SEMIGROUPS OF SINGULAR ENDOMORPHISMS OF VECTOR SPACES

Robert J. H. Dawlings

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ABSTRACT

In 1967, J. A. Erdős showed, using a matrix theory approach, that the semigroup $Sing_n$ of singular endomorphisms of an n-dimensional vector space is generated by the set E of idempotent endomorphisms of rank $n - 1$. This thesis gives an alternative proof using a linear algebra and semigroup theory approach. It is also shown that not all the elements of E are needed to generate $Sing_n$. Necessary conditions for a subset of E to generate $Sing_n$ are found; these conditions are shown to be sufficient if the vector space is defined over a finite field. In this case, the minimum order of all subsets of E that generate $Sing_n$ is found. The problem of determining the number of subsets of E that generate $Sing_n$ and have this minimum order is considered; it is completely solved when the vector space is two-dimensional.

From the proof given by Erdős, it could be deduced that any element of $Sing_n$ could be expressed as the product of, at most, 2n elements of E. It is shown here that this bound may be reduced to n, and that this is best possible. It is also shown that, if E^+ is the set of all idempotents of $Sing_n$, then $(E^+)^{n-1}$ is strictly contained in $Sing_n$.

Finally, it is shown that Erdős's result cannot be extended to the semigroup Sing of continuous singular endomorphisms of a separable Hilbert space. The subsemigroup of Sing generated by the idempotents of Sing is determined and is, clearly, strictly contained in Sing.

DECLARATION

I declare that the following thesis is a record of research carried out by me, that the thesis is my own composition, and that it has not been accepted previously in application for a higher degree.

Robert J. H. Dawlings

DECLARATION

I declare that I was admitted in October 1977 under Court Ordinance General Number 12 as a full-time research student in the Department of Pure Mathematics.

Robert J. H. Dawlings

CERTIFICATE

I certify that Robert J. H. Dawlings has satisfied the conditions of the Resolution and Regulations and is thus qualified to submit the accompanying thesis in application for the degree of Doctor of Philosophy.

John M. Howie

PREFACE

I would like to thank the Science Research Council for their financial support over the past three years. Also, my thanks are due to Forrest and Grinsell Foundation for very generous grants to enable me to attend a conference at Tulane University, U.S.A., and to spend three months working at Monash University, Australia. I am very grateful to the Department of Pure Mathematics at Monash University for making my stay there so pleasant.

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CONTENTS

INTRODUCTION

If M is a mathematical system and End(M) is the set of endomorphisms of M then End(M) forms a semigroup under composition of mappings. Since 1966 a number of papers have been written to determine the subsemigroup S_M of End(M) generated by the idempotents E_M of End(M) for different systems M .

In [8] the problem was solved when M is a finite set, in this case End(M) being the full transformation semigroup $\mathcal{T}_M$. Here the subsemigroup generated was found to be $(\mathcal{T}_M \backslash \mathcal{G}_M) \cup \{Id\}$ where $\mathcal{G}_M$ is the symmetric group on M and Id is the identity mapping on M .

In [9] M was taken to be a totally-ordered set. If M is finite then the semigroup $\mathcal{O}_M$ of order-preserving mappings of M was shown to be generated by the idempotents of $\mathcal{O}_M$. If M is infinite and has order type ω (i.e. is isomorphic to $\mathbb{N}$ with the natural order) then necessary and sufficient conditions for certain elements of $\mathcal{O}_M$ to be idempotent generated were also given in [9].

In [14] this was generalised to an arbitrary well-ordered set and then in [17] to an arbitrary totally-ordered set.

Having ascertained the subsemigroup S_M generated by the idempotents E_M in these cases, various further questions arise. The most obvious is, are all the elements of E_M required to generate S_M ? If not, then the question arises of how small the order of a generating subset of E_M may be. From this the problem arises of ascertaining the number of ways it is possible to choose subsets of E_M that generate S_M and have this minimum order. In the case of M being a finite set, these questions have been solved in [8] and [11].

In any semigroup of endomorphisms of M we have

$$S_M = \langle E_M \rangle = \bigcup_{n=1}^{\infty} E_M^n$$

where $E_M \subseteq E_M^2 \subseteq E_M^3 \subseteq \dots$, and so for each element α of S_M there is a least integer $g(\alpha)$ such that

$$\alpha \in E_M^{g(\alpha)} .$$

The problem of ascertaining $g(\alpha)$ has been partially solved and the problem of finding $\sup\{g(\alpha) : \alpha \in S_M\}$ completely solved in [12] (and reported in [13]) for the case of M being a finite set. Comparable results may be deduced from [14] if M is a well-ordered set or a finite totally-ordered set.

In Chapter 1 I shall consider all these questions when M is an n-dimensional vector space V over a field F . Rather than consider the subsemigroup generated by E_V , I have considered the subsemigroup generated by $E_V \backslash \{I\}$ where I is the identity mapping. This restriction is of trivial effect since $\langle E_V \rangle = \langle E_V \backslash \{I\} \rangle \cup \{I\}$. It has already been shown, in [7], that $E_V \backslash \{I\}$ generates Sing_n , the semigroup of singular endomorphisms of an n-dimensional vector space. A more illuminating proof of this result is given as Theorem 1.4.9 . If F is finite, then the minimum order of a generating set of idempotents is found at Corollary 1.5.7 . An upper bound for the number of ways of choosing a generating set of idempotents with this minimal order is obtained in Lemma 1.7.7 , Lemma 1.7.15 and Lemma 1.7.18 . The final two questions are solved, for an arbitrary field, in Theorem 1.8.7 and Theorem 1.8.8 .

In Chapter 2 I shall determine S_H where H is a separable Hilbert space and $\mathrm{End}(H)$ is the semigroup of continuous linear mappings of H to itself.

Throughout this thesis the semigroup notation used shall be as in [5] and [10]. $\mathcal{V}$ will always denote an n-dimensional vector space (n finite) over a field F and $\mathcal{H}$ will denote a separable Hilbert space. $Sing_n$ will denote the semigroup of singular endomorphisms of $\mathcal{V}$ and Sing will denote the semigroup of singular continuous endomorphisms of $\mathcal{H}$. (Note that an element of Sing may have a null-space consisting solely of $\{\underline{0}\}$, for a continuous endomorphism α of $\mathcal{H}$ is singular if there does not exist a continuous endomorphism β of $\mathcal{H}$ such that $\alpha\beta$ is the identity mapping on $\mathcal{H}$.) PF^0_{n-1} will denote the principal factor of $Sing_n$ containing those elements of rank $n-1$ whereas PF_{n-1} will be the set of non-zero elements of PF^0_{n-1}. In Chapter 1 E will denote the idempotents in the set PF_{n-1}, in Chapter 2 E will denote all the idempotents of Sing. If α is an element of $Sing_n$ or Sing then the null-space of α will be denoted by $\mathcal{N}_\alpha$ and the range of α by $\mathcal{R}_\alpha$. At times $Sing_n$ will purposely be confused with the semigroup of singular n x n matrices. Throughout, elements of $Sing_n$ [Sing] will be written on the right of elements of $\mathcal{V}$ [$\mathcal{H}$].

CHAPTER 1

THE SEMIGROUP OF SINGULAR ENDOMORPHISMS OF A FINITE DIMENSIONAL VECTOR SPACE

§1 PRELIMINARIES

The first eleven lemmas are probably well known and are included here for the sake of completeness. The proofs of the first three, being elementary, are omitted.

1.1 LEMMA Let $\alpha,\beta \in \mathrm{Sing}_n$. Then $N_\alpha \subseteq N_{\alpha\beta}$ and $R_{\alpha\beta} \subseteq R_\beta$.

1.2 LEMMA Let $\alpha,\beta \in \mathrm{Sing}_n$. Then α,β and $\alpha\beta$ all have the same rank if and only if $N_\alpha = N_{\alpha\beta}$ and $R_{\alpha\beta} = R_\beta$.

1.3 LEMMA (Exercise 2.2.6 in [5]) Let $\alpha,\beta \in \mathrm{Sing}_n$. Then:

(i) $\alpha \mathcal{L} \beta$ if and only if $R_\alpha = R_\beta$

(ii) $\alpha \mathcal{R} \beta$ if and only if $N_\alpha = N_\beta$

(iii) $\alpha \mathcal{D} \beta$ if and only if α and β have the same rank

(iv) $\alpha \mathcal{D} \beta$ if and only if $\alpha \mathcal{J} \beta$.

1.4 LEMMA If $\varepsilon \in E$ then $N_\varepsilon \cap R_\varepsilon = \{\underline{0}\}$ and $V = N_\varepsilon \oplus R_\varepsilon$.

PROOF Let $\underline{x} \in N_\varepsilon \cap R_\varepsilon$. Then $\underline{0} = \underline{x}\varepsilon = \underline{x}$. So $N_\varepsilon \cap R_\varepsilon = \{\underline{0}\}$. Also, for all $\underline{x}$ in V, $\underline{x} \in N_\varepsilon + R_\varepsilon$ since $\underline{x} = (\underline{x}-\underline{x}\varepsilon) + \underline{x}\varepsilon$. So $V = N_\varepsilon + R_\varepsilon$ and hence $V = N_\varepsilon \oplus R_\varepsilon$.

1.5 LEMMA Let $\alpha,\beta \in \mathrm{Sing}_n$ be of rank r. Then $\alpha\beta$ is of rank r if and only if $R_\alpha \cap N_\beta = \{\underline{0}\}$.

PROOF Suppose first that $\alpha\beta$ is of rank r. Let

$\underline{x} \in R_\alpha \cap N_\beta$. Then there exists an element $\underline{y}$ in V such that $\underline{y}\alpha = \underline{x}$. Now $\underline{x}\beta = \underline{0}$ and so $\underline{y}\alpha\beta = \underline{0}$, i.e. $\underline{y} \in N_{\alpha\beta} = N_\alpha$ (by Lemma 1.2). So $\underline{x} = \underline{y}\alpha = \underline{0}$. Hence $R_\alpha \cap N_\beta = \{\underline{0}\}$.

Conversely suppose that $R_\alpha \cap N_\beta = \{\underline{0}\}$. Let $\underline{x} \in N_{\alpha\beta}$. Then $\underline{x}\alpha\beta = \underline{0}$ and so $\underline{x}\alpha \in N_\beta$. Hence $\underline{x}\alpha \in R_\alpha \cap N_\beta = \{\underline{0}\}$ by hypothesis. So $\underline{x} \in N_\alpha$. Thus we have $N_{\alpha\beta} \subseteq N_\alpha$. But (by Lemma 1.1) $N_\alpha \subseteq N_{\alpha\beta}$ and so $N_\alpha = N_{\alpha\beta}$. Hence $\alpha\beta$ is of the same rank as α , namely r .

<u>1.6 LEMMA</u> Every element of Sing_n of rank r has a (semigroup) inverse of rank r . Consequently Sing_n is regular.

<u>PROOF</u> Let α be an element of Sing_n of rank r . By [5 , Exercise 2.2.6] there exists an endomorphism β of V (not necessarily singular) such that $\alpha\beta\alpha = \alpha$. Now consider the element $\beta' = \beta\alpha\beta$. Clearly the rank of β' is less than or equal to the rank of α . But $\alpha\beta'\alpha = \alpha(\beta\alpha\beta)\alpha = (\alpha\beta\alpha)\beta\alpha = \alpha\beta\alpha = \alpha$ and so the rank of α is less than or equal to the rank of β' . Thus α and β' have the same rank. Also β' is an inverse of α for $\beta'\alpha\beta' = (\beta\alpha\beta)\alpha(\beta\alpha\beta)$ $= \beta(\alpha\beta\alpha)\beta\alpha\beta = \beta(\alpha\beta\alpha)\beta = \beta\alpha\beta = \beta'$. Thus β' is an inverse of α of rank r .

<u>1.7 LEMMA</u> Let $\alpha, \beta \in PF^0_{n-1}$. Then:

(i) $\alpha \mathcal{L} \beta$ if and only if $R_\alpha = R_\beta$

(ii) $\alpha \mathcal{R} \beta$ if and only if $N_\alpha = N_\beta$.

<u>PROOF</u> (i) By [10, Lemma II.4.1] $\alpha \mathcal{L} \beta$ if and only if there exist inverses α' and β' (of α and β respectively) in PF^0_{n-1} such that $\alpha'\alpha = \beta'\beta$. Now considering $\alpha, \alpha', \beta, \beta'$ as elements of Sing_n we still have that α' is an inverse of α , β' is an inverse

of β and that $\alpha'\alpha = \beta'\beta$. Thus (by [10, Lemma II.4.1]) α and β are $\mathcal{L}$-equivalent in Sing_n. So (by Lemma 1.3) $R_\alpha = R_\beta$.

Conversely, if $R_\alpha = R_\beta$ then (by Lemma 1.3) α and β are $\mathcal{L}$-equivalent in Sing_n. So (by [10, Lemma II.4.1]) there exist inverses α' and β' (of α and β respectively) in Sing_n such that $\alpha'\alpha = \beta'\beta$. By [10, Lemma II.3.5] α' and α are $\mathcal{D}$-equivalent in Sing_n, and β' and β are $\mathcal{D}$-equivalent in Sing_n. Thus (by Lemma 1.3) rank α' = rank α and rank β' = rank β. Thus $\alpha', \beta' \in PF^0_{n-1}$. So (by [10, Lemma II.4.1]) α and β are $\mathcal{L}$-equivalent in PF^0_{n-1}.

The proof of (ii) is dual to the proof of (i).

<u>1.8 LEMMA</u> PF^0_{n-1} is a completely 0-simple semigroup.

<u>PROOF</u> By [5, Lemma 2.39] PF^0_{n-1} is either 0-simple or null. PF^0_{n-1} is not null since it contains the $n \times n$ idempotent matrix

$$\begin{bmatrix} 0 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix}$$

of rank $n-1$. So PF^0_{n-1} is 0-simple. To show that PF^0_{n-1} is completely 0-simple, it will suffice to show that PF^0_{n-1} contains a primitive idempotent [10, Theorem III.3.1]. Let $\varepsilon, \phi \in PF^0_{n-1}$ be non-zero idempotents with $\varepsilon \leq \phi$. Then $\varepsilon = \varepsilon\phi = \phi\varepsilon$. So $N_\varepsilon = N_{\phi\varepsilon}$ and $R_\varepsilon = R_{\varepsilon\phi}$. But (by Lemma 1.2) $N_{\phi\varepsilon} = N_\phi$ and $R_{\varepsilon\phi} = R_\phi$. Thus $N_\varepsilon = N_\phi$ and $R_\varepsilon = R_\phi$. Hence (by Lemma 1.7) $\varepsilon \mathcal{L} \phi$ and $\varepsilon \mathcal{R} \phi$, i.e. $\varepsilon \mathcal{H} \phi$. But since each $\mathcal{H}$-class contains at most one idempotent [10, Corollary II.2.6] we have $\varepsilon = \phi$. So every non-zero idempotent of PF^0_{n-1} is primitive and PF^0_{n-1} contains a non-zero idempotent (as already shown). Hence PF^0_{n-1} is a completely 0-simple semigroup.

1.9 LEMMA (exercise 7.7.5 in [6]) Let $\alpha, \beta, \gamma \in PF^0_{n-1}$. Then $\alpha\beta\gamma = 0$ if and only if $\alpha\beta = 0$ or $\beta\gamma = 0$.

1.10 LEMMA Let $\alpha, \beta \in PF^0_{n-1}$. Then $\alpha\beta \neq 0$ if and only if there exists a non-zero idempotent $\varepsilon \in PF^0_{n-1}$ such that $\alpha \mathcal{L} \varepsilon$ and $\varepsilon \mathcal{R} \beta$.

PROOF If $\alpha\beta \neq 0$ then rank $\alpha\beta$ is $n - 1$, as are the ranks of α and β. So (by Lemma 1.2 and Lemma 1.3) $\alpha\beta \mathcal{L} \beta$ and $\alpha\beta \mathcal{R} \alpha$. Thus by Green's Lemmas [10, Lemma II.2.1] $\gamma \mapsto \gamma\beta$ and $\gamma \mapsto \gamma\beta'$ (where $\alpha = (\alpha\beta)\beta'$) are mutually inverse $\mathcal{R}$-class preserving bijections from L_α onto $L_{\alpha\beta}$ and $L_{\alpha\beta}$ onto L_α respectively. Thus $\beta\beta'$ is a non-zero element of PF^0_{n-1}, i.e. $\beta\beta'$ has rank $n - 1$. So (by Lemma 1.2 and Lemma 1.3) $\beta\beta' \mathcal{R} \beta$ and $\beta\beta' \mathcal{L} \beta'$. But $R_{\beta'} = R_\alpha$ and so $\beta' \mathcal{L} \alpha$. Thus $\beta\beta' \in L_\alpha \cap R_\beta$. Also since $\gamma \mapsto \gamma\beta'\beta$ is the identity mapping on $L_{\alpha\beta}$ we have $\beta = \beta\beta'\beta$. Hence $\beta\beta'$ is idempotent and so $L_\alpha \cap R_\beta$ contains an idempotent.

Conversely if $L_\alpha \cap R_\beta$ contains a non-zero idempotent ε we have that $\varepsilon\beta = \beta$ since an idempotent acts as a left identity within its $\mathcal{R}$-class. So by Green's Lemma $\gamma \mapsto \gamma\beta$ is a bijective $\mathcal{R}$-class preserving mapping from L_α onto L_β. Thus $\alpha\beta \in L_\beta \cap R_\alpha$. Thus $\alpha\beta \mathcal{D} \varepsilon$ and so $\alpha\beta$ has the same rank as ε, i.e. $\alpha\beta \neq 0$.

1.11 LEMMA Let $\alpha \in PF^0_{n-1}$. Then $N_\alpha \cap R_\alpha = \{\underline{0}\}$ if and only if there exists an idempotent $\varepsilon \in PF^0_{n-1}$ such that $\alpha \mathcal{H} \varepsilon$.

PROOF Suppose first that $N_\alpha \cap R_\alpha = \{\underline{0}\}$. Let $\underline{x} \in N_{\alpha^2}$. Then $\underline{x}\alpha^2 = \underline{0}$, i.e. $\underline{x\alpha} \in R_\alpha \cap N_\alpha$. Thus $\underline{x\alpha} = \underline{0}$ by hypothesis. Hence $\underline{x} \in N_\alpha$. Consequently $N_{\alpha^2} \subseteq N_\alpha$. But (Lemma 1.1) $N_\alpha \subseteq N_{\alpha^2}$ and so $N_\alpha = N_{\alpha^2}$. Thus (Lemma 1.3) $\alpha \mathcal{R} \alpha^2$. Also, since $\dim N_\alpha = \dim N_{\alpha^2}$ we have $\dim R_\alpha = \dim R_{\alpha^2}$. But $R_{\alpha^2} \subseteq R_\alpha$ and so $R_\alpha = R_{\alpha^2}$. Thus (by Lemma 1.3) $\alpha \mathcal{L} \alpha^2$. Hence $\alpha \mathcal{H} \alpha^2$. So (by [10, Theorem II.2.5]) H_α is a group. Thus H_α contains an idempotent.

Conversely (by Lemma 1.4) $N_\varepsilon \cap R_\varepsilon = \{\underline{0}\}$. Since (by Lemma 1.3) $N_\varepsilon = N_\alpha$ and $R_\varepsilon = R_\alpha$ we have $N_\alpha \cap R_\alpha = \{\underline{0}\}$.

1.12 THEOREM Let $\varepsilon, \phi \in PF^0_{n-1}$ be non-zero idempotents, and suppose that $R_\varepsilon \cap N_\phi = \{\underline{0}\}$ which (by Lemma 1.5) implies $\varepsilon\phi \neq 0$. Then $\varepsilon\phi$ is idempotent if and only if either:

(i) $\varepsilon\phi = \phi$ which happens if and only if $N_\varepsilon = N_\phi$ or

(ii) $\varepsilon\phi = \varepsilon$ which happens if and only if $R_\varepsilon = R_\phi$.

PROOF Suppose first that $\varepsilon\phi$ is idempotent and that $N_\varepsilon \neq N_\phi$. Let $\underline{x} \in V$. Then (by Lemma 1.4) for some $\underline{r} \in R_\varepsilon$ and some $\underline{n} \in N_\varepsilon$ we have

$$\underline{x}\varepsilon\phi = \underline{r} + \underline{n} \ . \qquad (1)$$

So

$$\underline{x}\varepsilon\phi\varepsilon = \underline{r} \ .$$

So substituting for $\underline{r}$ in (1) we have

$$\underline{x}\varepsilon\phi = \underline{x}\varepsilon\phi\varepsilon + \underline{n} \ .$$

Thus

$$\underline{x}\varepsilon\phi = \underline{x}\varepsilon\phi^2 = \underline{x}\varepsilon\phi\varepsilon\phi + \underline{n}\phi .$$

But since we have assumed that $\varepsilon\phi$ is idempotent this implies that $\underline{n} \in N_\phi$. But since both N_ε and N_ϕ are one-dimensional and we have assumed that $N_\varepsilon \neq N_\phi$ we have $N_\varepsilon \cap N_\phi = \{\underline{0}\}$. Thus $\underline{n} = \underline{0}$. Hence, from (1), $\underline{x}\varepsilon\phi \in R_\varepsilon$. But this holds for all $\underline{x}$ in V and so $R_{\varepsilon\phi} \subseteq R_\varepsilon$. But since $\varepsilon\phi \neq 0$ we have that $\dim R_{\varepsilon\phi} = n - 1$. Thus $R_{\varepsilon\phi} = R_\varepsilon$. Also (by Lemma 1.1) $R_{\varepsilon\phi} \subseteq R_\phi$ and so $R_{\varepsilon\phi} = R_\phi$. Thus $R_\varepsilon = R_\phi$. So if $\varepsilon\phi$ is idempotent then either $N_\varepsilon = N_\phi$ or $R_\varepsilon = R_\phi$.

We shall now show the equivalence in condition (i). Suppose that $\varepsilon\phi = \phi$. Then $N_{\varepsilon\phi} = N_\phi$. But $N_{\varepsilon\phi} \supseteq N_\varepsilon$ and $\dim N_{\varepsilon\phi} = \dim N_\varepsilon$ since $\varepsilon\phi$ and ε both have rank $n - 1$. Thus $N_{\varepsilon\phi} = N_\varepsilon$ and so $N_\phi = N_\varepsilon$. Conversely, suppose that $N_\phi = N_\varepsilon$. Then (by Lemma 1.3) $\varepsilon \mathcal{R} \phi$. But an idempotent acts as a left identity within its own $\mathcal{R}$-class and so $\varepsilon\phi = \phi$.

The proof of the equivalence in (ii) is dual.

It is immediate that if condition (i) or condition (ii) holds then $\varepsilon\phi$ is idempotent.

§2 STROKE PRODUCTS

The purpose of this section is to introduce a new notation for elements of E (i.e. the idempotents of $Sing_n$ of rank $n - 1$ or equivalently the non-zero idempotents of PF^0_{n-1}) and for the $\mathcal{H}$-classes of PF^0_{n-1} (and so for the $\mathcal{H}$-classes of the top $\mathcal{J}$-class of $Sing_n$). This new notation will make future results much clearer.

If $\varepsilon \in E$ then if we are to describe ε by giving its null-space and its range we have to give one vector for its null-space and $n - 1$ vectors to determine its range. Similarly to denote any $\mathcal{H}$-class in PF^0_{n-1} by giving vectors that determine the null-space and the range of elements in that H-class we again have to specify n vectors. This is somewhat cumbersome and nothing is saved from merely giving the matrix relative to some basis of any element in that H-class. The notation to be developed will reduce the number of vectors it is necessary to state to determine a particular H-class of PF^0_{n-1} or a particular element of E to just two.

2.1 DEFINITION Let ξ,χ be automorphisms of the field F such that $(\chi\xi^{-1})^2$ is the identity mapping. Let $\underline{a} = (a_1,a_2,\ldots,a_n)$ and $\underline{b} = (b_1,b_2,\ldots,b_n)$ be elements of V. The (ξ,χ)-stroke product of $\underline{a}$ with $\underline{b}$ is denoted by $\langle\underline{a}|\underline{b}\rangle_{(\xi,\chi)}$, or simply by $\langle\underline{a}|\underline{b}\rangle$, and is defined by

$$\langle\underline{a}|\underline{b}\rangle = \sum_{i=1}^{n}(a_i\xi)(b_i\chi) .$$

Clearly, if ξ is the identity and χ sends an element to its complex conjugate, then $\langle\cdot|\cdot\rangle$ is the normal inner product on an n-dimensional vector space over the field of complex numbers (or real numbers).

We shall regard ξ and χ as fixed in advance and shall not normally make explicit reference to them in definitions and statements.

2.2 DEFINITION If $\underline{a} = (a_1,a_2,\ldots,a_n)$ and $\underline{b} = (b_1,b_2,\ldots,b_n)$ are elements of V we shall say that $\underline{a}$ and $\underline{b}$ are perpendicular if $\langle\underline{a}|\underline{b}\rangle = 0$. This definition is reasonable since $\langle\underline{a}|\underline{b}\rangle = 0$ if and only if

$$\sum_{i=1}^{n}(a_i\xi)(b_i\chi) = 0 ,$$

i.e. if and only if

$$\left(\sum_{i=1}^{n}(a_i\xi)(b_i\chi)\right)\xi^{-1}\chi = 0 ,$$

i.e. if and only if

$$\sum_{i=1}^{n}(a_i\chi)(b_i\chi\xi^{-1}\chi) = 0 ,$$

i.e. if and only if

$$\sum_{i=1}^{n}(a_i\chi)(b_i\xi) = 0 ,$$

i.e. if and only if

$$\langle \underline{b} | \underline{a} \rangle = 0 .$$

If A is a subset of V, we shall define the perpendicular of A to be $A^{\perp} = \{\underline{x} \in V : \langle \underline{x} | \underline{a} \rangle = 0 \ (\forall\, \underline{a} \in A)\}$.

It is worth noting that in general A and $A^{\perp}$ are not disjoint. For example, if V is the two-dimensional vector space over the complex numbers and ξ and χ are both the identity mapping, then $(1,i) \in \langle(1,i)\rangle^{\perp}$ where $\langle(1,i)\rangle$ denotes the space generated by the vector $(1,i)$. Another simple example is obtained by taking V as the two-dimensional vector space over $\mathbb{Z}_2$, and ξ and χ as the identity mapping; then $(1,1) \in \langle(1,1)\rangle^{\perp}$.

It should also be noted that if A is any subset of $\mathcal{V}$ then $A^{\perp}$ is a subspace of $\mathcal{V}$.

2.3 LEMMA Let $\mathcal{V}$ be an n-dimensional vector space over the field F , and let $\mathcal{U}$ be a subspace of $\mathcal{V}$. Then $\dim \mathcal{U} = n - \dim \mathcal{U}^{\perp}$.

PROOF If A is an $m \times n$ matrix of rank r then $\{\underline{x} \in F^n : \underline{x}A = 0\}$ is a subspace of F^n of dimension $n - r$.

Now let $\dim \mathcal{U} = r$ and let $\{\underline{u}_1,\ldots,\underline{u}_r\}$ be a basis for $\mathcal{U}$, where $\underline{u}_i = (u_i^{(1)},u_i^{(2)},\ldots,u_i^{(n)})$. Then $\underline{x} = (x^{(1)},x^{(2)},\ldots,x^{(n)}) \in \mathcal{U}^{\perp}$ if and only if $\langle\underline{x}|\underline{u}_i\rangle = 0$ for $i = 1, 2, \ldots, r$, i.e. if and only if $(\underline{x}\xi)A = \underline{0}$ where $A = ((\underline{u}_1\chi)^T,(\underline{u}_2\chi)^T,\ldots,(\underline{u}_r\chi)^T)$ is an $n \times r$ matrix and $\underline{x}\xi = (x^{(1)}\xi,x^{(2)}\xi,\ldots,x^{(n)}\xi)$. Since the r columns are linearly independent, it follows that $\dim \mathcal{U}^{\perp} = n - r$.

2.4 LEMMA Let $\mathcal{U}$ and $\mathcal{W}$ be subspaces of $\mathcal{V}$. Then

(i) $(\mathcal{U}^{\perp})^{\perp} = \mathcal{U}$ and

(ii) if $\mathcal{U} \subset \mathcal{W}$ then $\mathcal{W}^{\perp} \subset \mathcal{U}^{\perp}$.

PROOF (i) Clearly $\mathcal{U} \subseteq (\mathcal{U}^{\perp})^{\perp}$. Since (by Lemma 2.3) $\dim (\mathcal{U}^{\perp})^{\perp} = n - \dim \mathcal{U}^{\perp} = n - (n-\dim\mathcal{U}) = \dim \mathcal{U}$ we have that $(\mathcal{U}^{\perp})^{\perp} = \mathcal{U}$.

(ii) Let $\underline{x} \in \mathcal{W}^{\perp}$. Then $\langle\underline{x}|\underline{w}\rangle = 0$ for all $\underline{w} \in \mathcal{W}$. So certainly $\langle\underline{x}|\underline{u}\rangle = 0$ for all $\underline{u} \in \mathcal{U}$ since $\mathcal{U} \subset \mathcal{W}$. Thus $\underline{x} \in \mathcal{U}^{\perp}$ and so $\mathcal{W}^{\perp} \subset \mathcal{U}^{\perp}$.

2.5 NOTATION Since every element in any particular R-class of PF_{n-1}^{0} has the same one-dimensional null-space we can label the R-classes of PF_{n-1}^{0} in the obvious way with an element of $\mathcal{V}$ that generates this one-dimensional subspace of V . Similarly, the L-classes of PF^0_{n-1} could be labelled in the obvious way with $n - 1$ elements of V that generate the common range. But since if $\dim \mathsf{U} = n - 1$ we have (by Lemma 2.3) that $\dim \mathsf{U}^{\perp} = 1$, it follows that we can label the L-classes of PF^0_{n-1} in an obvious way with an element of V that generates the one-dimensional subspace of V perpendicular to the common range of the elements in that L-class. Thus if α is a non-zero element of PF^0_{n-1} such that $\mathsf{N}_\alpha = \langle\underline{n}\rangle$ and $\mathsf{R}^{\perp}_\alpha = \langle\underline{r}\rangle$ then we can label the L-class containing α by $L_{\underline{r}}$, the R-class containing α by $R_{\underline{n}}$ and the H-class containing α by $H_{\underline{n},\underline{r}}$. As $H_{\underline{n},\underline{r}}$ is rather unwieldy this will usually be denoted by $[\underline{n}:\underline{r}]$. It is clear that $[\underline{n}:\underline{r}]$ denotes exactly one H-class for any choice of $\underline{n}$ and $\underline{r}$ in V (the fact that $[\underline{n}:\underline{r}]$ represents at least one H-class of PF^0_{n-1} is a result of [5, Exercise 2.2.6]). It is also clear that for any non-zero scalars λ and μ we have $[\underline{n}:\underline{r}] = [\lambda\underline{n}:\mu\underline{r}]$.

Having adopted this notation, it is then reasonable to introduce the following: If $[\underline{n}:\underline{r}]$ is a group H-class of PF^0_{n-1} we shall denote the idempotent in $[\underline{n}:\underline{r}]$ by $(\underline{n}:\underline{r})$. $(\underline{n}:\underline{r})$ is clearly unique since no H-class contains more than one idempotent.

With this notation we have a very simple way of telling if a particular H-class of PF^0_{n-1} contains an idempotent.

<u>2.6 LEMMA</u> $[\underline{n}:\underline{r}]$ is a group H-class of PF^0_{n-1} if and only if $\langle\underline{n}|\underline{r}\rangle \neq 0$.

<u>PROOF</u> Suppose that $[\underline{n}:\underline{r}]$ is a group H-class. Then $[\underline{n}:\underline{r}]$ contains the idempotent $\varepsilon = (\underline{n}:\underline{r})$. Now (by Lemma 1.4) $\mathsf{N}_\varepsilon \cap \mathsf{R}_\varepsilon = \{\underline{0}\}$ and since $\underline{n} \in \mathsf{N}_\varepsilon$ and $\underline{n} \neq \underline{0}$ we have $\underline{n} \notin \mathsf{R}_\varepsilon = (\mathsf{R}^{\perp}_\varepsilon)^{\perp}$. But since

$\underline{r} \in R_\varepsilon^\perp$ and $R_\varepsilon^\perp$ is one-dimensional we have $\langle \underline{n} | \underline{r} \rangle \neq 0$. Conversely, suppose $\langle \underline{n} | \underline{r} \rangle \neq 0$. Now there exists an element $\alpha \in PF_{n-1}^0$ such that $N_\alpha = \langle \underline{n} \rangle$ and $R_\alpha = \langle \underline{r} \rangle^\perp$ (by the comments of Notation 2.5). Since $\langle \underline{n} | \underline{r} \rangle \neq 0$ we have $\lambda \underline{n} \notin (R_\alpha^\perp)^\perp = R_\alpha$ for any non-zero scalar λ in F , i.e. $R_\alpha \cap N_\alpha = \{\underline{0}\}$. So (by Lemma 1.11) there exists an idempotent ε in PF_{n-1}^0 such that $\alpha \mathcal{H} \varepsilon$. Clearly $R_\varepsilon^\perp = R_\alpha^\perp = \langle \underline{r} \rangle$ and $N_\varepsilon = N_\alpha = \langle \underline{n} \rangle$ (by Lemma 1.7) and so $\varepsilon = (\underline{n}:\underline{r})$, i.e. $[\underline{n}:\underline{r}]$ contains an idempotent and so is a group $\mathcal{H}$-class.

This alternative notation for $\mathcal{H}$-classes of PF_{n-1}^0 enables us to rewrite Lemma 1.10 as:

<u>2.7 LEMMA</u> Let α and β be elements of PF_{n-1}^0 in $[\underline{n}_1:\underline{r}_1]$ and $[\underline{n}_2:\underline{r}_2]$ respectively. Then $\alpha\beta \neq 0$ if and only if $\langle \underline{n}_2 | \underline{r}_1 \rangle \neq 0$.

<u>PROOF</u> By Lemma 1.10, $\alpha\beta \neq 0$ if and only if there exists an idempotent ε in PF_{n-1}^0 such that $\alpha \mathcal{L} \varepsilon$ and $\varepsilon \mathcal{R} \beta$. Clearly $\alpha \in L_{\underline{r}_1}$ and $\beta \in R_{\underline{n}_2}$. Thus $\alpha\beta \neq 0$ if and only if there exists an idempotent ε in $L_{\underline{r}_1} \cap R_{\underline{n}_2} = [\underline{n}_2:\underline{r}_1]$, i.e. if and only if $[\underline{n}_2:\underline{r}_1]$ is a group $\mathcal{H}$-class. But (by Lemma 2.6) this happens if and only if $\langle \underline{n}_2 | \underline{r}_1 \rangle \neq 0$.

§3 <u>PRODUCTS OF THREE IDEMPOTENTS OF RANK n − 1</u>

The purpose of this section is to determine when the product of three idempotents of rank n − 1 is itself an idempotent of rank n − 1 . Lemma 3.1, Lemma 3.2, Lemma 3.12 and Theorem 3.14 give necessary and sufficient conditions for this to happen. It is in the case of Theorem 3.14 only that the product generates a new idempotent of rank $n - 1$.

Throughout this section we shall be changing backwards and forwards between the two notations for non-zero idempotents and H-classes other than $\{0\}$ of PF^0_{n-1} so we shall adopt the following conventions:

$$N_i = N_{\varepsilon_i} = \langle \underline{n}_i \rangle \ , \ \ R_i^{\perp} = R_{\varepsilon_i}^{\perp} = \langle \underline{r}_i \rangle$$

and so

$$\varepsilon_i = (\underline{n}_i : \underline{r}_i) \in [\underline{n}_i : \underline{r}_i] = H_{\varepsilon_i} \ .$$

We first dispose of a very trivial lemma which is included only for the sake of completeness since it does give sufficient conditions for the product of three idempotents of PF_{n-1} to be an idempotent of PF_{n-1} .

<u>3.1 LEMMA</u> Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be idempotents of PF_{n-1} . If

(i) $\varepsilon_1, \varepsilon_2, \varepsilon_3$ have a common null-space then $\varepsilon_1\varepsilon_2\varepsilon_3 = \varepsilon_3$; or

(ii) $\varepsilon_1, \varepsilon_2, \varepsilon_3$ have a common range then $\varepsilon_1\varepsilon_2\varepsilon_3 = \varepsilon_1$.

<u>PROOF</u> This is immediate from Theorem 1.12.

This is equivalent to:

<u>3.2 LEMMA</u> Let $\underline{n}_1, \underline{n}_2, \underline{n}_3, \underline{r}_1, \underline{r}_2$ and $\underline{r}_3$ be elements of V such that $\langle \underline{n}_i | \underline{r}_i \rangle \neq 0 \ (i=1,2,3)$. If

(i) $\langle \underline{n}_1 \rangle = \langle \underline{n}_2 \rangle = \langle \underline{n}_3 \rangle$ then $(\underline{n}_1 : \underline{r}_1)(\underline{n}_2 : \underline{r}_2)(\underline{n}_3 : \underline{r}_3) = (\underline{n}_3 : \underline{r}_3)$; or

(ii) $\langle \underline{r}_1 \rangle = \langle \underline{r}_2 \rangle = \langle \underline{r}_3 \rangle$ then $(\underline{n}_1 : \underline{r}_1)(\underline{n}_2 : \underline{r}_2)(\underline{n}_3 : \underline{r}_3) = (\underline{n}_1 : \underline{r}_1)$.

We may now concentrate on the case when the three idempotents $\varepsilon_1,\varepsilon_2,\varepsilon_3$ of PF^0_{n-1} in the product $\varepsilon_1\varepsilon_2\varepsilon_3$ have neither a common null-space nor a common range.

<u>3.3 LEMMA</u> Let $\varepsilon_1,\varepsilon_2,\varepsilon_3$ be non-zero idempotents of PF^0_{n-1}. If $\varepsilon_1\varepsilon_2\varepsilon_3$ is a non-zero idempotent and $\varepsilon_1,\varepsilon_2,\varepsilon_3$ do not have a common range then $\dim(N_1+N_2+N_3) \le 2$.

<u>PROOF</u> Let $\underline{x} \in V$. Then, by Lemma 1.4, there exists $\underline{s}_1 \in R_1$ and $\underline{m}_1 \in N_1$ such that

$$\underline{x}\varepsilon_1\varepsilon_2\varepsilon_3 = \underline{s}_1 + \underline{m}_1 . \qquad (+)$$

Again by Lemma 1.4 there exists $\underline{s}_2 \in R_2$ and $m_2 \in N_2$ such that $\underline{s}_1 = \underline{s}_2 + \underline{m}_2$. Thus

$$\underline{x}\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_1 = (\underline{s}_1+\underline{m}_1)\varepsilon_1 = \underline{s}_1 = \underline{s}_2 + \underline{m}_2 .$$

Hence $\underline{x}\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_1\varepsilon_2 = \underline{s}_2$. Thus $\underline{s}_1 = \underline{x}\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_1\varepsilon_2 + \underline{m}_2$. Now substituting this for $\underline{s}_1$ in (+) we obtain

$$\underline{x}\varepsilon_1\varepsilon_2\varepsilon_3 = \underline{x}\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_1\varepsilon_2 + \underline{m}_2 + \underline{m}_1 .$$

Thus

$$\underline{x}\varepsilon_1\varepsilon_2\varepsilon_3 = \underline{x}\varepsilon_1\varepsilon_2\varepsilon_3^2 = \underline{x}(\varepsilon_1\varepsilon_2\varepsilon_3)^2 + (\underline{m}_1+\underline{m}_2)\varepsilon_3 .$$

But we have assumed that $\varepsilon_1\varepsilon_2\varepsilon_3$ is idempotent and so $(\underline{m}_1+\underline{m}_2)\varepsilon_3 = \underline{0}$, i.e. $\underline{m}_1 + \underline{m}_2 \in N_3$. The elements $\underline{m}_1$ and $\underline{m}_2$ depend of course on the choice of the original element $\underline{x}$. If there exists an $\underline{x} \in V$ such that $\underline{m}_1 + \underline{m}_2 \ne \underline{0}$ then $\underline{m}_1 + \underline{m}_2$ generates N_3 (since N_3 is one-dimensional) and the result is immediate. If $\underline{m}_1 + \underline{m}_2 = \underline{0}$ for all choices of $\underline{x}$ in V then we have two cases to consider: (i) there exists an $\underline{x} \in V$ such that $\underline{m}_1 = -\underline{m}_2 \neq \underline{0}$ and (ii) for all choices of $\underline{x} \in V$ we have $\underline{m}_1 = \underline{m}_2 = \underline{0}$.

If case (i) occurs then it is clear that $N_1 = N_2$ and so again the result is immediate.

We shall now show that case (ii) cannot occur. Suppose that case (ii) does occur; then

$$\underline{x}\varepsilon_1\varepsilon_2\varepsilon_3 = \underline{s}_1 + \underline{m}_1 = \underline{s}_1 \in R_1 .$$

But since this holds for all $\underline{x}$ in V and (by Lemma 1.2) the range of $\varepsilon_1\varepsilon_2\varepsilon_3$ is the same as the range of ε_3 we have $R_3 \subseteq R_1$. Since $\dim R_3 = \dim R_1$ we thus have $R_3 = R_1$. Also

$$\underline{x}\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_1 = \underline{s}_2 + \underline{m}_2 = s_2 \in R_2$$

and so, by an argument similar to the above, $R_1 = R_2$. Thus $R_1 = R_2 = R_3$ which contradicts the hypothesis of the lemma. So, as claimed, case (ii) cannot occur.

Using the alternative notation for idempotents of PF_{n-1} this lemma may be stated as follows:

3.4 LEMMA Let $\underline{n}_1, \underline{n}_2, \underline{n}_3, \underline{r}_1, \underline{r}_2$ and $\underline{r}_3$ be elements of V such that $\langle \underline{n}_1 | \underline{r}_3 \rangle \neq 0$ and $\langle \underline{n}_i | \underline{r}_i \rangle \neq 0$ $(i=1,2,3)$. If $(\underline{n}_1 : \underline{r}_1)(\underline{n}_2 : \underline{r}_2)(\underline{n}_3 : \underline{r}_3) = (\underline{n}_1 : \underline{r}_3)$ and $\dim \langle \{\underline{r}_1, \underline{r}_2, \underline{r}_3\} \rangle \geq 2$ then $\dim \langle \{\underline{n}_1, \underline{n}_2, \underline{n}_3\} \rangle \leq 2$.

3.5 LEMMA Suppose $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are idempotents of PF^0_{n-1}. Then the following are equivalent:

(i) $\varepsilon_1\varepsilon_2\varepsilon_3$ is a non-zero idempotent of PF^0_{n-1}

(ii) $\varepsilon_2\varepsilon_3\varepsilon_1$ is a non-zero idempotent of PF^0_{n-1}

(iii) $\varepsilon_3\varepsilon_1\varepsilon_2$ is a non-zero idempotent of PF^0_{n-1} .

PROOF Clearly if we can show that (i) implies (ii) then we are able to modify the proof to obtain (ii) implies (iii) and (iii) implies (i).

Suppose that $\varepsilon_1\varepsilon_2\varepsilon_3$ is a non-zero idempotent of PF^0_{n-1} . Then $\varepsilon_1\varepsilon_2\varepsilon_3$ has rank $n - 1$ and so $n - 1 = \text{rank } \varepsilon_1\varepsilon_2\varepsilon_3$ $= \text{rank } \varepsilon_1(\varepsilon_2\varepsilon_3\varepsilon_1)\varepsilon_2\varepsilon_3 \le \text{rank } \varepsilon_2\varepsilon_3\varepsilon_1 \le n - 1$. Thus $\varepsilon_2\varepsilon_3\varepsilon_1$ is non-zero in PF^0_{n-1} . Also since $\varepsilon_2\varepsilon_3\varepsilon_1$ has rank $n - 1$ then ε_1 has rank $n - 1$. Now since the range of $\varepsilon_2\varepsilon_3\varepsilon_1$ is contained in R_1 we have that the range of $\varepsilon_2\varepsilon_3\varepsilon_1$ is R_1 . Now, by Lemma 1.2, $\varepsilon_1\varepsilon_2\varepsilon_3$ has null-space N_1 and range R_3 and so, by Lemma 1.4, $V = N_1 \oplus R_3$. Let $\underline{r}_1 \in R_1$, then there exist $\underline{r}_3 \in R_3$ and $\underline{n}_1$ in N_1 such that $\underline{r}_1 = (\underline{r}_3+\underline{n}_1)\varepsilon_1$, i.e. such that $\underline{r}_1 = \underline{r}_3\varepsilon_1$. Hence $\underline{r}_1\varepsilon_2\varepsilon_3\varepsilon_1 = \underline{r}_3(\varepsilon_1\varepsilon_2\varepsilon_3)\varepsilon_1$ $= \underline{r}_3\varepsilon_1$ since $\varepsilon_1\varepsilon_2\varepsilon_3$ acts identically on its range. But $\underline{r}_3\varepsilon_1 = r_1$ and so $\underline{r}_1\varepsilon_2\varepsilon_3\varepsilon_1 = \underline{r}_1$ for all $\underline{r}_1 \in R_1$. Hence $\varepsilon_2\varepsilon_3\varepsilon_1$ acts identically on its range and so is idempotent.

3.6 LEMMA Let U and W be subspaces of V . Then $(U \cap W)^\perp = U^\perp + W^\perp$.

PROOF Clearly $U \cap W \subseteq U$ and $U \cap W \subseteq W$ so, by Lemma 2.4, $U^\perp \subseteq (U \cap W)^\perp$ and $W^\perp \subseteq (U \cap W)^\perp$. Thus $U^\perp + W^\perp \subseteq (U \cap W)^\perp$.

Also $U^\perp \subseteq U^\perp + W^\perp$ and $W^\perp \subseteq U^\perp + W^\perp$ and so, by Lemma 2.4, $(U^\perp + W^\perp)^\perp \subseteq (U^\perp)^\perp = U$ and $(U^\perp + W^\perp)^\perp \subseteq (W^\perp)^\perp = W$. Thus $(U^\perp + W^\perp)^\perp \subseteq U \cap W$. So, again by Lemma 2.4, $(U \cap W)^\perp \subseteq ((U^\perp + W^\perp)^\perp)^\perp = U^\perp + W^\perp$. Thus

$(U \cap W)^{\perp} = U^{\perp} + W^{\perp}$.

3.7 LEMMA Let T, U, W be subspaces of V . Then $\dim V = \dim (T^{\perp}+U^{\perp}+W^{\perp}) + \dim (T \cap U \cap W)$.

PROOF By an obvious extension of Lemma 3.6, $(T \cap U \cap W)^{\perp} = T^{\perp} + U^{\perp} + W^{\perp}$. Also, by Lemma 2.3, $\dim V = \dim (T \cap U \cap W) + \dim (T \cap U \cap W)^{\perp}$. The result is now immediate.

3.8 LEMMA Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be non-zero idempotents of PF^{0}_{n-1} . If $\varepsilon_1\varepsilon_2\varepsilon_3$ is a non-zero idempotent and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ do not have a common null-space then $\dim (R_1 \cap R_2 \cap R_3) \geq n - 2$.

PROOF Suppose the result does not hold. By Lemma 3.7, we have $\dim (R_1 \cap R_2 \cap R_3) \geq n - 3$ and so we have $\dim (R_1 \cap R_2 \cap R_3) = n - 3$, and $\dim (R_1 \cap R_2) = \dim (R_2 \cap R_3) = \dim (R_3 \cap R_1) = n - 2$. Let $A = \{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_{n-3}\}$ be a basis of $R_1 \cap R_2 \cap R_3$ and extend A to bases $A \cup \{\underline{b}_1\}$ of $R_2 \cap R_3$, $A \cup \{\underline{b}_2\}$ of $R_3 \cap R_1$ and $A \cup \{\underline{b}_3\}$ of $R_1 \cap R_2$. Clearly $\underline{b}_i \notin R_i$.

Now consider $B = A \cup \{\underline{b}_1, \underline{b}_2\}$. Clearly $\langle B \rangle \subseteq R_3$ since $B \subseteq R_3$. Also it is clear that $R_2 \cap R_3 \subset \langle B \rangle$ since $\langle A \cup \{\underline{b}_1\} \rangle = R_2 \cap R_3$ and $\underline{b}_2 \notin R_2 \cap R_3$. Hence $n - 2 = \dim (R_2 \cap R_3) < \dim \langle B \rangle \leq \dim R_3 = n - 1$. Thus $\dim \langle B \rangle = \dim R_3$ and so B spans R_3 . Since B contains exactly $n - 1 = \dim R_3$ elements, B is a basis for R_3 .

Now consider $C = B \cup \{\underline{b}_3\}$. Clearly $R_3 \subset \langle C \rangle$ since $R_3 = \langle B \rangle$ and $\underline{b}_3 \notin R_3$. Hence $\dim \langle C \rangle = n$. Thus C spans V and, since C contains exactly n elements, is a basis for V .

Let $\underline{n}_1$ be a non-zero vector of N_1 . Then

$$\underline{n}_1 = \sum_{i=1}^{n-3}\lambda_i\underline{a}_i + \lambda_{n-2}\underline{b}_1 + \lambda_{n-1}\underline{b}_2 + \lambda_n\underline{b}_3 \ . \qquad (+)$$

Now, $\varepsilon_1\varepsilon_2\varepsilon_3$ acts identically on its range which, by Lemma 1.2, is R_3. So $\varepsilon_1\varepsilon_2\varepsilon_3$ acts identically on $A \cup \{\underline{b}_1,\underline{b}_2\}$. Also $\varepsilon_1\varepsilon_2$ acts identically on $\underline{b}_3$ since $\underline{b}_3 \in R_1 \cap R_2$. So acting on (+) by $\varepsilon_1\varepsilon_2\varepsilon_3$ we obtain

$$\underline{0} = \sum_{i=1}^{n-3}\lambda_i\underline{a}_i + \lambda_{n-2}\underline{b}_1 + \lambda_{n-1}\underline{b}_2 + \lambda_n\underline{b}_3\varepsilon_3 \ .$$

Subtracting this from (+) gives

$$\underline{n}_1 = \lambda_n\underline{b}_3 - \lambda_n\underline{b}_3\varepsilon_3 \ .$$

Hence

$$\underline{n}_1\varepsilon_3 = \lambda_n\underline{b}_3\varepsilon_3 - \lambda_n\underline{b}_3\varepsilon_3^2 = \underline{0}$$

since ε_3 is idempotent. Thus $\underline{n}_1 \in N_3$ and so $N_1 \subseteq N_3$. But, since N_1 and N_3 have the same dimension, this implies $N_1 = N_3$.

Similarly since we know (Lemma 3.5) that $\varepsilon_2\varepsilon_3\varepsilon_1$ is a non-zero idempotent, we may express a non-zero element $\underline{n}_2$ of N_2 as

$$\underline{n}_2 = \sum_{i=1}^{n-3}\mu_i\underline{a}_i + \mu_{n-2}\underline{b}_1 + \mu_{n-1}\underline{b}_2 + \mu_n\underline{b}_1$$

and act on this by $\varepsilon_2\varepsilon_3\varepsilon_1$ to obtain $N_2 = N_1$.

Hence $N_1 = N_2 = N_3$ which is contrary to the hypothesis. Thus $\dim (R_1 \cap R_2 \cap R_3) \neq n - 3$ and so the result holds.

3.9 LEMMA Let $\underline{n}_1,\underline{n}_2,\underline{n}_3,\underline{r}_1,\underline{r}_2$ and $\underline{r}_3$ be elements of V such that $\langle\underline{n}_1|\underline{r}_1\rangle,\langle\underline{n}_2|\underline{r}_2\rangle,\langle\underline{n}_3|\underline{r}_3\rangle$ and $\langle\underline{n}_1|\underline{r}_3\rangle$ are all non-zero. If $(\underline{n}_1:\underline{r}_1)(\underline{n}_2:\underline{r}_2)(\underline{n}_3:\underline{r}_3) = (\underline{n}_1:\underline{r}_3)$ and $\dim \langle\{\underline{n}_1,\underline{n}_2,\underline{n}_3\}\rangle \geq 2$ then

$\dim \langle\{\underline{r}_1,\underline{r}_2,\underline{r}_3\}\rangle \le 2$.

<u>PROOF</u> By virtue of Lemma 3.7, this result is identical to Lemma 3.8 using the alternative notation for idempotents of PF^0_{n-1} .

It is now immediate from Lemma 3.3 and Lemma 3.8 that:

<u>3.10 LEMMA</u> Let $\varepsilon_1,\varepsilon_2,\varepsilon_3$ be non-zero idempotents of PF^0_{n-1} . If $\varepsilon_1\varepsilon_2\varepsilon_3$ is idempotent and $\varepsilon_1,\varepsilon_2,\varepsilon_3$ have neither a common range nor a common null-space then $\dim (N_1+N_2+N_3) = 2$ and $\dim (R_1\cap R_2\cap R_3) = n - 2$.

It is also immediate, from Lemma 3.4 and Lemma 3.9, or direct from Lemma 3.10, that:

<u>3.11 LEMMA</u> Let $\underline{n}_1,\underline{n}_2,\underline{n}_3,\underline{r}_1,\underline{r}_2$ and $\underline{r}_3$ be elements of V such that $\langle\underline{n}_1|\underline{r}_1\rangle,\langle\underline{n}_2|\underline{r}_2\rangle,\langle\underline{n}_3|\underline{r}_3\rangle$ and $\langle\underline{n}_1|\underline{r}_3\rangle$ are all non-zero. If:

(i) $(\underline{n}_1:\underline{r}_1)(\underline{n}_2:\underline{r}_2)(\underline{n}_3:\underline{r}_3) = (\underline{n}_1:\underline{r}_3)$

(ii) $\dim \langle\{\underline{r}_1,\underline{r}_2,\underline{r}_3\}\rangle \ge 2$ and

(iii) $\dim \langle\{\underline{n}_1,\underline{n}_2,\underline{n}_3\}\rangle \ge 2$

then

(i) $\dim \langle\{\underline{n}_1,\underline{n}_2,\underline{n}_3\}\rangle = 2$ and

(ii) $\dim \langle\{\underline{r}_1,\underline{r}_2,\underline{r}_3\}\rangle = 2$.

The conditions given in Lemma 3.10 and Lemma 3.11 are not sufficient conditions for the product of three non-zero idempotents in PF^0_{n-1} to be a non-zero idempotent if the three idempotents have neither a common range nor a common null-space. To obtain sufficient conditions it is necessary to consider two different cases. The more interesting case is where the three null-spaces are distinct and the three ranges are distinct (i.e. where for $i,j = 1,2,3$ and $i \neq j$ then $N_i \neq N_j$ and $R_i \neq R_j$ or equivalently $\langle \underline{n}_i \rangle \neq \langle \underline{n}_j \rangle$ and $\langle \underline{r}_i \rangle \neq \langle \underline{r}_j \rangle$). This will be dealt with from Lemma 3.13 to the end of the section. Firstly we shall consider the case where two of the null-spaces are the same or two of the ranges are the same (i.e. where for some $i,j = 1,2,3$ and $i \neq j$ we have $N_i = N_j$ or $R_i = R_j$ or equivalently $\langle \underline{n}_i \rangle = \langle \underline{n}_j \rangle$ or $\langle \underline{r}_i \rangle = \langle \underline{r}_j \rangle$).

<u>3.12 LEMMA</u> Let $\underline{n}_1, \underline{n}_2, \underline{n}_3, \underline{r}_1, \underline{r}_2$ and $\underline{r}_3$ be elements of V such that:

(i) $\langle \underline{n}_1 | \underline{r}_1 \rangle, \langle \underline{n}_2 | \underline{r}_2 \rangle, \langle \underline{n}_3 | \underline{r}_3 \rangle$ and $\langle \underline{n}_1 | \underline{r}_3 \rangle$ are all non-zero

(ii) $\dim \langle \{\underline{n}_1, \underline{n}_2, \underline{n}_3\} \rangle \geq 2$ and

(iii) $\dim \langle \{\underline{r}_1, \underline{r}_2, \underline{r}_3\} \rangle \geq 2$.

Let $\alpha = (\underline{n}_1 : \underline{r}_1)(\underline{n}_2 : \underline{r}_2)(\underline{n}_3 : \underline{r}_3)$. If:

(iv) $\langle \underline{n}_1 \rangle, \langle \underline{n}_2 \rangle$ and $\langle \underline{n}_3 \rangle$ are not all distinct or

(iv') $\langle \underline{r}_1 \rangle, \langle \underline{r}_2 \rangle$ and $\langle \underline{r}_3 \rangle$ are not all distinct then

$\alpha = (\underline{n}_1 : \underline{r}_3)$ if and only if one of the following holds:

(a) $\langle \underline{n}_1 \rangle = \langle \underline{n}_2 \rangle$ and $\langle \underline{r}_2 \rangle = \langle \underline{r}_3 \rangle$ in which case $\alpha = (\underline{n}_2 : \underline{r}_2)$

(b) $\langle \underline{n}_2 \rangle = \langle \underline{n}_3 \rangle$ and $\langle \underline{r}_3 \rangle = \langle \underline{r}_1 \rangle$ in which case $\alpha = (\underline{n}_1 : \underline{r}_1)$

(c) $\langle \underline{n}_3 \rangle = \langle \underline{n}_1 \rangle$ and $\langle \underline{r}_1 \rangle = \langle \underline{r}_2 \rangle$ in which case $\alpha = (\underline{n}_3 : \underline{r}_3)$.

<u>PROOF</u> By Lemma 3.11 we have from (ii) and (iii) that $\dim \langle \{\underline{n}_1, \underline{n}_2, \underline{n}_3\} \rangle = \dim \langle \{\underline{r}_1, \underline{r}_2, \underline{r}_3\} \rangle = 2$.

Assume first that condition (iv) holds and that $\alpha = (\underline{n}_1 : \underline{r}_3)$. Then we have either:

(a') $\langle \underline{n}_1 \rangle = \langle \underline{n}_2 \rangle$

(b') $\langle \underline{n}_2 \rangle = \langle \underline{n}_3 \rangle$ or

(c') $\langle \underline{n}_3 \rangle = \langle \underline{n}_1 \rangle$.

(a') $\langle \underline{n}_1 \rangle = \langle \underline{n}_2 \rangle$ implies (by Theorem 1.12) that $(\underline{n}_1 : \underline{r}_1)(\underline{n}_2 : \underline{r}_2) = (\underline{n}_2 : \underline{r}_2)$. So (again by Theorem 1.12) $\alpha = (\underline{n}_1 : \underline{r}_3) = (\underline{n}_2 : \underline{r}_3)$ if and only if $\langle \underline{n}_2 \rangle = \langle \underline{n}_3 \rangle$ or $\langle \underline{r}_2 \rangle = \langle \underline{r}_3 \rangle$. But if $\langle \underline{n}_2 \rangle = \langle \underline{n}_3 \rangle$ then $\dim \langle \{\underline{n}_1, \underline{n}_2, \underline{n}_3\} \rangle = 1$ which is a contradiction. Thus $\langle \underline{r}_2 \rangle = \langle \underline{r}_3 \rangle$ which is result (a).

(b') $\langle \underline{n}_2 \rangle = \langle \underline{n}_3 \rangle$. Now (by Lemma 3.5), α is a non-zero idempotent if and only if $\varepsilon_2 \varepsilon_3 \varepsilon_1$ is a non-zero idempotent. But, by (a'), we have that $\varepsilon_2 \varepsilon_3 \varepsilon_1$ is a non-zero idempotent only if $\langle \underline{r}_3 \rangle = \langle \underline{r}_1 \rangle$. This is result (b).

(c') $\langle \underline{n}_3 \rangle = \langle \underline{n}_1 \rangle$. Again (by Lemma 3.5), α is a non-zero idempotent if and only if $\varepsilon_3 \varepsilon_1 \varepsilon_2$ is a non-zero idempotent. But, by (a'), we have that $\varepsilon_3 \varepsilon_1 \varepsilon_2$ is a non-zero idempotent only if $\langle \underline{r}_1 \rangle = \langle \underline{r}_2 \rangle$. This is result (c).

If, instead, we assume that condition (iv') holds and that $\alpha = (\underline{n}_1 : \underline{r}_3)$ then by a similar argument we again obtain (a), (b) and (c).

If (a), (b) or (c) hold, then, using Theorem 1.12, it is obvious that $\alpha = (\underline{n}_1 : \underline{r}_3)$.

Here again, as in Lemma 3.1, we have failed to generate a new non-zero idempotent of PF^0_{n-1} . The remainder of this section is concerned with the case when there are distinct null-spaces and distinct ranges for the three non-zero idempotents of PF^0_{n-1} in the product. It is in this case alone that the product of three non-zero idempotents of PF^0_{n-1} can produce a new non-zero idempotent of PF^0_{n-1} .

<u>3.13 LEMMA</u> Let $\underline{n}_1, \underline{n}_2, \underline{n}_3, \underline{r}_1, \underline{r}_2$ and $\underline{r}_3$ be elements of V

and $\nu_1, \nu_2, \nu_3, \rho_1, \rho_2$ and ρ_3 be elements of F such that

$$\nu_1\underline{n}_1 + \nu_2\underline{n}_2 + \nu_3\underline{n}_3 = \underline{0}$$

and

$$\rho_1\underline{r}_1 + \rho_2\underline{r}_2 + \rho_3\underline{r}_3 = \underline{0}$$

then the following are all equal:

(i) $\langle\nu_1\underline{n}_1|\rho_1\underline{r}_1\rangle + \langle\nu_1\underline{n}_1|\rho_2\underline{r}_2\rangle + \langle\nu_2\underline{n}_2|\rho_2\underline{r}_2\rangle$,

(ii) $\langle\nu_2\underline{n}_2|\rho_2\underline{r}_2\rangle - \langle\nu_1\underline{n}_1|\rho_3\underline{r}_3\rangle$

(iii) $\langle\nu_2\underline{n}_2|\rho_2\underline{r}_2\rangle + \langle\nu_2\underline{n}_2|\rho_3\underline{r}_3\rangle + \langle\nu_3\underline{n}_3|\rho_3\underline{r}_3\rangle$

(iv) $\langle\nu_3\underline{n}_3|\rho_3\underline{r}_3\rangle - \langle\nu_2\underline{n}_2|\rho_1\underline{r}_1\rangle$

(v) $\langle\nu_3\underline{n}_3|\rho_3\underline{r}_3\rangle + \langle\nu_3\underline{n}_3|\rho_1\underline{r}_1\rangle + \langle\nu_1\underline{n}_1|\rho_1\underline{r}_1\rangle$

(vi) $\langle\nu_1\underline{n}_1|\rho_1\underline{r}_1\rangle - \langle\nu_3\underline{n}_3|\rho_2\underline{r}_2\rangle$.

PROOF We shall only show that (i) = (ii) = (iii) since the remaining equalities follow in an identical manner.

Since $\rho_1\underline{r}_1 + \rho_2\underline{r}_2 + \rho_3\underline{r}_3 = \underline{0}$ we have

$$\langle\nu_1\underline{n}_1|\rho_1\underline{r}_1\rangle = \langle\nu_1\underline{n}_1|-\rho_2\underline{r}_2 - \rho_3\underline{r}_3\rangle$$

$$= -\langle\nu_1\underline{n}_1|\rho_2\underline{r}_2\rangle - \langle\nu_1\underline{n}_1|\rho_3\underline{r}_3\rangle .$$

Thus

$$\langle\nu_1\underline{n}_1|\rho_1\underline{r}_1\rangle + \langle\nu_1\underline{n}_1|\rho_2\underline{r}_2\rangle + \langle\nu_2\underline{n}_2|\rho_2\underline{r}_2\rangle = -\langle\nu_1\underline{n}_1|\rho_3\underline{r}_3\rangle + \langle\nu_2\underline{n}_2|\rho_2\underline{r}_2\rangle ,$$

i.e. (i) = (ii).

Since $\nu_1\underline{n}_1 + \nu_2\underline{n}_2 + \nu_3\underline{n}_3 = \underline{0}$ we have

$$\langle \nu_1 \underline{n}_1 | \rho_3 \underline{r}_3 \rangle = \langle -\nu_2 \underline{n}_2 - \nu_3 \underline{n}_3 | \rho_3 \underline{r}_3 \rangle$$

$$= -\langle \nu_2 \underline{n}_2 | \rho_3 \underline{r}_3 \rangle - \langle \nu_3 \underline{n}_3 | \rho_3 \underline{r}_3 \rangle .$$

Thus

$$\langle \nu_2 \underline{n}_2 | \rho_2 \underline{r}_2 \rangle - \langle \nu_1 \underline{n}_1 | \rho_3 \underline{r}_3 \rangle = \langle \nu_2 \underline{n}_2 | \rho_2 \underline{r}_2 \rangle + \langle \nu_2 \underline{n}_2 | \rho_3 \underline{r}_3 \rangle + \langle \nu_3 \underline{n}_3 | \rho_3 \underline{r}_3 \rangle ,$$

i.e. (ii) = (iii).

<u>3.14 THEOREM</u> Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be idempotent endomorphism of rank $n - 1$ of an n-dimensional vector space V over an arbitrary field F. Suppose that $\langle \underline{n}_1 \rangle, \langle \underline{n}_2 \rangle$ and $\langle \underline{n}_3 \rangle$ are distinct (where $\langle \underline{n}_i \rangle = N_{\varepsilon_i}$) and that $\langle \underline{r}_i \rangle, \langle \underline{r}_2 \rangle$ and $\langle \underline{r}_3 \rangle$ are distinct (where $\langle \underline{r}_i \rangle^{\perp} = R_{\varepsilon_i}$). Then $\varepsilon_1 \varepsilon_2 \varepsilon_3$ is an idempotent endomorphism of rank $n - 1$ if and only if there exist non-zero elements $\nu_1, \nu_2, \nu_3, \rho_1, \rho_2$ and ρ_3 of F such that:

(i) $\nu_1 \underline{n}_1 + \nu_2 \underline{n}_2 + \nu_3 \underline{n}_3 = \underline{0}$

(ii) $\rho_1 \underline{r}_1 + \rho_2 \underline{r}_2 + \rho_3 \underline{r}_3 = \underline{0}$ and

(iii) $\langle \nu_1 \underline{n}_1 | \rho_1 \underline{r}_1 \rangle + \langle \nu_1 \underline{n}_1 | \rho_2 \underline{r}_2 \rangle + \langle \nu_2 \underline{n}_2 | \rho_2 \underline{r}_2 \rangle = 0$.

Before starting the proof of this result, it is worth noting that the asymmetry of condition (iii) is only apparent. As given the left hand side does not contain an explicit reference to $\underline{n}_3$ or $\underline{r}_3$; however, Lemma 3.13 gives alternative forms of this which omit $\underline{n}_1$, and $\underline{r}_1$ or $\underline{n}_2$ and $\underline{r}_2$.

There are also several technical lemmas which would best be proved now rather than in the body of the proof.

<u>3.15 LEMMA</u> If we assume the conditions of the theorem and that $\varepsilon_1\varepsilon_2\varepsilon_3$ is idempotent, then there exist non-zero elements $\nu_1, \nu_2, \nu_3, \rho_1, \rho_2, \rho_3$ of F such that:

(i) $\nu_1\underline{n}_1 + \nu_2\underline{n}_2 + \nu_3\underline{n}_3 = \underline{0}$

(ii) $\rho_1\underline{r}_1 + \rho_2\underline{r}_2 + \rho_3\underline{r}_3 = \underline{0}$.

<u>PROOF</u> Since $\varepsilon_1\varepsilon_2\varepsilon_3$ is idempotent, we have (by Lemma 3.11) that $\dim \langle\{\underline{n}_1,\underline{n}_2,\underline{n}_3\}\rangle = \dim \langle\{\underline{r}_1,\underline{r}_2,\underline{r}_3\}\rangle = 2$. Since, by hypothesis, $\langle\underline{n}_1\rangle, \langle\underline{n}_2\rangle$ and $\langle\underline{n}_3\rangle$ are distinct and $\langle\underline{r}_1\rangle, \langle\underline{r}_2\rangle$ and $\langle\underline{r}_3\rangle$ are distinct, we have the result.

<u>3.16 LEMMA</u> Assuming the conditions of the theorem and that conditions (i), (ii) and (iii) hold, then $\varepsilon_1\varepsilon_2\varepsilon_3$ has rank $n - 1$ and belongs to a group H-class.

<u>PROOF</u> Since $\varepsilon_3 = (\underline{n}_3:\underline{r}_3)$ we have (by Lemma 2.6) that $\langle\underline{n}_3|\underline{r}_3\rangle \neq 0$. But, by (i) and (ii),

$$\begin{aligned}\langle\nu_3\underline{n}_3|\rho_3\underline{r}_3\rangle &= \langle-\nu_2\underline{n}_2-\nu_1 n_1|-\rho_2\underline{r}_2-\rho_1 r_1\rangle \\ &= \langle\nu_2\underline{n}_2|\rho_2\underline{r}_2\rangle + \langle\nu_2\underline{n}_2|\rho_1\underline{r}_1\rangle + \langle\nu_1\underline{n}_1|\rho_2\underline{r}_2\rangle + \langle\nu_1\underline{n}_1|\rho_1\underline{r}_1 \\ &= \langle\nu_2\underline{n}_2|\rho_1\underline{r}_1\rangle \quad \text{(by (iii))}\end{aligned}$$

Thus $\langle\underline{n}_2|\underline{r}_1\rangle \neq 0$ and so (by Lemma 2.7) $\varepsilon_1\varepsilon_2$ has rank $n - 1$. Similarly (but using also Lemma 3.13), since $\varepsilon_3 = (\underline{n}_3:\underline{r}_3)$ we have $\varepsilon_2\varepsilon_3$ has rank $n - 1$.

Thus (by Lemma 1.9), $\varepsilon_1\varepsilon_2\varepsilon_3$ has rank $n - 1$.

Again, by the above argument, since $\varepsilon_2 = (\underline{n}_2:\underline{r}_2)$ we have that $\langle\nu_1\underline{n}_1|\rho_3\underline{r}_3\rangle \neq 0$, i.e. that $[\underline{n}_1:\underline{r}_3]$ is a group H-class. Now (by Lemma 1.2)

$\varepsilon_1\varepsilon_2\varepsilon_3$ has null-space $\dot{N}_1$ and range R_3 (since we have already shown that $\varepsilon_1\varepsilon_2\varepsilon_3$ has rank $n - 1$). Thus $\varepsilon_1\varepsilon_2\varepsilon_3 \in [\underline{n}_1 : \underline{r}_3]$ and so $\varepsilon_1\varepsilon_2\varepsilon_3$ belongs to a group H-class.

<u>3.17 LEMMA</u> Given the conditions of the theorem, suppose that there exist non-zero elements $\nu_1, \nu_2, \nu_3, \rho_1, \rho_2$ and ρ_3 of F such that:

(i) $\nu_1\underline{n}_1 + \nu_2\underline{n}_2 + \nu_3\underline{n}_3 = \underline{0}$

(ii) $\rho_1\underline{r}_1 + \rho_2\underline{r}_2 + \rho_3\underline{r}_3 = \underline{0}$

and also that:

(iii) $\langle\underline{n}_2|\underline{r}_3\rangle = 0$

(iv) $\langle\underline{n}_1|\underline{r}_3\rangle \neq 0$

(v) $\varepsilon_1\varepsilon_2\varepsilon_3$ has rank $n - 1$.

Then there exist non-zero elements λ_1, λ_3 of F such that

$$\langle\lambda_1\underline{n}_1|\rho_1\underline{r}_1\rangle + \langle\lambda_3\underline{n}_3|\rho_2\underline{r}_2\rangle = 0 .$$

Furthermore, $\varepsilon_1\varepsilon_2\varepsilon_3$ is idempotent if and only if $\lambda_3\nu_1 + \lambda_1\nu_3 = 0$.

<u>PROOF</u> By (ii) and by the conditions of the theorem, we have that $\dim (R_1^\perp + R_2^\perp + R_3^\perp) = 2$. Thus, by Lemma 3.7, $\dim (R_1 \cap R_2 \cap R_3) = n - 2$. So there exists a basis $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_{n-2}\}$ of $R_1 \cap R_2 \cap R_3$. Since $\langle\underline{n}_2|\underline{r}_3\rangle = 0$, we have that $\underline{n}_2 \in R_3$. But $\underline{n}_2 \notin R_1 \cap R_2 \cap R_3$ for otherwise we would have $n_2 \in R_2$ contrary to Lemma 1.4. Thus $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_{n-2}, \underline{n}_2\}$ is a basis for R_3 .

Now, since $N_3 \cap R_3 = \{\underline{0}\}$ (by Lemma 1.4), we have that $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_{n-2}, \underline{n}_2, \underline{n}_3\}$ is a basis for V . Thus there exist σ_2, σ_3 in F such that $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_{n-2}, \sigma_2\underline{n}_2 + \sigma_3\underline{n}_3\}$ is a basis for R_2 . Now, if $\sigma_2 = 0$, then we would have $\underline{n}_3 \in R_2$, i.e. $\langle\underline{n}_3|\underline{r}_2\rangle = 0$ and so (by

Lemma 2.7) $\varepsilon_2\varepsilon_3$ would have rank less than $n - 1$. This is contrary to (v). Thus $\sigma_2 \neq 0$. If $\sigma_3 = 0$, then we would have $\underline{n}_2 \in \mathcal{R}_2$ contrary to Lemma 1.4. Thus $\sigma_3 \neq 0$. So, putting $\lambda_3 = \sigma_2^{-1}\sigma_3$, we have that $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_{n-2}, \underline{n}_2+\lambda_3\underline{n}_3\}$ is a basis for $\mathcal{R}_2$ where λ_3 is a non-zero element of F.

Now, by (iv), $\langle\underline{n}_1|\underline{r}_3\rangle \neq 0$. Thus (by Lemma 2.6), $[\underline{n}_1:\underline{r}_3]$ is a group H-class. So (by Lemma 1.4), $\mathcal{N}_1 \cap \mathcal{R}_3 = \{\underline{0}\}$. Thus $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_{n-2}, \underline{n}_2, \underline{n}_1\}$ is a basis for V. Hence there exist τ_1, τ_2 in F such that $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_{n-2}, \tau_1\underline{n}_1+\tau_2\underline{n}_2\}$ is a basis for $\mathcal{R}_1$. If $\tau_1 = 0$, then we would have $\underline{n}_2 \in \mathcal{R}_1$, i.e. that $\langle\underline{n}_2|\underline{r}_1\rangle = 0$. So (by Lemma 2.7), $\varepsilon_1\varepsilon_2$ would have rank less than $n - 1$. This contradicts (v) and so $\tau_1 \neq 0$. If $\tau_2 = 0$, then we would have $\underline{n}_1 \in \mathcal{R}_1$. This contradicts Lemma 1.4 and so $\tau_2 \neq 0$. So, putting $\lambda_1 = \tau_2^{-1}\tau_1$, we have that $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_{n-2}, \lambda_1\underline{n}_1+\underline{n}_2\}$ is a basis for $\mathcal{R}_1$ where λ_1 is a non-zero element of F.

Now since $\underline{n}_2 + \lambda_3\underline{n}_3 \in \mathcal{R}_2$ we have

$$\langle\underline{n}_2+\lambda_3\underline{n}_3|\underline{r}_2\rangle = 0 ,$$

i.e.

$$\langle\underline{n}_2|\rho_2\underline{r}_2\rangle + \langle\lambda_3\underline{n}_3|\rho_2\underline{r}_2\rangle = 0 .$$

So, by (ii),

$$\langle\underline{n}_2|-\rho_1\underline{r}_1-\rho_3\underline{r}_3\rangle + \langle\lambda_3\underline{n}_3|\rho_2\underline{r}_2\rangle = 0 .$$

Thus

$$-\langle\underline{n}_2|\rho_1\underline{r}_1\rangle - \langle\underline{n}_2|\rho_3\underline{r}_3\rangle + \langle\lambda_3\underline{n}_3|\rho_2\underline{r}_2\rangle = 0 .$$

But, by (iii), $\langle\underline{n}_2|\underline{r}_3\rangle = 0$. So

$$-\langle \underline{n}_2 | \rho_1 \underline{r}_1 \rangle + \langle \lambda_3 \underline{n}_3 | \rho_2 \underline{r}_2 \rangle = 0 \ . \tag{A}$$

Also, since $\underline{n}_2 + \lambda_1 \underline{n}_1 \in R_1$, we have

$$\langle \underline{n}_2 + \lambda_1 \underline{n}_1 | \underline{r}_1 \rangle = 0 \ ,$$

i.e.

$$\langle \underline{n}_2 | \rho_1 \underline{r}_1 \rangle + \langle \lambda_1 \underline{n}_1 | \rho_1 \underline{r}_1 \rangle = 0 \ .$$

Adding this to (A) gives

$$\langle \lambda_1 \underline{n}_1 | \rho_1 \underline{r}_1 \rangle + \langle \lambda_3 \underline{n}_3 | \rho_2 \underline{r}_2 \rangle = 0$$

as required.

Finally, $\varepsilon_1 \varepsilon_2 \varepsilon_3$ is idempotent if and only if it acts identically on a basis of its range. Now, by (v) and Lemma 1.2, $\varepsilon_1 \varepsilon_2 \varepsilon_3$ has range R_3 . $\varepsilon_1 \varepsilon_2 \varepsilon_3$ clearly acts identically on $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_{n-2}\}$ so it is idempotent if and only if it acts identically on $\underline{n}_2$.

Now,

$$\nu_1 \lambda_3 \underline{n}_2 = \nu_1 \lambda_3 (\underline{n}_2 + \lambda_1 \underline{n}_1) - \nu_1 \lambda_1 \lambda_3 \underline{n}_1$$

and so, since $\underline{n}_2 + \lambda_1 \underline{n}_1 \in R_1$,

$$\begin{aligned}
\nu_1 \lambda_3 \underline{n}_2 \varepsilon_1 &= \nu_1 \lambda_3 (\underline{n}_2 + \lambda_1 \underline{n}_1) \\
&= \nu_1 \lambda_3 \underline{n}_2 + \nu_1 \lambda_1 \lambda_3 \underline{n}_1 \\
&= \nu_1 \lambda_3 \underline{n}_2 - \nu_2 \lambda_1 \lambda_3 \underline{n}_2 - \nu_3 \lambda_1 \lambda_3 \underline{n}_3 \quad \text{(by (i))} \\
&= -\lambda_1 \nu_3 (\underline{n}_2 + \lambda_3 \underline{n}_3) + (\lambda_1 \nu_3 + \nu_1 \lambda_3 - \lambda_1 \lambda_3 \nu_2) \underline{n}_2 \ .
\end{aligned}$$

Thus, since $\underline{n}_2 + \lambda_3 \underline{n}_3 \in R_2$,

$$\nu_1\lambda_3\underline{n}_2\varepsilon_1\varepsilon_2 = -\lambda_1\nu_3(\underline{n}_2+\lambda_3\underline{n}_3) \ .$$

So, since $\underline{n}_2 \in R_3$,

$$\nu_1\lambda_3\underline{n}_2\varepsilon_1\varepsilon_2\varepsilon_3 = -\lambda_1\nu_3\underline{n}_2 \ .$$

Thus $\varepsilon_1\varepsilon_2\varepsilon_3$ acts identically on $\underline{n}_2$ if and only if $\nu_1\lambda_3 = -\lambda_1\nu_3$. Hence $\varepsilon_1\varepsilon_2\varepsilon_3$ is idempotent if and only if $\nu_1\lambda_3 + \lambda_1\nu_3 = 0$.

<u>3.18 LEMMA</u> Given the conditions of the theorem, suppose that there exist non-zero elements $\nu_1,\nu_2,\nu_3,\rho_1,\rho_2,\rho_3$ of F such that:

(i) $\nu_1\underline{n}_1 + \nu_2\underline{n}_2 + \nu_3\underline{n}_3 = \underline{0}$

(ii) $\rho_1\underline{r}_1 + \rho_2\underline{r}_2 + \rho_3\underline{r}_3 = \underline{0}$

and also that:

(iii) $\langle\underline{n}_1|\underline{r}_2\rangle \neq 0$

(iv) $\langle\underline{n}_2|\underline{r}_3\rangle \neq 0$

(v) $\langle\underline{n}_1|\underline{r}_3\rangle \neq 0$

(vi) $\varepsilon_1\varepsilon_2\varepsilon_3$ has rank $n - 1$.

Then there exist non-zero elements $\lambda_1,\lambda_2,\mu_1$ in F such that:

(A) $\langle\mu_1\underline{n}_1|\rho_2\underline{r}_2\rangle + \langle\lambda_1\underline{n}_1|\rho_1\underline{r}_1\rangle = 0$

(B) $\langle\lambda_2\underline{n}_2|\rho_2\underline{r}_2\rangle - \langle\mu_1\underline{n}_1|\rho_2\underline{r}_2\rangle = 0$.

Furthermore, $\varepsilon_1\varepsilon_2\varepsilon_3$ is idempotent if and only if $\lambda_2\mu_1\nu_1 - \lambda_1\lambda_2\nu_1 - \lambda_1\mu_1\nu_2 = 0$.

<u>PROOF</u> By (ii) and by the conditions of the theorem, we have that $\dim(R_1^\perp+R_2^\perp+R_3^\perp) = 2$. Thus, by Lemma 3.7, $\dim(R_1\cap R_2\cap R_3) = n - 2$. So there exists a basis $\{\underline{u}_1,\underline{u}_2,\ldots,\underline{u}_{n-2}\}$ of $R_1 \cap R_2 \cap R_3$. Extend

this to a basis $\{\underline{u}_1,\underline{u}_2,\ldots,\underline{u}_{n-2},\underline{x}\}$ of R_3.

Since, by (v), $\langle\underline{n}_1|\underline{r}_3\rangle \neq 0$, we have that $\underline{n}_1 \notin R_3$. Thus we can extend the basis of R_3 to a basis $\{\underline{u}_1,\underline{u}_2,\ldots,\underline{u}_{n-2},\underline{x},\underline{n}_1\}$ of V. Thus there exist σ_1,σ_2 of F such that $\{\underline{u}_1,\underline{u}_2,\ldots,\underline{u}_{n-2},\sigma_1\underline{x}+\sigma_2\underline{n}_1\}$ is a basis of R_1. Now, if $\sigma_1 = 0$, then we would have $\underline{n}_1 \in R_1$ which contradicts Lemma 1.4. Hence $\sigma_1 \neq 0$. If $\sigma_2 = 0$, then we would have $R_1 = R_3$ which contradicts the hypothesis of the theorem that $\langle\underline{r}_1\rangle$ and $\langle\underline{r}_3\rangle$ are distinct. Thus $\sigma_2 \neq 0$. If we now put $\lambda_1 = \sigma_2\sigma_1^{-1}$, then we obtain $\{\underline{u}_1,\underline{u}_2,\ldots,\underline{u}_{n-2},\underline{x}+\lambda_1\underline{n}_1\}$ to be a basis of R_1 where λ_1 is a non-zero element of F.

Since $\{\underline{u}_1,\underline{u}_2,\ldots,\underline{u}_{n-2},\underline{x},\underline{n}_1\}$ is a basis of V, there exist τ_1,τ_2 in F such that $\{\underline{u}_1,\underline{u}_2,\ldots,\underline{u}_{n-2},\tau_1\underline{x}+\tau_2\underline{n}_1\}$ is a basis for R_2. If $\tau_1 = 0$, then we have $\underline{n}_1 \in R_2$, i.e. $\langle\underline{n}_1|\underline{r}_2\rangle = 0$. But this contradicts (iii) and so $\tau_1 \neq 0$. If $\tau_2 = 0$, then $R_2 = R_3$ which again contradicts the hypothesis of the theorem. Thus $\tau_2 \neq 0$. If we now put $\mu_1 = \tau_2\tau_1^{-1}$, then we obtain $\{\underline{u}_1,\underline{u}_2,\ldots,\underline{u}_{n-2},\underline{x}+\mu_1\underline{n}_1\}$ to be a basis of R_2 where μ_1 is a non-zero element of F.

Since, by (iv), $\langle\underline{n}_2|\underline{r}_3\rangle \neq 0$, we have $\underline{n}_2 \notin R_3$. So we can extend the basis of R_3 to a basis $\{\underline{u}_1,\underline{u}_2,\ldots,\underline{u}_{n-2},\underline{x},\underline{n}_2\}$ of V. So there exist elements ω_1,ω_2 of F such that $\{\underline{u}_1,\underline{u}_2,\ldots,\underline{u}_{n-2},\omega_1\underline{x}+\omega_2\underline{n}_2\}$ is a basis for R_2. If $\omega_1 = 0$, then we would have $\underline{n}_2 \in R_2$ contradicting Lemma 1.4. So $\omega_1 \neq 0$. If $\omega_2 = 0$, then we would have $R_2 = R_3$ contradicting the hypothesis of the theorem. Thus $\omega_2 \neq 0$. If we now put $\lambda_2 = \omega_2\omega_1^{-1}$, then we obtain $\{\underline{u}_1,\underline{u}_2,\ldots,\underline{u}_{n-2},\underline{x}+\lambda_2\underline{n}_2\}$ to be a basis of R_2 where λ_2 is a non-zero element of F.

Since $\underline{x} + \mu_1\underline{n}_1 \in R_2$, we have

$$\langle\underline{x}+\mu_1\underline{n}_1|\underline{r}_2\rangle = 0 ,$$

i.e.

$$\langle \underline{x} | \rho_2 \underline{r}_2 \rangle + \langle \mu_1 \underline{n}_1 | \rho_2 \underline{r}_2 \rangle = 0 \ .$$

Thus, by (ii),

$$\langle \underline{x} | \rho_1 \underline{r}_1 \rangle + \langle \underline{x} | \rho_3 \underline{r}_3 \rangle + \langle \mu_1 \underline{n}_1 | \rho_1 \underline{r}_1 \rangle + \langle \mu_1 \underline{n}_1 | \rho_3 \underline{r}_3 \rangle = 0 \ .$$

But, since $\underline{x} \in R_3$, we have $\langle \underline{x} | \rho_3 \underline{r}_3 \rangle = 0$ and so

$$\langle \underline{x} | \rho_1 \underline{r}_1 \rangle + \langle \mu_1 \underline{n}_1 | \rho_1 \underline{r}_1 \rangle + \langle \mu_1 \underline{n}_1 | \rho_3 \underline{r}_3 \rangle = 0 \ ,$$

i.e.

$$\langle \underline{x} + \mu_1 \underline{n}_1 | \rho_1 \underline{r}_1 \rangle + \langle \mu_1 \underline{n}_1 | \rho_3 \underline{r}_3 \rangle = 0 \ .$$

So

$$\langle \underline{x} + \lambda_1 \underline{n}_1 + (\mu_1 - \lambda_1) \underline{n}_1 | \rho_1 \underline{r}_1 \rangle + \langle \mu_1 \underline{n}_1 | \rho_3 \underline{r}_3 \rangle = 0 \ ,$$

i.e.

$$\langle \underline{x} + \lambda_1 \underline{n}_1 | \rho_1 \underline{r}_1 \rangle + \langle (\mu_1 - \lambda_1) \underline{n}_1 | \rho_1 \underline{r}_1 \rangle + \langle \mu_1 \underline{n}_1 | \rho_3 \underline{r}_3 \rangle = 0 \ .$$

But, since $\underline{x} + \lambda_1 \underline{n}_1 \in R_1$, we have $\langle \underline{x} + \lambda_1 \underline{n}_1 | \rho_1 \underline{r}_1 \rangle = 0$ and so

$$\langle (\mu_1 - \lambda_1) \underline{n}_1 | \rho_1 \underline{r}_1 \rangle + \langle \mu_1 \underline{n}_1 | \rho_3 \underline{r}_3 \rangle = 0 \ ,$$

i.e.

$$\langle \mu_1 \underline{n}_1 | \rho_1 \underline{r}_1 + \rho_3 \underline{r}_3 \rangle - \langle \lambda_1 \underline{n}_1 | \rho_1 \underline{r}_1 \rangle = 0 \ .$$

Thus, by (ii),

$$\langle \mu_1 \underline{n}_1 | \rho_2 \underline{r}_2 \rangle + \langle \lambda_1 \underline{n}_1 | \rho_1 \underline{r}_1 \rangle = 0$$

which is (A).

Now, since $\underline{x} + \lambda_2\underline{n}_2 \in \mathcal{R}_2$ and $\underline{x} + \mu_1\underline{n}_1 \in \mathcal{R}_2$, we have $\lambda_2\underline{n}_2 - \mu_1\underline{n}_1 \in \mathcal{R}_2$, i.e.

$$<\lambda_2\underline{n}_2-\mu_1\underline{n}_1|\underline{r}_2> = 0 .$$

Thus

$$<\lambda_2\underline{n}_2|\rho_2\underline{r}_2> - <\mu_1\underline{n}_1|\rho_2\underline{r}_2> = 0$$

which is (B).

Now, by (vi) and Lemma 1.2, the range of $\varepsilon_1\varepsilon_2\varepsilon_3$ is $\mathcal{R}_3$. Thus $\varepsilon_1\varepsilon_2\varepsilon_3$ is idempotent if and only if it acts identically on a basis of $\mathcal{R}_3$. Clearly $\varepsilon_1\varepsilon_2\varepsilon_3$ acts identically on every element of $\{\underline{u}_1,\underline{u}_2,\ldots,\underline{u}_{n-2}\}$. Thus $\varepsilon_1\varepsilon_2\varepsilon_3$ is idempotent if and only if it acts identically on $\underline{x}$.

Now, $\mu_1\nu_1\underline{x} = \mu_1\nu_1(\underline{x}+\lambda_1\underline{n}_1) - \lambda_1\mu_1\nu_1\underline{n}_1$ and so, since $\underline{x} + \lambda_1\underline{n}_1 \in \mathcal{R}_1$,

$$\begin{aligned}\mu_1\nu_1\underline{x}\varepsilon_1 &= \mu_1\nu_1(\underline{x}+\lambda_1\underline{n}_1)\\ &= \lambda_1\nu_1(\underline{x}+\mu_1\underline{n}_1) + \mu_1\nu_1\underline{x} - \lambda_1\nu_1\underline{x}\\ &= \lambda_1\nu_1(\underline{x}+\mu_1\underline{n}_1) + \nu_1(\mu_1-\lambda_1)\underline{x} + \lambda_2\nu_1(\mu_1-\lambda_1)\underline{n}_2 - \lambda_2\nu_1(\mu_1-\lambda_1)\underline{n}_2\\ &= \lambda_1\nu_1(\underline{x}+\mu_1\underline{n}_1) + \nu_1(\mu_1-\lambda_1)(\underline{x}+\lambda_2\underline{n}_2) - \lambda_2\nu_1(\mu_1-\lambda_1)\underline{n}_2 .\end{aligned}$$

Since $(\underline{x}+\mu_1\underline{n}_1),(\underline{x}+\lambda_2\underline{n}_2) \in \mathcal{R}_2$, we then have

$$\begin{aligned}\mu_1\nu_1\underline{x}\varepsilon_1\varepsilon_2 &= \lambda_1\nu_1(\underline{x}+\mu_1\underline{n}_1) + \nu_1(\mu_1-\lambda_1)(\underline{x}+\lambda_2\underline{n}_2)\\ &= \mu_1\nu_1\underline{x} + \lambda_1\mu_1\nu_1\underline{n}_1 + \nu_1\lambda_2(\mu_1-\lambda_1)\underline{n}_2 .\end{aligned}$$

By (i), $\nu_1\underline{n}_1 = -\nu_2\underline{n}_2 - \nu_3\underline{n}_3$ and so

$$\mu_1\nu_1\underline{x}\varepsilon_1\varepsilon_2 = \mu_1\nu_1\underline{x} + (-\lambda_1\mu_1\nu_2+\nu_1\lambda_2\mu_1-\nu_1\lambda_2\lambda_1)\underline{n}_2 - \lambda_1\mu_1\nu_3\underline{n}_3 .$$

Since $\underline{x} \in R_3$, we now have

$$\mu_1\nu_1\underline{x}\varepsilon_1\varepsilon_2\varepsilon_3 = \mu_1\nu_1\underline{x} + (\lambda_2\mu_1\nu_1-\lambda_1\mu_1\nu_2-\lambda_1\lambda_2\nu_1)\underline{n}_2\varepsilon_2 .$$

Thus $\varepsilon_1\varepsilon_2\varepsilon_3$ is idempotent if and only if $(\lambda_2\mu_1\nu_1-\lambda_1\mu_1\nu_2-\lambda_1\lambda_2\nu_1)\underline{n}_2 \in N_3$. But $\underline{n}_2 \notin N_3$ since N_2 and N_3 are distinct and one-dimensional by hypothesis. Thus $\varepsilon_1\varepsilon_2\varepsilon_3$ is idempotent if and only if $\lambda_2\mu_1\nu_1 - \lambda_1\mu_1\nu_2 - \lambda_1\lambda_2\nu_1 = 0$.

We are now in a position to prove Theorem 3.14. We shall need to consider two separate cases:

(I) At least one of $\langle\underline{n}_1|\underline{r}_2\rangle, \langle\underline{n}_2|\underline{r}_3\rangle, \langle\underline{n}_3|\underline{r}_1\rangle$ is zero

(II) All of $\langle\underline{n}_1|\underline{r}_2\rangle, \langle\underline{n}_2|\underline{r}_3\rangle, \langle\underline{n}_3|\underline{r}_1\rangle$ are non-zero.

In considering case (I) it will suffice to consider

(I') $\langle\underline{n}_2|\underline{r}_3\rangle = 0$.

This is because if, instead, we had $\langle\underline{n}_1|\underline{r}_2\rangle = 0$ (and $\langle\underline{n}_2|\underline{r}_3\rangle \neq 0$), then, in the forward implication, we could, by virtue of Lemma 3.5, assume that $\varepsilon_3\varepsilon_1\varepsilon_2$ is idempotent and obtain (i), (ii) and

$$\langle\nu_3\underline{n}_3|\rho_3\underline{r}_3\rangle + \langle\nu_3\underline{n}_3|\rho_1\underline{r}_1\rangle + \langle\nu_1\underline{n}_1|\rho_1\underline{r}_1\rangle = 0 . \qquad (+)$$

But, by Lemma 3.13, this is equivalent to (iii).

For the reverse implication, we could (by Lemma 3.13) assume (+) and deduce that $\varepsilon_3\varepsilon_1\varepsilon_2$ is idempotent of rank $n - 1$. Again, by Lemma 3.5, this is equivalent to $\varepsilon_1\varepsilon_2\varepsilon_3$ being idempotent of rank $n - 1$.

A similar argument holds if we have $\langle\underline{n}_3|\underline{r}_1\rangle = 0$.

(I') Suppose first that $\varepsilon_1\varepsilon_2\varepsilon_3$ is idempotent of rank $n - 1$. We shall show that $\varepsilon_1, \varepsilon_2, \varepsilon_3$ satisfy all the conditions of Lemma 3.17.

By Lemma 3.15, there exist non-zero elements $\nu_1, \nu_2, \nu_3, \rho_1, \rho_2, \rho_3$ of F such that:

(i) $\nu_1 \underline{n}_1 + \nu_2 \underline{n}_2 + \nu_3 \underline{n}_3 = \underline{0}$

(ii) $\rho_1 \underline{r}_1 + \rho_2 \underline{r}_2 + \rho_3 \underline{r}_3 = \underline{0}$.

Condition (iii) is satisfied by the hypothesis of (I') that $\langle \underline{n}_2 | \underline{r}_3 \rangle = 0$.

By the assumption that $\varepsilon_1 \varepsilon_2 \varepsilon_3$ has rank $n - 1$ and by Lemma 1.2, we have that $\varepsilon_1 \varepsilon_2 \varepsilon_3$ has null-space N_1 and range R_3 . Thus $\varepsilon_1 \varepsilon_2 \varepsilon_3 \in [\underline{n}_1 : \underline{r}_3]$. But, by assumption, $\varepsilon_1 \varepsilon_2 \varepsilon_3$ is idempotent and so $[\underline{n}_1 : \underline{r}_3]$ is a group H-class. Thus, by Lemma 2.6, $\langle \underline{n}_1 | \underline{r}_3 \rangle \neq 0$. This is condition (iv).

We have assumed that $\varepsilon_1 \varepsilon_2 \varepsilon_3$ has rank $n - 1$ and so condition (v) is satisfied.

We may thus appeal to Lemma 3.17 to obtain that there exist non-zero elements λ_1, λ_3 of F such that

$$\langle \lambda_1 \underline{n}_1 | \rho_1 \underline{r}_1 \rangle + \langle \lambda_3 \underline{n}_3 | \rho_2 \underline{r}_2 \rangle = 0 . \tag{A}$$

Furthermore, since we have assumed that $\varepsilon_1 \varepsilon_2 \varepsilon_3$ is idempotent, we also have

$$\lambda_3 \nu_1 + \lambda_1 \nu_3 = 0 . \tag{B}$$

Now, multiplying (A) by $\nu_3 \xi$ gives

$$\langle \lambda_1 \nu_3 \underline{n}_1 | \rho_1 \underline{r}_1 \rangle + \langle \lambda_3 \nu_3 \underline{n}_3 | \rho_2 \underline{r}_2 \rangle = 0 .$$

So, by (B),

$$-\langle \lambda_3 \nu_1 \underline{n}_1 | \rho_1 \underline{r}_1 \rangle + \langle \lambda_3 \nu_3 \underline{n}_3 | \rho_2 \underline{r}_2 \rangle = 0 .$$

Dividing now by $\lambda_3 \xi$ (which is non-zero since $\lambda_3 \neq 0$), we have

$$-\langle \nu_1 \underline{n}_1 | \rho_1 \underline{r}_1 \rangle + \langle \nu_3 \underline{n}_3 | \rho_2 \underline{r}_2 \rangle = 0 .$$

But, by Lemma 3.13, this is equivalent to

$$\langle \nu_1 \underline{n}_1 | \rho_1 \underline{r}_1 \rangle + \langle \nu_1 \underline{n}_1 | \rho_2 \underline{r}_2 \rangle + \langle \nu_2 \underline{n}_2 | \rho_2 \underline{r}_2 \rangle = 0 .$$

This is condition (iii) of the theorem. We have already shown (Lemma 3.15) that conditions (i) and (ii) of the theorem hold and so we have proved the theorem one way for case (I').

Conversely, suppose conditions (i), (ii) and (iii) of the theorem hold. We shall again appeal to Lemma 3.17. Conditions (i) and (ii) of the lemma are clearly satisfied. Condition (iii) is again satisfied by the assumption of (I') that $\langle \underline{n}_2 | \underline{r}_3 \rangle = 0$. By Lemma 3.16, $\varepsilon_1 \varepsilon_2 \varepsilon_3$ has rank $n - 1$ and so condition (v) is fulfilled. But this also gives, with Lemma 1.2, that $\varepsilon_1 \varepsilon_2 \varepsilon_3$ has null-space $\mathcal{N}_1$ and range $\mathcal{R}_3$. Thus $\varepsilon_1 \varepsilon_2 \varepsilon_3 \in [\underline{n}_1 : \underline{r}_3]$. Lemma 3.16 also gives that $\varepsilon_1 \varepsilon_2 \varepsilon_3$ belongs to a group $\mathcal{H}$-class. Thus, by Lemma 2.6, $\langle \underline{n}_1 | \underline{r}_3 \rangle \neq 0$. Hence condition (iv) of Lemma 3.17 is satisfied. We are thus justified in using this lemma.

So there exist non-zero elements λ_1, λ_3 of F such that

$$\langle \lambda_1 \underline{n}_1 | \rho_1 \underline{r}_1 \rangle + \langle \lambda_3 \underline{n}_3 | \rho_2 \underline{r}_2 \rangle = 0 .$$

Multiplying by $\nu_3 \xi$ gives

$$\langle \lambda_1 \nu_3 \underline{n}_1 | \rho_1 \underline{r}_1 \rangle + \langle \lambda_3 \nu_3 \underline{n}_3 | \rho_2 \underline{r}_2 \rangle = 0 . \quad \text{(A)}$$

Now, by condition (iii) of the theorem and Lemma 3.13, we have

$$\langle \nu_1 \underline{n}_1 | \rho_1 \underline{r}_1 \rangle - \langle \nu_3 \underline{n}_3 | \rho_2 \underline{r}_2 \rangle = 0 .$$

Multiplying by $\lambda_3 \xi$ gives

$$\langle\lambda_3\nu_1\underline{n}_1|\rho_1\underline{r}_1\rangle + \langle\lambda_3\nu_3\underline{n}_3|\rho_2\underline{r}_2\rangle = 0 .$$

Adding this to (A) gives

$$\langle\lambda_1\nu_3\underline{n}_1|\rho_1\underline{r}_1\rangle + \langle\lambda_3\nu_1\underline{n}_1|\rho_1\underline{r}_1\rangle = 0 ,$$

i.e.

$$\langle(\lambda_1\nu_3+\lambda_3\nu_1)\underline{n}_1|\rho_1\underline{r}_1\rangle = 0 .$$

Since $\varepsilon_1 = (\underline{n}_1:\underline{r}_1)$, we have, by Lemma 2.6, that $\langle\underline{n}_1|\underline{r}_1\rangle \neq 0$. Thus, since $\rho_1 \neq 0$ by hypothesis,

$$\lambda_1\nu_3 + \lambda_3\nu_1 = 0 .$$

Now, appealing again to Lemma 3.17, we see that $\varepsilon_1\varepsilon_2\varepsilon_3$ is idempotent. We have already shown (Lemma 3.16) that $\varepsilon_1\varepsilon_2\varepsilon_3$ has rank $n - 1$. This completes the proof for case (I') and so also for case (I).

(II) Suppose first that $\varepsilon_1\varepsilon_2\varepsilon_3$ is idempotent of rank $n - 1$. We shall show that $\varepsilon_1,\varepsilon_2,\varepsilon_3$ satisfy all the conditions of Lemma 3.18.

By Lemma 3.15 there exist non-zero elements $\nu_1,\nu_2,\nu_3,\rho_1,\rho_2,\rho_3$ of F such that:

(i) $\nu_1\underline{n}_1 + \nu_2\underline{n}_2 + \nu_3\underline{n}_3 = \underline{0}$

(ii) $\rho_1\underline{r}_1 + \rho_2\underline{r}_2 + \rho_3\underline{r}_3 = \underline{0}$.

Conditions (iii) and (iv) are satisfied by the hypothesis of (II) that none of $\langle\underline{n}_1|\underline{r}_2\rangle,\langle\underline{n}_2|\underline{r}_3\rangle,\langle\underline{n}_3|\underline{r}_1\rangle$ are zero.

By the assumption that $\varepsilon_1\varepsilon_2\varepsilon_3$ has rank $n - 1$ and by Lemma 1.2, we have that $\varepsilon_1\varepsilon_2\varepsilon_3$ has null-space N_1 and range R_3 . Thus $\varepsilon_1\varepsilon_2\varepsilon_3 \in [\underline{n}_1:\underline{r}_3]$. But, by assumption, $\varepsilon_1\varepsilon_2\varepsilon_3$ is idempotent and so $[\underline{n}_1:\underline{r}_3]$ is a group H-class. Thus, by Lemma 2.6, $\langle\underline{n}_1|\underline{r}_3\rangle \neq 0$. This is condition (v).

We have assumed that $\varepsilon_1\varepsilon_2\varepsilon_3$ has rank $n - 1$ and so condition (vi) is also satisfied.

We may thus appeal to Lemma 3.18 to obtain that there exist non-zero elements $\lambda_1, \lambda_2, \mu_1$ of F such that

$$<\mu_1\underline{n}_1|\rho_2\underline{r}_2> + <\lambda_1\underline{n}_1|\rho_1\underline{r}_1> = 0 \tag{A}$$

and

$$<\lambda_2\underline{n}_2|\rho_2\underline{r}_2> - <\mu_1\underline{n}_1|\rho_2\underline{r}_2> = 0 \ . \tag{B}$$

Furthermore, since we have assumed $\varepsilon_1\varepsilon_2\varepsilon_3$ to be idempotent, we also have

$$\lambda_2\mu_1\nu_1 - \lambda_1\lambda_2\nu_1 - \lambda_1\mu_1\nu_2 = 0 \ . \tag{C}$$

We shall now eliminate λ_2 from equations (B) and (C). Equation (B) is equivalent to

$$<\lambda_2(\mu_1\nu_1-\lambda_1\nu_1)\underline{n}_2|\rho_2\underline{r}_2> - <\mu_1(\mu_1\nu_1-\lambda_1\nu_1)\underline{n}_1|\rho_2\underline{r}_2> = 0 \ .$$

Thus, using (C), we have

$$<\lambda_1\mu_1\nu_2\underline{n}_2|\rho_2\underline{r}_2> - <\mu_1(\mu_1\nu_1-\lambda_1\nu_1)\underline{n}_1|\rho_2\underline{r}_2> = 0 \ .$$

So, dividing by $\mu_1\xi$ (which is non-zero since $\mu_1 \neq 0$), we get

$$<\lambda_1\nu_2\underline{n}_2|\rho_2\underline{r}_2> - <\mu_1\nu_1\underline{n}_1|\rho_2\underline{r}_2> + <\lambda_1\nu_1\underline{n}_1|\rho_2\underline{r}_2> = 0 \ . \tag{D}$$

We shall now eliminate μ_1 from equations (A) and (D). Equation (A) is equivalent to

$$<\mu_1\nu_1\underline{n}_1|\rho_2\underline{r}_2> + <\lambda_1\nu_1\underline{n}_1|\rho_1\underline{r}_1> = 0 \ .$$

Adding this to (D) gives

$$<\lambda_1\nu_2\underline{n}_2|\rho_2\underline{r}_2> + <\lambda_1\nu_1\underline{n}_1|\rho_2\underline{r}_2> + <\lambda_1\nu_1\underline{n}_1|\rho_1\underline{r}_1> = 0 .$$

Now, dividing by $\lambda_1\xi$ (which is non-zero since $\lambda_1 \neq 0$), gives

$$<\nu_1\underline{n}_1|\rho_1\underline{r}_1> + <\nu_1\underline{n}_1|\rho_2\underline{r}_2> + <\nu_2\underline{n}_2|\rho_2\underline{r}_2> = 0 ,$$

which is condition (iii) of the theorem.

We have already shown (Lemma 3.15) that conditions (i) and (ii) of the theorem hold. So we have proved the theorem one way for case (II).

Conversely, suppose conditions (i), (ii) and (iii) of the theorem hold. We shall again appeal to Lemma 3.18. Conditions (i) and (ii) of the lemma are clearly satisfied. Conditions (iii) and (iv) are again satisfied by the assumption of (II) that none of $<\underline{n}_1|\underline{r}_2>$, $<\underline{n}_2|\underline{r}_3>$, $<\underline{n}_3|\underline{r}_1>$ are zero. By Lemma 3.16, $\varepsilon_1\varepsilon_2\varepsilon_3$ has rank $n - 1$ and so condition (vi) is fulfilled. But this also gives, with Lemma 1.2, that $\varepsilon_1\varepsilon_2\varepsilon_3$ has null-space N_1 and range R_3. Thus $\varepsilon_1\varepsilon_2\varepsilon_3 \in [\underline{n}_1:\underline{r}_3]$. Lemma 3.16 also gives that $\varepsilon_1\varepsilon_2\varepsilon_3$ belongs to a group H-class and so, by Lemma 2.6, we have $<\underline{n}_1|\underline{r}_3> \neq 0$. Hence condition (v) of Lemma 3.18 is satisfied. We are thus justified in using this lemma.

Thus there exist non-zero elements $\lambda_1, \lambda_2, \mu_1$ of F such that

$$<\mu_1\underline{n}_1|\rho_2\underline{r}_2> + <\lambda_1\underline{n}_1|\rho_1\underline{r}_1> = 0 \qquad \text{(A)}$$

$$<\lambda_2\underline{n}_2|\rho_2\underline{r}_2> - <\mu_1\underline{n}_1|\rho_2\underline{r}_2> = 0 . \qquad \text{(B)}$$

We shall now eliminate $\underline{r}_1$ from equation (A) and condition (iii) of the theorem.

Equation (A) is equivalent to

$$<\mu_1\nu_1\underline{n}_1|\rho_2\underline{r}_2> + <\lambda_1\nu_1\underline{n}_1|\rho_1\underline{r}_1> = 0 \qquad \text{(E)}$$

and (iii) is equivalent to

$$\langle \lambda_1 \nu_1 \underline{n}_1 | \rho_1 \underline{r}_1 \rangle + \langle \lambda_1 \nu_1 \underline{n}_1 | \rho_2 \underline{r}_2 \rangle + \langle \lambda_1 \nu_2 \underline{n}_2 | \rho_2 \underline{r}_2 \rangle = 0 .$$

Subtracting (E) from this gives

$$\langle \lambda_1 \nu_1 \underline{n}_1 | \rho_2 \underline{r}_2 \rangle + \langle \lambda_1 \nu_2 \underline{n}_2 | \rho_2 \underline{r}_2 \rangle - \langle \mu_1 \nu_1 \underline{n}_1 | \rho_2 \underline{r}_2 \rangle = 0 . \qquad \text{(F)}$$

We shall now eliminate $\underline{n}_2$ from equations (B) and (F). Equation (F) is equivalent to

$$\langle \lambda_2 (\lambda_1 - \mu_1) \nu_1 \underline{n}_1 | \rho_2 \underline{r}_2 \rangle + \langle \lambda_1 \lambda_2 \nu_2 \underline{n}_2 | \rho_2 \underline{r}_2 \rangle = 0 \qquad \text{(G)}$$

and equation (B) is equivalent to

$$\langle \lambda_1 \lambda_2 \nu_2 \underline{n}_2 | \rho_2 \underline{r}_2 \rangle - \langle \mu_1 \lambda_1 \nu_2 \underline{n}_1 | \rho_2 \underline{r}_2 \rangle .$$

Subtracting (G) from this gives

$$-\langle \mu_1 \lambda_1 \nu_2 \underline{n}_1 | \rho_2 \underline{r}_2 \rangle - \langle \lambda_2 (\lambda_1 - \mu_1) \nu_1 \underline{n}_1 | \rho_2 \underline{r}_2 \rangle = 0 ,$$

i.e.

$$\langle (\lambda_2 \mu_1 \nu_1 - \lambda_1 \lambda_2 \nu_1 - \lambda_1 \mu_1 \nu_2) \underline{n}_1 | \rho_2 \underline{r}_2 \rangle = 0 .$$

Since, by the hypothesis of (II), $\langle \underline{n}_1 | \underline{r}_2 \rangle \neq 0$ and, by the hypothesis of the theorem, $\rho_2 \neq 0$, we now have

$$\lambda_2 \mu_1 \nu_1 - \lambda_1 \lambda_2 \nu_1 - \lambda_1 \mu_1 \nu_2 = 0 .$$

Thus, appealing to Lemma 3.18 again, we see that $\varepsilon_1 \varepsilon_2 \varepsilon_3$ is idempotent. We have already shown (Lemma 3.16) that $\varepsilon_1 \varepsilon_2 \varepsilon_3$ has rank $n - 1$ and so the proof of case (II) is complete.

This also completes the proof of the theorem.

§4 GENERATING SETS OF IDEMPOTENTS 1: THE VECTOR SPACE V DEFINED OVER AN ARBITRARY FIELD F

In this section I shall give a new proof of a result due to J. A. Erdős [7]. The proof in [7] that $Sing_n$ is generated by E (the set of idempotents of $Sing_n$ of rank $n - 1$), depended entirely on results in matrix theory. This shed very little light on the structure of the semigroup. In the following proof we shall consider the chain

$$Sing_n \supseteq PF_{n-1} \supseteq E \cup H$$

where H denotes the set of elements in any $\mathcal{H}$-class (other than $\{0\}$) of PF^0_{n-1} . We shall show that each set is generated by the succeeding set, and then that E generates all the elements of one particular $\mathcal{H}$-class (other than $\{0\}$) of PF^0_{n-1} .

At the end of the section I shall obtain necessary conditions for a subset of E to generate $Sing_n$.

4.1 LEMMA PF_{n-1} generates $Sing_n$.

PROOF The proof is by induction on the nullity of elements of $Sing_n$. Suppose, as the hypothesis, that, if $\alpha \in Sing_n$ and the dimension of the null-space N_α of α is less than or equal to k , then $\alpha \in \langle PF_{n-1} \rangle$. Now let $\beta \in Sing_n$ be such that $\dim N_\beta = k + 1$. Let N_β have basis $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_{k+1}\}$ and extend this to a basis $\{\underline{u}_1, \ldots, \underline{u}_n\}$ of V . Let $\underline{v}$ be any element of V not in R_β . Now let $\beta_1, \beta_2 \in Sing_n$ be given by

$$\underline{u}_i\beta_1 = \begin{cases} \underline{u}_i & i \neq k+1 \\ \underline{0} & i = k+1 \end{cases}$$

and

$$\underline{u}_i\beta_2 = \begin{cases} \underline{u}_i\beta & i \neq k+1 \\ \underline{v} & i = k+1 \end{cases}$$

Clearly $\beta = \beta_1\beta_2$. Now, $\beta_1 \in PF_{n-1}$ and $\dim R_{\beta_2} = \dim R_\beta + 1$, i.e. $\dim N_{\beta_2} = k$. Thus $\beta_1, \beta_2 \in \langle PF_{n-1}\rangle$ and, consequently, $\beta \in \langle PF_{n-1}\rangle$. The induction process may be started since any element with nullity 1 belongs to PF_{n-1} .

Before proceeding to the next step in the chain, we shall need to know a few properties of the relation $\Pi(E')$ on a subset E' of E given by:

<u>4.2 DEFINITION</u> Let E' be a subset of E and $\phi, \gamma \in E'$. Then $(\phi,\gamma) \in \Pi(E')$ if there exist elements $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_q$ in E' such that $\phi\varepsilon_1\varepsilon_2 \ldots \varepsilon_q\gamma \in PF_{n-1}$.

In this section we shall only be concerned with $\Pi(E)$. It is, however, convenient to give the more general definition here.

It is obvious that $\Pi(E')$ is transitive for all subsets E' of E . Not so obvious is:

<u>4.3 LEMMA</u> Let E be the idempotents of rank $n-1$ of $Sing_n$. Then $\Pi(E)$ is the universal relation on E .

<u>PROOF</u> Let $(\underline{n}_1:\underline{r}_1)$ and $(\underline{n}_2:\underline{r}_2)$ be any two elements of E and suppose that $((\underline{n}_1:\underline{r}_1),(\underline{n}_2:\underline{r}_2)) \notin \Pi(E)$. Then, certainly, $(\underline{n}_1:\underline{r}_1)(\underline{n}:\underline{r})(\underline{n}_2:\underline{r}_2) = 0$ in PF^0_{n-1} for all elements $(\underline{n}:\underline{r})$ of E ,

i.e. (by Lemma 2.6) for all elements $\underline{n}$ and $\underline{r}$ of V such that $\langle \underline{n} | \underline{r} \rangle \neq 0$. Thus, by Lemma 1.9 and Lemma 2.6, either $\langle \underline{n} | \underline{r}_1 \rangle = 0$ or $\langle \underline{n}_2 | \underline{r} \rangle = 0$ for all $\underline{n}, \underline{r} \in V$ such that $\langle \underline{n} | \underline{r} \rangle \neq 0$.

Let us suppose that the vectors $\underline{r}_1$ and $\underline{n}_2$ have co-ordinates $\underline{r}_1 = (x^{(1)}, x^{(2)}, \ldots, x^{(n)})$ and $\underline{n}_2 = (y^{(1)}, y^{(2)}, \ldots, y^{(n)})$. Let $i = \min \{j : x^{(j)} \neq 0\}$ and define $\underline{n} = \underline{e}_i$, the vector with 1 in the i^{th} position and zeros elsewhere. Now, for each j in $\{1, 2, \ldots, n\}$ define

$$\underline{r}^{(j)} = \begin{cases} \underline{e}_i + \underline{e}_j & \text{if } i \neq j \\ \underline{e}_i & \text{if } i = j . \end{cases}$$

Then $\langle \underline{n} | \underline{r}^{(j)} \rangle = 1 \neq 0$ for all j and so, by the remark at the end of the last paragraph, we have either

$$\langle \underline{n} | \underline{r}_1 \rangle = 0 \quad \text{or} \quad \langle \underline{n}_2 | \underline{r}^{(j)} \rangle = 0 .$$

Since $\langle \underline{n} | \underline{r}_1 \rangle = x^{(i)} \neq 0$, this implies that $\langle \underline{n}_2 | \underline{r}^{(j)} \rangle = 0$. Moreover, this holds for each j in $\{1, 2, 3, \ldots, n\}$. Putting $j = i$ we obtain that $y^{(i)} = 0$; then for each $j \neq i$ we obtain $y^{(i)} + y^{(j)} = 0$, i.e. $y^{(j)} = 0$. Consequently $\underline{n}_2 = \underline{0}$ which contradicts the assumption of $(\underline{n}_2 : \underline{r}_2)$ being an idempotent.

In the terminology used by Byleen, Meakin and Pastijn in [4], Lemma 4.3 is equivalent to saying that the non-zero idempotents of PF^0_{n-1} are connected. However, if E' is a subset of E, then saying that $\Pi(E')$ is universal on E' is, in general, a weaker condition than saying that the elements of E' are connected.

4.4 LEMMA Let S be a completely 0-simple semigroup and let $a \in S$.

(i) If $e_1,\ldots,e_k$ are non-zero idempotents in S such that $e_1 \mathcal{L} a$ and $e_1e_2\ldots e_k \neq 0$, then the mapping $x \mapsto xe_2\ldots e_k$ $(x \in H_a)$ is a bijective mapping of H_a onto $R_a \cap L_{e_k}$.

(ii) If $e_1,e_2,\ldots,e_k$ are non-zero idempotents in S such that $e_1 \mathcal{R} a$ and $e_ke_{k-1}\ldots e_1 \neq 0$, then the mapping $x \mapsto e_ke_{k-1}\ldots e_2x$ $(x \in H_a)$ is a bijective mapping of H_a onto $R_{e_k} \cap L_a$.

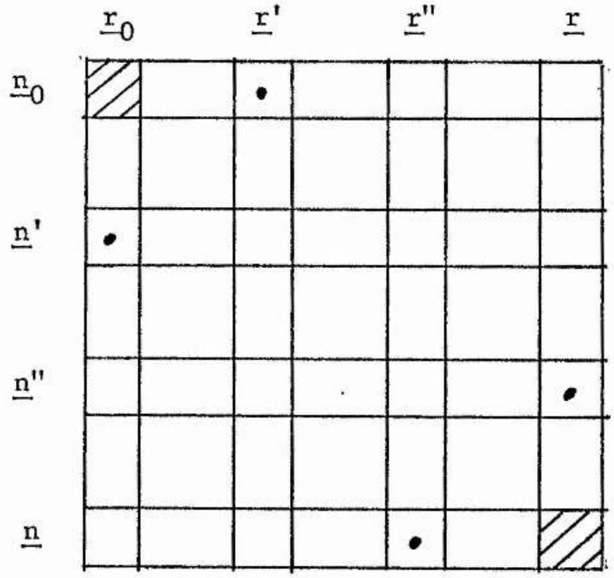
PROOF Both parts are immediate from the Rees representation theorem for completely 0-simple semigroups (see, e.g., [10, Theorem III.2.5]).

The next definition, although not needed in this section, is included now for convenience. It enables us to prove a more general version than required here of Lemma 4.6. This will be required in Section 5.

4.5 DEFINITION Let E' be a subset of the non-zero idempotents of PF^0_{n-1}. We shall say that E' covers [sparsely covers] PF^0_{n-1} if E' has non-empty intersection with [intersects in exactly one element] each non-zero $\mathcal{L}$-class and each non-zero $\mathcal{R}$-class of PF^0_{n-1}. We shall also say that E' covers PF_{n-1}.

4.6 LEMMA Let E' be a subset of the non-zero idempotents of PF^0_{n-1} such that E' covers PF^0_{n-1} and $\Pi(E')$ is the universal relation on E'. Let $[\underline{n}_0:\underline{r}_0]$ be any $\mathcal{H}$-class other than $\{0\}$ of PF^0_{n-1}. Then $E' \cup [\underline{n}_0:\underline{r}_0]$ generates PF^0_{n-1}.

PROOF Let $[\underline{n}:\underline{r}]$ be an arbitrary $\mathcal{H}$-class in PF^0_{n-1}.



Since E' covers PF^0_{n-1} there exist idempotents $(\underline{n}':\underline{r}_0)$, $(\underline{n}'':\underline{r})$, $(\underline{n}_0:\underline{r}')$ and $(\underline{n}:\underline{r}'')$ in E'. Since $\Pi(E')$ is universal, there exist $\varepsilon_1,\ldots,\varepsilon_q \in E'$ such that

$$(\underline{n}':\underline{r}_0)\varepsilon_1 \ldots \varepsilon_q(\underline{n}'':\underline{r}) \neq 0 .$$

By Lemma 4.4(i) it follows that

$$\alpha \mapsto \alpha\varepsilon_1 \ldots \varepsilon_q(\underline{n}'':\underline{r}) \qquad (\alpha \in [\underline{n}_0:\underline{r}_0])$$

is a bijection from $[\underline{n}_0:\underline{r}_0]$ onto $[\underline{n}_0:\underline{r}]$.

Equally, the universality of $\Pi(E')$ means that there exist $\varepsilon_1',\ldots,\varepsilon_p' \in E'$ such that

$$(\underline{n}:\underline{r}'')\varepsilon_p' \ldots \varepsilon_1'(\underline{n}_0:\underline{r}') \neq 0 .$$

By Lemma 4.4(ii) it follows that

$$\beta \mapsto (\underline{n}:\underline{r}'')\varepsilon_p' \ldots \varepsilon_1'\beta \qquad (\beta \in [\underline{n}_0:\underline{r}])$$

is a bijection from $[\underline{n}_0:\underline{r}]$ onto $[\underline{n}:\underline{r}]$.

Thus

$$\alpha \mapsto (\underline{n}:\underline{r}'')\varepsilon'_p \ldots \varepsilon'_1\alpha\varepsilon_1 \ldots \varepsilon_q(\underline{n}'':\underline{r}) \qquad (\alpha \in [\underline{n}_0:\underline{r}_0])$$

is a bijection from $[\underline{n}_0:\underline{r}_0]$ onto $[\underline{n}:\underline{r}]$. It follows that every element of $[\underline{n}:\underline{r}]$ lies in $\langle E' \cup [\underline{n}_0:\underline{r}_0]\rangle$. So $E' \cup [\underline{n}_0:\underline{r}_0]$ generates PF^0_{n-1}.

4.7 EXAMPLE Let V be the two-dimensional vector space over the field of two elements. Then $PF^0_1 = Sing_2$ and has structure

	(1,0)	(0,1)	(1,1)	
(1,0)	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	
(0,1)	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	
(1,1)	$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$\begin{bmatrix}1&0\\1&0\end{bmatrix}$	$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	
				$\begin{bmatrix}0&0\\0&0\end{bmatrix}$

In the notation of Lemma 4.6, let

$$E' = \left\{\begin{bmatrix}0&0\\0&1\end{bmatrix}, \begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\1&0\end{bmatrix}\right\}.$$

and $[\underline{n}:\underline{r}] = [(1,0):(1,0)] = \left\{\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$. Then E' is a cover for PF_1 (it is indeed a sparse cover).

We now show that $\Pi(E')$ is universal on E'.

$$\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\1&0\end{bmatrix}=\begin{bmatrix}0&0\\1&0\end{bmatrix}\in PF_1 \quad \text{so} \quad \left(\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}1&0\\1&0\end{bmatrix}\right)\in \Pi(E')$$

$$\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\1&0\end{bmatrix}\begin{bmatrix}1&1\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\1&1\end{bmatrix}\in PF_1 \quad \text{so} \quad \left(\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}1&1\\0&0\end{bmatrix}\right)\in \Pi(E')$$

$$\begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}0&1\\0&0\end{bmatrix}\in PF_1 \quad \text{so} \quad \left(\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)\in \Pi(E')$$

$$\begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\1&0\end{bmatrix}=\begin{bmatrix}1&0\\0&0\end{bmatrix}\in PF_1 \quad \text{so} \quad \left(\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}1&0\\1&0\end{bmatrix}\right)\in \Pi(E')$$

$$\begin{bmatrix}1&0\\1&0\end{bmatrix}\begin{bmatrix}1&1\\0&0\end{bmatrix}=\begin{bmatrix}1&1\\1&1\end{bmatrix}\in PF_1 \quad \text{so} \quad \left(\begin{bmatrix}1&0\\1&0\end{bmatrix},\begin{bmatrix}1&1\\0&0\end{bmatrix}\right)\in \Pi(E')$$

$$\begin{bmatrix}1&0\\1&0\end{bmatrix}\begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}0&1\\0&1\end{bmatrix}\in PF_1 \quad \text{so} \quad \left(\begin{bmatrix}1&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)\in \Pi(E')$$

Thus $\Pi(E')$ is universal on E'.

So, by Lemma 4.6, $E' \cup [(1,0):(1,0)]$ generates PF^0_{n-1}. Since $[(1,0):(1,0)] = \left\{\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\} \subset E'$ we have that E' generates PF^0_{n-1}.

We now verify this. We have already shown that E' generates all the elements of PF^0_1 except for $\begin{bmatrix}1&1\\0&0\end{bmatrix}$ and $\begin{bmatrix}0&0\\0&0\end{bmatrix}$. However, since

$$\begin{bmatrix}1&1\\0&0\end{bmatrix}=\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}1&1\\1&1\end{bmatrix}$$

and

$$\begin{bmatrix}0&0\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}1&1\\0&0\end{bmatrix},$$

we have that $\langle E'\rangle = PF^0_{n-1}$.

<u>4.8 LEMMA</u> The non-zero idempotents of PF^0_{n-1} generate the $\mathcal{H}$-class $H = [(1,0,0,\ldots,0):(0,1,0,0,\ldots,0)]$.

<u>PROOF</u> The proof is by induction on the dimension of the vector space. Suppose, as the induction hypothesis, that the lemma is true for PF^0_{n-2} . Then, since the non-zero idempotents of PF^0_{n-2} cover PF^0_{n-2} , we have, by Lemma 4.1, Lemma 4.3 and Lemma 4.6, that the idempotents of rank $n - 2$ of $\underset{\sim}{S}ing_{n-1}$ generate $Sing_{n-1}$.

Now, let $\alpha \in PF^0_{n-1}$ be an element of H . Then, relative to the standard basis, α has matrix

$$M_\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & 0 & a_{23} & a_{24} & \cdots & a_{2n} \\ a_{31} & 0 & a_{33} & a_{34} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & 0 & a_{n3} & a_{n4} & \cdots & a_{nn} \end{bmatrix} .$$

Now $M_\alpha = A_1 B$ where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & 1 & 0 & 0 & \cdots & 0 \\ a_{31} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \ddots & \\ a_{n1} & 0 & 0 & 0 & & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_{23} & a_{24} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & a_{34} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & a_{n3} & a_{n4} & \cdots & a_{nn} \end{bmatrix} .$$

Notice that A_1 is idempotent.

Now, $B = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & B' \end{array}\right]$ where B' is an $(n-1) \times (n-1)$ singular matrix. So, by the induction hypothesis, $B' = A_2'A_3' \ldots A_k'$ where A_i' $(i=2,\ldots,k)$ are idempotent $(n-1) \times (n-1)$ matrices. Thus the matrices

$$A_i = \left[\begin{array}{c|c} 1 & \underline{0} \\ \hline \underline{0} & A_i' \end{array}\right] \qquad (i=2,\ldots,k)$$

are idempotent $n \times n$ matrices with

$$A_iA_j = \left[\begin{array}{c|c} 1 & \underline{0} \\ \hline \underline{0} & A_i'A_j' \end{array}\right].$$

Hence $B = A_2A_3\ldots A_k$ and so

$$M_\alpha = A_1A_2\ldots A_k \text{ , a product of idempotents.}$$

All that remains now is to anchor the hypothesis by showing that every 2×2 matrix in the H-class $[(1,0):(0,1)]$ can be expressed as a product of idempotents. If $\alpha \in [(1,0):(0,1)]$, then, relative to the standard basis, α has matrix of the form $M_\alpha = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}$. But $M_\alpha = E_1E_2$ where $E_1 = \begin{bmatrix} 0 & 0 \\ a & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are both idempotent.

<u>4.9 THEOREM</u> (J. A. Erdös [7]) Let V be a finite dimensional vector space and let $Sing_n$ be the semigroup of singular endomorphisms of V. Let E be the set of idempotents of $Sing_n$ of rank $n-1$. Then E generates $Sing_n$.

<u>PROOF</u> This is immediate from Lemma 4.1, Lemma 4.3, Lemma 4.6 and Lemma 4.8.

We have already shown (Theorem 3.14) that E may be generated by a proper subset of E . Thus we know now that a proper subset of E will generate $Sing_n$. It is reasonable to ask how small a subset of E will suffice to generate $Sing_n$. The following two lemmas are used in Sections 5 and 6 where this problem is considered for the cases of F being a finite field and an infinite field respectively.

4.10 LEMMA If E' is a subset of E and E' generates $Sing_n$ then E' covers PF_{n-1} and $\Pi(E')$ is the universal relation on E' .

PROOF Let β be any element of PF_{n-1} . Since E' generates $Sing_n$, it certainly generates PF_{n-1} . Thus there exist elements $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_p \in E'$ such that $\beta = \varepsilon_1\varepsilon_2\ldots\varepsilon_p$. Now, since $\dim \beta = \dim \varepsilon_i$ $(i=1,2,\ldots,p)$, we have, by Lemma 1.2, that $N_\beta = N_{\varepsilon_1}$ and $R_\beta = R_{\varepsilon_p}$. Thus, by Lemma 1.3, $\beta \mathcal{R} \varepsilon_1$ and $\beta \mathcal{L} \varepsilon_p$. Hence both $R_\beta \cap E'$ and $L_\beta \cap E'$ are non-empty. Since β was chosen arbitrarily, it follows that E' covers PF_{n-1} .

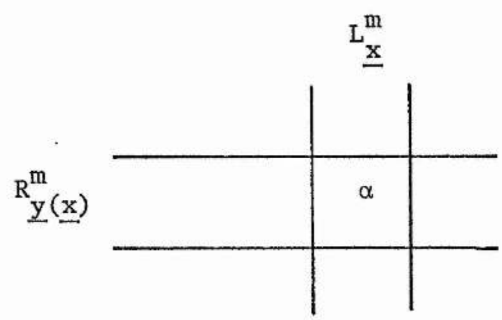
Now let $\phi,\gamma \in E'$, and let $\alpha \in R_\phi \cap L_\gamma$. Since E' generates α we have $\alpha = \varepsilon_1\varepsilon_2\ldots\varepsilon_p$ for some $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_p \in E'$. But, by Lemmas 1.2 and 1.3, $\varepsilon_1 \mathcal{R} \alpha$ and $\varepsilon_p \mathcal{L} \alpha$. Thus $\phi \mathcal{R} \varepsilon_1$ and $\gamma \mathcal{L} \varepsilon_p$. Hence $\phi\varepsilon_1 = \varepsilon_1$ and $\varepsilon_p\gamma = \varepsilon_p$. So $\alpha = \phi\varepsilon_1\varepsilon_2\ldots\varepsilon_p\gamma$, i.e. $\phi\varepsilon_1\varepsilon_2\ldots\varepsilon_p\gamma$ has rank $n - 1$. So $\phi\varepsilon_1\varepsilon_2\ldots\varepsilon_p\gamma \neq 0$ in PF^0_{n-1} . Since $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_p \in E'$, we have that $(\phi,\gamma) \in \Pi(E')$. Since ϕ and γ were chosen arbitrarily, it follows that $\Pi(E')$ is the universal relation on E'.

4.11 LEMMA There exists a sparse covering set E' for PF^0_{n-1} .

<u>PROOF</u> The proof is by induction on the dimension n of the vector space V. For clarity we shall denote the m-dimensional vector space by V_m.

We now define a set of representatives V_m' of the one-dimensional subspaces of V_m. So, for all non-zero $\underline{x}$ in V_m there exists a unique $\underline{y}$ in V_m' such that $\langle\underline{x}\rangle = \langle\underline{y}\rangle$. We shall denote by $L_{\underline{x}}^m$ [$R_{\underline{x}}^m$] the L-class [R-class] of PF_{m-1}^0 containing those elements with range perpendicular [null-space] $\langle\underline{x}\rangle$.

Now suppose, as the induction hypothesis, that there exists a sparse covering set E_m' of PF_{m-1}^0. Then there exists exactly one element α in $L_{\underline{x}}^m \cap E_m'$ for each $\underline{x} \in V_m'$. All the elements in R_α have, by Lemma 1.3, the same null-space, generated by a particular element of V_m'. If we denote this element by $\underline{y}(\underline{x})$, we have, in fact, defined a mapping $V_m' \to V_m'$ by $\underline{x} \mapsto \underline{y}(\underline{x})$. This mapping is characterised by $L_{\underline{x}}^m \cap R_{\underline{y}(\underline{x})}^m \cap E_m'$ is non-empty.



This mapping is clearly a bijection. Notice that there exists an idempotent, namely α, with null-space $\langle\underline{y}(\underline{x})\rangle$ and range $\langle\underline{x}\rangle^\perp$. Thus, by Lemma 2.6, $\langle\underline{y}(\underline{x})|\underline{x}\rangle \neq 0$.

If $\underline{x} = (x_1, x_2, \ldots, x_m)$ is an element of V_m' and $a \in F$, then denote by $(\underline{x}, a)$ the element of V_{m+1}' that generates the space $\langle(x_1, x_2, \ldots, x_m, a)\rangle$. We shall denote by $(\underline{0}, 1)$ the element of V_{m+1}' that generates the space $\langle(0, 0, \ldots, 0, 1)\rangle$. Clearly, these are all distinct and every element of V'_{m+1} may be denoted in this way. Notice that, if $\underline{y} = (y_1, y_2, \ldots, y_{m+1})$, then for some $\underline{x} \in V'_m \cup \{\underline{0}\}$ and some $\lambda, a \in F$, we have $(y_1, \ldots, y_m) = \lambda\underline{x}$ and $y_{m+1} = \lambda a$.

We shall now set up a bijection $\underline{\bar{y}} : V'_{m+1} \to V'_{m+1}$ such that $L_{(\underline{x},a)} \cap R_{\underline{\bar{y}}(\underline{x},a)}$ is a group H-class of PF^0_m for all $\underline{x} \in V'_m$ and all $a \in F$ and also $L_{(\underline{0},1)} \cap R_{(\underline{0},1)}$ is a group H-class of PF^0_m. It would be nice if $\bar{y}$ were the identity map. In some cases this would work (e.g. $F = \mathbb{R}$ and the stroke product being an inner product) but in general we do not have $\langle \underline{a} | \underline{a} \rangle \neq 0$ (see the comments following Definition 2.2) and so we are unable to guarantee that $[\underline{a}:\underline{a}]$ is a group H-class. It is logical to construct $\bar{y}$ so that for $\underline{x} \in V'_m$ and $a \in F$ we have $\underline{\bar{y}}(\underline{x},a) = (\underline{y}(x), z)$ for some $z \in F$. We need to have $\langle \underline{\bar{y}}(\underline{x},a) | (\underline{x},a) \rangle \neq 0$ and so we must have $\langle (\underline{y}(x), z) | (\underline{x},a) \rangle \neq 0$, i.e. $\langle \underline{y}(x) | \underline{x} \rangle + (z\xi)(a\chi) \neq 0$. Now, by the definition of $\underline{y}(\underline{x})$, we know that $\langle \underline{y}(\underline{x}) | \underline{x} \rangle \neq 0$. Thus, if $a \neq 0$, we need $z\xi \neq -\langle \underline{y}(\underline{x}) | \underline{x} \rangle (a\chi)^{-1}$ and, if $a = 0$, z may take any value we choose. Now, all we know for certainty about the field F is that it contains two elements, namely 0 and 1. Thus, if $a \neq 0$, we may put $z\xi = 1 - \langle \underline{y}(\underline{x}) | \underline{x} \rangle (a^{-1}\chi)$. This clearly satisfies $z\xi \neq -\langle \underline{y}(\underline{x}) | \underline{x} \rangle (a^{-1}\chi)$. Now, for a given $\underline{x}$, the only value that $1 - \langle \underline{y}(\underline{x}) | \underline{x} \rangle (a^{-1}\chi)$ $(a \neq 0)$ may not take is 1 since $\langle \underline{y}(\underline{x}) | \underline{x} \rangle \neq 0$. So if $a = 0$ we shall set $z = 1$. So we shall define the map $\underline{\bar{y}} : V'_{m+1} \to V'_{m+1}$ by

$$\underline{\bar{y}}(\underline{x},a) = \begin{cases} (\underline{y}(\underline{x}), b(\underline{x},a)) & \text{if } \underline{x} \in V'_m \\ (\underline{0},1) & \text{if } \underline{x} = 0 \text{ , } a = 1 \end{cases}$$

where

$$b(\underline{x},a) = \begin{cases} [1 - \langle \underline{y}(\underline{x}) | \underline{x} \rangle (a^{-1}\chi)]\xi^{-1} & a \neq 0 \\ 1\xi^{-1} = 1 & a = 0 \text{ .} \end{cases}$$

It is obvious that $\underline{\bar{y}}$ is an injection for, if $\underline{\bar{y}}(\underline{x},a) = \underline{\bar{y}}(\underline{x}',a')$ and $\underline{x} \neq \underline{0}$, then we would have $\underline{y}(\underline{x}) = \underline{y}(\underline{x}')$ and $b(\underline{x},a) = b(\underline{x}',a')$. But, since $\underline{y}$ is bijective, this implies $\underline{x} = \underline{x}'$ and $b(\underline{x},a) = b(\underline{x},a')$. This, in turn, implies $a = a'$. If $\underline{x} = \underline{0}$, then clearly $\underline{x}' = 0$ and so $a = a' = 1$ since $(\underline{0},a)$ and $(\underline{0},a') \in V'_{m+1}$.

We shall now show that $\underline{\bar{y}}$ is surjective. Let $(\underline{x},a) \in V'_{m+1}$. If $\underline{x} = \underline{0}$ and $a = 1$, then $\underline{\bar{y}}(\underline{0},1) = (\underline{x},a)$. So suppose $\underline{x} \neq \underline{0}$. Then $\underline{x} \in V'_m$. Since $\underline{y} : V'_m \to V'_m$ is bijective, $\underline{y}^{-1}(\underline{x})$ is defined and unique. If $a = 1$, then $\underline{\bar{y}}(\underline{y}^{-1}(x),0) = (\underline{x},a)$. So suppose $a \neq 1$. Then $a\chi \neq 1$ and so $\frac{1}{1-a\chi}$ is defined. Thus $\underline{\bar{y}}(\underline{y}^{-1}(\underline{x}),(\frac{<\underline{y}^{-1}(\underline{x})|\underline{x}>}{1-a\chi})\xi^{-1}) = (\underline{x},a)$. Hence $\underline{\bar{y}} : V'_{m+1} \to V'_{m+1}$ is surjective and, consequently, is bijective.

From the definition of $\underline{\bar{y}}$ we have that, for all $(\underline{x},a) \in V'_{m+1}$, $<\underline{\bar{y}}(\underline{x},a)|(\underline{x},a)> \neq 0$. Thus $L_{(\underline{x},a)} \cap R_{\underline{\bar{y}}(\underline{x},a)}$ contains an idempotent. Hence the set

$$E'_{m+1} = \{(\underline{\bar{y}}(\underline{x},a) : (\underline{x},a)) : (\underline{x},a) \in V'_{m+1}\}$$

is a sparse cover for PF^0_m.

It remains to show that we may anchor the induction at $m = 2$. Since, in this case, every one-dimensional subspace of V_2 may be generated by the vector $(0,1)$ or a vector of the form $(1,a)$, it is easy to see that the set

$$\{((1,(1-\frac{1}{a\chi})\xi^{-1}) : (1,a)) ; a \in F\backslash\{0\}\} \cup \{((1,1):(1,0)),((0,1):(0,1))\}$$

forms a sparse cover for PF^0_1.

§5 GENERATING SETS OF IDEMPOTENTS 2: THE VECTOR SPACE V DEFINED OVER A FINITE FIELD F

If the field F is finite then the semigroup $Sing_n$ is also finite. I shall show (Theorem 5.1) that in this case the necessary conditions for a subset E' of E to generate $Sing_n$ given in Lemma 4.10 are also sufficient conditions. From this I shall obtain the minimum number m such that there exists a subset E' of E that generates $Sing_n$ and has order m (Corollary 5.7).

5.1 THEOREM Let V be an n-dimensional vector space over a finite field F . Let $Sing_n$ be the semigroup of singular endomorphisms of V and let PF_{n-1} be the set of elements in $Sing_n$ with rank $n - 1$. Let E' be a subset of the idempotents of PF_{n-1} . Then E' generates $Sing_n$ if and only if $\Pi(E')$ is the universal relation on E' and E' covers PF_{n-1} .

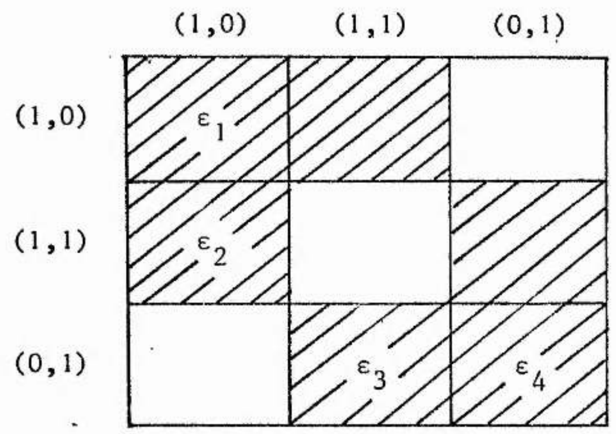
PROOF We already know (Lemma 4.10) that if E' generates $Sing_n$ then $\Pi(E')$ is universal on E' and that E' covers PF_{n-1} .

To show the converse it will suffice to show that E' generates E , the set of all idempotents in PF_{n-1} , for, by Theorem 4.9 and [7], we have that E generates $Sing_n$.

Let $\varepsilon \in E$. Since E' covers PF_{n-1} , there exist $\phi,\gamma \in E'$ such that $\phi \mathcal{R} \varepsilon$ and $\gamma \mathcal{L} \varepsilon$. Since $\Pi(E')$ is universal on E', we have that $(\phi,\gamma) \in \Pi(E')$. Hence there exist $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_p \in E'$ such that $\alpha = \phi\varepsilon_1\varepsilon_2\ldots\varepsilon_p\gamma$ has rank $n - 1$. Now, by Lemma 1.2, $N_\alpha = N_\phi$ and $R_\alpha = R_\gamma$. Thus, by Lemma 1.3, $\alpha \mathcal{R} \phi$ and $\alpha \mathcal{L} \gamma$. Hence $\alpha \mathcal{R} \varepsilon$ and $\alpha \mathcal{L} \varepsilon$, i.e. $\alpha \mathcal{H} \varepsilon$. Now, since F is finite, $Sing_n$ is finite and so certainly H_ε is finite. So α belongs to a finite group. Thus, for some integer $k \geq 1$, α^k is the identity of that group, i.e. $\alpha^k = \varepsilon$. Since α is a product of elements of E', we have that E' generates ε . But this holds for all elements of E and so E' generates E as required.

If a subset E' of the idempotents E covers PF_{n-1} it is not true in general that $\Pi(E')$ is universal on E' as the next example shows.

<u>5.2 EXAMPLE</u> If $F \simeq \mathbb{Z}_2$ and $n = 2$ then the structure of PF_1^0 is



where the shaded boxes contain idempotents and where

$$\varepsilon_1 = ((1,0):(1,0)) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\varepsilon_2 = ((1,1):(1,0)) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\varepsilon_3 = ((0,1):(1,1)) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$\varepsilon_4 = ((0,1):(0,1)) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} .$$

If $E' = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ then E' covers PF_1^0. However, $\Pi(E')$ is not universal on E'. To see this we shall compute $\langle E' \rangle$ and then apply Theorem 5.1.

Now $\langle E' \rangle = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, 0\}$. Clearly $\langle E' \rangle \neq PF_1^0$ for $((1,0):(1,1)) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \notin \langle E' \rangle$. Thus, since E' covers PF_1^0, we have, by Theorem 5.1, that $\Pi(E')$ is not universal on E'.

The next three lemmas and Theorem 5.6 will show that if F is any finite field then any sparse cover of PF_{n-1}^0 will generate $Sing_n$.

5.3 LEMMA If $|F| = q$, then the number of non-zero L-classes [R-classes] in PF_{n-1}^0 is $(q^n-1)/(q-1)$.

PROOF By Lemma 4.11 we know that there is a bijection between the elements of a sparse cover of PF_{n-1}^0 and the L-classes [R-classes] of PF_{n-1}^0. Thus there is a bijection between the L-classes and R-classes of PF_{n-1}^0. Since F is finite it follows that PF_{n-1}^0 is finite and so there are only finitely many L-classes [R-classes] in PF_{n-1}^0. Consequently there are the same number of L-classes as R-classes in PF_{n-1}^0.

By the comments following Definition 2.5, we know that there is a bijection between the one-dimensional subspaces of V and the non-zero L-classes of PF_{n-1}^0. Now the number of non-zero vectors in V is $q^n - 1$. However, for each $\underline{x}$ in V and for all non-zero scalars λ in F we have $\langle \underline{x} \rangle = \langle \lambda \underline{x} \rangle$. Hence there are $(q^n-1)/(q-1)$ one-dimensional subspaces in V.

<u>5.4 LEMMA</u> If $|F| = q$, then the number of idempotents in any non-zero L-class [R-class] of PF^0_{n-1} is q^{n-1}.

<u>PROOF</u> The number of idempotents in a given L-class L is the number of R-classes containing an idempotent in L. If the elements in L have range $\langle \underline{r} \rangle^{\perp}$ then, by Lemma 2.6, this is just $Q = |\{\langle \underline{n} \rangle : \langle \underline{n} | \underline{r} \rangle \neq 0\}|$ where $|X|$ denotes the order of the set X. Since the number of one-dimensional subspaces of V is, by Lemma 5.3, $(q^n-1)/(q-1)$, we have $Q = (q^n-1)/(q-1) - |\{\langle \underline{n} \rangle : \langle \underline{n} | \underline{r} \rangle = 0\}|$. But $\{\langle n \rangle : \langle \underline{n} | \underline{r} \rangle = 0\} = \{\langle \underline{n} \rangle : \underline{n} \in \langle \underline{r} \rangle^{\perp}\}$. Since, by Lemma 2.3, $\dim \langle \underline{r} \rangle^{\perp} = n - 1$, we have, by the proof of Lemma 5.3, that

$$|\{\langle \underline{n} \rangle : \underline{n} \in \langle \underline{r} \rangle^{\perp}\}| = (q^{n-1}-1)/(q-1) .$$

Thus $Q = (q^n-1)/(q-1) - (q^{n-1}-1)/(q-1) = q^{n-1}$ as required.

<u>5.5 LEMMA</u> If F is a finite field and E' is a sparse cover for PF^0_{n-1}, then $\Pi(E')$ is the universal relation on E'.

<u>PROOF</u> Let ϕ, γ be any two elements of E' and suppose that $\phi\Pi(E') \cap \gamma[\Pi(E')]^{-1}$ is empty. Since each L-class of PF^0_{n-1} contains q^{n-1} idempotents (Lemma 5.4) and E' is a sparse cover for PF^0_{n-1}, we know that there are exactly q^{n-1} elements ε_i of E' such that $\phi\varepsilon_i \neq 0$ in PF^0_{n-1} (Lemma 2.7). Hence $|\phi\Pi(E')| \geq q^{n-1}$. Similarly, since each R-class of PF^0_{n-1} contains q^{n-1} idempotents, we have that there exist exactly q^{n-1} elements ε_i' of E' such that $\varepsilon_i'\gamma \neq 0$ in PF^0_{n-1}. Thus $|\gamma[\Pi(E')]^{-1}| \geq q^{n-1}$. Now, since we have assumed $\phi\Pi(E') \cap \gamma[\Pi(E')]^{-1}$ is empty, we have

$$|\phi\Pi(E') \cup \gamma[\Pi(E')]^{-1}| = |\phi\Pi(E')| + |\gamma[\Pi(E')]^{-1}| \geq q^{n-1} + q^{n-1} = 2q^{n-1} .$$

But, since, by the proof of Lemma 5.3, we have $|E'| = (q^n-1)/(q-1)$ and $\phi\Pi(E') \cup \gamma[\Pi(E')]^{-1} \subseteq E'$, we then have

$$|\phi\Pi(E') \cup \gamma[\Pi(E')]^{-1}| \leq |E'| = (q^n-1)/(q-1) .$$

Thus

$$(q^n-1)/(q-1) \geq 2q^{n-1} ,$$

i.e.

$$q^n - 1 \geq 2q^n - 2q^{n-1} .$$

Hence

$$q^n - 2q^{n-1} + 1 \leq 0 ,$$

i.e.

$$q^{n-1}(q-2) \leq -1 . \qquad (+)$$

But, since $|F| = q$ and F is a field, we have that $q \geq 2$. Thus (+) is impossible. So there exists $\varepsilon \in \phi\Pi(E') \cap \gamma[\Pi(E')]^{-1}$, i.e. $(\phi,\varepsilon) \in \Pi(E')$ and $(\varepsilon,\gamma) \in \Pi(E')$. Thus $(\phi,\gamma) \in \Pi(E')$.

We now have:

5.6 THEOREM Let V be an n-dimensional vector space over a finite field F. Let $Sing_n$ denote the semigroup of singular endomorphisms of V and let PF_{n-1} be the set of elements of $Sing_n$ with rank $n - 1$. Then there exists a subset E' of the idempotents of PF_{n-1} such that E' is a sparse cover for PF_{n-1} and E' generates $Sing_n$. Further, any sparse cover for PF_{n-1} generates $Sing_n$.

PROOF By Lemma 4.11, there exists a sparse cover for PF_{n-1}. By Lemma 5.5, $\Pi(E')$ is the universal relation on any sparse cover E' and so, by Theorem 5.1, any sparse cover E' for PF_{n-1} generates $Sing_n$.

5.7 COROLLARY Let V be an n-dimensional vector space over a finite field $|F| = q$. Let $Sing_n$ be the semigroup of singular endomorphisms of $Sing_n$ and let E be the idempotents of $Sing_n$ of rank $n - 1$. Then

$$\min \{|E'| : E' \subseteq E , \langle E' \rangle = Sing_n\} = (q^n-1)/(q-1) .$$

PROOF This is immediate from Lemma 4.10, Lemma 5.3 and Theorem 5.6.

§6 GENERATING SETS OF IDEMPOTENTS 3: THE VECTOR SPACE V DEFINED OVER AN INFINITE FIELD F

In Lemma 4.10 we found necessary conditions for a subset of E to generate $Sing_n$. When F was finite we were able to show that these conditions were also sufficient (Theorem 5.1). Unfortunately this is not the case when F is infinite,as Example 6.1 will show. Despite this, we shall be able to obtain a theorem (Theorem 6.7) that is similar to Theorem 5.6, but much weaker. Before stating Theorem 6.7, we shall need two more definitions and three simple lemmas.

6.1 EXAMPLE Let $F \cong \mathbb{R}$, $\langle \cdot | \cdot \rangle$ be the stroke product defined by $x\xi = x\chi = x$ and let E' be the set of idempotents of the form $(\underline{a}:\underline{a})$. E' clearly covers PF_{n-1} . Also $\Pi(E')$ is universal on E' . To show this we shall consider any two idempotents $(\underline{a}:\underline{a})$ and $(\underline{b}:\underline{b})$ of E' . If $(\underline{a}:\underline{a})(\underline{b}:\underline{b})$ has rank less than $n - 1$, then we have (by Lemma 2.7) $<\underline{a}|\underline{b}> = 0$. Hence $<\underline{a}|\underline{a}+\underline{b}> = <\underline{a}|\underline{a}> \neq 0$ and $<\underline{a}+\underline{b}|\underline{b}> = <\underline{b}|\underline{b}> \neq 0$. Thus $(\underline{a}:\underline{a})(\underline{a}+\underline{b}:\underline{a}+\underline{b})$ and $(\underline{a}+\underline{b}:\underline{a}+\underline{b})(\underline{b}:\underline{b})$ have rank $n - 1$ (Lemma 2.7) and so $(\underline{a}:\underline{a})(\underline{a}+\underline{b}:\underline{a}+\underline{b})(\underline{b}:\underline{b})$ has rank $n - 1$ (Lemma 1.9). Thus $((\underline{a}:\underline{a}),(\underline{b}:\underline{b})) \in \Pi(E')$. So E' covers PF_{n-1} and $\Pi(E')$ is universal on E'.

Now let $\underline{x} \in V$ and $(\underline{a}:\underline{a})$ be any element of E' . Then $\underline{x} = \lambda\underline{a} + \underline{b}$ where $\lambda \in \mathbb{R}$ and $\underline{b} \in <\underline{a}>^{\perp}$ (by Lemma 1.4). Thus $\underline{x}(\underline{a}:\underline{a}) = \underline{b}$. So $<\underline{x}|\underline{x}> = <\lambda\underline{a}|\lambda\underline{a}> + <\underline{b}|\underline{b}> = <\lambda\underline{a}|\lambda\underline{a}> + <\underline{x}(\underline{a}:\underline{a})|\underline{x}(\underline{a}:\underline{a})>$. Thus, since $<\lambda\underline{a}|\lambda\underline{a}> \geq 0$ with equality occurring if and only if $\lambda\underline{a} = \underline{0}$, we have

$$<\underline{x}|\underline{x}> \geq <\underline{x}(\underline{a}:\underline{a})|\underline{x}(\underline{a}:\underline{a})> \tag{+}$$

with equality occurring if and only if $\underline{x} \in <\underline{a}>^{\perp}$.

Now let $(\underline{n}:\underline{r})$ be any idempotent of E not in E' and suppose that E' generates E . Then there exist $\underline{n}_1,\underline{n}_2,\ldots,\underline{n}_k$ in V such that

$$(\underline{n}:\underline{r}) = (\underline{n}:\underline{n})(\underline{n}_1:\underline{n}_1)(\underline{n}_2:\underline{n}_2) \ldots (\underline{n}_k:\underline{n}_k)(\underline{r}:\underline{r}) .$$

Now let $\underline{x} \in <\underline{r}>^{\perp}$. Then

$$\underline{x}(\underline{n}:\underline{r}) = \underline{x} . \tag{++}$$

But, by repeated applications of (+),

$$<\underline{x}|\underline{x}> \geq <\underline{x}(\underline{n}:\underline{n})|\underline{x}(\underline{n}:\underline{n})> \geq \ldots \geq <\underline{x}(\underline{n}:\underline{r})|\underline{x}(\underline{n}:\underline{r})>$$

with equality occurring at each stage if and only if

$$\underline{x} \in \langle\underline{n}\rangle^{\perp}\ ,\ x(\underline{n}:\underline{n}) \in \langle\underline{n}_1\rangle^{\perp}\ ,\ \underline{x}(\underline{n}:\underline{n})(\underline{n}_1:\underline{n}_1) \in \langle\underline{n}_2\rangle^{\perp}\ ,\ \ldots$$

$$\underline{x}(\underline{n}:\underline{n})(\underline{n}_1:\underline{n}_1)\ \ldots\ (\underline{n}_k:\underline{n}_k) \in \langle\underline{r}\rangle^{\perp}\ .$$

Since, by (++), equality does occur, we have $\underline{x} \in \langle\underline{n}\rangle^{\perp}$. This holds for all $x \in \langle\underline{r}\rangle^{\perp}$. Thus $\langle\underline{r}\rangle^{\perp} \subseteq \langle\underline{n}\rangle^{\perp}$. Now, since $\langle\underline{r}\rangle^{\perp}$ and $\langle\underline{n}\rangle^{\perp}$ have the same dimension, we have $\langle\underline{r}\rangle^{\perp} = \langle\underline{n}\rangle^{\perp}$, i.e. $\langle\underline{r}\rangle = \langle\underline{n}\rangle$. But, since we assumed $(\underline{n}:\underline{r}) \notin E'$, we have $\langle\underline{r}\rangle \neq \langle\underline{n}\rangle$. Thus E' does not generate E and so certainly does not generate $Sing_n$.

<u>6.2 DEFINITION</u> Let E be the set of idempotents of rank $n-1$ of $Sing_n$ and let A and B be subsets of E. Define $A_0 = A$ and $A_i = A_{i-1}^3 \cap E$ $(i=1,2,\ldots)$. Clearly, $A = A_0 \subseteq A_1 \subseteq A_2 \ldots$. We shall say that B is <u>A_i-accessible</u> if $B \subseteq A_{i+1}$ and A-obtainable if B is A_i-accessible for some $i \in \mathbb{N}$. Clearly, if B is A-obtainable, then A generates B.

<u>6.3 DEFINITION</u> Let E be the set of idempotents of rank $n-1$ of $Sing_n$ and let A be a subset of E. If $\varepsilon \in E$ is A-obtainable, we shall define the <u>height of ε from</u> A to be $h_A(\varepsilon) = \min\{m : \varepsilon \in A_m\}$.

The next three lemmas are trivial, but it is more convenient to place them here than include them in the proof of Theorem 6.7 where they will be called upon.

<u>6.4 LEMMA</u> If $\varepsilon_1,\varepsilon_2,\varepsilon_3$ are A-obtainable, i.e.

$$\varepsilon_1,\varepsilon_2,\varepsilon_3 \in \bigcup_{i=0}^{\infty} A_i\ ,$$

for some subset A of E and if $\varepsilon_1\varepsilon_2\varepsilon_3 \in E$, then

$$h_A(\varepsilon_1\varepsilon_2\varepsilon_3) \leq \max_{i=1,2,3} \{h_A(\varepsilon_i)\} + 1$$

<u>PROOF</u> Let $h = \max_{i=1,2,3} \{h_A(\varepsilon_i)\}$. Then $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in A_h$. Thus $\varepsilon_1\varepsilon_2\varepsilon_3 \in A_h^3 \cap E = A_{h+1}$. So $h_A(\varepsilon_1\varepsilon_2\varepsilon_3) \leq h + 1$.

<u>6.5 LEMMA</u> If $h_A(\varepsilon) = m$, for some subset A of E and some $\varepsilon \in E$, $\varepsilon = \varepsilon_1\varepsilon_2\varepsilon_3$, for some

$$\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \bigcup_{i=0}^{\infty} A_i \ ,$$

and $h_A(\varepsilon_i) < m$, then $\max_{i=1,2,3} \{h_A(\varepsilon_i)\} = m - 1$.

<u>PROOF</u> This is immediate from Lemma 6.4 .

<u>6.6 LEMMA</u> $A = \{\varepsilon \in \bigcup_{i=0}^{\infty} A_i : h_A(\varepsilon) = 0\}$ for all subsets A of E .

<u>PROOF</u> This is immediate from the definition of height.

<u>6.7 THEOREM</u> Let V be an n-dimensional vector space over an infinite field F . Let $Sing_n$ denote the semigroup of singular endomorphisms of V and let PF_{n-1} be the set of elements of $Sing_n$ with rank $n - 1$. Then there exists a subset A of the idempotents E in PF_{n-1} such that A is a sparse cover for PF_{n-1} and A generates $Sing_n$.

<u>PROOF</u> The proof is by induction on m in the following

hypothesis:

There exists a subset $A^{(m)}$ of the idempotents E_m in PF_{m-1} such that $A^{(m)}$ is a sparse cover for PF_{m-1} and E_m is $A^{(m)}$-obtainable

If $m = 1$, then the hypothesis is clearly true since E_1 consists solely of the zero map. So, putting $A^{(1)} = E_1$, we have that $A^{(1)}$ is a sparse cover for PF_0^0 and E_1 is $A^{(1)}$-obtainable.

Now suppose the hypothesis holds for $m = n - 1$. We shall show that it also holds for $m = n$. Adopting the notation of Lemma 4.11 let $A^{(m-1)} = \{(\underline{y}(\underline{x}):\underline{x}) : x \in V'_{n-1}\}$. As before, define the mapping $\underline{\bar{y}} : V'_n \to V'_n$ by

$$\underline{\bar{y}}(\underline{x},a) = \begin{cases} (\underline{y}(\underline{x}),b(\underline{x},a)) & \text{if } \underline{x} \in V'_{n-1} \\ (\underline{0},1) & \text{if } \underline{x} = \underline{0} \text{ and } a = 1 \end{cases}$$

where

$$b(\underline{x},a) = \begin{cases} [1 - \langle\underline{y}(\underline{x})|\underline{x}\rangle/(a\chi)]\xi^{-1} & a \neq 0 \\ 1 & a = 0 \end{cases}$$

The inverse of $\underline{\bar{y}}$ is given by

$$\underline{\bar{y}}^{-1}(\underline{x},a) = \begin{cases} (\underline{y}^{-1}(\underline{x}),c(\underline{x},a)) & \text{if } \underline{x} \in V'_{n-1} \\ (\underline{0},1) & \text{if } \underline{x} = \underline{0} \text{ and } a = 1 \end{cases}$$

where

$$c(x,a) = \begin{cases} [\langle\underline{x}|\underline{y}^{-1}(x)\rangle/(1-a\xi)]\chi^{-1} & \text{if } a \neq 1 \\ 0 & \text{if } a = 1 \end{cases}$$

From the proof of Lemma 4.11 we know that

$$D_0 = \{(\bar{y}(\underline{x},a):(\underline{x},a)) : (\underline{x},a) \in V_n'\}$$

forms a sparse cover for PF_{n-1}^0 . We shall show that E_n is D_0-obtainable.

In listing the possible idempotents $(\underline{n}:\underline{r})$ in E_n we may suppose that $\underline{n}$ and $\underline{r}$ are expressed as $(\underline{x},a)$ with $\underline{x} \in V_{n-1}'$ and $a \in F$ or as $(\underline{0},1)$. The four main cases are:

(A) $\underline{n} = (\underline{z},c)$ and $\underline{r} = (\underline{x},a)$ where $\underline{z},\underline{x} \in V_{n-1}'$

(B) $\underline{n} = (\underline{z},c)$ with $\underline{z} \in V_{n-1}'$ and $\underline{r} = (\underline{0},1)$

(C) $\underline{n} = (\underline{0},1)$ and $\underline{r} = (\underline{x},a)$ with $\underline{x} \in V_{n-1}'$

(D) $\underline{n} = \underline{r} = (\underline{0},1)$.

We may subdivide case (A) into subcases as follows:

(A1) $\underline{z} = \underline{y}(\underline{x})$, $c = b(\underline{x},a)$

(A2) $\underline{z} = \underline{y}(\underline{x})$, $c \neq b(\underline{x},a)$

(A3) $\underline{z} = \underline{y}(\underline{x})$, $a = 0$, $c = 1$

(A4) $\underline{z} = \underline{y}(\underline{x})$, $a \neq 0$, $c = 1$

(A5) $\underline{z} \neq \underline{y}(\underline{x})$, $\langle\underline{z}|\underline{x}\rangle \neq 0$

(A6) $\underline{z} \neq \underline{y}(\underline{x})$, $\langle\underline{z}|\underline{x}\rangle = 0$.

Case (B) may be subdivided into:

(B1) $\underline{n} = (\underline{z},c)$, $\underline{r} = (\underline{0},1)$, $c \neq 1$

(B2) $\underline{n} = (\underline{z},c)$, $\underline{r} = (\underline{0},1)$, $c = 1$.

In cases (A1) and (D) we have that $(\underline{n},\underline{r}) \in D_0$. The remaining elements of E_n may thus be divided into eight classes as follows. The reason for the order of the listing will become apparent as the proof progresses.

$$D_1 = \{((\underline{y}(\underline{x}),c):(\underline{x},0)) : c \neq 1 , \underline{x} \in V_{n-1}'\} \quad \text{(case (A3))}$$

$$D_2 = \{((\underline{0},1):(\underline{x},a)) : \underline{x} \in V_{n-1}'\} \quad \text{(case (C))}$$

$$D_3 = \{((\underline{y}(\underline{x}),1):(\underline{0},1)) \ : \ \underline{x} \in V'_{n-1}\} \quad \text{(case (B2))}$$

$$D_4 = \{((\underline{y}(\underline{x}),a):(\underline{0},1)) \ : \ a \neq 1 \ , \ \underline{x} \in V'_{n-1}\} \quad \text{(case (B1))}$$

$$D_5 = \{((\underline{y}(\underline{x}),1):(\underline{x},a)) \ : \ a \neq 0 \ , \ \underline{x} \in V'_{n-1}\} \quad \text{(case (A4))}$$

$$D_6 = \{((\underline{y}(\underline{x}),b):(\underline{x},a)) \ : \ a \neq 0 \ , \ b \neq 1 \ , \ \underline{x} \in V'_{n-1} \ , \ b \neq b(\underline{x},a)\} \quad \text{(case (A2))}$$

$$D_7 = \{((\underline{y}(\underline{z}),b):(\underline{x},a)) \ : \ \underline{x},\underline{z} \in V'_{n-1} \ , \ \underline{x} \neq \underline{z} \ , \ \langle \underline{y}(\underline{z}) | \underline{x} \rangle \neq 0\} \quad \text{(case (A5))}$$

$$D_8 = \{((\underline{y}(\underline{z}),b):(\underline{x},a)) \ : \ \underline{x},\underline{z} \in V'_{n-1} \ , \ \underline{x} \neq \underline{z} \ , \ \langle \underline{y}(\underline{z}) | \underline{x} \rangle = 0\} \quad \text{(case (A6))}$$

By the construction of $D_0,\ldots,D_8$ we have that $D_i \cap D_j = \Phi$ if $i \neq j$ and that

$$E_n = \bigcup_{i=0}^{8} D_i \ .$$

We shall show, in eight stages, that D_i is D_0-obtainable $(i=1,2,\ldots,8)$.

We show first by using Theorem 3.14 that D_1 is D_0-accessible. More precisely we show that

$$((\underline{y}(\underline{x}),a):(\underline{x},0)) = (\underline{n}_1:\underline{r}_1)(\underline{n}_2:\underline{r}_2)(\underline{n}_3:\underline{r}_3)$$

where

$$\underline{n}_1 = (\underline{y}(\underline{x}),a) \qquad \underline{r}_1 = \bar{\underline{y}}^{-1}(\underline{y}(\underline{x}),a) = (\underline{x},[\langle \underline{y}(\underline{x}) | \underline{x} \rangle/(1-a\xi)]\chi^{-1})$$

$$\underline{n}_2 = \bar{\underline{y}}(\underline{0},1) = (\underline{0},1) \qquad \underline{r}_2 = (\underline{0},1)$$

$$\underline{n}_3 = \bar{\underline{y}}(\underline{x},0) = (\underline{y}(\underline{x}),1) \qquad \underline{r}_3 = (\underline{x},0)$$

Notice first that $<\underline{n}_1>, <\underline{n}_2>$ and $<\underline{n}_3>$ are all distinct, as are $<\underline{r}_1>$, $<\underline{r}_2>$ and $<\underline{r}_3>$. Now define

$$\nu_1 = 1 \qquad \rho_1 = -(1-a\xi)\chi^{-1}$$

$$\nu_2 = 1 - a \qquad \rho_2 = <\underline{y}(\underline{x})|\underline{x}>\chi^{-1}$$

$$\nu_3 = -1 \qquad \rho_3 = (1-a\xi)\chi^{-1}$$

Since, for D_1 , we have $a \neq 1$, it follows that $1 - a\xi \neq 0$. Thus all of these are non-zero. Also

$$\nu_1\underline{n}_1 + \nu_2\underline{n}_2 + \nu_3\underline{n}_3 = (\underline{y}(\underline{x}),a) + (1-a)(\underline{0},1) - (\underline{y}(\underline{x}),1) = (\underline{0},0) ,$$

$$\rho_1\underline{r}_1 + \rho_2\underline{r}_2 + \rho_3\underline{r}_3 = -(1-a\xi)\chi^{-1}(\underline{x},[<\underline{y}(\underline{x})|\underline{x}>/(1-a\xi)]\chi^{-1})$$
$$+ <\underline{y}(\underline{x})|\underline{x}>\chi^{-1}(\underline{0},1) + (1-a\xi)\chi^{-1}(\underline{x},0) = (\underline{0},0)$$

and, by Lemma 3.13,

$$<\nu_1\underline{n}_1|\rho_1\underline{r}_1> + <\nu_1\underline{n}_1|\rho_2\underline{r}_2> + <\nu_2\underline{n}_2|\rho_2\underline{r}_2> = <\nu_2\underline{n}_2|\rho_2\underline{r}_2> - <\nu_1\underline{n}_1|\rho_3\underline{r}_3>$$

$$= [(1-a)\xi][<\underline{y}(\underline{x})|\underline{x}>]<(\underline{0},1)|(\underline{0},1)> - [1\xi][(1-a\xi)]<(\underline{y}(\underline{x}),a)|(\underline{x},0)>$$

$$= [(1-a)\xi]<\underline{y}(\underline{x})|\underline{x}> - (1-a\xi)<\underline{y}(\underline{x})|\underline{x}>$$

$$= 0 \quad \text{since } \xi \text{ is an automorphism.}$$

We now show, again using Theorem 3.14, that D_2 is D_0-accessible. We show that

$$((\underline{0}:1):(\underline{x},a)) = (\underline{n}_1:\underline{r}_1)(\underline{n}_2:\underline{r}_2)(\underline{n}_3:\underline{r}_3)$$

where

$$\underline{n}_1 = (\underline{0},1) \qquad \underline{r}_1 = \bar{\underline{y}}^{-1}(\underline{0},1) = (\underline{0},1)$$

$$\underline{n}_2 = \bar{\underline{y}}(\underline{x},0) = (\underline{y}(\underline{x}),1) \qquad \underline{r}_2 = (\underline{x},0)$$

$$\underline{n}_3 = \bar{\underline{y}}(\underline{x},a) = (\underline{y}(\underline{x}),[1-\langle\underline{y}(\underline{x})|\underline{x}\rangle/(a\chi)]\xi^{-1}) \qquad \underline{r}_3 = (\underline{x},a)$$

Notice first that $\langle\underline{n}_1\rangle, \langle\underline{n}_2\rangle$ and $\langle\underline{n}_3\rangle$ are all distinct, as are $\langle\underline{r}_1\rangle, \langle\underline{r}_2\rangle$ and $\langle\underline{r}_3\rangle$. Now define

$$\nu_1 = \langle\underline{y}(\underline{x})|\underline{x}\rangle\xi^{-1} \qquad \rho_1 = a$$

$$\nu_2 = -a\chi\xi^{-1} \qquad \rho_2 = 1$$

$$\nu_3 = a\chi\xi^{-1} \qquad \rho_3 = -1$$

Since $((\underline{0},1):(\underline{x},a)) \in E$, we have, by Lemma 2.6, that $0 \neq \langle(\underline{0},1)|(\underline{x},a)\rangle = (1\xi)(a\chi) = a\chi$. Thus $\nu_1,\nu_2,\nu_3,\rho_1,\rho_2$ and ρ_3 are non-zero. Also

$$\begin{aligned}\nu_1\underline{n}_1 + \nu_2\underline{n}_2 + \nu_3\underline{n}_3 &= \langle\underline{y}(\underline{x})|\underline{x}\rangle\xi^{-1}(\underline{0},1) - a\chi\xi^{-1}(\underline{y}(\underline{x}),1) \\ &\quad + a\chi\xi^{-1}(\underline{y}(\underline{x}),[1-\langle\underline{y}(\underline{x})|\underline{x}\rangle/(a\chi)]\xi^{-1}) \\ &= (\underline{0},\langle\underline{y}(\underline{x})|\underline{x}\rangle\xi^{-1} - a\chi\xi^{-1} + (a\chi-\langle\underline{y}(\underline{x})|\underline{x}\rangle)\xi^{-1}) \\ &= (\underline{0},0)\end{aligned}$$

$$\rho_1\underline{r}_1 + \rho_2\underline{r}_2 + \rho_3\underline{r}_3 = a(\underline{0},1) + (\underline{x},0) - (\underline{x},a) = (\underline{0},0)$$

and, by Lemma 3.13,

$$\begin{aligned}&\langle\nu_1\underline{n}_1|\rho_1\underline{r}_1\rangle + \langle\nu_1\underline{n}_1|\rho_2\underline{r}_2\rangle + \langle\nu_2\underline{n}_2|\rho_2\underline{r}_2\rangle = \langle\nu_2\underline{n}_2|\rho_2\underline{r}_2\rangle - \langle\nu_1\underline{n}_1|\rho_3\underline{r}_3\rangle \\ &= -(a\chi)(1\xi)\langle(\underline{y}(\underline{x}),1)|(\underline{x},0)\rangle + \langle\underline{y}(\underline{x})|\underline{x}\rangle(1\xi)\langle(\underline{0},1)|(\underline{x},a)\rangle \\ &= -(a\chi)\langle\underline{y}(\underline{x})|\underline{x}\rangle + \langle\underline{y}(\underline{x})|\underline{x}\rangle(a\chi) \\ &= 0 .\end{aligned}$$

Next we show that D_3 is D_0-accessible, again using Theorem 3.14. In fact we show that

$$((\underline{y}(\underline{x}),1):(\underline{0},1)) = (\underline{n}_1:\underline{r}_1)(\underline{n}_2:\underline{r}_2)(\underline{n}_3:\underline{r}_3)$$

where

$$\underline{n}_1 = (\underline{y}(\underline{x}),1) \qquad \underline{r}_1 = \bar{\underline{y}}^{-1}(\underline{y}(\underline{x}),1) = (\underline{x},0)$$

$$\underline{n}_2 = \bar{\underline{y}}(\underline{x},1) = (\underline{y}(x),[1-\langle\underline{y}(\underline{x})|\underline{x}\rangle]\xi^{-1}) \qquad \underline{r}_2 = (\underline{x},1)$$

$$\underline{n}_3 = \bar{\underline{y}}(\underline{0},1) = (\underline{0},1) \qquad \underline{r}_3 = (\underline{0},1)$$

Notice first that, since $\langle\underline{y}(\underline{x})|\underline{x}\rangle \neq 0$, $\langle\underline{n}_1\rangle,\langle\underline{n}_2\rangle$ and $\langle\underline{n}_3\rangle$ are distinct, as are $\langle\underline{r}_1\rangle,\langle\underline{r}_2\rangle,\langle\underline{r}_3\rangle$. Now define

$$\nu_1 = -1 \qquad \rho_1 = 1$$

$$\nu_2 = 1 \qquad \rho_2 = -1$$

$$\nu_3 = \langle\underline{y}(\underline{x})|\underline{x}\rangle\xi^{-1} \qquad \rho_3 = 1$$

Now, all these are non-zero. Also

$$\begin{aligned}\nu_1\underline{n}_1 + \nu_2\underline{n}_2 + \nu_3\underline{n}_3 &= -(\underline{y}(\underline{x}),1) + (\underline{y}(\underline{x}),[1-\langle\underline{y}(\underline{x})|\underline{x}\rangle]\xi^{-1}) \\ &\quad + \langle\underline{y}(\underline{x})|\underline{x}\rangle\xi^{-1}(\underline{0},1) \\ &= (\underline{0},0) ,\end{aligned}$$

$$\begin{aligned}\rho_1\underline{r} + \rho_2\underline{r}_2 + \rho_3\underline{r}_3 &= (\underline{x},0) - (\underline{x},1) + (\underline{0},1) \\ &= (\underline{0},0)\end{aligned}$$

and, by Lemma 3.13,

$$\begin{aligned}\langle\nu_1\underline{n}_1|\rho_1\underline{r}_1\rangle + \langle\nu_1\underline{n}_1|\rho_2\underline{r}_2\rangle + \langle\nu_2\underline{n}_2|\rho_2\underline{r}_2\rangle &= \langle\nu_1\underline{n}_1|\rho_1\underline{r}_1\rangle - \langle\nu_3\underline{n}_3|\rho_2\underline{r}_2\rangle \\ &= -(1\xi)(1\chi)\langle(\underline{y}(\underline{x}),1)|(\underline{x},0)\rangle + \langle\underline{y}(\underline{x})|\underline{x}\rangle\langle(\underline{0},1)|(\underline{x},1)\rangle \\ &= -\langle\underline{y}(\underline{x})|\underline{x}\rangle + \langle\underline{y}(\underline{x})|\underline{x}\rangle = 0 .\end{aligned}$$

In the next step we show that D_4 is $(D_0 \cup D_1 \cup D_3)$-accessible. Since D_1 and D_3 have been shown to be D_0-accessible, we shall thus have that D_4 is D_0-obtainable. Again we use Theorem 3.14 to show that

$$((\underline{y}(\underline{x}),a):(\underline{0},1)) = (\underline{n}_1:\underline{r}_1)(\underline{n}_2:\underline{r}_2)(\underline{n}_3:\underline{r}_3)$$

where

$$\underline{n}_1 = (\underline{y}(\underline{x}),a) \qquad \underline{r}_1 = \bar{\underline{y}}^{-1}(\underline{y}(\underline{x}),a) = (\underline{x},[\langle\underline{y}(\underline{x})|\underline{x}\rangle/(1-a\xi)]\chi^{-1})$$

$$\underline{n}_2 = (\underline{y}(\underline{x}),(1-a+a^2)/a) \qquad \underline{r}_2 = (\underline{x},0)$$

$$\underline{n}_3 = (\underline{y}(\underline{x}),1) \qquad \underline{r}_3 = (\underline{0},1)$$

Notice that, since $((\underline{y}(\underline{x}),a):(\underline{0},1)) \in E$, we have, by Lemma 2.6, that $0 \neq \langle(\underline{y}(\underline{x}),a)|(\underline{0},1)\rangle = a\xi$. Thus $a \neq 0$ and so the definition of $\underline{n}_2$ is meaningful. Also, since $\langle\underline{y}(\underline{x})|\underline{x}\rangle \neq 0$ and, in D_4, $a \neq 1$, we have that $\langle\underline{n}_1\rangle, \langle\underline{n}_2\rangle$ and $\langle\underline{n}_3\rangle$ are distinct, as are $\langle\underline{r}_1\rangle, \langle\underline{r}_2\rangle$ and $\langle\underline{r}_3\rangle$. Now define

$$\nu_1 = a - 1 \qquad \rho_1 = -(1-a\xi)\chi^{-1}$$

$$\nu_2 = -a \qquad \rho_2 = (1-a\xi)\chi^{-1}$$

$$\nu_3 = 1 \qquad \rho_3 = \langle\underline{y}(\underline{x})|\underline{x}\rangle\chi^{-1}$$

Since $\langle\underline{y}(\underline{x})|\underline{x}\rangle \neq 0$ and, in D_4, $a \neq 1$, we have that all these are non-zero. Also

$$\begin{aligned}\nu_1\underline{n}_1 + \nu_2\underline{n}_2 + \nu_3\underline{n}_3 &= (a-1)(\underline{y}(\underline{x}),a) - a(\underline{y}(\underline{x}),(1-a+a^2)/a) + 1(\underline{y}(\underline{x}),1) \\ &= (\underline{0},a^2-a-1+a-a^2+1) \\ &= (\underline{0},0)\ ,\end{aligned}$$

$$\rho_1\underline{r}_1 + \rho_2\underline{r}_2 + \rho_3\underline{r}_3 = -(1-a\xi)\chi^{-1}(\underline{x},[\langle\underline{y}(\underline{x})|\underline{x}\rangle/(1-a\xi)]\chi^{-1})$$
$$+ (1-a\xi)\chi^{-1}(\underline{x},0) + \langle\underline{y}(\underline{x})|\underline{x}\rangle(\underline{0},1)$$
$$= (\underline{0},0)$$

and, by Lemma 3.13,

$$\langle\nu_1\underline{n}_1|\rho_1\underline{r}_1\rangle + \langle\nu_1\underline{n}_1|\rho_2\underline{r}_2\rangle + \langle\nu_2\underline{n}_2|\rho_2\underline{r}_2\rangle = \langle\nu_2\underline{n}_2|\rho_2\underline{r}_2\rangle - \langle\nu_1\underline{n}_1|\rho_3\underline{r}_3\rangle$$
$$= -(a\xi)(1-a\xi)\langle(\underline{y}(\underline{x}),(1-a+a^2)/a)|(\underline{x},0)\rangle$$
$$- [(a-1)\xi]\langle\underline{y}(\underline{x})|\underline{x}\rangle\langle(\underline{y}(\underline{x}),a)|(\underline{0},1)\rangle$$
$$= -(a\xi)[(1-a)\xi]\langle\underline{y}(\underline{x})|\underline{x}\rangle - [(a-1)\xi]\langle\underline{y}(\underline{x})|\underline{x}\rangle(a\xi)(1\chi)$$
$$= 0 .$$

To show that $(n_2:r_2) \in D_1$ we need only show that $(1-a+a^2)/a \neq 1$. But if $(1-a+a^2)/a = 1$, we would have $a = 1$ and this is excluded by D_4. $(\underline{n}_3:\underline{r}_3)$ clearly belongs to D_3.

Next we show that D_5 is $(D_0 \cup D_1 \cup D_3)$-accessible and hence D_0-obtainable. More precisely we show that

$$((\underline{y}(\underline{x}),1):(\underline{x},a)) = (\underline{n}_1:\underline{r}_1)(\underline{n}_2:\underline{r}_2)(\underline{n}_3:\underline{r}_3)$$

where

$$\underline{n}_1 = (\underline{y}(\underline{x}),1) \qquad \underline{r}_1 = (\underline{0},1)$$
$$\underline{n}_2 = (\underline{y}(\underline{x}),[a\chi/(a\chi+\langle\underline{y}(\underline{x})|\underline{x}\rangle)]\xi^{-1}) \qquad \underline{r}_2 = (\underline{x},0)$$
$$\underline{n}_3 = \bar{\underline{y}}(\underline{x},a) = (\underline{y}(\underline{x}),[1-\langle\underline{y}(\underline{x})|\underline{x}\rangle/(a\chi)]\xi^{-1}) \qquad \underline{r}_3 = (\underline{x},a)$$

Now, since $((\underline{y}(\underline{x}),1):(\underline{x},a)) \in E$, we have, by Lemma 2.6, that $0 \neq \langle(\underline{y}(\underline{x}),1)|(\underline{x},a)\rangle = \langle\underline{y}(\underline{x})|\underline{x}\rangle + (a\chi)$. Thus the definition of $\underline{n}_2$ is meaningful. Also, since $\langle\underline{y}(\underline{x})|\underline{x}\rangle \neq 0$, we have that $\langle\underline{n}_1\rangle,\langle\underline{n}_2\rangle$ and

$\langle \underline{n}_3 \rangle$ are distinct, as are $\langle \underline{r}_1 \rangle , \langle \underline{r}_2 \rangle$ and $\langle \underline{r}_3 \rangle$.

Now define

$$\nu_1 = \langle \underline{y}(\underline{x}) | \underline{x} \rangle \xi^{-1} \qquad \rho_1 = a$$

$$\nu_2 = -(a\chi + \langle \underline{y}(\underline{x}) | \underline{x} \rangle)\xi^{-1} \qquad \rho_2 = 1$$

$$\nu_3 = a\chi\xi^{-1} \qquad \rho_3 = -1 \ .$$

All of these are non-zero. Also

$$\begin{aligned}
\nu_1\underline{n}_1 + \nu_2\underline{n}_2 + \nu_3\underline{n}_3 &= \langle \underline{y}(\underline{x}) | \underline{x} \rangle \xi^{-1} (\underline{y}(\underline{x}), 1) \\
&\quad - (a\chi + \langle \underline{y}(\underline{x}) | \underline{x} \rangle)\xi^{-1} (\underline{y}(\underline{x}), [a\chi/(a\chi + \langle \underline{y}(\underline{x}) | \underline{x} \rangle)]\xi^{-1}) \\
&\quad + a\chi\xi^{-1}(\underline{y}(\underline{x}), [1 - \langle \underline{y}(\underline{x}) | \underline{x} \rangle/(a\chi)]\xi^{-1}) \\
&= (\underline{0}, \langle \underline{y}(\underline{x}) | \underline{x} \rangle \xi^{-1} - (a\chi\xi^{-1}) + [1 - \langle \underline{y}(\underline{x}) | \underline{x} \rangle]\xi^{-1}) \\
&= (\underline{0}, 0) \ ,
\end{aligned}$$

$$\begin{aligned}
\rho_1\underline{r}_1 + \rho_2\underline{r}_2 + \rho_3\underline{r}_3 &= a(\underline{0}, 1) + (\underline{x}, 0) - (\underline{x}, a) \\
&= (\underline{0}, 0) \ ,
\end{aligned}$$

and, by Lemma 3.13,

$$\begin{aligned}
\langle \nu_1\underline{n}_1 | \rho_1\underline{r}_1 \rangle + \langle \nu_1\underline{n}_1 | \rho_2\underline{r}_2 \rangle + \langle \nu_2\underline{n}_2 | \rho_2\underline{r}_2 \rangle &= \langle \nu_1\underline{n}_1 | \rho_1\underline{r}_1 \rangle - \langle \nu_3\underline{n}_3 | \rho_2\underline{r}_2 \rangle \\
&= \langle \underline{y}(\underline{x}) | \underline{x} \rangle . a\chi . \langle (\underline{y}(\underline{x}), 1) | (\underline{0}, 1) \rangle \\
&\quad - a\chi . 1\xi . \langle (\underline{y}(\underline{x}), [1 - \langle \underline{y}(\underline{x}) | \underline{x} \rangle/(a\chi)]\xi^{-1}) | (\underline{x}, 0) \rangle \\
&= a\chi . \langle \underline{y}(\underline{x}) | \underline{x} \rangle - a\chi . \langle \underline{y}(\underline{x}) | \underline{x} \rangle \\
&= 0 \ .
\end{aligned}$$

Clearly $(\underline{n}_1 : \underline{r}_1) \in D_3$ and $(\underline{n}_2 : \underline{r}_2) \in D_1$ since, by D_5 , $a \neq 0$.

To show that D_6 is D_0-obtainable, we show that D_6 is $(D_0 \cup D_1)$-accessible. In particular, we show that

$$((\underline{y}(\underline{x}),b):(\underline{x},a)) = (\underline{n}_1:\underline{r}_1)(\underline{n}_2:\underline{r}_2)(\underline{n}_3:\underline{r}_3)$$

where

$$\underline{n}_1 = (\underline{y}(\underline{x}),b) \qquad \underline{r}_1 = \bar{\underline{y}}^{-1}(\underline{y}(\underline{x}),b)$$

$$= (\underline{x},[\langle \underline{y}(\underline{x}) | \underline{x} \rangle / (1-b\xi)]\chi^{-1})$$

$$\underline{n}_2 = (\underline{y}(\underline{x}),[b\xi-1+(a\chi)/\langle(\underline{y}(\underline{x}),b)|(\underline{x},a)\rangle]\xi^{-1}) \quad \underline{r}_2 = (\underline{x},0)$$

$$\underline{n}_3 = \bar{\underline{y}}(\underline{x},a) = (\underline{y}(\underline{x}),[1-\langle \underline{y}(\underline{x}) | \underline{x} \rangle/(a\chi)]\xi^{-1}) \qquad \underline{r}_3 = (\underline{x},a)$$

Now in D_6 we have $b \neq 1$ and $a \neq 0$ and so the definitions of $\underline{r}_1$ and $\underline{n}_3$ are meaningful. Also, since $((\underline{y}(\underline{x}),b):(\underline{x},a)) \in E$, we have, by Lemma 2.6, that $<(\underline{y}(\underline{x}),b)|(\underline{x},a)> \neq 0$. Thus the definition of $\underline{n}_2$ is meaningful.

We now show that $<\underline{n}_1>, <\underline{n}_2>$ and $<\underline{n}_3>$ are distinct. Since $((\underline{y}(\underline{x}),b):(\underline{x},a)) \notin D_0$ we have $b \neq [1-<\underline{y}(\underline{x})|\underline{x}>/(a\chi)]\xi^{-1}$. Thus $<\underline{n}_1>$ and $<\underline{n}_3>$ are distinct. Now suppose that $<\underline{n}_1> = <\underline{n}_2>$. Then $b\xi = b\xi - 1 + (a\chi)/<(\underline{y}(\underline{x}),b)|(\underline{x},a)>$, i.e.

$$<\underline{y}(\underline{x})|\underline{x}> + b\xi.a\chi = a\chi .$$

But this implies

$$b\xi = 1 - <\underline{y}(\underline{x})|\underline{x}>/(a\chi)$$

which we have already shown to be false. Thus $<\underline{n}_1> \neq <\underline{n}_2>$. Finally we show that $<\underline{n}_2>$ and $<\underline{n}_3>$ are distinct. Suppose not, then

$$b\xi - 1 + (a\chi)/<(\underline{y}(\underline{x}),b)|(\underline{x},a)> = 1 - <\underline{y}(\underline{x})|\underline{x}>/(a\chi) ,$$

i.e.

$$b\xi - 2 + <\underline{y}(\underline{x})|\underline{x}>/(a\chi) + (a\chi)/(<\underline{y}(\underline{x})|\underline{x}>+b\xi.a\chi) = 0 .$$

But this would imply

$$a\chi.b\xi<\underline{y}(\underline{x})|\underline{x}> + (a\chi)^2(b\xi)^2 - 2(a\chi)<\underline{y}(\underline{x})|\underline{x}> - 2(a\chi)^2.b\xi$$
$$+ <\underline{y}(\underline{x})|\underline{x}>^2 + a\chi.b\xi<\underline{y}(\underline{x})|\underline{x}> + (a\chi)^2 = 0 ,$$

i.e.

$$(a\chi.b\xi-a\chi+<\underline{y}(\underline{x})|\underline{x}>)^2 = 0 .$$

Thus

$$b\xi = 1 - \langle \underline{y}(\underline{x}) | \underline{x} \rangle / (a\chi)$$

which we have already shown to be false. Thus $\langle \underline{n}_2 \rangle \neq \langle \underline{n}_3 \rangle$.

We now show that $\langle \underline{r}_1 \rangle, \langle \underline{r}_2 \rangle$ and $\langle \underline{r}_3 \rangle$ are distinct. Since $\langle \underline{y}(\underline{x}) | \underline{x} \rangle \neq 0$ and, in D_6 , $a \neq 0$, it is clear $\langle r_2 \rangle \neq \langle \underline{r}_3 \rangle$ and $\langle \underline{r}_2 \rangle \neq \langle \underline{r}_1 \rangle$. Now suppose $\langle \underline{r}_1 \rangle = \langle \underline{r}_3 \rangle$. Then

$$a\chi = \langle \underline{y}(\underline{x}) | \underline{x} \rangle / (1 - b\xi) ,$$

i.e.

$$a\chi - a\chi . b\xi = \langle \underline{y}(\underline{x}) | \underline{x} \rangle$$

and so

$$b\xi = 1 - \langle \underline{y}(\underline{x}) | \underline{x} \rangle / (a\chi)$$

which we have already shown to be false in D_6 .

Now define

$$\nu_1 = [a\chi(b\xi - 1) + \langle \underline{y}(\underline{x}) | \underline{x} \rangle]\xi^{-1} \qquad \rho_1 = -(1 - b\xi)\chi^{-1}a$$

$$\nu_2 = -[a\chi . b\xi + \langle \underline{y}(\underline{x}) | \underline{x} \rangle]\xi^{-1} \qquad \rho_2 = [a\chi(1 - b\xi) - \langle \underline{y}(\underline{x}) | \underline{x} \rangle]\chi^{-1}$$

$$\nu_3 = a\chi\xi^{-1} \qquad \rho_3 = \langle \underline{y}(\underline{x}) | \underline{x} \rangle \chi^{-1}$$

Now ν_1 and ρ_2 are non-zero otherwise we would have $b\xi = 1 - \langle \underline{y}(\underline{x}) | \underline{x} \rangle / a\chi$ contrary to the conditions of D_6 . $\nu_2 = -\langle (\underline{y}(\underline{x}), b) | (\underline{x}, a) \rangle \xi^{-1}$ is non-zero, by Lemma 2.6, since $((\underline{y}(\underline{x}), b) : (\underline{x}, a)) \in E$. ν_3 and ρ_1 are non-zero since, by the conditions of D_6 , $a \neq 0$ and $b \neq 1$. ρ_3 is non-zero, by Lemma 2.6, since $(\underline{y}(\underline{x}) : \underline{x})$ is an idempotent in PF_{n-2} .

Also

$$\nu_1\underline{n}_1 + \nu_2\underline{n}_2 + \nu_3\underline{n}_3 = [a\chi(b\xi-1)+\langle\underline{y}(\underline{x})|\underline{x}\rangle]\xi^{-1}(\underline{y}(\underline{x}),b)$$

$$- [a\chi.b\xi+\langle\underline{y}(\underline{x})|\underline{x}\rangle]\xi^{-1}(\underline{y}(\underline{x}),[b\xi-1+a\chi/\langle(\underline{y}(\underline{x}),b)|(\underline{x},a)\rangle]\xi^{-1})$$

$$+ a\chi\xi^{-1}(\underline{y}(\underline{x}),[1-\langle\underline{y}(\underline{x})|\underline{x}\rangle/(a\chi)]\xi^{-1})$$

$$= (\underline{0},[a\chi(b\xi)^2-a\chi.b\xi+\langle\underline{y}(\underline{x})|\underline{x}\rangle.b\xi-a\chi(b\xi)^2$$

$$-\langle\underline{y}(\underline{x})|\underline{x}\rangle b\xi+a\chi.b\xi+\langle\underline{y}(\underline{x})|\underline{x}\rangle-a\chi+a\chi-\langle\underline{y}(\underline{x})|\underline{x}\rangle]\xi^{-1})$$

$$= (\underline{0},0) ,$$

$$\rho_1\underline{r}_1 + \rho_2\underline{r}_2 + \rho_3\underline{r}_3 = -(1-b\xi)\chi^{-1}a(\underline{x},[\langle\underline{y}(\underline{x})|\underline{x}\rangle/(1-b\xi)]\chi^{-1})$$

$$+ [a\chi(1-b\xi)-\langle\underline{y}(\underline{x})|\underline{x}\rangle]\chi^{-1}(\underline{x},0) + \langle\underline{y}(\underline{x})|\underline{x}\rangle\chi^{-1}(\underline{x},a)$$

$$= (\underline{0},-a\langle\underline{y}(\underline{x})|\underline{x}\rangle+a\langle\underline{y}(\underline{x})|x\rangle)$$

$$= (\underline{0},0) ,$$

and, by Lemma 3.13,

$$\langle\nu_1\underline{n}_1|\rho_1\underline{r}_1\rangle + \langle\nu_1\underline{n}_1|\rho_2\underline{r}_2\rangle + \langle\nu_2\underline{n}_2|\rho_2\underline{r}_2\rangle = \langle\nu_2\underline{n}_2|\rho_2\underline{r}_2\rangle - \langle\nu_1\underline{n}_1|\rho_3\underline{r}_3\rangle$$

$$= -[a\chi.b\xi+\langle\underline{y}(\underline{x})|\underline{x}\rangle][a\chi(1-b\xi)-\langle\underline{y}(\underline{x})|\underline{x}\rangle]\langle\underline{y}(\underline{x})|\underline{x}\rangle$$

$$- [a\chi(b\xi-1)+\langle\underline{y}(\underline{x})|\underline{x}\rangle]\langle\underline{y}(\underline{x})|\underline{x}\rangle\langle(\underline{y}(\underline{x}),b)|(\underline{x},a)\rangle$$

$$= 0$$

since $\langle(\underline{y}(\underline{x}),b)|(\underline{x},a)\rangle = \langle\underline{y}(\underline{x})|\underline{x}\rangle + b\xi.a\chi$.

To show that any element $((\underline{y}(\underline{z}),b):(\underline{x},a))$ of D_7 is D_0-obtainable we must use induction on the height of the idempotent $(\underline{y}(\underline{z}):\underline{x})$ of PF_{n-2} from $A^{(n-1)}$. Suppose, as the induction hypothesis, that all elements of

$$\bigcup_{t=0}^{7} D_t$$

of the form $((\underline{y}_i, b) : (\underline{x}_j, a))$ are D_0'-obtainable if

$$h_{A^{(n-1)}}((y_i : x_j)) \leq k \ .$$

Now, if $k = 0$, we have, by Lemma 6.6, that $(\underline{y}_i : \underline{x}_j) \in A^{(n-1)}$. Thus $\underline{y}_i = \underline{y}(\underline{x}_j)$. Thus

$$((\underline{y}_i, b) : (\underline{x}_j, a)) \in \bigcup_{t=0}^{6} D_t \ .$$

But we have already shown that

$$\bigcup_{t=0}^{6} D_t$$

is D_0'-obtainable, so we may start the induction process.

Consider now some element $((\underline{y}_1, b) : (\underline{x}_3, a))$ of

$$\bigcup_{t=0}^{7} D_t$$

where

$$h_{A^{(n-1)}}((\underline{y}_1 : \underline{x}_3)) = k + 1 \ .$$

Then $(\underline{y}_1 : \underline{x}_3) = (\underline{y}_1 : \underline{x}_1)(\underline{y}_2 : \underline{x}_2)(\underline{y}_3 : \underline{x}_3)$ for some idempotents $(\underline{y}_i : \underline{x}_i)$ $(i=1,2,3)$ of E_{n-1} where

$$h_{A^{(n-1)}}((\underline{y}_i : \underline{x}_i)) \leq k$$

$(i=1,2,3)$. By Theorem 1.12 and Lemma 3.12, $\langle \underline{x}_1 \rangle, \langle \underline{x}_2 \rangle$ and $\langle \underline{x}_3 \rangle$ are distinct, as are $\langle \underline{y}_1 \rangle, \langle \underline{y}_2 \rangle$ and $\langle \underline{y}_3 \rangle$. So, by Theorem 3.14 and Lemma 3.13, there exist non-zero elements $\nu_1, \nu_2, \nu_3, \rho_1, \rho_2$ and ρ_3 of F such that:

(i) $\nu_1\underline{y}_1 + \nu_2\underline{y}_2 + \nu_3\underline{y}_3 = \underline{0}$

(ii) $\rho_1\underline{x}_1 + \rho_2\underline{x}_2 + \rho_3\underline{x}_3 = \underline{0}$

(iii) $\langle\nu_1\underline{y}_1|\rho_1\underline{x}_1\rangle - \langle\nu_3\underline{y}_3|\rho_2\underline{x}_2\rangle = 0$.

Now, we wish to find elements c , d , e and f of F such that $[(\underline{y}_1,b):(\underline{x}_1,c)]$, $[(\underline{y}_2,d):(\underline{x}_2,e)]$ and $[(\underline{y}_3,f):(\underline{x}_3,a)]$ are group H-classes and $((\underline{y}_1,b):(\underline{x}_3,a)) = ((\underline{y}_1,b):(\underline{x}_1,c))((\underline{y}_2,d):(\underline{x}_2,e))((\underline{y}_3,f):(\underline{x}_3,a))$, i.e., by Lemma 2.6, such that:

(1) $\langle\underline{y}_1|\underline{x}_1\rangle + b\xi.c\chi \neq 0$

(2) $\langle\underline{y}_2|\underline{x}_2\rangle + d\xi.e\chi \neq 0$

(3) $\langle\underline{y}_3|\underline{x}_3\rangle + f\xi.a\chi \neq 0$

and, by Theorem 3.14, Lemma 3.13 and (i), (ii) and (iii) above, such that:

(4) $\nu_1 b + \nu_2 d + \nu_3 f = 0$

(5) $\rho_1 c + \rho_2 e + \rho_3 a = 0$

(6) $(\nu_1 b)\xi.(\rho_1 c)\chi - (\nu_3 f)\xi.(\rho_2 e)\chi = 0$.

We first find two values that c may not take. From (1) we see that if $b \neq 0$ then we must choose $c \in F$ such that

$$c\chi \neq -\langle\underline{y}_1|\underline{x}_1\rangle/(b\xi) \; . \qquad \text{(A)}$$

Eliminating a from (5) and (6) gives

$$(\nu_1 b)\xi.(\rho_1 c)\chi + (\nu_3 f)\xi.(\rho_1 c)\chi + (\nu_3 f)\xi.(\rho_3 a)\chi = 0 \; ,$$

i.e.

$$(\nu_3 f)\xi.(\rho_1 c+\rho_3 a)\chi + (\nu_1 b)\xi.(\rho_1 c)\chi = 0 \; .$$

From this and (3) we see we must choose c such that

$$-\nu_3\xi.\langle \underline{y}_3|\underline{x}_3\rangle.(\rho_1 c+\rho_3 a)\chi + a\chi.(\nu_1 b)\xi.(\rho_1 c)\chi \neq 0 ,$$

i.e.

$$c\chi[a\chi.(b\nu_1/\nu_3)\xi - \langle \underline{y}_3|\underline{x}_3\rangle] \neq (a\rho_3/\rho_1)\chi.\langle \underline{y}_3|\underline{x}_3\rangle .$$

Thus if $a\chi.(b\nu_1/\nu_3)\xi - \langle \underline{y}_3|\underline{x}_3\rangle \neq 0$ we must choose c such that

$$c\chi \neq (a\rho_3/\rho_1)\chi.\langle \underline{y}_3|\underline{x}_3\rangle/[a\chi.(b\nu_1/\nu_3)\xi - \langle \underline{y}_3|\underline{x}_3\rangle] . \quad \text{(B)}$$

It is also convenient to choose e to be non-zero. Thus, from (5),

$$c \neq -a\rho_3/\rho_1 . \quad \text{(C)}$$

Since F is infinite we have no trouble satisfying these three conditions.

Suppose now that we have chosen an element c of F to satisfy conditions (A), (B), (C).

From (5) we have

$$\rho_2 e = -\rho_1 c - \rho_3 a .$$

So from (6) we have

$$f\xi = -(b\nu_1/\nu_3)\xi[(\rho_1 c)/(\rho_1 c+\rho_3 a)]\chi .$$

(This is defined since, by (C), $c \neq -a\rho_3/\rho_1$.) Thus, from (4),

$$d\xi = -(b\nu_1/\nu_2)\xi + (b\nu_1/\nu_2)\xi.(\rho_1 c)\chi/[(\rho_1 c+\rho_3 a)\chi]$$

$$= -(b\nu_1/\nu_2)\xi[(\rho_3 a)/(\rho_1 c+\rho_3 a)]\chi .$$

We now show that with these values of c , d , e and f ,

$[(\underline{y}_1,b):(\underline{x}_1,c)]$, $[(\underline{y}_2,d):(\underline{x}_2,e)]$ and $[(\underline{y}_3,f):(\underline{x}_3,a)]$ are group H-classes.

If $b = 0$, then $<(\underline{y}_1,b)|(\underline{x}_1,c)> = <\underline{y}_1|\underline{x}_1> \neq 0$ since $(\underline{y}_1:\underline{x}_1) \in E_{m-1}$.

If $b \neq 0$, then

$$\begin{aligned} <(\underline{y}_1,b)|(\underline{x}_1,c)> &= <\underline{y}_1|\underline{x}_1> + b\xi c\chi \\ &\neq <\underline{y}_1|\underline{x}_1> - <\underline{y}_1|\underline{x}_1> \quad \text{(by (A))} \\ &= 0 . \end{aligned}$$

Next,

$$\begin{aligned} <(\underline{y}_2,d)|(\underline{x}_2,e)> &= <\underline{y}_2|\underline{x}_2> + d\xi e\chi \\ &= <\underline{y}_2|\underline{x}_2> + (b\nu_1/\nu_2)\xi.(a\rho_3/\rho_2)\chi \\ &= [<\nu_2\underline{y}_2|\rho_2\underline{x}_2> + (b\nu_1)\xi.(a\rho_3)\chi]/[\nu_2\xi.\rho_2\chi] \\ &= [<\nu_1\underline{y}_1+\nu_3\underline{y}_3|\rho_1\underline{x}_1+\rho_3\underline{x}_3>+(b\nu_1)\xi.(a\rho_3)\chi]/[\nu_2\xi.\rho_2\chi] \\ &\qquad \text{(by (i) and (ii))} \\ &= [<\nu_1\underline{y}_1|\rho_1\underline{x}_1>+<\nu_1\underline{y}_1|\rho_3\underline{x}_3>+<\nu_3\underline{y}_3|\rho_1\underline{x}_1> \\ &\quad +<\nu_3\underline{y}_3|\rho_3\underline{x}_3>+(b\nu_1)\xi.(a\rho_3)\chi]/[\nu_2\xi.\rho_2\chi] \\ &= [<\nu_1\underline{y}_1|\rho_3\underline{x}_3>+(b\nu_1)\xi.(a\rho_3)\chi]/[\nu_2\xi.\rho_2\chi] \\ &\qquad \text{(by (iii) and Lemma 3.13)} \\ &= \nu_1\xi.\rho_3\chi.<(\underline{y}_1,b)|(\underline{x}_3,a)>/[\nu_2\xi.\rho_2\chi] \\ &\neq 0 \quad \text{since} \quad ((\underline{y}_1,b):(\underline{x}_3,a)) \in E_n . \end{aligned}$$

Finally,

$$<(\underline{y}_3,f)\,|\,(\underline{x}_3,a)> \; = \; <\underline{y}_3|\underline{x}_3> + f\xi . a\chi$$

$$= <\underline{y}_3|\underline{x}_3> - (b\nu_1/\nu_3)\xi[\rho_1 ac/(\rho_1 c+\rho_3 a)]\chi \; .$$

If $a\chi.(b\nu_1/\nu_3)\xi - <\underline{y}_3|\underline{x}_3> = 0$, then

$$<(\underline{y}_3,f)\,|\,(\underline{x}_3,a)> \; = \; a\chi.(b\nu_1/\nu_3)\xi - (b\nu_1/\nu_3)\xi[\rho_1 ac/(\rho_1 c+\rho_3 a)]\chi$$

$$= a\chi.(b\nu_1/\nu_3)\xi[1-(\rho_1 c/(\rho_1 c+\rho_3 a))\chi]$$

$$= a\chi.(b\nu_1/\nu_3)\xi[\rho_3 a/(\rho_1 c+\rho_3 a)]\chi$$

$$= <\underline{x}_3|\underline{y}_3>[\rho_3 a/(\rho_1 c+\rho_3 a)]\chi \; .$$

Now, if $a = 0$, then the assumption $a\chi.(b\nu_1/\nu_3)\xi - <\underline{y}_3|\underline{x}_3> = 0$ would give $<\underline{y}_3|\underline{x}_3> = 0$ contradicting $(\underline{y}_3:\underline{x}_3) \in E_{n-1}$. Thus $a \neq 0$. Hence, if $a\chi.(b\nu_1/\nu_3)\xi - <\underline{y}_3|\underline{x}_3> = 0$, then

$$<(\underline{y}_3,f)\,|\,(\underline{x}_3,a)> \; \neq 0 \; .$$

Now suppose $a\chi.(b\nu_1/\nu_3)\xi - <\underline{y}_3|\underline{x}_3> \neq 0$. By (B) we have chosen c such that

$$c\chi \neq (a\rho_3/\rho_1)\chi.<\underline{y}_3|\underline{x}_3>/[a\chi.(b\nu_1/\nu_3)\xi - <\underline{y}_3|\underline{x}_3>] \; .$$

Thus

$$<(\underline{y}_3,f)\,|\,(\underline{x}_3,a)> \; = \; [(\rho_1 c+\rho_3 a)\chi.<\underline{y}_3|\underline{x}_3> - (\rho_1 ac)\chi.(b\nu_1/\nu_3)\xi]/[(\rho_1 c+\rho_3 a)\chi]$$

$$= \frac{(\rho_1 c)\chi.[<\underline{y}_3|\underline{x}_3> - (a\chi).(b\nu_1/\nu_3)\xi] + (\rho_3 a)\chi.<\underline{y}_3|\underline{x}_3>}{(\rho_1 c+\rho_3 a)\chi}$$

$$\neq \frac{-(a\rho_3)\chi.<\underline{y}_3|\underline{x}_3> + (\rho_3 a)\chi.<\underline{y}_3|\underline{x}_3>}{(\rho_1 c+\rho_3 a)\chi}$$

$$= 0 \; .$$

We now show, using Theorem 3.14, that with these values of c, d, e and f

$$((\underline{y}_1,b):(\underline{x}_3,a)) = ((\underline{y}_1,b):(\underline{x}_1,c))((\underline{y}_2,d):(\underline{x}_2,e))((\underline{y}_3,f):(\underline{x}_3,a)) .$$

Since $\langle\underline{y}_1\rangle$, $\langle\underline{y}_2\rangle$ and $\langle\underline{y}_3\rangle$ are distinct, then also $\langle(\underline{y}_1,b)\rangle$, $\langle(\underline{y}_2,d)\rangle$ and $\langle(\underline{y}_3,f)\rangle$ are distinct. Also, since $\langle\underline{x}_1\rangle$, $\langle\underline{x}_2\rangle$ and $\langle\underline{x}_3\rangle$ are distinct, then $\langle(\underline{x}_1,c)\rangle$, $\langle(\underline{x}_2,e)\rangle$ and $\langle(\underline{x}_3,a)\rangle$ are distinct.

Now,

$$\begin{aligned}\nu_1(\underline{y}_1,b) + \nu_2(\underline{y}_2,d) + \nu_3(\underline{y}_3,f) &= (\underline{0},\nu_1 b+\nu_2 d+\nu_3 f) \quad \text{(by (iii))}\\ &= (\underline{0},\nu_1 b-b\nu_1.[\rho_3 a/(\rho_1 c+\rho_3 a)]\chi\xi^{-1}-b\nu_1.[\rho_1 c/(\rho_1 c+\rho_3 a)]\chi\xi^{-1})\\ &= (\underline{0},\nu_1 b.[1-(\rho_3 a/(\rho_1 c+\rho_3 a))-(\rho_1 c/(\rho_1 c+\rho_3 a))]\chi\xi^{-1})\\ &= (\underline{0},\nu_1 b.[(\rho_1 c+\rho_3 a-\rho_3 a-\rho_1 c)/(\rho_1 c+\rho_3 a)]\chi\xi^{-1})\\ &= (\underline{0},0) ,\end{aligned}$$

$$\begin{aligned}\rho_1(\underline{x}_1,c) + \rho_2(\underline{x}_2,e) + \rho_3(\underline{x}_3,a) &= (\underline{0},\rho_1 c+\rho_2 e+\rho_3 a) \quad \text{(by (iii))}\\ &= (\underline{0},\rho_1 c-\rho_1 c-\rho_3 a+\rho_3 a)\\ &= (\underline{0},0)\end{aligned}$$

and, by Lemma 3.13,

$$\begin{aligned}&\langle\nu_1(\underline{y}_1,b)|\rho_1(\underline{x}_1,c)\rangle + \langle\nu_1(\underline{y}_1,b)|\rho_2(\underline{x}_2,e)\rangle + \langle\nu_2(\underline{y}_2,d)|\rho_2(\underline{x}_2,e)\rangle\\ &= \langle\nu_1(\underline{y}_1,b)|\rho_1(\underline{x}_1,c)\rangle - \langle\nu_3(\underline{y}_3,f)|\rho_2(x_2,e)\rangle\\ &= \langle\nu_1\underline{y}_1|\rho_1\underline{x}_1\rangle + (\nu_1 b)\xi.(\rho_1 c)\chi - \langle\nu_3\underline{y}_3|\rho_2\underline{x}_2\rangle - (\nu_3 f)\xi.(\rho_2 e)\chi\\ &= (\nu_1 b)\xi.(\rho_1 c)\chi - (\nu_3 f)\xi.(\rho_2 e)\chi \quad \text{(by (iii) and Lemma 3.13)}\end{aligned}$$

$$= (\nu_1 b)\xi.(\rho_1 c)\chi - (\nu_1 b)\xi[\rho_1 c/(\rho_1 c+\rho_3 a)]\chi.(\rho_1 c+\rho_3 a)\chi$$

$$= (\nu_1 b)\xi.(\rho_1 c)\chi - (\nu_1 b)\xi.(\rho_1 c)\chi$$

$$= 0 .$$

So the induction step on the height of elements of

$$\bigcup_{t=0}^{7} D_t$$

holds. Hence every element of D_7 is D_0-obtainable.

Finally we show that D_8 is

$$(\bigcup_{i=0}^{7} D_i)\text{-accessible}$$

and so D_0-obtainable. Let $((\underline{y}(\underline{z}),b):(\underline{x},a)) \in D_8$. Notice first that since $((\underline{y}(\underline{z}),b):(\underline{x},a)) \in D_8$ we have, by Lemma 2.6, that

$$0 \neq \langle(\underline{y}(\underline{z}),b)|(\underline{x},a)\rangle = \langle\underline{y}(\underline{z})|\underline{x}\rangle + b\xi.a\chi$$

$$= b\xi.a\chi . \qquad \text{(D)}$$

We shall find $c,d \in F$, $\underline{n},\underline{r} \in V_n'$ and non-zero elements $\nu_1,\nu_2,\nu_3,\rho_1,\rho_2,\rho_3 \in F$ such that $[(\underline{y}(\underline{z}),b):(\underline{z},0)]$, $[(\underline{n}:\underline{r})]$ and $[(\underline{y}(\underline{x}),d):(\underline{x},a)]$ are group H-classes, $(n:r) \in \bigcup_{i=0}^{7} D_7$ and

$$\nu_1(\underline{y}(\underline{z}),b) + \nu_2\underline{n} + \nu_3(\underline{y}(\underline{x}),d) = (\underline{0},0)$$

$$\rho_1(\underline{z},0) + \rho_2\underline{r} + \rho_3(\underline{x},a) = (\underline{0},0)$$

$$\langle\nu_1(\underline{y}(\underline{z}),b)|\rho_1(\underline{z},0)\rangle + \langle\nu_1(\underline{y}(\underline{z}),b)|\rho_2\underline{r}\rangle + \langle\nu_2\underline{n}_2|\rho_2\underline{r}_2\rangle = 0$$

We shall start by putting $\rho_1 = \rho_3 = 1$. If $[\underline{n}:\underline{r}]$ is to be a group H-class we must have $<\underline{n}|\underline{r}> \neq 0$ by Lemma 2.6. Thus

$$0 \neq <\nu_1(\underline{y}(\underline{z}),b) + \nu_3(\underline{y}(\underline{x}),d)|\rho_1(\underline{z},0) + \rho_3(\underline{x},a)>$$

$$= <\nu_1\underline{y}(\underline{z})|\rho_1\underline{z}> + <\nu_1\underline{y}(\underline{z})|\rho_3\underline{x}> + (\nu_1 b)\xi.(\rho_3 a)\chi$$

$$+ <\nu_3\underline{y}(\underline{x})|\rho_1\underline{z}> + <\nu_3\underline{y}(\underline{x})|\rho_3\underline{x}> + (\nu_3 d)\xi.(\rho_3 a)\chi \ .$$

But, since $((\underline{y}(\underline{z}),b):(\underline{x},a)) \in D_8$, we have $<\underline{y}(\underline{z})|\underline{x}> = 0$. Also $\rho_1 = \rho_3 = 1$, so

$$0 \neq \nu_1\xi<\underline{y}(\underline{z})|\underline{z}> + \nu_1\xi.b\xi.a\chi + \nu_3\xi<\underline{y}(\underline{x})|\underline{z}>$$

$$+ \nu_3\xi<\underline{y}(\underline{x})|\underline{x}> + \nu_3\xi.d\xi.a\chi \ .$$

If we choose d such that

$$<\underline{y}(\underline{x})|\underline{z}> + <\underline{y}(\underline{x})|\underline{x}> + d\xi.a\chi \neq 0 \ ,$$

i.e. such that

$$d\xi \neq -[<\underline{y}(\underline{x})|\underline{x}> + <\underline{y}(\underline{x})|\underline{z}>]/(a\chi) \ , \tag{E}$$

then this inequality will be satisfied by putting

$$\nu_3\xi = -[<\underline{y}(\underline{x})|\underline{z}> + <\underline{y}(\underline{x})|\underline{x}> + d\xi.a\chi]^{-1} \tag{F}$$

and

$$\nu_1\xi = <\underline{y}(\underline{z})|\underline{z}>^{-1} \ . \tag{G}$$

Also, since we require $[(\underline{y}(\underline{x}),d):(\underline{x},a)]$ to be a group H-class, we must choose d such that

$$<(\underline{y}(\underline{x}),d)|(\underline{x},a)> \neq 0 \ ,$$

i.e. such that

$$d\xi \neq -\langle \underline{y}(\underline{x}) | \underline{x}\rangle / (a\chi) \qquad (\text{by (D)}, \ a\chi \neq 0\ ) \ . \tag{H}$$

Since F is infinite, we may choose $d \in F$ to satisfy conditions (E) and (H). If we then define ν_1 and ν_3 as in (F) and (G), define $\rho_1 = \rho_3 = 1$, $\langle\underline{n}\rangle = \langle\nu_1(\underline{y}(\underline{z}),b)+\nu_3(\underline{y}(\underline{x}),d)\rangle$, $\langle\underline{r}\rangle = \langle\underline{z}+\underline{x},a\rangle$ and ν_2,ρ_2 such that

$$\nu_2\underline{n} = -\nu_1(\underline{y}(\underline{z}),b) - \nu_3(\underline{y}(\underline{x}),d)$$

and

$$\rho_2\underline{r} = -(\underline{z}+\underline{x},a) \ ,$$

we can show that all the conditions of Theorem 3.14 apply to the product $((\underline{y}(\underline{z}),b):(\underline{z},0))(\underline{n}:\underline{r})((\underline{y}(\underline{x}),d):(\underline{x},a))$.

We first show that the null-spaces are distinct. Clearly, $\langle(\underline{y}(\underline{z}),b)\rangle \neq \langle(\underline{y}(\underline{x}),d)\rangle$ since $\underline{z} \neq \underline{x}$ and $\underline{x},\underline{z} \in V'_{n-1}$. From (F) and (G) it is obvious that ν_1 and ν_3 are non-zero, thus $\langle\underline{n}\rangle$ is distinct from $\langle(\underline{y}(\underline{z}),b)\rangle$ and $\langle(\underline{y}(\underline{x}),d)\rangle$.

The three ranges are distinct since $\underline{x} \neq \underline{z}$, since $\underline{x},\underline{z} \in V'_{n-1}$, and neither ρ_1 nor ρ_3 are zero.

Since $\underline{x}$ and $\underline{z}$ are distinct elements of V'_{n-1} and

$$\rho_2\underline{r} = -(\underline{z}+\underline{x},a) \ ,$$

it is clear that $\rho_2 \neq 0$.

Similarly, we have $\nu_2 \neq 0$. Now

$$\nu_1(\underline{y}(\underline{z}),b) + \nu_2\underline{n} + \nu_3(\underline{y}(\underline{x}),d) = (\underline{0},0) \ ,$$

$$\rho_1(\underline{z},0) + \rho_2\underline{r} + \rho_3(\underline{x},a) = (\underline{z},0) + \rho_2\underline{r} + (\underline{x},a) = (\underline{0},0)$$

and, by Lemma 3.13,

$$\langle \nu_1(\underline{y}(\underline{z}),b)|\rho_1(\underline{z},0)\rangle + \langle \nu_1(\underline{y}(\underline{z}),b)|\rho_2\underline{r}\rangle + \langle \nu_2\underline{n}|\rho_2\underline{r}\rangle$$

$$= \langle \nu_1(\underline{y}(\underline{z}),b)|\rho_1(\underline{z},0)\rangle - \langle \nu_3(\underline{y}(\underline{x}),d)|\rho_2\underline{r}\rangle$$

$$= \nu_1\xi.\langle \underline{y}(\underline{z})|\underline{z}\rangle + \nu_3\xi.\langle (\underline{y}(\underline{x}),d)|(\underline{z}+\underline{x},a)\rangle$$

$$= \nu_1\xi.\langle \underline{y}(\underline{z})|\underline{z}\rangle + \nu_3\xi[\langle \underline{y}(\underline{x})|\underline{z}\rangle + \langle \underline{y}(\underline{x})|\underline{x}\rangle + d\xi.a\chi]$$

$$= 1 + (-1) \quad \text{(by (F) and (G))}$$

$$= 0 .$$

Thus $((\underline{y}(\underline{z}),b):(\underline{x},a)) = ((\underline{y}(\underline{z}),b):(\underline{z},0))(\underline{n}:\underline{r})((\underline{y}(\underline{x}),d):(\underline{x},a))$. Clearly, $((\underline{y}(\underline{z}),b):(\underline{z},0))$ and $((\underline{y}(\underline{x}),d):(\underline{x},a))$ are elements of

$$\bigcup_{i=0}^{6} D_i .$$

It remains to show that

$$(\underline{n}:\underline{r}) \in \bigcup_{i=0}^{7} D_i .$$

To show this we need to consider the stroke product of the first $n - 1$ co-ordinates of $\underline{n}$ with the first $n - 1$ co-ordinates of $\underline{r}$.

$$\langle \nu_1\underline{y}(\underline{z}) + \nu_3\underline{y}(\underline{x})|\underline{z}+\underline{x}\rangle$$

$$= \nu_1\xi.\langle \underline{y}(\underline{z})|\underline{z}\rangle + \nu_3\xi.[\langle \underline{y}(\underline{x})|\underline{z}\rangle + \langle \underline{y}(\underline{x})|\underline{x}\rangle] \quad (\text{since } \langle \underline{y}(\underline{z})|\underline{x}\rangle = 0)$$

$$= 1 - [\langle \underline{y}(\underline{x})|\underline{z}\rangle + \langle \underline{y}(\underline{x})|\underline{x}\rangle]/[\langle \underline{y}(\underline{x})|\underline{z}\rangle + \langle \underline{y}(\underline{x})|\underline{x}\rangle + d\xi.a\chi]$$

(from (F) and (G))

$$= d\xi.a\chi/[\langle \underline{y}(\underline{x})|\underline{z}\rangle + \langle \underline{y}(\underline{x})|\underline{x}\rangle + d\xi.a\chi]$$

$$\neq 0 \quad \text{(by (D))} .$$

Thus $(\underline{n}:\underline{r}) \notin D_8$. But $(\underline{n}:\underline{r})$ is an idempotent and thus belongs to

$$\bigcup_{i=0}^{8} D_8 .$$

Hence

$$(\underline{n}:\underline{r}) \in \bigcup_{i=0}^{7} D_i .$$

So D_8 is

$$(\bigcup_{i=0}^{7} D_i)\text{-accessible}$$

and so D_0-obtainable.

Consequently,

$$E_n = \bigcup_{i=0}^{8} D_i$$

is D_0-obtainable. Since D_0 forms a sparse cover for PF_{n-1}, we have, by putting $A^{(n)} = D_0$, completed the induction step.

So, for all $m \in \mathbb{N}$, there exists a subset $A^{(m)}$ of the idempotents E_m in PF_{m-1} such that $A^{(m)}$ is a sparse cover for PF_{m-1} and E_m is $A^{(m)}$-obtainable. By the comments following Definition 6.2, we know that $A^{(m)}$ therefore generates E_m. But E_m generates $Sing_m$ (Theorem 4.9) and so $A^{(m)}$ generates $Sing_m$.

§7 GENERATING SETS OF IDEMPOTENTS 4: THE NUMBER OF GENERATING SETS OF MINIMUM ORDER WHEN V IS DEFINED OVER A FINITE FIELD F

In Section 5 we found the minimum order of a subset E' of the idempotents E of rank $n - 1$ such that E' generates $Sing_n$

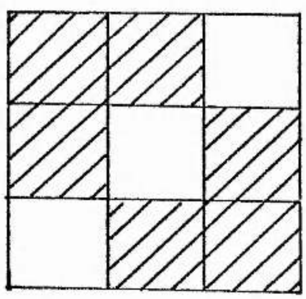
(Corollary 5.7). This section will be devoted to finding the number $W(q,n)$ of generating sets with this order. Theorem 7.7 will determine $W(q,n)$ when V is a two-dimensional vector space. Lemma 7.15 and Lemma 7.17 give upper bounds for $W(q,n)$ when $n \geq 3$. Lemma 7.18 (with subsidiary Lemmas 7.19 to 7.21) shows that the bound given in Lemma 7.15 is the better of the two.

If $n = 2$, then it is possible to determine $W(q,n)$ using what, in [1], are called rook polynomials.

7.1 DEFINITIONS Define an m-board B to be an $m \times m$ array of cells, an arbitrary number of which are coloured black and the rest coloured white.

Define the m-complement-board B' of B to be B with the colours of the cells interchanged.

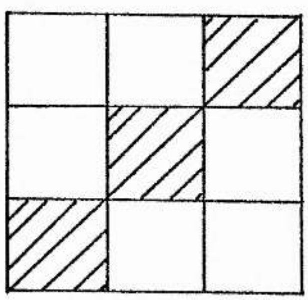
7.2 EXAMPLE Let B be the three-board



.

The three-complement-board of B is

$B' =$



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<u>7.3 DEFINITION</u> The <u>rook polynomial</u> R_B of an m-board B is

$$R_B = a_0 + a_1x + \dots + a_mx^m$$

where a_i is the number of ways of selecting i black squares from B such that no two are in the same row or column (i.e. the number of ways of placing i chess rooks on the black squares so that no two may take each other - they may, as in chess, pass over the white squares). Clearly, for all boards, $a_0 = 1$.

<u>7.4 EXAMPLE</u> In Example 7.2, the rook polynomial of the board B is $R_B = 1 + 6x + 9x^2 + 2x^3$ and the rook polynomial of B' is $R_{B'} = 1 + 3x + 3x^2 + x^3$.

<u>7.5 LEMMA</u> (Inclusion-Exclusion Principle) Let B be an m-board with rook polynomial

$$R_B = a_0 + a_1x + \dots + a_mx^m .$$

Let B' be the m-complement-board of B . The coefficient of x^m in the rook polynomial of B' is

$$\sum_{k=0}^{m}(-1)^k(m-k)!a_k .$$

<u>PROOF</u> See, for example, [1].

<u>7.6 DEFINITION</u> If $|F| = q$ we shall associate with $Sing_n$ an m-board $B(q,n)$ where $m = (q^n-1)/(q-1)$. We shall do this as follows:

Consider the egg-box of the $\mathcal{D}$-class of $Sing_n$ containing elements of PF_{n-1} . This has m rows and m columns. Colour the group $\mathcal{H}$-classes of this $\mathcal{D}$-class black and the non-group $\mathcal{H}$-classes white.

Clearly, $W(q,n)$ equals the coefficient of x^m in the rook polynomial of $B(q,n)$.

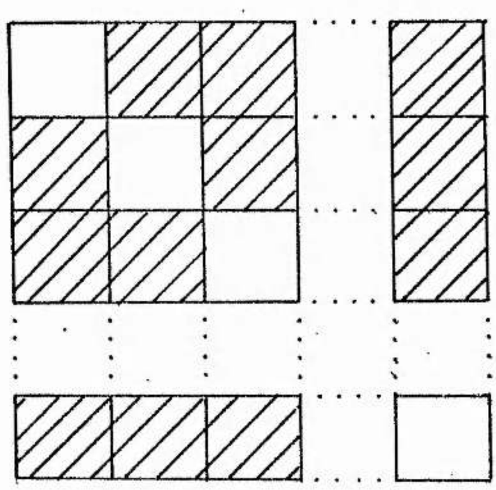
<u>7.7 THEOREM</u> Let V be a two-dimensional vector space over a finite field $|F| = q$. Let $Sing_2$ be the semigroup of singular endomorphisms of V and let E be the idempotents of $Sing_2$ of rank 1 . Let $W(q,2)$ be the order of the set

$$\{E' : E' \subset E \ , \ |E'| = (q^2-1)/(q-1) \ , \ \langle E' \rangle = Sing_2\} \ .$$

Then

$$W(q,2) = (q+1)! \sum_{k=0}^{q+1} \frac{(-1)^k}{k!} \ .$$

<u>PROOF</u> By the comments following Definition 7.6, all we need do is find the coefficient of x^m in the rook polynomial of the m-board $B(q,2)$ where $m = (q^2-1)/(q-1) = q + 1$. By the construction of the board $B(q,2)$ and by Lemma 5.4, each row and each column of $B(q,2)$ contains precisely q black cells and 1 white cell, i.e. $B(q,2)$ is of the form



Clearly, the rook polynomial of the m-complement-board B' of B(q,2) is

$$R_{B'} = \binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \dots + \binom{m}{m-1}x^{m-1} + \binom{m}{m}x^m .$$

Thus, by Lemma 7.5, the coefficient of x^m in the rook polynomial for B(q,n) is

$$\sum_{k=0}^{q+1}(-1)^k(q+1-k)!\,\frac{(q+1)!}{(q+1-k)!\,k!} ,$$

i.e.

$$W(q,2) = (q+1)!\sum_{k=0}^{q+1}(-1)^k/k! .$$

If $n \geq 3$ then the problem of determining the number of generating sets of minimum order becomes much harder. Upper bounds may be obtained from Theorem 4 of [3] and Theorem 10 of [16] (quoted here as Lemma 7.14 and Lemma 7.16). In Lemma 7.18 I shall show that the bound obtained from [3] is, in fact, better. Before quoting these results some further definitions are needed.

7.8 DEFINITION Let $A = (a_{ij})$ be an $n \times n$ matrix. The permanent of A, denoted Per(A), is defined to be $\sum_{\sigma\in G_n} a_{i,i\sigma}$ where G_n is the symmetric group on the set $\{1,2,\dots,n\}$.

7.9 DEFINITION A is an n square (0,1) matrix if A is an $n \times n$ matrix with entries in $\{0,1\}$.

Clearly, if A is an n square (0,1) matrix, then Per(A) is the number of ways of choosing n entries of A, each of which is 1, such that no two are from the same row or the same column of A . If we now construct the matrix $M(q,n)$ from the board $B(q,n)$ by putting the $(i,j)^{th}$ entry of $M(q,n)$ equal to 1 if the $(i,j)^{th}$ square of $B(q,n)$ is black and 0 otherwise, it is clear that $Per(M(q,n)) = W(q,n)$.

<u>7.10 DEFINITION</u> The <u>incidence matrix of a (v,k,λ) configuration</u> is a v square (0,1) matrix satisfying:

(i) every row and every column of A contains exactly k entries which are 1

(ii) any pair of columns [rows] of A both have entry 1 in the same row [column] for exactly λ rows [columns].

<u>7.11 EXAMPLE</u> The matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

is the incidence matrix of a (3,2,1) configuration. Also $Per(A) = 2$.

<u>7.12 DEFINITION</u> Let $A = (a_{ij})$ be an n x n matrix. A is <u>doubly stochastic</u> if $\sum_{i=1}^{n} a_{ij} = 1$ for all $j = 1,\ldots,n$ and $\sum_{j=1}^{n} a_{ij} = 1$ for all $i = 1,2,\ldots,n$.

<u>7.13 LEMMA</u> The matrix $M(q,n)$ is the incidence matrix of a (v,k,λ) configuration, where $v = (q^n-1)/(q-1)$, $k = q^{n-1}$ and $\lambda = q^{n-2}(q-1)$.

<u>PROOF</u> By the definition of $M(q,n)$ and $B(q,n)$, it is immediate that $v = (q^n-1)/(q-1)$. The number of 1's in each row of

$M(q,n)$ is precisely the number of black squares in each row of $B(q,n)$. But this is precisely the number of idempotents in each R-class of PF^0_{n-1}, i.e. there are precisely q^{n-1} 1's in each row of $M(q,n)$. Similarly, there are precisely q^{n-1} 1's in each column of $M(q,n)$. Thus $k = q^{n-1}$.

Now consider any two rows of $M(q,n)$. Let these correspond to the R-classes of PF^0_{n-1} containing elements with null-space $\langle \underline{x} \rangle$ or $\langle \underline{y} \rangle$. Now consider any L-class L of PF^0_{n-1} that intersects $R_{\langle \underline{x} \rangle}$ and $R_{\langle \underline{y} \rangle}$ in non-group H-classes. Clearly, L contains elements with range perpendicular in $\langle \underline{x},\underline{y} \rangle^{\perp}$, i.e. L must be labelled with any one-dimensional subspace of $\langle \underline{x},\underline{y} \rangle^{\perp}$. Since $\langle \underline{x},\underline{y} \rangle^{\perp}$ is of dimension $n - 2$ (Lemma 2.3), $\langle \underline{x},\underline{y} \rangle^{\perp}$ contains exactly $(q^{n-2}-1)/(q-1)$ one-dimensional subspaces (from the proof of Lemma 5.3). Thus, given any two rows of $M(q,n)$, there are exactly $(q^{n-2}-1)/(q-1)$ columns of $M(q,n)$ that contain the entry 0 in both of these rows. If we let the number of columns of $M(q,n)$ that contain the entry 1 in both these rows be c, then we have

$$\frac{q^{n-2}-1}{q-1} + q^{n-1} + q^{n-1} - c = \frac{q^{n}-1}{q-1} ,$$

i.e.

$$c = \frac{1}{q-1}\{q^{n-2}-1+2q^{n}-2q^{n-1}-q^{n}+1\}$$

$$= q^{n-2}(q-1) .$$

Similarly, given any two columns of $M(q,n)$, there are exactly $q^{n-2}(q-1)$ rows of $M(q,n)$ that contain the entry 1 in both of these rows. Thus $\lambda = q^{n-2}(q-1)$.

7.14 LEMMA (Marcus and Newman [16]) If A is the incidence matrix of a (v,k,λ) configuration, then

$$\mathrm{Per}(A) < v!\left(\frac{k-\theta}{v}\right)^v \sum_{r=0}^{v} \left(\frac{k\theta+\theta^2}{\lambda}\right)^r \frac{1}{r!} ,$$

where

$$\theta = (k-\lambda)^{1/2} .$$

It is now immediate that:

7.15 LEMMA If $\ell = (q^n-1)/(q-1)$, then

$$W(q,n) < \ell!\{(q^{n-1}-q^{(n-2)/2})/\ell\}^{\ell} \sum_{r=0}^{\ell}\left(\frac{q^{n/2}+1}{q-1}\right)^r \frac{1}{r!} .$$

In the following, we shall denote this upper bound for $W(q,n)$ by m .

7.16 LEMMA (Bregman [3]) If A is an n square $(0,1)$ matrix with exactly r_i 1's in row i, then

$$\mathrm{Per}(A) \le \prod_{i=1}^{n}[(r_i!)^{1/r_i}] .$$

From this it is immediate that:

7.17 LEMMA

$$W(q,n) \le (q^{n-1})!^{(q^n-1)/\{q^{n-1}(q-1)\}}$$

In the following, we shall denote this upper bound by b .

7.18 LEMMA For all q and all $n \geq 3$, $m < b$.

PROOF This is mostly through a series of technical lemmas. Eventually we shall show that $(\frac{m}{b})^{1/\ell} < 1$ where $\ell = (q^n-1)/(q-1)$. Throughout this section, the following abbreviations will be used:

$$k = q^{n-1}$$

and

$$c = q^{n/2} .$$

Since $\sum_{r=0}^{\ell} (\frac{c+1}{q-1})^r \frac{1}{r!} < \exp \frac{c+1}{q-1}$, we have

$$m < \ell!(\frac{c(c-1)}{q\ell})^{\ell} \exp \frac{c+1}{q-1} .$$

Thus

$$m^{1/\ell} < (\ell!)^{1/\ell} . \frac{c(c-1)}{q\ell} \{\exp(\frac{c+1}{q-1})\}^{1/\ell}$$

$$= (\ell!)^{1/\ell} . \frac{c(c-1)}{q\ell} \{\exp(\frac{1}{\ell} . \frac{c+1}{q-1})\}$$

$$= (\ell!)^{1/\ell} . \frac{c(c-1)}{q\ell} \exp (\frac{1}{c-1}) .$$

Also

$$b = (k!)^{\ell/k} .$$

Thus

$$(\frac{m}{b})^{1/\ell} < \frac{f(\ell)}{f(k)} . \frac{c(c-1)}{q\ell} \exp (\frac{1}{c-1}) ,$$

where $f(x) = (x!)^{1/x}$.

<u>7.19 LEMMA</u> If $x \geq 7$ and $f(x) = x!^{1/x}$, then

$$\frac{f(x+1)}{f(x)} < \exp\left\{\frac{1}{x+1} - \frac{\log(2\pi)}{2x(x+1)}\right\} .$$

<u>PROOF</u> Clearly,

$$\frac{f(x+1)}{f(x)} = \left(\frac{(x+1)!^{x}}{x!^{(x+1)}}\right)^{\frac{1}{x(x+1)}} = \left(\frac{(x+1)^{x}}{x!}\right)^{\frac{1}{x(x+1)}} .$$

But Stirling's formula (see e.g. [19]) gives

$$x! = (2\pi x)^{1/2} . x^{x} . e^{-x} . \exp\left\{\frac{\theta}{12(x+1)}\right\} ,$$

where $\theta \in (0,1)$. Thus

$$x! > (2\pi x)^{1/2} . x^{x} . e^{-x} .$$

So

$$\frac{f(x+1)}{f(x)} < (2\pi)^{\frac{-1}{2x(x+1)}} . T^{\frac{1}{(x+1)}} ,$$

where

$$T = e(x+1)/x^{1 + 1/(2x)} .$$

We shall now show that $T/e < 1$, i.e. that

$$g(x) = (x+1)/x^{1 + 1/(2x)} < 1$$

for $x \geq 7$.

By logarithmic differentiation, we find

$$\frac{g'(x)}{g(x)} = \frac{x(\log x - 3) + (\log x - 1)}{2x^{3}(x+1)} .$$

Now, since

$$\frac{g(x)}{2x^3(x+1)} > 0 \quad \text{for} \quad x \geq 7 ,$$

we have $g'(x) \geq 0$ if and only if

$$h(x) = x(\log x - 3) + (\log x - 1) \geq 0 .$$

Since $h(15) < 0$ and $h(16) > 0$, there exists an $x_0 \in (15,16)$ such that $h(x_0) = 0$.

Suppose first that $x \geq x_0$. Then

$$h'(x) = \log x + \frac{1}{x} - 2$$

$$> \log 15 - 2 > 0 .$$

Thus $h(x) \geq 0$ if $x \geq x_0$, i.e. $g'(x) \geq 0$ if $x \geq x_0$. Consequently, if $x \geq x_0$, then

$$g(x) < \lim_{y\to\infty} g(y)$$

$$= \lim_{y\to\infty} \frac{y+1}{y} . y^{-1/(2y)}$$

$$= 1 .$$

Now suppose that $7 \leq x < x_0$. We have $h''(x) = 1/x - 1/x^2 > 0$ since $x \geq 7$. Thus $h'(x) > h'(7) > 0$ if $x \geq 7$, i.e. $h(x) < h(x_0) = 0$ for $x \in [7,x_0)$. Hence $g'(x) < 0$ for $x \in [7,x_0)$. Consequently, if $7 \leq x < x_0$, then

$$g(x) \leq g(7) < 1 .$$

Thus, if $x \geq 7$, we have that $g(x) < 1$. Hence $T < e$, i.e.

$$\frac{f(x+1)}{f(x)} < \exp\left\{\frac{1}{x+1} - \frac{\log(2\pi)}{2x(x+1)}\right\}$$

as required.

7.20 LEMMA Let $x > y \geq 7$ and $f(x) = x!^{1/x}$. Then

$$\frac{f(x)}{f(y)} < \frac{x}{y} \exp\left\{-\frac{1}{2}\left(\frac{1}{y} - \frac{1}{x}\right)\log(2\pi)\right\} .$$

PROOF

$$\frac{f(x)}{f(y)} = \prod_{r=y}^{x-1} \frac{f(r+1)}{f(r)}$$

and so, by Lemma 7.19,

$$\frac{f(x)}{f(y)} < \prod_{r=y}^{x-1} \exp\left\{\frac{1}{r+1} - \frac{\log(2\pi)}{2r(r+1)}\right\}$$

$$= \exp\left\{\sum_{r=y}^{x-1} \frac{1}{r+1} - \frac{\log(2\pi)}{2} \sum_{r=y}^{x-1} \frac{1}{r(r+1)}\right\} .$$

Now,

$$\sum_{r=y}^{x-1} \frac{1}{r+1} < \int_y^x \frac{1}{z}\, dz = \log \frac{x}{y}$$

and

$$\sum_{r=y}^{x-1} \frac{1}{r(r+1)} = \frac{1}{y} - \frac{1}{x} .$$

Thus

$$\frac{f(x)}{f(y)} < \exp\left\{\log\frac{x}{y} - \frac{\log(2\pi)}{2}\left(\frac{1}{y} - \frac{1}{x}\right)\right\}$$

$$= \frac{x}{y} \exp\left\{-\frac{1}{2}\left(\frac{1}{y} - \frac{1}{x}\right) \log(2\pi)\right\} .$$

We return now to the proof of Lemma 7.18. Immediately before Lemma 7.19 we obtained

$$\left(\frac{m}{b}\right)^{1/\ell} < \frac{f(\ell)}{f(k)}\cdot\frac{c(c-1)}{q\ell}\cdot\exp\left(\frac{1}{c-1}\right)$$

where $f(x) = x!^{1/x}$.

Now, by Lemma 7.20, if $k \geq 7$, this gives

$$\left(\frac{m}{b}\right)^{1/\ell} < \frac{\ell}{k}\cdot\frac{c(c-1)}{q\ell}\cdot\exp\left\{\frac{1}{c-1} - \frac{1}{2}\left(\frac{1}{y}-\frac{1}{x}\right)\log(2\pi)\right\}$$

$$= \frac{c-1}{c}\cdot\exp\left\{\frac{1}{c-1} - \frac{1}{2}\left(\frac{1}{y}-\frac{1}{x}\right)\log(2\pi)\right\}.$$

Now, since $c = q^{n/2}$ and $n \geq 3$, we have $q \leq c^{2/3}$. Thus, if $k \geq 7$, we have

$$\left(\frac{m}{b}\right)^{1/\ell} < \frac{c-1}{c}\cdot\exp\left\{\frac{1}{c-1} + \frac{c^{2/3}-c^2}{2c^2(c^2-1)}\cdot\log(2\pi)\right\}.$$

<u>7.21 LEMMA</u> Let

$$g(x) = \frac{x-1}{x}\exp\left\{\frac{1}{x-1} + \frac{x^{2/3}-x^2}{2x^2(x^2-1)}\cdot\log(2\pi)\right\}.$$

If $x \geq 4$, then $g(x) < 1$.

<u>PROOF</u> By logarithmic differentiation, we have

$$\frac{g'(x)}{g(x)} = \frac{3x^4 - 5x^{8/3} + 2x^{2/3}}{3x^3(x^2-1)^2}\cdot\log(2\pi) - \frac{1}{x(x-1)^2}$$

$$> \frac{3x^4 - 5x^3 + 2}{3x^3(x^2-1)^2}\cdot\log(2\pi) - \frac{1}{x(x-1)^2}$$

$$= \frac{x^2 k(x) + 2\log(2\pi)}{3x^3(x^2-1)^2}$$

$$> \frac{k(x)}{3x(x^2-1)^2}$$

where

$$k(x) = 3(\log(2\pi) - 1)x^2 - (5\log(2\pi) + 6)x - 3 .$$

Now, $k(x)$ takes a minimum value when

$$x = \frac{5\log(2\pi) + 6}{6(\log(2\pi) - 1)}$$

$$= 3.02 \quad \text{(to three significant figures).}$$

Let the roots of $k(x) = 0$ be x_1 and x_2 where $x_1 \le x_2$. Then, to three significant figures, we have

$$x_1 = -0.191 \quad \text{and} \quad x_2 = 6.23 .$$

Hence $k(x) \ge 0$ for all $x \ge x_2$ and $k(x) < 0$ for all x in $[4, x_2)$. Hence, since $g(x) > 0$ if $x \ge 4$, we have

$$g'(x) \ge 0 \quad \text{if} \quad x \ge x_2$$

$$g'(x) < 0 \quad \text{if} \quad x \in [4, x_2) .$$

Thus

$$g(x) < \lim_{y \to \infty} g(y) \quad \text{if} \quad x \ge x_2$$

$$g(x) \le g(4) \qquad \text{if} \quad x \in [4, x_2) .$$

Hence

$$g(x) < 1 \quad \text{if} \quad x \geq 4$$

since

$$\frac{x^{2/3} - x^2}{2x^2(x^2-1)} \to 0 \quad \text{as} \quad x \to \infty$$

and

$$g(4) = 0.994 \quad \text{(to three significant figures).}$$

We now return again to the proof of Lemma 7.18 . Immediately prior to Lemma 7.21, we obtained

$$(\frac{m}{b})^{1/\ell} < g(c)$$

if $k \geq 7$, where $g(x)$ is as defined in Lemma 7.21 . We now have that, if $k \geq 7$ and $c \geq 4$, then

$$(\frac{m}{b})^{1/\ell} < 1 ,$$

i.e. we have $m < b$ if $k \geq 7$ and $c \geq 4$.

Now, since $q \geq 2$ and $n \geq 3$, we have

$$c = q^{n/2} \geq 4 \quad \text{if} \quad (n,q) \neq (3,2)$$

and

$$k = q^{n-1} \geq 7 \quad \text{if} \quad (n,q) \neq (3,2) .$$

Hence, if $(n,q) \neq (3,2)$, we have $m < b$.

Now, if $(n,q) = (3,2)$, we see, by direct calculation of the inequality immediately prior to Lemma 7.19, that

$$(\frac{m}{b})^{1/\ell} < 0.975 \quad \text{(to three significant figures)}.$$

Thus, in this case also we have $m < b$. This completes the proof of Lemma 7.18 .

7.22 TABLE This table evaluates the upper bound for $W(q,n)$ given in Lemma 7.18 . All the values are rounded up to four figures. The second number in each entry indicates the power of ten by which the first number must by multiplied.

q	n = 3	n = 4	n = 5
2	2.085 2	2.084 8	1.917 25
3	7.192 7	1.619 41	8.628 179
4	2.057 17	1.130 118	7.185 674
5	1.202 31	8.339 260	1.992 1846
7	7.997 72	1.372 842	1.510 8254
8	3.249 101	4.732 1332	6.165 14878
9	3.621 135	1.580 1993	7.740 24969

To give an idea of how good a bound Lemma 7.16 gives, it is worth noting that $W(2,3) = 144$ whereas, in the table, we have $W(2,3) \leq 208.5$.

§8 GRAVITY AND DEPTH

Let $\mathcal{T}_X$ be the full transformation semigroup on the finite set X and let α be an element of $\mathcal{T}_X$. In [8], the defect of α was defined to be the order of the set $X \backslash X_\alpha$. It is shown in [8] that the subsemigroup of $\mathcal{T}_X$ generated by the idempotents E^+ with non-zero defect is $\mathcal{T}_X \backslash \mathcal{G}_X$, where $\mathcal{G}_X$ is the symmetric group on the set X . In [13] the gravity of α was defined to be the least $g(\alpha) \in \mathbb{N}$ for which $\alpha \in E^{g(\alpha)}$, where E is the set of idempotents of defect 1 . The depth of $\langle E^+ \rangle = \mathcal{T}_X \backslash \mathcal{G}_X$ was defined, in [13], to be the least $\Delta \in \mathbb{N}$ such that $(E^+)^\Delta = \mathcal{T}_X \backslash \mathcal{G}_X$, where E^+ is the set of idempotents of non-zero defect. Formulae for $g(\alpha)$ and Δ were determined in [12] and reported in [13].

In this section, similar definitions for gravity and depth will be given, and the gravity of any element of Sing_n will be determined, as will the depth of Sing_n .

8.1 DEFINITIONS Let V be an n-dimensional vector space over the field F and let Sing_n denote the semigroup of singular endomorphisms of V . Let E denote the idempotents of Sing_n of rank $n - 1$ and E^+ denote all the idempotents of Sing_n .

Let $\alpha \in \mathrm{Sing}_n$. Since E generates Sing_n (Theorem 4.9), there exists an integer k such that $\alpha \in E^k$. The gravity of α is defined to be

$$g(\alpha) = \min \{k \in \mathbb{N} : \alpha \in E^k\} .$$

If there exists an integer k such that

$$(E^+)^k = \mathrm{Sing}_n ,$$

then the <u>depth</u> of Sing_n is defined to be

$$\Delta(\mathrm{Sing}_n) = \min \{k \in \mathbb{N} : (E^+)^k = \mathrm{Sing}_n\} ;$$

otherwise the depth of Sing_n is defined to be infinite.

If F is finite, then Sing_n is a finite semigroup. Thus the chain

$$E^+ \subseteq (E^+)^2 \subseteq (E^+)^3 \subseteq \ldots$$

cannot have infinitely many inclusions. Since E generates Sing_n and $E \subseteq E^+$, we know that this chain must become stationary at Sing_n . Thus, if F is finite, Sing_n has finite depth.

Before attempting to find the depth of Sing_n , or the gravity of any element of Sing_n , it is convenient to introduce some matrix notation and prove three technical lemmas.

<u>8.2 NOTATION</u> Denote by S_k the $k \times k$ matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & 0 & & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and by $E_i^{(n)}$ the $n \times n$ matrix

$$\left[\begin{array}{c|cc|c} I_{n-2-i} & 0 & & 0 \\ \hline & 0 & 1 & \\ 0 & & & 0 \\ & 0 & 1 & \\ \hline 0 & 0 & & I_i \end{array}\right]$$

where I_d denotes the $d \times d$ identity matrix ($n \geq 2$, $i \leq n-2$).

<u>8.3 LEMMA</u> Let A be the matrix

$$\left[\begin{array}{cccc|cc} & & & & 0 & 0 \\ & & & & 0 & 0 \\ & S_{n-2} & & & \vdots & \vdots \\ & & & & 0 & 0 \\ & & & & 1 & 0 \\ \hline 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{array}\right].$$

Then $A = E_1^{(n)} E_2^{(n)} \ldots E_{n-2}^{(n)}$ $(n \geq 3)$.

<u>PROOF</u> The proof is by induction on k in the formula

$$E_1^{(n)} E_2^{(n)} \ldots E_k^{(n)} = A_k$$

where

$$A_k = \left[\begin{array}{c|c|cc} I_{n-2-k} & 0 & 0 & \\ \hline & & 0 & 0 \\ 0 & S_k & \vdots & \vdots \\ & & 0 & 0 \\ & & 1 & 0 \\ \hline 0 & 0 & I_2 & \end{array}\right].$$

To show that the induction process may be started at $k = 1$, notice that

$$A_1 = \left[\begin{array}{c|c|cc} I_{n-3} & \underline{0} & \multicolumn{2}{c}{0} \\ \hline \underline{0} & 0 & 1 & 0 \\ \hline 0 & \underline{0} & \multicolumn{2}{c}{I_2} \end{array}\right] = E_1^{(n)} .$$

Now suppose the result is true for $k - 1$, i.e.

$$E_1^{(n)} E_2^{(n)} \ldots E_{k-1}^{(n)} = A_{k-1} .$$

Then

$$E_1^{(n)} E_2^{(n)} \ldots E_k^{(n)} = A_{k-1} E_k^{(n)}$$

$$= \left[\begin{array}{c|c|cc} I_{n-1-k} & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & 0 \\ 0 & S_{k-1} & \vdots & \vdots \\ & & 0 & 0 \\ & & 1 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{I_2} \end{array}\right] \left[\begin{array}{c|cc|c} I_{n-2-k} & \multicolumn{2}{c|}{0} & 0 \\ \hline 0 & 0 & 1 & 0 \\ & 0 & 1 & \\ \hline 0 & \multicolumn{2}{c|}{0} & I_k \end{array}\right] .$$

$$= \left[\begin{array}{c|cc|cccc|cc} I_{n-2-k} & \multicolumn{2}{c|}{0} & \multicolumn{4}{c|}{0} & \multicolumn{2}{c}{0} \\ \hline 0 & 1 & 0 & 0 & 0 & \ldots & 0 & \multicolumn{2}{c}{0} \\ & 0 & 0 & 1 & 0 & \ldots & 0 & & \\ \hline & & & & & & & 0 & 0 \\ 0 & \multicolumn{2}{c|}{0} & \multicolumn{4}{c|}{S_{k-2}} & \vdots & \vdots \\ & & & & & & & 0 & 0 \\ & & & & & & & 1 & 0 \\ \hline 0 & \multicolumn{2}{c|}{0} & \multicolumn{4}{c|}{0} & \multicolumn{2}{c}{I_2} \end{array}\right] \left[\begin{array}{c|cc|c} I_{n-2-k} & \multicolumn{2}{c|}{0} & 0 \\ \hline 0 & 0 & 1 & 0 \\ & 0 & 1 & \\ \hline 0 & \multicolumn{2}{c|}{0} & I_k \end{array}\right]$$

$$
= \left[\begin{array}{c|cc|cccc|cc}
I_{n-2-k} & \multicolumn{2}{c|}{0} & \multicolumn{4}{c|}{0} & \multicolumn{2}{c}{0} \\
\hline
\multirow{2}{*}{0} & 0 & 1 & 0 & 0 & \cdots & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\
 & 0 & 0 & 1 & 0 & \cdots & 0 & & \\
\hline
 & & & & & & & 0 & 0 \\
 & & & & & & & \vdots & \vdots \\
0 & \multicolumn{2}{c|}{0} & \multicolumn{4}{c|}{S_{k-2}} & 0 & 0 \\
 & & & & & & & 1 & 0 \\
\hline
0 & \multicolumn{2}{c|}{0} & \multicolumn{4}{c|}{0} & \multicolumn{2}{c}{I_2}
\end{array}\right]
$$

$$
= \left[\begin{array}{c|c|cc}
I_{n-2-k} & 0 & \multicolumn{2}{c}{0} \\
\hline
 & & 0 & 0 \\
 & & \vdots & \vdots \\
0 & S_k & 0 & 0 \\
 & & 1 & 0 \\
\hline
0 & 0 & \multicolumn{2}{c}{I_2}
\end{array}\right]
$$

$$
= A_k \; .
$$

Thus

$$
A_{n-2} = E_1^{(n)} E_2^{(n)} \cdots E_{n-2}^{(n)} \; .
$$

But

$$
A_{n-2} = \left[\begin{array}{c|cc}
 & 0 & 0 \\
 & \vdots & \vdots \\
S_{n-2} & 0 & 0 \\
 & 1 & 0 \\
\hline
0 & \multicolumn{2}{c}{I_2}
\end{array}\right]
$$

$$
= A \; ,
$$

so $E_1^{(n)} E_2^{(n)} \cdots E_{n-2}^{(n)} = A$.

8.4 LEMMA Let A and B be the $n \times n$ matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \ddots & & \\ 0 & 0 & 0 & 0 & 0 & \ddots & 1 \\ 0 & a_2 & a_3 & a_4 & a_5 & \cdots & a_n \end{bmatrix},$$

$$B = \left[\begin{array}{ccccc|c} & & I_{n-1} & & & \underline{0} \\ \hline a_2 & a_3 & a_4 & \cdots & a_n & 0 \end{array}\right].$$

Then

$$A = BE_0^{(n)}E_1^{(n)}\cdots E_{n-2}^{(n)} .$$

Notice that B and each $E_i^{(n)}$ $(i = 0, \ldots, n-2)$ are idempotent and have nullity 1.

PROOF By Lemma 8.3, we have

$$E_0^{(n)}E_1^{(n)}\cdots E_{n-2}^{(n)} = \left[\begin{array}{c|cc} I_{n-2} & & 0 \\ \hline 0 & 0 & 1 \\ & 0 & 1 \end{array}\right]\left[\begin{array}{c|cc} & 0 & 0 \\ & \vdots & \vdots \\ S_{n-2} & 0 & 0 \\ & 1 & 0 \\ \hline 0 & 1 & 0 \\ & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|cc} & 0 & 0 \\ S_{n-2} & \vdots & \vdots \\ & 0 & 0 \\ & 1 & 0 \\ \hline 0 & 0 & 1 \\ & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} S_{n-1} & \begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix} \\ \hline \underline{0} & 1 \end{array}\right] .$$

It is now clear that

$$BE_0^{(n)}E_1^{(n)}\ldots E_{n-2}^{(n)} = A .$$

<u>8.5 LEMMA</u> Let A be the $(n+1) \times (n+1)$ matrix

$$\left[\begin{array}{ccccccc|c} 0 & 1 & 0 & 0 & 0 & \ldots & 0 & \\ 0 & 0 & 1 & 0 & 0 & \ldots & 0 & \\ 0 & 0 & 0 & 1 & 0 & \ldots & 0 & \underline{0} \\ \vdots & \vdots & \vdots & & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & & 1 & \\ a_1 & a_2 & a_3 & a_4 & a_5 & \ldots & a_n & \\ \hline & & & \underline{0} & & & & 0 \end{array}\right] .$$

Then

$$A = DE_1^{(n+1)}E_2^{(n+1)}\ldots E_{n-1}^{(n+1)}G ,$$

where

$$D = \left[\begin{array}{c|cc} I_{n-1} & & 0 \\ \hline 0 & 1 & 1 \\ & 0 & 0 \end{array}\right] \text{ and } G = \left[\begin{array}{ccccc|c} & & I_n & & & \\ \hline a_1 & a_2 & \ldots & a_{n-1} & a_n-1 & \frac{0}{0} \end{array}\right].$$

Notice that D , G and each $E_i^{(n+1)}$ are idempotent and have nullity 1 .

PROOF By Lemma 8.3,

$$DE_1^{(n+1)}E_2^{(n+1)}\cdots E_{n-1}^{(n+1)} = \left[\begin{array}{c|cc} I_{n-1} & & 0 \\ \hline 0 & 1 & 1 \\ & 0 & 0 \end{array}\right] \left[\begin{array}{c|cc} & 0 & 0 \\ & 0 & 0 \\ S_{n-1} & \vdots & \vdots \\ & 0 & 0 \\ & 1 & 0 \\ \hline 0 & 1 & 0 \\ & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|cc} & 0 & 0 \\ & 0 & 0 \\ S_{n-1} & \vdots & \vdots \\ & 0 & 0 \\ & 1 & 0 \\ \hline 0 & 1 & 1 \\ & 0 & 0 \end{array}\right] .$$

Thus

$$DE_1^{(n+1)}E_2^{(n+1)}\cdots E_{n-1}^{(n+1)}G = \left[\begin{array}{c|cc} & 0 & 0 \\ & 0 & 0 \\ S_{n-1} & \vdots & \vdots \\ & 0 & 0 \\ & 1 & 0 \\ \hline 0 & 1 & 1 \\ & 0 & 0 \end{array}\right] \left[\begin{array}{cccc|c} & & I_n & & \underline{0} \\ \hline a_1 & \cdots & a_{n-1} & a_n-1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|cc} & & & 0 & 0 \\ & & & 0 & 0 \\ & S_{n-1} & & \vdots & \vdots \\ & & & 0 & 0 \\ & & & 1 & 0 \\ \hline a_1 & \cdots & a_{n-1} & a_n-1+1 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{array}\right]$$

$$= A .$$

We are now in a position to find an upper bound for the depth of $Sing_n$. This, of course, depends on n .

<u>8.6 LEMMA</u> Let V be an n-dimensional vector space and $Sing_n$ the semigroup of singular endomorphisms of V . Let E be the set of idempotents of $Sing_n$ of rank $n-1$ and let $\alpha \in Sing_n$. Then there exist $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in E$ such that $\alpha = \varepsilon_1\varepsilon_2\ldots\varepsilon_n$ and $V = R^{\perp}_{\varepsilon_1} + R^{\perp}_{\varepsilon_2} + \ldots + R^{\perp}_{\varepsilon_n}$.

<u>PROOF</u> Since every element α of $Sing_n$ is singular, we know that, relative to a suitable basis, α has matrix $M_\alpha = \text{diag}\,\{A_q, A_{q-1}, \ldots, A_1\}$, where each A_i is a $d_i \times d_i$ matrix of the form

$$A_i = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & & \ddots & & \\ 0 & 0 & 0 & 0 & & 1 \\ a_{i1} & a_{i2} & a_{i3} & a_{i4} & \ldots & a_{id_i} \end{bmatrix}$$

and A_1 is singular (this being the rational canonical form for a matrix; see, for example, [15]). It is thus sufficient to prove the theorem for matrices of the form M_α . We shall do this by induction on q .

Clearly, for all values of q , we have

$$n = \sum_{i=1}^{q} d_i$$

and, since A_1 is singular, $a_{11} = 0$.

Suppose first that $q = 1$. Then, using the notation of Lemma 8.4,

$$M_\alpha = BE_0^{(n)}E_1^{(n)}\dots E_{n-2}^{(n)} .$$

Letting $\underline{e}_i^{(n)}$ denote the i^{th} standard basis element of an n-dimensional space, notice that

$$\mathcal{R}_B^\perp = \langle \underline{e}_n^{(n)} \rangle ,$$

and, denoting the range of $E_i^{(n)}$ by $\mathcal{R}_i$,

$$\mathcal{R}_i^\perp = \langle \underline{e}_{n-i-1}^{(n)} \rangle \quad (i = 0, 1, \dots, n-2).$$

Thus

$$V = \mathcal{R}_B^\perp + \mathcal{R}_1^\perp + \mathcal{R}_2^\perp + \dots + \mathcal{R}_{n-1}^\perp ,$$

so we may anchor the induction process.

Now suppose the result holds if $q \le k-1$ and consider the matrix $M_\alpha = \text{diag}\,\{A_k, A_{k-1}, \dots, A_2, A_1\}$. By the hypothesis,

$$M = \text{diag}\,\{A_{k-1}, A_{k-2}, \dots, A_2, A_1\} = F_1F_2\dots F_t ,$$

where $t = d_1 + d_2 + \dots + d_{k-1}$, each F_i is idempotent and

$$\dim\,(\mathcal{R}_{F_1}^\perp + \mathcal{R}_{F_2}^\perp + \dots + \mathcal{R}_{F_t}^\perp) = t .$$

Thus $M_\alpha = F_1'F_2'\dots F_t'$ where

$$F_1' = \left[\begin{array}{c|c} A_k & 0 \\ \hline 0 & F_1 \end{array}\right] \text{ and } F_i' = \left[\begin{array}{c|c} I_{d_k} & 0 \\ \hline 0 & F_i \end{array}\right] \quad (i = 2, \dots, t).$$

Now, since (by the hypothesis) F_1 has nullity 1, there exists a basis

$\{\underline{u}_1\}$ for the null-space of F_1 and a basis $\{\underline{u}_2,\underline{u}_3,\ldots,\underline{u}_{n-d_k}\}$ for the range of F_1. By Lemma 1.4, $\{\underline{u}_1,\underline{u}_2,\ldots,\underline{u}_{n-d_k}\}$ forms a basis for the domain of F_1. Relative to this basis, F_1 has matrix

$$I'_{n-d_k} = \left[\begin{array}{c|c} 0 & \underline{0} \\ \hline \underline{0} & I_{n-d_k-1} \end{array}\right]$$

where I_i is the $i \times i$ identity matrix. Hence there exists an invertible matrix P such that $F_1 = P^{-1}I'_{n-d_k}P$. Thus $F'_1 = P_1^{-1}A'_kP_1$ where

$$P_1 = \left[\begin{array}{c|c} I_{d_k} & 0 \\ \hline 0 & P \end{array}\right] \text{ and } A'_k = \left[\begin{array}{c|c} A_k & 0 \\ \hline 0 & I'_{n-d_k} \end{array}\right].$$

Now, using the notation of Lemma 8.5,

$$\left[\begin{array}{c|c} A_k & \underline{0} \\ \hline \underline{0} & 0 \end{array}\right] = DE_1^{(d_k+1)}E_2^{(d_k+1)}\ldots E_{d_k-1}^{(d_k+1)}G .$$

Notice that

$$R_D^{\perp} = \langle \underline{e}_{d_k}^{(d_k+1)} - \underline{e}_{d_k+1}^{(d_k+1)} \rangle$$

$$R_G^{\perp} = \langle \underline{e}_{d_{k+1}}^{(d_k+1)} \rangle$$

and, denoting the range of $E_i^{(d_k+1)}$ by R_i,

$$R_i^{\perp} = \langle \underline{e}_{d_k-i}^{(d_k+1)} \rangle .$$

Thus

$$A_k' = H_0 H_1 \ldots H_{d_k},$$

where

$$H_0 = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & I_{n-d_k-1} \end{array}\right], \quad H_{d_k} = \left[\begin{array}{c|c} G & 0 \\ \hline 0 & I_{n-d_k-1} \end{array}\right]$$

and

$$H_i = \left[\begin{array}{c|c} E_i^{(d_k+1)} & 0 \\ \hline 0 & I_{n-d_k-1} \end{array}\right] \quad (i = 1,2,\ldots,d_k-1).$$

Notice that

$$R_{H_0}^{\perp} = \langle \underline{e}_{d_k}^{(n)} - e_{d_k+1}^{(n)} \rangle$$

$$R_{H_{d_k}}^{\perp} = \langle \underline{e}_{d_k+1}^{(n)} \rangle$$

and

$$R_{H_i}^{\perp} = \langle \underline{e}_{d_k-i}^{(n)} \rangle \quad (i = 1,2,\ldots,d_k-1).$$

Now $F_1' = H_0' H_1' \ldots H_{d_k}'$ where $H_i' = P_1^{-1} H_i P_1$.

We shall now find $R_{H_i'}^{\perp}$ $(i = 0,1,\ldots,d_k)$.

If $\underline{x} \in R_{H_i}^{\perp}$ $(\underline{x} \neq \underline{0})$, then we have $H_i \underline{x}^T = \underline{0}^T$. Hence $H_i'(P_1^{-1}\underline{x}^T) = \underline{0}^T$ and so $\langle (P_1^{-1}\underline{x}^T)^T \rangle \subseteq R_{H_i'}^{\perp}$. But, since $P_1^{-1}\underline{x}^T \neq \underline{0}^T$ and $R_{H_i}^{\perp}$ is one-dimensional, we have $\langle (P_1^{-1}\underline{x}^T)^T \rangle = R_{H_i}^{\perp}$. Thus, denoting P^{-1} by $(p_{i,j})_{1 \le i,j \le n-d_k}$, we have

$$R^{\perp}_{H'_0} = \langle \underline{e}^{(n)}_{d_k} - \sum_{i=1}^{n-d_k} p_{i1}\, \underline{e}^{(n)}_{i+d_k} \rangle$$

$$R^{\perp}_{H'_{d_k}} = \langle \sum_{i=1}^{n-d_k} p_{i1}\, \underline{e}^{(n)}_{i+d_k} \rangle$$

and

$$R^{\perp}_{H'_i} = \langle \underline{e}^{(n)}_{d_k - i} \rangle \quad (i = 1,2,\ldots,d_{k-1}).$$

Now, $R_{F'_1} \subseteq R_{H'_{d_k}}$ and so $R^{\perp}_{H'_{d_k}} \subseteq R^{\perp}_{F'_1}$.

Hence

$$\langle \sum_{i=1}^{n-d_k} p_{i1}\, \underline{e}^{(n)}_{i+d_k} \rangle \subseteq R^{\perp}_{F'_1} .$$

Thus

$$\langle \sum_{i=1}^{n-d_k} p_{i1}\, e_i^{(n-d_k)} \rangle \subseteq R^{\perp}_{F_1} .$$

But $R^{\perp}_{F_1}$ is one-dimensional and $\sum_{i=1}^{n-d_k} p_{i1}\, \underline{e}_i^{(n-d_k)} \neq \underline{0}$, so $\langle \sum_{i=1}^{n-d_k} p_{i1}\, \underline{e}_i^{(n-d_k)} \rangle = R^{\perp}_{F_1}$. Consequently, by the hypothesis,

$$\langle \sum_{i=1}^{n-d_k} p_{i1}\, \underline{e}_i^{(n-d_k)} \rangle + \langle R^{\perp}_{F_i} : i = 2,3,\ldots,t \rangle$$

$$= \langle \underline{e}_i^{(n-d_k)} : i = 1,2,\ldots,n-d_k \rangle .$$

Thus

$$R^{\perp}_{H'_{d_k}} + \langle R^{\perp}_{F'_i} : i = 2,3,\ldots,t \rangle = \langle \underline{e}_i^{(n)} : i = d_k+1,\ldots,n \rangle .$$

Now, since

$$<R^{\perp}_{H'_i} : i = 0,\ldots,d_k-1> \cap <\underline{e}_i^{(n)} : i = d_k+1,\ldots,n> = \{\underline{0}\}$$

and

$$\dim <R^{\perp}_{H'_i} : i = 0,1,\ldots,d_k-1> = d_k \ ,$$

we have

$$\dim (<R^{\perp}_{H'_i} : i = 0,1,\ldots,d_k>+<R^{\perp}_{F'_i} : i = 2,3,\ldots,t>) = t + d_k$$

$$= \sum_{i=1}^{k} d_i = n \ .$$

Thus $V = R^{\perp}_{H'_0} + R^{\perp}_{H'_1} + \ldots + R^{\perp}_{H'_{d_k}} + R^{\perp}_{F'_2} + R^{\perp}_{F'_3} + \ldots + R^{\perp}_{F'_t}$. But we also have

$$M_\alpha = F'_1 F'_2 \ldots F'_t$$

$$= H'_0 H'_1 \ldots H'_{d_k} F'_2 F'_3 \ldots F'_t$$

and $n = t + d_k$.

Hence the induction step holds.

From this it follows that n is an upper bound for the depth of $Sing_n$ and for the gravity of any element of $Sing_n$. In order to show that $\Delta(Sing_n) = n$, the following theorem (which is also interesting in its own right) is needed.

<u>8.7 THEOREM</u> Let $Sing_n$ denote the semigroup of singular endomorphisms of an n-dimensional vector space V and let E denote the idempotent elements of $Sing_n$ of rank $n - 1$. Let $\alpha \in Sing_n$. Then $\alpha \in E^g$ where $g = \dim \{\underline{x} \in V : \underline{x}\alpha = \underline{x}\}^{\perp}$. Also, if $\ell < g$,

then $\alpha \notin E^{\ell}$, i.e. the gravity of α is $\dim \{\underline{x} \in V : \underline{x}\alpha = \underline{x}\}^{\perp}$.

PROOF Suppose $X_{\alpha} = \{\underline{x} \in V : \underline{x}\alpha = \underline{x}\}$ has dimension d. Let $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_d\}$ be a basis for X_{α} and extend this to a basis $B = \{\underline{u}_1, \ldots, \underline{u}_n\}$ for V. Relative to this basis, α has matrix

$$M_{\alpha} = \left[\begin{array}{c|c} I_d & 0 \\ \hline P & M \end{array}\right],$$

where I_d is the $d \times d$ identity matrix and M is an $(n-d) \times (n-d)$ singular matrix.

By Lemma 8.6,

$$M = M_1 M_2 \ldots M_{n-d},$$

where each M_i is idempotent with nullity 1 $(i = 1, \ldots, n-d)$ and $\dim \langle\{\underline{r}_i : i = 1, \ldots, n-d\}\rangle = n - d$ where $\langle \underline{r}_i \rangle = R_{M_i}^{\perp}$ $(i = 1, 2, \ldots, n-d)$.

Thus $M_{\alpha} = N_1 N_1'$, where

$$N_1 = \left[\begin{array}{c|c} I_d & 0 \\ \hline P_1 & M_1 \end{array}\right] \text{ and } N_1' = \left[\begin{array}{c|c} I_d & 0 \\ \hline P_1' & M_1' \end{array}\right]$$

and where I_d is the $d \times d$ identity matrix, $P_1 = P - M_1 P$, $P_1' = P - \underline{r}_1^T \underline{a}_1$ for some arbitrary d-dimensional vector $\underline{a}_1$ and $M_1' = M_2 M_3 \ldots M_{n-d}$.

Similarly, $N_1' = N_2 N_2'$ where

$$N_2 = \left[\begin{array}{c|c} I_d & 0 \\ \hline P_2 & M_2 \end{array}\right] \text{ and } N_2' = \left[\begin{array}{c|c} I_d & 0 \\ \hline P_2' & M_2' \end{array}\right]$$

and where $P_2 = P_1' - M_2P_1'$, $P_2' = P_1' - \underline{r}_2^T\underline{a}_2$ for some arbitrary d-dimensional vector $\underline{a}_2$ and $M_2' = M_3M_4\ldots M_{n-d}$.

Continuing in this manner, we see that

$$M_\alpha = N_1N_2\ldots N_{n-d-1}N'_{n-d-1} .$$

Notice that each N_i $(i = 1,\ldots,n-d-1)$ is idempotent with nullity 1 and so is an element of E . Now

$$N'_{n-d-1} = \left[\begin{array}{c|c} I_d & 0 \\ \hline P'_{n-d-1} & M'_{n-d-1} \end{array}\right]$$

$$= \left[\begin{array}{c|c} I_d & 0 \\ \hline P'_{n-d-1} & M_{n-d} \end{array}\right].$$

Thus $N'_{n-d-1} \in E$ if and only if $M_{n-d}P'_{n-d-1} = [0]$, i.e. if and only if $P'_{n-d-1} = \underline{r}_{n-d}^T\underline{a}_{n-d}$ for some d-dimensional vector $\underline{a}_{n-d}$. But

$$\begin{aligned} P'_{n-d-1} &= P'_{n-d-2} - \underline{r}_{n-d-1}^T\underline{a}_{n-d-1} \\ &= P'_{n-d-3} - \underline{r}_{n-d-2}^T\underline{a}_{n-d-2} - \underline{r}_{n-d-1}^T\underline{a}_{n-d-1} \\ &= \ldots \\ &= P_1' - \sum_{i=2}^{n-d-1} \underline{r}_i^T\underline{a}_i \\ &= P - \sum_{i=1}^{n-d-1} \underline{r}_i^T\underline{a}_i . \end{aligned}$$

Thus $N'_{n-d-1} \in E$ if and only if

$$P = \sum_{i=1}^{n-d}\underline{r}_i^T\underline{a}_i \qquad (+)$$

Now, we already know that $\dim \langle\{\underline{r}_i : i = 1,\ldots,n-d\}\rangle = n - d$ and that

P is an $(n-d) \times d$ matrix. Hence we may choose the vectors $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_{n-d}$ in such a way that (+) holds, i.e. such that $N'_{n-d-1} \in E$. Hence $M_\alpha \in E^{n-d}$. But, by Lemma 2.3, $\dim X_\alpha^\perp = n - \dim X_\alpha$. Thus $g = n - d$ and so $M_\alpha \in E^g$.

Now suppose that $\ell < g$ and $\alpha \in E^\ell$. Then there exist elements $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell$ of E such that

$$\alpha = \varepsilon_1 \varepsilon_2 \ldots \varepsilon_\ell .$$

Since $V = N_{\varepsilon_j} \oplus R_{\varepsilon_j}$ $(j = 1,2,\ldots,\ell)$ we may define, for each $\underline{u}_i$ in B, an element $\underline{m}_{i,1} \in N_{\varepsilon_1}$ and an element $\underline{s}_{i,1} \in R_{\varepsilon_1}$ such that $\underline{u}_i = \underline{m}_{i,1} + \underline{s}_{i,1}$. We may then define, inductively, elements $\underline{m}_{i,j} \in N_{\varepsilon_j}$ and elements $\underline{s}_{i,j} \in R_{\varepsilon_j}$ satisfying $\underline{s}_{i,j-1} = \underline{m}_{i,j} + \underline{s}_{i,j}$ $(j = 2,3,\ldots,\ell)$. Thus

$$\underline{s}_{i,\ell} = \underline{u}_i - \sum_{j=1}^{\ell} \underline{m}_{i,j} . \qquad (+)$$

Now $\underline{u}_i = \underline{m}_{i1} + \underline{s}_{i1}$ and so

$$\underline{u}_i \varepsilon_1 = \underline{s}_{i,1} = \underline{m}_{i,2} + \underline{s}_{i,2} .$$

Thus

$$\underline{u}_i \varepsilon_1 \varepsilon_2 = \underline{s}_{i,2} = \underline{m}_{i,3} + \underline{s}_{i,3} .$$

Continuing in this way, we clearly obtain

$$\underline{u}_i \alpha = \underline{u}_i \varepsilon_1 \varepsilon_2 \ldots \varepsilon_\ell = s_{i,\ell} .$$

So, using (+), we have

$$\underline{u}_i \alpha = \underline{u}_i - \sum_{j=1}^{\ell} \underline{m}_{i,j} ,$$

i.e.

$$\underline{u}_i - \underline{u}_i\alpha = \sum_{j=1}^{\ell} \underline{m}_{i,j} \ .$$

But each N_{ε_i} is generated by a single element of V, $\underline{n}_i$ say. Thus, for each $\underline{m}_{i,j}$, there exists a scalar $\lambda_{i,j}$ such that $\underline{m}_{i,j} = \lambda_{i,j}\underline{n}_j$. Thus

$$\underline{u}_i - \underline{u}_i\alpha = \sum_{j=1}^{\ell} \lambda_{i,j}\underline{n}_j \qquad (i = 1,2,\ldots,n).$$

Now, clearly, $\dim \langle\{\sum_{j=1}^{\ell} \lambda_{i,j}\underline{n}_j : i = 1,2,\ldots,n\}\rangle \le \ell$. Thus

$$\dim \langle\{\underline{u}_i - \underline{u}_i\alpha : i = 1,2,\ldots,n\}\rangle \le \ell \ .$$

Now, the basis B was chosen so that

$$\underline{u}_1 - \underline{u}_1\alpha = \underline{u}_2 - \underline{u}_2\alpha = \ldots = \underline{u}_d - \underline{u}_d\alpha = \underline{0} \ .$$

Thus

$$\dim \langle\{\underline{u}_i - \underline{u}_i\alpha : i = d+1,\ldots,n\}\rangle \le \ell \ .$$

But $n - d = g$ and $\ell < g$. Hence there exist scalars $\mu_{d+1},\ldots,\mu_n$ (not all zero) such that

$$\mu_{d+1}(\underline{u}_{d+1} - \underline{u}_{d+1}\alpha) + \ldots + \mu_n(\underline{u}_n - \underline{u}_n\alpha) = \underline{0} \ ,$$

i.e.

$$\sum_{j=d+1}^{n} \mu_j\underline{u}_j - \sum_{j=d+1}^{n} \mu_j\underline{u}_j\alpha = \underline{0} \ .$$

Thus

$$\sum_{j=d+1}^{n} \mu_j\underline{u}_j \in X_\alpha \ .$$

Hence there exist scalars $\nu_1,\nu_2,\ldots,\nu_d$ such that

$$\sum_{j=1}^{d}\nu_j\underline{u}_j = \sum_{j=d+1}^{n}\mu_j\underline{u}_j .$$

But this is a contradiction since $\{\underline{u}_1,\ldots,\underline{u}_n\}$ forms a basis for V and not all the μ_j are zero. Thus $\alpha \notin E^{\ell}$.

<u>8.8 THEOREM</u> Let $Sing_n$ denote the semigroup of singular endomorphisms of an n-dimensional vector space V and let E^+ denote the set of idempotents of $Sing_n$. Then the depth of $Sing_n$ is n (i.e. $(E^+)^n = Sing_n$ and if $\ell < n$ then $(E^+)^{\ell} \neq Sing_n$).

<u>PROOF</u> By Lemma 8.6, we know that $E^n = Sing_n$, where E denotes the idempotents of $Sing_n$ of rank $n-1$. Since $E \subseteq E^+$, we thus have $\Delta(Sing_n) \leq n$.

By Lemma 1.1,

$$\Delta(Sing_n) \geq \max\{g(\alpha) : \alpha \in PF_{n-1}\} .$$

By Lemma 8.7, the element

$$\left[\begin{array}{c|c} \underline{0} & \underline{0} \\ \hline I_{n-1} & \underline{0} \end{array}\right]$$

of PF_{n-1} has gravity n . Hence $\Delta(Sing_n) \geq n$. Consequently, $\Delta(Sing_n) = n$.

<u>8.9 COROLLARY</u> Let $\varepsilon_1,\varepsilon_2 : V \to V$ be idempotent singular endomorphisms of an n-dimensional vector space V . If ε_1 has rank $n-k_1$, ε_2 has rank $n-k_2$, and $\varepsilon_1\varepsilon_2$ has rank $n-k_1-k_2$ $(n \geq k_1+k_2)$, then $\varepsilon_1\varepsilon_2$ is an idempotent endomorphism.

PROOF Since ε_1 is idempotent of rank $n - k_1$, it follows that $\dim \{\underline{x} \in V : \underline{x}\varepsilon_1 = \underline{x}\} = n - k_1$. Thus $g(\varepsilon_1) = k_1$. Similarly, $g(\varepsilon_2) = k_2$. Consequently, $g(\varepsilon_1\varepsilon_2) \leq k_1 + k_2$.

Now let $d = \dim \{\underline{x} \in V : \underline{x}\varepsilon_1\varepsilon_2 = \underline{x}\}$; then $g(\varepsilon_1\varepsilon_2) = n - d$. Thus $n - d \leq k_1 + k_2$, i.e. $d \geq n - k_1 - k_2$. But $\varepsilon_1\varepsilon_2$ has rank $n - k_1 - k_2$, so, by necessity, $d \leq n - k_1 - k_2$. Thus $d = n - k_1 - k_2$, i.e.

$$\dim \{\underline{x} \in V : \underline{x}\varepsilon_1\varepsilon_2 = \underline{x}\} = \dim R_{\varepsilon_1\varepsilon_2} .$$

Also $\{\underline{x} \in V : \underline{x}\varepsilon_1\varepsilon_2 = \underline{x}\} \subseteq R_{\varepsilon_1\varepsilon_2}$ and so $\{\underline{x} \in V : \underline{x}\varepsilon_1\varepsilon_2 = \underline{x}\} = R_{\varepsilon_1\varepsilon_2}$. Thus $\varepsilon_1\varepsilon_2$ acts identically on its range and so is idempotent.

CHAPTER 2

THE SEMIGROUP OF SINGULAR CONTINUOUS ENDOMORPHISMS OF A SEPARABLE HILBERT SPACE

§1 BASIC DEFINITIONS AND RESULTS

This section gives the basic definitions and lemmas that will be used in the final two sections. As most of these results are well known, I have omitted many proofs and given instead suitable references.

1.1 DEFINITION A pre-Hilbert space is a complex vector space $\mathbb{P}$ together with a map, called an inner product, $\langle\cdot|\cdot\rangle : \mathbb{P} \times \mathbb{P} \to \mathbb{C}$ satisfying the following properties:

(1) $\langle \underline{x}|\underline{y}\rangle = \overline{\langle \underline{y}|\underline{x}\rangle}$ $(\forall \underline{x},\underline{y} \in \mathbb{P})$

(2) $\langle \underline{x}+\underline{y}|\underline{z}\rangle = \langle \underline{x}|\underline{z}\rangle + \langle \underline{y}|\underline{z}\rangle$ $(\forall \underline{x},\underline{y},\underline{z} \in \mathbb{P})$

(3) $\langle \lambda\underline{x}|\underline{y}\rangle = \lambda\langle \underline{x}|\underline{y}\rangle$ $(\forall \underline{x},\underline{y} \in \mathbb{P}\ ,\ \forall\lambda \in \mathbb{C})$

(4) $\langle \underline{x}|\underline{x}\rangle > 0$ $(\forall \underline{x} \in \mathbb{P}\ ,\ \underline{x} \neq \underline{0})$

1.2 DEFINITION A Hilbert space is a complete pre-Hilbert space, i.e. a pre-Hilbert space in which every cauchy sequence is convergent.

A separable Hilbert space is a Hilbert space which has a countable basis.

1.3 DEFINITION A linear subspace of a separable Hilbert space $\mathbb{H}$ is a subset $\mathbb{A}$ of $\mathbb{H}$ such that, if $\underline{x},\underline{y} \in \mathbb{H}$ and $\lambda,\mu \in \mathbb{C}$, then $\lambda\underline{x} + \mu\underline{y} \in \mathbb{A}$.

1.4 DEFINITION A closed linear subspace of a separable Hilbert space $\mathbb{H}$ is a linear subspace $\mathbb{A}$ of $\mathbb{H}$ such that, if $(\underline{x}_n)_{n\in\mathbb{N}}$ is a sequence of elements in $\mathbb{A}$ with limit $\underline{x}$ in $\mathbb{H}$, then $\underline{x}$ belongs to $\mathbb{A}$. The closure of any subset $\mathbb{B}$ of $\mathbb{H}$, denoted by $\overline{\mathbb{B}}$, is the smallest closed linear subspace of $\mathbb{H}$ containing $\mathbb{B}$.

1.5 LEMMA (Theorem II.5.1 [2]) Let $<\cdot|\cdot>$ denote an inner product on a separable Hilbert space $\mathbb{H}$. Then, for each $\underline{x}$ in $\mathbb{H}$, the mappings $<\cdot|\underline{x}> : \mathbb{H} \to \mathbb{C}$ and $<\underline{x}|\cdot> : \mathbb{H} \to \mathbb{C}$ are continuous. The first mapping is also linear, while the second has the 'conjugate linear' property given by $<\underline{x}|\lambda\underline{y}+\mu\underline{z}> = \bar{\lambda}<\underline{x}|\underline{y}> + \bar{\mu}<\underline{y}|\underline{z}>$ $(\forall\ \underline{x},\underline{y},\underline{z} \in \mathbb{H})$ $(\forall\ \lambda,\mu \in \mathbb{C})$.

1.6 DEFINITION Let $\mathbb{A}$ be a subset of a separable Hilbert space $\mathbb{H}$. $\mathbb{A}^{\perp}$ will denote the set $\{\underline{x} \in \mathbb{H} : <\underline{x}|\underline{a}> = 0\ (\forall \underline{a} \in \mathbb{A})\}$.

1.7 LEMMA (§53 [17]) Let $\mathbb{A}$ be a subset of a separable Hilbert space $\mathbb{H}$. Then $\mathbb{A}^{\perp}$ is a closed linear subspace of $\mathbb{H}$.

1.8 LEMMA (Theorem III.6.2 [2]) If $\mathbb{A}$ is a closed linear subspace of a Hilbert space $\mathbb{H}$, then $\mathbb{H} = \mathbb{A} \oplus \mathbb{A}^{\perp}$ and $\mathbb{A} = \mathbb{A}^{\perp\perp}$.

1.9 LEMMA (Corollary III.6.1 [17]) If $\mathbb{A}$ is any subset of a separable Hilbert space, then $\overline{\mathbb{A}} = \mathbb{A}^{\perp\perp}$.

1.10 LEMMA (Theorem 53C [2]) If $\mathbb{A}$ and $\mathbb{B}$ are any closed linear subspaces of a separable Hilbert space $\mathbb{H}$ such that $\mathbb{A} \perp \mathbb{B}$, then the set $\mathbb{A} \oplus \mathbb{B}$ is also a closed linear subspace of $\mathbb{H}$.

1.11 LEMMA If $\mathbb{A}$ and $\mathbb{B}$ are linear subspaces of a separable Hilbert space $\mathbb{H}$, then:

$$\text{(i)}\quad (\mathbb{A}+\mathbb{B})^{\perp} = \mathbb{A}^{\perp} \cap \mathbb{B}^{\perp}$$

and

$$\text{(ii)}\quad (\overline{\mathbb{A}}\cap\overline{\mathbb{B}})^{\perp} = \overline{\mathbb{A}^{\perp} + \mathbb{B}^{\perp}}\ .$$

PROOF (i) Clearly, $\mathbb{A} \subseteq \mathbb{A} + \mathbb{B}$ and $\mathbb{B} \subseteq \mathbb{A} + \mathbb{B}$. Thus $\mathbb{A}^\perp \supseteq (\mathbb{A}+\mathbb{B})^\perp$ and $\mathbb{B}^\perp \supseteq (\mathbb{A}+\mathbb{B})^\perp$. Hence $(\mathbb{A}+\mathbb{B})^\perp \subseteq \mathbb{A}^\perp \cap \mathbb{B}^\perp$.

Now, if $\underline{x} \in \mathbb{A}^\perp \cap \mathbb{B}^\perp$, then $<\underline{x}|\underline{a}> = 0$ $(\forall \underline{a} \in \mathbb{A})$ and $<\underline{x}|\underline{b}> = 0$ $(\forall \underline{b} \in \mathbb{B})$. Thus $<\underline{x}|\underline{a}+\underline{b}> = 0$ $(\forall \underline{a} \in \mathbb{A}, \forall \underline{b} \in \mathbb{B})$, i.e. $\underline{x} \in (\mathbb{A}+\mathbb{B})^\perp$. Hence $\mathbb{A}^\perp \cap \mathbb{B}^\perp \subseteq (\mathbb{A}+\mathbb{B})^\perp$, and so $\mathbb{A}^\perp \cap \mathbb{B}^\perp = (\mathbb{A}+\mathbb{B})^\perp$.

(ii) From (i) we have

$$[\mathbb{A}^\perp]^\perp \cap [\mathbb{B}^\perp]^\perp = ([\mathbb{A}^\perp]+[\mathbb{B}^\perp])^\perp ,$$

i.e.

$$\overline{\mathbb{A}} \cap \overline{\mathbb{B}} = (\mathbb{A}^\perp+\mathbb{B}^\perp)^\perp .$$

Thus

$$(\overline{\mathbb{A}} \cap \overline{\mathbb{B}})^\perp = (\mathbb{A}^\perp+\mathbb{B}^\perp)^{\perp\perp}$$

$$= \overline{(\mathbb{A}^\perp+\mathbb{B}^\perp)}$$

1.12 LEMMA Let $\mathbb{A}$ and $\mathbb{B}$ be closed linear subspaces of a separable Hilbert space $\mathbb{H}$ such that $\mathbb{A} \subseteq \mathbb{B}$. Then $\mathbb{B} = \mathbb{A} \oplus (\mathbb{B} \cap \mathbb{A}^\perp)$.

PROOF Since $\mathbb{B}$ is a closed subspace of $\mathbb{H}$, it is a Hilbert space itself. Since $\mathbb{A}$ is closed in $\mathbb{H}$, it is also closed in $\mathbb{B}$. So, by Lemma 1.8,

$$\mathbb{B} = \mathbb{A} \oplus (\mathbb{B} \cap \mathbb{A}^\perp) .$$

1.13 DEFINITION Let $\alpha \in$ Sing. The adjoint α^* of α is defined to be the unique mapping in Sing such that $<\underline{x}|\underline{y}\alpha^*> = <\underline{x}\alpha|\underline{y}>$ for all $\underline{x},\underline{y}$ in $\mathbb{H}$.

<u>1.14 LEMMA</u> If $\varepsilon \in E$ then $\varepsilon^* \in E$.

<u>PROOF</u> For any $\alpha, \beta \in \text{Sing}$, we have $(\alpha\beta)^* = \beta^*\alpha^*$ (see Theorem 56A [17]). So, putting $\varepsilon = \alpha = \beta$ gives $\varepsilon^* = (\varepsilon^2)^* = \varepsilon^*\varepsilon^*$.

<u>1.15 LEMMA</u> Let $\alpha \in \text{Sing}$. Then $R_\alpha^\perp = N_{\alpha^*}$ and $N_\alpha^\perp = \overline{R}_{\alpha^*}$.

<u>PROOF</u> Let $\underline{x} \in R_\alpha^\perp$. Then $\langle \underline{x} | \underline{y}\alpha \rangle = 0$ $(\forall \underline{y} \in H)$. Thus $\langle \underline{x}\alpha^* | \underline{y} \rangle = 0$ $(\forall \underline{y} \in H)$, i.e. $\underline{x}\alpha^* \in H^\perp = \{\underline{0}\}$. Thus $R_\alpha^\perp \subseteq N_{\alpha^*}$. Conversely, if $\underline{x} \in N_{\alpha^*}$, then $\langle \underline{x}\alpha^* | \underline{y} \rangle = 0$ $(\forall y \in H)$. Thus $\langle \underline{x} | \underline{y}\alpha \rangle = 0$ $(\forall y \in H)$, i.e. $\underline{x} \in R_\alpha^\perp$. Thus $N_{\alpha^*} \subseteq R_\alpha^\perp$. Hence $N_{\alpha^*} = R_\alpha^\perp$.

Similarly, $R_{\alpha^*}^\perp = N_{\alpha^{**}} = N_\alpha$. Thus $N_\alpha^\perp = R_{\alpha^*}^{\perp\perp} = \overline{R}_{\alpha^*}$.

<u>1.16 LEMMA</u> Let A and B be closed linear subspaces of a separable Hilbert space H .

(i) If $\dim A = \dim B$, then A is isomorphic to B

(ii) If $\dim A < \dim B$, then there exists a closed linear subspace C of B such that A is isomorphic to C .

<u>PROOF</u> This is immediate from Theorem II.9.1 of [2].

<u>1.17 LEMMA</u> (Theorem IV.7.2 [2]) Let $\alpha \in \text{Sing}$. Then N_α is a closed linear subspace of H .

<u>1.18 LEMMA</u> If $\varepsilon \in E$, then R_ε is a closed linear subspace of H .

<u>PROOF</u> Since ε is linear, R_ε is clearly a linear subspace of H . Let $(\underline{x}_n)_{n \in \mathbb{N}}$ be a sequence of elements of R_ε with limit $\underline{x}$

in $\mathcal{H}$. Since ε is idempotent, $\underline{x}_n\varepsilon = x_n$ $(n = 1,2,\ldots)$. Thus, since ε is continuous,

$$\underline{x}\varepsilon = (\lim \underline{x}_n)\varepsilon = \lim (\underline{x}_n\varepsilon) = \lim \underline{x}_n = \underline{x} .$$

Thus $\underline{x} \in \mathcal{R}_\varepsilon$.

1.19 LEMMA Let $\varepsilon \in E$. Then $\mathcal{H} = \mathcal{R}_\varepsilon + \mathcal{N}_\varepsilon$ and $\mathcal{R}_\varepsilon \cap \mathcal{N}_\varepsilon = \{\underline{0}\}$.

PROOF Let $\underline{x} \in \mathcal{H}$. Then $\underline{x} = \underline{x}\varepsilon + (\underline{x}-\underline{x}\varepsilon) \in \mathcal{R}_\varepsilon + \mathcal{N}_\varepsilon$. Suppose $\underline{x} \in \mathcal{R}_\varepsilon \cap \mathcal{N}_\varepsilon$. Then $\underline{0} = \underline{x}\varepsilon = \underline{x}$. Thus $\mathcal{R}_\varepsilon \cap \mathcal{N}_\varepsilon = \{\underline{0}\}$.

1.20 LEMMA Let $\mathcal{A}$ be a subspace of a separable Hilbert space $\mathcal{H}$. Then $\dim \mathcal{A} = \dim \overline{\mathcal{A}}$.

PROOF Suppose first that $\dim \mathcal{A} < \aleph_0$. Then $\mathcal{A}$ has finite dimension and so is closed. Thus $\mathcal{A} = \overline{\mathcal{A}}$. If $\dim \mathcal{A}$ has infinite dimension, then, since $\mathcal{A} \subseteq \overline{\mathcal{A}} \subseteq \mathcal{H}$, we have $\dim \mathcal{A} \leq \dim \overline{\mathcal{A}} \leq \dim \mathcal{H}$, i.e. $\aleph_0 \leq \dim \overline{\mathcal{A}} \leq \aleph_0$. Thus $\dim \mathcal{A} = \dim \overline{\mathcal{A}}$.

1.21 LEMMA Let $\mathcal{A}$ be a linear subspace and $\mathcal{B}$ a closed linear subspace of a separable Hilbert space $\mathcal{H}$. Then

$$\overline{\overline{\mathcal{A}} + \mathcal{B}} = \overline{\mathcal{A} + \mathcal{B}} .$$

PROOF Clearly $\overline{\mathcal{A}} + \mathcal{B} \supseteq \mathcal{A} + \mathcal{B}$, and so

$$\overline{\overline{\mathcal{A}} + \mathcal{B}} \supseteq \overline{\mathcal{A} + \mathcal{B}} .$$

Let $\underline{x}$ be an element of $\overline{\overline{\mathcal{A}} + \mathcal{B}}$. Then there exists a sequence

$(\underline{x}_i)_{i\in\mathbb{N}}$ in $\overline{A} + B$ with limit $\underline{x}$. Hence there exist sequences $(\underline{a}_i)_{i\in\mathbb{N}}$ in $\overline{A}$ and $(\underline{b}_i)_{i\in\mathbb{N}}$ in B such that $\underline{x}_i = \underline{a}_i + \underline{b}_i$. Now, for each element $\underline{a}_i$, there exists a sequence $(\underline{a}_{ij})_{j\in\mathbb{N}}$ in A with limit $\underline{a}_i$ such that $\| \underline{a}_i - \underline{a}_{ij} \| \le 1/2^j$.

Now,

$$\begin{aligned} \| \underline{x} - \underline{a}_{ii} - \underline{b}_i \| &= \| \underline{x} - \underline{a}_i - \underline{b}_i + \underline{a}_i - \underline{a}_{ii} \| \\ &\le \| \underline{x} - \underline{a}_i - \underline{b}_i \| + \| \underline{a}_i - \underline{a}_{ii} \| \\ &\to 0 + 0 \quad \text{as} \quad i \to \infty . \end{aligned}$$

Consequently, the sequence $(\underline{a}_{ii}+\underline{b}_i)_{i\in\mathbb{N}}$ has limit $\underline{x}$. Thus $\underline{x} \in \overline{A + B}$.

<u>1.22 LEMMA</u> Let A be a subspace of a separable Hilbert space H and let α be a linear mapping from A to H . Then the following are equivalent:

(i) α is a continuous mapping

(ii) there exists a constant M such that $\| \underline{x}\alpha \| \le M \| \underline{x} \|$ for all $\underline{x}$ in H .

<u>PROOF</u> This is immediate from Theorem IV.7.3 of [2].

§2 SOME TECHNICAL RESULTS

2.1 LEMMA Let $\mathbb{A}$ be a linear subspace of $\mathbb{H}$. If α is a continuous linear map from $\mathbb{A}$ to $\mathbb{H}$, then there exists a unique continuous linear map $\bar{\alpha}$ from $\overline{\mathbb{A}}$ to $\mathbb{H}$ that coincides with α on $\mathbb{A}$..

PROOF Let $\underline{x}$ be any point of $\overline{\mathbb{A}}$. Then there exists a sequence $(\underline{x}_n)_{n\in\mathbb{N}}$ in $\mathbb{A}$ with limit $\underline{x}$. Define $\alpha' : \overline{\mathbb{A}} \to \mathbb{H}$ by $\underline{x}\alpha' = \lim (\underline{x}_n\alpha)$.

Let $\underline{y}$ be any point of $\overline{\mathbb{A}}$. Then there exists a sequence $(\underline{y}_n)_{n\in\mathbb{N}}$ of $\mathbb{A}$ with limit $\underline{y}$. Let λ,μ be any elements of $\mathbb{C}$. Then

$$
\begin{aligned}
(\lambda\underline{x}+\mu\underline{y})\alpha' &= \lim [(\lambda\underline{x}_n+\mu\underline{y}_n)\alpha] \\
&= \lim [\lambda(\underline{x}_n\alpha)+\mu(\underline{y}_n\alpha)] \quad \text{since } \alpha \text{ is linear} \\
&= \lambda \lim (\underline{x}_n\alpha) + \mu \lim (\underline{y}_n\alpha) \\
&= \lambda(\underline{x}\alpha') + \mu(\underline{y}\alpha') .
\end{aligned}
$$

Thus α' is linear.

Now let $\underline{x} \in \overline{\mathbb{A}}$. Then there exists a sequence $(\underline{x}_n)_{n\in\mathbb{N}}$ in $\mathbb{A}$ with limit $\underline{x}$. By Lemma 1.22, there exists an $M \geq 0$ such that $\|\underline{x}_n\alpha\| \leq M\|\underline{x}_n\|$. Thus

$$\lim \|\underline{x}_n\alpha\| \leq M \lim \|\underline{x}_n\| ,$$

i.e.

$$\|\underline{x}\alpha'\| = \|\lim(\underline{x}_n\alpha)\| \leq M\|\lim \underline{x}_n\| = M\|\underline{x}\| .$$

Hence, by Lemma 1.22, α' is continuous.

Now, suppose that α_1 is another continuous linear map from $\overline{\mathbb{A}}$

to $\mathcal{H}$ that coincides with α on $\mathcal{A}$. If $\underline{x}$ is any point of $\overline{\mathcal{A}}$, then there exists a sequence $(\underline{x}_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ with limit $\underline{x}$. Thus $\underline{x}\alpha' = \lim (x_n\alpha) = \lim (\underline{x}_n\alpha_1) = \underline{x}\alpha_1$. Thus $\overline{\alpha} = \alpha' = \alpha_1$, and so $\overline{\alpha}$ is unique.

<u>2.2 LEMMA</u> Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be any closed linear subspaces of $\mathcal{H}$ such that $\mathcal{A}_1 \cap \mathcal{A}_2 = \{\underline{0}\}$. If α_1 and α_2 are continuous linear maps from $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively to $\mathcal{H}$, then there exists a unique linear map $(\alpha_1+\alpha_2)$ from $\mathcal{A}_1 + \mathcal{A}_2$ to $\mathcal{H}$ that coincides with α_1 on $\mathcal{A}_1$ and α_2 on $\mathcal{A}_2$.

<u>PROOF</u> If $\underline{x} \in \mathcal{A}_1 + \mathcal{A}_2$, then $\underline{x} = \underline{a}_1 + \underline{a}_2$ for some $\underline{a}_1 \in \mathcal{A}_1$ and some $\underline{a}_2 \in \mathcal{A}_2$. Define $(\alpha_1+\alpha_2) : \mathcal{A}_1 + \mathcal{A}_2 \to \mathcal{H}$ by

$$\underline{x}(\alpha_1+\alpha_2) = \underline{a}_1\alpha_1 + \underline{a}_2\alpha_2 .$$

It is immediate that $(\alpha_1+\alpha_2)$ is continuous, linear and unique.

<u>2.3 LEMMA</u> Let $\mathcal{A}$ and $\mathcal{B}$ be closed linear subspaces of a separable Hilbert space $\mathcal{H}$. Then $\dim \mathcal{B} \doteq \dim (\mathcal{A}\cap\mathcal{B}) + \dim [\mathcal{A}^{\perp}\cap(\mathcal{A}+\mathcal{B})]$.

<u>PROOF</u> Define the mapping $\beta : \mathcal{H} \to \mathcal{H}$ by $\beta = \alpha_1 \oplus \alpha_2$, where $\alpha_1 : \mathcal{A} \to \mathcal{H}$ is the zero mapping and $\alpha_2 : \mathcal{A}^{\perp} \to \mathcal{H}$ is the identity mapping. Let β_1 be the restriction of β to the linear subspace $\mathcal{B}$. Clearly, β_1 is continuous and linear, and so

$$\dim \mathcal{B} = \dim \mathcal{N}_{\beta_1} + \dim \mathcal{R}_{\beta_1} .$$

Clearly, $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{N}_{\beta_1}$. Suppose $\underline{x} \in \mathcal{N}_{\beta_1}$. Then $\underline{x} = \underline{a} + \underline{p}$ for some $\underline{a} \in \mathcal{A}$ and some $\underline{p} \in \mathcal{A}^{\perp}$ with $\underline{p} = \underline{a}\beta_1 + \underline{p}\beta_1 = \underline{x}\beta_1 = \underline{0}$, i.e. $\underline{x} \in \mathcal{A}$.

But, by definition, $\underline{x} \in \mathcal{B}$. Thus $N_{\beta_1} \subseteq \mathcal{A} \cap \mathcal{B}$. Hence $N_{\beta_1} = \mathcal{A} \cap \mathcal{B}$.

Now suppose $\underline{x} \in R_{\beta_1}$. Then there exists $\underline{y} \in \mathcal{B}$ such that $\underline{y}\beta_1 = \underline{x}$. But $\underline{y} = \underline{a} + \underline{p}$ for some $\underline{a} \in \mathcal{A}$ and some $\underline{p} \in \mathcal{A}^{\perp}$. Thus

$$\underline{x} = \underline{y}\beta_1 = \underline{a}\beta + \underline{p}\beta = \underline{p} .$$

Also, $\underline{p} = \underline{y} - \underline{a} \in \mathcal{B} + \mathcal{A}$. Thus $\underline{p} \in \mathcal{A}^{\perp} \cap (\mathcal{A}+\mathcal{B})$. Hence $R_{\beta_1} \subseteq \mathcal{A}^{\perp} \cap (\mathcal{A}+\mathcal{B})$.

Conversely, suppose $\underline{x} \in \mathcal{A}^{\perp} \cap (\mathcal{A}+\mathcal{B})$. Then $\underline{x} = \underline{a} + \underline{b}$ for some $\underline{a} \in \mathcal{A}$ and some $\underline{b} \in \mathcal{B}$. Thus $\underline{x} - \underline{a} \in \mathcal{B}$ and

$$(\underline{x}-\underline{a})\beta_1 = \underline{x} - \underline{a}\beta = \underline{x} .$$

Thus $\underline{x} \in R_{\beta_1}$, i.e. $\mathcal{A}^{\perp} \cap (\mathcal{A}+\mathcal{B}) \subseteq R_{\beta_1}$. Thus $R_{\beta_1} = \mathcal{A}^{\perp} \cap (\mathcal{A}+\mathcal{B})$.

Consequently,

$$\dim \mathcal{B} = \dim (\mathcal{A}\cap\mathcal{B}) + \dim [\mathcal{A}^{\perp}\cap(\mathcal{A}+\mathcal{B})] .$$

<u>2.4 LEMMA</u> Let α and β be continuous endomorphisms of $\mathcal{H}$. Then

$$N_{\alpha\beta} = N_{\alpha} \oplus \{\underline{x} \in N_{\alpha}^{\perp} : \underline{x}\alpha \in N_{\beta}\} .$$

<u>PROOF</u> We shall first establish that $\mathcal{A} = \{\underline{x} \in N_{\alpha}^{\perp} : \underline{x}\alpha \in N_{\beta}\}$ is a closed linear subspace of $\mathcal{H}$. Suppose $\underline{x},\underline{y} \in \mathcal{A}$ and $\lambda,\mu \in \mathbb{C}$. Then $\lambda\underline{x} + \mu\underline{y} \in N_{\alpha}^{\perp}$ since $N_{\alpha}^{\perp}$ is a linear subspace (Lemma 1.7). Also, $(\lambda\underline{x}+\mu\underline{y})\alpha = \lambda(\underline{x}\alpha) + \mu(\underline{y}\alpha) \in N_{\beta}$ since N_{β} is a linear subspace (Lemma 1.17). Thus $\lambda\underline{x} + \mu\underline{y} \in \mathcal{A}$, and so $\mathcal{A}$ is a linear subspace of $\mathcal{H}$. Now let $(\underline{x}_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{A}$ with limit $\underline{x}$ in $\mathcal{H}$. Since $N_{\alpha}^{\perp}$ is closed, $\underline{x} \in N_{\alpha}^{\perp}$. Also, since $\underline{x}_i\alpha \in N_{\beta}$, we have $\underline{x}_i\alpha\beta = \underline{0}$ $(i = 1,2,\ldots)$. Thus $\underline{x}_i \in N_{\alpha\beta}$ $(i = 1,2,\ldots)$. Since $N_{\alpha\beta}$ is closed

(Lemma 1.17), we have $\underline{x} \in N_{\alpha\beta}$. Thus $\underline{x}\alpha\beta = \underline{0}$, i.e. $\underline{x}\alpha \in N_\beta$. Thus $\underline{x} \in A$, and so A is closed.

Now, let $\underline{x}$ be any element of $N_{\alpha\beta}$. Then (by Lemma 1.8 and Lemma 1.17) $\underline{x} = \underline{n} + \underline{p}$ for some $\underline{n} \in N_\alpha$ and some $\underline{p} \in N_\alpha^\perp$. So

$$\underline{0} = \underline{x}\alpha\beta = (\underline{n}+\underline{p})\alpha\beta = \underline{0}\beta + \underline{p}\alpha\beta = \underline{p}\alpha\beta .$$

Thus $\underline{p}\alpha \in N_\beta$. Hence

$$N_{\alpha\beta} \subseteq N_\alpha \oplus \{\underline{x} \in N_\alpha^\perp : \underline{x}\alpha \in N_\beta\} .$$

Now, let $y \in N_\alpha \oplus \{\underline{x} \in N_\alpha^\perp : \underline{x}\alpha \in N_\beta\}$. Then $\underline{y} = \underline{n} + \underline{a}$ for some $\underline{n} \in N_\alpha$ and some $\underline{a} \in \{\underline{x} \in N_\alpha^\perp : \underline{x}\alpha \in N_\beta\}$. So
$\underline{y}\alpha\beta = (\underline{n}+\underline{a})\alpha\beta = \underline{n}\alpha\beta + \underline{a}\alpha\beta = (\underline{a}\alpha)\beta = \underline{0}$. So $N_\alpha \oplus \{\underline{x} \in N_\alpha^\perp : \underline{x}\alpha \in N_\beta\} \subseteq N_{\alpha\beta}$. Thus the result holds.

2.5 LEMMA Let α be a continuous endomorphism of H . Define $\alpha_1 : N_\alpha^\perp \to R_\alpha$ by $\underline{x}\alpha_1 = \underline{x}\alpha$. Then α_1 is a continuous linear bijection.

PROOF Since α is continuous and linear, it follows that α_1 is also continuous and linear.

To show that α_1 is injective, consider an element $\underline{x}$ of N_{α_1} . Then $\underline{0} = \underline{x}\alpha_1 = \underline{x}\alpha$, i.e. $\underline{x} \in N_\alpha$. But $N_\alpha \cap N_\alpha^\perp = \{\underline{0}\}$ and α_1 is only defined on $N_\alpha^\perp$. Thus $\underline{x} = \underline{0}$, and so α_1 is injective.

To show that α_1 is surjective, consider an element $\underline{x}$ of R_α . Then there exists an element $\underline{y}$ of H such that $\underline{y}\alpha = \underline{x}$. But (by Lemma 2.8 and Lemma 2.17) $\underline{y} = \underline{n} + \underline{p}$ for some $\underline{n} \in N_\alpha$ and some $\underline{p} \in N_\alpha^\perp$. So

$$\underline{x} = \underline{y}\alpha = (\underline{n}+\underline{p})\alpha = \underline{p}\alpha = \underline{p}\alpha_1 .$$

2.6 LEMMA If α and β are continuous endomorphisms of H, then

$$\dim \{\underline{x} \in N_\alpha^\perp : \underline{x}\alpha \in N_\beta\} = \dim (R_\alpha \cap N_\beta) .$$

PROOF Since, by Lemma 2.5, $\alpha_1 = \alpha|_{N_\alpha^\perp}$ is a bijective linear mapping from $N_\alpha^\perp$ to R_α, we have
$\dim (R_\alpha \cap N_\beta) = \dim [(R_\alpha \cap N_\beta)\alpha_1^{-1}] = \dim \{\underline{x} \in N_\alpha^\perp : \underline{x}\alpha \in N_\beta\}$.

2.7 LEMMA Let $\alpha \in$ Sing and $\varepsilon \in E$. If $\dim N_\alpha = \aleph_0$, then $\dim N_{\varepsilon\alpha} = \aleph_0$.

PROOF If $\dim N_\varepsilon = \aleph_0$, then the result is immediate from Lemma 2.4. So, suppose $\dim N_\varepsilon < \aleph_0$.

Define a map $\theta : N_\alpha \to N_\varepsilon$ by $x\theta = x - x\varepsilon$. θ is clearly linear and so

$$\dim N_\alpha = \dim N_\theta + \dim R_\theta .$$

$$\leq \dim N_\theta + \dim N_\varepsilon .$$

Now, since $\dim N_\alpha = \aleph_0$ and $\dim N_\varepsilon < \aleph_0$, this gives $\dim N_\theta = \aleph_0$. Thus there exist infinitely many linearly independent elements of N_α satisfying $\underline{x}\theta = \underline{0}$, i.e. satisfying $\underline{x}\varepsilon = \underline{x}$. But each of these elements is in N_α. Thus there are infinitely many linearly independent elements satisfying $\underline{x}\varepsilon\alpha = \underline{x}\alpha = \underline{0}$. Thus $\dim N_{\varepsilon\alpha} = \aleph_0$.

§3 THE SUBSEMIGROUP GENERATED BY THE IDEMPOTENTS

In this section we determine the subsemigroup generated by the idempotents E of the semigroup Sing of singular continuous endomorphism of a separable Hilbert space $\mathcal{H}$.

We shall need one further concept before proceeding.

3.1 DEFINITION Let $\alpha \in \langle E\rangle$. Define the length, $\ell(\alpha)$, of α to be $\min\{n : \alpha \in E^n\}$.

3.2 LEMMA Let $\tau \in \langle E\rangle$; then $\dim \mathcal{N}_\tau = \dim \mathcal{R}_\tau^\perp$.

PROOF The proof is by induction on the length of elements of $\langle E\rangle$. We shall show first that the result is true for elements of $\langle E\rangle$ of length 1, i.e. for elements of E .

Let $\varepsilon \in E$ and define a mapping $\theta : \mathcal{R}_\varepsilon^\perp \to \mathcal{N}_\varepsilon$ by $\underline{x}\theta = \underline{x} - \underline{x}\varepsilon$. θ is injective. To see this, notice that if $\underline{x}\theta = \underline{0}$ for $\underline{x}$ in $\mathcal{R}_\varepsilon^\perp$, then $\underline{x} = \underline{x}\varepsilon \in \mathcal{R}_\varepsilon$; hence $\underline{x} = \underline{0}$ since $\mathcal{R}_\varepsilon \cap \mathcal{R}_\varepsilon^\perp = \{\underline{0}\}$. Also, θ is surjective. To see this, notice that if $\underline{n} \in \mathcal{N}_\varepsilon$, then $\underline{n} = \underline{r} + \underline{s}$ for some $\underline{r} \in \mathcal{R}_\varepsilon$ and some $\underline{s} \in \mathcal{R}_\varepsilon^\perp$ (by Lemma 1.8 and Lemma 1.17), i.e. $\underline{n}\varepsilon = \underline{r}\varepsilon + \underline{s}\varepsilon$, and hence $\underline{0} = \underline{r} + \underline{s}\varepsilon$. Now, substituting for $\underline{r}$ in $\underline{n} = \underline{r} + \underline{s}$ gives $\underline{n} = \underline{s} - \underline{s}\varepsilon$, i.e. $\underline{n} = \underline{s}\theta$ where $\underline{s} \in \mathcal{R}_\varepsilon^\perp$. Hence θ is a bijection. Since θ is also linear, we have $\dim \mathcal{N}_\varepsilon = \dim \mathcal{R}_\varepsilon^\perp$. So we may start the induction process.

Now, let $\eta \in \langle E\rangle$ have length n and assume the result holds for all elements of $\langle E\rangle$ with length less than n . Now, there exists an $\varepsilon \in E$ and a $\tau \in \langle E\rangle$ of length $n - 1$ such that $\eta = \varepsilon\tau$.

Suppose first that $\dim \mathcal{N}_\tau = \aleph_0$. Then, by the hypothesis, $\dim \mathcal{R}_\tau^\perp = \aleph_0$. Now, $\mathcal{R}_{\varepsilon\tau} \subseteq \mathcal{R}_\tau$, and so $\mathcal{R}_\tau^\perp \subseteq \mathcal{R}_{\varepsilon\tau}^\perp$. So $\dim \mathcal{R}_{\varepsilon\tau}^\perp = \aleph_0$.

By Lemma 2.7, $\dim N_\tau = \aleph_0$ implies $\dim N_{\varepsilon\tau} = \aleph_0$. So $\dim N_\eta = \dim R_\eta^\perp$.

Now suppose that $\dim N_\tau < \aleph_0$. By Lemma 2.4,

$$N_{\varepsilon\tau} = N_\varepsilon \oplus \{\underline{x} \in N_\varepsilon^\perp : \underline{x}\varepsilon \in N_\tau\} ,$$

and, by Lemma 2.6,

$$\dim \{\underline{x} \in N_\varepsilon^\perp : \underline{x}\varepsilon \in N_\tau\} = \dim (R_\varepsilon \cap N_\tau) .$$

So,

$$\dim N_{\varepsilon\tau} = \dim N_\varepsilon + \dim (R_\varepsilon \cap N_\tau) .$$

Now, $(\varepsilon\tau)^* = \tau^*\varepsilon^*$ and so, by Lemma 1.15, $R_{\varepsilon\tau}^\perp = N_{\tau^*\varepsilon^*}$. Now, again by Lemma 2.4 and Lemma 2.6,

$$\dim N_{\tau^*\varepsilon^*} = \dim N_{\tau^*} + \dim (R_{\tau^*} \cap N_{\varepsilon^*})$$

$$\leq \dim N_{\tau^*} + \dim (\overline{R}_{\tau^*} \cap N_{\varepsilon^*}) .$$

Hence, by Lemma 1.15,

$$\dim R_{\varepsilon\tau}^\perp \leq \dim R_\tau^\perp + \dim (N_\tau^\perp \cap R_\varepsilon^\perp) . \qquad (+)$$

Since $R_\varepsilon^\perp \cap N_\tau^\perp$ is a closed subspace of $R_\varepsilon^\perp$, we have, by Lemma 1.12, that

$$R_\varepsilon^\perp = (R_\varepsilon^\perp \cap N_\tau^\perp) \oplus [R_\varepsilon^\perp \cap (R_\varepsilon^\perp \cap N_\tau^\perp)^\perp] .$$

So, by Lemma 1.11, Lemma 1.17 and Lemma 1.18,

$$R_\varepsilon^\perp = (R_\varepsilon^\perp \cap N_\tau^\perp) \oplus [R_\varepsilon^\perp \cap \overline{(R_\varepsilon + N_\tau)}]$$

$$\supseteq (R_\varepsilon^\perp \cap N_\tau^\perp) \oplus [R_\varepsilon^\perp \cap (R_\varepsilon + N_\tau)] .$$

Now, by Lemma 2.3, we have

$$\dim [R_\varepsilon^\perp \cap (R_\varepsilon + N_\tau)] = \dim N_\tau - \dim (R_\varepsilon \cap N_\tau) .$$

(This is defined since we have assumed $\dim N_\tau < \aleph_0$.) So

$$\dim R_\varepsilon^\perp \geq \dim (R_\varepsilon^\perp \cap N_\tau^\perp) + \dim N_\tau - \dim (R_\varepsilon \cap N_\tau) .$$

Thus, substituting for $\dim (R_\varepsilon^\perp \cap N_\tau^\perp)$ in (+), gives

$$\dim R_{\varepsilon\tau}^\perp \leq \dim R_\tau^\perp + \dim R_\varepsilon^\perp - \dim N_\tau + \dim (R_\varepsilon \cap N_\tau) .$$

By the induction hypothesis, $\dim R_\tau^\perp = \dim N_\tau$ and $\dim R_\varepsilon^\perp = \dim N_\varepsilon$, and so

$$\dim R_{\varepsilon\tau}^\perp \leq \dim N_\varepsilon + \dim (R_\varepsilon \cap N_\tau) .$$

But, by Lemma 2.4 and Lemma 2.6,

$$\dim N_{\varepsilon\tau} = \dim N_\varepsilon + \dim (R_\varepsilon \cap N_\tau) .$$

Thus, $\dim R_{\varepsilon\tau}^\perp \leq \dim N_{\varepsilon\tau}$, i.e. $\dim R_\eta^\perp \leq \dim N_\eta$.

Similarly, we may obtain the inequality $\dim R_{\eta*}^\perp \leq \dim N_{\eta*}$. So, by Lemma 1.15, $\dim N_\eta \leq \dim R_\eta^\perp$. Thus, $\dim N_\eta = \dim R_\eta^\perp$.

3.3 LEMMA Let $\alpha \in \mathrm{Sing}$ and be such that $\dim N_\alpha = \dim R_\alpha^\perp = \aleph_0$. Then $\alpha \in \langle E \rangle$.

PROOF By Lemma 1.11, Lemma 1.12 and Lemma 1.21,

$$R_\alpha^\perp = (R_\alpha^\perp \cap N_\alpha) \oplus [R_\alpha^\perp \cap \overline{(R_\alpha + N_\alpha^\perp)}] .$$

Since $R_\alpha^\perp = \aleph_0$, it follows that at least one of $R_\alpha^\perp \cap N_\alpha$ and $R_\alpha^\perp \cap \overline{(R_\alpha + N_\alpha^\perp)}$ must have infinite dimension. We must consider the two cases separately.

(a) $\dim (R_\alpha^\perp \cap N_\alpha) = \aleph_0$. Since $H = N_\alpha \oplus N_\alpha^\perp$, we may define a mapping $\varepsilon_1 \in E$ by $\varepsilon_1 = \eta_1 \oplus \eta_2$, where

$$\underline{x}\eta_1 = \underline{0} \quad (\underline{x} \in N_\alpha)$$

and

$$\underline{x}\eta_2 = \underline{x} \quad (\underline{x} \in N_\alpha^\perp) \ .$$

By Lemma 1.16, there exists an isomorphism θ from $N_\alpha^\perp$ to a closed subspace A of $R_\alpha^\perp \cap N_\alpha$. Since $H = N_\alpha \oplus N_\alpha^\perp$, we may define a mapping $\varepsilon_2 \in E$ by $\varepsilon_2 = \gamma \oplus \theta$, where

$$\underline{x}\gamma = \underline{x} \quad (\underline{x} \in N_\alpha) \ .$$

Since $H = A \oplus A^\perp$, we may define a mapping $\varepsilon_3 \in \text{Sing}$ by $\varepsilon_3 = \delta_1 \oplus \delta_2$, where

$$\underline{x}\delta_1 = \underline{x}\theta^{-1}\alpha \quad (\underline{x} \in A)$$

and

$$\underline{x}\delta_2 = \underline{x} \quad (\underline{x} \in A^\perp) \ .$$

Since $A \subseteq R_\alpha^\perp \cap N_\alpha = (\overline{R}_\alpha + N_\alpha^\perp)^\perp$, it follows that $R_\alpha \subseteq \overline{R}_\alpha + N_\alpha^\perp \subseteq A^\perp$. Thus $\varepsilon_3 \in E$.

We now show that $\alpha = \varepsilon_1\varepsilon_2\varepsilon_3$. To verify this, consider any element $\underline{x}$ in H . Now, $\underline{x} = \underline{n} + \underline{p}$ for some $\underline{n} \in N_\alpha$ and some $p \in N_\alpha^\perp$, and so

$$\underline{x}\varepsilon_1\varepsilon_2\varepsilon_3 = (\underline{n}+\underline{p})\varepsilon_1\varepsilon_2\varepsilon_3 = \underline{p}\varepsilon_2\varepsilon_3 = (p\theta)\varepsilon_3 = (\underline{p}\theta)\theta^{-1}\alpha$$

$$= \underline{p}\alpha = \underline{p}\alpha + \underline{n}\alpha = (\underline{p}+\underline{n})\alpha = \underline{x}\alpha \ .$$

(b) $\dim [R_\alpha^\perp \cap \overline{(R_\alpha + N_\alpha^\perp)}] = \aleph_0$. Since $H = N_\alpha \oplus N_\alpha^\perp$, we may define a mapping $\varepsilon_1 \in E$ by $\varepsilon_1 = \eta_1 \oplus \eta_2$, where

$$\underline{x}\eta_1 = \underline{0} \quad (\underline{x} \in N_\alpha)$$

and

$$\underline{x}\eta_2 = \underline{x} \quad (\underline{x} \in N_\alpha^\perp) \ .$$

By Lemma 1.16, there exists an isomorphism θ from $N_\alpha^\perp$ to a closed linear subspace A of N_α . Since $H = N_\alpha \oplus N_\alpha^\perp$, we may define a mapping $\varepsilon_2 \in E$ by $\varepsilon_2 = \theta \oplus \gamma$, where

$$\underline{x}\gamma = \underline{x} \quad (\underline{x} \in N_\alpha) \ .$$

Again, by Lemma 1.16, there exists an isomorphism ϕ from N_α to $R_\alpha^\perp \cap \overline{(R_\alpha + N_\alpha^\perp)}$. Since

$$N_\alpha \cap [R_\alpha^\perp \cap \overline{(R_\alpha + N_\alpha^\perp)}] = (N_\alpha \cap R_\alpha^\perp) \cap \overline{(R_\alpha + N_\alpha^\perp)}$$

$$= (N_\alpha \cap R_\alpha^\perp) \cap (R_\alpha^\perp \cap N_\alpha)^\perp$$

$$= \{\underline{0}\} \ ,$$

we may define (by Lemma 2.1 and Lemma 2.2) a mapping δ_1 from $B = \overline{N_\alpha + [R_\alpha^\perp \cap \overline{(R_\alpha + N_\alpha^\perp)}]}$ to H by $\delta_1 = \overline{\phi + \delta}$, where

$$\underline{x}\delta = \underline{x} \quad (\underline{x} \in R_\alpha^\perp \cap \overline{(R_\alpha + N_\alpha^\perp)}) \ .$$

Now, by Lemma 2.2, we may define a mapping $\varepsilon_3 \in E$ by $\varepsilon_3 = \delta_1 \oplus \delta_2$, where

$$\underline{x}\delta_2 = \underline{x} \quad (\underline{x} \in \mathbb{B}^{\perp}) \ .$$

Since $\mathbb{A}\phi$ is a closed linear subspace of $\mathbb{H}$ (ϕ being an isomorphism), we may define a mapping ε_4 from $\mathbb{H}$ to $\mathbb{H}$ by $\varepsilon_4 = \mu_1 \oplus \mu_2$, where

$$\underline{x}\mu_1 = \underline{x}\phi^{-1}\theta^{-1}\alpha \quad (\underline{x} \in \mathbb{A}\phi)$$

and

$$\underline{x}\mu_2 = \underline{x} \quad (\underline{x} \in (\mathbb{A}\phi)^{\perp}) \ .$$

Since $\mathbb{A}\phi \subseteq \mathbb{R}_\alpha^{\perp} \cap \overline{(\mathbb{R}_\alpha + \mathbb{N}_\alpha^{\perp})}$, we have that $(\mathbb{A}\phi)^{\perp} \supseteq \overline{\mathbb{R}}_\alpha \oplus (\mathbb{R}_\alpha^{\perp} \cap \mathbb{N}_\alpha)$. Thus, $\mathbb{R}_\alpha \subseteq (\mathbb{A}\phi)^{\perp}$ and so $\varepsilon_4 \in E$.

We now show that $\alpha = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$. To verify this, let $\underline{x}$ be any element of $\mathbb{H}$. Then $\underline{x} = \underline{n} + \underline{p}$ for some $\underline{n} \in \mathbb{N}_\alpha$ and some $\underline{p} \in \mathbb{N}_\alpha^{\perp}$. So

$$\begin{aligned}\underline{x}\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 &= (\underline{n}+\underline{p})\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = \underline{p}\varepsilon_2\varepsilon_3\varepsilon_4 = (\underline{p}\theta)\varepsilon_3\varepsilon_4 \\ &= (\underline{p}\theta\phi)\varepsilon_4 = (\underline{p}\theta\phi)\phi^{-1}\theta^{-1}\alpha = \underline{p}\alpha \\ &= \underline{n}\alpha + \underline{p}\alpha = (\underline{n}+\underline{p})\alpha = \underline{x}\alpha \ .\end{aligned}$$

<u>3.4 DEFINITION</u> Let $\alpha \in$ Sing. Define the <u>stable set</u> $\mathbb{X}_\alpha$ <u>of</u> α to be $\{\underline{x} \in \mathbb{H} : \underline{x}\alpha = \underline{x}\}$.

<u>3.5 LEMMA</u> Let $\alpha \in$ Sing. $\mathbb{X}_\alpha$ is a closed linear subspace of $\mathbb{H}$.

<u>PROOF</u> Since α is a linear mapping, $\mathbb{X}_\alpha$ is easily seen to be a linear subspace of $\mathbb{H}$. Now, let $(x_i)_{i\in\mathbb{N}}$ be a sequence in $\mathbb{X}_\alpha$

with limit $\underline{x}$ in $\mathcal{H}$. Then

$$\underline{x}\alpha = (\lim \underline{x}_i)\alpha = \lim (\underline{x}_i\alpha)$$

since α is continuous. But

$$\lim (\underline{x}_i\alpha) = \lim (\underline{x}_i) = \underline{x} \ .$$

So $\underline{x}\alpha = \underline{x}$, i.e. $\underline{x} \in \mathcal{X}_\alpha$.

<u>3.6 LEMMA</u> Let $\alpha \in \langle E\rangle$, then either $\dim \mathcal{N}_\alpha = \aleph_0$ or $\dim \mathcal{X}_\alpha^\perp < \aleph_0$.

<u>PROOF</u> The proof is by induction on the length, $\ell(\alpha)$, of α . We show first that the result holds for elements of length 1 , i.e. for elements of E . If $\alpha \in E$, then $\mathcal{X}_\alpha = \mathcal{R}_\alpha$ and, by Lemma 3.2, $\dim \mathcal{N}_\alpha = \dim \mathcal{R}_\alpha^\perp$. Thus $\dim \mathcal{N}_\alpha = \dim \mathcal{X}_\alpha^\perp$. Either $\dim \mathcal{N}_\alpha = \aleph_0$ or $\dim \mathcal{N}_\alpha < \aleph_0$. If the latter holds, then clearly $\dim \mathcal{X}_\alpha^\perp < \aleph_0$. Thus the result holds for elements of length 1 .

Now suppose the result holds for elements of $\langle E\rangle$ with length strictly less than n . Let $\eta \in \langle E\rangle$ with $\ell(\eta) = n$. Then $\eta = \tau\varepsilon$ where $\tau \in \langle E\rangle$, $\ell(\tau) = n - 1$ and $\varepsilon \in E$. Suppose $\dim \mathcal{N}_\eta < \aleph_0$. Let $\underline{x} \in \mathcal{R}_\varepsilon \cap \mathcal{X}_\tau$. Then $\underline{x}\tau = \underline{x}$ and $\underline{x}\varepsilon = \underline{x}$. So

$$\underline{x}\eta = (\underline{x}\tau)\varepsilon = \underline{x}\varepsilon = \underline{x} \ .$$

Thus $\underline{x} \in \mathcal{X}_\eta$. So $\mathcal{R}_\varepsilon \cap \mathcal{X}_\tau \subseteq \mathcal{X}_\eta$. Hence, by Lemma 1.11 and Lemma 1.18, $\mathcal{X}_\eta^\perp \subseteq \overline{\mathcal{R}_\varepsilon^\perp + \mathcal{X}_\tau^\perp}$. Thus, by Lemma 1.20,

$$\begin{aligned} \dim \mathcal{X}_\eta^\perp &\le \dim (\overline{\mathcal{R}_\varepsilon^\perp + \mathcal{X}_\tau^\perp}) = \dim (\mathcal{R}_\varepsilon^\perp + \mathcal{X}_\tau^\perp) \\ &\le \dim \mathcal{R}_\varepsilon^\perp + \dim \mathcal{X}_\tau^\perp \ . \end{aligned} \tag{+}$$

Now, $R_\eta = R_{\tau\varepsilon} \subseteq R_\varepsilon$. So $\dim R_\varepsilon^\perp \le \dim R_\eta^\perp = \dim N_\eta$ by Lemma 3.2. Now we have assumed that $\dim N_\eta < \aleph_0$, and so $\dim R_\varepsilon^\perp < \aleph_0$. Also, $\dim N_\tau \le \dim N_{\tau\varepsilon} = \dim N_\eta < \aleph_0$. So, by the induction hypothesis, $\dim X_\tau^\perp < \aleph_0$. But we have already shown (at (+)) that

$$\dim X_\eta^\perp \le \dim R_\varepsilon^\perp + \dim X_\tau^\perp .$$

Thus $\dim X_\eta^\perp < \aleph_0$ as required.

<u>3.7 LEMMA</u> Let $\alpha \in \text{Sing}$. If $\dim X_\alpha^\perp < \aleph_0$ and $\dim N_\alpha = \dim R_\alpha^\perp$, then $\alpha \in \langle E \rangle$.

<u>PROOF</u> We show first that the null-space of α is non-trivial and that the closed linear subspace $X_\alpha^\perp + (X_\alpha^\perp)\alpha$ is invariant under α .

Since $X_\alpha^\perp$ is a finite dimensional linear subspace of H , we have that $(X_\alpha^\perp)\alpha$ is a finite dimensional linear subspace of H . Thus $X_\alpha^\perp + (X_\alpha^\perp)\alpha$ is a finite dimensional linear subspace of H , and so is closed.

Now, let $\underline{v} \in X_\alpha^\perp + (X_\alpha^\perp)\alpha$. Then $\underline{v} = \underline{p} + \underline{p}'$ for some $\underline{p} \in X_\alpha^\perp$ and some $\underline{p}' \in (X_\alpha^\perp)\alpha$. Now, $\underline{p}' = \underline{x} + \underline{y}$ for some $\underline{x} \in X_\alpha^\perp$ and some $\underline{y} \in X_\alpha$. Thus

$$\underline{v} = \underline{p} + \underline{x} + \underline{y} .$$

Hence

$$\begin{aligned} \underline{v}\alpha &= \underline{p}\alpha + \underline{x}\alpha + \underline{y} \\ &= (\underline{p}+\underline{x})\alpha + (\underline{y}+\underline{x}) - \underline{x} \in (X_\alpha^\perp)\alpha + X_\alpha^\perp . \end{aligned}$$

Thus $X_\alpha^\perp + (X_\alpha^\perp)\alpha$ is invariant under α .

Now, let α_1 be the restriction of α to the closed linear subspace $X_\alpha^\perp + (X_\alpha^\perp)\alpha$. Then $\alpha = \alpha_1 \oplus \alpha_2$, where α_2 is defined by

$$\underline{x}\alpha_2 = \underline{x} \quad (\underline{x} \in [X_\alpha^\perp + (X_\alpha^\perp)\alpha]^\perp).$$

Now, suppose that $N_\alpha = \{\underline{0}\}$. Then, certainly, $N_{\alpha_1} = \{\underline{0}\}$ and so α_1 is an automorphism of $X_\alpha^\perp + (X_\alpha^\perp)\alpha$. Hence there exists a (group theoretic) inverse α_1^{-1} of α_1 such that $\alpha_1\alpha_1^{-1} = \alpha_1^{-1}\alpha_1$ and $\alpha_1\alpha_1^{-1}$ is the identity map on $X_\alpha^\perp + (X_\alpha^\perp)\alpha$. By defining $\alpha' = \alpha_1^{-1} \oplus \alpha_2$, we see that $\alpha\alpha'$ is the identity map on H . Since this contradicts the hypothesis that $\alpha \in \text{Sing}$, we have that $N_\alpha \neq \{\underline{0}\}$.

Since $N_\alpha \cap X_\alpha = \{\underline{0}\}$, we may define a mapping $\varepsilon \in E$ by $\varepsilon = \overline{(\gamma_1+\gamma_2)} \oplus \gamma_3$, where

$$\underline{x}\gamma_1 = \underline{0} \quad (\underline{x} \in N_\alpha) ,$$

$$\underline{x}\gamma_2 = \underline{x} \quad (\underline{x} \in X_\alpha)$$

and

$$\underline{x}\gamma_3 = \underline{x} \quad (\underline{x} \in (N_\alpha + X_\alpha)^\perp) .$$

By Lemma 1.11 and Lemma 1.12,

$$X_\alpha^\perp = [X_\alpha^\perp \cap N_\alpha^\perp] \oplus (X_\alpha^\perp \cap [X_\alpha^\perp \cap N_\alpha^\perp]^\perp)$$

$$= [X_\alpha + N_\alpha]^\perp \oplus (X_\alpha^\perp \cap \overline{[X_\alpha + N_\alpha]}) .$$

Thus we may define a map $\delta \in \text{Sing}$ by $\delta = \phi_1 \oplus \phi_2 \oplus \phi_3$, where

$$\underline{x}\phi_1 = \underline{x}\alpha \quad (\underline{x} \in [X_\alpha + N_\alpha]^\perp)$$

$$\underline{x}\phi_2 = \underline{0} \quad (\underline{x} \in X_\alpha^\perp \cap \overline{[X_\alpha + N_\alpha]})$$

and

$$\underline{x}\phi_3 = \underline{x} \quad (\underline{x} \in X_\alpha) .$$

We now show that $\alpha = \varepsilon\delta$. Let $\underline{y}$ be any element of H . Then $\underline{y} = \underline{n} + \underline{x} + \underline{p}$ for some $\underline{n} \in N_\alpha$, some $\underline{x} \in X_\alpha$ and some $\underline{p} \in (N_\alpha + X_\alpha)^\perp$. So

$$\underline{y}\varepsilon\delta = (\underline{n}+\underline{x}+\underline{p})\varepsilon\delta = (\underline{x}+\underline{p})\delta = \underline{x} + \underline{p}\alpha = \underline{x}\alpha + \underline{p}\alpha$$

since $\underline{x} \in X_\alpha$. So

$$\underline{y}\varepsilon\delta = (\underline{x}+\underline{p})\alpha = \underline{n}\alpha + (\underline{x}+\underline{p})\alpha = (\underline{n}+\underline{x}+\underline{p})\alpha = \underline{y}\alpha$$

since $\underline{n} \in N_\alpha$. Thus $\varepsilon\delta = \alpha$.

Now, let δ' be the restriction of δ to the closed linear subspace $X_\alpha^\perp + (X_\alpha^\perp)\alpha$. Since $X_\alpha^\perp + (X_\alpha^\perp)\alpha$ is invariant under α , we have that δ' is an endomorphism of $X_\alpha^\perp + (X_\alpha^\perp)\alpha$. Since $N_\alpha \neq \{\underline{0}\}$ and $X_\alpha \cap N_\alpha = \{\underline{0}\}$, we have that

$$X_\alpha^\perp \cap \overline{[X_\alpha + N_\alpha]} \neq \{\underline{0}\} ,$$

i.e.

$$N_{\delta'} \neq \{\underline{0}\} .$$

Since $X_\alpha^\perp + (X_\alpha^\perp)\alpha$ has finite dimension, n say, we have that $\delta' \in \mathrm{Sing}_n$. Hence, by Theorem 1.4.9, $\delta' = \varepsilon_1'\varepsilon_2'\ldots\varepsilon_m'$ where each ε_i' $(i = 1,2,\ldots,m)$ is an idempotent of Sing_n .

Now, since $H = [X_\alpha^\perp + (X_\alpha^\perp)\alpha] \oplus [X_\alpha^\perp + (X_\alpha^\perp)\alpha]^\perp$, we may define $\varepsilon_i : H \to H$ by $\varepsilon_i = \varepsilon_i' \oplus \iota$ $(i = 1,2,\ldots,m)$, where

$$\underline{x}\iota = \underline{x} \quad (\underline{x} \in [X_\alpha^\perp + (X_\alpha^\perp)\alpha]^\perp) .$$

Thus, $\delta = \varepsilon_1\varepsilon_2\ldots\varepsilon_m$ where each $\varepsilon_i \in E$. Hence $\alpha = \varepsilon\varepsilon_1\varepsilon_2\ldots\varepsilon_m \in \langle E\rangle$.

3.8 THEOREM Let $\mathcal{H}$ be a separable Hilbert space, Sing the set of singular continuous endomorphisms of $\mathcal{H}$ and E the set of idempotent elements of Sing . If $\alpha \in$ Sing , define X_α to be the set $\{\underline{x} \in \mathcal{H} : \underline{x}\alpha = \underline{x}\}$. Then $\langle E\rangle = I \cup F$ where

$$I = \{\alpha \in \text{Sing} : \dim N_\alpha = \dim R_\alpha^\perp = \aleph_0\}$$

and

$$F = \{\alpha \in \text{Sing} : \dim N_\alpha = \dim R_\alpha^\perp \text{ , } \dim X_\alpha^\perp < \aleph_0\} \text{ .}$$

PROOF By Lemma 3.3 and Lemma 3.7, we have $I \cup F \subseteq \langle E\rangle$.

Now, let $\alpha \in \langle E\rangle$. Then, by Lemma 3.2, $\dim N_\alpha = \dim R_\alpha^\perp$. Also, by Lemma 3.6, either $\dim N_\alpha = \aleph_0$ or $\dim X_\alpha^\perp < \aleph_0$, i.e. $\alpha \in I \cup F$. Thus $\langle E\rangle \subseteq I \cup F$, and so $\langle E\rangle = I \cup F$.

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