his work in logarithms, to one who knows how much he had already accomplished in the field of original research before 1614 there is more than a touch of irony in the undoubted fact that his memory is so largely logarithmic that this one part of his work overshadows everything else he ever did, and even calls forth from Mark Napier + the gibe that he is a mere computer, a satellite of Napier. To me his achievments in logarithms are considerable, but not to be taken as his most notable contribution to mathematics.

I give the facts, the reader can judge for himself.

There can be little doubt that the discovery of logarithms changed the whole tenor of Briggs's life, for as soon as Napier's "Descriptio" came into his hands, he devoted himself almost entirely to logarithms. Because of the large part they played in Briggs's life/

\+ Mark Napier.

life after 1614 I have taken considerable pains to gather all the information I could concerning their early history, but after culling as much as possible from all the sources I have been able to lay my hands on, I find that there is little to add to the admirable account given by J. W. L. Glaisher in the eleventh edition of the Encyclopedia Britannica from which I shall freely quote, because all my researches merely go to verify and confirm the facts given there.

Attempts at Shorter Methods. - Prosthaphaersis.

Towards the end of the 16th century the improvement in Trigonometrical Tables gave to the astronomer a means of accurate calculation which his predecessors had never dreamed possible, but unfortunately the means of actual computation had not made corresponding progress. The greater accuracy in the tables resulted in calculations becoming so laborious and tedious that shorter and more commodious methods became an absolute necessity. The first attempt at this is found in the process called "prosthaphaersis", which consists of the use of the formula $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$, by means of which the multiplication of two sines is reduced to the addition or subtraction of two tabular results taken from a table of Sines. The method seems to be due to Wittich of Breslau, who was assistant for a short time to Tycho Brahe, and it was used by them in their calculations in 1582. It is evident however, that this could not be a good method of practically effecting multiplications unless the quantities to be multiplied were sines on account of the labour of the interpolations although it satisfies the condition equally with logarithms of enabling multiplication to be performed by the aid of a table of single entry.

Early History of Logarithms. Probable Date of Invention.

There is evidence that Napier deliberately set himself the task of devising shorter methods of calculation and that his logarithms/

logarithms were the product of many years of study and thought. His logarithms were not the only product of this attempt, but they are by far the most important. Sir John Leslie in his "Philosophy of Arithmetic" writes +:- "It will be admitted that artificial helps may prove useful in laborious and protracted multiplication by sparing the exercise of memory and preventing the attention from being overstrained. Of this description are the Rods or Bones which we owe to the early studies of the great Napier whose life, devoted to the improvement of the science of calculation, was crowned by the invention of logarithms, the noblest conquest ever achieved by man." And the invention of logarithms has been accorded to John Napier, Fear of Merchiston with a unanimity which is rare with regard to important scientific discoveries. These are usually not so much individual conquests but rather the grand result of a succession of victories under separate leaders. The invention of logarithms presents a different aspect, for with the exception of the tables of Justus Byrgius, there seems to have been no other mathematician of the time whose mind had conceived the principle on which logarithms depend, and no partial anticipations of the discovery are met with in previous writers. They were the result of an unaided isolated speculation and unlooked for when they appeared.

There is indirect evidence++ that Napier was occupied with logarithms some twenty years before his other avocations added to the labour of the computation, and his own diffidence suffered him to give them to the world. In a letter to P. Crügerus from Kepler, dated 9th September 1624 it is distinctly stated that some Scotsman in 1594, in a letter to Tycho Brahe gave him some hope of the logarithms, and although Kepler probably never saw this letter, of which no trace has been found in spite of search, he had been so closely associated with Tycho Brahe in his work that he would be likely to be correct in any assertion of this kind. In connection/

+ Mark Napier. ++ Mark Napier.

connection with Kepler's statement, the story told by Anthony Wood in the Athenae Oxonienses is of some importance.

"It must be now known that one, Dr. Craig, a Scotchman .... coming out of Denmark into his own country, called upon John Naper, Baron of Merchiston, near Edinburgh, and told him among other discoveries of a new invention in Denmark (by Longomontanus as 'tis said) to save the tedious multiplication and division in astronomical calculations. Naper being solicitous to know farther of him concerning this matter he could give no other account of it than that it was by proportional numbers. Which hint Naper taking, he desired him at his return to call upon him again. Craig after some weeks had passed, did so and Naper then showed him a rude draught of what he called Canon Mirabiles Logarithmorum. Which draught, with some alteration, he printing in 1614, it came forthwith into the hands of our author Briggs and into those of Will Oughtred, from whom the relation of this matter came."

Mark Napier has shown that this account is false, and it should be pointed out first that there is no reference whatever to it in the works of Oughtred and secondly that Longomontanus made no claim to the invention of logarithms although he lived for some thirty years after their first publication. The new invention in Denmark is in all probability the process we have called "Prosthaphaersis". The story, however, gives valuable information by connecting Dr. Craig with Napier and Longomontanus who was Tycho Brahe's assistant. This Dr. Craig was John Craig, third son of Sir Thomas Craig of Riccarton, who was one of the colleagues of Napier's father, in the office of Justice-Depute. Betwixt John Craig and John Napier a friendship grew up of which the source is not to be doubted for he was an excellent mathematician. In a letter dated 2nd September 1608, John Napier writes to his eldest son Archibold who was serving the King - "Ye sall make my commendatiouns to Doctor Craig ...... "

There are extant three letters from Dr. Craig to Tycho Brahe, Which/

which prove that he was on the most friendly and confidential footing with him. The first of these letters commences - "About the beginning of last winter that magnificent man Sir William Stuart delivered to me your letter and the book you sent." Now Mark Napier found a mathematical work of Tycho Brahe's in the Library of Edinburgh University which bears upon the first blank leaf a manuscript sentence in Latin to the following effect.

"To Doctor John Craig of Edinburgh in Scotland, a most illustrious man and highly gifted with varied and excellent learning, Professor of Medicine and exceedingly skilled in Mathematics Tycho Brahe hath sent this gift and with his own hand hath written this at Uraniburg 2nd November 1588."

As Sir William Stuart was sent to Denmark to arrange the preliminaries of King James's marriage and returned to Edinburgh on 15th November 1588, it would seem highly probable that this was the volume referred to by Craig. In any case it is certain that Craig was a friend and correspondent of Tycho Brahe's, and it is probable that he was the "some Scotsman" Kepler refers to. It may be inferred therefore that as early as 1594 Napier had communicated to some one, probably John Craig, his hope of being able to effect a simplification in the processes of Arithmetic.

Everything then tends to show that the invention of logarithms was the result of many years of labour and thought, undertaken with this special object and it would seem that Napier had seen some prospect of success nearly twenty years before the publication of the Descriptio. Mark Napier has successfully shown, I think, that no mere hint with regard to the use of proportional numbers could have been of any service to him, but it is possible, and I think probable, that the news brought by Craig of the difficulties placed in the progress of astronomy by the labour of its calculations may have stimulated him to persevere in his efforts.

The Descriptio Published.

In/

The Descriptio Published.

In 1614 appeared the work, which, Glaisher says, + in the history of British science can be placed as second only to Newton's Principia, and Napier in the preface writes -

"Seeing that there is nothing (right well beloved students in the mathematics) that is so troublesome to mathematical practice nor that doth more molest and hinder calculations, than the multiplications, divisions, square and cubical extractions of great numbers, which besides the tedious expense of time are the most part subject to many slippery errors. I began therefore to consider in my mind by what certain and ready art I might remove these hindrances, and having thought upon many things to this purpose I found at length some excellent brief rules to be treated of perhaps hereafter. But amongst all none more profitable than this, which together with the hard and tedious multiplications, divisions, and extraction of roots, doth also cast away even the very numbers themselves that are to be multiplied, divided and resolved into roots, and putteth other numbers in their place which perform as much as they can do only by addition and subtraction, division by two or division by three, which secret invention being (as all other good things are) so much the better as it shall be the more common, I thought good heretofore to set forth in Latin for the public use of Mathematicians." Immediately the Descriptio was published it attracted the attention of the two most eminent English Mathematicians then living, Edward Wright and Henry Briggs. The former, so well known for his work in navigation, at once saw the value of logarithms as an aid to that science and lost no time in preparing a translation which he submitted to Napier himself for his approval. He, having carefully perused it, at once gave it his approbation, but before returning it to Wright added to his preface as follows:-

"But now some of our countrymen in this island, well affected to these studies, and the more public good, procured a most learned mathematician to translate the same into our vulgar English tongue, who/

+ Encyc. Brit. vol. 19 p. 172.

who after he had finished it, sent a copy of it to me, to be seen and considered on by myself. I, having most willingly and gladly done the same, find it to be most exact and precisely conformable to my mind and the original. Therefore it may please you who are inclined to these studies, to receive it from me and the translator with as much good will as we recommend it unto you." He further inserted in this translation a very important note which was not in the original, a note to which we shall require to refer later, and which reads -

"But because the addition and subtraction of these former numbers may seem somewhat painfull, I intend (if it shall please God) in a second edition, to set out such logarithms as shall make those numbers above written to fall upon decimal numbers 100,000,000, 200,000,000, 300,000,000 etc. which are easie to be added or abated to or from any number."

Shortly after the return of the translation Wright died (1615) and the publication was left in the hands of his son Samuel Wright and Henry Briggs, who wrote a short "Preface to the Reader."

As has been pointed out already Henry Briggs welcomed the Descriptio with something more than enthusiasm. We have Briggs's own statements in his letter to Ussher, and Dr. Thomas Smith describes the ardour with which Briggs studied the book in the following terms:- "He regarded this with pleasure, carried it in his bosom, his hands, and his heart : he read it again and again with the very greediest of eyes and the most attentive of minds."

The Invention of Decimal Logarithms.

Briggs saw at once that the logarithms could be improved, and in his lectures to his students at Gresham College he proposed the change which was to revolutionise logarithms, namely, that instead of the ratio of 10 to 1 being 2·3025851 as Napier had it, it would be far more convenient to have it 1, so that he is the first to enunciate the value of such a scale, Not only did he communicate it/

it to the public in his lectures, but it was the subject of a communication from him to Napier himself,+ whom he visited in 1615 and again in 1616. On the first visit the change of system was discussed, and Napier said that he had already thought of the change and pointed out a further improvement, viz. that the characteristics of numbers greater than unity should be positive and not negative as suggested by Briggs. In view of the controversy regarding the invention of decimal logarithms, i.e. the new kind proposed, we must carefully peruse what evidence there is, and refer in more detail to the invention itself, examining the claims of Napier and Briggs to the capital improvement involved in the change from Napier's original logarithms to decimal logarithms.

The Descriptio contained only an explanation of the use of the logarithms without any account of the manner in which the canon was constructed. In an "Admonitio" on the seventh page Napier states that although in that place the mode of construction should have been explained, he proceeds at once to the use of logarithms, preferring to await the judgment and censure of the learned before giving the construction. Then in another "Admonitio" on the last page of the tables of some copies of the Descriptio he writes - "Seeing that the calculation of this table which ought to have been perfected by the labour and pains of many calculators has been finished by the operation and industry of one alone, it is not surprising if many errors have crept into them. I beseech you, benevolent readers, pardon these whether caused by the weariness of computation or an oversight of the press. As for me declining health and weightier matters have prevented my adding the last finish. But if I shall understand that the use of this invention proves acceptable to the learned, I will perhaps, shortly give (God willing) the philosophy and method either of amending this canon or of constructing a new one upon a better plan, so that through the diligence of many calculators, a canon more highly finished and accurate than the work of a single individual could effect may at length see the light. Nothing is perfect at its birth."

+ Arith. Log.

Napier did not live to carry out his intention, but before his death in 1617 he published a small work entitled Rabdologia in which he gives the chronological order of his inventions in a dedicatory epistle to the Earl of Dunfermline, High Chancellor of Scotland, which I quote because of its contents.

"The difficulty and prolixity of calculation, the weariness of which is so apt to deter from the study of Mathematics, I have always with what powers and little genius I possess laboured to eradicate, and with that end in view I published of late years the canon of logarithms wrought out by myself a long time ago, which casting aside the natural numbers, and the more difficult operations performed by them, substitutes in their place others affording the same results by means of easy addition, subtractions, bisections and trisections. Of which logarithms indeed I have now found out another species much superior to the former, and intend, if God shall grant me longer life and the possession of health, to make known the method of constructing, as well as the manner of using them. But the actual computation of the new canon I have left, on account of the infirmity of my bodily health to those versant in such studies, and especially to that truly most learned man Henry Briggs, public professor of Geometry in London, my most beloved friend. In the mean time however, for the sake of those who prefer to work with the natural numbers as they stand, I have excogitated three other compendious modes of which the first is by means of numerating rods and these I have called Rabdologia...... "

He states he publishes them lest some one else does so, since their use is becoming fairly common.

This dedication clearly discloses Napier's intentions, but he not living to finish what he had undertaken in relation to the new logarithms, the whole devolved on Briggs. After Napier's death, Briggs prepared his Constructio for publication, but did not stop there, making several additions as will appear from the contents themselves among which appear -

Lucubrationes/

Lucubrationes aliquot doctissimi D. Henrici Briggii in Appendicem praemissam.
Annotationes aliquot doctissimi D. Henrici Briggii in propositiones praemissas.

The Constructio was published by Napier's son Robert in 1619, who pays tribute to Briggs in a letter prefacing the volume.

"....... You have then (benevolent reader) the doctrine of the construction of Logarithms, which here he calls artificial numbers, for he had this treatise beside him composed for several years before he invented the word Logarithms - most copiously unfolded and their nature, accidences and various adaptations to their natural numbers perspicuously demonstrated. I have also thought good to subjoin to the construction itself a certain appendix concerning the method of forming another and more excellent species of Logarithms to which the <u>inventor</u> himself alludes in his epistle prefixed to the Rabdologia, and in which the logarithm of unity is 0. The treatise which comes last is that which tending to the utmost perfection of his logarithmic trigonometry was the fruit of his latest toil, namely, certain very remarkable propositions for resolving spherical triangles without the necessity of dividing them into quadrantal or rectangular triangles, and which are absolutely general. These indeed he intended to have reduced to order and to have successfully demonstrated had not death snatched him from us too soon. I have also published some lucubrations upon these propositions and upon the new species of logarithms by that most excellent mathematician Henry Briggs, public professor in London, who undertook most willingly the very severe labour of calculating this canon, in consequence of the singular affection that existed between him and my father of illustrious memory - the method and construction and explanation of its use being left to the inventor himself. But now since he has been called from this life the whole burden of the business seems to have fallen on the shoulders of the most learned Briggs as if it were his peculiar destiny to adorn this Sparta."

It/

It will be seen from these accounts therefore that Napier claimed for himself the invention of decimal logarithms and that his son Robert above repeats the same claim.

The Hutton - Mark Napier Controversy.

It has been necessary to relate the above because unfortunately there is not the same unity of opinion over the invention of decimal logarithms as there was over the first invention of logarithms. Indeed Wingate, who was the first to bring out a publication of Briggian Logarithms on the Continent (Paris 1625), in a small tract published in 1626 attributes the invention of decimal logarithms solely to Briggs, writing + "John Napier, Baron of Merchiston in Scotland hath due right to challenge the first invention of logarithms in general. Then to Master Henry Briggs, professor of Geometry in the University of Oxford is duly attributed the invention and fabricke of that kind of logarithms which are far more expedite than those of Master Napier's invention." Moreover, Charles Hutton ++ not only champions the claim of Briggs, but in doing so accuses Napier of deliberately ignoring the part played by Briggs, and of keeping silent to secure for himself an honour which he knew should have gone elsewhere. Hutton bases his accusation on his interpretation of a passage in the preface to Briggs's Logarithmorum Chilias Prima, which he published in 1617, and which he translates - "Why these logarithms differ from those set forth by their illustrious inventor of ever respectful memory, in his Canon Mirificus, <u>it is to be hoped</u> his posthumous work will shortly make appear." Hutton reads into this a complaint on the part of Briggs at there being no acknowledgement of the share he had played in the alteration.

Now I think there can be little doubt that Hutton has interpreted all Briggs's statements with regard to the invention of decimal logarithms in a manner clearly contrary to their meaning and entirely unfair to Napier. This naturally has led to retaliation on the part of the supporters of Napier, and indeed Mark Napier, who wrote the/

+ Ward ++ Math. Intro. pp. 34-5

the "Memoirs of John Napier", not only successfully refutes Hutton but he himself falls into the opposite extreme of abusing Briggs (and Wright), and trying to reduce him to the level of a mere computer. In his attempt to grab the honour and the glory for his illustrious ancestor he overlooks all the inventive genius of Briggs. He calls both Briggs and Wright satellites of Napier and in his view of Briggs's capacities he would rank with that inferior class of scientific men who possess power sufficient to act upon principles already discovered but have not themselves the intellectual resources for establishing original principles. Nothing could be more unfair to Briggs and it seems to me that Mark Napier either never studied Briggs's work and was therefore in no position to judge or if he did, he was so blinded by prejudice that he was most unwilling to concede merit even where honour was due. It is difficult to arrive at a correct estimate of the facts, but there is no reason to doubt the statements of either Napier or Briggs in relationship to the matter. I have already given the statements from the Napier point of view, and therefore it may be proper to hear in what manner Briggs himself relates this matter which he does in the address to the reader in his Arithmetica Logarithmica.

"That these logarithms are different from those which that celebrated man Baron Merchiston gave in his Canon Mirificus is not to be wondered at. For I, when I set forth the doctrine of these to my listeners in London, publicly, at Gresham College, considered that it would be much more convenient if the logarithm of the whole sine were kept 0 (as in the Canon Mirifici), but that the logarithm of the tenth part of that whole sine namely $5^\circ\ 44'\ 21''$ be 10000000000. Indeed I wrote immediately to the author himself about it, and as soon as the time of the year and my vacation from my public office of teaching allowed, I proceeded to Edinburgh, where being most kindly received by him, I resided for a whole month. But when the discussion on the change of these things took place between us he said that he himself had felt and desired the same thing for some little time but nevertheless had taken the

trouble/

trouble to publish those which he had already prepared, until he might prepare others more suitable should time and health allow. He was of the opinion however that the change should be made in this way : that 0 should be the logarithm of unity and 10000000000 that of the whole sine, which I could not but own to be by far the most convenient. Therefore on his exhortation I began to think in earnest of the calculation of the latter, having rejected all those which I had previously prepared, and in the following Summer I set out for Edinburgh a second time, to show him those particular ones which I show here. And gladly would I have done the same for a third summer, if God had willed to spare him to us for so long."

These extracts contain all the original statements made by Napier, Robert Napier and Briggs which have reference to the origin of decimal logarithms. It seems rather strange that the relations of Napier and Briggs should have formed the matter for controversy, since, as will have been seen, their statements are all in perfect agreement. To the very end there existed the very warmest friendship between Napier and Briggs, and their intentions with regard to the construction and publication of decimal logarithms show that they worked in perfect harmony, and all the writings of Briggs show the greatest admiration and reverence for Napier. No doubt the invention of decimal logarithms occurred both to Napier and Briggs independently, but in my opinion the greater credit must go to Briggs, for not only did he first announce the change, but he actually undertook and completed tables of the new logarithms. I cannot agree with Hutton that Briggs felt aggrieved in the matter : all his subsequent conduct is contrary to such an assumption. His own account written seven years after Napier's death shows absolutely no trace of injured feeling but rather a sense of satisfaction at having been of service, and he even goes out of his way to explain that he abandoned his own proposed alteration in favour of Napier's on recognising that the latter's was the more convenient, + rejecting the tables that he had already calculated, and beginning others/

\+ Arith. Log. (preface).

others of the new species. Further, on Napier's death he left some of his manuscripts to be sent to Briggs, and Briggs as before stated went to a great deal of trouble in assisting Robert Napier to edit the "Constructio" in 1619. Indeed in the Arithmetica Logarithmica he recounts that "These numbers were first invented by the most excellent John Neper, Baron of Merchiston, and the same were transformed, and the foundations and use of them illustrated with his approbation by Henry Briggs," and later in the account of the change previously quoted, he seems more anxious that justice should be done to Napier than to himself. Surely not the attitude of a man aggrieved! On the other hand Napier received Briggs most hospitably, and refers to him as "amico mihi lohge charissimo".

It would appear to be nearer the truth to say that each recognised the worth of the other, and finding a common interest in their desire to improve this new means of calculation, with no thought of individual claims of discovery they settled down, not as rivals for the honour of invention but as willing workers only too ready to co-operate, and to pool their ideas and resources in a combined effort to attain their object. That Briggs contributed the more was rather the accident of circumstances than deliberate intention.

CHAPTER VIII.

The Arithmetica Logarithmica.

Now to the Arithmetica Logarithmica itself:-

Following the lead of Napier, Briggs dedicates his work to Charles, Prince of Wales, and it is interesting to note just how highly Briggs appraised logarithms, for he says that "as mathematical arts hold chief place in the arts, so in them the method of logarithms excels all others, whether we have regard to shrewdness of invention or excellence of use." The part of the preface concerning/

concerning the change from Naperian to decimal logarithms has already been quoted, but he further states that he now understands the method of differences much better than when he first used them over twenty years before (i.e. circa 1600.)

Definition and Properties of Logarithms.

In the first Chapter he defines Logarithms in general and illustrates some properties of them. In the second he remarks that out of all the possible systems that which has the logarithm of unity 0 is the most convenient, and by far the easiest of application. He further points out that the logarithms of all numbers are either those numbers called indices and which are accustomed to be associated in the minds of arithmetical writers with numbers in continued proportion from unity, and which show the distance of the latter from unity, or are proportional to these indices. This is of considerable interest in showing that Briggs was near in thought to the modern basis of logarithms, and shows a distinct change from the original idea of a logarithm given by Napier, where he defines the logarithm of a sine in terms of moving points. It draws attention to that correspondence between an arithmetical series and a geometrical series, which had been known and freely treated of by previous writers as far back as Archimedes, and yet had never suggested the invention of logarithms to any one of them. On the basis of 0 being the logarithm of unity he exemplifies the laws of logarithms for a product and a quotient. Then having settled the logarithm of unity, to fix his system the next point is to endeavour to find some other number and assign to it some suitable logarithm. Upon this choice depends the kind of logarithms we are dealing with. Napier by his choice unwittingly made his system depend upon the number we now denote by 'e', though of course of this he had no knowledge, as neither Briggs nor Napier had any thought or idea of a base. All that they knew was that having assigned to two nmubers suitable logarithms the others were no longer a matter of choice, but were determined from the two chosen,/

POWERS OF 2.	INDICES.	NUMBER OF DIGITS OR LOGARITHMS.
2	1	1
4	2	1
16	4	2
256	8	3
1024	10	4
10486	20	7
10995	40	13
12089	80	25
12676	100	31
16069	200	62
25823	400	121
66680	800	241
10715	1000	302
11481	2000	603
13182	4000	1205
14377	8000	2409
19950	10000	3011
39803	20000	6021
15843	40000	12042
25099	80000	24083
99900	100000	30103
99801	200000	60206
99601	400000	120412
99204	800000	240824
99006	1000000	301030
98023	2000000	602060
96085	4000000	1204120
92323	8000000	2408240
90498	10000000	3010300
81899	20000000	6020600
67075	40000000	12041200
44990	80000000	24082400
36846	100000000	30103000
13577	200000000	60206000
18433	400000000	120411999
33977	800000000	240823997
46129	1000000000	301029996.

chosen, and as stated Napier by his choice made his logarithm to a base depending on 'e', while Briggs made his to the base 10. Of the remaining logarithms required some are rational i.e. exact, and some are irrational i.e. approximate but can be found to any degree of accuracy demanded. Briggs then defines what he means by the characteristic of a logarithm, the name being due to him, and he establishes that in his system numbers of the same digit sequence have logarithms differing only in the characteristic. Having defined logarithms in general and Briggian logarithms in particular he passes to the actual construction of the latter, which he says [+] can be found chiefly from two methods recorded by their famous inventor Baron Merchiston in an appendix which he added to his book on the construction of his canon of logarithms. It should be carefully noted that Briggs ascribes their invention to Napier, but I think there can be little doubt that these methods were the joint work of Napier and Briggs, and the direct outcome of their discussions on the new system of logarithms.

First Method Of Calculating Logarithms.

To illustrate the methods he enunciates five propositions on continued proportionals. His first method makes use of the fact that in decimal logarithms the characteristic is always one less than the number of digits in the integral part of the number concerned. Thus to find log. 2, by convenient stages he raises 2 to the power whose index is the number giving the logarithm of 10, and he calculates, as on the opposite page, that this power is a number which has 301029996 digits, whence he concludes that $301029995 < \log. 2 < 301029996$, the process being continued to any required degree of accuracy.

Method of Continued Means.

The second general method of finding the logarithms of prime numbers is the method of "continued means."

It was shown that the logarithms of all numbers in the same series/

+ Arith. Log. Ch. 5

series of continued proportionals as unity and ten are rational, and that they can be found very easily by the rule of local position. Therefore are sought out a large number of continued means between 10 and unity, first by finding the square root of ten, the square root of the square root, and so on until the process has been repeated fifty-four times. The logarithms of these means are placed opposite them and are got from that of 10 by repeated bisection. He notices that after a certain time the significant digits of the means are the halves of those of the previous mean, just as their logarithms are halves of the previous logarithms, and therefore concludes what within the limits in which this happens, the logarithms of numbers are proportional to the significant digits of the numbers themselves. The last of the means was 1·000000000000000127819..... and its logarithm found by bisection 0·000000000000000055511....., from which by the proportion $\frac{127819\cdots}{1} = \frac{55511\cdots}{x}$ he discovers the logarithm of 1·00000,00000, 00000,1 to be 0·0000000000000000043429....., and therefore if a number is composed of unity together with significant figures placed after 15 ciphers the logarithm of that number is 0·0000000000000000 43429..... multiplied by the significant figures above mentioned.

But as the extraction of such roots for any number is very tedious Briggs with his usual fertility of mind devised some ingenious contrivances to lighten that labour. He proceeds thus:-

By continued multiplication of the number by itself or powers of itself, find a power of the number which begins with unity followed by one or more zeros, e.g. in the case of 2, the tenth power is 1024. Divide this by the appropriate power of 10 to make it 1·024. Then by the method previously stated find the logarithm of 1·024, 47 means being required to bring it within the limits required. The logarithm of this last mean is found by simple proportion, and from it that of 1·024 either by doubling it 47 times or multiplying it by 2^{47}. Then from log 1·024 is found log 1024 and hence log 2.

The/

	1	00776	96					
1.	1	00387	72833	36962	455663	84655	1	
2.	1	00193	64661	36946	61675	87022	9	
3.	1	00096	79146	39099	01728	89072	0	
4.	1	00048	38402	68846	62985	49253	5	A.
5	1	00024	18908	78824	68563	80872	7	A
		24	19201	34423	31492	74626	7	$\frac{1}{2}$A
			292	55598	62928	93754	0	B
6	1	00012	09381	26397	13459	43919	4	A.
		12	09454	39412	34281	90436	3	$\frac{1}{2}$A
			73	13015	20822	46516	9	B
			73	13899	65732	23438	5	$\frac{1}{4}$B
				884	44909	76921	5	C
7.	1	00006	04672	35055	30968	01600	5	A
		6	04690	63198	56729	71959	7	$\frac{1}{2}$A
			18	28143	25761	70359	2	B
			18	28253	80205	61629	2	$\frac{1}{4}$B
				110	54443	91270	0	C
				110	55613	72115	2	$\frac{1}{8}$C
					1169	80845	2	D.
8	1	00003	02331	60505	65775	96479	4	A.
		3	02336	17527	65484	00800	2	$\frac{1}{2}$A
			4	57021	99708	04320	8	B
			4	57035	81440	42589	8	$\frac{1}{4}$B
				13	81732	38269	0	C
				13	81805	48908	7	$\frac{1}{8}$C
					73	10639	7	D
					73	11302	8	$\frac{1}{16}$D
						663	1	E.
9	1	00001	51164	65999	05672	95048	8	A
		1.	51165	80252	82887	98239	7	$\frac{1}{2}$A
			1	14253	77215	03190	9	B
Hitherto the			1	14255	49927	01080	2	$\frac{1}{4}$B
smaller differences are				1	72711	97889	3	C
found by subtracting				1	72716	54783	6	$\frac{1}{8}$C
the larger from the parts of					4	56894	3	D
the like preceding ones.					4	56915	0	$\frac{1}{16}$D
						20	7	E
						20	7	$\frac{1}{32}$E.
Here the greater differences remain							65	$\frac{1}{32}$E
after subtracting the smaller from						28555	89	$\frac{1}{16}$D
the parts of the difference of the next						28555	24	D
preceding number.					21588	99736	16	$\frac{1}{8}$C
					21588	71180	92	C
				28563	44303	75797	72	$\frac{1}{4}$B.
				28563	22715	04616	80	B
			75582	32999	52836	47524	40	$\frac{1}{2}$A
10.	1	00000	75582	04436	30121	42907	60	A.
							2	$\frac{1}{32}$E
						1784	70	$\frac{1}{16}$D
						1784	68	D.
					2698	58897	62	$\frac{1}{8}$C
					2698	57112	94	C
				7140	80678	76154	20	$\frac{1}{4}$B
				7140	77980	19041	26	B
			37791	02218	15060	71453	80	$\frac{1}{2}$A
11	1	00000	37790	95077	37080	52412	54	A.

The real difficulty of this continued mean method lies in the labour of calculating the respective means, and while the method stated above does decrease the work somewhat, Briggs himself devised the method described in Chapter VIII in which the irksomeness of the work is very much diminished by the method of Differences. This method is entirely due to Briggs and is but one further instance of that inventiveness and originality which he brought to all his work and which has never been adequately recognised.

CHAPTER IX.

A Difference Method For Calculating Means

Briggs's description of the method.

He illustrates the method by the very means which he obtained by the straightforward extraction of roots in finding the logarithm of six. "The first digit on the left of these numbers" he says, "denotes unity : the others express the fraction to be added to unity, as was shown in Chapter VI. The fraction to be added to unity I call the First Difference, that is to say the difference of the given number and unity, denoted by A. The numbers placed immediately below A are the halves of the first differences of those immediately preceding and are marked $\frac{1}{2}$A. From these halves are subtracted the first differences written above, and the remainders are written below, and, marked with the letter B are called the Second Differences. Again the numbers placed below B are the quarters of the immediately preceding second differences and are marked $\frac{1}{4}$B, from which the numbers B having been taken leave the Third Differences C, and these third ones having been subtracted from $\frac{1}{8}$th of the preceding third ones marked C leave the Fourth Differences D. In the same way are found the Fifth Differences E, the sixth F, the seventh G etc. by subtracting the fourth difference D/

D from $\frac{1}{16}$ of the preceding fourth difference, the fifth E from $\frac{1}{32}$ the preceding fifth and the sixth F from $\frac{1}{64}$ of the preceding sixth etc."

All these differences are got very easily from the continued series of continued mean numbers. Their use however is not of much service, the work being too laborious, unless three or more zeros follow unity, in which case differences lessen the greater part of the work undertaken. From the continued mean numbers the differences are produced until the differences are less than the limits of accuracy required, and after that these differences produce the continued mean numbers themselves.

This method of Differences, of course, was by no means Briggs's first use of the Differential Method, his process of quinquesection being used for sines some twenty years before, but this method is peculiar to the finding of continued means, and Hutton remarks + that this method of generating logarithms by Differences in addition to the methods of the Trigonometria Britannica, he believes to be the first instances there are to be found of making such use of differences, and show that Briggs was the inventor of the Differential method, a conclusion which all my researches certainly confirm.

As usual Briggs states his rules but gives no indication whatever as to the line of reasoning or as to the circumstances which led him to the discovery of these rules, so that any theory which may be put forward is a matter of mere conjecture. I think it highly probable, however, as also does Charles Hutton, that the fact which would first suggest a difference method to him would be the result of observation, and once on the track he would be led to pursue his method from further observation of his results. It should be kept in mind that Briggs was already well aware of the power of differences, and his mind would be receptive to any clue suggestive of such a procedure.

In the construction of so many continued means between unity and the number proposed, especially when the number was of the form unity followed by a number of ciphers, Briggs, with his keen arithmetical/

+Math. Intro. p. 65.

arithmetical mind, could scarcely have failed to note that the continued means approach always nearer and nearer to halves of the preceding means, (he certainly noted this much as is evident from Chapter VI), which fact would become the more obvious when placed together under each other. Indeed he must have noted that as many of the significant figures of each decimal part as there are ciphers between them and the integer 1 agree with the halves of those above them. Having observed this evident approximation he would subtract each of the decimal parts, called by him A, from half the preceding ones, and by comparing these second differences B he would observe that they were approximately continued proportionals of common ratio $\frac{1}{4}$. Therefore subtracting each second difference from $\frac{1}{4}$ of the previous, he would get a series of third differences C, which by observation he would find to be again roughly continued proportionals, this time of common ratio $\frac{1}{8}$. Again taking the difference between each C and $\frac{1}{8}$ of the preceding one he would find the fourth differences D which he would discover to be nearly continued proportionals of common ratio $\frac{1}{16}$ and so on the fifth E, the sixth F etc. differences being nearly continued proportionals of common ratio $\frac{1}{32}$ and $\frac{1}{64}$.

These observations having been made, they very naturally and clearly suggested to him the notion and method of constructing all the remaining means from the differences of a few of the first, found by extracting the roots in the usual way. The convenience of the method is demonstrated by the specimen quoted, where after the differences vanish, the remaining means are constructed from the preceding differences, and it will be evident that in proceeding the trouble will become less and less, because the differences gradually vanish.

Briggs's Curious Equations.

Had this been all that Briggs had accomplished in this method we must have given him credit for an ingenious method, paying tribute to his powers of observation but little to his mathematical genius.

The rules given above are all that are necessary for the successful application of the method, yet Briggs does not rest there, but gives certain curious equations which demonstrate that he must have had a keener appreciation of and insight into the procedure stated than mere observation of approximate proportionals would indicate. I think it likely that the method came first as a result of observation, and the insight into it later as the result of a critical investigation of it. These equations give an ingenious but not an obvious method of finding the differences B,C,D,E etc. belonging to any number from the number itself independent of any of the preceding means, and the advantages of being able to do this are very evident.

These equations are -

$$B = \tfrac{1}{2} A^2$$

$$C = \tfrac{1}{2} A^3 + \tfrac{1}{8} A^4 +$$

$$D = \tfrac{7}{8} A^4 + \tfrac{7}{8} A^5 + \tfrac{7}{16} A^6 + \tfrac{1}{8} A^7 + \tfrac{1}{64} A^8$$

$$E = 2\tfrac{5}{8} A^5 + 7 A^6 + 10\tfrac{15}{16} A^7 + 12\tfrac{69}{128} A^8 + 11\tfrac{11}{64} A^9 + 7\tfrac{105}{128} A^{10}$$

$$F = 13\tfrac{9}{16} A^6 + 81\tfrac{3}{8} A^7 + 296\tfrac{87}{128} A^8 + 834\tfrac{43}{128} A^9 + 1953\tfrac{155}{256} A^{10}$$ [BRIGGS :- $1953\frac{285}{512}$

$$G = 122\tfrac{1}{16} A^7 + 1510\tfrac{67}{128} A^8 + 11475\tfrac{73}{128} A^9 + 68371\tfrac{244}{256} A^{10}$$ $68372\frac{79}{2048}$.

$$H = 1937\tfrac{95}{128} A^8 + 47151\tfrac{93}{128} A^9 + 406845\tfrac{37}{128} A^{10}$$ $406845\frac{1493}{8192}$

$$I = 54902\tfrac{89}{128} A^9 + 2558465\tfrac{77}{128} A^{10}$$ $2558465\frac{23587}{32768}$

$$K = 2805527\tfrac{187}{256} A^{10}$$ 2805527.].

where **A** is the decimal part added to unity or what was previously called the first difference of the number whose differences are sought.

Briggs successfully demonstrates that these lead to the same results as those obtained by the longer method. The problem we are faced with is this :- One can see how Briggs could easily have arrived at his method merely from observation, but one cannot see how these curious equations could be obtained without a deep insight into the analysis of the problem, an analysis difficult enough with our/

	A	A^2	A^3	A^4	A^5	A^6	A^7	A^8	A^9	A^{10}
16	16	120	560	1820	4368	8008	11440	12870	11440	8008
	8	60	280	910	2184	4004	5720	6435	5720	4004
8	8	28	56	70	56	28	8	1		
	4	14	28	35	28	14	4	$\frac{1}{2}$		
4	4	6	4	1						
	2	3	2	$\frac{1}{2}$						
2.	2	1								
	1	$\frac{1}{2}$								
A.1.	1									
	·	32	224	840	2128	3976	5712	6434	5720	4004
	·	8	56	210	532	994	1428	$1608\frac{1}{2}$	1430	1001
	·	8	24	34	28	14	4	$\frac{1}{2}$		
	·	2	6	$8\frac{1}{2}$	7	$3\frac{1}{2}$	1	$\frac{1}{8}$		
	·	2	2	$\frac{1}{2}$						
	·	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{8}$						
B.	·	$\frac{1}{2}$								
	·	·	32	176	504	980	1424	1608	1430	1001
	·	·	4	22	63	$122\frac{1}{2}$	178	201	$178\frac{3}{4}$	$125\frac{1}{8}$
	·	·	4	8	7	$3\frac{1}{2}$	1	$\frac{1}{8}$		
	·	·	$\frac{1}{2}$	1	$\frac{7}{8}$	$\frac{7}{16}$	$\frac{1}{8}$	$\frac{1}{64}$		
C.	·	·	$\frac{1}{2}$	$\frac{1}{8}$						
	·	·	·	14	56	119	177	$200\frac{7}{8}$	$178\frac{3}{4}$	$125\frac{1}{8}$
	·	·	·	$\frac{7}{8}$	$3\frac{1}{2}$	$7\frac{7}{16}$	$11\frac{1}{16}$	$12\frac{71}{128}$	$11\frac{11}{64}$	$7\frac{105}{128}$
D.	·	·	·	$\frac{7}{8}$	$\frac{7}{8}$	$\frac{7}{16}$	$\frac{1}{8}$	$\frac{1}{64}$		
E.	·	·	·	·	$2\frac{5}{8}$	7.	$10\frac{15}{16}$	$12\frac{69}{128}$	$11\frac{11}{64}$	$7\frac{105}{128}$
	A	A^2	A^3	A^4	A^5	A^6	A^7	A^8	A^9	A^{10}

our improved algebraic notation, but one very much more difficult without a suitable notation, a handicap from which Briggs suffered severely as has been frequently noted.

There are two methods however that I can think of whereby Briggs might have obtained these equations, the second of which I will give in the fuller detail as being in my opinion the much likelier of the two, and also my own work.

Suggested Methods Of Construction.

For the first of these methods I am very much indebted to G. J. Lidstone, Esq., who kindly communicated it to me.

FIRST METHOD.

In the first block of figures on the opposite page there have been written down (from the Abacus Panchrestus) the coefficients of the powers of A up to A^{10} in the expansions of $(1 + A)^{2^x}$ for $x = 0, 1, 2, 3, 4$. The figures in red ink are found by dividing the preceding line by 2.

Taking differences as indicated by Briggs we get the figures in black in the second block. The red figures are found by dividing the preceding line by 4.

Taking differences again, as indicated, and dividing by 8 we get the red ink figures : and so on dividing by 16 at the next stage.

The last line of each block gives Briggs's coefficients in his A, B, C, D and E as indicated.

To get F, G, ... K we should have to begin with the expansions of $(1 + A)^{5/2}$ up to A^{10}, and when this goes beyond the limits of the Abacus each power must be found by squaring the previous one. The figures would become very large.

OBJECTIONS TO FIRST METHOD

Now/

OBJECTIONS TO FIRST METHOD

Now at first sight this looks a very plausible and attractive theory, viz. that Briggs merely applied to these algebraic expressions (using a method of detached coefficients) the rules he applied to the numbers in practice, but closer investigation shows that this method has too many disadvantages to be tenable. Had he gone only to the equation giving E I would have been inclined to support this theory, since only the coefficients of powers up to $(1 + A)^{16}$ are required for this, and all these coefficients are in the Abacus Panchrestus, but Briggs actually gives the equation for K which by this method would mean starting with the coefficients in the expansion of $(1 + A)^{512}$ as far as A^{10}, some of which contain twenty one digits. Even such large numbers would not render the work prohibitive as far as Briggs was concerned, for he very frequently worked with large ones, and calculation was normally to him the very breath of life. But at this time he had little time to spare on unnecessary calculations, and, be it noted, the time he spent in finding K was wasted since from his example he never actually made use of such a remote difference. It seems to me that Briggs was merely running true to form by generalising his results well beyond the limit arising in practice, and since we noticed before that he curbed this tendency when time was pressing we can only suppose that the generalisation did not entail an excessive amount of extra time and labour. If this be granted then it argues against the probability of the method proposed, where the differences as far as E are easily derived, but those beyond E demand an amount of laborious calculation which would deter even a Briggs if it were not actually essential.

It has been stated that by this method the coefficients of $(1 + A)^{512}$ would be required. Now the Abacus Panchrestus gives the coefficients of the expansions of the powers up to the 16th, and therefore it would be necessary to work out independently the coefficients of the powers of A up to A^{10} in the expansions of the 32nd, 64th, 128th, 256th and 512th powers of 1 + A, and considering the/

the large numbers involved, that in itself would have been no mean feat. When, however, it is coupled with the labour of the subsequent divisions and subtractions, which from trial among the earlier differences I have found to be far greater than would appear on the surface, I can only conclude that the method though a possible one is an improbable one.

SECOND METHOD OF CONSTRUCTION PROPOSED BY MYSELF.

The second method makes use of a very simple but important iterative property, which allows of infinite extension and which I could scarcely imagine Briggs overlooking, namely, that if the r th difference is $f(A)$, then the $(r+1)$th difference is $\frac{f(2A+A^2)}{2^r} - f(A)$.

The first being known (given = A), the second can be found, hence the third, hence the fourth etc. This method does not involve large powers, indeed nothing beyond the tenth, since Briggs does not use any power in his equation beyond A^{10} though obviously the expressions do not end there.

It is my purpose to show that the equations are true and to indicate a purely arithmetical way of arriving at the result, for I feel convinced that Briggs's process must be arithmetical.

Suppose it is required to find the 2nd, 3rd, 4th ... differences B, C, D, E etc. for a given number 1 + A say, where A is the first difference.

Let the corresponding differences for the preceding means be A_1, B_1, C_1, D_1, etc., A_2, B_2, C_2, D_2....., and so on.

These are the differences for the numbers -

$1+A_1$, $1+A_2$, $1+A_3$, $1+A_4$ .... i.e. of

$(1+A)^2$, $(1+A)^4$, $(1+A)^8$, $(1+A)^{16}$ .... since the numbers themselves are formed by continuously taking square roots.

We have then,

$$A = \underline{\underline{A}} \qquad A_1 = (1+A)^2 - 1 = 2A + A^2 \qquad A_2 = (1+A_1)^2 - 1 = 2A_1 + A_1^2$$

$$\therefore\ B = \tfrac{1}{2}A_1 - A = A + \tfrac{1}{2}A^2 - A = \underline{\underline{\tfrac{1}{2}A^2}}$$

$$B_1 = \tfrac{1}{2}A_2 - A_1 = A_1 + \tfrac{1}{2}A_1^2 - A_1 = \tfrac{1}{2}A_1^2 = \tfrac{1}{2}(2A+A^2)^2$$

$$\therefore\ C = \tfrac{1}{4}B_1 - B = \tfrac{1}{8}A_1^2 - \tfrac{1}{2}A^2 = \tfrac{1}{8}(4A^2+4A^3+A^4) - \tfrac{1}{2}A^2 = \underline{\underline{\tfrac{1}{2}A^3 + \tfrac{1}{8}A^4}}$$

$$C_1 = \tfrac{1}{4}B_2 - B_1 = \tfrac{1}{2}A_1^3 + \tfrac{1}{8}A_1^4 = \tfrac{1}{2}(2A+A^2)^3 + \tfrac{1}{8}(2A+A^2)^4$$

$$\therefore\ D = \tfrac{1}{8}C_1 - C = \tfrac{1}{16}(2A+A^2)^3 + \tfrac{1}{64}(2A+A^2)^4 - \tfrac{1}{2}A^3 - \tfrac{1}{8}A^4$$

$$= \tfrac{1}{16}(8A^3 + 12A^4 + 6A^5 + A^6) + \tfrac{1}{64}(16A^4 + 32A^5 + 24A^6 + 8A^7 + A^8) - \tfrac{1}{2}A^3 - \tfrac{1}{8}A^4$$

$$= A^3\left(\tfrac{8}{16} - \tfrac{1}{2}\right) + A^4\left(\tfrac{12}{16} + \tfrac{16}{24} - \tfrac{1}{8}\right) + A^5\left(\tfrac{6}{16} + \tfrac{32}{64}\right) + A^6\left(\tfrac{1}{16} + \tfrac{24}{64}\right) + A^7\left(\tfrac{8}{64}\right) + A^8\left(\tfrac{1}{64}\right).$$

$$= \underline{\underline{\tfrac{7}{8}A^4 + \tfrac{7}{8}A^5 + \tfrac{7}{16}A^6 + \tfrac{1}{8}A^7 + \tfrac{1}{64}A^8}}$$

$$D_1 = \tfrac{7}{8}A_1^4 + \tfrac{7}{8}A_1^5 + \tfrac{7}{16}A_1^6 + \tfrac{1}{8}A_1^7 + \tfrac{1}{64}A_1^8$$

$$\therefore\ E = \tfrac{1}{16}D_1 - D$$

$$= \tfrac{7}{128}(2A+A^2)^4 + \tfrac{7}{128}(2A+A^2)^5 + \tfrac{7}{256}(2A+A^2)^6 + \tfrac{1}{128}(2A+A^2)^7 + \tfrac{1}{1024}(2A+A^2)^8$$
$$- \tfrac{7}{8}A^4 - \tfrac{7}{8}A^5 - \tfrac{7}{16}A^6 - \tfrac{1}{8}A^7 - \tfrac{1}{64}A^8$$

$$= \tfrac{7}{128}(16A^4 + 32A^5 + 24A^6 + 8A^7 + A^8) + \tfrac{7}{128}(32A^5 + 80A^6 + 80A^7 + 40A^8 + 10A^9 + A^{10})$$
$$+ \tfrac{7}{256}(64A^6 + 192A^7 + 240A^8 + 160A^9 + 60A^{10} + \ldots\ldots) + \tfrac{1}{128}(128A^7 + 448A^8 + 672A^9$$
$$+ 560A^{10} + \ldots\ldots) + \tfrac{1}{1024}(256A^8 + 1024A^9 + 1792A^{10} + \ldots\ldots)$$
$$- \tfrac{7}{8}A^4 - \tfrac{7}{8}A^5 - \tfrac{7}{16}A^6 - \tfrac{1}{8}A^7 - \tfrac{1}{64}A^8$$

$$= \underline{\underline{2\tfrac{5}{8}A^5 + 7A^6 + 10\tfrac{15}{16}A^7 + 12\tfrac{69}{128}A^8 + 11\tfrac{11}{64}A^9 + 7\tfrac{105}{128}A^{10}}}$$

etc.

If E, F, G..... were worked out in full, the working would be very heavy since F would go to A^{32}, G to A^{64}, H to A^{128} ......
But since A is small it is quite unnecessary to go to such high powers. Indeed Briggs does not go beyond A^{10} in his equations which cuts down the work of formation very considerably, and in the example he gives it is found that the required degree of accuracy is obtained by considering powers of A only to the sixth.

These results for B, C, D, E agree with Briggs's equations, and the higher differences could be obtained in the same way.

It will be seen that when one difference, D say, has been found in terms of powers of A, D_1, is the same function of A_1, i.e. of $(2A + A^2)$ and/

Coefficients in Briggs's Equations

	A	A^2	A^3	A^4	A^5	A^6	A^7	A^8	A^9	A^{10}	A	B	C	D	E	F	G	H	I	K
①	2	1									1									
②	·	4	4	1								$\frac{1}{2}$								
③	·	·	8	12	6	1							$\frac{1}{2}$							
④	·	·	·	16	32	24	8	1					$\frac{1}{8}$	$\frac{7}{8}$						
⑤	·	·	·	·	32	80	80	40	10	1				$\frac{7}{8}$	$2\frac{5}{8}$					
⑥	·	·	·	·	·	64	192	240	160	60				$\frac{7}{16}$	7	$13\frac{9}{16}$				
⑦	·	·	·	·	·	·	128	448	672	560				$\frac{1}{8}$	$10\frac{15}{16}$	$81\frac{3}{8}$	$122\frac{1}{16}$			
⑧	·	·	·	·	·	·	·	256	1024	1792				$\frac{1}{64}$	$12\frac{69}{128}$	$296\frac{87}{128}$	$1510\frac{67}{128}$	$1937\frac{95}{128}$		
⑨	·	·	·	·	·	·	·	·	512	2304					$11\frac{11}{64}$	$834\frac{43}{128}$	$11475\frac{73}{128}$	$47151\frac{93}{128}$	$54902\frac{89}{128}$	
⑩	·	·	·	·	·	·	·	·	·	1024					$7\frac{105}{128}$	$1953\frac{155}{256}$	$68371\frac{244}{256}$	$406845\frac{37}{128}$	$2558465\frac{77}{128}$	$2805527\frac{187}{256}$

NOTE: ① Table on left gives the coefficients of the powers of A in the expansion of the powers of $(2A+A^2)$.

② Table can be extended indefinitely (if it is wished to go beyond A^{10}) without expanding by Binomial Theorem.

The number in any square = 2 (left upper diagonal number) + (number on its left in same row.)

Table on right gives coefficients in the differences A, B, C, D etc., the indices of the powers of A being given by the number of the row as shown in first column. These coefficients are got thus (e.g. E):- Take the coefficients for D already found. The first is $\frac{7}{8}$ in row ④, and the number in the leading diagonal is 16, while the last is $\frac{1}{64}$ in row ⑧. We take (mentally) 16 from all the leading diagonal numbers in the rows from ④ to ⑧ inclusive, and then divide all the numbers in this block by 16. The coefficient of (say) A^5 in E is now found by multiplying the resultant numbers in the "A^5" column by those coefficients of D which lie in the same row with them, and taking the sums of the products. Similarly for the coefficients of A^6, A^7 etc.

Then working in the same way from the coefficients of E we get those of F, and so on.

e.g. Coefficient of A^5 in $E = \frac{32}{16}\times\frac{7}{8} + \frac{32-16}{16}\times\frac{7}{8} = 1\frac{6}{8} + \frac{7}{8} = \underline{2\frac{5}{8}}$; of $A^6 = \frac{24}{16}\times\frac{7}{8} + \frac{80}{16}\times\frac{7}{8} + \frac{64-16}{16}\times\frac{7}{16} = \frac{21}{16} + 4\frac{3}{8} + \frac{21}{16} = \underline{7}$

and since the next difference E depends only on D, and D it can be easily found.

Although I have taken the trouble to derive each difference showing the detailed working, the real problem is one of iteration, for when the r^{th} difference is $f(A)$, the $(r+1)^{th}$ difference is $\frac{f(A_1)}{2^r} - f(A) = \frac{f(2A+A^2)}{2^r} - f(A)$, and we have merely to be able to form $f(2A + A^2)$ when $f(A)$ is known, and to evaluate $f(2A + A^2)$. This is what I did in deriving the equations above, but the whole working may be performed arithmetically in the following very simple manner and I am inclined to think that this method must approach closely to that actually used by Briggs, and that it is a much more probable one than the first.

It may be, of course, that Briggs used neither of these methods. I have tried to discover evidence either for or against the one or the other from the fact that his coefficients of A^{10} in F and the subsequent differences are wrong. In F it should be $1953\frac{155}{256}$ instead of $1953\frac{285}{512}$. I used the correct value to calculate the true coefficients of the later differences, but I also tried using the false value given by Briggs, and found that neither the first nor the second methods led to the false values in Briggs's equations. Could a method be found where the use of the false value in F leads to the false values in the later differences, the probability would be that we had happened upon the very method used by Briggs. It should be remarked that while the coefficients of A^{10} in F, G, etc. are wrong, the error is very small indeed being in the fractional part in each case.

CHAPTER X.

The Arithmetica Logarithmica (Contd.)

Calculation of logarithm of a prime number.

In Chapter IX Briggs observes that after finding the logarithms of/

of 1, 2, 3, 5, 10, and of all those which come from the multiplication or division of these among themselves, it remains for us to seek the logarithms of the remaining primes and indicates another method of so doing. It would seem from this that Briggs's method was to find the logarithm of a composite number from the sum of the logarithms of its factors, logarithms which would be already known, and therefore only the logarithms of prime numbers required to be actually computed. The method is almost the same as before, the logarithm being calculated by the method of continued means already explained in Chapters VII and VIII. The real merit lies in the ingenious way in which Briggs prepares the ground for the application of the continued means method by showing a clever way of obtaining a number beginning with unity and a number of ciphers which is related to the prime number under discussion.

The method is as follows:- Find three products having a common difference 1, such that two of them are the products of factors whose logarithms are already known, while the third is that produced from the prime number whose logarithm is required multiplied by itself or by some other factor whose logarithm is known. Then the product of the greatest and least will differ from the square of the mean one by unity, and if the smaller of these two resultant products be divided into the larger the quotient will be 1 followed by a small decimal beginning with a number of ciphers. The logarithm of this number is then found by the method of continued means and thence the logarithm of the prime number is found.

For instance, if it be proposed to find the logarithm of 7, we have $6 \times 8 = 48$, $7 \times 7 = 49$ and $5 \times 10 = 50$, and $49 \times 49 = 2401$ while $48 \times 50 = 2400$. If the smaller divides the greater, the quotient will be $1{\cdot}00041\frac{2}{3}$ or $1{\cdot}0004166666666666$ whose logarithm is found by the continued means method, and this logarithm added to that of 2400 will give that of 2401 which divided by 4 will produce the logarithm of 7.

By this method can be found the logarithms of all prime numbers, among which in the first instance those which are less than 100 suggest/

suggest themselves, and Briggs adds a table giving the primes in order together with the composite numbers by means of which the logarithms sought may most conveniently be obtained.

The Logarithm of a Fraction.

Having given methods for finding the logarithms of all whole numbers, he now turns his attention to fractions, and after justifying their inclusion, teaches how to find their logarithms viz. by subtracting that of the denominator from that of the numerator, the result being either abundant or defective i.e. positive or negative according as the fraction is greater or less than unity.

Calculation of Proportional Parts.

In Chapter XI Briggs observes that in addition to the table of logarithms he has given their differences which not only show any errors which may have crept in, but also give the proportional part whenever the occasion demands greater accuracy than the table itself gives. This proportional part however is never absolutely perfect because while the absolute numbers everywhere increase equally the logarithms increase with unequal and diminishing increments. Therefore if for any absolute number the logarithm of the tabular number immediately smaller requires to be increased by proportional parts, that part sought by means of the intervening difference will always be less than it should be, but on the contrary, if from the given logarithm the absolute number is sought it will always be increased more than it should by the proportional part. This disadvantage is by no means peculiar to logarithms, but occurs in the case of sines, tangents and secants and in general in all tables of numbers where the differences on the one hand are equal and on the other hand unequal.

Nevertheless the less the inequality of the differences, the less will be the deviation from the truth, and for that reason the proportional part is farther from the true value in the earlier chiliads/

chiliads than in the later ones.

From this property, Briggs, ever fruitful in happy expedients to overcome natural difficulties, devises a way of throwing the proportional parts to be found from the numbers or the logarithms, near the end of the table no matter where they happen naturally to fall.
His method is as follows:-

Neglect the characteristic of any given logarithm, whose absolute number it is required to find, and find the arithmetical complement of the decimal part by subtracting it from the logarithm of 10 : next find in the table the logarithm immediately smaller than this arithmetical complement together with its absolute number. To the logarithm thus found add the given one, and the sum must be a logarithm necessarily falling near the end of the table. The absolute number, corrected by the proportional part, of this logarithm is then found and being near the end of the table the result will not be very inaccurate. This number is then divided by the absolute number above found, and the quotient will be the required result, a result much more accurate than if it had been got from the proportional part of the difference where it naturally happened to fall. The reason for the method is evident from the nature of logarithms.

I had often speculated and wondered at the chiliads actually calculated by Briggs. Having computed the logarithms of numbers from 1 - 20,000, why should he not have carried straight on instead of jumping next to those from 90,000 to 100,000? I think the explanation is to be found in the above, since it explains why Briggs should want those chiliads near the end of the table. In the first place it is merely the digit sequence that is required to be found, and by looking up the absolute number in these final chiliads corresponding to the logarithm recorded which is just smaller than the arithmetical complement found he gets at once more digits than he would get say looking up the table near the end of the 10th chiliad (say), where the differences are fairly equal, though/

though larger than near the end. Then the differences being more regular and smaller give a more accurate proportional part. I think, therefore, that it was this necessity for arriving at a more accurate result by proportional parts which led him to jump from the twentieth chiliad to the ninety-first and those following.

Not content with the above simplification, Briggs tries to diminish the working of its application even farther. It is remarked that the divisor when taken as the number corresponding to the tabular logarithm immediately smaller than the arithmetical complement may happen to be a large prime number, thus making the division an awkward and laborious one, and so to overcome this disadvantage Briggs suggests that we use instead the next less composite number which has small factors and its logarithm, and for the more easy finding of convenient composite numbers he selects those between 1000 and 10,000 which are suitable and subjoins a list of them together with their logarithms and their component factors.

The processes of decisection and quinquesection as applied to logarithms have already been dealt with, but it should be added Briggs gives in Chapter XII a method of getting the proportional part which is merely the application of the Gregory or Gauss formula for equal second differences, namely $u_x = u_0 + x\,\delta u_{1/2} + \frac{x(x-1)}{\underline{|2}}\,\delta^2 u_0.$

CHAPTER XI

The Radix Method.

It was shown in Chapter XI how a number met with in the Chiliads could be increased by the proportional part so that the first twelve significant digits are accurate. In Chapter XIV he proceeds to/

to describe how both logarithms and anti-logarithms can be found to a far greater degree of accuracy, by a method which is now universally known as the "radix method", and is recognised as one of the very best general methods for finding logarithms. That Briggs considered it an important one and a simple one we can infer from the fact that it was by this method that he calculated the logarithms of the primary sines, which in turn became primary parts, the repeated quinquesection of which gave the whole canon of logarithmic sines.

The method in its simplest form consists of resolving the number whose logarithm is required into factors, the first being the appropriate power of ten to make the second lie between unity and ten, The second factor is further, resolved into the number given by the digit in the unit's place and a third factor which of necessity must be of the form unity together with a decimal. This last is resolved into factors of the type $1 \pm \frac{n}{10^r}$ where n is one of the nine digits.

This resolution can always be done and fairly simply at that. Briggs takes factors of the type $1 + \frac{n}{10^r}$, so that all that is required to obtain the logarithm of any number is a table giving the logarithms of units, $1 \cdot n$, $1 \cdot 0n$, $1 \cdot 00n$, $1 \cdot 000n$, ... Briggs gives such a table, the logarithms being expressed to fifteen places of decimals, and so thereby he was able to express the logarithm of any number to 15 places of decimals and conversely, he was able to find the true antilogarithm of any logarithm if it could be written down by fourteen or fewer digits.

The earliest account of the "radix method" + is in "An Appendix to the Logarithms" which appears in the second (1618) edition of Edward Wright's translation of Napier's Descriptio, which among other things describes a new and ready way for the exact finding of lines and logarithms as are not to be precisely found in the canons. We are not told by what author, but most authorities seem to agree that it was probably by Oughtred, on what/

+ Ball.

what grounds I have failed to discover. Hutton, however, says that it was probably by Briggs, but again no grounds are given for such a statement, although Briggs took such a large part in and was so closely identified with the publication of the original translation (he wrote a preface and certainly contributed some things to the publication as well as editing it) that it would not be the least surprising if he were the author, and I think it is likelier to be Briggs than Oughtred. Be that as it may, there is no doubt that Briggs fully developed the method in his Arithmetica Logarithmica, and I think he is fully entitled to be acclaimed as its inventor, a claim which most historians are willing to allow.

The method has frequently been re-discovered and given in various forms, the names of Flower and Weddle being closely associated with it. Indeed the application of the method to finding logarithms is usually known as Weddle's method, and to the finding of antilogarithms as Hearn's method, but both in their essential features are entirely due to Briggs, and give us but further proof of his originality and inventiveness.

In practice Briggs modifies the method slightly both in the finding of a logarithm and of an antilogarithm. In the case of logarithms he proceeds as follows:-

He takes the number given by the first four digits of the given one, and consequently has a number between 1,000 and 10,000 so that its logarithm is always to be found in the chiliads. The restriction to the first four digits is obviously due to the gap between 20,000 and 90,000 and I do not doubt that if the given number were such that the first five digits gave a number whose logarithm is in the chiliads i.e. between 10,000 and 20,000 or 90,000 and 100,000, he would take the number of five digits as his first factor. However he takes the first four in his examples, and breaks the given number up into factors the first of which is determined by the first four digits and the others being of the form $1 + \frac{n}{10^r}$, where n is one of the nine digits, and r has a value/

value between 1 and 15. The logarithm of his first factor he gets from the chiliads and those of the other factors from the table of radices which he had formed, and since the given number is the product of the factors, the required logarithm is the sum of the logarithms of the factors.

The procedure in the case of finding the absolute number is just the converse :- He finds the logarithm in the chiliads which is nearest to, but smaller than the given logarithm, and notes its antilogarithm. This tabular logarithm is taken from the given one, and the difference compared with those logarithms in his table of radices. That which is nearest to and less than this remainder is noted, and its antilogarithm taken from the table. The process is repeated, the given logarithm being thus broken up into the sum of logarithms, the first of which is in the chiliads and the others in the table of radices. The antilogarithms of these being read off, the required antilogarithm is the product of these separate factors, and since all the factors except the first are of the form $1 + \frac{n}{10^r}$, the multiplication is very easily carried out.

It should be noted that this method depends upon the existence of some other method of calculating logarithms, since it requires that certain basic logarithms should have been already found, but once they have been found, the method gives very accurate results without a great deal of labour. Briggs in all probability calculated the basic logarithms by the method of continued means or by one of his interpolation methods, though the first seems the likelier since the form of the given numbers are peculiarly suited to the conditions of that method.

Briggs here brings the account of the calculation of logarithms to an end, and turns his attention to their many uses and applications, which, however, concern us little in this account.

Reply to Mark Napier's Criticism of Briggs.

Let it be recalled that Briggs's memory is so largely logarithmic that Mark Napier was moved to infer that had there been no logarithms Briggs would not have been remembered at all, but had there been no Briggs the calculation of logarithms would have gone on just the same in the hands of men like Gunter and Kepler. It is true that Gunter's interest was great and that Kepler's was nearly as great as that of Briggs himself, but I strongly suspect that as far as Englishmen were concerned it was the undoubted enthusiasm of Briggs which communicated itself to the others, and that it was something of this that Barrow had in mind when he declares that had it not been for Briggs logarithms might have lain neglected and unknown. Mark Napier is, of course, most unfair to Briggs all along, and yet, while to John Napier must go the credit and the honour of the discovery of the general principles of logarithms, enough of Briggs's work has been given to show that no less honour must go to him for the part he played in improving logarithms and for the successful completion of the colossal task of calculating them. Mark Napier's gibe of his being a mere computer most certainly cannot be allowed. Briggs stands on an equal footing with John Napier in promulgating the earliest methods of calculating decimal logarithms, methods which he at a later date, by his own mathematical genius, so vastly improved that the merit in the improvement equals if it does not exceed that of the original discovery. So thoroughly did Briggs understand logarithms, and so sound were the principles which he himself discovered, and upon which he himself based new methods that he seems to have anticipated all the essential principles used by later calculators who adopted purely arithmetical methods.

Yet in spite of the many ingenious devices and contrivances used by Briggs in connection with logarithms, I think it must be granted that the greatest is his use of Differential methods, a discovery of his which had nothing to do with logarithms, and one which/

which had been made about fifteen years before logarithms were given to the world. In laying the foundation of all later work on central differences Briggs is worthy of a high place in mathematical history quite independent of his achievements in logarithms.

CHAPTER XII.

The Binomial Theorem.

How wide of the mark Mark Napier was will be further exemplified by my claim that Briggs has every right to the credit of having discovered the Binomial Theorem.

This claim is based upon the eighth chapter of the Trigonometria Britannica, wherein Briggs describes the construction of the figurate numbers, and inserts a large table of them which he called the Abacus Panchrestus, because of its many uses. The table has already been given and its construction described, together with an explanation of its use for giving the coefficients in the expression of the multiple chord in terms of the simple one.

Another very important property, perhaps its chief, is that the numbers along a diagonal constitute the coefficients in the expansion of any power of a binomial, and it is to this table and to this property I now wish to refer.

Early History of The Arithmetical Triangle.

Briggs was by no means the first person to discover this table, nor the first one to recognise that the diagonal numbers were the binomial coefficients. The table was discovered and rediscovered by quite a number of mathematicians, and there is nothing to show that the discoveries were not entirely independent of each other. The property is mentioned by Vieta, and certainly among the/

the pioneers using the table must be included Stifel, Cardan, Stevinus, Napier, Vieta, Briggs, Oughtred, Mercator and Pascal.

It may seem strange therefore that on the strength of this table there should be claimed for Briggs the discovery of the Binomial Theorem, when obviously so many before him had discovered the table, a table "the contemplation of which" Hutton has remarked, "has probably been attended with the invention and extension of some of our most curious discoveries in mathematics."

The claim, however, is based upon the fact that all those previous to Briggs, while able to generate the table, had to do so in its entirety in order to find any particular coefficients. Thus to find the coefficients of the sixth power say they had to start from the beginning and generate those of the second, third, fourth and fifth powers before they could find those of the sixth, so that the generation with them was performed mechanically according to certain rules of addition, without any general principle being grasped. Briggs however states certain proportional properties of the table whereby he can generate the coefficients of any power without any knowledge whatever of the previous ones, and thus gives a general rule for the formation of binomial coefficients.

Briggs's Statement of the Theorem.

Referring to the Abacus Panchrestus, which I have shown, Briggs states his rule :-

"Any number whatever is to its Diagonal number going up towards the left as the vertical of the first is to the Marginal of the second". Thus the numbers in column A are to their diagonal numbers in B (ascending) as 2 to the marginal of the second e.g. $2 : 11 = 12 : 66$, $2 : 9 = 10 : 45$.

Hence it follows that the numbers lying at the right margin and the rest immediately adjacent can be found and continued as far as one wishes. The whole table need not be written down from/

from the beginning."

Thus for the ninth power :- 9 being the coefficient of the second term (that of the first always being 1), to find the third coefficient we have 2 : 8 = 9 : ? i.e.36, for the fourth 3 : 7 = 36 : ? (=84) , for the fifth 4 : 6 = 84 : ? (=126) and so on for the rest, from which it is obvious that the coefficients are inversely proportional to the vertical numbers 1, 2, 3, 4 ..... and directly proportional to the ascending numbers in column A starting from 9, so that the actual coefficients are got from the continued multiplication of the fractions $\frac{9}{1}, \frac{8}{2}, \frac{7}{3}, \frac{6}{4}, \frac{5}{5}, \frac{4}{6}, \frac{3}{7}, \frac{2}{8}, \frac{1}{9}$. Had Briggs had had a suitable notation such as our present algebraic notation, he could scarcely have failed to give the general case of the r^{th} term, showing as above that for the n^{th} power the co-efficients are got from the continued multiplication of $\frac{n}{1}, \frac{n-1}{2}, \frac{n-2}{3}$ $\frac{n-3}{4}, \ldots\ldots, \frac{1}{n}$, which, of course, is the Binomial Theorem as stated at the present day. It is clear therefore that Briggs was in full possession of that principle which is called the Binomial Theorem, and that his law brought him as close to its present form as the notation of his day rendered possible.

Briggs's Title to Honour of Invention.

Now Briggs is the first person to state this rule and if being first in the field with a discovery be ample justification to the title of discoverer, there can be no doubt whatever as to the justness of Briggs's claim. His discovery was every bit as much the result of the speculation of one single man as were logarithms, there being nothing in the works of those who had already treated the Arithmetical Triangle to give him any clue to this rule of generating the coefficients of any power one from the other. Nor indeed must the significance and the importance of his discovery have been recognised, for although it was made about 1600, the theorem could not have been treated of by later writers, for it took nearly 70 more years for it to become established and widely known/

known, the genius of Sir Isaac Newton being responsible not only for the expansions of integral powers as Briggs had given it, but also for that of fractional powers, thereby bringing the Theorem to perfection. I do not see how the claim of Briggs can be disputed, for although he gives his principle in arithmetical language it is done so explicitly and so simply that from his rule it is obvious that as we proceed from diagonal number to diagonal number the numerators (i.e. the marginal numbers) of the multipliers are decreasing by unity while the denominators (i.e. the vertical numbers) are increasing by unity. The only thing Briggs lacked was the modern algebraic notation which would express the multipliers as $\frac{n}{1}, \frac{n-1}{2}, \frac{n-2}{3}, \frac{n-3}{4}, \ldots\ldots\ldots\ldots, \frac{1}{n}$.
Newton, of course, had an improved algebraic notation, and if he were acquainted with the works of Briggs, which I have seen suggested but have not been able to establish, there would be little merit in his Statement of the theorem for positive integers, though undoubtedly the greatest merit in his discovery of its extension to fractional indices.

Dr. Wallis seems to have been totally unaware of Briggs's discovery + for in his Algebra he fully ascribes the theorem to Newton. Indeed the theorem has been called Sir Isaac Newton's Binomial Theorem, and he valued it so highly that it was engraved on his tomb as one of his greatest discoveries, although, as has been shown, its leading principle originated with Briggs. Nor was Newton the only claimant for John Bernoulli disputed the invention of the theorem with Newton himself and then gave the discovery of it to Pascal on the strength of his Arithmetical Triangle, and this be it marked inspite of the fact that Pascal was not born until long after Briggs had taught the theorem. Even Mark Napier, prejudiced as he was against Briggs, and having tried his utmost to wrest the laurels from him, has to admit that Hutton ascribes the Theorem to Briggs with far more justification than Bernoulli to Pascal. He is not however willing to prefer Briggs's claim to Newton's, and, while not disputing Briggs's being first in the field with/

+ Math. Intro.

with the statement of the general principle, he awards the credit to Newton for his improvement and extension of it. He says that "In the hands of Newton the binomial table of Stifellius, Napier, Briggs and Pascal was expanded into the Binomial Theorem par excellence" and that therefore Newton must be acknowledged as "The General who won the victory and therefore wears the laurels." But Mark Napier is far from consistent, for by the very argument he uses in preferring the claims of Napier to those of Justus Byrgius in the matter of logarithms, Napier being first in the field with his discovery by a matter of six years and further by his exaltation of the first statement of a general principle with his contempt for whose who merely improve it, as in the case of Napier and Briggs (" a mere computer") he should acclaim Briggs as the inventor of this elegant Theorem. Instead of which, he grudgingly credits Briggs with the observation of the law of proportion among the entries in the table, minimises its importance, and shows himself to be highly chagrined that his illustrious ancestor had not recorded the law in his writings. Had Napier done so, I have not the slightest doubt Mark Napier would have proved conclusively that John Napier invented the Binomial Theorem.

Mark Napier's Case in Favour of John Napier.

But let us see what Mark Napier himself writes + on the subject:-

"Napier had the triangular table of Binomial coefficients, which he claimed teaches the rules of finding the supplements of all radicates and roots.........

It is a curious fact that Napier's friend Henry Briggs gave a table of the same description, and Dr. Hutton when noticing this says "This is the first mention I have seen made of this law of the coefficients of the powers of a binomial commonly called Sir Isaac Newton's Binomial Theorem, although it is very evident that Sir Isaac was not the first inventor of it : the part of it properly belonging to him seems to be only the extending of it to fractional indices/

+ Mark Napier.

indices which was, indeed, an immediate effect of the general method of denoting all roots like powers with fractional exponents, the theorem being not at all altered......... "

Notwithstanding the many long and delightful discussions that must have passed between Henry Briggs and the Baron of Merchiston upon their favourite topics, there seems to be no grounds for alleging that the former had borrowed his idea from his illustrious friend. We have also ventured to call him a satellite of Napier's and fairly enough as his memory is chiefly logarithmic, and his persevering pilgrimages to the old tower in Scotland is an ample justification for that epithet. But Briggs has evinced in his two logarithmic works a mind capable of great mathematical conceptions (a belated tribute! and a change of front!). (The kind assistance of an Oxford friend enabled me to ascertain with tolerable certainty, that there are no traces among Briggs's papers preserved at that University of a correspondence between him and Merchiston : probably he found the Baron a better host than a correspondent).

In reference to the arithmetical triangle Briggs appears to have been the first to point out a particular law of that configuration which brought him as close to the Binomial Theorem as the notation of his day rendered possible.

I have looked anxiously but in vain through Napier's manuscript to discover some expressions indicative of his observation of this important law of proportion actually existing in the table he had formed. There is however, no question that his triangle is what would now be called a table of coefficients of the powers of a binomial which he framed for its most important application, that of extracting roots. In doing this he was certainly at the confines of the Binomial Theorem.

Had he only recorded the observation of Briggs it must have been admitted that he had actually stated the leading principle of that elegant theorem, which is engraved upon the tomb of Newton as one of his greatest discoveries. The observation which leaves that laurel with Briggs (and which Napier may have seen though he did/

did not state it) amounts to this, that by a certain law of proportion existing between the figures of the diagram, by which law he points out, all the terms of the binomial quantity could be successively deduced or raised from the second term (the coefficients of the first and second terms being always known) without the necessity of finding the intermediate and preceding powers."

Mark Napier seems to be in no doubt as to ths importance of the law and the honour attached to its invention. He would fain have planted so fine a laurel upon the brow of Napier, and failing to do so, cannot refrain from hinting that Napier may have known the law without stating it. I am not surprised that Mark Napier could find neither grounds for alleging that Briggs borrowed his ideas from Napier nor any trace of a correspondence between them, especially on this subject, considering that Briggs had already discovered his theorem some fifteen years before he ever heard of Napier! A point which Mark Napier seems to have missed. The argument that Napier perhaps saw the law without stating it can be dismissed without a second thought as a case of wishful thinking: since it has absolutely no basis in fact, and the argument would apply equally well to all those who dealt with the Arithmetical Triangle even before Napier. To conclude I would reiterate that I do not see how the claim that Briggs was the inventor of the Binomial Theorem can be disputed, and even if Hutton does seem a bit anxious on occasion to claim overmuch for Briggs, I do think that this time he has ample grounds for his claim.

CHAPTER XIII.

Decimal Notation.

There are so many who took part in the invention of decimal fractions and the improvement of its notation, that it seems well nigh impossible to single out one person and award him the credit. The/

The topic arises in any case only because a historian of the standing of Prof. W. W. Rouse Ball gives it as his considered opinion + that the introduction of our decimal notation was due to Briggs, a conclusion which also has the backing of W. F. Sheppard.++ There is no doubt that Briggs used decimal fractions, not merely as a convenient notation for stating results but also as an operative form, and he seems to have done so in all his work, as is evident from his earliest researches. Furthermore such a notation as Briggs uses must have been far from common, if known at all, for he considers it necessary in his preface to the Arithmetica Logarithmica to explain it to his readers, the inference being that it was unknown to them. Now the use of decimals by Briggs is understandable and it would probably happen early in his career, for we have seen elsewhere that right from the beginning he revelled in decimal and centesimal division, preferring the decimal division of the radius, the degree, and being in favour even of that of the quadrant. Briggs certainly did not invent decimal fractions but he would be undoubtedly one of the first to consider them, the whole system coinciding so much in principle with his usual line of thought that it would be sure to have his whole-hearted support. It would not therefore be the least surprising if indeed our decimal notation was due to Briggs, for I do not see from what source Briggs could have got his notation. He certainly had it early in the seventeenth century, using it in an operative form as we do at present for multiplication and division as witnessed by the examples throughout his Trigonometria Britannica, which although not published till after his death was written by him in the first decade of the seventeenth century.

Early History of Decimal Fractions and Notation.

The whole problem of decimals, their invention and notation is a curious one. One would have expected that after the Arabic notation had been thoroughly mastered the extension to decimal fractions would have followed as a matter of course. But as has been/

+ Ball. ++ Encyc. Brit. Vol 2 p.535.

been pointed out,[+] it is curious to think how much science had attempted in physical research and how deeply numbers had been pondered before it was perceived that the all powerful simplicity of the Arabic notation was as valuable and as manageable in an infinite descending as in an infinitely ascending progression. The extension seems so simple and so obvious to us that it is remarkable to reflect that the invention and notation are neither the result of one mind nor even of one age, but that they came by almost imperceptible degrees, their nature but vaguely grasped at first, and so little understood that no suitable notation was invented even when the theory became clear.

Historians themselves are by no means agreed to whom should be ascribed the honour of the invention of decimal fractions or the notation. The divergence of opinion is mainly due to different standards of judgment, but in my own mind there seems little doubt that the invention of the fractions themselves must go either to Christoff Rudolff (c. 1530) or to Stevin, both of whom understood the nature of the fractions, and knew how to operate with them as well as merely to write them as various predecessors had done. D. E. Smith says [++] of Stevin that his treatment left little further to be done except to improve the symbolism which improvement was carried out by Justus Byrgius according to some, by Kepler, Beyer, Napier, and the English followers of Napier according to others. Byrgius seems to have quite a good claim to the credit, but he was himself undecided as to the best method of representing the fractions and it does not appear that he could operate with them. If it be enough simply to have used a point or a comma then precedence in their use must go to Byrgius, but if something more should be required, and I think it proper that an understanding of the use of the notation as an operative form should be demanded, then we must consider Napier and Briggs to be strong candidates for the honour. Let us examine the facts.

[+] Mark Napier. [++] Smith. Vol II.

Napier's and Briggs's Claims to Invention of Decimal Notation.

We know that Napier had had in mind before the end of the sixteenth century a complete survey of the means of calculating and in such a survey notation was almost bound to come under review. In his Descriptio published in 1614 however he made no explicit use of decimal fractions, the sine and the logarithm each being a line of so many units. But in his Rabdologia, Napier gives an "Admonitio pro Decimali Arithmetica" in which he commends the fractions of Stevin and gives an example of their use viz. the division of 861094 by 432, the quotient being written both as 1993,273 and 1993, 2'7"3'". J. W. L. Glaisher observes + "that this simple instance of the use of the decimal point in the midst of an arithmetical process, if it stood alone would not suffice to establish a claim for its introduction, as the real introducer of the decimal point is the person who first saw that a point or line as separator was all that was required to distinguish between integers and fractions, and used it as a permanent notation and not merely in the course of performing an arithmetical operation. The decimal point is however used systematically in the Constructio (1619), there being perhaps two hundred decimal points together in the book. The decimal point is defined, and 25·803 stated to mean $25\frac{803}{1000}$; also 10·502 is multiplied by 3·216 the answer being given as 33·774432, while in addition decimals occur not attached to integers. It would seem, therefore, that Napier was in possession of all the conventions and attributes that enable the decimal point to complete so symmetrically our system of notation, viz. (1) he saw that a point or separatrix was quite enough to separate integers from decimals, and that no signs were required to indicate primes, seconds etc: (2) he used ciphers after the decimal point and preceding the first significant figure and (3) he had no objection to a decimal standing itself without an integer."

Now on the surface this presents a strong case for Napier, and Mark Napier also was at great pains to claim the honour of our decimal/

+ Encyc. Brit. Vol 19 p. 175.

decimal notation for his ancestor, but in spite of that I shall attempt to show that Briggs has an even stronger claim.

It is granted forthwith that Briggs's notation is not quite so convenient as that of Napier, for where Napier used a point Briggs drew a bar under the decimal fraction, although in operating with them the bar was often left out, and the value of the decimal shown by position.

Now all that has been claimed for Napier in his command of the attributes of our notation is equally true of Briggs as can be shown from his works, but they apply with even greater force to him because he was <u>using</u> them in the first decade of the seventeenth century. He seems to have fixed his notation early on and uses it consistently throughout his works. This would give Briggs priority in its use, and since he everywhere displays as great a command of it as Napier, he has therefore a better claim to the credit of its introduction.

The issue however is not just so easy as would appear. There can be little doubt that both Napier and Briggs had command of our decimal notation, so that the greater honour must go to him who can be shown to have used it first, which I have stated above, in my opinion is Briggs. Napier's claim I think it will be admitted, stands or falls on the evidence of the Constructio published in 1619.

Now it is significant that Napier does <u>not</u> use decimals in his Descriptio,+ published in 1614, which would be most remarkable were he at this time in possession of that command of them which Glaisher would have us believe. Further in the 1616 translation of the Desceiptio use is made of the decimal point, and if this is due to Napier, for the translation was sent to him for his perusal and approval, the inference is that between 1614 and 1616 he had become acquainted with this new notation, which inference seems best to fit the facts. Also it was during this very period that he was visited by Briggs, who undoubtedly at that time was in full command of his own notation, and it should be kept in mind that it was/

+ Encyc. Brit. vol 19 p. 175.

was not unlikely that the nature of decimal fractions would come up for discussion between them in 1615 when Briggs came to discuss decimal logarithms since decimal fractions become almost a necessity in such calculations. This would further explain why in 1617 Napier still used both notations, the comma and the apices, which argues at this stage a sort of indecision in notation and that the habit of using the point had not yet been established. This brings us to the Constructio, which Rouse Ball believes was written after Briggs's stay with Napier, and for the publication of which Briggs was largely responsible. He infers that Briggs revised it, and holds that it confirms the view that the invention was due to Briggs and was communicated by him to Napier.

Now Rouse Ball is wrong in this first belief, for it is pointed out both by Robert Napier and by John Napier himself that the Constructio was written long before it was published, indeed he had it by him before the Descriptio was given to the world, and the use of "artificial number" in place of "logarithm" shows that it is the product of a period before the word logarithm was invented by him. It might be therefore that the decimal notation of the Constructio dates from that early period, but I can scarcely conceive that to be the case, else why did it not appear at all in the "Descriptio", a later work, and only in a very imperfect state, being non-operative, in the Rabdologia, which is an even later work?

The evidence certainly seems to me to point to the decimal notation of the Constructio, on which Napier's claim is based, being an addition after 1614: but granted this I am still left wondering, for if after 1614 Napier revised his early work in the matter of notation, why should he not have taken the opportunity to replace "artificial number" by the word "logarithm" which had become well known and commonly accepted. Nor do I think it likely that Briggs was responsible for it in preparing the Constructio for publication, for in the Lucubrations which he added to the Constructio he writes decimal fractions in his own notation without explanation, using the bar, whereas the rest of the book has the point.

I/

I think therefore that on the evidence of his undoubted early use of the decimal notation Briggs is entitled to the claim of having invented it. There is no doubt that Napier had an equal command of it, but everything points to its being of a much later date. One cannot overlook the fact that no use is made of decimals in the Descriptio and only an imperfect use in the Rabdologia, while Napier is not entirely free of the suspicion that he received his knowledge of it from Briggs.

CHAPTER XIV

Miscellaneous Facts.

Napier's Analogies.

The four formulae, known as Napier's Analogies were for use in the solution of oblique spherical triangles, and were published in the Constructio in 1619. They are the formulae

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \frac{C}{2}. : \qquad \tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cdot \cot \frac{C}{2}.$$

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \cdot \cot \frac{c}{2}. : \qquad \tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \cdot \cot \frac{c}{2}.$$

It should be remarked however that only one of the four is actually given by Napier, the other three being added by Briggs in his Annotations to the above work. The work left by Napier is rough and unfinished, and it is uncertain whether he knew the other three results or not.

J. W. L. Glaisher, however, points out that all four analogies may properly be called by his name, since the other three are so easily deducible from Napier's results.

The Law of Continuity. +

The law or principle of continuity seems to have been tacitly assumed as axiomatic by even the most learned Geometers, and was first/

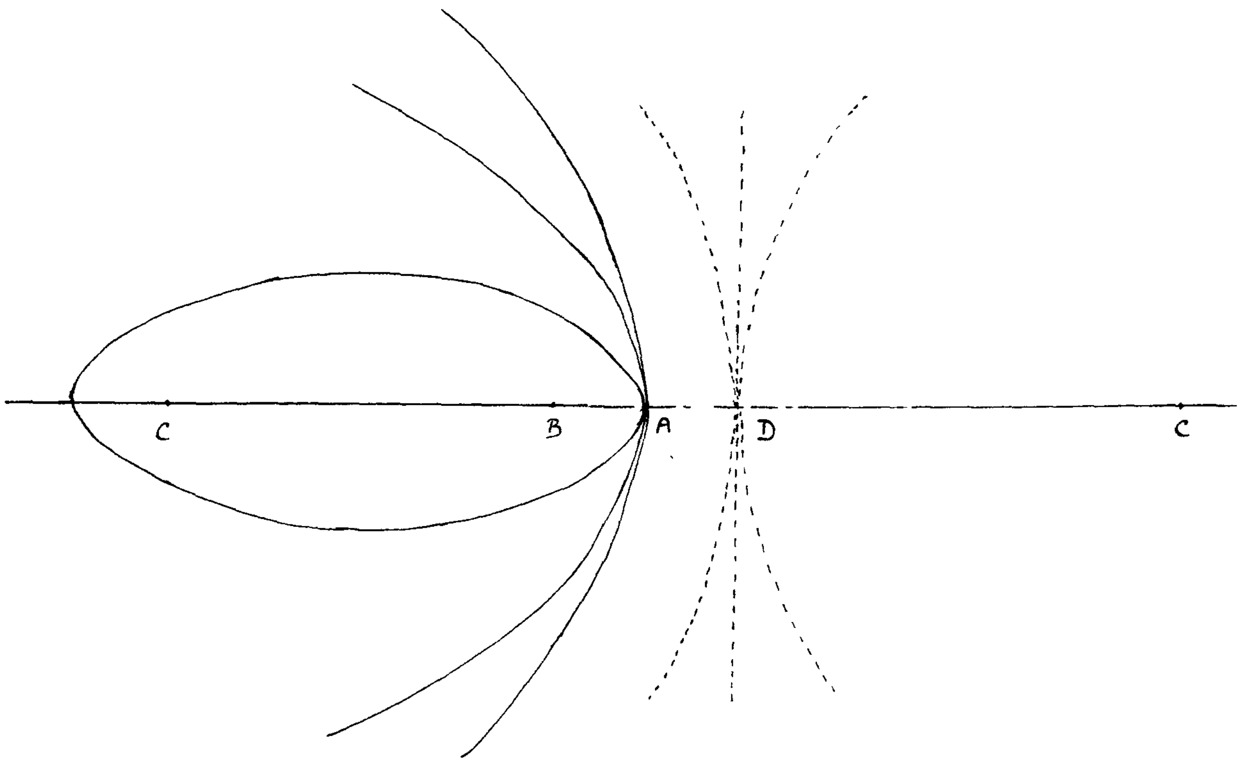
+ Encyc. Brit. vol. 11 p. 674.

first enunciated under that name by Gottfried W. Leibnitz, though he was not the first to be aware of the nature of the principle. It had previously been stated under another name by Johann Kepler in Chapter IV. 4. of his "Ad Vitellionem paralipomena quibus astronomiae pars optica traditur" (Frankfurt, 1604).

"Of sections of the cone", he says, "there are five species from the "recta linea" or line pair to the circle. From the line-pair we pass through an infinity of hyperbolas to the parabola, and thence through an infinity of ellipses to the circle. Related to the sections are certain remarkable points which have no name, but which Kepler calls foci. The circle has one focus at the centre, an ellipse or hyperbola two foci equidistant from the centre. The parabola has one focus within it, and another, the "caecus focus", which may be imagined to be <u>at infinity</u> on the axis <u>within or without the curve</u>. The line from it to any point of the section is parallel to the axis. To carry out the analogy we must speak paradoxically, and say that the line-pair likewise has foci, which in this case coalesce as in the circle and fall upon the lines themselves; for our geometrical terms should be subject to analogy. Kepler dearly loves analogies, and they are to be especially regarded in geometry as, by the use of "however absurd expressions", classing extreme limiting forms with an infinity of intermediate cases and placing the whole essence of a thing clearly before the eyes."

Here then we find formulated by Kepler the doctrine of the concurrence of parallels at a single point at infinity, and the principle of continuity (under the name analogy) in relation to the infinitely great. Such conceptions so strikingly propounded in a famous work could not escape the notice of contemporary mathematicians, and it is rather striking to find that Briggs, though occupied with the many formidable tasks previously described, must have kept himself informed of any new developments in the subject which he taught. At any rate Briggs wrote to Kepler from Merton College, Oxford, and in his letter, dated "10 Cal. Martii 1625" suggested/

suggested improvements in the Ad Vitellionem paralipomena, giving the following construction:-



"Draw a line CBADC, and let an ellipse, a parabola, and a hyperbola have B and A for focus and vertex. Let CC be the other foci of the ellipse and hyperbola. Make AD equal to AB, and with centres CC and radius in each case equal to CD describe circles. Then any point of the ellipse is equidistant from the focus B and one circle, and any point of the hyperbola from the focus B and the other circle. Any point of the parabola, in which the second focus is missing or infinitely distant, is equidistant from the focus B and the line through D which we call the directrix, this taking the place of either circle when its centre C is at infinity, and every line CP being then parallel to the axis."

Thus Briggs, and we know not how many learned geometers who have left no record, had already taken up the new doctrine in geometry in its author's lifetime.

CHAPTER XV

Conclusion.

This really brings to an end my account of the works of Briggs. As may be seen from what has been written he seems to have had a hand in many of the most important discoveries of Mathematics, discoveries which were thought to have been of a much later date. His works exhibit a mind full of fruitful and original ideas and capable of the greatest mathematical concepts, and it seems remarkable to me that a man of his undoubted calibre should have gained such scanty recognition, while such as has been given him should be for a part of his work which is not the most noteworthy. His Angular Sections, his Differential Methods, his Binomial Theorem all display mathematical ability and acumen of the highest order, and all of these had been established by him before his association with Napier in the matter of logarithms. Certainly his contributions in this branch of Mathematics alone stake for him a claim for a place in the History of Mathematics, but considering all that he accomplished in the field of original research I am far from satisfied that his association with logarithms was the best thing that could have happened to him, though undoubtedly it was in its benefit to mankind. Had attention not been so focussed on his logarithmic work that it overshadowed all the rest, the latter would probably have had a better chance of being known and Briggs a better chance of having his real worth recognised. It appears to me that Briggs's discoveries were too far ahead of his time to be appreciated by his contemporaries, and by the time mathematicians might have done so they were in possession of a notation, so improved that they were no longer interested in works without it, partly because of their purely arithmetical nature, and partly because the new notation had opened up for them wide fields to be conquered, and/

and they were too busy with the conquest. The lack of a suitable algebraic notation is the tragedy of Briggs, as it was of his times, for with one he might have interpreted his results in the neat, concise, and general form that is now possible, and in all probability the many facts he himself discovered would have been known by his name instead of that of some later worker in the same field. I feel that had he only possessed an analytic notation, we might truthfully have said of him what he wrote of Napier -

"for what is there which we may not have hoped for at his hands."

APPENDIX I

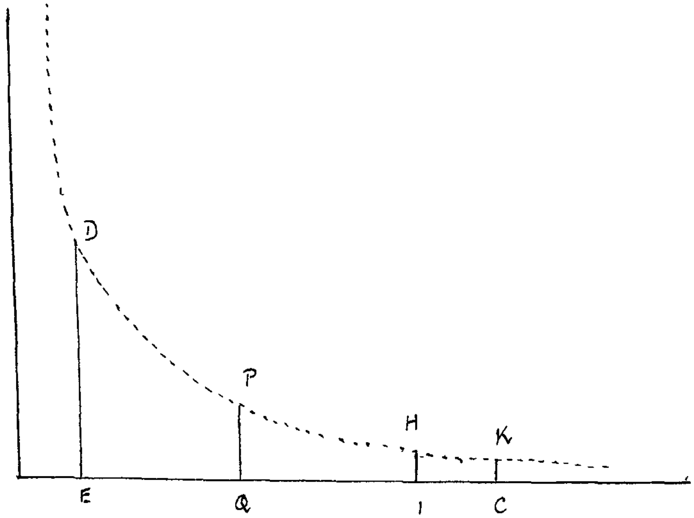
Later Developments

As has been shown the methods used by Briggs in the calculation of his logarithms were entirely arithmetical, and these methods continued to be employed by his successors, not being superseded until the introduction of infinite series into mathematics effected a great change both in the modes of calculation and the treatment of the subject.

The Logarithmic Nature of Hyperbolic Areas.

The first step in the advance towards a logarithmic series was taken when the Flemish mathematician Grégoire de Saint-Vincent (1584-1667) discovered that the area contained under a rectangular hyperbola, one asymptote and two ordinates parallel to the other asymptote increases in arithmetical progression as the distance between the ordinates (the one nearer to the centre being kept fixed) increases in geometrical progression. The discovery was purely geometric and was published in his chief mathematical treatise entitled Opus Geometricum Quadraturae Circuli et Sectionum Coni (Antwerp 1647), a large folio of over 1000 pages in which he gives a great number of new theorems on the properties of the circle and the conic sections, geometrical progressions and volumes of solids of revolution.

In view of the importance of this proposition, Proposition 125 in his book, it may be of historical interest to give it as St. Vincent himself stated it, the figure and the lettering being exactly as in the original.



The curve DPHK is a rectangular hyperbola referred to its centre as origin and its asymptotes as axes. DE, PQ, HI and KC are any four ordinates parallel to one of the asymptotes.

St. Vincent writes "I say that the surface DEQP contains the surface HICK as many times as the ratio $\frac{HI}{KC}$ multiplies to give the ratio $\frac{DE}{PQ}$"

i.e. if $\frac{DEQP}{HICK} = k$, the "k" is such that $\frac{DE}{PQ} = \left(\frac{HI}{KC}\right)^k$.

[This is easily proved to be correct: for let the equation of the rectangular hyperbola be $y = \frac{1}{x}$, and let $OE = a$, $OQ = b$, $OI = c$ and $OC = d$ where O is the centre and the origin

$$\therefore \; DEQP = \int_a^b \frac{1}{x} \cdot dx = \log b - \log a = \log \frac{b}{a}.$$

$$HICK = \int_c^d \frac{1}{x} \cdot dx. = \log d - \log c = \log \frac{d}{c}$$

$$\therefore \text{ If } DEQP = k . HICK,$$

$$\log \frac{b}{a} = k \log \frac{d}{c} = \log\left(\frac{d}{c}\right)^k$$

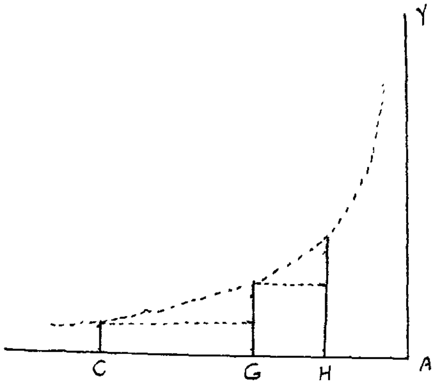
$$\therefore \frac{b}{a} = \left(\frac{d}{c}\right)^k.$$

But $ED = \frac{1}{a}$, $QP = \frac{1}{b}$, $IH = \frac{1}{c}$ and $CK = \frac{1}{d}$

$$\therefore \frac{b}{a} = \frac{DE}{PQ} \text{ and } \frac{d}{c} = \frac{HI}{KC}$$

$$\therefore \underline{\underline{\frac{DE}{PQ} = \left(\frac{HI}{KC}\right)^k}} \quad]$$

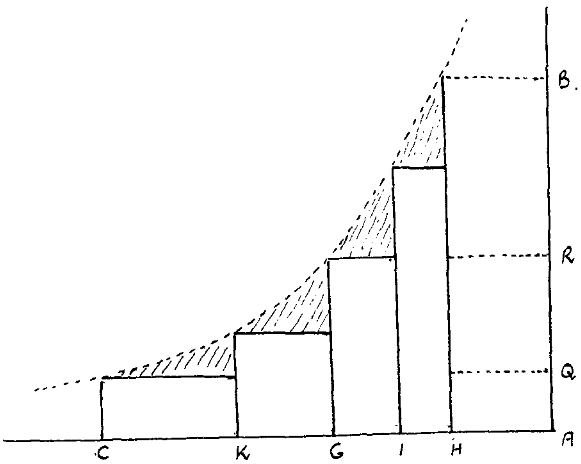
St. Vincent's proof of course was geometrical, and was made to depend on showing that if ordinates be drawn for abscissae in geometrical progression, the hyperbolic areas between successive ordinates are equal. This he did by the Method of Exhaustions, thus:



Let there be a rectangular hyperbola with AC and AY as asymptotes, and let AC and AH be two given abscissae with AG their geometric mean. At C, G, and H erect ordinates parallel to AY, and construct inscribed rectangles on CG and CH as shown.

St. Vincent then proved that these two rectangles on either side of the mean are equal. This having been established, it can be shown by exhausting the areas, that the hyperbolic areas on each side of the mean are equal.

Thus let n geometric means be inserted between both AC and AG and AG and AH, and let ordinates be drawn corresponding to these abscissae.



The inscribed rectangles are drawn as before, and since each abscissa is the geometric mean to those on either side of it, it follows from the first part that all these $(2n + 2)$ rectangles will be equal in area.

Therefore the sum of the $(n + 1)$ rectangles between the ordinates at C and G equals the sum of the $(n + 1)$ rectangles between the ordinates at G and H.

But the hyperbolic area on CG differs from the sum of the $(n + 1)$ rectangles by the area of the shaded triangles, the area of which is/

is less than the width of the rectangle of greatest width times QR which is finite and independent of n.

Therefore by taking $n \to \infty$, the greatest width $\to 0$ and the error in taking the sum of the rectangles as the hyperbolic area $\to 0$. Similarly for the hyperbolic area on GH, Thus. the hyperbolic area on CG = the hyperbolic area on GH, and so the hyperbolic area between two ordinates is bisected by the ordinate through the geometric mean of the two abscissae. Hence if ordinates be drawn at distances from A increasing in geometric progression, the hyperbolic areas between successive ordinates will be equal, so that the hyperbolic areas, by the continual addition of the equal parts, increase arithmetically as the corresponding abscissae increase geometrically.

That this property was analogous to a system of logarithms was at once pointed out [+] by the French mathematician, Marin Mersenne (1588-1648), and the same analogy was remarked upon and illustrated soon after by Christian Huygens (1625-95) and many others who used it to show how to square the hyperbolic spaces by means of logarithms. In addition Fermat, whose formulation of the problem of quadratures is akin to our definition of a definite integral, described his method of integration as a logarithmic method, [++] so that it is clear that the relation between the quadrature of the hyperbola and logarithms was well understood although it was not expressed analytically.

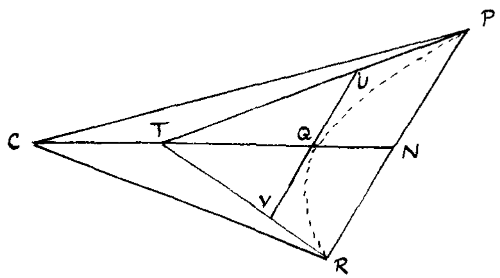
+ Math. Intro. p.84. ++ Encyc. Brit. vol 14 p.539.

James Gregory's Quadrature of the Rectangular Hyperbola.

The idea of an analytical series for a logarithm certainly did in time occur to James Gregory (1638-75), but before that happened he, unlike those before mentioned who saw in this property a means of squaring the hyperbola by means of logarithms, saw in it rather a method of computing logarithms by squaring the hyperbola, and he made use of it to calculate the logarithm of any number. His method of doing so was to calculate arithmetically the hyperbolic area concerned by approximating to it by a series of inscribed and circumscribed polygons, which area then gave the required logarithm.

His mode of working was described by him in his Vera Circuli et Hyperbolae Quadratura (Padua, 1667), and is based upon the Archimedean Method of Exhaustions : indeed this attempt of Gregory's may be regarded as about the last in which this method was used for quadratures, all later attempts (even by Gregory himself) being analytical in nature.

His method depends upon the geometrical properties of central conics, from which he established the law for deriving the areas of his successive inscribed and circumscribed polygons, a law which is merely the Archimedean Algorithm.



Let PR be any chord of a conic, centre C, and let PN = NR.

Let PT and RT be tangents, so that T, the point of intersection, will lie on the diameter CN, which also cuts the conic in Q. The tangent at Q will therefore be parallel to PR.

Further/

Further, $CT.CN = CQ^2$

i.e. $\dfrac{CT}{CQ} = \dfrac{CQ}{CN}$

But $\dfrac{CT}{CQ} = \dfrac{\Delta CPT}{\Delta CPQ}$ and $\dfrac{CQ}{QN} = \dfrac{\Delta CPQ}{\Delta CPN}$ (Δs. of same altitude are to each other as their bases.)

$$\frac{\Delta CPT}{\Delta CPQ} = \frac{\Delta CPQ}{\Delta CPN}$$

But since CN is a median of ΔCPR

$\Delta CPT = \Delta CRT$, $\Delta CPQ = \Delta CRQ$, $\Delta CPN = \Delta CRN$

$\Delta CPR = 2\ \Delta CPN = A$ (say).

fig. $CPTRC = 2\ \Delta CPT = B$ (say).

fig. $CPQRC = 2\ \Delta CPQ = C$ (say).

$$\frac{B}{C} = \frac{C}{A}$$

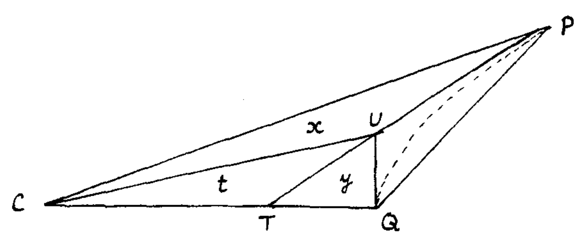
$$C^2 = AB \quad \text{i.e.} \quad C = \sqrt{A.B}.$$

This "A" is the first "inscribed" triangle of the sector.

and "C" is the second "inscribed" polygon.

"B" is the first "circumscribed" polygon of tangents.

Let the tangent at Q cut PT at U and RT at V, so that figure CPUVRC (= D say) is the second "circumscribed" polygon of tangents.



Then CU produced bisects PQ and CV produced bisects QR, for the same reason over again as for CQ bisecting PR.

since CU is a median of ΔCPQ

$\Delta CPU = \Delta CUQ$. Similarly $\Delta CRV = \Delta CVQ$.

But/

But in Δ TPR, TN is a median and UV is parallel to PR

$$UQ = QV$$

$$\Delta\, CUQ = \Delta\, CVQ \qquad (\text{ CQ is a median of } \Delta\, CUV \text{ })$$

$$D = \text{fig. } CPUVRC = 4\Delta CPU.$$

$$\frac{D}{B} = \frac{4\Delta CPU}{2\Delta CPT} = \frac{2\Delta CPU}{\Delta CPT}$$

$$\frac{D}{B} = \frac{2x}{x+t}$$, calling the areas x, y, t as shown

But $\frac{B}{C} = \frac{CT}{CQ} = \frac{t}{t+y} = \frac{t}{x} \qquad (\ \Delta CUQ = \Delta CUP\ )$

$$\frac{B+C}{C} = \frac{t+x}{x}$$

$$\frac{D}{B} = \frac{2C}{B+C} \qquad \text{i.e. } D = \frac{2B.C.}{B+C}$$

D is the harmonic mean of B and C.

Gregory now formed a convergent series

A C E G I K ....

B D F H J L ....

where C = geometric mean of A and B

D = harmonic mean of B and C

Likewise E = geometric mean of C and D

F = harmonic mean of D and E etc.

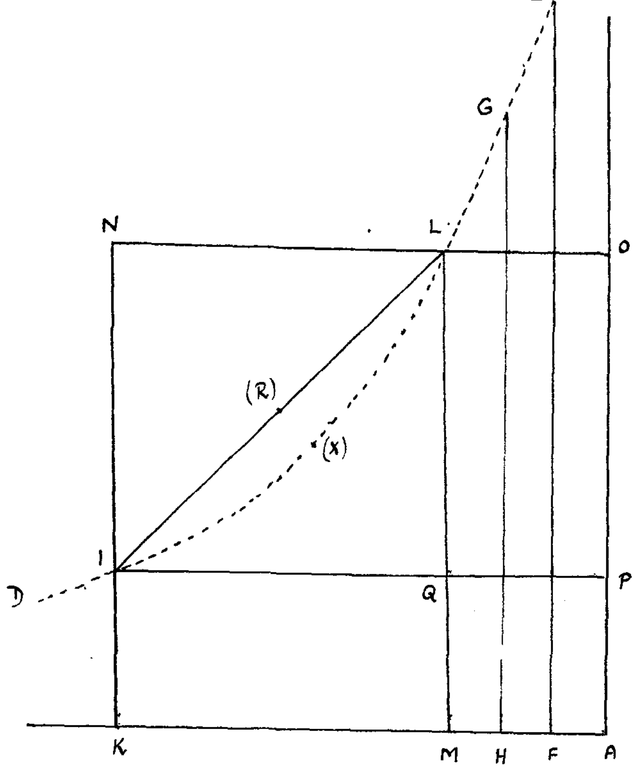
The A,C,E .... for a hyperbola decrease and are ΔCPR, the next polygon CPQRC and so on.

The B,D,F .... for a hyperbola increase and are the polygons of tangents CPTRC, CPUQVRC and so on.

Both sets converge and tend to the area of the sector.

The same is true for the circle and the ellipse, but in this case A is less than B.

In dealing with any hyperbolic sector, therefore, the only difficulty would be the finding of areas A and B, since from them all the others could be calculated, and we shall see how Gregory accomplished this by considering his calculation of the value of $\log_e 10$. Once more for the sake of historical interest I adhere to the original figure and lettering given by Gregory.



Given. DILGE a rectangular hyperbola, centre A and asymtotes AMK and APO.

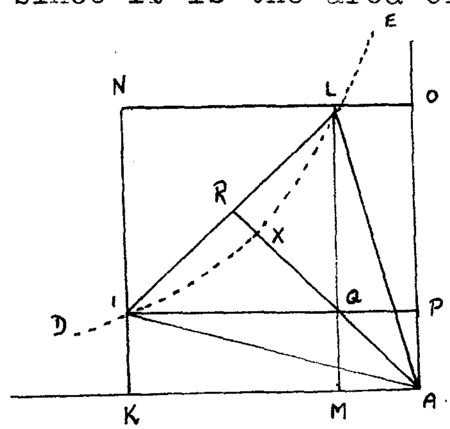
$AM = KI = 10^{12}$

$AK = ML = 10^{13}$

(It follows therefore that if R is the mid-point of LI and if AR cuts the hyperbola at X, then both R and X are on the diagonal of the square AKNO, and that X is the vertex of the hyperbola.)

It is required to find the area of the hyperbolic space LIKM, which is known to be log 10.

It has first to be proved that hyperbolic space LIKM = sector AIL, since it is the area of the sector which can be found.



Proof. Join IA and LA.

Δ IKA = $\frac{1}{2}$ rect IKAP

Δ LMA = $\frac{1}{2}$ rect LMAO

But rect LMAO = rect IKAP

Δ LMA = Δ IKA

Fig. IKAL − Δ LMA = Fig. IKAL − Δ IKA

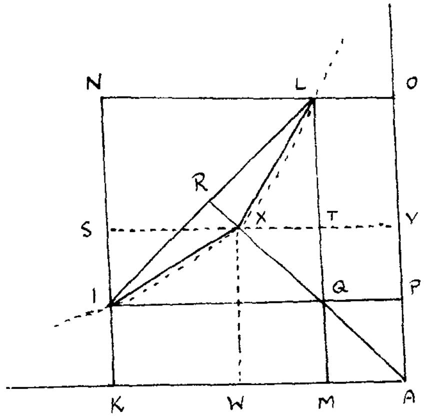
i.e. hyperbolic area LIKM = sector AIL.

Also. trap. LIKM = Δ AIL (adding IXL to each)

But Δ AIL = A

A = trap. LIKM = $\frac{1}{2}$ (rect.IKMQ + rect.NKML) ...... (1)

Instead of finding B, Gregory then proceeds to find C which really amounts to the same thing, since from A and C, B can be found using $A.B = C^2$.



C = fig. AIXLA

= fig. LXIKM

Draw SXTV parallel to KMA.

Since $\Delta LXQ = \Delta IXQ$ (X is on median of ILQ.)

and $\Delta IXQ = \frac{1}{2}$ rect. SIQT.

C = pentagon LXIKM

= rect IKMQ + rect SIQT

= rect SKMT.

But since X is the vertex of the hyperbola XWAV is a square and square XWAV = rect IKAP.

XW^2 = rect. AK.KI

KS^2 = KN.KI (Given AK = ML = KN)

$$\frac{KN}{KS} = \frac{KS}{KI}$$

$$\frac{\text{Rect NKML}}{\text{Rect SKMT}} = \frac{\text{Rect SKMT}}{\text{Rect IKMQ}}$$ (Rectangles of same altitude are to each other as their bases)

C is the geometric mean between rect NKML and rect. IKMQ

Now rect IKMQ = IK.KM = $10^{12} \times 9.10^{12} = 9.10^{24}$

rect NKML = ML.KM = $10^{13} \times 9.10^{12} = 9.10^{25}$

$$A = \tfrac{1}{2}(9.10^{24} + 9.10^{25}) = 49\cdot 5 \times 10^{24}$$

$$C = \sqrt{9.10^{24} \times 9.10^{25}}$$

$$= 9.10^{24} \times \sqrt{10}.$$

$$= 28460498941515413987990042$$

Knowing A and C, Gregory was then able to calculate B and from these primary values, he calculated the areas of the "inscribed" and "circumscribed" polygons got by repeated bisection according to the law previously established.

He repeated his bisection until he had up to 2^{20} sides, and he thus arrived at two expressions for the area of the sector (and therefore of $\log_e 10$), one from the "inscribed" polygons slightly greater than the sector and one from the polygon of tangents slightly less. The result of the former he found to be/

be 23025850929940456840178704 and of the latter
23025850929940456840178681 so that

$\log_e 10$ lies between 23025850929940456840178704
and 23025850929940456840178681

The idea of an infinite series for the area of a hyperbolic space occurred not only to Gregory but to many others of the same period, but the many difficulties of the problem proved too much for them. However, in 1668 partial success was achieved by Lord Brouncker who gave the area of the hyperbola $y = \frac{1}{x}$ from $x = 1$ to $x = 2$ by the series $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} + \cdots\cdots$ (Phil. Trans. Vol 1 No 34 Page 645) and in the same year the efforts of Nicholas Mercator (1640-87), a native of Holstein, were crowned with success when he published his chief work Logarithmotechnia giving the series

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \cdots\cdots\cdots$$

It will be noticed that Brouncker's result is merely a special case of the more general result given by Mercator, namely the case $x = 1$, so that his series is really a series for $\log_e 2$.

Mercator's Logarithmic Series.

To understand how Mercator happened upon this series one must bear in mind that since the logarithmic nature of the hyperbolic area was clearly understood, the real stumbling block was the quadrature of the rectangular hyperbola, which up till then no one had succeeded in effecting.

The person who paved the way for Mercator was undoubtedly John Wallis (1616-1703) + who in his elaborate work the Arithmetica Infinitorum (1656) came very near to solving the whole question of quadratures, one which had defied the attempts of even the best mathematicians of that century. Not that Wallis was in any way a pioneer in the subject, for the quadrature of many curves had been effected before he tackled the problem, and indeed Cavalieri, Fermat,/

+ Scott. Ch. 4 pp. 27-8 62-3.

Fermat, Roberval and Torricelli had all shown more or less independently that in the family of curves given by $y = x^n$ the area under the curve bore to that of the rectangle of the same base and altitude the ratio $\frac{1}{n+1}$, i.e the area from x_1 to x_2 was $\left[\frac{x^{n+1}}{n+1}\right]_{x_1}^{x_2}$. So long as the power was a whole number, the methods established were sufficient. The real merit of Wallis's work lay in his preception that the relation persisted for all values of n, be they positive, fractional or even negative, and he was able to generalise under one comprehensive law the work of his predecessors, showing that the areas of all curves may be found whose ordinate is expressed by any power of the abscissa, - 1 excepted since in this case $\frac{1}{n+1}$ becomes $\frac{1}{0}$, and Wallis did not know how to interpret this result.

But this case of $n = -1$ is the case of the rectangular hyperbola and that very one which was required in connection with logarithms, hence the reason for the long delay in the quadrature of this curve.

Now Wallis's next step was to show that this method of finding areas applies with equal validity to cases more complex, i.e. those in which the ordinate is equal to a compound expression such as $y = a + bx + cx^2 + dx^3$, where it is evident that we can assume the ordinate to be identical with the sum of the several ordinates a, bx, cx^2, dx^3 to each of which his ordinary rule for quadratures can be applied. Indeed Wallis had high hopes of being able to effect the quadrature of the circle in this way, but failed owing to his having no means of expanding the ordinate of the circle in terms of the abscissa x.

It was to this part of the Arithmetica Infinitorum that Mercator without a doubt owed the invention of his well known series. Wallis himself was well on the way to success in the quadrature of the rectangular hyperbola, and had even observed that particular fact which gave Mercator his triumph, namely that although the ordinate of the hyperbola is $\frac{1}{x}$ when the centre of the/

the hyperbola is taken as origin, by taking the origin at a point on the asymptote a unit's distance from the centre the ordinate is $\frac{1}{1+x}$, so that if that could be expressed as a series in terms of x the quadrature could be effected by his ordinary rule. He was unable to do this however, and therefore his efforts were fruitless.

But Mercator saw further than Wallis and succeeded in expanding the ordinate into an infinite series by means of common division, as Wallis himself had done in the case of fractions of the form $\frac{1}{1-x}$. Then considering each term of this series separately as representing a particular ordinate, he applied to it Wallis's method for curves whose ordinates are expressed by a single term, the sum of the partial areas so obtained giving him the value of the whole area.

By ordinary division he would get $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots\cdots$ which is a geometrical progression and convergent if $x < 1$

Therefore the area will be $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots\cdots\cdots\cdots$

But owing to the logarithmic nature of the area it is known to be log (1+x) hence, $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots\cdots\cdots$ which was the final step in the process of achieving a logarithmic series.

The series was also published in the Phil. Trans. Vol 1. No 38 Page 741. In addition to the singular result, the series is remarkable for two other reasons, namely, + that it was the first example given to the world of obtaining the quadrature of a curve by expanding its ordinate into an infinite series, and that the division of 1+x into 1 is claimed to be the first example of algebraic division done by the method used at the present day. Mercator himself describes the process step by step as a thing that was new or uncommon, but Charles Hutton ++ remarks that that method of division had been taught before by Dr. Wallis in his Opus Arithmeticum.

The novelty of the invention caused it to be received with general applause, and John Collins (1625-83) hastened to send Mercator's/

+ L.U.K. ++ Math. Intro. p.93.

Mercator's book to his friend Isaac Barrow, who in turn communicated it to his friend Newton. [+]

This happening had rather unexpected repercussions, for on perusing the book Newton immediately recognised in it the fundamental idea contained in original and unpublished works of his own, which he had completed a few years earlier. He himself tells us in his Commonplace Book, [++] "I borrowed Wallis's works and by consequence made these annotations out of Schooten and Wallis in the winter between the years 1664 and 1665. At such time I found the method of Infinite Series and in summer 1665, being forced from Cambridge by the plague, I computed the area of the Hyperbola at Boothby in Lincolnshire, to two and fifty figures by the same method." To evince that he had not borrowed from Mercator, and that his work was entirely original he immediately presented the manuscript in which he had explained his own methods to Barrow, this being his treatise, Analysis per aequationes numero terminorum infinitas.

Barrow was struck with astonishment [+++] at seeing such a rich collection of analytical discoveries, the importance of which was far greater than this particular one which then excited such general admiration, and he was probably even more surprised at their young author having been able to keep them so profoundly secret, and at his indifference in the matter of publication.

Barrow deemed his discovery too important to be kept to himself and immediately wrote to Collins about it, without disclosing the author's name. The glowing terms of Barrow's communication stirred Collins to entreat Barrow to procure for him the sight of such a precious manuscript. He, with Newton's permission, duly granted the request, and on 31st July 1669 Barrow sent the manuscript to Collins with no clue to the author's identity beyond saying that he was a friend staying at Cambridge, who had a powerful genius for such matters It was in a subsequent letter on 20th August that Barrow,/

[+] L.U.K. [++] Encyc. Brit. vol 19 p.583. [+++] et sequ. L.U.K.

Barrow, having expressed his pleasure at hearing the favourable opinion Collins had formed of the paper, revealed + that "the name of the author is Newton, a fellow of our college, and a young man, who is only in his second year since he took the degree of master of arts, and who, with an unparalleled genius, has made very great progress in this branch of mathematics."

Collins before returning the work took a copy of it, which being found after his death, among his papers, and published in 1711 has determined beyond dispute by the date it bore, at what period Newton made the memorable discovery of expansion by series and of the method of fluxions.

+ Encyc. Brit. vol 19 p. 584.

APPENDIX II

Although they have little to do with Briggs, there are one or two points which are of interest and which are a direct result of my researches in connection with Briggs, and I propose to give them here.

I. A new expression for $\dfrac{\sin n\theta}{\sin (n-1)\theta}$ and for $\dfrac{\sin n\theta}{\sin \theta}$.

In the eleventh chapter of the Trigonometria Britannica, Briggs in his alternative method of deriving the expression for the multiple chord in terms of the simple chord, proves a property from which, expressed in modern notation with $S_n = \sin n\theta$ can easily be derived.

$$\frac{S_2}{S_1} = \frac{S_1 + S_3}{S_2} = \frac{S_2 + S_4}{S_3} = \frac{S_3 + S_5}{S_4} = \cdots\cdots\cdots = \frac{S_{n-2} + S_n}{S_{n-1}}$$

Now let each ratio $= k$, where k in this case $= \dfrac{\sin 2\theta}{\sin\theta} = 2\cos\theta$.

Therefore we have

$$\frac{S_2}{S_1} = k$$

$$\frac{S_1}{S_2} + \frac{S_3}{S_2} = k \qquad \therefore \frac{S_3}{S_2} = k - \frac{S_1}{S_2} = k - \frac{1}{k}.$$

$$\frac{S_2}{S_3} + \frac{S_4}{S_3} = k \qquad \frac{S_4}{S_3} = k - \frac{S_2}{S_3} = k - \cfrac{1}{k - \cfrac{1}{k}}$$

$$\frac{S_3}{S_4} + \frac{S_5}{S_4} = k \qquad \frac{S_5}{S_4} = k - \frac{S_3}{S_4} = k - \cfrac{1}{k - \cfrac{1}{k - \cfrac{1}{k}}}$$

Similarly for $\dfrac{S_6}{S_5}$, $\dfrac{S_7}{S_6}$ etc.

$$\therefore \frac{S_n}{S_{n-1}} = k - \frac{1}{k-}\ \frac{1}{k-}\ \frac{1}{k-}\ \frac{1}{k-}\ \frac{1}{k-} \cdots\cdots \text{ to } (n-1) \text{ terms of } k.$$

$$\therefore \frac{\sin n\theta}{\sin (n-1)\theta} = 2\cos\theta - \frac{1}{2\cos\theta -}\ \frac{1}{2\cos\theta -}\ \frac{1}{2\cos\theta -}\ \frac{1}{2\cos\theta -} \cdots \text{ to } (n-1) \text{ terms.}$$

2).
$$\frac{S_n}{S_1} = \frac{S_2}{S_1}\cdot\frac{S_3}{S_2}\cdot\frac{S_4}{S_3}\cdot\frac{S_5}{S_4}\cdots\cdots\frac{S_n}{S_{n-1}}$$

$$= k.\left(k-\frac{1}{k}\right).\left(k-\frac{1}{k-\frac{1}{k}}\right)\left(k-\frac{1}{k-\frac{1}{k-\frac{1}{k}}}\right)\cdots\cdots \text{ to } n-1 \text{ factors.}$$

$$\therefore\ \frac{\sin n\theta}{\sin\theta} = 2\cos\theta.\left(2\cos\theta-\frac{1}{2\cos\theta}\right)\left(2\cos\theta-\frac{1}{2\cos\theta-\frac{1}{2\cos\theta}}\right)\left(2\cos\theta-\frac{1}{2\cos\theta-\frac{1}{2\cos\theta-\frac{1}{2\cos\theta}}}\right)$$

$$\cdots\cdots\cdots \text{ to } (n-1) \text{ factors.}$$

$$\therefore\ \frac{\sin n\theta}{\sin\theta} = \prod_{r=1}^{r=n-1}\left(2\cos\theta - \frac{1}{2\cos\theta-}\ \frac{1}{2\cos\theta-}\ \frac{1}{2\cos\theta-}\cdots\cdots \text{ to } r \text{ terms}\right)$$

3) These equal ratios can be generated from the scale of relation

$$\frac{S_{n-2}+S_n}{S_{n-1}} = k \qquad \text{where } S_0 = 0$$

$$\text{i.e.} \qquad S_n - kS_{n-1} + S_{n-2} = 0$$

Now obviously what has been proved above can be proved for the terms of any series obeying the above scale of relation, having any value and provided $S_0 = 0$

This scale of relation is of particular interest as James Gregory has shown that the roots of a Pellian equation obey just such a scale, and therefore new roots may be calculated from those already got.

Thus if $u_0, u_1, u_2, \cdots\cdots$ represent terms of a series obeying the scale of relation $u_n - ku_{n-1} + u_{n-2} = 0$, u_0 being equal to 0, then

$$\frac{u_n}{u_{n-1}} = k - \frac{1}{k-}\ \frac{1}{k-}\ \frac{1}{k-}\ \frac{1}{k-}\cdots\cdots \text{ to } (n-1) \text{ terms.}$$

and

$$\frac{u_n}{u_1} = \prod_{r=1}^{r=n-1}\left(k - \frac{1}{k-}\ \frac{1}{k-}\ \frac{1}{k-}\ \frac{1}{k-}\cdots\cdots \text{ to } r \text{ terms}\right).$$

APPENDIX III

Franciscus Vieta's Angular Sections.

II. I perused Vieta's Angular Sections carefully in order to compare with them the work of Briggs on this subject.

It is remarkable that Vieta states a method for expressing $\cos n\theta$ and $\sin n\theta$ in terms of $\cos\theta$ and $\sin\theta$, a method which he certainly knew before 1579 and which is nothing more nor less than De Moivre's Theorem for a positive integral exponent, viz. $\cos n\theta + i \sin n\theta = (\cos\theta + i \sin\theta)^n$ without the use of the symbol i, but giving in simple and explicit terms the rule of signs which is mechanically achieved by the use of i.

Now nowhere have I seen this pointed out, although it was at least a century later before Demoivre gave his Theorem to the world. It may be that the 1579 work of Vieta was rare and unknown in England, but it would be contained in the Angular Sections of Vieta published in Paris in 1615 and later in the Schooten edition of Vieta's collected works published in Leyden in 1646, so that it is really surprising that the first discovery of this well known theorem has never been credited to Vieta.

Vieta's Statement of De Moivre's Theorem.

Vieta has stated the theorem so clearly that I simply quote what he writes so that the reader can judge for himself whether or not Vieta is entitled to the honour of the discovery.

"Let there be two right angled triangles in which the acute angle of the first is a submultiple of the acute angle of the second: The sides of the second will have the following proportions.

The hypotenuse is proportional to that power of the hypotenuse of the first determined by the submultiple : the conditioned power is/

Right Angled Triangles.

	Simple Angle			Multiple Angle.		
		Sides about Right Angle				
	Hypot.	Base	Perpendicular.	Hypot.	Base	Perpendicular.
	Z	D	B.			
					Double.	
Twice	Z^2	D^2 $2D.B.$ B^2		Z^2	D^2 $-B^2$	$2D.B.$
					Treble.	
Triple	Z^3	D^3 $3D^2.B.$ $3D\,B^2$ B^3		Z^3	D^3 $-3D.B^2$	$3D^2.B.$ $-B^3$
					Four Times.	
Fourth	Z^4	D^4 $4D^3B.$ $6D^2B^2$ $4DB^3$ B^4		Z^4	D^4 $-6D^2.B^2$ $+B^4$	$4D^3B$ $-4D.B^3$
					Five Times.	
Fifth	Z^5	D^5 $5D^4B$ $10D^3B^2$ $10D^2.B^3$ $5D.B^4$ B^5		Z^5	D^5 $-10D^3.B^2$ $+5D.B^4$	$5D^4.B$ $-10D^2B^3$ $+B^5.$

is that which complies in degree with the size of the multiple, viz. the square for the second, the cube for the third, the fourth for the fourth and so on ad infinitum.

As regards the proportion of the sides about the right angle, it is produced from the base and perpendicular of the first as a binomial radix, in the power and the degrees as well : and the several homogenerus products are set forth in pairs, the first of which is positive, the next negative.

The base of the second (i.e. the multiple angle) is proportional to the first of these parts, the perpendicular to the other."

He then gives examples showing the relative proportions of the three sides where the multiple angle is twice, triple, quadruple, and quintuple the simple angle. He then proves the truth in these cases, and adds that the sides of the triangles containing multiple angles come from analogy with those proved.

For simplicity he further adds - "what has been determined by this method is set out more clearly in the table on the opposite page."

Could there be a simpler explanation of Demoivre's Theorem, and it was given ninety years before Demoivre was born!

Not content with this explanation, he gives it in geometrical terms, saying - "Let there be a number of right angled triangles in which the acute angle of the second is twice that of the first, of the third three times, of the fourth four times, and so on in natural sequence without a break.

Moreover let the perpendicular of the first triangle be taken as the first of the proportionals, its base as the second, and let the series be continued (i.e. the series is a geometrical progression first term sine θ and common ratio cotan θ).

He then gives the value of $\frac{\text{base}}{\text{perpendicular}}$ for the second, third, fourth etc. but I quote only one or two to show the law of formation "e.g. in the fourth, $\frac{\text{base}}{\text{perpendicular}} = \frac{\text{fifth} - 6\text{ third} + \text{first}}{4\text{ fourth} - 4\text{ second.}}$

in/

in the seventh, $\dfrac{\text{base}}{\text{perpendicular}} = \dfrac{\text{Eighth} - 21 \text{ sixth} + 35 \text{ fourth} - 7 \text{ second}}{7 \text{ seventh} - 35 \text{ fifth} + 21 \text{ third} - \text{first.}}$
and so to infinity, the proportionals being divided successively into pairs according to the sequence in the series, the first pair being positive, the next negative, and so on : the coefficients are taken as they are in the sequence of terms in the systematic generation of powers, from which they are found. All of these things are clear from an inspection of the table above."

Now this expressed in our modern notation where cos and sin may be taken for base and perpendicular, and which we shall denote by c and s, is simply as follows:-

Take, $(c+s)^n = c^n + \binom{n}{1} c^{n-1}s + \binom{n}{2} c^{n-2}s^2 + \binom{n}{3} c^{n-3}s^3 + \cdots\cdots + \binom{n}{r} c^{n-r}s^r + \cdots\cdots + s^n$

Group these in pairs, starting with first pair positive, and alternate pairs having different signs, then

$\cos n\theta$ = expression formed from algebraic sum of first terms in each pair, $\sin n\theta$ = expression formed from algebraic sum of second terms in each pair.

i.e. $\cos n\theta = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^4\theta - \binom{n}{6}\cos^{n-6}\theta\sin^6\theta + \cdots\cdots$

and $\sin n\theta = \binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{3}\cos^{n-3}\theta\sin^3\theta + \binom{n}{5}\cos^{n-5}\theta\sin^5\theta - \binom{n}{7}\cos^{n-7}\theta\sin^7\theta + \cdots\cdots$

and in addition

$$\frac{\cos n\theta}{\sin n\theta} = \cot n\theta = \frac{\cot^n\theta - \binom{n}{2}\cot^{n-2}\theta + \binom{n}{4}\cot^{n-4}\theta - \binom{n}{6}\cot^{n-6}\theta + \cdots\cdots}{\binom{n}{1}\cot^{n-1}\theta - \binom{n}{3}\cot^{n-3}\theta + \binom{n}{5}\cot^{n-5}\theta - \binom{n}{7}\cot^{n-7}\theta + \cdots\cdots}$$

so that it is quite obvious that these three well known results are due to Vieta, got by a method which was thought not to have been discovered until at least a century later.

This neglect of Vieta's discovery is strangely reminiscent of the fate which befell a large part of Briggs's work.

It is strange though that Charles Hutton, who was certainly familiar with the rest of Vieta's work should have been unacquainted with the above, but so it must have been, for he ascribes both results and method to Herman and the Bernoullis all of whom lived long after the time of Vieta.

INDEX

<u>Index, contd</u>.

Index, contd.

APPENDIX IV.

Since the completion and printing of the text of my Thesis, I have come upon "Newton's Interpolation Formulas" by Duncan C. Fraser, M.A., F.I.A., which has raised two points which are of such direct interest in relation to my research that I have inserted this new appendix in order to deal with them.

1. In pages 38-40 I advanced the theory that Briggs's rules for quinquesection were derived from his consideration of the properties of the sine difference table which were known to him. This speculation imagined Briggs reasoning as on page 40 (q.v.), discovering his relationships by careful inspection and investigation and confirming their generality of application by the process of trial and error.

Although this was pure speculation, it seemed a likely method to me. It cannot be denied however, that it would have strengthened my case very considerably, if only I could have quoted a truly authentic case of this method having been adopted by some mathematician of repute. Unfortunately at the time of writing I was unable to do so, and the speculation had to stand on its own merits.

That it was a happy one would now seem probable, for it would appear that the method indicated was certainly used, even after the time of Briggs by a most famous and eminent mathematician, none other than the celebrated Gottfried Leibnitz, who himself describes in detail exactly what he did to attain a certain result. This he does in a letter to Oldenburg (mentioned also on page 23), dated February 1672/3, an extract from which is quoted by Fraser on pages 35-40 of his book.

We owe the description to the fact that Leibnitz felt that, in a conversation with Pell on the difference table for cubes, he had laid himself open to the suspicion that "he had tried to appropriate the/

the credit of another man's ideas by suppressing the name of the discoverer" and was constrained "to take pains that no suspicion should remain in their minds .... by showing his rough notes, in which not only his discovery, but also the manner and the occasion of the discovery appear." He writes:-

"From my papers it appears that the occasion of the discovery was as follows: I was seeking a method of finding the difference of every kind of powers; just as it is known that the differences of the square numbers are the odd numbers..........

My mind being fixed on these ideas, as in the case of square numbers the differences are the odd numbers, so also I enquired what might be the differences of the cubes: and since these appeared to be irregular I sought the differences of the differences, until I found the third differences to be all sixes. This observation produced another. For I saw that the terms and the successive differences were generated from the preceding differences in the same way as all the successive terms arise from the primary differences, which I call on that account the generating differences, namely, in this case 0,1,6,6. Having come to this conclusion it remained to find by what kind of addition or multiplication, or combination of these, the successive terms could be produced from the generating differences. And <u>thus by solution and experiment</u> I perceived the first term, 0, to be composed of the first generating difference, 0, taken once, or by itself; the second term, 1, to be composed of the first generating difference, 0, taken once, and the second, 1, taken once; the third term, 8, of the first generating difference, 0, taken once, the second, 1, taken twice, and the third, 6, taken once; for $0 \times 1 + 1 \times 2 + 6 \times 1 = 8$; the fourth term, 27, of the first generating difference, 0, taken once, the second, 1, taken three times: the third, 6, taken three times, and the fourth, 6, taken once: for $0 \times 1 + 1 \times 3 + 6 \times 3 + 6 \times 1 = 27$ <u>and further calculation proved to me that this was general</u>.

This was the occasion of my observation, far otherwise from Mouton's/

Mouton's way of approaching it."

This admitted use by Leibnitz, and a similar use (seen later) by Newton, of this method I suggested, would indicate that mathematicians were not above making use of it, and since Leibnitz and Newton were probably by no means the first to do so, my contention that Briggs used the method no longer belongs to mere speculation but is raised to the realm of probability.

2. On page 97 in connection with Briggs's claim to the Binomial Theorem I wrote "Newton, of course, had an improved algebraic notation, and if he were acquainted with the work of Briggs, which I have seen suggested but have not been able to establish, there would be little merit in his statement of the theorem for positive integers but undoubtedly the greatest merit in his discovery of its extension to fractional indices."

It is this very point of Newton being acquainted with Briggs's work which I now wish to discuss, for Fraser puts forward [+]a reasoned and convincing argument in favour of such an acquaintance, basing his conclusions on the known mathematical interest of the times in logarithms and differences, and above all on a direct comparison of the methods of decisection formulated by Newton and Briggs. These are so nearly identical, that, although the evidence is very largely circumstantial, since nowhere does Newton mention Briggs's name or admit any indebtedness to him, it can scarcely be doubted that Newton found the source of his method in the work of Briggs, in which case he must have known the Arithmetica Logarithmica which contained it.

This is not to be greatly wondered at, for ever since the discovery of the logarithmic nature of the hyperbolic area, the whole question of logarithms had been very much in the air, and Newton was so interested in the matter that "in the summer of 1665, being forced from Cambridge by the plague he computed the area of the hyperbola at Boothby, Lincolnshire, to two and fifty figures." This latter fact caused him to write [++] "I am ashamed to say to what/

[+] Fraser pp 57-9. [++] Fraser p 48.

what a number of decimal places I carried these calculations being then at leisure. For, indeed, I took too much pleasure at that time in these investigations."

It seems justifiable to conclude that anyone as interested as that would have recourse to what had already been written on the subject, and in that connection nothing had yet superseded the the work of Briggs. Consequently Fraser argues that it is a natural supposition that when Newton went into the country in 1665 he took Briggs's Arithmetica Logarithmica with him. Granted that he did so, I would go further and say it was the first or London edition that he took with him. This apparently trifling point is rather important, for Fraser establishes his link between the two men chiefly on the strength of the resemblance between their methods of Decisection, and that of Briggs appears only in the first edition, so that if this was not the one known to Newton, Fraser's argument loses its point.

As was previously stated (vide page 58), the second or Batavian edition was published by Adrian Vlacq, unknown to Briggs, and therein he suppressed the two chapters on Decisection and Quinquesection which had appeared in the London edition, an omission which called forth a strong protest from Briggs in his Trigonometria Britannica, where he repeated his rules for Quinquesection.

We have strong grounds to believe that it was this very fact which deprived James Gregory of a knowledge of what Briggs had done in the field of differences, for Professor Turnbull has shown + that it is highly probable that it was to the second edition of the Arithmetica Logarithmica only that Gregory had access, and thus he was cut off from a wealth of material which was available to Newton, material which I am sure would have given him much room for meditation and thought, had he known it.

Again, it is very probable that Newton's Volume would be a first edition, for, although Professor Turnbull argues such were rare, which probably was the case in Scotland which is the place he refers/

+ Turnbull, p 166.

refers to, I scarcely think such would be the case round London, since, although the work is now very scarce and costly, it is said + that (in Briggs's time) surplus copies were hawked in the streets of London at eighteen pence each, and quite a number of these must have survived to the time of Newton.

The real issue however as far as my work is concerned is whether or not Newton established his Binomial Theorem as a result of his consideration of the arithmetical statement of it by Briggs, since I have shown there would have been little merit in so doing.

Let us consider the matter further, for in a letter to Oldenburg, dated 24th October 1676, a letter which was to be communicated to Leibnitz, Newton has himself placed on record the method by which he arrived at the Binomial Theorem, and it is of interest to note that he also adopts the method of trying to extract simply by the inspection of known results the general law inherent in them, and then by trial and error proving its correctness.

Newton writes as follows:-
"At the beginning of my mathematical studies, when I had fallen in with the works of our celebrated Wallis, and came to consider the series by the interpolation of which he brings out the areas of the circle and of the hyperbola, since in the series of curves whose basis or common axis is x and whose ordinates are

$(1-x^2)^{\frac{0}{2}}, \quad (1-x^2)^{\frac{1}{2}}, \quad (1-x^2)^{\frac{2}{2}}, \quad (1-x^2)^{\frac{3}{2}}, \quad (1-x^2)^{\frac{4}{2}}, \quad (1-x^2)^{\frac{5}{2}}, \quad$ etc.

if the areas of the alternate curves, which are

$x, \quad x - \frac{1}{3}x^3, \quad x - \frac{2}{3}x^3 + \frac{1}{5}x^5, \quad x - \frac{3}{3}x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7, \quad$ etc.

can be interpolated we should have the areas of the intermediate curves, the first of which, $(1-x^2)^{\frac{1}{2}}$, is the circle: I <u>noted</u> for these interpolations that in every case the first term was x, that the second terms $\frac{0}{3}x^3, \frac{1}{3}x^3, \frac{2}{3}x^3, \frac{3}{3}x^3$ etc. were in A.P.; and accordingly the two first terms of the series to be interpolated must be

$x - \frac{1}{2}\cdot\frac{x^3}{3}, \quad x - \frac{3}{2}\cdot\frac{x^3}{3}, \quad x - \frac{5}{2}\cdot\frac{x^3}{3},$ etc.

For the insertion of the remaining terms I considered that the denominators 1,3,5,7 etc. were in A.P. and so the numerical coefficients/

\+ Nature, No 2882 Vol. 115 Page 111.

coefficients had to be investigated for the numerators only. But in the given alternate areas these were the figures which express the powers of the number 11

namely 1; 1,1; 1,2,1; 1,3,3,1; 1 4,6,4,1; etc.

Then I set myself to enquire how in these groups of figures when the first two terms of a group were given the rest could be derived, and I found that assuming the second figure to be m, the rest would be produced by the continuous multiplication of the terms

$$\frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \quad \text{etc.} \; . \; . \; . \; . \; . \; .$$

I applied this rule therefore to obtain the intermediate series, and since for the circle the second term was $\frac{1}{2} \cdot \frac{x^3}{3}$, I put m = $\frac{1}{2}$.

. . . .

This was my first entrance into these speculations, which would have quite passed out of my memory, had I not cast my eyes on certain memoranda a few weeks ago.

But when I had obtained these results I soon began to consider that the terms $(1-x^2)^{0/2}$, $(1-x^2)^{2/2}$, $(1-x^2)^{4/2}$, $(1-x^2)^{6/2}$ could be interpolated in the same manner as the areas generated by them; and for this nothing more was necessary than the omission of the denominators 1,3,5,7 etc. in the terms expressing the areas.

That is to say, the coefficients of the terms of the quantity to be interpolated $(1-x^2)^{1/2}$ or $(1-x^2)^{3/2}$, or generally $(1-x^2)^m$ arise from the continuous multiplication of the terms of this series

$$\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \; . \; . \; . \; . \; .$$

So, for example

$$(1-x^2)^{\frac{1}{2}} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 \text{ etc.}$$

$$(1-x^2)^{3/2} = 1 - \frac{3}{2}x^2 + \frac{3}{8}x^4 + \frac{1}{16}x^6 \text{ etc.}$$

In this way therefore the general reduction of radical expressions into infinite series became known to me, ...

For in order that I might prove these operations I multiplied $1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6$ etc into itself and the result was $(1-x^2)$, the remaining terms to infinity vanishing through the continuation of the series."

It/

It is evident that Newton reduced the problem to the finding of the general law underlying the formation of the numbers 1; 1,1; 1,2,1; 1,3,3,1; 1,4,6,4,1.

This was the crux of the whole problem, a problem which Briggs by his law of proportionality had solved arithmetically, and yet this is the very matter wherein I stated there would be little credit to Newton were he acquainted with Briggs's previous work in it. I still hold to that, basing my opinion on the reasons stated in the Thesis.

Now Briggs stated his law of proportionality so simply and concisely that I do not see how Newton had he known of it could have failed to derive his expression $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$ etc. from it.+

But from Newton's own statement it seems evident that he was confronted by a definite problem in finding the general law of formation, and that it was by his own endeavours he succeeded in solving it without indebtedness to any other person. I can only conclude therefore that Newton was in no way obliged to Briggs for his discovery, and consequently that he was ignorant of at least this part of Briggs's work.

This admission however is in no way inconsistent with the conclusion reached by Fraser to the effect that Newton must have been acquainted with Briggs's Arithmetica Logarithmica, a conclusion which is in all likelihood true, for it was in this work that Briggs gave his method of decisection, and as has been stated that of Newton resembles it so closely that there can be little doubt that Newton derived his method from a knowledge of Briggs.

The Arithmetica Logarithmica however contains neither the Abacus Panchrestus nor its proportionality property on which rests Briggs's title to the honour of discovering the Binomial Theorem, so that Newton's apparent ignorance of them does not necessarily mean that he was unacquainted with Briggs's work but only that he had no knowledge of that work which contained those parts above-mentioned. This work was his other famous publication, the Trigonometria/

+ See pp. 95-6.

Trigonometria Britannica.

I am forced therefore to the conclusion that

(1) Newton's discovery of the Binomial Theorem was quite independent of that of Briggs.

(2) Newton was conversant with the work done by Briggs on logarithms but not with the work done by him on the generation of the Binomial Coefficients.

(3) Newton had a knowledge of Briggs's Arithmetica Logarithmica (1624) but not of his Trigonometria Britannica.