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The paper investigates prices and deadweight loss in multi-product monopoly (MPM) with linear interrelated demands and constant marginal costs. We show that with commonly used models for linear demand such as Bowley demand as well as for vertically or horizontally differentiated demand the price for each good is independent of demand cross-effects and of the characteristics and number of other goods. This contrasts with the prevailing view that prices critically depend on demand cross-effects. We also show that for these linear models the deadweight loss due to MPM monopoly amounts to half the total monopoly profit.

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1. Introduction

Most firms do not only sell one, but many interrelated products. For example supermarkets sell a multitude of substitute, complement or independent goods. Airlines and railway companies sell tickets with different conditions for the same route and oil corporations sell gasoline in petrol stations that differ by their locations.

This paper examines multi-product monopoly (MPM) facing linear demand for differentiated goods and constant unit costs and shows that optimal prices and welfare loss can be expressed in a very simple way: As in the textbook case of a single product monopoly, *the monopoly price of each good is the average of its own inverse demand intercept and its own marginal cost, and is thus independent of the characteristics of other products, the interactions between products and the number of products sold.* In contrast, a common view in economics as well as marketing is that monopoly prices critically depend on cross demand effects[2]. In particular, a somewhat prevalent misconception is that MPM prices should be lower for complements and higher for substitutes, relative to independent goods (see Section 2). Though intuitively plausible along a seemingly natural line of argument, this conclusion is actually invalid!

We obtain these elementary results for MPM facing three commonly used linear demand structures, corresponding to the three examples cited above: The standard Bowley (1924) or Shubik (1959) models for demand with heterogeneous products, vertically (quality) differentiated products and horizontally (spatially) differentiated products. As seen below, this multitude of demand models is motivated by the diversity

[2] For example Reibstein and Gatignon (1984) argue in a seminal marketing paper that "The optimal price is extremely sensitive to the inclusion or the exclusion of the cross-elasticities" (p.266).

of economic settings where the issue of MPM pricing has been historically analyzed, often by founding fathers of modern industrial economics.

Our result on MPM prices is interesting in its own right, but it can also be used to address a number of further questions. In particular, we show that this result helps to solve the complex but important problem of deadweight loss in multiproduct monopoly. In a single product monopoly with linear demand deadweight loss is half the monopoly profit. Exploiting the property that prices of existing goods do not change when a new product is added to a product line, we can show that monopoly profit and the deadweight loss always rise proportionally. Consequently deadweight loss in MPM will also be half the monopoly profit regardless of how many, or what types of, products are added. This surprising result holds in all the models of linear demand we consider, despite fundamentally different welfare functions.

Based on an extensive literature search going back to the beginnings of neo-classic economics, we believe that these simple properties of linear MPM have not been fully uncovered.[3] While the complete solution of MPM pricing seems to have largely eluded economists' attention, some features of our results have emerged in the marketing literature. Shugan and Desiraju (2001) show that monopoly prices of two vertically

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[3] The first formal analysis of the MPM goes back to Wicksell (1901, 1934). Edgeworth (1925) analyzes railway fares of different classes but does not give explicit solutions. Hotelling (1932) provides a numerical example with first and second-class railway tickets. Robinson (1933) formally solves the problem of a monopolist selling in different markets, but explicitly excludes price interdependence such as in "the case of first- and third-class railway fares, analyzed by Edgeworth" (Robinson, 1933, p. 181). Coase (1946) goes beyond Robison's analysis and examines monopoly prices for two interrelated products using verbal and graphical arguments but again does not provide a mathematical solution. Holton (1957) considers the MPM problem of a supermarket selling interrelated products arguing that, "supermarket operators do indeed establish prices with not only price elasticities but cross elasticities in mind". Finally Selten (1970) formally addresses the problem of a multiproduct monopolist facing linear demand but does not provide the simple properties of monopoly prices.

differentiated products do not depend on each other's costs. Moorthy (2005) and Besanko et al. (2005) show that with linear demand MPM prices do not respond to cost changes of other products. Neither of these studies derives the general solution for prices or explores the full scope of MPM pricing.[4]

The present results may be useful in various contexts. A direct practical implication is that even in the presence of strong product interactions, neglecting such relations is part of good pricing practice for a monopolist. Cross-subsidization will not be optimal for a non-regulated monopolist under linear demand (Baumol, Panzar and Willig, 1982).[5] Besides its managerial relevance this insight also provides a theoretical justification for research in industrial organization and quantitative marketing that analyzes retail prices in a single product context, a priori for tractability reasons.

Our welfare results could serve as a simple and practical benchmark helping antitrust authorities estimate the social loss of MPM. If demand functions can be considered as approximately linear, it is not necessary to analyze in detail every product's price elasticity and cross-elasticities. The social cost of MPM can simply be estimated by looking at the firm's profit. For example this approach could help to evaluate the social cost of a local retail monopoly. Likewise the deadweight loss caused by a railway monopoly can be estimated from the firm's profit without having to analyze the qualities and prices of the different tickets offered.

[4] Several authors have pointed out the similarity of the MPM problem to the Ramsey tax (Ramsey 1927), which maximizes social welfare for a certain level of tax revenue. Yet the Ramsey problem is not identical to unconstrained monopoly profit maximization. For example Ten Raa (2009) shows that the structure of monopoly prices often differs from that induced by the Ramsey tax.

[5] Note that it might be optimal to price goods below marginal costs if their demand intercept is negative i.e. if they cannot be sold independently without complement goods.

In the present setting, social efficiency can be restored by a standard per-unit subsidy scheme, the implementation of which is facilitated by two helpful features. The scheme is based on minimal informational requirements and the firm has a direct incentive to truthfully disclose its unit costs to the policy maker.

Finally our results have implications for joint profit maximization by oligopoly firms. Jointly maximizing the total profit is mathematically equivalent to the MPM price problem. Our findings indicate that with linear demand, even if products exhibit strong interdependence, oligopoly firms do not need any information about their competitors' products and costs in order to set the prices that jointly maximize the total profit.

The paper is organized as follows: Section 2 illustrates our main result with a simple example and explains misconceptions regarding monopoly prices. Section 3 shows that MPM prices are independent of product interactions for general linear MPM. In Section 4, this result is applied to three common models of demand with interdependent products. Section 5 analyzes the relation between the deadweight loss and monopoly profits. Section 6 examines an efficiency-restoring subsidy scheme. Section 7 briefly concludes.

2. Illustrative two-good example

An elementary fallacy in basic monopoly theory holds that a firm selling two complementary products will charge less for each than when the two products are sold independently. Alternatively, it claims that the monopoly price of a given good is lower when it is sold alone than in case it is sold together with a substitute. In its most succinct form, this invalid view can be presented within the standard two-good paradigm.

Consider a representative consumer with utility function $U(x_1,x_2) = a(x_1 + x_2) - 0.5b(x_1^2 + x_2^2) + gx_1x_2 + y$, where y is income, and $|g| < b$. This gives rise to the standard symmetric inverse demand function $p_i = a - bx_i + gx_j$ (Bowley, 1924). The direct demand is then $x_i = a/(b - g) - (bp_i + gp_j)/(b^2 - g^2)$. As in Singh and Vives (1984) or Amir and Jin (2001), this can be written as $x_i = \alpha - \beta p_i + \gamma p_j$, with $\alpha = a/(b - g)$, $\beta = b/(b^2 - g^2)$ and $\gamma = -g/(b^2 - g^2)$. However, it is important to observe that the constants α, β and γ are not autonomous. In contrast, a, b and g are, but for the restriction that $|g| < b$.

Using the demand functions in the form $x_i = \alpha - \beta p_i + \gamma p_j$, $i, j = 1, 2$, and unit cost c for both products, one obtains both monopoly prices as $p^* = 0.5[c + \alpha/(\beta - \gamma)]$. Then, so goes the fallacy, this price is higher with substitute goods ($\gamma > 0$) and lower with complements ($\gamma < 0$), relative to the case of independent goods ($\gamma = 0$). This would be correct if α and β remained constant when γ changes, which however is not the case. Indeed, using the relations between Greek and Roman letters, we obtain $\alpha/(\beta - \gamma) = a$, which is the intercept of inverse demand, or the consumer's willingness to pay at zero consumption i.e., $\partial U(0,0)/\partial x_i$, indeed a primitive constant.[6] Hence, expressing the prices with the parameters of the inverse demand function, we obtain $p^* = 0.5(c + a)$, which is also the optimal price for a monopolist selling only good i (facing inverse demand $p_i = a - bx_i$ and unit cost c). With linear demand, a monopolist selling two goods sets each price as if it were the only good sold. In other words, pricing is completely independent of the (substitute/complement) relationships between the two goods. A basic intuition for this result is given in the more formal treatment of Section 4.1.

In fact this property extends to some settings without linear demand functions. For instance, consider the inverse demand function $p_i = a - bx_i^{\sigma} - rx_i^{0.5(\sigma-1)}x_j^{0.5(\sigma+1)}$, $\sigma > 0$. The

[6] In contrast, α is the quantity demanded under zero prices. In the presence of substitutes (complements) one would expect it to be lower (higher), relative to the case of independent products.

optimal price is $(\sigma a + c)/(1 + \sigma)$, which is the same as in the case where each product is sold in a separate market with independent demand function $p_i = a - bx_i^{\sigma}$.

On the other hand, the monopolist's optimal outputs do depend on product relationships. Indeed, to maximize total monopoly profit, the optimal output is given by $x^* = 0.5(a - c)/(b - g)$, which is higher for complements ($g > 0$) and lower for substitutes ($g < 0$), relative to independent products ($g = 0$). When two perfect substitutes are sold (i.e. $g = -b$) we obtain the optimal output as half of the usual monopoly optimal output. This is all fully in line with standard economic intuition.[7]

With the two standard ways of solving the monopoly problem using direct or indirect demand functions, the simple expression for optimal prices can be easily overseen. Allen (1938) uses the direct demand function to solve for optimal prices in the linear two-good example but fails to see that the solution can be simplified if expressed in terms of the parameters of the indirect demand function. Selten (1970, p. 52) discovered that in a general linear MPM the optimal quantities can be "expressed in a surprisingly simple way" as half of the socially optimal level[8] but he too does not derive our solution for monopoly prices. Selten's result does not yield simple comparative statics, as optimal quantities still depend in a complex way on cross demand effects.

[7] If output is expressed in terms of the parameters of the direct demand function we obtain $x^* = 0.5[\alpha - (\beta - \gamma)c]$, which seems to imply that output is higher for "substitutes" ($\gamma > 0$). In particular, if two goods are close to perfect substitutes, the monopoly will sell at least twice as much as when it sells one good alone, clearly violating standard intuition.

[8] A similar result has been obtained by Ramsey (1927) for revenue maximizing taxes in a competitive market with linear demand, but the connection to Selten's result has not been recognized. In fact the similarity breaks down when marginal costs are not constant.

Similarly, when the indirect demand functions are used to solve for the optimal monopoly quantities, our simple solutions will appear only if one then substitutes these quantities back into the demand function. Varian (2006, p. 455) uses this approach for solving a numerical example without identifying the simple formula for prices.

Another line of reasoning often used to justify this invalid view in settings with general non-linear demand is as follows. For such settings, one derives the optimal Lerner index, $(p_i - c_i)/p_i$, as $1/\varepsilon_{ii} - \sum_{j\neq i}(p_j - c_j)x_j\varepsilon_{ij}/(\varepsilon_{ii}p_ix_i)$, where ε_{ii} and ε_{ij} are the price elasticity and cross-elasticity.[9] When every cross elasticity ε_{ij} is zero, we obtain the single-good monopoly condition $(p_i - c_i)/p_i = 1/\varepsilon_{ii}$. When the goods are substitutes, we have $\varepsilon_{ij} < 0$, and the price p_i would appear to be higher than the corresponding price in a separate (single-good) market.[10] Again this argument would be correct if the elasticity ε_{ii} were to remain the same as new products are added. In the presence of substitute goods, the quantity demanded for a given product will fall and the value of ε_{ii} will rise. A higher ε_{ii} offsets the impact of ε_{ij}'s, and pushes p_i in the opposite direction. To further elaborate on this point, we provide in the Appendix an explicit example with a non-linear demand function with the property that complements are optimally priced higher (not lower) than the corresponding independent goods would be.

On a historical note, this invalid view might have partly originated from the fact that the term "monopoly" had in earlier days also been used to refer to multiple firms

9 See e.g., Tirole (1988, p. 70).

10 One such example is in Betancourt (2004, p. 94), which carries out this analysis and concludes: *"The latter implies that if two items are gross substitutes (ε_{21} >0), the price of item 1 will be higher in the multiproduct setting than it would have been in the single product one* (…)*. On the other hand if they are gross complements* (ε_{21} *<0), the price of item 1 in the multiproduct setting will be lower than it would have been in the single product one."*

selling differentiated goods. For instance, comparing a multiproduct monopoly to what we now call a duopoly selling complementary or substitute products, Cournot (1838) and Allen (1938) refer to this duopoly as "two independent monopolists" (Allen, 1938, p. 361; Cournot, 2001, p. 80). Indeed, relative to monopoly prices, prices set by a duopoly are lower for substitutes and usually higher for complements, but this is not the question addressed in this paper. Instead, we compare the monopoly price of a given product when sold alone to its price when sold together with a complement or substitute by the same monopolist.

3. Monopoly prices in a general linear model

We start by analyzing MPM pricing for a general linear demand model, which will be shown in Section 4 to encompass three commonly used but quite different linear demand structures. We refrain at this stage from specifying a precise microeconomic model of interdependent demand, as we know of none that nests all of our applications.

We consider a monopoly firm selling n products with constant marginal costs. Prices, quantities and marginal cost are denoted by p_i, x_i, and c_i respectively, $i = 1,\ldots, n$. The corresponding vectors for all n products are written as bold $\boldsymbol{p}$, $\boldsymbol{x}$ and $\boldsymbol{c}$. The linear demand function is specified by a constant Jacobian matrix $\partial\boldsymbol{x}/\partial\boldsymbol{p} = \boldsymbol{A}$, and a constant $n\times 1$ vector $\boldsymbol{\alpha}$, representing the vector of quantity demanded when all prices are zero, as

$$\boldsymbol{x}(\boldsymbol{p}) = \boldsymbol{\alpha} + \boldsymbol{A}\boldsymbol{p} \tag{1}$$

When $\boldsymbol{p} = \boldsymbol{c}$, we get the socially optimal output $\boldsymbol{x}(\boldsymbol{c})$. We assume $\boldsymbol{A}$ is negative definite and symmetric, i.e., its elements $a_{ij} = a_{ji}$ for all i, j. The diagonal elements of $\boldsymbol{A}$ are all negative, i.e., $a_{ii} < 0$ for all i, as the demand for every good is downward sloping.

The off-diagonal elements, however, can be positive, negative or zero, according to products being substitutes, complements or independent. Given the demand function (1) and marginal costs vector $\boldsymbol{c} = (c_1,\ldots, c_n)$, we can write the monopoly profit as

$$\pi(\boldsymbol{p}) = (\boldsymbol{p} - \boldsymbol{c})'(\boldsymbol{\alpha} + \boldsymbol{A}\boldsymbol{p}) \qquad (2)$$

We will demonstrate that the monopoly prices can be expressed in a simple way using the vector of demand intercepts $\boldsymbol{p}^0$, which is the (minimal) price vector that exactly reduces demand for all products to zero. As matrix $\boldsymbol{A}$ is invertible, this vector is uniquely defined by $\boldsymbol{x}(\boldsymbol{p}^0) = \boldsymbol{\alpha} + \boldsymbol{A}\boldsymbol{p}^0 = \boldsymbol{0}$, i.e., $\boldsymbol{p}^0 = -\boldsymbol{A}^{-1}\boldsymbol{\alpha}$.

PROPOSITION 1: The profit-maximizing prices for MPM with constant marginal costs, facing a linear demand function with a symmetric and negative definite Jacobian matrix, can be expressed as $\boldsymbol{p}^* = 0.5(\boldsymbol{p}^0 + \boldsymbol{c})$ *where* $\boldsymbol{p}^0$ *is the demand intercept. Under these prices only half of the socially optimal quantity of every good is sold.*

Proof: see Appendix A.

The impact of all the parameters of the demand function (1) on the monopoly price vector $\boldsymbol{p}^*$ is summarized in the demand intercept $\boldsymbol{p}^0$. This is similar to the single product case with linear demand, where the monopoly price does not depend on the slope. Hence, Proposition 1 can be interpreted as a generalized version of the solution to a single-good or a two-good monopoly (see Section 2). It implies that, in general linear monopoly, only 50% of a product's cost change is passed on to its price, and that a cost change for one product does not affect the prices of other products. This conclusion corroborates the findings by Moorthy (2005) and Besanko et al. (2005).

The optimal price vector satisfies the key property at hand - independence of inter-product relations (substitutes or complements) - whenever $\boldsymbol{p}^0$ has the same property. The latter in turn depends on the microeconomic model invoked to derive demand. As seen in Section 2, for the standard quadratic utility with two products, $\boldsymbol{p}^0$ indeed has the desired property. In Section 4, we argue that this property holds for three widely used distinct models of product differentiation in industrial organization.

4. Independent pricing in three linear models

In this section we show that the result derived in Section 3 can be applied to MPM facing three commonly used models of linear demand for differentiated products, each having its own microeconomic foundation. In all three cases we obtain simple profit-maximizing prices, which are independent of other products and product relations.

4.1 Bowley demand function for heterogeneous products

We first look at one of the standard models for heterogeneous products, a generalized Bowley-type demand with a mixture of substitute, complement and independent goods. The representative consumer's utility function is $h + \boldsymbol{a}'\boldsymbol{x} - 0.5\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x}$, where h is the numeraire good whose price is 1, $\boldsymbol{x}$ is the consumption bundle of the monopoly products, $\boldsymbol{a}$ is an $n\times1$ positive vector and $\boldsymbol{B}$ is an $n\times n$ matrix. Without loss of generality, let $\boldsymbol{B}$ be symmetric. We assume it to be positive definite so that the utility function is concave. The consumer chooses $\boldsymbol{x}$ to maximize utility subject to a budget constraint $h + \boldsymbol{p}'\boldsymbol{x} = m$. The first-order condition of utility maximization, $\boldsymbol{a} - \boldsymbol{B}\boldsymbol{x} - \boldsymbol{p} = \boldsymbol{0}$ yields the demand function:

$$\boldsymbol{x}(\boldsymbol{p}) = \boldsymbol{B}^{-1}(\boldsymbol{a} - \boldsymbol{p}) \tag{3}$$

This demand function (3) follows our general version (1) with $\boldsymbol{B}^{-1}\boldsymbol{a} = \boldsymbol{\alpha}$, and $\boldsymbol{B}^{-1} = -\boldsymbol{A}$. To ensure an interior solution we require the following condition.

ASSUMPTION 1: $\boldsymbol{B}^{-1}(\boldsymbol{a} - \boldsymbol{c}) > \boldsymbol{0}$.

Assumption 1 implies that when all prices are equal to marginal costs, demand is positive for every product. This ensures that the demand function (3) is valid under the monopoly price. As $\boldsymbol{B}$ is symmetric and positive definite the demand function (3) satisfies the requirement of Proposition 1. The vector $\boldsymbol{p}^0$ of maximum prices is easy to determine. As $\boldsymbol{B}^{-1}$ has full rank, $\boldsymbol{x}(\boldsymbol{p})$ is zero when $\boldsymbol{a} - \boldsymbol{p} = \boldsymbol{0}$, so $\boldsymbol{p}^0 = \boldsymbol{a}$.

PROPOSITION 2: The MPM price for good i is $p_i^* = 0.5(c_i + a_i)$.

As a_i and c_i pertain only to good i, the inter-product relationships do not affect the optimal price. Observing that a_i can be interpreted as the marginal utility of product i when consumption is zero, it is intuitive that it should not depend on product relations. If the monopolist can estimate the value of a_i, he can choose the optimal price of good i easily, independently of how many goods he sells and how large their cross-elasticities are, in full contrast to the conclusions reached by Holton (1957).

If the monopoly sells each good in an independent market, good i's demand function will reduce to $(a_i - p_i)/b_i$ and the optimal price will be $0.5(c_i + a_i)$, which is identical to the MPM price. So the MPM achieves optimal price coordination when it acts as if it were selling n products in n separate markets. Product interdependence does not have any influence whatsoever on optimal pricing.

We now provide a simple economic intuition behind this result. Adding a complement (substitute) good generates two interacting and opposite effects. The first is that the demand for an existing product shifts out (in), thereby pushing the price higher (lower). However, a higher (lower) price would lower the demand for the new product.

A special feature of the linear model is that these two effects exactly cancel each other out, leading to the stand-alone optimal monopoly prices (this is not robust for non-linear cases, as may be seen in the example in the Appendix). Typically, the elementary error in the literature was accompanied by a flawed intuition that limited consideration only to the second effect above, completely ignoring the first countervailing effect.

From a quantitative standpoint, to see why the two effects exactly cancel out, we show that, under linear demand, the condition MR = MC remains valid under the original price. With just one product, $\mathrm{MR}_1 = p_1 - x_1 \dfrac{\partial p_1}{\partial x_1}$. With n products, $\mathrm{MR}_1 = p_1 - \sum_{i=1}^{n} x_i \dfrac{\partial p_i}{\partial x_1}$. The inverse demand function implies $\sum_{i=1}^{n} x_i \dfrac{\partial p_i}{\partial x_1} = \sum_{i=1}^{n} b_{i1} x_i = a_1 - p_1$. So for the same p_1, $\mathrm{MR}_1 = c_1$ holds, implying thereby that there is no need to change p_1.

One cannot a priori apply Proposition 2 to situations such as "*first- and third-class railway fares*" analyzed by Edgeworth (1925) and Hotelling (1932). Demand for vertically differentiated products relies on different micro foundations wherein the demand intercept $\boldsymbol{p}^0$ might not be independent of inter-product relations. Nevertheless, we now show that our result does cover vertically differentiated products.

4.2 Vertically differentiated products

We consider a model of n (≥ 2) vertically differentiated products where product i has quality q_i.[11] Without loss of generality, let $q_{i+1} > q_i$ for all i, so that q_n indicates the highest quality and q_1 the lowest. There is a continuum of consumers indexed by θ, which

[11] See Mussa and Rosen (1978), Gabszewicz and Thisse (1979), and Shaked and Sutton (1982).

is uniformly distributed on [0, 1]. Each consumer θ purchases at most one good. If he buys good i at price p_i, he obtains a surplus of $\theta q_i - p_i$. Each consumer chooses the product with the highest surplus, provided it is non-negative. Mussa and Rosen (1978) study the monopoly price problem in a more general model where the monopolist also determines quality. They do not obtain explicit price solutions given non-linear demand.

We need some basic assumptions. To ensure that every product has positive demand, we assume that the marginal cost $c(q_i)$ of a product of quality q_i increases with q_i, while the consumer benefit increases more (the latter is to avoid some technical difficulties). In addition, marginal cost increases with quality at an increasing rate.

ASSUMPTION 2: *For any q,* $0 < c'(q) < 1$ *and* $c''(q) > 0$.

We can determine the demand for a given good with quality q_i by identifying the highest and lowest type of consumers buying this good. The marginal consumer who is indifferent between buying product 1 and buying nothing gets a surplus $\theta_1 q_1 - p_1 = 0$, so all consumers with an index lower than $\theta_1 \equiv p_1/q_1$ will not buy any product. For consumer θ_i indifferent between buying products i and $i-1$ we have $\theta_i q_{i-1} - p_{i-1} = \theta_i q_i - p_i$, so $\theta_i \equiv (p_i - p_{i-1})/(q_i - q_{i-1})$. If $\theta_i < \theta_{i+1}$ for all $i < n$, and $\theta_n < 1$, we obtain positive demand for all goods as $x_i = \theta_{i+1} - \theta_i$ for $i < n$ and $x_n = 1 - \theta_n$. We will show that these conditions hold at the MPM prices. Substituting θ_is into these demand functions we get:

$$x_1 = \frac{p_2 - p_1}{q_2 - q_1} - \frac{p_1}{q_1}, \qquad x_n = 1 - \frac{p_n - p_{n-1}}{q_n - q_{n-1}},$$

$$x_i = \frac{p_{i+1} - p_i}{q_{i+1} - q_i} - \frac{p_i - p_{i-1}}{q_i - q_{i-1}} \quad \text{for } 1 < i < n \tag{4}$$

(4) is linear and has a symmetric Jacobian matrix, which is negative definite so Proposition 1 applies. To find the MPM price the only information we need is the vector of demand intercepts $\boldsymbol{p}^0$. One can see that the demand for each product is zero when $p_i = q_i$ for all i. So $\boldsymbol{p}^0$ is equal to the vector of product qualities $\boldsymbol{q}$. Given Assumption 2 price vector $0.5(\boldsymbol{q} + \boldsymbol{c})$ lead to positive demand for all products.

PROPOSITION 3: The price for good i in MPM with vertically differentiated products is $p_i^* = 0.5(c_i + q_i)$.

Proof: see Appendix B.

The monopoly price is simply the average of a product's quality and cost. It is independent of other products' characteristics. Hence, the prices for "*first- and third-class railway fares*" only depend on the quality and cost of the service offered, not on those of other classes. Again the prices are the same as the single good monopoly price, i.e. the price if the monopoly only offers one class of tickets. In this case demand is $x_i = 1 - p_i/q_i$, and the optimal price is $0.5(c_i + q_i)$, which is identical to the MPM price.

According to Proposition 1 the monopoly only sells half the quantities sold in a competitive market. In a model of vertically differentiated products every consumer acquires at most one product. This means that compared to a competitive market, in monopoly some consumers switch to lower quality goods and in total fewer consumers will be served. While in the previous model each consumer buys half of the quantity of the social optimum, here the number of customers being served falls by half.

4.3 Horizontally differentiated products

We finally analyze a model of spatially (horizontally) differentiated products. The Hotelling model and its various extensions have been widely used to analyze oligopoly competition and location choices. Tirole (1988) discusses spatial discrimination by a monopolist selling one product (Tirole, 1988, p. 140). However, little seems to be known about how a monopolist sets prices for a fixed number of horizontally differentiated products with predetermined locations. We will show that these prices are again independent of the features of other products.

We construct an extended version of the Hotelling model[12], which can be nested in our linear framework. Our model can be visualized as a star-shaped city with n (≥ 2) selling locations owned by a monopolist. The city has $n - 1$ roads radiating from the center and stretching indefinitely into suburbs. There is one shop at the city center and one branch shop along each road with one unit distance from the center. We do not address the question of how to choose locations but simply examine how a MPM sets profit-maximizing prices at these different shops. We assume that the central shop offers consumers a value v_1 at a price p_1, while branches offer v_i at p_i, for $i > 1$. Consumers reside along each road with uniform density. Each consumer incurs a unit travel cost τ, and maximizes his surplus $v_i - p_i - \tau s$, where s is distance.[13]

To ensure an interior solution where every shop has a positive demand under MPM prices, we need certain conditions. On the one hand the shops' net values need to be

[12] For the original Hotelling model with two firms located at the ends of [0, 1], the monopoly prices are $p_i = 0.25(3v_i + c_i + v_j - c_j - 2\tau)$, which are not independent, as the monopolist tries to take all surplus from the marginal consumers and there is no interior solution for prices. However, if the line extends beyond point 1, it becomes our special case with n = 2. If both sides extend, the prices are independent too.

[13] Chen and Riordan (2007) analyze an oligopoly version of this model with full symmetry across firms, and hence no firm at the center. Since their model covers the Hotelling one as a special case, price independence does not apply.

sufficiently high relative to the travel cost so that all consumers between the center and branch shops are covered. On the other hand, the differences between the net values of the center and branch shops should be sufficiently small so that every shop can sell something. These requirements lead to the following conditions.

ASSUMPTION 3. $|v_1 - c_1 - v_i + c_i| < \tau < 0.2(v_1 - c_1 + v_i - c_i)$ *for all* $i > 1$.

In equilibrium no shop can charge a price higher than the value it offers, so we have $p_i < v_i$ for all i. If a branch shop can sell anything, we must have $v_i - p_i + \tau > v_1 - p_1$. Under these conditions we can derive the demand functions by identifying marginal consumers indifferent between buying at the center or a branch shop and those indifferent between a branch and buying nothing. For the former marginal consumers, we have $v_1 - p_1 - \tau y_i = v_i - p_i - \tau(1 - y_i)$, where y_i is the distance to the center. Thus demand for the central shop $y_i = 0.5(v_1 - p_1 + p_i - v_i + \tau)/\tau$. Shop i serves the remaining $1 - y_i$ customers, but also attracts clients from the suburb up to a distance z_i, which is determined by $v_i - p_i - \tau z_i = 0$, so $z_i = (v_i - p_i)/\tau$. If $0 < y_i < 1$ for all $i > 1$, the demand function for the center $x_1 = \sum_{i=2}^{n} y_i$, and for branch shop i, $x_i = 1 - y_i + z_i$, i.e.

$$x_1 = \frac{n-1}{2\tau}(\tau + v_1 - p_1) - \sum_{i=2}^{n} \frac{v_i - p_i}{2\tau},$$

$$x_i = \frac{\tau + 3v_i - 3p_i - v_1 + p_1}{2\tau} \qquad \text{for } i > 1. \tag{5}$$

Again (5) is linear in prices and the Jacobian matrix is symmetric. We also prove that this matrix is negative definite. One can verify that the demand for every good is zero when $p_1^0 = v_1 + 2\tau$ and $p_i^0 = v_i + \tau$ for any $i > 1$.[14] With these demand intercepts we can apply Proposition 1 and obtain the monopoly prices. Assumption 3 ensures that the demands for all products are positive under these prices.

PROPOSITION 4: The MPM prices with horizontally differentiated products are

$$p_1^* = 0.5(v_1 + c_1) + \tau, \text{ and } p_i^* = 0.5(v_i + c_i + \tau) \text{ for } i > 1.$$

Proof: see Appendix C.

The monopoly prices cannot be characterized by a single formula here, as the center shop differs from the others. Nonetheless, all prices again only depend on shop-specific parameters, not on other shops' values or costs. In fact, this property can be generalized to a model with different distances between the center and branch shops.[15]

As in the other cases, every shop sells only half of the socially optimal quantity. This is surprising as the market always covers all consumers between the center and branch shops. Only suburban residents stop buying due to monopoly pricing.

In the previous two models, every price is equal to the "naïve" monopoly prices, charged in independent markets or for a single product monopoly. In this case, if a branch shop is the only seller along its road, its price would be $0.5(v_i + c_i + \tau)$, which is again exactly the MPM price. However, if the central shop is the only seller, its price would be $0.5(v_1 + c_1)$, lower than the MPM price by τ. This result indicates that the MPM price is not always equal to separate monopoly prices. Nonetheless, for $n \geq 2$, the introduction of

[14] These hypothetical prices lie outside the permissible price range as demand should vanish when $p_i \geq v_i$.
[15] If we let s_i be the distance between the center and shop i, and normalize the average distance to 1, p_i^* will change slightly, with τ multiplied by $(s_i + 2)/3$, while p_1^* remains the same.

any new product/road will not affect the existing prices. In this sense we can still say that the MPM prices are independent of each other.

5. Welfare loss

Estimating the deadweight loss in MPM with complex product relations might at first sight appear quite challenging. In this section we will show that the optimal prices determined in the previous section can be used to establish a simple relation between deadweight loss and monopoly profits. Since profits are usually observable, this relation provides an easy way to estimate the social loss caused by MPM.

Again our result can be understood as a generalization of a well-known property of the textbook single-product monopoly with linear demand: Deadweight loss equals half the monopoly profit. This relation remains valid in our three MPM models. This is unexpected because the welfare functions are fundamentally different across the three models and unlike the profit function cannot be presented in a unified framework.

In the first model we substitute demand $\boldsymbol{x} = \boldsymbol{B}^{-1}(\boldsymbol{a} - \boldsymbol{p})$ to express consumer surplus as $0.5(\boldsymbol{a} - \boldsymbol{p})'\boldsymbol{x}$. With vertically differentiated products consumers purchasing product i obtain utility $q_i \int_{\theta_i}^{\theta_{i+1}} \theta d\theta = 0.5 q_i(\theta_{i+1}^2 - \theta_i^2)$, with $\theta_{n+1} \equiv 1$. The total utility of all consumers is $0.5 \sum_1^n q_i\ (\theta_{i+1}^2 - \theta_i^2)$. In the case of horizontally differentiated products, along each road i, the utility from the center is $\int_0^{y_i} (v_1 - \tau s) ds = v_1 y_i - 0.5\tau y_i^2$, where y_i is the demand for the center. Consumers who purchase from shop i obtain utility $\int_0^{1-y_i} (v_i - \tau s) ds$ +

$\int_0^{z_i}(v_i - \tau s)ds = v_i(1-y_i) - 0.5\tau(1-y_i)^2 + v_i z_i - 0.5\tau z_i^2$, where z_i equals $(v_i - p_i)/\tau$. Adding them together we obtain the utility along road i.

We can show that under monopoly pricing, the consumer surplus is equal to half of the profit in the first and second models, and is half of the profit $- n\tau$ in the third model. Furthermore, we find a uniform relation between deadweight loss and the monopoly profits, identical to the single product case, despite the fundamental structural differences in the welfare functions explained above.

PROPOSITION 5: In all three MPM models, the deadweight loss is equal to half the monopoly profit.

Proof: see Appendix D.

The simple relationship known from the linear single product monopoly survives in multiproduct monopoly. The intuition can be best explained for the first model. In the single good case, the profit rectangle is twice the deadweight loss triangle with the same height and base. With multi-products, when all outputs proportionally increase beyond the monopoly levels, the marginal utility of each good falls, differently from its demand curve, but still linearly. When all goods double, the marginal utility is equal to the marginal cost for every good. Again we have a triangle with the same height and base as the profit. So the monopoly profit is twice the deadweight loss, for each good as well as for the total. As long as demand and cost functions are all linear, the relation between the deadweight loss and firm profits remains unchanged regardless of how many products or what types of goods are introduced.[16]

6. Efficiency-restoring subsidies

Monopoly leads to insufficient levels of production, a social inefficiency that might be reversed by negative taxation, i.e. government subsidy (see e.g., Amir, Maret and Troege, 2004). In theory a multi-product monopoly can be induced to produce socially optimal output levels, if the government subsidizes sales. This government expenditure can then be offset by a lump-sum tax, e.g. in the form of a franchise fee, if a balanced budget is desired. To this end, two practical problems need to be solved: 1) find the optimal subsidies, 2) find the right level of the lump-sum tax. Without knowing the market conditions such as the demand functions, both problems present serious challenges to the policy maker. Demsetz' (1968) idea of franchise bidding can take away the monopoly profit, but cannot eliminate the inefficiency. In the present linear model, however, we can achieve the efficient outcome with limited information.

We assume a multi-product monopolist sets optimal prices as described in Section 4.1. The policy maker can observe the price p_i and cost c_i of each product, and know the total profit π. Then the policy maker can set a unit subsidy $s_i = 2(p_i - c_i)$ for every product. So the monopolist's perceived net unit cost becomes $c_i - s_i = 3c_i - 2p_i$. (We assume the subsidy is paid according to sales, so the monopolist cannot make profits without sales when the net cost is negative.) Given this subsidy s_i, the monopoly price p_i' will be 0.5(a_i

[16] Again linearity is not always necessary. For example, given the inverse demand function in section 2, the deadweight loss is equal to the profit multiplied by $\sigma/(1 + \sigma)$, if either one or two goods are sold.

$+ c_i - s_i) = 0.5(a_i + 3c_i - 2p_i)$. Substituting the monopoly price $p_i = 0.5(a_i + c_i)$, the new price will be equal to the (original) marginal cost c_i, i.e., $p_i' = c_i$.

Given marginal cost pricing, consumers will demand the socially optimal output levels, which are twice the monopoly quantities. For each unit of a product, the profit is just the subsidy $s_i = 2(p_i - c_i)$, which is twice the monopoly price margin. Therefore, the new total profit is four times of the original one, i.e. $\pi' = 4\pi$. This should be charged as the franchise fee, in which case the monopoly's net profit is zero, the government budget is balanced, and consumers receive maximal surplus, as in perfect competition.

PROPOSITION 6: With a subsidy $s_i = 2(p_i - c_i)$ for each product and a lump-sum tax 4π, we can restore the competitive outcome with a balanced budget.

The information requirements for this subsidy scheme are relatively low. Given the principal result of our paper, it will not come as a surprise that the subsidy does not require estimates of cross-elasticities. However, even the additional information that is necessary to determine this subsidy can be easily obtained, if the government can observe the behavior of the un-subsidized MPM. In particular, non-subsidized prices are usually observable and information about profit is available to the tax authority. Only cost information tends to be imperfectly known to the policy maker, and can thus be potentially misreported by the monopolist (this feature typically leads to the analysis of a game of asymmetric information in related settings, e.g., Laffont and Tirole, 1993). Fortunately, this needs not be the case here, provided only the total subsidy is bounded from above by the lump-sum tax corresponding to four times the profit of the non-subsidized monopoly. Indeed, we show that, even if the monopolist is free to distort information about costs to determine the subsidy level, it will truthfully report its costs.

PROPOSITION 7: *Given the above subsidy/lump-sum-taxation scheme, with the total subsidy limited to the lump-sum tax, then the monopolist has an incentive to truthfully report its costs.*

Proof: see Appendix E.

While this solution is ideal for consumers, it may be too harsh for the monopolist. If instead the government prefers to allow a positive profit, it may choose a subsidy $s_i = 2r(p_i - c_i)$, with $0 \leq r \leq 1$. Then the new price $p_i' = 0.5(a_i + c_i) - r(p_i - c_i) = 0.5[a_i + c_i - r(a_i - c_i)]$. Substituting this into the demand function $\boldsymbol{x} = \boldsymbol{B}^{-1}(\boldsymbol{a} - \boldsymbol{p}')$, the corresponding demand vector is $0.5(1 + r)\boldsymbol{B}^{-1}(\boldsymbol{a} - \boldsymbol{c})$, which is $(1 + r)$ times the monopoly demand. The unit profit for each product is $p_i' - c_i + s_i = 0.5(1 + r)(a_i - c_i) = (1 + r)(p_i - c_i)$. So the total profit is $(1 + r)^2$ times the original one, i.e. $(1 + r)^2\pi$. But the total subsidy is only $2r(1 + r)\pi$. Hence, if the franchise fee is equal to the subsidy, the monopoly still retains a positive profit of $(1 - r^2)\pi$. Of course this solution is not socially optimal as p_i' is higher than c_i, unless $r = 1$. The remaining welfare loss is then $0.5(1 - r)^2\pi$. Summarizing, we have established the following result.

PROPOSITION 8: *With a subsidy* $s_i = 2r(p_i - c_i)$ *and a lump-sum tax* $2r(1 + r)\pi$, *the monopoly profit is* $(1 - r^2)\pi$ *and welfare loss is* $0.5(1 - r)^2\pi$.

Notice that the welfare loss is no longer half of the profit, but $0.5(1 - r)/(1 + r)\pi$. The government can choose r to balance social efficiency and monopoly profit.

7. Concluding remarks

The paper analyzes pricing and welfare effects of MPM with linear demand and cost functions. Our main result is that the MPM price for each good depends only on the marginal cost and the inverse demand intercept of that good, with the relationship to, and the number of, other goods being immaterial. This conclusion is at odds with much literature, old and new in industrial organization and marketing, stressing the role of substitute/complement products and cross-elasticities in MPM pricing.

Our pricing result can be used to show that deadweight loss in monopoly is half of the MPM profit. In other words, relations known from the simple one-product textbook linear model generalize verbatim to three workhorse linear models of interdependent products: heterogeneous, vertically and horizontally differentiated.

Due to their basic nature, the results presented here can be relevant in a wide range of contexts, covering theoretical and policy issues in fields as different as antitrust theory, regulation, spatial economics and marketing. Our welfare results are potentially useful in regulatory design. We have demonstrated how these results can not only be used to estimate deadweight loss in complex situations, but also help to design simple welfare restoring subsidy schemes with low informational requirements.

While we limited our analysis mainly to linear demand, the examples in the paper indicate that our main insight is not limited exclusively to linear demand, which is special only insofar as it leads to the two effects of adding a substitute or complement product to an existing product line being clearly identified and exactly canceling out. We hope that this paper might lead to renewed interest in the topic of monopoly pricing, which was addressed by economists in the early years, but seems to have been largely ignored in recent times.

Appendix A: We differentiate the profit function (2), $(\boldsymbol{p} - \boldsymbol{c})'(\boldsymbol{\alpha} + \boldsymbol{Ap})$, and obtain $d\pi/d\boldsymbol{p} = \boldsymbol{\alpha} + \boldsymbol{A}'(\boldsymbol{p} - \boldsymbol{c}) + \boldsymbol{Ap} = \boldsymbol{0}$. Since $\boldsymbol{A}$ is symmetric, we have $\boldsymbol{\alpha} + \boldsymbol{A}(2\boldsymbol{p} - \boldsymbol{c}) = \boldsymbol{0}$. The Hessian matrix of the profit function is equal to $2\boldsymbol{A}$, which is negative definite, so the second-order condition holds. Then the optimal price can be solved from the first-order condition, as $\boldsymbol{p}^* = 0.5(\boldsymbol{c} - \boldsymbol{A}^{-1}\boldsymbol{\alpha})$.

If we plug $-\boldsymbol{A}^{-1}\boldsymbol{\alpha}$ into the demand function (1), we get $\boldsymbol{x} = \boldsymbol{0}$. So $-\boldsymbol{A}^{-1}\boldsymbol{\alpha}$ is the demand intercept vector $\boldsymbol{p}^0$. The optimal price $\boldsymbol{p}^*$ can be written as $0.5(\boldsymbol{c} + \boldsymbol{p}^0)$.

Putting $\boldsymbol{p}^*$ into the demand function (1), we get $\boldsymbol{x}^* = 0.5(\boldsymbol{\alpha} + \boldsymbol{Ac})$. When $\boldsymbol{p} = \boldsymbol{c}$, we get the socially optimal output $\boldsymbol{\alpha} + \boldsymbol{Ac}$, which is twice of the monopoly output $\boldsymbol{x}^*$. ||

Appendix B: (i) We first show that Jacobian matrix $\partial\boldsymbol{x}/\partial\boldsymbol{p}$ is symmetric and negative definite. As $\partial x_i/\partial p_{i+1} = 1/(q_{i+1} - q_i) = \partial x_{i+1}/\partial p_i$ for all i, and $\partial x_i/\partial p_j = 0$ for any $j \neq i$ and $|j - i| > 1$, the matrix is indeed symmetric.

To show it is negative definite, we see the sum of the first row of $\partial\boldsymbol{x}/\partial\boldsymbol{p}$ is equal to $-1/q_1$, and the sum of every other row is zero. Hence the matrix has a quasi-dominant diagonal and must be negative definite (McKenzie 1960, Theorem 2).

To complete the proof, we need to show $\boldsymbol{x}(\boldsymbol{c}) > \boldsymbol{0}$. For $x_1 \geq 0$, we need to show $\frac{c_2 - c_1}{q_2 - q_1} \geq \frac{c_1}{q_1}$, or $\frac{c_2}{q_2} \geq \frac{c_1}{q_1}$. This holds since $c''(q) > 0$. For $x_n \geq 0$, we must have $c_n - c_{n-1} \leq q_n - q_{n-1}$. This is true given $c'(q) < 1$.

For $1 < i < n$, $x_i \geq 0$ holds if $\frac{c_{i+1} - c_i}{q_{i+1} - q_i} \geq \frac{c_i - c_{i-1}}{q_i - q_{i-1}}$. To prove this, we write $c_{i+1} - c_i$ as $(q_{i+1} - q_i)c'(\omega_i)$ and $c_i - c_{i-1} = (q_i - q_{i-1})c'(\omega_{i-1})$, where $q_{i-1} \leq \omega_{i-1} \leq q_i \leq \omega_i \leq q_{i+1}$. As $c''(q) > 0$, $\omega_{i-1} \leq \omega_i$, we get $c'(\omega_i) \geq c'(\omega_{i-1})$, so $x_i \geq 0$.

Finally, we show that no consumer receives a negative surplus under $\boldsymbol{p}^*$. The marginal consumer buying from good 1 receives a zero surplus. For $i > 1$, the marginal consumer $\theta_i = (p_i - p_{i-1})/(q_i - q_{i-1})$, receives a positive surplus if $\theta_i q_i \geq p_i$ or $p_i q_{i-1} \geq p_{i-1} q_i$. Using p_i^* and p_{i-1}^*, it becomes $c_i/q_i \geq c_{i-1}/q_{i-1}$, which holds given $c''(q) > 0$. ||

Appendix C: (i) We first show the Jacobian matrix $\partial \boldsymbol{x}/\partial \boldsymbol{p}$ is symmetric and negative definite. As $\partial x_i/\partial p_1 = \partial x_1/\partial p_i = 0.5/\tau$ for all $i > 1$, and $\partial x_i/\partial p_j = 0$ for i and $j \neq 1$, it is indeed symmetric. Moreover since $\partial x_1/\partial p_1 = -0.5(n-1)/\tau$, $\partial x_i/\partial p_i = -1.5/\tau$ for $i > 1$, the sum of the first row of $\partial \boldsymbol{x}/\partial \boldsymbol{p}$ is 0, and the sum of any other row is $-1/\tau < 0$. By McKenzie (1960), $\partial \boldsymbol{x}/\partial \boldsymbol{p}$ must be negative definite.

We then need to show $\boldsymbol{x}(\boldsymbol{c}) > \boldsymbol{0}$. For $x_1 \geq 0$, it suffices to show $\tau + v_1 - c_1 \geq v_i - c_i$. For $x_i \geq 0$, we need $v_i - c_i + 3\tau \geq v_1 - c_1$. Assumption 3 guarantees both of them.

Finally, every marginal consumer must receive a non-negative surplus. For a consumer indifferent between the center and shop i, her surplus from the center is $v_1 - p_1 - \tau y_i = v_1 - p_1 - 0.5(v_1 - p_1 - v_i + p_i + \tau) = 0.25(v_1 - c_1 + v_i - c_i - 5\tau)$. It is positive given Assumption 3. A marginal consumer outside of shop i receives a zero surplus. ||

Appendix D: (i) Heterogeneous goods: As $\boldsymbol{p}^* = 0.5(\boldsymbol{a} + \boldsymbol{c})$, we get CS* $= 0.5(\boldsymbol{a} - \boldsymbol{p}^*)'\boldsymbol{x}^* = 0.5(\boldsymbol{p}^* - \boldsymbol{c})'\boldsymbol{x}^* = 0.5\pi^*$, Social welfare is CS* + $\pi^* = 1.5\pi^*$. The maximum welfare is $0.5(\boldsymbol{a} - \boldsymbol{c})'\boldsymbol{x} = 2(\boldsymbol{p}^* - \boldsymbol{c})'\boldsymbol{x}^* = 2\pi^*$. So the deadweight loss is $0.5\pi^*$.

(ii) Vertically differentiated products: We write the twice utility $2u = \sum_1^n q_i\,(\theta_{i+1}^2 - \theta_i^2)$. Regrouping the summation items yields $q_n - q_1\theta_1^2 - \sum_2^n \theta_i^2\,(q_i - q_{i-1})$. As $\theta_1 \equiv p_1/q_1$ and $\theta_i \equiv (p_i - p_{i-1})/(q_i - q_{i-1})$, this becomes $q_n - \theta_1 p_1 - \sum_2^n \theta_i\,(p_i - p_{i-1})$. Regrouping the summation again, this changes to $q_n + \sum_1^{n-1} p_i\,(\theta_{i+1} - \theta_i) - \theta_n p_n$. As $\theta_{i+1} - \theta_i = x_i$, we get $2u = \sum_1^n p_i x_i + q_n - p_n$. We write $q_n - p_n = q_n - \sum_2^n (p_i - p_{i-1}) - p_1 = q_n - \sum_2^n \theta_i (q_i - q_{i-1}) - \theta_1 q_1$. Regrouping the summation items, we get $q_n - p_n = q_n + \sum_1^{n-1} q_i\,(\theta_{i+1} - \theta_i) - \theta_n q_n = \sum_1^n q_i x_i$. So $u = 0.5\sum_1^n (p_i + q_i)x_i = 0.5(\boldsymbol{p} + \boldsymbol{q})'\boldsymbol{x}$.

As $\boldsymbol{p}^* = 0.5(\boldsymbol{q} + \boldsymbol{c})$, CS* $= u - \boldsymbol{p}'\boldsymbol{x} = 0.5(\boldsymbol{q} - \boldsymbol{p})'\boldsymbol{x} = 0.5(\boldsymbol{p}^* - \boldsymbol{c})'\boldsymbol{x}^* = 0.5\pi^*$. Thus social welfare under monopoly is $1.5\pi^*$. The maximum welfare CS($\boldsymbol{c}$) is $0.5(\boldsymbol{q} - \boldsymbol{c})'\boldsymbol{x} = 2(\boldsymbol{p}^* - \boldsymbol{c})'\boldsymbol{x}^* = 2\pi^*$. So the deadweight loss due to monopoly prices is $0.5\pi^*$.

(iii) Horizontally differentiated products: The utility obtained by consumers along one road is $u_i = (v_1 - 0.5\tau y_i)y_i + [v_i - 0.5\tau(1 - y_i)](1 - y_i) + [v_i - 0.5\tau z_i]z_i$, where $z_i = (v_i - p_i)/\tau$ and $y_i = 0.5(v_1 - p_1 - v_i + p_i + \tau)/\tau$. Hence $CS_i = u_i - p_1 y_i - p_i(1 - y_i + z_i)$

$= [3(v_1 - p_1) + v_i - p_i - \tau]y_i/4 + [3(v_i - p_i) + v_1 - p_1 - \tau](1 - y_i)/4 + (v_i - p_i)z_i/2$

$= (v_1 - p_1)y_i/2 + (v_i - p_i)(1 - y_i + z_i)/2 + (v_1 - p_1 + v_i - p_i - \tau)/4.$

$= 0.5(v_1 - p_1 + 2\tau)y_i + 0.5(v_i - p_i + \tau)(1 - y_i + z_i) - \tau.$

Under monopoly prices, $v_1 - p_1 + 2\tau = p_1^* - c_1$ and $v_i - p_i + \tau = p_i^* - c_i$, so $CS = 0.5\pi^* - n\tau$ and social welfare is $1.5\pi^* - n\tau$. When $\boldsymbol{p} = \boldsymbol{c}$, the maximum $CS_i(\boldsymbol{c}) = (v_1 - c_1 + 2\tau)y_i^* + (v_i - c_i + \tau)(1 - y_i^* + z_i^*) - \tau$, so the maximum welfare is $2\pi^* - n\tau$. Hence the deadweight loss due to monopoly prices must be $0.5\pi^*$. ||

Appendix E: Let d_i be the reported cost for each product by the monopolist. Then the government sets $s_i = 2(p_i - d_i)$, so the new monopoly price $p_i' = 0.5(a_i + c_i - s_i) = d_i$, and the price margin $p_i' - c_i + s_i = a_i - d_i$. The corresponding demand is $\boldsymbol{B}^{-1}(\boldsymbol{a} - \boldsymbol{d})$. Then the total profit will be $(\boldsymbol{a} - \boldsymbol{d})'\boldsymbol{B}^{-1}(\boldsymbol{a} - \boldsymbol{d})$. The total subsidy will be $(\boldsymbol{a} + \boldsymbol{c} - 2\boldsymbol{d})'\boldsymbol{B}^{-1}(\boldsymbol{a} - \boldsymbol{d})$.

Let the monopolist choose $\boldsymbol{d}$ to maximize $L = (\boldsymbol{a} - \boldsymbol{d})'\boldsymbol{B}^{-1}(\boldsymbol{a} - \boldsymbol{d})$, subject to the total subsidy not exceeding 4π, i.e. $W = (\boldsymbol{a} + \boldsymbol{c} - 2\boldsymbol{d})'\boldsymbol{B}^{-1}(\boldsymbol{a} - \boldsymbol{d}) - (\boldsymbol{a} - \boldsymbol{c})'\boldsymbol{B}^{-1}(\boldsymbol{a} - \boldsymbol{c}) \leq 0$.

If $\boldsymbol{d} = \boldsymbol{c}$, we have $L = W = 0$. Now we proceed by contradiction and let $\boldsymbol{d} = \boldsymbol{c} + \boldsymbol{\Delta}$ such that $L > 0$ and $W \leq 0$. Using the Taylor expansion for L and W around $\boldsymbol{c}$, we need

$$L = -2\boldsymbol{\Delta}'\boldsymbol{B}^{-1}(\boldsymbol{a} - \boldsymbol{c}) + \boldsymbol{\Delta}'\boldsymbol{B}^{-1}\boldsymbol{\Delta} > 0 \quad \text{and} \quad W = -3\boldsymbol{\Delta}'\boldsymbol{B}^{-1}(\boldsymbol{a} - \boldsymbol{c}) + 2\boldsymbol{\Delta}'\boldsymbol{B}^{-1}\boldsymbol{\Delta} \leq 0$$

For $W \leq 0$, we need $3\boldsymbol{\Delta}'\boldsymbol{B}^{-1}(\boldsymbol{a} - \boldsymbol{c}) \geq 2\boldsymbol{\Delta}'\boldsymbol{B}^{-1}\boldsymbol{\Delta}$, which is possible only when $\boldsymbol{\Delta}'\boldsymbol{B}^{-1}(\boldsymbol{a} - \boldsymbol{c}) > 0$. But then for $L > 0$, we need $\boldsymbol{\Delta}'\boldsymbol{B}^{-1}\boldsymbol{\Delta} > 2\boldsymbol{\Delta}'\boldsymbol{B}^{-1}(\boldsymbol{a} - \boldsymbol{c})$. There is a contradiction. Hence it is optimal to declare true cots, i.e. $\boldsymbol{d} = \boldsymbol{c}$. ||

Appendix F: The following is an example of MPM pricing with non-linear demand, showing that the conventional wisdom may actually be strictly reversed.

Consider a two-good monopolist with zero costs facing a demand function $p_i = a - bx_i^\sigma - rx_j$, $i, j = 1, 2$, $\sigma > 0$. Profit is $\pi = (a - bx_1^\sigma - rx_2)x_1 + (a - bx_2^\sigma - rx_1)x_2$.

The first-order condition for x_i implies: $a - b(\sigma + 1)(x^*)^{\sigma} - 2rx^* = 0$. (i)

The monopoly price is $p^* = a - b(x^*)^{\sigma} - rx^* = \sigma b(x^*)^{\sigma} + rx^*$. (ii)

In two independent markets, demand and profit are:

$$p_i = a - bx_i^{\sigma} \qquad \text{and} \qquad \pi = (a - bx_i^{\sigma})x_i.$$

The first order condition for x_i is $a - b(\sigma + 1)(x^0)^{\sigma} = 0$. (iii)

The separate price satisfies $p^0 = a - b(x^0)^{\sigma} = \sigma b(x^0)^{\sigma}$. (iv)

Subtracting (iv) from (ii), we get: $p^* - p^0 = \sigma b[(x^*)^{\sigma} - (x^0)^{\sigma}] + rx^*$. (v)

Subtracting (i) from (iii), we get: $b(\sigma + 1)[(x^*)^{\sigma} - (x^0)^{\sigma}] = -2rx^*$. (vi)

Substituting (vi) into (v), we obtain:

$$p^* - p^0 = \frac{1-\sigma}{1+\sigma} rx^*. \qquad \text{(vii)}$$

If $\sigma < 1$, (vii) implies that $p^* > p^0$ if goods are substitutes ($r > 0$) and $p^* < p^0$ if goods are complements ($r < 0$), which is in line with conventional wisdom.

However, if $\sigma > 1$, (vii) implies that $p^* > p^0$ if goods are complements ($r < 0$), and $p^* < p^0$ if goods are substitutes ($r > 0$), in total violation of conventional wisdom. ||

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