

INVOLUTIVE AUTOMORPHISMS AND REAL FORMS
OF KAC-MOODY ALGEBRAS

Stefan Clarke

A Thesis Submitted for the Degree of PhD
at the
University of St Andrews



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AND REAL FORMS
OF KAC-MOODY ALGEBRAS**

A thesis submitted to the University of St. Andrews
for the degree of Doctor of Philosophy

by Stefan Clarke



Department of Physics and Astronomy

University of St. Andrews

February 1996

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Professor J.F. Cornwell
Supervisor

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Abstract

Involutive automorphisms of complex affine Kac-Moody algebras (in particular, their conjugacy classes within the group of all automorphisms) and their compact real forms are studied, using the matrix formulation which was developed by Cornwell. The initial study of the $A_\ell^{(1)}$ series of affine untwisted Kac-Moody algebras is extended to include the complex affine untwisted Kac-Moody algebras $B_\ell^{(1)}$, $C_\ell^{(1)}$ and $D_\ell^{(1)}$. From the information obtained, explicit bases for real forms of these Kac-Moody algebras are then constructed. A scheme for naming some real forms is suggested. Further work is included which examines the involutive automorphisms and the real forms of $A_2^{(2)}$ and the algebra $G_2^{(1)}$ (which is based upon an exceptional simple Lie algebra). The work involving the algebra $A_2^{(2)}$ is part of work towards extending the matrix formulation to twisted Kac-Moody algebras. The analysis also acts as a practical test of this method, and from it we may infer different ways of using the formulation to eventually obtain a complete picture of the conjugacy classes of the involutive automorphisms of all the affine Kac-Moody algebras.

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1 Involutive automorphisms of Kac-Moody algebras and their real forms

1.1 Introduction

In Lie algebra theory, the reverse process of complexification was examined at an early date by both Cartan and Gantmacher. That is, they were engaged in the task of finding real Lie algebras whose complexifications were given complex Lie algebras. The method suggested by Cartan (see [5]) is of particular interest and will be discussed in greater detail later in this thesis. In the work of Cartan and of Gantmacher [16,17], there is an extremely strong connection between the real forms of a given semi-simple complex Lie algebra and the involutive automorphisms of the compact real form of the same semi-simple Lie algebra.

In recent years, the finite-dimensional Lie algebras have been supplemented by a new class of infinite-dimensional Lie algebras (known better as Kac-Moody Lie algebras, or simply as Kac-Moody algebras). It had been shown by Serre that each semi-simple Lie algebra could be found by starting with a Cartan matrix, effectively reversing the process by which the Lie algebras had been classified by Cartan. Kac [22] and Moody [25], [26], [27], working independently, generalised Serre's method to produce the algebras now known as Kac-Moody algebras. Of these infinite-dimensional algebras, the most widely-studied are the affine Kac-Moody algebras. The others, known as indefinite Kac-Moody algebras, are not as well-known or studied. The Kac-Moody algebras have been further supplemented by the study of Kac-Moody superalgebras, although these are not within the scope of this present work.

To a large extent, the study of the affine Kac-Moody algebras has mirrored the study of the finite dimensional Lie algebras which has already taken place. This is not surprising, given the evolution of the Kac-Moody algebras. There are, therefore, many properties and definitions which have been “carried over” from Lie algebra theory to Kac-Moody theory, with little or no modification in some cases. These include, amongst others, the property of compactness, roots, and a symmetric bilinear form (analogous to the Killing form). These will be mentioned in more detail in the remainder of this chapter. Importantly, the notion of a real form of an affine Kac-Moody algebra may be introduced, and the various processes which have been used to find (and identify) real forms of Lie algebras may, where possible, be adapted and used to generate real forms of the affine complex Kac-Moody algebras. The method which is, perhaps, the most elegant and simple is that suggested by the work of Cartan. In order to use this, the involutive automorphisms of the compact real form must be known. Hence, a knowledge of the automorphisms of the affine Kac-Moody algebras is desirable, and also a method by which they can readily be found and analysed. The automorphisms of affine Kac-Moody algebras have been investigated by a number of authors, notably Kobayashi [23] and Levstein [24]. In addition, Gorman *et al* [19] have considered those automorphisms which are “Cartan-preserving” for both the untwisted and twisted Kac-Moody algebras. The Cartan-preserving algebras are important because each conjugacy class of involutive automorphisms contains one or more Cartan-preserving automorphism (see Levstein).

A general method for the analysis of automorphisms of affine Kac-Moody algebras was developed by Cornwell, and used to investigate the conjugacy classes of the involutive automorphisms of the algebras $A_\ell^{(1)}$ for $\ell \geq 1$. For a full account of this, see [8]. This method, the “matrix formulation”, is the one which will be employed throughout this thesis. In addition to the classification of involutive automorphisms and the generation of real forms, the analysis will serve to test the matrix formulation for other affine Kac-Moody algebras, thus identifying any limitations or problems which require further investigation. Work in similar areas has been undertaken by a number of

others. Andruskiewitsch [1] has investigated the forms of Kac-Moody algebras that are “almost compact”, having in an earlier paper presented a construction of k -forms of symmetrisable derived Kac-Moody algebras, where k is a field of characteristic zero with algebraic closure \bar{k} . Berman and Pianzola [2] have, likewise, investigated generators for real forms, with some of their work involving “non-Euclidean” (or non-affine) algebras. It is worth mentioning briefly the implications that these algebras have for other fields. The strong connection with the Virasoro algebras and conformal field theory is well-documented, with Goddard and Olive’s review [18] on Kac-Moody and Virasoro algebras being a prime example of this fact. The part played by the Kac-Moody algebras is also discussed in the work of (amongst others) Bernard [3], Bouwknegt [4], Font [15] and Walton [30].

The second section of this chapter will recall some definitions, properties and observations pertinent to the rest of the thesis. Subsequent chapters will analyse the affine Kac-Moody algebras as described above. Chapter 2 is concerned with the complex untwisted algebra $A_1^{(1)}$, and the following chapter with the complex untwisted algebra $A_2^{(1)}$. Chapter 4 examines the remaining algebras of the series $A_\ell^{(1)}$. The basis for much of the work in these three chapters is the published work of Cornwell [8-11]. The following chapters investigate, respectively, the affine complex untwisted algebras $B_\ell^{(1)}$, $C_\ell^{(1)}$ and $D_\ell^{(1)}$. The algebra $D_4^{(1)}$ is sufficiently different from the others in the series $D_\ell^{(1)}$ to merit inclusion in a chapter of its own. Chapter 9 is concerned with the exceptional Kac-Moody algebra $G_2^{(1)}$. We then encounter some of the twisted affine Kac-Moody algebras. Following an initial discussion of the twisted algebras in general, an investigation of $A_2^{(2)}$ takes place. This study is intended to find out to what extent the matrix formulation (which is formulated specifically for untwisted algebras) is suitable for adaptation to the twisted algebras.

1.2 Lie algebras and untwisted affine Kac-Moody algebras

Since we are extending concepts of a Lie algebraic origin, it is customary to adopt notation from Lie theory when examining the affine Kac-Moody algebras. The following information is contained in greater detail in [12-14] and in other places in the literature, and is placed here for convenience in this thesis. We assume that $\tilde{\mathcal{L}}^{(0)}$ is a semi-simple complex Lie algebra, unless otherwise stated.

Let $\tilde{\mathcal{L}}^{(0)}$ be of complex dimension n^0 , with basis elements $a_1^0, a_2^0, \dots, a_{n^0}^0$. Recall that the structure constants c_{ij}^k are defined in terms of the Lie product ($[,]$) by

$$[a_i^0, a_j^0] = \sum_{k=1}^{n^0} c_{ij}^k a_k^0. \quad (1.2.1)$$

In addition, the adjoint representation of $\tilde{\mathcal{L}}^{(0)}$ is defined in terms of the Lie product (or commutator). This is the representation of $\tilde{\mathcal{L}}^{(0)}$ in which every $a^0 \in \tilde{\mathcal{L}}^{(0)}$ has as its representative the $n^0 \times n^0$ matrix $\mathbf{ad}(a^0)$ defined by

$$[a^0, a_j^0] = \sum_{k=1}^{n^0} \left\{ \mathbf{ad}(a^0) \right\}_{kj} a_k^0 \quad (j = 1, 2, \dots, n^0), \quad (1.2.2)$$

The Killing form $B^{(0)}(a^0, b^0)$ is then defined by

$$B^{(0)}(a^0, b^0) = \text{tr} \left\{ \mathbf{ad}(a^0) \mathbf{ad}(b^0) \right\}, \quad (1.2.3)$$

where $\text{tr}(\mathbf{A})$ is the trace of the matrix \mathbf{A} . The Killing form is a non-degenerate symmetric bilinear form.

The Cartan subalgebra of the algebra $\tilde{\mathcal{L}}^{(0)}$ is defined to be a subalgebra $\mathcal{H}^{(0)}$ of $\tilde{\mathcal{L}}^{(0)}$ which is such that

1. It is a maximal Abelian subalgebra of $\tilde{\mathcal{L}}^{(0)}$

2. For all $h^0 \in \mathcal{A}^{(0)}$, the matrix $\mathbf{ad}(h^0)$ is completely reducible.

It is well-documented in the literature that each semi-simple complex Lie algebra $\tilde{\mathcal{L}}^{(0)}$ does possess at least one Cartan subalgebra. Furthermore, it is well-known that each Cartan subalgebra of $\tilde{\mathcal{L}}^{(0)}$ may be mapped into any other Cartan subalgebra of $\tilde{\mathcal{L}}^{(0)}$ by an automorphism of $\tilde{\mathcal{L}}^{(0)}$. Hence, all of the Cartan subalgebras have the same dimension, ℓ^0 , which is called the rank of $\tilde{\mathcal{L}}^{(0)}$. The subalgebra $\mathcal{A}^{(0)}$ may be assumed to have the basis $h_1^0, h_2^0, \dots, h_{\ell^0}^0$. The general element of $\mathcal{A}^{(0)}$ is given by

$$h^0 = \sum_{j=1}^{\ell^0} \mu_j^0 h_j^0 \quad (\mu_j^0 \in \mathbb{C}). \quad (1.2.4)$$

Now, since $\mathcal{A}^{(0)}$ is Abelian, all irreducible representations of it are one-dimensional, and so the matrices $\mathbf{ad}(h_j^0)$ (for $j=1,2,\dots,\ell^0$) are simultaneously diagonalisable.

Since a similarity transformation corresponds to a change of basis, there thus exists a set of basis elements $h_1^0, \dots, h_{\ell^0}^0, a_1^0, \dots, a_{n^0-\ell^0}^0$ (of $\tilde{\mathcal{L}}^{(0)}$) such that

$$[h_j^0, a_k^0] = \alpha_k^0(h_j^0) a_k^0 \quad (\alpha_k^0(h_j^0) \in \mathbb{C}). \quad (1.2.5)$$

Then, with h^0 given by (1.2.4), the linear functional α_k^0 is defined on $\mathcal{A}^{(0)}$ by

$$\alpha_k^0(h^0) = \sum_{j=1}^{\ell^0} \mu_j^0 \alpha_k^0(h_j^0). \quad (1.2.6)$$

Each linear functional α_k^0 ($k=1,\dots,n^0-\ell^0$) defined in this manner is called a non-zero root of $\tilde{\mathcal{L}}^{(0)}$. For any non-zero root α^0 of $\tilde{\mathcal{L}}^{(0)}$, those elements $a_{\alpha^0}^0$ which are such that $[h^0, a_{\alpha^0}^0] = \alpha^0(h^0) a_{\alpha^0}^0$ form a subspace (denoted by $\tilde{\mathcal{L}}_{\alpha^0}^{(0)}$) which is known as the root subspace corresponding to the root α^0 . The algebra $\tilde{\mathcal{L}}^{(0)}$ is the vector space direct sum of the Cartan subalgebra and of the root subspaces corresponding to all non-zero roots of $\tilde{\mathcal{L}}^{(0)}$. It is always the case that, if α^0 is a root of $\tilde{\mathcal{L}}^{(0)}$, then so is $-\alpha^0$. One fact which is less immediately obvious is that each non-zero root of $\tilde{\mathcal{L}}^{(0)}$ may be written as

$$\alpha^0 = \sum_{k=1}^{\ell^0} \lambda_k^\alpha \beta_k^0, \quad (1.2.7)$$

where the quantities λ_k^α are either all non-negative integers or non-positive integers, and β_k^0 (for $k = 1, \dots, \ell^0$) form a set of non-zero roots called the simple roots. The root α^0 is called either positive (if the integers λ_k^α are all non-negative) or called negative otherwise. It will be assumed implicitly from now on that the set $\alpha_1^0, \dots, \alpha_{\ell^0}^0$ is a set of simple roots. The set of non-zero roots of $\tilde{\mathcal{L}}^{(0)}$ will be denoted by Δ^0 . The set of positive roots is denoted by Δ_+^0 and the set of negative roots by Δ_-^0 . The group of linear transformations that sends the set of roots of an algebra to itself is called the group of rotations of that algebra. One particular rotation to which we assign a name is the so-called ‘‘Cartan involution’’, for which the corresponding rotation of the roots is the transformation $\tau(\alpha_k) = -\alpha_k$.

To each root α^0 of Δ^0 , a unique element $h_{\alpha^0}^0$ is associated according to the prescription that $B^{(0)}(h_{\alpha^0}^0, h^0) = \alpha^0(h^0)$ for all elements h^0 of $\mathcal{H}^{(0)}$. The elements $h_{\alpha_j^0}^0$ (for $j = 1, \dots, \ell^0$) form a basis for $\mathcal{H}^{(0)}$. The basis element of the root subspace $\tilde{\mathcal{L}}_{\alpha^0}^{(0)}$ is denoted by $e_{\alpha^0}^0$. Clearly, there is a certain amount of choice regarding the basis elements. However, the Weyl canonical form provides a set of commutation relations for these basis elements, and it is this particular canonical form of the basis that will be used in this thesis. The basis elements of the Weyl canonical form satisfy the following set of commutation relations.

1. For $j, k = 1, 2, \dots, \ell^0$, the elements $h_{\alpha_j^0}^0, h_{\alpha_k^0}^0$ satisfy $[h_{\alpha_j^0}^0, h_{\alpha_k^0}^0] = 0$.
2. The basis elements $e_{\alpha^0}^0, e_{-\alpha^0}^0$ are chosen such that $B^{(0)}(e_{\alpha^0}^0, e_{-\alpha^0}^0) = -1$. In addition, with $\alpha^0 = \sum_{j=1}^{\ell^0} k_j^{\alpha^0} \alpha_j^0$, they satisfy

$$[e_{\alpha^0}^0, e_{-\alpha^0}^0] = -\sum_{j=1}^{\ell^0} k_j^{\alpha^0} h_{\alpha_j^0}^0.$$

3. For any $h^0 \in \mathcal{H}^{(0)}$ and any non-zero root α^0 , we have

$$\left[h^0, e_{\alpha^0}^0 \right] = \alpha^0(h^0) e_{\alpha^0}^0. \quad (1.2.8)$$

4. For any non-zero roots α^0, β^0 where $\alpha^0 + \beta^0$ is not a root (zero or non-zero), we have

$$\left[e_{\alpha^0}^0, e_{\beta^0}^0 \right] = 0. \quad (1.2.9)$$

Where $\alpha^0 + \beta^0$ is a non-zero root, then $\left[e_{\alpha^0}^0, e_{\beta^0}^0 \right] = N_{\alpha^0, \beta^0} e_{\alpha^0 + \beta^0}^0$. The convention associated with the Weyl canonical form is that N_{α^0, β^0} is non-zero, and

$$N_{\alpha^0, \beta^0} = N_{-\alpha^0, -\beta^0}. \quad (1.2.10)$$

If Γ is a representation of the compact real form of $\tilde{\mathcal{L}}^{(0)}$, then it may be assumed that it provides a representation of the compact real form by anti-Hermitian matrices. As a consequence of this, it may be assumed that $\Gamma(h_{\alpha^0}^0)$ is Hermitian, and that $\Gamma(e_{\alpha^0}^0) = -\Gamma^*(e_{-\alpha^0}^0)$ for each non-zero root α^0 .

The above is a brief resume of the structure of a general semi-simple complex Lie algebra. If we retain the definitions and conventions, then we may construct a complex untwisted affine Kac-Moody algebra $\tilde{\mathcal{L}}^{(1)}$ from the semi-simple complex Lie algebra $\tilde{\mathcal{L}}^{(0)}$ in a straightforward manner. The precise method of construction is given in [8], and this yields an algebra $\tilde{\mathcal{L}}^{(1)}$ whose general element has the form (given in terms of the basis elements of $\tilde{\mathcal{L}}$)

$$\sum_{j \in \mathbb{Z}} \sum_{p=1}^{n^0} \mu_{jp} t^j \otimes a_p^0 + \mu_c c + \mu_d d, \quad (1.2.11)$$

where only a finite number of the quantities μ_{jp} are non-zero. The basis elements will be assumed to be basis elements of the compact real form of $\tilde{\mathcal{L}}^{(0)}$, whose basis elements, in terms of the Weyl canonical basis elements of $\tilde{\mathcal{L}}^{(0)}$ are

$$ih_{\alpha_j^0} \quad (j=1, 2, \dots, \ell^0), \quad (1.2.12)$$

$$\left. \begin{aligned} & \left(e_{\alpha^0}^0 + e_{-\alpha^0}^0 \right) \\ & i \left(e_{\alpha^0}^0 - e_{-\alpha^0}^0 \right) \end{aligned} \right\} \text{ for each non - zero root } \alpha^0. \quad (1.2.13)$$

The quantities c and d (called the central charge and the scaling element) are introduced to be such that, for all $\alpha^0, \beta^0 \in \tilde{\mathcal{L}}^{(0)}$, they satisfy

$$\begin{aligned} [t^j \otimes a^0, t^k \otimes b^0] &= t^{j+k} \otimes [a^0, b^0] + j\delta^{j+k,0} B^{(0)}(a^0, b^0)c, \\ [d, t^j \otimes a^0] &= jt^j \otimes a^0, \\ [t^j \otimes a^0, c] &= 0, \\ [d, c] &= 0. \end{aligned} \quad (1.2.14)$$

It is sometimes the convention to add the superscript "0" to elements that belong to semi-simple Lie algebras in order to distinguish them from elements of their associated untwisted affine Kac-Moody algebra. This convention will not generally be repeated, unless it is to be made explicit that a^0 is a member of the semi-simple complex Lie algebra. In most cases, there is no possibility of confusion.

Following the definition of the Killing form for $\tilde{\mathcal{L}}^{(0)}$, a symmetric bilinear form $B^{(1)}(,)$ is defined for $\tilde{\mathcal{L}}^{(1)}$ by

$$\begin{aligned} B^{(1)}(t^j \otimes a^0, t^k \otimes b^0) &= \delta^{j+k,0} B^{(0)}(a^0, b^0), \\ B^{(1)}(t^j \otimes a^0, c) &= B^{(1)}(t^j \otimes a^0, d) = 0, \\ B^{(1)}(c, c) &= B^{(1)}(d, d) = 0, \\ B^{(1)}(d, c) &= 1. \end{aligned} \quad (1.2.15)$$

Similarly, a Cartan subalgebra $\mathcal{A}^{(1)}$ is defined for $\tilde{\mathcal{L}}^{(1)}$ by

$$\mathcal{A}^{(1)} = (\mathbb{C}c) \oplus (\mathbb{C}d) \oplus \sum_{k=1}^{\ell^0} \mathbb{C} \left(t^0 \otimes h_{\alpha_k^0}^0 \right). \quad (1.2.16)$$

If α^0 is any linear functional on $\mathcal{H}^{(0)}$, then its extension to $\mathcal{H}^{(1)}$ is defined to be the linear functional α , which is such that $\alpha(t^0 \otimes h^0) = \alpha^0(h^0)$ (for $h^0 \in \mathcal{H}^{(0)}$) and $\alpha(c) = \alpha(d) = 0$. One important linear functional defined on $\mathcal{H}^{(1)}$ is δ , defined by

$$\delta(t^0 \otimes h^0) = \delta(c) = 0 \quad (\forall h^0 \in \mathcal{H}^{(0)}), \quad (1.2.17)$$

$$\delta(d) = 1. \quad (1.2.18)$$

For any linear function α defined on $\mathcal{H}^{(1)}$, we define an element $h_\alpha^{(1)} \in \mathcal{H}^{(1)}$ such that for all $h \in \mathcal{H}^{(1)}$, $B^{(1)}(h, h_\alpha^{(1)}) = \alpha(h)$. The functional δ is important because the roots of the complex affine untwisted Kac-Moody algebra $\tilde{\mathcal{L}}^{(1)}$ are expressed in terms of it. In fact, $h_\delta^{(1)} = c$. The real roots of $\tilde{\mathcal{L}}^{(1)}$ consist of extensions of the positive roots of $\tilde{\mathcal{L}}^{(0)}$, together with the roots of the form $j\delta + \alpha$, where α is the extension of a non-zero root of $\tilde{\mathcal{L}}^{(0)}$ and j is a natural number. The imaginary roots are all of the form $j\delta$, where j is a natural number. A linear functional α_0 on $\mathcal{H}^{(1)}$ is defined by $\alpha_0 = \delta - \alpha_H$, where α_H is the extension of the highest root of $\tilde{\mathcal{L}}^{(0)}$.

A symmetric bilinear form $\langle, \rangle^{(1)}$ may be defined on $\mathcal{H}^{(1)}$. Let α, β be extensions of linear functionals on $\mathcal{H}^{(0)}$. We then let $\langle \alpha, \beta \rangle^{(1)} = \langle \alpha^0, \beta^0 \rangle^{(0)}$. Furthermore, $\langle \delta, \alpha \rangle^{(1)}, \langle \delta, \delta \rangle^{(1)}$ are zero. With this symmetric bilinear form, we are able to define the generalised Cartan matrix $\mathbf{A}^{(1)}$, whose index set is taken to be $0, 1, \dots, \ell^0$. If the Cartan matrix of $\tilde{\mathcal{L}}^{(0)}$ is denoted by $\mathbf{A}^{(0)}$, then

$$\left(\mathbf{A}^{(1)}\right)_{jk} = \left(\mathbf{A}^{(0)}\right)_{jk} \quad (j, k = 1, 2, \dots, \ell^0), \quad (1.2.18)$$

$$\left(\mathbf{A}^{(1)}\right)_{0k} = -\frac{2\langle \alpha_H^0, \alpha_k^0 \rangle^{(0)}}{\langle \alpha_H^0, \alpha_H^0 \rangle^{(0)}} \quad (k = 1, 2, \dots, \ell^0), \quad (1.2.19)$$

$$\left(\mathbf{A}^{(1)}\right)_{k0} = -\frac{2\langle \alpha_H^0, \alpha_k^0 \rangle^{(0)}}{\langle \alpha_k^0, \alpha_k^0 \rangle^{(0)}} \quad (k = 1, 2, \dots, \ell^0), \quad (1.2.20)$$

and $\left(\mathbf{A}^{(1)}\right)_{00} = 2$.

Throughout the rest of this thesis, notation will be used which is now defined. We define $e_{j\delta+\alpha}$ (where $\alpha \in \Delta^0$) and $e_{j\delta}^k$ (where $1 \leq k \leq \ell$) by

$$\begin{aligned} e_{j\delta+\alpha} &= t^j \otimes e_\alpha^0 \quad (j \in \mathbf{Z} \setminus \{0\}), \\ e_{j\delta}^k &= it^j \otimes h_{\alpha_k}^0 \quad (j \in \mathbf{Z} \setminus \{0\}, 1 \leq k \leq \ell). \end{aligned} \quad (1.2.21)$$

The ‘‘compact involution’’ is used in the definition of the compact real form of a complex affine Kac-Moody algebra. Let us consider an untwisted complex affine Kac-Moody algebra $\tilde{\mathcal{L}}^{(1)}$. The general element a of $\tilde{\mathcal{L}}^{(1)}$ has already been given in (1.2.6). The compact involution ϕ_C of $\tilde{\mathcal{L}}^{(1)}$ is then defined by

$$\phi_C(a) = \sum_{j \in \mathbf{Z}} \sum_{p=1}^{n^0} \mu_{jp}^* (t^{-j} \otimes a_p^0) - \mu_c^* c - \mu_d^* d. \quad (1.2.22)$$

The compact real form ($\mathcal{L}_C^{(1)}$) of the complex untwisted affine Kac-Moody algebra $\tilde{\mathcal{L}}^{(1)}$ is then defined to consist of those elements a of $\tilde{\mathcal{L}}^{(1)}$ which satisfy $\phi_C(a) = a$. Thus, in terms of the Weyl canonical basis elements of $\mathcal{L}^{(0)}$, the basis elements of $\mathcal{L}_C^{(1)}$ are given by the following, which are numbered (1.2.23)

$$\begin{aligned} &i(h_{\alpha_k}) \quad (1 \leq k \leq \ell), \\ &(e_{j\delta}^k + e_{-j\delta}^k) \quad (1 \leq k \leq \ell, j = 1, 2, \dots), \\ &i(e_{j\delta}^k - e_{-j\delta}^k) \quad (1 \leq k \leq \ell, j = 1, 2, \dots), \\ &(e_\alpha + e_{-\alpha}) \quad (\alpha \in \Delta_+^0), \\ &i(e_\alpha - e_{-\alpha}) \quad (\alpha \in \Delta_+^0), \\ &(e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \quad (\alpha \in \Delta^0, j = 1, 2, \dots), \\ &i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \quad (\alpha \in \Delta^0, j = 1, 2, \dots), \\ &ic, \\ &id. \end{aligned}$$

The definition of the compact real form of a complex twisted affine Kac-Moody algebra will be explained later in the thesis, when twisted algebras are being examined.

At this point, some terms used in the thesis, together with some non-standard notation and shorthand will be explained. The first items are the terms “offdiag”, “dsum” and “offsum”. The expression “diag{ a, b, \dots, y, z }” is well-known shorthand for diagonal matrices. We define

$$\text{offdiag}\{a, b, \dots, y, z\} = \begin{bmatrix} 0 & 0 & \dots & 0 & a \\ 0 & 0 & \dots & b & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & y & \dots & 0 & 0 \\ z & 0 & \dots & 0 & 0 \end{bmatrix}. \quad (1.2.24)$$

Similarly, (for square matrices $\mathbf{a}, \mathbf{b}, \dots, \mathbf{y}, \mathbf{z}$) we define

$$\text{dsum}\{\mathbf{a}, \mathbf{b}, \dots, \mathbf{y}, \mathbf{z}\} = \begin{bmatrix} \mathbf{a} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{b} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{y} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{z} \end{bmatrix}, \quad (1.2.25)$$

$$\text{offsum}\{\mathbf{a}, \mathbf{b}, \dots, \mathbf{y}, \mathbf{z}\} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{a} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{b} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{y} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{z} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \end{bmatrix}. \quad (1.2.26)$$

The expression $\mathbf{1}_k$ will be used for the $k \times k$ unit matrix, and $\mathbf{K}_j = \text{offdiag}\{1, 1, \dots, 1, 1\}$ and is a $j \times j$ matrix.

1.3 The matrix formulation of Kac-Moody algebras

It was stated previously that one of the secondary intentions of this work was to investigate the extent to which the "matrix formulation" is sufficient for determining the conjugacy classes of the involutive automorphisms within the group of all automorphisms of an untwisted complex affine Kac-Moody algebra. This formulation was developed by Cornwell (see [8]) and later applied to the algebras $A_\ell^{(1)}$, $B_\ell^{(1)}$ and $C_\ell^{(1)}$ (for $\ell \geq 1$) (see [6-11]). The formulation in question enables all of the automorphisms of an untwisted complex affine Kac-Moody algebra to be examined. It extends related work of Levstein [24] concerning the "derived subalgebra" $\tilde{\mathcal{L}}'$, which is defined to be $[\mathcal{L}, \mathcal{L}]$. In the matrix formulation, the automorphisms are of four different types, namely 1a, 1b, 2a and 2b. A detailed explanation of the derivation of the formulation, together with sundry formulae relating to the conditions for an automorphisms to be conjugate, and for the products of automorphisms, is contained in [8]. An account of the matrix formulation is now given, with only the most relevant points, definitions and conditions being given. Let Γ be a faithful irreducible representation (of dimension d_Γ) of the complex semi-simple Lie algebra $\tilde{\mathcal{L}}^{(0)}$, with γ such that $\text{tr} \left\{ \Gamma(a^0) \Gamma(b^0) \right\} = \gamma B^{(0)}(a^0, b^0)$ for all $a^0, b^0 \in \tilde{\mathcal{L}}^{(0)}$. The number γ is called the Dynkin index of the representation Γ . It then follows that the general element (given in (1.2.11)) of the affine untwisted Kac-Moody algebra $\tilde{\mathcal{L}}$ may be expressed in the form

$$\mathbf{a}(t) + \mu_c c + \mu_d d, \quad (1.3.1)$$

where the first term in the expression (1.3.1) is given by

$$\mathbf{a}(t) = \sum_{j \in \mathbb{Z}} \sum_{p=1}^{n^0} \mu_{jp} t^j \Gamma(a_p^0). \quad (1.3.2)$$

Now, each automorphism may be expressed in terms of a matrix $\mathbf{U}(t)$, a non-zero complex parameter u and an arbitrary complex parameter ξ . The matrix $\mathbf{U}(t)$ is assumed to be invertible, with both it and its inverse having Laurent polynomial entries. These are often written as a triple in the form $\{\mathbf{U}(t), u, \xi\}$. On occasions where only the matrix quantity $\mathbf{U}(t)$ is of concern, we will often refer to the “automorphism generated by $\mathbf{U}(t)$ ”. The automorphisms are specified by their actions upon the “matrix part”, their actions upon c and their actions upon d . These may be summarised as follows:

1. The action on the matrix part $\mathbf{a}(t)$. In general, the automorphism $\phi = \{\mathbf{U}(t), u, \xi\}$ has the following effect upon a matrix part $\mathbf{a}(t)$:

$$\phi(\mathbf{a}(t)) = \mathbf{U}(t)\theta(\mathbf{a}(t))\mathbf{U}(t)^{-1} + \frac{1}{\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \theta(\mathbf{a}(t)) \right) \right\} c, \quad (1.3.3)$$

where the quantity $\theta(\mathbf{a}(t))$ is defined differently for different types of automorphism by

$$\theta(\mathbf{a}(t)) = \begin{cases} \mathbf{a}(ut) & \text{(for type 1a automorphisms)} \\ -\tilde{\mathbf{a}}(ut) & \text{(for type 1b automorphisms)} \\ \mathbf{a}(ut^{-1}) & \text{(for type 2a automorphisms)} \\ -\tilde{\mathbf{a}}(ut^{-1}) & \text{(for type 2b automorphisms).} \end{cases} \quad (1.3.4)$$

It is assumed that the quantity $\mathbf{U}(t)\theta(\mathbf{a}(t))\mathbf{U}(t)^{-1}$ does represent a member of the affine Kac-Moody algebra $\tilde{\mathcal{L}}^{(1)}$.

2. The action on the elements c and d . This is summarised by

$$\begin{aligned} \phi(c) &= \mu c, \\ \phi(d) &= \mu(\Phi(\mathbf{U}(t)) + \xi c + d), \end{aligned} \quad (1.3.5)$$

where $\mu = 1$ for type 1a and type 1b automorphisms, $\mu = -1$ for type 2a and type 2b automorphisms, and

$$\Phi(\mathbf{U}(t)) = \left\{ -t \frac{d\mathbf{U}(t)}{dt} \mathbf{U}(t)^{-1} + \frac{1}{d_{\Gamma}} \text{tr} \left(t \frac{d\mathbf{U}(t)}{dt} \mathbf{U}(t)^{-1} \right) \mathbf{1}_{d_{\Gamma}} \right\}. \quad (1.3.6)$$

(The notation “tr” in the above refers to the “trace” of the matrix in question). It was also noted in the general theory of the matrix formulation that two triples $\{\mathbf{U}(t), u, \xi\}$ and $\{\mathbf{U}'(t), u', \xi'\}$ are identical if, and only if $u = u'$, $\xi = \xi'$ and $\mathbf{U}'(t) = \eta t^k \mathbf{U}(t)$, where η is some non-zero complex number and k is an integer. It follows, therefore, that a matrix $\mathbf{U}(t)$ which generates an automorphism (of a complex untwisted affine Kac-Moody algebra) is arbitrary up to a common multiplicative factor of the form ηt^k . This fact will not always be stated explicitly in the main body of the thesis. An automorphism corresponding to the triple $\{\mathbf{U}(t), u, \xi\}$ is involutive if certain conditions (dependent upon the type of automorphism) are satisfied. For example, a type 1a automorphism corresponding to the triple $\{\mathbf{U}(t), u, \xi\}$ is involutive if, and only if the following are all satisfied:

$$\begin{aligned} \mathbf{U}(t)\mathbf{U}(ut) &= \eta t^k \mathbf{1}, \\ u^2 &= 1, \\ \xi &= -\frac{1}{2\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \Phi(\mathbf{U}(t)) \right) \right\}, \end{aligned} \tag{1.3.7}$$

where η is some non-zero complex number and k is an integer. The expression “Res” is shorthand for the residue of a function, being the coefficient of t^{-1} in the Laurent expansion of the function concerned. Similarly, the type 1b automorphism corresponding to the triple $\{\mathbf{U}(t), u, \xi\}$ is involutive if, and only if the following are all satisfied:

$$\begin{aligned} \mathbf{U}(t)\tilde{\mathbf{U}}(ut) &= \eta t^k \mathbf{1}, \\ u^2 &= 1, \\ \xi &= \frac{1}{2\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \tilde{\Phi}(\mathbf{U}(t)) \right) \right\}. \end{aligned} \tag{1.3.8}$$

The type 2a automorphism corresponding to the triple $\{\mathbf{U}(t), u, \xi\}$ is involutive if, and only if the following are all satisfied:

$$\begin{aligned} \mathbf{U}(t)\mathbf{U}(ut^{-1}) &= \eta t^k \mathbf{1}, \\ \xi &= -\frac{1}{2\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \Phi(\mathbf{U}(t^{-1})) \right) \right\}. \end{aligned} \quad (1.3.9)$$

The type 2b automorphism corresponding to the triple $\{\mathbf{U}(t), u, \xi\}$ is involutive if, and only if the following are satisfied:

$$\begin{aligned} \mathbf{U}(t)\tilde{\mathbf{U}}(ut^{-1}) &= \eta t^k \mathbf{1}, \\ \xi &= \frac{1}{2\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \tilde{\Phi}(\mathbf{U}(t^{-1})) \right) \right\}. \end{aligned} \quad (1.3.10)$$

Detailed expressions also exist for the products of automorphisms of different types, and also for the inverses of automorphisms. It is not necessary for this present analysis to use these expressions explicitly, and they are not given explicitly. If required, one should consult the original paper on the matrix formulation. It is required, however, to give the necessary and sufficient conditions for two automorphisms to be conjugate. If two automorphisms ϕ_1 and ϕ_2 are conjugate, then they are necessarily of the same type. In the following, ϕ_1 has the corresponding triple $\{\mathbf{U}_1(t), u_1, \xi_1\}$, ϕ_2 the corresponding triple $\{\mathbf{U}_2(t), u_2, \xi_2\}$, and another automorphism ϕ has the corresponding triple $\{\mathbf{S}(t), s, \xi\}$. The conditions are the necessary and sufficient ones for the equality $\phi_1 = \phi \circ \phi_2 \circ \phi^{-1}$ to hold. It will be assumed that η is any non-zero complex number, and k is any integer.

The automorphisms ϕ_1 and ϕ_2 are both of type 1a. There are four sets conditions for them to be conjugate, depending upon which type the automorphism ϕ is. If ϕ is of type 1a, 1b, 2a, 2b respectively, then the sets of conditions are:

$$\left. \begin{aligned} \eta t^k \mathbf{U}_1(t) &= \mathbf{S}(t)\mathbf{U}_2(st)\mathbf{S}(u_2t)^{-1} \\ u_1 &= u_2 \end{aligned} \right\} \phi \text{ is of type 1a,} \quad (1.3.11)$$

$$\left. \begin{aligned} \eta t^k \mathbf{U}_1(t) &= \mathbf{S}(t)\tilde{\mathbf{U}}_2(st)^{-1}\mathbf{S}(u_2t)^{-1} \\ u_1 &= u_2 \end{aligned} \right\} \phi \text{ is of type 1b,} \quad (1.3.12)$$

$$\left. \begin{aligned} \eta t^k \mathbf{U}_1(t) &= \mathbf{S}(t) \mathbf{U}_2(st^{-1}) \mathbf{S}(u_2^{-1}t)^{-1} \\ u_1 &= u_2^{-1} \end{aligned} \right\} \phi \text{ is of type 2a,} \quad (1.3.13)$$

$$\left. \begin{aligned} \eta t^k \mathbf{U}_1(t) &= \mathbf{S}(t) \tilde{\mathbf{U}}_2(st^{-1})^{-1} \mathbf{S}(u_2^{-1}t)^{-1} \\ u_1 &= u_2^{-1} \end{aligned} \right\} \phi \text{ is of type 2b.} \quad (1.3.14)$$

When the two automorphisms under consideration are both of type 1b, then the conditions for their conjugacy become:

$$\left. \begin{aligned} \eta t^k \mathbf{U}_1(t) &= \mathbf{S}(t) \tilde{\mathbf{U}}_2(st) \mathbf{S}(u_2t) \\ u_1 &= u_2 \end{aligned} \right\} \phi \text{ is of type 1a,} \quad (1.3.15)$$

$$\left. \begin{aligned} \eta t^k \mathbf{U}_1(t) &= \mathbf{S}(t) \tilde{\mathbf{U}}_2(st)^{-1} \tilde{\mathbf{S}}(u_2t) \\ u_1 &= u_2 \end{aligned} \right\} \phi \text{ is of type 1b,} \quad (1.3.16)$$

$$\left. \begin{aligned} \eta t^k \mathbf{U}_1(t) &= \mathbf{S}(t) \mathbf{U}_2(st^{-1}) \tilde{\mathbf{S}}(u_2^{-1}t) \\ u_1 &= u_2^{-1} \end{aligned} \right\} \phi \text{ is of type 2a,} \quad (1.3.17)$$

$$\left. \begin{aligned} \eta t^k \mathbf{U}_1(t) &= \mathbf{S}(t) \tilde{\mathbf{U}}_2(st^{-1})^{-1} \tilde{\mathbf{S}}(u_2^{-1}t) \\ u_1 &= u_2^{-1} \end{aligned} \right\} \phi \text{ is of type 2b.} \quad (1.3.18)$$

When the automorphisms under consideration are of type 2a, then the conditions for their conjugacy become

$$\left. \begin{aligned} \eta t^k \mathbf{U}_1(t) &= \mathbf{S}(t) \mathbf{U}_2(st) \mathbf{S}(s^{-2}u_2t^{-1})^{-1} \\ u_1 &= s^{-2}u_2 \end{aligned} \right\} \phi \text{ is of type 1a,} \quad (1.3.19)$$

$$\left. \begin{aligned} \eta t^k \mathbf{U}_1(t) &= \mathbf{S}(t) \tilde{\mathbf{U}}_2(st)^{-1} \mathbf{S}(s^{-2}u_2t^{-1})^{-1} \\ u_1 &= s^{-2}u_2 \end{aligned} \right\} \phi \text{ is of type 1b,} \quad (1.3.20)$$

$$\left. \begin{aligned} \eta t^k \mathbf{U}_1(t) &= \mathbf{S}(t) \mathbf{U}_2(st^{-1}) \mathbf{S}(s^2u_2^{-1}t^{-1})^{-1} \\ u_1 &= s^2u_2^{-1} \end{aligned} \right\} \phi \text{ is of type 2a,} \quad (1.3.20)$$

$$\left. \begin{aligned} \eta^k \mathbf{U}_1(t) &= \mathbf{S}(t) \tilde{\mathbf{U}}_2(st^{-1})^{-1} \mathbf{S}(s^2 u_2^{-1} t)^{-1} \\ u_1 &= s^2 u_2^{-1} \end{aligned} \right\} \phi \text{ is of type 2b.} \quad (1.3.21)$$

When the automorphisms ϕ_1 and ϕ_2 are of type 2b, then the conditions for their being conjugate to one another become:

$$\left. \begin{aligned} \eta^k \mathbf{U}_1(t) &= \mathbf{S}(t) \mathbf{U}_2(st) \tilde{\mathbf{S}}(s^{-2} u_2 t^{-1}) \\ u_1 &= s^{-2} u_2 \end{aligned} \right\} \phi \text{ is of type 1a,} \quad (1.3.22)$$

$$\left. \begin{aligned} \eta^k \mathbf{U}_1(t) &= \mathbf{S}(t) \tilde{\mathbf{U}}_2(st)^{-1} \tilde{\mathbf{S}}(s^{-2} u_2 t^{-1}) \\ u_1 &= s^{-2} u_2 \end{aligned} \right\} \phi \text{ is of type 1b,} \quad (1.3.23)$$

$$\left. \begin{aligned} \eta^k \mathbf{U}_1(t) &= \mathbf{S}(t) \mathbf{U}_2(st^{-1}) \tilde{\mathbf{S}}(s^2 u_2^{-1} t^{-1}) \\ u_1 &= s^2 u_2^{-1} \end{aligned} \right\} \phi \text{ is of type 2a,} \quad (1.3.24)$$

$$\left. \begin{aligned} \eta^k \mathbf{U}_1(t) &= \mathbf{S}(t) \tilde{\mathbf{U}}_2(st^{-1})^{-1} \tilde{\mathbf{S}}(s^2 u_2^{-1} t) \\ u_1 &= s^2 u_2^{-1} \end{aligned} \right\} \phi \text{ is of type 2b.} \quad (1.3.25)$$

In practice, the preceding equations may be greatly simplified. Since the involutive automorphisms are the ones that will be studied, certain constraints may be imposed upon the values of u and s . For example, take an involutive automorphism of type 1a. It follows from the involutiveness conditions given previously, that $u^2 = 1$. It was noted in [8] that every class of type 1a involutive automorphisms for which $u = 1$ was disjoint (that is, non-conjugate) from every class of type 1a involutive automorphisms for which $u = -1$. The same conclusion is reached for the conjugacy classes of the type 1b involutive automorphisms.

However, when the involutive automorphisms under consideration are of type 2a or type 2b, then [8] notes that u is completely arbitrary, and will thus be chosen to take the value unity. It follows therefore, that the involutive automorphisms are studied thus: firstly the type 1a involutive automorphisms (with $u = 1$ and $u = -1$ being

examined separately). Then the type 1b involutive automorphisms (with $u = 1$ and $u = -1$ being examined separately). Finally the type 2a and then the type 2b involutive automorphisms (with $u = 1$) in both cases. One interesting feature is that, if the representation Γ is equivalent to its “contragredient representation”, then the type 1a automorphisms and the type 1b automorphisms coincide, and the type 2a and the type 2b automorphisms also coincide. That is, if there exists a non-singular $d_\Gamma \times d_\Gamma$ matrix C which is such that

$$-\bar{\Gamma}(a^0) = C^{-1}\Gamma(a^0)C \quad (\forall a^0 \in \tilde{\mathcal{L}}^{(0)}). \quad (1.3.26)$$

In these cases, the type 1b automorphisms and the type 2b automorphisms may be disregarded. Otherwise, automorphisms are only conjugate to other automorphisms of the same type.

1.4 Automorphisms of the compact real form and real forms generated from them

There is a convenient condition upon the matrix $\mathbf{U}(t)$ such that, if $\mathbf{U}(t)$ satisfies the condition, then the restriction (to the compact real form $\mathcal{L}_{\mathbb{C}}^{(1)}$) of the automorphism corresponding to the triple $\{\mathbf{U}(t), 1, \xi\}$ is also an automorphism of $\mathcal{L}_{\mathbb{C}}^{(1)}$. In general, we will be using representations of semi-simple Lie algebras that are real. Thus, in general, the "matrix part" $\mathbf{a}(t)$ of an element of the compact real form $\mathcal{L}_{\mathbb{C}}^{(1)}$ will satisfy $\bar{\mathbf{a}}^*(t^{-1}) = -\mathbf{a}(t)$. Suppose for the moment that $\{\mathbf{U}(t), 1, \xi\}$ corresponds to a type 1a automorphism. Then the effect of this automorphism (which we call ϕ) is as follows:

$$\phi(\mathbf{a}(t)) = \mathbf{U}(t)\mathbf{a}(ut)\mathbf{U}(t)^{-1} + \frac{1}{\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{a}(ut) \right) \right\} c. \quad (1.4.1)$$

We assume that $\mathbf{a}(t)$ is the matrix representative of an element of the compact real form $\mathcal{L}_{\mathbb{C}}^{(1)}$. Thus, $\bar{\mathbf{a}}^*(t^{-1}) = -\mathbf{a}(t)$. If the restriction of ϕ to $\mathcal{L}_{\mathbb{C}}^{(1)}$ is an automorphism of $\mathcal{L}_{\mathbb{C}}^{(1)}$, then we must have that $\bar{\mathbf{b}}^*(t^{-1}) = -\mathbf{b}(t)$, where $\mathbf{b}(t) = \mathbf{U}(t)\mathbf{a}(ut)\mathbf{U}(t)^{-1}$. This implies that

$$\bar{\mathbf{U}}^*(t^{-1})^{-1} \bar{\mathbf{a}}^*(ut^{-1}) \bar{\mathbf{U}}(t^{-1}) = -\mathbf{U}(t)\mathbf{a}(ut)\mathbf{U}(t)^{-1}. \quad (1.4.2)$$

Since $\bar{\mathbf{a}}^*(t^{-1}) = -\mathbf{a}(t)$, we may rearrange this to give

$$\mathbf{U}(t)^{-1} \bar{\mathbf{U}}^*(t^{-1})^{-1} \mathbf{a}(ut) = \mathbf{a}(ut)\mathbf{U}(t)^{-1} \bar{\mathbf{U}}^*(t^{-1})^{-1}. \quad (1.4.3)$$

This allows us to use the result of Schur's lemma (see appendix B) to infer that

$$\mathbf{U}(t)^{-1} \bar{\mathbf{U}}^*(t^{-1})^{-1} = f(t)\mathbf{1}. \quad (1.4.4)$$

However, since $\mathbf{U}(t)$ and its inverse are Laurent polynomial matrices, we may infer that $\bar{\mathbf{U}}^*(t^{-1})\mathbf{U}(t) = \alpha t^{\beta} \mathbf{1}$ (where α is a non-zero complex number and β is an integer).

If $\tilde{\mathcal{L}}$ is a complex affine Kac-Moody algebra and if \mathcal{L}_0 is a real Lie subalgebra of $\tilde{\mathcal{L}}$, then \mathcal{L}_0 is a real form of $\tilde{\mathcal{L}}$ if the complexification of \mathcal{L}_0 is isomorphic to $\tilde{\mathcal{L}}$. In the special case where $\tilde{\mathcal{L}}$ is finite-dimensional, it should be remembered that $\dim_{\mathbf{R}} \mathcal{L}_0 = \dim_{\mathbf{C}} \tilde{\mathcal{L}}$, although where $\tilde{\mathcal{L}}$ is of complex affine type these quantities are, of course, infinite. It may also be recalled that every element of $\tilde{\mathcal{L}}$ may be represented in the form: $a + ib$, where a, b are elements of \mathcal{L}_0 . Moreover, the representation is unique. This property also holds when $\tilde{\mathcal{L}}$ is a complex affine Kac-Moody algebra.

The real forms of the semi-simple Lie algebras were found and classified by Cartan. The non-compact real forms were all created from the compact real form by studying the involutive automorphisms of the compact real form and applying Cartan's method. This method may be summarised thus:

Let ψ be an involutive automorphism of the compact real form $\mathcal{L}_{\mathbf{C}}^{(0)}$ of $\tilde{\mathcal{L}}^{(0)}$, a semi-simple complex Lie algebra of dimension n^0 . Define an eigenvector of ψ to be an element a_p^0 of $\mathcal{L}_{\mathbf{C}}^{(0)}$ that satisfies $\psi(a_p^0) = \lambda_p^0 a_p^0$ and let the quantity λ_p^0 be called the eigenvalue corresponding to the eigenvector a_p^0 . Since the automorphism ψ is involutive, the eigenvalues take the values ± 1 . If $\{a_p^0\}_{p=1}^{n^0}$ is a basis of $\mathcal{L}_{\mathbf{C}}^{(0)}$ consisting of eigenvectors of ψ (with eigenvalues taking the values $1, -1$) then $\{b_p^0\}_{p=1}^{n^0}$ is a basis of a real form of $\tilde{\mathcal{L}}^{(0)}$, where

$$b_p^0 = \begin{cases} a_p^0 & \text{when } \lambda_p^0 = 1 \\ ia_p^0 & \text{when } \lambda_p^0 = -1. \end{cases} \quad (1.4.5)$$

Furthermore, all the real forms of $\tilde{\mathcal{L}}^{(0)}$ may be obtained in this way.

This theorem of Cartan and the method of generating real forms that is suggested by it will be discussed further, with particular respect to infinite-dimensional Lie algebras and their real forms. It is well-known that each real form of a complex semi-simple Lie algebra gives rise to an involutive automorphism of the compact real form of that

Lie algebra. For an account of this (and of the real forms of semi-simple Lie algebras in general) see, for example [28]. This property also holds for Kac-Moody algebras, as will be seen below. We call a basis of the compact real form of an algebra in which every element is an eigenvector of the automorphism ψ (with eigenvalues ± 1) a “ ψ -eigenvector basis”.

Let \mathcal{L}_C be the compact real form of the complex affine Kac-Moody algebra $\tilde{\mathcal{L}}$, defined with respect to the compact involution ϕ_C , given by (1.2.22). Now, if $\psi_{\mathcal{L}_C}$ is an automorphism of the compact real form \mathcal{L}_C , then we may extend it to an automorphism $\psi_{\tilde{\mathcal{L}}}$ of $\tilde{\mathcal{L}}$ by the definition $\psi_{\tilde{\mathcal{L}}}(\mu a) = \mu \psi_{\mathcal{L}_C}(a)$ for all complex numbers μ , and all $a \in \mathcal{L}_C$. In fact, if $\psi_{\mathcal{L}_0}$ is an automorphism of any real form \mathcal{L}_0 , then this automorphism may be extended to the whole of $\tilde{\mathcal{L}}$ by letting $\psi_{\tilde{\mathcal{L}}}(\mu a) = \mu \psi_{\mathcal{L}_0}(a)$ for all complex numbers μ and all $a \in \mathcal{L}_0$.

It is useful to note here that, when \mathcal{L}_0 is any real subalgebra of \mathcal{L} , every element of the compact real form \mathcal{L}_C may be expressed in the form $c + d$, where $c \in (\mathcal{L}_C \cap \mathcal{L}_0)$ and $d \in (\mathcal{L}_C \cap i\mathcal{L}_0)$. Proof of this is rather simple. Let $\{a\}$ denote the set which contains only the element a . What we wish to show is that

$$\bigcup_{a,b} \{a+b\} = \mathcal{L}_C \quad \text{where } a \in (\mathcal{L}_C \cap \mathcal{L}_0), b \in (\mathcal{L}_C \cap i\mathcal{L}_0), \quad (1.4.6)$$

where the double summation in (1.4.6) is over all elements $a \in (\mathcal{L}_C \cap \mathcal{L}_0)$ and $b \in (\mathcal{L}_C \cap i\mathcal{L}_0)$. Now it follows for each $a \in (\mathcal{L}_C \cap \mathcal{L}_0)$ and $b \in (\mathcal{L}_C \cap i\mathcal{L}_0)$ that $a \in \mathcal{L}_C$ and $b \in \mathcal{L}_C$. Equation (1.4.6) may thus be re-written as

$$\bigcup_{a,b} \{a+b\} = \mathcal{L}_C \cap \left(\bigcup_{\substack{c \in \mathcal{L}_0 \\ d \in i\mathcal{L}_0}} \{c+d\} \right). \quad (1.4.7)$$

Now it is clear that the summation of sets in the right-hand side of (1.4.7) is equal to $\tilde{\mathcal{L}}$ itself, and so we have that

$$\bigcup_{a,b} \{a+b\} = \mathcal{L}_C \cap \tilde{\mathcal{L}} = \mathcal{L}_C \quad (a \in \mathcal{L}_C \cap \mathcal{L}_0, b \in \mathcal{L}_C \cap i\mathcal{L}_0). \quad (1.4.8)$$

Now, let \mathcal{L}_1 and \mathcal{L}_2 be two isomorphic real forms of $\tilde{\mathcal{L}}$. There exists, therefore, a homomorphism ϕ from \mathcal{L}_1 onto \mathcal{L}_2 which is injective, surjective and such that $\phi(\mathcal{L}_1) = \mathcal{L}_2$. The homomorphism ϕ may be extended to $\tilde{\mathcal{L}}$, in which case it becomes an automorphism ($\phi_{\tilde{\mathcal{L}}}$) of $\tilde{\mathcal{L}}$. Clearly, the image $\phi_{\tilde{\mathcal{L}}}(\mathcal{L}_C)$ of the compact real form is itself a compact real form of $\tilde{\mathcal{L}}$, and all of the compact real forms of a particular complex affine Kac-Moody algebra are isomorphic. Thus, there exists some bijective homomorphism ρ such that $\rho(\phi_{\tilde{\mathcal{L}}}(\mathcal{L}_C)) = \mathcal{L}_C$. This may also be written as $\phi_{\tilde{\mathcal{L}}}(\mathcal{L}_C) = \rho^{-1}(\mathcal{L}_C)$, and we may infer that ρ may be extended into an automorphism $\rho_{\tilde{\mathcal{L}}}$ of $\tilde{\mathcal{L}}$. Consider then the automorphism $\rho_{\tilde{\mathcal{L}}}\phi_{\tilde{\mathcal{L}}}$ of $\tilde{\mathcal{L}}$. The images of \mathcal{L}_1 and \mathcal{L}_2 under this automorphism are clearly isomorphic to \mathcal{L}_1 , \mathcal{L}_2 respectively, although the image of \mathcal{L}_C is \mathcal{L}_C itself. There exists, therefore, an automorphism of $\tilde{\mathcal{L}}$ which maps an isomorphic image of \mathcal{L}_1 bijectively into an isomorphic image of \mathcal{L}_2 and which preserves the compact real form \mathcal{L}_C . Since \mathcal{L}_1 and \mathcal{L}_2 have been specified only up to isomorphism, it will be assumed from now on that the automorphism $\phi_{\tilde{\mathcal{L}}}$ maps \mathcal{L}_1 into \mathcal{L}_2 and preserves the compact real form of $\tilde{\mathcal{L}}$.

We will now proceed to demonstrate that the isomorphic real forms \mathcal{L}_1 and \mathcal{L}_2 give rise to automorphisms of the compact real form \mathcal{L}_C , and moreover, these automorphisms are conjugate. We begin by defining two mappings $\psi_{\mathcal{L}_j}$ (for $j = 1, 2$) as follows:

$$\begin{aligned} \psi_{\mathcal{L}_j}(a) &= a & \text{when } a \in \mathcal{L}_C \cap \mathcal{L}_j, \\ \psi_{\mathcal{L}_j}(a) &= -a & \text{when } a \in \mathcal{L}_C \cap i\mathcal{L}_j. \end{aligned} \tag{1.4.9}$$

The mapping $\psi_{\mathcal{L}_j}$ is defined only upon the sets $(\mathcal{L}_C \cap \mathcal{L}_j)$ and $(\mathcal{L}_C \cap i\mathcal{L}_j)$. However, it has been demonstrated that every element of \mathcal{L}_C may be written as the sum of an element from the first of these sets and an element from the other. Hence, the mapping $\psi_{\mathcal{L}_j}$ extends by linearity to the whole of \mathcal{L}_C , and thus extended, $\psi_{\mathcal{L}_j}$ is an automorphism of \mathcal{L}_C , since it maps \mathcal{L}_C onto itself, and preserves the operation of commutation. The automorphism $\psi_{\mathcal{L}_j}$ may be extended to the whole of $\tilde{\mathcal{L}}$, and this automorphism is called $\psi_{\tilde{\mathcal{L}}_j}$. (The subscript "j" is retained in order to denote that the

automorphism $\psi_{\tilde{\mathcal{L}}_j}$ is associated with the real form \mathcal{L}_j). The automorphisms $\psi_{\mathcal{L}_j}$ (of \mathcal{L}_C) and $\psi_{\tilde{\mathcal{L}}_j}$ (of $\tilde{\mathcal{L}}$) are clearly of order two.

Every element a of \mathcal{L}_2 is the image (under $\phi_{\tilde{\mathcal{L}}}$) of some b in \mathcal{L}_1 . It follows, therefore, that

$$\begin{aligned} \psi_{\tilde{\mathcal{L}}_2}(\phi_{\tilde{\mathcal{L}}}(b)) &= \phi(b) & \phi_{\tilde{\mathcal{L}}}(b) &\in \mathcal{L}_C \cap \mathcal{L}_2, \\ \psi_{\tilde{\mathcal{L}}_2}(\phi_{\tilde{\mathcal{L}}}(b)) &= -\phi(b) & \phi_{\tilde{\mathcal{L}}}(b) &\in \mathcal{L}_2 \cap i\mathcal{L}_C, \\ \phi_{\tilde{\mathcal{L}}}^{-1}(\psi_{\tilde{\mathcal{L}}_2}(\phi_{\tilde{\mathcal{L}}}(b))) &= b & b &\in \phi_{\tilde{\mathcal{L}}}^{-1}(\mathcal{L}_2 \cap \mathcal{L}_C), \\ \phi_{\tilde{\mathcal{L}}}^{-1}(\psi_{\tilde{\mathcal{L}}_2}(\phi_{\tilde{\mathcal{L}}}(b))) &= -b & b &\in \phi_{\tilde{\mathcal{L}}}^{-1}(\mathcal{L}_2 \cap i\mathcal{L}_C). \end{aligned} \tag{1.4.10}$$

It should be recalled that $\phi_{\tilde{\mathcal{L}}}(\mathcal{L}_1) = \mathcal{L}_2$ and that $\phi_{\tilde{\mathcal{L}}}(\mathcal{L}_C) = \mathcal{L}_C$. The equations (1.4.10) therefore become the following:

$$\begin{aligned} \psi_{\tilde{\mathcal{L}}_1}(b) &= \phi_{\tilde{\mathcal{L}}}^{-1}(\psi_{\tilde{\mathcal{L}}_2}(\phi_{\tilde{\mathcal{L}}}(b))) = b & \text{where } b &\in \mathcal{L}_C \cap \mathcal{L}_1, \\ \psi_{\tilde{\mathcal{L}}_1}(b) &= \phi_{\tilde{\mathcal{L}}}^{-1}(\psi_{\tilde{\mathcal{L}}_2}(\phi_{\tilde{\mathcal{L}}}(b))) = -b & \text{where } b &\in \mathcal{L}_C \cap i\mathcal{L}_1. \end{aligned} \tag{1.4.11}$$

Thus $\psi_{\tilde{\mathcal{L}}_1} = \phi_{\tilde{\mathcal{L}}}^{-1} \circ \psi_{\tilde{\mathcal{L}}_2} \circ \phi_{\tilde{\mathcal{L}}}$, and this implies that the automorphisms $\psi_{\tilde{\mathcal{L}}_1}$ and $\psi_{\tilde{\mathcal{L}}_2}$ are conjugate within the group of all automorphisms of $\tilde{\mathcal{L}}$. It follows also that the restrictions of these automorphisms (to \mathcal{L}_C) are conjugate within the group of all automorphisms of the compact real form.

Thus, every real form \mathcal{L}_j of a Kac-Moody algebra $\tilde{\mathcal{L}}$ gives rise to an involutive automorphism $\psi_{\mathcal{L}_j}$ of the compact real form of that algebra. The sets $(\mathcal{L}_C \cap \mathcal{L}_j)$ and $(\mathcal{L}_C \cap i\mathcal{L}_j)$ are determined uniquely by the real form \mathcal{L}_j , (which is itself unique up to isomorphism), and the automorphism $\psi_{\mathcal{L}_j}$ is determined by $(\mathcal{L}_C \cap \mathcal{L}_j)$ and $(\mathcal{L}_C \cap i\mathcal{L}_j)$. Furthermore, the involutive automorphism $\psi_{\mathcal{L}_j}$ may be extended into the involutive automorphism $\psi_{\tilde{\mathcal{L}}_j}$ (which is an automorphism of $\tilde{\mathcal{L}}$).

We have shown also that isomorphic real forms $(\mathcal{L}_1$ and $\mathcal{L}_2)$ of the same algebra generate conjugate isomorphisms $(\psi_{\mathcal{L}_1}$ and $\psi_{\mathcal{L}_2})$ of \mathcal{L}_C . Thus, attention may be restricted to non-conjugate involutive automorphisms of Kac-Moody algebras.

In subsequent chapters, we will investigate the involutive automorphisms of various complex affine Kac-Moody algebras. Real forms of the Kac-Moody algebras will then be obtained by using Cartan's method, which has been explained above. Suppose that the involutive automorphism being analysed is $\phi(= \{ \mathbf{U}(t), u, \xi \})$, and is of type 1a. This automorphism has the following action upon the elements of the Kac-Moody algebra:

$$\phi(\mathbf{a}(ut)) = \mathbf{U}(t)\mathbf{a}(ut)\mathbf{U}(t)^{-1} + \frac{1}{\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{a}(ut) \right) \right\} c, \quad (1.4.12)$$

$$\phi(ic) = ic, \quad (1.4.13)$$

$$\phi(id) = id + \xi ic + i\Phi(\mathbf{U}(t)), \quad (1.4.14)$$

as given in the general theory of the matrix formulation. Let \mathcal{L}_0 be a real form generated (according to Cartan's method) by the automorphism ϕ . It is immediately clear that \mathcal{L}_0 contains all real multiples of the operator ic . Consider those elements (belonging to \mathcal{L}_C) of the form $(\mathbf{a}(t) + \lambda ic)$ that are mapped by ϕ either to themselves or to their negatives. That is, those elements that are eigenvectors of the involutive automorphism ϕ . In those cases where the associated eigenvalue is unity, this means that

$$\phi(\mathbf{a}(t) + \lambda ic) = \mathbf{a}(t) + \lambda ic, \quad (1.4.15)$$

$$\Rightarrow \mathbf{a}(t) + \lambda ic = \mathbf{U}(t)\mathbf{a}(ut)\mathbf{U}(t)^{-1} + \frac{1}{\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{a}(ut) \right) \right\} c + \lambda ic, \quad (1.4.16)$$

from which it follows that $\mathbf{a}(t) = \mathbf{U}(t)\mathbf{a}(ut)\mathbf{U}(t)^{-1}$, λ is arbitrary and

$$\text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{a}(ut) \right) \right\} = 0. \quad (1.4.17)$$

Thus, in the case $\phi(\mathbf{a}(t)) = \mathbf{a}(t)$, with $\mathbf{a}(t) \in \mathcal{L}_0, \mathcal{L}_C$, it should be noted that

$$\bar{\mathbf{a}}^*(t^{-1}) = -\mathbf{a}(t), \quad (1.4.18)$$

$$-\mathbf{U}(t)\bar{\mathbf{a}}^*(t^{-1})\mathbf{U}(t)^{-1} = \mathbf{a}(t). \quad (1.4.19)$$

Consider then the case where $\phi(\mathbf{a}(t) + \lambda ic) = -\mathbf{a}(t) - \lambda ic$, (with λ arbitrary, real).

This implies that

$$-\mathbf{a}(t) - \lambda ic = \mathbf{U}(t)\mathbf{a}(ut)\mathbf{U}(t)^{-1} + \frac{1}{\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{a}(ut) \right) \right\} c + \lambda ic, \quad (1.4.20)$$

$$\Rightarrow \mathbf{a}(t) = -\mathbf{U}(t)\mathbf{a}(ut)\mathbf{U}(t)^{-1}, \quad (1.4.21)$$

$$-2\lambda i = \frac{1}{\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{a}(ut) \right) \right\}, \quad (1.4.22)$$

$$\Rightarrow \lambda = \frac{i}{2\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{a}(ut) \right) \right\}, \quad (1.4.23)$$

and λ is real. In this case, therefore, the real form \mathcal{L}_0 contains the element $(i\mathbf{a}(t) - \lambda c)$. Now, if $\mathbf{b}(t) = i\mathbf{a}(t)$ (so that $\mathbf{b}(t) \in \mathcal{L}_0$), then

$$-\mathbf{U}(t)\tilde{\mathbf{b}}^*(t^{-1})\mathbf{U}(t)^{-1} = \mathbf{b}(t), \quad (1.4.24)$$

remembering, of course, that $-\tilde{\mathbf{a}}^*(t^{-1}) = \mathbf{a}(t)$. Suppose, then, that an element of \mathcal{L}_c of the form $(\mathbf{a}(t) + \lambda ic + id)$ is mapped by ϕ either to itself or to its negative. Upon consideration of the action of ϕ upon the operator id (when ϕ is of type 1a), it becomes clear that the element in question cannot be mapped to its negative by ϕ .

Thus

$$id + \lambda ic + \mathbf{a}(t) = \left\{ \begin{array}{l} id + \lambda ic + \xi ic + \frac{1}{\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{a}(ut) \right) \right\} c \\ + i\Phi(\mathbf{U}(t)) + \mathbf{U}(t)\mathbf{a}(ut)\mathbf{U}(t)^{-1} \end{array} \right\}. \quad (1.4.25)$$

Thus, the following equalities hold:

$$\begin{aligned} \mathbf{a}(t) &= i\Phi(\mathbf{U}(t)) + \mathbf{U}(t)\mathbf{a}(ut)\mathbf{U}(t)^{-1}, \\ 0 &= \xi i + \frac{1}{\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{a}(ut) \right) \right\}. \end{aligned} \quad (1.4.26)$$

Recall, however, that ξ satisfies

$$\xi i = -\frac{i}{2\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \Phi(\mathbf{U}(ut)) \right) \right\}, \quad (1.4.27)$$

$$\Rightarrow \frac{1}{\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{a}(ut) \right) \right\} = \frac{1}{2\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} i\Phi(\mathbf{U}(ut)) \right) \right\}, \quad (1.4.28)$$

$$\text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{a}(ut) \right) \right\} = \left\{ \begin{array}{l} \frac{1}{2} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{a}(t) \right) \right\} \\ -\frac{1}{2} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{a}(ut) \right) \right\} \end{array} \right\}. \quad (1.4.29)$$

It follows then, that if $u = 1$

$$\text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{a}(ut) \right) \right\} = 0. \quad (1.4.30)$$

Thus, if $\phi(= \{\mathbf{U}(t), 1, \xi\})$ is a type 1a involutive automorphism, then the elements of \mathcal{L}_0 , the real form generated by Cartan's method, are all real linear combinations of the following:

1. elements $\mathbf{a}(t)$ where $\mathbf{a}(t) = \mathbf{U}(t)\mathbf{a}(t)\mathbf{U}(t)^{-1} = -\mathbf{U}(t)\bar{\mathbf{a}}^*(t^{-1})\mathbf{U}(t)^{-1}$.
2. elements $(i\mathbf{a}(t) - \lambda c)$ where $\mathbf{a}(t) = -\mathbf{U}(t)\mathbf{a}(t)\mathbf{U}(t)^{-1}$ and λ is real, with

$$\lambda = \frac{i}{2\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{a}(t) \right) \right\}. \quad (1.4.31)$$

It has been seen already that, if $\mathbf{b}(t) = i\mathbf{a}(t)$, where $\mathbf{a}(t)$ is of this form, then $\mathbf{b}(t) = -\mathbf{U}(t)\bar{\mathbf{b}}^*(t^{-1})\mathbf{U}(t)^{-1}$.

3. elements μic , where μ is an arbitrary real number.
4. elements $(id + \mathbf{m}(t))$, where $\mathbf{m}(t)$ is a matrix part such that

$$\{\mathbf{m}(t) - \mathbf{U}(t)\mathbf{m}(t)\mathbf{U}(t)^{-1}\} = i\Phi(\mathbf{U}(t)), \quad (1.4.32)$$

$$\text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{m}(t) \right) \right\} = 0. \quad (1.4.33)$$

Thus, the general element of \mathcal{L}_0 may be expressed in the form

$$\mathbf{a}(t) + (\alpha + i\beta)c + \mu(id + \mathbf{m}(t)), \quad (1.4.34)$$

where, in (1.4.34), $\mathbf{a}(t) = -\mathbf{U}(t)\tilde{\mathbf{a}}^*(t^{-1})\mathbf{U}(t)^{-1}$,

$$\alpha = -\frac{1}{2\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{a}(t) \right) \right\}, \quad (1.4.35)$$

and the quantities β and μ are arbitrary real parameters. The above analysis may be developed further. Let the matrix $\mathbf{m}(t)$ be expressed in the following way:

$$\mathbf{m}(t) = \mathbf{a}(t) + \mathbf{b}(t), \quad (1.4.36)$$

where $\mathbf{a}(t)$ and $\mathbf{b}(t)$ satisfy $\mathbf{a}(t) = \mathbf{U}(t)\mathbf{a}(t)\mathbf{U}(t)^{-1}$, and $\mathbf{b}(t) = -\mathbf{U}(t)^{-1}\mathbf{b}(t)\mathbf{U}(t)^{-1}$.

Then

$$\begin{aligned} \phi(id + \mathbf{m}(t)) &= id + \mathbf{m}(t) = id + \mathbf{a}(t) + \mathbf{b}(t) \\ &= \left\{ \begin{aligned} &id + \xi ic + i\Phi(\mathbf{U}(t)) + \mathbf{U}(t)\mathbf{a}(t)\mathbf{U}(t)^{-1} + \mathbf{U}(t)\mathbf{b}(t)\mathbf{U}(t)^{-1} \\ &+ \frac{1}{\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{a}(t) \right) \right\} + \frac{1}{\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{b}(t) \right) \right\} \end{aligned} \right\} \quad (1.4.37) \end{aligned}$$

This implies that the quantity $\mathbf{a}(t)$ is arbitrary, so it may be supposed that $\mathbf{a}(t) = \mathbf{0}$.

Moreover, it implies that

$$\mathbf{b}(t) = \frac{i}{2} \Phi(\mathbf{U}(t)), \quad (1.4.38)$$

$$\xi i = -\frac{1}{2\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} i\Phi(\mathbf{U}(t)) \right) \right\}, \quad (1.4.39)$$

the second of which is, of course, the same as in the results obtained by Cornwell in the general theory of the matrix formulation [8]. Similar expressions and conditions may be derived for real forms generated by other automorphism types.

1.5 Real forms of Kac-Moody algebras generated from real forms of semi-simple Lie algebras

In this section we suggest a number of ways in which real forms of a complex untwisted affine Kac-Moody algebra $\tilde{\mathcal{L}}^{(1)}$ may be obtained by examining the real forms $\mathcal{L}^{(0)}$ of the associated semi-simple Lie algebra $\tilde{\mathcal{L}}^{(0)}$. Let $a_1^0, a_2^0, \dots, a_n^0$ be a basis for $\mathcal{L}^{(0)}$. Consider then the set which is spanned by elements of the following form:

$$\begin{aligned}
 & t^0 \otimes a_p^0, \\
 & t^j \otimes a_p^0 + t^{-j} \otimes a_p^0, \\
 & i(t^j \otimes a_p^0 - t^{-j} \otimes a_p^0), \\
 & ic, \\
 & id,
 \end{aligned} \tag{1.5.1}$$

where $j = 1, 2, \dots$ and $p = 1, 2, \dots, n^0$. The commutators of the above elements are all such that the structure constants are all real. Hence, a real affine Kac-Moody algebra has been generated from a real semi-simple Lie algebra. We call this method 1.

The second method for creating real Kac-Moody algebras from real semi-simple Lie algebras involves our taking the following elements:

$$\begin{aligned}
 & t^0 \otimes a_p^0, \\
 & t^j \otimes a_p^0, \\
 & c, \\
 & d,
 \end{aligned} \tag{1.5.2}$$

where $j = \mathbf{Z} \setminus \{0\}$ and $p = 1, 2, \dots, n^0$.

The third method works by taking as basis elements those given below:

$$\begin{aligned}
& t^0 \otimes a_p^0, \\
& (t^j \otimes a_p^0 + t^{-j} \otimes a_p^0) \quad (j \text{ is even}), \\
& i(t^j \otimes a_p^0 + t^{-j} \otimes a_p^0) \quad (j \text{ is odd}), \\
& i(t^j \otimes a_p^0 - t^{-j} \otimes a_p^0) \quad (j \text{ is even}), \\
& (t^j \otimes a_p^0 - t^{-j} \otimes a_p^0) \quad (j \text{ is odd}), \\
& ic, \\
& id.
\end{aligned} \tag{1.5.3}$$

The fourth method forms a real affine Kac-Moody algebra from the following basis elements:

$$\begin{aligned}
& t^0 \otimes a_p^0, \\
& t^j \otimes a_p^0 \quad (j \text{ is even}), \\
& it^j \otimes a_p^0 \quad (j \text{ is odd}), \\
& ic, \\
& id.
\end{aligned} \tag{1.5.4}$$

These methods of creating real affine Kac-Moody algebras are introduced mainly for the purposes of naming some of the real forms of the complex untwisted affine Kac-Moody algebras that will be found subsequently. For example, if $\mathcal{L}^{(0)}$ is $sl(2, \mathbf{R})$, for example, and a real affine Kac-Moody algebra has been formed by using the second method, then the resulting real affine Kac-Moody algebra will be called $sl_{(2)}(2, \mathbf{R})$. This method will not, however, provide us with a system for naming all affine Kac-Moody algebras.

2 Involutive automorphisms and real forms of the Kac-Moody algebra $A_1^{(1)}$

2.1 Introduction

In this chapter I shall be examining the real forms of affine Kac-Moody algebras (with some particular reference to the algebra $A_1^{(1)}$) and topics that arise during this analysis.

The information of the previous chapter may now be used to investigate the involutive automorphisms of the compact real form of $A_1^{(1)}$. A study of the involutive automorphisms of the complex algebra $A_1^{(1)}$ can be found in Cornwell [9], and that work forms the basis of this section. Recall that [9] uses the following representation of $A_1^{(1)}$:

$$\begin{aligned}\Gamma\left(h_{\alpha_1^0}^0\right) &= \mathbf{h}_{\alpha_1^0}^0 = \frac{1}{4} \text{diag}\{1, -1\}, \\ \Gamma\left(e_{\alpha_1^0}^0\right) &= \mathbf{e}_{\alpha_1^0}^0 = \frac{1}{2} \text{offdiag}\{1, 0\}, \\ \Gamma\left(e_{-\alpha_1^0}^0\right) &= \mathbf{e}_{-\alpha_1^0}^0 = \frac{1}{2} \text{offdiag}\{0, -1\}.\end{aligned}\tag{2.1.1}$$

with a basis of the compact real form being constructed in the manner described in chapter 1. We work through the analysis of [9], relating it explicitly to the compact real form.

In the following sections, the involutive Cartan-preserving automorphisms will be examined, with particular reference to their conjugacy classes within the group of all automorphisms of the compact real form of $A_1^{(1)}$. It should be noted firstly that there are only 2 root-preserving transformations of A_1 , namely $\tau(\alpha_1) = \pm\alpha_1$, and the

most general Laurent matrix $\mathbf{U}(t)$ such that $\mathbf{U}(t)\mathbf{h}_{\alpha_1}^0\mathbf{U}(t)^{-1} = \mathbf{h}_{\alpha_1}^0$ is given by $\mathbf{U}(t) = \text{diag}\{1, \eta t^k\}$ (after removal of any common factor of the form λt^m). Similarly, the most general Laurent matrix $\mathbf{U}(t)$ such that $\mathbf{U}(t)\mathbf{h}_{\alpha_1}^0\mathbf{U}(t)^{-1} = -\mathbf{h}_{\alpha_1}^0$ is, (after removal of common factors) of the form $\mathbf{U}(t) = \text{offdiag}\{1, \eta t^k\}$. In both of the above cases, η is non-zero, but is not necessarily real. However, for these matrices to generate automorphisms of the compact real form, they must satisfy the conditions derived in the first chapter. This requires that $|\eta| = 1$.

2.2 Type 1a involutive automorphisms of $A_1^{(1)}$ with $u = 1$

The involutive Cartan-preserving automorphisms of type 1a with $u = 1$ are those generated by the following matrices:

$$\begin{aligned} U(t) &= \mathbf{1}_2, \\ U(t) &= \text{diag}\{1, -1\}, \\ U(t) &= \text{offdiag}\{1, \eta t^k\}. \end{aligned} \tag{2.2.1}$$

It should be remembered that $|\eta| = 1$. In the complex algebra $A_1^{(1)}$, there were three conjugacy classes of type 1a involutive automorphisms with $u = 1$. This subsection demonstrates explicitly using the matrix formulation that the same conclusion may be reached for the compact real form. In general, the matrices $S(t)$ contained in [9] generate automorphisms of the compact real form, with the same method being used. Clearly the involutive automorphism that is generated by $\mathbf{1}_2$ belongs in a conjugacy class of its own, and this class is called (A). The only automorphism in this class is the identity automorphism. Note also that $\{S(t), u, \xi\}$ is an automorphism of the compact real form, where

$$S(t) = \begin{bmatrix} 1 & \eta^{-\frac{1}{2}} t^{-\frac{1}{2}k} \\ 1 & -\eta^{-\frac{1}{2}} t^{-\frac{1}{2}k} \end{bmatrix}, \tag{2.2.2}$$

and this matrix also satisfies the following equation:

$$S(t) \begin{bmatrix} 0 & 1 \\ \eta t^k & 0 \end{bmatrix} S(t)^{-1} = \eta^{\frac{1}{2}} t^{\frac{1}{2}k} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{2.2.3}$$

so that the automorphism $\{\text{diag}\{1, -1\}, 1, 0\}$ belongs to the same conjugacy class as the automorphisms $\{\text{offdiag}\{1, \eta t^k\}, 1, -k^2\}$ on those occasions when k is even. This

class is called (B). Now, for odd k , when $S(t) = \text{diag}\left\{1, -i\eta^{-\frac{1}{2}t} t^{-\frac{1}{2}(k-1)}\right\}$, an automorphism is generated by $S(t)$. Moreover, it satisfies

$$S(t) \begin{bmatrix} 0 & 1 \\ \eta t^k & 0 \end{bmatrix} S(t)^{-1} = i\eta^{\frac{1}{2}} t^{\frac{1}{2}(k-1)} \begin{bmatrix} 0 & 1 \\ -t & 0 \end{bmatrix}. \quad (2.2.4)$$

Hence, all of the automorphisms under consideration that are in neither (A) nor (B) are mutually conjugate. It follows from the analysis (in [9]) of the complex algebra $A_1^{(1)}$ that these automorphisms fall into a third class, which is called (C). (The corresponding automorphisms in $A_1^{(1)}$ are not conjugate, so they cannot be conjugate in the compact real form of $A_1^{(1)}$). It should be noted that the type 1a automorphisms $\{\text{offdiag}\{1, t\}, 1, \xi\}$ and $\{\text{offdiag}\{1, -t\}, 1, \xi\}$ are mutually conjugate. This is because the matrices $U_1(t) (= \text{offdiag}\{1, t\})$ and $U_2(t) (= \text{offdiag}\{1, -t\})$ are such that $U_1(t) = U_2(st)$ (with $s = -1$). Investigation of the conjugacy conditions given in chapter 1, section 3 then imply that the two involutive automorphisms are in fact conjugate.

The identity automorphism is, as has been noted, the only possible representative for the conjugacy class (A). For the class (B), take as representative the automorphism $\phi_B = \{\text{offdiag}\{1, 1\}, 1, 0\}$, and for the class (C), take as representative the automorphism $\phi_C = \{\text{offdiag}\{1, t\}, 1, \xi\}$. It is clear that the Cartan's method, when applied to the identity automorphism, gives rise to the compact real form of the algebra $A_1^{(1)}$, which may also be referred to as $su_{(1)}(2)$.

A basis of the compact real form of $A_1^{(1)}$ composed entirely of eigenvectors of ϕ_B (with eigenvalues 1 or -1) is given below, together with the associated eigenvalues:

ih_{α_1}	with eigenvalue	-1,	
$(e_{\alpha_1} + e_{-\alpha_1})$	"	-1,	
$i(e_{\alpha_1} - e_{-\alpha_1})$	"	1,	
$(e_{j\delta}^1 + e_{-j\delta}^1)$	"	-1,	
$i(e_{j\delta}^1 - e_{-j\delta}^1)$	"	-1,	
$(e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1} + e_{j\delta-\alpha_1} + e_{-j\delta+\alpha_1})$	"	-1,	
$(e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1} - e_{j\delta-\alpha_1} - e_{-j\delta+\alpha_1})$	"	1,	
$i(e_{j\delta+\alpha_1} - e_{-j\delta-\alpha_1} + e_{j\delta-\alpha_1} - e_{-j\delta+\alpha_1})$	"	-1,	
$i(e_{j\delta+\alpha_1} - e_{-j\delta-\alpha_1} - e_{j\delta-\alpha_1} + e_{-j\delta+\alpha_1})$	"	1,	(2.2.5)
ic	"	1,	
id	"	1.	

It is thus possible to create a basis for a real form of $A_1^{(1)}$ that is generated by the automorphism ϕ_B . Using Cartan's method, the following basis elements are obtained.

$$\begin{aligned}
& -h_{\alpha_1}, \\
& i(e_{j\delta}^1 + e_{-j\delta}^1), \\
& -(e_{j\delta}^1 - e_{-j\delta}^1), \\
& i(e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1} + e_{j\delta-\alpha_1} + e_{-j\delta+\alpha_1}), \\
& (e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1} - e_{j\delta-\alpha_1} - e_{-j\delta+\alpha_1}), \\
& -(e_{j\delta+\alpha_1} - e_{-j\delta-\alpha_1} + e_{j\delta-\alpha_1} - e_{-j\delta+\alpha_1}), \\
& i(e_{j\delta+\alpha_1} - e_{-j\delta-\alpha_1} - e_{j\delta-\alpha_1} + e_{-j\delta+\alpha_1}), \\
& ic, \\
& id.
\end{aligned} \tag{2.2.6}$$

In the above, $j=0,1,\dots$. At first glance, this does not appear to correspond in an obvious manner to any of the real forms of $A_1^{(1)}$ generated from real forms of A_1 . Recall, however, the real form $sl_{(1)}(2, \mathbf{R})$, whose basis elements $\{a_p\}_{p=1}^3$ may be

defined by $a_1 = -(e_{\alpha_1^0}^0 - e_{-\alpha_1^0}^0)$, $a_2 = (e_{\alpha_1^0}^0 + e_{-\alpha_1^0}^0)$ and $a_3 = -2h_{\alpha_1^0}^0$. We then define a mapping θ as follows:

$$\begin{aligned}\theta(a_1) &= i(e_{\alpha_1^0}^0 + e_{-\alpha_1^0}^0) = b_1, \\ \theta(a_2) &= i(-e_{\alpha_1^0}^0 + e_{-\alpha_1^0}^0) = b_2, \\ \theta(a_3) &= 2h_{\alpha_1^0}^0 = b_3.\end{aligned}\tag{2.2.7}$$

The mapping θ may be extended so that it applies to the infinite-dimensional algebra $sl_{(1)}(2, \mathbf{R})$. It may be extended in the following fashion:

$$\begin{aligned}\theta(t^k \otimes a_p) &= t^k \otimes b_p, \\ \theta(ic) &= ic, \\ \theta(id) &= id,\end{aligned}\tag{2.2.8}$$

where $p = 1, 2, 3$. It is easily verified that, for a, b belonging to $sl_{(1)}(2, \mathbf{R})$, the equation $\theta([a, b]) = [\theta(a), \theta(b)]$ holds. Thus the extension of θ is actually an isomorphism of $sl_{(1)}(2, \mathbf{R})$ with the real form generated by the conjugacy class (B).

For the representative automorphism of the conjugacy class (C), a basis of the compact real form of $A_1^{(1)}$ that consists of eigenvectors (with the associated eigenvalues 1 or -1) is given by the following:

$i(h_{\alpha_1} - \frac{1}{2}c)$	with eigenvalue	-1,	
$(e_{j\delta + \alpha_1} + e_{-j\delta - \alpha_1} + e_{k\delta - \alpha_1} + e_{-j\delta + \alpha_1})$	"	-1,	
$(e_{j\delta + \alpha_1} + e_{-j\delta - \alpha_1} - e_{k\delta - \alpha_1} - e_{-j\delta + \alpha_1})$	"	1,	
$i(e_{j\delta + \alpha_1} - e_{-j\delta - \alpha_1} + e_{k\delta - \alpha_1} - e_{-j\delta + \alpha_1})$	"	-1,	
$i(e_{j\delta + \alpha_1} - e_{-j\delta - \alpha_1} - e_{k\delta - \alpha_1} + e_{-j\delta + \alpha_1})$	"	1,	(2.2.9)
ic	"	1,	
$i(d + h_{\alpha_1} - \frac{1}{2}c)$	"	1,	

where in the above, $j = 0, 1, \dots$ and $k = j + 1$. When the same process is applied to the automorphism ϕ_C , a basis of a real form of $A_1^{(1)}$ is obtained. These basis elements are the following:

$$\begin{aligned}
 & h_{\alpha_1} - \frac{1}{2}c, \\
 & i(e_{j\delta}^1 + e_{-j\delta}^1) \quad j = 1, 2, \dots, \\
 & (e_{j\delta}^1 - e_{-j\delta}^1) \quad j = 1, 2, \dots,
 \end{aligned} \tag{2.2.10}$$

$$\begin{aligned}
 & i(e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1} + e_{k\delta-\alpha_1} + e_{-k\delta+\alpha_1}), \\
 & (e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1} - e_{k\delta-\alpha_1} - e_{-k\delta+\alpha_1}), \\
 & (e_{j\delta+\alpha_1} - e_{-j\delta-\alpha_1} + e_{k\delta-\alpha_1} - e_{-k\delta+\alpha_1}), \\
 & i(e_{j\delta+\alpha_1} - e_{-j\delta-\alpha_1} - e_{k\delta-\alpha_1} - e_{-k\delta+\alpha_1}), \\
 & ic, \\
 & i(d + h_{\alpha_1} - \frac{1}{2}c).
 \end{aligned} \tag{2.2.11}$$

As was the case previously, $k = j + 1$ and $j = 0, 1, 2, \dots$, except where otherwise indicated.

2.3 Type 1a involutive automorphisms of $A_1^{(1)}$ with $u = -1$

As in the previous subsection, this illustrates briefly that the results found for the complex case are the same for the compact real form. For the algebra $A_1^{(1)}$ it has been shown that there is only one such conjugacy class within the group of all automorphisms. The involutive Cartan-preserving automorphisms under consideration are the following:

$$U(t) = \mathbf{1}_2 \quad u = -1 \quad \xi = 0, \quad (2.3.1)$$

$$U(t) = \text{diag}\{1, -1\} \quad u = -1 \quad \xi = 0, \quad (2.3.2)$$

$$U(t) = \text{offdiag}\{1, \eta t^k\} \quad u = -1 \quad \xi = -k^2, \quad (2.3.3)$$

where $|\eta| = 1$ and k is even. In fact, the analysis of [9] makes use of only two different matrices to prove that there is only one such conjugacy class. These matrices (referred to as $S(t)$) also satisfy the conditions necessary for $\{S(t), s, \xi\}$ to be an automorphism of the compact real form of $A_1^{(1)}$. Thus, in this case, there is also just one conjugacy class of type 1a involutive automorphisms with $u = -1$. This class will be referred to as (D), and its representative will be the type 1a involution $\{\mathbf{1}_2, -1, 0\}$. A basis of the compact real form, consisting of eigenvectors of this representative automorphism (with the eigenvalues being ± 1) is given by the following:

$$\begin{aligned}
& (e_{j\delta}^1 + e_{-j\delta}^1), \\
& i(e_{j\delta}^1 - e_{-j\delta}^1), \\
& (e_{j\delta \pm \alpha_1} + e_{-j\delta \mp \alpha_1}), \\
& i(e_{j\delta \pm \alpha_1} - e_{-j\delta \mp \alpha_1}), \\
& ic, \\
& id.
\end{aligned} \tag{2.3.4}$$

The elements ic, id each have the eigenvalue 1, although each of the other eigenvectors has the eigenvalue $(-1)^j$. In the above, $j = 0, 1, \dots$. Thus, the conjugacy class (D) has a real form associated with it, for which a suitable basis is given hereunder:

$$\left. \begin{aligned}
& (e_{j\delta}^1 + e_{-j\delta}^1) \\
& i(e_{j\delta}^1 - e_{-j\delta}^1) \\
& (e_{j\delta \pm \alpha_1} + e_{-j\delta \mp \alpha_1}) \\
& i(e_{j\delta \pm \alpha_1} - e_{-j\delta \mp \alpha_1})
\end{aligned} \right\} \text{for } j = 0, 2, 4, \dots, \tag{2.3.5}$$

$$\left. \begin{aligned}
& i(e_{j\delta}^1 + e_{-j\delta}^1) \\
& (e_{j\delta}^1 - e_{-j\delta}^1) \\
& i(e_{j\delta \pm \alpha_1} + e_{-j\delta \mp \alpha_1}) \\
& (e_{j\delta \pm \alpha_1} - e_{-j\delta \mp \alpha_1})
\end{aligned} \right\} \text{for } j = 1, 3, 5, \dots, \tag{2.3.6}$$

$$\begin{aligned}
& ic, \\
& id.
\end{aligned} \tag{2.3.7}$$

This real form is $su_{(3)}(2,0)$. This concludes the analysis of the type 1a Cartan-preserving involutive automorphisms.

2.4 Type 2a involutive automorphisms of $A_1^{(1)}$ with $u = 1$

From [9] and from the previous sections, it follows that the involutive Cartan-preserving automorphisms of the compact real form of $A_1^{(1)}$ are generated by the following matrices:

$$\begin{aligned} \mathbf{U}(t) &= \text{diag}\{1, t^k\}, \\ \mathbf{U}(t) &= \text{diag}\{1, -t^k\}, \\ \mathbf{U}(t) &= \text{offdiag}\{1, \eta\} \quad (|\eta| = 1). \end{aligned} \tag{2.4.1}$$

This section will show that the results of the complex case also hold for the compact real form, this result being obtained using the matrix formulation. Note firstly that the matrix $\mathbf{S}(t) = \text{diag}\{1, t^j\}$ generates an automorphism of the compact real form (as well as an automorphism of the complex algebra). Thus, with $s = \pm 1$,

$$\mathbf{S}(t) \text{diag}\{1, \eta t^k\} \mathbf{S}(t^{-1})^{-1} = \text{diag}\{1, \eta^s t^{k+2j}\}. \tag{2.4.2}$$

Unlike the analysis of the type 1a automorphisms, the analysis of the type 2a automorphisms has to be amended slightly from the analysis of the complex algebra. This is because the matrix $\mathbf{S}(t)$, defined by

$$\mathbf{S}(t) = \begin{bmatrix} 1 & \eta^{-\frac{1}{2}} \\ -\eta^{-\frac{1}{2}} & 1 \end{bmatrix} \tag{2.4.3}$$

does not generate an automorphism of the compact real form of $A_1^{(1)}$, except in the special case where $\eta = 1$. In fact, if $\eta = -1$ and $\sqrt{\eta} = i$ then $\mathbf{S}(t)$ is singular. So, for the special case $\eta = 1$, we have

$$\mathbf{S}(t) \mathbf{K}_2 \mathbf{S}(t^{-1})^{-1} = \text{diag}\{1, -1\}, \tag{2.4.4}$$

although this does not completely solve the problem. With $S = \text{diag} \left\{ \eta^{-\frac{1}{2}}, 1 \right\}$, then $\bar{S}^* S = \mathbf{1}_2$, hence $\{S, s, 0\}$ is an automorphism of the compact real form of $A_1^{(1)}$. Moreover, this satisfies $SK_2 S^{-1} = \eta^{-\frac{1}{2}} \text{offdiag} \{1, \eta\}$, and this is sufficient to prove that the results for the compact real form are the same as for the complex algebra $A_1^{(1)}$. That is, there are three conjugacy classes of involutive Cartan-preserving automorphisms of the compact real form of $A_1^{(1)}$. These will be called (E), (F) and (G). Representatives of (E), (F), (G) are the automorphisms corresponding to the triples

$$\begin{aligned} & \{\text{diag} \{1, t, 1, 1\}, \\ & \{\text{diag} \{1, 1, 1, 0\}, \\ & \{\text{diag} \{1, -1, 1, 0\}, \end{aligned} \tag{2.4.5}$$

respectively. Now, a basis of the compact real form consisting of eigenvectors of the representative of (E) (with the associated eigenvalues $1, -1$) is given below, together with the relevant eigenvalues:

$$\begin{aligned} & i \left(h_{\alpha_1} - \frac{1}{2} c \right) && 1, \\ & \left(e_{j\delta}^1 + e_{-j\delta}^1 \right) && 1 \quad (j \neq 0), \\ & i \left(e_{j\delta}^1 - e_{-j\delta}^1 \right) && -1 \quad (j \neq 0), \\ & \left(e_{j\delta + \alpha_1} + e_{-j\delta - \alpha_1} + e_{k\delta - \alpha_1} + e_{-k\delta + \alpha_1} \right) && 1, \\ & \left(e_{j\delta + \alpha_1} + e_{-j\delta - \alpha_1} - e_{k\delta - \alpha_1} - e_{-k\delta + \alpha_1} \right) && -1, \\ & i \left(e_{j\delta + \alpha_1} - e_{-j\delta - \alpha_1} + e_{k\delta - \alpha_1} - e_{-k\delta + \alpha_1} \right) && -1, \\ & i \left(e_{j\delta + \alpha_1} - e_{-j\delta - \alpha_1} - e_{k\delta - \alpha_1} + e_{-k\delta + \alpha_1} \right) && 1, \\ & ic && -1, \\ & i \left(d + h_{\alpha_1} - \frac{1}{2} c \right) && 1. \end{aligned} \tag{2.4.6}$$

In the above, $j = 0, 1, \dots$ and $k = j + 1$, (unless otherwise stated). When Cartan's method is then applied to these eigenvectors, the following basis elements are obtained:

$$\begin{aligned}
& i\left(h_{\alpha_1} - \frac{1}{2}c\right), \\
& \left. \begin{aligned} & \left(e_{j\delta}^1 + e_{-j\delta}^1\right) \\ & \left(e_{j\delta}^1 - e_{-j\delta}^1\right) \end{aligned} \right\} j \neq 0, \\
& \left(e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1} + e_{k\delta-\alpha_1} + e_{-k\delta+\alpha_1}\right), \\
& i\left(e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1} - e_{k\delta-\alpha_1} - e_{-k\delta+\alpha_1}\right), \\
& \left(e_{j\delta+\alpha_1} - e_{-j\delta-\alpha_1} + e_{k\delta-\alpha_1} - e_{-k\delta+\alpha_1}\right), \\
& i\left(e_{j\delta+\alpha_1} - e_{-j\delta-\alpha_1} - e_{k\delta-\alpha_1} + e_{-k\delta+\alpha_1}\right), \\
& c, \\
& \left(d + h_{\alpha_1} - \frac{1}{2}c\right),
\end{aligned} \tag{2.4.7}$$

where in the above, $j = 0, 1, \dots$ unless otherwise stated, and $k = j + 1$. This real form is not instantly recognisable as a real form generated from a real form of A_1 . Consider the matrix parts of the above, and also the matrix parts of real, linear combinations of them, excluding any linear combinations involving the element $\left(d + h_{\alpha_1} - \frac{1}{2}c\right)$. These matrix parts satisfy

$$\tilde{\mathbf{g}}\mathbf{a}^*(t)\mathbf{g}^{-1} = -\mathbf{a}(t) \quad \text{where } \mathbf{g} = \text{diag}\{1, t\}. \tag{2.4.8}$$

Thus, if $\mathbf{a}(t)$ satisfies the above equation, then a general element of this real form may be expressed as

$$\mathbf{a}(t) + (\lambda + i\mu)c + \eta d + \frac{1}{4}\eta \text{diag}\{1, -1\}, \tag{2.4.9}$$

where λ, μ are arbitrary, real numbers, and μ is such that

$$2 \text{Res}\{\text{tr}(\mathbf{h}\mathbf{a}(t))\} = i\mu \quad \text{and } \mathbf{h} = \text{diag}\{0, 1\}. \tag{2.4.10}$$

For the conjugacy class (F), there is a basis of the compact real form consisting of eigenvectors of the representative automorphism of (F) (with the associated

eigenvalues 1 or -1). The members of such a basis, together with their associated eigenvalues, are

$$\begin{aligned}
 & \left(e_{j\delta}^1 + e_{-j\delta}^1 \right) && 1, \\
 & i \left(e_{j\delta}^1 - e_{-j\delta}^1 \right) && -1, \\
 & \left(e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1} + e_{j\delta-\alpha_1} + e_{-j\delta+\alpha_1} \right) && 1, \\
 & \left(e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1} - e_{j\delta-\alpha_1} - e_{-j\delta+\alpha_1} \right) && -1, \\
 & i \left(e_{j\delta+\alpha_1} - e_{-j\delta-\alpha_1} + e_{j\delta-\alpha_1} - e_{-j\delta+\alpha_1} \right) && -1, \\
 & i \left(e_{j\delta+\alpha_1} - e_{-j\delta-\alpha_1} - e_{j\delta-\alpha_1} + e_{-j\delta+\alpha_1} \right) && 1, \\
 & ic && -1, \\
 & id && -1,
 \end{aligned} \tag{2.4.11}$$

where in the above, $j = 0, 1, \dots$. Cartan's method, when applied to this basis of the compact real form yields the following basis of a real form of $A_1^{(1)}$:

$$\begin{aligned}
 & \left(e_{j\delta}^1 + e_{-j\delta}^1 \right), \\
 & \left(e_{j\delta}^1 - e_{-j\delta}^1 \right) \quad (j \neq 0), \\
 & \left(e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1} + e_{j\delta-\alpha_1} + e_{-j\delta+\alpha_1} \right), \\
 & i \left(e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1} - e_{j\delta-\alpha_1} - e_{-j\delta+\alpha_1} \right), \\
 & \left(e_{j\delta+\alpha_1} - e_{-j\delta-\alpha_1} + e_{j\delta-\alpha_1} - e_{-j\delta+\alpha_1} \right), \\
 & i \left(e_{j\delta+\alpha_1} - e_{-j\delta-\alpha_1} - e_{j\delta-\alpha_1} + e_{-j\delta+\alpha_1} \right), \\
 & c, \\
 & d,
 \end{aligned} \tag{2.4.12}$$

where $j = 0, 1, \dots$ unless otherwise stated. This is, in fact, the real form $su_{(2)}(2)$. Finally, consider the representative of the conjugacy class (G). The following is a list of eigenvectors of this automorphism together with their respective eigenvalues.

$$\left. \begin{array}{l} (e_{j\delta}^1 + e_{-j\delta}^1) \\ (e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1} - e_{j\delta-\alpha_1} - e_{-j\delta+\alpha_1}) \\ i(e_{j\delta+\alpha_1} - e_{-j\delta-\alpha_1} + e_{j\delta-\alpha_1} - e_{-j\delta+\alpha_1}) \end{array} \right\} \text{with eigenvalue } 1, \quad (2.4.13)$$

$$\left. \begin{array}{l} i(e_{j\delta}^1 - e_{-j\delta}^1) \text{ with } j \neq 0 \\ (e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1} - e_{j\delta-\alpha_1} - e_{-j\delta+\alpha_1}) \\ i(e_{j\delta+\alpha_1} - e_{-j\delta-\alpha_1} + e_{j\delta-\alpha_1} - e_{-j\delta+\alpha_1}) \\ ic \\ id \end{array} \right\} \text{with eigenvalue } -1. \quad (2.4.14)$$

It follows then, that a basis for a real form of $A_1^{(1)}$ is given by the elements listed below:

$$\begin{aligned} & (e_{j\delta}^1 + e_{-j\delta}^1), \\ & (e_{j\delta}^1 - e_{-j\delta}^1) \text{ for } j \neq 0, \\ & i(e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1} + e_{j\delta-\alpha_1} + e_{-j\delta+\alpha_1}), \\ & (e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1} - e_{j\delta-\alpha_1} - e_{-j\delta+\alpha_1}), \\ & i(e_{j\delta+\alpha_1} - e_{-j\delta-\alpha_1} + e_{j\delta-\alpha_1} - e_{-j\delta+\alpha_1}), \\ & (e_{j\delta+\alpha_1} - e_{-j\delta-\alpha_1} - e_{j\delta-\alpha_1} + e_{-j\delta+\alpha_1}), \\ & c, \\ & d, \end{aligned} \quad (2.4.15)$$

where $j = 0, 1, \dots$ unless otherwise specified. This real form is readily identifiable, and is the real form $su_{(2)}(1,1)$, which is generated from the real form of the Lie algebra A_1 in the manner explained in chapter 1.

3 Involutive automorphisms and real forms of the Kac-Moody algebra $A_2^{(1)}$

3.1 Introduction

This chapter examines the conjugacy classes of the compact real form of the affine Kac-Moody algebra $A_2^{(1)}$; in particular the conjugacy classes of the involutive Cartan-preserving automorphisms within the group of all automorphisms of $A_2^{(1)}$. On the whole, this chapter will be based upon [10], which uses the theory of the matrix formulation, in order to determine the conjugacy classes of the involutions for the complex affine algebra $A_2^{(1)}$, rather than its compact real form. In general, the analysis for the complex algebra requires little modification, in order to work for the compact real form, although in certain circumstances, a reasonable amount of modification is needed.

Recall that, with the representation of A_2 specified previously, the type 1b and type 2b automorphisms are distinct from the type 1a and the type 2a automorphisms respectively, whereas this was not the case for the algebra A_1 . It should also be noted that the group \mathcal{R} , the group of root-preserving transformations of A_2 , consists of the following elements (which are grouped by conjugacy class within the group \mathcal{R}):

- (1) The identity root-transformation,
- (2) The Weyl reflections $S_{\alpha_1^0}^0, S_{\alpha_2^0}^0, S_{\alpha_1^0 + \alpha_2^0}^0$,
- (3) The involutions $\rho^0, (S_{\alpha_1^0}^0 \circ \tau_{Cartan}^0), (S_{\alpha_2^0}^0 \circ \tau_{Cartan}^0)$, where $\rho^0(\alpha_1^0) = \alpha_2^0$ and $\rho^0(\alpha_2^0) = \alpha_1^0$.

(4) The Cartan involution τ_{Cartan}^0 ,

(5) The root transformations $\left(S_{\alpha_1}^0 \circ S_{\alpha_2}^0\right), \left(S_{\alpha_2}^0 \circ S_{\alpha_1}^0\right)$,

(6) The root transformations $\left(S_{\alpha_1}^0 \circ \rho^0\right), \left(S_{\alpha_2}^0 \circ \rho^0\right)$.

In fact, only the first four of these conjugacy classes consist of involutive root-preserving transformations (including the identity root-preserving transformation as involutive); the other two conjugacy classes are thus of no further interest.

3.2 Type 1a involutive automorphisms of $A_2^{(1)}$ with $u = 1$

Consider the conditions obtained earlier in this chapter. That is to say, those that may be used for determining whether or not $\{U(t), u, \xi\}$ is an automorphism of the compact real form of the Kac-Moody algebra in question. When these constraints are applied to the Cartan-preserving automorphisms of type 1a (with $u = 1$) given in [10], the following automorphisms are left to be analysed:

$$U(t) = \text{diag}\{1, \lambda_2, \lambda_3\} \quad u = 1 \quad \xi = 0, \quad (3.2.1)$$

$$U(t) = \begin{bmatrix} 0 & 1 & 0 \\ \eta^2 t^{2k} & 0 & 0 \\ 0 & 0 & \eta t^k \end{bmatrix} \quad u = 1 \quad \xi = 0, \quad (3.2.2)$$

where $\lambda_2^2 = \lambda_3^2 = |\eta| = 1$. In the complex case, there are two conjugacy classes of type 1a involutions with $u = 1$, one being the conjugacy class that contains only the identity automorphism, the other being the conjugacy class that includes all of the other type 1a involutions. That this is the case was established by proving conjugacy via the following type 1a automorphisms:

$$S(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad s = 1, \quad (3.2.3)$$

$$S(t) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad s = 1. \quad (3.2.4)$$

In each of the above cases, it may be noted that $S(t)$ satisfies the conditions derived previously, so that $\{S(t), 1, \xi\}$ is an automorphism of the compact real form of $A_2^{(1)}$,

rather than just being an automorphism of the complex algebra. Thus, there are again two conjugacy classes, which will be called (A) and (B). The class (A) is the one consisting solely of the identity automorphism, whilst (B) contains the remaining automorphisms, notably the representative $\{\text{diag}\{1,1,-1\},1,0\}$. The real form that corresponds to the conjugacy class (A) is, of course, the compact real form of $A_2^{(1)}$ itself. For the conjugacy class (B), there exists a basis of the compact real form composed entirely of eigenvectors (with the associated eigenvalues being 1,-1). The members of such a basis set are listed below:

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \\ i(e_{j\delta}^a - e_{-j\delta}^a) \end{array} \right\} \text{where } a = 1,2; j = 0,1,\dots; \text{ (eigenvalue 1),} \quad (3.2.5)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1}) \\ i(e_{j\delta+\alpha_1} - e_{-j\delta-\alpha_1}) \\ ic \\ id \end{array} \right\} \text{(eigenvalue 1),} \quad (3.2.6)$$

$$\left. \begin{array}{l} (e_{j\delta+\beta} + e_{-j\delta-\beta}) \\ i(e_{j\delta+\beta} - e_{-j\delta-\beta}) \end{array} \right\} \text{where } \beta = \alpha_2, \alpha_1 + \alpha_2; \text{ (eigenvalue } -1), \quad (3.2.7)$$

where, unless otherwise restricted, the parameter j is permitted to take all integral values. Using Cartan's method on this set of basis elements yields the following basis of a real form of $A_2^{(1)}$:

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \\ i(e_{j\delta}^a - e_{-j\delta}^a) \end{array} \right\} \text{where } a = 1,2; j = 0,1,\dots, \quad (3.2.8)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1}), \\ i(e_{j\delta+\alpha_1} - e_{-j\delta-\alpha_1}), \end{array} \right\} \quad (3.2.9)$$

$$\left. \begin{array}{l} i(e_{j\delta+\beta} + e_{-j\delta-\beta}) \\ (e_{j\delta+\beta} - e_{-j\delta-\beta}) \end{array} \right\} \text{where } \beta = \alpha_2, \alpha_1 + \alpha_2, \quad (3.2.10)$$

$$\begin{array}{l} ic, \\ id. \end{array} \quad (3.2.11)$$

Again, except where otherwise specified, the quantity j is an arbitrary integer. The elements of this real form are all such that the matrix parts satisfy

$$\mathbf{a}(t) = -\mathbf{U}\tilde{\mathbf{a}}^* \mathbf{U}^{-1}, \quad \text{tr}(\mathbf{a}(t)) = 0, \quad (3.2.12)$$

where the matrix \mathbf{U} is given by $\mathbf{U} = \text{diag}\{1, 1, -1\}$. This is, in fact, the real form identified previously as $su_{(1)}(2, 1)$. The elements of this real form are such that their matrix parts $\mathbf{a}(t)$ satisfy $\mathbf{a}(t) = -\mathbf{g}\tilde{\mathbf{a}}^* \mathbf{g}^{-1}$, where $\text{tr}(\mathbf{a}(t)) = 0$, and $\mathbf{g} = \text{diag}\{1, -1, 1\}$.

3.3 Type 1a involutive automorphisms of $A_2^{(1)}$ with $u = -1$

In the case of the complex algebra $A_2^{(1)}$, there is only one conjugacy class of such automorphisms within the group of all automorphisms of $A_2^{(1)}$. For the compact real form of $A_2^{(1)}$, the analysis previously undertaken in [10] reduces to the study of the type 1a involutive automorphisms generated by the following matrices:

$$\mathbf{U}(t) = \text{diag}\{1, \lambda_2, \lambda_3\}, \quad (3.3.1)$$

$$\mathbf{U}(t) = \begin{bmatrix} 0 & 1 & 0 \\ (-1)^k \eta^2 t^{2k} & 0 & 0 \\ 0 & 0 & \eta t^k \end{bmatrix}, \quad (3.3.2)$$

where $\lambda_2^2 = \lambda_3^2 = 1$. These do all belong to just one conjugacy class, which will be called (C). Proof that they are all mutually conjugate follows from the fact that the type 1a automorphisms of $A_2^{(1)}$ specified by

$$\mathbf{U}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad s=1, \quad (3.3.3)$$

$$\mathbf{U}(t) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad s=1, \quad (3.3.4)$$

$$\mathbf{U}(t) = \begin{bmatrix} 0 & t & 0 \\ t & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad s=1, \quad (3.3.5)$$

are all also automorphisms of the compact real form of $A_2^{(1)}$. These are automorphisms used in [10] to prove conjugacy. Hence they also help to prove that

all of the automorphisms currently under consideration belong to just one conjugacy class, (C). Take as a representative automorphism the type 1a automorphism $\psi (= \{1_3, 1, 0\})$. A basis of the compact real form consisting of eigenvectors of ψ is given below

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \\ i(e_{j\delta}^a - e_{-j\delta}^a) \end{array} \right\} \text{where } a = 1, 2; j = 0, 1, \dots, \quad (3.3.6)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \text{where } \alpha = \alpha_1, \alpha_2, \alpha_1 + \alpha_2; j \in \mathbf{Z}, \quad (3.3.7)$$

$$\begin{array}{l} ic, \\ id. \end{array} \quad (3.3.8)$$

The eigenvalues associated with the operators ic , id in the above both take the value 1. For each of the other basis elements, the associated eigenvalue is $(-1)^j$. Note also that e_0^k is to be interpreted as ih_{α_k} . A basis of a real form of $A_2^{(1)}$ is thus given by the set of elements listed hereunder:

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \\ i(e_{j\delta}^a - e_{-j\delta}^a) \end{array} \right\} \text{where } a = 1, 2; j = 0, 2, 4, \dots, \quad (3.3.9)$$

$$\left. \begin{array}{l} i(e_{j\delta}^a + e_{-j\delta}^a) \\ (e_{j\delta}^a - e_{-j\delta}^a) \end{array} \right\} \text{where } a = 1, 2; j = 1, 3, 5, \dots, \quad (3.3.10)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \text{where } \alpha = \alpha_1, \alpha_2, \alpha_1 + \alpha_2; j = 0, \pm 2, \pm 4, \dots, \quad (3.3.11)$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \text{where } \alpha = \alpha_1, \alpha_2, \alpha_1 + \alpha_2; j = \pm 1, \pm 3, \pm 5, \dots, \quad (3.3.12)$$

$$ic, \quad (3.3.13)$$

$$id. \tag{3.3.14}$$

The matrix parts of the elements of the abovementioned real form satisfy

$$\mathbf{a}(t) = -\tilde{\mathbf{a}}^* (-t^{-1}), \quad \text{tr}(\mathbf{a}(t)) = 0. \tag{3.3.15}$$

This real form is , therefore, the real form $su_{(3)}(3,0)$.

3.4 Type 1b involutive automorphisms of $A_2^{(1)}$ with $u=1$

Since the representation of A_2 in use is not equivalent to its “contragredient” representation, the type 1b automorphisms are distinct from the type 1a automorphisms. (The same is also true of the type 2a and type 2b automorphisms). It will be demonstrated briefly that, as in the case of $A_2^{(1)}$, there exists only one conjugacy class of such involutions within the group of all automorphisms of the compact real form of $A_2^{(1)}$. The only automorphisms that need to be considered are those type 1b involutions (with $u=1$) generated by the matrices of the following forms:

$$U(t) = \text{diag}\{1, \eta_2 t^{k_2}, \eta_3 t^{k_3}\}, \quad (3.4.1)$$

$$U(t) = \text{diag}\{1, \eta t^k, 1\}, \quad (3.4.2)$$

where $|\eta| = |\eta_2| = |\eta_3| = 1$. All of the automorphisms that are used to establish the result for the complex algebra are such that their restrictions to the compact real form of $A_2^{(1)}$ are automorphisms. Hence there is only one conjugacy class (within the group of automorphisms of the compact real form of $A_2^{(1)}$) of type 1b involutive, Cartan-preserving automorphisms (with $u=1$). This class will be referred to as the conjugacy class (D), and its representative may be taken to be the type 1b ($u=1$) involutive automorphism generated by the matrix $\mathbf{1}_3$. A basis of the compact real form consisting of eigenvectors is the following:

$$\left. \begin{array}{l}
\left. \begin{array}{l}
(e_{j\delta}^a + e_{-j\delta}^1) \\
i(e_{j\delta}^a - e_{-j\delta}^1)
\end{array} \right\} \text{where } j = 0, 1, \dots; a \in \{1, 2\} \\
\left. \begin{array}{l}
(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta-\alpha} - e_{-j\delta+\alpha}) \\
i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta-\alpha} - e_{-j\delta+\alpha})
\end{array} \right\} \text{where } j \in \mathbf{Z}; \alpha \in \{\alpha_1, \alpha_2, \alpha_3\}
\end{array} \right\} \text{eigenvalues } -1,$$

(3.4.3)

$$\left. \begin{array}{l}
\left. \begin{array}{l}
(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta-\alpha} + e_{-j\delta+\alpha}) \\
i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta-\alpha} - e_{-j\delta+\alpha})
\end{array} \right\} \text{where } j \in \mathbf{Z}; \alpha \in \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} \\
ic, id
\end{array} \right\} \text{eigenvalues } 1,$$

A basis of a real form of $A_2^{(1)}$ corresponding to this automorphism of the compact real form is provided by the following elements:

$$\left. \begin{array}{l}
\left. \begin{array}{l}
i(e_{j\delta}^a + e_{-j\delta}^a) \\
(e_{j\delta}^a - e_{-j\delta}^a)
\end{array} \right\} \text{where } a \in \{1, 2\}; j = 0, 1, \dots, \\
\left. \begin{array}{l}
(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta-\alpha} + e_{-j\delta+\alpha}) \\
i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta-\alpha} - e_{-j\delta+\alpha}) \\
i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta-\alpha} - e_{-j\delta+\alpha}) \\
(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta-\alpha} + e_{-j\delta+\alpha})
\end{array} \right\} j \in \mathbf{Z}; \alpha \in \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}, \quad (3.4.4) \\
ic, \\
id.
\end{array} \right\}$$

Upon inspection, it becomes clear that this is the real form generated (by the first method mentioned previously) from the Lie algebra with generators

$$h_{\alpha_1}, h_{\alpha_2}, e_{\alpha}, e_{-\alpha} \text{ (where } \alpha \in \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}). \quad (3.4.5)$$

So this is the algebra which was previously given the name $sl_{(1)}(3, \mathbf{R})$. Notice that the matrix parts are all traceless and satisfy $\mathbf{a}(t) = \mathbf{a}^*(t^{-1})$.

3.5 Type 1b involutive automorphisms of $A_2^{(1)}$ with $u = -1$

In order to show that the results for the complex case also apply to the compact real form, it is necessary to consider only the type 1b involutive automorphisms with $u = -1$ which are generated by the following matrices:

$$\begin{aligned} \mathbf{U}(t) &= \text{diag}\{1, \eta_2 t^{k_2}, \eta_3 t^{k_3}\}, \\ \mathbf{U}(t) &= \text{offdiag}\{1, \eta t^k, (-1)^k\}, \end{aligned} \quad (3.5.1)$$

where $|\eta| = |\eta_2| = |\eta_3| = 1$, and the quantities k_1 and k_2 are even integers. Now the following two type 1a automorphisms of $A_2^{(1)}$ are also automorphisms of its compact real form:

$$\mathbf{S}(t) = \text{diag}\{\eta t^k, 1, 1\} \quad s = 1, \quad (3.5.2)$$

$$\mathbf{S}(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & \eta_3^{-\frac{1}{2}} (-1)^{\frac{k_3}{4}} t^{-\frac{k_3}{2}} \\ 0 & \sqrt{2} \eta^{-\frac{1}{2}} (-1)^{\frac{k_2}{4}} t^{-\frac{k_2}{2}} & 0 \\ -i & 0 & \eta_3^{-\frac{1}{2}} (-1)^{\frac{k_3}{4}} t^{-\frac{k_3}{2}} \end{bmatrix} \quad s = 1. \quad (3.5.3)$$

Thus, it follows that, (with the constraints upon η , η_2 , η_3 , k_2 , k_3 still applying), there is only one conjugacy class of type 1b involutive Cartan-preserving automorphisms of the compact real form of $A_2^{(1)}$ (with $u = -1$). This class is called (E), and its representative automorphism ψ is the type 1b involutive automorphism generated by the matrix $\mathbf{1}_3$, for which $u = -1$. There exists a basis of the compact real form of $A_2^{(1)}$, all of whose elements are eigenvectors of ψ . The members of such a basis, together with their respective eigenvalues, are listed below:

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \\ i(e_{j\delta}^a - e_{-j\delta}^a) \end{array} \right\} \text{where } j \in \mathbb{N}^0; a \in \{1, 2\}; \text{associated eigenvalue } (-1)^{j+1}, \quad (3.5.4)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta-\alpha} + e_{-j\delta+\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta-\alpha} - e_{-j\delta+\alpha}) \end{array} \right\} \text{where } j \in \mathbb{Z}; \alpha \in \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}; \text{eigenvalue } (-1)^j, \quad (3.5.5)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta-\alpha} - e_{-j\delta+\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta-\alpha} + e_{-j\delta+\alpha}) \end{array} \right\} \text{where } j \in \mathbb{Z}; \alpha \in \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}; \text{eigenvalue } (-1)^{j+1}, \quad (3.5.6)$$

$$ic \quad \text{with associated eigenvalue } 1, \quad (3.5.7)$$

$$id \quad \text{with associated eigenvalue } 1. \quad (3.5.8)$$

A basis for a real form of the Kac-Moody algebra $A_2^{(1)}$ is therefore provided by

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \\ i(e_{j\delta}^a - e_{-j\delta}^a) \end{array} \right\} \text{where } j \in \{1, 3, 5, \dots\}; a \in \{1, 2\}, \quad (3.5.9)$$

$$\left. \begin{array}{l} i(e_{j\delta}^a + e_{-j\delta}^a) \\ (e_{j\delta}^a - e_{-j\delta}^a) \end{array} \right\} \text{where } j \in \{2, 4, 6, \dots\}; a \in \{1, 2\}, \quad (3.5.10)$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta-\alpha} + e_{-j\delta+\alpha}) \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta-\alpha} - e_{-j\delta+\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta-\alpha} - e_{-j\delta+\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta-\alpha} + e_{-j\delta+\alpha}) \end{array} \right\} \text{where } j \in \{1, 3, 5, \dots\}; \alpha \in \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}, \quad (3.5.11)$$

$$\left. \begin{aligned} & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta-\alpha} + e_{-j\delta+\alpha} \right) \\ & i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta-\alpha} - e_{-j\delta+\alpha} \right) \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta-\alpha} - e_{-j\delta+\alpha} \right) \\ & \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta-\alpha} + e_{-j\delta+\alpha} \right) \end{aligned} \right\} \text{where } j \in \{2, 4, 6, \dots\}; \alpha \in \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\},$$

(3.5.12)

ic,

(3.5.13)

id.

The matrix parts of these elements satisfy the condition $\mathbf{a}(t) = \mathbf{a}^*(-t^{-1})$, which means that the real form generated is, in fact, the real form $sl_{(3)}(3, \mathbf{R})$.

3.6 Type 2a involutive automorphisms of $A_2^{(1)}$ with $u = 1$

In the algebra $A_2^{(1)}$ there are three conjugacy classes of type 2a involutive automorphisms with $u = 1$. This is also the case for its compact real form. That this is the case follows from the fact that the following type 1a automorphisms of $A_2^{(1)}$ are also automorphisms of the compact real form:

$$S(t) = \text{diag}\{1, t^{-K_2}, t^{-K_3}\} \quad s = 1, \quad (3.6.1)$$

$$S(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad s = 1, \quad (3.6.2)$$

$$S(t) = \text{offdiag}\{1, 1, 1\} \quad s = 1, \quad (3.6.3)$$

$$S(t) = \frac{1}{2} \begin{bmatrix} t+1 & t-1 & 0 \\ t-1 & t+1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad s = 1, \quad (3.6.4)$$

$$S(t) = \text{offdiag}\{1, 1, t\} \quad s = 1, \quad (3.6.5)$$

$$S(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \eta_3 t^{K_3} & 1 & 0 \\ \eta_3 t^{K_3} & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \quad s = 1, \quad (3.6.6)$$

where, in the above, K_2 and K_3 are arbitrary and $|\eta_3| = 1$. This demonstrates that certain automorphisms are conjugate to certain others, following the analysis [10]. That automorphisms from different conjugacy classes are indeed non-conjugate does not need proving, it being a necessary consequence of the analysis of the complex

algebra $A_2^{(1)}$. The involutive Cartan-preserving automorphisms to be examined are (following the analysis contained in [10]) those generated by the following matrices:

$$U(t) = \text{diag}\{1, \eta_2 t^{k_2}, \eta_3 t^{k_3}\}, \quad (3.6.7)$$

$$U(t) = \begin{bmatrix} 0 & 1 & 0 \\ \eta^2 & 0 & 0 \\ 0 & 0 & \eta t^k \end{bmatrix}, \quad (3.6.8)$$

with $|\eta| = 1$. There are, therefore, three conjugacy classes, which are called (F), (G) and (H). Take as representative of the class (F) the type 2a involutive automorphism ψ_F generated by the matrix $\mathbf{1}_3$. An eigenvector-basis of the compact real form (consisting of eigenvectors of ψ_F) is shown below:

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \quad \text{eigenvalue } 1 \\ i(e_{j\delta}^a - e_{-j\delta}^a) \quad \text{eigenvalue } -1 \end{array} \right\} \text{where } j \in \mathbb{N}^0; a \in \{1, 2\}, \quad (3.6.9)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta-\alpha} + e_{-j\delta+\alpha}) \quad \text{eigenvalue } 1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta-\alpha} - e_{-j\delta+\alpha}) \quad \text{eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta-\alpha} - e_{-j\delta+\alpha}) \quad \text{eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta-\alpha} + e_{-j\delta+\alpha}) \quad \text{eigenvalue } 1 \end{array} \right\} \text{where } j \in \mathbb{Z}; \alpha \in \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}, \quad (3.6.10)$$

$$ic \quad \text{with eigenvalue } -1, \quad (3.6.11)$$

$$id \quad \text{with eigenvalue } -1. \quad (3.6.12)$$

This automorphism corresponds to the following basis for a real form of $A_2^{(1)}$:

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \\ (e_{j\delta}^a - e_{-j\delta}^a) \end{array} \right\} \text{where } j \in \mathbb{N}^0; a \in \{1, 2\}, \quad (3.6.13)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta-\alpha} + e_{-j\delta+\alpha}) \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta-\alpha} - e_{-j\delta+\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta-\alpha} - e_{-j\delta+\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta-\alpha} + e_{-j\delta+\alpha}) \end{array} \right\} \text{where } j \in \mathbf{Z}; \alpha \in \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}, \quad (3.6.14)$$

$$ic, \quad (3.6.15)$$

$$id. \quad (3.6.16)$$

These elements are such that their matrix parts satisfy $\mathbf{a}(t) = -\tilde{\mathbf{a}}^*(t)$, which means that the real form whose basis is given above is the algebra $su_{(2)}(3,0)$. The representative automorphism for the conjugacy class (G) is the automorphism ψ_G , which is the type 2b involutive automorphism (with $u = 1$) generated by the matrix $\text{diag}\{1, -1, 1\}$. The following is a basis of the compact real form of $A_2^{(1)}$ consisting of eigenvectors of ψ_G :

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \text{ with eigenvalue } 1 \\ i(e_{j\delta}^a - e_{-j\delta}^a) \text{ with eigenvalue } -1 \end{array} \right\} \text{where } j \in \mathbf{N}^0; a \in \{1, 2\}, \quad (3.6.17)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta-\alpha} + e_{-j\delta+\alpha}) \text{ with eigenvalue } -1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta-\alpha} - e_{-j\delta+\alpha}) \text{ with eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta-\alpha} - e_{-j\delta+\alpha}) \text{ with eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta-\alpha} + e_{-j\delta+\alpha}) \text{ with eigenvalue } -1 \end{array} \right\} \text{where } j \in \mathbf{Z}; \alpha \in \{\alpha_1, \alpha_2\}, \quad (3.6.18)$$

$$\left. \begin{array}{l} (e_{j\delta+\beta} + e_{-j\delta-\beta} + e_{j\delta-\beta} + e_{-j\delta+\beta}) \text{ with eigenvalue } 1 \\ (e_{j\delta+\beta} + e_{-j\delta-\beta} - e_{j\delta-\beta} - e_{-j\delta+\beta}) \text{ with eigenvalue } -1 \\ i(e_{j\delta+\beta} - e_{-j\delta-\beta} + e_{j\delta-\beta} - e_{-j\delta+\beta}) \text{ with eigenvalue } -1 \\ i(e_{j\delta+\beta} - e_{-j\delta-\beta} - e_{j\delta-\beta} + e_{-j\delta+\beta}) \text{ with eigenvalue } 1 \end{array} \right\} \text{where } j \in \mathbf{Z}; \beta = \alpha_1 + \alpha_2, \quad (3.6.19)$$

$$\begin{aligned}
ic & \text{ with eigenvalue } -1, \\
id & \text{ with eigenvalue } -1.
\end{aligned}
\tag{3.6.20}$$

Cartan's method supplies the following basis of a real form of the Kac-Moody algebra $A_2^{(1)}$, generated by the automorphism ψ_G :

$$\left. \begin{aligned}
& \left(e_{j\delta}^a + e_{-j\delta}^a \right) \\
& \left(e_{j\delta}^a - e_{-j\delta}^a \right)
\end{aligned} \right\} \text{ where } j \in \mathbb{N}^0; a \in \{1, 2\},
\tag{3.6.21}$$

$$\left. \begin{aligned}
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta-\alpha} + e_{-j\delta+\alpha} \right) \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta-\alpha} - e_{-j\delta+\alpha} \right) \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta-\alpha} - e_{-j\delta+\alpha} \right) \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta-\alpha} + e_{-j\delta+\alpha} \right)
\end{aligned} \right\} \text{ where } j \in \mathbb{Z}; \alpha \in \{\alpha_1, \alpha_2\},
\tag{3.6.22}$$

$$\left. \begin{aligned}
& \left(e_{j\delta+\beta} + e_{-j\delta-\beta} + e_{j\delta-\beta} + e_{-j\delta+\beta} \right) \\
& i \left(e_{j\delta+\beta} + e_{-j\delta-\beta} - e_{j\delta-\beta} - e_{-j\delta+\beta} \right) \\
& \left(e_{j\delta+\beta} - e_{-j\delta-\beta} + e_{j\delta-\beta} - e_{-j\delta+\beta} \right) \\
& i \left(e_{j\delta+\beta} - e_{-j\delta-\beta} - e_{j\delta-\beta} + e_{-j\delta+\beta} \right)
\end{aligned} \right\} \text{ where } j \in \mathbb{Z}; \beta = \alpha_1 + \alpha_2,
\tag{3.6.23}$$

$$\begin{aligned}
& c, \\
& d.
\end{aligned}
\tag{3.6.24}$$

The matrix parts of elements of this real form are such that they satisfy

$$\mathbf{a}(t) = -\mathbf{g}\bar{\mathbf{a}}^*(t)\mathbf{g}^{-1} \quad \text{where } \mathbf{g} = \text{diag}\{-1, 1, -1\}.
\tag{3.6.25}$$

At first glance, this does not appear to correspond to a real form generated by one of the real forms of A_2 . However, the real Lie algebra $su(2, 1)$ is isomorphic to the real Lie algebra $su(1, 2)$, which has as basis set the following:

$$ih_{\alpha_1},
\tag{3.6.26}$$

$$ih_{\alpha_2},
\tag{3.6.27}$$

$$i(e_{\alpha} + e_{-\alpha}) \quad \text{where } \alpha \in \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\},
\tag{3.6.28}$$

$$(e_\alpha - e_{-\alpha}) \quad \text{where } \alpha \in \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}, \quad (3.6.29)$$

so the real form in question is generated from the real form $su(1,2)$, and is, in fact, the real form $su_{(2)}(1,2)$. The matrix parts of this real form are all traceless, and satisfy

$$\mathbf{a}(t) = -\mathbf{g}\tilde{\mathbf{a}}^*(t)\mathbf{g}^{-1} \quad \text{where } \mathbf{g} = \text{diag}\{1, -1, 1\}. \quad (3.6.30)$$

The representative automorphism of the conjugacy class (H) is the automorphism ψ_H , where ψ_H is the type 2a automorphism for which

$$\mathbf{U}(t) = \text{diag}\{1, 1, t\} \quad u = 1 \quad \xi = 2. \quad (3.6.31)$$

A basis of the compact real form of $A_2^{(1)}$ (in which each basis element is an eigenvector of ψ_H) is given below:

$$\left. \begin{array}{l} ih_{\alpha_1} \\ i\left(h_{\alpha_2} - \frac{1}{2}c\right) \end{array} \right\} \text{with eigenvalue } 1, \quad (3.6.32)$$

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \text{ with eigenvalue } 1 \\ i(e_{j\delta}^a - e_{-j\delta}^a) \text{ with eigenvalue } -1 \end{array} \right\} \text{where } j \in \mathbf{N}^0; a \in \{1, 2\}, \quad (3.6.33)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta-\alpha} + e_{-j\delta+\alpha}) \text{ with eigenvalue } 1, \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta-\alpha} - e_{-j\delta+\alpha}) \text{ with eigenvalue } -1, \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta-\alpha} - e_{-j\delta+\alpha}) \text{ with eigenvalue } -1, \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta-\alpha} + e_{-j\delta+\alpha}) \text{ with eigenvalue } 1, \end{array} \right\} \text{where } j \in \mathbf{Z}; \alpha = \alpha_1, \quad (3.6.34)$$

$$\left. \begin{array}{l} (e_{j\delta+\beta} + e_{-j\delta-\beta} + e_{k\delta-\beta} + e_{-k\delta+\beta}) \text{ with eigenvalue } 1, \\ (e_{j\delta+\beta} + e_{-j\delta-\beta} - e_{k\delta-\beta} - e_{-k\delta+\beta}) \text{ with eigenvalue } -1, \\ i(e_{j\delta+\beta} - e_{-j\delta-\beta} + e_{k\delta-\beta} - e_{-k\delta+\beta}) \text{ with eigenvalue } -1, \\ i(e_{j\delta+\beta} - e_{-j\delta-\beta} - e_{k\delta-\beta} + e_{-k\delta+\beta}) \text{ with eigenvalue } 1, \end{array} \right\} j \in \mathbf{Z}; k = j + 1; \beta \in \{\alpha_2, \alpha_1 + \alpha_2\}, \quad (3.6.35)$$

$$\begin{aligned}
ic & \text{ with eigenvalue } -1, \\
i(d + h_{\alpha_1} + 2h_{\alpha_2}) & \text{ with eigenvalue } -1.
\end{aligned} \tag{3.6.36}$$

The corresponding basis of a real form of $A_2^{(1)}$ is thus given by Cartan's method, and is

$$\begin{aligned}
ih_{\alpha_1}, \\
i\left(h_{\alpha_2} - \frac{1}{2}c\right),
\end{aligned} \tag{3.6.37}$$

$$\left. \begin{aligned}
(e_{j\delta}^a + e_{-j\delta}^a) \\
(e_{j\delta}^a - e_{-j\delta}^a)
\end{aligned} \right\} \text{ where } j \in \mathbb{N}^0; a \in \{1, 2\}, \tag{3.6.38}$$

$$\left. \begin{aligned}
(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta-\alpha} + e_{-j\delta+\alpha}) \\
i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta-\alpha} - e_{-j\delta+\alpha}) \\
(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta-\alpha} - e_{-j\delta+\alpha}) \\
i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta-\alpha} + e_{-j\delta+\alpha})
\end{aligned} \right\} \text{ where } j \in \mathbb{Z}; \alpha = \alpha_1, \tag{3.6.39}$$

$$\left. \begin{aligned}
(e_{j\delta+\beta} + e_{-j\delta-\beta} + e_{k\delta-\beta} + e_{-k\delta+\beta}) \\
i(e_{j\delta+\beta} + e_{-j\delta-\beta} - e_{k\delta-\beta} - e_{-k\delta+\beta}) \\
(e_{j\delta+\beta} - e_{-j\delta-\beta} + e_{k\delta-\beta} - e_{-k\delta+\beta}) \\
i(e_{j\delta+\beta} - e_{-j\delta-\beta} - e_{k\delta-\beta} + e_{-k\delta+\beta})
\end{aligned} \right\} j \in \mathbb{Z}; k = j + 1; \beta \in \{\alpha_2, \alpha_1 + \alpha_2\}, \tag{3.6.40}$$

$$\begin{aligned}
c, \\
(d + h_{\alpha_1} + 2h_{\alpha_2}).
\end{aligned} \tag{3.6.41}$$

The identification of this real form is not as straightforward as with the previous real forms encountered. It is clear that neither d nor id nor ic may be elements of this real form. Suppose, for the moment, that the real form lacks the element $(d + h_{\alpha_1} + 2h_{\alpha_2})$. The matrix parts of these elements (namely, the ones that do not contain any real linear multiples of the aforementioned element) are such that

$$\mathbf{a}(t) = -\tilde{\mathbf{g}}^*(t)\mathbf{g}^{-1} \quad \text{where } \mathbf{g} = \text{diag}\{1, 1, t\}. \tag{3.6.42}$$

It follows, therefore, that a typical element of this real form may be expressed as

$$\mathbf{a}(t) + (\lambda + i\mu)c + \eta(d + h_{\alpha_1} + 2h_{\alpha_2}), \quad (3.6.43)$$

where $\mathbf{a}(t)$ satisfies the condition described above, and λ , μ , η are all real quantities, with the quantity μ satisfying

$$3\text{tr}(\mathbf{h}\mathbf{a}(t)) = \mu i \quad \text{where } \mathbf{h} = \text{diag}\{0,0,1\}, \quad (3.6.44)$$

and the quantities λ and η are arbitrary.

3.7 Type 2b involutive automorphisms of $A_2^{(1)}$ with $u = 1$

The paper [10] examined the type 2b involutive automorphisms of $A_2^{(1)}$ (with $u = 1$) which were generated by the following matrices:

$$U(t) = \text{diag}\{1, \eta_2, \eta_3\}, \quad (3.7.1)$$

$$U(t) = \text{offdiag}\{1, \eta t^k, t^{2k}\}, \quad (3.7.2)$$

Those automorphisms included in the above whose restrictions to the compact real form are automorphisms of it are those for which $|\eta| = |\eta_2| = |\eta_3| = 1$. It is not surprising that the result for the compact real form is the same as for the complex algebra $A_2^{(1)}$, and that results is obtainable using the matrix formulation. That is, all of the type 2b involutive automorphisms (with $u = 1$) are mutually conjugate, belonging to a conjugacy class called (I). To see explicitly that this is the case, consider the following type 1a automorphisms, which are used to establish the results in [10] for the complex Kac-Moody algebra:

$$S(t) = \text{diag}\{1, \eta_2, \eta_3\} \quad s = 1, \quad (3.7.3)$$

$$S(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i\eta t^k & 0 & i \\ 0 & \sqrt{2} & 0 \\ \eta t^k & 0 & 1 \end{bmatrix} \quad s = 1. \quad (3.7.4)$$

The restrictions of these automorphisms to the compact real form of $A_2^{(1)}$ are automorphisms of the compact real form, and so the results for the compact real form are the same as for the complex Kac-Moody algebra. The representative automorphism for the sole conjugacy class may be taken to be the involutive automorphism ψ_I , which is the type 2b automorphism $\{1_3, 1, 0\}$. An eigenvector

basis of the compact real form of $A_2^{(1)}$ (with respect to the automorphism (ψ_1)) is the one given hereunder:

$$\begin{aligned} & (e_{j\delta}^a + e_{-j\delta}^a) \text{ where } j \in \mathbf{N}^0; a \in \{1, 2\} \quad \text{eigenvalue } -1, \\ & i(e_{j\delta}^a - e_{-j\delta}^a) \text{ where } j \in \mathbf{N}; a \in \{1, 2\} \quad \text{eigenvalue } 1, \end{aligned} \quad (3.7.5)$$

$$\left. \begin{aligned} & (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \quad \text{eigenvalue } 1 \\ & i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \quad \text{eigenvalue } -1 \end{aligned} \right\} j \in \mathbf{Z}; \alpha \in \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}, \quad (3.7.6)$$

$$\left. \begin{aligned} & ic \\ & id \end{aligned} \right\} \text{eigenvalue } -1. \quad (3.7.7)$$

Thus, the basis for a real form of $A_2^{(1)}$ is, as suggested by Cartan's theorem, given by

$$\begin{aligned} & i(e_{j\delta}^a + e_{-j\delta}^a) \text{ where } j \in \mathbf{N}^0; a \in \{1, 2\}, \\ & i(e_{j\delta}^a - e_{-j\delta}^a) \text{ where } j \in \mathbf{N}; a \in \{1, 2\}, \end{aligned} \quad (3.7.8)$$

$$\left. \begin{aligned} & (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ & (e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{aligned} \right\} j \in \mathbf{Z}; \alpha \in \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}, \quad (3.7.9)$$

$$\begin{aligned} & c, \\ & d. \end{aligned} \quad (3.7.10)$$

This basis is the basis of the real form previously given the name $sl_{(2)}(3, \mathbf{R})$. The general element is of the form

$$\mathbf{a}(t) + \lambda c + \mu d, \quad (3.7.11)$$

where λ and μ are arbitrary real numbers, and the matrix part $\mathbf{a}(t)$ is traceless and satisfies

$$\bar{\mathbf{a}}^*(t) = \mathbf{a}(t). \quad (3.7.12)$$

4 Involutive automorphisms and real forms of $A_\ell^{(1)}$ where $\ell > 2$

4.1 Introduction

In previous chapters, the real forms of the affine Kac-Moody algebras $A_1^{(1)}$ and $A_2^{(1)}$ have been investigated. This chapter extends the work of those chapters (and of [11]) to the Kac-Moody algebras $A_\ell^{(1)}$, where $\ell > 2$. Note firstly that the representation of A_ℓ being used is Γ , which is specified by the following:

$$\Gamma\left(h_{\alpha_j^0}^0\right) = \{2(\ell+1)\}^{-1}(\mathbf{e}_{j,j} - \mathbf{e}_{j+1,j+1}), \quad (4.1.1)$$

$$\left. \begin{aligned} \Gamma\left(e_{\alpha^0}^0\right) &= \{2(\ell+1)\}^{-\frac{1}{2}} \mathbf{e}_{p,q} \\ \Gamma\left(e_{-\alpha^0}^0\right) &= -\{2(\ell+1)\}^{-\frac{1}{2}} \mathbf{e}_{q,p} \end{aligned} \right\} \text{ where } 1 \leq p < q \leq \ell+1, \quad (4.1.2)$$

and where the root α^0 is given in terms of the simple roots by

$$\alpha^0 = \sum_{j=p}^{q-1} \alpha_j^0, \quad (4.1.3)$$

with $\mathbf{e}_{j,k}$ being the $(\ell+1) \times (\ell+1)$ matrix whose elements are defined by the equation

$$\left(\mathbf{e}_{j,k}\right)_{rs} = \delta_{jr} \delta_{ks}. \quad (4.1.4)$$

The Kac-Moody algebras $A_\ell^{(1)}$ are closely associated with the complex simple Lie algebras A_ℓ . The real forms of A_ℓ are given below:

1. The real Lie algebra $sl(\ell+1, \mathbf{R})$. This is the real Lie algebra of real, traceless $(\ell+1) \times (\ell+1)$ matrices. A convenient matrix basis for this real form is provided by the matrices

$$\begin{aligned} & (\mathbf{e}_{p,p} - \mathbf{e}_{p+1,p+1}) \quad p = 1, \dots, \ell, \\ & \mathbf{e}_{p,q} \quad 1 \leq p, q \leq \ell+1; p \neq q. \end{aligned} \quad (4.1.5)$$

It may be noted that each of these basis elements is a real multiple of the basis matrices of the representation Γ defined previously.

2. The real Lie algebra $su(p, q)$. This is the algebra of $(\ell+1) \times (\ell+1)$ traceless matrices \mathbf{a} which also satisfy

$$\mathbf{a} = -\tilde{\mathbf{g}} \mathbf{a}^* \mathbf{g}^{-1} \quad \text{where } \mathbf{g} = \text{dsum} \{ \mathbf{1}_p, -\mathbf{1}_{\ell+1-p} \}. \quad (4.1.6)$$

The precise choice of the matrix \mathbf{g} is, to a certain extent, arbitrary. In particular circumstances, one realisation of the algebra may be preferred to another. In the general case in which $\mathbf{g} = \text{diag} \{ \eta_1, \eta_2, \dots, \eta_\ell, \eta_{\ell+1} \}$, where $\eta_1^2 = \eta_2^2 = \dots = \eta_\ell^2 = \eta_{\ell+1}^2 = 1$.

In this case, a convenient matrix basis for the real form is given by

$$i(\mathbf{e}_{j,j} - \mathbf{e}_{j+1,j+1}) \quad j = 1, 2, \dots, \ell, \quad (4.1.7)$$

$$\left. \begin{aligned} & (\mathbf{e}_{p,q} - \mathbf{e}_{q,p}) \\ & i(\mathbf{e}_{p,q} + \mathbf{e}_{q,p}) \end{aligned} \right\} \text{where } \eta_p \eta_q = 1; 1 \leq p < q \leq \ell+1, \quad (4.1.8)$$

$$\left. \begin{aligned} & (\mathbf{e}_{p,q} + \mathbf{e}_{q,p}) \\ & i(\mathbf{e}_{p,q} - \mathbf{e}_{q,p}) \end{aligned} \right\} \text{where } \eta_p \eta_q = -1; 1 \leq p < q \leq \ell+1, \quad (4.1.9)$$

3. The real Lie algebra $su^*(\ell+1)$. In this case, the quantity ℓ is assumed to be odd. The Lie algebra may be realised as the algebra of traceless $(\ell+1) \times (\ell+1)$ matrices \mathbf{a} that satisfy

$$\mathbf{a} = \mathbf{J} \mathbf{a} \mathbf{J}^{-1} \quad \text{where } \mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{1}_{\frac{1}{2}(\ell+1)} \\ -\mathbf{1}_{\frac{1}{2}(\ell+1)} & \mathbf{0} \end{bmatrix}. \quad (4.1.10)$$

A convenient matrix basis for this real Lie algebra is provided by the following:

$$i(\mathbf{e}_{j,j} - \mathbf{e}_{k,k}) \quad \text{where } 1 \leq j \leq \frac{1}{2}(\ell+1), \quad (4.1.11)$$

$$\left. \begin{array}{l} (\mathbf{e}_{p,q} + \mathbf{e}_{r,s}) \\ i(\mathbf{e}_{p,q} - \mathbf{e}_{r,s}) \end{array} \right\} 1 \leq p, q \leq \frac{1}{2}(\ell+1); p \neq q; r - p = s - q = \frac{1}{2}(\ell+1), \quad (4.1.12)$$

$$\left. \begin{array}{l} (\mathbf{e}_{p,q} - \mathbf{e}_{r,s}) \\ i(\mathbf{e}_{p,q} + \mathbf{e}_{r,s}) \end{array} \right\} 1 \leq p \leq \frac{1}{2}(\ell+1) < q \leq (\ell+1); r - p = q - s = \frac{1}{2}(\ell+1). \quad (4.1.13)$$

The conjugacy classes of the involutive automorphisms of $A_\ell^{(1)}$ have been investigated previously (see [24], [11]), although it is the latter which forms the basis for much of this present work. In general, for $A_1^{(1)}$ and $A_2^{(1)}$, the number of conjugacy classes of involutive Cartan-preserving automorphisms of the compact real form is the same as that quantity for the relevant complex Kac-Moody algebra. This section will examine $A_\ell^{(1)}$ in much the same fashion as $A_1^{(1)}$ and $A_2^{(1)}$ were examined in the preceding sections. The work of [11] is adapted where necessary, with some repetition of material. This information is then used as the basis for an investigation into the real forms of $A_\ell^{(1)}$, generalising Cartan's theorem to the affine Kac-Moody algebras.

Consider the group \mathcal{R} , defined to be the group of rotations of the roots of A_ℓ . The conjugacy classes of the members of the group \mathcal{R} fall naturally into 6 sets. These may be summarised thus:

1. This is the set that contains only the identity root transformation.
2. This contains those conjugacy classes for which one may take as representatives the rotations $S_{\alpha_1^0}^0, (S_{\alpha_1^0}^0 \circ S_{\alpha_3^0}^0), \dots, (S_{\alpha_1^0}^0 \circ S_{\alpha_3^0}^0 \circ \dots \circ S_{\alpha_k^0}^0)$, (where $k = (\ell - 2)$ if ℓ is odd, and $k = (\ell - 1)$ if ℓ is even).

3. This set contains the class whose representative may be taken to be the root transformation $(S_{\alpha_1}^0 \circ S_{\alpha_3}^0 \circ \dots \circ S_{\alpha_\ell}^0)$. In this case, naturally, the quantity ℓ is assumed to be odd.

4. This is the set which contains only the single root transformation τ_{Cartan}^0 , which is specified by $\tau_{Cartan}^0(\alpha_j^0) = -\alpha_j^0$ (for $1 \leq j \leq \ell$).

5. This set contains conjugacy classes whose representatives may be taken to be the products of the rotations listed in 2 with the rotation τ_{Cartan}^0 that has just been specified.

6. This set is the one whose members include the rotation $(S_{\alpha_1}^0 \circ S_{\alpha_3}^0 \circ \dots \circ S_{\alpha_\ell}^0) \circ \tau_{Cartan}^0$ and the rotation ρ^0 , for which $\rho^0(\alpha_j^0) = \alpha_{\ell+1-j}^0$ (with $1 \leq j \leq \ell$). In practice, it is the second of these that proves to be a more useful representative automorphism.

In these 6 sets, the representative rotation of the roots of A_ℓ will be called τ_j , where j (from 1 to 6) is the number of the set in question.

4.2 Type 1a involutive automorphisms of $A_\ell^{(1)}$ with $u = 1$

It may be recalled from [11] that there are no type 1a automorphisms corresponding to the root transformations of sets 4, 5 and 6 given in the previous section. We examine therefore, those corresponding to the other three rotations, each in turn.

Automorphisms corresponding to the rotation τ_1

It is also the case that the type 1a involutive automorphisms that correspond to the identity root transformation are precisely those automorphisms corresponding to the triples $\{\mathbf{U}, 1, 0\}$ for which

$$\mathbf{U} = \text{diag}\{1, \eta_2, \dots, \eta_{\ell+1}\} \quad \text{with } \eta_2^2 = \eta_3^2 = \dots = \eta_{\ell+1}^2 = 1. \quad (4.2.1)$$

With \mathbf{U} so defined, $\tilde{\mathbf{U}}^* \mathbf{U} = \mathbf{1}_{\ell+1}$, so that $\{\mathbf{U}, 1, 0\}$ is an automorphism of the compact real form of $A_\ell^{(1)}$, as well as being an automorphism of the complex algebra $A_\ell^{(1)}$. The order of the elements $\eta_2, \eta_3, \dots, \eta_{\ell+1}$ is arbitrary, so that \mathbf{U} may be assumed to be of the form $\mathbf{U} = \text{dsum}\{\mathbf{1}_a, -\mathbf{1}_{\ell+1-a}\}$. The results (and analysis) thus follow those of $A_\ell^{(1)}$, and require no further modification. The automorphisms currently under consideration fall into $\left(\left[\frac{1}{2}(\ell+1)\right]+1\right)$ mutually disjoint conjugacy classes. It follows from the analysis regarding $A_\ell^{(1)}$ that these classes are, in fact, disjoint. The conjugacy classes thus specified will be referred to as $(A)^{(k)}$, where $1 \leq k \leq \left(\left[\frac{1}{2}(\ell+1)\right]+1\right)$, and for each value of k the representative automorphism is taken to be ψ_{A^k} , where

$$\psi_{A^k} = \left\{ \text{dsum}\{\mathbf{1}_{\ell-k}, \mathbf{1}_k, \mathbf{1}_1\}, 1, 0 \right\}. \quad (4.2.2)$$

For each such representative, there exists a basis of the compact real form consisting entirely of eigenvectors of ψ_{A^k} with eigenvalues ± 1 . The roots $\{\alpha\}$ are expressed in terms of the simple roots by the formula

$$\alpha^0 = \sum_{j=p}^{q-1} \alpha_j^0. \quad (4.2.3)$$

A (ψ_{A^k}) -eigenvector basis of the compact real form of $A_\ell^{(1)}$ (for $k > 0$), together with eigenvalues, is given below:

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \\ i(e_{j\delta}^a - e_{-j\delta}^a) \end{array} \right\} \text{with } 1 \leq j \leq \ell; \text{ eigenvalue } 1, \quad (4.2.4)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \left\{ \begin{array}{l} (1 \leq p \leq \ell - k < q \leq \ell) \\ (\ell - k < p \leq \ell; q = \ell + 1) \end{array} \right\} \text{with eigenvalue } -1, \quad (4.2.5)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \text{other values of } \{p, q\}; q > p; \text{ eigenvalue } 1, \quad (4.2.6)$$

$$\begin{array}{l} ic \text{ eigenvalue } 1, \\ id \text{ eigenvalue } 1. \end{array} \quad (4.2.7)$$

The corresponding basis of a real form of $A_\ell^{(1)}$ is thus given by the following elements:

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \\ i(e_{j\delta}^a - e_{-j\delta}^a) \end{array} \right\} a = 1, 2, \dots, \ell, \quad (4.2.8)$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \left\{ \begin{array}{l} (1 \leq p \leq \ell - k < q \leq \ell) \\ (\ell - k < p \leq \ell; q = \ell + 1) \end{array} \right\}, \quad (4.2.9)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \text{other values of } \{p, q\}; q > p, \quad (4.2.10)$$

$$\begin{array}{l} ic, \\ id. \end{array} \quad (4.2.11)$$

However, this is not the most convenient basis for a real form, and the choice of representative was not the most promising. A more useful choice for the representative automorphism of $(A)^{(k)}$ is the automorphism $\psi = \left\{ \text{dsum} \{ \mathbf{1}_{\ell+1-k}, -\mathbf{1}_k \}, 1, 0 \right\}$. The elements of a (ψ) -eigenvector basis of the compact real form are listed here, together with their associated eigenvalues:

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \\ i(e_{j\delta}^a - e_{-j\delta}^a) \end{array} \right\} j \in \mathbf{N}^0; a \in \{1, 2, \dots, \ell\}; \text{eigenvalue } 1, \quad (4.2.12)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \left\{ \begin{array}{l} 1 \leq p < q \leq \ell+1-k \\ \ell+1-k < p < q \leq \ell+1 \end{array} \right\} j \in \mathbf{Z}; \text{eigenvalue } 1, \quad (4.2.13)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} 1 \leq p \leq (\ell+1-k) < q \leq \ell+1; j \in \mathbf{Z}; \text{eigenvalue } -1, \quad (4.2.14)$$

$$ic \quad \text{eigenvalue } 1, \quad (4.2.15)$$

$$id \quad \text{eigenvalue } 1.$$

A more convenient real form basis is thus provided by the following set of elements:

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \\ i(e_{j\delta}^a - e_{-j\delta}^a) \end{array} \right\} j \in \mathbf{N}^0; a \in \{1, 2, \dots, \ell\}, \quad (4.2.16)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \left\{ \begin{array}{l} 1 \leq p < q \leq \ell+1-k \\ \ell+1-k < p < q \leq \ell+1 \end{array} \right\} j \in \mathbf{Z}, \quad (4.2.17)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} 1 \leq p \leq (\ell+1-k) < q \leq \ell+1; j \in \mathbf{Z}, \quad (4.2.18)$$

$$ic, \quad (4.2.19)$$

$$id.$$

It is easily verified that the matrix parts of the elements of this real form satisfy

$$\mathbf{a}(t) = -\mathbf{g}\bar{\mathbf{a}}^*(t^{-1})\mathbf{g}^{-1} \quad \text{where } \mathbf{g} = \text{dsum} \{ \mathbf{1}_{\ell+1-k}, -\mathbf{1}_k \}, \quad (4.2.20)$$

and it is also clear that the real form in question is, in fact, the real form $su_{(1)}(\ell + 1 - k, k)$. For the special case $k = 0$ this agrees with what we already know, given that the compact real form of $A_\ell^{(1)}$ is the algebra $su_{(1)}(\ell + 1, 0)$.

Involutive automorphisms corresponding to the rotation τ_2

For $A_\ell^{(1)}$, it is the case that each automorphism corresponding to the rotation τ_2 is conjugate to at least one automorphism that corresponds to the identity rotation. This is also the case for $su_{(1)}(\ell + 1, 0)$, its compact real form.

Consider a typical automorphism for which the associated root transformation is in set (2). If the associated root transformation of A_ℓ is $S_{\alpha_1}^0 \circ S_{\alpha_3}^0 \circ \dots \circ S_{\alpha_r}^0$, then the involutive automorphisms corresponding to it are generated by matrices $U(t)$ of the form

$$U(t) = \text{dsum}\{Z_1(t), Z_3(t), \dots, Z_r(t), T\}, \quad (4.2.21)$$

where the submatrices $Z(t)$ and T are given by

$$Z(t) = \begin{bmatrix} 0 & \eta_j t^{k_j} \\ \eta_j^{-1} t^{-k_j} & 0 \end{bmatrix} \quad \eta_j \eta_j^* = 1, \quad (4.2.22)$$

$$T = \text{diag}\{\eta_{r+2}, \eta_{r+3}, \dots, \eta_\ell, 1\} \quad \eta_m^2 = 1; r+2 \leq m \leq \ell. \quad (4.2.23)$$

It only remains to be seen whether or not the analysis of [11] is still applicable when restricted to the compact real form $su_{(1)}(\ell + 1, 0)$. Let the matrix $S(t)$ be given by

$$S(t) = \text{dsum}\{X_1(t), X_3(t), \dots, X_r(t), \mathbf{1}_{\ell-r}\} \quad X_j = \begin{bmatrix} 1 & \eta t^{k_j} \\ -1 & \eta t^{k_j} \end{bmatrix}, \quad (4.2.24)$$

where the quantity j is, of course, odd. Then the following hold:

$$\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{\ell+1} \quad \mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(t)^{-1} = \text{dsum}\{\mathbf{W}_{1,r}, \mathbf{T}\}, \quad (4.2.25)$$

where $\mathbf{W}_{1,r}$ is an $(r+1) \times (r+1)$ diagonal matrix whose elements are alternately 1, -1.

Thus, each automorphism corresponding to a rotation in set (2) is, in fact, conjugate to an automorphism corresponding to the identity root transformation. No new conjugacy classes, or indeed any new real forms are thus produced.

Involutive automorphisms corresponding to the rotation τ_3

The results for the corresponding automorphisms of $\mathbf{A}_\ell^{(1)}$ are that these automorphisms fall into two separate conjugacy classes. One of these classes contains automorphisms corresponding to the identity root transformation, so that its members belong to a conjugacy class already identified. The other class contains the involutive automorphism generated by the matrix $\text{offsum}\left\{\mathbf{K}_{\frac{1}{2}(\ell+1)}, t\mathbf{K}_{\frac{1}{2}(\ell+1)}\right\}$ (remember that ℓ is odd in this case). For $su_{(1)}(\ell+1, 0)$, no adaptation is necessary. The automorphisms $\{\mathbf{S}(t), s, \xi\}$ used (in [11]) to establish conjugacy of certain involutive automorphisms are all such that their restrictions to $su_{(1)}(\ell+1, 0)$ are automorphisms (when the conjugate automorphisms of $\mathbf{A}_\ell^{(1)}$ are extensions of automorphisms of $su_{(1)}(\ell+1, 0)$).

The new conjugacy class that is identified will be called (B). Its representative automorphism is the involution ψ , which is the type 1a automorphism corresponding to the triple $\left\{\text{offsum}\left\{\mathbf{K}_{\frac{1}{2}(\ell+1)}, t\mathbf{K}_{\frac{1}{2}(\ell+1)}\right\}, 1, -\frac{1}{4}(\ell+1)^2\right\}$. The elements of a (ψ) -eigenvector basis of $su_{(1)}(\ell+1, 0)$, together with the associated eigenvalue (being ± 1)

are listed hereunder:

$$i(h_{\alpha_k} + h_{\alpha_{\ell+1-k}}) \quad \text{eigenvalue } -1; k \in \left\{1, 2, \dots, \frac{1}{2}(\ell+1)\right\}, \quad (4.2.26)$$

$$i(h_{\alpha_k} - h_{\alpha_{\ell+1-k}}) \text{ eigenvalue } 1; k \in \{1, 2, \dots, \frac{1}{2}(\ell-1)\}, \quad (4.2.27)$$

$$i\left(h_{\alpha_{\frac{1}{2}(\ell+1)}} - \frac{1}{2}c\right) \text{ eigenvalue } -1, \quad (4.2.28)$$

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a + e_{j\delta}^{\ell+1-a} + e_{-j\delta}^{\ell+1-a}) \text{ eigenvalue } -1 \\ (e_{j\delta}^a + e_{-j\delta}^a - e_{j\delta}^{\ell+1-a} - e_{-j\delta}^{\ell+1-a}) \text{ eigenvalue } 1 \\ i(e_{j\delta}^a - e_{-j\delta}^a + e_{j\delta}^{\ell+1-a} - e_{-j\delta}^{\ell+1-a}) \text{ eigenvalue } -1 \\ i(e_{j\delta}^a - e_{-j\delta}^a - e_{j\delta}^{\ell+1-a} + e_{-j\delta}^{\ell+1-a}) \text{ eigenvalue } 1 \end{array} \right\} a \in \{1, 2, \dots, \frac{1}{2}(\ell+1)\}, \quad (4.2.29)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\beta} + e_{j\delta-\beta}) \text{ eigenvalue } -1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\beta} - e_{j\delta-\beta}) \text{ eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\beta} - e_{j\delta-\beta}) \text{ eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\beta} + e_{j\delta-\beta}) \text{ eigenvalue } -1 \end{array} \right\} \left[\begin{array}{l} \alpha = \sum_{k=p}^{q-1} \alpha_k; \beta = \sum_{k=\ell+2-q}^{\ell+1-p} \alpha_k \\ 1 \leq p < q \leq \frac{1}{2}(\ell+1); j \in \mathbb{Z} \end{array} \right], \quad (4.2.30)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\beta} + e_{-k\delta-\beta}) \text{ eigenvalue } -1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\beta} - e_{-k\delta-\beta}) \text{ eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\beta} - e_{-k\delta-\beta}) \text{ eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\beta} + e_{-k\delta-\beta}) \text{ eigenvalue } -1 \end{array} \right\} \left[\begin{array}{l} \alpha = \sum_{m=p}^{q-1} \alpha_m; \beta = \sum_{m=\ell+2-q}^{\ell+1-p} \alpha_m; k = -(j+1) \\ 1 \leq p \leq \frac{1}{2}(\ell+1) < q \leq (\ell+1) \end{array} \right], \quad (4.2.31)$$

$$\begin{array}{l} ic \text{ eigenvalue } 1, \\ i\left(d + \frac{1}{2} \left[(\ell+1) \sum_{m=1}^{\frac{1}{2}(\ell-1)} m(h_{\alpha_m} + h_{\alpha_{\ell+1-m}}) \right] + \frac{1}{4}(\ell+1)^2 h_{\alpha_{\frac{1}{2}(\ell+1)}} \right) \text{ eigenvalue } 1. \end{array} \quad (4.2.32)$$

This suggests the following basis for a real form of the Kac-Moody algebra $A_\ell^{(1)}$:

$$\left. \begin{array}{l} (h_{\alpha_k} + h_{\alpha_{\ell+1-k}}) \\ i(h_{\alpha_k} - h_{\alpha_{\ell+1-k}}) \end{array} \right\} \text{ where } j \in \{1, 2, \dots, \frac{1}{2}(\ell-1)\}, \quad (4.2.33)$$

$$\left(h_{\alpha_{\frac{1}{2}(\ell+1)}} - \frac{1}{2}c \right), \quad (4.2.34)$$

$$\begin{aligned}
& i \left(e_{j\delta}^a + e_{-j\delta}^a + e_{j\delta}^{\ell+1-a} + e_{-j\delta}^{\ell+1-a} \right) \quad a \in \left\{ 1, 2, \dots, \frac{1}{2}(\ell+1) \right\}; j \in \mathbf{N}, \\
& \left(e_{j\delta}^a + e_{-j\delta}^a - e_{j\delta}^{\ell+1-a} - e_{-j\delta}^{\ell+1-a} \right) \quad a \in \left\{ 1, 2, \dots, \frac{1}{2}(\ell-1) \right\}; j \in \mathbf{N}, \\
& \left(e_{j\delta}^a - e_{-j\delta}^a + e_{j\delta}^{\ell+1-a} - e_{-j\delta}^{\ell+1-a} \right) \quad a \in \left\{ 1, 2, \dots, \frac{1}{2}(\ell+1) \right\}; j \in \mathbf{N}, \\
& i \left(e_{j\delta}^a - e_{-j\delta}^a - e_{j\delta}^{\ell+1-a} + e_{-j\delta}^{\ell+1-a} \right) \quad a \in \left\{ 1, 2, \dots, \frac{1}{2}(\ell-1) \right\}; j \in \mathbf{N},
\end{aligned} \tag{4.2.35}$$

$$\begin{aligned}
& \left. \begin{aligned}
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\beta} + e_{j\delta-\beta} \right) \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\beta} - e_{j\delta-\beta} \right) \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\beta} - e_{j\delta-\beta} \right) \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\beta} + e_{j\delta-\beta} \right)
\end{aligned} \right\} \left\{ \begin{aligned}
& \alpha = \sum_{k=p}^{q-1} \alpha_k; \beta = \sum_{k=\ell+2-q}^{\ell+1-p} \alpha_k \\
& 1 \leq p < q \leq \frac{1}{2}(\ell+1); j \in \mathbf{Z}
\end{aligned} \right\}, \tag{4.2.36}
\end{aligned}$$

$$\begin{aligned}
& \left. \begin{aligned}
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\beta} + e_{-k\delta-\beta} \right) \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\beta} - e_{-k\delta-\beta} \right) \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\beta} - e_{-k\delta-\beta} \right) \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\beta} + e_{-k\delta-\beta} \right)
\end{aligned} \right\} \left\{ \begin{aligned}
& \alpha = \sum_{m=p}^{q-1} \alpha_m; \beta = \sum_{m=\ell+2-q}^{\ell+1-p} \alpha_m; k = -(j+1) \\
& 1 \leq p \leq \frac{1}{2}(\ell+1) < q \leq (\ell+1)
\end{aligned} \right\}, \tag{4.2.37}
\end{aligned}$$

ic,

$$i \left(d + \frac{1}{2} \left\{ (\ell+1) \sum_{m=1}^{\frac{1}{2}(\ell-1)} m \left(h_{\alpha_m} + h_{\alpha_{\ell+1-m}} \right) \right\} + \frac{1}{4} (\ell+1)^2 h_{\alpha_{\frac{1}{2}(\ell+1)}} \right). \tag{4.2.38}$$

The most general element of this real form may be expressed in the form

$$\mathbf{a}(t) + (\lambda + i\mu)c + vid + v\mathbf{b}, \tag{4.2.39}$$

where $\mathbf{a}(t)$ is a matrix that satisfies the condition

$$\mathbf{a}(t) = -\mathbf{g}(t)\bar{\mathbf{a}}^*(t^{-1})\mathbf{g}(t)^{-1}, \tag{4.2.40}$$

where $\mathbf{g} = \text{offsum} \left\{ \mathbf{K}_{\frac{1}{2}(\ell+1)}, i\mathbf{K}_{\frac{1}{2}(\ell+1)} \right\}$. The matrix \mathbf{b} is the representative of the element added to id in (4.2.38). Clearly, the quantities μ and v are arbitrary real numbers, whereas the real number λ is such that

$$\lambda = -(\ell+1) \text{Res} \left\{ \text{tr} \left(t^{-1} \mathbf{a}(t) \text{dsum} \left\{ \mathbf{1}_{\frac{1}{2}(\ell+1)}, \mathbf{0}_{\frac{1}{2}(\ell+1)} \right\} \right) \right\}. \tag{4.2.41}$$

This expression for λ agrees with the expression for it derived earlier, since the derived expression for the quantity λ is

$$\begin{aligned} \lambda &= -\frac{1}{2\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \mathbf{a}(t) \right) \right\}, \\ &= -(\ell + 1) \text{Res} \left\{ \text{tr} \left(t^{-1} \text{dsum} \left\{ \mathbf{1}_{\frac{1}{2}(\ell+1)}, \mathbf{0}_{\frac{1}{2}(\ell+1)} \right\} \mathbf{a}(t) \right) \right\}. \end{aligned} \tag{4.2.42}$$

4.3 Type 1a involutive automorphisms of $A_\ell^{(1)}$ with $u = -1$

It will be shown now that, as in other cases, the type 1a involutive automorphisms with $u = -1$ are all mutually conjugate.

Automorphisms corresponding to the identity rotation

To see this briefly, recall (from [11]) that the type 1a involutive automorphisms (with $u = -1$) corresponding to the identity rotation are precisely those automorphisms generated by the matrices

$$\mathbf{U} = \text{diag}\{1, \eta_2, \eta_3, \dots, \eta_{\ell+1}\} \quad (\eta_2^2 = \eta_3^2 = \dots = \eta_{\ell+1}^2 = 1). \quad (4.3.1)$$

It requires little verification to see that each of these involutive automorphisms is also a type 1a (with $u = -1$) automorphism of $su_{(1)}(\ell + 1, 0)$, the compact real form. For each such matrix \mathbf{U} , let another associated matrix $\mathbf{S}(t)$ be diagonal, and defined so that

$$(\mathbf{S}(t))_{j,j} = \begin{cases} t & \text{if } \eta_j = -1 \\ 1 & \text{if } \eta_j = 1 \end{cases} \quad \text{for } 2 \leq j \leq \ell + 1. \quad (4.3.2)$$

Then $\mathbf{S}(t)$ satisfies the following equations:

$$\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{2\ell+1}, \quad \mathbf{S}(t)\mathbf{1}_{2\ell+1}\mathbf{S}(-t)^{-1} = \mathbf{U}, \quad (4.3.3)$$

which means that all of the type 1a involutions under consideration (namely those corresponding to the identity root transformation) do belong to just one conjugacy

class. This conjugacy class is called (C). Taking as representative of (C) the type 1a automorphism $\{1_{2\ell+1}, 1, 0\}$, gives the following eigenvector-basis of $su_{(1)}(\ell+1, 0)$:

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \quad j \in \mathbf{N}^0 \\ i(e_{j\delta}^a - e_{-j\delta}^a) \quad j \in \mathbf{N} \end{array} \right\} \text{eigenvalue } (-1)^j, \quad (4.3.4)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} j \in \mathbf{Z}; \alpha \in \Delta^0; \text{eigenvalue } (-1)^j, \quad (4.3.5)$$

$$\left. \begin{array}{l} ic \\ id \end{array} \right\} \text{eigenvalue } 1. \quad (4.3.6)$$

This implies, of course, that a basis for the real form associated with the conjugacy class (C) is provided by the following elements:

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \quad j \in \mathbf{N}^0 \\ i(e_{j\delta}^a - e_{-j\delta}^a) \quad j \in \mathbf{N} \end{array} \right\} \text{when } j \text{ is even,} \quad (4.3.7)$$

$$\left. \begin{array}{l} i(e_{j\delta}^a + e_{-j\delta}^a) \quad j \in \mathbf{N}^0 \\ (e_{j\delta}^a - e_{-j\delta}^a) \quad j \in \mathbf{N} \end{array} \right\} \text{when } j \text{ is odd,} \quad (4.3.8)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} j \text{ is even; } \alpha \text{ is a root of } A_\ell \quad (4.3.9)$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} j \text{ is odd; } \alpha \text{ is a root of } A_\ell, \quad (4.3.10)$$

$$\begin{array}{l} ic, \\ id. \end{array} \quad (4.3.11)$$

This real form is, as has been noted previously, the real form $su_{(3)}(\ell+1, 0)$, whose matrix parts are traceless, and satisfy

$$\mathbf{a}(t) = -\tilde{\mathbf{a}}^*(-t^{-1}). \quad (4.3.12)$$

Automorphisms corresponding to the rotation τ_2

All of these automorphisms belong to the conjugacy class (C), and thus the associated real form is $su_{(3)}(\ell+1, 0)$. The most general rotation of the roots of A_ℓ belonging to set (2) is

$$S_{\alpha_1}^0 \circ S_{\alpha_3}^0 \circ \dots \circ S_{\alpha_r}^0, \quad (4.3.13)$$

and the automorphisms corresponding to this root transformation are generated by the matrices $\mathbf{U}(t)$ of the form

$$\mathbf{U}(t) = \text{dsum}\{\mathbf{Z}_1(t), \mathbf{Z}_3(t), \dots, \mathbf{Z}_r(t), \mathbf{T}\}, \quad (4.3.14)$$

where the submatrices $\mathbf{Z}_m(t)$ and \mathbf{T} are defined as follows:

$$\mathbf{Z}_m(t) = \begin{bmatrix} 0 & \eta_m t^{k_m} \\ (-1)^{k_m} \eta_m^{-1} t^{-k_m} & 0 \end{bmatrix} \quad (\eta_m \eta_m^* = 1; m \text{ is odd}), \quad (4.3.15)$$

$$\mathbf{T} = \text{diag}\{\eta_{r+2}, \eta_{r+3}, \dots, \eta_\ell, 1\} \quad (\eta_{r+2}^2 = \eta_{r+3}^2 = \dots = \eta_\ell^2 = 1). \quad (4.3.16)$$

Then, for each matrix $\mathbf{U}(t)$, a matrix $\mathbf{S}(t)$ is defined thus

$$\mathbf{S}(t) = \text{dsum}\{\mathbf{X}_1(t), \mathbf{X}_3(t), \dots, \mathbf{X}_r(t), \mathbf{1}_{\ell-r}\} \quad \text{where } \mathbf{X}_m(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \eta_m^{-1} t^{-k_m} & \eta_m t^{k_m} \\ -\eta_m^{-1} t^{-k_m} & \eta_m t^{k_m} \end{bmatrix}. \quad (4.3.17)$$

This matrix $\mathbf{S}(t)$ is such that it satisfies simultaneously

$$\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{\ell+1}, \quad \mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(-t)^{-1} = \text{dsum}\{\mathbf{W}_{1,r+1}, \mathbf{T}\}. \quad (4.3.18)$$

Hence all of the type 1a involutive automorphisms with $u = -1$ that correspond to the root transformations in set (2) belong to the conjugacy class (C), as was indicated previously.

Automorphisms corresponding to the rotation τ_3

It remains only to show that these involutive automorphisms belong to the conjugacy class (C). In general, the automorphisms that are associated with this root transformation are those generated by the matrices of the general form

$$\left. \begin{aligned} \mathbf{U}(t) &= \text{offdiag}\{1, \eta_2 t^{k_2}, \dots, \eta_\ell t^{k_\ell}, \eta_{\ell+1} t^{k_{\ell+1}}\} \quad (\eta_m \eta_m^* = 1; 2 \leq m \leq \ell + 1), \\ \eta_{\ell+1} &= \eta_m \eta_{\ell+2-m} (-1)^{k_m} \\ k_{\ell+1} &= k_m + k_{\ell+2-m} = \text{some even integer} \end{aligned} \right\} 2 \leq m \leq \frac{1}{2}(\ell + 1), \quad (4.3.19)$$

For any matrix $\mathbf{U}(t)$ of the above form, a diagonal matrix $\mathbf{S}(t)$ may be defined by

$$(\mathbf{S}(t))_{j,j} = \begin{cases} \eta_{\ell+2-j}^{\frac{1}{2}} \eta_j^{-\frac{1}{2}} t^{\frac{1}{2}(k_{\ell+2-j} - k_j)} & (1 \leq j \leq \frac{1}{2}(\ell + 1)), \\ 1 & (\frac{1}{2}(\ell + 3) \leq j \leq \ell), \end{cases} \quad (4.3.20)$$

with the consequence that $\bar{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{\ell+1}$ and furthermore that

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(-t)^{-1} = \eta_{\ell+1}^{\frac{1}{2}} t^{\frac{1}{2}(k_{\ell+1})} \begin{bmatrix} \mathbf{0} & \mathbf{K}_{\frac{1}{2}(\ell+1)} \\ (-1)^{\frac{1}{2}(k_{\frac{1}{2}(\ell+1)} - k_{\frac{1}{2}(\ell+3)})} \mathbf{K}_{\frac{1}{2}(\ell+1)} & \mathbf{0} \end{bmatrix}. \quad (4.3.21)$$

Note that the quantity $\left(k_{\frac{1}{2}(\ell+1)} - k_{\frac{1}{2}(\ell+3)}\right)$ is actually even, and thus perfectly divisible by 2. There are thus only two cases that need to be considered further, namely

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(-t)^{-1} = \eta_{\frac{1}{2}(\ell+1)}^{\frac{1}{2}} t^{\frac{1}{2}k_{\frac{1}{2}(\ell+1)}} \begin{bmatrix} \mathbf{0} & \mathbf{K}_{\frac{1}{2}(\ell+1)} \\ \mathbf{K}_{\frac{1}{2}(\ell+1)} & \mathbf{0} \end{bmatrix}, \quad (4.3.22)$$

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(-t)^{-1} = \eta_{\frac{1}{2}(\ell+1)}^{\frac{1}{2}k_{\frac{1}{2}(\ell+1)}} t^{\frac{1}{2}k_{\frac{1}{2}(\ell+1)}} \begin{bmatrix} \mathbf{0} & \mathbf{K}_{\frac{1}{2}(\ell+1)} \\ -\mathbf{K}_{\frac{1}{2}(\ell+1)} & \mathbf{0} \end{bmatrix}. \quad (4.3.23)$$

Let the matrix \mathbf{Z} be given by $\mathbf{Z} = \text{dsum}\left\{i\mathbf{1}_{\frac{1}{2}(\ell+1)}, \mathbf{1}_{\frac{1}{2}(\ell+1)}\right\}$, so that $\tilde{\mathbf{Z}}^* \mathbf{Z} = \mathbf{1}_{\ell+1}$, and $\{\mathbf{Z}, s, \xi\}$ is an automorphism of $su_{(1)}(\ell+1, 0)$. Then it can be seen that

$$\mathbf{Z} \begin{bmatrix} \mathbf{0} & \mathbf{K}_{\frac{1}{2}(\ell+1)} \\ -\mathbf{K}_{\frac{1}{2}(\ell+1)} & \mathbf{0} \end{bmatrix} \mathbf{Z}^{-1} = i \begin{bmatrix} \mathbf{0} & \mathbf{K}_{\frac{1}{2}(\ell+1)} \\ \mathbf{K}_{\frac{1}{2}(\ell+1)} & \mathbf{0} \end{bmatrix}. \quad (4.3.24)$$

Hence, all of the involutive automorphisms of type 1a with $u = -1$ that correspond to root transformations in set 3 do belong to just one conjugacy class. That this class is the class (C) follows because, if

$$\mathbf{S} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{K}_{\frac{1}{2}(\ell+1)} & \mathbf{1}_{\frac{1}{2}(\ell+1)} \\ \mathbf{1}_{\frac{1}{2}(\ell+1)} & -\mathbf{K}_{\frac{1}{2}(\ell+1)} \end{bmatrix}, \quad (4.3.25)$$

then $\tilde{\mathbf{S}}^* \mathbf{S} = \mathbf{1}_{\ell+1}$ and the following matrix equation holds:

$$\mathbf{S} \mathbf{K}_{\ell+1} \mathbf{S}^{-1} = \text{dsum}\left\{\mathbf{1}_{\frac{1}{2}(\ell+1)}, -\mathbf{1}_{\frac{1}{2}(\ell+1)}\right\}, \quad (4.3.26)$$

which is sufficient to complete the analysis.

4.4 Type 1b involutive automorphisms $A_\ell^{(1)}$ with $u = 1$

It should be remembered that there are no type 1b automorphisms which correspond to the rotations τ_j for $j = 1, 2, 3$. This is also the case for type 2b automorphisms.

Automorphisms corresponding to the rotation τ_4

The analysis of $A_\ell^{(1)}$ contained in [11] shows that, when ℓ is even, all of the type 1b involutions corresponding to the root transformation τ_{Cartan}^0 (with $u = 1$) are mutually conjugate. When ℓ is odd, then each such involutive automorphism is conjugate either to the type 1b involutive automorphism generated by $\mathbf{1}_{\ell+1}$, or to that generated by $\text{dsum}\{\mathbf{1}_\ell, t\mathbf{1}_1\}$. For $A_\ell^{(1)}$, the proof involves generalisation of work on $A_2^{(1)}$, and similar generalisation of work on $su_{(1)}(3, 0)$ will be of use here.

The automorphisms that are being analysed in this subsection are those involutions of type 1b with $u = 1$ which are generated by the matrices

$$\mathbf{U}(t) = \text{diag}\{1, \eta_2 t^{k_2}, \dots, \eta_{\ell+1} t^{k_{\ell+1}}\} \quad (\eta_m \eta_m^* = 1; 2 \leq m \leq \ell + 1). \quad (4.4.1)$$

A diagonal matrix $\mathbf{S}(t)$ is defined (for each different matrix $\mathbf{U}(t)$ of the above form) by

$$\mathbf{S}(t) = \text{diag}\left\{1, \eta_2^{-\frac{1}{2}} t^{-\theta_2}, \eta_3^{-\frac{1}{2}} t^{-\theta_3}, \dots, \eta_{\ell+1}^{-\frac{1}{2}} t^{-\theta_{\ell+1}}\right\}, \quad (4.4.2)$$

where $\theta_2, \theta_3, \dots, \theta_{\ell+1}$ are arbitrary integers. Then $\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{\ell+1}$, and

$$\mathbf{S}(t) \text{diag}\{1, \eta_2 t^{k_2}, \dots, \eta_{\ell+1} t^{k_{\ell+1}}\} \tilde{\mathbf{S}}(t) = \text{diag}\{1, t^{k_2 - 2\theta_2}, \dots, t^{k_{\ell+1} - 2\theta_{\ell+1}}\}. \quad (4.4.3)$$

The quantities $\theta_2, \theta_3, \dots, \theta_{\ell+1}$ are, as has been remarked upon already, arbitrary integers. Thus, it may be assumed that $(k_m - 2\theta_m)$ takes either the value 1 or the value 0. Consider the case where $\mathbf{U}(t)$ has a diagonal entries taking the value 1 (with the remainder of the diagonal entries taking the value t). Since the order of the index set of $\mathbf{U}(t)$ is itself arbitrary, it follows that every automorphism $\{\mathbf{U}(t), 1, \xi\}$ (where $\mathbf{U}(t)$ is of the above form) is conjugate to some type 1b involutive automorphism generated by a matrix $\mathbf{U}'(t)$ of the form

$$\mathbf{U}'(t) = \text{dsum}\{\mathbf{1}_a, t\mathbf{1}_{\ell+1-2a}, \mathbf{1}_a\}. \quad (4.4.4)$$

Then with a matrix $\mathbf{S}(t)$ defined by the expression

$$\mathbf{S}(t) = \frac{1}{2} \begin{bmatrix} (t+1)\mathbf{1}_a & \mathbf{0} & -i(t-1)\mathbf{K}_a \\ \mathbf{0} & 2\mathbf{1}_{\ell+1-2a} & \mathbf{0} \\ i(t-1)\mathbf{K}_a & \mathbf{0} & (t+1)\mathbf{1}_a \end{bmatrix}, \quad (4.4.5)$$

the two matrix equations given below hold

$$\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{\ell+1}, \quad \mathbf{S}(t)\mathbf{U}(t)\tilde{\mathbf{S}}(t) = t\mathbf{1}_{\ell+1}. \quad (4.4.6)$$

Hence, when $\mathbf{U}(t)$ has an even number of diagonal elements taking the value 1, the resultant type 1b involutive automorphism is conjugate to the type 1b involutive automorphism generated by the matrix $\mathbf{1}_{\ell+1}$. This conjugacy class is called $(D)^{(0)}$. Consider now the case when $\mathbf{U}(t)$ has an odd number of diagonal entries taking the value 1 (with the remainder taking the value t). In the case where ℓ is even then $\mathbf{U}(t)$ may be suitably re-ordered, and the preceding analysis then implies that $\{\mathbf{U}(t), 1, \xi\}$ belongs to $(D)^{(0)}$. In the case where ℓ is odd, then re-ordering of the index set of the matrix $\mathbf{U}(t)$ and adaptation of the previous analysis lead to the conclusion that the automorphisms

$$\{\mathbf{U}(t), 1, \xi\} \text{ and } \{\text{dsum}\{\mathbf{1}_\ell, t\mathbf{1}_1\}, 1, \xi\} \quad (4.4.7)$$

are, in fact, mutually conjugate. Furthermore, the automorphism $\{\text{dsum}\{\mathbf{1}_\ell, t\mathbf{1}_1\}, 1, \xi\}$ does not belong to the conjugacy class $(D)^{(0)}$ when ℓ is odd. If it did, then there would exist some matrix $S(t)$ such that

$$S(t)\mathbf{1}_{\ell+1}\tilde{S}(t) = \lambda t^\mu \text{dsum}\{\mathbf{1}_\ell, t\mathbf{1}_1\}. \quad (4.4.8)$$

The determinants of both sides of this matrix equation may be taken. Remembering that $\det(S(t)) = \alpha t^\beta$, this provides the equality

$$\alpha^2 t^{2\beta} = (\lambda t^\mu)^{\ell+1} t, \quad (4.4.9)$$

which means that the left-hand side contains only an even index of t , whereas the right-hand side contains only an odd index of t . Thus, for odd values of ℓ , there exists a conjugacy class $(D)^{(1)}$, for which no corresponding conjugacy class exists when ℓ is even. The chosen representative automorphism for the conjugacy class $(D)^{(1)}$ is the type 1b involution $\{\text{dsum}\{\mathbf{1}_\ell, t\mathbf{1}_1\}, 1, \xi\}$.

An eigenvector-basis of $su_{(1)}(\ell+1, 0)$ (with respect to the representative automorphism of the class $(D)^{(0)}$, generated by the matrix $\mathbf{1}_{\ell+1}$) is given below, together with the associated eigenvalues.

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \quad j \in \mathbf{N}^0 \\ i(e_{j\delta}^a - e_{-j\delta}^a) \quad j \in \mathbf{N} \end{array} \right\} \text{with } a \in \{1, 2, \dots, \ell\}; \text{ eigenvalue } -1, \quad (4.4.10)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \quad \text{eigenvalue } 1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \quad \text{eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \quad \text{eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \quad \text{eigenvalue } 1 \end{array} \right\} j \in \mathbf{N}^0; \alpha \text{ is a positive root of } A_\ell, \quad (4.4.11)$$

$$\left. \begin{array}{l} ic \\ id \end{array} \right\} \text{eigenvalue } 1. \quad (4.4.12)$$

The corresponding real form of $su_{(1)}(\ell + 1, 0)$ that is generated by following Cartan's theorem has the following basis elements:

$$\left. \begin{array}{l} i(e_{j\delta}^a + e_{-j\delta}^a) \quad j \in \mathbf{N}^0 \\ (e_{j\delta}^a - e_{-j\delta}^a) \quad j \in \mathbf{N} \end{array} \right\} \text{where } a \in \{1, 2, \dots, \ell\}, \quad (4.4.13)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \end{array} \right\} \text{where } \alpha \text{ is a positive root of } A_\ell, \quad (4.4.14)$$

$$\begin{array}{l} ic, \\ id. \end{array} \quad (4.4.15)$$

It is easily verified that this real form is, in fact $sl_{(1)}(\ell + 1, \mathbf{R})$. For this real form, the matrix parts are traceless and satisfy the condition $\mathbf{a}(t) = -\tilde{\mathbf{a}}^*(t^{-1})$.

When ℓ is odd, there is a separate conjugacy class $(D)^{(1)}$ exists, and a suitable representative for it is the type 1b involutive automorphism $\{\text{dsum}\{\mathbf{1}_\ell, t\mathbf{1}_1\}, 1, -\ell\}$. An eigenvector-basis of $su_{(1)}(\ell + 1, 0)$ (with respect to this representative automorphism) is given below:

$$\left. \begin{array}{l} ih_{\alpha_k} \\ i(h_{\alpha_\ell} - \frac{1}{2}c) \end{array} \right\} 1 \leq k \leq \ell - 1; \text{ eigenvalue } -1, \quad (4.4.16)$$

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \\ i(e_{j\delta}^a - e_{-j\delta}^a) \end{array} \right\} 1 \leq a \leq \ell; j \in \mathbf{N}; \text{ eigenvalue } -1, \quad (4.4.17)$$

$$\left. \begin{array}{l}
(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \quad \text{eigenvalue } 1 \\
(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \quad \text{eigenvalue } -1 \\
i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \quad \text{eigenvalue } -1 \\
i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \quad \text{eigenvalue } 1
\end{array} \right\} \text{where } \alpha = \sum_{m=p}^{q-1} \alpha_m; 1 \leq p < q \leq \ell; j \in \mathbf{N}^0,$$

(4.4.18)

$$\left. \begin{array}{l}
(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \quad \text{eigenvalue } 1 \\
(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \quad \text{eigenvalue } -1 \\
i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \quad \text{eigenvalue } -1 \\
i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \quad \text{eigenvalue } 1
\end{array} \right\} \left\{ \begin{array}{l}
\text{where } \alpha = \sum_{m=p}^{q-1} \alpha_m; 1 \leq p \leq \ell; q = \ell + 1 \\
j \in \mathbf{N}^0; k = -(j+1)
\end{array} \right\},$$

(4.4.19)

$$i \left(d + \sum_{m=1}^{\ell} j h_{\alpha_m} \right) \left. \right\} \text{eigenvalue } 1.$$

(4.4.20)

This implies that the basis of a real form associated with this involutive automorphism is given by the following set of elements:

$$\left. \begin{array}{l}
h_{\alpha_k} \\
(h_{\alpha_l} - \frac{1}{2}c)
\end{array} \right\} 1 \leq k \leq \ell - 1,$$

(4.4.21)

$$\left. \begin{array}{l}
i(e_{j\delta}^a + e_{-j\delta}^a) \\
(e_{j\delta}^a - e_{-j\delta}^a)
\end{array} \right\} 1 \leq a \leq \ell; j \in \mathbf{N},$$

(4.4.22)

$$\left. \begin{array}{l}
(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\
i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\
(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\
i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha})
\end{array} \right\} \text{where } \left\{ \begin{array}{l}
\alpha = \sum_{m=p}^{q-1} \alpha_m; 1 \leq p < q \leq \ell \\
j \in \mathbf{N}^0
\end{array} \right\},$$

(4.4.23)

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \end{array} \right\} \text{where } \left\{ \begin{array}{l} \alpha = \sum_{m=p}^{q-1} \alpha_m; 1 \leq p \leq \ell; q = \ell + 1 \\ j \in \mathbf{N}^0; k = -(j+1) \end{array} \right\}, \quad (4.4.24)$$

$$i \left(d + \sum_{m=1}^{\ell} j h_{\alpha_m} \right). \quad (4.4.25)$$

Clearly, this real form is not encountered whenever ℓ is even, at least not as a real form that is non-isomorphic to other real forms that have previously been identified. There is not, for example, any such real form for $A_2^{(1)}$ that is not isomorphic to the real forms of $A_2^{(1)}$ that were found previously. The most general element of this real form can be represented in the form

$$\mathbf{a}(t) + (\lambda + i\mu)c + \nu i \left(d + \sum_{m=1}^{\ell} m h_{\alpha_m}^0 \right), \quad (4.4.26)$$

where the quantities μ and ν are arbitrary, and λ satisfies

$$\lambda = (\ell + 1) \text{Res} \left\{ \text{tr} \left(\mathbf{a}(t) \text{dsum} \left\{ \mathbf{0}_{\ell}, t^{-1} \mathbf{1}_1 \right\} \right) \right\}. \quad (4.4.27)$$

The matrix parts $\mathbf{a}(t)$ are all traceless, and also satisfy the condition

$$\mathbf{a}(t) = \mathbf{g} \tilde{\mathbf{a}}^* (t^{-1}) \mathbf{g}^{-1}, \quad (4.4.28)$$

where $\text{dsum} \left\{ \mathbf{1}_{\ell}, t \mathbf{1}_1 \right\}$.

Automorphisms corresponding to the rotation τ_5

The results for $su_{(1)}(\ell + 1, 0)$ are the same as those for $A_{\ell}^{(1)}$. Namely, that each type 1b involution (with $u = 1$) corresponding to a rotation in set (5) is conjugate to some other type 1b involution whose corresponding root transformation is in set (4). Recall that the most general root transformation of this type is

$$S_{\alpha_1}^0 \circ S_{\alpha_3}^0 \circ \dots \circ S_{\alpha_r}^0 \circ \tau_{Cartan}^0, \quad (4.4.29)$$

and the most general type 1b involutive automorphism of $su_{(1)}(\ell + 1, 0)$ (with $u = 1$) is generated by the matrix $\mathbf{U}(t)$, where

$$\mathbf{U}(t) = \text{dsum}\{\mathbf{Z}_1(t), \mathbf{Z}_3(t), \dots, \mathbf{Z}_r(t), \mathbf{T}(t)\}, \quad (4.4.30)$$

with the submatrices being defined by

$$\mathbf{Z}_m(t) = \eta_m t^{k_m} \mathbf{K}_2 \quad \text{odd } j; 1 \leq j \leq r; \eta_m \eta_m^* = 1, \quad (4.4.31)$$

$$\mathbf{T}(t) = \text{diag}\{\eta_{r+2} t^{k_{r+2}}, \dots, \eta_{\ell} t^{k_{\ell}}, 1\} \quad r + 2 \leq m \leq \ell; \eta_m \eta_m^* = 1. \quad (4.4.32)$$

Each such automorphism is conjugate to an involution of type 1b with $u = 1$, but corresponding to the Cartan involution. To see this, let the matrix \mathbf{S} be defined by

$$\mathbf{S} = \text{dsum}\{\mathbf{X}_1, \mathbf{X}_3, \dots, \mathbf{X}_r, \mathbf{1}_{\ell-r}\} \left(\mathbf{X}_m = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}; 1 \leq m \leq r; m \text{ odd} \right). \quad (4.4.33)$$

It is easily verified that \mathbf{S} satisfies $\tilde{\mathbf{S}}^* \mathbf{S} = \mathbf{1}_{\ell+1}$, and also

$$\mathbf{S}\mathbf{U}(t)\tilde{\mathbf{S}} = \mathbf{U}'(t), \quad (4.4.34)$$

where the type 1b involutive automorphism $\{\mathbf{U}'(t), 1, \xi\}$ corresponds to the Cartan involution, as was to be demonstrated.

Automorphisms corresponding to the rotation τ_6

The results of the analysis of $A_\ell^{(1)}$ suggest that some of these automorphisms will belong to a new conjugacy class. The general root transformation of A_ℓ in this set is given by

$$\rho^0(\alpha_k^0) = \alpha_{\ell+1-k}^0 \quad 1 \leq k \leq \ell. \quad (4.4.35)$$

The most general involutive automorphisms (of type 1b with $u = 1$) corresponding to this root transformation are those generated by the matrices $U(t)$ of the form

$$\begin{aligned} U(t) &= \text{offdiag}\{1, \eta_2 t^{k_2}, \dots, \eta_{\ell+1} t^{k_{\ell+1}}\}, \\ \eta_2 \eta_2^* &= \eta_3 \eta_3^* = \dots = \eta_\ell \eta_\ell^* = 1 = \eta_{\ell+1}^2, \\ \eta_{\ell+2-m} &= \eta_m \eta_{\ell+1}, \\ k_{\ell+2-m} &= k_m. \end{aligned} \quad (4.4.36)$$

Then, for each matrix $U(t)$ of this form, we define a diagonal matrix $S(t)$ by

$$(S(t))_{m,m} = \begin{cases} 1 & (m = 1, \ell + 1), \\ \eta_m^{-\frac{1}{2}} t^{-k_m} & (2 \leq m \leq \frac{1}{2}(\ell + 1)), \\ \eta_{\ell+2-m}^{-\frac{1}{2}} & (\frac{1}{2}(\ell + 3) \leq m \leq \ell). \end{cases} \quad (4.4.37)$$

As defined, $S(t)$ satisfies $\tilde{S}^*(t^{-1})S(t) = \mathbf{1}_{\ell+1}$, and also the conjugacy relation

$$S(t)U(t)\tilde{S}(t) = \text{offsum}\left\{\mathbf{K}_{\frac{1}{2}(\ell+1)}, \eta_{\ell+1} \mathbf{K}_{\frac{1}{2}(\ell+1)}\right\}. \quad (4.4.38)$$

There are, thus, only two special case to consider separately, namely $\eta_{\ell+1} = \pm 1$.

Recall from [11] that there exists a matrix $V(t)$, where

$$V(t) = \begin{bmatrix} \mathbf{1}_{\frac{1}{2}(\ell+1)} & \mathbf{K}_{\frac{1}{2}(\ell+1)} \\ \mathbf{K}_{\frac{1}{2}(\ell+1)} & -\mathbf{1}_{\frac{1}{2}(\ell+1)} \end{bmatrix} \quad (\tilde{V}^*(t^{-1})V(t) = 2\mathbf{1}_{\ell+1}), \quad (4.4.39)$$

and $\mathbf{V}(t)$ is such that

$$\mathbf{V}(t)\mathbf{K}_{\ell+1}\tilde{\mathbf{V}}(t) = 2\text{dsum}\left\{\mathbf{1}_{\frac{1}{2}(\ell+1)}, -\mathbf{1}_{\frac{1}{2}(\ell+1)}\right\}. \quad (4.4.40)$$

The type 1b involutive automorphism generated by the matrix $\text{dsum}\left\{\mathbf{1}_{\frac{1}{2}(\ell+1)}, -\mathbf{1}_{\frac{1}{2}(\ell+1)}\right\}$ has already been investigated, and seen to belong to the conjugacy class (D)⁽⁰⁾. It also follows from the analysis of $\mathbf{A}_\rho^{(1)}$ that there does not exist any suitable matrix $\mathbf{S}(t)$ that satisfies

$$\mathbf{S}(t)\mathbf{1}_{\ell+1}\tilde{\mathbf{S}}(t) = \eta t^k \begin{bmatrix} \mathbf{0} & \mathbf{K}_{\frac{1}{2}(\ell+1)} \\ -\mathbf{K}_{\frac{1}{2}(\ell+1)} & \mathbf{0} \end{bmatrix}, \quad (4.4.41)$$

since the left-hand side is symmetric, and the right-hand side is anti-symmetric. Thus, for $su_{(1)}(\ell+1,0)$, there exists a conjugacy class (E), separate from all of the other conjugacy classes that have already been discussed. The representative automorphism of the class (E) is the type 1b involutive automorphism $\left\{\text{offsum}\left\{\mathbf{K}_{\frac{1}{2}(\ell+1)}, -\mathbf{K}_{\frac{1}{2}(\ell+1)}\right\}, 1, 0\right\}$. The elements of an eigenvector-basis of $su_{(1)}(\ell+1,0)$ associated with this representative automorphism are given below, together with their respective eigenvalues. Note that the quantities α and β are defined in all of the cases below, by the equations

$$\alpha = \sum_{m=p}^{q-1} \alpha_m, \quad \beta = \sum_{\ell+2-q}^{\ell+1-p} \alpha_m. \quad (4.4.42)$$

The elements of the eigenvector-basis are

$$\begin{aligned} \left(e_{j\delta}^a + e_{-j\delta}^a + e_{j\delta}^{\ell+1-a} + e_{-j\delta}^{\ell+1-a}\right) & \text{ eigenvalue } 1, \\ \left(e_{j\delta}^a + e_{-j\delta}^a - e_{j\delta}^{\ell+1-a} - e_{-j\delta}^{\ell+1-a}\right) & \text{ eigenvalue } -1, \end{aligned} \quad (4.4.43)$$

where $a \in \left\{1, 2, \dots, \frac{1}{2}(\ell+1)\right\}; j \in \mathbf{N}^0$,

$$\begin{aligned}
i\left(e_{j\delta}^a - e_{-j\delta}^a + e_{j\delta}^{\ell+1-a} - e_{-j\delta}^{\ell+1-a}\right) & \text{ eigenvalue } 1, \\
i\left(e_{j\delta}^a - e_{-j\delta}^a - e_{j\delta}^{\ell+1-a} + e_{-j\delta}^{\ell+1-a}\right) & \text{ eigenvalue } -1,
\end{aligned} \tag{4.4.44}$$

where $a \in \{1, 2, \dots, \frac{1}{2}(\ell+1)\}; j \in \mathbb{N}$,

$$\begin{aligned}
\left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta+\beta} + e_{-j\delta-\beta}\right) & \text{ eigenvalue } -1, \\
\left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta+\beta} - e_{-j\delta-\beta}\right) & \text{ eigenvalue } 1, \\
i\left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta+\beta} - e_{-j\delta-\beta}\right) & \text{ eigenvalue } -1, \\
i\left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta+\beta} + e_{-j\delta-\beta}\right) & \text{ eigenvalue } 1,
\end{aligned} \tag{4.4.45}$$

where either $1 \leq p < q \leq \frac{1}{2}(\ell+1)$ or $\frac{1}{2}(\ell+1) < p < q \leq (\ell+1)$,

$$\begin{aligned}
\left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta+\beta} + e_{-j\delta-\beta}\right) & \text{ eigenvalue } 1, \\
\left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta+\beta} - e_{-j\delta-\beta}\right) & \text{ eigenvalue } -1, \\
i\left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta+\beta} - e_{-j\delta-\beta}\right) & \text{ eigenvalue } 1, \\
i\left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta+\beta} + e_{-j\delta-\beta}\right) & \text{ eigenvalue } -1,
\end{aligned} \tag{4.4.46}$$

where $1 \leq p \leq \frac{1}{2}(\ell+1) < q \leq (\ell+1)$,

$$\left. \begin{array}{l} ic \\ id \end{array} \right\} \text{ eigenvalue } 1. \tag{4.4.47}$$

So, the associated basis for a real form of the Kac-Moody algebra $A_\ell^{(1)}$ is provided by the elements

$$\left. \begin{array}{l} \left(e_{j\delta}^a + e_{-j\delta}^a + e_{j\delta}^{\ell+1-a} + e_{-j\delta}^{\ell+1-a}\right) \\ i\left(e_{j\delta}^a + e_{-j\delta}^a - e_{j\delta}^{\ell+1-a} - e_{-j\delta}^{\ell+1-a}\right) \end{array} \right\} a \in \{1, 2, \dots, \frac{1}{2}(\ell+1)\}; j \in \mathbb{N}^0. \tag{4.4.48}$$

$$\left. \begin{array}{l} i\left(e_{j\delta}^a - e_{-j\delta}^a + e_{j\delta}^{\ell+1-a} - e_{-j\delta}^{\ell+1-a}\right) \\ \left(e_{j\delta}^a - e_{-j\delta}^a - e_{j\delta}^{\ell+1-a} + e_{-j\delta}^{\ell+1-a}\right) \end{array} \right\} a \in \{1, 2, \dots, \frac{1}{2}(\ell+1)\}; j \in \mathbb{N}, \tag{4.4.49}$$

$$\begin{aligned}
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta+\beta} + e_{-j\delta-\beta}), \\
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta+\beta} - e_{-j\delta-\beta}), \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta+\beta} - e_{-j\delta-\beta}), \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta+\beta} + e_{-j\delta-\beta}),
\end{aligned} \tag{4.4.50}$$

where either $1 \leq p < q \leq \frac{1}{2}(\ell + 1)$ or $\frac{1}{2}(\ell + 1) < p < q \leq (\ell + 1)$,

$$\left. \begin{aligned}
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta+\beta} + e_{-j\delta-\beta}) \\
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta+\beta} - e_{-j\delta-\beta}) \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta+\beta} - e_{-j\delta-\beta}) \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta+\beta} + e_{-j\delta-\beta})
\end{aligned} \right\} 1 \leq p \leq \frac{1}{2}(\ell + 1) < q \leq (\ell + 1), \tag{4.4.51}$$

where $1 \leq p \leq \frac{1}{2}(\ell + 1) < q \leq (\ell + 1)$,

$$\begin{aligned}
& ic, \\
& id.
\end{aligned} \tag{4.4.52}$$

The matrix parts of the elements of this algebra are such that they are traceless and satisfy the equation

$$\mathbf{a}(t) = \mathbf{g}\mathbf{a}^*(t^{-1})\mathbf{g}^{-1} \quad \text{where } \mathbf{g} = \text{offsum} \left\{ \mathbf{K}_{\frac{1}{2}(\ell+1)}, -\mathbf{K}_{\frac{1}{2}(\ell+1)} \right\}. \tag{4.4.53}$$

4.5 Type 1b involutive automorphisms of $A_\ell^{(1)}$ with $u = -1$

Automorphisms corresponding to the rotation τ_4

The type 1b ($u = -1$) involutive automorphisms that correspond to the rotation τ_4 are generated by matrices $U(t)$ of the following form:

$$U(t) = \text{diag}\{1, \eta_2 t^{k_2}, \dots, \eta_{\ell+1} t^{k_{\ell+1}}\} \quad k_2, k_{\ell+1} \text{ are even}; \eta_m \eta_m^* = 1 \text{ for } 2 \leq m \leq \ell + 1. \quad (4.5.1)$$

Moreover, all of these are mutually conjugate. For each matrix $U(t)$ of the above form, let a diagonal matrix $S(t)$ be defined by

$$(S(t))_{m,m} = \begin{cases} 1 & m = 1 \\ \eta_m^{-\frac{1}{2}} t^{-\frac{1}{2}k_m} & 2 \leq m \leq \ell + 1; k_2 \equiv 0 \pmod{4} \\ i\eta_m^{-\frac{1}{2}} t^{-\frac{1}{2}k_m} & 2 \leq m \leq \ell + 1; k_2 \equiv 2 \pmod{4} \end{cases}, \quad (4.5.2)$$

The matrix $S(t)$ satisfies $\tilde{S}^*(t^{-1})S(t) = \mathbf{1}_{\ell+1}$, so it generates an automorphism of $su_{(1)}(\ell + 1, 0)$, and is such that

$$S(t)U(t)\tilde{S}(-t) = \mathbf{1}_{\ell+1}. \quad (4.5.3)$$

Thus, all of the type 1b involutive automorphisms with $u = -1$ that correspond to the root transformation τ_{Cartan}^0 are mutually conjugate. The conjugacy class to which they all belong is called (F), and its representative is taken to be the type 1b involutive automorphism (with $u = -1$) that is generated by the matrix $\mathbf{1}_{\ell+1}$. A basis of $su_{(1)}(\ell + 1, 0)$, all of whose members are eigenvectors of this representative automorphism, is given below, together with the associated eigenvalues:

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \quad j \in \mathbf{N}_{\text{even}}^0 \\ i(e_{j\delta}^a - e_{-j\delta}^a) \quad j \in \mathbf{N}_{\text{even}} \end{array} \right\} a \in \{1, 2, \dots, \ell\}; \text{ eigenvalue } -1, \quad (4.5.4)$$

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \quad j \in \mathbf{N}_{\text{odd}} \\ i(e_{j\delta}^a - e_{-j\delta}^a) \quad j \in \mathbf{N}_{\text{odd}} \end{array} \right\} a \in \{1, 2, \dots, \ell\}; \text{ eigenvalue } 1, \quad (4.5.5)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \quad \text{eigenvalue } 1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \quad \text{eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \quad \text{eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \quad \text{eigenvalue } 1 \end{array} \right\} j \in \mathbf{N}_{\text{even}}; \alpha \in \Delta_+^0, \quad (4.5.6)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \quad \text{eigenvalue } -1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \quad \text{eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \quad \text{eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \quad \text{eigenvalue } -1 \end{array} \right\} j \in \mathbf{N}_{\text{odd}}; \alpha \in \Delta_+^0, \quad (4.5.7)$$

$$\left. \begin{array}{l} ic \\ id \end{array} \right\} \text{ eigenvalue } 1. \quad (4.5.8)$$

The terms $\mathbf{N}_{\text{even}}^0$, \mathbf{N}_{even} , \mathbf{N}_{odd} refer to the non-negative even integers, the positive even integers and the positive odd integers, respectively. Cartan's theorem implies that a basis of a real form of $A_\ell^{(1)}$ corresponding to the conjugacy class (E) is provided by the elements

$$\left. \begin{array}{l} i(e_{j\delta}^a + e_{-j\delta}^a) \quad j \in \mathbf{N}_{\text{even}}^0 \\ (e_{j\delta}^a - e_{-j\delta}^a) \quad j \in \mathbf{N}_{\text{even}} \end{array} \right\} a \in \{1, 2, \dots, \ell\}, \quad (4.5.9)$$

$$\left. \begin{array}{l} (e_{j\delta}^a + e_{-j\delta}^a) \quad j \in \mathbf{N}_{\text{odd}} \\ i(e_{j\delta}^a - e_{-j\delta}^a) \quad j \in \mathbf{N}_{\text{odd}} \end{array} \right\} a \in \{1, 2, \dots, \ell\}, \quad (4.5.10)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \end{array} \right\} j \in \mathbf{N}_{\text{even}}; \alpha \in \Delta_+^0, \quad (4.5.11)$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \end{array} \right\} j \in \mathbf{N}_{\text{odd}}; \alpha \in \Delta_+^0, \quad (4.5.12)$$

$$ic, \quad (4.5.13)$$

id.

This real form is obtainable from the real form $sl(\ell+1, \mathbf{R})$ of A_ℓ . In fact, it is the real form $sl_{(3)}(\ell+1, \mathbf{R})$ of $A_\ell^{(1)}$, in which the matrix parts are traceless and satisfy

$$\mathbf{a}(t) = \mathbf{a}^* \left(-t^{-1} \right). \quad (4.5.14)$$

Automorphisms corresponding to the rotation τ_5

All of these automorphisms are members of the conjugacy class (F), which has just been investigated. The most general automorphism corresponding to a root transformation in this set is generated by a matrix $\mathbf{U}(t)$ of the form

$$\mathbf{U}(t) = \text{dsum} \{ \mathbf{Z}_1(t), \mathbf{Z}_3(t), \dots, \mathbf{Z}_r(t), \mathbf{T}(t) \}, \quad (4.5.15)$$

where the submatrices are such that

$$\mathbf{Z}_m(t) = \begin{bmatrix} 0 & \eta_m t^{k_m} \\ (-1)^{k_m} \eta_m t^{k_m} & 0 \end{bmatrix} \quad \eta_m \eta_m^* = 1; m \in \{1, 3, \dots, r\}, \quad (4.5.16)$$

$$\mathbf{T}(t) = \text{diag}\{\eta_{r+2} t^{k_{r+2}}, \eta_{r+3} t^{k_{r+3}}, \dots, \eta_\ell t^{k_\ell}, 1\} \quad \eta_m \eta_m^* = 1; r+2 \leq m \leq \ell. \quad (4.5.17)$$

For each such matrix $\mathbf{U}(t)$, let another matrix $\mathbf{S}(t)$ be defined by

$$\mathbf{S}(t) = \text{dsum}\{\mathbf{W}_1(t), \mathbf{W}_3(t), \dots, \mathbf{W}_r(t), \mathbf{1}_{\ell-r}\}, \quad (4.5.18)$$

where

$$\mathbf{W}_m(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & t^{\text{deg } k_m} \\ t^{\text{deg } k_m} & -t^{2 \text{deg } k_m} \end{bmatrix}, \quad j \in \{1, 3, \dots, r\}, \quad (4.5.19)$$

where $\text{deg } k_m$ has the value of the parity of k_m ; namely 0 if k_m is even, and 1 if k_m is odd. Then

$$\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{\ell+1}, \quad \mathbf{S}(t)\mathbf{U}(t)\tilde{\mathbf{S}}(-t) = \mathbf{U}'(t), \quad (4.5.20)$$

where the matrix $\mathbf{U}'(t)$ is such that $\{\mathbf{U}'(t), -1, \xi\}$ does, in fact, belong to the conjugacy class (F).

Automorphisms corresponding to the rotation τ_ℓ

It can be verified that, for $su_{(1)}(\ell+1, 0)$, all of these automorphisms are mutually conjugate, and lie in the conjugacy class (F) (the corresponding automorphisms for $A_\ell^{(1)}$ are all conjugate, and are mutually conjugate to all of the other type 1b involutions with $u = -1$). In this case, it is clear that the quantity ℓ is an odd integer, since the root transformation of A_ℓ in question is ρ^0 , where

$$\rho^0(\alpha_k^0) = \alpha_{\ell+1-k}^0 \quad 1 \leq k \leq \ell. \quad (4.5.21)$$

The type 1b involutive automorphisms corresponding to this root transformation are those that are generated by matrices $\mathbf{U}(t)$ of the following form

$$\mathbf{U}(t) = \text{offdiag} \left\{ 1, \eta_2 t^{k_2}, \dots, \eta_{\ell+1} t^{k_{\ell+1}} \right\}, \quad \left. \begin{array}{l} \eta_{\ell+1}^2 = 1; k_{\ell+1} = 0 \\ k_m = k_{\ell+2-m}; \eta_m = \eta_{\ell+2-m}; 2 \leq m \leq \frac{1}{2}(\ell+1) \end{array} \right\}. \quad (4.5.22)$$

For each matrix $\mathbf{U}(t)$, the matrix $\mathbf{S}(t)$ is the diagonal matrix defined by

$$(\mathbf{S}(t))_{m,m} = \begin{cases} 1 & m = 1, \ell+1 \\ \eta_m^{-1} t^{-k_m} & 2 \leq m \leq \frac{1}{2}(\ell+1) \\ 1 & \frac{1}{2}(\ell+3) \leq m \leq \ell \end{cases}, \quad (4.5.23)$$

which is such that $\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{\ell+1}$, and also

$$\mathbf{S}(t)\mathbf{U}(t)\tilde{\mathbf{S}}(-t) = \text{offsum} \left\{ \mathbf{K}_{\frac{1}{2}(\ell+1)}, \eta_{\ell+1} \mathbf{K}_{\frac{1}{2}(\ell+1)} \right\}. \quad (4.5.24)$$

This is slightly different from the analysis of $A_\ell^{(1)}$. There are two distinct cases; namely $\eta_{\ell+1} = 1$ and $\eta_{\ell+1} = -1$. However, it is easily shown that these are all mutually conjugate. For, with $\mathbf{S}(t) = \text{dsum} \left\{ \mathbf{1}_{\frac{1}{2}(\ell+1)}, -t \mathbf{1}_{\frac{1}{2}(\ell+1)} \right\}$, it is the case that $\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{\ell+1}$, and

$$\mathbf{S}(t) \text{offsum} \left\{ \mathbf{K}_{\frac{1}{2}(\ell+1)}, -\mathbf{K}_{\frac{1}{2}(\ell+1)} \right\} \tilde{\mathbf{S}}(-t) = t \mathbf{K}_{\ell+1}, \quad (4.5.25)$$

so that only the case $\eta_{\ell+1} = 1$ need be studied. Let the matrix \mathbf{V} be given by

$$\mathbf{V} = \begin{bmatrix} \mathbf{1}_{\frac{1}{2}(\ell+1)} & \mathbf{K}_{\frac{1}{2}(\ell+1)} \\ \mathbf{K}_{\frac{1}{2}(\ell+1)} & -\mathbf{1}_{\frac{1}{2}(\ell+1)} \end{bmatrix}, \quad (4.5.26)$$

so that the following matrix equations hold:

$$\tilde{\mathbf{V}}^* \mathbf{V} = 2 \mathbf{1}_{\ell+1}, \quad \mathbf{V} \mathbf{K}_{\ell+1} \tilde{\mathbf{V}} = 2 \text{dsum} \left\{ \mathbf{1}_{\frac{1}{2}(\ell+1)}, -\mathbf{1}_{\frac{1}{2}(\ell+1)} \right\}. \quad (4.5.27)$$

It follows, therefore, that all of the type 1b involutions corresponding to the root transformation τ_6 (with $u = -1$) are mutually conjugate, and belong, moreover, to the conjugacy class (F). The corresponding real form has therefore been discussed previously.

4.6 Type 2a involutive automorphisms of $A_\ell^{(1)}$ with $u = 1$

Automorphisms corresponding to the rotation τ_1

The results here do not differ from those obtained for the complex algebra $A_\ell^{(1)}$. The automorphisms in question are those that are generated by the matrices $U(t)$, where

$$U(t) = \text{diag}\{1, \eta_2 t^{k_2}, \dots, \eta_{\ell+1} t^{k_{\ell+1}}\} \quad \eta_m^2 = 1; 2 \leq m \leq \ell + 1. \quad (4.6.1)$$

Let the diagonal matrix $S(t)$ be defined so that

$$(S(t))_{m,m} = \begin{cases} 1 & m = 1 \\ t^{\theta_m} & 2 \leq m \leq \ell + 1 \end{cases}, \quad (4.6.2)$$

where the integers θ_m , for $2 \leq m \leq \ell + 1$ are arbitrary. The matrix $S(t)$ then satisfies the condition $\tilde{S}^*(t^{-1})S(t) = \mathbf{1}_{\ell+1}$, and is such that

$$S(t)U(t)S(t^{-1})^{-1} = \text{diag}\{1, \eta_2 t^{k_2+2\theta_2}, \dots, \eta_{\ell+1} t^{k_{\ell+1}+2\theta_{\ell+1}}\}. \quad (4.6.3)$$

Since each of the quantities θ_m is arbitrary, it may be supposed that $\{\theta_m\}_{m=2}^{\ell+1}$ have all been chosen so that

$$S(t)U(t)S(t^{-1})^{-1} = \text{diag}\{1, \eta_2 t^{\text{deg}k_2}, \dots, \eta_{\ell+1} t^{\text{deg}k_{\ell+1}}\}. \quad (4.6.4)$$

Furthermore, since the ordering of the index sets is arbitrary, it follows that each of these automorphisms is conjugate to an automorphism generated by a matrix $U(t)$ of the form

$$U(t) = \text{dsum}\{\mathbf{1}_a, -\mathbf{1}_b, t\mathbf{1}_c, -t\mathbf{1}_d\}, \quad (4.6.5)$$

where $a + b + c + d = \ell + 1$. It may be supposed that $c \geq d$. Note that if $\mathbf{U}(t)$ were such that $c < d$, then $\mathbf{U}(-t)$ would be such that $c \geq d$. The automorphisms $\{\mathbf{U}(t), 1, \xi\}$ and $\{\mathbf{U}(-t), 1, \xi\}$ are conjugate, because $\mathbf{U}(-t) = \mathbf{U}(st)|_{s=-1}$. Thus, the assumption may be made. Consider then, two matrices of this form, $\mathbf{U}_1(t)$ and $\mathbf{U}_2(t)$, where

$$\mathbf{U}_1(t) = \text{diag}\{1, \eta_2 t^{k_2}, \dots, \eta_r t^{k_r}, 1, -1, \eta_{r+3} t^{k_{r+3}}, \dots, \eta_{\ell+1} t^{k_{\ell+1}}\}, \quad (4.6.6)$$

$$\mathbf{U}_2(t) = \text{diag}\{1, \eta_2 t^{k_2}, \dots, \eta_r t^{k_r}, t, -t, \eta_{r+3} t^{k_{r+3}}, \dots, \eta_{\ell+1} t^{k_{\ell+1}}\}. \quad (4.6.7)$$

The type 2a automorphisms $\{\mathbf{U}_1(t), 1, \xi\}$ and $\{\mathbf{U}_2(t), 1, \xi\}$ are conjugate. To show this, let a matrix $\mathbf{S}(t)$ be given by

$$\mathbf{S}(t) = \text{dsum}\left\{\mathbf{1}_r, \frac{1}{2}\begin{bmatrix} t+1 & t-1 \\ t-1 & t+1 \end{bmatrix}, \mathbf{1}_{\ell-r-1}\right\}. \quad (4.6.8)$$

Thus, $\bar{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{\ell+1}$ and $\mathbf{S}(t)\mathbf{U}_1(t)\mathbf{S}(t^{-1})^{-1} = \mathbf{U}_2(t)$. Thus, the values of a, b have both been reduced by 1, whilst those of c, d have both been increased by 1. The above analysis implies that the involutive automorphism $\{\mathbf{U}(t), 1, \xi\}$ is conjugate to the automorphism $\{\mathbf{U}'(t), 1, \xi\}$, where

$$\mathbf{U}'(t) = \text{dsum}\{\mathbf{1}_a, -\mathbf{1}_b, t\mathbf{1}_{c-d}, \mathbf{1}_d, -\mathbf{1}_d\}. \quad (4.6.9)$$

Clearly, with suitable re-ordering of the index sets, this means that $\{\mathbf{U}(t), 1, \xi\}$ and $\{\mathbf{U}''(t), 1, \xi\}$ are conjugate type 2a involutions, where

$$\mathbf{U}''(t) = \text{dsum}\{\mathbf{1}_{a'}, -\mathbf{1}_{b'}, t\mathbf{1}_{c'}\} \quad a' + b' + c' = \ell + 1. \quad (4.6.10)$$

It may be further assumed that $c' \leq \ell - \left[\frac{\ell}{2}\right]$. In any instances where this is not so, a simple re-ordering of the index set, together with the other analysis of this subsection will recast the matrix in question into the required form. Thus, the involutive automorphisms $\{\mathbf{U}''(t), 1, \xi\}$ fall into a number of different conjugacy classes, with disjoint classes corresponding, essentially, to different values of the quantities a', b', c' . The parameter c' can take any of the values $0, 1, \dots, \left(\ell - \left[\frac{1}{2}\ell\right]\right)$, thus leaving

$\ell + 1 - c'$ other diagonal elements. The parameter b' may then take values such that $0 \leq b' \leq \left[\frac{1}{2}(\ell + 1 - c')\right]$. Define $(G)^{(m,n)}$ to be the conjugacy class that contains the type 2a involutive automorphism $\{U(t), 1, \xi\}$, where the matrix $U(t)$ is given by

$$U(t) = \text{dsum}\{\mathbf{1}_{\ell+1-m-n}, -\mathbf{1}_n, \mathbf{1}_m\}. \quad (4.6.11)$$

It follows from [11] that, with the choice of b', c' restricted to the above values, the conjugacy classes $(G)^{(c', b')}$ are non-conjugate for different values of (b', c') . Consider the automorphism $\{U(t), 1, \xi\}$, where

$$U(t) = \text{dsum}\{\mathbf{1}_a, -\mathbf{1}_{b-a}, t\mathbf{1}_{\ell+1-b}\}. \quad (4.6.12)$$

This automorphism may be taken as representative of $(G)^{(c', b')}$, with $\ell + 1 - b = c'$ and $b - a = b'$. A basis of $su_{(1)}(\ell + 1, 0)$ consisting of eigenvectors of this involutive automorphism is given below. The general positive root α of A_ℓ is written in the form

$$\alpha = \sum_{j=p}^q \alpha_j^0, \quad (4.6.13)$$

where $1 \leq p \leq q \leq \ell$. The elements of the basis (together with their eigenvalues) are given below.

$$\left. \begin{array}{l} ih_{\alpha_k} \\ i(h_{\alpha_b} - \frac{1}{2}c) \end{array} \right\} \begin{array}{l} 1 \leq k \leq \ell; k \neq b \\ \end{array} \left. \vphantom{\begin{array}{l} ih_{\alpha_k} \\ i(h_{\alpha_b} - \frac{1}{2}c) \end{array}} \right\} \text{eigenvalue } 1, \quad (4.6.14)$$

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \\ i(e_{j\delta}^k - e_{-j\delta}^k) \end{array} \right\} \begin{array}{l} \text{eigenvalue } 1 \\ \text{eigenvalue } -1 \end{array} \left. \vphantom{\begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \\ i(e_{j\delta}^k - e_{-j\delta}^k) \end{array}} \right\} 1 \leq k \leq \ell, \quad (4.6.15)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \quad \text{eigenvalue } 1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \quad \text{eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \quad \text{eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \quad \text{eigenvalue } -1 \end{array} \right\} j \in \mathbb{N}^0, \quad (4.6.16)$$

where either $1 \leq p < q \leq a$, $a < p < q \leq b$ or $b < p < q \leq \ell + 1$,

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \quad \text{eigenvalue } -1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \quad \text{eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \quad \text{eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \quad \text{eigenvalue } 1 \end{array} \right\} j \in \mathbb{N}^0, \quad (4.6.17)$$

where $1 \leq p \leq a < q \leq b$,

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \quad \text{eigenvalue } 1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \quad \text{eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \quad \text{eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \quad \text{eigenvalue } -1 \end{array} \right\} j \in \mathbb{N}^0, \quad (4.6.18)$$

where $k = -j - 1; 1 \leq p \leq a; b < q \leq \ell + 1$,

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \quad \text{eigenvalue } -1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \quad \text{eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \quad \text{eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \quad \text{eigenvalue } 1 \end{array} \right\} j \in \mathbb{N}^0, \quad (4.6.19)$$

where $k = -j - 1; a < p \leq b < q \leq \ell + 1$

$$\left. \begin{array}{l} ic \\ i \left(d + (\ell + 1 - b) \sum_{m=1}^b mh_{\alpha_{\ell+1-m}} + b \sum_{m=1}^b mh_{\alpha_{\ell+1-m}} \right) \end{array} \right\} \text{eigenvalue } -1. \quad (4.6.20)$$

The corresponding real form is, therefore, the one for which a basis is given by

$$ih_{\alpha_k} \quad 1 \leq k \leq \ell; k \neq b, \quad (4.6.21)$$

$$i\left(h_{\alpha_b} - \frac{1}{2}c\right),$$

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \\ (e_{j\delta}^k - e_{-j\delta}^k) \end{array} \right\} 1 \leq k \leq \ell, \quad (4.6.22)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \end{array} \right\} j \in \mathbb{N}^0 \left\{ \begin{array}{l} 1 \leq p < q \leq a \\ a < p < q \leq b \\ b < p < q \leq \ell + 1 \end{array} \right\}, \quad (4.6.23)$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \end{array} \right\} j \in \mathbb{N}^0; 1 \leq p \leq a < q \leq b, \quad (4.6.24)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \end{array} \right\} j \in \mathbb{N}^0; k = -j - 1; 1 \leq p \leq a; b < q \leq \ell + 1, \quad (4.6.25)$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \end{array} \right\} j \in \mathbb{N}^0; k = -j - 1; a < p \leq b < q \leq \ell + 1, \quad (4.6.26)$$

$$c, \quad (4.6.27)$$

$$\left(d + (\ell + 1 - b) \sum_{m=1}^b mh_{\alpha_{\ell+1-m}} + b \sum_{m=1}^b mh_{\alpha_{\ell+1-m}} \right).$$

In those cases where the matrix $\mathbf{U}(t)$ is t -independent, then the real form in question is isomorphic to $su_{(2)}(p, q)$, where $\mathbf{U}(t)$ is diagonal with p elements taking the value 1, and the remainder (q in number) taking the value -1 .

Automorphisms corresponding to the rotation τ_2

Each one of these involutive automorphisms belongs to one of the conjugacy classes that were examined in the previous subsection. The root transformations in question may be generalised thus

$$S_{\alpha_1}^0 \circ \dots \circ S_{\alpha_m}^0 \circ \dots \circ S_{\alpha_r}^0 \quad m, r \text{ both odd}; 1 \leq m, r < \ell, \quad (4.6.28)$$

and the automorphisms associated with these root transformations are those that are generated by the matrices $\mathbf{U}(t)$, where

$$\mathbf{U}(t) = \text{dsum}\{\mathbf{Z}_1(t), \dots, \mathbf{Z}_m(t), \dots, \mathbf{Z}_r(t), \mathbf{T}(t)\}, \quad (4.6.29)$$

where the submatrices in this expression are themselves defined by the expressions

$$\mathbf{Z}_m(t) = \begin{bmatrix} 0 & \eta_m t^{k_m} \\ \eta_m^{-1} t^{k_m} & 0 \end{bmatrix} \quad \eta_m \eta_m^* = 1; 1 \leq m \leq r, \quad (4.6.30)$$

$$\mathbf{T}(t) = \text{diag}\{\eta_{r+2} t^{k_{r+2}}, \dots, \eta_\ell t^{k_\ell}, 1\} \quad \eta_q^2 = 1; r+2 \leq q \leq \ell. \quad (4.6.31)$$

Let the matrix \mathbf{S} be defined as

$$\mathbf{S} = \text{dsum}\{\mathbf{X}_1, \dots, \mathbf{X}_m, \dots, \mathbf{X}_r, \mathbf{1}_{\ell-r}\} \quad \mathbf{X}_m = \begin{bmatrix} 1 & \eta_m \\ -\eta_m^{-1} & 1 \end{bmatrix}; 1 \leq m \leq r; m \text{ odd}. \quad (4.6.32)$$

Thus, $\tilde{\mathbf{S}}^* \mathbf{S} = 2\mathbf{1}_{\ell+1}$ and furthermore

$$\mathbf{SU}(t)\mathbf{S}^{-1} = \mathbf{U}'(t), \quad (4.6.33)$$

where the type 2a involutive automorphism $\{\mathbf{U}'(t), 1, \xi\}$ corresponds to the identity rotation, and has therefore already been examined, and belongs to one of the conjugacy classes $(\mathbf{G})^{(a,b)}$, for suitable a, b .

Automorphisms corresponding to the rotation τ_3

As with the complex Kac-Moody algebra $A_\ell^{(1)}$, each of these involutive automorphisms will be shown to be conjugate to a type 2a involution that corresponds to the identity root transformation. For $su_{(1)}(\ell+1, 0)$, the involutive automorphisms are generated by the matrices $\mathbf{U}(t)$, where

$$\mathbf{U}(t) = \text{offdiag}\left\{1, \eta_2 t^{k_2}, \dots, \eta_\ell t^{k_\ell}, \eta_{\ell+1}\right\}, \quad k_m = k_{\ell+2-m}; 2 \leq m \leq \frac{1}{2}(\ell+1),$$

$$\eta_{\ell+1} = \eta_\ell \eta_2 = \eta_{\ell-1} \eta_3 = \dots = \eta_{\frac{1}{2}(\ell+3)} \eta_{\frac{1}{2}(\ell+1)}, \quad (4.6.34)$$

$$\eta_2 \eta_2^* = \eta_3 \eta_3^* = \dots = \eta_{\ell+1} \eta_{\ell+1}^* = 1.$$

With $\mathbf{U}(t)$ as defined above, let $\mathbf{S}(t)$ be the diagonal matrix whose elements are specified by

$$(\mathbf{S}(t))_{m,m} = \begin{cases} \eta_{\ell+1}^{\frac{1}{2}} & m = 1 \\ \eta_{\ell+1}^{\frac{1}{2}} \eta_m^{-1} t^{-k_m} & 2 \leq m \leq \frac{1}{2}(\ell+1) \\ 1 & \frac{1}{2}(\ell+3) \leq m \leq \ell+1 \end{cases}, \quad (4.6.35)$$

so that $\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{\ell+1}$, and finally

$$\mathbf{S}(t)\mathbf{K}_{\ell+1}\mathbf{S}(t^{-1})^{-1} = \text{dsum}\left\{\mathbf{1}_{\frac{1}{2}(\ell+1)}, -\mathbf{1}_{\frac{1}{2}(\ell+1)}\right\}. \quad (4.6.36)$$

This concludes the analysis of this section.

4.7 Type 2b involutive automorphisms of $A_\ell^{(1)}$ with $u = 1$

Automorphisms corresponding to the rotation τ_4

These automorphisms are the ones that are generated by the matrices $U(t)$ of the form

$$U(t) = \text{diag}\{1, \eta_2, \dots, \eta_{\ell+1}\} \quad \eta_m \eta_m^* = 1; 2 \leq m \leq \ell + 1. \quad (4.7.1)$$

Thus, with the diagonal matrix $S(t)$ given by

$$(S(t))_{m,m} = \begin{cases} 1 & m = 1 \\ \eta_m^{-\frac{1}{2}} & 2 \leq m \leq \ell + 1 \end{cases}, \quad (4.7.2)$$

it follows that $\tilde{S}^*(t^{-1})S(t) = \mathbf{1}_{\ell+1}$, and similarly that

$$S(t)U(t)\tilde{S}(t^{-1}) = \mathbf{1}_{\ell+1}. \quad (4.7.3)$$

Thus, all of these involutions are mutually conjugate. The conjugacy class to which they all belong may be called (H). Its representative is taken to be the type 2a involutive automorphism that is generated by the matrix $\mathbf{1}_{\ell+1}$. A basis of $su_{(1)}(\ell + 1, 0)$, all of whose elements are eigenvectors of this representative automorphism, is given below

$$\begin{aligned} & \left(e_{j\delta}^k + e_{-j\delta}^k \right) \quad j \in \mathbf{N}^0; 1 \leq k \leq \ell; \text{eigenvalue } -1, \\ & i \left(e_{j\delta}^k - e_{-j\delta}^k \right) \quad j \in \mathbf{N}; 1 \leq k \leq \ell; \text{eigenvalue } 1, \end{aligned} \quad (4.7.4)$$

$$\begin{aligned} & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} \right) \quad j \in \mathbf{Z}; \alpha \in \Delta_+^0; \text{eigenvalue } -1, \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} \right) \quad j \in \mathbf{Z}^*; \alpha \in \Delta_+^0; \text{eigenvalue } 1, \end{aligned} \quad (4.7.5)$$

$$\left. \begin{array}{l} ic \\ id \end{array} \right\} \text{eigenvalue } -1. \quad (4.7.6)$$

In the above, \mathbf{Z}^* denotes the set of non-zero integers. Therefore, a basis of a real form of $\mathbf{A}_\ell^{(1)}$ is given by the following elements

$$\left. \begin{array}{l} i(e_{j\delta}^k + e_{-j\delta}^k) \quad j \in \mathbf{N}^0 \\ i(e_{j\delta}^k - e_{-j\delta}^k) \quad j \in \mathbf{N} \end{array} \right\} 1 \leq k \leq \ell, \quad (4.7.7)$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \quad j \in \mathbf{Z} \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \quad j \in \mathbf{Z}^* \end{array} \right\} \alpha \in \Delta_+^0, \quad (4.7.8)$$

$$\begin{array}{l} c, \\ d. \end{array} \quad (4.7.9)$$

It is clear that this real form is the real form $sl_{(2)}(\ell + 1, \mathbf{R})$, whose matrix parts satisfy

$$\mathbf{a}(t) = \mathbf{a}^*(t). \quad (4.7.10)$$

Automorphisms corresponding to the rotation τ_5

Each of these automorphisms is conjugate to some automorphism for which the corresponding root transformation is τ_{Cartan}^0 , and these automorphisms were discussed in the previous subsection. The automorphisms that are being investigated are those that are generated by the matrices $\mathbf{U}(t)$, where

$$\mathbf{U}(t) = \text{dsum}\{\mathbf{Z}_1(t), \dots, \mathbf{Z}_m(t), \dots, \mathbf{Z}_r(t), \mathbf{T}\}, \quad (4.7.11)$$

$$\mathbf{Z}_m(t) = \begin{bmatrix} 0 & \eta_m t^{k_m} \\ \eta_m t^{-k_m} & 0 \end{bmatrix} \quad \eta_m \eta_m^* = 1; 1 \leq m \leq r, \quad (4.7.12)$$

$$\mathbf{T} = \text{diag}\{\eta_{r+2}, \dots, \eta_{\ell+1}\}. \quad (4.7.13)$$

Then, define a matrix $\mathbf{S}(t)$ by

$$\mathbf{S}(t) = \text{dsum}\{\mathbf{X}_1(t), \dots, \mathbf{X}_m(t), \dots, \mathbf{X}_r(t), \mathbf{1}_{\ell-r}\} \quad \mathbf{X}_m = \frac{1}{\sqrt{5}} \begin{bmatrix} \eta_m t^{-k_m} & 2 \\ 2\eta_m t^{-k_m} & -1 \end{bmatrix}; 1 \leq m \leq r, \quad (4.7.14)$$

which is such that $\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{\ell+1}$, and

$$\mathbf{S}(t)\mathbf{U}(t)\tilde{\mathbf{S}}(t^{-1}) = \mathbf{U}'(t), \quad (4.7.15)$$

where the matrix $\mathbf{U}'(t)$ is such that $\{\mathbf{U}'(t), 1, \xi\}$ is a type 2b involutive automorphism that corresponds to the root transformation τ_{Cartan}^0 . Thus, all of the automorphisms to be studied in this subsection belong to the conjugacy class (H).

Automorphisms corresponding to the rotation τ_ℓ

Recall that these automorphisms correspond to a root transformation that itself is associated only with odd values of ℓ . These automorphisms fall into two disjoint conjugacy classes, and these classes are disjoint from all of the other classes that have been encountered previously. As expected, this is the same as the conclusion reached for $A_\ell^{(1)}$. The automorphisms in question are those generated by the matrices

$$\begin{aligned} \mathbf{U}(t) &= \text{offdiag}\{1, \eta_2 t^{k_2}, \dots, \eta_{\ell+1} t^{k_{\ell+1}}\}, \\ \eta_{\ell+1}^2 &= \eta_m \eta_m^* = 1; (2 \leq m \leq \ell), \\ k_{\ell-m} &= k_{\ell+1} - k_{m+2}; \eta_{\ell-m} = \eta_{\ell+1} \eta_{m+2}; (0 \leq m \leq \frac{1}{2}(\ell - 3)). \end{aligned} \quad (4.7.16)$$

A diagonal matrix $\mathbf{S}(t)$ may be defined by

$$(\mathbf{S}(t))_{m,m} = \begin{cases} 1 & m = 1 \\ \eta_m^{-1} t^{-k_m} & 2 \leq m \leq \frac{1}{2}(\ell + 1) \\ t^{\frac{1}{2}(\deg(\ell+1) - k_{\ell+1})} & \frac{1}{2}(\ell + 3) \leq m \leq \ell + 1 \end{cases}. \quad (4.7.17)$$

This means that the following hold

$$\begin{aligned} \tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) &= \mathbf{1}_{\ell+1}, \\ \mathbf{S}(t)\mathbf{U}(t)\tilde{\mathbf{S}}(t^{-1}) &= t^{\frac{1}{2}(k_{\ell+1} - \deg k_{\ell+1})} \text{offdiag} \left\{ \mathbf{K}_{\frac{1}{2}(\ell+1)}, \eta_{\ell+1} t^{\deg k_{\ell+1}} \mathbf{K}_{\frac{1}{2}(\ell+1)} \right\}. \end{aligned} \quad (4.7.18)$$

This shows that all of the automorphisms in question fall in to two conjugacy classes. That these classes are disjoint follows from the analysis of the involutive automorphisms of $A_\ell^{(1)}$. The two classes will be called $(\mathbf{I})^{(1)}$ and $(\mathbf{I})^{(i)}$. The representative of $(\mathbf{I})^{(i)}$ will be the type 2b automorphism generated by the matrix offsum $\left\{ \mathbf{K}_{\frac{1}{2}(\ell+1)}, -\mathbf{K}_{\frac{1}{2}(\ell+1)} \right\}$, whereas the representative of the class $(\mathbf{I})^{(1)}$ will be the type 2b involution generated by the matrix offsum $\left\{ \mathbf{K}_{\frac{1}{2}(\ell+1)}, t\mathbf{K}_{\frac{1}{2}(\ell+1)} \right\}$. Bases of $su_{(1)}(\ell + 1, 0)$ exist in which the basis elements are all eigenvectors of one of these representative automorphisms. A basis of $su_{(1)}(\ell + 1, 0)$ in which each basis element is an eigenvector of the representative of $(\mathbf{I})^{(1)}$ is given below, together with associated eigenvalues

$$\begin{aligned} & \left(e_{j\delta}^k + e_{-j\delta}^k + e_{j\delta}^{\ell+1-k} + e_{-j\delta}^{\ell+1-k} \right) \quad j \in \mathbf{N}^0; k \in \left\{ 1, \dots, \frac{1}{2}(\ell + 1) \right\} \quad \text{eigenvalue } 1, \\ & \left(e_{j\delta}^k + e_{-j\delta}^k - e_{j\delta}^{\ell+1-k} - e_{-j\delta}^{\ell+1-k} \right) \quad j \in \mathbf{N}^0; k \in \left\{ 1, \dots, \frac{1}{2}(\ell - 1) \right\} \quad \text{eigenvalue } -1, \\ & i \left(e_{j\delta}^k - e_{-j\delta}^k + e_{j\delta}^{\ell+1-k} - e_{-j\delta}^{\ell+1-k} \right) \quad j \in \mathbf{N}; k \in \left\{ 1, \dots, \frac{1}{2}(\ell + 1) \right\} \quad \text{eigenvalue } -1, \\ & i \left(e_{j\delta}^k - e_{-j\delta}^k - e_{j\delta}^{\ell+1-k} + e_{-j\delta}^{\ell+1-k} \right) \quad j \in \mathbf{N}; k \in \left\{ 1, \dots, \frac{1}{2}(\ell - 1) \right\} \quad \text{eigenvalue } 1, \end{aligned} \quad (4.7.19)$$

$$\begin{aligned} & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\beta} + e_{j\delta-\beta} \right) \quad j \in \mathbf{Z}; 1 \leq p < q \leq \frac{1}{2}(\ell + 1) \quad \text{eigenvalue } -1, \\ & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\beta} - e_{j\delta-\beta} \right) \quad j \in \mathbf{Z}; 1 \leq p < q \leq \frac{1}{2}(\ell + 1) \quad \text{eigenvalue } 1, \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\beta} - e_{j\delta-\beta} \right) \quad j \in \mathbf{Z}^\pm; 1 \leq p < q \leq \frac{1}{2}(\ell + 1) \quad \text{eigenvalue } -1, \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\beta} + e_{j\delta-\beta} \right) \quad j \in \mathbf{Z}^\pm; 1 \leq p < q \leq \frac{1}{2}(\ell + 1) \quad \text{eigenvalue } 1, \end{aligned} \quad (4.7.20)$$

$$\begin{aligned}
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\beta} + e_{j\delta-\beta}) \quad j \in \mathbf{Z}; 1 \leq p \leq \frac{1}{2}(\ell+1) < q \leq \ell+1 \quad \text{eigenvalue } 1, \\
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\beta} - e_{j\delta-\beta}) \quad j \in \mathbf{Z}; 1 \leq p \leq \frac{1}{2}(\ell+1) < q \leq \ell+1 \quad \text{eigenvalue } -1, \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\beta} - e_{j\delta-\beta}) \quad j \in \mathbf{Z}^\pm; 1 \leq p \leq \frac{1}{2}(\ell+1) < q \leq \ell+1 \quad \text{eigenvalue } 1, \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\beta} + e_{j\delta-\beta}) \quad j \in \mathbf{Z}^\pm; 1 \leq p \leq \frac{1}{2}(\ell+1) < q \leq \ell+1 \quad \text{eigenvalue } -1,
\end{aligned} \tag{4.7.21}$$

$$\left. \begin{array}{l} ic \\ id \end{array} \right\} \text{eigenvalue } -1, \tag{4.7.22}$$

where, in the above the roots are given by

$$\alpha = \sum_{m=p}^{q-1} \alpha_m, \quad \beta = \sum_{m=\ell+2-q}^{\ell+1-p} \alpha_m. \tag{4.7.23}$$

It is clear then, that the following is a basis of a real form of $A_\ell^{(1)}$

$$\begin{aligned}
& (e_{j\delta}^k + e_{-j\delta}^k + e_{j\delta}^{\ell+1-k} + e_{-j\delta}^{\ell+1-k}) \quad j \in \mathbf{N}^0; k \in \{1, \dots, \frac{1}{2}(\ell+1)\}, \\
& i(e_{j\delta}^k + e_{-j\delta}^k - e_{j\delta}^{\ell+1-k} - e_{-j\delta}^{\ell+1-k}) \quad j \in \mathbf{N}^0; k \in \{1, \dots, \frac{1}{2}(\ell-1)\}, \\
& (e_{j\delta}^k - e_{-j\delta}^k + e_{j\delta}^{\ell+1-k} - e_{-j\delta}^{\ell+1-k}) \quad j \in \mathbf{N}; k \in \{1, \dots, \frac{1}{2}(\ell+1)\}, \\
& i(e_{j\delta}^k - e_{-j\delta}^k - e_{j\delta}^{\ell+1-k} + e_{-j\delta}^{\ell+1-k}) \quad j \in \mathbf{N}; k \in \{1, \dots, \frac{1}{2}(\ell-1)\},
\end{aligned} \tag{4.7.24}$$

$$\begin{aligned}
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\beta} + e_{j\delta-\beta}) \quad j \in \mathbf{Z}; 1 \leq p < q \leq \frac{1}{2}(\ell+1), \\
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\beta} - e_{j\delta-\beta}) \quad j \in \mathbf{Z}; 1 \leq p < q \leq \frac{1}{2}(\ell+1), \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\beta} - e_{j\delta-\beta}) \quad j \in \mathbf{Z}^\pm; 1 \leq p < q \leq \frac{1}{2}(\ell+1), \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\beta} + e_{j\delta-\beta}) \quad j \in \mathbf{Z}^\pm; 1 \leq p < q \leq \frac{1}{2}(\ell+1),
\end{aligned} \tag{4.7.25}$$

$$\begin{aligned}
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\beta} + e_{j\delta-\beta}) \quad j \in \mathbf{Z}; 1 \leq p \leq \frac{1}{2}(\ell+1) < q \leq \ell+1, \\
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\beta} - e_{j\delta-\beta}) \quad j \in \mathbf{Z}; 1 \leq p \leq \frac{1}{2}(\ell+1) < q \leq \ell+1, \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\beta} - e_{j\delta-\beta}) \quad j \in \mathbf{Z}^\pm; 1 \leq p \leq \frac{1}{2}(\ell+1) < q \leq \ell+1, \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\beta} + e_{j\delta-\beta}) \quad j \in \mathbf{Z}^\pm; 1 \leq p \leq \frac{1}{2}(\ell+1) < q \leq \ell+1,
\end{aligned} \tag{4.7.26}$$

$$\begin{aligned}
& c, \\
& d.
\end{aligned} \tag{4.7.27}$$

The matrix parts of this real form are traceless and such that they satisfy

$$\mathbf{a}(t) = \mathbf{U}(t)\mathbf{a}^*(t^{-1})\mathbf{U}(t)^{-1}, \quad (4.7.28)$$

where $\mathbf{U}(t) = \text{offsum} \left\{ \mathbf{K}_{\frac{1}{2}(\ell+1)}, -\mathbf{K}_{\frac{1}{2}(\ell+1)} \right\}$. The representative automorphism of the conjugacy class $(\mathbf{I})^{(\ell)}$ is such that there exists a basis of $su_{(1)}(\ell+1, 0)$ in which each basis element is an eigenvector of this automorphism. The elements below form such a basis. Note that the expressions for the roots α and β are unchanged.

$$\left. \begin{array}{l} i(h_{\alpha_k} + h_{\alpha_{\ell+1-k}}) \quad \text{eigenvalue } 1 \\ i(h_{\alpha_k} - h_{\alpha_{\ell+1-k}}) \quad \text{eigenvalue } -1 \end{array} \right\} 1 \leq k \leq \frac{1}{2}(\ell-1), \quad (4.7.29)$$

$$i\left(h_{\frac{1}{2}(\ell+1)} - \frac{1}{2}c\right) \quad \text{eigenvalue } 1, \quad (4.7.30)$$

$$\begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k + e_{j\delta}^{\ell+1-k} + e_{-j\delta}^{\ell+1-k}) \quad j \in \mathbf{N}^0; k \in \left\{1, \dots, \frac{1}{2}(\ell+1)\right\} \quad \text{eigenvalue } 1, \\ (e_{j\delta}^k + e_{-j\delta}^k - e_{j\delta}^{\ell+1-k} - e_{-j\delta}^{\ell+1-k}) \quad j \in \mathbf{N}^0; k \in \left\{1, \dots, \frac{1}{2}(\ell-1)\right\} \quad \text{eigenvalue } -1, \\ i(e_{j\delta}^k - e_{-j\delta}^k + e_{j\delta}^{\ell+1-k} - e_{-j\delta}^{\ell+1-k}) \quad j \in \mathbf{N}; k \in \left\{1, \dots, \frac{1}{2}(\ell+1)\right\} \quad \text{eigenvalue } -1, \\ i(e_{j\delta}^k - e_{-j\delta}^k - e_{j\delta}^{\ell+1-k} + e_{-j\delta}^{\ell+1-k}) \quad j \in \mathbf{N}; k \in \left\{1, \dots, \frac{1}{2}(\ell-1)\right\} \quad \text{eigenvalue } 1, \end{array} \quad (4.7.31)$$

$$\begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\beta} + e_{j\delta-\beta}) \quad j \in \mathbf{Z}; 1 \leq p < q \leq \frac{1}{2}(\ell+1) \quad \text{eigenvalue } -1, \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\beta} - e_{j\delta-\beta}) \quad j \in \mathbf{Z}; 1 \leq p < q \leq \frac{1}{2}(\ell+1) \quad \text{eigenvalue } 1, \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\beta} - e_{j\delta-\beta}) \quad j \in \mathbf{Z}^\pm; 1 \leq p < q \leq \frac{1}{2}(\ell+1) \quad \text{eigenvalue } -1, \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\beta} + e_{j\delta-\beta}) \quad j \in \mathbf{Z}^\pm; 1 \leq p < q \leq \frac{1}{2}(\ell+1) \quad \text{eigenvalue } 1, \end{array} \quad (4.7.32)$$

$$\begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\beta} + e_{-k\delta-\beta}) \quad j \in \mathbf{Z} \quad \text{eigenvalue } -1, \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\beta} - e_{-k\delta-\beta}) \quad j \in \mathbf{Z} \quad \text{eigenvalue } 1, \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\beta} - e_{-k\delta-\beta}) \quad j \in \mathbf{Z}^\pm \quad \text{eigenvalue } -1, \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\beta} + e_{-k\delta-\beta}) \quad j \in \mathbf{Z}^\pm \quad \text{eigenvalue } 1, \end{array} \quad (4.7.33)$$

where, in the above, $k = -j - 1; 1 \leq p \leq \frac{1}{2}(\ell+1) < q \leq \ell+1$,

$$i \left(d + \frac{(\ell+1)}{2} \left\{ \sum_{m=1}^{\frac{1}{2}(\ell-1)} m (h_{\alpha_m} + h_{\alpha_{\ell+1-m}}) \right\} + \frac{(\ell+1)^2}{4} h_{\alpha_{\frac{1}{2}(\ell+1)}} \right) \Bigg\} \text{eigenvalue } -1, \quad (4.734)$$

Hence, the basis for an associated real form is given by

$$\left. \begin{array}{l} i(h_{\alpha_k} + h_{\alpha_{\ell+1-k}}) \\ (h_{\alpha_k} - h_{\alpha_{\ell+1-k}}) \end{array} \right\} 1 \leq k \leq \frac{1}{2}(\ell-1), \quad (4.735)$$

$$i \left(h_{\alpha_{\frac{1}{2}(\ell+1)}} - \frac{1}{2}c \right), \quad (4.736)$$

$$\begin{aligned} & (e_{j\delta}^k + e_{-j\delta}^k + e_{j\delta}^{\ell+1-k} + e_{-j\delta}^{\ell+1-k}) \quad j \in \mathbf{N}; k \in \{1, \dots, \frac{1}{2}(\ell+1)\}, \\ & i(e_{j\delta}^k + e_{-j\delta}^k - e_{j\delta}^{\ell+1-k} - e_{-j\delta}^{\ell+1-k}) \quad j \in \mathbf{N}; k \in \{1, \dots, \frac{1}{2}(\ell-1)\}, \\ & (e_{j\delta}^k - e_{-j\delta}^k + e_{j\delta}^{\ell+1-k} - e_{-j\delta}^{\ell+1-k}) \quad j \in \mathbf{N}; k \in \{1, \dots, \frac{1}{2}(\ell+1)\}, \\ & i(e_{j\delta}^k - e_{-j\delta}^k - e_{j\delta}^{\ell+1-k} + e_{-j\delta}^{\ell+1-k}) \quad j \in \mathbf{N}; k \in \{1, \dots, \frac{1}{2}(\ell-1)\}, \end{aligned} \quad (4.737)$$

$$\begin{aligned} & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\beta} + e_{j\delta-\beta}) \quad 1 \leq p < q \leq \frac{1}{2}(\ell+1) \quad j \in \mathbf{Z}, \\ & (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\beta} - e_{j\delta-\beta}) \quad 1 \leq p < q \leq \frac{1}{2}(\ell+1) \quad j \in \mathbf{Z}, \\ & (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\beta} - e_{j\delta-\beta}) \quad 1 \leq p < q \leq \frac{1}{2}(\ell+1) \quad j \in \mathbf{Z}^\pm, \\ & i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\beta} + e_{j\delta-\beta}) \quad 1 \leq p < q \leq \frac{1}{2}(\ell+1) \quad j \in \mathbf{Z}^\pm, \end{aligned} \quad (4.738)$$

$$\begin{aligned} & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\beta} + e_{-k\delta-\beta}) \quad j \in \mathbf{Z}; k = -j-1; 1 \leq p \leq \frac{1}{2}(\ell+1) < q \leq \ell+1 \quad j \in \mathbf{Z}, \\ & (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\beta} - e_{-k\delta-\beta}) \quad j \in \mathbf{Z}; k = -j-1; 1 \leq p \leq \frac{1}{2}(\ell+1) < q \leq \ell+1 \quad j \in \mathbf{Z}, \\ & (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\beta} - e_{-k\delta-\beta}) \quad j \in \mathbf{Z}^\pm; k = -j-1; 1 \leq p \leq \frac{1}{2}(\ell+1) < q \leq \ell+1 \quad j \in \mathbf{Z}^\pm, \\ & i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\beta} + e_{-k\delta-\beta}) \quad j \in \mathbf{Z}^\pm; k = -j-1; 1 \leq p \leq \frac{1}{2}(\ell+1) < q \leq \ell+1 \quad j \in \mathbf{Z}^\pm, \end{aligned} \quad (4.739)$$

$$c, \quad \left(d + \frac{(\ell+1)}{2} \left\{ \sum_{m=1}^{\frac{1}{2}(\ell-1)} m (h_{\alpha_m} + h_{\alpha_{\ell+1-m}}) \right\} + \frac{(\ell+1)^2}{4} h_{\alpha_{\frac{1}{2}(\ell+1)}} \right). \quad (4.740)$$

The general element of this real form may be expressed in the form

$$\mathbf{a}(t) + (\lambda + i\mu)c + \eta(d + \mathbf{m}(t)), \quad (4.741)$$

where the quantities λ and η are arbitrary real numbers, and $\mathbf{a}(t)$ is traceless and satisfies

$$\mathbf{a}(t) = \mathbf{U}(t) \mathbf{a}^* \mathbf{U}(t)^{-1}. \quad (4.7.42)$$

5 Involutive automorphisms and real forms of $B_\ell^{(1)}$

5.1 Introduction

In this chapter, the conjugacy classes of the involutive automorphisms of the affine Kac-Moody algebra $B_\ell^{(1)}$ will be investigated. Using the results obtained, real forms of $B_\ell^{(1)}$ will then be constructed using Cartan's method. The basis for this chapter is [6]. The simple Lie algebra B_ℓ is naturally of great importance when investigating the affine Kac-Moody that is based upon it. The generalised Cartan matrix for $B_\ell^{(1)}$, together with other information relevant to B_ℓ and to $B_\ell^{(1)}$, is contained within the appendices. Note that, for $B_\ell^{(1)}$, the imaginary root δ is given by

$$\delta = \alpha_0 + \alpha_1 + 2(\alpha_2 + \cdots + \alpha_\ell), \quad (5.1.1)$$

which means that the operator c is given by the expression

$$c = h_{\alpha_0} + h_{\alpha_1} + 2(h_{\alpha_2} + \cdots + h_{\alpha_\ell}). \quad (5.1.2)$$

The choice of the representation Γ of the Lie algebra B_ℓ has also to be made. The Lie algebra is the complexification of (among others) the pseudo-orthogonal algebras $so(p, q)$ (where $p + q = 2\ell + 1$). The algebra $so(p, q)$ may be realised as the set of real traceless $(2\ell + 1) \times (2\ell + 1)$ matrices \mathbf{a} which satisfy

$$\tilde{\mathbf{a}}\mathbf{g} + \mathbf{g}\mathbf{a} = \mathbf{0}, \quad (5.1.3)$$

where \mathbf{g} is a $(2\ell + 1)$ -dimensional diagonal matrix with p entries taking the value 1 and with $q (= 2\ell + 1 - p)$ entries taking the value -1 . In fact, this realisation of B_ℓ is not the precise one which will be used. Instead, an explicit realisation of dimension

$(2\ell+1)$ in which the elements of the Cartan subalgebra of B_ℓ are represented by diagonal matrices will be the one used. Each positive root of B_ℓ may be expressed in one of the following forms

$$\alpha^0 = \begin{cases} \sum_{p=r}^{\ell} \alpha_p^0 & (1 \leq r \leq \ell), \\ \sum_{p=r}^{s-1} \alpha_p^0 + 2 \sum_{p=r}^{\ell} \alpha_p^0 & (1 \leq r < s \leq \ell), \\ \sum_{p=r}^{s-1} \alpha_p^0 & (1 \leq r < s \leq \ell). \end{cases} \quad (5.1.4)$$

The members of the Cartan subalgebra of B_ℓ are represented by

$$\Gamma(h_{\alpha_k}^0) = \begin{cases} \{2(2\ell-1)\}^{-1} \{\mathbf{X}_k - \mathbf{X}_{k+1}\} & (1 \leq k \leq \ell-1), \\ \{2(2\ell-1)\}^{-1} \{\mathbf{X}_\ell\} & (k = \ell), \end{cases} \quad (5.1.5)$$

The elements $e_{\alpha^0}^0$, where α^0 can be any positive root of B_ℓ , are represented by the following (which are equations 5.1.6):

$$\Gamma(e_{\alpha^0}^0) = \frac{1}{\sqrt{2(2\ell-1)}} \{(-1)^{\ell+r} \mathbf{e}_{r,\ell+1} + \mathbf{e}_{\ell+1,2\ell+2-r}\} \text{ where } \alpha^0 = \sum_{j=r}^{\ell} \alpha_j^0,$$

$$\Gamma(e_{\alpha^0}^0) = \frac{1}{\sqrt{2(2\ell-1)}} \{(-1)^{r+s+1} \mathbf{e}_{r,2\ell+2-s} + \mathbf{e}_{s,2\ell+2-r}\} \text{ where } \alpha^0 = \left\{ \sum_{j=r}^{s-1} \alpha_j^0 + 2 \sum_{j=s}^{\ell} \alpha_j^0 \right\},$$

$$\Gamma(e_{\alpha^0}^0) = \frac{1}{\sqrt{2(2\ell-1)}} \{(-1)^{r+s+1} \mathbf{e}_{r,s} - \mathbf{e}_{2\ell+2-s,2\ell+2-r}\} \text{ where } \alpha^0 = \sum_{j=r}^{s-1} \alpha_j^0.$$

In 5.1.6, the matrix \mathbf{X}_p (for $1 \leq p \leq \ell$) is the diagonal matrix of dimension $(2\ell+1)$ whose diagonal elements are specified by

$$(\mathbf{X}_p)_{j,j} = \begin{cases} 1 & j = p, \\ -1 & j = 2\ell+2-p, \\ 0 & \text{otherwise.} \end{cases} \quad (5.1.7)$$

If α^0 is any positive non-zero root of B_ℓ , then

$$\Gamma(e_{\alpha^0}^0) = -\tilde{\Gamma}^*(e_{-\alpha^0}^0). \quad (5.1.8)$$

Thus, the remaining basis elements of the representation may be found from those already given.

The real forms of B_ℓ are the real Lie algebras $so(p, q)$, where $p + q = 2\ell + 1$, and $p > q$. The algebra $so(p, q)$ may be realised as the set of real $(2\ell + 1)$ -dimensional matrices \mathbf{a} which satisfy

$$\mathbf{a}\mathbf{g} + \mathbf{g}\tilde{\mathbf{a}} = \mathbf{0}, \quad (5.1.9)$$

where \mathbf{g} is a $(2\ell + 1)$ -dimensional diagonal matrix with p entries taking the value 1, and the remainder taking the value -1 . Clearly, the explicit realisation being used here does not satisfy this. However, let the matrix \mathbf{g} be defined by

$$\mathbf{g} = \text{offdiag}\{1, -1, \dots, -1, 1\}. \quad (5.1.10)$$

That is, \mathbf{g} has non-zero elements only on the minor diagonal, and these take values which are alternately 1 and -1 . Then, the matrices of the realisation used all satisfy

$$\mathbf{a}\mathbf{g} + \mathbf{g}\tilde{\mathbf{a}} = \mathbf{0}, \quad (5.1.11)$$

but with this *new* choice of \mathbf{g} . It should be noted that this representation is equivalent to its contragredient representation. For, it is easily verified that

$$\tilde{\Gamma} = -\mathbf{g}\Gamma\mathbf{g}^{-1}, \quad (5.1.12)$$

which demonstrates this. The Dynkin index of this representation is given by

$$\gamma = \{2(2\ell - 1)\}^{-1}. \quad (5.1.13)$$

The compact real form of $B_\ell^{(1)}$ will be studied subsequently, rather than $B_\ell^{(1)}$ itself. A matrix basis of the compact real form is easily constructed from the matrix representatives of the basis elements given above. Recall that, in terms of the Weyl canonical basis elements of B_ℓ , a basis of the compact real form is given by

$$\left. \begin{aligned} ih_{\alpha_k} & \quad (1 \leq k \leq \ell), \\ (e_{\alpha} + e_{-\alpha}) \\ i(e_{\alpha} - e_{-\alpha}) \end{aligned} \right\} \alpha \in \Delta_+. \quad (5.1.14)$$

Note that the superscript "0" is now being omitted. Thus, a basis of the compact real form of $B_{\ell}^{(1)}$ is obtainable from these basis elements, and in terms of the basis elements of the Weyl canonical form, the members of such a basis are

$$\left. \begin{aligned} (e_{j\delta}^k + e_{-j\delta}^k) & \quad j \in \mathbf{N}^0 \\ i(e_{j\delta}^k - e_{-j\delta}^k) & \quad j \in \mathbf{N} \end{aligned} \right\} (1 \leq k \leq \ell), \quad (5.1.15)$$

$$\left. \begin{aligned} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{aligned} \right\} j \in \mathbf{Z}; \alpha \in \Delta_+^0, \quad (5.1.16)$$

$$\begin{aligned} ic, \\ id. \end{aligned} \quad (5.1.17)$$

The matrix parts of the matrix representatives of elements of the compact real form of $B_{\ell}^{(1)}$ all satisfy

$$\tilde{\mathbf{a}}^*(t^{-1}) = -\mathbf{a}(t). \quad (5.1.18)$$

In addition, they also satisfy

$$\mathbf{g}\tilde{\mathbf{a}}(t)\mathbf{g}^{-1} = -\mathbf{a}(t). \quad (5.1.19)$$

Consider the automorphism $\{\mathbf{S}(t), s, \xi\}$ of $B_{\ell}^{(1)}$. It is of some interest to know under what circumstances the restriction of this automorphism to the compact real form is itself an automorphism of the compact real form. Let the automorphism $\{\mathbf{S}(t), s, \xi\}$ be denoted by ϕ , and suppose for the present that it is of type 1a. It follows then that

$$\phi(\mathbf{a}(t)) = \mathbf{S}(t)\mathbf{a}(ut)\mathbf{S}(t)^{-1} + \frac{1}{\gamma} \text{Res} \left\{ \text{tr} \left(\mathbf{S}(t)^{-1} \frac{d\mathbf{S}(t)}{dt} \mathbf{a}(ut) \right) \right\} c. \quad (5.1.20)$$

If ϕ is an automorphism of the compact real form of $B_\ell^{(1)}$, then the matrix $\mathbf{b}(t)$, where $\mathbf{b}(t) = \mathbf{S}(t)\mathbf{a}(ut)\mathbf{S}(t)^{-1}$, must satisfy

$$\mathbf{g}\tilde{\mathbf{b}}(t)\mathbf{g}^{-1} = -\mathbf{b}(t). \quad (5.1.21)$$

Expanding this in full implies that

$$\mathbf{g}\tilde{\mathbf{S}}(t)^{-1}\tilde{\mathbf{a}}(ut)\tilde{\mathbf{S}}(t)\mathbf{g}^{-1} = -\mathbf{S}(t)\mathbf{a}(ut)\mathbf{S}(t)^{-1}. \quad (5.1.22)$$

However, since it is the case that $\mathbf{a}(t) = -\mathbf{g}\tilde{\mathbf{a}}(t)\mathbf{g}^{-1}$, this means that

$$\begin{aligned} -\mathbf{g}\tilde{\mathbf{S}}(t)^{-1}\{-\mathbf{g}^{-1}\mathbf{a}(ut)\mathbf{g}\}\tilde{\mathbf{S}}(t)\mathbf{g}^{-1} &= \mathbf{S}(t)\mathbf{a}(ut)\mathbf{S}(t)^{-1}, \\ \mathbf{g}\tilde{\mathbf{S}}(t)^{-1}\mathbf{g}^{-1}\mathbf{a}(ut)\mathbf{g}\tilde{\mathbf{S}}(t)\mathbf{g}^{-1} &= \mathbf{S}(t)\mathbf{a}(ut)\mathbf{S}(t)^{-1}, \end{aligned} \quad (5.1.23)$$

and, with suitable re-arrangement, this becomes

$$\mathbf{a}(ut)\mathbf{g}\tilde{\mathbf{S}}(t)\mathbf{g}^{-1}\mathbf{S}(t) = \mathbf{g}\tilde{\mathbf{S}}(t)\mathbf{g}^{-1}\mathbf{S}(t)\mathbf{a}(ut). \quad (5.1.24)$$

This holds for all non-zero values of t , and $\mathbf{S}(t)$ is assumed to be a Laurent polynomial matrix. Schur's lemma (see appendix B) may thus be modified in this case, and it is shown to be a necessary condition that

$$\mathbf{g}\tilde{\mathbf{S}}(t)\mathbf{g}^{-1}\mathbf{S}(t) = \alpha t^\beta \mathbf{1}_{2\ell+1}, \quad (5.1.25)$$

where α is a non-zero constant complex number, and β is an integer. Suppose then, that ϕ is of type 2a rather than type 1a. If the preceding analysis is then repeated, the same conclusion is reached, namely that the matrix $\mathbf{S}(t)$ satisfies the above condition. Noting that $\mathbf{g}^{-1} = \mathbf{g}$, and pre-multiplying (5.1.25) by \mathbf{g} , it becomes

$$\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = \alpha t^\beta \mathbf{g}. \quad (5.1.26)$$

5.2 Supplementary notation

We define several "general forms" of matrices which will be used in the rest of this chapter. Firstly, the matrices \mathbf{D}_j and \mathbf{D}_j^0 are defined by the following (which are the equations (5.2.1))

$$\left. \begin{aligned} \mathbf{D}_j &= \text{diag} \left\{ \lambda_j t^{\mu_j}, \dots, \lambda_\ell t^{\mu_\ell}, 1, \lambda_\ell t^{-\mu_\ell}, \dots, \lambda_j t^{-\mu_j} \right\} \\ \mathbf{D}_j^0 &= \text{diag} \left\{ \lambda_j, \dots, \lambda_\ell, 1, \lambda_\ell, \dots, \lambda_j \right\} \end{aligned} \right\} \quad (\lambda_q^2 = 1; j \leq q \leq \ell).$$

The general form \mathbf{F}_j is defined such that

$$\mathbf{F}_j = \text{offdiag} \left\{ \lambda_j t^{\mu_j}, \dots, \lambda_\ell t^{\mu_\ell}, 1, \lambda_\ell^{-1} t^{-\mu_\ell}, \dots, \lambda_j^{-1} t^{-\mu_j} \right\}. \quad (5.2.2)$$

\mathbf{F}_j^e is a variant on the general form which is defined to be such that the integers μ_m (for $j \leq m \leq \ell$) are all even. Similarly, \mathbf{F}_j^0 is a variant on the general form \mathbf{F}_j which is such that μ_m is zero (for $j \leq m \leq \ell$).

The general form $\mathbf{C}_{j,k}$ and the corresponding general form $\mathbf{C}'_{j,k}$ are both defined by

$$\mathbf{C}_{j,k} = \text{diag} \left\{ \lambda_j t^{\mu_j}, \dots, \lambda_k t^{\mu_k} \right\}, \quad (5.2.3)$$

$$\mathbf{C}'_{j,k} = \text{diag} \left\{ \lambda_k^{-1} t^{-\mu_k}, \dots, \lambda_j^{-1} t^{-\mu_j} \right\}, \quad (5.2.4)$$

where $\lambda_m^2 = 1$ (for $j \leq m \leq k$). The forms $\mathbf{C}_{j,k}^0$ and $\mathbf{C}'_{j,k}^0$ are the special cases of $\mathbf{C}_{j,k}$ and $\mathbf{C}'_{j,k}$ respectively where μ_m is zero for $j \leq m \leq k$. The matrices \mathbf{E}_j , \mathbf{E}'_j , \mathbf{E}_j^0 and \mathbf{E}'_j^0 are defined to be $\mathbf{C}_{1,j}$, $\mathbf{C}'_{1,j}$, $\mathbf{C}_{1,j}^0$ and $\mathbf{C}'_{1,j}^0$ respectively.

The matrix $\mathbf{L}_{j,k}$ is defined for values of j, k such that $(k - j)$ is even. In tandem with this, a matrix $\mathbf{L}'_{j,k}$ is defined for the same values of j, k . Both matrices are defined by

$$\begin{aligned}\mathbf{L}_{j,k} &= \text{dsum}\{\lambda_j, \dots, \lambda_q, \dots, \lambda_k\}, \\ \mathbf{L}'_{j,k} &= -\text{dsum}\{\lambda_k, \dots, \lambda_q, \dots, \lambda_j\},\end{aligned}\tag{5.2.5}$$

$$\text{where } \lambda_q = \begin{bmatrix} 0 & \lambda_q t^{\mu_q} \\ \lambda_q^{-1} t^{-\mu_q} & 0 \end{bmatrix}, \text{ for } (k - q) \text{ even; } j \leq q \leq k.$$

In a similar fashion, other matrices are defined (for values of j and k such that $(k - j)$ is even) by the equations

$$\begin{aligned}\mathbf{M}_{j,k} &= \text{dsum}\{\mu_j, \dots, \mu_q, \dots, \mu_k\}, \\ \mathbf{M}'_{j,k} &= -\text{dsum}\{\mu_k^{-1}, \dots, \mu_q^{-1}, \dots, \mu_j^{-1}\},\end{aligned}\tag{5.2.6}$$

$$\text{where } \mu_q = \begin{bmatrix} 0 & \lambda_q t^{\mu_q} \\ (-1)^{\mu_q} \lambda_q^{-1} t^{-\mu_q} & 0 \end{bmatrix}, \text{ for } (k - q) \text{ even; } j \leq q \leq k,$$

$$\begin{aligned}\mathbf{N}_{j,k} &= \text{dsum}\{\nu_j, \dots, \nu_q, \dots, \nu_k\}, \\ \mathbf{N}'_{j,k} &= -\text{dsum}\{\nu_k^{-1}, \dots, \nu_q^{-1}, \dots, \nu_j^{-1}\},\end{aligned}\tag{5.2.7}$$

$$\text{where } \nu_q = \begin{bmatrix} 0 & \lambda_q t^{\mu_q} \\ \lambda_q^{-1} t^{\mu_q} & 0 \end{bmatrix}, \text{ for } (k - q) \text{ even; } j \leq q \leq k.$$

5.3 The Weyl group of $B_\ell^{(1)}$

It is advantageous to have some information about the group of root-preserving transformations of B_ℓ . This is because every conjugacy class of involutive automorphisms of $B_\ell^{(1)}$ contains an involutive automorphism that is Cartan-preserving. Furthermore, each Cartan-preserving involutive automorphism induces an involutive rotation (τ) of the roots of B_ℓ . The group \mathcal{R} , which is the group of rotations of the roots of B_ℓ , coincides with \mathcal{W} , the Weyl group of B_ℓ . The conjugacy classes of involutions of the Weyl group of B_ℓ are well-known, and it is required only to take a representative of each of them. In general, the number of such classes rises as the integer ℓ rises. The following list is, however, an exhaustive list of such representatives, which are grouped into a number of "types", each type having a common basic form. The representatives may be grouped into eight such types. In a number of these, parameters are introduced, together with bounds for their values. As the parameters take all of their possible values, so are all of the representatives obtained.

(1) The only rotation of this type is the identity rotation which is, of course, such that

$$\tau(\alpha_k^0) = \alpha_k^0 \quad (1 \leq k \leq \ell). \quad (5.3.1)$$

(2) This type has only the single representative for which

$$\tau(\alpha_k^0) = -\alpha_k^0 \quad (1 \leq k \leq \ell). \quad (5.3.2)$$

(3) In this case the rotation τ is the one for which

$$\begin{aligned}
\tau(\alpha_m^0) &= \alpha_m^0 & (1 \leq m \leq q-2; q \neq 2), \\
\tau(\alpha_{q-1}^0) &= \alpha_{q-1}^0 + 2 \left(\sum_{j=q}^{\ell} \alpha_j^0 \right), \\
\tau(\alpha_n^0) &= -\alpha_n^0 & (q \leq n \leq \ell).
\end{aligned} \tag{5.3.3}$$

In this case, the parameter q takes all values such that $2 \leq q \leq \ell$.

(4) The most general rotation of this type is that given by

$$\begin{aligned}
\tau(\alpha_m^0) &= -\alpha_m^0 & (\text{for } m \text{ odd}; 1 \leq m \leq q), \\
\tau(\alpha_n^0) &= \alpha_{n-1}^0 + \alpha_n^0 + \alpha_{n+1}^0 & (\text{for } n \text{ even}; 1 < n < q; q \neq 1), \\
\tau(\alpha_{q+1}^0) &= \alpha_q^0 + \alpha_{q+1}^0, \\
\tau(\alpha_p^0) &= \alpha_p^0 & (\text{for } q+2 \leq p \leq \ell).
\end{aligned} \tag{5.3.4}$$

The parameter q takes all odd values such that $1 \leq q \leq \ell$.

(5) For this type, the representative rotation is the one for which (equations (5.3.5))

$$\begin{aligned}
\tau(\alpha_m^0) &= \alpha_m^0 & (\text{for } 1 \leq m \leq q-2), \\
\tau(\alpha_{q-1}^0) &= \alpha_{q-1}^0 + \alpha_q^0, \\
\tau(\alpha_n^0) &= -\alpha_n^0 & (\text{for } q \leq n \leq \ell; (\ell - n) \text{ is even}), \\
\tau(\alpha_p^0) &= \alpha_{p-1}^0 + \alpha_p^0 + \alpha_{p+1}^0 & (\text{for } q < p < \ell - 1; (\ell - p) \text{ is odd}), \\
\tau(\alpha_{\ell-1}^0) &= \alpha_{\ell-2}^0 + \alpha_{\ell-1}^0 + 2\alpha_{\ell}^0.
\end{aligned}$$

The parameter q is such that $(\ell - q)$ is even, and $1 < q < \ell$.

(6) In this case, the representative rotation is specified by

$$\begin{aligned}
\tau(\alpha_m^0) &= -\alpha_m^0 & (\text{for } m \text{ odd}; 1 \leq m \leq \ell), \\
\tau(\alpha_n^0) &= \alpha_{n-1}^0 + \alpha_n^0 + \alpha_{n+1}^0 & (\text{for } n \text{ even}; 1 < n < \ell), \\
\tau(\alpha_{\ell-1}^0) &= \alpha_{\ell-2}^0 + \alpha_{\ell-1}^0 + 2\alpha_{\ell}^0.
\end{aligned} \tag{5.3.6}$$

(7) The rotation in this case is the one for which (equations (5.3.7))

$$\begin{aligned}
\tau(\alpha_m^0) &= -\alpha_m^0 && (\text{for } m \text{ odd; } 1 \leq m \leq q-2), \\
\tau(\alpha_n^0) &= \alpha_{n-1}^0 + \alpha_n^0 + \alpha_{n+1}^0 && (\text{for } n \text{ even; } 1 < n < q-2), \\
\tau(\alpha_{q-1}^0) &= \alpha_{q-2}^0 + \alpha_{q-1}^0 + 2\left(\sum_{j=q}^{\ell} \alpha_j^0\right), \\
\tau(\alpha_p^0) &= -\alpha_p^0 && (\text{for } q \leq p \leq \ell).
\end{aligned}$$

The parameter q is odd, and also such that $1 \leq q \leq \ell - 1$.

(8) In this final type of representative automorphism, the general rotation is

$$\begin{aligned}
\tau(\alpha_m^0) &= -\alpha_m^0 && (\text{for } m \text{ odd; } 1 \leq m \leq q), \\
\tau(\alpha_n^0) &= \alpha_{n-1}^0 + \alpha_n^0 + \alpha_{n+1}^0 && (\text{for } n \text{ even; } 1 < n < q), \\
\tau(\alpha_{q+1}^0) &= \alpha_q^0 + \alpha_{q+1}^0, \\
\tau(\alpha_p^0) &= \alpha_p^0 && (\text{for } q+2 \leq p \leq r-2), \\
\tau(\alpha_{r-1}^0) &= \alpha_{r-1}^0 + 2\left(\sum_{j=r}^{\ell} \alpha_j^0\right), \\
\tau(\alpha_s^0) &= -\alpha_s^0 && (\text{for } r \leq s \leq \ell).
\end{aligned} \tag{5.3.8}$$

The parameters q and r take all values such that q is odd, and $r - q \geq 3$.

5.4 The involutive automorphisms of $B_\ell^{(1)}$

The first task of this section is to state which involutive automorphisms are being investigated. For each of the eight rotations, the general form of Laurent polynomial matrix $U(t)$ is sought such that

$$U(t)h_{\alpha_k}^0 U(t)^{-1} = h_{\tau(\alpha_k)}^0 \quad (\text{for } 1 \leq k \leq \ell), \quad (5.4.1)$$

$$\begin{aligned} U(t)U(ut) &= \alpha t^\beta \mathbf{1}_{2\ell+1} \quad (\text{for type 1a automorphisms}) \\ U(t)U(u^{-1}t) &= \alpha t^\beta \mathbf{1}_{2\ell+1} \quad (\text{for type 2a automorphisms}). \end{aligned} \quad (5.4.2)$$

where α is some non-zero complex number, and β is an integer. It is found that, for the type 1a automorphisms with $u = 1$, these matrices are the following (which are listed according to the rotation in question, numbered 1 to 8)

$$\begin{aligned} (1) \quad U(t) &= D_1^0, \\ (2) \quad U(t) &= F_1, \\ (3) \quad U(t) &= \text{dsum}\{E_{q-1}^0, F_q, E_{q-1}'\}, \\ (4) \quad U(t) &= \text{dsum}\{L_{1,q}, D_{q+2}^0, L_{1,q}'\}, \\ (5) \quad U(t) &= \text{dsum}\{E_{q-1}^0, L_{q,\ell-2}, F_\ell, L_{q,\ell-2}', E_{q-1}'\}, \\ (6) \quad U(t) &= \text{dsum}\{L_{1,\ell-2}, F_\ell, L_{1,\ell-2}'\}, \\ (7) \quad U(t) &= \text{dsum}\{L_{1,q-2}, F_q, L_{1,q-2}'\}, \\ (8) \quad U(t) &= \text{dsum}\{L_{1,q}, C_{q+2,r-1}^0, F_r, C_{q+2,r-1}'^0, L_{q,1}'\}. \end{aligned} \quad (5.4.3)$$

The quantities q, r in the equations (5.4.3) are precisely those parameters included in the listing (in section 5.3) of the eight types of representative rotation. Each type 1a involutive automorphism (with $u = 1$) is conjugate to at least one other automorphism $\{U(t), 1, \xi\}$, where $U(t)$ is of one of the general forms (1) to (8) given in (5.4.3).

For the type 1a automorphisms with $u = -1$, the matrices in question are the following:

$$\begin{aligned}
 (1) \quad & \mathbf{U}(t) = \mathbf{D}_1^0, \\
 (2) \quad & \mathbf{U}(t) = \mathbf{F}_1^e, \\
 (3) \quad & \mathbf{U}(t) = \text{dsum}\{\mathbf{E}_{q-1}^0, \mathbf{F}_q^e, \mathbf{E}'_{q-1}\}, \\
 (4) \quad & \mathbf{U}(t) = \text{dsum}\{\mathbf{M}_{1,q}, \mathbf{D}_{q+2}^0, \mathbf{M}'_{1,q}\}, \\
 (5) \quad & \mathbf{U}(t) = \text{dsum}\{\mathbf{E}_{q-1}^0, \mathbf{M}_{q,\ell-2}, \mathbf{F}_\ell^e, \mathbf{M}'_{q,\ell-2}, \mathbf{E}'_{q-1}\}, \\
 (6) \quad & \mathbf{U}(t) = \text{dsum}\{\mathbf{M}_{1,\ell-2}, \mathbf{F}_\ell^e, \mathbf{M}'_{1,\ell-2}\}, \\
 (7) \quad & \mathbf{U}(t) = \text{dsum}\{\mathbf{M}_{1,q-2}, \mathbf{F}_q^e, \mathbf{M}'_{1,q-2}\}, \\
 (8) \quad & \mathbf{U}(t) = \text{dsum}\{\mathbf{M}_{1,q}, \mathbf{C}_{q+2,r-1}^0, \mathbf{F}_r^e, \mathbf{C}'_{q+2,r-1}, \mathbf{M}'_{1,q}\}.
 \end{aligned} \tag{5.4.4}$$

For the type 2a automorphisms with $u = 1$, the matrices in question are the following:

$$\begin{aligned}
 (1) \quad & \mathbf{U}(t) = \mathbf{D}_1, \\
 (2) \quad & \mathbf{U}(t) = \mathbf{F}_1^0, \\
 (3) \quad & \mathbf{U}(t) = \text{dsum}\{\mathbf{E}_{q-1}, \mathbf{F}_q^0, \mathbf{E}'_{q-1}\}, \\
 (4) \quad & \mathbf{U}(t) = \text{dsum}\{\mathbf{N}_{1,q}, \mathbf{D}_{q+2}, \mathbf{N}'_{1,q}\}, \\
 (5) \quad & \mathbf{U}(t) = \text{dsum}\{\mathbf{E}_{q-1}, \mathbf{N}_{q,\ell-2}, \mathbf{F}_\ell^0, \mathbf{N}'_{q,\ell-2}, \mathbf{E}'_{q-1}\}, \\
 (6) \quad & \mathbf{U}(t) = \text{dsum}\{\mathbf{N}_{1,\ell-2}, \mathbf{F}_\ell^0, \mathbf{N}'_{1,\ell-2}\}, \\
 (7) \quad & \mathbf{U}(t) = \text{dsum}\{\mathbf{N}_{1,q-2}, \mathbf{F}_q^0, \mathbf{N}'_{1,q-2}\}, \\
 (8) \quad & \mathbf{U}(t) = \text{dsum}\{\mathbf{N}_{1,q}, \mathbf{C}_{q+2,r-1}, \mathbf{F}_r^0, \mathbf{C}'_{q+2,r-1}, \mathbf{N}'_{1,q}\}.
 \end{aligned} \tag{5.4.5}$$

We recall, of course, that each matrix $\mathbf{U}(t)$ has an arbitrary factor of the form αt^β , where α is a non-zero complex number and β is an integer. This will be noted, and matrices will be taken to be unique only up to this arbitrary factor.

5.5 Matrix transformations and conjugate automorphisms

This section of the chapter will give details of a number of conjugacy relations which are useful in the main body of the work. They demonstrate that certain pairs of automorphisms are conjugate. Therefore, unnecessary repetition of some material is avoided. In this section, \mathbf{H}_j indicates an arbitrary $j \times j$ matrix. In all cases, it will also be assumed that the parameters p, q, r are such that $2(p+q)+r = 2\ell + 1$, where

1. Let the matrix $\mathbf{S}_{j,k}$ be the matrix defined (for $1 \leq j < k \leq \ell$) by

$$(\mathbf{S}_{j,k})_{m,n} = \begin{cases} (i)^{\deg(k-j)} & m = n; m \neq j, k, j', k', \\ 0 & m = n = j, k, j', k', \\ 1 & (m,n) = (j,k), (k,j), (j',k'), (k',j'), \\ 0 & \text{for other values of } m,n. \end{cases} \quad (5.5.1)$$

where the quantities j', k' are defined by $(j', k') = (2\ell + 2 - j, 2\ell + 2 - k)$. The matrix $\mathbf{S}_{j,k}$ as defined satisfies the following:

$$\begin{aligned} \tilde{\mathbf{S}}_{j,k}^* \mathbf{S}_{j,k} &= \mathbf{1}_n, \\ \tilde{\mathbf{S}}_{j,k} \mathbf{g} \mathbf{S}_{j,k} &= \mathbf{g}, \\ \mathbf{S}_{j,k} \mathbf{U}(t) \mathbf{S}_{j,k}^{-1} &= \mathbf{U}'(t), \end{aligned} \quad (5.5.2)$$

where the matrix $\mathbf{U}'(t)$ is obtained from the matrix $\mathbf{U}(t)$ by exchanging the (j) th and (k) th rows (and columns) and also by exchanging the (j') th row (and column) with the (k') th row (and column).

2. Let the matrix $\mathbf{U}(t)$ be of the following form:

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{H}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}'_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_p \end{bmatrix}, \quad (5.5.3)$$

where the quantities p, q, r are such that $2(p+q) + r = 2\ell + 1$. The submatrices \mathbf{B}_q and \mathbf{B}'_q are given by

$$\begin{aligned} \mathbf{B}_q &= \text{offdiag} \left\{ \lambda_{p+1} t^{k_{p+1}}, \dots, \lambda_{p+q} t^{k_{p+q}} \right\}, \\ \mathbf{B}'_q &= \text{offdiag} \left\{ \lambda_{p+q}^{-1} t^{-k_{p+q}}, \dots, \lambda_{p+1}^{-1} t^{-k_{p+1}} \right\}. \end{aligned} \quad (5.5.4)$$

Suppose for the moment that $\{\mathbf{U}(t), 1, \xi\}$ is a type 1a automorphism. Then a matrix $\mathbf{S}(t)$ may be defined by

$$\mathbf{S}(t) = \text{dsum} \left\{ \mathbf{1}_p, \mathbf{x}(t), \mathbf{1}_r, \mathbf{x}'(t), \mathbf{1}_p \right\}, \quad (5.5.5)$$

where the submatrices $\mathbf{x}(t), \mathbf{x}'(t)$ are defined by the following (which is (5.5.6))

$$\left. \begin{aligned} \mathbf{x}(t) &= \text{diag} \left\{ \lambda_{p+1}^{\frac{1}{2}} t^{\mu_{p+1}}, \dots, \lambda_{p+q}^{\frac{1}{2}} t^{\mu_{p+q}} \right\} \\ \mathbf{x}'(t) &= \text{diag} \left\{ \lambda_{p+q}^{-\frac{1}{2}} t^{-\mu_{p+q}}, \dots, \lambda_{p+1}^{-\frac{1}{2}} t^{-\mu_{p+1}} \right\} \end{aligned} \right\} \mu_j = \frac{1}{2} (k_j - \deg k_j).$$

The matrix $\mathbf{S}(t)$ that has just been defined satisfies

$$\begin{aligned} \tilde{\mathbf{S}}(t) \mathbf{g} \mathbf{S}(t) &= \mathbf{1}_{2\ell+1}, \\ \tilde{\mathbf{S}}^*(t^{-1}) \mathbf{S}(t) &= \mathbf{1}_{2\ell+1}. \end{aligned} \quad (5.5.7)$$

Now let another matrix $\mathbf{U}'(t)$ be given by

$$\mathbf{U}'(t) = \begin{bmatrix} \mathbf{H}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{d}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{d}'_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_p \end{bmatrix}, \quad (5.5.8)$$

where $\mathbf{d}_q = \text{offdiag} \{ t^{\text{deg} k_{p+1}}, \dots, t^{\text{deg} k_{p+q}} \}$, and $\mathbf{d}'_q = \text{offdiag} \{ t^{-\text{deg} k_{p+q}}, \dots, t^{-\text{deg} k_{p+1}} \}$. It is then clear that $\mathbf{S}(t)\mathbf{U}'(t)\mathbf{S}(t)^{-1} = \mathbf{U}(t)$.

3. The analysis in this example is similar to that in the previous example.. Let the matrix $\mathbf{U}(t)$ be given by

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{H}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}'_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_p \end{bmatrix}, \quad (5.5.9)$$

where the quantities p, q, r are as in the previous subsection. The submatrices $\mathbf{A}_p, \mathbf{A}'_p$ are arbitrary matrices of dimension $p \times p$, and \mathbf{C}_r is an arbitrary matrix of dimension $r \times r$. The submatrices \mathbf{B}_q and \mathbf{B}'_q are given by equations (5.5.9), which are

$$\left. \begin{aligned} \mathbf{B}_q &= \text{offdiag} \left\{ \lambda_{p+1} t^{k_{p+1}}, \dots, \lambda_{p+q} t^{k_{p+q}} \right\} \\ \mathbf{B}'_q &= \text{offdiag} \left\{ \lambda_{p+q}^{-1} t^{-k_{p+q}}, \dots, \lambda_{p+1}^{-1} t^{-k_{p+1}} \right\} \end{aligned} \right\} k_j \text{ is even ; } p+1 \leq j \leq p+q.$$

Now let the matrix $\mathbf{S}(t)$ be defined as in the previous subsection and the matrix $\mathbf{U}'(t)$ given by

$$\mathbf{U}'(t) = \begin{bmatrix} \mathbf{H}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{d}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{d}'_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_p \end{bmatrix}, \quad (5.5.10)$$

where the submatrices are defined by

$$\begin{aligned} \mathbf{d}_q &= \text{offdiag} \left\{ (-1)^{\frac{1}{2}k_{p+1}}, \dots, (-1)^{\frac{1}{2}k_{p+q}} \right\}, \\ \mathbf{d}'_q &= \text{offdiag} \left\{ (-1)^{\frac{1}{2}k_{p+q}}, \dots, (-1)^{\frac{1}{2}k_{p+1}} \right\}. \end{aligned} \quad (5.5.11)$$

Then, with all of these matrices thus defined, it follows that

$$\mathbf{S}(t)\mathbf{U}'(t)\mathbf{S}(-t)^{-1} = \mathbf{U}(t). \quad (5.5.12)$$

4. This follows closely the two previous examples. Let the matrix $\mathbf{U}(t)$ be defined by

$$\mathbf{U}(t) = \text{dsum} \left\{ \mathbf{H}_{q-1}, \mathbf{F}_\ell^0, \mathbf{H}_{q-1} \right\}, \quad (5.5.13)$$

and let the matrix $\mathbf{U}'(t)$ be defined by

$$\mathbf{U}'(t) = \text{dsum} \left\{ \mathbf{H}_p, \mathbf{K}_{2\ell+1-2p}, \mathbf{H}_p \right\}. \quad (5.5.14)$$

The matrix $\mathbf{S}(t)$ is defined by

$$\mathbf{S}(t) = \text{dsum} \left\{ \mathbf{1}_{q-1}, \text{diag} \left\{ \lambda_q^{\frac{1}{2}}, \dots, \lambda_\ell^{\frac{1}{2}}, 1, \lambda_\ell^{-\frac{1}{2}}, \dots, \lambda_q^{-\frac{1}{2}} \right\}, \mathbf{1}_{q-1} \right\}. \quad (5.5.15)$$

If $\lambda_j \lambda_j^* = 1$ (for $q \leq j \leq \ell$) then $\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{2\ell+1}$. It is also the case that

$$\begin{aligned} \tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) &= \mathbf{g}, \\ \mathbf{S}(t)\mathbf{U}'(t)\mathbf{S}(t^{-1})^{-1} &= \mathbf{U}(t). \end{aligned} \quad (5.5.16)$$

5. In this case, let the matrix $U(t)$ be of the general form

$$U(t) = \text{dsum}\{\mathbf{H}_p, \mathbf{C}_{p+1, p+q}, \mathbf{H}_{r,r}, \mathbf{C}'_{p+1, p+q}, \mathbf{H}_p\}, \quad (5.5.17)$$

and let the matrix $U'(t)$ be given by

$$U'(t) = \text{dsum}\{\mathbf{H}_p, \mathbf{B}_q, \mathbf{H}_r, \mathbf{B}'_q, \mathbf{H}_p\}, \quad (5.5.18)$$

with the submatrices $\mathbf{B}_q, \mathbf{B}'_q$ being obtained from $\mathbf{C}_{p+1, p+q}, \mathbf{C}'_{p+1, p+q}$ respectively by replacing k_j with $\text{deg } k_j$ for $p+1 \leq j \leq p+q$. Then let the matrix $S(t)$ be given by

$$\begin{aligned} S(t) &= \text{dsum}\{\mathbf{1}_p, \mathbf{X}(t), \mathbf{1}_r, \mathbf{X}(t), \mathbf{1}_p\}, \\ \mathbf{X}(t) &= \text{diag}\left\{t^{\frac{1}{2}(k_{p+1} - \text{deg } k_{p+1})}, \dots, t^{\frac{1}{2}(k_{p+q} - \text{deg } k_{p+q})}\right\}, \\ \mathbf{X}'(t) &= \text{diag}\left\{t^{\frac{1}{2}(\text{deg } k_{p+q} - k_{p+q})}, \dots, t^{\frac{1}{2}(\text{deg } k_{p+q} - k_{p+1})}\right\}. \end{aligned} \quad (5.5.19)$$

It is clear that this satisfies $\tilde{S}(t)gS(t) = g$, also $\tilde{S}^*(t^{-1})S(t) = \mathbf{1}_n$, and most importantly

$$S(t)U'(t)S(t^{-1})^{-1} = U(t). \quad (5.5.20)$$

6. Let the matrix $U(t)$ be given by the equation

$$U(t) = \begin{bmatrix} \mathbf{H}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_p \end{bmatrix}, \quad (5.5.21)$$

where $2p + 4 + r = n$. Then, with the definition $\mathbf{x} = \text{diag}\{1, -1\}$, a matrix $U'(t)$ may be defined thus:

$$U'(t) = \text{dsum}\{\mathbf{H}_p, \mathbf{x}, \mathbf{H}_r, -\mathbf{x}, \mathbf{H}_p\}. \quad (5.5.22)$$

A matrix \mathbf{S} (which is independent of t) may be defined by

$$\mathbf{S} = \begin{bmatrix} \mathbf{1}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{t}_1 & \mathbf{0} & \mathbf{t}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{t}_3 & \mathbf{0} & \mathbf{t}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_p \end{bmatrix}, \quad (5.5.23)$$

where the submatrices are all given by the following equations (5.2.24):

$$\mathbf{t}_1 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \quad \mathbf{t}_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \mathbf{t}_3 = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad \mathbf{t}_4 = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

These matrices that have been thus defined together satisfy the following:

$$\begin{aligned} \tilde{\mathbf{S}} \mathbf{g} \mathbf{S} &= \mathbf{g}, \\ \tilde{\mathbf{S}}^* \mathbf{S} &= \mathbf{1}_n, \\ \mathbf{S} \mathbf{U}'(t) \mathbf{S}^{-1} &= \mathbf{U}(t). \end{aligned} \quad (5.5.25)$$

7. Let the matrix $\mathbf{U}(t)$ be given by

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{H}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_r & \mathbf{0} & \mathbf{H}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_r & \mathbf{0} & \mathbf{H}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_p \end{bmatrix}, \quad (5.5.26)$$

where the parameters p, r are such that $2(p+r)+3 = 2\ell+1$. The matrix $\mathbf{U}'(t)$ is then defined to be

$$U'(t) = \begin{bmatrix} \mathbf{H}_p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1}_1 & 0 \\ 0 & 0 & \mathbf{H}_r & 0 & \mathbf{H}_r & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{1}_1 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{H}_r & 0 & \mathbf{H}_r & 0 & 0 \\ 0 & \mathbf{1}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{H}_p \end{bmatrix}, \quad (5.5.27)$$

where the quantities p, r are as defined previously. The matrix S is given by

$$S = \begin{bmatrix} \mathbf{1}_p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{i}{2}\mathbf{1}_1 & 0 & \frac{i}{\sqrt{2}}\mathbf{1}_1 & 0 & \frac{i}{2}\mathbf{1}_1 & 0 \\ 0 & 0 & \mathbf{1}_r & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}}\mathbf{1}_1 & 0 & 0 & 0 & \frac{1}{\sqrt{2}}\mathbf{1}_1 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1}_r & 0 & 0 \\ 0 & -\frac{i}{2}\mathbf{1}_1 & 0 & \frac{1}{\sqrt{2}}\mathbf{1}_1 & 0 & -\frac{i}{2}\mathbf{1}_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1}_p \end{bmatrix}, \quad (5.5.28)$$

where the integer r is even. When the integer r is odd, then

$$S = \begin{bmatrix} \mathbf{1}_p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\mathbf{1}_1 & 0 & \frac{i}{\sqrt{2}}\mathbf{1}_1 & 0 & \frac{1}{2}\mathbf{1}_1 & 0 \\ 0 & 0 & \mathbf{1}_r & 0 & 0 & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}}\mathbf{1}_1 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}}\mathbf{1}_1 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1}_r & 0 & 0 \\ 0 & \frac{1}{2}\mathbf{1}_1 & 0 & -\frac{i}{\sqrt{2}}\mathbf{1}_1 & 0 & \frac{1}{2}\mathbf{1}_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1}_p \end{bmatrix}. \quad (5.5.29)$$

In both cases (r being odd or even), the following all hold:

$$\begin{aligned}
\tilde{\mathbf{S}}\mathbf{g}\mathbf{S} &= \mathbf{g}, \\
\tilde{\mathbf{S}}^*\mathbf{S} &= \mathbf{1}_{2\ell+1}, \\
\mathbf{S}\mathbf{U}'(t)\mathbf{S} &= \mathbf{U}(t).
\end{aligned}
\tag{5.5.30}$$

8. Let the matrix $\mathbf{U}(t)$ be given by

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{H}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & t\mathbf{K}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & t^{-1}\mathbf{K}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_p \end{bmatrix},
\tag{5.5.31}$$

where the quantities p, r are such that $2p + r + 4 = 2\ell + 1$. Also, let the matrix $\mathbf{U}'(t)$ be given by

$$\mathbf{U}'(t) = \text{dsum} \{ \mathbf{H}_p, \mathbf{1}_1, -\mathbf{1}_1, \mathbf{H}_r, -\mathbf{1}_1, \mathbf{1}_1, \mathbf{H}_p \}.
\tag{5.5.32}$$

It follows that, if the matrix $\mathbf{S}(t)$ is defined by

$$\mathbf{S}(t) = \begin{bmatrix} \mathbf{1}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{t}_1 & \mathbf{0} & \mathbf{t}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{t}_3 & \mathbf{0} & \mathbf{t}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_p \end{bmatrix},
\tag{5.5.33}$$

where the submatrices are defined by the following (which are numbered (5.5.34)):

$$\mathbf{t}_1 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \quad \mathbf{t}_2 = \frac{t}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \mathbf{t}_3 = \frac{t^{-1}}{2} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad \mathbf{t}_4 = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix},$$

then the following equations all hold:

$$\begin{aligned}
\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) &= \mathbf{g}, \\
\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) &= \mathbf{1}_{2\ell+1}, \\
\mathbf{S}(t)\mathbf{U}'(t)\mathbf{S}(t)^{-1} &= \mathbf{U}(t).
\end{aligned}
\tag{5.5.35}$$

9. Suppose that the matrix $\mathbf{U}(t)$ is of the form given by

$$\mathbf{U}(t) = \text{dsum}\{\mathbf{H}_{m-1}, \mathbf{L}_{m,n}, \mathbf{H}_{2\ell+1-2n-2}, \mathbf{L}'_{m,n}, \mathbf{H}_{m-1}\}.
\tag{5.5.36}$$

It may first be noted that the values of the quantities $\lambda_j t^{\mu_j}$ (for $m \leq j \leq n$) are arbitrary, and may be conveniently taken to be unity. To see this, let the matrix $\mathbf{S}(t)$ be defined by

$$\mathbf{S}(t) = \text{dsum}\{\mathbf{1}_{m-1}, \mathbf{X}(t), \mathbf{1}_{2\ell-1-2n}, \mathbf{X}'(t), \mathbf{1}_{m-1}\},
\tag{5.5.37}$$

where

$$\begin{aligned}
\mathbf{X}(t) &= \text{diag}\{\lambda_m^{-1}t^{-\mu_m}, 1, \dots, \lambda_j^{-1}t^{-\mu_j}, 1, \dots, \lambda_n^{-1}t^{-\mu_n}, 1\}, \\
\mathbf{X}'(t) &= \text{diag}\{1, \lambda_n t^{\mu_n}, \dots, 1, \lambda_j t^{\mu_j}, \dots, 1, \lambda_m t^{\mu_m}\}.
\end{aligned}
\tag{5.5.38}$$

The matrix $\mathbf{S}(t)$ satisfies

$$\begin{aligned}
\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(t)^{-1} &= \mathbf{U}'(t), \\
\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) &= \mathbf{g},
\end{aligned}
\tag{5.5.39}$$

where the matrix $\mathbf{U}'(t)$ is defined by

$$\begin{aligned}
\mathbf{U}'(t) &= \text{dsum}\{\mathbf{H}_{m-1}, \mathbf{a}, \mathbf{H}_{2\ell-2n-1}, -\mathbf{a}, \mathbf{H}_{m-1}\}, \\
\mathbf{a} &= \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.
\end{aligned}
\tag{5.5.40}$$

10. In a similar fashion to the example 9, let the matrix $\mathbf{U}(t)$ be given by

$$\mathbf{U}(t) = \text{dsum}\{\mathbf{H}_{m-1}, \mathbf{M}_{m,n}, \mathbf{H}_{2\ell+1-2n-2}, \mathbf{M}'_{m,n}, \mathbf{H}_{m-1}\}. \quad (5.5.41)$$

It may be shown, as in the preceding example, that the quantities $\lambda_j t^{\mu_j}$ ($m \leq j \leq n$) are arbitrary, and may be assumed therefore, to have the value 1. Let the matrix $\mathbf{S}(t)$ be defined (as in the previous example) by

$$\mathbf{S}(t) = \text{dsum}\{\mathbf{1}_{m-1}, \mathbf{X}(t), \mathbf{1}_{2\ell-1-2n}, \mathbf{X}'(t), \mathbf{1}_{m-1}\}, \quad (5.5.42)$$

where

$$\begin{aligned} \mathbf{X}(t) &= \text{diag}\{\lambda_m^{-1} t^{-\mu_m}, 1, \dots, \lambda_j^{-1} t^{-\mu_j}, 1, \dots, \lambda_n^{-1} t^{-\mu_n}, 1\}, \\ \mathbf{X}'(t) &= \text{diag}\{1, \lambda_n t^{\mu_n}, \dots, 1, \lambda_j t^{\mu_j}, \dots, 1, \lambda_m t^{\mu_m}\}. \end{aligned} \quad (5.5.43)$$

It is clear from the previous example that the matrix $\mathbf{S}(t)$ does, in fact, generate an automorphism of the compact real form of $\mathbf{B}_\ell^{(1)}$. In addition, the matrix also satisfies

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(-t)^{-1} = \text{dsum}\{\mathbf{H}_{m-1}, \mathbf{a}, \mathbf{H}_{2\ell-1-2n}, -\mathbf{a}, \mathbf{H}_{m-1}\}, \quad (5.5.44)$$

where the matrix \mathbf{a} is as defined in (5.5.40).

11. This follows closely on from the two preceding examples. In this case, however, the matrix $\mathbf{U}(t)$ is given by

$$\mathbf{U}(t) = \text{dsum}\{\mathbf{H}_{m-1}, \mathbf{N}_{m,n}, \mathbf{H}_{2\ell+1-2n-2}, \mathbf{N}'_{m,n}, \mathbf{H}_{m-1}\}. \quad (5.5.45)$$

The matrix $\mathbf{S}(t)$ retains precisely the form it had in the two preceding examples.

Therefore,

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(t^{-1})^{-1} = \text{dsum}\{\mathbf{H}_{m-1}, \mathbf{a}, \mathbf{H}_{2\ell-1-2n}, -\mathbf{a}, \mathbf{H}_{m-1}\}. \quad (5.5.46)$$

12. Let the matrices $\mathbf{U}(t)$ and $\mathbf{U}'(t)$ be given by

$$\begin{aligned} \mathbf{U}(t) &= \text{dsum}\{\mathbf{H}_{m-1}, \mathbf{W}_{m,n+1}, \mathbf{H}_{2\ell-1-2n}, -\mathbf{W}_{m,n+1}, \mathbf{H}_{m-1}\}, \\ \mathbf{U}'(t) &= \text{dsum}\{\mathbf{H}_{m-1}, \mathbf{a}, \mathbf{H}_{2\ell-1-2n}, -\mathbf{a}, \mathbf{H}_{m-1}\}, \end{aligned} \quad (5.5.47)$$

where in (5.5.47), the matrix $\mathbf{W}_{m,n+1}$ is an $(n+2-m) \times (n+2-m)$ diagonal matrix whose non-zero entries are alternately $1, -1, \dots$. The matrix \mathbf{a} is as defined in (5.5.40). A matrix $\mathbf{V}(t)$ may then be defined by

$$\mathbf{V}(t) = \text{dsum}\{\mathbf{1}_{m-1}, \mathbf{x}, \mathbf{1}_{2\ell-1-2n}, \mathbf{x}, \mathbf{1}_{m-1}\}, \quad (5.5.48)$$

where the submatrix is defined by

$$\mathbf{x} = \frac{i}{\sqrt{2}} \begin{bmatrix} 1 & 1 & \cdots & 0 & 0 \\ 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 1 & -1 \end{bmatrix} \quad (5.5.49)$$

This matrix satisfies

$$\begin{aligned} \tilde{\mathbf{V}}(t)\mathbf{g}\mathbf{V}(t) &= \mathbf{g}, \\ \tilde{\mathbf{V}}^*(t^{-1})\mathbf{V}(t) &= \mathbf{1}_{2\ell+1}, \\ \mathbf{V}(t)\mathbf{U}'(t)\mathbf{V}(t)^{-1} &= \mathbf{U}(t), \\ \mathbf{V}(t)\mathbf{U}'(t)\mathbf{V}(-t)^{-1} &= \mathbf{U}(t), \\ \mathbf{V}(t)\mathbf{U}'(t)\mathbf{V}(t^{-1})^{-1} &= \mathbf{U}(t). \end{aligned} \quad (5.5.50)$$

13. Let s be some arbitrary non-zero complex number, and consider the mapping $\mathbf{U}(t) \mapsto \mathbf{U}(st)$. It should be noted that, for the matrices $\mathbf{U}(t)$ which will be examined in this chapter, such a mapping is tantamount to keeping $\mathbf{U}(t)$ the same, except for changing the values of the coefficients λ_j . The previous examples may then be used, and it can be inferred from these that, if we are examining whether or not certain type 1a involutive automorphisms are conjugate via an automorphism corresponding to the triple $\{\mathbf{S}(t), s, \xi\}$, that it may be assumed that $s = 1$ (without loss of generality).

5.6 Type 1a involutive automorphisms of $B_\ell^{(1)}$ with $u = 1$

The information contained in the previous section implies that each type 1a involutive automorphism (with $u = 1$) is conjugate to at least one involution $\{\mathbf{U}(t), 1, \xi\}$, where $\mathbf{U}(t)$ is of one of the following two forms:

$$\mathbf{U}(t) = \mathbf{U}_A(t) = \text{dsum}\{\mathbf{1}_{N_+}, -\mathbf{1}_{N_-}, \mathbf{1}_1, -\mathbf{1}_{N_-}, \mathbf{1}_{N_+}\}, \quad (5.6.1)$$

$$\mathbf{U}(t) = \mathbf{U}_B(t) = \text{dsum}\{\mathbf{1}_p, -\mathbf{1}_{\ell-p-1}, \text{offdiag}\{t, 1, t^{-1}\}, -\mathbf{1}_{\ell-p-1}, \mathbf{1}_p\}. \quad (5.6.2)$$

In the above, N_+ and N_- are non-negative integer parameters such that $N_+ + N_- = \ell$, and p is some non-negative integer parameter which is less than ℓ . Consider firstly the case where $\mathbf{U}(t)$ is of the general form $\mathbf{U}_A(t)$ given in (5.6.1). There are only $(\ell + 1)$ distinct matrices of this form. Denote by \mathbf{U}_j° the matrix of this form for which $N_- = j$, and suppose that there exists some Laurent matrix $\mathbf{S}(t)$ such that

$$\mathbf{S}(t)\mathbf{U}_j^\circ\mathbf{S}(t)^{-1} = \lambda t^\mu \mathbf{U}_k^\circ \quad (\text{for } j \neq k), \quad (5.6.3)$$

where λ is a non-zero complex number and μ is some integer. It is easily verified that there does not exist any non-singular matrix $\mathbf{S}(t)$ such that this holds true. The type 1a automorphisms (with $u = 1$) that are generated by the matrices \mathbf{U}_j° (for $1 \leq j \leq \ell$) fall into $(\ell + 1)$ disjoint classes. These classes will be called $(A)^{(b)}$ (for $1 \leq b \leq \ell$), with the class $(A)^{(b)}$ defined to be that which contains the type 1a automorphism $\{\mathbf{U}_b^\circ, 1, 0\}$.

Now let $\mathbf{U}(t)$ be of the second form, namely the form $\mathbf{U}_B(t)$. There are only ℓ distinct matrices of this form, and hence only ℓ involutive automorphisms generated by them. Denote by $\mathbf{U}_m(t)$ the matrix given by (5.6.2) with $p = m$. It will firstly be demonstrated that the involutive automorphisms generated by these matrices

belong to ℓ non-conjugate conjugacy classes (one class corresponding to each distinct value of p). If this were not the case, then there would exist a Laurent polynomial matrix $\mathbf{S}(t)$, a non-zero complex number λ and an integer μ such that

$$\mathbf{S}(t)\mathbf{U}_m(t)\mathbf{S}(t)^{-1} = \lambda t^\mu \mathbf{U}_n(t). \quad (5.6.4)$$

A brief inspection of the special case $t = 1$ reveals that, for $m \neq n$, this is not possible. It may still be the case that some of these automorphisms may belong to a conjugacy class $(\mathbf{A})^{(b)}$ for a suitable value of b . In fact, this is not the case, and this will now be shown. Proof of this requires use of the lemma given in appendix B. It should also be noted that there does not exist any Laurent matrix $\mathbf{M}(t)$ which satisfies

$$\mathbf{M}(t)\mathbf{U}_j^\circ \mathbf{M}(t) = \lambda t^\mu \mathbf{U}_k^\circ \quad (1 \leq j < k \leq \ell). \quad (5.6.5)$$

Now, the results of the previous section imply that there exists some t -independent matrix \mathbf{N} which satisfies

$$\mathbf{N}\mathbf{U}_m(1)\mathbf{N}^{-1} = \alpha \mathbf{U}_{\ell-m}^\circ \quad (\alpha \neq 0). \quad (5.6.6)$$

There are two possibilities to consider. Firstly, that the automorphism $\{\mathbf{U}_m(t), 1, \xi\}$ belongs to the conjugacy class $(\mathbf{A})^{(m)}$, or secondly, that it belongs to some class non-conjugate to all of those that have already been identified. It is the second of these which is the one which actually holds. Let the two matrices $\mathbf{U}_C(t)$ and $\mathbf{U}_D(t)$ be defined by

$$\begin{aligned} \mathbf{U}_C(t) &= \text{dsum} \left\{ \mathbf{1}_{\ell-m-1}, -\mathbf{1}_m, \text{offdiag} \left\{ t, 1, t^{-1} \right\}, -\mathbf{1}_m, \mathbf{1}_{\ell-m-1} \right\} \\ \mathbf{U}_D(t) &= \text{dsum} \left\{ -\mathbf{1}_{\ell-m-1}, \mathbf{1}_m, \text{diag} \left\{ -1, 1, -1 \right\}, \mathbf{1}_m, -\mathbf{1}_{\ell-m-1} \right\}. \end{aligned} \quad (5.6.7)$$

Assume, by way of obtaining a contradiction, that

$$\begin{aligned} \mathbf{S}(t)\mathbf{U}_D\mathbf{S}(t)^{-1} &= \lambda t^\mu \mathbf{U}_C(t) \\ \tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) &= \alpha t^\beta \mathbf{g}. \end{aligned} \quad (5.6.8)$$

Some information may be found about the precise form of λt^μ . Examining determinants of both sides of the first of the equations (5.6.8) implies that

$$\lambda^{2\ell+1} = -1 \quad \mu = 0. \quad (5.6.9)$$

If this equation is then re-arranged, then it becomes

$$\mathbf{S}(t)\mathbf{U}_2 = \lambda\mathbf{U}_1(t)\mathbf{S}(t). \quad (5.6.10)$$

If attempts are made to find the general form of $\mathbf{S}(t)$ by solving this equation, we find that a necessary condition is that $\lambda^2 = 1$. It then follows that $\lambda = -1$, since $(1)^{2\ell+1} = -1$. Now there exists a matrix $\mathbf{R}(t)$ which satisfies

$$\mathbf{R}(t)\mathbf{U}_2(t)\mathbf{R}(t)^{-1} = -\mathbf{U}_1(t), \quad (5.6.11)$$

However, the matrix $\mathbf{R}(t)$ does not satisfy $\tilde{\mathbf{R}}(t)\mathbf{g}\mathbf{R}(t) = f(t)\mathbf{g}$. Instead, it satisfies

$$\tilde{\mathbf{R}}(t)\mathbf{g}\mathbf{R}(t) = \text{dsum} \{ \mathbf{1}_{\ell-1}, t^{-1}\mathbf{1}_2, \mathbf{1}_\ell \}. \quad (5.6.12)$$

A suitable form for the matrix $\mathbf{R}(t)$ is

$$\mathbf{R}(t) = \begin{bmatrix} \mathbf{t}_1 & \mathbf{0} & \mathbf{t}_2 \\ \mathbf{0} & \mathbf{v}(t) & \mathbf{0} \\ \mathbf{t}_3 & \mathbf{0} & \mathbf{t}_4 \end{bmatrix}, \quad (5.6.13)$$

where the submatrices \mathbf{t}_j ($1 \leq j \leq 4$) are defined by

$$\begin{aligned} \mathbf{t}_1 &= \frac{1}{\sqrt{2}} \text{diag} \{ 1, i, 1, i, \dots \}, \\ \mathbf{t}_2 &= \frac{1}{\sqrt{2}} \text{offdiag} \{ i, -1, i, -1, \dots \}, \\ \mathbf{t}_3 &= \frac{1}{\sqrt{2}} \text{offdiag} \{ \dots, i, 1, i, 1 \}, \\ \mathbf{t}_4 &= \frac{1}{\sqrt{2}} \text{diag} \{ \dots, 1, -i, 1, -i \}, \end{aligned} \quad (5.6.14)$$

and the submatrix $\mathbf{v}(t)$ is defined by

$$\mathbf{v}(t) = \frac{\alpha}{\sqrt{2}} \begin{bmatrix} i & -1 & 0 \\ 0 & 0 & -1 \\ it^{-1} & t^{-1} & 0 \end{bmatrix}, \quad (5.6.15)$$

where $\alpha = 1$ when ℓ is even, and is equal to i when ℓ is odd. Hence, the conditions of the lemma are satisfied, and the lemma implies that

$$\mathbf{S}(t) = \mathbf{R}(t)\mathbf{Q}(t). \quad (5.6.16)$$

The matrix $\mathbf{Q}(t)$ is necessarily a Laurent polynomial matrix. Furthermore, it must also satisfy $\mathbf{Q}(t)\mathbf{U}_2\mathbf{Q}(t)^{-1} = \eta\mathbf{U}_2$. This condition may only be satisfied when $\eta = 1$, and when this is the case the matrix $\mathbf{Q}(t)$ has the form given below. (The submatrices \mathbf{G}_k^j and \mathbf{H}_k^j are taken to represent arbitrary matrices with j rows and k columns).

$$\mathbf{Q}(t) = \begin{bmatrix} \mathbf{H}_{\ell-m-1}^{\ell-m-1} & \mathbf{0} & \mathbf{H}_1^{\ell-m-1} & \mathbf{0} & \mathbf{H}_1^{\ell-m-1} & \mathbf{0} & \mathbf{H}_{\ell-m-1}^{\ell-m-1} \\ \mathbf{0} & \mathbf{G}_m^m & \mathbf{0} & \mathbf{G}_1^m & \mathbf{0} & \mathbf{G}_m^m & \mathbf{0} \\ \mathbf{H}_{\ell-m-1}^1 & \mathbf{0} & \mathbf{H}_1^1 & \mathbf{0} & \mathbf{H}_1^1 & \mathbf{0} & \mathbf{H}_{\ell-m-1}^1 \\ \mathbf{0} & \mathbf{G}_m^1 & \mathbf{0} & \mathbf{G}_1^1 & \mathbf{0} & \mathbf{G}_m^1 & \mathbf{0} \\ \mathbf{H}_{\ell-m-1}^1 & \mathbf{0} & \mathbf{H}_1^1 & \mathbf{0} & \mathbf{H}_1^1 & \mathbf{0} & \mathbf{H}_{\ell-m-1}^1 \\ \mathbf{0} & \mathbf{G}_m^m & \mathbf{0} & \mathbf{G}_1^m & \mathbf{0} & \mathbf{G}_m^m & \mathbf{0} \\ \mathbf{H}_{\ell-m-1}^{\ell-m-1} & \mathbf{0} & \mathbf{H}_1^{\ell-m-1} & \mathbf{0} & \mathbf{H}_1^{\ell-m-1} & \mathbf{0} & \mathbf{H}_{\ell-m-1}^{\ell-m-1} \end{bmatrix}, \quad (5.6.17)$$

Now $\mathbf{Q}(t)$ is obviously decomposable, and with a suitable re-ordering of its index set could be written in the form

$$\mathbf{Q}'(t) = \begin{bmatrix} \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{bmatrix}, \quad (5.6.18)$$

with \mathbf{H} and \mathbf{G} being given by

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{\ell-m-1}^{\ell-m-1} & \mathbf{H}_1^{\ell-m-1} & \mathbf{H}_1^{\ell-m-1} & \mathbf{H}_{\ell-m-1}^{\ell-m-1} \\ \mathbf{H}_{\ell-m-1}^1 & \mathbf{H}_1^1 & \mathbf{H}_1^1 & \mathbf{H}_{\ell-m-1}^1 \\ \mathbf{H}_{\ell-m-1}^1 & \mathbf{H}_1^1 & \mathbf{H}_1^1 & \mathbf{H}_{\ell-m-1}^1 \\ \mathbf{H}_{\ell-m-1}^{\ell-m-1} & \mathbf{H}_1^{\ell-m-1} & \mathbf{H}_1^{\ell-m-1} & \mathbf{H}_{\ell-m-1}^{\ell-m-1} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{G}_m^m & \mathbf{G}_1^m & \mathbf{G}_m^m \\ \mathbf{G}_m^1 & \mathbf{G}_1^1 & \mathbf{G}_m^1 \\ \mathbf{G}_m^1 & \mathbf{G}_1^1 & \mathbf{G}_m^1 \end{bmatrix}, \quad (5.6.19)$$

It follows that, since $\mathbf{Q}(t)$ is a Laurent polynomial matrix, its submatrices \mathbf{G} and \mathbf{H} are themselves Laurent polynomial matrices. Furthermore, by hypothesis

$$\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = \alpha t^\beta \mathbf{g}, \quad (5.6.20)$$

which in turn means that

$$\tilde{\mathbf{Q}}(t)\tilde{\mathbf{R}}(t)\mathbf{g}\mathbf{R}(t)\mathbf{Q}(t) = \alpha t^\beta \mathbf{g}. \quad (5.6.21)$$

Consideration of determinants of the expressions on both sides of (5.6.21) gives the necessary condition

$$|\mathbf{Q}(t)|^2 |\mathbf{R}(t)|^2 = (\alpha t^\beta)^{2\ell+1}, \quad (5.6.22)$$

which implies that the integer β is even. Further substitution gives

$$\tilde{\mathbf{Q}}(t) \text{dsum} \{ \mathbf{1}_{\ell-1}, t^{-1} \mathbf{1}_2, \mathbf{1}_\ell \} \mathbf{Q}(t) = \alpha t^\beta \mathbf{g}''. \quad (5.6.23)$$

The matrix \mathbf{g}'' may be written in the following form

$$\mathbf{g}'' = \text{offdiag} \{ \gamma_1, \dots, \gamma_\ell, 1, \gamma_\ell, \dots, \gamma_1 \}, \quad (5.6.24)$$

where $\gamma_j^2 = 1$ for $1 \leq j \leq \ell$, and the matrix \mathbf{g}' may be expressed in terms of the non-zero elements of \mathbf{g}'' by

$$\mathbf{g}' = \text{offdiag} \{ \gamma_1, \dots, \gamma_{\ell-m-1}, \gamma_\ell, \gamma_\ell, \gamma_{\ell-m-1}, \dots, \gamma_1 \}. \quad (5.6.25)$$

Recalling the definition of the matrix \mathbf{H} given above, it follows that

$$\tilde{\mathbf{H}} \text{dsum} \{ \mathbf{1}_{\ell-m-1}, t^{-1} \mathbf{1}_1, \mathbf{1}_{\ell-m} \} \mathbf{H} = \alpha t^\beta \mathbf{g}'. \quad (5.6.26)$$

A contradiction has now been encountered, since the determinant of the left-hand side of (5.6.27) is a multiple of an odd power of t , whilst the determinant of the right-hand side is a multiple of an even power of t . Thus, in addition to the conjugacy classes that have already been identified, there are ℓ more conjugacy classes of type 1a involutive automorphisms with $u = 1$. These classes will be called $(\mathbf{B})^{(j)}$ (for

$0 \leq j \leq \ell - 1$). The conjugacy class $(B)^{(j)}$ (for a particular value of j) will be defined to be that conjugacy class which contains the type 1a automorphism $\{U_j(t), 1, \xi\}$.

where

$$U_j(t) = \text{dsum} \left\{ \mathbf{1}_{\ell-j-1}, -\mathbf{1}_j, \text{offdiag} \left\{ t, 1, t^{-1} \right\}, -\mathbf{1}_j, \mathbf{1}_{\ell-j-1} \right\}. \quad (5.6.27)$$

Let the automorphism $\psi_{A^{(b)}}$ be the type 1a involutive automorphism given by $\left\{ \text{dsum} \left\{ \mathbf{1}_{\ell-b}, -\mathbf{1}_b, \mathbf{1}_1, -\mathbf{1}_b, \mathbf{1}_{\ell-b} \right\}, 1, 0 \right\}$. Clearly, $\psi_{A^{(b)}}$ belongs to the conjugacy class $(A)^{(b)}$, and may be taken as the representative automorphism of the class. The class $(A)^{(0)}$ contains only the identity automorphism, and so it will be assumed that $b \neq 0$. A basis of the compact real form of $B_\ell^{(1)}$ which consists entirely of eigenvectors of $\psi_{A^{(b)}}$ (together with the respective eigenvalues) is given below (unless specified otherwise, j is allowed to take any whole value)

$$\left. \begin{array}{l} \left(e_{j\delta}^k + e_{-j\delta}^k \right) \quad j \in \mathbf{N}^0 \\ i \left(e_{j\delta}^k - e_{-j\delta}^k \right) \quad j \in \mathbf{N} \end{array} \right\} 1 \leq k \leq \ell; \text{ eigenvalue } 1, \quad (5.6.28)$$

$$\left. \begin{array}{l} \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} \right) \\ i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} \right) \end{array} \right\} j \in \mathbf{Z}; \text{ eigenvalue } 1, \quad (5.6.29)$$

where, in the above

$$\alpha = \left[\begin{array}{l} \sum_{p=r}^{\ell} \alpha_p \quad (1 \leq r \leq \ell - b) \\ \sum_{p=r}^{s-1} \alpha_p \quad \left\{ \begin{array}{l} (1 \leq r < s \leq \ell - b) \\ (\ell - b < r < s \leq \ell) \end{array} \right. \\ \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell} \alpha_p \quad \left\{ \begin{array}{l} (1 \leq r < s \leq \ell - b) \\ (\ell - b < r < s \leq \ell) \end{array} \right. \end{array} \right], \quad (5.6.30)$$

$$\left. \begin{array}{l} \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} \right) \\ i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} \right) \end{array} \right\} j \in \mathbf{Z}; \text{ eigenvalue } -1, \quad (5.6.31)$$

where, in the above

$$\alpha = \left\{ \begin{array}{ll} \sum_{p=r}^{\ell} \alpha_p & (\ell - b \leq r \leq \ell) \\ \sum_{p=r}^{s-1} \alpha_p & (1 \leq r \leq \ell - b < s \leq \ell) \\ \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell} \alpha_p & (1 \leq r \leq \ell - b < s \leq \ell) \end{array} \right\}, \quad (5.6.32)$$

$$\left. \begin{array}{l} ic \\ id \end{array} \right\} \text{eigenvalue 1.} \quad (5.6.33)$$

It follows, therefore, that a basis of a real form of $B_{\ell}^{(1)}$ is given by

$$\left. \begin{array}{ll} (e_{j\delta}^k + e_{-j\delta}^k) & j \in \mathbf{N}^0 \\ i(e_{j\delta}^k - e_{-j\delta}^k) & j \in \mathbf{N} \end{array} \right\} 1 \leq k \leq \ell, \quad (5.6.34)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} j \in \mathbf{Z}, \quad (5.6.35)$$

where, in the above

$$\alpha = \left\{ \begin{array}{ll} \sum_{p=r}^{\ell} \alpha_p & (1 \leq r \leq \ell - b) \\ \sum_{p=r}^{s-1} \alpha_p & \left\{ \begin{array}{l} (1 \leq r < s \leq \ell - b) \\ (\ell - b < r < s \leq \ell) \end{array} \right\} \\ \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell} \alpha_p & \left\{ \begin{array}{l} (1 \leq r < s \leq \ell - b) \\ (\ell - b < r < s \leq \ell) \end{array} \right\} \end{array} \right\}, \quad (5.6.36)$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} j \in \mathbf{Z}, \quad (5.6.37)$$

where, in the above

$$\alpha = \left. \begin{array}{l} \sum_{p=r}^{\ell} \alpha_p \quad (\ell - b \leq r \leq \ell) \\ \sum_{p=r}^{s-1} \alpha_p \quad (1 \leq r \leq \ell - b < s \leq \ell) \\ \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell} \alpha_p \quad (1 \leq r \leq \ell - b < s \leq \ell) \end{array} \right\}, \quad (5.6.38)$$

$$\begin{array}{l} ic, \\ id. \end{array} \quad (5.6.39)$$

The representative automorphism of the conjugacy class $(B)^{(a)}$ is the type Ia automorphism $\psi_{B^{(a)}}$, which is specified by $\psi_{B^{(a)}} = \{U_{B^{(a)}}, 1, 0\}$, and $U_{B^{(a)}} = \text{dsum} \{ \mathbf{1}_{\ell-1-a}, -\mathbf{1}_a, \text{offdiag} \{t, 1, t^{-1}\}, -\mathbf{1}_a, \mathbf{1}_{\ell-1-a} \}$. There exists a basis of the compact real form of $B_\ell^{(1)}$, of which every basis element is an eigenvector of the representative automorphism $\psi_{B^{(a)}}$ (with eigenvalue 1 or -1). Such a basis is listed below:

$$\left. \begin{array}{l} ih_{\alpha_k} \quad (1 \leq k \leq \ell - 2) \\ i(h_{\alpha_{\ell-1}} + h_{\alpha_\ell}) \end{array} \right\} \text{eigenvalue 1,} \quad (5.6.40)$$

$$i(h_{\alpha_\ell} + \frac{1}{2}c) \quad \text{eigenvalue -1,} \quad (5.6.41)$$

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \\ i(e_{j\delta}^k - e_{-j\delta}^k) \end{array} \right\} j \in \mathbf{N}; 1 \leq k \leq \ell - 2; \text{eigenvalue 1,} \quad (5.6.42)$$

$$\left. \begin{array}{l} (e_{j\delta}^{\ell-1} + e_{-j\delta}^{\ell-1} + e_{j\delta}^\ell + e_{-j\delta}^\ell) \\ i(e_{j\delta}^{\ell-1} - e_{-j\delta}^{\ell-1} + e_{j\delta}^\ell - e_{-j\delta}^\ell) \end{array} \right\} j \in \mathbf{N}; \text{eigenvalue 1,} \quad (5.6.43)$$

$$\left. \begin{array}{l} (e_{j\delta}^\ell + e_{-j\delta}^\ell) \\ i(e_{j\delta}^\ell - e_{-j\delta}^\ell) \end{array} \right\} j \in \mathbf{N}; \text{eigenvalue -1,} \quad (5.6.44)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} j \in \mathbf{Z}; \text{eigenvalue 1,} \quad (5.6.45)$$

where, in the above

$$\alpha = \left[\begin{array}{l} \sum_{p=r}^{\ell} \alpha_p \quad (1 \leq r \leq \ell-1-a) \\ \sum_{p=r}^{s-1} \alpha_p \quad \left\{ \begin{array}{l} (1 \leq r < s \leq \ell-1-a) \\ ((\ell-1-a < r < s \leq \ell-1)) \end{array} \right. \\ \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell} \alpha_p \quad \left\{ \begin{array}{l} (1 \leq r < s \leq \ell-1-a) \\ ((\ell-1-a < r < s \leq \ell-1)) \end{array} \right. \end{array} \right], \quad (5.6.46)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} j \in \mathbf{Z}; \text{ eigenvalue } -1, \quad (5.6.47)$$

where, in the above

$$\alpha = \left[\begin{array}{l} \sum_{p=r}^{\ell} \alpha_p \quad (\ell-1-a < r \leq \ell-1) \\ \sum_{p=r}^{s-1} \alpha_p \quad (1 \leq r \leq \ell-1-a < s \leq \ell-1) \\ \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell} \alpha_p \quad (1 \leq r \leq \ell-1-a < s \leq \ell-1) \end{array} \right], \quad (5.6.48)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha_t} + e_{-j\delta-\alpha_t} + e_{-(j-1)\delta+\alpha_t} + e_{(j-1)\delta-\alpha_t}) \text{ eigenvalue } -1, \\ (e_{j\delta+\alpha_t} + e_{-j\delta-\alpha_t} - e_{-(j-1)\delta+\alpha_t} - e_{(j-1)\delta-\alpha_t}) \text{ eigenvalue } 1, \\ i(e_{j\delta+\alpha_t} - e_{-j\delta-\alpha_t} + e_{-(j-1)\delta+\alpha_t} - e_{(j-1)\delta-\alpha_t}) \text{ eigenvalue } 1, \\ i(e_{j\delta+\alpha_t} - e_{-j\delta-\alpha_t} - e_{-(j-1)\delta+\alpha_t} + e_{(j-1)\delta-\alpha_t}) \text{ eigenvalue } -1, \end{array} \right\} \quad (5.6.49)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(j+1)\delta+\beta} + e_{-(j+1)\delta-\beta}) \text{ eigenvalue } 1, \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{(j+1)\delta+\beta} - e_{-(j+1)\delta-\beta}) \text{ eigenvalue } -1, \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{(j+1)\delta+\beta} - e_{-(j+1)\delta-\beta}) \text{ eigenvalue } 1, \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{(j+1)\delta+\beta} + e_{-(j+1)\delta-\beta}) \text{ eigenvalue } -1, \end{array} \right\} \quad (5.6.50)$$

where, in the above

$$\alpha = \sum_{p=r}^{\ell-1} \alpha_p \quad \beta = \sum_{p=r}^{\ell-1} \alpha_p + 2\alpha_{\ell} \quad (1 \leq r \leq \ell-1-a), \quad (5.6.51)$$

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(j+1)\delta+\beta} + e_{-(j+1)-\beta} \right) \text{ eigenvalue } -1, \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{(j+1)\delta+\beta} - e_{-(j+1)-\beta} \right) \text{ eigenvalue } 1, \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{(j+1)\delta+\beta} - e_{-(j+1)-\beta} \right) \text{ eigenvalue } -1, \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{(j+1)\delta+\beta} + e_{-(j+1)-\beta} \right) \text{ eigenvalue } 1,
\end{aligned} \tag{5.6.52}$$

where, in the above

$$\alpha = \sum_{p=r}^{\ell-1} \alpha_p \quad \beta = \sum_{p=r}^{\ell-1} \alpha_p + 2\alpha_\ell \quad (\ell-1-a \leq r \leq \ell-1), \tag{5.6.53}$$

$$\left. \begin{aligned} & ic \\ & i(d - (2\ell-1)h_{\alpha_\ell}) \end{aligned} \right\} \text{ eigenvalue } 1. \tag{5.6.54}$$

The basis of a real form of $B_\ell^{(1)}$ which corresponds to the automorphism $\psi_{B^{(a)}}$ (being generated from the above eigenvalues of it, according to Cartan's method) is given by

$$\begin{aligned} & ih_{\alpha_k} \quad (1 \leq k \leq \ell-2), \\ & i(h_{\alpha_{\ell-1}} + h_{\alpha_\ell}), \end{aligned} \tag{5.6.55}$$

$$(h_{\alpha_\ell} + \frac{1}{2}c), \tag{5.6.56}$$

$$\left. \begin{aligned} & (e_{j\delta}^k + e_{-j\delta}^k) \\ & i(e_{j\delta}^k - e_{-j\delta}^k) \end{aligned} \right\} j \in \mathbb{N}; 1 \leq k \leq \ell-2, \tag{5.6.57}$$

$$\left. \begin{aligned} & (e_{j\delta}^{\ell-1} + e_{-j\delta}^{\ell-1} + e_{j\delta}^\ell + e_{-j\delta}^\ell) \\ & i(e_{j\delta}^{\ell-1} - e_{-j\delta}^{\ell-1} + e_{j\delta}^\ell - e_{-j\delta}^\ell) \end{aligned} \right\} j \in \mathbb{N}, \tag{5.6.58}$$

$$\left. \begin{aligned} & i(e_{j\delta}^\ell + e_{-j\delta}^\ell) \\ & (e_{j\delta}^\ell - e_{-j\delta}^\ell) \end{aligned} \right\} j \in \mathbb{N}, \tag{5.6.59}$$

$$\begin{aligned} & (e_{j\delta+\alpha} + e_{-j\delta-\alpha}), \\ & i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}), \end{aligned} \tag{5.6.60}$$

where, in the above

$$\alpha = \left[\begin{array}{l} \sum_{p=r}^{\ell} \alpha_p \quad (1 \leq r \leq \ell-1-a) \\ \sum_{p=r}^{s-1} \alpha_p \quad \left\{ \begin{array}{l} (1 \leq r < s \leq \ell-1-a) \\ (\ell-1-a < r < s \leq \ell-1) \end{array} \right. \\ \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell} \alpha_p \quad \left\{ \begin{array}{l} (1 \leq r < s \leq \ell-1-a) \\ (\ell-1-a < r < s \leq \ell-1) \end{array} \right. \end{array} \right], \quad (5.6.61)$$

$$\begin{aligned} & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}), \\ & (e_{j\delta+\alpha} - e_{-j\delta-\alpha}), \end{aligned} \quad (5.6.62)$$

where, in the above

$$\alpha = \left[\begin{array}{l} \sum_{p=r}^{\ell} \alpha_p \quad (\ell-1-a < r \leq \ell-1) \\ \sum_{p=r}^{s-1} \alpha_p \quad (1 \leq r \leq \ell-1-a < s \leq \ell-1) \\ \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell} \alpha_p \quad (1 \leq r \leq \ell-1-a < s \leq \ell-1) \end{array} \right], \quad (5.6.63)$$

$$\begin{aligned} & i(e_{j\delta+\alpha_\ell} + e_{-j\delta-\alpha_\ell} + e_{-(j-1)\delta+\alpha_\ell} + e_{(j-1)\delta-\alpha_\ell}), \\ & (e_{j\delta+\alpha_\ell} + e_{-j\delta-\alpha_\ell} - e_{-(j-1)\delta+\alpha_\ell} - e_{(j-1)\delta-\alpha_\ell}), \\ & i(e_{j\delta+\alpha_\ell} - e_{-j\delta-\alpha_\ell} + e_{-(j-1)\delta+\alpha_\ell} - e_{(j-1)\delta-\alpha_\ell}), \\ & (e_{j\delta+\alpha_\ell} - e_{-j\delta-\alpha_\ell} - e_{-(j-1)\delta+\alpha_\ell} + e_{(j-1)\delta-\alpha_\ell}), \end{aligned} \quad (5.6.64)$$

$$\begin{aligned} & (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(j+1)\delta+\beta} + e_{-(j+1)\delta-\beta}), \\ & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{(j+1)\delta+\beta} - e_{-(j+1)\delta-\beta}), \\ & i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{(j+1)\delta+\beta} - e_{-(j+1)\delta-\beta}), \\ & (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{(j+1)\delta+\beta} + e_{-(j+1)\delta-\beta}), \end{aligned} \quad (5.6.65)$$

where, in the above

$$\alpha = \sum_{p=r}^{\ell-1} \alpha_p \quad \beta = \sum_{p=r}^{\ell-1} \alpha_p + 2\alpha_\ell \quad (1 \leq r \leq \ell-1-a), \quad (5.6.66)$$

$$\begin{aligned}
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(j+1)\delta+\beta} + e_{-(j+1)-\beta} \right), \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{(j+1)\delta+\beta} - e_{-(j+1)-\beta} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{(j+1)\delta+\beta} - e_{-(j+1)-\beta} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{(j+1)\delta+\beta} + e_{-(j+1)-\beta} \right),
\end{aligned} \tag{5.6.67}$$

$$\alpha = \sum_{p=r}^{\ell-1} \alpha_p \quad \beta = \sum_{p=r}^{\ell-1} \alpha_p + 2\alpha_\ell \quad (\ell-1-a \leq r \leq \ell-1), \tag{5.6.68}$$

$$\begin{aligned}
& ic, \\
& i \left(d - (2\ell-1)h_{\alpha_\ell} \right).
\end{aligned} \tag{5.6.69}$$

5.7 Type 1a involutive automorphisms of $B_\ell^{(1)}$ with $u = -1$

For each of the eight representative rotations of the roots of B_ℓ , there exist matrices of a general form $U(t)$, such that the type 1a automorphism $\{U(t), -1, \xi\}$ induces the corresponding rotation on the extensions of those roots to $B_\ell^{(1)}$. The general form of $U(t)$ for each of these rotations is given below

$$\begin{aligned}
 (1) \quad U(t) &= D_1^0, \\
 (2) \quad U(t) &= F_1^e, \\
 (3) \quad U(t) &= \text{dsum}\{E_{q-1}^0, F_q^e, E_{q-1}'^0\}, \\
 (4) \quad U(t) &= \text{dsum}\{M_{1,q}, D_{q+2}^0, M_{1,q}'\}, \\
 (5) \quad U(t) &= \text{dsum}\{E_{q-1}^0, M_{q,\ell-2}, F_\ell^e, M_{q,\ell-2}', E_{q-1}'^0\}, \\
 (6) \quad U(t) &= \text{dsum}\{M_{1,\ell-2}, F_\ell^e, M_{1,\ell-2}'\}, \\
 (7) \quad U(t) &= \text{dsum}\{M_{1,q-2}, F_q^e, M_{1,q-2}'\}, \\
 (8) \quad U(t) &= \text{dsum}\{M_{1,q}, C_{q+2,r-1}^0, F_r^e, C_{q+2,r-1}'^0, M_{1,q}'\}.
 \end{aligned} \tag{5.7.1}$$

The matrix transformations that are given in section 5.5 imply that each automorphism $\{U(t), -1, \xi\}$, (where $U(t)$ is of one of the above forms), is itself conjugate to $\{U, -1, 0\}$, where U is of the general form D_1^0 . Actually, all of these automorphisms are mutually conjugate, as will now be shown. Suppose that

$$U = \text{diag}\{\lambda_1, \dots, \lambda_\ell, 1, \lambda_\ell, \dots, \lambda_1\} \quad (\lambda_j^2 = 1; 1 \leq j \leq \ell). \tag{5.7.2}$$

The quantities μ_j (for $1 \leq j \leq \ell$) are defined by

$$\mu_j = \begin{cases} 0 & \lambda_j = 1, \\ 1 & \lambda_j = -1, \end{cases} \tag{5.7.3}$$

and the matrix $S(t)$ is then defined by

$$S(t) = \text{diag}\{t^{\mu_1}, \dots, t^{\mu_\ell}, 1, t^{-\mu_\ell}, \dots, t^{-\mu_1}\}. \tag{5.7.4}$$

That the type 1a involutive automorphisms (with $u = -1$) are all conjugate is a consequence of the relations

$$\begin{aligned}\bar{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) &= \mathbf{g}, \\ \bar{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) &= \mathbf{1}_{2\ell+1}, \\ \mathbf{S}(t)\mathbf{1}_{2\ell+1}\mathbf{S}(-t)^{-1} &= \mathbf{U}(t).\end{aligned}\tag{5.7.5}$$

There is thus only one conjugacy class of type 1a involutive automorphisms with $u = -1$, and this class will be called (C). The basis of the compact real form given below consists entirely of eigenvectors of the automorphism ψ_C , where ψ_C is the type 1a automorphism $\{\mathbf{1}_{2\ell+1}, -1, 0\}$.

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \quad j \in \mathbf{N}^0 \\ i(e_{j\delta}^k - e_{-j\delta}^k) \quad j \in \mathbf{N} \end{array} \right\} 1 \leq k \leq \ell; \text{ eigenvalue } (-1)^j,\tag{5.7.6}$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} j \in \mathbf{Z}; \text{ eigenvalue } (-1)^j,\tag{5.7.7}$$

$$\left. \begin{array}{l} ic \\ id \end{array} \right\} \text{ eigenvalue } 1.\tag{5.7.8}$$

A basis of a real form of $B_\ell^{(1)}$, obtained by applying Cartan's method, is provided by the elements

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \quad j \in \mathbf{N}^0 \\ i(e_{j\delta}^k - e_{-j\delta}^k) \quad j \in \mathbf{N} \end{array} \right\} 1 \leq k \leq \ell; j \text{ even},\tag{5.7.9}$$

$$\left. \begin{array}{l} i(e_{j\delta}^k + e_{-j\delta}^k) \quad j \in \mathbf{N}^0 \\ (e_{j\delta}^k - e_{-j\delta}^k) \quad j \in \mathbf{N} \end{array} \right\} 1 \leq k \leq \ell; j \text{ odd},\tag{5.7.10}$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} j \in \mathbf{Z}; j \text{ even},\tag{5.7.11}$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} j \in \mathbf{Z}; j \text{ odd}, \quad (5.7.12)$$

ic,

id. (5.7.13)

The “matrix parts” of these elements are all such that

$$\begin{aligned} \mathbf{g}\tilde{\mathbf{a}}(t)\mathbf{g}^{-1} &= -\mathbf{a}(t), \\ \tilde{\mathbf{a}}^*(-t^{-1}) &= -\mathbf{a}(t). \end{aligned} \quad (5.7.14)$$

5.8 Type 2a involutive automorphisms of $B_\ell^{(1)}$ with $u = 1$

To begin this section, it is necessary to note precisely which automorphisms are under consideration. To each of the eight rotations which were given previously, there exist matrices of a general form $U(t)$ such that a matrix $U(t)$ (which corresponds to the rotation τ) satisfies

$$U(t)h_{\alpha_k^0}^0 U(t)^{-1} = h_{\tau(\alpha_k^0)}^0. \quad (5.8.1)$$

In addition, the matrices $U(t)$ have also to satisfy the involutiveness condition, which for type 2a automorphisms (with $u = 1$) is the requirement that

$$U(t)U(t^{-1}) = \alpha t^\beta \mathbf{1}_{2\ell+1}, \quad (5.8.2)$$

where the complex number α is non-zero, and β is an integer. Furthermore, if $\{U(t), 1, \xi\}$ is also an automorphism of the compact real form, then

$$\tilde{U}^*(t^{-1})U(t) = \lambda t^\mu \mathbf{1}_{2\ell+1}, \quad (5.8.3)$$

where λ is a non-zero complex number. The general matrices $U(t)$ for the eight representative rotations are given below

- (1) $U(t) = \mathbf{D}_1$,
- (2) $U(t) = \mathbf{F}_1^0$,
- (3) $U(t) = \text{dsum}\{\mathbf{E}_{q-1}, \mathbf{F}_q^0, \mathbf{E}'_{q-1}\}$,
- (4) $U(t) = \text{dsum}\{\mathbf{N}_{1,q}, \mathbf{D}_{q+2}, \mathbf{N}'_{1,q}\}$,
- (5) $U(t) = \text{dsum}\{\mathbf{E}_{q-1}, \mathbf{N}_{q,\ell-2}, \mathbf{F}_\ell^0, \mathbf{N}'_{q,\ell-2}, \mathbf{E}'_{q-1}\}$,
- (6) $U(t) = \text{dsum}\{\mathbf{N}_{1,\ell-2}, \mathbf{F}_\ell^0, \mathbf{N}'_{1,\ell-2}\}$,
- (7) $U(t) = \text{dsum}\{\mathbf{N}_{1,q-2}, \mathbf{F}_q^0, \mathbf{N}'_{1,q-2}\}$,
- (8) $U(t) = \text{dsum}\{\mathbf{N}_{1,q}, \mathbf{C}_{q+2,r-1}, \mathbf{F}_r^0, \mathbf{C}'_{q+2,r-1}, \mathbf{N}'_{1,q}\}$.

(5.8.4)

The matrix transformations contained in section 5.5 imply that any type 2a involutive automorphism $\{U(t), 1, \xi\}$, where $U(t)$ is of one of the forms (1) to (8) given above, is conjugate to a type 2a automorphism $\{U(t), 1, \xi\}$, where $U(t)$ is of the form

$$U(t) = \text{dsum} \left\{ t \mathbf{1}_{n_+}, \mathbf{1}_{n_+}, -t \mathbf{1}_{n_-}, \mathbf{1}_1, -t^{-1} \mathbf{1}_{n_-}, \mathbf{1}_{n_+}, t^{-1} \mathbf{1}_{n_+} \right\}. \quad (5.8.5)$$

The quantities n_+, n_+^t, n_-, n_-^t are all such that

$$n_+ + n_+^t + n_- + n_-^t = \ell. \quad (5.8.6)$$

It will now be shown that the automorphisms $\{U(t), 1, \xi\}$ (where $U(t)$ is of the form given in (5.8.5)) belong to $\frac{1}{2}(\ell+1)(\ell+2)$ different conjugacy classes. Suppose that two matrices $U_1(t)$ and $U_2(t)$ are such that

$$\begin{aligned} U_1(t) &= \text{dsum} \left\{ t \mathbf{1}_a, \mathbf{1}_b, -\mathbf{1}_c, -t \mathbf{1}_d, \mathbf{1}_1, -t^{-1} \mathbf{1}_d, -\mathbf{1}_c, \mathbf{1}_b, t^{-1} \mathbf{1}_a \right\}, \\ U_2(t) &= \text{dsum} \left\{ t \mathbf{1}_\alpha, \mathbf{1}_\beta, -\mathbf{1}_\gamma, -t \mathbf{1}_\delta, \mathbf{1}_1, -t^{-1} \mathbf{1}_\delta, -\mathbf{1}_\gamma, \mathbf{1}_\beta, t^{-1} \mathbf{1}_\alpha \right\}, \end{aligned} \quad (5.8.7)$$

where $a, b, c, d, \alpha, \beta, \gamma, \delta$ are all such that

$$a + b = \alpha + \beta, \quad a - d = \alpha - \delta. \quad (5.8.8)$$

It can be shown that the type 2a involutive automorphism $\{U_1(t), 1, \xi\}$ is conjugate to the involutive automorphism $\{U_2(t), 1, \xi\}$. To prove that this is the case, it is sufficient to prove that it is possible to increase both a and d by the same arbitrary number. In fact, let the matrix $S(t)$ be given by

$$S(t) = \text{dsum} \left\{ \mathbf{1}_p, \mathbf{a}(t), \mathbf{1}_{2\ell-2p-3}, \mathbf{a}'(t), \mathbf{1}_p \right\}, \quad (5.8.9)$$

where $p = a + b - 1$, and the submatrices are given by

$$\begin{aligned} \mathbf{a}(t) &= \frac{1}{2} \begin{bmatrix} 1+t & 1-t \\ 1-t & 1+t \end{bmatrix}, \\ \mathbf{a}'(t) &= \frac{1}{2} \begin{bmatrix} 1+t^{-1} & t^{-1}-1 \\ t^{-1}-1 & 1+t^{-1} \end{bmatrix}. \end{aligned} \quad (5.8.10)$$

Now let the matrix $\mathbf{U}'(t)$ be defined by the following (equation (5.8.11)):

$$\mathbf{U}'(t) = \text{dsum} \left\{ t\mathbf{1}_a, \mathbf{1}_{b-1}, t\mathbf{1}_1, -t\mathbf{1}_1, -\mathbf{1}_{c-1}, -t\mathbf{1}_d, \mathbf{1}_1, -t^{-1}\mathbf{1}_d, -\mathbf{1}_{c-1}, -t^{-1}\mathbf{1}_1, t^{-1}\mathbf{1}_1, \mathbf{1}_{b-1}, t^{-1}\mathbf{1}_a \right\}.$$

It can be deduced from section 5.5 that the matrix $\mathbf{U}'(t)$ is equivalent to the matrix $\mathbf{U}''(t)$, where $\mathbf{U}''(t)$ is given in the following (equation (5.8.12)):

$$\mathbf{U}''(t) = \text{dsum} \left\{ t\mathbf{1}_{a+1}, \mathbf{1}_{b-1}, -\mathbf{1}_{c-1}, -t\mathbf{1}_{d+1}, \mathbf{1}_1, -t^{-1}\mathbf{1}_{d+1}, -\mathbf{1}_{c-1}, \mathbf{1}_{b-1}, t^{-1}\mathbf{1}_{a+1} \right\}.$$

Now, the matrix $\mathbf{S}(t)$ satisfies

$$\begin{aligned} \tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) &= \mathbf{g}, \\ \tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) &= \mathbf{1}_{2\ell+1}, \\ \mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(t^{-1})^{-1} &= \mathbf{U}'(t). \end{aligned} \tag{5.8.13}$$

Hence, the two type 2a involutive automorphisms $\{\mathbf{U}_1(t), 1, \xi\}$ and $\{\mathbf{U}_2(t), 1, \xi\}$ are mutually conjugate. Return now to the form of $\mathbf{U}(t)$ given by

$$\mathbf{U}(t) = \text{dsum} \left\{ t\mathbf{1}_{n_+^t}, \mathbf{1}_{n_+}, -t\mathbf{1}_{n_-^t}, \mathbf{1}_1, -t^{-1}\mathbf{1}_{n_-^t}, \mathbf{1}_{n_+}, t^{-1}\mathbf{1}_{n_+^t} \right\}. \tag{5.8.14}$$

Let $N_+ = n_+^t + n_+$, and let $N_- = n_-^t + n_-$. The preceding analysis implies that the quantities n_-^t and n_+^t may both be increased (or decreased) by the same arbitrary integer. However, for a particular $\mathbf{U}(t)$, although n_-^t and n_+^t are arbitrary (to a certain degree), their difference remains constant, as do the values of N_+ and N_- . Consider then the value of $(n_+^t - n_-^t)$. It follows that, for a matrix $\mathbf{U}(t)$ whose values of N_+ and N_- are known, the quantity $(n_+^t - n_-^t)$ is such that $-N_- \leq (n_+^t - n_-^t) \leq N_+$. Thus, $(n_+^t - n_-^t)$ may take any one of $(\ell + 1)$ distinct values (for a given set of values for N_+ and N_-). Now consider the matrix $\mathbf{U}(t)$, where

$$\mathbf{U}(t) = \text{dsum} \left\{ \mathbf{1}_j, -\mathbf{1}_k, -t\mathbf{1}_m, \mathbf{1}_1, -t^{-1}\mathbf{1}_m, -\mathbf{1}_k, \mathbf{1}_j \right\}. \tag{5.8.15}$$

It follows from the matrix transformations in section 5.5 that the matrix $\mathbf{U}(-t)$ is equivalent to the matrix $\mathbf{U}'(t)$, which is given by

$$\mathbf{U}'(t) = \text{dsum} \{ t \mathbf{1}_m, \mathbf{1}_j, -\mathbf{1}_k, \mathbf{1}_l, -\mathbf{1}_k, \mathbf{1}_j, t^{-1} \mathbf{1}_m \}. \quad (5.8.16)$$

Thus, it may be assumed that $(n_+^t - n_-^t)$ takes a non-negative value. Thus, each type 2a involutive Cartan-preserving automorphism (with $u = 1$) is conjugate to some automorphism $\{ \mathbf{U}_{a,b}(t), 1, \xi \}$, where

$$\mathbf{U}_{a,b}(t) = \text{dsum} \{ t \mathbf{1}_a, \mathbf{1}_{b-a}, -\mathbf{1}_{\ell-b}, \mathbf{1}_1, -\mathbf{1}_{\ell-b}, \mathbf{1}_{b-a}, t^{-1} \mathbf{1}_a \}, \quad (5.8.17)$$

where $0 \leq a \leq \ell$, and $0 \leq a \leq b$. Let $(D)^{(a,b)}$ be the conjugacy class which contains the type 2a involutive automorphism $\{ \mathbf{U}_{a,b}(t), 1, \xi \}$ where a, b take the values given above. It will now be shown that $(D)^{(a,b)}$ is the same class as $(D)^{(c,d)}$ only if $a = c, b = d$. Suppose firstly that there exists some matrix $\mathbf{S}(t)$ which is such that

$$\mathbf{S}(t) \mathbf{U}_{a,c}(st^{\pm 1}) \mathbf{S}(t^{-1})^{-1} = \lambda t^\mu \mathbf{U}_{a,d}(t). \quad (5.8.18)$$

The only permitted values for s are $1, -1$. With $s = 1$, consider the special case $t = -1$. It is then a necessary condition that

$$\mathbf{S}(-1) \mathbf{U}_{a,c}(-1) \mathbf{S}(-1)^{-1} = \lambda t^\mu \mathbf{U}_{a,d}(-1), \quad (5.8.19)$$

and this implies that $c = d$. When $s = -1$, the substitution $t = -1$ implies that

$$\mathbf{S}(-1) \mathbf{U}_{a,c}(1) \mathbf{S}(-1)^{-1} = \lambda t^\mu \mathbf{U}_{a,d}(-1). \quad (5.8.20)$$

Once again, this implies that $c = d$. To conclude, suppose that there exists some matrix $\mathbf{S}(t)$ which satisfies

$$\mathbf{S}(t) \mathbf{U}_{a,b}(st^{\pm 1}) \mathbf{S}(t^{-1})^{-1} = \lambda t^\mu \mathbf{U}_{c,d}, \quad (5.8.21)$$

where $a \neq c$. If $s = 1$, and if $t = 1$, then a necessary condition is that $a = c$, which contradicts our original hypothesis. Consider then the possibility that $s = -1$. With $t = 1$, a necessary condition is that $a = c - d$, whereas with $t = -1$, it is necessary that $c = a - b$. Thus, $a \leq c$ and $c \leq a$, which means that $a = c$, which is not possible.

Let the matrix $U_{a,b}(t)$ be defined by

$$U_{a,b}(t) = \text{dsum} \{ \mathbf{1}_{a-b}, -\mathbf{1}_{\ell-a}, t\mathbf{1}_b, \mathbf{1}_1, t^{-1}\mathbf{1}_b, -\mathbf{1}_{\ell-a}, \mathbf{1}_{a-b} \}. \quad (5.8.22)$$

The type 2a involutive automorphism generated by $U_{a,b}(t)$ belongs (in the notation of the preceding analysis) to the conjugacy class $(D)^{(b,a)}$. There is listed below a basis of the compact real form of $B_\ell^{(1)}$, in which every basis element is an eigenvector of the type 2a involutive automorphism generated by $U_{a,b}(t)$.

$$\left. \begin{array}{l} ih_{\alpha_k} \quad (1 \leq k \leq \ell; k \neq \ell - b + 1) \\ i(h_{\alpha_{\ell-b+1}} - c) \quad (b \neq 0) \end{array} \right\} \text{eigenvalue } 1, \quad (5.8.23)$$

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \quad \text{eigenvalue } 1 \\ i(e_{j\delta}^k - e_{-j\delta}^k) \quad \text{eigenvalue } -1 \end{array} \right\} 1 \leq k \leq \ell; j \neq 0, \quad (5.8.24)$$

$$\begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \quad \text{eigenvalue } 1, \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \quad \text{eigenvalue } -1, \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \quad \text{eigenvalue } 1, \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \quad \text{eigenvalue } -1, \end{array} \quad (5.8.25)$$

where, in (5.8.25) above

$$\alpha = \left[\begin{array}{l} \sum_{p=r}^{\ell} \alpha_p \quad (1 \leq r \leq a-b) \\ \sum_{p=r}^{s-1} \alpha_p \quad \left[\begin{array}{l} (1 \leq r < s \leq a-b) \\ (a-b < r < s \leq \ell-b) \\ (\ell-b < r < s \leq \ell) \end{array} \right] \\ \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell} \alpha_p \quad \left[\begin{array}{l} (1 \leq r < s \leq a-b) \\ (a-b < r < s \leq \ell-b) \end{array} \right] \end{array} \right], \quad (5.8.26)$$

$$\begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \quad \text{eigenvalue } -1, \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \quad \text{eigenvalue } 1, \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \quad \text{eigenvalue } -1, \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \quad \text{eigenvalue } 1, \end{array} \quad (5.8.27)$$

where, in (5.8.27) above

$$\alpha = \left[\begin{array}{l} \sum_{p=r}^{\ell} \alpha_p \quad (a-b < r \leq \ell-b) \\ \sum_{p=r}^{s-1} \alpha_p \quad (1 \leq r \leq a-b < s \leq \ell-b) \\ \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell} \alpha_p \quad (1 \leq r \leq a-b < s \leq \ell-b) \end{array} \right], \quad (5.8.28)$$

$$\begin{aligned} & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right) \text{ eigenvalue } 1, \\ & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right) \text{ eigenvalue } -1, \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right) \text{ eigenvalue } 1, \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right) \text{ eigenvalue } -1, \end{aligned} \quad (5.8.29)$$

where, in (5.8.29) above

$$\alpha = \sum_{p=r}^{s-1} \alpha_p \quad (1 \leq r \leq a-b; \ell-b < s \leq \ell), \quad (5.8.30)$$

$$\begin{aligned} & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right) \text{ eigenvalue } -1, \\ & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right) \text{ eigenvalue } 1, \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right) \text{ eigenvalue } -1, \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right) \text{ eigenvalue } 1, \end{aligned} \quad (5.8.31)$$

where, in (5.8.31) above

$$\alpha = \sum_{p=r}^{s-1} \alpha_p \quad (a-b < r \leq \ell-b < s \leq \ell), \quad (5.8.32)$$

$$\begin{aligned} & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(-j+1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right) \text{ eigenvalue } 1, \\ & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{(-j+1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right) \text{ eigenvalue } -1, \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{(-j+1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right) \text{ eigenvalue } 1, \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{(-j+1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right) \text{ eigenvalue } -1, \end{aligned} \quad (5.8.33)$$

where, in (5.8.33) above

$$\alpha = \left. \begin{array}{l} \sum_{p=r}^{\ell} \alpha_p \quad (\ell - b < r \leq \ell) \\ \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell} \alpha_p \quad (1 \leq r \leq a - b; \ell - b < s \leq \ell) \end{array} \right\}, \quad (5.8.34)$$

$$\begin{array}{l} \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(-j+1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right) \text{ eigenvalue } -1, \\ \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{(-j+1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right) \text{ eigenvalue } 1, \\ i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{(-j+1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right) \text{ eigenvalue } -1, \\ i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{(-j+1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right) \text{ eigenvalue } 1, \end{array} \quad (5.8.35)$$

where, in (5.8.35) above

$$\alpha = \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell} \alpha_p \quad (a - b < r \leq \ell - b < s \leq \ell), \quad (5.8.36)$$

$$\begin{array}{l} \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(-j+2)\delta+\alpha} + e_{(j-2)\delta-\alpha} \right) \text{ eigenvalue } 1, \\ \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{(-j+2)\delta+\alpha} - e_{(j-2)\delta-\alpha} \right) \text{ eigenvalue } -1, \\ i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{(-j+2)\delta+\alpha} - e_{(j-2)\delta-\alpha} \right) \text{ eigenvalue } 1, \\ i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{(-j+2)\delta+\alpha} + e_{(j-2)\delta-\alpha} \right) \text{ eigenvalue } -1, \end{array} \quad (5.8.37)$$

where, in (5.8.37) above

$$\alpha = \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell} \alpha_p \quad (\ell - b < r < s \leq \ell), \quad (5.8.38)$$

$$\left. \begin{array}{l} ic \\ i \left(d - (2\ell - 1) \sum_{p=1}^b p h_{\alpha_{\ell-b+p}} \right) \quad (b \neq 0) \end{array} \right\} \text{ eigenvalue } -1. \quad (5.8.39)$$

When $b = 0$, then id is an eigenvector, with associated eigenvalue -1 . A basis of a real form is thus provided by the elements

$$\begin{array}{l} ih_{\alpha_k} \quad (1 \leq k \leq \ell; k \neq \ell - b + 1 \text{ if } b \neq 0), \\ i(h_{\alpha_{\ell-b+1}} - c) \quad (b \neq 0), \end{array} \quad (5.8.40)$$

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \\ (e_{j\delta}^k - e_{-j\delta}^k) \end{array} \right\} 1 \leq k \leq \ell; j \neq 0, \quad (5.8.41)$$

$$\begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \end{array} \quad (5.8.42)$$

where, in (5.8.42) above

$$\alpha = \left[\begin{array}{l} \sum_{p=r}^{\ell} \alpha_p \quad (1 \leq r \leq a-b) \\ \sum_{p=r}^{s-1} \alpha_p \quad \left\{ \begin{array}{l} (1 \leq r < s \leq a-b) \\ (a-b < r < s \leq \ell-b) \\ (\ell-b < r < s \leq \ell) \end{array} \right. \\ \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell} \alpha_p \quad \left\{ \begin{array}{l} (1 \leq r < s \leq a-b) \\ (a-b < r < s \leq \ell-b) \end{array} \right. \end{array} \right], \quad (5.8.43)$$

$$\begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \end{array} \quad (5.8.44)$$

where, in (5.8.44) above

$$\alpha = \left[\begin{array}{l} \sum_{p=r}^{\ell} \alpha_p \quad (a-b < r \leq \ell-b) \\ \sum_{p=r}^{s-1} \alpha_p \quad (1 \leq r \leq a-b < s \leq \ell-b) \\ \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell} \alpha_p \quad (1 \leq r \leq a-b < s \leq \ell-b) \end{array} \right], \quad (5.8.45)$$

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right),
\end{aligned} \tag{5.8.46}$$

where, in (5.8.46) above

$$\alpha = \sum_{p=r}^{s-1} \alpha_p \quad (1 \leq r \leq a-b; \ell-b < s \leq \ell), \tag{5.8.47}$$

$$\begin{aligned}
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right),
\end{aligned} \tag{5.8.48}$$

where, in (5.8.48) above

$$\alpha = \sum_{p=r}^{s-1} \alpha_p \quad (a-b < r \leq \ell-b < s \leq \ell), \tag{5.8.49}$$

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right),
\end{aligned} \tag{5.8.50}$$

where, in (5.8.50) above

$$\alpha = \left\{ \begin{array}{l} \sum_{p=r}^{\ell} \alpha_p \quad (\ell-b < r \leq \ell) \\ \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell} \alpha_p \quad (1 \leq r \leq a-b; \ell-b < s \leq \ell) \end{array} \right\}, \tag{5.8.51}$$

$$\begin{aligned}
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(-j+1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{(-j+1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{(-j+1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{(-j+1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right),
\end{aligned} \tag{5.8.52}$$

where, in (5.8.52) above

$$\alpha = \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell} \alpha_p \quad (a-b < r \leq \ell - b < s \leq \ell), \tag{5.8.53}$$

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(-j+2)\delta+\alpha} + e_{(j-2)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{(-j+2)\delta+\alpha} - e_{(j-2)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{(-j+2)\delta+\alpha} - e_{(j-2)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{(-j+2)\delta+\alpha} + e_{(j-2)\delta-\alpha} \right),
\end{aligned} \tag{5.8.54}$$

where, in (5.8.54) above

$$\alpha = \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell} \alpha_p \quad (\ell - b < r < s \leq \ell), \tag{5.8.55}$$

$$\begin{aligned}
& c, \\
& \left(d - (2\ell - 1) \sum_{p=1}^b p h_{\alpha_{t-b+p}} \right) \quad (b \neq 0), \\
& d \quad \quad \quad (b = 0).
\end{aligned} \tag{5.8.56}$$

6 Involutive automorphisms and real forms of $C_\ell^{(1)}$

6.1 Introduction

This chapter will examine the involutive automorphisms of the affine Kac-Moody algebra $C_\ell^{(1)}$ and of its compact real form. Then, in the same manner as the preceding chapters, the real forms of the complex affine Kac-Moody algebra will be examined. It should be recalled firstly that the simple complex Lie algebra C_ℓ may be realised as the set of $(2\ell \times 2\ell)$ matrices \mathbf{a} which satisfy

$$\bar{\mathbf{a}}\mathbf{J} + \mathbf{J}\mathbf{a} = \mathbf{0}, \quad \text{where } \mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{1}_\ell \\ -\mathbf{1}_\ell & \mathbf{0} \end{bmatrix}. \quad (6.1.1)$$

The Kac-Moody algebra $C_\ell^{(1)}$ is based upon the simple Lie algebra C_ℓ , so this realisation is the starting-point for a realisation of $C_\ell^{(1)}$. It may be recalled that the positive roots of C_ℓ can be written in the form

$$\left. \begin{array}{l} (\varepsilon_r - \varepsilon_s) \\ (\varepsilon_r + \varepsilon_s) \end{array} \right\} \text{ for } 1 \leq r < s \leq \ell, \quad (6.1.2)$$

where the quantities ε_r (for $1 \leq r \leq \ell$) are defined in terms of the simple roots of C_ℓ by

$$\varepsilon_r = \left\{ \begin{array}{ll} \sum_{p=r}^{\ell-1} \alpha_p + \frac{1}{2} \alpha_\ell & (\text{for } 1 \leq r \leq \ell - 1) \\ \frac{1}{2} \alpha_\ell & (\text{for } r = \ell) \end{array} \right\}. \quad (6.1.3)$$

The imaginary root δ for the Kac-Moody algebra $C_\ell^{(1)}$ is given by

$$\delta = \alpha_0 + 2(\alpha_1 + \dots + \alpha_{\ell-1}) + \alpha_\ell, \quad (6.1.4)$$

and hence the operator c is such that

$$c = h_{\alpha_0} + 2(h_{\alpha_1} + \dots + h_{\alpha_{\ell-1}}) + h_{\alpha_\ell}. \quad (6.1.5)$$

It is necessary at this point to define some notation which will be employed throughout the analysis relating to the Kac-Moody algebra $C_\ell^{(1)}$. The matrix $e_{r,s}$ (for $1 \leq r, s \leq \ell$) is defined to be the $\ell \times \ell$ matrix whose elements are defined by

$$(e_{r,s})_{j,k} = \delta_{jr} \delta_{ks}. \quad (6.1.6)$$

Using this definition, other matrices are then defined by

$$\left. \begin{aligned} \mathbf{X}_{r,s} &= \text{dsum} \{ \mathbf{e}_{r,s}, -\mathbf{e}_{r,s} \} \\ \mathbf{Y}_{r,s} &= \text{offsum} \{ (\mathbf{e}_{r,s} + \mathbf{e}_{s,r}), \mathbf{0} \} \end{aligned} \right\} \text{ (for } 1 \leq r, s \leq \ell). \quad (6.1.7)$$

The matrix realisation used in this case is the one which is defined by

$$\Gamma(h_{\alpha_k}) = \begin{cases} \{4(\ell+1)\}^{-1} (\mathbf{X}_{k,k} - \mathbf{X}_{k+1,k+1}) & \text{(for } 1 \leq k \leq \ell-1), \\ \{2(\ell+1)\}^{-1} (\mathbf{X}_{\ell,\ell}) & \text{(for } k = \ell), \end{cases} \quad (6.1.8)$$

$$\Gamma(e_{\varepsilon_r - \varepsilon_s}) = \{4(\ell+1)\}^{-\frac{1}{2}} \mathbf{X}_{r,s} \quad \text{(for } 1 \leq r < s \leq \ell), \quad (6.1.9)$$

$$\Gamma(e_{\varepsilon_r - \varepsilon_s}) = \{4(\ell+1)(1 + \delta_{rs})\}^{-\frac{1}{2}} \mathbf{Y}_{r,s} \quad \text{(for } 1 \leq r < s \leq \ell).$$

The Dynkin index γ of this representation is $\{2(\ell+1)\}^{-1}$. In addition, the representation is equivalent to its contragredient representation, since

$$\tilde{\Gamma} = -\mathbf{J}\Gamma\mathbf{J}^{-1}. \quad (6.1.10)$$

It is not necessary, therefore, to examine the type 1b or the type 2b automorphisms of the Kac-Moody algebra, since in this case they coincide with the type 1a and the type 2a automorphisms respectively.

As with the other Kac-Moody algebras that are being investigated, it is necessary to know precisely which matrices generate automorphisms of the algebra. In the present case, consider a matrix $\mathbf{S}(t)$, and let

$$\mathbf{b}(t) = \mathbf{S}(t)\mathbf{a}(st)\mathbf{S}(t)^{-1}. \quad (6.1.11)$$

For the matrix $\mathbf{S}(t)$ to generate a type 1a automorphism, it is necessary that

$$\tilde{\mathbf{b}}(t)\mathbf{J} + \mathbf{J}\mathbf{b}(t) = \mathbf{0}. \quad (6.1.12)$$

This condition may be re-arranged so that

$$\tilde{\mathbf{S}}(t)^{-1}\tilde{\mathbf{a}}(st)\tilde{\mathbf{S}}(t)\mathbf{J} + \mathbf{J}\mathbf{S}(t)\mathbf{a}(st)\mathbf{S}(t)^{-1} = \mathbf{0}. \quad (6.1.13)$$

Then, since $\tilde{\mathbf{a}}(st) = -\mathbf{J}\mathbf{a}(st)\mathbf{J}^{-1} = \mathbf{J}\mathbf{a}(st)\mathbf{J}$, it follows that

$$\begin{aligned} \tilde{\mathbf{S}}(t)^{-1}\mathbf{J}\mathbf{a}(st)\mathbf{J}\tilde{\mathbf{S}}(t)\mathbf{J} &= -\mathbf{J}\mathbf{S}(t)\mathbf{a}(st)\mathbf{S}(t)^{-1}, \\ \mathbf{a}(st)\mathbf{J}\tilde{\mathbf{S}}(t)\mathbf{J}\mathbf{S}(t) &= \mathbf{J}\tilde{\mathbf{S}}(t)\mathbf{J}\mathbf{S}(t)\mathbf{a}(st). \end{aligned} \quad (6.1.14)$$

Schur's lemma (as stated in the appendix) may then be used, and this then implies, where $\mathbf{S}(t)$ is a Laurent polynomial matrix, that

$$\mathbf{J}\tilde{\mathbf{S}}(t)\mathbf{J}\mathbf{S}(t) = \alpha t^\beta, \quad (6.1.15)$$

where α is a non-zero complex number, and the quantity β is an integer. This may be re-arranged so that the requirement is that

$$\tilde{\mathbf{S}}(t)\mathbf{J}\mathbf{S}(t) = \lambda t^\mu \mathbf{J}, \quad (6.1.16)$$

with λ being non-zero. This analysis may also be repeated, assuming that $\mathbf{S}(t)$ generates a type 2a automorphism, with precisely the same conclusion, namely the same condition (6.1.16) which $\mathbf{S}(t)$ must satisfy in order to generate an automorphism.

6.2 The Weyl group of C_ℓ

The involutive automorphisms are being examined, and in particular the Cartan-preserving involutive automorphisms, as investigated by Gorman *et al* [19]. Each Cartan-preserving involutive automorphism has been shown to correspond to a rotation of the roots of the corresponding simple Lie algebra, in this case C_ℓ . Therefore, if a complete set of conjugacy class representatives is obtained for the classes of involutive rotations (within the group of all rotations), then only the involutive automorphisms corresponding to these representatives will need to be examined. In the case of the simple Lie algebra C_ℓ , the group of rotations coincides with the Weyl group, which is denoted \mathcal{W} . The conjugacy classes of involutive members of the Weyl group of C_ℓ are well-known. The choice of representatives is completely arbitrary. Using the algorithm developed by Richardson([29]), representatives were chosen such that each representative fell into one of a set number of types. There follows a list of such representatives. Where there is more than one rotation of a particular type, they are distinguished by a number of integer parameters. There are nine different types of representative, some being associated only with odd or even values of ℓ . A representative, τ_j , is given for each of these types below.

1. This type contains only one representative rotation, which is such that

$$\tau_1(\alpha_k^0) = \alpha_k^0 \quad (\text{for } 1 \leq k \leq \ell). \quad (6.2.1)$$

2. This type contains only one representative rotation, which is such that

$$\tau_2(\alpha_k^0) = -\alpha_k^0 \quad (\text{for } 1 \leq k \leq \ell). \quad (6.2.2)$$

3. For this type, the integer parameter q is such that $1 < q < \ell$, and the rotation is

$$\begin{aligned} \tau_3(\alpha_m^0) &= \alpha_m^0 && (\text{for } 1 \leq m < q-1), \\ \tau_3(\alpha_{q-1}^0) &= \alpha_{q-1}^0 + 2(\alpha_q^0 + \cdots + \alpha_{\ell-1}^0) + \alpha_\ell^0, \\ \tau_3(\alpha_n^0) &= -\alpha_n^0 && (\text{for } q \leq n \leq \ell). \end{aligned} \quad (6.2.3)$$

4. For this type, there are two parameters, q and r , which are such that q is odd, $\ell \neq r$, and $r - q > 2$. The representative rotation is then specified by

$$\begin{aligned} \tau_4(\alpha_m^0) &= -\alpha_m^0 && \left\{ \begin{array}{l} m \text{ odd}; 1 \leq m \leq q \\ r \leq m \leq \ell \end{array} \right\}, \\ \tau_4(\alpha_n^0) &= \alpha_{n-1}^0 + \alpha_n^0 + \alpha_{n+1}^0 && n \text{ even}; 1 < n < q, \\ \tau_4(\alpha_{q+1}^0) &= \alpha_q^0 + \alpha_{q+1}^0, \\ \tau_4(\alpha_p^0) &= \alpha_p^0 && q+2 \leq p \leq r-2, \\ \tau_4(\alpha_{r-1}^0) &= \alpha_{r-1}^0 + 2(\alpha_r^0 + \cdots + \alpha_{\ell-1}^0) + \alpha_\ell^0. \end{aligned} \quad (6.2.4)$$

5. In this case, q is odd, and such that $q < \ell - 2$. The representative rotation is

$$\begin{aligned} \tau_5(\alpha_m^0) &= -\alpha_m^0 && \left\{ \begin{array}{l} m \text{ odd}; 1 \leq m \leq q \\ q+2 \leq m \leq \ell \end{array} \right\}, \\ \tau_5(\alpha_n^0) &= \alpha_{n-1}^0 + \alpha_n^0 + \alpha_{n+1}^0 && n \text{ even}; 1 < m < q, \\ \tau_5(\alpha_{q+1}^0) &= \alpha_q^0 + \alpha_{q+1}^0 + 2(\alpha_{q+2}^0 + \cdots + \alpha_{\ell-1}^0) + \alpha_\ell^0. \end{aligned} \quad (6.2.5)$$

6. For this type, the parameter q is odd, and such that $q \neq \ell - 1$ when ℓ is even. The rotation is given by

$$\begin{aligned}
 \tau_6(\alpha_m^0) &= -\alpha_m^0 & m \text{ odd}; 1 \leq m \leq q, \\
 \tau_6(\alpha_n^0) &= \alpha_{n-1}^0 + \alpha_n^0 + \alpha_{n+1}^0 & n \text{ even}; 1 < n < q, \\
 \tau_6(\alpha_{q+1}^0) &= \alpha_q^0 + \alpha_{q+1}^0, \\
 \tau_6(\alpha_p^0) &= \alpha_p^0 & q+2 \leq p \leq \ell.
 \end{aligned} \tag{6.2.6}$$

7. In this case q is such that $q \neq 1, 2$ and $\ell - q$ is even. The representative rotation is given by

$$\begin{aligned}
 \tau_7(\alpha_m^0) &= \alpha_m^0 & 1 \leq m \leq q-2, \\
 \tau_7(\alpha_{q-1}^0) &= \alpha_{q-1}^0 + \alpha_q^0, \\
 \tau_7(\alpha_n^0) &= -\alpha_n^0 & (\ell - n) \text{ even}; q \leq n \leq \ell, \\
 \tau_7(\alpha_p^0) &= \alpha_{p-1}^0 + \alpha_p^0 + \alpha_{p+1}^0 & (\ell - n) \text{ odd}; q < n < \ell.
 \end{aligned} \tag{6.2.7}$$

8. In this case, ℓ is assumed to be odd. The rotation is specified by

$$\begin{aligned}
 \tau_8(\alpha_m^0) &= -\alpha_m^0 & (\ell - m) \text{ even}; 1 \leq m \leq \ell, \\
 \tau_8(\alpha_n^0) &= \alpha_{n-1}^0 + \alpha_n^0 + \alpha_{n+1}^0 & (\ell - n) \text{ odd}; 1 < m < \ell.
 \end{aligned} \tag{6.2.8}$$

9. In this case, ℓ is assumed to be even. The rotation is specified by

$$\begin{aligned}
 \tau_9(\alpha_m^0) &= -\alpha_m^0 & m \text{ odd}; 1 \leq m < \ell, \\
 \tau_9(\alpha_n^0) &= \alpha_{n-1}^0 + \alpha_n^0 + \alpha_{n+1}^0 & n \text{ even}; 1 < m < \ell, \\
 \tau_9(\alpha_\ell^0) &= 2\alpha_{\ell-1}^0 + \alpha_\ell^0.
 \end{aligned} \tag{6.2.9}$$

6.3 Supplementary notation

Some shorthand notation for matrices will now be introduced. This notation will make the subsequent analysis of this chapter easier. The notation applies only to this chapter, and should not be confused with similar shorthand for other affine Kac-Moody algebras. It should be recalled that matrices $\mathbf{e}_{p,q}$, $\mathbf{X}_{p,q}$ and $\mathbf{Y}_{p,q}$ have already been defined earlier in this chapter. The notation introduced will not be, in general, for specific matrices, but will be for general forms of matrices. That is, a piece of notation will be taken to represent all matrices of a given form.

The notation $\mathbf{D}_{j,k}$ will be introduced to represent all $(k-j+1) \times (k-j+1)$ matrices which are of the form given by

$$\mathbf{D}_{j,k} = \text{diag} \left\{ 1, \lambda_{j+1} t^{\mu_{j+1}}, \dots, \lambda_k t^{\mu_k} \right\} \quad \left(\lambda_q^2 = 1 \text{ for } j < p \leq k \right). \quad (6.3.1)$$

A slight variation upon this is the notation $\mathbf{D}_{j,k}^0$. The general form $\mathbf{D}_{j,k}^0$ is the same as the general form $\mathbf{D}_{j,k}$, but is such that $\mu_p = 0$ for $j < p \leq k$.

The general form $\mathbf{G}_{j,k}$ is, like $\mathbf{D}_{j,k}$, defined to be an $(\ell-k+1) \times (\ell-k+1)$ diagonal matrix. In this case, the form is defined by

$$\mathbf{G}_{j,k} = \text{diag} \left\{ \lambda_j t^{\mu_j}, \dots, \lambda_k t^{\mu_k} \right\}. \quad (6.3.2)$$

The form $\mathbf{D}_{j,k}$ may be considered to be a special form of $\mathbf{G}_{j,k}$, although it merits special consideration, and is thus given its own notation. There are variations of the form $\mathbf{G}_{j,k}$, which will now be defined. The form $\mathbf{G}_{j,k}^{\text{even}}$ is identical to $\mathbf{G}_{j,k}$ with the additional constraint that μ_p be even (for $j \leq p \leq k$). In a similar fashion, $\mathbf{G}_{j,k}^{\text{odd}}$

(respectively $\mathbf{G}_{j,k}^0$) is defined to be identical to $\mathbf{G}_{j,k}$ with the additional constraint that μ_p be odd (respectively, that μ_p be zero), where $j \leq p \leq k$. Furthermore, the form $\hat{\mathbf{G}}_{j,k}$ is derived from the form $\mathbf{G}_{j,k}$ by letting $\lambda_j t^{\mu_j} = 1$ (with $\hat{\mathbf{G}}_{j,k}^{\text{even}}$, $\hat{\mathbf{G}}_{j,k}^{\text{odd}}$ and $\hat{\mathbf{G}}_{j,k}^{\text{zero}}$ being derived from $\mathbf{G}_{j,k}^{\text{even}}$, $\mathbf{G}_{j,k}^{\text{odd}}$ and $\mathbf{G}_{j,k}^{\text{zero}}$ respectively by letting $\lambda_j t^{\mu_j} = 1$ in each).

Another general form to be defined is that which is denoted by $\mathbf{L}_{j,k}$, where $(k-j)$ is an even integer. The general form is given by

$$\mathbf{L}_{j,k} = \text{dsum}\{\lambda_j, \dots, \lambda_q, \dots, \lambda_k\} \quad \text{where } (q-j) \text{ is even,} \quad (6.3.3)$$

and where the 2×2 submatrix λ_q (for $(q-j)$ even) is defined by

$$\lambda_q = \begin{bmatrix} 0 & \lambda_q t^{\mu_q} \\ \lambda_q^{-1} t^{-\mu_q} & 0 \end{bmatrix}. \quad (6.3.4)$$

Similarly, the forms $\mathbf{M}_{j,k}$ and $\mathbf{N}_{j,k}$ may be defined (when $(k-j)$ is an even integer) by

$$\mathbf{M}_{j,k} = \text{dsum}\{\mu_j, \dots, \mu_q, \dots, \mu_k\}, \quad (6.3.5)$$

$$\mathbf{N}_{j,k} = \text{dsum}\{\nu_j, \dots, \nu_q, \dots, \nu_k\}, \quad (6.3.6)$$

where the 2×2 submatrices μ_q and ν_q are defined (for $(q-j)$ even) by

$$\mu_q = \begin{bmatrix} 0 & \lambda_q t^{\mu_q} \\ (-1)^{\mu_q} \lambda_q^{-1} t^{-\mu_q} & 0 \end{bmatrix}, \quad (6.3.7)$$

$$\nu_q = \begin{bmatrix} 0 & \lambda_q t^{\mu_q} \\ \lambda_q^{-1} t^{\mu_q} & 0 \end{bmatrix}. \quad (6.3.8)$$

There are three other general forms, which are similar to, but slightly different from the general forms $\mathbf{L}_{j,k}$, $\mathbf{M}_{j,k}$ and $\mathbf{N}_{j,k}$. They are called $\hat{\mathbf{L}}_{1,k}$, $\hat{\mathbf{M}}_{1,k}$ and $\hat{\mathbf{N}}_{1,k}$. It is assumed for all of them that k is odd. They are defined by

$$\hat{\mathbf{L}}_{1,k} = \text{dsum} \left\{ \hat{\lambda}_1, \dots, \hat{\lambda}_q, \dots, \hat{\lambda}_k \right\}, \quad (6.3.9)$$

$$\hat{\mathbf{M}}_{1,k} = \text{dsum} \left\{ \hat{\mu}_1, \dots, \hat{\mu}_q, \dots, \hat{\mu}_k \right\}, \quad (6.3.10)$$

$$\hat{\mathbf{N}}_{1,k} = \text{dsum} \left\{ \hat{\nu}_1, \dots, \hat{\nu}_q, \dots, \hat{\nu}_k \right\}. \quad (6.3.11)$$

In each of the above, the 2×2 submatrices $\hat{\lambda}_q$ (or $\hat{\mu}_q$ or $\hat{\nu}_q$) are defined only for odd values of q . They are given by

$$\hat{\lambda}_1 = \begin{bmatrix} 0 & 1 \\ \lambda_2 t^{\mu_2} & 0 \end{bmatrix}, \quad (6.3.12)$$

$$\hat{\mu}_1 = \begin{bmatrix} 0 & 1 \\ \lambda_2 t^{\mu_2} & 0 \end{bmatrix} \quad (\mu_2 \text{ is even}), \quad (6.3.13)$$

$$\hat{\nu}_1 = \begin{bmatrix} 0 & 1 \\ \lambda_2 & 0 \end{bmatrix}, \quad (6.3.14)$$

$$\hat{\lambda}_q = \begin{bmatrix} 0 & \lambda_q t^{\mu_q} \\ \lambda_2 \lambda_q^{-1} t^{\mu_2 - \mu_q} & 0 \end{bmatrix}, \quad (6.3.15)$$

$$\hat{\mu}_q = \begin{bmatrix} 0 & \lambda_q t^{\mu_q} \\ (-1)^{\mu_q} \lambda_2 \lambda_q^{-1} t^{\mu_2 - \mu_q} & 0 \end{bmatrix}, \quad (6.3.16)$$

$$\hat{\nu}_q = \begin{bmatrix} 0 & \lambda_q t^{\mu_q} \\ \lambda_2 \lambda_q^{-1} t^{\mu_2} & 0 \end{bmatrix}. \quad (6.3.17)$$

One other variation upon the general form $\hat{\mathbf{L}}_{1,k}$ is the general form $\mathbf{L}_{1,k}^0$. This is defined so that $\mathbf{L}_{1,k}^0$ is of the form $\hat{\mathbf{L}}_{1,k}$, but is such that $\mu_2 = 1$, $\mu_q = 0$ (for q odd, $1 \leq q \leq k$).

The next definition is not that of a general form of matrix. Let \mathbf{W}_j be the $j \times j$ diagonal matrix defined by

$$\mathbf{W}_j = \text{diag}\{1, -1, \dots\}. \quad (6.3.18)$$

Clearly, if j is even, then

$$\mathbf{W}_j = \text{diag}\{1, -1, \dots, 1, -1\}. \quad (6.3.19)$$

6.4 Involutive automorphisms of $C_\ell^{(1)}$

Introduction

In section 2 of this chapter, representatives of the conjugacy classes of the involutive members of the Weyl group of C_ℓ were given. For each such representative Weyl group element we require the corresponding involutive automorphisms of type 1a (with $u = \pm 1$). In each case, it is sufficient to give the matrix $\mathbf{U}(t)$ which generates the automorphism in question. Thus, for a Weyl group element τ_j , the matrix $\mathbf{U}(t)$ will satisfy

$$\mathbf{U}(t)\mathbf{h}_{\alpha_k}^0\mathbf{U}(t)^{-1} = \mathbf{h}_{\tau_j(\alpha_k)}^0 \quad (\text{for } 1 \leq k \leq \ell). \quad (6.4.1)$$

We recall that the conditions imposed upon $\mathbf{U}(t)$, in order that it generates an *involutive* automorphism are that

$$\mathbf{U}(t)\mathbf{U}(ut) = \lambda t^\mu \mathbf{1}_{2\ell} \quad \text{for type 1a automorphisms,} \quad (6.4.2)$$

$$\mathbf{U}(t)\mathbf{U}(ut^{-1}) = \lambda t^\mu \mathbf{1}_{2\ell} \quad \text{for type 2a automorphisms.} \quad (6.4.3)$$

The following subsections give lists of such matrices for the type 1a and type 2a involutive automorphisms which correspond to each of the Weyl group elements listed in section 2. In each case, use will be made of the general forms of matrices which were defined previously.

Type 1a involutive automorphisms with $u = 1$

For each of the representative Weyl group elements τ_j (where $1 \leq j \leq 9$), the following list gives the matrices $U(t)$ which generate type 1a involutive automorphisms (with $u = 1$) corresponding to those Weyl group transformations.

$$U(t) = \text{dsum}\{\mathbf{D}_{1,\ell}^0, \mathbf{D}_{1,\ell}^0\}, \quad (6.4.4)$$

$$U(t) = \text{dsum}\{\mathbf{D}_{1,\ell}^0, -\mathbf{D}_{1,\ell}^0\}, \quad (6.4.5)$$

$$U(t) = \text{offsum}\{\hat{\mathbf{G}}_{1,\ell}, \lambda_{\ell+1} t^{\mu_{\ell+1}} \hat{\mathbf{G}}_{1,\ell}^{-1}\}, \quad (6.4.6)$$

$$U(t) = \begin{bmatrix} \mathbf{D}_{1,q-1}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}_{q,\ell} \\ \mathbf{0} & \mathbf{0} & -\mathbf{D}_{1,q-1}^0 & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_{q,\ell}^{-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.4.7)$$

$$U(t) = \begin{bmatrix} \mathbf{L}_{1,q} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{q+2,r-1}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}_{r,\ell} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\tilde{\mathbf{L}}_{1,q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{D}_{q+2,r-1}^0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_{r,\ell}^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.4.8)$$

$$U(t) = \begin{bmatrix} \mathbf{L}_{1,q}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}_{q+2,\ell} \\ \mathbf{0} & \mathbf{0} & -\tilde{\mathbf{L}}_{1,q}^0 & \mathbf{0} \\ \mathbf{0} & \lambda_2 t^{\mu_2} \mathbf{G}_{q+2,\ell}^{-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.4.9)$$

$$U(t) = \text{dsum}\{\mathbf{L}_{1,q}, \mathbf{D}_{q+2,\ell}^0, \tilde{\mathbf{L}}_{1,q}, \mathbf{D}_{q+2,\ell}^0\}, \quad (6.4.10)$$

$$U(t) = \text{dsum}\{\mathbf{L}_{1,q}, \mathbf{D}_{q+2,\ell}^0, -\tilde{\mathbf{L}}_{1,q}, -\mathbf{D}_{q+2,\ell}^0\}, \quad (6.4.11)$$

$$U(t) = \begin{bmatrix} \mathbf{D}_{1,q-1}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{q,\ell-2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}_{\ell,\ell} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{D}_{1,q-1}^0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\tilde{\mathbf{L}}_{q,\ell-2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_{\ell,\ell}^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.4.12)$$

$$U(t) = \begin{bmatrix} \mathbf{L}_{1,\ell-2}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}_{\ell,\ell} \\ \mathbf{0} & \mathbf{0} & -\mathbf{L}_{1,\ell-2}^0 & \mathbf{0} \\ \mathbf{0} & \lambda_2 t^{\mu_2} \mathbf{G}_{\ell,\ell}^{-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.4.13)$$

$$U(t) = \text{dsum} \left\{ \mathbf{L}_{1,\ell-1}^0, \tilde{\mathbf{L}}_{1,\ell-1}^0 \right\}, \quad (6.4.14)$$

$$U(t) = \text{dsum} \left\{ \mathbf{L}_{1,\ell-1}^0, -\tilde{\mathbf{L}}_{1,\ell-1}^0 \right\}, \quad (6.4.15)$$

Type 1a involutive automorphisms with $u = -1$

Following on from the previous subsection, this subsection will give various matrices which give rise to involutive automorphisms of $C_\ell^{(1)}$, and which correspond to the Weyl group elements given in section 6.2. In this case, however, the automorphisms are of type 1a with $u = -1$. The automorphism-generating matrices are given in the following list.

$$U(t) = \text{dsum} \left\{ \mathbf{D}_{1,\ell}^0, \mathbf{D}_{1,\ell}^0 \right\}, \quad (6.4.16)$$

$$U(t) = \text{dsum} \left\{ \mathbf{D}_{1,\ell}^0, -\mathbf{D}_{1,\ell}^0 \right\}, \quad (6.4.17)$$

$$U(t) = \text{offsum} \left\{ \hat{G}_{1,\ell}^{\text{even}}, \lambda_{\ell+1} t^{\mu_{\ell+1}} \left(\hat{G}_{1,\ell}^{\text{even}} \right)^{-1} \right\}, \quad (6.4.18)$$

$$U(t) = \begin{bmatrix} D_{1,q-1}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & G_{q,\ell}^{\text{even}} \\ \mathbf{0} & \mathbf{0} & -D_{1,q-1}^0 & \mathbf{0} \\ \mathbf{0} & (G_{q,\ell}^{\text{even}})^{-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.4.19)$$

$$U(t) = \begin{bmatrix} D_{1,q-1}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & G_{q,\ell}^{\text{odd}} \\ \mathbf{0} & \mathbf{0} & -D_{1,q-1}^0 & \mathbf{0} \\ \mathbf{0} & (G_{q,\ell}^{\text{odd}})^{-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.4.20)$$

$$U(t) = \begin{bmatrix} M_{1,q} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_{q+2,r-1}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & G_{r,\ell}^{\text{even}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\tilde{M}_{1,q}^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -D_{q+2,r-1}^0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (G_{r,\ell}^{\text{even}})^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.4.21)$$

$$U(t) = \begin{bmatrix} M_{1,q} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_{q+2,r-1}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & G_{r,\ell}^{\text{odd}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\tilde{M}_{1,q}^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -D_{q+2,r-1}^0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (G_{r,\ell}^{\text{odd}})^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.4.22)$$

$$U(t) = \begin{bmatrix} \hat{M}_{1,q} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & G_{q+2,\ell}^{\text{even}} \\ \mathbf{0} & \mathbf{0} & -(\hat{M}_{1,q})^{-1} & \mathbf{0} \\ \mathbf{0} & \lambda_2 t^{\mu_2} (G_{q+2,\ell}^{\text{even}})^{-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.4.23)$$

$$U(t) = \begin{bmatrix} \hat{\mathbf{M}}_{1,q} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}_{q+2,\ell}^{\text{odd}} \\ \mathbf{0} & \mathbf{0} & -(\hat{\mathbf{M}}_{1,q})^{\sim} & \mathbf{0} \\ \mathbf{0} & \lambda_{-2} t^{\mu_2} (\mathbf{G}_{q+2,\ell}^{\text{odd}})^{-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.4.24)$$

$$U(t) = \text{dsum} \left\{ \mathbf{M}_{1,q}, \mathbf{D}_{q+2,\ell}^0, \tilde{\mathbf{M}}_{1,q}^{-1}, \mathbf{D}_{q+2,\ell}^0 \right\}, \quad (6.4.25)$$

$$U(t) = \text{dsum} \left\{ \mathbf{M}_{1,q}, \mathbf{D}_{q+2,\ell}^0, -\tilde{\mathbf{M}}_{1,q}^{-1}, -\mathbf{D}_{q+2,\ell}^0 \right\}, \quad (6.4.26)$$

$$U(t) = \begin{bmatrix} \mathbf{D}_{1,q-1}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{q,\ell-2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}_{\ell,\ell}^{\text{even}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{D}_{1,q-1}^0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\tilde{\mathbf{M}}_{q,\ell-2}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -(\mathbf{G}_{\ell,\ell}^{\text{even}})^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.4.27)$$

$$U(t) = \begin{bmatrix} \mathbf{D}_{1,q-1}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{q,\ell-2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}_{\ell,\ell}^{\text{odd}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{1,q-1}^0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{\mathbf{M}}_{q,\ell-2}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -(\mathbf{G}_{\ell,\ell}^{\text{odd}})^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.4.28)$$

$$U(t) = \begin{bmatrix} \hat{\mathbf{M}}_{1,\ell-2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}_{\ell,\ell}^{\text{even}} \\ \mathbf{0} & \mathbf{0} & -\lambda_2 t^{\mu_2} (\hat{\mathbf{M}}_{1,\ell-2}^{-1})^{\sim} & \mathbf{0} \\ \mathbf{0} & \lambda_2 t^{\mu_2} (\mathbf{G}_{\ell,\ell}^{\text{even}})^{-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.4.29)$$

$$U(t) = \begin{bmatrix} \hat{\mathbf{M}}_{1,\ell-2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}_{\ell,\ell}^{\text{odd}} \\ \mathbf{0} & \mathbf{0} & \lambda_2 t^{\mu_2} (\hat{\mathbf{M}}_{1,\ell-2}^{-1})^{\sim} & \mathbf{0} \\ \mathbf{0} & -\lambda_2 t^{\mu_2} (\mathbf{G}_{\ell,\ell}^{\text{odd}})^{-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.4.30)$$

$$U(t) = \text{dsum} \left\{ \hat{\mathbf{M}}_{1,\ell-1}, \lambda_2 t^{\mu_2} \left(\hat{\mathbf{M}}_{1,\ell-1} \right)^{\sim} \right\} \quad (6.4.31)$$

$$U(t) = \text{dsum} \left\{ \hat{\mathbf{M}}_{1,\ell-1}, -\lambda_2 t^{\mu_2} \left(\hat{\mathbf{M}}_{1,\ell-1} \right)^{\sim} \right\} \quad (6.4.32)$$

Type 2a involutive automorphisms with $u = 1$

The type 2a involutive automorphisms (with $u = 1$) which correspond to the Weyl transformations given in section 6.2 are generated by the matrices given in the following list:

$$U(t) = \text{dsum} \left\{ \mathbf{D}_{1,\ell}, \mathbf{D}_{1,\ell} \right\}, \quad (6.4.33)$$

$$U(t) = \text{dsum} \left\{ \mathbf{D}_{1,\ell}, -\mathbf{D}_{1,\ell} \right\}, \quad (6.4.34)$$

$$U(t) = \begin{bmatrix} \mathbf{0} & \hat{\mathbf{G}}_{1,\ell}^{\text{zero}} \\ \lambda_{\ell+1} t^{\mu_{\ell+1}} \left(\hat{\mathbf{G}}_{1,\ell}^{\text{zero}} \right)^{-1} & \mathbf{0} \end{bmatrix}, \quad (6.4.35)$$

$$U(t) = \begin{bmatrix} \mathbf{D}_{1,q-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & t^{\mu_q} \mathbf{G}_{q,\ell}^{\text{zero}} \\ \mathbf{0} & \mathbf{0} & -t^{2\mu_q} \left(\mathbf{D}_{1,q-1} \right)^{-1} & \mathbf{0} \\ \mathbf{0} & t^{\mu_q} \left(\mathbf{G}_{q,\ell}^{\text{zero}} \right)^{-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.4.36)$$

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{N}_{1,q} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{q+2,r-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & t^{\mu_r} \mathbf{G}_{r,\ell}^{\text{zero}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -t^{2\mu_r} (\tilde{\mathbf{N}}_{1,q})^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -t^{2\mu_r} (\mathbf{D}_{q+2,r-1})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & t^{\mu_r} (\mathbf{G}_{r,\ell}^{\text{zero}})^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.437)$$

$$\mathbf{U}(t) = \begin{bmatrix} \hat{\mathbf{N}}_{1,q} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & t^{\mu_\ell} \mathbf{G}_{q+2,\ell}^{\text{zero}} \\ \mathbf{0} & \mathbf{0} & -\lambda_2 t^{2\mu_\ell} \left((\hat{\mathbf{N}}_{1,q})^{-1} \right)^\sim & \mathbf{0} \\ \mathbf{0} & \lambda_2 t^{2\mu_\ell} (\mathbf{G}_{q+2,\ell}^{\text{zero}})^{-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.438)$$

$$\mathbf{U}(t) = \text{dsum} \left\{ \mathbf{N}_{1,q}, \mathbf{D}_{q+2,\ell}, t^{\mu_{\ell+1}} (\tilde{\mathbf{N}}_{1,q})^{-1}, t^{\mu_{\ell+1}} \mathbf{D}_{q+2,\ell} \right\}, \quad (6.439)$$

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{D}_{1,q-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_{q,\ell-2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}_{\ell,\ell}^{\text{zero}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -t^{2\mu_\ell} (\mathbf{D}_{1,q-1})^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -(\tilde{\mathbf{N}}_{q,\ell-2})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & t^{2\mu_\ell} (\mathbf{G}_{\ell,\ell}^{\text{zero}})^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.440)$$

$$\mathbf{U}(t) = \begin{bmatrix} \hat{\mathbf{N}}_{1,\ell-2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & t^{\mu_\ell} \mathbf{G}_{\ell,\ell}^{\text{zero}} \\ \mathbf{0} & \mathbf{0} & -\lambda_2 t^{2\mu_\ell} (\hat{\mathbf{N}}_{1,\ell-2}) & \mathbf{0} \\ \mathbf{0} & \lambda_2 t^{\mu_\ell} \mathbf{G}_{\ell,\ell}^{\text{zero}} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.441)$$

$$\mathbf{U}(t) = \text{dsum} \left\{ \hat{\mathbf{N}}_{1,\ell-1}, \hat{\mathbf{N}}_{1,\ell-1} \right\}, \quad (6.442)$$

$$\mathbf{U}(t) = \text{dsum} \left\{ \hat{\mathbf{N}}_{1,\ell-1}, -\hat{\mathbf{N}}_{1,\ell-1} \right\}. \quad (6.443)$$

6.5 Some matrix transformations

In this section, certain useful matrix transformations will be given. In the subsequent analysis, they will help to show that certain assumptions are valid, thus simplifying the work.

1. We are dealing (for the Kac-Moody algebra $C_\ell^{(1)}$) with $2\ell \times 2\ell$ matrices.

The index set of such matrices may be written as

$$I = \{1, 2, \dots, \ell, \ell + 1, \dots, 2\ell\}. \quad (6.5.1)$$

Let the integers j and k be such that $1 \leq j < k \leq \ell$. Then let two matrices $\mathbf{U}(t)$ and $\mathbf{V}(t)$ be such that $\mathbf{V}(t)$ is obtained from $\mathbf{U}(t)$ by exchanging j and k (and also $(j + \ell)$ and $(k + \ell)$) in the index set given above. Then define a t -independent matrix \mathbf{S} by

$$S_{ab} = \begin{cases} 1 & \text{if } a = b; a \neq j, k, j + \ell, k + \ell, \\ 0 & \text{if } a = b; a = j, k, j + \ell, k + \ell, \\ 1 & \text{if } (a, b) = (j, k), (k, j), (j + \ell, k + \ell), (k + \ell, j + \ell), \\ 0 & \text{otherwise.} \end{cases} \quad (6.5.2)$$

The matrix \mathbf{S} is such that it satisfies

$$\tilde{\mathbf{S}}\mathbf{J}\mathbf{S} = \mathbf{J}, \quad (6.5.3)$$

$$\tilde{\mathbf{S}}^*\mathbf{S} = \mathbf{1}_{2\ell}, \quad (6.5.4)$$

$$\mathbf{S}\mathbf{U}(t)\mathbf{S}^{-1} = \mathbf{V}(t). \quad (6.5.5)$$

Thus, if the matrices $U(t)$ and $V(t)$ generate involutive automorphisms of $C_t^{(1)}$ of the same type (and with the same value of u), then $\{U(t), u, \xi\}$ and $\{V(t), u, \xi\}$ are conjugate to each other. In effect, this means that there is some degree of choice in the ordering of the elements of the index set.

2. In a similar fashion to the preceding example, let $U(t)$ and $V(t)$ generate automorphisms of $C_t^{(1)}$ of the same type, and with the same value of u . Furthermore, suppose that $V(t)$ is obtained from $U(t)$ by exchanging the elements j and $(j + \ell)$ in the index set (which we have seen is given by $\{1, 2, \dots, 2\ell\}$). In this case, a matrix S (which is t -independent) is given by

$$S_{ab} = \begin{cases} 1 & \text{if } a = b; a \neq j, j + \ell, \\ 0 & \text{if } a = b; a = j, j + \ell, \\ i & \text{if } (a, b) = (j, j + \ell), (j + \ell, j), \\ 0 & \text{otherwise.} \end{cases} \quad (6.5.6)$$

It may be verified that

$$\tilde{S}JS = J, \quad (6.5.7)$$

$$\tilde{S}^*S = \mathbf{1}_{2t}, \quad (6.5.8)$$

$$SU(t)S^{-1} = V(t). \quad (6.5.9)$$

Therefore, the automorphisms $\{U(t), u, \xi\}$ and $\{V(t), u, \xi\}$ are mutually conjugate.

Once again, this implies a degree of choice in the elements of the index set.

3. Let $U(t)$ be of the form given by (6.4.8) and let $V(t)$ be obtained from $U(t)$ by replacing $L_{1,q}$ with W_{q+1} . We define a matrix $S(t)$ by

$$S(t) = \text{dsum} \left\{ V_1, \dots, V_{\frac{1}{2}(q+1)}, \mathbf{1}_{\ell-q-1}, V'_1, \dots, V'_{\frac{1}{2}(q+1)}, \mathbf{1}_{\ell-q-1} \right\}, \quad (6.5.10)$$

where the submatrices in (6.5.10) are given by

$$\mathbf{V}_j = \frac{1}{\sqrt{2}} \begin{bmatrix} \lambda_{2j-1} t^{\mu_{2j-1}} & \lambda_{2j-1} t^{\mu_{2j-1}} \\ 1 & -1 \end{bmatrix}, \quad (6.5.11)$$

$$\mathbf{V}'_j = \frac{1}{\sqrt{2}} \begin{bmatrix} \lambda_{2j-1}^{-1} t^{-\mu_{2j-1}} & \lambda_{2j-1}^{-1} t^{-\mu_{2j-1}} \\ 1 & -1 \end{bmatrix}. \quad (6.5.12)$$

It then follows that

$$\tilde{\mathbf{S}}(t)\mathbf{J}\mathbf{S}(t) = \mathbf{J}, \quad (6.5.13)$$

$$\mathbf{S}(t)\mathbf{V}(t)\mathbf{S}(t)^{-1} = \mathbf{U}(t). \quad (6.5.14)$$

Furthermore, if $\mathbf{U}(t)$ is such that $\tilde{\mathbf{U}}^*(t^{-1})\mathbf{U}(t) = \lambda t^\mu \mathbf{1}_{2\ell}$, it follows that $\mathbf{S}(t)$ is such that

$$\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \alpha t^\beta \mathbf{1}_{2\ell}. \quad (6.5.15)$$

Thus, the type 1a involutive automorphisms $\{\mathbf{U}(t), 1, \xi\}$ and $\{\mathbf{V}(t), 1, \xi\}$ are conjugate.

This may be adapted slightly for other matrices of the forms given in (6.4.4)-(6.4.15).

Thus, if $\mathbf{U}(t)$ is of one of these forms, it may be assumed (without loss of generality)

that the submatrix $\mathbf{L}_{j,k}$ is replaced by \mathbf{W}_{k-j+2} . Furthermore, this analysis may also be

modified for type 1a involutive automorphisms with $\mu = -1$ or for the type 2a

involutive automorphisms with $\mu = 1$. In those cases, the submatrices $\mathbf{M}_{j,k}$ or $\mathbf{N}_{j,k}$

may be replaced by \mathbf{W}_{k-j+2} .

4. Let the matrix $\mathbf{U}(t)$ be of the form

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{H}_{q-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}_{q,\ell} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_{q,\ell}^{-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.5.16)$$

where \mathbf{H}_{q-1} represents an arbitrary $(q-1) \times (q-1)$ matrix. The matrix $\mathbf{V}(t)$ is

obtained from $\mathbf{U}(t)$ by letting $\lambda_j t^{\mu_j} = t^{\deg \mu_j}$ (for $q \leq j \leq \ell$). We define $\mathbf{S}(t)$ by

$$\mathbf{S}(t) = \text{dsum}\{\mathbf{1}_{q-1}, \mathbf{t}, \mathbf{1}_{q-1}, \mathbf{t}^{-1}\}, \quad (6.5.17)$$

where the submatrix \mathbf{t} is given by

$$\mathbf{t} = \text{diag} \left\{ \lambda_q^{\frac{1}{2}} t^{\sigma_1}, \dots, \lambda_\ell^{\frac{1}{2}} t^{\sigma_\ell} \right\}, \quad (6.5.18)$$

and the quantity σ_j is defined to be such that $\sigma_j = \frac{1}{2}(\text{deg } \mu_j - \mu_j)$. The type 1a involutive automorphisms $\{\mathbf{U}(t), 1, \xi\}$ and $\{\mathbf{V}(t), 1, \xi\}$ are conjugate, since

$$\tilde{\mathbf{S}}(t) \mathbf{J} \mathbf{S}(t) = \mathbf{J}, \quad (6.5.19)$$

$$\mathbf{S}(t) \mathbf{V}(t) \mathbf{S}(t)^{-1} = \mathbf{U}(t). \quad (6.5.20)$$

Furthermore, if $\mathbf{U}(t)$ satisfies $\tilde{\mathbf{U}}^*(t^{-1}) \mathbf{U}(t) = \lambda t^\mu \mathbf{1}_{2\ell}$ (and hence if $\mathbf{V}(t)$ satisfies $\tilde{\mathbf{V}}^*(t^{-1}) \mathbf{V}(t) = \lambda t^\mu \mathbf{1}_{2\ell}$), then

$$\tilde{\mathbf{S}}^*(t^{-1}) \mathbf{S}(t) = \alpha t^\beta \mathbf{1}_{2\ell}. \quad (6.5.21)$$

The element $\lambda_j t^{\mu_j}$ (for $q \leq j \leq \ell$) may therefore be replaced by $t^{\text{deg } \mu_j}$, since $\{\mathbf{U}(t), 1, \xi\}$ and $\{\mathbf{V}(t), 1, \xi\}$ are conjugate. Similar results follow for type 1a involutive automorphisms with $u = -1$, and for type 2a involutive automorphisms with $u = 1$.

5. Let $\mathbf{U}(t)$ be of the form (6.4.9). The matrix $\mathbf{V}(t)$ is derived from $\mathbf{U}(t)$ by letting $\lambda_j t^{\mu_j} = 1$ (for j such that $3 \leq j \leq q$ and $(q-j)$ is even), and by letting $\lambda_j t^{\mu_j} = t^{\text{deg } \mu_j}$ (for $j = 2, q+2 \leq j \leq \ell$). We define $\mathbf{S}(t)$ by

$$\mathbf{S}(t) = \text{dsum} \left\{ \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{-1} \right\}, \quad (6.5.22)$$

where the submatrices \mathbf{t}_j ($j = 1, 2, 3$) are given by

$$\mathbf{t}_1 = \text{diag} \left\{ \lambda_2^{-\frac{1}{2}} t^{-\sigma_2}, 1, \lambda_2^{-\frac{1}{2}} \lambda_3 t^{\mu_3 - \sigma_2}, 1, \dots, \lambda_2^{-\frac{1}{2}} \lambda_q t^{\mu_q - \sigma_2}, 1 \right\}, \quad (6.5.23)$$

$$\mathbf{t}_2 = \lambda_2^{-\frac{1}{2}} \text{diag} \left\{ \lambda_{q+2}^{\frac{1}{2}} t^{\sigma_{q+2}}, \dots, \lambda_\ell^{\frac{1}{2}} t^{\sigma_\ell} \right\}, \quad (6.5.24)$$

$$\mathbf{t}_3 = \text{diag} \left\{ \lambda_2^{\frac{1}{2}} \lambda_2^{-1} t^{\sigma_2}, \lambda_2^{\frac{1}{2}} \lambda_3^{-1} t^{-\mu_3} t^{\sigma_2}, \dots, \lambda_2^{\frac{1}{2}} \lambda_q^{-1} t^{-\mu_q} t^{\sigma_2} \right\}. \quad (6.5.25)$$

The matrix $\mathbf{S}(t)$ is such that the following hold:

$$\tilde{\mathbf{S}}(t) \mathbf{J} \mathbf{S}(t) = t^{\sigma_2} \mathbf{J}, \quad (6.5.26)$$

$$\mathbf{S}(t)\mathbf{V}(t)\mathbf{S}(t)^{-1} = t^{\sigma_2}\mathbf{U}(t). \quad (6.5.27)$$

Furthermore, if $\tilde{\mathbf{U}}^*(t^{-1})\mathbf{U}(t) = \lambda t^\mu \mathbf{1}_{2\ell}$, then $\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \alpha t^\beta \mathbf{1}_{2\ell}$ (where α, λ are both non-zero complex numbers). Thus, the type 1a involutive automorphisms $\{\mathbf{U}(t), 1, \xi\}$ and $\{\mathbf{V}(t), 1, \xi\}$ are conjugate. A similar conclusion is reached when $\mathbf{U}(t)$ is given by (6.4.13), which is itself a special case of (6.4.9). This analysis may be adapted slightly for the type 1a involutive automorphisms (with $u = -1$) and for the type 2a involutive automorphisms (with $u = 1$).

6. Let $\mathbf{U}(t)$ be of the form (6.4.36), and let $\mathbf{V}(t)$ be obtained from $\mathbf{U}(t)$ by letting $\mu_q = 0$. If we let $\mathbf{S}(t) = \text{dsum}\{\mathbf{1}_\ell, t^{-\mu_q} \mathbf{1}_\ell\}$, then

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(t^{-1})^{-1} = \mathbf{V}(t), \quad (6.5.28)$$

$$\tilde{\mathbf{S}}(t)\mathbf{J}\mathbf{S}(t) = \mathbf{J}, \quad (6.5.29)$$

$$\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{2\ell}. \quad (6.5.30)$$

Thus, the type 2a involutive automorphisms $\{\mathbf{U}(t), 1, \xi\}$ and $\{\mathbf{V}(t), 1, \xi\}$ are conjugate. Similarly, if $\mathbf{U}(t)$ is given by (6.4.37) (or (6.4.38), (6.4.40), (6.4.41)), then the type 2a involutive automorphism $\{\mathbf{U}(t), 1, \xi\}$ is conjugate to $\{\mathbf{V}(t), 1, \xi\}$, where $\mathbf{V}(t)$ is derived from $\mathbf{U}(t)$ by letting $\mu_r = 0$ (or $\mu_\ell = 0$).

7. Let $\mathbf{U}(t)$ be given by

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{H}_{\ell-q} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{a} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_{\ell-q} & \mathbf{0} \\ \mathbf{0} & \mathbf{a}^{-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.5.31)$$

where $\mathbf{H}_{\ell-q}$ is used to represent an arbitrary $(\ell - q) \times (\ell - q)$ matrix, and where

$$\mathbf{a} = \text{diag}\{t^{\mu_{\ell-q+1}}, \dots, t^{\mu_\ell}\}. \quad (6.5.32)$$

The matrix $\mathbf{V}(t)$ is given by $\mathbf{V}(t) = \text{dsum}\{\mathbf{H}_{\ell-q}, \mathbf{1}_q, \mathbf{H}_{\ell-q}, -\mathbf{1}_\ell\}$, and $\mathbf{S}(t)$ by

$$\mathbf{S}(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2}\mathbf{1}_{t-q} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_q & \mathbf{0} & -\mathbf{a} \\ \mathbf{0} & \mathbf{0} & \sqrt{2}\mathbf{1}_{t-q} & \mathbf{0} \\ \mathbf{0} & \mathbf{a}^{-1} & \mathbf{0} & \mathbf{1}_q \end{bmatrix}. \quad (6.5.33)$$

We have, therefore, that

$$\tilde{\mathbf{S}}(t)\mathbf{J}\mathbf{S}(t) = \mathbf{J}, \quad (6.5.34)$$

$$\mathbf{S}(t)\mathbf{V}(t)\mathbf{S}(t)^{-1} = \mathbf{U}(t). \quad (6.5.35)$$

Thus, the type 1a involutive automorphisms $\{\mathbf{U}(t), 1, \xi\}$ and $\{\mathbf{V}(t), 1, \xi\}$ are conjugate.

8. The mapping $\mathbf{U}(t) \mapsto \mathbf{U}(st)$ is important within the matrix formulation. However, for all of the matrices $\mathbf{U}(t)$ that we are studying, this mapping is such that $\mathbf{U}(st)$ is of the same form as $\mathbf{U}(t)$, but such that the coefficients λ_j might be different. Thus, for type 1a automorphisms, the contents of section 5 imply that the assumption $s = 1$ may be made.

6.6 Type 1a involutive automorphisms of $C_\ell^{(1)}$ with $u = 1$

To each of the rotations τ_j (for $1 \leq j \leq 9$) which were defined previously, there exists a general form of matrix $U(t)$ which is such that

$$U(t) \left\{ \mathbf{h}_{\alpha_k}^0 \right\} U(t)^{-1} = \mathbf{h}_{\tau_j(\alpha_k)}^0. \quad (6.6.1)$$

Then, for each such form of $U(t)$, those matrices of the form which also satisfy

$$\begin{aligned} U(t)^2 &= \alpha t^\beta \mathbf{1}_{2\ell} & (\alpha \neq 0), \\ \bar{U}^*(t^{-1})U(t) &= \lambda t^\mu \mathbf{1}_{2\ell} & (\lambda \neq 0), \end{aligned} \quad (6.6.2)$$

may also be found. The matrices yielded by this process all generate involutive automorphisms of the compact real form of $C_\ell^{(1)}$ of type 1a with $u = 1$. However, it is not necessary to analyse all of these matrices in order to find all of the conjugacy classes of the automorphisms they generate. That this is the case follows on from the matrix relations which are given in the section 6.5. These imply that many of the automorphism are conjugate to others under consideration, and thus reduce the number of separate automorphisms which need to be examined. In fact, using the information contained in the section 6.5, the following matrices are obtained

$$U(t) = \text{dsum} \{ \mathbf{1}_a, -\mathbf{1}_{\ell-a}, \mathbf{1}_a, -\mathbf{1}_{\ell-a} \}, \quad (6.6.3)$$

$$U(t) = \text{dsum} \{ \mathbf{1}_\ell, -\mathbf{1}_\ell \}, \quad (6.6.4)$$

$$U(t) = \text{offsum} \{ \mathbf{1}_\ell, t\mathbf{1}_\ell \}, \quad (6.6.5)$$

$$U(t) = \begin{bmatrix} \mathbf{L}_{1,q}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{\ell-q-1} \\ \mathbf{0} & \mathbf{0} & -\bar{\mathbf{L}}_{1,q}^0 & \mathbf{0} \\ \mathbf{0} & t\mathbf{1}_{\ell-q-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.6.6)$$

(Note that the definition of $\mathbf{L}_{1,q}^0$ does imply the t -dependence in (6.6.6))

$$U(t) = \text{dsum} \{ \mathbf{L}_{1,\ell-1}^0, \mathbf{L}_{1,\ell-1}^0 \}, \quad (6.6.7)$$

$$U(t) = \text{dsum} \{ \mathbf{L}_{1,\ell-1}^0, -\bar{\mathbf{L}}_{1,\ell-1}^0 \}. \quad (6.6.8)$$

Suppose then, that \mathbf{U}_b is of the form given by (6.6.3), with

$$\mathbf{U}_b = \text{dsum} \{ \mathbf{1}_{\ell-b}, -\mathbf{1}_b, \mathbf{1}_{\ell-b}, -\mathbf{1}_b \}. \quad (6.6.9)$$

The conjugacy class $(A)^{(b)}$ may be defined to be that conjugacy class which contains the type 1a involutive automorphism $\{ \mathbf{U}_b, 1, 0 \}$. It should first be noted that j may be assumed to be such that $1 \leq b \leq \lfloor \frac{\ell}{2} \rfloor$. For, if $\lfloor \frac{\ell}{2} \rfloor < b \leq \ell$, then section 6.5 implies that \mathbf{U}_b is equivalent to some matrix $\mathbf{U}_{b'}$, where $1 \leq b' \leq \lfloor \frac{\ell}{2} \rfloor$. Now, with $1 \leq b < c \leq \lfloor \frac{\ell}{2} \rfloor$, suppose that there exists some Laurent polynomial matrix $\mathbf{S}(t)$ which satisfies

$$\mathbf{S}(t)\mathbf{U}_c\mathbf{S}(t)^{-1} = \lambda t^\mu \mathbf{U}_b, \quad (6.6.10)$$

where $\mathbf{U}_c = \text{dsum} \{ \mathbf{1}_{\ell-c}, -\mathbf{1}_c, \mathbf{1}_{\ell-c}, -\mathbf{1}_c \}$. It is easily verified that there does not exist any Laurent polynomial matrix $\mathbf{S}(t)$ which satisfies these conditions for $1 \leq b < c \leq \lfloor \frac{\ell}{2} \rfloor$. Thus, for $1 \leq b < c \leq \lfloor \frac{\ell}{2} \rfloor$, the conjugacy classes $(A)^{(j)}$ and $(A)^{(k)}$ are not conjugate. The class $(A)^{(0)}$ is clearly the conjugacy class containing only the identity automorphism. For each b with $1 \leq b \leq \ell$, there exists a basis of the compact real form of $C_\ell^{(1)}$, all of whose members are eigenvectors of the representative automorphism given previously. Such a basis (with respect to the representative of the conjugacy class $(A)^{(b)}$) is given below

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \quad 1 \leq k \leq \ell, j \in \mathbf{Z} \\ i(e_{j\delta}^k - e_{-j\delta}^k) \quad 1 \leq k \leq \ell, j \in \mathbf{Z}^\pm \end{array} \right\} \text{eigenvalue } 1, \quad (6.6.11)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \quad j \in \mathbf{Z} \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \quad j \in \mathbf{Z}^\pm \end{array} \right\} \text{eigenvalue } 1, \quad (6.6.12)$$

where the quantity α in (6.6.12) above is given by the following:

$$\alpha = \begin{cases} \sum_{p=r}^{s-1} \alpha_p & \{1 \leq r < s \leq \ell - b, \\ & \{\ell - b < r < s \leq \ell, \\ \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell-1} \alpha_p + \alpha_\ell & \{1 \leq r \leq s \leq \ell - b, \\ & \{\ell - b < r \leq s \leq \ell, \end{cases} \quad (6.6.13)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \quad j \in \mathbf{Z} \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \quad j \in \mathbf{Z}^\pm \end{array} \right\} \text{eigenvalue } -1, \quad (6.6.14)$$

where the quantity α in (6.6.14) above is given by the following:

$$\alpha = \begin{cases} \sum_{p=r}^{s-1} \alpha_p & \{1 \leq r \leq \ell - b < s \leq \ell, \\ \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell-1} \alpha_p + \alpha_\ell & \{1 \leq r \leq \ell - b < s \leq \ell, \end{cases} \quad (6.6.15)$$

$$\left. \begin{array}{l} ic \\ id \end{array} \right\} \text{eigenvalue } 1. \quad (6.6.16)$$

Thus, a basis for a real form of $C_\ell^{(1)}$ is provided by the following set of elements

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \quad 1 \leq k \leq \ell, j \in \mathbf{Z}, \\ i(e_{j\delta}^k - e_{-j\delta}^k) \quad 1 \leq k \leq \ell, j \in \mathbf{Z}^\pm, \end{array} \right\} \quad (6.6.17)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \quad j \in \mathbf{Z}, \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \quad j \in \mathbf{Z}^\pm, \end{array} \right\} \quad (6.6.18)$$

where the quantity α in (6.6.18) above is given by the following:

$$\alpha = \begin{cases} \sum_{p=r}^{s-1} \alpha_p & \{1 \leq r < s \leq \ell - b, \\ \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell-1} \alpha_p + \alpha_\ell & \{\ell - b < r < s \leq \ell, \\ & \{1 \leq r \leq s \leq \ell - b, \\ & \{\ell - b < r \leq s \leq \ell, \end{cases} \quad (6.6.19)$$

$$\begin{aligned} & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \quad j \in \mathbf{Z}, \\ & (e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \quad j \in \mathbf{Z}^\pm, \end{aligned} \quad (6.6.20)$$

where the quantity α in the above is given by the following:

$$\alpha = \begin{cases} \sum_{p=r}^{s-1} \alpha_p & \{1 \leq r \leq \ell - b < s \leq \ell, \\ \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell-1} \alpha_p + \alpha_\ell & \{1 \leq r \leq \ell - b < s \leq \ell, \end{cases} \quad (6.6.21)$$

$$\begin{aligned} & ic, \\ & id. \end{aligned} \quad (6.6.22)$$

Let the matrix $\mathbf{U}(t)$ be given by $\mathbf{U}(t) = \text{dsum}\{\mathbf{1}_\ell, -\mathbf{1}_\ell\}$. The conjugacy class (B) is defined to be that which contains as representative the type 1a involutive automorphism $\{\mathbf{U}(t), 1, 0\}$. It is required firstly to show that (B) is disjoint from the conjugacy classes (A)^(j) (for $1 \leq j \leq \lfloor \frac{\ell}{2} \rfloor$). Suppose that this were not the case, and that there did exist some Laurent polynomial matrix $\mathbf{S}(t)$ such that

$$\mathbf{S}(t) \text{dsum}\{\mathbf{1}_\ell, -\mathbf{1}_\ell\} \mathbf{S}(t)^{-1} = \lambda t^\mu \mathbf{U}_j \quad (6.6.23)$$

for some value of j . Consideration of the determinants of both sides of this equation implies that ℓ must be even, and furthermore that $j = \frac{1}{2}\ell$. However, since the above equation holds for all values of t , it must hold in particular for $t = 1$. Upon making the substitution $t = 1$ in the above equation, it becomes clear that the matrix $\mathbf{S}(1)$ cannot be chosen so that

$$\tilde{\mathbf{S}}(1) \mathbf{J} \mathbf{S}(1) = \lambda \mathbf{J}. \quad (6.6.24)$$

Thus, the conjugacy class (B) is disjoint (that is, distinct) from the conjugacy classes (A)^(j) for $1 \leq j \leq \lfloor \frac{\ell}{2} \rfloor$. The representative automorphism for the class (B) is the type 1a automorphism $\{U, 1, 0\}$, where $U = \text{dsum}\{1_\ell, -1_\ell\}$. The action of this automorphism upon a basis of the compact real form of $C_\ell^{(1)}$ (the members of which are all eigenvectors of the automorphism) is summarised below

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \quad 1 \leq k \leq \ell, j \in \mathbf{N}^0 \\ i(e_{j\delta}^k - e_{-j\delta}^k) \quad 1 \leq k \leq \ell, j \in \mathbf{N} \end{array} \right\} \text{eigenvalue } 1, \quad (6.6.25)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \alpha = \sum_{p=r}^{s-1} \alpha_p; 1 \leq r < s \leq \ell; \text{ eigenvalue } 1, \quad (6.6.26)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \alpha = \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell-1} \alpha_p + \alpha_\ell; 1 \leq r \leq s \leq \ell; \text{ eigenvalue } -1, \quad (6.6.27)$$

$$\left. \begin{array}{l} ic \\ id \end{array} \right\} \text{eigenvalue } 1. \quad (6.6.28)$$

Therefore, the corresponding basis for a real form of the Kac-Moody algebra $C_\ell^{(1)}$ is

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \quad 1 \leq k \leq \ell, j \in \mathbf{N}^0, \\ i(e_{j\delta}^k - e_{-j\delta}^k) \quad 1 \leq k \leq \ell, j \in \mathbf{N}, \end{array} \right\} \quad (6.6.29)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \alpha = \sum_{p=r}^{s-1} \alpha_p; 1 \leq r < s \leq \ell, \quad (6.6.30)$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \alpha = \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell-1} \alpha_p + \alpha_\ell; 1 \leq r \leq s \leq \ell, \quad (6.6.31)$$

$$\left. \begin{array}{l} ic, \\ id. \end{array} \right\} \quad (6.6.32)$$

Consider now the type 1a involutive automorphism $\{U(t), 1, \xi\}$, where $U(t) = \text{offsum}\{1_\ell, t1_\ell\}$. Consideration of the determinant of the matrix $U(t)$ leads to the conclusion that the type 1a involutive automorphism $\{U(t), 1, -\frac{\ell}{2}(\ell + 1)\}$ does not belong to any of the conjugacy classes that have already been identified. Suppose that the automorphism in question were a member of one of these classes. For each of the previously-identified classes, a representative automorphism may be taken to be $\{V, 1, 0\}$, where V is some t -independent matrix. This supposes that there exists a Laurent polynomial matrix $S(t)$ which is such that

$$S(t)VS(t)^{-1} = \lambda t^\mu U(t). \quad (6.6.33)$$

Consideration of the determinants of both sides of this equation implies that $\mu = -\frac{1}{2}$, which is a contradiction. Thus, the type 1a involutive automorphism $\{U(t), 1, -\frac{\ell}{2}(\ell + 1)\}$ belongs to a conjugacy class called (C) which is disjoint from all of the other conjugacy classes that have been identified previously. The representative automorphism of the class (C) is the automorphism $\{U(t), 1, -\frac{\ell}{2}(\ell + 1)\}$. There follow eigenvectors of this automorphism, which together form a basis of the compact real form of $C_\ell^{(1)}$:

$$\left. \begin{array}{l} ih_{\alpha_k} \\ i(h_{\alpha_k} - \frac{1}{2}c) \end{array} \right\} \begin{array}{l} (1 \leq k \leq \ell - 1) \\ \end{array} \left. \right\} \text{eigenvalue } -1, \quad (6.6.34)$$

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \\ i(e_{j\delta}^k - e_{-j\delta}^k) \end{array} \right\} j \in \mathbf{N}; \text{ eigenvalue } -1, \quad (6.6.35)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \text{ eigenvalue } 1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \text{ eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \text{ eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \text{ eigenvalue } -1 \end{array} \right\} \alpha = \sum_{p=r}^{s-1} \alpha_p; (1 \leq r < s \leq \ell), \quad (6.6.36)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \text{ eig. } -1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \text{ eig. } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \text{ eig. } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \text{ eig. } -1 \end{array} \right\} \left[\alpha = \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell-1} \alpha_p + \alpha_\ell; (1 \leq r < s \leq \ell-1) \right],$$

(where eig. stands for eigenvalue) (6.6.37)

$$\left. i c \left(d + (\ell + 1) \left(\sum_{p=1}^{\ell-1} p h_{\alpha_p} + \frac{\ell}{2} h_{\alpha_\ell} \right) \right) \right\} \text{ eigenvalue } 1. \quad (6.6.38)$$

Thus, the basis of the corresponding real form suggested by Cartan's theorem is given by

$$\begin{aligned} & h_{\alpha_k} \quad (1 \leq k \leq \ell-1), \\ & (h_{\alpha_\ell} - \frac{1}{2}c), \end{aligned} \quad (6.6.39)$$

$$\left. \begin{array}{l} i(e_{j\delta}^k + e_{-j\delta}^k) \\ (e_{j\delta}^k - e_{-j\delta}^k) \end{array} \right\} j \in \mathbf{N}, \quad (6.6.40)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \end{array} \right\} \alpha = \sum_{p=r}^{s-1} \alpha_p; (1 \leq r < s \leq \ell), \quad (6.6.41)$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \end{array} \right\} \left[\alpha = \sum_{p=r}^{s-1} \alpha_p + 2 \sum_{p=s}^{\ell-1} \alpha_p + \alpha_\ell; (1 \leq r < s \leq \ell-1) \right], \quad (6.6.42)$$

$$i c, \quad i \left(d + (\ell + 1) \left(\sum_{p=1}^{\ell-1} p h_{\alpha_p} + \frac{\ell}{2} h_{\alpha_\ell} \right) \right), \quad (6.6.43)$$

Now let $\mathbf{U}(t)$ be given by

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{L}_{1,q}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{\ell-q-1} \\ \mathbf{0} & \mathbf{0} & -\tilde{\mathbf{L}}_{1,q}^0 & \mathbf{0} \\ \mathbf{0} & t\mathbf{1}_{\ell-q-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (6.6.44)$$

The type 1a involutive automorphism $\{\mathbf{U}(t), \mathbf{1}, \xi\}$ will be shown to belong to the conjugacy class (C). To see this, let $\mathbf{S}(t)$ be given by

$$\mathbf{S}(t) = \begin{bmatrix} \mathbf{V}_1 & \mathbf{0} & \mathbf{V}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{\ell-q-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{V}_3 & \mathbf{0} & \mathbf{V}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{\ell-q-1} \end{bmatrix}, \quad (6.6.45)$$

where the submatrices are defined by

$$\begin{aligned} \mathbf{V}_1 &= \text{dsum}\{\mathbf{Q}_1^1, \mathbf{Q}_3^1, \dots, \mathbf{Q}_q^1\} & \mathbf{Q}_j^1 &= \begin{bmatrix} 1 & i \\ \frac{1}{\sqrt{2}}t & -\frac{1}{\sqrt{2}}it \end{bmatrix}, \\ \mathbf{V}_2 &= \text{dsum}\{\mathbf{Q}_1^2, \mathbf{Q}_3^2, \dots, \mathbf{Q}_q^2\} & \mathbf{Q}_j^2 &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}i \\ 1 & i \end{bmatrix}, \\ \mathbf{V}_3 &= \text{dsum}\{\mathbf{Q}_1^3, \mathbf{Q}_3^3, \dots, \mathbf{Q}_q^3\} & \mathbf{Q}_j^3 &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}i \\ -1 & i \end{bmatrix}, \\ \mathbf{V}_4 &= \text{dsum}\{\mathbf{Q}_1^4, \mathbf{Q}_3^4, \dots, \mathbf{Q}_q^4\} & \mathbf{Q}_j^4 &= \begin{bmatrix} 1 & -i \\ -\frac{1}{\sqrt{2}}t^{-1} & -\frac{1}{\sqrt{2}}it^{-1} \end{bmatrix}. \end{aligned} \quad (6.6.46)$$

The matrix $\mathbf{S}(t)$ as defined satisfies

$$\begin{aligned} \tilde{\mathbf{S}}(t)\mathbf{J}\mathbf{S}(t) &= \mathbf{J}, \\ \mathbf{S}(t)\text{offsum}\{\mathbf{1}_\ell, t\mathbf{1}_\ell\}\mathbf{S}(t)^{-1} &= \mathbf{U}(t). \end{aligned} \quad (6.6.47)$$

This shows that the involutive automorphism in question does belong to the conjugacy class (C).

The involutive automorphism $\{U(t), 1, \xi\}$, where

$$U(t) = \text{dsum}\{\mathbf{L}_{1,\ell-1}^0, \tilde{\mathbf{L}}_{1,\ell-1}^0\}, \quad (6.6.48)$$

belongs to a conjugacy class disjoint from any of the other conjugacy classes which have already been identified. Consideration of the determinant of $U(t)$ is sufficient to imply that $\{U(t), 1, \xi\}$ does not belong to $(A)^{(b)}$ (for any value of b) or to (B). The two possibilities are that it belongs to (C), or to some other class, not yet considered. If it were to belong to (C), then there would be some Laurent polynomial matrix $S(t)$ which satisfied

$$\begin{aligned} S(t)\text{dsum}\{\tilde{\mathbf{L}}_{1,\ell-1}^0, \mathbf{L}_{1,\ell-1}^0\}S(t)^{-1} &= \lambda t^\mu \text{offsum}\{\mathbf{1}_\ell, t\mathbf{1}_\ell\}, \\ \tilde{S}(t)\mathbf{J}S(t) &= \alpha t^\beta \mathbf{J}. \end{aligned} \quad (6.6.49)$$

This is assumed to hold for all non-zero values of t , and thus for the particular value $t = 1$. Let the matrix \mathbf{T} be obtained from the matrix $\mathbf{L}_{1,\ell-1}^0$ by putting $t = 1$. Now, whilst it has been shown already (see section 6.5) that there exists a t -independent matrix \mathbf{R} , which is such that

$$\begin{aligned} \mathbf{R}\text{dsum}\{\mathbf{T}, \tilde{\mathbf{T}}\}\mathbf{R}^{-1} &= \text{dsum}\{\mathbf{W}_{1,\ell-1}, \mathbf{W}_{1,\ell-1}\}, \\ \tilde{\mathbf{R}}\mathbf{J}\mathbf{R} &= \mathbf{J}, \end{aligned} \quad (6.6.50)$$

it has also been seen that there does not exist any t -independent matrix \mathbf{Q} such that both of the following hold

$$\begin{aligned} \mathbf{Q}\text{dsum}\{\mathbf{W}_{1,\ell-1}, \mathbf{W}_{1,\ell-1}\}\mathbf{Q}^{-1} &= \beta \text{offsum}\{\mathbf{1}_\ell, \mathbf{1}_\ell\}, \\ \tilde{\mathbf{Q}}\mathbf{J}\mathbf{Q} &= \gamma \mathbf{J}. \end{aligned} \quad (6.6.51)$$

Thus, $\{U(t), 1, \xi\}$ belongs to a new class, which is to be called (D). The representative automorphism of the conjugacy class (D) is the type 1a involutive automorphism generated by the matrix $U(t)$, where

$$\mathbf{U}(t) = \text{dsum} \left\{ \mathbf{V}_1(t), \dots, \mathbf{V}_m(t), \dots, \mathbf{V}_{\ell-1}, \tilde{\mathbf{V}}_1(t), \dots, \tilde{\mathbf{V}}_m(t), \dots, \tilde{\mathbf{V}}_{\ell-1}(t) \right\},$$

$$\mathbf{V}_m(t) = \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix} (1 \leq m \leq \ell-1; m \text{ odd}). \quad (6.6.52)$$

A basis of the compact real form of $C_r^{(1)}$ exists in which every basis element is an eigenvector of this representative with eigenvalue ± 1 . Such a basis is given below.

In the following, the quantities r and s are such that $1 \leq r < s \leq \ell$.

$$\begin{aligned} i \left(h_{\alpha_k} - \frac{1}{2}c \right) & \quad 1 \leq k < \ell; k \text{ odd; eigenvalue } -1, \\ i \left(\frac{1}{2}h_{\alpha_{k-1}} + h_{\alpha_k} + \frac{1}{2}h_{\alpha_{k+1}} \right) & \quad 1 < k < \ell; k \text{ even; eigenvalue } 1, \\ i \left(h_{\alpha_{\ell-1}} + h_{\alpha_\ell} \right) & \quad \text{eigenvalue } 1, \end{aligned} \quad (6.6.53)$$

$$\begin{aligned} (e_{j\delta}^k + e_{-j\delta}^k) & \quad 1 \leq k < \ell; k \text{ odd; eigenvalue } -1, \\ \left(\frac{1}{2}(e_{j\delta}^{k-1} + e_{-j\delta}^{k-1}) + (e_{j\delta}^k + e_{-j\delta}^k) + \frac{1}{2}(e_{j\delta}^{k+1} + e_{-j\delta}^{k+1}) \right) & \quad 1 < k < \ell; k \text{ even; eigenvalue } 1, \\ (e_{j\delta}^{\ell-1} + e_{-j\delta}^{\ell-1}) + (e_{j\delta}^\ell + e_{-j\delta}^\ell) & \quad \text{eigenvalue } 1, \end{aligned} \quad (6.6.54)$$

$$\begin{aligned} i(e_{j\delta}^k - e_{-j\delta}^k) & \quad 1 \leq k < \ell; k \text{ odd; eigenvalue } -1, \\ i \left(\frac{1}{2}(e_{j\delta}^{k-1} - e_{-j\delta}^{k-1}) + (e_{j\delta}^k - e_{-j\delta}^k) + \frac{1}{2}(e_{j\delta}^{k+1} - e_{-j\delta}^{k+1}) \right) & \quad 1 < k < \ell; k \text{ even; eigenvalue } 1, \\ i(e_{j\delta}^{\ell-1} - e_{-j\delta}^{\ell-1}) + i(e_{j\delta}^\ell - e_{-j\delta}^\ell) & \quad \text{eigenvalue } 1, \end{aligned} \quad (6.6.55)$$

$$\left. \begin{aligned} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) & \quad \text{eigenvalue } -1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) & \quad \text{eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) & \quad \text{eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) & \quad \text{eigenvalue } -1 \end{aligned} \right\} \alpha = \alpha_r; r \text{ odd}; k = -j-1, \quad (6.6.56)$$

$$\begin{array}{l}
(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta+\beta} + e_{-j\delta-\beta}) \quad \text{eigenvalue } 1 \\
(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta+\beta} - e_{-j\delta-\beta}) \quad \text{eigenvalue } -1 \\
i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta+\beta} - e_{-j\delta-\beta}) \quad \text{eigenvalue } 1 \\
i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta+\beta} + e_{-j\delta-\beta}) \quad \text{eigenvalue } -1
\end{array} \left\{ \begin{array}{l} \alpha = \sum_{p=r}^{s-1} \alpha_p \quad \beta = \sum_{p=r+1}^s \alpha_p \\ r, s \text{ odd} \end{array} \right\},$$

(6.6.57)

$$\begin{array}{l}
(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\beta} + e_{-k\delta-\beta}) \quad \text{eigenvalue } 1 \\
(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\beta} - e_{-k\delta-\beta}) \quad \text{eigenvalue } -1 \\
i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\beta} - e_{-k\delta-\beta}) \quad \text{eigenvalue } 1 \\
i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\beta} + e_{-k\delta-\beta}) \quad \text{eigenvalue } -1
\end{array} \left\{ \begin{array}{l} \alpha = \sum_{p=r}^{s-1} \alpha_p \quad \beta = \sum_{p=r+1}^{s-2} \alpha_p \\ r \text{ odd}, s \text{ even}; s \neq r+1; k = j+1 \end{array} \right\},$$

(6.6.58)

$$\begin{array}{l}
(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\beta} + e_{-k\delta-\beta}) \quad \text{eigenvalue } 1 \\
(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\beta} - e_{-k\delta-\beta}) \quad \text{eigenvalue } -1 \\
i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\beta} - e_{-k\delta-\beta}) \quad \text{eigenvalue } 1 \\
i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\beta} + e_{-k\delta-\beta}) \quad \text{eigenvalue } -1
\end{array} \left\{ \begin{array}{l} \alpha = \varepsilon_r + \varepsilon_s \quad \beta = \varepsilon_{r+1} + \varepsilon_{s+1} \\ r, s \text{ odd}; k = j+1 \end{array} \right\},$$

(6.6.59)

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \alpha = \varepsilon_r + \varepsilon_s; r \text{ odd}, s = r+1; \text{ eigenvalue } 1,$$

(6.6.60)

$$\begin{array}{l}
(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta+\beta} + e_{-j\delta-\beta}) \quad \text{eigenvalue } 1 \\
(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta+\beta} - e_{-j\delta-\beta}) \quad \text{eigenvalue } -1 \\
i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta+\beta} - e_{-j\delta-\beta}) \quad \text{eigenvalue } 1 \\
i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta+\beta} + e_{-j\delta-\beta}) \quad \text{eigenvalue } -1
\end{array} \left\{ \begin{array}{l} \alpha = \varepsilon_r + \varepsilon_s \quad \beta = \varepsilon_{r+1} + \varepsilon_{s-1} \\ r \text{ odd}, s \text{ even}; s \neq r+1 \end{array} \right\},$$

(6.6.61)

$$i \left(d + (\ell + 1) \sum_{\substack{p=1 \\ p \text{ odd}}}^{\ell-1} h_{\alpha_p} \right) \left. \vphantom{\sum} \right\} \text{eigenvalue } 1.$$

(6.6.62)

Therefore, a basis of the real form is given by the following elements:

$$\begin{aligned} & \left(h_{\alpha_k} - \frac{1}{2}c \right) \quad 1 \leq k < \ell; k \text{ odd}, \\ & i \left(\frac{1}{2}h_{\alpha_{k-1}} + h_{\alpha_k} + \frac{1}{2}h_{\alpha_{k+1}} \right) \quad 1 < k < \ell; k \text{ even}, \\ & i \left(h_{\alpha_{\ell-1}} + h_{\alpha_\ell} \right), \end{aligned} \quad (6.6.63)$$

$$\begin{aligned} & i \left(e_{j\delta}^k + e_{-j\delta}^k \right) \quad 1 \leq k < \ell; k \text{ odd}, \\ & \left(\frac{1}{2} \left(e_{j\delta}^{k-1} + e_{-j\delta}^{k-1} \right) + \left(e_{j\delta}^k + e_{-j\delta}^k \right) + \frac{1}{2} \left(e_{j\delta}^{k+1} + e_{-j\delta}^{k+1} \right) \right) \quad 1 < k < \ell; k \text{ even}, \\ & \left(e_{j\delta}^{\ell-1} + e_{-j\delta}^{\ell-1} \right) + \left(e_{j\delta}^\ell + e_{-j\delta}^\ell \right), \end{aligned} \quad (6.6.64)$$

$$\begin{aligned} & \left(e_{j\delta}^k - e_{-j\delta}^k \right) \quad 1 \leq k < \ell; k \text{ odd}, \\ & i \left(\frac{1}{2} \left(e_{j\delta}^{k-1} - e_{-j\delta}^{k-1} \right) + \left(e_{j\delta}^k - e_{-j\delta}^k \right) + \frac{1}{2} \left(e_{j\delta}^{k+1} - e_{-j\delta}^{k+1} \right) \right) \quad 1 < k < \ell; k \text{ even}, \\ & i \left(e_{j\delta}^{\ell-1} - e_{-j\delta}^{\ell-1} \right) + i \left(e_{j\delta}^\ell - e_{-j\delta}^\ell \right), \end{aligned} \quad (6.6.65)$$

$$\begin{aligned} & \left. \begin{aligned} & i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha} \right) \\ & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha} \right) \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha} \right) \\ & \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha} \right) \end{aligned} \right\} \alpha = \alpha_r; r \text{ odd}; k = -j - 1, \end{aligned} \quad (6.6.66)$$

$$\begin{aligned} & \left. \begin{aligned} & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta+\beta} + e_{-j\delta-\beta} \right) \\ & i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta+\beta} - e_{-j\delta-\beta} \right) \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta+\beta} - e_{-j\delta-\beta} \right) \\ & \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta+\beta} + e_{-j\delta-\beta} \right) \end{aligned} \right\} \left[\begin{aligned} & \alpha = \sum_{p=r}^{s-1} \alpha_p \quad \beta = \sum_{p=r+1}^s \alpha_p \\ & r, s \text{ odd} \end{aligned} \right], \end{aligned} \quad (6.6.67)$$

$$\begin{aligned} & \left. \begin{aligned} & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\beta} + e_{-k\delta-\beta} \right) \\ & i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\beta} - e_{-k\delta-\beta} \right) \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\beta} - e_{-k\delta-\beta} \right) \\ & \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\beta} + e_{-k\delta-\beta} \right) \end{aligned} \right\} \left[\begin{aligned} & \alpha = \sum_{p=r}^{s-1} \alpha_p \quad \beta = \sum_{p=r+1}^{s-2} \alpha_p \\ & r \text{ odd}, s \text{ even}; s \neq r + 1; k = j + 1 \end{aligned} \right], \end{aligned} \quad (6.6.68)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\beta} + e_{-k\delta-\beta}) \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\beta} - e_{-k\delta-\beta}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\beta} - e_{-k\delta-\beta}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\beta} + e_{-k\delta-\beta}) \end{array} \right\} \left[\begin{array}{l} \alpha = \varepsilon_r + \varepsilon_s \quad \beta = \varepsilon_{r+1} + \varepsilon_{s+1} \\ r, s \text{ odd}; k = j+1 \end{array} \right], \quad (6.6.69)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \alpha = \varepsilon_r + \varepsilon_s; r \text{ odd}, s = r+1, \quad (6.6.70)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta+\beta} + e_{-j\delta-\beta}) \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta+\beta} - e_{-j\delta-\beta}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta+\beta} - e_{-j\delta-\beta}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta+\beta} + e_{-j\delta-\beta}) \end{array} \right\} \left[\begin{array}{l} \alpha = \varepsilon_r + \varepsilon_s \quad \beta = \varepsilon_{r+1} + \varepsilon_{s-1} \\ r \text{ odd}, s \text{ even}; s \neq r+1 \end{array} \right], \quad (6.6.71)$$

$$ic, \quad i \left(d + (\ell+1) \sum_{\substack{p=1 \\ p \text{ odd}}}^{\ell-1} h_{\alpha_p} \right). \quad (6.6.72)$$

The final automorphism to be considered in this section is the type 1a involutive automorphism $\{U(t), 1, \xi\}$, where

$$U(t) = \text{dsum} \{L_{1,\ell-1}^0, -\tilde{L}_{1,\ell-1}^0\}. \quad (6.6.73)$$

This automorphism belongs to the conjugacy class (C). To see this, let the matrix $S(t)$ be defined by

$$S(t) = \begin{bmatrix} \mathbf{V}_1 & \mathbf{V}_2 \\ \mathbf{V}_3 & \mathbf{V}_4 \end{bmatrix}, \quad (6.6.74)$$

and let the submatrices be defined by

$$\begin{aligned}
V_1 &= \text{dsum}\{Q_1^1, Q_3^1, \dots, Q_{\ell-1}^1\} & Q_j^1 &= \begin{bmatrix} 1 & i \\ \frac{1}{\sqrt{2}}t & -\frac{1}{\sqrt{2}}it \end{bmatrix}, \\
V_2 &= \text{dsum}\{Q_1^2, Q_3^2, \dots, Q_{\ell-1}^2\} & Q_j^2 &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}i \\ 1 & i \end{bmatrix}, \\
V_3 &= \text{dsum}\{Q_1^3, Q_3^3, \dots, Q_{\ell-1}^3\} & Q_j^3 &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}i \\ -1 & i \end{bmatrix}, \\
V_4 &= \text{dsum}\{Q_1^4, Q_3^4, \dots, Q_{\ell-1}^4\} & Q_j^4 &= \begin{bmatrix} 1 & -i \\ -\frac{1}{\sqrt{2}}t^{-1} & -\frac{1}{\sqrt{2}}it^{-1} \end{bmatrix}.
\end{aligned} \tag{6.6.75}$$

The matrix $S(t)$ satisfies

$$\begin{aligned}
S(t) \text{offsum}\{1_\ell, t1_\ell\} S(t)^{-1} &= U(t), \\
\tilde{S}(t)JS(t) &= J,
\end{aligned} \tag{6.6.76}$$

and so the type 1a involutive automorphism $\{U(t), 1, \xi\}$ does belong to the conjugacy class (C).

6.7 Type 1a involutive automorphisms of $C_\ell^{(1)}$ with $u = -1$

The involutive automorphisms of type 1a (with $u = -1$) may be found for each of the representative rotations τ_j that were defined previously. However, the information contained in section 6.5 shows that some of these are mutually conjugate, so that a full picture of the conjugacy class structure of the involutive automorphisms may be found by analysing only a subset of those automorphisms which were to be investigated. In fact, each type 1a involutive Cartan-preserving automorphism (with $u = -1$) is conjugate to at least one of the type 1a involutive automorphisms generated by the following matrices

$$\mathbf{U}(t) = \text{dsum}\{\mathbf{D}_{1,\ell}^0, \mathbf{D}_{1,\ell}^0\} \quad (6.7.1)$$

$$\mathbf{U}(t) = \text{dsum}\{\mathbf{D}_{1,\ell}^0, -\mathbf{D}_{1,\ell}^0\}. \quad (6.7.2)$$

Let the matrix $\mathbf{U}(t)$ be as given in (6.7.1). For each matrix $\mathbf{U}(t)$ of the general form given in (6.7.1), a set of integers s_j (for $1 \leq j \leq \ell$) is defined by letting

$$s_j = \begin{cases} 1 & \text{if } \lambda_j = -1, \\ 0 & \text{if } \lambda_j = 1. \end{cases} \quad (6.7.3)$$

Then, for each such matrix $\mathbf{U}(t)$, a matrix $\mathbf{S}(t)$ is defined by

$$\mathbf{S}(t) = \text{diag}\{t^{s_1}, t^{s_2}, \dots, t^{s_\ell}, t^{-s_1}, t^{-s_2}, \dots, t^{-s_\ell}\}, \quad (6.7.4)$$

and this matrix satisfies the following equations

$$\begin{aligned} \tilde{\mathbf{S}}(t)\mathbf{J}\mathbf{S}(t) &= \mathbf{J}, \\ \mathbf{S}(t)\mathbf{1}_{2\ell}\mathbf{S}(-t)^{-1} &= \mathbf{U}(t). \end{aligned} \quad (6.7.5)$$

Thus, all of the type 1a involutive automorphisms $\{\mathbf{U}(t), -1, 0\}$, where $\mathbf{U}(t)$ is given by (6.7.1), are mutually conjugate. The class which contains all of them will be called (E). The involutive automorphisms $\{\mathbf{U}(t), -1, 0\}$, where $\mathbf{U}(t)$ is given by (6.7.2), also belong to (E). To see this, let $\mathbf{S}(t)$ be defined by

$$\mathbf{S}(t) = \text{diag} \left\{ t^{s_1}, t^{s_2}, \dots, t^{s_\ell}, t^{1-s_1}, t^{1-s_2}, \dots, t^{1-s_\ell} \right\}, \quad (6.7.6)$$

where the quantities $\{s_j\}_{j=1}^{j=\ell}$ are as defined in (6.7.3). Then $\mathbf{S}(t)$ satisfies

$$\begin{aligned} \bar{\mathbf{S}}(t)\mathbf{J}\mathbf{S}(t) &= \mathbf{J}, \\ \mathbf{S}(t)\mathbf{1}_{2\ell}\mathbf{S}(-t)^{-1} &= \mathbf{U}(t). \end{aligned} \quad (6.7.7)$$

Thus, all of the type 1a involutive Cartan-preserving automorphisms (with $u = -1$) are mutually conjugate, and belong to the conjugacy class which has already been called (E). The conjugacy class (E) contains the type 1a automorphism $\{\mathbf{1}_{2\ell}, -1, 0\}$, which may be taken as its representative. The representative is such that the following elements form a basis for the compact real form of $C_\ell^{(1)}$, in which each element is an eigenvector of the representative type 1a involution $\{\mathbf{1}_{2\ell}, -1, 0\}$

$$\left. \begin{aligned} & \left(e_{j\delta}^k + e_{-j\delta}^k \right) \quad j \in \mathbf{N}^0 \\ & i \left(e_{j\delta}^k - e_{-j\delta}^k \right) \quad j \in \mathbf{N} \end{aligned} \right\} 1 \leq k \leq \ell; \text{ eigenvalue } (-1)^j, \quad (6.7.8)$$

$$\left. \begin{aligned} & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} \right) \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} \right) \end{aligned} \right\} j \in \mathbf{Z}; \alpha \in \Delta^0; \text{ eigenvalue } (-1)^j, \quad (6.7.9)$$

$$\left. \begin{aligned} & ic \\ & id \end{aligned} \right\} \text{ eigenvalue } 1. \quad (6.7.10)$$

The basis, therefore, for a real form of the Kac-Moody algebra $C_\ell^{(1)}$ is given by the following elements

$$\left. \begin{aligned} & \left(e_{j\delta}^k + e_{-j\delta}^k \right) \quad j \in \mathbf{N}^0 \\ & i \left(e_{j\delta}^k - e_{-j\delta}^k \right) \quad j \in \mathbf{N} \end{aligned} \right\} 1 \leq k \leq \ell; j \text{ even}, \quad (6.7.11)$$

$$\left. \begin{array}{l} i(e_{j\delta}^k + e_{-j\delta}^k) j \in \mathbf{N}^0 \\ (e_{j\delta}^k - e_{-j\delta}^k) j \in \mathbf{N} \end{array} \right\} 1 \leq k \leq \ell; j \text{ odd}, \quad (6.7.12)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} j \in \mathbf{Z}; \alpha \in \Delta^0; j \text{ even}, \quad (6.7.13)$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} j \in \mathbf{Z}; \alpha \in \Delta^0; j \text{ odd}, \quad (6.7.14)$$

$$\begin{array}{l} ic, \\ id. \end{array} \quad (6.7.15)$$

The elements of this real form are such that the matrix parts $\mathbf{a}(t)$ are all traceless and satisfy

$$\begin{aligned} \mathbf{a}(t) &= -\tilde{\mathbf{a}}^*(-t^{-1}), \\ \mathbf{a}(t) &= -\mathbf{J}\tilde{\mathbf{a}}(t)\mathbf{J}^{-1}. \end{aligned} \quad (6.7.16)$$

6.8 Type 2a involutive automorphisms of $C_\ell^{(1)}$ with $u = 1$

The analysis of the previous sections implies that each involutive Cartan-preserving type 2a automorphism (with $u = 1$) is conjugate to a type 2a involutive automorphism generated by at least one of the following

$$U(t) = \text{dsum}\{\mathbf{1}_a, -\mathbf{1}_b, t\mathbf{1}_c, -t\mathbf{1}_d, \mathbf{1}_a, -\mathbf{1}_b, t^{-1}\mathbf{1}_c, -t^{-1}\mathbf{1}_d\}, \quad (6.8.1)$$

$$U(t) = \text{dsum}\{\mathbf{1}_{\ell-1-a}, -\mathbf{1}_a, t\mathbf{1}_{\ell-1-a}, -t\mathbf{1}_a\}, \quad (6.8.2)$$

$$U(t) = \text{dsum}\{\mathbf{1}_{\ell-a}, t\mathbf{1}_a, -\mathbf{1}_{\ell-a}, -t\mathbf{1}_a\}, \quad (6.8.3)$$

$$U(t) = \text{dsum}\{\mathbf{D}_{1,\ell}, -t\mathbf{D}_{1,\ell}^{-1}\}, \quad (6.8.4)$$

Let $U(t)$ be given by (6.8.1). A function θ may be defined, whose argument is the matrix $U(t)$ and whose value is an ordered quadruplet. The function is defined by

$$\theta(U(t)) = \{a, b, c, d\}. \quad (6.8.5)$$

The set $(F)^{(j,k)}$ may be defined to be that set which contains the type 2a involution $\{U(t), 1, \xi\}$ (and all involutive automorphisms conjugate to it) where $U(t)$ is of the form given in (6.8.1), and where

$$\theta(U(t)) = \{a, b, c, d\}, \quad (6.8.6)$$

with $b + c = j$ and $c - d = k$. It follows from the previous analysis of this chapter that j may be assumed to be such that $0 \leq j \leq \lfloor \frac{\ell}{2} \rfloor$. It must be first demonstrated that *all* of the automorphisms within each set $(F)^{(j,k)}$ are mutually conjugate. To show that this is the case, let two matrices $U(t)$ and $V(t)$ be of the form (6.8.1) but such that

$$\theta(U(t)) = \{n_+, n_-, n_+^t, n_-^t\}, \quad (6.8.7)$$

$$\theta(\mathbf{V}(t)) = \{n_+ - 1, n_m - 1, n_+^t + 1, n_-^t - 1\}. \quad (6.8.8)$$

A matrix $\mathbf{S}(t)$ will now be defined by

$$\mathbf{S}(t) = \begin{bmatrix} \mathbf{1}_{n_+ - 1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}(1+t)\mathbf{1}_2 & \mathbf{0} & \frac{i}{2}(1-t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_{\ell-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{i}{2}(1-t^{-1}) & \mathbf{0} & \frac{1}{2}(1+t^{-1})\mathbf{1}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{\ell-n_+ - 1} \end{bmatrix}. \quad (6.8.9)$$

This matrix satisfies

$$\begin{aligned} \tilde{\mathbf{S}}(t)\mathbf{J}\mathbf{S}(t) &= \mathbf{J}, \\ \mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(t^{-1})^{-1} &= \mathbf{V}'(t), \end{aligned} \quad (6.8.10)$$

where $\mathbf{V}'(t)$ may be obtained from $\mathbf{V}(t)$ by one of the matrix transformations given previously. Thus, the type 2a involutive automorphisms (with $u = 1$) generated by $\mathbf{U}(t)$ and $\mathbf{V}(t)$ are mutually conjugate. The set $(\mathbf{F})^{(j,k)}$ has thus been shown to consist entirely of mutually conjugate automorphisms. It should be noted that the type 2a automorphism $\{\mathbf{U}(t), 1, \xi\}$ is conjugate to the type 2a automorphism $\{\mathbf{U}(-t), 1, \xi\}$. This, together with the aforementioned matrix transformation (that is, the transformation involving the matrix given by (6.8.9)) implies that each of the involutive automorphisms under consideration is conjugate to an involutive automorphism $\{\mathbf{U}(t), 1, \xi\}$, where

$$\mathbf{U}(t) = \text{dsum} \left\{ \mathbf{1}_{n_+}, -\mathbf{1}_{n_-}, t\mathbf{1}_{n_+^t}, \mathbf{1}_{n_+}, -\mathbf{1}_{n_-}, t^{-1}\mathbf{1}_{n_+^t} \right\}. \quad (6.8.11)$$

It has already been remarked that the quantity j may be assumed to take values such that $0 \leq j \leq \left\lfloor \frac{\ell}{2} \right\rfloor$. It then follows that $0 \leq k \leq \left(\left\lfloor \frac{\ell}{2} \right\rfloor - j \right)$. It remains to be seen whether or not any of the conjugacy classes $(\mathbf{F})^{(j,k)}$ coincide. Let the quantities a, b, c and d be such that $0 \leq a, c \leq \left\lfloor \frac{\ell}{2} \right\rfloor$ and $0 \leq b \leq \left(\left\lfloor \frac{\ell}{2} \right\rfloor - a \right)$ and $0 \leq d \leq \left(\left\lfloor \frac{\ell}{2} \right\rfloor - c \right)$. Suppose, by way of obtaining a contradiction, that the conjugacy classes $(\mathbf{F})^{(a,b)}$ and $(\mathbf{F})^{(c,d)}$ coincide

(for $(a,b) \neq (c,d)$). This implies that, for all non-zero t , there exists a Laurent polynomial matrix $S(t)$ such that

$$S(t)U_{a,b}(st^{\pm 1})S(t^{-1})^{-1} = \lambda t^\mu U_{c,d}(t), \quad (6.8.12)$$

where the matrices $U_{a,b}(t)$ and $U_{c,d}$ are defined by

$$\begin{aligned} U_{a,b}(t) &= \text{dsum} \{ \mathbf{1}_{\ell-a-b}, t\mathbf{1}_b, -\mathbf{1}_a, \mathbf{1}_{\ell-a-b}, t^{-1}\mathbf{1}_b, -\mathbf{1}_a \}, \\ U_{c,d}(t) &= \text{dsum} \{ \mathbf{1}_{\ell-c-d}, t\mathbf{1}_d, -\mathbf{1}_c, \mathbf{1}_{\ell-c-d}, t^{-1}\mathbf{1}_d, -\mathbf{1}_c \}. \end{aligned} \quad (6.8.13)$$

The integer s may take either of the values 1 or -1 , whilst t takes any non-zero complex value. If the substitutions $s = 1$ and $t = 1$ are made in (6.8.12), then we have as a necessary condition that $a + c = \ell$. Since the quantities a and c are assumed to satisfy

$$0 \leq a, c \leq \left[\frac{\ell}{2} \right], \quad (6.8.14)$$

it follows that ℓ must be even, and $a = c = \left[\frac{\ell}{2} \right]$. In that case, the substitutions $t = -1$, $s = 1$ may be made in (6.8.12). This implies that $b = d$, which contradicts the original hypothesis. It must be the case, therefore, that $s = -1$. However, upon substituting the values $t = 1, -1$ into the equation (6.8.12), it becomes clear that $a = c$ and $b = d$. Thus, the conjugacy classes $(F)^{(a,b)}$ and $(F)^{(c,d)}$ have trivial intersection when $(a,b) \neq (c,d)$. We have seen therefore, that each type 2a involutive automorphism $\{U(t), 1, \xi\}$, where $U(t)$ is given by (6.8.1), belongs to a conjugacy class $(F)^{(j,k)}$, where j and k are such that $0 \leq j \leq \left[\frac{\ell}{2} \right]$, $0 \leq k \leq \left(\left[\frac{\ell}{2} \right] - j \right)$.

The type 2a involutive automorphism $\psi (= \{U(t), 1, \xi\})$, where the matrix $U(t) = \text{dsum} \{ \mathbf{1}_{\ell-a-b}, -\mathbf{1}_a, t\mathbf{1}_b, \mathbf{1}_{\ell-a-b}, -\mathbf{1}_a, t^{-1}\mathbf{1}_b \}$, is taken to be the representative automorphism of the conjugacy class $(F)^{(a,b)}$. A basis of the compact real form, in which each basis element is an eigenvector of ψ , is given below

$$\left. \begin{array}{l} i\left(h_{\alpha_t - \frac{1}{2}c}\right) \\ i\left(h_{\alpha_t + \frac{1}{2}c}\right) \\ ih_{\alpha_k} \text{ otherwise} \end{array} \right\} \begin{array}{l} b \neq 0 \\ \\ \end{array} \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{eigenvalue } 1, \quad (6.8.15)$$

$$\left. \begin{array}{l} \left(e_{j\delta}^k + e_{-j\delta}^k\right) \text{ eigenvalue } 1 \\ i\left(e_{j\delta}^k - e_{-j\delta}^k\right) \text{ eigenvalue } -1 \end{array} \right\} j \in \mathbb{N}; 1 \leq k \leq l, \quad (6.8.16)$$

$$\left. \begin{array}{l} \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}\right) \text{ eigenvalue } 1 \\ \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}\right) \text{ eigenvalue } -1 \\ i\left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}\right) \text{ eigenvalue } 1 \\ i\left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}\right) \text{ eigenvalue } -1 \end{array} \right\} j \in \mathbb{N}^0, \quad (6.8.17)$$

where, in (6.8.17), α is such that

$$\alpha = \left[\begin{array}{l} \left[\begin{array}{l} 1 \leq r < s \leq l - a - b \\ \varepsilon_r - \varepsilon_s, l - a - b < r < s \leq l - b \\ l - b < r < s \leq l \\ 1 \leq r \leq s \leq l - a - b \\ \varepsilon_r + \varepsilon_s, l - a - b < r \leq s \leq l - b \\ l - b < r \leq s \leq l \end{array} \right] \end{array} \right], \quad (6.8.18)$$

$$\left. \begin{array}{l} \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}\right) \text{ eigenvalue } -1 \\ \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}\right) \text{ eigenvalue } 1 \\ i\left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}\right) \text{ eigenvalue } -1 \\ i\left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}\right) \text{ eigenvalue } 1 \end{array} \right\} j \in \mathbb{N}^0, \quad (6.8.19)$$

where, in (6.8.19), α is such that

$$\alpha = \{\varepsilon_r \pm \varepsilon_s \quad (1 \leq r \leq l - a - b < s \leq l - b)\}, \quad (6.8.20)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \text{ eigenvalue } 1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \text{ eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \text{ eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \text{ eigenvalue } -1 \end{array} \right\} j \in \mathbb{N}^0; k = -j - 1, \quad (6.8.21)$$

where, in (6.8.21), α is such that

$$\alpha = \{\varepsilon_r - \varepsilon_s \quad (1 \leq r \leq \ell - a - b; \ell - b < s \leq \ell)\}, \quad (6.8.22)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \text{ eigenvalue } -1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \text{ eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \text{ eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \text{ eigenvalue } 1 \end{array} \right\} j \in \mathbb{N}^0; k = -j - 1, \quad (6.8.23)$$

where, in (6.8.23), α is such that

$$\alpha = \{\varepsilon_r - \varepsilon_s \quad (\ell - a - b < r \leq \ell - b < s \leq \ell)\}, \quad (6.8.24)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \text{ eigenvalue } 1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \text{ eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \text{ eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \text{ eigenvalue } -1 \end{array} \right\} j \in \mathbb{N}^0; k = -j + 1, \quad (6.8.25)$$

where, in (6.8.25), α is such that

$$\alpha = \{\varepsilon_r + \varepsilon_s \quad (1 \leq r \leq \ell - a - b; \ell - b < s \leq \ell)\}, \quad (6.8.26)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \text{ eigenvalue } -1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \text{ eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \text{ eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \text{ eigenvalue } 1 \end{array} \right\} j \in \mathbb{N}^0; k = -j + 1, \quad (6.8.27)$$

where, in (6.8.27), α is such that

$$\alpha = \{\varepsilon_r + \varepsilon_s \quad (\ell - a - b < r \leq \ell - b < s \leq \ell)\}, \quad (6.8.28)$$

$$ic, \quad \left(d - 2i(\ell + 1) \left\{ \sum_{\substack{p=1 \\ b \neq 0}}^{b-1} ph_{\alpha_{\ell-b-p}} + \frac{b}{2} h_{\alpha_\ell} \right\} \right) \text{ eigenvalue } -1. \quad (6.8.29)$$

Cartan's theorem gives the following basis of a real form of $C_\ell^{(1)}$:

$$\left. \begin{array}{l} i(h_{\alpha_{\ell-b}} - \frac{1}{2}c) \\ i(h_{\alpha_\ell} + \frac{1}{2}c) \\ ih_{\alpha_k} \end{array} \right\} \begin{array}{l} b \neq 0; k = \ell, \ell - b \\ \text{otherwise} \end{array} \left. \vphantom{\begin{array}{l} i(h_{\alpha_{\ell-b}} - \frac{1}{2}c) \\ i(h_{\alpha_\ell} + \frac{1}{2}c) \\ ih_{\alpha_k} \end{array}} \right\} 1 \leq k \leq \ell, \quad (6.8.30)$$

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \\ (e_{j\delta}^k - e_{-j\delta}^k) \end{array} \right\} j \in \mathbb{N}; 1 \leq k \leq \ell, \quad (6.8.31)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \end{array} \right\} j \in \mathbb{N}^0, \quad (6.8.32)$$

where, in (6.8.32), α is such that

$$\alpha = \left\{ \begin{array}{l} \varepsilon_r - \varepsilon_s \quad \left\{ \begin{array}{l} 1 \leq r < s \leq \ell - a - b \\ \ell - a - b < r < s \leq \ell - b \\ \ell - b < r < s \leq \ell \end{array} \right. \\ \varepsilon_r + \varepsilon_s \quad \left\{ \begin{array}{l} 1 \leq r \leq s \leq \ell - a - b \\ \ell - a - b < r \leq s \leq \ell - b \\ \ell - b < r \leq s \leq \ell \end{array} \right. \end{array} \right\}, \quad (6.8.33)$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \end{array} \right\} j \in \mathbb{N}^0, \quad (6.8.34)$$

where, in (6.8.34), α is such that

$$\alpha = \{ \varepsilon_r \pm \varepsilon_s \quad (1 \leq r \leq \ell - a - b < s \leq \ell - b) \}, \quad (6.8.35)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \end{array} \right\} j \in \mathbf{N}^0; k = -j - 1, \quad (6.8.36)$$

where, in (6.8.36), α is such that

$$\alpha = \{ \varepsilon_r - \varepsilon_s \quad (1 \leq r \leq \ell - a - b; \ell - b < s \leq \ell) \}, \quad (6.8.37)$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \end{array} \right\} j \in \mathbf{N}^0; k = -j - 1, \quad (6.8.38)$$

where, in (6.8.39), α is such that

$$\alpha = \{ \varepsilon_r - \varepsilon_s \quad (\ell - a - b < r \leq \ell - b < s \leq \ell) \}, \quad (6.8.39)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \end{array} \right\} j \in \mathbf{N}^0; k = -j + 1, \quad (6.8.40)$$

where, in (6.8.40), α is such that

$$\alpha = \{ \varepsilon_r + \varepsilon_s \quad (1 \leq r \leq \ell - a - b; \ell - b < s \leq \ell) \}, \quad (6.8.41)$$

$$\left. \begin{aligned} & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \\ & (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ & (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ & i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \end{aligned} \right\} j \in \mathbb{N}^0; k = -j + 1, \quad (6.8.42)$$

where, in (6.8.42), α is such that

$$\alpha = \{\varepsilon_r + \varepsilon_s \quad (\ell - a - b < r \leq \ell - b < s \leq \ell)\}, \quad (6.8.43)$$

$$c, \quad \left(d - 2i(\ell + 1) \cdot \left[\sum_{p=1}^{b-1} p h_{\alpha_{\ell-b-p}} + \frac{b}{2} h_{\alpha_\ell} \right] \right). \quad (6.8.44)$$

The automorphisms to be examined next are those type 2a involutive automorphisms $\{U(t), 1, \xi\}$, where $U(t)$ is given by (6.8.2). There are only $(\ell + 1)$ distinct automorphisms under consideration, each different one corresponding to a different value of a in the equation (6.8.2). Thus, the type 2a involutive automorphisms which correspond to (6.8.2) belong to at most $(\ell + 1)$ disjoint conjugacy classes. However, if we re-order the index set of $U(t)$ and multiply by a common factor, it becomes clear that there are at most $(1 + \lfloor \frac{\ell}{2} \rfloor)$ disjoint conjugacy classes. That is to say that a may be assumed to be such that

$$0 \leq a \leq \lfloor \frac{\ell}{2} \rfloor. \quad (6.8.45)$$

The conjugacy class $(G)^{(j)}$ is defined to be that class which contains the type 2a involutive automorphism $\{U_j(t), 1, \xi\}$, where $0 \leq j \leq \lfloor \frac{\ell}{2} \rfloor$, and

$$U_j(t) = \text{dsum} \{ \mathbf{1}_{\ell-1-j}, -\mathbf{1}_j, t \mathbf{1}_{\ell-1-a}, -t \mathbf{1}_j \}. \quad (6.8.46)$$

We need to demonstrate that the conjugacy classes $(G)^{(j)}$ (for $0 \leq j \leq \lfloor \frac{\ell}{2} \rfloor$) are all mutually disjoint, and that they are also disjoint from the conjugacy classes $(F)^{(a,b)}$,

which were discussed previously. Consider firstly the possibility that a conjugacy class $(G)^{(j)}$ (for some suitable value of j) coincides with a class $(F)^{(a,b)}$ (for suitable a, b). This would require that there existed a Laurent polynomial matrix $S(t)$ such that

$$S(t)U_j(st^{\pm})S(t^{-1})^{-1} = \lambda t^{\mu}U_{a,b}(t), \quad (6.8.47)$$

where $U_{a,b}(t)$ is of the form given in (6.8.1). Substitutions may always be made so that $st = 1$, and this implies that

$$S(s)U_j(1)S(s)^{-1} = \alpha U_{a,b}(s). \quad (6.8.48)$$

It is possible to find matrices $S(s)$ (for $s = \pm 1$) which satisfy (6.8.18). However, a brief inspection will reveal that $S(s)$ fails to satisfy $\tilde{S}(s)JS(s) = \lambda J$. It remains only to show that each of the conjugacy classes $(G)^{(j)}$ is disjoint from all of the others (for $0 \leq j \leq \lfloor \frac{\ell}{2} \rfloor$). We suppose that the opposite is true, so that there exists some matrix $S(t)$ which satisfies

$$S(t)U_j(st^{\pm 1})S(t^{-1})^{-1} = \lambda t^{\mu}U_k \quad (j \neq k). \quad (6.8.49)$$

Suppose firstly that $s = 1$. If we then put $t = 1$ into the equation (6.8.19), then this implies that $j = k$, which is a contradiction. Similarly, if $s = -1$, then we put $t = -1$. Again, this yields a contradiction. Thus, the conjugacy class $(G)^{(j)}$ is disjoint from $(G)^{(k)}$ whenever $0 \leq j < k \leq \lfloor \frac{\ell}{2} \rfloor$.

The automorphism ψ_{G^b} , which is the type 2a involutive automorphism generated by the matrix $U(t)$, where $U(t) = \text{dsum}\{\mathbf{1}_{\ell-b}, -\mathbf{1}_b, t\mathbf{1}_{\ell-b}, -t\mathbf{1}_b\}$, may be taken to be the representative of the conjugacy class $(G)^{(b)}$. The action of ψ_{G^b} upon a basis of the compact real form (consisting entirely of eigenvectors of ψ_{G^b}) is summarised below

$$\left. \begin{array}{l} ih_{\alpha_k} \quad (1 \leq k \leq \ell-1) \\ i(h_{\alpha_\ell} - \frac{1}{2}c) \end{array} \right\} \text{eigenvalue } 1, \quad (6.8.50)$$

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \text{ eigenvalue } 1 \\ i(e_{j\delta}^k - e_{-j\delta}^k) \text{ eigenvalue } -1 \end{array} \right\} 1 \leq k \leq \ell, \quad (6.8.51)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \text{ eigenvalue } 1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \text{ eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \text{ eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \text{ eigenvalue } -1 \end{array} \right\} j \in \mathbb{N}^0, \quad (6.8.52)$$

(where in (6.8.52), $\alpha = \varepsilon_r - \varepsilon_s$ and either $1 \leq r < s \leq \ell - b$ or $\ell - b < r < s \leq \ell$),

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \text{ eigenvalue } -1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \text{ eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \text{ eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \text{ eigenvalue } 1 \end{array} \right\} j \in \mathbb{N}^0, \quad (6.8.53)$$

(where in (6.8.53), $\alpha = \varepsilon_r - \varepsilon_s$ ($1 \leq r \leq \ell - b < s \leq \ell$)),

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \text{ eigenvalue } 1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \text{ eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \text{ eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \text{ eigenvalue } -1 \end{array} \right\} j \in \mathbb{N}^0; k = -j - 1, \quad (6.8.54)$$

(where in (6.8.54), $\alpha = \varepsilon_r + \varepsilon_s$ and either $1 \leq r \leq s \leq \ell - b$ or $\ell - b < r \leq s \leq \ell$),

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \text{ eigenvalue } -1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \text{ eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \text{ eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \text{ eigenvalue } 1 \end{array} \right\} j \in \mathbb{N}^0; k = -j - 1, \quad (6.8.55)$$

(where in (6.8.55), $\alpha = \varepsilon_r + \varepsilon_s$ ($\{1 \leq r \leq \ell - b < s \leq \ell\}$),

$$i \left(d + (\ell + 1) \left\{ \sum_{p=1}^{\ell-1} p h_{\alpha_p} + \frac{\ell}{2} h_{\alpha_\ell} \right\} \right) \text{ eigenvalue } -1. \quad (6.8.56)$$

Thus, the basis of a real form is provided by the following elements

$$ih_{\alpha_k} \quad (1 \leq k \leq \ell - 1), \quad (6.8.57)$$

$$i\left(h_{\alpha_\ell} - \frac{1}{2}c\right),$$

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \\ (e_{j\delta}^k - e_{-j\delta}^k) \end{array} \right\} 1 \leq k \leq \ell, \quad (6.8.58)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \end{array} \right\} j \in \mathbb{N}^0, \quad (6.8.59)$$

where in (6.8.59) above, $\alpha = \varepsilon_r - \varepsilon_s$ and either $1 \leq r < s \leq \ell - b$ or $\ell - b < r < s \leq \ell$,

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \end{array} \right\} j \in \mathbb{N}^0, \quad (6.8.60)$$

where in (6.8.60) above, $\alpha = \varepsilon_r - \varepsilon_s$ ($1 \leq r \leq \ell - b < s \leq \ell$),

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \end{array} \right\} j \in \mathbb{N}^0; k = -j - 1, \quad (6.8.61)$$

where in (6.8.61) above, $\alpha = \varepsilon_r + \varepsilon_s$ and either $1 \leq r \leq s \leq \ell - b$ or $\ell - b < r \leq s \leq \ell$,

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}) \end{array} \right\} j \in \mathbb{N}^0; k = -j - 1, \quad (6.8.62)$$

where in (6.8.62) above, $\alpha = \varepsilon_r + \varepsilon_s$ $\{1 \leq r \leq \ell - b < s \leq \ell\}$,

$$c, \left(d + (\ell + 1) \left[\sum_{p=1}^{\ell-1} p h_{\alpha_p} + \frac{\ell}{2} h_{\alpha_\ell} \right] \right). \quad (6.8.63)$$

The next automorphisms to be considered are those which are generated by the matrices of the form given in (6.8.3). In fact, all of these automorphisms are mutually conjugate, as will now be demonstrated. Let $U_j(t)$ be of the form (6.8.3), but with $a = j$. A matrix $S(t)$ is then defined by

$$S(t) = \begin{bmatrix} \mathbf{1}_{\ell-j-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}(1+t)\mathbf{1}_1 & \mathbf{0} & \frac{i}{2}(1-t)\mathbf{1}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_{\ell-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{i}{2}(1-t^{-1})\mathbf{1}_1 & \mathbf{0} & \frac{1}{2}(1+t^{-1})\mathbf{1}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{j+2} \end{bmatrix}. \quad (6.8.64)$$

The matrix $S(t)$ satisfies

$$\tilde{S}(t)JS(t) = J, \quad (6.8.65)$$

$$\tilde{S}^*(t^{-1})S(t) = \mathbf{1}_{2\ell}, \quad (6.8.66)$$

$$S(t)U_j(t)S(t^{-1})^{-1} = U_{j+1}(t). \quad (6.8.67)$$

Thus, all of the type 2a involutive automorphisms (with $u = 1$) generated by the matrices (6.8.3) are mutually conjugate. The class containing them is called (H). It may be shown (in a manner to the preceding demonstrations) that the conjugacy class (H) is disjoint from all of the other classes which have been identified in this section. Suppose that (H) coincided with $(F)^{(j,k)}$, for suitable values of j, k . This would imply that there existed some matrix $S(t)$ which satisfied

$$S(t) \text{dsum}\{\mathbf{1}_\ell, -\mathbf{1}_\ell\} S(t^{-1})^{-1} = \lambda t^\mu \text{dsum}\{\mathbf{1}_{\ell-j-k}, -\mathbf{1}_j, t\mathbf{1}_k, \mathbf{1}_{\ell-j-k}, -\mathbf{1}_j, t^{-1}\mathbf{1}_k\}. \quad (6.8.68)$$

The substitution $t = 1$ may be made in (6.8.24). Although we can find a matrix $S(1)$ which satisfies (6.8.24) (with $t = 1$), it may be verified that the form of $S(1)$ necessary for this to hold is such that the equation

$$\tilde{S}(1)JS(1) = \alpha J \quad (6.8.69)$$

cannot be satisfied. The conjugacy class (H) is disjoint, therefore, from the classes $(F)^{(j,k)}$. Consider then the possibility that (H) coincides with the class $(G)^{(b)}$ (for some value of b). This requires the existence of a matrix $S(t)$ such that

$$S(t) \text{dsum}\{\mathbf{1}_\ell, -\mathbf{1}_\ell\} S(t^{-1})^{-1} = \lambda t^\mu \text{dsum}\{\mathbf{1}_{\ell-b}, -\mathbf{1}_b, t\mathbf{1}_{\ell-b}, -t\mathbf{1}_b\}. \quad (6.8.70)$$

We put $t = -1$ into (6.8.26). As in the previous case, it is possible to find a matrix $S(-1)$ which satisfies (6.8.26) (with $t = -1$), although a brief inspection will demonstrate that no matrix which is capable of satisfying (6.8.26) (with $t = -1$) is also capable of satisfying

$$\tilde{S}(-1)JS(-1) = \alpha J. \quad (6.8.71)$$

Thus, the class (H) is disjoint from all other classes that have already been identified.

The representative automorphism of the conjugacy class (H) is taken to be the automorphism ψ_H , which is the type 2a involution $\{\text{dsum}\{\mathbf{1}_\ell, -\mathbf{1}_\ell\}, 1, 0\}$. The action of the automorphism ψ_H upon a basis of the compact real form of $C_\ell^{(1)}$ (a basis whose elements are all eigenvectors of ψ_H) is outlined below:

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \quad j \in \mathbf{N}^0; \text{ eigenvalue } 1 \\ i(e_{j\delta}^k - e_{-j\delta}^k) \quad j \in \mathbf{N}; \text{ eigenvalue } -1 \end{array} \right\} 1 \leq k \leq \ell, \quad (6.8.72)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \text{ eigenvalue } 1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \text{ eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \text{ eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \text{ eigenvalue } -1 \end{array} \right\} j \in \mathbb{N}^0; \alpha = \varepsilon_r - \varepsilon_s, (1 \leq r < s \leq \ell),$$

(6.8.73)

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \text{ eigenvalue } -1 \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \text{ eigenvalue } 1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \text{ eigenvalue } -1 \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \text{ eigenvalue } 1 \end{array} \right\} j \in \mathbb{N}^0; \alpha = \varepsilon_r + \varepsilon_s, (1 \leq r \leq s \leq \ell),$$

(6.8.74)

$$\left. \begin{array}{l} ic \\ id \end{array} \right\} \text{ eigenvalue } -1.$$

(6.8.75)

Cartan's method gives, therefore, the following basis of a real form of the Kac-Moody algebra $C_\ell^{(1)}$

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \quad j \in \mathbb{N}^0 \\ (e_{j\delta}^k - e_{-j\delta}^k) \quad j \in \mathbb{N} \end{array} \right\} 1 \leq k \leq \ell,$$

(6.8.76)

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \end{array} \right\} j \in \mathbb{N}^0; \alpha = \varepsilon_r - \varepsilon_s, (1 \leq r < s \leq \ell),$$

(6.8.77)

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \end{array} \right\} j \in \mathbb{N}^0; \alpha = \varepsilon_r + \varepsilon_s, (1 \leq r \leq s \leq \ell),$$

(6.8.78)

$$\begin{array}{l} ic, \\ id. \end{array}$$

(6.8.79)

7 Involutive automorphisms and real forms of $D_\ell^{(1)}$ where $\ell > 4$

7.1 Introduction

The usual realisation of $so(p, q)$ is as the algebra of real $(p + q) \times (p + q)$ matrices \mathfrak{a} which satisfy

$$\tilde{\mathfrak{a}}\mathfrak{g} + \mathfrak{g}\mathfrak{a} = \mathbf{0} \quad (\mathfrak{g} = \text{dsum}\{\mathbf{1}_p, -\mathbf{1}_q\}). \quad (7.1.1)$$

However, this is not the most convenient realisation for this present analysis. This was also the case when the complexification of $so(p, q)$ was $B_\ell^{(1)}$. A more convenient explicit realisation may be obtained by constructing the representation $\Gamma\{1, 0, \dots, 0\}$.

The representation may be specified by the following:

$$\Gamma(h_{\alpha_k}) = \frac{1}{\ell(\ell-1)} (\mathbf{e}_{k,k} - \mathbf{e}_{k+1,k+1} + \mathbf{e}_{2\ell-k, 2\ell-k} - \mathbf{e}_{2\ell+1-k, 2\ell+1-k}) \quad (k \neq \ell), \quad (7.1.2)$$

$$\Gamma(h_{\alpha_\ell}) = \frac{1}{\ell(\ell-1)} (\mathbf{e}_{\ell-1, \ell-1} + \mathbf{e}_{\ell, \ell} - \mathbf{e}_{\ell+1, \ell+1} - \mathbf{e}_{\ell+2, \ell+2}), \quad (7.1.3)$$

$$\Gamma(e_{\varepsilon_r + \varepsilon_s}) = \frac{1}{\{\ell(\ell-1)\}^{\frac{1}{2}}} (\mathbf{e}_{r, 2\ell+1-s} + (-1)^{r+s+1} \mathbf{e}_{s, 2\ell+1-r}), \quad (7.1.4)$$

$$\Gamma(e_{\varepsilon_r - \varepsilon_s}) = \frac{1}{\{\ell(\ell-1)\}^{\frac{1}{2}}} (\mathbf{e}_{r, s} + (-1)^{r+s+1} \mathbf{e}_{2\ell+1-s, 2\ell+1-r}). \quad (7.1.5)$$

The quantities $\varepsilon_r \pm \varepsilon_s$ are the positive roots of D_ℓ , which are summarised as follows: we let ℓ linear functionals ε_r (with $1 \leq r \leq \ell$) be defined so that $\alpha_r = \varepsilon_r - \varepsilon_{r+1}$ (for

$1 \leq r \leq \ell - 1$) and $\alpha_\ell = \varepsilon_{\ell-1} + \varepsilon_\ell$. The pattern of the roots may then be expressed as $\varepsilon_r \pm \varepsilon_s$ (with $1 \leq r < s \leq \ell$). Recall also that the representation satisfies

$$\Gamma(e_\alpha) = -\tilde{\Gamma}^*(e_{-\alpha}). \quad (7.1.6)$$

This representation, therefore, is in terms of matrices \mathbf{a} which satisfy

$$\tilde{\mathbf{a}}\mathbf{g} + \mathbf{g}\mathbf{a} = \mathbf{0} \quad (\mathbf{g} = \text{offdiag}\{1, -1, \dots, -1, 1\}). \quad (7.1.7)$$

It may be shown then, that the matrix $\mathbf{U}(t)$ generates an automorphism of $D_\ell^{(1)}$ only if the following condition holds:

$$\tilde{\mathbf{U}}(t)\mathbf{g}\mathbf{U}(t) = \alpha t^\beta \mathbf{g} \quad (\alpha \in \mathbb{C}; \alpha \neq 0). \quad (7.1.8)$$

This representation will also be used for the special case $\ell = 4$. In general, the Dynkin index of such a representation is given by

$$\gamma = \frac{2}{\ell(\ell-1)}, \quad (7.1.9)$$

so that when $\ell = 4$, the Dynkin index is $\frac{1}{6}$.

In some cases in this chapter, it will be useful to consider the representation $\Gamma\{0, 0, \dots, 1\}$ of D_ℓ . If required, one may construct an explicit realisation of this representation using the methods outlined previously. It is not necessary to give them explicitly here. In fact we need only make the observation that the representation is not equivalent to its contragredient representation.

7.2 The group of rotations of D_ℓ

The group of rotations of the roots of the simple Lie algebra D_ℓ is called R and has the following semi-direct product structure:

$$R = \mathcal{W} \ltimes \mathcal{S}, \quad (7.2.1)$$

where \mathcal{W} is the Weyl group of the Lie algebra in question. The group \mathcal{S} has a different structure when $\ell = 4$ from when $\ell > 4$. When $\ell > 4$, it is the case that

$$\mathcal{S} = \{1, \tau\} \quad \text{where} \quad \begin{cases} \tau(\alpha_{\ell-1}^0) = \alpha_\ell^0, \\ \tau(\alpha_\ell^0) = \alpha_{\ell-1}^0, \\ \tau(\alpha_m^0) = \alpha_m^0 \quad (m = 1, 2, \dots, \ell - 2). \end{cases} \quad (7.2.2)$$

When $\ell = 4$, the group \mathcal{S} is slightly more complicated, and this case is dealt with separately in the following chapter. Returning to the case in hand, it may be seen that the members of the Weyl group are such that

$$\tau \circ S_{\alpha_{\ell-1}^0}^0 \circ \tau = S_{\alpha_\ell^0}^0. \quad (7.2.3)$$

Hence, a generating set S for the group R is provided by

$$S = \left\{ S_{\alpha_1^0}^0, S_{\alpha_2^0}^0, \dots, S_{\alpha_{\ell-1}^0}^0, \tau \right\}. \quad (7.2.4)$$

Now, each member of the set S is a reflection in the sense that it fixes pointwise a hyperplane in the space spanned by $\{\alpha_j^0\}_{j=1}^{\ell}$ and sends vectors orthogonal to this hyperplane to their negatives. Thus, the system $\{R, S\}$ is a Coxeter system (see [20]). The conjugacy classes of the involutions are therefore found most readily by using the algorithm developed by Richardson [29]. The Coxeter graph of $\{R, S\}$ is

$$\begin{array}{c}
 \text{---} \circ \text{---} \circ \text{---} \dots \text{---} \circ \text{---} \circ \text{---} \\
 S_{\alpha_1^0}^0 \quad S_{\alpha_2^0}^0 \quad S_{\alpha_3^0}^0 \quad S_{\alpha_{\ell-2}^0}^0 \quad S_{\alpha_{\ell-1}^0}^0 \quad \tau
 \end{array} \tag{7.2.5}$$

which is of the type B_ℓ . Hence the group \mathcal{R} for D_ℓ is isomorphic to the Weyl group of B_ℓ for $\ell \neq 4$. Although the number of conjugacy classes of involutions increases as ℓ increases, the class representatives may be chosen so that they fall into a constant number of types, thus simplifying subsequent analysis. Such representatives will be given below. In some of them, various parameters are employed, together with the conditions that these parameters must satisfy. In this manner are all of the conjugacy classes of involutions included.

- (1) The single representative of this type is

$$\tau_1(\alpha_k^0) = \alpha_k^0 \quad (1 \leq k \leq \ell). \tag{7.2.6}$$

- (2) The single representative of this type is

$$\tau_1(\alpha_k^0) = -\alpha_k^0 \quad (1 \leq k \leq \ell). \tag{7.2.7}$$

- (3) The representatives in this type are of the following form:

$$\begin{aligned}
 \tau_3(\alpha_m^0) &= \alpha_m^0 \quad (1 \leq m \leq q-2; q \neq 2), \\
 \tau_3(\alpha_{q-1}^0) &= \alpha_{q-1}^0 + 2(\alpha_q^0 + \dots + \alpha_{\ell-2}^0) + \alpha_{\ell-1}^0 + \alpha_\ell^0, \\
 \tau_3(\alpha_n^0) &= -\alpha_n^0 \quad (q \leq n \leq \ell).
 \end{aligned} \tag{7.2.8}$$

where in the above, the parameter q is permitted to take integer values in the range $2 \leq q \leq \ell - 2$.

- (4) The representatives of this type are of the following form:

$$\begin{aligned}
\tau_4(\alpha_m^0) &= \alpha_m^0 & (1 \leq m \leq \ell - 3), \\
\tau_4(\alpha_{\ell-2}^0) &= \alpha_{\ell-2}^0 + \alpha_{\ell-1}^0 + \alpha_{\ell}^0, \\
\tau_4(\alpha_{\ell-1}^0) &= -\alpha_{\ell-1}^0, \\
\tau_4(\alpha_{\ell}^0) &= -\alpha_{\ell}^0.
\end{aligned} \tag{7.2.9}$$

(5) The general representative of this type is

$$\begin{aligned}
\tau_5(\alpha_m^0) &= -\alpha_m^0 & (m \text{ odd}; 1 \leq m \leq q), \\
\tau_5(\alpha_n^0) &= \alpha_{n-1}^0 + \alpha_n^0 + \alpha_{n+1}^0 & (n \text{ even}; 1 < n < q), \\
\tau_5(\alpha_{q+1}^0) &= \alpha_q^0 + \alpha_{q+1}^0, \\
\tau_5(\alpha_r^0) &= \alpha_r^0 & (q < r \leq \ell).
\end{aligned} \tag{7.2.10}$$

where the parameter q is permitted to take odd values, but does not take any of the values $\ell, \ell - 2$ (when ℓ is odd), or $\ell - 1$ (when ℓ is even).

(6) The general representative of this type is

$$\begin{aligned}
\tau_6(\alpha_m^0) &= -\alpha_m^0 & (m \text{ odd}; 1 \leq m \leq \ell - 2), \\
\tau_6(\alpha_n^0) &= \alpha_{n-1}^0 + \alpha_n^0 + \alpha_{n+1}^0 & (n \text{ even}; 1 < m < \ell - 2), \\
\tau_6(\alpha_{\ell-1}^0) &= \alpha_{\ell-2}^0 + \alpha_{\ell-1}^0, \\
\tau_6(\alpha_{\ell}^0) &= \alpha_{\ell-2}^0 + \alpha_{\ell}^0.
\end{aligned} \tag{7.2.11}$$

where, in the above, the quantity ℓ is assumed to be odd.

(7) The representative transformation of this type is given by

$$\begin{aligned}
\tau_7(\alpha_m^0) &= \alpha_m^0 & (1 \leq j \leq q - 2; q \neq 1, 2), \\
\tau_7(\alpha_{q-1}^0) &= \alpha_{q-1}^0 + \alpha_q^0 & (q \neq 1), \\
\tau_7(\alpha_n^0) &= -\alpha_n^0 & ((\ell - n) \text{ even}; q \leq n \leq \ell - 2), \\
\tau_7(\alpha_p^0) &= \alpha_{p-1}^0 + \alpha_p^0 + \alpha_{p+1}^0 & ((\ell - p) \text{ odd}; q < p < \ell - 2), \\
\tau_7(\alpha_{\ell-1}^0) &= \alpha_{\ell-2}^0 + \alpha_{\ell}^0, \\
\tau_7(\alpha_{\ell}^0) &= \alpha_{\ell-2}^0 + \alpha_{\ell-1}^0.
\end{aligned} \tag{7.2.12}$$

where, in the above, the parameter q is permitted to take values such that $(\ell - q)$ is even and $0 < q \leq \ell - 2$.

(8) The representative of this type is of the following form:

$$\begin{aligned}\tau_8(\alpha_m^0) &= -\alpha_m^0 & (m \text{ odd}; 1 \leq m \leq \ell - 2), \\ \tau_8(\alpha_n^0) &= \alpha_{n-1}^0 + \alpha_n^0 + \alpha_{n+1}^0 & (n \text{ even}; 1 < n < \ell - 2), \\ \tau_8(\alpha_{\ell-1}^0) &= \alpha_{\ell-2}^0 + \alpha_{\ell}^0, \\ \tau_8(\alpha_{\ell}) &= \alpha_{\ell-2}^0 + \alpha_{\ell-1}^0.\end{aligned}\tag{7.2.13}$$

where ℓ takes an odd value in this particular case.

(9) In this case, the representative transformations are all of the form

$$\begin{aligned}\tau_9(\alpha_m^0) &= -\alpha_m^0 & (m \text{ odd}; 1 \leq m \leq \ell - 1), \\ \tau_9(\alpha_n^0) &= \alpha_{n-1}^0 + \alpha_n^0 + \alpha_{n+1}^0 & (n \text{ even}; 1 < n < \ell - 1), \\ \tau_9(\alpha_{\ell}^0) &= \alpha_{\ell}^0.\end{aligned}\tag{7.2.14}$$

where ℓ takes an even value in this particular case.

$$\begin{aligned}\tau_{10}(\alpha_m^0) &= -\alpha_m^0 & \left. \begin{array}{l} 1 \leq m \leq q; m \text{ odd} \\ r \leq m \leq \ell \end{array} \right\}, \\ \tau_{10}(\alpha_n^0) &= \alpha_{n-1}^0 + \alpha_n^0 + \alpha_{n+1}^0 & (n \text{ even}; 1 < n < q), \\ \tau_{10}(\alpha_{q+1}^0) &= \alpha_q^0 + \alpha_{q+1}^0, \\ \tau_{10}(\alpha_p^0) &= \alpha_p^0 & (q + 1 < p < r - 1; r \neq q + 2), \\ \tau_{10}(\alpha_{r-1}^0) &= \alpha_2^0 + \dots + \alpha_{r-1}^0 + 2(\alpha_r^0 + \dots + \alpha_{\ell-2}^0) + \alpha_{\ell-1}^0 + \alpha_{\ell}^0.\end{aligned}\tag{7.2.15}$$

where the parameters q, r are such that q is odd, $r - q \geq 2$, and $r < \ell - 1$.

(11) For this type the most general transformation may be expressed in the form given below:

$$\begin{aligned}
\tau_{11}(\alpha_m^0) &= -\alpha_m^0 & (1 \leq m \leq q; m \text{ odd}), \\
\tau_{11}(\alpha_n^0) &= \alpha_{n-1}^0 + \alpha_n^0 + \alpha_{n+1}^0 & (1 < n < q; n \text{ even}), \\
\tau_{11}(\alpha_{q+1}^0) &= \alpha_q^0 + \alpha_{q+1}^0, \\
\tau_{11}(\alpha_p^0) &= \alpha_p^0 & (q+1 < p < \ell-2), \\
\tau_{11}(\alpha_{\ell-2}^0) &= \alpha_{\ell-2}^0 + \alpha_{\ell-1}^0 + \alpha_{\ell}^0, \\
\tau_{11}(\alpha_{\ell-1}^0) &= -\alpha_{\ell-1}^0, \\
\tau_{11}(\alpha_{\ell}^0) &= -\alpha_{\ell}^0.
\end{aligned} \tag{7.2.16}$$

where the parameter q takes odd values such that $q+2 < \ell-1$.

(12) In this case, the most general transformation is that for which

$$\begin{aligned}
\tau_{12}(\alpha_m^0) &= -\alpha_m^0 & (1 \leq m \leq \ell-3; m \text{ odd}), \\
\tau_{12}(\alpha_n^0) &= \alpha_{n-1}^0 + \alpha_n^0 + \alpha_{n+1}^0 & (1 < n < \ell-3; n \text{ even}), \\
\tau_{12}(\alpha_{\ell-2}^0) &= \alpha_{\ell-3}^0 + \alpha_{\ell-2}^0 + \alpha_{\ell-1}^0 + \alpha_{\ell}^0, \\
\tau_{12}(\alpha_{\ell-1}^0) &= -\alpha_{\ell-1}^0, \\
\tau_{12}(\alpha_{\ell}^0) &= -\alpha_{\ell}^0.
\end{aligned} \tag{7.2.17}$$

where, in this case, ℓ takes an even value.

7.3 Supplementary notation

We will introduce some notational shorthand here, in order to simplify the analysis that is to follow. The shorthand introduced is applicable only to algebras of the $D_\ell^{(1)}$ series of affine complex Kac-Moody algebras. Much of the notation introduced is similar to the notation introduced for the Kac-Moody algebras $B_\ell^{(1)}$ and $C_\ell^{(1)}$. In particular, we define general forms of matrices. These do not specify individual matrices, but specify a set (often infinite) of matrices which fall into a general pattern.

1. We define $\mathbf{D}_{j,k}$ and $\hat{\mathbf{D}}_{j,k}$ to be diagonal $(k-j+1) \times (k-j+1)$ matrices such that

$$\mathbf{D}_{j,k} = \text{diag} \left\{ \lambda_j t^{\mu_j}, \dots, \lambda_k t^{\mu_k} \right\} \quad (\lambda_q^2 = 1 \text{ for } j \leq q \leq k), \quad (7.3.1)$$

$$\hat{\mathbf{D}}_{j,k} = \text{diag} \left\{ \lambda_k^{-1} t^{-\mu_k}, \dots, \lambda_j^{-1} t^{-\mu_j} \right\} \quad (\lambda_q^2 = 1 \text{ for } j \leq q \leq k). \quad (7.3.2)$$

There are also other general forms defined, which are similar to $\mathbf{D}_{j,k}$ and $\hat{\mathbf{D}}_{j,k}$. The forms $\mathbf{D}_{j,k}^{(0)}$ and $\hat{\mathbf{D}}_{j,k}^{(0)}$ are defined to be the same as the general forms $\mathbf{D}_{j,k}$ and $\hat{\mathbf{D}}_{j,k}$, but such that $\mu_q = 0$ (for $j \leq q \leq k$). In a similar manner, we define $\mathbf{D}_{j,k}^{(1)}$ and $\hat{\mathbf{D}}_{j,k}^{(1)}$ (or $\mathbf{D}_{j,k}^{(2)}$ and $\hat{\mathbf{D}}_{j,k}^{(2)}$ respectively) to be of the general forms $\mathbf{D}_{j,k}$ and $\hat{\mathbf{D}}_{j,k}$, but with the additional constraint that μ_q be odd (or even, respectively) for $j \leq q \leq k$.

2. The form of matrix $\mathbf{F}_{j,k}$ and the form $\hat{\mathbf{F}}_{j,k}$ (where, in both cases, $k > j$) are defined by

$$\mathbf{F}_{j,k} = \text{offdiag} \left\{ \lambda_j t^{\mu_j}, \dots, \lambda_k t^{\mu_k} \right\}, \quad (7.3.3)$$

$$\hat{\mathbf{F}}_{j,k} = (\mathbf{F}_{j,k})^{-1}.$$

The only constraint upon these is the requirement that $\lambda_q \neq 0$ (for $j \leq q \leq k$).

3. We define the forms $\mathbf{L}_{j,k}$ and $\hat{\mathbf{L}}_{j,k}$ (where $k - j$ is even, and greater than zero) by

$$\mathbf{L}_{j,k} = \text{dsum}\{\lambda_j, \dots, \lambda_q, \dots, \lambda_k\}, \quad (7.3.4)$$

$$\hat{\mathbf{L}}_{j,k} = \text{dsum}\{\lambda_k, \dots, \lambda_q, \dots, \lambda_j\}, \quad (7.3.5)$$

where the submatrix λ_q is defined (for q such that $(q - j)$ is even and positive) by

$$\lambda_q = \begin{bmatrix} 0 & \lambda t^{\mu_q} \\ \lambda_q^{-1} t^{-\mu_q} & 0 \end{bmatrix}. \quad (7.3.6)$$

4. The general form $\mathbf{L}'_{j,k}$ is similar to the general form $\mathbf{L}_{j,k}$, which we have seen defined. It, together with the form $\hat{\mathbf{L}}'_{j,k}$, is defined (for $(k - j)$ even) by

$$\mathbf{L}'_{j,k} = \text{dsum}\{\lambda'_j, \dots, \lambda'_q, \dots, \lambda'_k\}, \quad (7.3.7)$$

$$\hat{\mathbf{L}}'_{j,k} = \text{dsum}\{\hat{\lambda}'_k, \dots, \hat{\lambda}'_q, \dots, \hat{\lambda}'_j\}, \quad (7.3.8)$$

where the 2×2 submatrices λ'_q and $\hat{\lambda}'_q$ are defined (for $(q - j)$ even, $j \leq q \leq k$) by

$$\lambda'_q = \begin{bmatrix} 0 & \lambda_q t^{\mu_q} \\ -\lambda_j \lambda_q^{-1} t^{\mu_j - \mu_q} & 0 \end{bmatrix}, \quad (7.3.9)$$

$$\hat{\lambda}'_q = (\lambda'_q)^{-1}. \quad (7.3.10)$$

5. The general form $\mathbf{L}''_{j,k}$ is similar to the general form $\mathbf{L}_{j,k}$, which we have seen defined. It, together with the form $\hat{\mathbf{L}}''_{j,k}$, is defined (for $(k - j)$ even) by

$$\mathbf{L}''_{j,k} = \text{dsum}\{\lambda''_j, \dots, \lambda''_q, \dots, \lambda''_k\}, \quad (7.3.11)$$

$$\hat{\mathbf{L}}''_{j,k} = \text{dsum}\{\hat{\lambda}''_k, \dots, \hat{\lambda}''_q, \dots, \hat{\lambda}''_j\}, \quad (7.3.12)$$

where the 2×2 submatrices λ''_q and $\hat{\lambda}''_q$ are defined (for $(q - j)$ even, $j \leq q \leq k$) by

$$\lambda'_q = \begin{bmatrix} 0 & \lambda_q t^{\mu_q} \\ \lambda_j \lambda_q^{-1} t^{\mu_j - \mu_q} & 0 \end{bmatrix}, \quad (7.3.13)$$

$$\hat{\lambda}''_q = (\lambda''_q)^{-1}. \quad (7.3.14)$$

6. The general form $\mathbf{M}_{j,k}$ and its corresponding general form $\hat{\mathbf{M}}_{j,k}$ are defined for values of j and k such that $(k - j)$ is even. They are specified by

$$\mathbf{M}_{j,k} = \text{dsum} \{ \mu_j, \dots, \mu_q, \dots, \mu_k \}, \quad (7.3.15)$$

$$\hat{\mathbf{M}}_{j,k} = \text{dsum} \{ \hat{\mu}_k, \dots, \hat{\mu}_q, \dots, \hat{\mu}_j \}. \quad (7.3.16)$$

The 2×2 submatrices μ_q and $\hat{\mu}_q$ are defined (for $(q - j)$ is even, and $j \leq q \leq k$) by

$$\mu_q = \begin{bmatrix} 0 & \lambda_q t^{\mu_q} \\ (-1)^{\mu_q} \lambda_q^{-1} t^{-\mu_q} & 0 \end{bmatrix}, \quad (7.3.17)$$

$$\hat{\mu}_q = \begin{bmatrix} 0 & (-1)^{\mu_q} \lambda_q t^{\mu_q} \\ \lambda_q^{-1} t^{-\mu_q} & 0 \end{bmatrix}. \quad (7.3.18)$$

7. The general form $\mathbf{M}'_{1,k}$ and the corresponding general form $\hat{\mathbf{M}}'_{1,k}$ are defined (for odd values of k) by

$$\mathbf{M}'_{1,k} = \text{dsum} \{ \mu'_1, \dots, \mu'_q, \dots, \mu'_k \}, \quad (7.3.19)$$

$$\hat{\mathbf{M}}'_{1,k} = \text{dsum} \{ \hat{\mu}'_k, \dots, \hat{\mu}'_q, \dots, \hat{\mu}'_1 \}, \quad (7.3.20)$$

where the 2×2 submatrices μ'_q and $\hat{\mu}'_q$ are defined (for odd values of q) by

$$\mu'_q = \begin{bmatrix} 0 & \lambda_q t^{\mu_q} \\ (-1)^{1+\mu_\epsilon+\mu_q} \lambda_1 \lambda_q^{-1} t^{\mu_1} & 0 \end{bmatrix}, \quad (7.3.21)$$

$$\hat{\mu}'_q = \begin{bmatrix} 0 & (-1)^{1+\mu_\epsilon+\mu_q} \lambda_1^{-1} \lambda_q t^{-\mu_1} \\ \lambda_q^{-1} t^{-\mu_q} & 0 \end{bmatrix}, \quad (7.3.22)$$

and the quantity μ_1 is even.

8. The general form $\mathbf{M}_{1,\ell-1}''$ and the corresponding general form $\hat{\mathbf{M}}_{1,\ell-1}''$ are defined (for even values of ℓ) by

$$\mathbf{M}_{1,\ell-1}'' = \text{dsum} \{ \boldsymbol{\mu}_1'', \dots, \boldsymbol{\mu}_q'', \dots, \boldsymbol{\mu}_{\ell-1}'' \}, \quad (7.3.23)$$

$$\hat{\mathbf{M}}_{1,\ell-1}'' = \text{dsum} \{ \hat{\boldsymbol{\mu}}_{\ell-1}'', \dots, \hat{\boldsymbol{\mu}}_q'', \dots, \hat{\boldsymbol{\mu}}_1'' \}, \quad (7.3.24)$$

where the 2×2 submatrices $\boldsymbol{\mu}_q''$ and $\hat{\boldsymbol{\mu}}_q''$ are defined (for odd values of q) by

$$\boldsymbol{\mu}_q'' = \begin{bmatrix} 0 & \lambda_q t^{\mu_q} \\ (-1)^{\mu_q + \lambda_2} \lambda_1 \lambda_q^{-1} t^{\mu_1 - \mu_q} & 0 \end{bmatrix}, \quad (7.3.25)$$

$$\hat{\boldsymbol{\mu}}_q'' = \begin{bmatrix} 0 & (-1)^{\lambda_2 + \mu_q} \lambda_1^{-1} \lambda_q t^{\mu_q - \mu_1} \\ \lambda_q^{-1} t^{-\mu_q} & 0 \end{bmatrix}, \quad (7.3.26)$$

the quantity μ_1 is even, and $\lambda_2 = \pm 1$.

9. The general form $\mathbf{N}_{j,k}$ and its corresponding general form $\hat{\mathbf{N}}_{j,k}$ are defined for values of j and k such that $(k - j)$ is even. They are specified by

$$\mathbf{N}_{j,k} = \text{dsum} \{ \mathbf{v}_j, \dots, \mathbf{v}_q, \dots, \mathbf{v}_k \}, \quad (7.3.27)$$

$$\hat{\mathbf{N}}_{j,k} = \text{dsum} \{ \hat{\mathbf{v}}_k, \dots, \hat{\mathbf{v}}_q, \dots, \hat{\mathbf{v}}_j \}. \quad (7.3.28)$$

The 2×2 submatrices \mathbf{v}_q and $\hat{\mathbf{v}}_q$ are defined (for $(q - j)$ is even, and $j \leq q \leq k$) by

$$\mathbf{v}_q = \begin{bmatrix} 0 & \lambda_q t^{\mu_q} \\ \lambda_q^{-1} t^{\mu_q} & 0 \end{bmatrix}, \quad (7.3.29)$$

$$\hat{\mathbf{v}}_q = \begin{bmatrix} 0 & \lambda_q t^{-\mu_q} \\ \lambda_q^{-1} t^{-\mu_q} & 0 \end{bmatrix}. \quad (7.3.30)$$

10. The general form $\mathbf{N}'_{1,k}$ and the corresponding general form $\hat{\mathbf{N}}'_{1,k}$ are defined (for odd values of k) by

$$\mathbf{N}'_{j,k} = \text{dsum} \{ \mathbf{v}'_j, \dots, \mathbf{v}'_q, \dots, \mathbf{v}'_k \}, \quad (7.3.31)$$

$$\hat{\mathbf{N}}'_{j,k} = \text{dsum} \{ \hat{\mathbf{v}}'_k, \dots, \hat{\mathbf{v}}'_q, \dots, \hat{\mathbf{v}}'_j \}, \quad (7.3.32)$$

where the 2×2 submatrices \mathbf{v}'_q and $\hat{\mathbf{v}}'_q$ are defined (for odd values of q) by

$$\mathbf{v}'_q = \begin{bmatrix} 0 & \lambda_q t^{\mu_q} \\ \lambda_1 \lambda_q^{-1} t^{\mu_q} & 0 \end{bmatrix}, \quad (7.3.33)$$

$$\hat{\mathbf{v}}'_q = \begin{bmatrix} 0 & \lambda_1^{-1} \lambda_q t^{-\mu_q} \\ \lambda_q^{-1} t^{-\mu_q} & 0 \end{bmatrix}. \quad (7.3.34)$$

11. The general form $\mathbf{N}''_{1,\ell-1}$ and the corresponding general form $\hat{\mathbf{N}}''_{1,\ell-1}$ are defined (for even values of ℓ) by

$$\mathbf{N}''_{j,k} = \text{dsum} \{ \mathbf{v}''_j, \dots, \mathbf{v}''_q, \dots, \mathbf{v}''_k \}, \quad (7.3.35)$$

$$\hat{\mathbf{N}}''_{j,k} = \text{dsum} \{ \hat{\mathbf{v}}''_k, \dots, \hat{\mathbf{v}}''_q, \dots, \hat{\mathbf{v}}''_j \}. \quad (7.3.36)$$

The 2×2 submatrices \mathbf{v}''_q and $\hat{\mathbf{v}}''_q$ are defined (for odd values of q , and $1 \leq q \leq \ell - 1$) by

$$\mathbf{v}''_q = \begin{bmatrix} 0 & \lambda_q t^{\mu_q} \\ -\lambda_1 \lambda_q^{-1} t^{\mu_q} & 0 \end{bmatrix}, \quad (7.3.37)$$

$$\hat{\mathbf{v}}''_q = \begin{bmatrix} 0 & -\lambda_1^{-1} \lambda_q t^{-\mu_q} \\ \lambda_q^{-1} t^{-\mu_q} & 0 \end{bmatrix}. \quad (7.3.38)$$

7.4 Involutive automorphisms of $D_\ell^{(1)}$

We will now specify the involutive Cartan-preserving automorphisms of $D_\ell^{(1)}$ that are to be examined. It is sufficient to give the matrices that generate them. We proceed systematically through the representative root transformations which were given previously in this chapter. For each such root transformation (which is τ_j , for a suitable value of j), we find the most general matrix $U(t)$ which satisfies

$$U(t) \left\{ \mathfrak{h}_{\alpha_k}^0 \right\} U(t)^{-1} = i \mathfrak{h}_{\tau_j(\alpha_k)}^0 \quad (\text{for } 1 \leq k \leq \ell). \quad (7.4.1)$$

We then impose a condition on $U(t)$ which depends according to the type of automorphisms being sought. For the type 1a automorphisms, this condition (which may be termed the involutiveness condition) is that

$$U(t)U(-t) = \lambda t^\mu \mathbf{1}_{2\ell} \quad (\lambda \neq 0). \quad (7.4.2)$$

For the type 2a automorphisms, the corresponding involutiveness condition is that

$$U(t)U(t^{-1}) = \lambda t^\mu \mathbf{1}_{2\ell} \quad (\lambda \neq 0). \quad (7.4.3)$$

In addition, the matrix $U(t)$ generates an automorphisms of the compact real form of $D_\ell^{(1)}$ only if it satisfies

$$\tilde{U}^*(t^{-1})U(t) = \lambda t^\mu \mathbf{1}_{2\ell} \quad (\lambda \neq 0). \quad (7.4.4)$$

We now give a list of matrices (each of which is called $U(t)$) which has been obtained by following the procedure referred to above. This procedure is followed three times in all, once for the type 1a involutive automorphisms with $u = 1$, once for the type 1a involutive automorphisms with $u = -1$, and once for the type 2a involutive automorphisms with $u = 1$.

The type 1a involutive automorphisms (with $u = 1$) which are generated by following the procedure outlined above are generated by the following matrices:

$$\text{dsum} \left\{ \mathbf{D}_{1,\ell}^{(0)}, \lambda_1 \hat{\mathbf{D}}_{1,\ell}^{(0)} \right\}, \quad (7.4.5)$$

$$\text{offsum} \left\{ \mathbf{F}_{1,\ell}, \lambda_1 t^{\mu_1} \hat{\mathbf{F}}_{1,\ell} \right\}, \quad (7.4.6)$$

$$\begin{bmatrix} \mathbf{1}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{2,\ell} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{F}}_{2,\ell} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_1 \end{bmatrix}, \quad (7.4.7)$$

$$\begin{bmatrix} \mathbf{1}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{2,q}^{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{F}_{q+1,\ell} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \hat{\mathbf{F}}_{q+1,\ell} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_{2,q}^{(0)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_1 \end{bmatrix}, \quad (7.4.8)$$

$$\text{dsum} \left\{ \mathbf{L}_{1,q}, \mathbf{D}_{q+2,\ell-1}^{(0)}, \lambda_\ell \mathbf{1}_1, \mathbf{1}_1 \hat{\mathbf{D}}_{q+2,\ell-1}^{(0)}, -\lambda_\ell \hat{\mathbf{L}}_{1,q} \right\}, \quad (7.4.9)$$

(where $\lambda_\ell^2 = 1$ in (7.4.9)),

$$\begin{bmatrix} \mathbf{1}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{2,q-1}^{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{L}_{q,\ell-2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{F}_{\ell,\ell} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{F}}_{\ell,\ell} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\hat{\mathbf{L}}_{q,\ell-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_{2,q-1}^{(0)} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_1 \end{bmatrix}, \quad (7.4.10)$$

$$\begin{bmatrix} \mathbf{L}'_{1,\ell-2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{\ell,\ell} & \mathbf{0} \\ \mathbf{0} & -\lambda_1 t^{\mu_1} \hat{\mathbf{F}}_{\ell,\ell} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \lambda_1 t^{\mu_1} \hat{\mathbf{L}}'_{1,\ell-2} \end{bmatrix}, \quad (7.4.11)$$

$$\begin{bmatrix} \mathbf{L}_{1,q} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_{q+3,r-1}^{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{F}_{r,\ell} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{F}}_{r,\ell} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_{q+3,r-1}^{(0)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_1 & \mathbf{0} \\ \mathbf{0} & -\hat{\mathbf{L}}_{1,q} \end{bmatrix}, \quad (7.4.12)$$

$$\begin{bmatrix} \mathbf{L}'_{1,q} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{q+2,\ell} & \mathbf{0} \\ \mathbf{0} & -\lambda_1 t^{-\mu_1} \hat{\mathbf{F}}_{q+2,\ell} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \lambda_1 t^{\mu_1} \hat{\mathbf{L}}'_{1,q} \end{bmatrix}, \quad (7.4.13)$$

$$\text{dsum} \left\{ \mathbf{L}'_{1,\ell-1}, \lambda_1 t^{\mu_1} \hat{\mathbf{L}}'_{1,\ell-1} \right\}, \quad (7.4.14)$$

$$\text{dsum} \left\{ \mathbf{L}''_{1,\ell-1}, \lambda_1 t^{\mu_1} \hat{\mathbf{L}}''_{1,\ell-1} \right\}. \quad (7.4.15)$$

If we follow the procedure described earlier for finding type 1a involutive automorphisms (with $u = -1$), then we obtain the following list of matrices (which are denoted by $\mathbf{U}(t)$, and which generate representative automorphisms):

$$\text{dsum} \left\{ \mathbf{D}_{1,\ell}^{(0)}, \lambda_1 \hat{\mathbf{D}}_{1,\ell}^{(0)} \right\}, \quad (7.4.16)$$

$$\text{offsum} \left\{ \mathbf{F}_{1,\ell}^{(2)}, \lambda_1 t^{\mu_1} \mathbf{F}_{1,\ell}^{(2)} \right\} \quad \mu_1 \text{ is even,} \quad (7.4.17)$$

$$\begin{bmatrix} \mathbf{1}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{2,\ell}^{(2)} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{F}}_{2,\ell}^{(2)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_1 \end{bmatrix}, \quad (7.4.18)$$

$$\begin{bmatrix} -\mathbf{1}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{2,\ell}^{(1)} & \mathbf{0} \\ \mathbf{0} & -\hat{\mathbf{F}}_{2,\ell}^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_1 \end{bmatrix}, \quad (7.4.19)$$

$$\begin{bmatrix} \mathbf{1}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{2,\ell-2}^{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{F}_{\ell-1,\ell}^{(2)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \hat{\mathbf{F}}_{\ell-1,\ell}^{(2)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_{2,\ell-2}^{(0)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_1 \end{bmatrix}, \quad (7.4.20)$$

$$\begin{bmatrix} -\mathbf{1}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{2,\ell-2}^{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{F}_{\ell-1,\ell}^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\hat{\mathbf{F}}_{\ell-1,\ell}^{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\hat{\mathbf{D}}_{2,\ell-2}^{(0)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_1 \end{bmatrix}, \quad (7.4.21)$$

$$\text{dsum} \left\{ \mathbf{M}_{1,q}, \mathbf{D}_{q+2,\ell}^{(0)}, \lambda_\ell \hat{\mathbf{D}}_{q+2,\ell}^{(0)}, -\lambda_\ell \hat{\mathbf{M}}_{1,q} \right\}, \quad (7.4.22)$$

$$\begin{bmatrix} \mathbf{D}_{1,q-1}^{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{q,\ell-2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{F}_{\ell,\ell}^{(2)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \lambda_1 \hat{\mathbf{F}}_{\ell,\ell}^{(2)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\lambda_1 \hat{\mathbf{M}}_{q,\ell-2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \lambda_1 \hat{\mathbf{D}}_{1,q-1}^{(0)} \end{bmatrix}, \quad (7.4.23)$$

$$\begin{bmatrix} \mathbf{M}'_{1,q} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{q+2,\ell}^{(2)} & \mathbf{0} \\ \mathbf{0} & -\lambda_1 t^{\mu_1} \hat{\mathbf{F}}_{q+2,\ell}^{(2)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \lambda_1 t^{\mu_1} \hat{\mathbf{M}}'_{1,q} \end{bmatrix}, \quad (7.4.24)$$

$$\begin{bmatrix} \mathbf{M}'_{1,q} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{q+2,\ell}^{(1)} & \mathbf{0} \\ \mathbf{0} & -\lambda_1 t^{\mu_1} \hat{\mathbf{F}}_{q+2,\ell}^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \lambda_1 t^{\mu_1} \hat{\mathbf{M}}'_{1,q} \end{bmatrix}, \quad (7.4.25)$$

$$\text{dsum} \left\{ \mathbf{M}''_{1,\ell-2}, \lambda_1 t^{\mu_1} \mathbf{M}''_{1,\ell-2} \right\}. \quad (7.4.26)$$

We follow the same procedure, but for the type 2a involutive automorphisms with $u = 1$. In this manner, we find the following list of matrices, each of which is referred to as $\mathbf{U}(t)$:

$$\text{dsum} \left\{ \mathbf{D}_{1,\ell}, \lambda_1 t^{\mu_1} \hat{\mathbf{D}}_{1,\ell} \right\}, \quad (7.4.27)$$

$$\text{offsum} \left\{ \mathbf{F}_{1,\ell}^{(0)}, \lambda_1 \hat{\mathbf{F}}_{1,\ell}^{(0)} \right\}, \quad (7.4.28)$$

$$\begin{bmatrix} t^{\mu_1} \mathbf{1}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & t^{\frac{1}{2}\mu_1} \mathbf{F}_{2,\ell}^{(0)} & \mathbf{0} \\ \mathbf{0} & t^{\frac{1}{2}\mu_1} \hat{\mathbf{F}}_{2,\ell}^{(0)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_1 \end{bmatrix}, \quad (7.4.29)$$

(where, in (7.4.29), μ_1 is even)

$$\begin{bmatrix} t^{\mu_1} \mathbf{1}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{2,q} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & t^{\frac{1}{2}\mu_1} \mathbf{F}_{q+1,\ell}^{(0)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & t^{\frac{1}{2}\mu_1} \hat{\mathbf{F}}_{q+1,\ell}^{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & t^{\frac{1}{2}\mu_1} \hat{\mathbf{D}}_{2,q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_1 \end{bmatrix}, \quad (7.4.30)$$

(where, in (7.4.30), μ_1 is even),

$$\text{dsum} \left\{ \mathbf{N}_{1,q}, \mathbf{D}_{q+2,\ell-1}, \lambda_\ell t^{\mu_\ell} \mathbf{1}_1, \mathbf{1}_1, \lambda_\ell t^{\mu_\ell} \hat{\mathbf{D}}_{q+2,\ell-1}, -\lambda_\ell t^{\mu_\ell} \hat{\mathbf{N}}_{1,q} \right\}, \quad (7.4.31)$$

(where $\lambda_\ell^2 = 1$ in (7.4.31)),

$$\begin{bmatrix} t^{\mu_1} \mathbf{1}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{2,q-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{N}_{q,\ell-2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & t^{\frac{1}{2}\mu_1} \mathbf{F}_{\ell,\ell} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & t^{\frac{1}{2}\mu_1} \hat{\mathbf{F}}_{\ell,\ell} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\lambda_1 t^{\mu_1} \hat{\mathbf{N}}_{q,\ell-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \lambda_1 t^{\mu_1} \hat{\mathbf{D}}_{2,q-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_1 \end{bmatrix}, \quad (7.4.32)$$

(where μ_1 is even in (7.4.32)),

$$\begin{bmatrix} \mathbf{N}'_{1,\ell-2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & t^{\frac{1}{2}\mu_1} \mathbf{F}_{\ell,\ell}^{(0)} & \mathbf{0} \\ \mathbf{0} & -\lambda_1 t^{\frac{1}{2}\mu_1} \hat{\mathbf{F}}_{\ell,\ell}^{(0)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \lambda_1 t^{\mu_1} \hat{\mathbf{N}}_{1,\ell-2} \end{bmatrix}, \quad (7.4.33)$$

(where μ_1 is even in (7.4.33)),

$$\begin{bmatrix} \mathbf{N}_{1,q} & \mathbf{0} \\ \mathbf{0} & t^{\mu_\ell} \mathbf{1}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_{q+3,r-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & t^{\mu_\ell} \mathbf{F}_{r,\ell} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & t^{\mu_\ell} \hat{\mathbf{F}}_{r,\ell} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{D}}_{q+3,r-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_1 & \mathbf{0} \\ \mathbf{0} & -\hat{\mathbf{N}}_{1,q} \end{bmatrix}, \quad (7.4.34)$$

$$\text{dsum} \left\{ \mathbf{N}'_{1,\ell-1}, \lambda_1 t^{\mu_1} \hat{\mathbf{N}}'_{1,\ell-1} \right\}, \quad (7.435)$$

$$\text{dsum} \left\{ \mathbf{N}''_{1,\ell-1}, \lambda_1 t^{\mu_1} \mathbf{N}''_{1,\ell-1} \right\}. \quad (7.436)$$

7.5 Matrix transformations

In this section, we give some matrix transformations which will come in useful later on in the chapter. They have the effect of reducing the number of matrices (and thus the automorphisms generated by them) since they demonstrate that automorphisms generated by matrices of certain forms are mutually conjugate. Since we are engaged in the task of finding non-conjugate real forms, this means that only one involutive automorphism from each conjugacy class is required. In some of the following examples, the matrix transformation given in that example is applicable only to automorphisms of one type, or for only one value of u . In many of them, however, the examples (and their conclusions) hold good for other types of automorphisms with little or no modification. Some of the following give such modification, but in most cases such modification is little more than trivial, and is not listed explicitly.

1. This matrix transformation relates to the index set of the matrices which generate automorphisms of $D_\ell^{(1)}$. Let the integers j and k be such that $1 \leq j < k \leq \ell$. Let $U(t)$ and $V(t)$ be matrices which generate automorphisms of $D_\ell^{(1)}$, and which are such that $V(t)$ may be obtained from $U(t)$ by exchanging its j th and k th rows and columns (and also its $(2\ell+1-j)$ th and $(2\ell+1-k)$ th rows and columns). We define a matrix $S(t)$ by

$$(S(t))_{m,n} = \begin{cases} (i)^{\deg(k-j)} & m = n; m \neq j, k, j', k', \\ 0 & m = n; m = j, k, j', k', \\ 1 & (m, n) = (j, k), (k, j), (j', k'), (k', j'), \\ 0 & \text{for other values of } m, n. \end{cases} \quad (7.5.1)$$

The quantities j' and k' are given by $(2\ell+1-j)$ and $(2\ell+1-k)$ respectively. We then see that

$$\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{2\ell},$$

$$\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = \mathbf{g}, \quad (7.5.2)$$

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(t)^{-1} = \mathbf{V}(t). \quad (7.5.3)$$

Thus, for all types of automorphism, the automorphisms $\{\mathbf{U}(t), u, \xi\}$ and $\{\mathbf{V}(t), u, \xi\}$ are conjugate.

2. Let the matrix $\mathbf{U}(t)$ be of the form given by

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{H}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{p+1,\ell} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{F}}_{p+1,\ell} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_p \end{bmatrix}, \quad (7.5.4)$$

where \mathbf{H}_p is used to represent a completely arbitrary $p \times p$ matrix. If $\{\mathbf{U}(t), 1, \xi\}$ is a type 1a involutive automorphism, then let $\sigma_j = \frac{1}{2}(\deg \mu_j - \mu_j)$ and define the matrix $\mathbf{S}(t)$ by

$$\mathbf{S}(t) = \text{dsum}\{\mathbf{1}_p, \mathbf{x}(t), \mathbf{x}'(t), \mathbf{1}_p\}, \quad (7.5.5)$$

where the submatrices are given by

$$\mathbf{x}(t) = \text{diag}\left\{\lambda_{p+1}^{\frac{1}{2}} t^{-\sigma_{p+1}}, \dots, \lambda_\ell^{\frac{1}{2}} t^{-\sigma_\ell}\right\}, \quad (7.5.6)$$

$$\mathbf{x}'(t) = \text{diag}\left\{\lambda_\ell^{-\frac{1}{2}} t^{\sigma_\ell}, \dots, \lambda_{p+1}^{-\frac{1}{2}} t^{\sigma_{p+1}}\right\}, \quad (7.5.7)$$

We now define the matrix $\mathbf{V}(t)$ by

$$\mathbf{V}(t) = \begin{bmatrix} \mathbf{H}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{y}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{y}^{-1}(t) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_p \end{bmatrix}, \quad (7.5.8)$$

where the submatrix is given by $\mathbf{y}(t) = \text{offdiag}\{t^{\text{deg}\mu_{p+1}}, \dots, t^{\text{deg}\mu_\ell}\}$. The matrix $\mathbf{S}(t)$ satisfies

$$\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = \mathbf{g}, \quad (7.5.9)$$

$$\mathbf{S}(t)\mathbf{V}(t)\mathbf{S}(t)^{-1} = \mathbf{U}(t). \quad (7.5.10)$$

Furthermore, if $\tilde{\mathbf{U}}^*(t^{-1})\mathbf{U}(t) = \lambda t^\mu \mathbf{1}_{2\ell}$ (for some non-zero complex number λ), then we also have that $\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \alpha t^\beta \mathbf{1}_{2\ell}$, where α is some non-zero complex number.

3. The previous matrix transformation is able to help demonstrate that certain type 1a involutive automorphisms are conjugate in the case where $u = 1$. When $u = -1$, it has to be adapted slightly. Let the matrix $\mathbf{U}(t)$ be given by

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{H}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{p+1,\ell}^{(2)} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{F}}_{p+1,\ell}^{(2)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_p \end{bmatrix}, \quad (7.5.11)$$

where the matrix \mathbf{H}_p is as defined previously. We define the matrix $\mathbf{V}(t)$ by

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{H}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{y} & \mathbf{0} \\ \mathbf{0} & \mathbf{y}^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_p \end{bmatrix}, \quad (7.5.12)$$

where $\mathbf{y} = \text{offdiag}\{(-1)^{\frac{1}{2}\mu_{p+1}}, \dots, (-1)^{\frac{1}{2}\mu_\ell}\}$. The matrix $\mathbf{S}(t)$ is defined by

$$\mathbf{S}(t) = \text{dsum}\{\mathbf{1}_p, \mathbf{x}(t), \mathbf{x}'(t), \mathbf{1}_p\}, \quad (7.5.13)$$

where the submatrices $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ are given by

$$\mathbf{x}(t) = \text{diag}\left\{\lambda_{p+1}^{\frac{1}{2}} t^{-\sigma_{p+1}}, \dots, \lambda_\ell^{\frac{1}{2}} t^{-\sigma_\ell}\right\}, \quad (7.5.14)$$

$$\mathbf{x}'(t) = \text{diag}\left\{\lambda_\ell^{-\frac{1}{2}} t^{\sigma_\ell}, \dots, \lambda_{p+1}^{-\frac{1}{2}} t^{\sigma_{p+1}}\right\}. \quad (7.5.15)$$

If $\tilde{\mathbf{U}}^*(t^{-1})\mathbf{U}(t) = \lambda t^\mu \mathbf{1}_{2\ell}$, it then follows that $\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \alpha t^\beta \mathbf{1}_{2\ell}$ (where α and λ are both non-zero complex numbers). Thus,

$$\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = \mathbf{g}, \quad (7.5.16)$$

$$\mathbf{S}(t)\mathbf{V}(t)\mathbf{S}(-t)^{-1} = \mathbf{U}(t). \quad (7.5.17)$$

The type 1a involutive automorphisms $\{\mathbf{U}(t), -1, \xi\}$ and $\{\mathbf{V}(t), -1, \xi\}$ are therefore conjugate.

4. This transformation follows the two previous examples closely. Let the matrix $\mathbf{U}(t)$ be defined by

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{H}_{q-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{q,\ell}^{(0)} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{q,\ell}^{(0)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_{q-1} \end{bmatrix}. \quad (7.5.18)$$

Similarly, let the matrix $\mathbf{V}(t)$ be defined by

$$\mathbf{V}(t) = \text{dsum}\{\mathbf{H}_{q-1}, \mathbf{K}_{2\ell+2-2q}, \mathbf{H}_{q-1}\}. \quad (7.5.19)$$

The matrix $\mathbf{S}(t)$ is defined by

$$\mathbf{S}(t) = \text{dsum}\left\{\mathbf{1}_{q-1}, \text{diag}\left\{\lambda_q^{\frac{1}{2}}, \dots, \lambda_\ell^{\frac{1}{2}}, 1, \lambda_\ell^{-\frac{1}{2}}, \dots, \lambda_q^{-\frac{1}{2}}\right\}, \mathbf{1}_{q-1}\right\}. \quad (7.5.20)$$

If $\lambda_j \lambda_j^* = 1$ (for $q \leq j \leq \ell$) then $\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{2\ell+1}$. It is also clear that

$$\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = \mathbf{g}, \quad (7.5.21)$$

$$\mathbf{S}(t)\mathbf{V}(t)\mathbf{S}(t^{-1})^{-1} = \mathbf{U}(t). \quad (7.5.22)$$

Therefore, the type 2a automorphisms $\{\mathbf{U}(t), 1, \xi\}$ and $\{\mathbf{V}(t), 1, \xi\}$ are conjugate.

5. In this case, let the matrix $\mathbf{U}(t)$ be of the form given by

$$\mathbf{U}(t) = \text{dsum} \left\{ \mathbf{H}_{p-1}, \mathbf{D}_{p,q}, \mathbf{H}_{2\ell-2q}, \lambda \hat{\mathbf{D}}_{p,q}, \mathbf{H}_{p-1} \right\}, \quad (7.5.23)$$

where $\lambda \neq 0$, and let the matrix $\mathbf{V}(t)$ be of a form similar to that of $\mathbf{U}(t)$, namely

$$\mathbf{V}(t) = \text{dsum} \left\{ \mathbf{H}_{p-1}, \mathbf{B}_{p,q}, \mathbf{H}_{2\ell-2q}, \lambda \hat{\mathbf{B}}_{p,q}, \mathbf{H}_{p-1} \right\}, \quad (7.5.24)$$

with the submatrices $\mathbf{B}_{p,q}, \hat{\mathbf{B}}_{p,q}$ being obtained from $\mathbf{D}_{p,q}, \hat{\mathbf{D}}_{p,q}$ respectively by replacing μ_j with $\deg \mu_j$ for $p \leq j \leq q$. Then let the matrix $\mathbf{S}(t)$ be given by

$$\mathbf{S}(t) = \text{dsum} \left\{ \mathbf{1}_{p-1}, \mathbf{X}(t), \mathbf{1}_r, \mathbf{X}(t), \mathbf{1}_{p-1} \right\}, \quad (7.5.25)$$

with the submatrices being defined by

$$\mathbf{X}(t) = \text{diag} \left\{ t^{-\sigma_p}, \dots, t^{-\sigma_q} \right\}, \quad (7.5.26)$$

$$\mathbf{X}'(t) = \text{diag} \left\{ t^{\sigma_q}, \dots, t^{\sigma_p} \right\}. \quad (7.5.27)$$

It is clear that this satisfies $\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = \mathbf{g}$, also $\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_n$, and most importantly

$$\mathbf{S}(t)\mathbf{V}(t)\mathbf{S}(t^{-1})^{-1} = \mathbf{U}(t). \quad (7.5.28)$$

Thus, the type 2a automorphisms $\{\mathbf{U}(t), 1, \xi\}$ and $\{\mathbf{V}(t), 1, \xi\}$ are conjugate.

6. Let the matrix $\mathbf{U}(t)$ be given by the equation

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{H}_{p-2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_{2\ell-2p} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_{p-2} \end{bmatrix}. \quad (7.5.29)$$

Then, with the definition $\mathbf{x} = \text{diag} \{1, -1\}$, a matrix $\mathbf{V}(t)$ may be defined thus:

$$\mathbf{V}(t) = \text{dsum} \left\{ \mathbf{H}_{p-2}, \mathbf{x}, \mathbf{H}_{2\ell-2p}, -\mathbf{x}, \mathbf{H}_{p-2} \right\}. \quad (7.5.30)$$

We define a matrix \mathbf{S} (which is independent of t) by

$$S = \begin{bmatrix} \mathbf{1}_{p-2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{t}_1 & \mathbf{0} & \mathbf{t}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{t}_3 & \mathbf{0} & \mathbf{t}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{p-2} \end{bmatrix}, \quad (7.5.31)$$

where the submatrices are all given by the following:

$$\mathbf{t}_1 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \quad \mathbf{t}_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad (7.5.32)$$

$$\mathbf{t}_3 = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad \mathbf{t}_4 = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}. \quad (7.5.33)$$

These matrices that have been defined satisfy the following:

$$\tilde{S}gS = g, \quad (7.5.34)$$

$$\tilde{S}^*S = \mathbf{1}_{2\ell}, \quad (7.5.35)$$

$$S\mathbf{V}(t)S^{-1} = \mathbf{U}(t). \quad (7.5.36)$$

Thus, the automorphisms $\{\mathbf{U}(t), u, \xi\}$ and $\{\mathbf{V}(t), u, \xi\}$ are conjugate.

7. Let the matrix $\mathbf{U}(t)$ be given by

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{H}_{p-2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & t\mathbf{K}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_{2\ell-2p} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & t^{-1}\mathbf{K}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_{p-2} \end{bmatrix}. \quad (7.5.37)$$

Also, let the matrix $\mathbf{V}(t)$ be given by

$$\mathbf{V}(t) = \text{dsum} \{ \mathbf{H}_{p-2}, \mathbf{1}_1, -\mathbf{1}_1, \mathbf{H}_{2\ell-2p}, -\mathbf{1}_1, \mathbf{1}_1, \mathbf{H}_{p-2} \}. \quad (7.5.38)$$

Let the matrix $S(t)$ be defined by

$$S(t) = \begin{bmatrix} \mathbf{1}_{p-2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{t}_1 & \mathbf{0} & \mathbf{t}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{t}_3 & \mathbf{0} & \mathbf{t}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{p-2} \end{bmatrix}, \quad (7.5.39)$$

where the submatrices are defined by

$$\mathbf{t}_1 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \quad \mathbf{t}_2 = \frac{t}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad (7.5.40)$$

$$\mathbf{t}_3 = \frac{t^{-1}}{2} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad \mathbf{t}_4 = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}. \quad (7.5.41)$$

It follows that the following equations all hold:

$$\tilde{S}(t) \mathbf{g} S(t) = \mathbf{g}, \quad (7.5.42)$$

$$\tilde{S}^*(t^{-1}) S(t) = \mathbf{1}_n, \quad (7.5.43)$$

$$S(t) \mathbf{V}(t) S(t)^{-1} = \mathbf{U}(t). \quad (7.5.44)$$

Thus, the type 1a automorphisms $\{\mathbf{U}(t), 1, \xi\}$ and $\{\mathbf{V}(t), 1, \xi\}$ are conjugate.

8. Suppose that the matrix $\mathbf{U}(t)$ is of the form given by

$$\mathbf{U}(t) = \text{dsum} \left\{ \mathbf{H}_{m-1}, \mathbf{L}_{m,n}, \mathbf{H}_{2\ell+1-2n-2}, \lambda \hat{\mathbf{L}}_{m,n}, \mathbf{H}_{m-1} \right\}, \quad (7.5.45)$$

where the quantity λ is some non-zero complex number. We let the matrix $\mathbf{V}(t)$ be defined by

$$\mathbf{V}(t) = \text{dsum} \left\{ \mathbf{H}_{m-1}, \mathbf{a}, \mathbf{H}_{2\ell-2n}, \lambda \mathbf{a}, \mathbf{H}_{m-1} \right\}, \quad (7.5.46)$$

where \mathbf{a} is obtained from $\mathbf{L}_{m,n}$ by letting $\lambda_q t^{\mu_q} = 1$ for $m \leq q \leq n$. A matrix $\mathbf{S}(t)$ is defined by

$$\mathbf{S}(t) = \text{dsum}\{\mathbf{1}_{m-1}, \mathbf{X}(t), \mathbf{1}_{2\ell-1-2n}, \mathbf{X}'(t), \mathbf{1}_{m-1}\}, \quad (7.5.47)$$

where

$$\mathbf{X}(t) = \text{diag}\{\lambda_m^{-1}t^{-\mu_m}, 1, \dots, \lambda_j^{-1}t^{-\mu_j}, 1, \dots, \lambda_n^{-1}t^{-\mu_n}, 1\}, \quad (7.5.48)$$

$$\mathbf{X}'(t) = \text{diag}\{1, \lambda_n t^{\mu_n}, \dots, 1, \lambda_j t^{\mu_j}, \dots, 1, \lambda_m t^{\mu_m}\}. \quad (7.5.49)$$

The matrix $\mathbf{S}(t)$ satisfies

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(t)^{-1} = \mathbf{V}(t), \quad (7.5.50)$$

$$\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = \mathbf{g}. \quad (7.5.51)$$

If $\lambda_q^* \lambda_q = 1$ for $m \leq q \leq n$, then $\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{2\ell}$. Thus, if $\mathbf{U}(t)$ and $\mathbf{V}(t)$ generate type 1a automorphisms (with $u = 1$), then $\{\mathbf{U}(t), 1, \xi\}$ and $\{\mathbf{V}(t), 1, \xi\}$ are conjugate. This also holds when the automorphisms in question are extensions to $D_\ell^{(1)}$ of automorphisms of the compact real form of $D_\ell^{(1)}$.

9. Let the matrices $\mathbf{U}(t)$ and $\mathbf{V}(t)$ be given by

$$\mathbf{U}(t) = \text{dsum}\{\mathbf{H}_{m-1}, \mathbf{W}_{n+2-m}, \mathbf{H}_{2\ell-2-2n}, \lambda\mathbf{W}_{n+2-m}, \mathbf{H}_{m-1}\}, \quad (7.5.52)$$

$$\mathbf{V}(t) = \text{dsum}\{\mathbf{H}_{m-1}, \mathbf{a}, \mathbf{H}_{2\ell-1-2n}, \lambda\mathbf{a}, \mathbf{H}_{m-1}\}, \quad (7.5.53)$$

where \mathbf{a} is obtained from $\mathbf{L}_{m,n}$ by letting $\lambda_q t^{\mu_q} = 1$ (for $m \leq q \leq n$), and λ is some non-zero complex number. The submatrix \mathbf{W}_k (for appropriate k) is a $(k \times k)$ diagonal matrix whose diagonal entries are alternately $1, -1, \dots$. A matrix \mathbf{S} may then be defined by

$$\mathbf{S} = \text{dsum}\{\mathbf{1}_{m-1}, \mathbf{x}, \mathbf{1}_{2\ell-1-2n}, \mathbf{x}, \mathbf{1}_{m-1}\}, \quad (7.5.54)$$

where the submatrix \mathbf{x} is defined by

$$\mathbf{x} = \frac{i}{\sqrt{2}} \begin{bmatrix} 1 & 1 & \cdots & 0 & 0 \\ 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 1 & -1 \end{bmatrix}. \quad (7.5.55)$$

This matrix satisfies

$$\tilde{\mathbf{S}}\mathbf{g}\mathbf{S} = \mathbf{g}, \quad (7.5.56)$$

$$\tilde{\mathbf{S}}^*\mathbf{S} = \mathbf{1}_{2\ell+1}, \quad (7.5.57)$$

$$\mathbf{S}\mathbf{V}(t)\mathbf{S}^{-1} = \mathbf{U}(t). \quad (7.5.58)$$

Thus, if $\mathbf{U}(t)$ and $\mathbf{V}(t)$ generate automorphisms (either of $D_\ell^{(1)}$ or of its compact real form), then the automorphisms $\{\mathbf{U}(t), u, \xi\}$ and $\{\mathbf{V}(t), u, \xi\}$ are conjugate.

10. Let $\mathbf{U}(t)$ be of the form given by

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{L}'_{1,q} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{q+2,\ell} & \mathbf{0} \\ \mathbf{0} & -\lambda_1 t^{\mu_1} \hat{\mathbf{F}}_{q+2,\ell} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \lambda_1 t^{\mu_1} \hat{\mathbf{L}}'' \end{bmatrix}. \quad (7.5.59)$$

We define a matrix $\mathbf{S}(t)$ according to

$$\mathbf{S}(t) = \text{dsum} \{ \mathbf{s}_1, \dots, \mathbf{s}_j, \dots, \mathbf{s}_q, \mathbf{t}_1, \mathbf{t}_2, \mathbf{s}'_q, \dots, \mathbf{s}_j, \dots, \mathbf{s}_1 \}, \quad (7.5.60)$$

where the submatrices are defined by

$$\mathbf{s}_j = \begin{bmatrix} \lambda_1^{\frac{1}{2}} \lambda_j^{-1} t^{\sigma_1 - \mu_j} & 0 \\ 0 & 1 \end{bmatrix} \quad (j \text{ odd}, 1 \leq j \leq q), \quad (7.5.61)$$

$$\mathbf{s}'_j = \begin{bmatrix} \lambda_1^{\frac{1}{2}} t^{\sigma_1} & 0 \\ 0 & \lambda_j t^{\mu_j} \end{bmatrix} \quad (j \text{ odd}, 1 \leq j \leq q), \quad (7.5.62)$$

$$\mathbf{t}_1 = t^{\sigma_1} \mathbf{1}_{\ell-1-q}, \quad (7.5.63)$$

$$\mathbf{t}_2 = \lambda_1^{\frac{1}{2}} \mathbf{1}_{\ell-1-q}. \quad (7.5.64)$$

The matrix $\mathbf{S}(t)$ is such that $\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = \lambda_1^{\frac{1}{2}} t^{\sigma_1} \mathbf{g}$. It also satisfies $\tilde{\mathbf{S}}(t^{-1})^* \mathbf{S}(t) = \mathbf{1}_{2\ell}$ (provided that $\lambda_k^* \lambda_k = 1$ for $1 \leq k \leq \ell$) and

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(t)^{-1} = \lambda_1^{\frac{1}{2}} t^{\sigma_1} \mathbf{V}(t), \quad (7.5.65)$$

where $\mathbf{V}(t)$ is obtained from $\mathbf{U}(t)$ by putting $\lambda_j t^{\mu_j} = 1$ (for j odd, $1 \leq j \leq q$) and $\lambda_1 t^{\mu_1} = t^{\deg \mu_1}$. Furthermore, let us retain $\mathbf{S}(t)$ as defined, and let $\mathbf{U}(t)$ be given by

$$\mathbf{U}(t) = \begin{bmatrix} \mathbf{L}_{1,q}'' & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{q+2,\ell} & \mathbf{0} \\ \mathbf{0} & -\lambda_1 t^{\mu_1} \hat{\mathbf{F}}_{q+2,\ell} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \lambda_1 t^{\mu_1} \hat{\mathbf{L}}_{1,q}'' \end{bmatrix}. \quad (7.5.66)$$

Define $\mathbf{V}(t)$ to be obtained from $\mathbf{U}(t)$ by putting $\lambda_j t^{\mu_j} = 1$ (for j odd, $1 \leq j \leq q$) and $\lambda_1 t^{\mu_1} = t^{\deg \mu_1}$. The matrix $\mathbf{S}(t)$ satisfies all of the properties already mentioned, plus

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(t)^{-1} = \lambda_1^{\frac{1}{2}} t^{\sigma_1} \mathbf{V}(t), \quad (7.5.67)$$

11. Let the matrices $\mathbf{U}(t)$ and $\mathbf{V}(t)$ be given by

$$\mathbf{U}(t) = \text{dsum} \left\{ \mathbf{L}'_{1,\ell-1}, \lambda_1 t^{\mu_1} \hat{\mathbf{L}}'_{1,\ell-1} \right\}, \quad (7.5.68)$$

$$\mathbf{V}(t) = \text{dsum} \left\{ \mathbf{L}''_{1,\ell-1}, \lambda_1 t^{\mu_1} \mathbf{L}''_{1,\ell-1} \right\}, \quad (7.5.69)$$

where ℓ is even. It should be noted that $\mathbf{U}(-t) = \mathbf{V}(t)$, and hence that the automorphisms $\{\mathbf{U}(t), u, \xi\}$ and $\{\mathbf{V}(t), u, \xi\}$ are conjugate.

12. Finally, it should be noted that, in the same way as was shown for both $\mathbf{B}_\ell^{(1)}$ and $\mathbf{C}_\ell^{(1)}$, the quantity s has no bearing upon the conjugacy classes, and may be taken to be unity.

7.6 Type 1a involutive automorphisms of $D_\ell^{(1)}$ with $u = 1$

The type 1a involutions with $u = 1$ will now be examined, in order to determine their conjugacy classes within the group of all automorphisms of $D_\ell^{(1)}$. This will be done by taking automorphisms which correspond to each of the representative root transformations given previously. The analysis contained in the previous section implies that not all of the involutive automorphisms which were given earlier must be studied. In fact, we need only consider the type 1a involutive automorphisms which are generated by the following matrices (each of which is called $U(t)$):

$$\text{dsum}\{\mathbf{1}_{\ell-b}, -\mathbf{1}_{2b}, \mathbf{1}_{\ell-b}\}, \quad (7.6.1)$$

$$\text{dsum}\{-\mathbf{1}_{\ell-b}, \mathbf{1}_b, -\mathbf{1}_b, \mathbf{1}_{\ell-b}\}, \quad (7.6.2)$$

$$\text{offsum}\{\mathbf{K}_{\ell-b}, t\mathbf{K}_b, t^{-1}\mathbf{K}_b, \mathbf{K}_{\ell-b}\}, \quad (7.6.3)$$

$$\text{offsum}\{t\mathbf{K}_{\ell-b}, \mathbf{K}_b, t\mathbf{K}_b, \mathbf{K}_b\}, \quad (7.6.4)$$

$$\begin{bmatrix} \mathbf{1}_a & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_b & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_c & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & t\mathbf{K}_d & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & t^{-1}\mathbf{K}_d & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_c & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{1}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_a \end{bmatrix}, \quad (7.6.5)$$

$$\begin{bmatrix} \mathbf{a}(t) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{\ell-1-q} & \mathbf{0} \\ \mathbf{0} & -t\mathbf{K}_{\ell-1-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{a}(-t) \end{bmatrix} \quad (7.6.6)$$

In (7.6.6), the matrix $\mathbf{a}(t)$ is defined (for odd values) of q by

$$\mathbf{a}(t) = \text{dsum}\{\mathbf{b}_1, \dots, \mathbf{b}_{\frac{1}{2}q}\} \left(\mathbf{b}_1 \cdots \mathbf{b}_q = \begin{bmatrix} 0 & t \\ -1 & 0 \end{bmatrix} \right). \quad (7.6.7)$$

We will start by investigating those type 1a involutive automorphisms $\{\mathbf{U}(t), 1, \xi\}$, where the matrix $\mathbf{U}(t)$ is of the form (7.6.1). The matrices of this form all satisfy $\bar{\mathbf{U}}^*(t^{-1})\mathbf{U}(t) = \mathbf{1}_{2\ell}$, and so the restrictions of these automorphisms to the compact real form of $D_\ell^{(1)}$ are automorphisms of the compact real form. We may assume that $0 \leq b \leq [\frac{1}{2}\ell]$. For, upon re-ordering of the index set of $\mathbf{U}(t)$, and removing a factor of -1 (if necessary), we obtain a matrix $\mathbf{U}'(t)$, equivalent to $\mathbf{U}(t)$, given by

$$\mathbf{U}'(t) = \text{dsum}\{\mathbf{1}_{a'}, -\mathbf{1}_{2b'}, \mathbf{1}_{a'}\}, \quad (7.6.8)$$

for which $0 \leq b' \leq [\frac{1}{2}\ell]$. We then define the conjugacy class $(A)^{(b)}$ to be that conjugacy class which contains the type automorphism $\{\mathbf{U}_b, 1, 0\}$, where

$$\mathbf{U}_b = \text{dsum}\{\mathbf{1}_{\ell-b}, -\mathbf{1}_{2b}, \mathbf{1}_{\ell-b}\} \quad (0 \leq b \leq [\frac{1}{2}\ell]). \quad (7.6.9)$$

It is easily verified that the classes $(A)^{(b)}$ and $(A)^{(c)}$ are disjoint whenever $0 \leq b < c \leq [\frac{1}{2}\ell]$. Consider the contrary, so that the following holds:

$$\mathbf{S}(t)\mathbf{U}_b\mathbf{S}(t)^{-1} = \lambda t^\mu \mathbf{U}_c \quad (b \neq c). \quad (7.6.10)$$

Consideration of determinants implies that $\mu = 0$, whilst rearranging the matrix equation implies that $b + c = \ell$ must hold. Since $b \neq c$, and $0 \leq b, c \leq [\frac{1}{2}\ell]$, there is a contradiction, and the original hypothesis must be false. Thus, for $D_\ell^{(1)}$ (and its compact real form), the involutive automorphisms that correspond to the identity root transformation fall into $(1 + [\frac{1}{2}\ell])$ mutually disjoint conjugacy classes. These classes

are the classes $(A)^{(b)}$, for $0 \leq b \leq [\frac{1}{2}\ell]$. The representative automorphism of the class $(A)^{(b)}$ is, as has been noted already, ψ_{A^b} , where $\psi_{A^b} = \{\text{dsum}\{\mathbf{1}_{\ell-b}, -\mathbf{1}_{2b}, \mathbf{1}_{\ell-b}\}, 1, 0\}$. Suppose next that the type 1a involutive automorphisms $\{\mathbf{U}(t), 1, \xi\}$ are being considered, where $\mathbf{U}(t)$ is of the form (7.6.2). As with the automorphisms which we have just investigated, it may be assumed that $0 \leq b \leq [\frac{1}{2}\ell]$. This follows from the arbitrary nature of the index set, as has been noted already. We will now show that none of these automorphisms is a member of a conjugacy class $(A)^{(b)}$ (for any value of b). To show this, suppose that the converse is true, so that, for some Laurent polynomial matrix $\mathbf{S}(t)$

$$\mathbf{S}(t)\text{dsum}\{\mathbf{1}_{\ell-a}, -\mathbf{1}_{2a}, \mathbf{1}_{\ell-a}\}\mathbf{S}(t)^{-1} = \lambda t^\mu \mathbf{U}_b, \quad (7.6.11)$$

$$\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = \alpha t^\beta \mathbf{g}, \quad (7.7.12)$$

where $\mathbf{U}_b = \text{dsum}\{-\mathbf{1}_b, \mathbf{1}_{\ell-b}, -\mathbf{1}_{\ell-b}, \mathbf{1}_b\}$. If (7.6.8) is examined, and the traces of both sides of it are calculated, it becomes clear that $2\ell - 4b = 0$, which implies that ℓ is even, and that $b = \frac{1}{2}\ell$. In fact, if the (7.6.9) is then rearranged and a general form for $\mathbf{S}(t)$ obtained, it can be seen that $\mathbf{S}(t)$ cannot solve (7.6.12). That is, the equation

$$\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = \alpha t^\beta \mathbf{g}, \quad (7.6.13)$$

does not hold. The conjugacy class which contains the type 1a representative automorphism $\{\text{dsum}\{-\mathbf{1}_a, \mathbf{1}_{\ell-a}, -\mathbf{1}_{\ell-a}, \mathbf{1}_a\}, 1, 0\}$ will be called $(B)^{(a)}$. Suppose that the conjugacy classes $(B)^{(a)}$ and $(B)^{(b)}$ possess a common automorphism (for $0 \leq a < b \leq [\frac{1}{2}\ell]$). Thus, we are supposing that there exists some Laurent polynomial matrix $\mathbf{S}(t)$ which satisfies

$$\mathbf{S}(t)\mathbf{U}_a\mathbf{S}(t)^{-1} = \alpha t^\beta \mathbf{U}_b, \quad (7.6.14)$$

$$\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = \lambda t^\mu \mathbf{g}, \quad (7.6.15)$$

where $\mathbf{U}_z = \text{dsum}\{-\mathbf{1}_z, \mathbf{1}_{\ell-z}, -\mathbf{1}_{\ell-z}, \mathbf{1}_z\}$. It is assumed that $0 \leq a < b \leq [\frac{1}{2}\ell]$. Since the matrices \mathbf{U}_a and \mathbf{U}_b are t -independent, we may also assume that $\mathbf{S}(t)$ is t -

independent. If we investigate the determinants of the left-hand and right-hand sides of (7.6.14), then we infer firstly that $\beta = 0$. If we rearrange (7.6.14) and attempt to find a form for $S(t)$, then we find that $\alpha = \pm 1$. It is easily verified that, if $\alpha = \pm 1$, then there does not exist any Laurent polynomial matrix $S(t)$ which simultaneously satisfies (7.6.14) and (7.6.15). For, with $\alpha = 1$, the general form of $S(t)$ which satisfies (7.6.14) is given by

$$S(t) = \begin{bmatrix} \mathbf{h}_{pp} & \mathbf{h}_{pq} & \mathbf{0} & \mathbf{h}_{pr} & \mathbf{h}_{pq} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{h}_{qr} & \mathbf{0} & \mathbf{h}_{qq} & \mathbf{h}_{qp} \\ \mathbf{0} & \mathbf{0} & \mathbf{h}_{rr} & \mathbf{0} & \mathbf{h}_{rq} & \mathbf{h}_{rp} \\ \mathbf{h}_{rp} & \mathbf{h}_{rq} & \mathbf{0} & \mathbf{h}_{rr} & \mathbf{0} & \mathbf{0} \\ \mathbf{h}_{qp} & \mathbf{h}_{qq} & \mathbf{0} & \mathbf{h}_{qr} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{h}_{pr} & \mathbf{0} & \mathbf{h}_{pq} & \mathbf{h}_{pp} \end{bmatrix}, \quad (7.6.16)$$

where \mathbf{h}_{mn} (for $m, n = p, q, r$) represents an arbitrary $(m \times n)$ matrix, with $p = a$, $q = b - a$, and $r = \ell - b$. It is clear that $S(t)$ of this form does not satisfy (7.6.15). A similar conclusion is reached upon putting $\alpha = -1$. Thus, we have conjugacy classes $(B)^{(a)}$ for $0 \leq a \leq \lfloor \frac{1}{2} \ell \rfloor$. The representative automorphism of this class is the type 1a automorphism $\{U_{a,1,0}\}$, which we call ψ_{B^a} .

The representative automorphism of the conjugacy class $(A)^{(b)}$ is such that the following eigenvectors of it form a basis of the compact real form of $D_\ell^{(1)}$:

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \\ i(e_{j\delta}^k - e_{-j\delta}^k) \end{array} \right\} 1 \leq k \leq \ell; \text{ eigenvalue } 1, \quad (7.6.17)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \end{array} \right\} \left\{ \alpha = \varepsilon_r \pm \varepsilon_s \left(\begin{array}{l} 1 \leq r < s \leq \ell - b \\ \ell - b < r < s \leq \ell \quad (b \neq 0) \end{array} \right) \right\} \text{ eigenvalue } 1, \quad (7.6.18)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \end{array} \right\} \alpha = \varepsilon_r \pm \varepsilon_s; 1 \leq r \leq \ell - b < s \leq \ell; (b \neq 0); \text{ eigenvalue } -1, \quad (7.6.19)$$

$$\left. \begin{array}{l} ic \\ id \end{array} \right\} \text{eigenvalue 1.} \quad (7.6.20)$$

Thus, a basis for a real form of $D_\ell^{(1)}$ is supplied by the following elements:

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \\ i(e_{j\delta}^k - e_{-j\delta}^k) \end{array} \right\} 1 \leq k \leq \ell, \quad (7.6.21)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \left\{ \alpha = \varepsilon_r \pm \varepsilon_s \left(\begin{array}{l} 1 \leq r < s \leq \ell - b \\ \ell - b < r < s \leq \ell \quad (b \neq 0) \end{array} \right) \right\}, \quad (7.6.22)$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \alpha = \varepsilon_r \pm \varepsilon_s; 1 \leq r \leq \ell - b < s \leq \ell; (b \neq 0), \quad (7.6.23)$$

$$\begin{array}{l} ic, \\ id. \end{array} \quad (7.6.24)$$

The representative automorphism of the conjugacy class $(B)^{(a)}$ is such that there exists the following basis of the compact real form of $D_\ell^{(1)}$, each element of which is an eigenvector of the representative, with associated eigenvalue ± 1 :

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \quad j \in \mathbb{N}^0 \\ i(e_{j\delta}^k - e_{-j\delta}^k) \quad j \in \mathbb{N} \end{array} \right\} 1 \leq k \leq \ell, \text{ eigenvalue } 1, \quad (7.6.25)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \text{eigenvalue } 1, \quad (7.6.26)$$

(in (7.6.26) $\alpha = \varepsilon_r + \varepsilon_s$ and either $1 \leq r < s \leq a$ or $a < r < s \leq \ell$),

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \text{eigenvalue } -1, \quad (7.6.27)$$

(in (7.6.27) $\alpha = \varepsilon_r + \varepsilon_s$ and $1 \leq r \leq a < s \leq \ell$),

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \text{eigenvalue } 1, \quad (7.6.28)$$

(in (7.6.28) $\alpha = \varepsilon_r - \varepsilon_s$ and $1 \leq r \leq a < s \leq \ell$),

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \text{eigenvalue } -1, \quad (7.6.29)$$

(in (7.6.29) $\alpha = \varepsilon_r - \varepsilon_s$ and either $1 \leq r < s \leq a$ or $a < r < s \leq \ell$),

$$\left. \begin{array}{l} ic \\ id \end{array} \right\} \text{eigenvalue } 1. \quad (7.6.30)$$

The corresponding basis of a real form of $D_\ell^{(1)}$ is therefore given by

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \quad j \in \mathbb{N}^0 \\ i(e_{j\delta}^k - e_{-j\delta}^k) \quad j \in \mathbb{N} \end{array} \right\} 1 \leq k \leq \ell, \quad (7.6.31)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}), \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}), \end{array} \right\} \quad (7.6.32)$$

(in (7.6.32) $\alpha = \varepsilon_r + \varepsilon_s$ and either $1 \leq r < s \leq a$ or $a < r < s \leq \ell$),

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}), \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha}), \end{array} \right\} \quad (7.6.33)$$

(in (7.6.33) $\alpha = \varepsilon_r + \varepsilon_s$ and $1 \leq r \leq a < s \leq \ell$),

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}), \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}), \end{array} \right\} \quad (7.6.34)$$

(in (7.6.34) $\alpha = \varepsilon_r - \varepsilon_s$ and $1 \leq r \leq a < s \leq \ell$),

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}), \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha}), \end{array} \right\} \quad (7.6.35)$$

(in the above $\alpha = \varepsilon_r - \varepsilon_s$ and either $1 \leq r < s \leq a$ or $a < r < s \leq \ell$)

$$\begin{aligned} &ic, \\ &id. \end{aligned} \tag{7.6.36}$$

We will now consider the type 1a involutive automorphisms $\{U(t), 1, \xi\}$, where $U(t)$ is given by (7.6.3). There are two cases to consider. Firstly, the case ℓ is even, and the case ℓ is odd. Let it be assumed that ℓ is even, so that if $a + b = \ell$, then either a, b are both even, or they are both odd. In both instances it is possible to 'reduce' b by 2. That is, the two automorphisms generated by the matrices $\text{offsum}\{K_a, tK_b, t^{-1}K_b, K_a\}$ and $\text{offsum}\{K_{a+2}, tK_{b-2}, t^{-1}K_{b-2}, K_{a+2}\}$ (where $a + b = \ell$) are conjugate. That this is the case may be seen by examining examples 2, 6, and 7 of section 5 of this chapter. It follows that attention may be confined to the two type 1a involutive automorphisms generated by the matrices

$$U(t) = K_{2\ell}, \tag{7.6.37}$$

$$U(t) = \text{offsum}\{K_{\ell-1}, tK_1, t^{-1}K_1, K_{\ell-1}\}. \tag{7.6.38}$$

Now the automorphism $\{K_{2\ell}, 1, 0\}$ belongs to the conjugacy class $(A)^{(\frac{1}{2}\ell)}$. To see this, let the matrix $S(t)$ be given by

$$S(t) = \frac{1}{2} \begin{bmatrix} a_1 & 0 & \cdots & 0 & 0 & \cdots & 0 & b_1 \\ 0 & a_2 & \cdots & 0 & 0 & \cdots & b_2 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & a_{\frac{1}{2}\ell} & b_{\frac{1}{2}\ell} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{\frac{1}{2}\ell} & d_{\frac{1}{2}\ell} & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & \ddots & \vdots & \vdots \\ 0 & c_2 & \cdots & 0 & 0 & \cdots & d_2 & 0 \\ c_1 & 0 & \cdots & 0 & 0 & \cdots & 0 & d_1 \end{bmatrix}, \tag{7.6.39}$$

where the submatrices are defined (for $1 \leq j \leq \frac{1}{2}\ell$) by

$$\mathbf{a}_j = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \quad \mathbf{b}_j = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad (7.6.40)$$

$$\mathbf{c}_j = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad \mathbf{d}_j = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}. \quad (7.6.41)$$

This matrix is such that the following all hold:

$$\mathbf{S}(t)\mathbf{K}_{2\ell}\mathbf{S}(t)^{-1} = \text{diag}\{1, -1, \dots, -1, 1\}, \quad (7.6.42)$$

$$\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{2\ell}, \quad (7.6.43)$$

$$\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = \mathbf{g}. \quad (7.6.44)$$

The type 1a involutive automorphisms of the form $\phi(\{ \mathbf{U}(t), 1, \xi \})$, where $\mathbf{U}(t) = \text{offsum}\{ \mathbf{K}_{\ell-1}, t\mathbf{K}_1, t^{-1}\mathbf{K}_1, \mathbf{K}_{\ell-1} \}$ have still to be considered. Suppose that the following holds:

$$\mathbf{S}(t)\text{dsum}\{ \mathbf{1}_{\ell-b}, -\mathbf{1}_{2b}, \mathbf{1}_{\ell-b} \}\mathbf{S}(t)^{-1} = \lambda t^\mu \mathbf{U}(t). \quad (7.6.45)$$

Investigation of the traces (and determinants) of both sides of the above equation implies that $b = \frac{1}{2}\ell$, $\mu = 0$, and $\lambda^{2\ell} = 1$. Thus, the automorphisms in question belong either to the class $(A)^{(\frac{1}{2}\ell)}$, to one of the the classes $(B)^{(a)}$, or to some as yet unnamed class. Let the involutive automorphisms ϕ_j (for $j = 1, 2, 3$) be such that

$$\phi_1 = \{ \mathbf{U}(t), 1, \xi \}, \quad (7.6.46)$$

$$\phi_2 \text{ is a representative of the class } (A)^{(\frac{1}{2}\ell)}, \quad (7.6.47)$$

$$\phi_3 \text{ belongs to } (B)^{(a)} \text{ (for some value of } a \text{)}. \quad (7.6.48)$$

Now consider for the moment the highest-weight representation $\Gamma\{0, 0, \dots, 1\}$ of D_ℓ , which may be realised by the method given in [13] (chapter 16). If we were to use this as our representation, we would find that ϕ_1 is of type 1b, whereas ϕ_2, ϕ_3 are both of type 1a. Thus, the involutive automorphism ϕ_1 belongs to a new conjugacy class,

which will be called (C)^{even}. We will take ϕ_1 to be the representative of this class. (It should be noted that we are using the original representation, and so ϕ_1 is of type 1a). After finding a basis of the compact real form of $D_\ell^{(1)}$ (all of whose members are eigenvectors of ψ) and applying Cartan's method, the following basis of a real form of $D_\ell^{(1)}$ is obtained:

$$\begin{aligned} & h_{\alpha_k} \quad (1 \leq k \leq \ell - 2), \\ & \left(h_{\alpha_{\ell-1}} - \frac{1}{2}c \right), \\ & \left(h_{\alpha_\ell} + \frac{1}{2}c \right), \end{aligned} \tag{7.6.49}$$

$$\begin{aligned} & \left. \begin{aligned} & i(e_{j\delta}^k + e_{-j\delta}^k) \quad j \in \mathbb{N} \\ & (e_{j\delta}^k - e_{-j\delta}^k) \quad j \in \mathbb{N} \end{aligned} \right\} 1 \leq k \leq \ell, \end{aligned} \tag{7.6.50}$$

$$\begin{aligned} & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \\ & (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\ & i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\ & (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \end{aligned} \tag{7.6.51}$$

(where, in (7.6.51) above $j \in \mathbb{N}^0$, $\alpha = \varepsilon_r \pm \varepsilon_s$, $1 \leq r < s \leq \ell - 1$, $(r + s)$ is odd),

$$\begin{aligned} & (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \\ & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\ & (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\ & i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \end{aligned} \tag{7.6.52}$$

(where, in (7.6.52) above $j \in \mathbb{N}^0$, $\alpha = \varepsilon_r \pm \varepsilon_s$, $1 \leq r < s \leq \ell - 1$, $(r + s)$ is even),

$$\begin{aligned} & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha}), \\ & (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha}), \\ & i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha}), \\ & (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha}), \end{aligned} \tag{7.6.53}$$

(where, in (7.6.53) above $j \in \mathbb{N}^0$, $k = -j - 1$, $\alpha = \varepsilon_r \pm \varepsilon_s$, $1 \leq r \leq \ell - 1$, $s = \ell$, r is odd),

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{k\delta+\alpha} + e_{-k\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{k\delta+\alpha} - e_{-k\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{k\delta+\alpha} - e_{-k\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{k\delta+\alpha} + e_{-k\delta-\alpha} \right),
\end{aligned} \tag{7.6.54}$$

(where, in 7.6.54) above $j \in \mathbf{N}^0$, $k = -j - 1$, $\alpha = \varepsilon_r \pm \varepsilon_s$, $1 \leq r \leq \ell - 1$, $s = \ell$, r is even),

$$\begin{aligned}
& ic, \\
& i \left(d - \frac{1}{2} \ell (\ell - 1) h_{\alpha_i} \right).
\end{aligned} \tag{7.6.55}$$

Let it now be assumed that ℓ is odd, so that, if $a + b = \ell$, then one of a, b is necessarily even, and the other one odd. As with the case ℓ is even, attention may be confined to a small number of automorphisms, in order to ascertain all of the conjugacy classes. Recall that $\mathbf{U}(t)$ is of the form

$$\mathbf{U}(t) = \text{offsum} \left\{ \mathbf{K}_a, t \mathbf{K}_b, t^{-1} \mathbf{K}_b, \mathbf{K}_a \right\} \quad (a + b = \ell). \tag{7.6.56}$$

In this case, where ℓ is odd, via a number of suitable transformations (which are given in section 5 of this chapter) it is found that $\{\mathbf{U}(t), 1, \xi\}$ is conjugate to the type 1a involutive automorphism $\{\mathbf{K}_{2\ell}, 1, 0\}$. The automorphism $\{\mathbf{K}_{2\ell}, 1, 0\}$ does not belong to any of the conjugacy classes $(\mathbf{A})^{(b)}$ (for $0 \leq b \leq \lfloor \frac{1}{2} \ell \rfloor$), nor does it belong to any of the conjugacy classes $(\mathbf{B})^{(a)}$. If it did belong to any of these classes, then there would be some $\mathbf{S}(t)$ such that

$$\mathbf{S}(t) \mathbf{U}(t) \mathbf{S}(t)^{-1} = \lambda t^\mu \mathbf{K}_{2\ell}. \tag{7.6.57}$$

It is clear immediately that a necessary condition for this to hold is that the matrix $\mathbf{U}(t)$ is traceless. However, if $\{\mathbf{U}(t), 1, 0\}$ belongs to $(\mathbf{A})^{(b)}$ then $\mathbf{U}(t)$ is cannot be traceless (for odd values of ℓ). Suppose then that the automorphism $\{\mathbf{K}_{2\ell}, 1, 0\}$ belongs to the class $(\mathbf{B})^{(a)}$, so that the following hold:

$$\mathbf{S}(t) \text{dsum} \{-\mathbf{1}_a, \mathbf{1}_{\ell-a}, -\mathbf{1}_{\ell-a}, \mathbf{1}_a\} \mathbf{S}(t)^{-1} = \lambda t^\mu \mathbf{K}_{2\ell}, \quad (7.6.58)$$

$$\tilde{\mathbf{S}}(t) \mathbf{g} \mathbf{S}(t) = \alpha t^\beta \mathbf{g}. \quad (7.6.59)$$

Note that, if one puts $\mathbf{S}'(t) = \mathbf{S}(t) \text{dsum} \{\mathbf{1}_\ell, \alpha^{-1} t^{-\beta} \mathbf{1}_\ell\}$, then

$$\mathbf{S}'(t) \text{dsum} \{-\mathbf{1}_a, \mathbf{1}_{\ell-a}, -\mathbf{1}_a, \mathbf{1}_{\ell-a}\} \mathbf{S}'(t)^{-1} = \lambda t^\mu \mathbf{K}_{2\ell}, \quad (7.6.60)$$

$$\tilde{\mathbf{S}}'(t) \mathbf{g} \mathbf{S}'(t) = \mathbf{g}, \quad (7.6.61)$$

so that, in effect, the assumption $\alpha t^\beta = 1$ may be made. Similarly, we may assume that the matrix $\mathbf{S}(t)$ is t -independent. If determinants of both sides of the first of these equations are evaluated, then it becomes clear that $\lambda^{2\ell} = 1, \mu = 0$. In fact, upon re-arranging and attempting to solve the equation for $\mathbf{S}(t)$, it is found that $\lambda^2 = 1$ is necessary. Now there does exist a matrix $\mathbf{R}(t)$, for which

$$\mathbf{R}(t) \text{dsum} \{-\mathbf{1}_a, \mathbf{1}_{\ell-a}, -\mathbf{1}_{\ell-a}, \mathbf{1}_a\} \mathbf{R}(t)^{-1} = \text{offsum} \{\mathbf{K}_a, -\mathbf{K}_{2\ell-2a}, \mathbf{K}_a\}, \quad (7.6.62)$$

$$\tilde{\mathbf{R}}(t) \mathbf{g} \mathbf{R}(t) = \mathbf{1}_{2\ell}. \quad (7.6.63)$$

Such a matrix is given by

$$\mathbf{R}(t) = \begin{bmatrix} i & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & i & 0 \\ 0 & 0 & i & 0 & \dots & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 & 0 & \dots & i & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & i & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & \dots & i & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & \dots & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & i & 0 \\ -i & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix} \quad (7.6.64)$$

(We recall that the type Ia automorphisms $\{\mathbf{K}_{2\ell}, 1, 0\}$ and $\{\text{offsum}\{\mathbf{K}_a, -\mathbf{K}_{2\ell-2a}, \mathbf{K}_a\}, 1, 0\}$ are conjugate). If the lemma (see chapter 1) is used, then it implies that (for $\lambda = 1$) the required form of $\mathbf{S}(t)$ is given by

$$\mathbf{S}(t) = \mathbf{R}(t)\mathbf{Q}(t), \quad (7.6.65)$$

where $\mathbf{R}(t)$ is as defined already, and $\mathbf{Q}(t)$ satisfies

$$\mathbf{Q}(t)\mathbf{U}_a\mathbf{Q}(t)^{-1} = \mathbf{U}_a, \quad (7.6.66)$$

where $\mathbf{U}_a = \text{dsum}\{-\mathbf{1}_a, \mathbf{1}_{\ell-a}, -\mathbf{1}_{\ell-a}, \mathbf{1}_a\}$. Now, the most general form of $\mathbf{Q}(t)$ satisfying this is given by

$$\mathbf{Q}(t) = \begin{bmatrix} \mathbf{h}_{aa} & \mathbf{0} & \mathbf{h}_{ab} & \mathbf{0} \\ \mathbf{0} & \mathbf{h}_{bb} & \mathbf{0} & \mathbf{h}_{ba} \\ \mathbf{h}_{ba} & \mathbf{0} & \mathbf{h}_{bb} & \mathbf{0} \\ \mathbf{0} & \mathbf{h}_{ab} & \mathbf{0} & \mathbf{h}_{aa} \end{bmatrix}, \quad (7.6.67)$$

where \mathbf{h}_{mn} represents an arbitrary $(m \times n)$ matrix, and $b = \ell - a$. Since the hypothesis is that $\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = \mathbf{g}$, it follows that

$$\tilde{\mathbf{Q}}(t)\mathbf{Q}(t) = \mathbf{g}, \quad (7.6.68)$$

and it is clear that with $\mathbf{Q}(t)$ taking the above form, this cannot be the case. Thus, for the case $\lambda = 1$, there does not exist a matrix $\mathbf{S}(t)$ with the required properties. Consider the case $\lambda = -1$, and suppose that there exists a matrix $\mathbf{S}(t)$ with the following properties:

$$\mathbf{S}(t)\mathbf{U}_a\mathbf{S}(t)^{-1} = -\mathbf{K}_{2\ell}, \quad (7.6.69)$$

$$\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = \alpha t^\beta \mathbf{g}. \quad (7.6.70)$$

If the definition $\mathbf{V}(t) = \mathbf{S}(t)\mathbf{K}_{2\ell}$ is made, then

$$\mathbf{V}(t)\mathbf{U}_a\mathbf{V}(t)^{-1} = \text{offsum}\{\mathbf{K}_a, -\mathbf{K}_{2\ell-2a}, \mathbf{K}_a\}, \quad (7.6.71)$$

$$\bar{V}(t)gV(t) = \alpha t^\beta g. \quad (7.6.72)$$

and this is not possible, as it is the case $\lambda = 1$ again, which has already been examined. Hence the automorphism $\{\mathbf{K}_{2\ell}, 1, 0\}$ (for odd values of ℓ) does not belong to any of the previously identified conjugacy classes. The class which contains it will be called $(C)^{(\text{odd})}$. The following basis elements of the compact real form of $D_\ell^{(1)}$ are also eigenvectors of $\{\mathbf{K}_{2\ell}, 1, 0\}$ (and their eigenvalues are given):

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \quad j \in \mathbf{N}^0 \\ i(e_{j\delta}^k - e_{-j\delta}^k) \quad j \in \mathbf{N} \end{array} \right\} \text{eigenvalue } -1, \quad (7.6.73)$$

$$\begin{array}{ll} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) & \text{eigenvalue } -1, \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) & \text{eigenvalue } 1, \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) & \text{eigenvalue } 1, \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) & \text{eigenvalue } -1, \end{array} \quad (7.6.74)$$

where, in (7.6.74) above, $j \in \mathbf{N}^0$, $\alpha = \varepsilon_r \pm \varepsilon_s$, $1 \leq r < s \leq \ell$, $(r+s)$ is odd,

$$\begin{array}{ll} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) & \text{eigenvalue } 1, \\ (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) & \text{eigenvalue } -1, \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) & \text{eigenvalue } -1, \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) & \text{eigenvalue } 1, \end{array} \quad (7.6.75)$$

where, in (7.6.75) above $j \in \mathbf{N}^0$, $\alpha = \varepsilon_r \pm \varepsilon_s$, $1 \leq r < s \leq \ell$, $(r+s)$ is even,

$$\left. \begin{array}{l} ic \\ id \end{array} \right\} \text{eigenvalue } 1. \quad (7.6.76)$$

Hence, the basis of a real form of $D_\ell^{(1)}$ (generated by this automorphism) is provided by the following:

$$\begin{array}{ll} i(e_{j\delta}^k + e_{-j\delta}^k) & j \in \mathbf{N}^0, 1 \leq k \leq \ell, \\ (e_{j\delta}^k - e_{-j\delta}^k) & j \in \mathbf{N}, 1 \leq k \leq \ell, \end{array} \quad (7.6.77)$$

$$\begin{aligned}
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \\
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}),
\end{aligned} \tag{7.6.78}$$

where in (7.6.78) above $j \in \mathbf{N}^0$, $\alpha = \varepsilon_r \pm \varepsilon_s$, $1 \leq r < s \leq \ell$, $(r+s)$ is odd,

$$\begin{aligned}
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \\
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}),
\end{aligned} \tag{7.6.79}$$

where, in (7.6.79) above $j \in \mathbf{N}^0$, $\alpha = \varepsilon_r \pm \varepsilon_s$, $1 \leq r < s \leq \ell$, $(r+s)$ is even,

$$\begin{aligned}
& ic \\
& id.
\end{aligned} \tag{7.6.80}$$

We now consider the automorphisms $\{\mathbf{U}(t), 1, \xi\}$, where $\mathbf{U}(t)$ is given by (7.6.4). Investigation of the determinant of $\mathbf{U}(t)$ implies that the type 1a automorphism $\{\mathbf{U}(t), 1, \xi\}$ cannot belong to any of the conjugacy classes $(A)^{(b)}$, $(B)^{(a)}$, $(C)^{\text{even}}$ or $(C)^{\text{odd}}$. Thus there exist other conjugacy classes with $\{\mathbf{U}(t), 1, \xi\}$ as their representatives (with $\mathbf{U}(t)$ given by (7.6.4)), and these classes will be called $(D)^{(b)}$. The representative automorphism of the class $(D)^{(b)}$ is the type 1a involutive automorphism ψ , where $\psi = \{\text{offsum}\{i\mathbf{K}_{\ell-b}, \mathbf{K}_b, i\mathbf{K}_b, \mathbf{K}_{\ell-b}\}, 1, \xi\}$. The effect of ψ upon a basis of the compact real form of $\mathbf{D}_\ell^{(1)}$ is illustrated below

$$\left. \begin{aligned}
& ih_{\alpha_k} \\
& i\left(h_{\alpha_{\ell-a}} + \frac{1}{2}c\right)
\end{aligned} \right\} \begin{matrix} (k \neq \ell - a) \\ \text{eigenvalue} \end{matrix} -1, \tag{7.6.81}$$

$$\left. \begin{aligned}
& (e_{j\delta}^k + e_{-j\delta}^k) \\
& i(e_{j\delta}^k - e_{-j\delta}^k)
\end{aligned} \right\} j \in \mathbf{N}, 1 \leq k \leq \ell, \text{ eigenvalue} -1, \tag{7.6.82}$$

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right) \text{ eigenvalue } 1, \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right) \text{ eigenvalue } -1, \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right) \text{ eigenvalue } -1, \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right) \text{ eigenvalue } 1,
\end{aligned} \tag{7.6.83}$$

(where, in (7.6.83) above, either: (i) $\alpha = \varepsilon_r - \varepsilon_s$, with $(r+s)$ even, and either $1 \leq r < s \leq \ell - a$ or $\ell - a < r < s \leq \ell$, or: (ii) $\alpha = \varepsilon_r + \varepsilon_s$, with $(r+s)$ even and $1 \leq r \leq \ell - a < s \leq \ell$),

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right) \text{ eigenvalue } -1, \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right) \text{ eigenvalue } 1, \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right) \text{ eigenvalue } 1, \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right) \text{ eigenvalue } -1,
\end{aligned} \tag{7.6.84}$$

(where, in (7.6.84) above, either: (i) $\alpha = \varepsilon_r - \varepsilon_s$, with $(r+s)$ odd, and either $1 \leq r < s \leq \ell - a$ or $\ell - a < r < s \leq \ell$, or: (ii) $\alpha = \varepsilon_r + \varepsilon_s$, with $(r+s)$ odd and $1 \leq r \leq \ell - a < s \leq \ell$),

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(-j+1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right) \text{ eigenvalue } 1, \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{(-j+1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right) \text{ eigenvalue } -1, \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{(-j+1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right) \text{ eigenvalue } -1, \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{(-j+1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right) \text{ eigenvalue } 1,
\end{aligned} \tag{7.6.85}$$

(where, in (7.6.85) above, either: (i) $\alpha = \varepsilon_r - \varepsilon_s$, with $(r+s)$ even, and either $1 \leq r \leq \ell - a < s \leq \ell$ or: (ii) $\alpha = \varepsilon_r + \varepsilon_s$, with $(r+s)$ even and $1 \leq r < s \leq \ell - a$),

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(-j+1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right) \text{ eigenvalue } -1, \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{(-j+1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right) \text{ eigenvalue } 1, \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{(-j+1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right) \text{ eigenvalue } 1, \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{(-j+1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right) \text{ eigenvalue } -1,
\end{aligned} \tag{7.6.86}$$

(where, in (7.6.86) above, either: (i) $\alpha = \varepsilon_r - \varepsilon_s$, with $(r+s)$ odd, and either $1 \leq r \leq \ell - a < s \leq \ell$ or: (ii) $\alpha = \varepsilon_r + \varepsilon_s$, with $(r+s)$ odd and $1 \leq r < s \leq \ell - a$),

$$\begin{aligned} & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(-j-1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right) \text{ eigenvalue } 1, \\ & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{(-j-1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right) \text{ eigenvalue } -1, \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{(-j-1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right) \text{ eigenvalue } -1, \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{(-j-1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right) \text{ eigenvalue } 1, \end{aligned} \quad (7.6.87)$$

(where, in (7.6.87) above, $\alpha = \varepsilon_r + \varepsilon_s$, with $(r+s)$ even and $\ell - a < r < s \leq \ell$),

$$\begin{aligned} & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(-j-1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right) \text{ eigenvalue } -1, \\ & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{(-j-1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right) \text{ eigenvalue } 1, \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{(-j-1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right) \text{ eigenvalue } 1, \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{(-j-1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right) \text{ eigenvalue } -1, \end{aligned} \quad (7.6.88)$$

(where, in (7.6.88) above, $\alpha = \varepsilon_r + \varepsilon_s$, with $(r+s)$ odd and $\ell - a < r < s \leq \ell$),

$$ic \text{ eigenvalue } 1, \quad (7.6.89)$$

$$id - \frac{i\ell(\ell-1)}{4} \left\{ \begin{array}{l} \sum_{p=1}^{\ell-a} p h_{\alpha_p} + \sum_{p=1}^{a-2} (\ell-a-p) h_{\alpha_{\ell-a+p}} \\ + \frac{(\ell-2a+2)}{2} h_{\alpha_{\ell-1}} + \frac{(\ell-2a)}{2} h_{\alpha_\ell} \end{array} \right\} \quad a > 2, \text{ eigenvalue } 1, \quad (7.6.90)$$

$$id - \frac{i\ell(\ell-1)}{4} \left\{ \begin{array}{l} \sum_{p=1}^{\ell-2} p h_{\alpha_p} + \frac{(\ell-2)}{2} h_{\alpha_{\ell-1}} \\ + \frac{(\ell-4)}{2} h_{\alpha_\ell} \end{array} \right\} \quad a = 2, \text{ eigenvalue } 1, \quad (7.6.91)$$

$$id - \frac{i\ell(\ell-1)}{4} \left\{ \begin{array}{l} \sum_{p=1}^{\ell-2} p h_{\alpha_p} + \frac{\ell}{2} h_{\alpha_{\ell-1}} \\ + \frac{(\ell-2)}{2} h_{\alpha_\ell} \end{array} \right\} \quad a = 1, \text{ eigenvalue } 1, \quad (7.6.92)$$

$$id - \frac{i}{4} \ell(\ell-1) \left\{ \begin{array}{l} \sum_{p=2}^{\ell-2} p h_{\alpha_p} + \frac{(\ell-2)}{2} h_{\alpha_{\ell-1}} \\ + \frac{\ell}{2} h_{\alpha_\ell} \end{array} \right\} \quad a=0, \text{ eigenvalue } 1. \quad (7.6.93)$$

Therefore, a basis of a real form of $D_\ell^{(1)}$ (which is generated by this automorphism) is supplied by the following elements:

$$\begin{aligned} & h_{\alpha_k} \quad (k \neq \ell - a), \\ & \left(h_{\alpha_{\ell-a}} + \frac{1}{2}c \right), \end{aligned} \quad (7.6.94)$$

$$\begin{aligned} & i \left(\begin{array}{l} e_{j\delta}^k + e_{-j\delta}^k \\ e_{j\delta}^k - e_{-j\delta}^k \end{array} \right) \left\{ j \in \mathbb{N}, 1 \leq k \leq \ell, \right. \\ & \left. \right\} \end{aligned} \quad (7.6.95)$$

$$\begin{aligned} & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right), \\ & i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right), \\ & \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right), \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right), \end{aligned} \quad (7.6.96)$$

(where, in (7.6.96) above, either: (i) $\alpha = \varepsilon_r - \varepsilon_s$, with $(r+s)$ even, and either $1 \leq r < s \leq \ell - a$ or $\ell - a < r < s \leq \ell$, or: (ii) $\alpha = \varepsilon_r + \varepsilon_s$, with $(r+s)$ even and $1 \leq r \leq \ell - a < s \leq \ell$),

$$\begin{aligned} & i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right), \\ & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right), \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right), \\ & \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right), \end{aligned} \quad (7.6.97)$$

(where, in (7.6.97) above, either: (i) $\alpha = \varepsilon_r - \varepsilon_s$, with $(r+s)$ odd, and either $1 \leq r < s \leq \ell - a$ or $\ell - a < r < s \leq \ell$, or: (ii) $\alpha = \varepsilon_r + \varepsilon_s$, with $(r+s)$ odd and $1 \leq r \leq \ell - a < s \leq \ell$),

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(-j+1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{(-j+1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{(-j+1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{(-j+1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right),
\end{aligned} \tag{7.6.98}$$

(where, in (7.6.98) above, either: (i) $\alpha = \varepsilon_r - \varepsilon_s$, with $(r+s)$ even, and either $1 \leq r \leq \ell - a < s \leq \ell$ or: (ii) $\alpha = \varepsilon_r + \varepsilon_s$, with $(r+s)$ even and $1 \leq r < s \leq \ell - a$),

$$\begin{aligned}
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(-j+1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{(-j+1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{(-j+1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{(-j+1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right),
\end{aligned} \tag{7.6.99}$$

(where, in (7.6.99) above, either: (i) $\alpha = \varepsilon_r - \varepsilon_s$, with $(r+s)$ odd, and either $1 \leq r \leq \ell - a < s \leq \ell$ or: (ii) $\alpha = \varepsilon_r + \varepsilon_s$, with $(r+s)$ odd and $1 \leq r < s \leq \ell - a$),

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(-j-1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{(-j-1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{(-j-1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{(-j-1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right),
\end{aligned} \tag{7.6.100}$$

(where, in (7.6.100) above, $\alpha = \varepsilon_r + \varepsilon_s$, with $(r+s)$ even and $\ell - a < r < s \leq \ell$),

$$\begin{aligned}
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(-j-1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{(-j-1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{(-j-1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{(-j-1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right),
\end{aligned} \tag{7.6.101}$$

(where, in (7.6.101) above, $\alpha = \varepsilon_r + \varepsilon_s$, with $(r+s)$ odd and $\ell - a < r < s \leq \ell$),

$$ic, \tag{7.6.102}$$

$$id - \frac{i\ell(\ell-1)}{4} \left\{ \begin{array}{l} \sum_{p=1}^{\ell-a} ph_{\alpha_p} + \sum_{p=1}^{a-2} (\ell-a-p)h_{\alpha_{\ell-a+p}} \\ + \frac{(\ell-2a+2)}{2}h_{\alpha_{\ell-1}} + \frac{(\ell-2a)}{2}h_{\alpha_{\ell}} \end{array} \right\} \quad a > 2, \quad (7.6.103)$$

$$id - \frac{i\ell(\ell-1)}{4} \left\{ \begin{array}{l} \sum_{p=1}^{\ell-2} ph_{\alpha_p} + \frac{(\ell-2)}{2}h_{\alpha_{\ell-1}} \\ + \frac{(\ell-4)}{2}h_{\alpha_{\ell}} \end{array} \right\} \quad a = 2, \quad (7.6.104)$$

$$id - \frac{i\ell(\ell-1)}{4} \left\{ \begin{array}{l} \sum_{p=1}^{\ell-2} ph_{\alpha_p} + \frac{\ell}{2}h_{\alpha_{\ell-1}} \\ + \frac{(\ell-2)}{2}h_{\alpha_{\ell}} \end{array} \right\} \quad a = 1, \quad (7.6.105)$$

$$id - \frac{i}{4}\ell(\ell-1) \left\{ \begin{array}{l} \sum_{p=2}^{\ell-2} ph_{\alpha_p} + \frac{(\ell-2)}{2}h_{\alpha_{\ell-1}} \\ + \frac{\ell}{2}h_{\alpha_{\ell}} \end{array} \right\} \quad a = 0. \quad (7.6.106)$$

The next automorphisms which we examine are the type 1a involutive automorphisms $\{\mathbf{U}(t), 1, \xi\}$, where $\mathbf{U}(t)$ is given by (7.6.5). One observation which should be made is that the variables c and d in (7.6.5) may each be restricted to the values ± 1 . A second observation is that the matrices given by (7.6.3) are equivalent to matrices of this form (in the sense that they generate isomorphic involutive automorphisms). The matrix transformations in section 7.5 demonstrate that this is the case. There are 4 cases to consider, each case corresponding to a different value of the ordered pair (c, d) . The case $c = d = 0$ merits no further consideration, since the matrix $\mathbf{U}(t)$ in this case is of the form (7.6.1). Let us now consider the case $(c, d) = (1, 0)$. Comparison of the determinant of $\mathbf{U}(t)$ with that of $\mathbf{V}(t)$ (where $\mathbf{V}(t)$ is given by (7.6.4) implies that the type 1a involutive automorphism $\{\mathbf{U}(t), 1, \xi\}$ cannot belong to any of the conjugacy classes which have been called $(D)^{(j)}$. It will now be shown that the automorphism in question is not a member of any of the conjugacy classes $(A)^{(j)}$

or $(B)^{(k)}$ (for any values of j or k). Suppose to the contrary, so that there is some Laurent polynomial matrix $S(t)$ which satisfies

$$S(t) \text{dsum}\{1_{\ell-1-b}, -1_b, K_2, -1_b, 1_{\ell-1-b}\} S(t)^{-1} = \lambda t^\mu V(t), \quad (7.6.107)$$

$$\tilde{S}(t) g S(t) = \alpha t^\beta g, \quad (7.6.108)$$

where $V(t)$ is given by either (7.6.1) or (7.6.2). The determinant of the matrix $U(t)$ is -1 , whilst the determinant of $V(t)$ is unity. This implies that $\lambda^{2\ell} = -1$, and $\mu = 0$. However, we may re-arrange the equation to obtain

$$S(t)U(t) = \lambda V(t)S(t). \quad (7.6.109)$$

If we attempt to solve this for the unknown matrix $S(t)$, then it becomes clear that $\lambda^2 = 1$. This is a contradiction, since λ would then have to be both real and imaginary. Thus, the type 1a involutive automorphism $\{U(t), 1, 0\}$ does not belong to any of the conjugacy classes $(A)^{(j)}$ or $(B)^{(k)}$ (for any values of j or k). We define the conjugacy class $(C)^{(b,1)}$ to be the conjugacy class which contains the type 1a involutive automorphism $\{U(t), 1, 0\}$, where

$$U(t) = \text{dsum}\{1_{\ell-1-b}, -1_b, K_2, -1_b, 1_{\ell-1-b}\}. \quad (7.6.110)$$

It remains to be seen whether $(C)^{(b,1)}$ has any members in common with $(C)^{(c,1)}$, where $0 \leq b < c \leq \lceil \frac{1}{2}(\ell - 1) \rceil$. Suppose that some Laurent polynomial matrix $S(t)$ exists for which

$$S(t)U_b(t)S(t)^{-1} = \lambda t^\mu U_c(t), \quad (7.6.111)$$

where $U_z(t) = \text{dsum}\{1_{\ell-1-z}, -1_z, K_2, -1_z, 1_{\ell-1-z}\}$. If we re-arrange this, and attempt to find the matrix $S(t)$, we find that $\lambda^2 = 1$ and $\mu = 0$. Furthermore, examination of the traces of the matrices $U_b(t)$ and $U_c(t)$ implies that

$$(\ell - 1 - 2b) = \lambda(\ell - 1 - 2c). \quad (7.6.112)$$

Since, by hypothesis, the quantities $(\ell - 1 - 2b)$ and $(\ell - 1 - 2c)$ are both non-negative, we have that $b = c$, which disagrees with the original hypothesis. Thus, the classes $(C)^{(b,1)}$ and $(C)^{(c,1)}$ do not coincide whenever $0 \leq b < c \leq \lfloor \frac{1}{2}(\ell - 1) \rfloor$.

A basis of the compact real form of $D_\ell^{(1)}$ exists in which each basis element is an eigenvector (with eigenvalue ± 1) of the automorphism $\left\{ \text{dsum} \left\{ \mathbf{1}_{\ell-b}, -\mathbf{1}_b, \mathbf{K}_2, -\mathbf{1}_b, \mathbf{1}_{\ell-b} \right\}, \mathbf{1}, \mathbf{0} \right\}$, which is the representative of the conjugacy class $(C)^{(b,1)}$. The elements of such a basis, together with their respective eigenvalues are given below:

$$\left. \begin{array}{l} \left(e_{j\delta}^k + e_{-j\delta}^k \right) \\ i \left(e_{j\delta}^k - e_{-j\delta}^k \right) \end{array} \right\} 1 \leq k \leq \ell - 2, \text{ eigenvalue } 1, \quad (7.6.113)$$

$$\left. \begin{array}{l} \left(e_{j\delta}^{\ell-1} + e_{-j\delta}^{\ell-1} + e_{j\delta}^\ell + e_{-j\delta}^\ell \right) \\ i \left(e_{j\delta}^{\ell-1} - e_{-j\delta}^{\ell-1} + e_{j\delta}^\ell - e_{-j\delta}^\ell \right) \end{array} \right\} \text{ eigenvalue } 1, \quad (7.6.114)$$

$$\left. \begin{array}{l} \left(e_{j\delta}^{\ell-1} + e_{-j\delta}^{\ell-1} - e_{j\delta}^\ell - e_{-j\delta}^\ell \right) \\ i \left(e_{j\delta}^{\ell-1} - e_{-j\delta}^{\ell-1} - e_{j\delta}^\ell + e_{-j\delta}^\ell \right) \end{array} \right\} \text{ eigenvalue } -1, \quad (7.6.115)$$

$$\left. \begin{array}{l} \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} \right) \\ i \left(e_{j\delta-\alpha} - e_{-j\delta-\alpha} \right) \end{array} \right\} \text{ eigenvalue } 1, \quad (7.6.116)$$

(in (7.6.116) above $\alpha = \varepsilon_r \pm \varepsilon_s$ with either $1 \leq r < s \leq \ell - 1 - b$ or $\ell - b - 1 < r < s \leq \ell - 1$),

$$\left. \begin{array}{l} \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} \right) \\ i \left(e_{j\delta-\alpha} - e_{-j\delta-\alpha} \right) \end{array} \right\} \text{ eigenvalue } -1, \quad (7.6.117)$$

(in (7.6.117) above $\alpha = \varepsilon_r \pm \varepsilon_s$ with $1 \leq r \leq \ell - 1 - b < s \leq \ell - 1$),

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta+\beta} + e_{-j\delta-\beta} \right) \text{ eigenvalue } 1, \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta+\beta} - e_{-j\delta-\beta} \right) \text{ eigenvalue } -1, \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta+\beta} - e_{-j\delta-\beta} \right) \text{ eigenvalue } 1, \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta+\beta} + e_{-j\delta-\beta} \right) \text{ eigenvalue } -1,
\end{aligned} \tag{7.6.118}$$

(in (7.6.118) above $\alpha = \varepsilon_r - \varepsilon_\ell$, $\beta = \varepsilon_r + \varepsilon_\ell$, where $1 \leq r \leq \ell - 1 - b$),

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta+\beta} + e_{-j\delta-\beta} \right) \text{ eigenvalue } -1, \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta+\beta} - e_{-j\delta-\beta} \right) \text{ eigenvalue } 1, \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta+\beta} - e_{-j\delta-\beta} \right) \text{ eigenvalue } -1, \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta+\beta} + e_{-j\delta-\beta} \right) \text{ eigenvalue } 1,
\end{aligned} \tag{7.6.119}$$

(in (7.6.119) above $\alpha = \varepsilon_r - \varepsilon_\ell$, $\beta = \varepsilon_r + \varepsilon_\ell$, where $\ell - 1 - b \leq r \leq \ell - 1$),

$$\left. \begin{array}{l} ic \\ id \end{array} \right\} \text{ eigenvalue } 1. \tag{7.6.120}$$

Therefore, the basis of a real form of $D_\ell^{(1)}$ is provided by the following elements:

$$\left. \begin{array}{l} \left(e_{j\delta}^k + e_{-j\delta}^k \right) \\ i \left(e_{j\delta}^k - e_{-j\delta}^k \right) \end{array} \right\} (1 \leq k \leq \ell - 2), \tag{7.6.121}$$

$$\begin{aligned}
& \left(e_{j\delta}^{\ell-1} + e_{-j\delta}^{\ell-1} + e_{j\delta}^\ell + e_{-j\delta}^\ell \right), \\
& i \left(e_{j\delta}^{\ell-1} - e_{-j\delta}^{\ell-1} + e_{j\delta}^\ell - e_{-j\delta}^\ell \right),
\end{aligned} \tag{7.6.122}$$

$$\begin{aligned}
& i \left(e_{j\delta}^{\ell-1} + e_{-j\delta}^{\ell-1} - e_{j\delta}^\ell - e_{-j\delta}^\ell \right), \\
& \left(e_{j\delta}^{\ell-1} - e_{-j\delta}^{\ell-1} - e_{j\delta}^\ell + e_{-j\delta}^\ell \right),
\end{aligned} \tag{7.6.123}$$

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} \right), \\
& i \left(e_{j\delta-\alpha} - e_{-j\delta-\alpha} \right),
\end{aligned} \tag{7.6.124}$$

(in (7.6.124) above $\alpha = \varepsilon_r \pm \varepsilon_s$ with either $1 \leq r < s \leq \ell - 1 - b$ or $\ell - b - 1 < r < s \leq \ell - 1$),

$$\begin{aligned} & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}), \\ & (e_{j\delta-\alpha} - e_{-j\delta-\alpha}), \end{aligned} \tag{7.6.125}$$

(in (7.6.125) above $\alpha = \varepsilon_r \pm \varepsilon_s$ with $1 \leq r \leq \ell - 1 - b < s \leq \ell - 1$),

$$\begin{aligned} & (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta+\beta} + e_{-j\delta-\beta}), \\ & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta+\beta} - e_{-j\delta-\beta}), \\ & i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta+\beta} - e_{-j\delta-\beta}), \\ & (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta+\beta} + e_{-j\delta-\beta}), \end{aligned} \tag{7.6.126}$$

(in (7.6.126) above $\alpha = \varepsilon_r - \varepsilon_\ell$, $\beta = \varepsilon_r + \varepsilon_\ell$, where $1 \leq r \leq \ell - 1 - b$),

$$\begin{aligned} & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta+\beta} + e_{-j\delta-\beta}), \\ & (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta+\beta} - e_{-j\delta-\beta}), \\ & (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta+\beta} - e_{-j\delta-\beta}), \\ & i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta+\beta} + e_{-j\delta-\beta}), \end{aligned} \tag{7.6.127}$$

(in (7.6.127) above $\alpha = \varepsilon_r - \varepsilon_\ell$, $\beta = \varepsilon_r + \varepsilon_\ell$, where $\ell - 1 - b \leq r \leq \ell - 1$),

$$\begin{aligned} & ic, \\ & id. \end{aligned} \tag{7.6.128}$$

The case $(c, d) = (0, 1)$ presents no difficulties. Let $\mathbf{U}(t)$ and $\mathbf{V}(t)$ be defined by

$$\mathbf{U}(t) = \text{dsum}\{\mathbf{1}_{\ell-1-b}, -\mathbf{1}_b, \mathbf{K}_2, -\mathbf{1}_b, \mathbf{1}_{\ell-1-b}\}, \tag{7.6.129}$$

$$\mathbf{V}(t) = \text{dsum}\{\mathbf{1}_{\ell-1-b}, -\mathbf{1}_b, \text{offdiag}\{t, t^{-1}\}, -\mathbf{1}_b, \mathbf{1}_{\ell-1-b}\}. \tag{7.6.130}$$

The matrix $\mathbf{S}(t)$, which is given by $\mathbf{S}(t) = \text{dsum}\{t\mathbf{1}_\ell, \mathbf{1}_\ell\}$, is such that

$$\tilde{\mathbf{S}}(t)\mathbf{gS}(t) = t\mathbf{g}, \tag{7.6.131}$$

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(t)^{-1} = \mathbf{V}(t). \tag{7.6.132}$$

Thus, the case $(c, d) = (0, 1)$ does not present us with any conjugacy classes which are different from those which we have already encountered.

Now consider the case $(c, d) = (1, 1)$. We are concerned therefore with the matrix $U(t)$, where

$$U(t) = \text{dsum} \left\{ \mathbf{1}_a, -\mathbf{1}_b, \text{offsum} \left\{ 1, t, t^{-1}, 1 \right\}, -\mathbf{1}_b, \mathbf{1}_a \right\}, \quad (7.6.133)$$

for suitable a, b . As in the previous case, it is clear from the determinant of $U(t)$ that the type 1a involutive automorphism $\{U(t), 1, \xi\}$ does not belong to the conjugacy class $(D)^{(k)}$ (for any value of k). Consider the possibility that $\{U(t), 1, \xi\}$ is a member of the conjugacy class $(A)^{(j)}$ for some appropriate value of j . Therefore, we suppose that

$$S(t)U(t)S(t)^{-1} = \lambda t^\mu \text{dsum} \left\{ \mathbf{1}_{\ell-j}, -\mathbf{1}_{2j}, \mathbf{1}_{\ell-j} \right\}, \quad (7.6.134)$$

$$S(t)\mathbf{g}S(t) = \alpha t^\beta \mathbf{g}. \quad (7.6.135)$$

Re-arrangement of this equation implies (as we have seen in the previous case) that $\lambda = \pm 1$ and $\mu = 0$. If we evaluate the traces of both sides of the above, then it is seen that

$$(\ell - 2 - b) = (\ell - 2j), \quad (7.6.136)$$

which means that $j = b + 1$. Now, there are countless matrices $S(t)$ which are such that

$$S(t)U(t)S(t)^{-1} = \pm \text{dsum} \left\{ \mathbf{1}_{\ell-j}, -\mathbf{1}_{2j}, \mathbf{1}_{\ell-j} \right\}. \quad (7.6.137)$$

However, upon finding matrices which satisfy (7.6.137), it is invariably found to be the case that such matrices do not satisfy $\tilde{S}(t)\mathbf{g}S(t) = \alpha t^\beta \mathbf{g}$. With the chosen representation of D_ℓ , it is difficult to prove that there do not exist any matrices $S(t)$ which satisfy (7.6.137) and $\tilde{S}(t)\mathbf{g}S(t) = \alpha t^\beta \mathbf{g}$. This problem is not insoluble, however. As in similar problems earlier in this chapter, we consider the highest-weight

representation $\Gamma\{0,0,\dots,1\}$ of D_ℓ . Consider the type 1a involutive automorphism $\phi_1 (= \{U(t), 1, \xi\})$, where

$$U(t) = \text{dsum}\{1_a, -1_b, \text{offdiag}\{1, t, t^{-1}, 1\}, -1_b, 1_a\}. \quad (7.6.138)$$

Clearly this is of type 1a when the original representation $\Gamma\{1,0,\dots,0\}$ is in use. When the alternative representation $\Gamma\{0,0,\dots,1\}$ is used, it is easily verified that ϕ_1 is now of type 1b. It is also very easy to verify that ϕ_2 (where ϕ_2 is a suitable representative of the conjugacy class $(A)^{(j)}$) is of type 1a under the alternative representation. We infer (from consideration of the alternative representation) that there does not exist any $(2\ell \times 2\ell)$ Laurent polynomial matrix $S(t)$ which satisfies

$$S(t)U(t)S(t)^{-1} = (\lambda)\text{dsum}\{1_{\ell-j}, -1_{2j}, 1_{\ell-j}\}, \quad (7.6.139)$$

$$\tilde{S}(t)gS(t) = \alpha t^\beta, \quad (7.6.140)$$

remembering of course that

$$\begin{aligned} j &= b + 1, \\ \lambda &= \pm 1. \end{aligned} \quad (7.6.141)$$

We must also enquire about whether or not the type 1a (in terms of the original representation) involutive automorphism $\{U(t), 1, \xi\}$ belongs to the conjugacy class $(B)^{(j)}$ (for $0 \leq j \leq [\frac{1}{2}\ell]$). If this were the case, then there would exist some Laurent polynomial matrix $S(t)$ which satisfies

$$S(t)U(t)S(t)^{-1} = \lambda t^\mu \text{dsum}\{1_{\ell-j}, -1_j, 1_j, -1_{\ell-j}\}, \quad (7.6.142)$$

$$\tilde{S}(t)gS(t) = \alpha t^\beta g. \quad (7.6.143)$$

These hold for all non-zero values of t . In particular, they hold for $t = 1$. Putting $t = 1$ implies that

$$S(1)U(1)S(1)^{-1} = \lambda \text{dsum}\{1_{\ell-j}, -1_j, 1_j, -1_{\ell-j}\}. \quad (7.6.144)$$

It can be inferred from analysis earlier in this section that there does not exist any t -independent matrix \mathbf{m} which satisfies both of the following:

$$\mathbf{m}\mathbf{U}(1)\mathbf{m}^{-1} = \lambda \text{dsum}\{\mathbf{1}_{\ell-j}, -\mathbf{1}_j, \mathbf{1}_j, -\mathbf{1}_{\ell-j}\} \quad (7.6.145)$$

$$\tilde{\mathbf{m}}\mathbf{g}\mathbf{m} = \alpha t^\beta \mathbf{g}. \quad (7.6.146)$$

One further possibility remains. That is, that the type 1a involutive automorphism $\{\mathbf{U}(t), 1, \xi\}$ belongs to the class $(C)^{(j,1)}$ for some suitable value of j . Let us assume for the moment that this is the case, so that the following hold:

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(t)^{-1} = \lambda t^\mu \mathbf{V}(t), \quad (7.6.147)$$

$$\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = \alpha t^\beta \mathbf{g}, \quad (7.6.148)$$

where $\mathbf{V}(t) = \text{dsum}\{\mathbf{1}_{\ell-1-j}, -\mathbf{1}_j, \mathbf{K}_2, -\mathbf{1}_j, \mathbf{1}_{\ell-1-j}\}$. Comparison of the determinants of $\mathbf{U}(t)$ and $\mathbf{V}(t)$ implies that $\lambda^{2\ell} = -1$ and $\mu = 0$. However, upon re-arranging the above to give

$$\mathbf{S}(t)\mathbf{U}(t) = \lambda \mathbf{V}(t)\mathbf{S}(t), \quad (7.6.149)$$

and attempting to solve this and find a general form for $\mathbf{S}(t)$, it becomes clear that $\lambda^2 = 1$. There is a contradiction, since λ cannot be both real and purely imaginary. It appears therefore, that there is some conjugacy class for which the representative automorphism may be taken to be the type 1a involution $\{\mathbf{U}(t), 1, \xi\}$, where

$$\mathbf{U}(t) = \text{dsum}\{\mathbf{1}_{\ell-2-b}, -\mathbf{1}_b, \text{offdiag}\{1, t, t^{-1}, 1\}, -\mathbf{1}_b, \mathbf{1}_{\ell-2-b}\}. \quad (7.6.150)$$

We call this class $(C)^{(b,2)}$. The real form which is generated by this representative automorphism has the following basis elements:

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \\ i(e_{j\delta}^k - e_{-j\delta}^k) \end{array} \right\} j \in \mathbf{Z}, 1 \leq k < \ell - 2, \quad (7.6.151)$$

$$\left. \begin{aligned} & \left(e_{j\delta}^{\ell-2} + e_{-j\delta}^{\ell-2} - \frac{1}{2} \left(e_{j\delta}^{\ell-1} + e_{-j\delta}^{\ell-1} + e_{j\delta}^{\ell} + e_{-j\delta}^{\ell} \right) \right) \\ & i \left(e_{j\delta}^{\ell-2} - e_{-j\delta}^{\ell-2} - \frac{1}{2} \left(e_{j\delta}^{\ell-1} - e_{-j\delta}^{\ell-1} + e_{j\delta}^{\ell} - e_{-j\delta}^{\ell} \right) \right) \end{aligned} \right\} j \in \mathbb{N}^0, \quad (7.6.152)$$

$$\left. \begin{aligned} & i \left(e_{j\delta}^k + e_{-j\delta}^k \right) \\ & \left(e_{j\delta}^k - e_{-j\delta}^k \right) \end{aligned} \right\} j \in \mathbb{N}, k = \ell - 1, \ell, \quad (7.6.153)$$

$$\left. \begin{aligned} & \left(h_{\alpha_{\ell-1}} - \frac{1}{2}c \right), \\ & \left(h_{\alpha_{\ell}} + \frac{1}{2}c \right), \end{aligned} \right\} \quad (7.6.154)$$

$$\left. \begin{aligned} & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} \right), \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} \right), \end{aligned} \right\} \quad (7.6.155)$$

(in equation (7.6.155) above, $\alpha = \varepsilon_r \pm \varepsilon_s$, and either $1 \leq r < s \leq \ell - 2 - b$ or $\ell - 2 - b < r < s \leq \ell - 2 - b$),

$$\left. \begin{aligned} & i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} \right), \\ & \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} \right), \end{aligned} \right\} \quad (7.6.156)$$

(in (7.6.156) above, $\alpha = \varepsilon_r \pm \varepsilon_s$ and $1 \leq r \leq \ell - 2 - b < s \leq \ell - 2$),

$$\left. \begin{aligned} & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta+\beta} + e_{-j\delta-\beta} \right), \\ & i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta+\beta} - e_{-j\delta-\beta} \right), \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta+\beta} - e_{-j\delta-\beta} \right), \\ & \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta+\beta} + e_{-j\delta-\beta} \right), \end{aligned} \right\} \quad (7.6.157)$$

(in (7.6.157) above, $\alpha = \varepsilon_r - \varepsilon_{\ell-1}$, $\beta = \varepsilon_r + \varepsilon_{\ell-1}$ and $1 \leq r \leq \ell - 2 - b$),

$$\left. \begin{aligned} & i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta+\beta} + e_{-j\delta-\beta} \right), \\ & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta+\beta} - e_{-j\delta-\beta} \right), \\ & \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta+\beta} - e_{-j\delta-\beta} \right), \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta+\beta} + e_{-j\delta-\beta} \right), \end{aligned} \right\} \quad (7.6.158)$$

(in (7.6.158) above, $\alpha = \varepsilon_r - \varepsilon_{\ell-1}$, $\beta = \varepsilon_r + \varepsilon_{\ell-1}$ and $\ell - 2 - b < r \leq \ell - 2$),

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(j+1)\delta+\beta} + e_{-(j+1)\delta-\beta} \right), \\
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{(j+1)\delta+\beta} - e_{-(j+1)\delta-\beta} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{(j+1)\delta+\beta} - e_{-(j+1)\delta-\beta} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{(j+1)\delta+\beta} + e_{-(j+1)\delta-\beta} \right),
\end{aligned} \tag{7.6.159}$$

(in (7.6.159) above, $\alpha = \varepsilon_r - \varepsilon_\ell$, $\beta = \varepsilon_r + \varepsilon_\ell$ and $1 \leq r \leq \ell - 2 - b$),

$$\begin{aligned}
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{(j+1)\delta+\beta} + e_{-(j+1)\delta-\beta} \right), \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{(j+1)\delta+\beta} - e_{-(j+1)\delta-\beta} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{(j+1)\delta+\beta} - e_{-(j+1)\delta-\beta} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{(j+1)\delta+\beta} + e_{-(j+1)\delta-\beta} \right),
\end{aligned} \tag{7.6.160}$$

(in (7.6.160) above, $\alpha = \varepsilon_r - \varepsilon_\ell$, $\beta = \varepsilon_r + \varepsilon_\ell$ and $1 \leq r \leq \ell - 2 - b$),

$$\begin{aligned}
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta+\alpha} \right), \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta+\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta+\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta+\alpha} \right),
\end{aligned} \tag{7.6.161}$$

(in (7.6.161) above, $\alpha = \varepsilon_{\ell-1} - \varepsilon_\ell$),

$$\begin{aligned}
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta+\alpha} \right), \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta+\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta+\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta+\alpha} \right),
\end{aligned} \tag{7.6.162}$$

(in (7.6.162) above, $\alpha = \varepsilon_{\ell-1} + \varepsilon_\ell$),

$$\begin{aligned}
& ic, \\
& i \left(d + \frac{1}{4} \ell(\ell-1) \{ h_{\alpha_{\ell-1}} - h_{\alpha_\ell} \} \right).
\end{aligned} \tag{7.6.163}$$

It remains for us to examine the type 1a involutive automorphisms $\{\mathbf{U}(t), 1, \xi\}$, where $\mathbf{U}(t)$ is given by (7.6.6). In fact, all such automorphisms belong to the same conjugacy class, as will now be shown. Let the matrix $\mathbf{S}(t)$ be given by

$$\mathbf{S}(t) = \begin{bmatrix} \mathbf{a} & \mathbf{0} & \mathbf{0} & \mathbf{b} \\ \mathbf{0} & \mathbf{1}_{\ell-1-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -i\mathbf{1}_{\ell-1-q} & \mathbf{0} \\ \mathbf{c} & \mathbf{0} & \mathbf{0} & \mathbf{d} \end{bmatrix}, \quad (7.6.164)$$

where the submatrices are given (for odd values of q) by

$$\mathbf{a} = \text{dsum}\{\mathbf{a}_1, \dots, \mathbf{a}_{\frac{1}{2}q}\} \left(\mathbf{a}_1 = \dots = \mathbf{a}_{\frac{1}{2}q} = \frac{i}{\sqrt{2}} \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \right), \quad (7.6.165)$$

$$\mathbf{b} = \text{offsum}\{\mathbf{b}_1, \dots, \mathbf{b}_{\frac{1}{2}q}\} \left(\mathbf{b}_1 = \dots = \mathbf{b}_{\frac{1}{2}q} = \frac{i}{\sqrt{2}} \begin{bmatrix} -t^{-1} & 0 \\ t^{-1} & 0 \end{bmatrix} \right), \quad (7.6.166)$$

$$\mathbf{c} = \text{offsum}\{\mathbf{c}_1, \dots, \mathbf{c}_{\frac{1}{2}q}\} \left(\mathbf{c}_1 = \dots = \mathbf{c}_{\frac{1}{2}q} = -\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right), \quad (7.6.167)$$

$$\mathbf{d} = \text{dsum}\{\mathbf{d}_1, \dots, \mathbf{d}_{\frac{1}{2}q}\} \left(\mathbf{d}_1 = \dots = \mathbf{d}_{\frac{1}{2}q} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -t^{-1} \\ 0 & t^{-1} \end{bmatrix} \right). \quad (7.6.168)$$

This satisfies $\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = i\mathbf{g}$, $\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{2\ell}$ and also

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(t)^{-1} = (i)\text{offsum}\{\mathbf{K}_\ell, t\mathbf{K}_\ell\}. \quad (7.6.169)$$

This concludes the analysis of the type 1a involutive automorphisms with $u = 1$.

7.7 Type 1a involutive automorphisms of $D_\ell^{(1)}$ with $u = -1$

We will now examine the type 1a involutions with $u = -1$, finding out about the conjugacy classes to which they belong. The matrix transformations contained in section 5 of this chapter (modified, where necessary) demonstrate that only a subset of the involutive automorphisms given in section 4 need to be considered. In fact, we will be examining only those type 1a involutive automorphisms (with $u = -1$) which are generated by the following matrices:

$$\text{dsum}\{\mathbf{1}_{\ell-a}, -\mathbf{1}_{2a}, \mathbf{1}_{\ell-a}\}, \quad (7.7.1)$$

$$\text{dsum}\{\mathbf{1}_{\ell-a}, -\mathbf{1}_a, \mathbf{1}_a, -\mathbf{1}_{\ell-a}\}, \quad (7.7.2)$$

$$\text{offsum}\{\mathbf{K}_{\ell-a}, -\mathbf{K}_{2a}, \mathbf{K}_{\ell-a}\}, \quad (7.7.3)$$

$$\text{offsum}\{\mathbf{K}_{\ell-a}, -\mathbf{K}_a, \mathbf{K}_a, -\mathbf{K}_{\ell-a}\}, \quad (7.7.4)$$

$$\text{dsum}\{\mathbf{1}_a, -\mathbf{1}_b, \mathbf{K}_{2c}, -\mathbf{1}_b, \mathbf{1}_a\}, \quad (7.7.5)$$

$$\begin{bmatrix} -\mathbf{1}_a & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_b & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & t\mathbf{K}_c & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -t^{-1}\mathbf{K}_c & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{1}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_a \end{bmatrix}, \quad (7.7.6)$$

$$\text{dsum}\left\{s_1, \dots, s_{\frac{1}{2}(\ell)}, -s_{\frac{1}{2}(\ell)}, \dots, -s_1\right\} \left(s_j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right). \quad (7.7.7)$$

We begin with those type 1a automorphisms $\{U(t), -1, 0\}$, where $U(t)$ is given by (7.7.1). Let the matrix $S(t)$ be defined by

$$S(t) = \text{dsum} \{ \mathbf{1}_{\ell-a}, t\mathbf{1}_a, t^{-1}\mathbf{1}_a, \mathbf{1}_{\ell-a} \}. \quad (7.7.8)$$

This satisfies $\tilde{S}^*(t^{-1})S(t) = \mathbf{1}_{2\ell}$, $\tilde{S}(t)gS(t) = g$ and $S(t)\mathbf{1}_{2\ell}S(-t)^{-1} = U(t)$. Hence, all of the type 1a involutive automorphisms (with $u = -1$) which are generated by matrices of the form (7.7.1) are conjugate. Let the class to which they belong be called (E).

Now we examine those type 1a involutive automorphisms $\{U(t), -1, \xi\}$, where $U(t)$ is given by (7.7.2). In this case, we let the matrix $S(t)$ be defined by

$$S(t) = \text{dsum} \{ \mathbf{1}_{\ell-a}, t\mathbf{1}_a, \mathbf{1}_a, t\mathbf{1}_{\ell-a} \}. \quad (7.7.9)$$

This is such that $\tilde{S}(t)gS(t) = tg$, $\tilde{S}^*(t^{-1})S(t) = \mathbf{1}_{2\ell}$, and $S(t)\mathbf{1}_{2\ell}S(-t)^{-1} = U(t)$. Thus, all of these involutive automorphisms also belong to the conjugacy class (E). We take the type 1a involutive automorphism ψ_E (where $\psi_E = \{\mathbf{1}_{2\ell}, -1, 0\}$) as representative of the conjugacy class (E). The process for generating bases of real forms from involutive automorphisms then generates the following basis from the automorphism ψ_E :

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \\ i(e_{j\delta}^k - e_{-j\delta}^k) \end{array} \right\} j \text{ is even, } 1 \leq k \leq \ell, \quad (7.7.10)$$

$$\left. \begin{array}{l} i(e_{j\delta}^k + e_{-j\delta}^k) \\ (e_{j\delta}^k - e_{-j\delta}^k) \end{array} \right\} j \text{ is odd, } 1 \leq k \leq \ell, \quad (7.7.11)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \alpha \text{ is a root of } D_\ell, j \text{ is even,} \quad (7.7.12)$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \alpha \text{ is a root of } D_\ell, j \text{ is odd,} \quad (7.7.13)$$

$$\begin{array}{l} ic, \\ id. \end{array} \quad (7.7.14)$$

In a similar fashion, it can be shown that all of the type 1a involutive automorphisms $\{U(t), -1, \xi\}$ (where $U(t)$ is given by (7.7.3)) are mutually conjugate. We define $S(t)$ by

$$S(t) = \text{dsum}\{1_{\ell-a}, t1_a, t^{-1}1_a, 1_{\ell-a}\}. \quad (7.7.15)$$

We have already seen that this matrix generates automorphisms of $D_\ell^{(1)}$ and of its compact real form. In addition, it satisfies

$$S(t)K_{2\ell}S(t)^{-1} = \text{offsum}\{K_{\ell-a}, -K_{2a}, K_{\ell-a}\}. \quad (7.7.16)$$

We have not shown whether or not these involutive automorphisms belong to the conjugacy class (E). There are two distinct cases to consider, corresponding to odd and even values of ℓ . Let us take the case where ℓ is even. There is nothing left to demonstrate in this case, since the matrix transformations in section 5 illustrate clearly that the type 1a involutive automorphisms $\{K_{2\ell}, -1, 0\}$ and $\{1_{2\ell}, -1, 0\}$ are conjugate to each other. However, when ℓ is odd, this is not the case. If the type 1a involutive automorphism $\phi_1 (= \{K_{2\ell}, -1, 0\})$ belongs to the conjugacy class (E), then there must exist some Laurent polynomial matrix $S(t)$ which satisfies both of the following:

$$S(t)1_{2\ell}S(-t)^{-1} = \lambda t^\mu K_{2\ell}, \quad (7.7.17)$$

$$\tilde{S}(t)gS(t) = \alpha t^\beta g. \quad (7.7.18)$$

The type 1a involutive automorphism ϕ_2 is defined to be $\{1_{2\ell}, -1, 0\}$. It does not appear to be possible, in general, to find a matrix $S(t)$ which satisfies (7.7.18) and (7.7.19). However, proving that this is impossible for all values of ℓ using matrix methods is difficult. Consider the algebra $D_3^{(1)}$, which is isomorphic to the algebra $A_3^{(1)}$, which has been investigated already. We wish to investigate whether or not the two type 1a involutive automorphisms (of $D_3^{(1)}$) $\{1_6, -1, 0\}$ and $\{K_6, -1, 0\}$ are conjugate. These correspond to the root transformations $\rho_1(\alpha_k^0) = \alpha_k^0$ (where $k = 1, 2, 3$) and $\rho_2(\alpha_k^0) = -\alpha_k^0$ (where $k = 1, 2, 3$) respectively. (The root

transformations are given in terms of the roots of D_3 . The Dynkin diagrams of D_3 and A_3 are identical under the re-labelling

$$\sigma(\alpha_1^0) = \alpha_1^0, \quad (7.7.19)$$

$$\sigma(\alpha_2^0) = \alpha_3^0, \quad (7.7.20)$$

$$\sigma(\alpha_3^0) = \alpha_2^0. \quad (7.7.21)$$

Under this re-labelling, the root transformations ρ_1 and ρ_2 are preserved unaltered. Now, when considered in terms of the notation and conventions of $A_3^{(1)}$, the automorphisms corresponding to the root transformation ρ_1 and the automorphisms corresponding to the root transformation ρ_2 are of different types (of types 1a and 1b). Thus, for the specific case $\ell = 3$, it is possible to show that no matrix $S(t)$ can exist, which satisfies the conditions outlined previously. It is possible to prove this conclusively because of the readily-available comparison with the algebra $A_3^{(1)}$. For larger values of ℓ , there is no isomorphism with another complex affine Kac-Moody algebra which allows such a comparison. However, we can show that, for odd values of ℓ larger than 3, the type 1a involutive automorphisms ϕ_1 and ϕ_2 are not conjugate. To do this, we consider the highest-weight representation $\Gamma\{0,0,\dots,1\}$ of $D_\ell^{(1)}$ (with ℓ odd). If we generate the representation in question (using the method given in chapter 16 of [13]), we see that it is not equivalent to its contragredient representation. Thus, the type 1a and the type 1b automorphisms of $D_\ell^{(1)}$ are distinct. Returning to the two involutive automorphisms ϕ_1 and ϕ_2 , it is easily verified that, with the new representation of $D_\ell^{(1)}$, one of them is of type 1a (with $u = -1$) and the other is of type 1b (with $u = -1$). Hence they cannot be conjugate.

So, for odd values of ℓ , there is a new conjugacy class, which we will call (F). The representative automorphism of the class (F) is the type 1a involutive automorphism ψ_F , where $\{U, -1, 0\}$ with $U = \text{dsum}\{\mathbf{1}_{\ell-1}, \mathbf{K}_2, \mathbf{1}_{\ell-1}\}$. This choice of representative automorphism is justified by later results. A basis of a real form of $D_\ell^{(1)}$

(associated with ψ_F) which is generated by following Cartan's method is provided by the following elements:

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \\ i(e_{j\delta}^k - e_{-j\delta}^k) \end{array} \right\} j \text{ is even, } 1 \leq k \leq \ell - 1, \quad (7.7.22)$$

$$\left. \begin{array}{l} i(e_{j\delta}^k + e_{-j\delta}^k) \\ (e_{j\delta}^k - e_{-j\delta}^k) \end{array} \right\} j \text{ is odd, } 1 \leq k \leq \ell - 1, \quad (7.7.23)$$

$$\left. \begin{array}{l} (e_{j\delta}^{\ell-1} + e_{-j\delta}^{\ell-1} + e_{j\delta}^{\ell} + e_{-j\delta}^{\ell}) \\ i(e_{j\delta}^{\ell-1} + e_{-j\delta}^{\ell-1} - e_{j\delta}^{\ell} - e_{-j\delta}^{\ell}) \\ i(e_{j\delta}^{\ell-1} - e_{-j\delta}^{\ell-1} + e_{j\delta}^{\ell} - e_{-j\delta}^{\ell}) \\ (e_{j\delta}^{\ell-1} - e_{-j\delta}^{\ell-1} - e_{j\delta}^{\ell} + e_{-j\delta}^{\ell}) \end{array} \right\} j \text{ is even, } \quad (7.7.24)$$

$$\left. \begin{array}{l} i(e_{j\delta}^{\ell-1} + e_{-j\delta}^{\ell-1} + e_{j\delta}^{\ell} + e_{-j\delta}^{\ell}) \\ (e_{j\delta}^{\ell-1} + e_{-j\delta}^{\ell-1} - e_{j\delta}^{\ell} - e_{-j\delta}^{\ell}) \\ (e_{j\delta}^{\ell-1} - e_{-j\delta}^{\ell-1} + e_{j\delta}^{\ell} - e_{-j\delta}^{\ell}) \\ i(e_{j\delta}^{\ell-1} - e_{-j\delta}^{\ell-1} - e_{j\delta}^{\ell} + e_{-j\delta}^{\ell}) \end{array} \right\} j \text{ is odd, } \quad (7.7.25)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \alpha = \varepsilon_r \pm \varepsilon_s, 1 \leq r < s \leq \ell - 1, j \text{ is even, } \quad (7.7.26)$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \alpha = \varepsilon_r \pm \varepsilon_s, 1 \leq r < s \leq \ell - 1, j \text{ is odd, } \quad (7.7.27)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta+\beta} + e_{-j\delta-\beta}) \\ i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta+\beta} - e_{-j\delta-\beta}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta+\beta} - e_{-j\delta-\beta}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta+\beta} + e_{-j\delta-\beta}) \end{array} \right\} j \text{ is even, } \quad (7.7.28)$$

(in (7.7.28) above, $\alpha = \varepsilon_r - \varepsilon_\ell$, $\beta = \varepsilon_r + \varepsilon_\ell$, and $1 \leq r \leq \ell - 1$),

$$\left. \begin{aligned} & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{j\delta+\beta} + e_{-j\delta-\beta} \right) \\ & i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{j\delta+\beta} - e_{-j\delta-\beta} \right) \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{j\delta+\beta} - e_{-j\delta-\beta} \right) \\ & \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{j\delta+\beta} + e_{-j\delta-\beta} \right) \end{aligned} \right\} j \text{ is odd,} \quad (7.7.29)$$

(in (7.7.29) above, $\alpha = \varepsilon_r - \varepsilon_\ell$, $\beta = \varepsilon_r + \varepsilon_\ell$, and $1 \leq r \leq \ell - 1$),

$$\begin{aligned} & ic, \\ & id. \end{aligned} \quad (7.7.30)$$

The type 1a involutive automorphisms $\{\mathbf{U}(t), -1, 0\}$, where $\mathbf{U}(t)$ is given by (7.7.4) are conjugate to the type 1a involutive automorphism $\{\mathbf{K}_{2\ell}, -1, 0\}$. Hence, they all belong to either the class (E) or the class (F), depending upon whether ℓ is even or odd. To see this, let $\mathbf{S}(t)$ be defined by

$$\mathbf{S}(t) = \text{dsum}\{\mathbf{1}_{\ell-a}, t\mathbf{1}_a, \mathbf{1}_a, t\mathbf{1}_{\ell-a}\}. \quad (7.7.31)$$

This is such that $\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{2\ell}$, $\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = t\mathbf{g}$, and $\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(-t)^{-1} = \mathbf{K}_{2\ell}$.

We will now consider those type 1a involutive automorphisms $\{\mathbf{U}(t), -1, 0\}$, where $\mathbf{U}(t)$ is given by (7.7.5). Using the matrix transformations of section 5, together with some of the examples given in this section, we may confine our interest to those matrices which are of the form

$$\mathbf{U}(t) = \text{dsum}\{\mathbf{1}_{\ell-1}, \mathbf{K}_2, \mathbf{1}_{\ell-1}\}. \quad (7.7.32)$$

In the case where ℓ is odd, it follows from previous analysis (in section 5 and elsewhere in this section) that $\{\mathbf{U}(t), -1, 0\}$ is conjugate to the type 1a involutive automorphism $\{\mathbf{K}_{2\ell}, -1, 0\}$. Thus, when ℓ is odd, no new conjugacy classes are encountered. However, when ℓ is even we note that the matrix $\mathbf{U}(t)$ is of the same form as the matrix which generated the representative automorphism (for odd ℓ). The class (F) was previously defined only for odd values of ℓ . In fact, it can be extended, so that (F) is the class which contains the representative type 1a automorphism

$\{U(t), -1, 0\}$, where $U(t) = \text{dsum}\{\mathbf{1}_{\ell-1}, \mathbf{K}_2, \mathbf{1}_{\ell-1}\}$. This is the case for both odd and even values of ℓ .

Let the matrix $U(t)$ be given by (7.7.6). It may be assumed that $U(t)$ is of the form

$$U(t) = \begin{bmatrix} -\mathbf{1}_{\ell-a} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & t\mathbf{K}_a & \mathbf{0} \\ \mathbf{0} & -t^{-1}\mathbf{K}_a & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{\ell-a} \end{bmatrix}. \quad (7.7.33)$$

If we let $S(t) = \text{dsum}\{t^{-1}\mathbf{1}_\ell, \mathbf{1}_\ell\}$ (which satisfies $\tilde{S}(t)\mathbf{g}S(t) = t^{-1}\mathbf{g}$ and $\tilde{S}^*(t^{-1})S(t) = \mathbf{1}_{2\ell}$) then

$$S(t)U(t)S(-t)^{-1} = \text{dsum}\{\mathbf{1}_{\ell-a}, \mathbf{K}_{2a}, \mathbf{1}_{\ell-a}\}. \quad (7.7.34)$$

Thus, any such automorphisms belong to classes which have already been identified.

The final automorphism to be examined in this section is the type Ia automorphism $\{U(t), -1, 0\}$, where $U(t)$ is given by (7.7.7). Let $S(t)$ be defined by

$$S(t) = \text{dsum}\{\mathbf{x}_1, \dots, \mathbf{x}_\ell\} \left(\mathbf{x}_1 = \dots = \mathbf{x}_\ell = \frac{1}{\sqrt{2}} \begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} \right). \quad (7.7.35)$$

This is such that $\tilde{S}(t)\mathbf{g}S(t) = i\mathbf{g}$, $\tilde{S}^*(t^{-1})S(t) = \mathbf{1}_{2\ell}$, and also

$$S(t)U(t)S(-t)^{-1} = (-i)\text{dsum}\{\mathbf{y}_1, \dots, \mathbf{y}_{\frac{1}{2}\ell}, -\mathbf{y}_1, \dots, -\mathbf{y}_{\frac{1}{2}\ell}\}, \quad (7.7.36)$$

where \mathbf{y} is the 2×2 matrix given by $\mathbf{y} = \text{diag}\{1, -1\}$. Thus, the automorphism in question belongs to the conjugacy class (E).

7.8 Involutive automorphisms of type 2a with $u = 1$

In this section we examine the type 2a involutive automorphisms of $D_\ell^{(1)}$ using the matrix formulation, and use the results obtained to find real forms of $D_\ell^{(1)}$. As was the case with the type 1a involutive automorphisms, this approach is not entirely conclusive. This method of investigating the automorphisms, whilst giving conclusive results for other algebras, is not entirely satisfactory for the algebra $D_\ell^{(1)}$ with the representation of D_ℓ which is being employed. Clearly, we need only to consider a subset of those type 2a involutive automorphisms which were listed in section 4, since the matrix transformations of section 5 may be used to infer that certain automorphisms are conjugate to others. A suitable subset of involutive automorphisms is that set of type 2a involutive automorphisms $\{U(t), 1, \xi\}$, where $U(t)$ is given by one of the following:

$$\text{dsum}\{1_a, -1_b, t1_c, -t1_d, -t^{-1}1_d, t^{-1}1_c, -1_b, 1_a\}, \quad (7.8.1)$$

$$\text{dsum}\{-1_a, 1_b, -t1_c, t1_d, -t^{-1}1_d, t^{-1}1_c, -1_b, 1_a\}, \quad (7.8.2)$$

$$\text{dsum}\{t1_a, -1_b, 1_c, -t1_d, -1_d, t1_c, -t1_b, 1_a\}, \quad (7.8.3)$$

$$\text{dsum}\{-t1_a, -1_b, 1_c, t1_d, -1_d, -t1_c, t1_b, 1_a\}, \quad (7.8.4)$$

$$\mathbf{K}_{2\ell}, \quad (7.8.5)$$

$$\text{dsum}\{1_a, -1_b, t1_c, -t1_d, \mathbf{K}_2, -t^{-1}1_d, t^{-1}1_c, -1_b, 1_a\}. \quad (7.8.6)$$

We examine those type 2a involutive automorphisms $\{U(t), 1, \xi\}$, where $U(t)$ is of the form given by (7.8.1). We may show that the difference between the quantities c and

d is of more importance than the absolute values of those quantities. To see this, we define a matrix $\mathbf{S}(t)$ by the following:

$$\mathbf{S}(t) = \text{dsum} \left\{ \mathbf{1}_{a-1}, \frac{1}{2} \begin{bmatrix} 1+t & 1-t \\ 1-t & 1+t \end{bmatrix}, \mathbf{1}_{2\ell-2-2a}, \frac{1}{2} \begin{bmatrix} 1+t^{-1} & t^{-1}-1 \\ t^{-1}-1 & 1+t^{-1} \end{bmatrix}, \mathbf{1}_{a-1} \right\}, \quad (7.8.7)$$

where the quantity a is as given in (7.8.1). This matrix is such that $\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = \mathbf{g}$, $\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_{2\ell}$, and

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(t^{-1})^{-1} = \mathbf{V}(t), \quad (7.8.8)$$

where the matrix $\mathbf{V}(t)$ is given by

$$\text{dsum} \left\{ \mathbf{1}_{a-1}, t\mathbf{1}_1, -t\mathbf{1}_1, -\mathbf{1}_{b-1}, t\mathbf{1}_c, -t\mathbf{1}_d, -t^{-1}\mathbf{1}_d, t^{-1}\mathbf{1}_c, -\mathbf{1}_{b-1}, -t^{-1}\mathbf{1}_1, t^{-1}\mathbf{1}_1, \mathbf{1}_{a-1} \right\}. \quad (7.8.9)$$

It is clear that the type 2a involutive automorphism $\{\mathbf{V}(t), 1, \xi\}$ is conjugate to the type 2a involutive automorphism $\{\mathbf{W}(t), 1, \xi\}$, where

$$\mathbf{W}(t) = \text{dsum} \left\{ \mathbf{1}_{a-1}, -\mathbf{1}_{b-1}, t\mathbf{1}_{c+1}, -t\mathbf{1}_{d+1}, -t^{-1}\mathbf{1}_{d+1}, t^{-1}\mathbf{1}_{c+1}, -\mathbf{1}_{b-1}, \mathbf{1}_{a-1} \right\}. \quad (7.8.10)$$

This implies that we need only consider those automorphisms for which the associated matrix $\mathbf{U}(t)$ is of one of the following forms

$$\mathbf{U}_1(t) = \text{dsum} \left\{ \mathbf{1}_a, -\mathbf{1}_b, t\mathbf{1}_c, t^{-1}\mathbf{1}_c, -\mathbf{1}_b, \mathbf{1}_a \right\}, \quad (7.8.11)$$

$$\mathbf{U}_2(t) = \text{dsum} \left\{ \mathbf{1}_a, -\mathbf{1}_b, -t\mathbf{1}_c, -t^{-1}\mathbf{1}_c, -\mathbf{1}_b, \mathbf{1}_a \right\}. \quad (7.8.12)$$

However, since $\mathbf{U}_1(t) = \mathbf{U}_2(-t)$, it follows that each type 2a involutive automorphism $\{\mathbf{U}_2(t), 1, \xi\}$ is conjugate to some automorphism $\{\mathbf{U}_1(t), 1, \xi\}$. Thus, it may be supposed that $\mathbf{U}(t)$ is of the form $\mathbf{U}_1(t)$ defined above. We define the conjugacy class $(\mathbf{G})^{(q,r)}$ to be that class which contains the type 2a involutive automorphism $\{\mathbf{U}(t), 1, \xi\}$, where $\mathbf{U}(t) = \text{dsum} \left\{ \mathbf{1}_p, -\mathbf{1}_q, t\mathbf{1}_r, t^{-1}\mathbf{1}_r, -\mathbf{1}_q, \mathbf{1}_p \right\}$ (with $p + q + r = \ell$). The quantity r is able to take any of the values $0, \dots, \ell - q - 1$, and the quantity q may take

any of the values $0, \dots, \ell - 1$. However, the matrix transformations in section 5 can be used to help us infer that it is only necessary to assume that q and r are such that

$$q = 0, \dots, \left[\frac{1}{2} \ell \right], \quad r = 0, \dots, \left[\frac{1}{2} (\ell - q - 1) \right]. \quad (7.8.13)$$

We take the type 2a involutive automorphism generated by $U(t)$, where $U(t) = \text{dsum} \{ \mathbf{1}_p, -\mathbf{1}_q, t \mathbf{1}_r, t^{-1} \mathbf{1}_r, -\mathbf{1}_q, \mathbf{1}_p \}$ to be the representative of the conjugacy class $(G)^{(q,r)}$. Using Cartan's theorem, the following list contains a basis for a real form of $D_\ell^{(1)}$:

$$\begin{aligned} & i h_{\alpha_k} \quad (1 \leq k \leq \ell, \text{ and if } r \neq 0 \text{ then } k \neq \ell - r), \\ & i \left(h_{\alpha_{\ell-r}} - \frac{1}{2} c \right) \quad (r \neq 0), \\ & i \left(h_{\alpha_\ell} + \frac{1}{2} c \right) \quad (r = 1), \\ & i \left(h_{\alpha_\ell} + c \right) \quad (r = 2), \end{aligned} \quad (7.8.14)$$

$$\left. \begin{aligned} & \left(e_{j\delta}^k + e_{-j\delta}^k \right) \\ & \left(e_{j\delta}^k - e_{-j\delta}^k \right) \end{aligned} \right\} 1 \leq k \leq \ell, j > 0, \quad (7.8.15)$$

$$\begin{aligned} & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right), \\ & i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right), \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right), \\ & \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right), \end{aligned} \quad (7.8.16)$$

(where in the above, $\alpha = \varepsilon_a \pm \varepsilon_b$ with either $1 \leq a < b \leq p$ or $p < a < b \leq p+q$, and also $\alpha = \varepsilon_a - \varepsilon_b$ with $p+q < a < b \leq \ell$),

$$\begin{aligned} & i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right), \\ & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right), \\ & \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right), \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right), \end{aligned} \quad (7.8.17)$$

(where in the above, $\alpha = \varepsilon_a \pm \varepsilon_b$ and $1 \leq a \leq p < b \leq p+q$),

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right),
\end{aligned} \tag{7.8.18}$$

(where in the above, $\alpha = \varepsilon_a - \varepsilon_b$ and $1 \leq a \leq p, p+q < b \leq \ell$),

$$\begin{aligned}
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right),
\end{aligned} \tag{7.8.19}$$

(where in the above, $\alpha = \varepsilon_a - \varepsilon_b$ and $p < a \leq p+q < b \leq \ell$),

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right),
\end{aligned} \tag{7.8.20}$$

(where in the above, $\alpha = \varepsilon_a + \varepsilon_b$ and $1 \leq a \leq p, p+q < b \leq \ell$),

$$\begin{aligned}
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right),
\end{aligned} \tag{7.8.21}$$

(where in the above, $\alpha = \varepsilon_a + \varepsilon_b$ and $p < a \leq p+q < b \leq \ell$),

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} + e_{(j-2)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} - e_{(j-2)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} - e_{(j-2)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} + e_{(j-2)\delta-\alpha} \right),
\end{aligned} \tag{7.8.22}$$

(where, in the above, $\alpha = \varepsilon_a + \varepsilon_b$ with $p + q < a < b \leq \ell$),

$$c, \tag{7.8.23}$$

$$d - \frac{1}{2} \ell (\ell - 1) \left\{ \sum_{\substack{j=1 \\ r \geq 2}}^{r-2} j h_{\alpha_{\ell-r+j}} - \frac{1}{2} (r-2) h_{\alpha_{\ell-1}} - \frac{1}{2} (r-4) h_{\alpha_{\ell}} \right\}$$

We look next at those automorphisms which are generated by the matrices of the general form (7.8.2). The analysis dealing with the matrices of the general form (7.8.1) is also applicable here, to some extent. It implies that we need only examine those type 2a involutive automorphisms for which the matrix $\mathbf{U}(t)$ has the general form

$$\text{dsum} \left\{ -\mathbf{1}_a, \mathbf{1}_b, t \mathbf{1}_c, -t^{-1} \mathbf{1}_c, -\mathbf{1}_b, \mathbf{1}_a \right\}. \tag{7.8.24}$$

We define the conjugacy class $(\mathbf{H})^{(q,r)}$ to be that class which contains (as representative) the type 2a involutive automorphism $\{\mathbf{U}(t), 1, \xi\}$, where

$$\mathbf{U}(t) = \text{dsum} \left\{ -\mathbf{1}_p, \mathbf{1}_q, t \mathbf{1}_r, -t^{-1} \mathbf{1}_r, -\mathbf{1}_q, \mathbf{1}_p \right\} \quad (p + q + r = \ell). \tag{7.8.25}$$

The real form of $D_\ell^{(1)}$ which is generated by applying Cartan's method to the representative of the class $(\mathbf{H})^{(q,r)}$ may be generated by the following elements:

$$\begin{aligned}
& i h_{\alpha_k} \quad (1 \leq k \leq \ell, \text{ and if } r \neq 0 \text{ then } k \neq \ell - r), \\
& i \left(h_{\alpha_{\ell-r}} - \frac{1}{2} c \right) \quad (r \neq 0), \\
& i \left(h_{\alpha_{\ell}} + \frac{1}{2} c \right) \quad (r = 1), \\
& i \left(h_{\alpha_{\ell}} + c \right) \quad (r = 2),
\end{aligned} \tag{7.8.26}$$

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \\ (e_{j\delta}^k - e_{-j\delta}^k) \end{array} \right\} 1 \leq k \leq \ell, j > 0, \quad (7.8.27)$$

$$\begin{aligned} & (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \\ & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\ & i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\ & (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \end{aligned} \quad (7.8.28)$$

(where, in (7.8.28) above $\alpha = \varepsilon_a - \varepsilon_b$ with either $1 \leq a < b \leq p$, $p < a < b \leq p+q$ or $p+q < a < b \leq \ell$, and also $\alpha = \varepsilon_a + \varepsilon_b$ with $1 \leq a \leq p < b \leq p+q$),

$$\begin{aligned} & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \\ & (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\ & (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\ & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \end{aligned} \quad (7.8.29)$$

(where, in (7.8.29) above $\alpha = \varepsilon_a - \varepsilon_b$ with $1 \leq a \leq p < b \leq p+q$ and also $\alpha = \varepsilon_a + \varepsilon_b$ with either $1 \leq a < b \leq p$ or $p < a < b \leq p+q$),

$$\begin{aligned} & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha}), \\ & (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha}), \\ & (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha}), \\ & i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha}), \end{aligned} \quad (7.8.30)$$

(where, in (7.8.30) above $\alpha = \varepsilon_a - \varepsilon_b$ with $1 \leq a \leq p$, $p+q < b \leq \ell$),

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right),
\end{aligned} \tag{7.8.31}$$

(where, in (7.8.31) above $\alpha = \varepsilon_a - \varepsilon_b$ with $p < a \leq p+q < b \leq \ell$),

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right),
\end{aligned} \tag{7.8.32}$$

(where, in (7.8.32) above $\alpha = \varepsilon_a + \varepsilon_b$ with $1 \leq a \leq p$, $p+q < b \leq \ell$),

$$\begin{aligned}
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right)
\end{aligned} \tag{7.8.33}$$

(where, in (7.8.33) above $\alpha = \varepsilon_a + \varepsilon_b$ with $p < a \leq p+q < b \leq \ell$),

$$\begin{aligned}
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} + e_{(j-2)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} - e_{(j-2)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} - e_{(j-2)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} + e_{(j-2)\delta-\alpha} \right),
\end{aligned} \tag{7.8.34}$$

(where, in (7.8.34) above $\alpha = \varepsilon_a + \varepsilon_b$ with $p+q < a < b \leq \ell$),

c ,

$$d - \frac{1}{2} \ell(\ell - 1) \left\{ \sum_{\substack{j=1 \\ r \geq 2}}^{r-2} j h_{\alpha_{t-r+j}} - \frac{1}{2}(r-2)h_{\alpha_{t-1}} - \frac{1}{2}(r-4)h_{\alpha_t} \right\} \quad (7.8.35)$$

We follow on from this, and examine the type 2a involutive automorphisms $\{U(t), 1, \xi\}$, where $U(t)$ is of the form (7.8.3). It follows from the analysis of this section, and also the analysis of section 5, that each of these automorphisms is conjugate to a type 2a involutive automorphism $\{U(t), 1, \xi\}$, where

$$U(t) = \text{dsum} \{t\mathbf{1}_a, -\mathbf{1}_b, \mathbf{1}_c, t\mathbf{1}_c, -t\mathbf{1}_b, \mathbf{1}_a\}. \quad (7.8.36)$$

Define the conjugacy class $(J)^{(q,r)}$ to be that conjugacy class which contains the type 2a involutive automorphism $\{U(t), 1, \xi\}$, where $U(t) = \text{dsum} \{t\mathbf{1}_p, -\mathbf{1}_q, \mathbf{1}_r, t\mathbf{1}_r, -t\mathbf{1}_q, \mathbf{1}_p\}$ and of course, $p + q + r = \ell$. We take the type 2a involutive automorphism $\{U(t), 1, \xi\}$ to be the representative automorphism of the conjugacy class $(J)^{(q,r)}$. Thus, the following elements generate a real form of $D_\ell^{(1)}$, and are obtained by using Cartan's method:

$$\begin{aligned} ih_{\alpha_k} & \quad (1 \leq k \leq \ell, k \neq p), \\ i\left(h_{\alpha_p} + \frac{1}{2}c\right), \end{aligned} \quad (7.8.37)$$

$$\begin{aligned} \left. \begin{aligned} & \left(e_{j\delta}^k + e_{-j\delta}^k \right) \\ & i\left(e_{j\delta}^k - e_{-j\delta}^k \right) \end{aligned} \right\} 1 \leq k \leq \ell, j \geq 0, \end{aligned} \quad (7.8.38)$$

$$\begin{aligned} & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right), \\ & i\left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right), \\ & i\left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right), \\ & \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right), \end{aligned} \quad (7.8.39)$$

(where, in (7.8.39) above $\alpha = \varepsilon_a - \varepsilon_b$ with $1 \leq a < b \leq p$, $p < a < b \leq p+q$ or $p+q < a < b \leq \ell$, and also $\alpha = \varepsilon_a + \varepsilon_b$ with $1 \leq a \leq p$, $p+q < b \leq \ell$),

$$\begin{aligned}
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}), \\
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}), \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}), \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}),
\end{aligned} \tag{7.8.40}$$

(where, in (7.8.40) above $\alpha = \varepsilon_a - \varepsilon_b$ with $1 \leq a \leq p < b \leq p+q$),

$$\begin{aligned}
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}), \\
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}), \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}), \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}),
\end{aligned} \tag{7.8.41}$$

(where in (7.8.41) above $\alpha = \varepsilon_a - \varepsilon_b$ with $1 \leq a \leq p, p+q < b \leq \ell$ and $1 \leq a < b \leq p$),

$$\begin{aligned}
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \\
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}),
\end{aligned} \tag{7.8.42}$$

(where in (7.8.42) above $\alpha = \varepsilon_a - \varepsilon_b$ with $p < a \leq p+q < b \leq \ell$ and $\alpha = \varepsilon_a + \varepsilon_b$ with $1 \leq a \leq p < b \leq p+q$),

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right),
\end{aligned} \tag{7.8.43}$$

(where, in (7.8.43) above $\alpha = \varepsilon_a + \varepsilon_b$ with either $p < a < b \leq p+q$ or $p+q < a < b \leq \ell$),

$$\begin{aligned}
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right),
\end{aligned} \tag{7.8.44}$$

(where in (7.8.44) above $\alpha = \varepsilon_a + \varepsilon_b$ with $p < a \leq p+q < b \leq \ell$),

c,

$$d - \frac{\ell(\ell-1)}{4} \left\{ \sum_{j=1}^p j h_{\alpha_j} + \sum_{\substack{j=1 \\ p \neq \ell-1}}^{\ell-2-p} (p-j) h_{\alpha_{p+j}} \right. \\
\left. + \frac{1}{2}(2p+2-\ell) \{ h_{\alpha_{\ell-1}} + h_{\alpha_\ell} \} \right\}. \tag{7.8.45}$$

The last element in this list is only applicable when $p \neq \ell-1, \ell$. For the special cases $p = \ell-1, \ell$, this element should be replaced with

$$\begin{aligned}
d - \frac{\ell(\ell-1)}{4} \left\{ \sum_{j=1}^{\ell-2} j h_{\alpha_j} + \frac{1}{2} \ell h_{\alpha_{\ell-1}} + \frac{1}{2} (\ell-2) h_{\alpha_\ell} \right\} & \text{ (for } p = \ell-1), \\
d - \frac{\ell(\ell-1)}{4} \left\{ \sum_{j=1}^{\ell-2} j h_{\alpha_j} + \frac{1}{2} (\ell-2) h_{\alpha_{\ell-1}} + \frac{1}{2} \ell h_{\alpha_\ell} \right\} & \text{ (for } p = \ell).
\end{aligned} \tag{7.8.46}$$

The type 2a involutive automorphisms which are generated by the matrices of the form given in (7.8.4) present no difficulties. Let $U(t)$ be of the form (7.8.4).

Consider $U(-t)$, which is of the form (7.8.3). Since the type 2a involutive automorphism $\{U(t), 1, \xi\}$ is always conjugate to the type 2a involutive automorphism $\{U(-t), 1, \xi\}$, it follows that the automorphisms under consideration belong to classes which have already been identified.

The type 2a involutive automorphisms which are generated by matrices of the form (7.8.5) are examined separately, depending upon whether or not ℓ is even or odd. In the case where ℓ is even, it follows that the automorphism in question belongs to one of the classes identified previously, namely $(G)^{(\frac{1}{2}\ell)}$. This is because $K_{2\ell}$ is t -independent, and the transformations given in section 5 thus apply without any modification, implying that the type 2a involutive automorphisms $\{K_{2\ell}, 1, 0\}$ and $\left\{dsum\left\{1_{\frac{1}{2}\ell}, -1_{\ell}, 1_{\frac{1}{2}\ell}\right\}, 1, 0\right\}$ are conjugate. In the case where ℓ is odd, this is not quite the case. The main question to be considered is whether or not there exists a Laurent polynomial matrix $S(t)$ which is such that

$$S(t)K_{2\ell}S(t^{-1})^{-1} = \lambda t^{\mu} U(t), \quad (7.8.47)$$

where $U(t)$ is given by one of (7.8.1)-(7.8.4). In fact, from previous analysis, we know that, when ℓ is odd, the two type 2a involutive automorphisms $\{K_{2\ell}, 1, 0\}$ and $\left\{dsum\left\{1_{\frac{1}{2}(\ell-1)}, -1_{\frac{1}{2}(\ell-1)}, K_2, -1_{\frac{1}{2}(\ell-1)}, 1_{\frac{1}{2}(\ell-1)}\right\}, 1, 0\right\}$ are conjugate. The matrix

$$dsum\left\{1_{\frac{1}{2}(\ell-1)}, -1_{\frac{1}{2}(\ell-1)}, K_2, -1_{\frac{1}{2}(\ell-1)}, 1_{\frac{1}{2}(\ell-1)}\right\} \quad (7.8.48)$$

is, in fact a special case of the matrix given in (7.8.6). We will thus proceed with the involutive automorphisms which are generated by matrices of the general form (7.8.6). It is clear from earlier analysis in this section that we may confine our attention to those automorphisms for which the associated matrices $U(t)$ are given by

$$dsum\left\{1_a, -1_b, t1_c, K_2, t^{-1}1_c, -1_b, 1_a\right\}. \quad (7.8.49)$$

We wish firstly to investigate whether or not these automorphisms belong to any of the conjugacy classes which have been identified already. Let us suppose then that there exists some Laurent polynomial matrix $\mathbf{S}(t)$ which satisfies

$$\mathbf{S}(t)\mathbf{U}(st^{\pm 1})\mathbf{S}(t^{\pm 1})^{-1} = \lambda t^\mu \mathbf{V}(t), \quad (7.8.50)$$

$$\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = \alpha t^\beta \mathbf{g}, \quad (7.8.51)$$

where α and λ are non-zero complex numbers, and $\mathbf{V}(t)$ is given by one of (7.8.1)-(7.8.4). Note that the quantity s may take only the values ± 1 . It is always possible to choose the value of t such that $st = 1$, by putting $t = s$. Let us, therefore, put $t = s$ into the above equations. We can now deduce from the analysis of the type 1a involutive automorphisms (with $u = 1$) that there does not exist a matrix $\mathbf{S}(s)$ which satisfies both of

$$\mathbf{S}(s)\mathbf{U}(1)\mathbf{S}(s)^{-1} = \lambda' \mathbf{V}(s), \quad (7.8.52)$$

$$\tilde{\mathbf{S}}(s)\mathbf{g}\mathbf{S}(s) = \alpha' \mathbf{g}. \quad (7.8.53)$$

We require that $\mathbf{S}(t)$ exists for all non-zero values of t , and so it has been demonstrated that $\mathbf{S}(t)$ does not exist with the necessary properties. Next we examine whether or not there exist any matrices $\mathbf{S}(t)$ which satisfy

$$\mathbf{S}(t)\mathbf{U}_{b,c}\mathbf{S}(t^{-1})^{-1} = \lambda t^\mu \mathbf{U}_{p,q}, \quad (7.8.54)$$

$$\tilde{\mathbf{S}}(t)\mathbf{g}\mathbf{S}(t) = \alpha t^\beta \mathbf{g}, \quad (7.8.55)$$

where the quantities α and λ are both non-zero, and

$$\mathbf{U}_{y,z} = \text{dsum} \{ \mathbf{1}_x, -\mathbf{1}_y, t\mathbf{1}_z, \mathbf{K}_2, t^{-1}\mathbf{1}_z, -\mathbf{1}_y, \mathbf{1}_x \}. \quad (7.8.56)$$

Recall that the type 2a automorphism $\{\mathbf{U}(t), u, \xi\}$ and the type 2a automorphism $\{\mathbf{U}(s, t), u, \xi'\}$ (where $s_1^2 = 1$) are conjugate. We may thus put $t = s$, and infer that

$$\mathbf{S}(s)\mathbf{V}_1\mathbf{S}(s)^{-1} = \lambda' \mathbf{V}_2, \quad (7.8.57)$$

where \mathbf{V}_1 and \mathbf{V}_2 are given by

$$\mathbf{V}_1 = \text{dsum} \left\{ \mathbf{1}_{\ell-1-b-c}, -\mathbf{1}_b, \mathbf{1}_c, \mathbf{K}_2, \mathbf{1}_c, -\mathbf{1}_b, \mathbf{1}_{\ell-1-b-c} \right\}, \quad (7.8.58)$$

$$\mathbf{V}_2 = \text{dsum} \left\{ \mathbf{1}_{\ell-1-p-q}, -\mathbf{1}_p, \mathbf{1}_q, \mathbf{K}_2, \mathbf{1}_q, -\mathbf{1}_p, \mathbf{1}_{\ell-1-p-q} \right\}. \quad (7.8.59)$$

It may be seen, if we examine section 6 of this chapter, that there cannot exist a matrix $\mathbf{S}(s)$ which (for $s^2 = 1$) satisfies

$$\mathbf{S}(s)\mathbf{V}_1\mathbf{S}(s)^{-1} = \lambda'\mathbf{V}_2, \quad (7.8.60)$$

and is also such that $\tilde{\mathbf{S}}(s)\mathbf{g}\mathbf{S}(s) = \alpha\mathbf{g}$. We conclude therefore, that there is no matrix $\mathbf{S}(t)$, and that the type 2a involutive automorphisms $\{\mathbf{U}_{b,c}, 1, \xi_1\}$ and $\{\mathbf{U}_{p,q}, 1, \xi_2\}$ are non-conjugate. Let the class containing the type 2a involution $\{\mathbf{U}_{b,c}, 1, \xi_1\}$ be called $(\mathbf{K})^{(b,c)}$. The integers b and c are such that $0 \leq b \leq \left[\frac{1}{2}(\ell-1)\right]$ and $0 \leq c \leq \left[\frac{1}{2}(\ell-1-b)\right]$. (For those classes where b and c lie outside these ranges, the various matrix transformations outlined in this and previous sections imply that the classes in question are conjugate to the classes for which b and c do lie within these ranges). Cartan's method may be applied, and a real form of $\mathbf{D}_\ell^{(1)}$ generated. For the conjugacy class $(\mathbf{K})^{(q,r)}$, we take as the representative the type 2a involutive automorphism $\{\mathbf{U}_{q,r}, 1, \xi\}$, where $\mathbf{U}_{q,r} = \text{dsum} \left\{ \mathbf{1}_p, -\mathbf{1}_q, t\mathbf{1}_r, \mathbf{K}_2, t^{-1}\mathbf{1}_r, -\mathbf{1}_q, \mathbf{1}_p \right\}$ and $p+q+r = \ell-1$. The real form obtained using Cartan's method may be generated by the following elements:

$$\begin{aligned} ih_{\alpha_k} & \quad (1 \leq k \leq \ell-2, k \neq \ell-r-2), \\ i\left(h_{\alpha_{\ell-r-2}} - \frac{1}{2}c\right) & \quad (r \neq 0), \end{aligned} \quad (7.8.61)$$

$$\begin{aligned} i\left(h_{\alpha_{\ell-1}} + h_{\alpha_\ell} + c\right) & \quad (r \neq 0), \\ i\left(h_{\alpha_{\ell-1}} + h_{\alpha_\ell}\right) & \quad (r = 0), \end{aligned} \quad (7.8.62)$$

$$\left(h_{\alpha_{\ell-1}} - h_{\alpha_\ell}\right), \quad (7.8.63)$$

$$\left. \begin{aligned} \left(e_{j\delta}^k + e_{-j\delta}^k\right) \\ \left(e_{j\delta}^k - e_{-j\delta}^k\right) \end{aligned} \right\} 1 \leq k \leq \ell-2, j > 0, \quad (7.8.64)$$

$$\left. \begin{aligned} & (e_{j\delta}^{\ell-1} + e_{-j\delta}^{\ell-1} + e_{j\delta}^{\ell} + e_{-j\delta}^{\ell}) \\ & (e_{j\delta}^{\ell-1} - e_{-j\delta}^{\ell-1} + e_{j\delta}^{\ell} - e_{-j\delta}^{\ell}) \end{aligned} \right\} j > 0, \quad (7.8.65)$$

$$\left. \begin{aligned} & i(e_{j\delta}^{\ell-1} + e_{-j\delta}^{\ell-1} - e_{j\delta}^{\ell} - e_{-j\delta}^{\ell}) \\ & i(e_{j\delta}^{\ell-1} - e_{-j\delta}^{\ell-1} - e_{j\delta}^{\ell} + e_{-j\delta}^{\ell}) \end{aligned} \right\} j > 0, \quad (7.8.66)$$

$$\left. \begin{aligned} & (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \\ & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\ & i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\ & (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \end{aligned} \right\} \quad (7.8.67)$$

(where, in (7.8.67) above $\alpha = \varepsilon_a \pm \varepsilon_b$ (with one of $1 \leq a < b \leq p$ or $p < a < b \leq p+q$), and also $\alpha = \varepsilon_a - \varepsilon_b$ (with $p+q < a < b \leq \ell-2$)),

$$\left. \begin{aligned} & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \\ & (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\ & (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\ & i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \end{aligned} \right\} \quad (7.8.68)$$

(where, in (7.8.68) above $\alpha = \varepsilon_a \pm \varepsilon_b$ (with $1 \leq a \leq p < b \leq p+q$)),

$$\left. \begin{aligned} & (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha}), \\ & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha}), \\ & i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha}), \\ & (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha}), \end{aligned} \right\} \quad (7.8.69)$$

(where, in (7.8.69) above $\alpha = \varepsilon_a - \varepsilon_b$ (with $1 \leq a \leq p, p+q < b \leq \ell-2$)),

$$\begin{aligned}
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j+1)\delta+\alpha} - e_{(j+1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j+1)\delta+\alpha} + e_{(j+1)\delta-\alpha} \right),
\end{aligned} \tag{7.8.70}$$

(where, in (7.8.70) above $\alpha = \varepsilon_a - \varepsilon_b$ (with $p < a \leq p+q < b \leq \ell-2$)),

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right),
\end{aligned} \tag{7.8.71}$$

(where, in (7.8.71) above $\alpha = \varepsilon_a + \varepsilon_b$ (with $1 \leq a \leq p, p+q < b \leq \ell-2$)),

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right),
\end{aligned} \tag{7.8.72}$$

(where, in (7.8.72) above $\alpha = \varepsilon_a + \varepsilon_b$ (with $p < a \leq p+q < b \leq \ell-2$)),

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} + e_{-(j-2)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} - e_{-(j-2)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} - e_{-(j-2)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} + e_{-(j-2)\delta-\alpha} \right),
\end{aligned} \tag{7.8.73}$$

(where, in (7.8.73) above $\alpha = \varepsilon_a + \varepsilon_b$ (with $p+q < a < b \leq \ell-2$)),

$c,$

$$d - \frac{1}{4} \left\{ \sum_{j=1}^{r-1} j h_{\alpha_{t-r+1+j}} + \frac{1}{2}(1+r) \{ h_{\alpha_{t-1}} + h_{\alpha_t} \} \right\} \quad (r > 0), \tag{7.8.74}$$

$$d \quad (r = 0).$$

This concludes the analysis of the involutive automorphisms and real forms of the affine complex Kac-Moody algebra $D_\ell^{(1)}$ (where it is assumed that $\ell \geq 5$). For the first time, the method we have been employing has encountered difficulties which make the results inconclusive. It is not necessarily the case that the different classes which have been identified are all non-conjugate. That is, an approach using solely the given matrix formulation has limitations that were not evident for earlier algebras which were studied in the same manner. If we were to go on purely empirical evidence, then there are certain pairs of automorphisms which are non-conjugate, although it is difficult to show conclusively that this is the case. It appears to be the case that, for certain affine complex Kac-Moody algebras (and for certain representations of their corresponding simple Lie algebras), the best result that can be obtained is an upper limit on the number of conjugacy classes of the involutive Cartan-preserving automorphisms (within the group of all automorphisms of the complex affine Kac-Moody algebra concerned).

8 Involutive automorphisms and real forms of $D_4^{(1)}$

8.1 Differences between $D_4^{(1)}$ and the general case $D_\ell^{(1)}$ ($\ell \geq 5$)

This chapter will examine the involutive automorphisms of the complex affine Kac-Moody algebra. The reason for the inclusion of $D_4^{(1)}$ in its own chapter is the fact that the Dynkin diagram of D_4 is markedly different from the Dynkin diagrams of the other Lie algebras $D_\ell^{(1)}$. This means that the group of rotations of the roots of D_4 is rather different from the corresponding group when $\ell \neq 4$. Each Cartan-preserving automorphism corresponds, as is well-known, to a rotation of the roots of the corresponding simple Lie algebra, in this case D_4 . The group R of rotations of the roots of D_4 has the semi-direct product structure

$$R = \mathcal{W} \ltimes \mathcal{S}, \quad (8.1.1)$$

where \mathcal{W} is the Weyl group of D_4 and \mathcal{S} is the group specified by

$$\mathcal{S} = \{1, \tau_1, \tau_3, \tau_4\}, \quad (8.1.2)$$

where the rotations $\{\tau_j\}_{j=1,3,4}$ are defined by the following:

$$\left. \begin{aligned} \tau_j(\alpha_j^0) &= \alpha_j^0 \\ \tau_j(\alpha_2^0) &= \alpha_2^0 \\ \tau_j(\alpha_p^0) &= \alpha_r^0 \end{aligned} \right\} j \in \{1, 3, 4\}; p, r \notin \{j, 2\}; p \neq r; 1 \leq p, r \leq 4. \quad (8.1.3)$$

In fact, a generating set for R is the set S , where $S = \{S_{\alpha_1^0}^0, S_{\alpha_2^0}^0, \tau_1, \tau_3\}$. These generators are such that each fixes pointwise a hyperplane in the four-dimensional space that is spanned by $\{\alpha_j\}_{j=1}^{j=4}$, and also sends a non-zero vector (orthogonal to this hyperplane) to its negative. Thus, the

generating elements are all “reflections” in the sense used by Humphreys (see [20]), and $\{R, S\}$ is a Coxeter system with the Coxeter graph

$$\begin{array}{c}
 4 \\
 \circ \text{---} \circ \text{---} \circ \text{---} \circ \\
 S_{\alpha_2} \quad S_{\alpha_1} \quad \tau_1 \quad \tau_3
 \end{array} \tag{8.1.4}$$

The algorithm developed by Richardson [29] shows that there are 8 \mathcal{W} -equivalent subsets of S that satisfy the “(-1)-condition”. Hence there are 8 conjugacy classes of involutions within R and representatives $\{\rho_j\}_{j=1}^{j=8}$ of these are given below.

$$\rho_1(\alpha_k^0) = \alpha_k^0 \quad (1 \leq k \leq 4), \tag{8.1.5}$$

$$\rho_2(\alpha_k^0) = -\alpha_k^0 \quad (1 \leq k \leq 4), \tag{8.1.6}$$

$$\begin{aligned}
 \rho_3(\alpha_1^0) &= \alpha_1^0 + \alpha_2^0, \\
 \rho_3(\alpha_2^0) &= -\alpha_2^0, \\
 \rho_3(\alpha_3^0) &= \alpha_2^0 + \alpha_3^0, \\
 \rho_3(\alpha_4^0) &= \alpha_2^0 + \alpha_4^0,
 \end{aligned} \tag{8.1.7}$$

$$\begin{aligned}
 \rho_4(\alpha_1^0) &= -\alpha_1^0, \\
 \rho_4(\alpha_2^0) &= \alpha_1^0 + \alpha_2^0 + \alpha_3^0 + \alpha_4^0, \\
 \rho_4(\alpha_3^0) &= -\alpha_3^0, \\
 \rho_4(\alpha_4^0) &= -\alpha_4^0,
 \end{aligned} \tag{8.1.8}$$

$$\begin{aligned}
 \rho_5(\alpha_1^0) &= \alpha_3^0, \\
 \rho_5(\alpha_2^0) &= \alpha_2^0, \\
 \rho_5(\alpha_3^0) &= \alpha_1^0, \\
 \rho_5(\alpha_4^0) &= \alpha_4^0,
 \end{aligned}
 \tag{8.1.9}$$

$$\begin{aligned}
 \rho_6(\alpha_1^0) &= \alpha_2^0 + \alpha_3^0, \\
 \rho_6(\alpha_2^0) &= -\alpha_2^0, \\
 \rho_6(\alpha_3^0) &= \alpha_1^0 + \alpha_2^0, \\
 \rho_6(\alpha_4^0) &= \alpha_2^0 + \alpha_4^0,
 \end{aligned}
 \tag{8.1.10}$$

$$\begin{aligned}
 \rho_7(\alpha_1^0) &= -\alpha_1^0, \\
 \rho_7(\alpha_2^0) &= \alpha_1^0 + \alpha_2^0 + \alpha_3^0, \\
 \rho_7(\alpha_3^0) &= -\alpha_3^0, \\
 \rho_7(\alpha_4^0) &= \alpha_4^0,
 \end{aligned}
 \tag{8.1.11}$$

$$\begin{aligned}
 \rho_8(\alpha_1^0) &= -\alpha_1^0, \\
 \rho_8(\alpha_2^0) &= -\alpha_2^0, \\
 \rho_8(\alpha_3^0) &= -\alpha_3^0, \\
 \rho_8(\alpha_4^0) &= \alpha_1^0 + 2\alpha_2^0 + \alpha_3^0 + \alpha_4^0.
 \end{aligned}
 \tag{8.1.12}$$

8.2 The involutive automorphisms of $D_4^{(1)}$

It may be easily verified that there does not exist any non-singular matrix $U(t)$ which satisfies the following set of conditions:

$$\begin{aligned}
 U(t)h_{\alpha_1}^0 U(t)^{-1} &= h_{\alpha_3}^0, \\
 U(t)h_{\alpha_2}^0 U(t)^{-1} &= h_{\alpha_2}^0, \\
 U(t)h_{\alpha_3}^0 U(t)^{-1} &= h_{\alpha_1}^0, \\
 U(t)h_{\alpha_4}^0 U(t)^{-1} &= h_{\alpha_4}^0.
 \end{aligned}
 \tag{8.2.1}$$

Thus, there are no automorphisms of $D_4^{(1)}$ that correspond to the rotation ρ_5 , or to any other involution in \mathcal{R} that is conjugate to it. Similarly, there are no automorphisms of $D_4^{(1)}$ that correspond to the rotations ρ_6 , ρ_7 or ρ_8 . This leaves only the rotations ρ_j , where $1 \leq j \leq 4$. It is possible, for each of these rotations, to find the most general type of Laurent polynomial matrix $U(t)$ such that

$$U(t)h_{\alpha_k}^0 U(t)^{-1} = h_{\rho_j(\alpha_k)}^0 \quad (1 \leq k \leq 4).
 \tag{8.2.2}$$

Then, we may find those matrices of each of these four forms that satisfy $U(t)^2 = \eta t^k \mathbf{1}_8$ (where η is a non-zero complex number and k is an integer). The automorphisms corresponding to the triples $\{U(t), 1, \xi\}$ (where $U(t)^2 = \eta t^k \mathbf{1}_8$) form a set of type 1a involutive automorphisms with $u = 1$ to which all other such involutive automorphisms are conjugate. However, the matrices $U(t)$ yielded by this process are all of the form studied in the previous chapter. The same is true for the type 1a involutive automorphisms with $u = -1$ and the type 2a involutive automorphisms with $u = 1$. This implies that the involutive automorphisms of $D_4^{(1)}$ do not, in fact, need to be studied apart from the involutive automorphisms of $D_\ell^{(1)}$. The analysis of the preceding chapter is also applicable for the special case $\ell = 4$.

9 Involutive automorphisms and real forms of $G_2^{(1)}$

9.1 Introduction

The Kac-Moody algebra in question is the one whose associated finite-dimensional simple Lie algebra is the exceptional algebra G_2 . The fundamental representation $\Gamma\{0,1\}$ is of lowest dimension, namely seven, and a realisation of G_2 may be obtained by following the procedure given in [13] (chapter 16). This realisation is given by the following, which are equations (9.1.1):

$$\begin{aligned} \mathbf{h}_{\alpha_1}^0 &= \frac{1}{8} \text{diag}\{0, 1, -1, 0, 1, -1, 0\}, \\ \mathbf{h}_{\alpha_2}^0 &= \frac{1}{24} \text{diag}\{1, -1, 2, 0, -2, 1, -1\}, \\ \mathbf{e}_{\alpha_1}^0 &= \frac{1}{\sqrt{8}} \{\mathbf{e}_{2,3} + \mathbf{e}_{5,6}\}, \\ \mathbf{e}_{\alpha_2}^0 &= \frac{1}{\sqrt{24}} \{\mathbf{e}_{1,2} + \sqrt{2}\mathbf{e}_{3,4} + \sqrt{2}\mathbf{e}_{4,5} + \mathbf{e}_{6,7}\}, \\ \mathbf{e}_{\alpha_1^0 + \alpha_2^0}^0 &= \frac{1}{\sqrt{24}} \{\mathbf{e}_{1,3} - \sqrt{2}\mathbf{e}_{2,4} + \sqrt{2}\mathbf{e}_{4,6} - \mathbf{e}_{5,7}\}, \\ \mathbf{e}_{\alpha_1^0 + 2\alpha_2^0}^0 &= \frac{1}{\sqrt{24}} \{-\sqrt{2}\mathbf{e}_{1,4} + \mathbf{e}_{2,5} + \mathbf{e}_{3,6} - \sqrt{2}\mathbf{e}_{4,7}\}, \\ \mathbf{e}_{\alpha_1^0 + 3\alpha_2^0}^0 &= \frac{1}{\sqrt{8}} \{\mathbf{e}_{1,5} - \mathbf{e}_{3,7}\}, \\ \mathbf{e}_{2\alpha_1^0 + 3\alpha_2^0}^0 &= \frac{1}{\sqrt{8}} \{\mathbf{e}_{1,6} + \mathbf{e}_{2,7}\}. \end{aligned}$$

In the above, it is assumed that, for negative roots

$$\mathbf{e}_{-\alpha} = -\tilde{\mathbf{e}}_{-\alpha}^* \quad (9.1.2)$$

The matrix $\mathbf{e}_{j,k}$ is that 7×7 matrix whose only non-zero entry is in the j th row and the k th column, and which takes the value 1. It may be seen that, if \mathbf{a} is a member of this matrix realisation of G_2 , then it satisfies

$$\tilde{\mathbf{a}}\mathbf{g} + \mathbf{g}\mathbf{a} = \mathbf{0}, \quad (9.1.3)$$

where \mathbf{g} is the matrix defined by

$$\mathbf{g} = \text{offdiag}\{1, -1, 1, -1, 1, -1, 1\}. \quad (9.1.4)$$

However, the converse is not always true. That is, there are matrices \mathbf{a} which satisfy the above condition, but which are not members of this matrix realisation. The Kac-Moody algebra $G_2^{(1)}$ is constructed from the Lie algebra in the usual way, so that each element of $G_2^{(1)}$ has a matrix part $\mathbf{a}(t)$ given by

$$\mathbf{a}(t) = \sum_{j \in \mathbb{Z}} t^j \otimes \mathbf{a}_j \quad (\mathbf{a}_j \in \Gamma(G_2)), \quad (9.1.5)$$

where only a finite number of the matrices $\{\mathbf{a}_j\}_{j \in \mathbb{Z}}$ are non-zero. (The matrices \mathbf{a}_j are the representatives of the elements of G_2 under the representation Γ , specified in (9.1.1)). Thus, a member of the Kac-Moody algebra $G_2^{(1)}$ may be expressed in the form

$$\mathbf{a}(t) + \lambda c + \mu d, \quad (9.1.6)$$

where λ, μ are arbitrary complex numbers.

It is important to know which matrices $\mathbf{S}(t)$ generate automorphisms of $G_2^{(1)}$. At present, there is only one condition which the matrix parts of members of the Kac-Moody algebra must satisfy. In fact, this condition is identical to the condition which matrix parts of elements of $B_3^{(1)}$ must satisfy. It follows therefore, from the analysis of $B_\ell^{(1)}$ (with particular reference to the special case $\ell = 3$), that a necessary condition for $\mathbf{S}(t)$ to generate an automorphism of $G_2^{(1)}$ is that

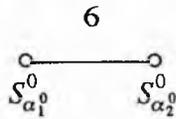
$$\tilde{\mathbf{S}}(t) \mathbf{g} \mathbf{S}(t) = \lambda t^\mu \mathbf{g} \quad (\lambda \neq 0). \quad (9.1.7)$$

This is not a sufficient condition however, since the original condition on the matrix parts of members was not a sufficient condition. Note that the members \mathbf{a} of the matrix realisation of G_2 all satisfy

$$-\tilde{\mathbf{a}} = \mathbf{g} \mathbf{a} \mathbf{g}^{-1}, \quad (9.1.8)$$

so that this representation (the explicit matrix realisation) is equivalent to its own contragredient representation. Thus, the type 1b automorphisms coincide with the type 1a automorphisms, and the type 2b automorphisms all coincide with the type 2a automorphisms. Attention may thus be confined to the type 1a and type 2a automorphisms.

The group \mathcal{R} of rotations of the roots of G_2 coincides with \mathcal{W} , the Weyl group of G_2 . The Coxeter graph of \mathcal{W} is given in the figure below.



The generating set \mathcal{S} is, in this case the set $\{S_{\alpha_1}^0, S_{\alpha_2}^0\}$. This Coxeter graph is positive definite of the type $I_2(6)$ (see Humphreys [20]). There are three non-trivial subsets of the set S which satisfy the (-1) -condition (as defined in [29]) and hence there are three conjugacy classes of order 2 rotations of the roots of G_2 . The three subsets of S in question are $\{S_{\alpha_1}^0\}$, $\{S_{\alpha_2}^0\}$ and $\{S_{\alpha_1}^0, S_{\alpha_2}^0\}$. Including the identity rotation, the conjugacy class representatives of the involutive rotations are given by

$$\tau_1(\alpha_k^0) = \alpha_k^0 \quad (1 \leq k \leq \ell), \tag{9.1.9}$$

$$\tau_2(\alpha_k^0) = -\alpha_k^0 \quad (1 \leq k \leq \ell), \tag{9.1.10}$$

$$\tau_3(\alpha_1^0) = -\alpha_1^0, \tag{9.1.11}$$

$$\tau_3(\alpha_2^0) = \alpha_1^0 + \alpha_2^0,$$

$$\tau_4(\alpha_1^0) = \alpha_1^0 + 3\alpha_2^0, \tag{9.1.12}$$

$$\tau_4(\alpha_2^0) = -\alpha_2^0.$$

It is known that there are only two non-isomorphic real forms of the simple Lie algebra G_2 . Naturally, one of these is the compact real form, which is denoted CG_2 , and a non-compact real form, which is denoted NG_2 . It follows therefore, that there are only two conjugacy classes of the involutive automorphisms of the compact real

form of G_2 (including the identity automorphism). Hence, all of the order-two automorphisms of the compact real form of G_2 are mutually conjugate, with only the identity being non-conjugate. It follows that, for the simple Lie algebra G_2 , every order two automorphism is conjugate to some automorphism which corresponds to the identity root transformation, but which is not the identity automorphism.

9.2 Type 1a involutive automorphisms of $G_2^{(1)}$ with $u = 1$

It has been remarked upon already that each automorphism of order two is conjugate to some involutive automorphism for which the corresponding rotation of the roots of G_2 is the identity rotation. The most general 7×7 Laurent polynomial matrix $U(t)$ such that

$$\begin{aligned} U(t)h_{\alpha_1}^0 U(t)^{-1} &= h_{\alpha_1}^0, \\ U(t)h_{\alpha_2}^0 U(t)^{-1} &= h_{\alpha_2}^0, \\ \tilde{U}(t)gU(t) &= \lambda t^\mu g, \end{aligned} \quad (9.2.1)$$

is the matrix which is given by

$$U(t) = \text{diag}\{\lambda_1 t^{\mu_1}, \lambda_2 t^{\mu_2}, \lambda_3 t^{\mu_3}, 1, \lambda_3^{-1} t^{-\mu_3}, \lambda_2^{-1} t^{-\mu_2}, \lambda_1^{-1} t^{-\mu_1}\}. \quad (9.2.2)$$

Not every matrix of this form is such that it generates an automorphism of $G_2^{(1)}$. If one considers the quantities

$$U(t)bU(t)^{-1}, \quad (9.2.3)$$

where b is the matrix representative of one of the basis elements of G_2 , then it becomes clear that the matrix $U(t)$ generates an automorphism of $G_2^{(1)}$ if, and only if

$$\lambda_1 = \lambda_2 \lambda_3 \quad \mu_1 = \mu_2 + \mu_3. \quad (9.2.4)$$

The involutiveness condition for type 1a automorphisms (with $u = 1$) is that the matrix $U(t)$ must satisfy $U(t)^2 = \alpha t^\beta \mathbf{1}_7$, where α is some non-zero complex number. Thus, the matrices $U(t)$ of the given general form which satisfy this condition are

$$\begin{aligned}
U_1(t) &= \mathbf{1}_7, \\
U_2(t) &= \text{diag}\{-1, 1, -1, 1, -1, 1, -1\}, \\
U_3(t) &= \text{diag}\{-1, -1, 1, 1, 1, -1, -1\}, \\
U_4(t) &= \text{diag}\{1, -1, -1, 1, -1, -1, 1\}.
\end{aligned} \tag{9.2.5}$$

It should be noticed that each of these matrices satisfies

$$\tilde{U}_j^*(t^{-1})U_j(t) = \mathbf{1}_7 \quad (1 \leq j \leq 4).$$

The type 1a automorphism $\{U_1(t), 1, 0\}$ is the identity automorphism, which belongs in its own conjugacy class, which is called (A). It will now be seen that the automorphisms generated by the other three of these matrices are mutually conjugate.

Firstly, let the matrix S be defined by

$$S = \text{dsum}\{\mathbf{1}_1, i\mathbf{K}_2, \mathbf{1}_1, i\mathbf{K}_2, \mathbf{1}_1\}. \tag{9.2.6}$$

This matrix is such that it satisfies

$$\begin{aligned}
\tilde{S}gS &= g, \\
\tilde{S}^*S &= \mathbf{1}_7, \\
SU_2S^{-1} &= U_2.
\end{aligned} \tag{9.2.7}$$

Moreover, it is easily verified that the matrix S does generate an automorphism of $G_2^{(1)}$ (and, consequently, of its compact real form). Let a second matrix V be defined by $V = \text{dsum}\{i\mathbf{K}_2, i\mathbf{K}_3, i\mathbf{K}_2\}$. This then satisfies

$$\begin{aligned}
VU_4V^{-1} &= U_2, \\
\tilde{V}gV &= -g, \\
\tilde{V}^*V &= \mathbf{1}_7.
\end{aligned} \tag{9.2.8}$$

Furthermore, the matrix V does generate an automorphism of $G_2^{(1)}$ and of its compact real form. The conjugacy class which contains all of the involutive automorphism considered so far (apart from the identity automorphism) will be called (B). There are

therefore, only two conjugacy classes of the type 1a involutions (with $u = 1$) that correspond to the identity rotation. Thus, there are only two conjugacy classes of type 1a involutive automorphisms with $u = 1$.

The conjugacy class (A) contains only the identity automorphism, as has been seen already. The real form of $G_2^{(1)}$ which corresponds to it is therefore the compact real form itself. The class (B) contains, amongst others, the representative automorphism ψ_B , where $\psi_B = \{U, 1, 0\}$ and $U = \text{diag}\{-1, 1, -1, 1, -1, 1\}$. A basis of the compact real form of $G_2^{(1)}$ exists in which each basis element is an eigenvector of ψ_B (with eigenvalues 1 or -1). Such a basis, together with the associated eigenvalues, is given by the elements

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \quad j \in \mathbf{N}^0 \\ i(e_{j\delta}^k - e_{-j\delta}^k) \quad j \in \mathbf{N} \end{array} \right\} 1 \leq k \leq 2, \text{ eigenvalue } 1, \quad (9.2.9)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} j \in \mathbf{Z}; \alpha = \alpha_1 + \alpha_2, \alpha_1 + 3\alpha_2; \text{ eigenvalue } 1, \quad (9.2.10)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} j \in \mathbf{Z}; \alpha = \alpha_1, \alpha_2, \alpha_1 + 2\alpha_2, 2\alpha_1 + 3\alpha_2; \text{ eigenvalue } -1, \quad (9.2.11)$$

$$\left. \begin{array}{l} ic \\ id \end{array} \right\} \text{ eigenvalue } 1. \quad (9.2.12)$$

Thus, applying Cartan's method to this basis, the basis of a real form of $G_2^{(1)}$ is obtained, consisting of the elements

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \quad j \in \mathbf{N}^0 \\ i(e_{j\delta}^k - e_{-j\delta}^k) \quad j \in \mathbf{N} \end{array} \right\} 1 \leq k \leq 2, \quad (9.2.13)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} j \in \mathbf{Z}; \alpha = \alpha_1 + \alpha_2, \alpha_1 + 3\alpha_2, \quad (9.2.14)$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} j \in \mathbf{Z}; \alpha = \alpha_1, \alpha_2, \alpha_1 + 2\alpha_2, 2\alpha_1 + 3\alpha_2, \quad (9.2.15)$$

$$ic \quad (9.2.16)$$

id.

The real form corresponding to the conjugacy class (A) is, as has been mentioned, the compact real form of $G_2^{(1)}$. This is generated from the real form CG_2 by the first method of generating real affine Kac-Moody algebras from the real forms of semi-simple Lie algebras. This real form may be denoted by $CG_{2(1)}$. The real form corresponding to the conjugacy class (B) is isomorphic to the real form generated from the non-compact real form of G_2 by the first method. This real form will be called $NG_{2(1)}$.

9.3 Type 1a involutive automorphisms of $G_2^{(1)}$ with $u = -1$

The involutive automorphisms corresponding to the identity root transformation are the only ones that will be examined here. The most general form of a 7×7 Laurent polynomial matrix $U(t)$ which is such that matrices of this form generate automorphisms of $G_2^{(1)}$, and which also satisfy

$$U(t)h_{\alpha_k}^0 U(t)^{-1} = h_{\alpha_k}^0 \quad (k=1,2), \quad (9.3.1)$$

$$U(t)U(-t) = \alpha t^\beta \mathbf{1}_7 \quad (\alpha \neq 0). \quad (9.3.2)$$

is the matrix $U(t)$ where

$$U(t) = \text{diag}\{\lambda_2 \lambda_3, \lambda_2, \lambda_3, 1, \lambda_3^{-1}, \lambda_2^{-1}, \lambda_2^{-1} \lambda_3^{-1}\} \quad (\lambda_2^2 = \lambda_3^2 = 1). \quad (9.3.3)$$

It may also be noted that all of the matrices of this general form satisfy $\tilde{U}^*(t^{-1})U(t) = \mathbf{1}_7$. In fact, all of the type 1a involutive automorphisms $\{U(t), -1, 0\}$ are conjugate (both within the group of all automorphisms of $G_2^{(1)}$ and within the group of automorphisms of its compact real form). We show this by firstly defining $S(t)$ by

$$S(t) = \text{diag}\{t^{s_2+s_3}, t^{s_2}, t^{s_3}, 1, t^{-s_3}, t^{-s_2}, t^{-s_2-s_3}\}, \quad (9.3.4)$$

where s_j is defined (for $j = 2, 3$) by

$$s_j = \begin{cases} 0 & (\lambda_j = 1) \\ 1 & (\lambda_j = -1). \end{cases} \quad (9.3.5)$$

This satisfies $\tilde{S}^*(t^{-1})S(t) = \mathbf{1}_7$ and also $S(t)\mathbf{1}_7 S(-t)^{-1} = U(t)$. This proves that all of the involutive automorphisms under consideration are mutually conjugate, belonging to a class which we shall call (C). It is easy to show that a basis of a real form of $G_2^{(1)}$

corresponding to the type 1a involutive automorphism $\{1_7, -1, 0\}$ is given by the following elements:

$$\left. \begin{array}{l} (e_{j\delta}^k + e_{-j\delta}^k) \\ i(e_{j\delta}^k - e_{-j\delta}^k) \end{array} \right\} 1 \leq k \leq \ell, j \text{ is even,} \quad (9.3.6)$$

$$\left. \begin{array}{l} i(e_{j\delta}^k + e_{-j\delta}^k) \\ (e_{j\delta}^k - e_{-j\delta}^k) \end{array} \right\} 1 \leq k \leq \ell, j \text{ is odd,} \quad (9.3.7)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ i(e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \alpha \text{ is a root of } G_2, j \text{ is even,} \quad (9.3.8)$$

$$\left. \begin{array}{l} i(e_{j\delta+\alpha} + e_{-j\delta-\alpha}) \\ (e_{j\delta+\alpha} - e_{-j\delta-\alpha}) \end{array} \right\} \alpha \text{ is a root of } G_2, j \text{ is odd,} \quad (9.3.9)$$

$$ic, \quad (9.3.10)$$

id.

This real form of the complex affine Kac-Moody algebra $G_2^{(1)}$ is obtained from the compact real form of the simple Lie algebra G_2 by the third method, and is thus the real form called $CG_{2(3)}$.

9.4 Type 2a involutive automorphisms of $G_2^{(1)}$ with $u = 1$

We examine those type 2a involutive automorphisms which correspond to the identity root transformation. The most general type 2a involutive automorphism with $u = 1$ that corresponds to the identity root transformation corresponds to the triple $\{\mathbf{U}(t), 1, \xi\}$, where $\mathbf{U}(t)$ is a 7×7 Laurent polynomial matrix of the form given by the following equation (numbered (9.4.1)):

$$\mathbf{U}(t) = \text{diag}\left\{\lambda_2\lambda_3t^{\mu_2+\mu_3}, \lambda_2t^{\mu_2}, \lambda_3t^{\mu_3}, 1, \lambda_3^{-1}t^{-\mu_3}, \lambda_2^{-1}t^{-\mu_2}, \lambda_2^{-1}\lambda_3^{-1}t^{-\mu_2-\mu_3}\right\},$$

where $\lambda_2^2 = \lambda_3^2 = 1$. We will now define a matrix $\mathbf{S}(t)$ such that $\mathbf{S}(t) = \text{diag}\left\{t^{-\sigma_2-\sigma_3}, t^{-\sigma_2}, t^{-\sigma_3}, 1, t^{\sigma_3}, t^{\sigma_2}, t^{\sigma_2+\sigma_3}\right\}$, where $\sigma_j = \frac{1}{2}(\text{deg}(\mu_j) - \mu_j)$ (for $j = 2, 3$). We also define the matrix $\mathbf{V}(t)$ to be the matrix obtained from $\mathbf{U}(t)$ by letting $\mu_j = \text{deg}(\mu_j)$ for $j = 2, 3$. These matrices satisfy

$$\mathbf{S}(t)\mathbf{V}(t)\mathbf{S}(t^{-1})^{-1} = \mathbf{U}(t), \quad (9.4.2)$$

$$\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_7, \quad (9.4.3)$$

and $\mathbf{S}(t)$ does generate automorphisms of $G_2^{(1)}$. We need only consider 16 automorphisms (since the quantities λ_2 , λ_3 , $\text{deg}(\mu_2)$ and $\text{deg}(\mu_3)$ each take one of two possible values). This number may be further reduced by noting a few relations which help to demonstrate that certain pairs of automorphisms are conjugate.

Recall firstly that the type 2a involutive automorphism $\{\mathbf{U}(t), 1, \xi\}$ and the type 2a involutive automorphism $\{\mathbf{U}(-t), 1, \xi'\}$ are conjugate. Certain pairs of matrices are connected in this way, and thus only one from each pair need be considered.

Secondly, let the matrix \mathbf{S} be given by $\mathbf{S} = \text{dsum}\{\mathbf{1}_1, i\mathbf{K}_2, \mathbf{1}_1, i\mathbf{K}_2, \mathbf{1}_1\}$. Then, if $\mathbf{U}(t)$ is a diagonal matrix, then

$$\mathbf{S}\mathbf{U}(t)\mathbf{S}^{-1} = \mathbf{U}'(t), \quad (9.4.4)$$

where $\mathbf{U}'(t)$ is obtained from $\mathbf{U}(t)$ by exchanging the second and third rows and columns, and also the fifth and sixth rows and columns. In addition, $\tilde{\mathbf{S}}^* \mathbf{S} = \mathbf{1}_7$.

Thirdly, let the matrix \mathbf{S} be given by $\mathbf{S} = \text{dsum}\{\mathbf{K}_2, \mathbf{K}_3, -\mathbf{K}_2\}$. We again take $\mathbf{U}(t)$ and $\mathbf{U}'(t)$ to be diagonal matrices, but this time with $\mathbf{U}'(t)$ being obtained from $\mathbf{U}(t)$ by exchanging the first and second rows and columns, the sixth and seventh rows and columns, and the third and fifth rows and columns. Once again

$$\mathbf{S}\mathbf{U}(t)\mathbf{S}^{-1} = \mathbf{U}'(t), \quad (9.4.5)$$

with $\tilde{\mathbf{S}}^* \mathbf{S} = \mathbf{1}_7$ also being satisfied.

These matrix transformations imply that we need only consider the type 2a involutive automorphisms which are generated by the following matrices:

$$\mathbf{U}_1(t) = \mathbf{1}_7, \quad (9.4.6)$$

$$\mathbf{U}_2(t) = \text{dsum}\{-\mathbf{1}_2, \mathbf{1}_3, -\mathbf{1}_2\}, \quad (9.4.7)$$

$$\mathbf{U}_3(t) = \text{dsum}\{t\mathbf{1}_2, \mathbf{1}_3, t^{-1}\mathbf{1}_2\}, \quad (9.4.8)$$

$$\mathbf{U}_4(t) = \text{diag}\{-t, t, -1, 1, -1, t^{-1}, -t^{-1}\}, \quad (9.4.9)$$

$$\mathbf{U}_5(t) = \text{diag}\{t^2, t, t, 1, t^{-1}, t^{-1}, t^{-2}\}, \quad (9.4.10)$$

$$\mathbf{U}_6(t) = \text{diag}\{-t^2, -t, t, 1, t^{-1}, -t^{-1}, -t^{-2}\}. \quad (9.4.11)$$

It is clear that the type 2a involutive automorphism $\{\mathbf{1}_7, 1, 0\}$ is not conjugate to the type 2a involutive automorphism $\{\mathbf{U}_j(t), 1, \xi\}$, where $2 \leq j \leq 6$. (A necessary condition would be that both $\mathbf{U}_j(1)$ and $\mathbf{U}_j(-1)$ were multiples of the 7×7 unit matrix, and this is obviously not the case). We define conjugacy classes (D), (E), (F), (G), (H) and (J) to be the conjugacy classes which contain the type 2a involutions generated by $\mathbf{U}_j(t)$, where j takes values 1, 2, 3, 4, 5 and 6 respectively. For each of these classes, we use

Cartan's theorem to obtain basis elements for the corresponding real forms. It does not appear to be possible to arrive at completely conclusive results using only the matrix formulation. In this case, the exceptional nature of the simple Lie algebra G_2 means that it is not possible to arrive at a convenient criterion for testing whether or not the matrix $U(t)$ generates an automorphism of $G_2^{(1)}$. It is possible to show that certain pairs of the type 2a involutive automorphisms with $u = 1$ generated by $U_j(t)$, where $1 \leq j \leq 6$ are not conjugate, although it will be presumed, in the absence of proof that any pairs of automorphisms are conjugate, that all of these automorphisms belong to different classes, namely (D), (E), (F), (G), (H), (J).

For the class (D), the corresponding real form is generated by the following elements:

$$\begin{aligned} & ih_{\alpha_k} \quad (1 \leq k \leq 2), \\ & (e_{j\delta}^k + e_{-j\delta}^k) \quad (1 \leq k \leq 2, j = 1, 2, \dots), \\ & (e_{j\delta}^k - e_{-j\delta}^k) \quad (1 \leq k \leq 2, j = 1, 2, \dots), \end{aligned} \quad (9.4.13)$$

$$\left. \begin{aligned} & (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \\ & i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ & i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}) \\ & (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}) \end{aligned} \right\} \alpha \in \Delta^0, j \in \mathbb{N}^0, \quad (9.4.14)$$

$$\begin{aligned} & c, \\ & d. \end{aligned} \quad (9.4.15)$$

The real form of $G_2^{(1)}$ which corresponds to the conjugacy class (E) has the following basis elements:

$$\begin{aligned} & ih_{\alpha_k} \quad (1 \leq k \leq 2), \\ & (e_{j\delta}^k + e_{-j\delta}^k) \quad (1 \leq k \leq 2, j = 1, 2, \dots), \\ & (e_{j\delta}^k - e_{-j\delta}^k) \quad (1 \leq k \leq 2, j = 1, 2, \dots), \end{aligned} \quad (9.4.16)$$

$$\left. \begin{aligned} & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right) \\ & i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right) \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right) \\ & \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right) \end{aligned} \right\} \alpha = \alpha_2, 2\alpha_1 + 3\alpha_2, \quad (9.4.17)$$

$$\left. \begin{aligned} & i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right) \\ & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right) \\ & \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right) \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right) \end{aligned} \right\} \alpha \in \Delta_+^0, \alpha \neq \alpha_2, 2\alpha_1 + 3\alpha_2, \quad (9.4.18)$$

$$\begin{aligned} & c, \\ & d. \end{aligned} \quad (9.4.19)$$

The real form of $G_2^{(1)}$ corresponding to the class (F) is generated by the following elements:

$$i \left(h_{\alpha_1} + \frac{1}{2}c \right), \quad (9.4.20)$$

$$ih_{\alpha_2},$$

$$\begin{aligned} & \left(e_{j\delta}^k + e_{-j\delta}^k \right) \quad (1 \leq k \leq \ell, j = 1, 2, \dots), \\ & i \left(e_{j\delta}^k - e_{-j\delta}^k \right) \quad (1 \leq k \leq \ell, j = 1, 2, \dots), \end{aligned} \quad (9.4.21)$$

$$\begin{aligned} & \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right), \\ & i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right), \\ & i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha} \right), \\ & \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha} \right), \end{aligned} \quad (9.4.22)$$

where $\alpha = \alpha_2$ and $j \in \mathbb{N}^0$,

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right),
\end{aligned} \tag{9.4.23}$$

where $\alpha = \alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2$ and $j \in \mathbb{N}^0$,

$$\begin{aligned}
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} + e_{(j-2)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} - e_{(j-2)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} - e_{(j-2)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} + e_{(j-2)\delta-\alpha} \right),
\end{aligned} \tag{9.4.24}$$

where $\alpha = 2\alpha_1 + 3\alpha_2$ and $j \in \mathbb{N}^0$,

$$\begin{aligned}
& c, \\
& d - 12 \left(\frac{2}{3} h_{\alpha_1} + h_{\alpha_2} \right).
\end{aligned} \tag{9.4.25}$$

The real form corresponding to the conjugacy class (G) is generated by the following elements:

$$i \left(h_{\alpha_1} + \frac{1}{2} c \right), \tag{9.4.26}$$

$$ih_{\alpha_2},$$

$$\begin{aligned}
& \left(e_{j\delta}^k + e_{-j\delta}^k \right) \quad (1 \leq k \leq \ell, j = 1, 2, \dots), \\
& i \left(e_{j\delta}^k - e_{-j\delta}^k \right) \quad (1 \leq k \leq \ell, j = 1, 2, \dots),
\end{aligned} \tag{9.4.27}$$

$$\begin{aligned}
& i \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha} \right), \\
& i \left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha} \right),
\end{aligned} \tag{9.4.28}$$

where $\alpha = \alpha_1, \alpha_1 + 2\alpha_2$ and $j \in \mathbb{N}^0$,

$$\begin{aligned}
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \\
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}),
\end{aligned} \tag{9.4.29}$$

where $\alpha = \alpha_2$ and $j \in \mathbf{N}^0$,

$$\begin{aligned}
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}), \\
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}), \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}), \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}),
\end{aligned} \tag{9.4.30}$$

where $\alpha = \alpha_1 + \alpha_2, \alpha_1 + 3\alpha_2$ and $j \in \mathbf{N}^0$,

$$\begin{aligned}
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} + e_{(j-2)\delta-\alpha}), \\
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} - e_{(j-2)\delta-\alpha}), \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} - e_{(j-2)\delta-\alpha}), \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} + e_{(j-2)\delta-\alpha}),
\end{aligned} \tag{9.4.31}$$

where $\alpha = 2\alpha_1 + 3\alpha_2$ and $j \in \mathbf{N}^0$,

$$\begin{aligned}
& c, \\
& d - 12\left(\frac{2}{3}h_{\alpha_1} + h_{\alpha_2}\right).
\end{aligned} \tag{9.4.32}$$

The real form of $G_2^{(1)}$ which corresponds to the conjugacy class (H) is generated by the following elements:

$$\begin{aligned}
& ih_{\alpha_1}, \\
& i\left(h_{\alpha_2} + \frac{1}{2}c\right),
\end{aligned} \tag{9.4.33}$$

$$\begin{aligned}
& (e_{j\delta}^k + e_{-j\delta}^k) \quad (1 \leq k \leq \ell, j = 1, 2, \dots), \\
& i(e_{j\delta}^k - e_{-j\delta}^k) \quad (1 \leq k \leq \ell, j = 1, 2, \dots),
\end{aligned} \tag{9.4.34}$$

$$\begin{aligned}
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}), \\
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}), \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}),
\end{aligned} \tag{9.4.35}$$

where $\alpha = \alpha_1$ and $j \in \mathbf{N}^0$,

$$\begin{aligned}
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}), \\
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}), \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}), \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}),
\end{aligned} \tag{9.4.36}$$

where $\alpha = \alpha_2, \alpha_1 + \alpha_2$ and $j \in \mathbf{N}^0$,

$$\begin{aligned}
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} + e_{(j-2)\delta-\alpha}), \\
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} - e_{(j-2)\delta-\alpha}), \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} - e_{(j-2)\delta-\alpha}), \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} + e_{(j-2)\delta-\alpha}),
\end{aligned} \tag{9.4.37}$$

where $\alpha = \alpha_1 + 2\alpha_2$ and $j \in \mathbf{N}^0$,

$$\begin{aligned}
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-3)\delta+\alpha} + e_{(j-3)\delta-\alpha}), \\
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-3)\delta+\alpha} - e_{(j-3)\delta-\alpha}), \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-3)\delta+\alpha} - e_{(j-3)\delta-\alpha}), \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-3)\delta+\alpha} + e_{(j-3)\delta-\alpha}),
\end{aligned} \tag{9.4.38}$$

where $\alpha = \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2$ and $j \in \mathbf{N}^0$,

$$c, \tag{9.4.39}$$

$$d - 12(h_{\alpha_1} + 2h_{\alpha_2}).$$

The real form of $G_2^{(1)}$ which corresponds to the conjugacy class (J) is generated by the following elements:

$$ih_{\alpha_1}, \tag{9.4.40}$$

$$i\left(h_{\alpha_2} + \frac{1}{2}c\right),$$

$$\left(e_{j\delta}^k + e_{-j\delta}^k\right) \quad (1 \leq k \leq \ell, j = 1, 2, \dots), \tag{9.4.41}$$

$$i\left(e_{j\delta}^k - e_{-j\delta}^k\right) \quad (1 \leq k \leq \ell, j = 1, 2, \dots),$$

$$i\left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-j\delta+\alpha} + e_{j\delta-\alpha}\right),$$

$$\left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-j\delta+\alpha} - e_{j\delta-\alpha}\right), \tag{9.4.42}$$

$$\left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-j\delta+\alpha} - e_{j\delta-\alpha}\right),$$

$$i\left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-j\delta+\alpha} + e_{j\delta-\alpha}\right),$$

where $\alpha = \alpha_1$ and $j \in \mathbb{N}^0$,

$$\left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}\right),$$

$$i\left(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}\right), \tag{9.4.42}$$

$$i\left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}\right),$$

$$\left(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}\right),$$

where $\alpha = \alpha_2$ and $j \in \mathbb{N}^0$,

$$\begin{aligned}
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}), \\
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}), \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-1)\delta+\alpha} - e_{(j-1)\delta-\alpha}), \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-1)\delta+\alpha} + e_{(j-1)\delta-\alpha}),
\end{aligned} \tag{9.4.43}$$

where $\alpha = \alpha_1 + \alpha_2$ and $j \in \mathbb{N}^0$,

$$\begin{aligned}
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} + e_{(j-2)\delta-\alpha}), \\
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} - e_{(j-2)\delta-\alpha}), \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-2)\delta+\alpha} - e_{(j-2)\delta-\alpha}), \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-2)\delta+\alpha} + e_{(j-2)\delta-\alpha}),
\end{aligned} \tag{9.4.44}$$

where $\alpha = \alpha_1 + 2\alpha_2$ and $j \in \mathbb{N}^0$,

$$\begin{aligned}
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-3)\delta+\alpha} + e_{(j-3)\delta-\alpha}), \\
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-3)\delta+\alpha} - e_{(j-3)\delta-\alpha}), \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-3)\delta+\alpha} - e_{(j-3)\delta-\alpha}), \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-3)\delta+\alpha} + e_{(j-3)\delta-\alpha}),
\end{aligned} \tag{9.4.45}$$

where $\alpha = \alpha_1 + 3\alpha_2$ and $j \in \mathbb{N}^0$,

$$\begin{aligned}
& (e_{j\delta+\alpha} + e_{-j\delta-\alpha} + e_{-(j-3)\delta+\alpha} + e_{(j-3)\delta-\alpha}), \\
& i(e_{j\delta+\alpha} + e_{-j\delta-\alpha} - e_{-(j-3)\delta+\alpha} - e_{(j-3)\delta-\alpha}), \\
& i(e_{j\delta+\alpha} - e_{-j\delta-\alpha} + e_{-(j-3)\delta+\alpha} - e_{(j-3)\delta-\alpha}), \\
& (e_{j\delta+\alpha} - e_{-j\delta-\alpha} - e_{-(j-3)\delta+\alpha} + e_{(j-3)\delta-\alpha}),
\end{aligned} \tag{9.4.46}$$

where $\alpha = 2\alpha_1 + 3\alpha_2$ and $j \in \mathbb{N}^0$,

$$c, \tag{9.4.47}$$

$$d - 12(h_{\alpha_1} + 2h_{\alpha_2}).$$

The real form corresponding to the class (D) is generated from the compact real form of G_2 by using the second method of generating real forms of Kac-Moody algebras from the real forms of semi-simple Lie algebras. That is, the real form corresponding to the conjugacy class (D) is $CG_{2(2)}$. Similarly, the real form corresponding to the conjugacy class (E) is generated from a non-compact real form of G_2 using the second method. Thus, the real form corresponding to the class (E) is $NG_{2(2)}$.

Conclusions

This analysis of the exceptional complex untwisted affine Kac-Moody algebra $G_2^{(1)}$ is such that it gives only an insight into the full results regarding the involutive automorphisms, their conjugacy classes within the group of all automorphisms, and the real forms of $G_2^{(1)}$ that may be generated from them. The chief problem which is encountered is the fact that there is no convenient matrix condition which the elements of the matrix realisation of G_2 satisfy, and which is both necessary and sufficient. This problem will be discussed in a subsequent chapter.

10 Twisted affine Kac-Moody algebras

10.1 Introduction and construction of the twisted algebras

It is useful to recall the construction of the affine twisted Kac-Moody algebras from the untwisted affine Kac-Moody algebras. Let $\tilde{\mathcal{L}}^{(0)}$ be a simple complex Lie algebra, and let $\tilde{\mathcal{L}}^{(1)}$ be the untwisted Kac-Moody algebra which is constructed from it (in the manner described in the first chapter). It will be assumed that $\tilde{\mathcal{L}}^{(0)}$ possesses an outer automorphism (which will be referred to as ψ_τ , and which corresponds to the rotation τ of the roots of $\tilde{\mathcal{L}}^{(0)}$). It is well-known that the Lie algebras A_1 , B_ℓ , C_ℓ , E_7 , E_8 , F_4 and G_2 possess no outer automorphisms, so that attention may be restricted to the remaining simple complex Lie algebras. In general the rotation τ is such that

$$\tau^q = \text{id}_{\text{map}}, \quad (10.1.1)$$

where id_{map} denotes the identity mapping, and q takes one of the values 2 or 3. (The value 3 is attainable only when $\tilde{\mathcal{L}}^{(0)} = D_4$). Hence

$$(\psi_\tau)^q = \text{id}_{\text{map}}. \quad (10.1.2)$$

The simple Lie algebra $\tilde{\mathcal{L}}^{(0)}$ has the decomposition

$$\tilde{\mathcal{L}}^{(0)} = \sum_{p \in \mathcal{L}_q} \tilde{\mathcal{L}}_p^{0(q)}, \quad (10.1.3)$$

where $\tilde{\mathcal{L}}_p^{0(q)}$ is the eigenspace of ψ_τ corresponding to the eigenvalue $e^{(2\pi i/pq)}$. That is, a belongs to $\tilde{\mathcal{L}}_p^{0(q)}$ if, and only if

$$\psi_\tau(a) = e^{(2\pi i/pq)} a, \quad (10.1.4)$$

and all of the eigenvalues of ψ_τ are of the form $e^{(2\pi i l/q)}$, where p is one of the integers modulo q . The linear space $\tilde{\mathcal{L}}^{(q)}$ may then be defined by

$$\tilde{\mathcal{L}}^{(q)} = \sum_{\substack{j=0 \\ j \bmod q = p}}^{q-1} (t^j \otimes \tilde{\mathcal{L}}_p^{0(q)}) \oplus Cc \oplus Cd. \quad (10.1.5)$$

This linear space is an affine Kac-Moody algebra, which is referred to as a twisted algebra because of the rotation τ (which may be thought of as twisting the roots of $\tilde{\mathcal{L}}^{(0)}$). A more thorough account of the construction of affine twisted Kac-Moody algebras is contained in [14], using the same notation and conventions (in general) as here. As was the case with the untwisted algebras, it is possible to define various quantities for the twisted algebras which are defined also for the semi-simple Lie algebras, and the untwisted algebras. It is not necessary to have any knowledge of them here, and the reader may consult [14]. It should be noticed that $\tilde{\mathcal{L}}^{(q)}$ is a proper subalgebra of the untwisted algebra $\tilde{\mathcal{L}}^{(1)}$. For this present analysis, let the compact real form $\mathcal{L}_C^{(q)}$ of $\tilde{\mathcal{L}}^{(q)}$ be defined to be the intersection of $\tilde{\mathcal{L}}^{(q)}$ with $\mathcal{L}_C^{(1)}$. In this and the following chapter, certain topics relating to the twisted affine Kac-Moody algebras will be investigated, in particular the relationship between involutive automorphisms of the compact real forms and the real forms of the various twisted algebras. The basis for all of this will be the matrix formulation that was developed for the untwisted Kac-Moody algebras.

It is natural to enquire about the nature of the automorphisms of the twisted algebras. Consider in particular those automorphisms of a given twisted Kac-Moody algebra which are the restrictions to the twisted algebra of automorphisms of the corresponding untwisted algebra. Since $\tilde{\mathcal{L}}^{(q)}$ is a proper subalgebra of $\tilde{\mathcal{L}}^{(1)}$, it follows that some, though not all of the automorphisms of $\tilde{\mathcal{L}}^{(1)}$ are such that their restrictions to $\tilde{\mathcal{L}}^{(q)}$ are also automorphisms of this subalgebra. Suppose then that the untwisted algebra $\tilde{\mathcal{L}}^{(1)}$ possesses an automorphism ψ such that an element a of $\tilde{\mathcal{L}}^{(1)}$ belongs to the twisted subalgebra $\tilde{\mathcal{L}}^{(q)}$ if, and only if

$$\psi(a) = a. \quad (10.1.6)$$

In this case, the identification of the automorphisms of $\tilde{\mathcal{L}}^{(1)}$ that are extensions of automorphisms of $\tilde{\mathcal{L}}^{(q)}$ becomes easier, as will be seen in each individual case. Thus, we will be examining the automorphisms of the twisted algebras from the starting point of the automorphisms of the corresponding untwisted Kac-Moody algebra. The results obtained by this will be used to identify any possible problems, and ultimately to develop a full method of examining all of the automorphisms of the untwisted algebras, analogous to the matrix formulation, which has been used for untwisted algebras.

The special role played by the Cartan-preserving automorphisms of twisted Kac-Moody algebras has previously been studied by Gorman *et al.* [19]

11 Involutive automorphisms and real forms of $A_2^{(2)}$

11.1 Introduction

The twisted algebra $A_2^{(2)}$ is the first (and hence the least complicated) of the $A_{2\ell}^{(2)}$ series of twisted affine Kac-Moody algebras. It may be constructed from the untwisted algebra $A_2^{(1)}$ in the manner described previously. The outer automorphism ψ_τ is the one for which

$$\begin{aligned} \psi(h_{\alpha_k}) &= h_{\alpha_{3-k}} \quad (k=1,2), \\ \psi(e_{\pm\alpha_k}) &= e_{\pm\alpha_{3-k}} \quad (k=1,2), \\ \psi(e_{\pm(\alpha_1+\alpha_2)}) &= -e_{\pm(\alpha_1+\alpha_2)}, \end{aligned} \tag{11.1.1}$$

Thus, a set of basis elements for the twisted algebra consists of the following:

$$\left. \begin{aligned} t^j \otimes (h_{\alpha_1} + h_{\alpha_2}) \\ t^j \otimes (e_{\pm\alpha_1} + e_{\pm\alpha_2}) \end{aligned} \right\} j \in \mathbf{Z}; j \text{ even}, \tag{11.1.2}$$

$$\left. \begin{aligned} t^j \otimes (h_{\alpha_1} - h_{\alpha_2}) \\ t^j \otimes (e_{\pm\alpha_1} - e_{\pm\alpha_2}) \\ t^j \otimes (e_{\pm(\alpha_1+\alpha_2)}) \end{aligned} \right\} j \in \mathbf{Z}; j \text{ odd}, \tag{11.1.3}$$

$$\begin{aligned} c, \\ d. \end{aligned} \tag{11.1.4}$$

The elements of the twisted Kac-Moody algebra $A_2^{(2)}$ can be seen to be those elements of the untwisted algebra $A_2^{(1)}$ that are mapped to themselves by ψ , the type

1b automorphism of $A_2^{(1)}$ given by $\psi = \{\text{offdiag}\{1, -1, 1\}, -1, 0\}$. Let the matrix \mathbf{d} be defined by

$$\mathbf{d} = \text{offdiag}\{1, -1, 1\}. \quad (11.1.5)$$

Now, suppose that ϕ is some automorphism of $A_2^{(1)}$. The action of ϕ upon the matrix parts of elements is summarised below:

$$\phi(\mathbf{a}(t)) = \mathbf{U}(t)\theta(\mathbf{a}(t))\mathbf{U}(t)^{-1} + \frac{1}{\gamma} \text{Res}\left\{\text{tr}\left(\mathbf{U}(t)^{-1} \frac{d\mathbf{U}(t)}{dt} \theta(\mathbf{a}(ut))\right)\right\}c, \quad (11.1.6)$$

where the quantity $\theta(\mathbf{a}(t))$ varies according to the type of automorphism thus:

$$\theta(\mathbf{a}(t)) = \begin{cases} \mathbf{a}(ut) & (\phi \text{ is of type 1a}), \\ -\tilde{\mathbf{a}}(ut) & (\phi \text{ is of type 1b}), \\ \mathbf{a}(ut^{-1}) & (\phi \text{ is of type 2a}), \\ -\tilde{\mathbf{a}}(ut^{-1}) & (\phi \text{ is of type 2b}). \end{cases} \quad (11.1.7)$$

Now, the elements of $A_2^{(2)}$ are those whose matrix parts satisfy

$$\mathbf{a}(t) = -\mathbf{d}\tilde{\mathbf{a}}(-t)\mathbf{d}^{-1}, \quad (11.1.8)$$

where the matrix \mathbf{d} is as defined in (11.1.5). If the restriction of ϕ to $A_2^{(2)}$ (which is a subalgebra of $A_2^{(1)}$) is to be an automorphism of $A_2^{(2)}$, then certain conditions must be satisfied. Most notably, we must have

$$\mathbf{b}(t) = -\mathbf{d}\tilde{\mathbf{b}}(-t)\mathbf{d}^{-1}, \quad (11.1.9)$$

where $\mathbf{b}(t) = \mathbf{U}(t)\theta(\mathbf{a}(t))\mathbf{U}(t)^{-1}$. Since the quantity $\mathbf{a}(t)$ takes all possible values in the algebra $A_2^{(2)}$, it follows that, in (11.1.9), $\mathbf{b}(t)$ can take the value of any matrix part of $A_2^{(2)}$. Expansion and re-arrangement of (11.1.9) yield the following:

$$\left\{\mathbf{U}(t^{-1})\mathbf{d}\tilde{\mathbf{U}}(-t)\mathbf{d}\right\}\mathbf{a}(ut) = \mathbf{a}(ut)\left\{\mathbf{U}(t^{-1})\mathbf{d}\tilde{\mathbf{U}}(-t)\mathbf{d}\right\}. \quad (11.1.10)$$

Schur's lemma (see appendix B) can be applied to this, and it implies that the matrices $U(t)$ and d are such that

$$\bar{U}(-t)dU(t) = \alpha t^\beta d \quad (\alpha \in \mathbb{C} \setminus \{0\}; \beta \in \mathbb{Z}). \quad (11.1.11)$$

The compact real form of $A_2^{(2)}$ is defined to be the intersection of $A_2^{(2)}$ with the compact real form of $A_2^{(1)}$. A basis for the compact real form of $A_2^{(2)}$ is thus given by the following:

$$\left. \begin{array}{l} (e_{j\delta}^1 + e_{j\delta}^2 + e_{-j\delta}^1 + e_{-j\delta}^2) \quad j \in \mathbb{N}^0 \\ i(e_{j\delta}^1 + e_{j\delta}^2 - e_{-j\delta}^1 - e_{-j\delta}^2) \quad j \in \mathbb{N} \\ (e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2}) \\ i(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2}) \end{array} \right\} j \text{ even}, \quad (11.1.12)$$

$$\left. \begin{array}{l} (e_{j\delta}^1 - e_{j\delta}^2 + e_{-j\delta}^1 - e_{-j\delta}^2) \quad (j \in \mathbb{N}) \\ i(e_{j\delta}^1 - e_{j\delta}^2 - e_{-j\delta}^1 + e_{-j\delta}^2) \quad (j \in \mathbb{N}) \\ (e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2}) \\ i(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2}) \end{array} \right\} j \text{ odd}, \quad (11.1.13)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha_1+\alpha_2} + e_{-j\delta-\alpha_1-\alpha_2}) \\ i(e_{j\delta+\alpha_1+\alpha_2} - e_{-j\delta-\alpha_1-\alpha_2}) \end{array} \right\} \begin{array}{l} ic, \\ id. \end{array} \quad (11.1.14)$$

Now, the Cartan subalgebra $\mathcal{H}^{(2)}$ of $A_2^{(2)}$ is 3-dimensional, with a set of basis elements being $i(h_{\alpha_1} + h_{\alpha_2})$, ic and id . The Cartan-preserving automorphisms of $A_2^{(2)}$ correspond, therefore, to the root transformations

$$\tau(\alpha_1^0 + \alpha_2^0) = \pm(\alpha_1^0 + \alpha_2^0). \quad (11.1.15)$$

Thus, the most general 3×3 Laurent matrix $U(t)$ which satisfies

$$\begin{aligned}\tilde{\mathbf{U}}(t)\mathbf{d}\mathbf{U}(-t) &= \lambda t^\mu \mathbf{d} \quad (\lambda \in \mathbf{C}, \lambda \neq 0; \mu \in \mathbf{Z}), \\ \mathbf{U}(t)i\left\{\mathbf{h}_{\alpha_1}^0 + \mathbf{h}_{\alpha_2}^0\right\}\mathbf{U}(t)^{-1} &= i\left\{\mathbf{h}_{\alpha_1}^0 + \mathbf{h}_{\alpha_2}^0\right\},\end{aligned}\tag{11.1.16}$$

is given (subject to factorisation by a constant non-zero coefficient of the form αt^β , where $\alpha \in \mathbf{C}$, $\beta \in \mathbf{Z}$) by $\mathbf{U}(t) = \text{diag}\{\eta t^k, 1, (-1)^k \eta^{-1} t^{-k}\}$. Similarly, the most general 3×3 Laurent matrix $\mathbf{U}(t)$ which satisfies

$$\begin{aligned}\tilde{\mathbf{U}}(t)\mathbf{d}\mathbf{U}(-t) &= \lambda t^\mu \mathbf{d} \quad (\lambda \in \mathbf{C}, \lambda \neq 0; \mu \in \mathbf{Z}), \\ \mathbf{U}(t)i\left\{\mathbf{h}_{\alpha_1}^0 + \mathbf{h}_{\alpha_2}^0\right\}\mathbf{U}(t)^{-1} &= -i\left\{\mathbf{h}_{\alpha_1}^0 + \mathbf{h}_{\alpha_2}^0\right\},\end{aligned}\tag{11.1.17}$$

is of the form $\mathbf{U}(t) = \text{offdiag}\{\eta t^k, 1, (-1)^k \eta^{-1} t^{-k}\}$.

11.2 Type 1a involutive automorphisms of $A_2^{(2)}$ with $u = 1$

Clearly, these automorphisms are those generated by the matrices of the forms

$$\begin{aligned} U(t) &= \text{diag}\{\pm 1, 1, \pm 1\}, \\ U(t) &= \text{offdiag}\{\eta t^k, 1, \eta^{-1} t^{-k}\} \quad (k \text{ even}). \end{aligned} \tag{11.2.1}$$

Each of these matrices $U(t)$ generates an automorphism of the compact real form of $A_2^{(2)}$ provided that $\eta\eta^* = 1$. The identity automorphism is clearly in a class of its own, and this class will be called (A). The automorphism generated by the matrix $U(t) (= \text{diag}\{-1, 1, -1\})$ belongs to some other class, which will be called (B). Now, let the matrix $U(t)$ be given by

$$U(t) = \text{offdiag}\{\eta t^k, 1, \eta^{-1} t^{-k}\}, \tag{11.2.2}$$

and for each matrix $U(t)$ of this form, let the matrix $S(t)$ be defined by

$$S(t) = \text{diag}\left\{\left(i^{\deg(\frac{1}{2}k)} \eta^{-\frac{1}{2}} t^{-\frac{1}{2}k}\right), 1, \left(i^{\deg(\frac{1}{2}k)} \eta^{\frac{1}{2}} t^{\frac{1}{2}k}\right)\right\}. \tag{11.2.3}$$

Then, provided that $\eta\eta^* = 1$, the matrix $S(t)$ satisfies

$$\begin{aligned} \tilde{S}^*(t^{-1})S(t) &= \mathbf{1}_3, \\ \tilde{S}(t)dS(-t) &= \mathbf{d}, \\ S(t)\text{offdiag}\{\eta t^k, 1, \eta^{-1} t^{-k}\}S(t)^{-1} &= \mathbf{K}_3. \end{aligned} \tag{11.2.4}$$

Furthermore, there exists a matrix T which is given by

$$\mathbf{T} = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}, \quad (11.2.5)$$

and this matrix satisfies all of the following:

$$\begin{aligned} \tilde{\mathbf{T}}\mathbf{d}\mathbf{T} &= \mathbf{d}, \\ \tilde{\mathbf{T}}^*\mathbf{T} &= \mathbf{1}_3, \\ \mathbf{T}\text{diag}\{-1, 1, -1\}\mathbf{T}^{-1} &= -\mathbf{K}_3. \end{aligned} \quad (11.2.6)$$

Hence, all of the automorphisms under consideration (apart from the identity automorphism) are mutually conjugate, and thus belong to the conjugacy class (B). This is the case both for $A_2^{(2)}$ and its compact real form. In the case of the conjugacy class (A), the corresponding real form is, clearly, the compact real form itself. For the conjugacy class (B), the representative automorphism is taken to be the type 1a automorphism generated by the matrix $\mathbf{U}(t)$ ($= \text{offdiag}\{1, -1, 1\}, 1, 0$). Each of the basis elements of the compact real form (which were given previously) is already an eigenvector of this representative automorphism. Thus, a basis of a real form of $A_2^{(2)}$ is provided by the following elements:

$$\left. \begin{aligned} & \left(e_{j\delta}^1 + e_{j\delta}^2 + e_{-j\delta}^1 + e_{-j\delta}^2 \right) \quad (j \in \mathbf{N}^0) \\ & i \left(e_{j\delta}^1 + e_{j\delta}^2 - e_{-j\delta}^1 - e_{-j\delta}^2 \right) \quad (j \in \mathbf{N}) \\ & i \left(e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1} + e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_2} \right) \quad (j \in \mathbf{N}^0) \\ & \left(e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1} - e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_2} \right) \quad (j \in \mathbf{N}^0) \end{aligned} \right\} j \text{ even}, \quad (11.2.7)$$

$$\left. \begin{aligned}
 & \left(e_{j\delta}^1 - e_{j\delta}^2 + e_{-j\delta}^1 - e_{-j\delta}^2 \right) \\
 & i \left(e_{j\delta}^1 - e_{j\delta}^2 - e_{-j\delta}^1 + e_{-j\delta}^2 \right) \\
 & \left(e_{j\delta+\alpha_1} - e_{-j\delta-\alpha_1} - e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_2} \right) \\
 & i \left(e_{j\delta+\alpha_1} + e_{-j\delta-\alpha_1} - e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_2} \right) \\
 & \left(e_{j\delta+\alpha_1+\alpha_2} + e_{-j\delta-\alpha_1-\alpha_2} \right) \\
 & i \left(e_{j\delta+\alpha_1+\alpha_2} - e_{-j\delta-\alpha_1-\alpha_2} \right)
 \end{aligned} \right\} j \text{ odd,} \tag{11.2.8}$$

ic,

id.

(11.2.9)

11.3 Type 1a involutive automorphisms of $A_2^{(2)}$ with $u = -1$

The type 1a involutive automorphisms with $u = -1$ are those that are generated by the following matrices:

$$U(t) = \begin{Bmatrix} \text{diag}\{-1, 1, -1\} \\ \text{offdiag}\{\eta t^k, 1, (-1)^k \eta^{-1} t^{-k}\} \end{Bmatrix} \quad (11.3.1)$$

A matrix $U(t)$ of the form $\text{offdiag}\{\eta t^k, 1, (-1)^k \eta^{-1} t^{-k}\}$ generates an automorphism of the compact real form of $A_2^{(2)}$ only if $\eta \eta^* = 1$. Now, let the matrix $S(t)$ be given by

$$S(t) = \text{diag}\{-t, 1, t^{-1}\}. \quad (11.3.2)$$

This matrix satisfies $\tilde{S}(t)dS(-t) = \mathbf{d}$, and also the condition

$$S(t)\mathbf{1}_3 S(-t)^{-1} = \text{diag}\{-1, 1, -1\}, \quad (11.3.3)$$

which demonstrates that the type 1a automorphisms $\{\text{diag}\{-1, 1, -1\}, -1, 0\}$ and $\{\mathbf{1}_3, -1, 0\}$ are conjugate via the type 1a automorphism $\{S(t), s, \xi\}$, which is an automorphism of the compact real form of $A_2^{(2)}$. Consider the remaining automorphisms. Then, for every matrix $U(t)$ of the form

$$U(t) = \text{offdiag}\{\eta t^k, 1, (-1)^k \eta^{-1} t^{-k}\}, \quad (11.3.4)$$

let a matrix $S(t)$ be defined accordingly by

$$S(t) = \text{diag}\left\{\left(i^{\deg \frac{1}{2}(k-\text{deg}k)} \eta^{\frac{1}{2}} t^{\frac{1}{2}(k-\text{deg}k)}\right), 1, \left((-i)^{\deg \frac{1}{2}(k-\text{deg}k)} \eta^{-\frac{1}{2}} t^{-\frac{1}{2}(k-\text{deg}k)}\right)\right\}. \quad (11.3.5)$$

The matrix $S(t)$ as thus defined satisfies $\tilde{S}(t)dS(-t) = \mathbf{d}$, (provided that $\eta \eta^* = 1$), and also

$$S(t)\text{offdiag}\{t^{\text{deg}k}, 1, (-1)^k \eta^{-1} t^{-k}\} S(-t)^{-1} = \text{offdiag}\{\eta t^k, 1, (-1)^k \eta^{-1} t^{-k}\}. \quad (11.3.6)$$

Thus, the only other automorphisms that need be examined are type 1a involutions $\{\text{offdiag}\{t, 1, -t^{-1}\}, -1, \xi\}$ and $\{\mathbf{K}_3, -1, 0\}$. It then follows that

$$\mathbf{T}(t) \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{T}(-t)^{-1} \quad \text{where } \mathbf{T} = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}, \quad (11.3.7)$$

and it has already been noted (in the previous subsection) that $\{\mathbf{T}(t), s, \xi\}$ is an automorphism of the compact real form of $A_2^{(2)}$. Finally, let $\mathbf{V}(t)$ be the matrix given by

$$\mathbf{V}(t) = \begin{bmatrix} 1 & 0 & t^{-1} \\ 0 & (i\sqrt{2})t^{-1} & 0 \\ -t^{-1} & 0 & t^{-2} \end{bmatrix}. \quad (11.3.8)$$

which satisfies both $\tilde{\mathbf{V}}(t)d\mathbf{V}(t) = 2t^{-2}d$, and $\tilde{\mathbf{V}}^*(t^{-1})\mathbf{V}(t) = 2\mathbf{1}_3$ (which means that $\mathbf{V}(t)$ generates an automorphism of the compact real form). In fact, note that

$$\mathbf{V}(t) \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{V}(-t)^{-1} = - \begin{bmatrix} 0 & 0 & t \\ 0 & 1 & 0 \\ -t^{-1} & 0 & 0 \end{bmatrix}. \quad (11.3.9)$$

Thus, all of the type 1a (with $u = -1$) involutive Cartan-preserving automorphisms of $A_2^{(2)}$ (and of its compact real form) are mutually conjugate within the group of all automorphisms of the Kac-Moody algebra. For both $A_2^{(2)}$ and its compact real form, this conjugacy class will be called (C), and its representative will be the type 1a automorphism $\{\mathbf{1}_3, -1, 0\}$. Each of the following basis elements of the compact real form of $A_2^{(2)}$ is an eigenvector of this representative automorphism, with the given associated eigenvalue:

$$\left. \begin{array}{l} (e_{j\delta}^1 + e_{j\delta}^2 + e_{-j\delta}^1 + e_{-j\delta}^2) \quad j \in \mathbb{N}^0 \\ i(e_{j\delta}^1 + e_{j\delta}^2 - e_{-j\delta}^1 - e_{-j\delta}^2) \quad j \in \mathbb{N} \\ (e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2}) \\ i(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2}) \end{array} \right\} j \text{ even; eigenvalue } 1, \quad (11.3.10)$$

$$\left. \begin{array}{l} (e_{j\delta}^1 - e_{j\delta}^2 + e_{-j\delta}^1 - e_{-j\delta}^2) \quad (j \in \mathbb{N}) \\ i(e_{j\delta}^1 - e_{j\delta}^2 - e_{-j\delta}^1 + e_{-j\delta}^2) \quad (j \in \mathbb{N}) \\ (e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2}) \\ i(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2}) \end{array} \right\} j \text{ odd; eigenvalue } -1, \quad (11.3.11)$$

$$\left. \begin{array}{l} (e_{j\delta+\alpha_1+\alpha_2} + e_{-j\delta-\alpha_1-\alpha_2}) \\ i(e_{j\delta+\alpha_1+\alpha_2} - e_{-j\delta-\alpha_1-\alpha_2}) \\ ic \\ id \end{array} \right\} \text{eigenvalue } 1. \quad (11.3.12)$$

This means that one basis of a real form of $A_2^{(2)}$ corresponding to the conjugacy class (C) is the one given by the following basis elements:

$$\left. \begin{array}{l} (e_{j\delta}^1 + e_{j\delta}^2 + e_{-j\delta}^1 + e_{-j\delta}^2) \quad j \in \mathbb{N}^0 \\ i(e_{j\delta}^1 + e_{j\delta}^2 - e_{-j\delta}^1 - e_{-j\delta}^2) \quad j \in \mathbb{N} \\ (e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2}) \\ i(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2}) \end{array} \right\} j \text{ even,} \quad (11.3.13)$$

$$\left. \begin{array}{l} i(e_{j\delta}^1 - e_{j\delta}^2 + e_{-j\delta}^1 - e_{-j\delta}^2) \quad (j \in \mathbb{N}) \\ (e_{j\delta}^1 - e_{j\delta}^2 - e_{-j\delta}^1 + e_{-j\delta}^2) \quad (j \in \mathbb{N}) \\ i(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2}) \\ (e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2}) \\ i(e_{j\delta+\alpha_1+\alpha_2} + e_{-j\delta-\alpha_1-\alpha_2}) \\ (e_{j\delta+\alpha_1+\alpha_2} - e_{-j\delta-\alpha_1-\alpha_2}) \end{array} \right\} j \text{ odd,} \quad (11.3.14)$$

ic,

id.

(11.3.15)

11.4 Type 1b involutive automorphisms of $A_2^{(2)}$ with $u = 1$

The type 1b involutive Cartan-preserving automorphisms of $A_2^{(2)}$ are generated by the matrices $U(t)$, where

$$U(t) = \begin{cases} \text{diag}\{\eta t^k, 1, (-1)^k \eta^{-1} t^{-k}\}, \\ \text{offdiag}\{\pm 1, 1, \pm 1\}. \end{cases} \quad (11.4.1)$$

These matrices all generate automorphisms of the compact real form, except those for which $\eta\eta^* \neq 1$. Consider firstly those automorphisms for which $U(t)$ is of the form $\text{diag}\{\eta t^k, 1, (-1)^k \eta t^k\}$. Then, for each matrix $U(t)$ of this form, a matrix $S(t)$ may be defined by

$$S(t) = \text{diag}\left\{\eta^{\frac{1}{2}} t^{\frac{1}{2}(k-\text{deg}k)}, 1, (-1)^{\frac{1}{2}(k-\text{deg}k)} \eta^{-\frac{1}{2}} t^{-\frac{1}{2}(k-\text{deg}k)}\right\}. \quad (11.4.2)$$

When $\eta\eta^* = 1$, this matrix satisfies $\tilde{S}^*(t^{-1})S(t) = \mathbf{1}_3$. In addition, it is also such that $\tilde{S}(t)dS(-t) = \mathbf{d}$, and

$$S(t)\text{diag}\{t^{\text{deg}k}, 1, (-1)^k t^{-\text{deg}k}\}\tilde{S}(t) = \text{diag}\{\eta t^k, 1, (-1)^k \eta^{-1} t^{-k}\}. \quad (11.4.3)$$

Furthermore, with the matrix $V(t)$ defined by

$$V(t) = \begin{bmatrix} -\frac{1}{\sqrt{2}}t^{-1} & 0 & \frac{1}{\sqrt{2}} \\ 0 & i & 0 \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}t \end{bmatrix}, \quad (11.4.4)$$

the following conditions are satisfied:

$$\begin{aligned}
\tilde{\mathbf{V}}^*(t^{-1})\mathbf{V}(t) &= \mathbf{1}_3, \\
\tilde{\mathbf{V}}(t)d\mathbf{V}(-t) &= -d, \\
\mathbf{V}(t)\text{diag}\{t, 1, -t^{-1}\}\tilde{\mathbf{V}}(t) &= -\mathbf{K}_3.
\end{aligned}
\tag{11.4.5}$$

Recall that the matrix $\mathbf{T}(t)$ satisfies $\tilde{\mathbf{T}}(t)d\mathbf{T}(t) = d$, and $\mathbf{T}(t)\text{diag}\{-1, 1, -1\}\tilde{\mathbf{T}}(t) = -\mathbf{K}_3$, where

$$\mathbf{T}(t) = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}.
\tag{11.4.6}$$

Finally, where $\mathbf{R}(t) = \text{diag}\{t, 1, -t^{-1}\}$, it is the case that

$$\begin{aligned}
\tilde{\mathbf{R}}^*(t^{-1})\mathbf{R}(t) &= \mathbf{1}_3, \\
\mathbf{R}(t)d\mathbf{R}(-t) &= -d, \\
\mathbf{R}(t)\mathbf{1}_3\tilde{\mathbf{R}}(t) &= \text{offdiag}\{-1, 1, -1\}.
\end{aligned}
\tag{11.4.7}$$

Thus, for both $A_2^{(2)}$ and its compact real form, all of the involutive Cartan-preserving automorphisms of type 1b (with $u = 1$) are mutually conjugate, and belong to a conjugacy class which is called (D) (in the case of both the compact real form and the complex algebra). The representative automorphism of the conjugacy class (D) is the type 1b automorphism $\{\mathbf{K}_3, 1, 0\}$. A basis of a real form of $A_2^{(2)}$, generated from this automorphism by Cartan's method, is given by the following elements:

$$\left. \begin{aligned}
& \left(e_{j\delta}^1 + e_{j\delta}^2 + e_{-j\delta}^1 + e_{-j\delta}^2 \right) & j \in \mathbf{N}^0 \\
& i \left(e_{j\delta}^1 + e_{j\delta}^2 - e_{-j\delta}^1 - e_{-j\delta}^2 \right) & j \in \mathbf{N} \\
& i \left(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2} \right) & j \in \mathbf{Z} \\
& \left(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2} \right) & j \in \mathbf{Z}^\pm
\end{aligned} \right\} \text{even}, \tag{11.4.8}$$

$$\left. \begin{array}{l}
 i(e_{j\delta}^1 - e_{j\delta}^2 + e_{-j\delta}^1 - e_{-j\delta}^2) \quad j \in \mathbf{N} \\
 (e_{j\delta}^1 - e_{j\delta}^2 - e_{-j\delta}^1 + e_{-j\delta}^2) \quad j \in \mathbf{N} \\
 (e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2}) \quad j \in \mathbf{Z} \\
 i(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2}) \quad j \in \mathbf{Z} \\
 i(e_{j\delta+\alpha_1+\alpha_2} + e_{-j\delta-\alpha_1-\alpha_2}) \quad j \in \mathbf{Z} \\
 (e_{j\delta+\alpha_1+\alpha_2} - e_{-j\delta-\alpha_1-\alpha_2}) \quad j \in \mathbf{Z}
 \end{array} \right\} j \text{ odd,} \tag{11.4.9}$$

ic,

id.

$$(11.4.10)$$

11.5 Type 1b involutive automorphisms of $A_2^{(2)}$ with $u = -1$

The automorphisms in question are generated by the following matrices:

$$U(t) = \begin{cases} \text{diag}\{\eta t^k, 1, \eta^{-1} t^{-k}\} & (k \text{ even}), \\ \text{offdiag}\{\pm 1, 1, \pm 1\}. \end{cases} \quad (11.5.1)$$

Consider the first form of $U(t)$, where it is a diagonal matrix. The matrix $U(t)$ generates an automorphism of the compact real form only if $\eta\eta^* = 1$. Let a matrix $S(t)$ be defined so that

$$S(t) = \text{diag}\left\{i^{\deg \frac{1}{2}k} \eta^{\frac{1}{2}} t^{\frac{1}{2}k}, 1, i^{\deg \frac{1}{2}k} \eta^{-\frac{1}{2}} t^{-\frac{1}{2}k}\right\}, \quad (11.5.2)$$

which means that

$$\begin{aligned} \tilde{S}(t) dS(-t) &= d, \\ \tilde{S}^*(t^{-1}) S(t) &= \mathbf{1}_3 \quad (\text{if } \eta\eta^* = 1), \\ S(t) \mathbf{1}_3 \tilde{S}(-t) &= \text{diag}\{\eta t^k, 1, \eta^{-1} t^{-k}\}. \end{aligned} \quad (11.5.3)$$

Thus, all of the type 1b involutions (with $u = -1$) generated by matrices of the form $\text{diag}\{\eta t^k, 1, \eta^{-1} t^{-k}\}$ (with k even) belong to the same conjugacy class, which will be called (E). This class also contains the type 1b involution $\{\mathbf{1}_3, -1, 0\}$. This is the case both for $A_2^{(2)}$ and its compact real form. Consider next the automorphism $\{\text{offdiag}\{-1, 1, -1\}, 1, 0\}$. It will now be shown that this automorphism belongs in a conjugacy class other than (E). Suppose then that the automorphism $\psi(= \{\text{offdiag}\{-1, 1, -1\}, 1, 0\})$ does belong to the conjugacy class (E), so that, for all non-zero values of t , the following hold:

$$\left. \begin{aligned} \mathbf{S}(t)d\tilde{\mathbf{S}}(-t) &= \alpha t^\beta \mathbf{K}_3 \\ \tilde{\mathbf{S}}(t)d\mathbf{S}(-t) &= \lambda t^\mu \mathbf{d} \end{aligned} \right\} \text{non-zero } \alpha, \beta \in \mathbf{C}; \beta, \mu \in \mathbf{Z}. \quad (11.5.4)$$

(Note that $\text{offdiag}\{-1, 1, -1\} = -\mathbf{d}$). If the substitution $t = 1$ is made in the first of these conditions, and the substitution $t = -1$ is made in the second, then

$$\left. \begin{aligned} \mathbf{S}(1)d\tilde{\mathbf{S}}(-1) &= \alpha \mathbf{K}_3 \quad (\text{non-zero } \alpha), \\ \tilde{\mathbf{S}}(-1)d\mathbf{S}(1) &= \beta \mathbf{d} \quad (\text{non-zero } \beta), \end{aligned} \right\} \quad (11.5.5)$$

and together these two imply that

$$\left. \begin{aligned} \alpha \mathbf{K}_3 d\mathbf{S}(1) &= \mathbf{S}(1)d\tilde{\mathbf{S}}(-1)d\mathbf{S}(1), \\ \alpha \mathbf{K}_3 d\mathbf{S}(1) &= \mathbf{S}(1)d\{\beta \mathbf{d}\}, \\ \mathbf{S}(1) &= (\alpha \beta^{-1}) \text{diag}\{1, -1, 1\} \mathbf{S}(1). \end{aligned} \right\} \quad (11.5.6)$$

The assumption is being made that $\mathbf{S}(t)$ is invertible, and a Laurent polynomial matrix. A necessary condition for this is that $\mathbf{S}(1)$ is non-singular, although a brief check will verify that there does not exist any non-singular matrix $\mathbf{S}(1)$ that satisfies

$$\mathbf{S}(1) = (\alpha \beta^{-1}) \text{diag}\{1, -1, 1\} \mathbf{S}(1), \quad (11.5.7)$$

and thus the automorphism ψ does not belong to the conjugacy class (E). The representative automorphism for the conjugacy class (E) is the type 1b automorphism $\{\mathbf{K}_3, 1, 0\}$. Each of the basis elements of the compact real form of $A_2^{(2)}$ (given at the beginning of this section) is an eigenvector of this representative automorphism, with the associated eigenvalue in each case being 1 or -1 . Thus, the basis of a real form of $A_2^{(2)}$ is provided by the following elements (obtained using Cartan's theorem):

$$\left. \begin{aligned} &\left(\begin{array}{l} e_{j\delta}^1 + e_{j\delta}^2 + e_{-j\delta}^1 + e_{-j\delta}^2 \\ i(e_{j\delta}^1 + e_{j\delta}^2 - e_{-j\delta}^1 - e_{-j\delta}^2) \end{array} \right) \quad \begin{array}{l} j \in \mathbf{N}^0 \\ j \in \mathbf{N} \end{array} \\ &i \left(\begin{array}{l} e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2} \\ e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2} \end{array} \right) \quad \begin{array}{l} j \in \mathbf{Z} \\ j \in \mathbf{Z}^+ \end{array} \end{aligned} \right\} j \text{ even}, \quad (11.5.8)$$

$$\left. \begin{array}{l}
\left(e_{j\delta}^1 - e_{j\delta}^2 + e_{-j\delta}^1 - e_{-j\delta}^2 \right) \quad j \in \mathbf{N} \\
i \left(e_{j\delta}^1 - e_{j\delta}^2 - e_{-j\delta}^1 + e_{-j\delta}^2 \right) \quad j \in \mathbf{N} \\
i \left(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2} \right) \quad j \in \mathbf{Z} \\
\left(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2} \right) \quad j \in \mathbf{Z} \\
\left(e_{j\delta+\alpha_1+\alpha_2} + e_{-j\delta-\alpha_1-\alpha_2} \right) \quad j \in \mathbf{Z} \\
i \left(e_{j\delta+\alpha_1+\alpha_2} - e_{-j\delta-\alpha_1-\alpha_2} \right) \quad j \in \mathbf{Z}
\end{array} \right\} j \text{ odd,} \quad (11.5.9)$$

$$\begin{array}{l}
ic, \\
id.
\end{array} \quad (11.5.10)$$

In a similar fashion, it may be seen that each of the given basis elements of the compact real form is an eigenvector of the automorphism $\{-\mathbf{d}, -1, 0\}$, which is the representative of the conjugacy class (F). The corresponding basis of a real form of $A_2^{(2)}$ is thus given by the following:

$$\left. \begin{array}{l}
\left(e_{j\delta}^1 + e_{j\delta}^2 + e_{-j\delta}^1 + e_{-j\delta}^2 \right) \quad j \in \mathbf{N}^0 \\
i \left(e_{j\delta}^1 + e_{j\delta}^2 - e_{-j\delta}^1 - e_{-j\delta}^2 \right) \quad j \in \mathbf{N} \\
\left(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2} \right) \quad j \in \mathbf{Z} \\
i \left(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2} \right) \quad j \in \mathbf{Z}^\pm
\end{array} \right\} j \text{ even,} \quad (11.5.11)$$

$$\left. \begin{array}{l}
\left(e_{j\delta}^1 - e_{j\delta}^2 + e_{-j\delta}^1 - e_{-j\delta}^2 \right) \quad j \in \mathbf{N} \\
i \left(e_{j\delta}^1 - e_{j\delta}^2 - e_{-j\delta}^1 + e_{-j\delta}^2 \right) \quad j \in \mathbf{N} \\
\left(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2} \right) \quad j \in \mathbf{Z} \\
i \left(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2} \right) \quad j \in \mathbf{Z} \\
\left(e_{j\delta+\alpha_1+\alpha_2} + e_{-j\delta-\alpha_1-\alpha_2} \right) \quad j \in \mathbf{Z} \\
i \left(e_{j\delta+\alpha_1+\alpha_2} - e_{-j\delta-\alpha_1-\alpha_2} \right) \quad j \in \mathbf{Z}
\end{array} \right\} j \text{ odd,} \quad (11.5.12)$$

$$\begin{array}{l}
ic, \\
id.
\end{array} \quad (11.5.13)$$

11.6 Type 2a involutive automorphisms of $A_2^{(2)}$ with $u = 1$

Note firstly that the automorphisms in question are those type 2a automorphisms that are generated by matrices $U(t)$ of the form

$$U(t) = \begin{Bmatrix} \text{diag} \left\{ \pm t^k, 1, \left(\pm (-1)^k t^{-k} \right) \right\} \\ \text{offdiag} \left\{ \eta, 1, \eta^{-1} \right\} \end{Bmatrix} \quad (11.6.1)$$

Take the first of these forms, that is, let $U(t)$ be given by

$$U(t) = \text{diag} \left\{ \pm t^k, 1, \left(\pm (-1)^k t^{-k} \right) \right\}, \quad (11.6.2)$$

and let the matrix $S(t)$ be defined by

$$S(t) = \text{diag} \left\{ t^{\frac{1}{2}(k-\text{deg}k)}, 1, (-1)^{\text{deg}(k-\text{deg}k)} t^{-\frac{1}{2}(k-\text{deg}k)} \right\}. \quad (11.6.3)$$

This matrix is such that all of the following are satisfied:

$$\begin{aligned} \tilde{S}(t) \mathbf{d} S(-t) &= \mathbf{d}, \\ \tilde{S}^*(t^{-1}) S(t) &= \mathbf{1}_3, \end{aligned} \quad (11.6.4)$$

$$S(t) \text{diag} \left\{ \pm t^{\text{deg}k}, 1, \left(\pm (-1)^k t^{-\text{deg}k} \right) \right\} S(t^{-1})^{-1} = \text{diag} \left\{ \pm t^k, 1, (-1)^k \pm t^{-k} \right\}.$$

Thus, only the automorphisms that are generated by the following four matrices need be considered:

$$\begin{aligned} U_1(t) &= \mathbf{1}_3, \\ U_2(t) &= \text{diag} \{-1, 1, -1\}, \\ U_3(t) &= \text{diag} \{t, 1, -t^{-1}\}, \\ U_4(t) &= \text{diag} \{-t, 1, t^{-1}\}. \end{aligned} \quad (11.6.5)$$

It is an easy matter to see that the first of these is non-conjugate to any of the others.

Similarly, the last two matrices produce mutually conjugate automorphisms since

$$U_3(-t) = U_4(t). \quad (11.6.6)$$

The conjugacy class that contains the type 2a automorphism $\{1_3, 1, 0\}$ will be referred to as (G), and that which contains $\{\text{diag}\{-1, 1, -1\}, 1, 0\}$ will be called (H). The conjugacy class that contains the automorphism $\{U_3(t), 1, 0\}$ needs to be identified. The "usual" methods that may be used with matrix formulation to show that given automorphisms are disjoint are not applicable in this case. There is no immediately obvious reason why there should not exist any matrix $S(t)$ which satisfies

$$\begin{aligned} S(t)\text{diag}\{-1, 1, -1\}S(t^{-1})^{-1} &= \lambda t^\mu \text{diag}\{t, 1, -t^{-1}\}, \\ \tilde{S}(t)dS(-t) &= \alpha t^\beta d. \end{aligned} \quad (11.6.7)$$

In particular, by substituting the values $t = \pm 1$ into the above equations, the general forms of $S(1)$ and $S(-1)$ may be found. However, there is nothing inherent in the forms of these matrices that suggests that the two equations given above are not impossible to satisfy simultaneously. For this problematic example, it will be assumed that the type 2a involutive automorphisms $\{U_2(t), 1, 0\}$ and $\{U_3(t), 1, \xi\}$ are non-conjugate. Let the conjugacy "class" which contains $\{U_3(t), 1, \xi\}$ be called (I). A suitable representative of the conjugacy class (G) is the type 2a automorphism $\{1_3, 1, 0\}$. The given basis of the compact real form of $A_2^{(2)}$ is not a basis of eigenvectors of this automorphism. However, it is not difficult to obtain such a basis, and it follows that a basis of a real form associated with the conjugacy class (G) is provided by the elements given below

$$\left. \begin{aligned} & \left(e_{j\delta}^1 + e_{j\delta}^2 \right) \quad j \in \mathbf{Z} \\ & \left(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} + e_{j\delta-\alpha_1} + e_{j\delta-\alpha_2} \right) \quad j \in \mathbf{Z} \\ & i \left(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} - e_{j\delta-\alpha_1} - e_{j\delta-\alpha_2} \right) \quad j \in \mathbf{Z} \end{aligned} \right\} j \text{ even}, \quad (11.6.8)$$

$$\left. \begin{array}{l}
 (e_{j\delta}^1 - e_{j\delta}^2) \quad j \in \mathbf{Z} \\
 (e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} + e_{j\delta-\alpha_1} - e_{j\delta-\alpha_2}) \quad j \in \mathbf{Z} \\
 i(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} - e_{j\delta-\alpha_1} + e_{j\delta-\alpha_2}) \quad j \in \mathbf{Z} \\
 (e_{j\delta+\alpha_1+\alpha_2} + e_{j\delta-\alpha_1-\alpha_2}) \quad j \in \mathbf{Z} \\
 i(e_{j\delta+\alpha_1+\alpha_2} - e_{j\delta-\alpha_1-\alpha_2}) \quad j \in \mathbf{Z}
 \end{array} \right\} j \text{ odd,} \quad (11.6.9)$$

$$\begin{array}{l}
 c, \\
 d.
 \end{array} \quad (11.6.10)$$

Similarly, for the conjugacy class (H), the representative automorphism is the type 2a automorphism $\{\text{diag}\{1, -1, 1\}, 1, 0\}$. Following the method of Cartan, one obtains the following basis of a real form of the twisted algebra $A_2^{(2)}$:

$$\left. \begin{array}{l}
 (e_{j\delta}^1 + e_{j\delta}^2) \\
 i(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} + e_{j\delta-\alpha_1} + e_{j\delta-\alpha_2}) \\
 (e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} - e_{j\delta-\alpha_1} - e_{j\delta-\alpha_2})
 \end{array} \right\} j \in \mathbf{Z}; j \text{ even,} \quad (11.6.11)$$

$$\left. \begin{array}{l}
 (e_{j\delta}^1 - e_{j\delta}^2) \\
 i(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} + e_{j\delta-\alpha_1} - e_{j\delta-\alpha_2}) \\
 (e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} - e_{j\delta-\alpha_1} + e_{j\delta-\alpha_2}) \\
 (e_{j\delta+\alpha_1+\alpha_2} + e_{j\delta-\alpha_1-\alpha_2}) \\
 i(e_{j\delta+\alpha_1+\alpha_2} - e_{j\delta-\alpha_1-\alpha_2})
 \end{array} \right\} j \in \mathbf{Z}; j \text{ odd,} \quad (11.6.12)$$

$$\begin{array}{l}
 c, \\
 d.
 \end{array} \quad (11.6.13)$$

The representative automorphism of the conjugacy class (I) is taken to be the type 2a automorphism $\{\text{diag}\{t, 1, -t^{-1}\}, 1, 6\}$. The corresponding basis of a real form is given by the following elements:

$$\left. \begin{array}{l} (e_{j\delta}^1 + e_{j\delta}^2) \\ (e_{\pm(j\delta+\alpha_1)} + e_{\pm(j\delta+\alpha_2)} + e_{\mp(k\delta+\alpha_1)} - e_{\mp(k\delta+\alpha_2)}) \\ i(e_{\pm(j\delta+\alpha_1)} + e_{\pm(j\delta+\alpha_2)} - e_{\mp(k\delta+\alpha_1)} + e_{\mp(k\delta+\alpha_2)}) \end{array} \right\} j \in \mathbf{Z}; k = -j + 1; j \text{ even}, \quad (11.6.14)$$

$$\left. \begin{array}{l} (e_{j\delta}^1 - e_{j\delta}^2) \\ i(e_{\pm(j\delta+\alpha_1+\alpha_2)} + e_{\mp(k\delta+\alpha_1+\alpha_2)}) \\ (e_{\pm(j\delta+\alpha_1+\alpha_2)} - e_{\mp(k\delta+\alpha_1+\alpha_2)}) \end{array} \right\} j \in \mathbf{Z}; k = -j + 2; j \text{ odd}, \quad (11.6.15)$$

$$c, \quad (11.6.16)$$

$$d + 3(h_{\alpha_1} + h_{\alpha_2}).$$

11.7 Type 2b involutive automorphisms of $A_2^{(2)}$ with $u = 1$

The type 2b involutive automorphisms (with $u = 1$) are generated by the following matrices:

$$U(t) = \begin{cases} \text{diag}\{\eta, 1, \eta^{-1}\}, \\ \text{offdiag}\{\eta t^k, 1, \eta^{-1}(-t)^{-k}\} \quad (\eta^2 = (-1)^k). \end{cases} \quad (11.7.1)$$

Each of these matrices generates an automorphism of the compact real form if $\eta\eta^* = 1$.

Let the matrix $U(t)$ be of the first given type, that is

$$U(t) = \text{diag}\{\eta, 1, \eta^{-1}\}. \quad (11.7.2)$$

Then, for each matrix $U(t)$ of this form, let another matrix $S(t)$ be defined by

$$S(t) = \text{diag}\left\{\eta^{\frac{1}{2}}, 1, \eta^{-\frac{1}{2}}\right\}, \quad (11.7.3)$$

which means that the following equations all hold:

$$\begin{aligned} \tilde{S}(t)dS(-t) &= d, \\ \tilde{S}^*(t^{-1})S(t) &= \mathbf{1}_3 \quad (\text{if } \eta\eta^* = 1), \\ S(t)\mathbf{1}_3\tilde{S}(t^{-1}) &= U(t). \end{aligned} \quad (11.7.4)$$

Thus, all of the type 2b automorphisms $\{U(t), 1, 0\}$, where $U(t)$ is a diagonal matrix of the form (11.7.1), are mutually conjugate. The conjugacy class which contains them is called (J). Now, let $U(t)$ be given by

$$U(t) = \text{offdiag}\{\eta t^k, 1, \eta^{-1}(-t)^{-k}\}. \quad (11.7.5)$$

(Where the case k is even is being investigated, so that $\eta^2 = 1$). For each such $\mathbf{U}(t)$, let a matrix $\mathbf{V}(t)$ be defined by

$$\mathbf{V}(t) = \text{diag}\left\{t^{\frac{1}{2}k}, 1, (-1)^{\frac{1}{2}k} t^{-\frac{1}{2}k}\right\}, \quad (11.7.6)$$

which is such that all of the following are satisfied simultaneously:

$$\begin{aligned} \tilde{\mathbf{V}}(t) \mathbf{d} \mathbf{V}(-t) &= \mathbf{d}, \\ \tilde{\mathbf{V}}^*(t^{-1}) \mathbf{V}(t) &= \mathbf{1}_3, \\ \mathbf{V}(t) \text{offdiag}\left\{(-1)^{\frac{1}{2}k} \eta, 1, (-1)^{\frac{1}{2}k} \eta^{-1}\right\} \tilde{\mathbf{V}}(t^{-1}) &= \mathbf{U}(t). \end{aligned} \quad (11.7.7)$$

Therefore, attention may be restricted to those type 2b involutions generated by the matrices $\mathbf{U}(t) = \text{offdiag}\{\pm 1, 1, \pm 1\}$. In fact, with

$$\mathbf{T}(t) = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}, \quad (11.7.8)$$

it is clear that $\tilde{\mathbf{T}}(t) \mathbf{d} \mathbf{T}(-t) = \mathbf{d}$, also $\tilde{\mathbf{T}}^*(t^{-1}) \mathbf{T}(t) = \mathbf{1}_3$, and importantly

$$\mathbf{T}(t) \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tilde{\mathbf{T}}(t^{-1}) = - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \quad (11.7.9)$$

This means that only the automorphism generated by the matrix $\mathbf{U}(t)$ (given by $\text{offdiag}\{-1, 1, -1\}$) need be investigated, since all of the other type 2b involutive automorphisms under consideration belong to the class (J). It will now be shown that there does not exist any suitable Laurent matrix $\mathbf{S}(t)$ which satisfies (for non-zero complex numbers α and λ , and integers β and μ)

$$\begin{aligned} \mathbf{S}(t) \text{offdiag}\{-1, 1, -1\} \tilde{\mathbf{S}}(t^{-1}) &= \lambda t^\mu \mathbf{K}_3, \\ \tilde{\mathbf{S}}(t) \mathbf{d} \mathbf{S}(-t) &= \alpha t^\beta \mathbf{d}. \end{aligned} \quad (11.7.10)$$

It is assumed that this is true, with proof following by contradiction. Re-arranging implies that

$$\tilde{\mathbf{S}}(t)\mathbf{dS}(-t)\mathbf{d} = \alpha t^\beta \mathbf{1}_3, \quad (11.7.11)$$

and since $\mathbf{S}(t)$ is a Laurent matrix, it follows that

$$\tilde{\mathbf{S}}(t)^{-1} = (\alpha^{-1}t^{-\beta})\mathbf{dS}(-t)\mathbf{d}. \quad (11.7.12)$$

Re-arranging the other condition implies that, for all non-zero values of t

$$\begin{aligned} \mathbf{S}(t)\text{offdiag}\{-1, 1, -1\} &= \lambda t^\mu \mathbf{K}_3 \tilde{\mathbf{S}}(t^{-1})^{-1}, \\ (\lambda \alpha^{-1})t^{\mu+\beta} \mathbf{K}_3 \mathbf{dS}(-t^{-1})\mathbf{d} &= \mathbf{S}(t)\{-\mathbf{d}\}, \end{aligned} \quad (11.7.13)$$

and so this implies that an expression for the matrix $\mathbf{S}(t)$ is

$$\mathbf{S}(t) = -(\lambda \alpha^{-1})t^{\beta+\mu} \text{diag}\{1, -1, 1\}\mathbf{S}(-t^{-1}). \quad (11.7.14)$$

This must hold for all non-zero complex values of t . In particular, the equation $t = -t^{-1}$ implies that $t = \pm i$. Substituting the value $t = i$ yields

$$\mathbf{S}(i) = (A)\text{diag}\{1, -1, 1\}\mathbf{S}(i), \quad (11.7.15)$$

for some non-zero complex number A . It is easily shown that there does not exist a singular 3×3 matrix $\mathbf{S}(i)$ which satisfies this condition. Hence there is a conjugacy class (K) which contains the type 2b involutive automorphism $\{-\mathbf{d}, 1, 0\}$. Now consider those automorphisms for which the matrix $\mathbf{U}(t)$ is of the form

$$\mathbf{U}(t) = \text{offdiag}\{\eta t^k, 1, -\eta^{-1}t^{-k}\} \quad (k \text{ is odd}; \eta^2 = -1). \quad (11.7.16)$$

Let $\mathbf{S}(t) = \text{diag}\left\{t^{\frac{1}{2}(k-1)}, 1, (-1)^{\frac{1}{2}(k-1)}t^{-\frac{1}{2}(k-1)}\right\}$. It then follows that

$$\tilde{\mathbf{S}}(t)d\mathbf{S}(-t) = \mathbf{d},$$

$$\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_3,$$

$$\mathbf{S}(t) \begin{bmatrix} 0 & 0 & (-1)^{\frac{1}{2}(k-1)}\eta \\ 0 & 1 & 0 \\ -(-1)^{\frac{1}{2}(k-1)}\eta^{-1} & 0 & 0 \end{bmatrix} \tilde{\mathbf{S}}(t^{-1}) = \mathbf{U}(t). \quad (11.7.17)$$

Let $\mathbf{U}_1(t) = \text{offdiag}\{it, 1, it^{-1}\}$, and $\mathbf{U}_2(t) = \text{offdiag}\{-it, 1, -it^{-1}\}$. Since $\mathbf{U}_1(t) = \mathbf{U}_2(-t)$, it follows that $\{\mathbf{U}_1(t), 1, \xi\}$ and $\{\mathbf{U}_2(t), 1, \xi\}$ are conjugate automorphisms. In fact, both of these automorphisms belong to the conjugacy class (J). For, with

$$\mathbf{S}(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & it \\ 0 & \sqrt{2} & 0 \\ it^{-1} & 0 & 1 \end{bmatrix}, \quad (11.7.18)$$

the following equations are satisfied:

$$\tilde{\mathbf{S}}(t)d\mathbf{S}(t) = \mathbf{d},$$

$$\tilde{\mathbf{S}}^*(t^{-1})\mathbf{S}(t) = \mathbf{1}_3, \quad (11.7.19)$$

$$\mathbf{S}(t)\mathbf{1}_3\tilde{\mathbf{S}}(t^{-1}) = \text{offdiag}\{it, 1, it^{-1}\}.$$

The representative automorphism of the conjugacy class (J) is taken to be the type 2b automorphism $\{\mathbf{1}_3, 1, 0\}$. Using this automorphism as representative, each of the basis elements of the compact real form of $A_2^{(2)}$ (given some sections previously) is an eigenvector of it. The corresponding basis of a real form of $A_2^{(2)}$ is thus given by the following elements:

$$\left. \begin{array}{l} i(e_{j\delta}^1 + e_{j\delta}^2) \\ (e_{j\delta \pm \alpha_1} + e_{j\delta \pm \alpha_2}) \end{array} \right\} j \in \mathbf{Z}; j \text{ even}, \quad (11.7.20)$$

$$\left. \begin{array}{l} i(e_{j\delta}^1 - e_{j\delta}^2) \\ (e_{j\delta \pm \alpha_1} - e_{j\delta \pm \alpha_2}) \\ (e_{j\delta \pm \alpha_1 \pm \alpha_2}) \end{array} \right\} j \in \mathbf{Z}; j \text{ odd}, \quad (11.7.21)$$

$$\begin{array}{l} c, \\ d. \end{array} \quad (11.7.22)$$

Finally, the type 2b automorphism $\{\text{offdiag}\{-1,1,-1\},1,0\}$ is taken as the representative automorphism of the conjugacy class (K). In this instance, the given basis elements of $A_2^{(2)}$ are not eigenvectors. However, after finding such a basis of eigenvectors, one obtains the following basis of a real form of $A_2^{(2)}$:

$$\left. \begin{array}{l} (e_{j\delta}^1 + e_{j\delta}^2) \\ (e_{j\delta \pm \alpha_1} + e_{j\delta \pm \alpha_2} + e_{j\delta \mp \alpha_1} + e_{j\delta \mp \alpha_2}) \\ i(e_{j\delta \pm \alpha_1} + e_{j\delta \pm \alpha_2} - e_{j\delta \mp \alpha_1} - e_{j\delta \mp \alpha_2}) \end{array} \right\} j \in \mathbf{Z}; j \text{ even}, \quad (11.7.23)$$

$$\left. \begin{array}{l} i(e_{j\delta}^1 + e_{j\delta}^2) \\ i(e_{j\delta \pm \alpha_1} - e_{j\delta \pm \alpha_2} + e_{j\delta \mp \alpha_1} - e_{j\delta \mp \alpha_2}) \\ (e_{j\delta \pm \alpha_1} - e_{j\delta \pm \alpha_2} - e_{j\delta \mp \alpha_1} + e_{j\delta \mp \alpha_2}) \\ (e_{j\delta \pm \alpha_1 \pm \alpha_2} + e_{j\delta \mp \alpha_1 \mp \alpha_2}) \\ i(e_{j\delta \pm \alpha_1 \pm \alpha_2} - e_{j\delta \mp \alpha_1 \mp \alpha_2}) \end{array} \right\} j \in \mathbf{Z}; j \text{ odd}, \quad (11.7.24)$$

$$\begin{array}{l} c, \\ d. \end{array} \quad (11.7.25)$$

12 Conclusions

12.1 Initial objectives and results obtained

The initial objective of the work upon which this thesis was based was to extend previous work on untwisted affine Kac-Moody algebras. In particular, the involutive automorphisms of these algebras were to be investigated, and their conjugacy classes (within the group of all automorphisms of the algebra) determined. This work was undertaken using the matrix formulation which is explained in chapter 1, which allows automorphisms of untwisted Kac-Moody algebras to be expressed in terms of matrices whose entries are all Laurent polynomial matrices.

The original work of Cornwell [8] was extended in [9-11] to examine all of the $A_\ell^{(1)}$ series of affine Kac-Moody algebras. I extended this work to cover other affine Kac-Moody algebras, and some of this work is contained herein, and also published [6,7]. The identification of conjugacy classes then allowed real forms to be constructed, by adapting Cartan's work on the semi-simple Lie algebras.

The aim was ideally to obtain a complete picture of the conjugacy classes of the algebras being examined. However, some problems were encountered when the algebra $D_\ell^{(1)}$. The method employed initially was not sufficient for proving non-conjugacy of certain pairs of involutive automorphisms. However, the problems were removed when a second representation was introduced, and the problem re-evaluated in terms of the alternative representation.

Work was also begun on the twisted affine Kac-Moody algebras. Although the matrix formulation is specifically for untwisted algebras, with slight adaptation it provided some information about the conjugacy classes of $A_2^{(2)}$. Problems similar to the ones encountered for $D_\ell^{(1)}$ were found for $A_2^{(2)}$ also. In this case though, consideration of an alternative representation was not sufficient to solve the problem.

The “problematic” cases were dealt with by making the working assumption that involutive automorphisms were not conjugate, unless it could be proven otherwise. In this manner, real forms were not needlessly ignored.

In addition to the work done on $A_2^{(2)}$, a brief study was made of the exceptional affine Kac-Moody algebra $G_2^{(1)}$. The problems in this case were entirely expected. By its very nature, the algebra defies a concise characterisation in terms of matrices. It is more difficult therefore, to characterise the automorphisms of this algebra. Nevertheless, a limited examination was made of the involutive automorphisms and the corresponding real forms.

The problems which were experienced in the course of this work have suggested several possibilities which may result in further solutions. Future work will hopefully present a complete breakdown of the conjugacy classes of the involutive automorphisms of all affine Kac-Moody algebras.

Appendix A

This section of the appendix contains general information concerning the semi-simple Lie algebras, and also about the affine complex Kac-Moody algebras. In particular, it gives the Cartan matrices for the former and the generalised Cartan matrices for the latter.

The simple complex Lie algebra A_ℓ (where $\ell \geq 1$)

The simple complex Lie algebra A_ℓ is of dimension $(\ell + 1)^2 - 1$ and has $\frac{1}{2}\ell(\ell + 1)$ positive roots. In the present notation each positive root α may be expressed in the form

$$\alpha = \sum_{p=j}^k \alpha_p, \quad (\text{A.1})$$

where j, k are such that $1 \leq j \leq k \leq \ell$. The roots $\{\alpha_p\}_{p=1}^{\ell}$ are the simple roots of the algebra. The form $\langle \alpha, \beta \rangle$ (which is non-degenerate, symmetric and bilinear on the space of all linear functionals of \mathcal{A}) is specified by

$$\langle \alpha_j, \alpha_k \rangle = \begin{cases} (\ell + 1)^{-1} & j = k; 1 \leq j \leq \ell \\ -(\ell + 1)^{-1} & j = k \pm 1; 1 \leq j, k \leq \ell \\ 0 & \text{otherwise} \end{cases} \quad (\text{A.2})$$

The Cartan matrix of A_ℓ is an $(\ell \times \ell)$ matrix \mathbf{A} , where

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \tag{A.3}$$

The Dynkin diagram of A_ℓ is given in the diagram below.

$$\begin{array}{cccc} 1 & 1 & & 1 & 1 \\ \circ & \text{---} & \circ & \cdots & \text{---} & \circ & \text{---} & \circ \\ \alpha_1 & \alpha_2 & & \alpha_{\ell-1} & \alpha_\ell \end{array} \tag{A.4}$$

The simple complex Lie algebra B_ℓ (where $\ell \geq 1$)

The simple complex Lie algebra B_ℓ has a dimension of $\ell(2\ell + 1)$ and has ℓ^2 positive roots. In terms of the simple roots, each positive root α is given by

$$\alpha = \left\{ \begin{array}{l} \sum_{p=j}^{\ell} \alpha_p \quad (1 \leq j \leq \ell) \\ \sum_{p=j}^{k-1} \alpha_p + 2 \sum_{p=k}^{\ell} \alpha_p \quad (1 \leq j < k \leq \ell) \\ \sum_{p=j}^{k-1} \alpha_p \quad (1 \leq j < k \leq \ell) \end{array} \right\}. \tag{A.5}$$

The values of the bilinear symmetric form $\langle \alpha, \beta \rangle$ for the positive roots are given by

$$\langle \alpha_j, \alpha_k \rangle = \begin{cases} (2\ell - 1)^{-1} & (j = k; 1 \leq j \leq \ell) \\ \{2(2\ell - 1)\}^{-1} & (j = k = \ell) \\ -\{2(2\ell - 1)\}^{-1} & (j = k \pm 1; 1 \leq j, k \leq \ell) \\ 0 & \text{otherwise} \end{cases} \quad (\text{A.6})$$

The Cartan matrix \mathbf{A} of B_ℓ is the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -2 & 2 \end{bmatrix} \quad (\text{A.7})$$

The Dynkin diagram of B_ℓ is the diagram given below.

$$\begin{array}{cccc} 2 & 2 & & 2 & 1 \\ \circ & \text{---} & \cdots & \text{---} & \circ \\ \alpha_1 & \alpha_2 & & \alpha_{\ell-1} & \alpha_\ell \end{array} \quad (\text{A.8})$$

The simple complex Lie algebra C_ℓ (where $\ell \geq 1$)

The simple complex Lie algebra C_ℓ is of dimension $\ell(2\ell + 1)$, and has ℓ^2 positive roots. Each positive root α may be expressed in terms of the simple roots by

$$\alpha = \left\{ \begin{array}{ll} \sum_{p=j}^{k-1} \alpha_p & (1 \leq j < k \leq \ell) \\ \sum_{p=j}^{k-1} \alpha_p + 2 \sum_{p=k}^{\ell-1} \alpha_p + \alpha_\ell & (1 \leq j < k \leq \ell-1) \\ \sum_{p=j}^{\ell-1} \alpha_p + \alpha_\ell & (1 \leq j \leq \ell-1) \\ 2 \sum_{p=j}^{\ell-1} \alpha_p + \alpha_\ell & (1 \leq j \leq \ell-1) \\ \alpha_\ell & \end{array} \right. \quad (\text{A.9})$$

The values of the symmetric bilinear form $\langle \alpha, \beta \rangle$ are given (when α, β are both simple roots) below

$$\langle \alpha_j, \alpha_k \rangle = \left\{ \begin{array}{ll} \{2(\ell+1)\}^{-1} & (j = k; 1 \leq j \leq \ell-1) \\ (\ell+1)^{-1} & (j = k = \ell) \\ -\{4(\ell+1)\}^{-1} & (j = k \pm 1; 1 \leq j, k \leq \ell-1) \\ -\{2(\ell+1)\}^{-1} & (j = \ell-1, k = \ell \text{ and } j = \ell, k = \ell-1) \\ 0 & \text{otherwise} \end{array} \right. \quad (\text{A.10})$$

The Cartan matrix of the simple Lie algebra C_ℓ is given by

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \quad (\text{A.11})$$

The Dynkin diagram of C_ℓ is the diagram shown below.

$$\begin{array}{ccccccc} & 1 & & & 1 & & 2 \\ & \circ & \cdots & \cdots & \circ & \cdots & \circ \\ \alpha_1 & \alpha_2 & & & \alpha_{\ell-1} & & \alpha_\ell \end{array} \quad (\text{A.12})$$

The simple complex Lie algebra D_ℓ (where $\ell \geq 3$)

The dimension of D_ℓ is $\ell(2\ell - 1)$, and it has $\ell(\ell - 1)$ positive roots. In terms of the simple roots, a positive root α may be given by

$$\alpha = \left. \begin{array}{l} \left. \begin{array}{l} \sum_{p=j}^{k-1} \alpha_p + 2 \sum_{p=k}^{\ell-2} \alpha_p + \alpha_{\ell-1} + \alpha_\ell \\ \sum_{p=j}^{k-1} \alpha_p \end{array} \right\} (1 \leq j < k \leq \ell - 2) \\ \left. \begin{array}{l} \sum_{p=j}^{\ell-2} \alpha_p + \alpha_{\ell-1} + \alpha_\ell \\ \sum_{p=j}^{\ell-2} \alpha_p + \alpha_{\ell-1} \\ \sum_{p=j}^{\ell-2} \alpha_p + \alpha_\ell \\ \sum_{p=j}^{\ell-2} \alpha_p \end{array} \right\} (1 \leq j \leq \ell - 2) \end{array} \right\} \quad (\text{A.13})$$

The symmetric bilinear form $\langle \alpha, \beta \rangle$ is specified by

$$\langle \alpha_j, \alpha_k \rangle = \begin{cases} \{2(\ell - 1)\}^{-1} & (1 \leq j, k \leq \ell) \\ -\{4(\ell - 1)\}^{-1} & \begin{cases} j = k \pm 1; 1 \leq j, k \leq \ell - 3 \\ j = \ell - 2, k = \ell - 1, \ell \\ k = \ell - 2, j = \ell - 1, \ell \end{cases} \\ 0 & \text{otherwise} \end{cases} \quad (\text{A.14})$$

The Cartan matrix A is given by

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 2 \end{bmatrix} \tag{A.15}$$

The Dynkin diagram for the simple Lie algebra D_ℓ is shown in the diagram below.

$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & & \circ & & \\ & 1 & & 1 & & \alpha_{\ell-1} & \\ \circ & \circ & \cdots & \circ & & 1 & \\ \alpha_1 & \alpha_2 & & \alpha_{\ell-2} & & \circ & \\ & & & & & \alpha_\ell & \end{array} \tag{A.16}$$

The simple complex Lie algebra G_2

This simple Lie algebra is one of the exceptional ones. Its dimension is 14 and it has only six positive roots, which are α_1 , α_2 , $\alpha_1 + \alpha_2$, $\alpha_1 + 2\alpha_2$, $\alpha_1 + 3\alpha_2$ and $2\alpha_1 + 3\alpha_2$. The symmetric bilinear form $\langle \alpha, \beta \rangle$ has (when α and β are simple roots) the values given by

$$\langle \alpha_j, \alpha_k \rangle = \begin{cases} \left[\frac{1}{4} \quad (j=1, k=1) \right] \\ \left[-\frac{1}{8} \quad \begin{matrix} (j=1, k=2) \\ (j=2, k=1) \end{matrix} \right] \\ \left[\frac{1}{12} \quad (j=2, k=2) \right] \end{cases} \tag{A.17}$$

The Cartan matrix for G_2 is the 2×2 matrix \mathbf{A} , where

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \quad (\text{A.18})$$

The Dynkin diagram of G_2 is given in the figure below.

$$\begin{array}{cc} 3 & 1 \\ \circ & \text{---} & \circ \\ \alpha_1 & & \alpha_2 \end{array} \quad (\text{A.19})$$

The complex untwisted affine Kac-Moody algebra $A_1^{(1)}$

The corresponding simple Lie algebra for the complex untwisted affine Kac-Moody algebra $A_1^{(1)}$ is the Lie algebra A_1 . The root δ of $A_1^{(1)}$ is given in terms of the root α_0 and of the single simple root of A_1 by

$$\delta = \alpha_0 + \alpha_1. \quad (\text{A.20})$$

The generalised Cartan matrix for $A_1^{(1)}$ is the matrix \mathbf{A} , which is given by

$$\mathbf{A} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}. \quad (\text{A.21})$$

It is customary to number the rows and columns of this generalised Cartan matrix with the members of the index set $\{0,1\}$. The quantities $\langle \alpha_j, \alpha_k \rangle^{(1)}$, where $j=0,1$ are given by

$$\langle \alpha_j, \alpha_k \rangle = \frac{1}{4} (\mathbf{A})_{jk}. \quad (\text{A.22})$$

The generalised Dynkin diagram for this Kac-Moody algebra is displayed in the figure below:

$$\begin{array}{cc} \circ & \text{---} & \circ \\ \alpha_0 & & \alpha_1 \end{array} \quad (\text{A.23})$$

The complex untwisted affine Kac-Moody algebra $A_\ell^{(1)}$ (where $\ell \geq 2$)

In this case, the corresponding simple Lie algebra is A_ℓ . The root δ is given by

$$\delta = \sum_{p=0}^{\ell} \alpha_p. \tag{A.24}$$

The generalised Cartan matrix \mathbf{A} (which is indexed by the numbers $0, 1, \dots, \ell$) is the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -1 & 0 & 0 & \dots & 0 & -1 & 2 \end{bmatrix}. \tag{A.25}$$

For the special case $\ell = 2$, the generalised Cartan matrix is given by

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \tag{A.26}$$

whilst for the special case $\ell = 3$, it is given by

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}. \tag{A.27}$$

The quantities $\langle \alpha_j, \alpha_k \rangle^{(1)}$ (for $0 \leq j, k \leq \ell$) are given by

$$\langle \alpha_j, \alpha_k \rangle^{(1)} = \{2(\ell + 1)\}^{-1} (\mathbf{A})_{jk}. \quad (\text{A.28})$$

The generalised Dynkin diagram for $A_\ell^{(1)}$ is given in the diagram below.

$$\begin{array}{ccccccc} \circ & \circ & \dots & \circ & \circ & & \\ \alpha_0 & \alpha_1 & & \alpha_{\ell-1} & \alpha_\ell & & \end{array} \quad (\text{A.29})$$

The complex untwisted affine Kac-Moody algebra $B_\ell^{(1)}$ (where $\ell \geq 3$)

The corresponding simple Lie algebra is, in this case, the simple Lie algebra B_ℓ . The root δ is given by

$$\delta = \alpha_0 + \alpha_1 + 2 \sum_{p=2}^{\ell} \alpha_p. \quad (\text{A.30})$$

The generalised Cartan matrix is the matrix \mathbf{A} , which is given by

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -2 & 2 \end{bmatrix} \quad (\text{A.31})$$

For the special case $\ell = 3$, the generalised Cartan matrix is given by

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{bmatrix} \quad (\text{A.32})$$

The quantities $\langle \alpha_j, \alpha_k \rangle^{(1)}$, where $0 \leq j, k \leq \ell$, are given by

$$\langle \alpha_j, \alpha_k \rangle^{(1)} = (\mathbf{B})_{jk}, \quad (\text{A.33})$$

where the matrix \mathbf{B} is given by

$$\mathbf{B} = \frac{1}{2(2\ell - 1)} \begin{bmatrix} 2 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \quad (\text{A.34})$$

For the special case $\ell = 3$, the matrix \mathbf{B} is given by

$$\mathbf{B} = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad (\text{A.35})$$

The generalised Dynkin diagram for $B_\ell^{(1)}$ is given by the figure below.



The complex untwisted affine Kac-Moody algebra $C_\ell^{(1)}$ (where $\ell \geq 2$)

The simple Lie algebra which corresponds to this Lie algebra is the algebra C_ℓ . In this case, the root δ is given by

$$\delta = \alpha_0 + 2 \sum_{p=1}^{\ell-1} \alpha_p + \alpha_\ell. \quad (\text{A.37})$$

The generalised Cartan matrix is provided by

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}. \quad (\text{A.38})$$

For $\ell = 2$, the generalised Cartan matrix is given by

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}, \quad (\text{A.39})$$

whereas for $\ell = 3$, the generalised Cartan matrix is given by

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix}. \quad (\text{A.40})$$

The quantities $\langle \alpha_j, \alpha_k \rangle^{(1)}$ (for $0 \leq j, k \leq \ell$) are given by

$$\langle \alpha_j, \alpha_k \rangle^{(1)} = (\mathbf{B})_{jk}, \quad (\text{A.41})$$

where the matrix \mathbf{B} is

$$\mathbf{B} = \frac{1}{4(\ell+1)} \begin{bmatrix} 4 & -2 & 0 & 0 & \dots & 0 & 0 & 0 \\ -2 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & \dots & 0 & -2 & 4 \end{bmatrix}. \quad (\text{A.42})$$

Once again, for the special cases $\ell = 2, 3$ respectively, the matrix \mathbf{B} is given by

$$\mathbf{B} = \frac{1}{12} \begin{bmatrix} 4 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 4 \end{bmatrix}, \quad (\text{A.43})$$

$$\mathbf{B} = \frac{1}{16} \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix}. \quad (\text{A.44})$$

The generalised Dynkin diagram for $C_\ell^{(1)}$ is shown in the diagram below.

$$\begin{array}{ccccccc} \circ & \text{---} & \circ & \text{---} & \dots & \text{---} & \circ & \text{---} & \circ \\ \alpha_0 & & \alpha_1 & & & & \alpha_{\ell-1} & & \alpha_\ell \end{array} \quad (\text{A.45})$$

The complex untwisted affine Kac-Moody algebra $D_\ell^{(1)}$ (where $\ell \geq 4$)

The corresponding Lie algebra in this case is the simple Lie algebra D_ℓ . The root δ is given by the expression

$$\delta = \alpha_0 + \alpha_1 + 2 \sum_{p=2}^{\ell-2} \alpha_p + \alpha_{\ell-1} + \alpha_\ell. \tag{A.46}$$

For the affine untwisted algebra $D_\ell^{(1)}$, the generalised Cartan matrix \mathbf{A} is given by

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 2 \end{bmatrix}. \tag{A.47}$$

The generalised Cartan matrices for the special cases $\ell = 4, 5$ are given, respectively, by

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}, \tag{A.48}$$

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{bmatrix} \quad (\text{A.49})$$

The quantities $\langle \alpha_j, \alpha_k \rangle^{(1)}$ (for $0 \leq j, k \leq \ell$) are given by

$$\langle \alpha_j, \alpha_k \rangle^{(1)} = \{2(\ell - 1)\}^{-1} (\mathbf{A})_{jk}. \quad (\text{A.50})$$

The generalised Dynkin diagram for $D_\ell^{(1)}$ is given in the figure below.



The complex untwisted affine Kac-Moody algebra $G_2^{(1)}$

The corresponding simple Lie algebra is the algebra G_2 . It follows therefore, that the root δ is given by

$$\delta = \alpha_0 + 2\alpha_1 + 3\alpha_2. \quad (\text{A.52})$$

The generalised Cartan matrix for $G_2^{(1)}$ is given by

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{bmatrix}. \quad (\text{A.53})$$

Then, for j, k such that $j, k = 0, 1, 2$, the quantities $\langle \alpha_j, \alpha_k \rangle^{(1)}$ are given by

$$\langle \alpha_j, \alpha_k \rangle^{(1)} = (\mathbf{B})_{jk}, \quad (\text{A.54})$$

where the matrix \mathbf{B} is given by

$$\mathbf{B} = \frac{1}{24} \text{diag} \{3, 3, 1\} \mathbf{A}. \quad (\text{A.55})$$

The generalised Dynkin diagram for $G_2^{(1)}$ is given in the figure below.

$$\begin{array}{c} \circ \text{---} \circ \text{---} \circ \\ \alpha_0 \quad \alpha_1 \quad \alpha_2 \end{array} \quad (\text{A.56})$$

The complex twisted affine Kac-Moody algebra $A_2^{(2)}$

The corresponding simple Lie algebra is A_2 , whilst the rotation τ of the roots of A_2 (which is two-fold) is given by

$$\begin{aligned} \tau(\alpha_1) &= \alpha_2, \\ \tau(\alpha_2) &= \alpha_1. \end{aligned} \quad (\text{A.57})$$

The generalised Cartan matrix is given by

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}. \quad (\text{A.58})$$

The quantities $\langle \alpha_j, \alpha_k \rangle^{(2)}$ (for $j, k = 0, 1$), are given by

$$\langle \alpha_j, \alpha_k \rangle^{(2)} = (\mathbf{B})_{jk}, \quad (\text{A.59})$$

where \mathbf{B} is the 2×2 matrix defined by

$$\mathbf{B} = \frac{1}{2} \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \quad (\text{A.60})$$

The generalised Dynkin diagram for $A_2^{(2)}$ is shown in the underneath figure.

$$\begin{array}{c} \circ \text{---} \circ \\ \alpha_0 \quad \alpha_1 \end{array} \quad (\text{A.61})$$

Appendix B

Schur's lemma

The first theorem listed is a version of Schur's lemma. There are a few different ways in which the lemma may be stated, but the following is used here:

Let Γ be an irreducible representation (of dimension n) of a Lie algebra \mathcal{L} . If there exists a matrix \mathbf{B} of dimension n , which is such that

$$\Gamma(a)\mathbf{B} = \mathbf{B}\Gamma(a) \quad (\forall a \in \mathcal{L}), \quad (\text{B.1})$$

then the matrix \mathbf{B} must be a multiple of the unit matrix.

This lemma may also be used under certain circumstances for infinite-dimensional Lie algebras. Suppose that the matrix parts of members of an affine untwisted Kac-Moody algebra $\mathcal{L}^{(1)}$ are denoted by $\{\mathbf{a}(t)\}$. Then, suppose that there exists an n -dimensional polynomial matrix $\mathbf{B}(t)$ which is such that, for each matrix part $\mathbf{a}(t)$

$$\mathbf{a}(t)\mathbf{B}(t) = \mathbf{B}(t)\mathbf{a}(t). \quad (\text{B.2})$$

It follows then that, the matrix $\mathbf{B}(t)$ is such that

$$\mathbf{B}(t) = f(t)\mathbf{1}_n. \quad (\text{B.3})$$

This requires little proof. The untwisted affine Kac-Moody algebra contains its corresponding simple Lie algebra $\mathcal{L}^{(0)}$ as a proper subset. That is, the algebra $\mathcal{L}^{(1)}$ contains

$$\Gamma(a) \quad (\forall a \in \mathcal{L}^{(0)}). \quad (\text{B.4})$$

For a particular value of t , (say $t = t_1$), it follows that

$$\Gamma(a)\mathbf{B}(t_1) = \mathbf{B}(t_1)\Gamma(a) \quad (\forall \Gamma(a) \in \mathcal{L}^{(0)}). \quad (\text{B.5})$$

Since $\mathbf{B}(t_1)$ is a fixed matrix whose entries are merely complex numbers, the result of Schur's lemma (as given above) may be applied. This implies that, for each value t_j of t ,

$$\mathbf{B}(t_j) = \lambda_j \mathbf{1}_n. \quad (\text{B.6})$$

Since $\mathbf{B}(t)$ is a Laurent polynomial matrix, it may be inferred that

$$\mathbf{B}(t) = f(t) \mathbf{1}_n, \quad (\text{B.7})$$

and, moreover, it follows that $f(t) = \lambda t^h$, since the inverse of $\mathbf{B}(t)$ is itself a Laurent polynomial matrix.

Explicit realisations of irreducible representations

It is desirable in certain circumstances, to have explicit realisations of irreducible representations Γ . It may be assumed that the representation Γ is provided by anti-Hermitian matrices, which implies in turn that $\Gamma(h_\alpha)$ is Hermitian and

$$\Gamma(e_\alpha) = -\tilde{\Gamma}^*(e_{-\alpha}) \quad (\text{B.8})$$

for each $\alpha \in \Delta_+$. There is contained in chapter 16 of [13], a general procedure for obtaining such representations, in which each matrix $\Gamma(h)$ is diagonal. The process involved is only used a few times in the work, but need not be recalled here in full. The fact that $\Gamma(h)$ is diagonal is by far the most important fact that does need to be known, and reference should be made to [13] chapter 4 for further information.

Lemma

This proposition is used to help analyse certain pairs of automorphisms. We suppose that two matrices $\mathbf{U}_1(t)$ and $\mathbf{U}_2(t)$ are such that

$$\lambda_1 t^{\mu_1} \mathbf{U}_1(t) = \mathbf{R}(t) \mathbf{U}_2(t) \mathbf{R}(t)^{-1}, \quad (\text{B.9})$$

where $\mathbf{R}(t)$ is a Laurent polynomial matrix which satisfies the inequality

$$\tilde{\mathbf{R}}(t) \mathbf{g} \mathbf{R}(t) \neq \alpha t^\beta \mathbf{g}. \quad (\text{B.10})$$

Now, suppose that there does exist some Laurent polynomial matrix $\mathbf{S}(t)$ such that

$$\lambda_2 t^{\mu_2} \mathbf{U}_1(t) = \mathbf{S}(t) \mathbf{U}_2(t) \mathbf{S}(t)^{-1}, \quad (\text{B.11})$$

$$\tilde{\mathbf{S}}(t) \mathbf{g} \mathbf{S}(t) = \alpha t^\beta \mathbf{g}. \quad (\text{B.12})$$

Where α is some non-zero complex number, and β is an integer. The quantities λ_1, λ_2 are both non-zero complex numbers, but are not necessarily equal. It then follows that

$$\mathbf{S}(t) = \mathbf{R}(t) \mathbf{Q}(t), \quad (\text{B.13})$$

where the matrix $\mathbf{Q}(t)$ is such that

$$\mathbf{Q}(t)\mathbf{U}_2(t)\mathbf{Q}(t)^{-1} = \lambda_2\lambda_1^{-1}t^{\mu_2-\mu_1}\mathbf{U}_2(t). \quad (\text{B.14})$$

Proof

From (B.11), we have that $\mathbf{S}(t) = \lambda_2 t^{\mu_2} \mathbf{U}_1(t) \mathbf{S}(t) \mathbf{U}_2(t)^{-1}$, and we may substitute (B.9) into this, implying that

$$\mathbf{S}(t) = \lambda_2 \lambda_1^{-1} t^{\mu_2 - \mu_1} \mathbf{R}(t) \mathbf{U}_2(t) \mathbf{R}(t)^{-1} \mathbf{S}(t) \mathbf{U}_2(t)^{-1}. \quad (\text{B.15})$$

Hence, $\mathbf{S}(t) = \mathbf{R}(t) \mathbf{Q}(t)$ where $\mathbf{Q}(t) = \lambda_2 \lambda_1^{-1} t^{\mu_2 - \mu_1} \mathbf{U}_2(t) \mathbf{R}(t)^{-1} \mathbf{S}(t) \mathbf{U}_2(t)^{-1}$. It remains only to show that (B.14) holds. Direct substitution implies that

$$\mathbf{Q}(t) \mathbf{U}_2(t) \mathbf{Q}(t)^{-1} = \mathbf{U}_2(t) \mathbf{R}(t)^{-1} \mathbf{S}(t) \mathbf{U}_2(t) \mathbf{S}(t)^{-1} \mathbf{R}(t) \mathbf{U}_2(t)^{-1}. \quad (\text{B.16})$$

This equation is readily simplified using the information contained in (B.9) and (B.11), and this does indeed show that (B.14) holds.

Coxeter systems and Richardson's algorithm

The algorithm of Richardson [29] is a convenient way of parametrising the conjugacy classes of involutions within arbitrary Coxeter groups. The problem of parametrising all of the conjugacy classes of arbitrary Coxeter groups is a much more complex problem, to which at present there does not appear to be a satisfactory method. Coxeter systems are of interest since they include as special cases the Weyl groups. For reference, some information about Coxeter systems is included here. The notation and conventions are mainly those used by Humphreys [20], and this should be consulted for a broader knowledge of Coxeter systems.

We recall firstly the definition of a Coxeter system. This is a pair $(\mathcal{W}, \mathcal{S})$ that consists of a group \mathcal{W} and a set $\mathcal{S}(\subset \mathcal{W})$ of generators, in which the generators are subject only to relations of the form

$$(ss')^{m(s,s')} = 1.$$

Note that $m(s,s) = 1$ and that $m(s,s') \geq 2$ when $s \neq s'$. In all cases, $s, s' \in \mathcal{S}$. It is conventional to refer to the group \mathcal{W} as a Coxeter group.

Given a Coxeter system, we may represent the group \mathcal{W} as being generated by reflections. In this case, a reflection is a linear transformation which fixes some plane pointwise and sends some non-zero vector to its negative. We define the vector space V to have basis elements $\{\alpha_s | s \in \mathcal{S}\}$ which are in one-to-one correspondence with the generators of the Coxeter system. A symmetric bilinear form B is defined on V by

$$B(\alpha_s, \alpha_{s'}) = -\cos \frac{\pi}{m(s,s')}. \quad (\text{B.})$$

Then, for each $s \in \mathcal{S}$ we define a reflection $\sigma_s: V \rightarrow V$ (in the sense explained above) by

$$\sigma_s(\lambda) = \lambda - 2B(\alpha_s, \lambda)\alpha_s.$$

Clearly, each such reflection has order 2 in $GL(V)$. The reflection σ_s sends the vector α_s to its negative, whilst fixing pointwise the hyperplane orthogonal to this vector. We may define a mapping $\sigma: \mathcal{W} \rightarrow V$ by letting $\sigma(s) = \sigma_s$. It can be shown (see for example [20]) that σ is a homomorphism that preserves the form B on V . Moreover, for every pair $s, s' \in \mathcal{S}$, the order (in V) of $\sigma_s \sigma_{s'}$ is $m(s, s')$. The homomorphism σ is generally referred to as the geometric representation of the Coxeter system.

For a given Coxeter system, the root system Φ is defined to be the set of vectors $\{\alpha_s | s \in \mathcal{S}\}$. In the same way as the simple, positive and negative roots were

defined for the roots of Lie algebras, we may define simple, positive and negative roots for the Coxeter system.

The parabolic subgroup \mathcal{W}_I (of \mathcal{W}) is the subgroup generated by a given subset $I \subset \mathcal{S}$. Similarly, the subspace V_I is defined to be the space spanned by $\{\alpha_s | s \in I\}$.

For any $I \subset \mathcal{S}$, it may or may not be true that the operator $-1(\in \text{GL}(V_I))$ lies in \mathcal{W}_I . (Noting that the geometric representation of \mathcal{W}_I may be realised as the action of \mathcal{W}_I on V_I). When this is the case, we say that I satisfies the (-1) -condition, and we denote by w_I the operator $-1(\in \mathcal{W}_I)$. Two sets I, J which both satisfy the (-1) -condition are said to be \mathcal{W} -equivalent if there exists some $w \in \mathcal{W}$ which maps the set $\{\alpha_s | s \in I\}$ onto the set $\{\alpha_s | s \in J\}$. The main result of Richardson is the following theorem:

Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system, and \mathcal{J} be the set of all subsets of \mathcal{S} which satisfy the (-1) -condition. Then (a) Every involution $c \in \mathcal{W}$ is conjugate to w_J where $J \in \mathcal{J}$. (b) If $I, J \in \mathcal{J}$, then w_I and w_J are conjugate if and only if I and J are \mathcal{W} -equivalent.

Thus, the problem of parametrisation of conjugacy classes of involutions within a Coxeter group is equivalent to the problem of finding the \mathcal{W} -equivalence classes of all subsets of \mathcal{S} which satisfy the (-1) -condition.

The Algorithm

The algorithm thus determines when two subsets $J, K \in \mathcal{J}$ are \mathcal{W} -equivalent. If $J \in \mathcal{S}$ and $s \in \mathcal{S}$, then we let $L(s, J)$ denote the connected component of $J \cup \{s\}$ which contains s . The set $A(J)$ is the set of all $s \in \mathcal{S} \setminus \{J\}$ such that the Coxeter system $(\mathcal{W}_{L(s, J)}, L(s, J))$ is of finite type and does not satisfy the (-1) -condition. In fact, for $s \in (\mathcal{S} \setminus J)$, we have that $s \in A(J)$ if and only if the Coxeter graph of $L(s, J)$ is of one of the types $A_n (n > 1)$, D_{2n+1} , E_6 or $I_2(2p+1)$. Now if a subset I of \mathcal{S} is

such that the Coxeter group \mathcal{W}_I is finite, then a permutation ι_I may be defined upon the elements of I . With $s \in I$, the longest word w_I in \mathcal{W}_I is such that

$$w_I(\alpha_s) = -\alpha_{s'}, \quad \forall s \in I,$$

with $s' \in I$. The permutation ι_I is then defined by $\iota_I(s) = s'$.

Then, with $J \in \mathcal{S}$, $s \in A(J)$ and $s' = \iota_L(s)$, we define the set $K(s, J)$ by

$$K = K(s, J) = (J \cup \{s\} \setminus \{s'\}).$$

The sets J, K are then \mathcal{W} -equivalent. By working systematically through all of the subsets that satisfy the (-1) -condition, all of the \mathcal{W} -equivalence classes may be determined. The representative involution associated with the set $J \in \mathcal{J}$ is the element w_J , whose restriction to \mathcal{W}_J is the operator (-1) .

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