QUINE AND BOOLOS ON SECOND-ORDER LOGIC : AN EXAMINATION OF THE DEBATE

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Quine and Boolos on Second-Order Logic:
An Examination of the Debate

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Abstract

The aim of this thesis is to examine the debate between Quine and Boolos over the logical status of higher-order logic—with Quine taking the position that higher-logic is more properly understood as set theory and Boolos arguing in opposition that higher-order logic is of a genuinely logical character. My purpose here then will be to stay as neutral as possible over the question of whether or not higher-order logic counts as logic and to instead focus on the exposition of the debate itself as exemplified in the work of Quine and Boolos.

Chapter I will be a detailed consideration of Quine's conception of logic and its place within the wider context of his philosophy. Only once this backdrop is in place will I then examine his views on higher-order logic. In Chapter II, I turn to Boolos's response to Quine—his attempt to examine the extent to which we may want to count higher-order logic as logic and the extent to which we may want to count it as set theory. With each point Boolos raises, I attempt to give what I think would have been Quine's reply. Finally, in Chapter III, I consider Boolos's attempt to show that monadic second-order logic (MSOL) should be understood as pure logic as it does not commit us to the existence of classes, as we may take the standard interpretation of MSOL to do. I discuss here some of the major reactions to Boolos's plural interpretation (Resnik, Parsons, and Linnebo), and conclude with more speculative remarks on what Quine's own response might have been. Throughout this thesis, my primary method has been one of close textual analysis.

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Introduction

The aim of this thesis is to examine the debate between Quine and Boolos over the logical status of higher-order logic—with Quine taking the position that higher-logic is more properly understood as set theory and Boolos arguing in opposition that higher-order logic is of a genuinely logical character. Many philosophers, especially since Boolos's 1975 paper "On Second-Order Logic", have taken sides in the debate but very little, if any work has been done to explain what Quine's and Boolos's respective positions actually were. This has been particularly detrimental, I think, to our understanding of Quine where the exposition of his views on higher-order logic often gets little beyond alluding to his well-known aphorism that higher-order logic is set theory in sheep's clothing. My purpose here then will be to stay as neutral as possible over the question of whether or not higher-order logic counts as logic and to instead focus on the exposition of the debate itself as exemplified in the work of Quine and Boolos. Though, admittedly, through my work on this thesis, I have become increasingly convinced of the strength of Quine's position and that the burden of proof, so to speak, lies with those who want to claim that higher-order logic is logic in the same sense that first-order logic is.

Chapter I will focus on the exposition of Quine's conception of logic, particularly as he presents it in § 33 of *Word and Object*, "Aims and Claims of Regimentation". My aim will be to explicate how Quine conceives of logic from within his overall philosophical project of arriving at the clearest, simplest understanding of the world. I will then turn more directly to consider his criticisms of higher-order logic, again emphasizing how he develops these criticisms from within the broader framework of what he sees as the task of philosophy generally. Another related feature of Quine's conception of logic that I hope to bring out in this section is the way in which he relies on a fairly intuitive and traditional characteristic of logic, that it is self-evident or obvious, so as to carve out a body of theory that we can reasonably identify as logic. In many ways this tradition of logic is pre-Fregean though Frege himself also appealed to it when he offered a foundation for arithmetic grounded in logic. The aim will be to show that Quine's position is at least well-motivated and interesting even if we should ultimately decide to reject it. In fact, that we may decide not to accept Quine's characterization of logic will be consistent with what he claims to have shown in drawing the boundary of logic at first-order quantification theory. Ultimately, I think Quine is rejecting the view that we need to supply logic with a general philosophical account needs of it.

Chapter II will consider in detail the criticisms Boolos's 1975 criticisms of Quine's position and the kinds of responses that Quine could have offered. In many places it will seem that the two philosophers are talking past one another; that their respective positions in a sense rules out any common ground for debate. Yet, we will continue to see how Quine's position gives him a way to respond to nearly all of Boolos's criticisms. It seems that here Boolos is at a disadvantage. He appeals to Frege and the logicist tradition to suggest that higher-order logic is indeed well within the bounds of what may be traditionally thought of as logic. Though Boolos gives up the view that logic in this tradition has any claim to epistemic privilege; it suffers from the same epistemic debilities as set theory. Furthermore, he admits that higher-order logic is committed to the existence of sets, or at least subsets of the domain. It is hard to see how granting either of these features does not further Quine's own view of higher-order logic as a substantial mathematical theory, too substantial to be thought of as pure logic. A better strategy for Boolos might have been to reject this standard view of higher order logic as we will see him do in Chapter III.

This final chapter will consider Boolos's plural interpretation of monadic second-order logic (MSOL). Here, he appears to confront Quine more directly in that this interpretation of MSOL apparently shows a way in which higher-order logic makes no additional ontological commitments to sets and can be understood in terms of our ordinary English plural locutions. Because Quine himself wrote almost nothing on Boolos's plural interpretation for MSOL, this chapter takes on a slightly different structure than the previous two. It in includes a discussion of the major reactions to Boolos's plural interpretation, such as those found in the work of Michael Resnik, Charles Parsons, and Øystein Linnebo and concludes with some more speculative remarks on what Quine's own thoughts on this matter might have been. Throughout this thesis, my primary method has been one of close textual analysis.

Before we begin I would like to highlight two topics that I have not discussed outright in this thesis but that I think are implicitly at play throughout much of what follows. One is the issue of what is to count as set theory, or a set theory. Much of the debate as it is presented here is roughly over whether higher-order logic is to be counted as logic or as set theory. I have focused mostly on Quine's and Boolos's differing views of what logic is. However, it seems to me that they are also at odds over what is to count as set theory. Thus, the debate between them could have been approached from this side as well. The other is the issue of the relationship between logic and model theory and what the relevance of the one to the other is, if any thing at all. At least in his 1975 paper, Boolos's arguments often rely heavily on the model theory for higher-order logic, a topic with which Quine often shows minimal concern. I highlight this topic not merely because of its relevance to the Quine-Boolos debate over higher-order logic, but also because it is a topic currently receiving a great deal of consideration throughout the literature on the development of analytic philosophy and its interaction with mathematical logic.[1]

Unfortunately, I leave both of these issues for some future work.

[1] For an overview of the debate and references see Juliet Floyd, "Frege, Semantics, and the Double-Definition Stroke," in *The Story of Analytic Philosophy*, Anat Biletzki and Anat Matar, eds. (London: Routledge, 1998), pp. 141-66.

Chapter I: Quine on [Higher-Order] Logic

In his contribution to the 1988 Washington University conference on Quine's philosophy, Burton Dreben remarked of Quine's 1932 Ph.D. dissertation "The Logic of Sequences: a Generalization of *Principia Mathematica*", that the generalization itself is unimpressive as a technical contribution to the development of mathematical logic (a view with which Quine agrees), but "[t]he true significance of the dissertation lies elsewhere: It shows in full the independence and force of mind, the special, if not unique, logical concerns, and the deepest philosophical impulses that have characterized and governed Quine to this very day."[1] The aim of the present chapter is to bring out what these "special, if not unique, logical concerns" are in the context of Quine's criticisms of higher-order logic. What we will see is that the issue for Quine is in a sense very much a terminological one. Quine's aim in logic is clarity; once we have made explicit what we are doing when we do logic, how we apply the label 'logic' itself is of little consequence. However, being a terminological issue does not make it a trivial issue for the pursuit of clarity runs deep throughout Quine's philosophy. Indeed it could be identified as the driving force behind his entire philosophical outlook.

The reason these two aims are not to be distinguished is because part of simplifying and clarifying our science, our understanding of the world, is the extent to which we are able to paraphrase our scientific theory into a canonical notation that makes explicit the ontological commitments of the theory and the logical relations among its sentences, in addition to removing ambiguities of ordinary language from the theory generally. In section **I**, I will attempt to show how Quine implements this strategy in his chief philosophical work, the 1960 *Word and Object*.[2]

[1] Burton S. Dreben, "Quine," in *Perspectives on Quine*, Robert B. Barrett and Roger F. Gibson, eds. (Oxford: Basil Blackwell, 19??), p. 81; Quine's dissertation was published years later as W.V. Quine, *The Logic of Sequences: A Generalization of Principia Mathematica*, (New York: Garland Publishers, 1990); see in particular Quine's preface.
[2] W.V. Quine, *Word and Object* (Cambridge: MIT Press, 1960).

In sections **II.i** and **II.ii**, I will turn directly to Quine's criticisms of higher-order logic, particularly as they appear in his 1970 *Philosophy of Logic*[3] by addressing his attitude towards quantification over predicate letters, the completeness theorem, and his characterization of logical truths as obvious. Again we will see that Quine's aim is to dispel confusions and to make assumptions explicit. For if regimentation into canonical notation is to clarify and simplify our scientific theorizing, the canonical notation itself must aspire to this aim of clarity and simplicity. What we will see is that Quine's philosophy of logic gives expression to a certain Quinean view that philosophy is science gone self-reflective. In both science and philosophy (and so in logic), our aim is understanding, and the means for achieving this is to strive for the simplest and clearest systematization of the world we are capable of constructing.

I

Quine begins §33 of his *Word and Object* entitled "Aims and Claims of Regimentation" reflecting upon the useful purpose served by practical temporary departures from ordinary language.[4] These departures achieve many advantages but most important among them are understanding of the referential work of language, clarification of our conceptual scheme, and simplification of theory. Consider, for example, the use of parentheses. Quine remarks that to limit their value only to the resolution of ambiguities of grouping fails to recognize their far-reaching importance as they also allow for the iteration of identical constructions without requiring repeated variation of their expression so as to maintain grouping. In this way,

[3] W.V. Quine, *Philosophy of Logic*, 2nd ed. (Cambridge: Harvard University Press, 1986).

[4] W.V. Quine, *Word and Object*, pp. 157-61. In developing my understanding of Quine's conception of logic, I am much indebted to Thomas Ricketts's papers "Frege, Carnap, and Quine: Continuities and Discontinuities" in *Carnap Brought Home: The View from Jena*, Steve Awodey and Carsten Klein, eds. (Chicago and LaSalle: Open Court Publishing Company, 2004), pp. 181-202; and "Languages and Calculi" in *Logical Empiricism in North America*, Gary L. Hardcastle and Alan W. Richardson, eds. (Minneapolis: University of Minnesota Press, 2004), pp. 257-80.

parentheses allow us to minimize the number of basic constructions and the techniques required for their employment. They also allow for the possibility of subjecting both long and short expressions to a uniform algorithm and to argue by substitution of long expressions for short ones and vice versa without forcing a readjustment of context. Were it not "for parentheses or some alternative convention yielding the foregoing benefits," he remarks, "mathematics would not have come far."[5]

On Quine's view, the introduction of the logical notation for the truth-functional connectives and the quantifier-variable notation for generality not to differ in kind from other linguistic innovations used to simplify or clarify scientific theory. In fact, given these devices, augmented with classes and the predicate "$\in$" for class membership, Quine holds that his canonical notation is sufficient for the regimentation of the sentences of any scientific theory and the demonstration of the logical relationships between these sentences once so regimented.[6] Additionally, the quantifier-variable notation provides an objective standard by which to judge the ontological commitments of a particular theory, a significant advance over ordinary language with its tendency towards nominalization. The ontological commitments of a theory regimented into canonical notation are displayed by the range of the values of the bound variables.[7] For Quine, this simplification and clarification by means of a canonical logical notation is continuous with the aims of scientific theory generally:

> The same motives that impel scientists to seek ever simpler and clearer theories adequate to the subject matter of their special sciences are motives for simplification and clarification of the broader framework shared by all the sciences. Here the objective is called philosophical, because of the breadth of the framework concerned; but the motivation is the same. The quest of a simplest, clearest overall pattern of canonical notation is not to be distinguished from a quest of ultimate categories, a limning of the most

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[5] Ibid., p. 158; for a similar account see also "Logic as a Source of Syntactical Insights," in *The Way of Paradox and Other Essays*, rev. ed. (Cambridge: Harvard University Press, 1976), p. 44. An alternate convention is found in the logical notation used by the Polish logicians.

[6] W.V. Quine, "The Scope and Language of Science," in *Ways of Paradox*, p. 242-44.

[7] W.V. Quine, "On What There Is," in *From a Logical Point of View*, 2nd rev. ed. (Cambridge: Harvard University Press, 1980), pp. 12-19; "Scope and Language," pp. 242-44; *Word and Object*, pp. 242-43.

> general traits of reality. Nor let it be retorted that such constructions are conventional affairs not dictated by reality; for may not the same be said of a physical theory?[8]

Quine illustrates this project vividly in his review of Strawson's *Introduction to Logical Theory* by considering the situation of a formal logician who is also a scientist or mathematician.[9] The logician-scientist's interest in ordinary language lies in its utility for the pursuit of his scientific aims. If a departure from ordinary language will improve upon this utility, the logician-scientist has no qualms about doing so. He may, for example, introduce the notation '$\supset$' to replace 'if-then' of his ordinary language knowing full-well that '$\supset$' does not capture exactly what 'if-then' did, but such is not the purpose of the new notation. The scientist-logician is never under the illusion that his aim is synonymy between the ordinary and extraordinary language. Rather, '$\supset$' is meant to simplify the theory by increasing perspicuity and adding algorithmic facility. So long as '$\supset$' is fully adequate to fulfill the role that 'if-then' originally played in the scientific work at hand, the scientist-logician can get along without this more cumbersome piece of ordinary language.[10] Quine compares this technique to that of paraphrasing in ordinary language so as to remove ambiguities, the difference being that there the purpose is to facilitate communication and here it is the application of logical theory. However, for both purposes he sees synonymy not just as unnecessary, but as wholly misplaced:

> In neither case is synonymy to be claimed for the paraphrase. Synonymy, for sentences generally, is not a notion that we can readily make adequate sense of; and even if it were, it would be out of place in these cases. *If we paraphrase a sentence to resolve ambiguity, what we seek is not a synonymous sentence, but one that is more informative by dint of resisting some alternative interpretations.*[11]

Quine's comparison of his use of logical notation to that of paraphrasing in ordinary language should be emphasized as a distinctive philosophical departure from his predecessors in the

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[8] Quine, *Word and Object*, p. 161.
[9] P.F. Strawson, *Introduction to Logical Theory* (New York: Wiley, 1952); W.V. Quine, "Mr. Strawson on Logical Theory," in *Ways of Paradox*, p. 150.
[10] Ibid., p. 150.
[11] Quine, *Word and Object*, p. 159. (My emphasis)

analytic tradition as is brought out particularly well by contrasting his view of the analysis of the ordered pair with that of Russell.[12]

Quine explains that the ordered pair appears as a sort of defective noun for it has the peculiar feature of allowing two objects to be treated as one. In previous cases, he determined that defective nouns, such as attributes or propositions, proved undeserving of their claim to denote objects and so dismissed them as irreferential components of their containing phrases. Unlike these earlier cases though, the particular feature that makes the ordered pair seem defective is the very feature that gives it its utility; it is essential to its purpose that the ordered pair be treated as a single object. For example, this feature allows relations to be assimilated to classes by construing them as classes of ordered pairs. Without its objectual status, the ordered pair would be ineligible for class membership. The aim of philosophical analysis as Quine conceives of it is to make sense of how it is that the ordered pair can stand as a single object.

He explains that mathematicians have introduced the ordered pair by way of the condition

(1) If $<x,y> = <z,w>$ then $x = z$ and $y = w$

And hence, any already recognized object fulfilling this condition will fulfill the role of the ordered pair. Norbert Wiener offered the first such analysis in February 1914 defining the ordered pair as the class $\{\{x\}, \{y, \emptyset\}\}$.[13] Kazimierz Kuratowski followed in 1921 with the now more standard $\{\{x\}, \{x, y\}\}$. Both versions fulfill (1) while maintaining the important feature

[12] Ibid., § 53.
[13] Though arguably Felix Hausdorff might have produced $\{\{x, 1\}, \{y, 2\}\}$ first; see Akihiro Kanamori, "The Empty Set, the Singleton, and the Ordered Pair," *Bulletin of Symbolic Logic* 9(3) 2003, p. 291 n. 33.

that the apparent two objects of the pair are treated as a single object, a class.[14] Wiener's definition, Quine explains, captures what he sees as central to the task of philosophy:

> This construction is paradigmatic of what we are most typically up to when in a philosophical spirit we offer an "analysis" or "explication" of some hitherto inadequately formulated "idea" or expression. We do not claim synonymy. We do not claim to make clear and explicit what the users of the unclear expression had unconsciously in mind all along. We do not expose hidden meanings, as the words "analysis" and "explication" would suggest; we supply lacks. We fix on the particular functions of the unclear expression that make it worth troubling about, and then devise a substitute, clear and couched in terms to our liking, that fulfills those functions. Beyond those conditions of partial agreement, dictated by our interests and purposes, any traits of the explicans come under the head of "don't cares". Under this head we are free to allow the explicans all manner of novel connotations never associated with the explicandum.[15]

Wiener's construction of the ordered pair may be what Quine is most typically up to when he offers a philosophical analysis, but it is clearly not what the tradition had in mind. Russell, in a way, conceived of the aim of analysis to be precisely what Quine says it is not: to expose hidden meanings.[16] In his 1903 *Principles of Mathematics*, Russell states that the purpose of the logical analysis of mathematics is the discovery of the logical indefinables, or logical constants, "in order that the mind may have that kind of acquaintance with them which it has with redness or the taste of a pineapple."[17] The success of this reduction would then allow philosophy to provide an answer to the question, what does mathematics mean? Russell explains,

> Mathematics in the past was unable to answer, and Philosophy answered by introducing the totally irrelevant notion of mind. But now Mathematics is able to answer, so far at least as to reduce the whole of its propositions to certain fundamental notions of logic. At this point, the discussion must be resumed by Philosophy. I shall endeavour to indicate what are the fundamental notions involved, to prove at length that no others occur in mathematics, and to point out briefly the philosophical difficulties involved in the analysis of these notions.[18]

Later in life, reflecting back on his mathematical work, Russell wrote specifically of the Wiener-Kuratowski analysis,

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[14] The proof is relatively straight forward. See, for example, Herbert B. Enderton, *Elements of Set Theory* (London: Academic Press, 1977), p. 36.
[15] Quine, *Word and Object*, pp. 258-59.
[16] For a detailed account of Russell's various views of philosophical analysis see Peter Hylton, "Beginning with Analysis," in *Bertrand Russell and the Origins of Analytic Philosophy*, Ray Monk and Anthony Palmer, eds. (Bristol: Thoemmes Press, 1997), pp. 183-215.
[17] Bertrand Russell, *The Principles of Mathematics*, 1903, 2nd ed., (London: George Allen and Unwin Ltd., 1937), p. xv.
[18] Ibid., p. 4.

> I thought of relations in those days [circa 1900], almost exclusively as *intensions*. I thought of sentences such as, 'x precedes y', 'x is greater than y', 'x is north of y'. It seemed to me—as, indeed, it still seems—that, *although from the point of view of a formal calculus one can regard a relation as a set of ordered couples, it is the intension alone which gives unity to the set.*[19]

Elsewhere Russell is reported to have called the analysis "a trick".[20] This contrast between Russell and Quine over the significance of the Wiener-Kuratowski definition of the ordered pair helps to illustrate the extent to which Quine is willing to do away with traditional philosophical concerns so as to obtain clarity about a troublesome but useful notion of our current ongoing science. That Wiener's and Kuratowski's definitions differ is of no matter for Quine. They both fulfill condition (1) and where their differences lie has no effect on this; there is no hidden meaning, no essence, for the analysis to discover. What distinguishes Quine's attitude towards philosophical analysis is that he does not judge the success of the analysis by some preconceived philosophical notion of what the analysis should be, a guiding metaphysical assumption concerning the intensionality of relations as in Russell's case. Rather his guiding concern is whether the proposed analysis fulfills the condition that makes the ordered pair worthwhile for science. So long as the analysis does this, any other features of it may be consigned to the realm of don't cares.[21] For Quine

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[19] Bertrand Russell, *My Philosophical Development* (London: George Allen and Unwin Ltd., 1959), p. 87 (My emphasis); and of Quine, in particular, he wrote, "Professor Quine, for example, has produced systems which I admire greatly on account of their skill, but which I cannot feel to be satisfactory because they seem to be created *ad hoc* and not to be such as even the cleverest logician would have thought of if he had not known of the contradictions" (p. 80).

[20] W.D. Hart, "Clarity," in *The Analytic Tradition*, David Bell and Neil Cooper, eds., (Oxford: Basil Blackwell Ltd., 1990), p. 207.

[21] This of course is also the view Quine holds with regard to numbers. As he remarks in "Ontological Relativity,"

> So, though Russell was wrong in suggesting that numbers need more than their arithmetical properties, he was right in objecting to the definition of numbers as any things fulfilling arithmetic. The subtle point is that any progression will serve as a version of number so long and only so long as we stick to one and the same progression. Arithmetic is, in this sense, all there is to number: there is no saying absolutely what the numbers are; there is only arithmetic. (p. 45)

See "Ontological Relativity," in *Ontological Relativity and Other Essays*, (New York: Columbia University Press, 1969), pp. 26-68, and especially pp. 43-5; and "Reply to Charles Parsons," in *The Philosophy of W.V. Quine*, expanded ed., Lewis Edwin Hahn and Paul Arthur Schilpp, eds. (Chicago and LaSalle: Open Court Publishing Company, 1998), pp. 400-1. The classic development of this view of numbers is Paul Benacerraf, "What Numbers Could not Be," *The Philosophical Review* 74:1 (1965), pp. 47-73.

> *explication is elimination.* We have, to begin with, an expression or form of expression that is somehow troublesome. It behaves partly like a term but not enough so, or it is vague in ways that bother us, or it puts kinks in a theory or encourages one or another confusion. But also it serves certain purposes that are not to be abandoned. Then we find a way of accomplishing those same purposes through other channels, using other and less troublesome forms of expression. The old perplexities are resolved.[22]

It is in this vein that Quine considers the ordered pair to be a paradigm of philosophical analysis and philosophy of science to be philosophy enough.[23]

Though the simplification and clarification of scientific theory, alone, is of sufficient gain to recommend regimentation, the regimentation of scientific theory into canonical notation also aims at the clarification of the science of logic itself. Having regimented a theory, we can more easily identify the logical relationships between its sentences by applying logical theory to them, what Quine describes as "the systematic study of logical truths." Chief among such logical interdependencies among sentences is logical implication as Quine explains, "Logical implication is the central business of logic. Logical truth would be of little concern to us on its own account, but it is important as an avenue to implication. It is simpler to theorize about truth than implication because it is attributable to single sentences whereas implication relates sentences in pairs."[24]

The importance of logical implication to scientific theorizing is not to be understated. For Quine this logical relationship is "the lifeblood of theories" as it is what links a theory to its empirical checkpoints.[25] Having established a hypothesis, implication allows for its testing. One side of the implication, the theoretical, is made up of our backlog of accepted theory plus the hypothesis; this side does the implying. On the other side, the observational, is an implied

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[22] Quine, *Word and Object*, p. 260 (Quine's italics); For my account of the significance of the ordered pair in Quine's philosophy, particularly as it indicates a decisive contrast between the philosophical outlooks of Quine and Russell, I am indebted to Kanamori, "Empty Set," pp. 288-93 and to Hylton, "Beginning with Analysis," pp. 213-15.
[23] Quine, "Strawson," p. 151.
[24] W.V. Quine, "Grammar, Truth, and Logic," in *Philosophy and Grammar*, Stig Kanger and Sven Öhman, eds., (Dordrecht, Holland: D. Reidel Publishing Company, 1981), pp. 17; see also *Methods of Logic*, 1950, 4th ed. (Cambridge: Harvard University Press, 1982), p. 4; and *Philosophy of Logic* [1986], pp. vii, 48-9.
[25] W.V. Quine, *From Stimulus to Science* (Cambridge: Harvard University Press, 1995), p. 51.

generality of the form "Whenever this, that", Quine's observation categorical, available to the experimenter for direct testing. So long as our canonical notation includes the resources to represent the relation of implication between a body of scientific theory and an observation categorical, this notation will be sufficient for the needs of science.[26] And determining whether one sentence implies another does in fact require nothing beyond the resources of Quine's canonical notation, the truth functional connectives and quantification. Implication holds simply when the conditional formed from the two sentences is valid, when it is a logical truth.[27] This raises an important and illuminating question for Quine's conception of logic: how is he to characterize logical truth? The definition of logical truth, or validity, offered by most contemporary philosophers and logicians goes by way of model theory: a sentence is *logically true*, or *valid*, if every model in a model-theoretic semantics S is a model of the sentence.[28] However, in his *Philosophy of Logic*, Quine first defines logical truth in terms of sentence substitution: a logical truth is a sentence that yields only truths when we substitute sentences for its simple sentences.[29] He then notes that his substitutional definition can also be given by a two step method employing the notion of a *valid logical schema*.

He describes a schema as a sort of "dummy sentence" that depicts the logical structure of what could be an actual sentence, a sentence of the fully interpreted object language. The logical structure of a sentence is its composition in terms of truth functions, quantifiers, and variables. Sentences then, by Quine's account, are composed only of logical structure and predicates. A schema depicts the logical structure of a sentence by replacing the predicates with schematic

[26] W.V Quine, *Pursuit of Truth*, 1990, rev. ed. (Cambridge: Harvard University Press, 1992), pp. 9-10; I am indebted in this paragraph to Ricketts, "Frege, Carnap, and Quine," p. 198.
[27] Quine, *Methods* [1982], p. 46.
[28] For example see Stewart Shapiro, *Foundations without Foundationalism: A Case for Second-order Logic*, paperback ed., (Oxford: Clarendon Press, 2002), p. 6.
[29] Quine, *Philosophy of Logic* [1986], p. 50.

predicate letters 'F', 'G', etc. Unlike actual predicates, these schematic predicate letters are not part of the object language but serve instead only to diagrammatically mark positions where object language predicates could appear. For example, the sentence 'There is something that walks' can be rendered into canonical notation as '$(\exists x)$(x walks)'. Replacing the predicate '① walks' with a schematic predicate letter 'F' we have depicted its logical structure thus '$(\exists x)Fx$'. Quine then defines a logical schema as *valid* if every sentence obtainable from it by substitution of sentences for simple sentence schemata is true. Now, a *logical truth* is a truth obtainable by this substitution method from a valid logical schema.[30] He does not include separate substitution clauses for term letters, such as names or functions. Given identity as a logical primitive, name and function letters are superfluous; both can be paraphrased away by Russell's method of descriptions. This austere language of truth functions, quantifiers, variables, and predicates is enough though eliminable term letters may always be introduced for mere convenience.[31]

We should pause here to observe that Quine's substitutional definition will not work if he takes identity as a primitive logical predicate for then '$(\exists x)(\exists y)\neg x = y$' would count as a logical truth, and logic as traditionally conceived, and as Quine conceives it, does not pronounce on the number of objects in the world. Yet, if he does not take identity as a primitive logical predicate, truths of identity theory such as '$x = x$' or '$(\exists y)x = y$' would not count as logical truths as they could be falsified by substituting some other predicate for '='. This, too, seems an undesirable result as such truths are often considered logical truths. This tension, though, does not result from Quine's substitutional definition but from the identity predicate itself he explains, in that once identity is allowed as part of our genuinely logical vocabulary, some logical generalities

[30] Ibid., pp. 24, 49-50.

[31] Ibid., pp. 25-6; Quine, *Methods* [1982], pp. 274-77.

become directly expressible in the object language. Logic's concern extends in this way from talk of forms of sentences to the expression of genuine sentences.

Still, Quine thinks there are also reasons for wanting to include identity as part of the logical vocabulary. One is that, like quantification theory, there are complete proof procedures available for identity theory. Another is that identity theory, again like quantification theory, does not discriminate amongst objects in its application. Neither of these features holds for many branches of higher mathematics and, in particular, not for set theory.[32] He then offers a further consideration for favoring the inclusion of identity theory as part of logic:

> [F]or identity is marginal in a curious way. Namely, any theory with any finite number of other primitive predicates gets identity too as a bonus. For identity can be defined, or something to the same formal purpose, by exhaustion of those primitive predicates. For example, if the primitive predicates are '*P*', '*Q*', and a dyadic '*R*', we can define '$x = y$' as
> $\forall z(Px \equiv Py \,.\, Qx \equiv Qy \,.\, Rxz \equiv Ryz \,.\, Rzx \equiv Rzy)$.[33]

By defining identity this way, truths of identity theory gain the same schematic status as other logical generalizations. Ultimately, these considerations lead Quine to include identity theory as more appropriately part of logic than of some other higher branch of mathematics. What should be stressed here is that his initial considerations against counting identity as a primitive logical predicate are not to be understood merely as an ad hoc maneuver for securing his substitutional definition of logical truth. Identity presents difficulties for any view that sees logic as primarily concerned with form, and specifically for Quine, forms of sentences.

Though his two-step definition of logical truth comes to the same thing as the definition given in terms of sentence substitution the notion of a valid schema serves a further purpose as he explains, "Because of their freedom from subject matter, schemata are the natural medium for logical laws and proofs."[34] It is only after he gives his substitutional definition that Quine then

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[32] Quine, *Philosophy of Logic* [1986], pp. 61-4; Quine, *From Stimulus to Science*, p. 52.
[33] Quine, *From Stimulus to Science*, p. 52.
[34] Quine, *Philosophy of Logic* [1986], p. 51

also presents a more standard definition of logical truth in terms of models: a schema is *valid*, if it is satisfied by all its models, and, again, it is a *logical truth* if it is a sentence obtainable by substitution into a valid logical schema.[35] These definitions are equivalent provided that the language is rich enough for elementary number theory, a demand he thinks moderate. Where this is not the case, he thinks there is just as much reason to blame any discrepancies between the two definitions on the weakness of the language as on the substitutional definition of logical truth.[36]

Let us return now to consider Quine's preference for using schemata to express logical laws and proofs. Why should their freedom from subject matter make schemata the natural medium for logical laws and proofs? What role does topic neutrality have in Quine's conception of logic? An answer begins to emerge in a remark from his introduction to *Methods of Logic*. Criticizing a flawed attempt at locating the difference between logic and branches of higher mathematics, Quine writes,

> Logic and mathematics were coupled, in earlier remarks, as jointly enjoying a central position within the total system of discourse. Logic as commonly presented, and in particular as it will be presented in this book, seems to differ from mathematics in that in logic we talk about statements and their interrelationships, notably implication, whereas in mathematics we talk about abstract nonlinguistic things: numbers, functions, and the like. This contrast is in large part misleading. Logical truths, e.g., statements of the form 'If *p* and *q* then *q*', are not about statements; they may be about anything, depending on what statements we put in the blanks '*p*' and '*q*'. When we talk *about* such logical truths, and when we expound implications, we are indeed talking about statements; but so are we when we talk *about* mathematical truths.[37]

For Quine, logic, as we have observed, is the systematic study of logical truths so as to make perspicuous the logical interrelations between sentences, most importantly that of implication. Logic traces implications regardless of subject matter and to this end, does not show prejudice towards any particular subdomain of objects. The relation of implication applies impartially to

[35] Ibid., pp. 52.
[36] Ibid., pp. 53-5.
[37] Quine, *Methods* [1982], p. 5. (Quine's italics)

sentences regardless of subject matter. His use of schemata here brings the notion of form to the forefront of his philosophy of logic as any sentence whatsoever can be written in place of a schema of the same form. In this way, his logical notation treats of any subject matter whatsoever while taking no particular subdomain of objects as its own.

Thomas Ricketts has stressed that this conception of logic is in sharp contrast with the view of logic found in Quine's predecessors Frege and Carnap, that of logic as the maximally general science. I would add to this list also Russell. For Russell, logic's generality was a consequence of logic being about the most general features of reality, the logical indefinables or constants. "Logic," Russell wrote, "is concerned with the real world just as truly as zoology, though with its more abstract and general features."[38] In contrast, Quine holds that logic is best viewed not as a complete notation for a maximally general science of logical objects or constants, but as a partial notation for discourse on all subjects. Quine's conception of logic on this count resorts back to a variation of the pre-Fregean understanding of logic that rendered validity in terms of form. Quine formulates the logical laws then not as statements in the object language as Frege and Russell did but rather as generalizations over the forms of sentences. So to take the example from the above quoted passage, the logical law governing the valid schema there is stated: "A material conditional of a conjunction with one of its conjuncts is true." Here, we see also an important contrast between Quine and the pre-Fregean tradition in logic. The forms Quine speaks of are not forms of thought that somehow lie behind language but are instead forms of sentences rendered into his canonical notation.[39]

Within this context, the significance of Quine's having introduced the notion of logical truth by way of sentence substitution becomes easier to see. It is only after having given his

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[38] Bertrand Russell, *Introduction to Mathematical Philosophy* (London: George Allen & Unwin, Ltd., 1919), p. 169.
[39] Ricketts, "Frege, Carnap, and Quine," pp. 196.

substitutional definition of logical truth that he goes on to give a more standard model-theoretic account and to demonstrate its equivalence with his substitutional definition. He remarks that one benefit of the substitutional definition of logical truth is that it saves on ontology; it does not invoke the universe of sets but rests instead within a realm of sentences. Presenting logic model-theoretically encourages the view that logic is about mathematical structures, its models. Quine sees this as potentially misleading in that it may lead one to think that quantification theory presupposes significant mathematical power at the outset. His definition of logical truth in terms of sentence substitution, in a sense, legitimizes the appeal to a model-theoretic definition. It shows that sentence substitution is enough. As Quine makes the point, "There is philosophical comfort in the assurance that we can talk of logical validity and consistency without appealing to a limitless realm of abstract objects called classes. We feel that in talking of substitution of expressions we still keep our feet on the ground."[40] Given though that the restrictions on allowable substitutions for quantificational schemata become rather complicated, a model-theoretic definition can be easier to operate with in practice as Quine himself often does.[41]

To briefly conclude this account of Quine's conception of logic, I wish to draw attention to one further aspect of his view: the reciprocal containment of logic in language and language in logic.[42] In presenting his canonical notation, Quine often describes it as a departure from ordinary language, or as extraordinary language, but this should not be taken to indicate that logic is in some way external to our conceptual scheme, external to our language. He explains,

> Not that this logical language is independent of ordinary language. It has its roots in ordinary language, and these roots are not to be severed. Everyone, even to our hypothetical logician-scientist and his pupils' pupils, grows up in ordinary language, and can learn the logician-scientist's technical jargon, from '⊃' to '*dy*/*dx*' to 'neutrino', only by learning how, in principle at least, to paraphrase it into ordinary language. But for this purpose no extensive analysis of the logic of ordinary language is required. It is enough that

[40] Quine, *Methods* [1982], p. 212; Here I am indebted to conversations with Thomas Ricketts.
[41] For examples, see ibid., pp. 116, 173-74; for the restrictions on substitution see chapters 26 and 28.
[42] Here, I am again particularly indebted to Ricketts; see his "Languages and Calculi," p. 275-76.

> we show how to reduce the logical notations to a few primitive notations . . . and then explain just *these* in ordinary language, availing ourselves of ample paraphrases and scholia as needed for precision.[43]

Our language, as it stands and is used, includes the possibility for its regimentation by way of logical notation; logical notation is introduced piecemeal for the particular purposes of clarity, simplicity, and algorithmic power "as partial notations for discourse on all subjects."[44] And similarly, logic contains language, for it is by means of this regimentation that we are able to make perspicuous the logical relationship between any arbitrary sentences. Quine's point, as he concludes *Word and Object*, is that there is no privileged perspective from which our conceptual scheme can be described externally:

> The philosopher's task differs from the others' [other non-fiction researchers] . . . in detail; but in no such drastic way as those suppose who imagine for the philosopher a vantage point outside the conceptual scheme that he takes in charge. There is no such cosmic exile. He cannot study and revise the fundamental conceptual scheme of science and common sense without having some conceptual scheme, whether the same or another no less in need of philosophical scrutiny, in which to work. He can scrutinize and improve the system from within, appealing to coherence and simplicity; but this is the theoretician's method generally.[45]

Logic itself is part of this conceptual scheme of science, and like the other regions of science, it too is only understood from within this very conceptual scheme.

II.i

Quine's most cited criticisms of higher-order logic occur in his *Philosophy of Logic* in the chapter entitled "The Scope of Logic". Considering whether set theory belongs to logic, Quine remarks,

> Pioneers in modern logic viewed set theory as logic; thus Frege, Peano, and various of their followers, notably Whitehead and Russell. Frege, Whitehead, and Russell made a point of reducing mathematics to logic; Frege claimed in 1884 to have proved in this way, contrary to Kant, that the truths of arithmetic are analytic. But the logic capable of encompassing this reduction was logic inclusive of set theory.[46]

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[43] Quine, "Mr. Strawson," p. 150 (Quine's italics); see also *Word and Object*, p. 159.
[44] Quine, *Word and Object*, p. 160.
[45] Ibid., pp. 275-76; For the expression of a related view with regard to logic specifically see W.V. Quine, "Truth by Convention," in *Ways of Paradox*, pp. 105-06.
[46] Quine, *Philosophy of Logic* [1986], pp. 65-6.

Quine diagnoses this tendency to include set theory as part of logic as the result of a failure to see clearly the distinction between predication and membership, a confusion facilitated by an intermediate notion of attribution of attributes. His criticisms of higher-order logic can along these lines be viewed as continuous with the aims of his canonical notation sketched in the previous section. For Quine, the reason, at least in part, for paraphrasing a body of scientific theory into logical notation is the simplification and clarification of theory, and to obtain this objective, the logical notation itself must be a paradigm of clarity. Quine's critical discussion of higher-order logic contributes towards this end.

The confusion that Quine points to as leading to the seemingly innocent extension of ordinary, or classical, quantification theory by allowing quantification over predicate letters begins as a confusion of use and mention. As seen above, in the open sentence '*Fx*' of quantification theory, '*F*' is a schematic letter standing in place of a predicate. '*F*' and '*Fx*' are mere simulations of sentences and their parts, depictions of their logical structure. The schematic predicate letters do not name, or refer to, predicates, attributes, or sets but rather stand in place of unspecified predicates. It is the predicate expression itself, or a simulation thereof, that occurs in a sentence, not a name of it. This view of predicate letters is not held universally Quine adds: "Some logicians, however, have taken a contrary line, reading '*F*' as an attribute variable and '*Fx*' as '*x* has *F*'. Some, fond of attributes, have done this with their eyes open; others have been seduced into it by confusion."[47] The confused logician sees the predicate letter '*F*' as sometimes standing in place of an unspecified predicate and other times as naming an unspecified predicate. This confusion leads him to attribute noun status to '*F*' and so to arrive at the reading of '*Fx*' as '*x* has *F*'. Though, of course, not all logicians are confused over this

[47] Ibid., p. 66.

matter. Some prodigal logicians, he explains, embrace attributes without the confusion. In either case though, quantification over predicate letters leads to one or another sort of murkiness in logic which Quine wishes to dispel.

Central to this entire issue, he thinks, is allowing for quantification over predicate letters. It is worth quoting this passage at length, as it is one at which Boolos will specifically direct his attack:

> Consider first some ordinary quantifications: '$\exists x$ (x walks)', '$\forall x$ (x walks)', '$\exists x$ (x is prime)'. The open sentence after the quantifier shows 'x' in a position where a name could stand; a name of a walker, for instance, or of a prime number. The quantifications do not mean that names walk or are prime; what are said to walk or to be prime are things that could be named *by* names in those positions. To put the predicate letter 'F' in a quantifier, then, is to treat predicate positions suddenly as name positions, and hence to treat predicates as name of entities of some sort. The quantifier '$\exists F$' or '$\forall F$' says not that some or all predicates are thus and so, but that some or all entities of the sort named by predicates are thus and so.[48]

Quine's argument is that to take quantification over predicate letters as a mere extension of ordinary quantification theory is incoherent. Surely, these quantifications over predicate letters are not saying something about all predicates any more than ordinary quantifications say something about all names. But nor do these quantifications over predicate letters say something about the predicate expressions themselves. What would it even mean to render the quantifications '$\exists F$' or '$\forall F$' as 'There exists some is F' or 'For all is F'? Of course this rendering fails to capture precisely what a quantified schematic predicate letter would be like given that the *is* of predication is actually part of what 'F' simulates, but such imperfection is to be expected from an idea that began incoherent. Instead, by allowing these so-called higher-order quantifications, we are forced to take the predicate letter 'F' as a variable that takes some thing as its value. Quine's point is that when we allow quantification over predicate letters, the schematic predicate letters are no longer mere simulations of predicates. Instead, their status

[48] Ibid., pp. 66-7 (Quine's italics); A note to the reader: I have not standardized notation in quoting directly or in discussing particular passages. Rather, I kept with the notation used by the author in that context. I hope this will not confuse the reader.

changes from substitution taking schematic letters to value taking variables. In quantifying over predicate letters, we are not innocently extending our ordinary quantification theory but making it into a theory about some particular subdomain of objects. To reiterate his earlier point, the prodigal logician recognizes this move and takes the predicates to name attributes while the confused logician moves back and forth between the two options lacking a clear conception of the difference between schematic predicate letters and value taking variables.

The question naturally arises now as to why the prodigal logician should be grouped together with the confused logician. What is wrong with his explicit assumption of attributes as the values of predicate letters that have taken on the role of variables? Quine's response is that the notion of attribute is itself unclear for attributes lack a clear criterion of identity (and for Quine, there is no entity without identity). Unlike sets, which have their identity established by the law of extensionality—sets are the same when their members are the same—this does not hold for attributes. Compare the open sentence '*x* is a man' with the open sentence '*x* is a featherless biped'. If there were such things as attributes, and by Quine's lights there are not, these two open sentences may be said to have two different attributes as their intensions, or meanings. But to make sense of how to count attributes as the same and so also as different, we would first have to make sense of the relation of synonymy, a requirement that Quine does not think philosophy can meet.[49] In contrast, if we consider the two open sentences as determining sets, by the law of extensionality we can conclude that they do in fact determine the same set.

Quine next suggests rejecting attributes and taking the values of the predicate letters to be sets, but this too is confused in its own way. Indeed, this move would be little advance over the

[49] Quine addresses the question of synonymy in Chapter I of *Philosophy of Logic* though his most famous attack is in "Two Dogmas of Empiricism," in *From a Logical Point of View*, pp. 20-46. Detailed consideration of Quine's arguments against synonymy would take me too far a field for the purposes of this thesis.

confusion that Quine sees as leading to the very idea of treating predicate letters as quantifiable variables in the first place:

> But I deplore the use of predicate letters as quantified variables, even when the values are sets. Predicates have attributes as their "intensions" or meanings (or would if there were attributes), and they have sets as their extensions; but they are names of neither. Variables eligible for quantification therefore do not belong in predicate positions. They belong in name positions.[50]

And earlier he bemoaned attributes thus,

> My complaint is that questions of existence and reference are slurred over through failure to mark distinctions. Predicates are wanted in all sentences regardless of whether there are attributes to refer to, and the dummy predicate '*F*' is wanted generally for expository purposes without thought of its being a quantifiable variable taking attributes as values. If we are also going to quantify over attributes and refer to them, then clarity is served by using recognizable variables and distinctive names for the purpose and not mixing these up with the predicates.[51]

Taking predicate letters as value taking variables perpetuates the mistaken view that ordinary quantification theory was always about attributes or sets rather than a technique for making clear the logical relationships between statements concerning any subject matter whatsoever. The proper way to render attributes or sets into logical notation on Quine's view is with ordinary object variables as this does not obscure the distinction between variables and schematic predicate letters. Instead of reading 'Fx' as 'x has F' clarity in the science of logic is served by writing 'x has y', or with a distinctive attribute variable 'x has ζ'. And the same holds when one wants to admit sets as values of quantifiable variables writing instead '$x \in y$', or '$x \in \alpha$' using distinctive set variables. Whereas '$(\exists x)(Fx \,.\, Gx)$' is a logical schema, '$(\exists x)(x \in \alpha \,.\, x \in \beta)$ is an open sentence (what Quine calls its "set-theoretic analogue") about sets whose logical form is depicted by this logical schema. The set variables 'α', 'β', 'γ' etc. are eligible for quantification the same as any other object variables are.[52]

[50] Quine, *Philosophy of Logic* [1986], p. 67.
[51] Ibid., p. 28.
[52] Ibid., p. 51.

We might be inclined to ask now, given that attributes lack a clear criterion of identity, what would be the purpose of allowing quantification over attributes in the first place? Why not just start with sets? Here, Quine brings out more clearly how confusion over use and mention may lead a philosopher, namely Russell, to think that he is deriving set theory, and so mathematics, from narrowly logical beginnings. First, it will be helpful to briefly state the context in which Russell developed his theory.

Russell's paradox emerged from Frege's Basic Law V in combination with the substitution rule of his 1893 work, *Grundgesetze der Arithmetik* which allowed him to assert that for every function there is a logical object equivalent to it, what he called the function's value-range, or extension. Formally, in modern notation Basic Law V is

$$\{x : f(x)\} = \{x : g(x)\} \equiv (x)(f(x) = g(x)),$$

the extension of the concept 'f' is equal to the extension of the concept 'g' if and only if the function 'f' is extensionally equivalent to the function 'g'.[53] The paradox may look more familiar to contemporary readers rendered in terms of the unrestricted comprehension schema of naïve set theory:

$$(\exists y)(\forall x)(x \in y \equiv Fx)$$

where 'y' does not occur free in 'F'. Russell informed Frege of his discovery by letter on June 22, 1902, shortly before the second volume of *Grundgesetze* was to go to press:

> With regard to many particular questions, I find in your work discussions, distinctions, and definitions that one seeks in vain in the works of other logicians. Especially so far as function is concerned (§ 9 of your *Begriffsschrift*), I have been led on my own to views that are the same even in the details. There is just one point where I have encountered a difficulty. You state (p. 17 [p. 23 in van Heijenoort]) that a function, too, can act as the indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let *w* be the predicate: to be a predicate that cannot be predicated of itself. Can *w* be predicated of itself? From each answer the opposite follows. Therefore we must

[53] Michael Potter, *Reason's Nearest Kin: Philosophies of Arithmetic from Kant to Carnap* (New York: Oxford University Press, 2000), pp. 107, 112-13; Frege's Basic Law V can be found in Gottlob Frege, *The Basic Laws of Arithmetic: Exposition of the System*, Montgomery Furth, ed. and trans. (Berkeley: University of California Press, 1964), p. 72.

conclude that w is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definable collection [Menge] does not form a totality.[54]

Russell's point is that from the unrestricted comprehension principle, we can derive that there is a class of all classes which are not self-membered, and then ask of this class whether it is contained in itself. The answer is that if it is then it is not, and if it is not then it is and so, contradiction. Formally,

[1]	(1)	$(\exists y)(\forall x)(x \in y \equiv x \notin x)$	P
[1, 2]	(2)	$(\forall x)(x \in y \equiv x \notin x)$	(1)y EII
[1,2]	(3)	$y \in y \equiv y \notin y$	(2) UI
[1, 2]	(4)	$p \,.\, {\sim}p$	(3) TF; [2] EIE[55]

In his 1903 *Principles of Mathematics*, Russell gave lengthy treatment to the explication of the contradiction along with some inconclusive suggestions as to how it might be resolved, but it was not until his 1908 "Mathematical Logic as Based on the Theory of Types" that he discovered a solution.[56] What Russell saw as common to both of what we now identify as the set-theoretic and semantic paradoxes is the assumption of an illegitimate totality, a totality that would be enlarged by new members defined in terms of that very totality. For example, the collection of propositions will be supposed to contain a statement "all propositions are either true or false." But this is a meaningless statement for it would have to be about some already definite collection "all propositions" which is impossible if new propositions are created by statements about all propositions.[57] Russell's apparent insight here led him to formulate the Vicious Circle

[54] Bertrand Russell, "Letter to Frege," in Jean van Heijenoort, ed., *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931* (Cambridge: Harvard University Press, 1967), pp. 124-25; Gottlob Frege, *Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought*, 1879, in van Heijenoort, pp. 1-82.

[55] Warren Goldfarb presents this derivation of the contradiction in his "Deductive Logic," unpublished ms. (1994), p. 186; Quine presents a sketch of the derivation in "On Universals," *The Journal of Symbolic Logic* 12:3 (Sept. 1947), p. 78.

[56] Bertrand Russell, "Mathematical Logic as Based on the Theory of Types," 1908, in *Logic and Knowledge*, R.C. Marsh, ed. (London: George Allen and Unwin, Ltd., 1956), pp. 59-102.

[57] Alfred North Whitehead and Bertrand Russell, *Principia Mathematica*, 1910-1913, 2nd ed. (Cambridge: Cambridge University Press, 1960), p. 37.

Principle: "Whatever involves *all* of a collection must not be one of the collection." His theory of orders, or types, implements this principle.[58] Under his theory of orders (the intensional counterpart upon which we will see the extensional theory of types constructed), Russell stratifies the universe into levels with individuals at level 1, attributes of individuals at level 2, attributes of attributes of individuals at level 3 and so on. The paradox is then blocked by allowing the objects at each level of the hierarchy to hold only of the objects one level below. The intensional analogue to comprehension is then restricted as follows:

$$(\exists v)(\forall u)(vu \equiv Fu)$$

where 'v' is of level $n + 1$, 'u is of level n, and 'F' is a predicate that holds significantly over objects of the same level as 'u'. Thus Russell's theory of orders rules out self-predication as meaningless. This restriction then carries over to the extensional theory of types to block the set-theoretic paradoxes.

On Quine's reading, Russell's willingness to assume attributes results from his failure to appreciate the difference between schematically simulating predicates and quantifying over a particular kind of object, attributes.[59] With this confusion in place, the construction of set theory from attributes then proceeds by contextual definition as Quine presents in his *Set Theory and Its Logic.*[60] Russell's first move is to treat the notation for membership, '$\in$', as an alternative notation for the attribution of a predicative attribute, '$\varphi!x$'; a predicative attribute being an attribute of level $n + 1$ that holds of an object of level n:

'$x \in \varphi$' for '$\varphi!x$'.

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[58] Ibid., *Principia*, p. 37; Russell "Theory of Types," p. 63. (Russell's italics)
[59] Ibid., p. 68.
[60] W.V. Quine, *Set Theory and Its Logic*, 1963, rev. ed. (Cambridge: Harvard University Press, 1969), pp. 251-52; For Russell's original construction see Alfred North Whitehead and Bertrand Russell, *Principia Mathematica*, vol. I, 2nd ed. (Cambridge: Cambridge University Press, 1925) pp. 24-5, 71-8; he presents the contextual definition itself on p. 76; for an exposition of Russell's original account see Potter, *Reason's Nearest Kin*, pp. 147-50.

Then he provides his contextual definition for class abstraction:

$$\text{`}G\{x\colon Fx\}\text{' for `}(\exists\varphi)((\forall x)(\varphi!x \equiv Fx)\,.\,G\varphi)\text{'}$$

which defines '$G\{x\colon Fx\}$' to express that 'G' is true of some arbitrary predicative attribute φ such that '$(\exists\varphi)(\forall x)(\varphi!x \equiv Fx)$', in short that some predicative attribute 'φ' is extensionally equivalent to a predicate substitutable for the predicate 'F'. Quantification over classes is then defined

$$\text{`}(\forall\alpha)G\alpha\text{' for `}(\forall\varphi)G\{\text{x}\colon \varphi!x\}\text{', `}(\exists\alpha)G\alpha\text{' for `}(\exists\varphi)G\{x\colon \varphi!x\}\text{'.}$$

Finally, there is the law of extensionality,

$$(((\forall x)(x \in \alpha \equiv x \in \beta)\,.\,\alpha \in \kappa) \supset \beta \in \kappa),$$

for classes, but because they are only contextually defined, the law does not have to be assumed for attributes as well, for as shown above, extensionality is all that distinguishes classes from attributes. By this contextual definition then, Russell thought he had derived set theory from what might appear as an innocent extension of predicate logic. Quine, however, concludes his discussion of Russell's contextual definition remarking,

> Russell had ... a philosophical preference for attributes, and felt that in contextually defining classes on the basis of a theory of attributes he was explaining the obscurer in terms of the clearer. But this feeling was due to his failure to distinguish between propositional functions as predicates, or expressions, and propositional functions as attributes. Failing this, he could easily think that the notion of an attribute is clearer than that of a class; for that of a predicate is. But that of an attribute is less clear.[61]

It was the result of a failure to recognize distinctions between predicates and attributes that Russell thought that he had reduced mathematics to logic.

In their *Principles of Mathematical Logic*, Hilbert and Ackermann drop Russell's hierarchy of propositional functions and assume outright that the values of their predicate

[61] Quine, *Set Theory*, pp. 256-57; For perhaps Quine's earliest extended account of this criticism of Russell see his "Whitehead and the Rise of Modern Logic [1941]," in *Selected Logic Papers*, enl. ed. (Cambridge: Harvard University Press, 1995), pp. 3-36, especially pp. 19-24.

variables are extensional entities of a sort, classes or sets.[62] Here the problem as Quine sees it is no longer one of constructing the clearer upon the obscurer but that a misleading notational device remains in place:

> Followers of Hilbert have continued to quantify predicate letters, obtaining what they call a higher-order predicate calculus. The values of these variables are in effect sets; and this way of presenting set theory gives it a deceptive resemblance to logic. One is apt to feel that no abrupt addition to the ordinary logic of quantification has been made; just some more quantifiers, governing predicate letters already present.[63]

Echoing Russell's quantification over propositional functions, i.e., '$\forall\varphi$' and '$\exists\varphi$', they quantify predicate letters '$\forall F$' and '$\exists F$' continuing to encourage the idea that so-called higher-order predicate calculus is an innocent extension of ordinary predicate logic.

To illustrate his criticism Quine considers the axiom scheme of comprehension '$(\exists y)(\forall x)(x \in y \equiv Fx)$ which assumes a set $\{x: Fx\}$ determined by substituting an open sentence for 'Fx'. Here, it is worth noting some textual differences between the 1970 and 1986 editions of *Philosophy of Logic* as this will help clarify Quine's argument. In the earlier edition he explains the comprehension schema as

> the central hypothesis of set theory, and the one that has to be restrained in one way or another to avoid the paradoxes. This hypothesis falls dangerously out of sight in the so-called higher-order predicate calculus. It becomes '$(\exists G)\ (x)\ (Gx \equiv Fx)$', and thus evidently follows from the genuinely logical triviality '$(x)\ (Fx \equiv Fx)$' by an elementary logical inference. Set theory's staggering existential assumptions are cunningly hidden now in the tacit shift from schematic predicate letter to quantifiable set variable.[64]

Now, the importance of Quine's earlier argument for the incoherence of quantifying over schematic predicate letters reemerges. We saw there that having allowed quantification over predicate letters, the predicate letters could no longer be treated as merely standing in place of our ordinary predicates, e.g., '① walks', '① is prime', etc., but had to be rendered instead as standing in place of objects of some sort. Russell's theory of orders offered intensional objects,

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[62] David Hilbert and Wilhelm Ackermann, *Principles of Mathematical Logic*, Lewis M. Hammond et al., trans. (New York: Chelsea Publishing Company, 1950) translated from *Grundzüge der Theoretischen Logik*, 2nd ed. (Berlin: Julius Springer, 1938); they cover "the extended predicate calculus" in chapt. IV.
[63] Quine, *Philosophy of Logic* [1986], p. 68.
[64] W.V. Quine, *Philosophy of Logic* (Englewood Cliffs: Prentice-Hall, 1970) p. 68.

attributes, and now Hilbert and Ackermann offer extensional objects, classes or sets. Quine's complaint is that in doing so, they have not made this commitment to sets explicit but have rather presented their higher-order predicate calculus as misleadingly continuous with ordinary predicate logic. What Hilbert and Ackermann's higher-order predicate calculus actually states in its '$(\exists G)(x)\ (Gx \equiv Fx)$' is 'There is a set G such that for all objects x, x is a member of G if and only if x fulfills any condition F on x', rendered more explicitly in notation as '$(\exists y)(x)(x \in y \equiv Fx)$', the unrestricted, and so contradictory, axiom schema of comprehension.[65] Of course Hilbert and Ackermann's higher-order predicate calculus, through its hierarchy of types, avoids the set-theoretic paradoxes in a way analogous to Russell's hierarchy of orders.

What may leave Quine's criticism in this passage unclear is the phrase 'set theory's staggering ontology'. Given that the context of this passage is a discussion of Hilbert and Ackermann's higher-order predicate calculus, it is safe to say that the set-theoretic ontology is indeed staggering for the typically ambiguous comprehension schema will yield sets of individuals, sets of sets of individuals, sets of sets of sets of individuals and so on on up the hierarchy. Quine, whose primary interest in set theory has been in comparative set theory (comparative set theory being, roughly put, the study of what can be proved in a set theory according to the assumption of stronger and weaker axioms; much of contemporary set theory is more concerned with set theory's model theory), certainly knows that not all formulations of set theory include the same staggering ontology.[66] Indeed, a primary concern of his has been to

[65] For this way of reading Quine's criticism see his earlier, "Logic and the Reification of Universals," in *From a Logical Point of View*, p. 121 and the reference there back to the unrestricted comprehension schema R3 of "New Foundations for Mathematical Logic," also in *From a Logical Point of View*. It is in "New Foundations" that Quine presents his method of stratification for avoiding the set-theoretic paradoxes replacing R3 with R3′; see pp. 89-92 in particular.

[66] Quine's work in comparative set theory culminates with *Set Theory and its Logic*; for an account of Quine's work in the area of set theory see Joseph Ullian, "Quine and the Field of Mathematical Logic," in *The Philosophy of W. V. Quine*, expanded ed., Lewis Edwin Hahn and Paul Arthur Schilpp, eds. (Chicago and LaSalle: Open Court Publishing Company, 1998), pp. 569-89; *Set Theory and its Logic* is discussed on pp. 582-86.

investigate how little set theory must be assumed to still be sufficient for the needs of science. That he does recognize these differences in ontological commitments appears explicitly in his analogous discussion of the higher-order predicate calculus in *Set Theory and its Logic*. There he remarks that Hilbert and Ackermann also present subsystems of full type theory where the types terminate at a finite level. These theories they call the predicate calculus of *n*th order, and so a theory that allows quantification over classes of individuals and relations of individuals they call the second-order predicate calculus. They thus dub ordinary predicate logic "first-order predicate calculus." Quine views this practice of classifying logic and set theory by order as once again encouraging the view that quantification over predicate letters is a mere extension of ordinary predicate logic, a move that obscures the fact that quantification over predicate letters introduces to logic a specific kind of object, sets.

Turning back now to the 1986 edition of *Philosophy of Logic*, we find the following revision to the above quoted passage concerning comprehension:

> This hypothesis itself falls out of sight in the so-called higher-order predicate calculus. We get '$\exists G\, \forall x\, (Gx \leftrightarrow Fx)$', which evidently follows from the genuinely logical triviality '$\forall x\, (Fx \leftrightarrow Fx)$' by an elementary logical inference. There is no actual risk of paradox as long as the ranges of values of 'x' and 'G' are kept apart, but still a fair bit of set theory has slipped in unheralded.

Note first that Quine removes 'dangerously' from the first sentence, and second, he revises the last sentence explaining the restriction on comprehension needed to avoid the paradox and tempers his claim about "set theory's staggering existential assumptions". Both changes, we can surmise, are made in response to criticisms found in Boolos's 1975 paper "On Second-Order Logic" as we will see in the next chapter.[67] There, Boolos criticized that Quine's wording in the original passage may lead some readers to think that Russell's paradox had not been sufficiently

[67] George Boolos, "On Second-Order Logic," in *Logic, Logic, and Logic*, Richard Jeffrey, ed. (Cambridge: Harvard University Press, 1991), pp. 40.

eliminated from second-order logic. Quine's removal of "dangerously" from the text and his inclusion of an explanation of how the paradox can be avoided dispenses with this worry.

There is though also a way in which we could see Boolos's comments on Quine, not as criticism but as exposition. In a sense, Quine does think that higher-order logic is inconsistent, not as Russell, Hilbert and Ackermann, or Boolos present it of course, but as it stands prior to their introducing some ad hoc restriction in order to cope with the paradox. Without such a restriction "the genuinely logical triviality '$\forall x(Fx \leftrightarrow Fx)$' by an elementary logical inference," namely existential generalization over predicate letters, does generate Russell's paradox thus demonstrating the inconsistency of what we may want to call "naïve higher-order logic." Indeed, in his earlier work on the extension of quantification theory to quantification over predicate letters, this is precisely his order of explanation. In both the 1937 "New Foundations for Mathematical Logic" and the 1947 "On Universals", he initially extends quantification to predicate letters so that they have all the privileges of ordinary variables. The result is Russell's paradox. Only then does he add some restriction for blocking the paradox, some way of keeping the range of the predicate variables separate from that of the individual variables.[68] His purpose in presenting matters this way is to show that the extension of logic so as to include quantification over predicate letters deprives logic of any characterization as unconditioned, self-evident, or obvious, all descriptions that traditionally singled logic out from other sciences. It was precisely this trait that made the reduction of mathematics to logic so important to Quine's predecessors in the analytic tradition, a point of which I will say more below.[69]

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[68] Quine, "New Foundations for Mathematical Logic," pp. 89-92; "On Universals," pp. 78-9; see also "Logic and the Reification of Universals," pp. 121-25 and footnote 65 above.

[69] The classic exposition of the sort of type-theoretic machinery required to block the set-theoretic paradoxes is Alonzo Church, "A Formulation of the Simple Theory of Types," *Journal of Symbolic Logic* 5:2 (1940), pp. 56-68; see also Kazimierz Ajdukiewicz, "Syntactic Connexion," in *Polish Logic: 1920-1939*, Storrs McCall, ed. (Oxford: Clarendon Press, 1967), pp. 207-31.

On now to Quine's more moderate claim about the ontological commitments of higher-order logic. This revision indicates not an error in the earlier edition but only that Quine is no longer considering the entire type-theoretic hierarchy as he was in 1970 but only a second-order subsystem consisting of individuals and sets of individuals. One consequence of Boolos's paper was that it forced the debate over the status of higher-order predicate calculus to center on its second-order fragment. Though, from here, one may go on to argue this second-order fragment has nothing like the existential assumptions of the whole of set theory and so it is much closer to ordinary predicate logic than Quine would lead his readers to believe. Still, when he talks of keeping the ranges of the variables 'x' and 'G' apart, he is talking of two object variables, one over individuals and the other over sets. The existential assumptions of '$\exists G\, \forall x\, (Gx \equiv Fx)$' become clear when rendered into the less misleading notation '$\exists \alpha\, \forall x\, (x \in \alpha \equiv Fx)$'. The usual object variables 'x', 'y', 'z', etc. range over a domain of individuals, and the set variables, 'α', 'β', 'γ', etc. range over subsets of this domain. The comprehension principle then states only the existence of sets of individuals.

"Set theory" seems the appropriate label in a second way as well for the sets of this theory represent an application of the power set operation, and while only a single application, it is one of the central ingredients for generating the whole of Zermelo's iterative hierarchy of sets, and so is also a way of recognizing a particular subdomain of objects.[70] All of these considerations lead Quine to view the second-order fragment of higher-order predicate calculus as another chapter of comparative set theory.[71]

[70] Ignacio Jané raises another difficulty with the single application of power set here in that to understand the contents of the power set of a set of low rank we may need to appeal to sets of much higher rank, i.e. appeal to a much more powerful set theory; see his "A Critical Appraisal of Second-Order Logic," *History and Philosophy of Logic* 14 (1993), pp. 67-86, and especially pp. 75-6.
[71] Here I am indebted to conversations with Thomas Ricketts.

II.ii

In his attempt to carve out a "reasonable boundary" for what he calls 'logic', Quine does not limit his attention to classical quantification theory and what amounts to its supplementation by some fragment of set theory. He recognizes that this boundary may still seem arbitrary to some readers and so expands his considerations to a particular feature of classical quantification theory itself, namely its completeness: that all valid schemas of classical quantification theory are deducible within it.

He places this objection—that ordinary quantification theory is arbitrarily restrictive—into the context of Henkin's work on branching quantifiers.[72] To motivate this apparent extension of quantification theory, Quine considers the following example due essentially to Henkin:

(1) Each thing bears P to something y and each thing bears Q to something w such that Ryw.[73]

Quine then offers two possible ways for paraphrasing this sentence into ordinary quantification theory:

(2) $(x)(\exists y)(Pxy \,.\, (z)(\exists w)(Qzw \,.\, Ryw))$

(3) $(z)(\exists w)(Qzw \,.\, (x)(\exists y)(Pxy \,.\, Ryw))$.

But here he observes a difficulty: these two proposals are not equivalent. In (2) the choice of 'y' is independent of the choice of 'z' whereas in (3) the choice of 'y' is shown dependent on 'z'; and in (3) the choice of 'w' is independent of the choice of 'x' whereas in (2) the choice of 'w' is

[72] Quine, *Philosophy of Logic*, pp. 89-91; and "Existence and Quantification," in *Ontological Relativity and Other Essays* (New York: Columbia University Press, 1969), pp. 108-13. I will focus on Quine's more detailed discussion in "Existence and Quantification" though the conclusions he draws are the same in both texts. On Henkin's branching quantifiers see Leon Henkin, "Some Remarks on Infinitely Long Formulas," *Infinitistic Methods: Proceedings of the Symposium of Foundations of Mathematics, Warsaw 1959* (New York: Pergamon, 1961), pp. 167-83.

[73] Henkin, "Some Remarks on Infinitely Long Formulas," p. 181.

shown dependent on 'x'. In addition, he also provides a possible interpretation for 'P', 'Q', and 'R' in (1) that renders these dependencies altogether unnecessary by taking 'P' as 'is part of', 'Q' as 'contains', and 'R' as 'is bigger than' which yields

(4) Each thing is part of something y and each thing contains something w such that y is bigger than w.

Thus, it appears that the forced choice between (2) and (3) may be the fault of an arbitrarily restrictive quantificational notation. And in fact, this seems plausible as Henkin's initial solution to the dilemma, adding function letters as values of quantified variables, does avoid the prejudices of both (2) and (3):

(5) $(\exists f)(\exists g)(x)(z)\ (Pxf_x \,.\, Qzg_z \,.\, Rf_xg_z)$.[74]

But admitting function letters as values of bound variables yields a new prejudice not found in (2) or (3). We have now committed ourselves to higher-order mathematical objects, functions, whereas (2) and (3) made no such commitments to any particular brand of objects, mathematical or otherwise. (5) then does not share the elementary character of (2) and (3). Henkin attempts to avoid this higher-order commitment by introducing his branching quantifiers:

$$(6)\quad \begin{matrix}(x)(\exists y)\\ \\ (z)(\exists w)\end{matrix}\ (Pxy \,.\, Qzw \,.\, Ryw)$$

which eliminates the prejudices of both (2) and (3) and also of (5); the choice of 'y' depends only on 'x' and the choice of 'w' only on 'z' while apparently avoiding any commitments to functions

[74] Ibid., p. 181. This may seem a particularly natural way to expand ordinary quantification theory if the theory already contains function letters. However, as noted above, Quine does not include function letters in the austere notation of ordinary quantification theory. Function letters may be used of course as a notational convenience, but so long as we do not quantify over them, we have made no commitment to their existence. The function letters can always be paraphrased away by Russell's method of descriptions. Following Quine's view of unquantified class variables, we may think also of this use of function letters as only "a degenerate specimen" of the theory of functions given such a statement's equivalence to some statement of ordinary quantification theory without function letters. On the degenerate use of class variables see "Logic and the Reification of Universals," p. 114.

thus lending even further support to the view that our original quantification theory was overly restrictive in its notational resources.

Against this view though, Quine urges the completeness of quantification theory, that quantification theory yields complete proof procedures for proving both validity and inconsistency. Either procedure will serve both purposes in that a formula is valid if and only if its negation is inconsistent.[75] In the functionally existential annex however, including sentences such as (5) which is equivalent to (6), there are only complete proof procedures for inconsistency. Likewise, its functionally universal counterpart has only complete proof procedures for validity. The same procedure for proving validity or inconsistency will not apply generally in the extended theory of quantification because the negation of a functionally existential formula is equivalent to a functionally universal formula, not a functionally existential formula. Just the opposite holds in the case of functionally universal formulas. In fact, Quine notes that Craig has shown that the negation of a functionally existential formula is never equivalent to a functionally existential formula except where the functions were unnecessary to begin with, i.e., where the functionally existential formula was already equivalent to some first-order formula. This fact holds equally for functionally universal formulas.[76]

Yet we may remain unconvinced. Why limit logic to ordinary, or first-order, quantification theory? The answer lies in a point stressed in the first section of this paper—logic's concern with implication. In a passage *From Stimulus to Science*, Quine asks,

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[75] Quine, "Existence and Quantification," p. 111. The most relevant proof procedure in this context is based on Skolem's device of showing a formula inconsistent by taking its functional normal form and instantiating it so as to derive a truth functional contradiction from it. Quine presents the technique in "On a Proof Procedure for Quantification Theory [1954]," in *Selected Logic Papers*, pp. 196-204.

[76] William Craig, "Three Uses of the Herbrand-Gentzen Theorem," *Journal of Symbolic Logic* 22 (1957), pp. 269-85, specifically, p. 281. In his paper, Henkin reports a result of Ehrenfeucht's also demonstrating the incompleteness of quantification theory supplemented with branching quantifiers; see "Some Remarks on Infinitely Long Formulas," pp. 181-82. Quine no doubt presents the incompleteness of quantification theory supplemented with branching quantifiers in terms of quantified function letters to make more perspicuous the assumption of substantial mathematical power contained in Henkin's branching quantifiers.

> What defines implication? Elementary predicate logic is enough: the truth functions and quantification. . . . In our mathematical maturity we can encapsulate this logic in a complete formalization describable from scratch in a couple of pages. More briefly still, for those abreast of the jargon, it is as follows. To prove that a given set of premises implies a contemplated conclusion, prove that the premises are inconsistent with the negation of that conclusion. Do so by putting the premises and the negated conclusion into prenex form and then accumulating a truth functional inconsistency by persistent instantiation of the universal and existential quantifiers, taking care to use a new variable for each existential instantiation.
>
> Implication thus defined is all we need to mean by implication. The laws of set theory and the rest of mathematics can be ranged rather among the premises that are doing the implying, on a par with the laws and hypotheses of natural science.[77]

The method Quine describes here for making implications explicit is one of the complete proof procedures for ordinary quantification theory. "Thus," for Quine, "classical, unsupplemented quantification theory is on this score maximal: it is as far out as you can go and still have complete coverage of validity and inconsistency by the Skolem proof procedure."[78] Completeness, assures us that a precise account of implication is ready to hand, by mere description of one the complete proof procedures for classical quantification theory. Quine's account of implication does not rest on the truth of substantial mathematical suppositions such as the axiom of choice, expressible in quantification theory supplemented with quantifiable function letters, or the continuum hypothesis, expressible in quantification theory supplemented with quantifiable predicate letters.[79] Limiting logic to a complete theory, like his substitutional definition of logical truth, is another way in which we manage to keep our feet on the ground.[80] The functionally existential annex, what appeared to be a harmless supplementation of the theory of classical quantification, falls outside this boundary and gives us reason to fairly represent Henkin's branching quantifiers as committed to functions. "Fairly" in that Quine sees the lack of completeness to indicate extra-logical mathematical content in the branching quantifiers;

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[77] Quine, *Stimulus to Science*, pp. 51-2.

[78] Quine, "Existence and Quantification," p. 111-12.

[79] I will have more to say on this point in the following chapter.

[80] Quine, *Methods* [1982], p. 212; *Philosophy of Logic* [1986], pp. 56-8. Juliet Floyd, in conversation, has suggested to me another reason that Quine stresses the importance of completeness. Completeness allows him to maintain an extensional conception of logical implication; there is no need to bring in the modal notion of necessity. Again, a description of one of the complete proof procedures for ordinary quantification theory is all we need mean by 'implication'.

quantification over function letters makes this significant mathematical assumption explicit, he thinks, in a way that the branching quantifiers do not. The same can also be said for Quine's considerations of quantification over predicate letters construed as set variables as discussed above. By Gödel's incompleteness theorem, he remarks, "it follows that set theory, even the mere theory of sets of individuals, admits of no complete proof procedure. In this regard it is like most branches of mathematics."[81]

There is one final issue to take up bearing relevance to Quine's limiting quantification theory at the boundary of completeness. This is his characterization of logical truths as obvious or potentially obvious by a series of individually obvious steps. His choice of the word 'obvious' here is significant as a replacement for more traditional characterizations of logic as self-evident or analytic in that 'obvious' is meant to undercut the overtones of epistemological privilege in a foundational sense associated with the more traditional philosophical terminology as he makes clear in his 1954 "Carnap and Logical Truth":

> I have been using the vaguely psychological word 'obvious' non-technically, assigning it no explanatory value. My suggestion is merely that the linguistic doctrine of elementary logical truth likewise leaves explanation unbegun. I do not suggest that the linguistic doctrine is false and some doctrine of ultimate and inexplicable insight into the obvious traits of reality is true, but only that there is no real difference between these two pseudo-doctrines.[82]

For Quine, truth is truth, and nothing is gained for traditional epistemological aims by trying to distinguish the logical truths from other "ordinary" truths, if this distinction can even be made sense of at all.

Quine returns to this issue in *Philosophy of Logic* by connecting the obviousness of logical truths with translation.[83] He begins by describing a situation in which we are trying to translate some unknown language on the basis of observable behavior. Suppose that a native

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[81] Quine, *Stimulus to Science*, pp. 52.
[82] W.V. Quine, "Carnap and Logical Truth," in Paul Arthur Schilpp, ed., *The Philosophy of Rudolf Carnap* (LaSalle: Open Court Publishing Company, 1963), p. 390.
[83] Quine, *Philosophy of Logic* [1986], pp. 82-3.

were willing to assent to some compound sentence but not to one of its components, this would be reason not to construe the compound as conjunction. The same reasoning would lead us not to translate a sentence as an alternation if a native were willing to assent to a component but not the compound. In such a way, we impose our logic on the native so building it into our translation manual. He is quick to remark though that logic is not all that we build into our translation manual. If the native were unwilling to assent to a certain sentence in the rain, this would be evidence for not translating the sentence as 'It is raining'. In translating, we should aim to construe obvious sentences of the native language into obvious sentences of English. Quine's maxim for translation is "Save the obvious."[84]

Still he recognizes a difference between logic and other branches of science, for logic is more thoroughly obvious than these other branches, but again making his point, that 'obvious' is not meant to carry epistemological significance: "Preparatory to developing this point I must stress that I am using the word 'obvious' in an ordinary behavioral sense, with no epistemological overtones. When I call '1 + 1 = 2' obvious to a community I mean only that everyone, nearly enough, will unhesitatingly assent to it, for whatever reason. . . ."[85] Herein lies the difference between logic and the other sciences: all logical truths are obvious or potentially obvious by a series of individually obvious steps, and "[t]o say this is in effect just to repeat some remarks of Chapter 4: that the logic of quantification and identity admits of complete proof procedures, and some of these are procedures that generate sentences purely from visibly true sentences by steps that visibly preserve truth."[86]

Here Quine presents yet another reason to limit logic to unsupplemented quantification theory. As remarked in our previous considerations of completeness, the mere extension of

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[84] Ibid., p. 83.
[85] Ibid., p. 82.
[86] Ibid., p. 83.

quantification theory to branching quantifiers led to a loss of complete proof procedures and the same holds for set theory, even set theory limited to sets of individuals. There are truths in both of these extensions of quantification theory that are not obvious or obtainable by individually obvious steps. In the previous section, we saw a similar occurrence of a failure of obviousness with regard to the unrestricted comprehension principle. There, Quine focused not on the failure of completeness for higher-order logic, or set theory, but rather how the application of a supposedly elementary logical inference in a theory of classes or attributes led to Russell's paradox making his point that in set theory, "[c]ommon sense is bankrupt."[87] Now we see again, in considering completeness, how the logic of quantification is maximal and a solid and significant body of truths.

What Quine tries to show throughout his considerations of higher-order logic is that such extensions of ordinary quantification theory go beyond what intuitively, or traditionally, we would be willing to label 'logic'. In presenting classical quantification theory, he has brought out its continuity with traditional, or intuitive, views of logic. Hence, he sketches a conception of logic that has as its chief importance implication as such, rather than implication that shows prejudice towards some particular subdomain of objects whether they be attributes, sets, or functions. His focus, like that of the pre-Frege-Russell tradition in logic, is on form; logic itself has no objects which to call its own. Additionally, the importance Quine attaches to implication for logic, leads him to emphasize that "logic" lacking in complete proof procedures goes beyond the boundary of what he recognizes as logic for it fails to yield a notion of implication that is both precise and independent of substantial mathematical claims. Finally, we have seen how Quine appeals to the traditional description of logic as a self-evident body of truths. He maintains this view by redescribing logic as 'obvious', stripping away the traditional

[87] Quine, "Whitehead," p. 27; see also footnote 19.

epistemological overtones of 'self-evident', and indicating how ordinary quantification theory has this feature of obviousness, while other higher branches of mathematics do not.

Someone may still object to all of these considerations. What has Quine really proved? Why should Quine, a philosopher who has broken with so much of traditional philosophy, hold fast to any of these ideas in carving out a body of truths on which to bestow the honorific "logic"? In his essay "Existence and Quantification," Quine himself responded that he proved nothing.[88] To prove something, to show us what logic *really* is was not Quine's aim, nor had it ever been.[89] His purpose rather was to show how the logical tradition stemming from Frege and Russell failed to recognize important cleavages between ordinary predicate logic and so-called higher-order logic. This failure to draw a distinction between logic narrowly construed and logic broadly construed led to the view that mathematics had in essentials been reduced to logic, with all the epistemological privilege the tradition would associate with such a reduction. Quine's aim was to bring out how the move to quantification over predicate letters was a move far less innocent than it appeared for it allowed logic to presuppose the power of higher mathematics, namely set theory. As he explains,

> The reduction of classical mathematics to one or another so meager a conceptual basis was amazing and illuminating, but calling it a reduction of mathematics to logic—*logicism*, in a word—gave the wrong message. Logic was proverbially slight and trivial. Mathematics proverbially ranged from the profound to the impenetrable, and reduction of mathematics to logic challenged belief, as indeed it well might. The reduction was to the unbridled theory of classes, or set theory, which, far from being slight and trivial, is so strong as to tangle itself in paradox until bridled in one way or another. . . . But what it shows is that the startling reduction of mathematics is to something far richer than traditional logic. I prefer to limit the term

[88] W.V. Quine, "Existence and Quantification," p. 112.

[89] Of Quine's dissertation Dreben writes,

> Between 1916, when Sheffer began to teach at Harvard, and 1932 eleven dissertations were written on the nature of logical implication, the nature of logical systematization, or the nature of logical justification. Quine's however was not one of these. In a foreshadowing of his great debate with Carnap, he showed no sign of having shared his teachers' conviction that logic is *sui generis* and requires a general philosophical account.

In "Quine," p. 83.

> "logic" to the logic of truth functions, quantification, and identity, drawing the line at the reification of classes.[90]

We see here Quine not disparaging the reduction but only explaining that it was not a reduction to the obvious or self-evident foundation that Frege and Russell sought. Bestowing upon higher-order logic the label 'set theory' is not, then, in any way to show the reduction disrespect; it is rather to make clear what precisely the reduction was. Once, these assumptions have been made explicit, what we choose to call logic is merely a terminological issue.[91] What we see in Quine's identification of logic with classical quantification theory is his adherence to the ordered pair as a philosophical paradigm—his special, if not unique, concern with clarity and explicitness. We do not claim synonymy or to make explicit hidden meanings that the speaker had in mind all along, but rather, "devise a substitute, clear and couched in terms to our liking, that fulfills those functions." For Quine, logic's importance is its ability to make sense of and to make explicit the logical relation of implication, and classical quantification theory fulfills this function to Quine's satisfaction. The answer as to whether what he has presented is *really* logic is a non-starter for him. Indeed, in another context, we find just this sort of response to an objection of Kripke's:

> One of Kripke's moral precepts deplores "the tendency to propose technical criteria with the aim of excluding approaches one dislikes" (p. 410). He notes in illustration that I adopted a criterion of ontological reduction for no other reason than that it "includes well-known cases and excludes undesired cases." *I protest that mine was expressly a quest for an objective criterion agreeing with our intuitive sorting of cases. This is a proper and characteristically philosophical sort of quest, so long as one knows and says what one is doing.*[92]

Much the same can be said for Quine's conception of logic.

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[90] W.V. Quine, "Confessions of a Confirmed Extensionalist," in *Future Pasts: the Analytic Tradition in Twentieth-Century Philosophy*, Juliet Floyd and Sanford Shieh, eds. (New York: Oxford University Press, 2001) p. 219; similar considerations are found in Quine's "Epistemology Naturalized," in *Ontological Relativity*, pp. 69-70, 73.
[91] See for example W.V. Quine, "Carnap and Logical Truth," pp. 387-89; and his entry on "Predicate Logic," in *Quiddities: an Intermittently Philosophical Dictionary* (Cambridge: Harvard University Press, 1987), p. 158.
[92] W.V. Quine, "Responding Further to Kripke," in *Theories and Things* (Cambridge: Harvard University Press, 1981), p. 175. Quine's page reference is to Saul Kripke, "Is There a Problem About Substitutional Quantification?" in *Truth and Meaning*, Gareth Evans and John McDowell, eds. (Oxford: Oxford University Press, 1976), pp. 325-419. (My emphasis)

Chapter II: Boolos on Second-Order Logic

In this chapter we will consider George Boolos's response to Quine's criticisms of second-order logic. In his 1975 "On Second-Order Logic", Boolos sets out to call into question the view "commonly supposed that the arguments of Quine and others for not regarding second- (and higher-) order logic as logic are decisive. . . ." He describes his own interest in this dispute as one of a quasi-terminological nature, a deliberation over "the extent to which second-order logic is (or is to be counted as) logic, and the extent to which it is set theory." Whether second-order logic may bear the (honorific) label "logic" or that of "set theory" is of little significance. What is of significance to him are the reasons that can be offered in support of the two positions.[1]

From our considerations of Quine's conception of logic in the previous chapter, we see already that that the dialectic between Boolos and Quine will not be as straight-forward as we might have thought. For Quine himself is not among those who think that decisive arguments can be offered to decide the bounds of logic. The best we can do is to be explicit about how and what we are willing to apply the label "logic" to relying perhaps on certain intuitive characteristics we may think logic should have. As we have seen, this part of the terminological issue is of the greatest importance for Quine. What he has emphasized is that merely applying the label "logic" does not guarantee that what we apply it to will have the sorts of features traditionally thought to characterize logic. This is of particular significance in context of logicism where the reduction to something called "logic" was supposed to secure a solid foundation for all of mathematics. In contrast, Boolos's arguments proceed, mostly, by

[1] Boolos, "Second-Order Logic," p. 37; I think Boolos presents his own position here as far more neutral than it sometimes comes across in the body of his paper. To me, it seems, he is often presenting a defence of second-order logic as logic. Still, I have tried throughout this chapter to inform my interpretation of his position with his initial declaration of neutrality in mind.

comparing first-order logic, second-order logic, and set theory. However, he offers very little independent reason for us to consider why any of these theories should be considered logic further complicating a straightforward assessment as to the success of his arguments against Quine's position. With this in mind, the possibility that Boolos and Quine are talking past one another should always be retained as the proper assessment of this debate, and perhaps also one reason that Quine never responded to much of Boolos's attack.

I

Boolos begins his discussion by expressing his approval of Quine's having deplored two confusions concerning quantification over predicate letters—that '$(\exists F)$' and '$(\forall F)$' say that some or all predicates or predicate expressions are thus and so and that quantification over attributes has ontological advantages over quantification over sets. He rejects Quine's view though that quantification over predicate letters is to be deplored even when their values are taken to be sets because predicates are not names of their extensions. To make his point, Boolos returns to a passage from Quine, which we saw already in the previous chapter. Recall that Quine argued there,

> Consider first some ordinary quantifications: '$\exists x$ (x walks)', '$\forall x$ (x walks)', '$\exists x$ (x is prime)'. The open sentence after the quantifier shows 'x' in a position where a name could stand; a name of a walker, for instance, or of a prime number. The quantifications do not mean that names walk or are prime; what are said to walk or to be prime are things that could be named *by* names in those positions. To put the predicate letter 'F' in a quantifier, then, is to treat predicate positions suddenly as name positions, and hence to treat predicates as names of entities of some sort. The quantifier '$\exists F$' or '$\forall F$' says not that some or all predicates are thus and so, but that some or all entities of the sort named by predicates are thus and so.[2]

Boolos is willing to grant that predicates are not names but not that quantifiable variables do not belong in predicate positions. To make his point, he then poses this inverted passage against the quotation from Quine:

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[2] Quine, *Philosophy of Logic* [1986], pp. 66-7. (Quine's italics)

> Consider some extraordinary quantifications: '$(\exists F)$(Aristotle F)', '(F)(Aristotle F)', '$(\exists F)(17\ F)$'. The open sentence after the quantifier shows 'F' in a position where a predicate could stand; a predicate with an extension in which Aristotle, for instance, or 17 might be. The quantifications do not mean that Aristotle or 17 are in predicates; what Aristotle or 17 are said to be in are things that could be had *by* predicates in those positions. To put the variable 'x' in a quantifier, then, is to treat name positions suddenly as predicate positions, and hence to treat names as predicates with extensions of some sort. The quantifier '$(\exists x)$' or '(x)' says not that some or all names are thus and so, but that some or all extensions of the sort had by names are thus and so.

And concludes, supposing Quine had argued this instead, "we should have wanted to say that the last two statements were false and did not follow from what preceded them. It seems to me that the same ought to be said about the argument Quine actually gives."[3] Boolos explains that quantifying over 'F' may be to treat 'F' as having a range, but this does not mean that these predicates become names of any sort of entity. Quine's mistake is in supposing that because ordinary variables, individual variables, always occur where a name could stand, this must hold for all kinds of variables. The two concluding sentences of Boolos's parallel passage for extraordinary quantifications do indeed fail, but not, I think, in the way he intends them to.

Let us consider the two passages more carefully and see where they do or do not go wrong. First, we have Quine's ordinary quantification '$(\exists x)(x$ walks$)$'. In this sentence, 'x' can be instantiated by an object, any object of the domain, and we use a name to mention the object, so 'x' is in a position where a name could stand. For example we could use the name 'Aristotle' to mention the object, the person, Aristotle, and say 'Aristotle walks'. It would be difficult, really nonsensical, to try to place the person Aristotle himself into the sentence, but using a name to mention him easily dispenses with this awkwardness. The case of the predicate is different; we just have the predicate '① walks' itself occurring in the sentence, and this predicate has an extension, specifically the set of walkers, '$\{y\colon y$ walks$\}$', though nothing is said of this extension here. The occurrence of a predicate in a sentence commits us to no further entity of which it is a name. We could however re-write the sentence so that it does say something of the extension,

[3] Boolos, "Second-Order Logic," p. 38. (Boolos's italics)

the thing or object, that the predicate has thus: '$(\exists x)(x \in \{y: y \text{ walks}\})$'. Both renderings of this sentence have the logical form '$(\exists x)Fx$'. Except in this second version, the name '$\{y: y \text{ walks}\}$' names the set of walkers, $\{y: y \text{ walks}\}$, which is the extension of the predicate '① walks'. Aristotle is a member of this set: 'Aristotle $\in \{y: y \text{ walks}\}$' which invites additional ordinary objectual quantifications if we add a second sort of variable that ranges only over sets, such as '$(\exists \alpha)(\exists x)(x \in \alpha)$' and '$(\forall \alpha)(\exists x)(x \in \alpha)$'.

Comparing Boolos's parallel passage for predicates, here we have extraordinary quantifications such as '$(\exists F)$(Aristotle F)', '$(\forall F)$(Aristotle F)', and '$(\exists F)(17\ F)$'. Boolos explains that the open sentence shows 'F' in a predicate position, so here we could insert a predicate such as '① walks' that has the extension '$\{y: y \text{ walks}\}$' in which, for example, Aristotle might be. The extraordinary quantifications do not say Aristotle is in a predicate, but rather that Aristotle is in *things* that could be had by predicates in this position, and these things are extensions, or sets. But read with Boolos's locutions "for all things had by a predicate F such that" or "for some thing had by predicate F such that", the quantifications over predicate letters seem to actually be quantifications over extensions, or sets. By both Boolos and Quine's accounts then, the things had by predicates are extensions, or sets.

Now to assess the concluding two sentences of which Boolos thinks we should have wanted to say are false and do not follow from what preceded them: "To put the variable 'x' in a quantifier, then, is to treat name positions suddenly as predicate positions, and hence to treat names as predicates with extensions of some sort. The quantifier '$(\exists x)$' or '(x)' says not that some or all names are thus and so, but that some or all extensions of the sort had by names are thus and so." For example, existentially quantifying over 'x' we get something like '① is x' or '① x's', where a predicate such as '① aristotlizes' could go. When interpreted under Boolos's suggested locution for quantification over predicate letters, the extraordinary quantification does not force an incoherent reading. Instead, the quantification is over the extension of '① is x', over the predicate's extension, a set '$\{z: xz\}$'. The quantification treating the name position as a predicate position no longer treats indiscriminately among the objects of the domain but specifically of sets. Instead of saying "There is some object x such that x walks" as in Quine's name passage, the extraordinary quantification says 'There is some thing that the predicate x has such there is some thing that the predicate F has such that the thing that x has is included in the thing that F has' or alternatively, 'There is some set $\{z: xz\}$ such that there is some set $\{y: Fy\}$ such that $\{z: xz\}$ is included in $\{y: Fy\}$' or even more simply '$(\exists\alpha)(\exists\beta)(\alpha \subseteq \beta)$'.

What Boolos wants his readers to say is false and does not follow in his last two sentences is that names do not get turned into predicates with extensions of some sort when they are quantified over just because the extraordinary predicate letter quantifications quantify over the extensions had by predicates. This should lead us to the same conclusion about Quine's name passage—that just because some variables stand in name positions does not mean that all variables do. This does not work as Boolos would have liked. As we just saw, his predicate letters remain schematic, marking places where actual predicates could be substituted. What looks to be quantification over predicate letters, he explains, is quantification over the things had by predicates in these positions, over their extensions. Put this way no use of 'name' or any of its cognates is made. But this locution seems to obscure the fact that we are indeed quantifying over *the things* predicates have which nicely leads to a point Quine has repeatedly emphasized—that extending quantification to predicate letters is not an innocent extension of ordinary quantification theory but rather introduces sets to the quantification theory. These quantifications are fairly, and more honestly, shown by quantifying over sets, and these set variables occur in positions where names of sets could stand.

The apparently extraordinary quantifications over predicate letters then are not so extraordinary after all. They are just ordinary quantifications over a subdomain of objects, sets, and Quine certainly does not want to deny the legitimacy of these quantifications. In fact, this sort of quantification is already available in Quine's ordinary quantifications. Quantification over extensions is objectual quantification—quantification over places where names could stand—names of objects, specifically sets in the case of extensions. The variables 'x', 'y', 'z' will already admit sets as their values should we want to allow sets into our ontology. And if we want to distinguish set variables from individual variables we may admit the variables 'α', 'β', 'γ' to range separately over sets. This move leaves the schematic predicate letters available for the places where they are actually needed without any suggestion that they may also serve as quantifiable variables.[4] Quine and Boolos can agree so far as predicates not naming anything. What Quine will object to is Boolos's allowing predicate letters to do double duty so as not to make explicit the predicate variables' commitments to quantification over sets. Boolos wants to say that the predicate letter stands in a place where a predicate could go, so they are schematic in Quine's sense. But then when he quantifies over the predicate letter, he quantifies over the things predicates could have as their extensions. He does not add a distinct variable for these quantifications, so it is hard to understand the predicate letters otherwise than as now ranging over extensions, as being in positions where names could go, specifically names of sets.

What we should have wanted to say is false and does not follow from what preceded it is that names do not have extensions, but what we have seen as correct is that when quantifying

[4] For a discussion of just these sorts of issues see W.V. Quine, "Logic and the Reification of Universals," in *From a Logical Point of View*, pp. 112-13.

over extensions, the variable marks a place where a name could stand, a name of the things in the range of the variables. And we reach this correct conclusion whether we consider Quine's original passage or Boolos's inverted passage. A more promising strategy for Boolos perhaps would have been not to focus on Quine's view that quantified predicate letters say that some or all entities *named by* predicates are thus and so, but that quantified predicate letters do not force existential commitments to the extensions that these predicate variables could have.[5]

Boolos next turns his criticisms towards Quine's suggestion that rather than quantify over predicate letters, the logician who wants to admit these quantifications should instead replace the logical schema with its set-theoretic analogue and then quantify over the set variables. Boolos explains,

> In the same section of *Philosophy of Logic* Quine has some advice for the logician who want to admit sets as values of quantifiable variables and also wanted distinctive variables for sets. The logician should not, Quine says write 'Fx' and there upon quantify on 'F', but should instead write '$x \in \alpha$' and then, if he wishes, quantify on 'α'. The advantage of the new notation is thought to be its greater explicitness about the set-theoretic presuppositions of second-order logic.[6]

Boolos's explanation of Quine's suggestion is not quite right. In the passage Boolos refers to, Quine is instead further explicating the difference between schematic predicate letters and variables, as we saw in the previous chapter. Reiterating his point that predicates are not names of attributes or sets, he recommends instead replacing the predicate letters with ordinary variables and reading them as 'x has y', or 'x has ζ' using distinct attribute variables. The same can be done for the logician who rejects attributes in favor of taking the values of predicate letters to be sets writing instead '$x \in y$', or '$x \in \alpha$' using distinct set variables. Quine's recommended notation will of course have certain consequences for how the set- (or attribute-)

[5] Indeed this is the strategy we will see Boolos pursuing in the next chapter.
[6] Boolos, "Second-Order Logic," p. 39.

theoretic presuppositions of second-order logic are displayed, but this is not Quine's primary purpose here.

The way Boolos understands Quine's recommendation brings out an important difference between the ways these two philosophers regard second-order logic. Boolos, for the purposes of his present investigation into second- (or higher-) order logic, grants that second- (or higher-) order logic may be a distinct theory, perhaps independent of both logic and set theory, of which he can inquire into what extent it is like set theory and to what extent is it like first-order logic. Quine, in contrast, sees reasons to reject a distinct theory of higher-order logic identifying it instead as a way of formalizing set theory, and like all set theories, its set-theoretic commitments will vary depending on the strength of the set theory involved. As we have seen, it may include just individuals and sets of individuals or it may include the whole of the type-theoretic hierarchy (construed cumulatively or, as Quine does, non-cumulatively), individuals, sets of individuals, sets of sets of individuals and so on. This difference is worth noting as Boolos will place much emphasis on accounting for *the* set-theoretic presuppositions of second-order logic, whereas for Quine, they will vary according to how strong the higher-order logic, i.e., the set theory, is.

A related point can be made concerning the reason Boolos suggests as to why Quine may want to make explicit the set-theoretic presuppositions of second-order logic. He explains,

> In order to give a theory of truth for a first-order language which is materially adequate (in Tarski's sense) and in which such laws of truth as "The existential quantification of a true sentence is true" can be proved, it is not necessary to assume that the predicates of the language have extensions, although it does appear to be necessary to make this assumption on order to give such a theory for a second-order language.[7]

This does seem to be something like what Quine has in mind though he does not make the point with explicit reference to Tarski's theory of truth. Both Boolos and Quine understand predicates as having extensions; the difference is that in a first-order setting these extensions are not

[7] Ibid., p. 40; Boolos restates this point on p. 48.

quantified over.[8] We should be careful here however on the issue of model theory. Boolos's reference to Tarski may be misleading as to the source of Quine's criticisms. Quine's concerns over second-order logic are not concerns about its model-theoretic semantics though, as we have seen, he is concerned with how we are to interpret, or how we are to understand, what we are quantifying over when we allow quantification over predicate letters. He urges that ordinary individual variables or set variables replace the quantified predicate letters because these higher-order quantifications are not over predicates but over their extensions. His set-theoretic analogue makes explicit the departure from ordinary quantification theory in favor of a move towards a powerful mathematical theory.[9]

Boolos does offer some counter considerations for not accepting Quine's set-theoretic analogue notation, though. For one, he thinks Quine's notation does not capture certain aspects of logical form in the same striking way as the standard notation for second-order logic. In second-order logic, the definition of the strong ancestral is written '$(\forall F)[(\forall x)(aRx \supset Fx) \,.\, (\forall x)(\forall y)(Fx \,.\, xRy \supset Fy) \supset Fb]$' whereas in Quine's set-theoretic notation it is re-written as '$(\forall \alpha)[(\forall x)(aRx \supset x \in \alpha) \,.\, (\forall x)(\forall y)(x \in \alpha \,.\, xRy \supset y \in \alpha) \supset b \in \alpha]$'. Additionally, and more importantly from Boolos's perspective, is that Quine's notation results in a loss of validity or implication for some second order formulas: '$(\exists F)(\forall x)Fx$' is a valid second-order formula whereas '$(\exists \alpha)(\forall x)x \in \alpha$' is not, and '$x = y$' is logically implied by '$(\forall Y)(Yx \supset Yy)$' but not by

[8] Of course, as Quine points out, we are free to construe the schematic predicate letters by their set-theoretic analogues, but so long as we do not go on to quantify over the set variables, the set-theoretic analogues remain equivalent to logical schemas. '$(\exists x)(x \in \alpha \,.\, x \in \beta)$' is, as Quine describes it, "a degenerate specimen" of set theory as it is equivalent to the logical schema '$(\exists x)(Fx \,.\, Gx)$'. See W.V. Quine, "Logic and the Reification of Universals," p. 114; and Chapt. I, fn. 74.

[9] Indeed, the origins of Quine's claim that quantification over predicate letters amounts to quantification over sets can be found at least as early as his 1937 "New Foundations for Mathematical Logic". To the best of my knowledge, the model theory for higher-order logic was not given in its full generality until 1964 in David Kaplan and Richard Montague, "Foundations of Higher-Order Logic," in *Logic, Methodology, and Philosophy of Science: Proceedings of the 1964 International Congress*, Yehoshua Bar-Hillel, ed. (Amsterdam: North-Holland Publishing Company, 1965), pp. 101-11.

'$(\forall\alpha)(x \in \alpha \supset y \in \alpha)$'. There is reason to think that Quine will not feel the force of Boolos's counter considerations.

First, for Quine, '$(\forall F)[(\forall x)(aRx \supset Fx) \,.\, (\forall x)(\forall y)(Fx \,.\, xRy \supset Fy) \supset Fb]$' is not a logical form. Rather, this sentence recognizes a particular subdomain of objects in its quantified predicate letters, namely sets, and is properly construed as the sentence Quine suggests, a sentence of set theory where the set variables range over subsets of whatever the individual variables range over.[10] Similarly for the loss of validity or implication—for Quine, Boolos's second-order quantifications are just statements of set theory. This is made explicit when '$(\exists F)(\forall x)Fx$' is rewritten as '$(\exists\alpha)(\forall x)x \in \alpha$', a true statement in a type theoretic set theory (though as it stands, this statement is typically ambiguous) and in Quine's NF (New Foundations), and false in ZF (Zermelo-Fraenkel set theory). And again, '$x = y$' is not a logical implication as Quine understands it but is implied set-theoretically by '$(\forall\alpha)(x \in \alpha \supset y \in \alpha)$' where the sets range over subsets of individuals; in short in a type-theoretic set theory up to type 1. A proof of this implication in ZF will also require an appeal to the non-logical extensionality axiom, '$(\alpha)(\beta)((\forall x)(x \in \alpha \equiv x \in \beta) \supset \alpha = \beta)$' for the assurance that sets that have the same members are the same set. What we see emerging here more clearly is that there looks to be a lack of common ground for Quine and Boolos to stand on. Boolos recognizes a theory of higher-order logic which he can compare with both first-order logic and set theory in an attempt to establish what features higher-order logic shares with each of first-order logic and set theory. For him it is an open possibility as to whether logical implication or validity applies in higher-order logic. Quine on the other hand rejects that there is any such higher-order logic. What

[10] Boolos similarly confuses actual sentences with schemas on p. 51 calling '$(\exists x)(\exists y)\neg x = y$' both a sentence and a schema. There is nothing schematic about this formula though if '=' is counted as a logical symbol, as Boolos does. Quine, on the other hand, can count identity as schematic defining it by exhaustion of combinations of variables with a finite number of schematic predicate letters as seen in the previous chapter; see W.V. Quine, *Philosophy of Logic* [1986], pp. 63-4.

some philosophers have called "higher-order logic" is best construed as one of the ways that set theory can be formalized. Notions such as validity and logical implication do not apply as ways of characterizing truths of higher-order theories unless of course these higher-order validities or logical implications rely only on the resources of ordinary quantification theory for their validity or to do their implying.

Boolos now turns to Quine's remarks, discussed already in the preceding chapter, on the comprehension principle for higher-order logic. As we saw in our earlier discussion of this passage, Boolos first notes that Quine's objection here seems to be that the higher-order predicate calculus gestures at inconsistency, but as Boolos admits and as we have seen, Quine certainly does not think that higher-order logic is inconsistent, at least not as Boolos presents it. His point was instead that without imposing some ad hoc restriction on the comprehension axiom, the higher-order predicate calculus in its quantification over predicate letters does yield the inconsistent unrestricted comprehension principle. Quine takes this as one way to distinguish the thoroughgoing obviousness of ordinary quantification theory from the extra-logical mathematics of the higher-order predicate calculus.

Boolos goes on to consider how the existential assumptions of higher-order logic compare with those of set theory, how set theory's staggering existential assumptions are hidden in the shift from schematic predicate letter to quantifiable variable. Again, as shown in the earlier chapter, what Quine means by "staggering existential assumptions" will depend on the strength of the set theory being considered. In the 1970 edition of *Philosophy of Logic* and in *Set Theory and its Logic* he considered the entire type-theoretic hierarchy while in the 1986 edition he dropped the word "staggering" and considered only a theory of individuals and sets of individuals. Boolos adds a further observation. He points out that

> the validity of '$\exists X\forall x(Xx \equiv \neg x \in x)$', which certainly looks contradictory, would at any rate seem to demonstrate that their [higher-order predicate calculi's] existence assumptions must be regarded as "vast." A problem now arises: although '$\exists X\exists xXx$' and '$\exists X\forall xXx$' are also valid, '$\exists X\exists x\exists y(Xx \wedge Xy \wedge x \neq y)$' is not valid; it would seem that despite its affinities with set theory and its vast commitments, second-order logic is not committed to the existence of even a two-membered set.[11]

He proposes that consideration of the notion of validity in second-order logic will resolve this tension and "show a certain surprising weakness in second-order logic."[12]

A second-order sentence is valid when it is true under all its interpretations, and it follows from others when it is true under all its interpretations under which the other sentences are also true, he explains. And an interpretation for standard second-order logic, in which the second-order quantifiers range over *all* subsets of the range of the first-order quantifiers, is an ordered pair of a non-empty set, the domain, D and an assignment of an interpretation function I to each non-logical constant of the sentence that takes each of them to appropriate items constructed from D. The domain of the function is the set of all n-tuples of members of D if the constant is of degree n, and its range is a subset of D if the constant is a function constant and a subset of $\{T, F\}$ if it is a predicate constant. What Boolos points out as important in this notion of second-order interpretation is that it requires no explicit reference to a separate range for the second-order variables occurring in the sentence. An interpretation for a sentence of standard second-order logic is just the same thing as an interpretation for a sentence of first-order logic, an ordered pair $\langle D, I \rangle$. An existentially quantified sentence $\exists\alpha F(\alpha)$ is true under an interpretation then, when $F(\beta)$ is true under some interpretation J differing from I at most only on what J assigns to the constant β, where β does not occur in $\exists\alpha F(\alpha)$ and is of the same logical type as α. Like the notion of interpretation for a second-order language, the definition of truth in an interpretation requires no specific mention of what sort of variable—individual, sentential,

[11] Boolos, "On Second-Order Logic," p. 40.
[12] Ibid., p. 41.

function, or predicate—α is. Furthermore, when this definition is restricted to individual variables, Boolos observes, this is just a paraphrase of one standard account of truth in an interpretation.[13] While Quine has stressed the discontinuities of first-order logic with its higher-order counterpart, Boolos believes the continuity of the notion of an interpretation between first- and second-order languages shows otherwise. This continuity provides him with a straightforward and obvious way to extend the definition of truth in an interpretation and so also the standard first-order accounts of validity and consequence to second-order sentences.

Having shown how these notions extend to higher-order logic, Boolos can now offer an explanation of the validity of '$(\exists X)(\forall x)(Xx \equiv \neg x \in x)$'. It is valid simply because given any I, there will always be some appropriate J in which '$(\forall x)(Bx \equiv \neg x \in x)$' is true by assigning to 'B' the set of all objects in the domain D not bearing to themselves the relation that I assigns to '$\in$'. As the domain of I is a set, ZF's axiom of separation, '$(\forall z)(\exists y)(\forall x)(x \in y \,.\equiv.\, x \in z \,.\, Fx)$', guarantees the existence of such a subset of the domain. On the other hand, ZF offers no guarantee that there is a set of all sets, therefore the validity of '$(\exists X)(\forall x)(Xx \equiv \neg x \in x)$' does not assert the existence of a set of all non-self-membered sets which would yield Russell's paradox. Though Boolos observes trouble still arises if we suppose 'x ' to range over all sets and 'X' to range over sets of all objects that 'x' ranges over, and we interpret '$\in$' as membership. Because he takes the validity of '$(\exists X)(\forall x)(Xx \equiv \neg x \in x)$' to depend on a set-theoretic semantics done in ZF, this sentence now turns out false but not invalid, for in ZF there is no set, 'X', of all non-self-membered sets, 'x'; there is, as Boolos remarks, no interpretation whose domain contains all sets.

This way of blocking Russell's paradox leads to the surprising weakness Boolos finds in second-order logic. He explains, "Our difficulty is thus circumvented, but at some cost. We

[13] Ibid., p. 41.

must insist that we mean what we say when we say that a second-order sentence is valid if true under all its interpretations, and that an interpretation is an ordered pair of a *set* and an assignment of functions to constants."[14] As shown by the validity of '$(\exists X)(\forall x)(Xx \equiv \neg x \in x)$' and '$(\exists X)(\forall x)Xx$', second-order logic, unlike first-order logic, appears unavailable for formalizing discourse about certain sorts of objects, such as sets or ordinals, in case there is no *set* to which all the objects of that sort belong, i.e. when there is no domain D of objects of that sort that forms a set. However, ZF, for example, is often construed in first-order notation with the quantifiers supposedly ranging over all sets even though there is in ZF no set to which all sets belong. So to maintain the truth of '$(\exists X)(\forall x)(Xx \equiv \neg x \in x)$' and '$(\exists X)(\forall x)Xx$' as valid sentences according to the standard account of the conditions under which a sentence is true, '$\forall x$' cannot be taken to range over all sets in either sentence and cannot be taken to range over all ordinals in the second. For if '$\forall x$' ranges over all sets, we can generate the Russell paradox and if over all ordinals, the Burali-Forti. Under these circumstances then, second-order logic is unavailable for formalizing discourse about all sets or all ordinals, very much in contrast with first-order logic.

Quine, in response, would find all of Boolos's discussion of the natural extension of the notions of interpretation, truth in an interpretation, and validity to second-order logic completely beside the point. Quine thinks he has offered sufficient reasons not to recognize so-called higher order logic as logic, so these notions simply do not apply beyond first-order logic. In a footnote, Boolos observes, "In Part IV of [*Methods* (1972)], Quine extends the notion of validity to first-order sentences with identity and discusses higher-order logic at length, but does not describe the extension of the notion of validity to second-order logic."[15] Given what we have already seen of

[14] Ibid., p. 42. (Boolos's italics)
[15] Ibid., p. 42, fn 9; W.V. Quine, *Methods of Logic* [1972], 3rd ed. (New York: Holt, Rinehart, and Winston, Inc., 1972), pp. 209-56.

Quine's views of higher-order logic, it should come as no surprise that he does not extend validity to higher-order logic. For Quine, there is no higher-order validity because he understands higher-order logic to be properly construed as a theory of sets. Validity is a logical notion associated with ordinary quantification theory. There is no equivalent notion of validity for set theory, only the notion of being true in a set theory formalized according to certain axioms that yield set theories of greater and lesser strengths. Though, again, the axioms themselves and the theorems obtained from them can be characterized as valid, if they can already be shown valid by the resources of ordinary quantification theory alone.

Indeed, examining Part IV of *Methods*, we find less than a page covering higher-order logic if higher-order logic is to be understood simply as the extension of quantification theory to quantification over predicate letters, and Quine's remarks here focus on the sorts of confusions he sees involved in allowing such extensions of quantification theory. The chapter itself, Chapter 43, is entitled "Classes," and it initiates his discussion of the theory of classes, or sets, and then advances through the construction of the natural numbers and the ancestral culminating with Chapter 46 "Systems of Set Theory." He proposes his usual resolution to the unclarities of higher-order logic—a type-theoretic set theory with the ordinary (individual) variables 'x', 'y', 'z', . . . , the class variables 'α', 'β', 'γ', . . . , and the predicate '$\in$' for 'is a member of'. The variables 'x', 'y', etc. range over an unspecified universe U, while the class variables 'α', 'β', etc. range over a distinct but related universe U_1 of the subclasses of U.[16] The simple sentences of this theory are sentences of the form '$x \in \alpha$' where an ordinary variable stands to the left of '$\in$'

[16] So long as the class variables range over all subsets of the entities that the variables of U range over, Quine's type theory with separate but related domains is equivalent to Boolos's second-order logic with a single domain. See Shapiro, *Foundations without Foundationalism*, p. 74; Richard Montague shows how the model for higher-order logic with cumulative types is extended to higher-order logic with non-cumulative types, as Quine construes it. See Montague's "Set Theory and Higher-Order Logic," in *Formal Systems and Recursive Functions: Proceedings of the Eighth Logic Colloquium Oxford, July 1963*, J.N. Crossley and Michael Dummett, eds (Amsterdam: North-Holland Publishing Company, 1965), pp. 144-45, .

and a class variable stands to the right of '$\in$'. Hence, Quine blocks Russell's paradox not by appeal to ZF but by ruling out the specification of a set by the condition '$\neg ① \in ①$'. As Quine made the point in *Philosophy of Logic* [1986], the paradox is blocked so long as the ranges of the first- and second-order variables are kept apart. Complex sentences are then built up from these by quantification and the truth functions.[17] This is the theory that Quine discusses at length in Part IV of *Methods*, so we can only assume that this type-theoretic set theory is what Boolos refers to as "higher-order logic" in the footnote quoted above. And as we saw in the previous chapter, this is the theory, or some fragment of it, that Quine takes philosophers and logicians to mean by higher-order logic.[18]

Granted that Quine and Boolos's considerations are directed at the same theory, it is worth pausing to note the differences between their presentations of it. First, as we have already seen, Quine replaces the quantified predicate letters with class variables; second, he construes these class variables as ranging over a distinct but related universe U_I made up of the subclasses of U; and third, he introduces the notation '$\in$' for the membership predicate. All of these devices serve to distinguish set theory, i.e. Boolos's higher-order logic, from ordinary quantification theory. In contrast, Boolos presents higher-order logic, i.e. Quine's set theory, emphasizing its continuities with first-order logic. We have already seen him explain how the notion of interpretation does not change from first- to higher-order logic, noting in particular that there is no need to mention a separate domain for the second-order variables when the second-order variables are construed as ranging over all subsets of the domain. In addition, the definition of truth in an interpretation requires only a supplementary clause to handle the new

[17] Quine, *Methods* [1972], pp. 235-36.

[18] See for example W.V. Quine, *Set Theory and its Logic*, p. 258; "Response to Ullian," in *Philosophy of W.V. Quine*, p. 591-92.

second-order quantifiers, and again requires no specific mention of what sort of variable the quantifiers range over.

Boolos does not stress however that the second-order quantifiers range over subsets of the domain, and what assures this is that the function, a variable assignment function, assigned to each variable will take each first-order variable to a member of *D* and each second-order variable to a subset of *D*. The extension of the notion of truth in an interpretation to cover second-order quantification then yields the locutions 'there is a subset *U* of the universe' for '$\exists U$' and '*x* is an element of *U*' for '*Ux*' which essentially treats membership as logical.[19] Boolos's account of interpretation and truth in an interpretation, while accurate, draws attention away from the very features of second-order logic that Quine sees as crucial to distinguishing it so much from ordinary quantification theory, to the extent that it belongs with set theory. I take it that Boolos does not mean to deceive his readers by presenting second-order logic in the way he does. Rather, as we will see shortly, he just does not share the kinds of concerns that moved Quine to equate higher-order logic with set theory.

II

Boolos next turns to his most extended discussion of the ways in which second- (and higher-) order logic comprise existential commitments to sets. He initiates his discussion

[19] On the notion of "variable assignment" see Shapiro, *Foundations*, p. 72. While Shapiro also notes that the notion of interpretation does not change from first- to second-order logic, he makes much more explicit how first- and second-order logic differ in what values their variables take. He remarks on how second-order logic treats membership as logical on p. 6. Though he also thinks there is a distinction between what he calls 'logical sets' and the sets of the iterative hierarchy which use the non-logical '$\in$' to indicate membership; for criticism of this view see Jané, "A Critical Appraisal of Second-Order Logic," pp. 75-8. The locutions for reading second-order existential quantification and "predication" are given by John P. Burgess in his introduction to Boolos, *Logic, Logic, and Logic*, p. 7; Richard Montague's notation for higher-order logic uses a notation for higher-order logic that includes the logical constant 'η' for membership distinguishing it from the non-logical '$\in$'; see his "Set Theory and Higher-Order Logic," pp. 131-48. The more general issue here is the status of the copula between objects; for more on this see Peter Geach, "History of the Corruptions of Logic," in *Logic Matters* (Berkeley: University of California Press, 1980), pp. 44-61.

remarking that set theory, which he identifies here explicitly with ZF, does make staggering existence claims and that 'Quine maintains that higher-order logic involves "outright assumptions of sets the way [set theory] does."'[20] Boolos also immediately locates a difference between set theory and higher-order logic—that '$(\exists X)(\exists x)(\exists y)(Xx \,.\, Xy \,.\, x \neq y)$' is not valid. This sentence is false in all one-membered interpretations whereas all set theories (presumably ZF and its subsystems and extensions) agree that there is a two-membered set. The point may appear to be straightforward and obvious, and perhaps for this reason Boolos does not develop it further. Still, there are a couple of things in it to comment on that again illustrate how fundamentally different Boolos's understanding of higher-order logic is from that of Quine. First, Boolos shifts the context of Quine's remark in a slight but significant way. Quine states his comparison as between the theory of types and set theory, so what we see here again, as in Boolos's footnote on *Methods* Part IV, is Boolos freely equating the theory of types with higher-order logic. Second, Quine would not admit that every set theory agrees that there is a two-membered set, specifically one like the type-theoretic set theory he presents but that contains at level 0 only one individual and at level 1 only subsets of level 0 objects. Again, Boolos aims to bring out the continuity between first- and higher-order logic by showing how notions applicable to logic, such as interpretation and validity, carry over to higher-order logic. But as we have seen, Quine has no reason to extend these notions to higher-order logic because higher-order logic is more accurately understood as set theory.

[20] Boolos, "On Second-Order Logic," p. 44-5; Quine, *Set Theory and its Logic*, p. 258.

Now, let us continue with Boolos's main considerations on the relationship between higher-order logic and set theory. He begins with the claim that "in second-order logic one quantifies over sets,"[21] explaining,

> There are certain (second-order) sentences of any given language that will be classified by second-order logic as logical truths (i.e., as valid), even though they assert, under any interpretation of the language whose domain forms a set, the existence of certain sorts of sub*sets* of the domain. (The sort depends upon the interpretation.) '$\exists X\forall x(Xx \equiv \neg x \in x)$' and '$\exists X\forall x(Xx \equiv x = x)$' are two examples. Thus, unless there exist sets of the right sorts, these sentences will be false under certain interpretations [Boolos's italics].[22]

As we have already seen, since the domain of *I* is a set, ZF's axiom of separation guarantees that there will be a set of all the non-self-membered sets of the domain, though also by appeal to ZF, there is no set of all sets and so no set of all non-self-membered sets. Similarly, the axiom of separation guarantees a set of all the self-identical sets of the domain, and again by appeal to ZF, this set will not be the universal set but, like the domain itself, only some one set out of the vast number of sets that make up the so-called iterative hierarchy. It is on this feature of certain second-order sentences that Boolos thinks someone may object to their status as logical:

> The view that logic is "topic-neutral" is often adduced in support of this opinion: the idea is that the special sciences, such as astronomy, field theory, or set theory, have their own special subject matters, such as heavenly bodies, fields, or sets, but that logic is not about any sort of thing in particular, and, therefore, that it is no more in the province of logic to make assertions to the effect that sets of such-and-such sorts exist than to make claims about the existence of various types of planets. The subject matter of a particular science, what the science is about, is supposed to be determined by the range of the quantifiers in statements that formulate the assertions of the science; logic, however, is not supposed to have any special subject matter: there is neither any sort of thing that may not be quantified over, nor any sort that must be quantified over.[23]

To this view, Boolos admits he has no completely successful response though he does see some weaknesses with it. For one, he suggests that we should be suspicious of identifying the subject matter with the range of a theory's bound variables asking, "Is elementary arithmetic

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[21] Boolos, "On Second-Order Logic," p. 44. Boolos has the remark in quotations though provides no citation for it, so it is not clear if the remark is supposed to be from Quine. A previous quotation on the page does come from Quine, *Set Theory and its Logic*, p. 258, but this remark is not located there. In fact, it seems unlikely that this is meant to be a direct quote from Quine as he rarely ever refers just to second-order logic but instead uses the more all encompassing "higher-order logic."
[22] Ibid., p. 44.
[23] Ibid., p. 44.

really not *about* addition, but only *about* numbers [Boolos's italics]?" If we agree that arithmetic is in some sense about addition, then would not logic be about the notions of negation, conjunction, identity, and the notions expressed by the universal and existential quantifiers even though these notions are rarely ever quantified over? There is some truth to Boolos's suggestion, but it would seem to have little effect on a logician who thought logic is topic neutral. He may indeed grant the point that arithmetic is about addition, but what is addition other than a relationship holding between numbers? Now contrast this with the logical notions Boolos presents. They apply indiscriminately to any subject matter whatsoever, so while there may be a sense in which we can say that logic is about, or at least yields an understanding of how the logical vocabulary works, these notions still do not single out any particular subject matter in the way that arithmetical notions single out numbers.

A second consideration Boolos puts forward is that unlike the notions of planet or field, the notions of set, class, property, concept, and relation, etc. have often been treated as logical notions for reason of their general applicability—that these notions do not discriminate amongst the objects to which they apply. They treat of all objects on an equal footing; anything may belong to a set, have a property, or bear a relation. On these grounds, that second- or higher-order systems make some set or relation existence assertions does not seem sufficient to disqualify them as logical systems in the way that they would be disqualified if they classified as logical truths the existence of a planet with some number of satellites. Boolos concludes then that "[p]art 3 of the *Begriffsschrift* . . . where the definition of the ancestral was first given, is as much a part of a treatise on logic as are the first two parts; the first occurrence of a second-order quantifier in the *Begriffsschrift* no more disqualifies it from that point on as a work on logic than does the earlier use of the identity sign or the negation sign."[24]

Here, we see at least partially what motivates Boolos in his desire to construe higher-order logic as logic: logic for Boolos is that tradition in logic that might be deemed "Fregean". Let us review a little. So far in this chapter, we have seen many points of apparent dispute between Quine and Boolos as to whether higher-order logic ought to be considered continuous with its first-order counterpart and so also considered logic. Boolos, inclined towards the position that higher-order logic is logic, has shown various ways in which notions applied to first-order logic apply also to higher-order logic, either unchanged or in what he views as slightly modified or naturally extended ways. But against any of these extensions of first-order notions to higher-order logic, Quine always appears to have a response that denies Boolos's move, namely that notions such as interpretation or validity simply to not go over to higher-order logic because higher-order logic is more accurately described as set theory. Boolos and Quine, then, appear to be at an irreducible standoff.

Quine does have some further room to maneuver though, as we saw in the previous chapter he settles the boundary of logic at ordinary quantification theory because it possesses certain features that we might intuitively associate with logic. One such feature is logic's particular concern with implication as such; the expounding of implications does not rely on the specific subject matter of the sentences concerned. In this sense logic is topic-neutral. Another is the traditional view that logic is epistemically privileged, that it provides an irreproachable foundation for knowledge, and in particular for mathematical knowledge. Recall that Quine rejects this notion of epistemological privilege, attributing its origins to the fact that all logical truths are merely obvious or reachable by a number of obvious steps. Logic's obviousness is a

[24] Ibid., p. 45.

characteristic he does hold on to. He wishes to make explicit that the supposed reduction of mathematics to logic was a reduction to something far more powerful than traditional logic; specifically it was a reduction to logic inclusive of set theory. So again, we see the relevance of topic neutrality to logic. Topic neutrality assures that the assumption of sets and their accompanying mathematical power will not be built into logic. Limiting logic to ordinary quantification theory preserves, for Quine, a fairly intuitive or traditional conception of what counts as logic, a way of distinguishing it from other branches of mathematics or science. Ordinary quantification theory does not presuppose implication to depend upon particular sorts of objects nor misleadingly attribute epistemological privilege, or obviousness, to set theory.

Boolos, though, in his reference to Frege's construction of the ancestral in his *Begriffsschrift*, may now appear to acquire this same sort of room to maneuver that Quine has in motivating an intuitive or traditional conception of what logic is. Boolos appeals to Frege's authority on logic, authority gained as the figure most responsible for the development of modern mathematical logic as we know it, to motivate his view that higher-order logic is indeed part of logic. It is within Frege's logical work and the logicist tradition that notions such as set, class, property, concept, and relation play significant roles conceived as logical notions. Indeed, Boolos locates the purpose of his article precisely within this tradition in footnote on the opening page,

> My motive in taking up this issue [the logical status of higher-order logic] is that there is a way of associating a truth of second-order logic with each truth of arithmetic; this association can plausibly be regarded as a "reduction" of arithmetic to set theory. . . . I am inclined to think that the existence of this association is the heart of the best case that can made for Logicism and that unless second-order logic has *some* claim to be regarded as logic, Logicism must be considered to have failed totally. I see the reasons offered in this paper on behalf of this claim as part of a partial vindication of the logicist thesis. I don't believe we yet have an assessment that is as just as it could be of the extent to which Frege, Dedekind, and Russell succeeded in showing logic to be the ground of mathematical truth.[25]

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[25] Ibid., p. 37, fn. 1. (Boolos's italics)

Once this move has been made, Boolos is in a position to investigate the extent to which important notions in first-order logic such as interpretation and validity carry over to higher-order logic either directly or with minor or natural modifications.[26] He is not restricted by a conception of logic, such as Quine's, that does not admit sets, i.e. subsets of a given domain, as logical.

Yet, there is a further question to be raised about this move. As Boolos notes in the above quotation, the logicists aimed at a reduction of mathematics to logic so as to provide a foundation for mathematical truth. We have seen in the previous chapter that the thought that this reduction counted as a reduction to logic was one of the primary motivations in Quine's attempt to distinguish clearly between ordinary quantification theory and higher-order logic, or set theory. Certainly arithmetic, and most of the rest of mathematics, stands on firmer ground than does set theory itself, as evidenced for example by the discovery of Russell's paradox. The logicists' reduction does not achieve the epistemological aims they thought it did.

To this objection, Boolos simply concedes the point. He explains, by Gödelization and the completeness theorem, elementary arithmetic "Z" provides a suitable background for a theory of first-order validity coextensive with the usual model-theoretic notion of validity. First-order validity is definable in Z by Gödelization, and then, the validity each of the valid formulas, and only of the valid formulas, can be proved in this theory along with many general laws of validity. For second-order logic this does not hold. There is no way to prove each valid second-order sentence in elementary arithmetic, and second-order validity is not even definable in

[26] Though it should be stressed at this point that though Boolos appeals to the authority of Frege or the logicist tradition to encourage the idea that higher-order logic is to be counted as logic in the same sense that first-order logic is to be counted as logic so that notions such as interpretation or validity extend in their applicability to higher-order logic as well, this appeal does not mean that Frege and Russell had these semantic notions in their conception of logic. I do not mean to suggest here that Boolos thinks that they do (of course he may think this, but this is not the point here). Whether Frege and Russell did have a metatheoretic conception of logic is a highly contentious issue. Again, for an overview of the debate see Floyd, "Frege, Semantics, and the Double-Definition Stroke," pp. 141-66.

second-order arithmetic. While for each first-order sentence a fairly simple statement of arithmetic can be effectively associated with it that is true if and only if the sentence is valid, this association is not in the least possible for second-order sentences. Second-order validity is definable only in set theory (for Quine second-order validity would only be definable in a stronger set theory, if the notion of validity were applicable to second- and higher-order theories at all; that is beyond the validity already accounted for by ordinary quantification theory). This guarantees that set-theoretic truth does not reduce to second-order validity and that second-order validity does reduce to set-theoretic truth. There is no effective function that assigns second-order validities to all and only set-theoretic truths; nor is there such a function definable in set theory. For otherwise set-theoretic truth would be set-theoretically definable. Though, the function that assigns each second-order formula to the sentence of set theory asserting the formula's validity reduces second-order validity to set-theoretic truth. The notions of the series ⟨first-order validity, first-order arithmetical truth, second-order arithmetical truth, second-order validity, set-theoretic truth⟩ can always be reduced by effective functions to notions occurring later but never to notions occurring earlier in the series because they are in order of increasing strength.[27]

Boolos also notes here that there exist second-order sentences that are valid if and only if certain "highly problematical statements of set theory," such as the continuum hypothesis, are true. He concludes, "Thus the metatheory of second-order logic is hopelessly set-theoretic, and the notion of second-order validity possesses many if not all of the epistemic debilities of the notion of set-theoretic truth."[28] Here we see an illustration of the sort of maximality that Quine understands the completeness theorem to provide for first-order logic. Validity can be

[27] Boolos, "On Second-Order Logic," pp. 45-6.
[28] Ibid., p. 45.

characterized without appeal to the substantial mathematical theory of classes by merely describing one of the complete proof procedures for first-order validity.

Having conceded that higher-order logic lacks the epistemological privilege, self-evidence, or obviousness, of first-order logic, Boolos is now in a somewhat awkward position. He has shown how some second-order validities that assert the existence of particular subsets of the domain will depend upon the existence of sets of the appropriate sorts for their truth. And now he has also shown how second-order logic is tied up in many, if not all, of the epistemological debilities of set theory. Whereas Quine has singled out ordinary quantification theory, i.e. first-order logic, as corresponding to an intuitive and fairly uncontroversial conception of what logic may be like and what it aims to do, Boolos no longer has this option. He merely appeals to Frege and the logicist tradition that embraced as purely logical such notions as set, class, property, concept, and relation. In making this move, it is unclear as to how second- and higher-order logic are to be distinguished from set theory for it could just as easily be said of Frege's logic that it would have been more appropriately characterized as set theory, which is precisely what is at issue here between Quine and Boolos. Boolos could perhaps look to their differing strengths; second-order logic does not have the full power of set theory (ZF), but such comparisons of strength are common to the study of set theories in general.

Boolos thinks there is another way out though. Remarking on Quine's view that "the logic capable of encompassing [the reduction of mathematics to logic] was logic inclusive of set theory," he explains that this logic would have to count as valid some nontrivial theorems of set theory, and in particular some theorems of set theory asserting the existence of certain kinds of sets, where by "set theory" he means ZF or some subsystem or extension of it.[29] It looks as

[29] Ibid., p. 46; W.V. Quine, *Philosophy of Logic* [1986], p. 66. Boolos is not entirely clear throughout his article what he means by set theory. As noted earlier, in one place (p.43), he explicitly identifies set theory with ZF. It

though second-order logic does indeed do this for it counts both '$(\exists X)(\forall x)\neg Xx$' and '$(\exists X)(\forall x)(Xy \equiv y \subseteq x)$' as valid and these sentences look to assert the existence of the null and power sets. Against this view though, he cautions,

> It seems, however, that there is a serious difficulty in supposing that *any* second-order sentence asserts, for example, that there is a set with no members; it seems that no second-order sentence asserts the same thing as any theorem of set theory, and hence that not even the smallest fragment of set theory is, in this sense, included in second-order logic.[30]

The problem is that the range of the quantifiers must be fixed before a second- (or first-) order sentence can be said to have a determinate sense. For example, the sentence '$(\forall x)x = x$' only asserts that everything is self-identical once the range of 'everything' has been fixed. Prior to determining the range of everything, the sentence can only be said to assert "Everything in the domain, whatever the domain may be, is self-identical." Boolos then extends this observation to '$(\exists X)(\forall x)\neg Xx$'. What this sentence asserts "depends upon what the domain is supposed to be (and also upon how that domain is 'given' or described). But, whatever the domain may be, '$(\exists X)(\forall x)\neg Xx$' will assert that there is a subset of the domain to which none of its members belong."[31]

From this, he concludes that no valid second-order sentence asserts the same thing as any theorem of set theory. A second-order sentence, regardless of its validity, asserts something only with respect to an interpretation whose domain must be restricted so as not to include all sets on pain of paradox. To assert the same thing as a theorem of set theory, the domain would have to include all sets. The valid sentence '$(\exists X)(\forall x)Xx$' does not assert the existence of a universal set, which is false according to ZF, but only that there is a subset of the domain, whichever set this

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appears that many of his objections to Quine's view that higher-order logic includes set theory rely on equating "set theory" with ZF. Boolos criticisms in this section seem to make sense only when we understand him meaning ZF by "set theory".

[30] Boolos, "On Second-Order Logic," p. 46. (Boolos's italics)

[31] Ibid., pp. 46-7.

may be, of which everything in the domain is a member. The first-order quantifiers expressing the assertions of ZF on the other hand range over all sets and do not together form a single set. In summary, he expresses his point, "We have argued that the ranges of the variables in second-order sentences must be sets. If so, it is hard to see how any second-order sentence could express or assert what any theorem of ZF does, or that second-order logic counts as valid some significant theorems of set theory."[32]

Boolos does acknowledge one way in which second-order logic is clearly committed to asserting the existence of the empty set. Since the empty set is a subset of every set, and the domain itself is a set, and the empty set is the only set to which no members of the domain belong, '$(\exists X)(\forall x)\neg Xx$' asserts the existence of the empty set independently of any interpretation. By the same reasoning, higher- and higher-order logics will be committed to more and more sets in the following way. Second-order variables range over all subsets of the domain so at least over the empty set $\varnothing$. Third-order variables will then range over all subsets of what the second-order variables range over so over $\{\varnothing\}$. And fourth-order variables will then range over all subsets of what the third-order variables range over so over$\{\varnothing, \{\varnothing\}\}$. And fifth-order variables will then range over all subsets of what the fourth-order variables range over so over $\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$ and so on as each additional level of higher-order variables is introduced. Second-order logic itself, though, includes only this, what Boolos calls "modest", commitment to the empty set.[33]

I do not wish to belabor the point, but I will at least mention again briefly that Quine is unlikely to accept that second-order logic should have to count as *valid* some theorems of set theory, and in particular theorems asserting the existence of certain kinds of sets, in order for

[32] Ibid., p. 47.
[33] Ibid., p. 48.

second-order logic to be inclusive of set theory. Higher-order logic, as Quine sees it, is set theory so the notion of set-theoretic truth will be applicable to it rather than the logical notion of validity.

This point aside, the central issue for Boolos is that no sentence of second-order logic asserts the same thing as a theorem of set theory, ZF. For Quine the theory of types, or some variation of it such as Boolos's, is just another way of formalizing set theory, and so, it would hardly be surprising that no sentence of second-order logic would assert the same thing as a theorem of ZF. Where ZF has what is intuitively characterized as a cumulative hierarchy of sets, type theory, while also hierarchical, restricts its sets to discrete levels so that the sets of a level $n + 1$ are constructed only of subsets of the preceding level n. Accordingly, the comprehension axiom is restricted differently in each of the two theories. ZF blocks the set-theoretic paradoxes in its axiom of separation, '$(\forall z)(\exists y)(\forall x)(x \in y \,.\equiv.\, x \in z \,.\, Fx)$'. The members of the set y are not everything of which 'F' is true but everything in the set z of which 'F' is true. The method is one of relativization; we assume that every monadic predicate has an extension only within the given class z. And as shown in the previous section, the theory of types divides the universe into discreet levels where the variables 'x', 'y', 'z', etc. range over an unspecified universe U, the variables 'α', 'β', 'γ', etc. range over the universe U_1 composed of subsets of U, the variables 'κ', 'λ', 'μ', etc. range over the universe U_2 composed of subsets of U_1 and so on. The formulas are then restricted so that a formula of the form $u \in v$ only makes sense when the variable v is of one type higher than the variable u. Hence in the comprehension schema, '$(\exists x)(\forall y)(y \in x \equiv Fx)$', it is impossible to specify a set by the monadic predicate of non-self-membership, '① ∉ ①'. The notion of monadic predicate is itself restricted so as not to yield Russell's paradox.[34]

[34] For this account I am indebted to Goldfarb, "Deductive Logic," pp. 187-92.

There are some additional points to mention that further minimize the significance of the differences between the two theories. For one, Quine has presented a method for translating the many-sorted theory of types into a version of the single sorted ZF. It begins by adopting general variables and adding the predicate '$T_n x$' for 'x is of type n', followed by some simplifications, such as equating the empty sets at each level of the type-theoretic hierarchy with a single empty set, and then adding the Zermelo axioms, among them, power set, pairing, separation, sum, and extensionality and so obtaining a set theory of cumulative types. The logic is simplified and the many-sorted formulation is shown inessential to the theory of types considered merely as a theory of what sets there are.[35] Quine's own set theory, NF, combines the general variables of Zermelo's theory with Russell's type restrictions by demanding that each instance of the comprehension schema, '$(\exists x)(\forall y)(y \in x \equiv Fx)$', be "stratified" meaning that there must always be a way to index variables flanking '$\in$' in the formula 'Fx' so that they will be of consecutive type, of the form $n \in n + 1$. Thus stratification rules out the contradictory '$(\exists x)(\forall y)(y \in x \equiv y \notin y)$', but allows '$(\exists x)(\forall y)(y \in x \equiv y = y)$' for when '$y = y$' is fully expanded into the primitive notation of NF it becomes '$(z)(y \in z) \supset (y \in z)$'. Hence in NF the universe itself is a set, $V \in V$.[36]

One final feature I wish to point out of set theory and higher-order logic that also shows a striking similarity between these theories comes from Boolos's comment on how higher- and higher-order logics will be committed to more and more sets by way of second-order logic's commitment to the empty set. These increasing commitments to sets are in essentials the result of repeatedly applying the power set operation to the domain so as to obtain the requisite items

[35]See W.V. Quine, *Set Theory and its Logic*, pp. 266-86; "Unification of Universes in Set Theory," *Journal of Symbolic Logic* 21:3 (1956), pp. 267-79.
[36] W.V. Quine, "New Foundations for Mathematical Logic," pp. 80-101; see also *Set Theory and its Logic*, pp. 287-299.

for the additional $n + 1^{th}$-order quantifiers to range over.[37] And very much this same method generates the pure sets of ZF. In his 1971 article "The Iterative Conception of Set," Boolos describes the formation of the cumulative hierarchy by forming all subsets of each level of the hierarchy so as to obtain more and more sets.[38] Beginning with individuals, at stage one, form all possible collections of individuals, then at stage one form all possible collections of individuals and all possible collections of sets formed at stage zero, then at stage two form all possible collections of individuals, all possible collections of sets formed at stage zero, and all possible collections of sets formed at stage one and so on on up the ordinals. To obtain the pure sets of which ZF usually treats, the process is carried out in the same way only we assume that there are no individuals so at stage zero the only set formed is the empty set $\varnothing$, then at stage one we form $\{\varnothing\}$, at stage two we form $\{\varnothing, \{\varnothing\}\}$, at stage three we form $\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$ and so on. It is hard to see what fundamental difference distinguishes this process from the way in which higher- and higher-order logics will be committed to more and more sets.

Regardless of in what sense second-order logic can be said to count as valid some theorems of set theory asserting the existence of sets of a certain sort, Boolos concludes his discussion of second-order logic's commitment to the existence of sets referring back to his comment on how a materially adequate truth theory for a first-order language will differ from that of a second-order language:

> One sense, already noted, in which the use of second- but not first-order logic commits one to the existence of sets is this: If L_1 is the first-order fragment of an interpreted second-order language L_2 whose domain D contains no sets, then there are many logical truths of L_1 that claim the existence of objects in D with certain properties, but there are none that claim the existence of subsets of D; however, among the logical truths of L_2 there are many such: for each predicate of L_2 with one free individual variable, there is a logical truth of L_2 that asserts the existence of a subset of D that is the extension of the predicate.[39]

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[37] Shapiro, *Foundations without Foundationalism*, p. 138.
[38] George Boolos, "The Iterative Conception of Set," in *Logic, Logic, and Logic*, pp. 20-2.
[39] Boolos, "On Second-Order Logic," p. 48.

This is in a way an accurate rendering of how Quine views higher-order logic as inclusive of set theory to the extent that it brings to the fore that the quantified predicate letters of higher-order logic are no longer mere simulations of predicates. Though, again, it should be made clear that he does not conceive of this as resulting from the model theory for higher-order logic. Quine sees higher-order logic itself as set theory. Despite the differences between them, both ZF and something like higher-order logic with types are theories about sets.

III

Boolos next turns his discussion to second-order logic's greater expressive capacity with respect to first-order logic. He remarks that when we conjoin the first two "Peano postulates" for arithmetic, '$(\forall x)(sx \neq 0)$' for 'zero is not a successor' and '$(\forall x)(\forall y)(sx = sy \supset x = y)$' for 'uniqueness of immediate predecessor', replacing constants with variables and existentially closing, we get '$(\exists z)(\exists S)((\forall x)\, z \neq S(x) \,.\, (\forall x)(\forall y)(S(x) = S(y) \supset x = y))$', a sentence true only in interpretations with domains that are Dedekind infinite. We can apply the same method to the induction postulate, '$(F0 \,.\, (\forall x)(Fx \supset Fsx)) \supset Fx$', which yields '$(\exists z)(\exists S)(\forall X)(Xz \,.\, (\forall x)(Xx \supset XS(x)) \supset (\forall x)Xx)$', a sentence true only in interpretations with domains that are countable. Hence, the notions of infinity and countability are characterizable in second-order logic, unlike in first-order logic as shown by the compactness and Skolem-Löwenheim theorems. He observes that there are many other interesting notions such as well-ordering, progression, ancestral, and identity that also cannot be characterized in first-order logic.

Though second-order logic's greater expressive capacity alone is not reason enough to favor it as logic, for these notions can also be characterized in set theory. That second-order logic can characterize these notions may be all the more reason for equating it with set theory.

Boolos, however, does not appeal merely to second-order logic's greater expressive capacity as somehow determining its logical status. He continues on to point out that because of its greater expressive capacity, second-order logic can offer an explanation of the apparent inconsistency of certain infinite sets of statements, each of whose finite subsets is consistent. This again contrasts with compact first-order logic. Four examples of such sets of statements are {'Smith is an ancestor of Jones,' 'Smith is not a parent of Jones,' 'Smith is not a grandparent of Jones,' . . .}, {'It is not the case that there are infinitely many stars,' There are at least two stars,' 'There are at least three stars,' . . .}, {'R is a well-ordering,' 'a_1Ra_0,' 'a_2Ra_1,' 'a_3Ra_2,' . . .}, and {x is a natural number,' 'x is not zero,' 'x is not the successor of zero,' . . .}. When compared with the sets of statements {'Not: there are at least three stars,' 'Not: there are no stars,' 'Not: there is exactly one star,' 'Not: there are exactly two stars'} and {'R is a linear ordering,' 'a_0Ra_1,' 'a_1Ra_2,' 'Not: a_0Ra_2'}, which can be shown inconsistent by first-order logic.[40] In light of such differences in expressive capacity, Boolos thinks we should find first-order quantification theory too impoverished to be all there is to logic.

Consider the first of these two sets of sentences. The first sentence will be satisfied only in a model that has less than three elements, so in a model of zero, one, or two elements. The second sentence rules out that there are no stars; that is the domain is non-empty. But the next sentence rules out the second option and the last sentence rules out the third. Hence, this set of sentences is unsatisfiable and so also formally inconsistent by the completeness of first-order logic. By similar reasoning it might seem that logic should also be able to explain the inconsistency of the infinite set of sentences about stars that Boolos initially presents. Some model with a finite number of elements will certainly satisfy the first sentence. Some finite model will also satisfy each finite subset of this infinite set of sentences which by compactness

[40] Boolos, "On Second-Order Logic," pp. 48-9.

would yield the conclusion that this infinite set of sentences is satisfiable. Clearly this is not the case. In fact, the first of these sentences cannot be characterized by a first-order sentence, but it can be characterized by the second-order sentence '$(\forall f)\neg((\forall x)(\forall y)(fy = fy \supset x = y) \,.\, (\exists x)(\forall y)(fy \neq x))$'. Compactness thus fails for second-order logic as this set of infinite sentences shows. Drawing the boundary of logic at first-order logic may then again (as we saw first-order quantifier dependencies suggest in the previous chapter) appear to be arbitrarily restrictive. To show the inconsistency we could use second-order logic to express the infinite number of first-order sentences indicated by '. . .' thus '$(\exists f)((\forall x)(\forall y)(fy = fy \supset x = y) \,.\, (\exists x)(\forall y)(fy \neq x))$', a sentence clearly inconsistent with the first sentence of this infinite set.

Of course that we should think that *logic* alone should show the inconsistencies in the first four sets of sentences depends on the inconsistencies of the first four sets being of a strictly logical character. Quine rejects that they are on the grounds that what provides the second-order sentences with their greater expressive capacity, and so their capacity to explain the inconsistency in each of the first four sentences, is their assumption of sets (or in the specific case of the stars discussed above, functions, which can of course be defined in set theory). As we have seen, aside from his appeal to the fact that these inconsistencies can be explained in a system not unlike the one Frege used, Boolos has provided very little to counter Quine's view that these sets of sentences require more than just logic to show their respective inconsistencies. It is no detriment to Quine's conception of logic that notions such as well-ordering, progression, ancestral, (Dedekind) infinite, and denumerable fall outside the bounds of ordinary quantification theory. That ordinary quantification theory cannot characterize these notions tells us that they are "not that elementary, and that [they] can be expressed only with help of one or another *term*—epsilon, for instance. . . ."[41] They are notions of mathematics in a sense exclusive of logic.

This brings us in a natural way to consider the significance of the failure of the completeness theorem for higher-order logic, what Boolos notes as "hardly … one of second-order logic's happier features."[42] Boolos wishes to take issue here with the importance Quine attaches to the completeness of first-order logic as a characteristic that demarcates "an integrated domain of logical theory with bold and significant boundaries," a characteristic which the extension to higher-order logic lacks. Against this view, Boolos observes that though "[t]he existence of a sound and complete axiomatic proof procedure and the effectiveness of the notion of proof guarantee that the set of valid sentences of first-order logic is effectively generable; Church's theorem shows that it is not effectively decidable."[43] However, he continues that the monadic fragment of first-order logic with identity is decidable, and this feature carries over to monadic second-order logic. Why should we follow Quine then and favor completeness over decidability as determining the boundary between logic and mathematics, i.e. set theory?

Earlier Boolos remarked that

> although it is not hard to have some sympathy for the view that no notion of validity should be so extravagantly distant from the notion of proof, we should not forget that validity of a first-order sentence is just truth in all its interpretations. (The equation of first-order validity with provability effected by the completeness theorem would be miraculous if it weren't so familiar.)[44]

However miraculous this coincidence may be, that the two notions do coincide is significant for Quine in his move to limit logic to ordinary quantification theory. It is in this sense, as we saw in the previous chapter, that he views ordinary quantification theory as maximal in that "it is as

[41] W.V. Quine, *The Roots of Reference: The Paul Carus Lectures* (LaSalle: Open Court Publishers, 1974), p. 115; this partial quotation is in the context of Quine's discussion of the Skolem-Löwenheim theorem and logic's inability to distinguish between the notions of denumerable and indenumerable.
[42] Boolos, "On Second-Order Logic," p. 50,
[43] Quine, *Philosophy of Logic* [1986], p. 90; Boolos, "On Second-Order Logic," p. 50.
[44] Ibid., p. 46.

far out as you can go and still have complete coverage of validity and inconsistency by the Skolem proof procedure."[45] Decidability in contrast is not maximal in this way. Characterizing logic according to whether the theory is decidable or not would leave infinitely many validities and inconsistencies outside the scope of logic. So far as Quine sees logic's primary aim as the tracing of implications through the systematic study of logical truth, decidability would be an undesirable and arbitrary place to draw the boundary of logic.

But now it seems Boolos could make this same point against Quine. Confining logic to first-order logic leaves infinitely many higher-order validities and inconsistencies outside the scope of logic. After all, as Boolos characterizes validity as truth in all interpretations, the notion extends in a natural way to higher-order truths. Here we see another of the reasons that Quine attributes such significance to the coincidence of validity with provability in ordinary quantification theory. Though the notion of validity can be extended to higher-order logic, by the incompleteness theorem, higher-order logic will include "logical" truths that are not provable and so logical truths that are not obvious in the way that all truths of ordinary quantification theory are. As we have seen Quine emphasize, the completeness theorem assures us that the first-order validities can be specified mechanically in terms of proof. There is no need to appeal to the limitless realm of classes or even to notions of truth and satisfaction used in the substitutional and model-theoretic definitions of logical truth to specify the truths of ordinary quantification theory. On the other hand, Boolos has indicated that there will be a sentence of second-order logic whose validity will depend upon determining the truth of the continuum hypothesis. Truths of higher-order logic will not all be obvious, but in many cases will depend upon no small amount of mathematics to determine their truth. We see again how when the boundary of logic is drawn at ordinary quantification theory, logic is obvious through and

[45] Quine, "Existence and Quantification," pp. 111-12.

through unlike its higher-order counterpart. Of course we are free to give up this characterization of logic as obvious or self-evident, but then we also seem to have little reason for distinguishing higher-order logic from set theory in the first place.

In conclusion, Boolos remarks that the completeness theorem is not the sole feature that leads Quine to draw the boundary between logic and extra-logical mathematics where he does. Quine also cites, along with the completeness theorem, the "remarkable concurrence of diverse definitions of logical truth" as terminating the extent of logic at ordinary quantification theory. Boolos focuses specifically on Quine's substitutional definition of logical truth: a schema is provable if and only if it is valid, if and only if every substitution instance of it in any reasonably rich object language is true. Boolos remarks on two points about the definition, also, as we have seen, noted by Quine; that identity cannot be counted as logical, otherwise '$(\exists x)(\exists y)\neg x = y$' would count as a logical truth, and that for a language to be reasonably rich it must be rich enough for elementary number theory.[46]

Boolos thinks, "The theorem may be remarkable, but it is not," he thinks, "remarkably remarkable."[47] The reason for this, he explains, is that a distinction can be made between a weak completeness theorem and a strong completeness theorem. A weak completeness theorem shows that all valid sentences are provable and a strong completeness theorem shows that a sentence is provable from a set of sentences whenever it is a logical consequence of the set. Alternately, he states the strong completeness theorem as "a set of sentences is satisfiable if it lacks a refutation. (A refutation of a set of sentences is a proof of the negation of a conjunction of members of the

[46] Boolos, "On Second-Order Logic," p. 51.
[47] Ibid., p. 52.

set.)" Most proofs of weak completeness can be easily expanded to proofs of strong completeness.[48] However he remarks,

> It seems to me that the concurrence of the two accounts of the concept of logical truth cannot be called remarkably remarkable if their extensions to the relation of logical consequence do not occur. If there is a reasonably rich language and a set of sentences in that language which is satisfiable according to the usual account but which cannot be turned into a set of truths by (simultaneous, uniform) substitution of open sentences of the language, then the interest of the alternative definition of logical truth is somewhat diminished, for it is a definition that cannot be extended to kindred logical relations in the correct manner. And, as it happens, there is a satisfiable set of sentences of a reasonably rich language with this property.[49]

Boolos considers two first-order languages without identity L and M with predicate letters '*F*', '*Z*', '*S*', '*P*', '*T*', and '*G*' and variables ranging over the natural numbers. For both languages the predicate '*F*' is true of all natural numbers, '*Z*' is true of zero alone, '*S*', '*P*', and '*T*' are respectively 'successor', 'sum', and 'product'. L specifies that '*G*' is true of all natural numbers, and L is a reasonably rich language. Let A be the set of all Gödel numbers of truths of L. By Tarski's indefinability theorem, A is not definable in L. M specifies that '*G*' is true of all and only the members of A. Now, let B be the set of truths of M. B is satisfiable, but B cannot be turned into a set of truths of L by substitutions of open sentences of L for the predicate letters '*F*', '*Z*', '*S*', '*P*', '*T*', and '*G*'. For if this were possible, A would be recursive in L for the extensions of the open sentences substituted for '*Z*', '*S*' and '*G*'; and 'definable in L' is closed under 'recursive in'. Therefore, the set A would be definable in L which would violate Tarski's indefinability theorem. And Boolos shows that there is indeed a way to reinterpret the predicates '*Z*', '*S*' and '*G*' of M into predicates of L so that A would be recursive in L.[50]

Boolos suggests a fix to this difficulty by way of the compactness theorem. We could define satisfiability as *a set of sentences is satisfiable only when every conjunction of its members has a true substitution instance* since a set is satisfiable if and only if all of its finite

[48] Ibid., p. 52.
[49] Ibid., p. 52.
[50] Ibid., p. 53.

subsets are satisfiable. He concludes then that we have three accounts of satisfiability of sets of sentences, this one, truth in some model, and irrefutability. This concurrence Boolos does not think in the least remarkable though. What he does think remarkable about the concurrence of the two definitions of logical truth is that truth in all interpretations and truth of all instances is that "*both definitions have some antecedent plausibility as correct explications of a pre-theoretical notion of logical validity* ('truth regardless of what the non-logical words mean'). Conversely, he comments, "The definition of satisfiability of a set as 'truth of some instance of each conjunction of schemata in the set' has no such plausibility as an account of satisfiability. It even sounds wrong." [51]

There are a few points that we can raise in Quine's favor against Boolos's remarks. For one, Quine does not state that proof of the equivalence between provability, validity, and substitution is "remarkably remarkable", only remarkable, and Boolos agrees that it is indeed remarkable. What would make it remarkably remarkable would be if it easily extended to cover the case of strong completeness in addition to weak completeness. One way in which it seems that this feature may not matter a great deal to Quine is that he has continually focused on the notion of logical truth, or validity, as the route to implication, rather than looking directly to logical consequence. So far as his interest is logical truth, the weak completeness theorem is enough. And even in early editions of *Methods of Logic*, where Quine placed more emphasis on showing implications directly by a method of natural deduction, where it might have made more sense to prove the strong completeness theorem, the form of completeness he proved aimed at validity rather than logical consequence.[52] Quine, unlike Boolos, is content with his remarkable theorem.

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[51] Ibid., p. 52-3. (Boolos's italics)

[52] W.V. Quine, *Methods of Logic* [1958], 2nd rev. ed. (London: Routledge and Kegan Paul Ltd., 1970), p. 253.

A second and perhaps more important point is that Boolos does provide a way to extend the definition to logical consequence. Against it, he maintains that it lacks any plausibility as an account of the satisfiability of sets of sentences in a way that the substitutional and model-theoretic definitions of logical truth do not. Again though it should be emphasized that the account does work is more important than that it may sound wrong. And in fact it is not clear how wrong it actually does sound in the context of taking the substitutional definition of logical truth as the primary definition of logical truth. One could also think that the definition of something as trivial as logical truth sounds wrong in its appeal to the power of set theory. Quine himself admits both definitions of logical truth as legitimate but settles for substitution as it is ontologically more economical.

Boolos concludes his paper,

> One ought then to be wary of the claims that the concurrence of diverse definitions of logical truth is remarkable and that this concurrence suggests that classical quantificational logic is a "solid and significant unity." One of the definitions is a definition of logical truth only in virtue of a remarkable theorem about first-order logic; another cannot be generalized properly. Does classical quantificational logic then fail to be a significant and solid unity? Certainly not.[53]

Throughout this chapter I have tried to show the ways available to Quine for responding to Boolos's criticisms. I have also tried to bring out the ways in which they often seem to be arguing at cross purposes, that there may really be no common ground on which they can arbitrate their dispute over the logical status of higher-order logic or set theory. Boolos's concluding remark here does emphasize one of the ways in which Quine is at an advantage in this debate. He presents a conception of logic that is well-motivated in its appeal to intuitive and traditional views of what logic is and what its primary aims are. By way of this appeal, he settles the boundary of logic at ordinary quantification theory. Boolos comes closest to this sort of move in appealing to the logicist tradition stemming from Frege. However, it is the very logical

[53] Boolos, "On Second-Order Logic," p. 53.

status of Frege's logic that is in question, so the move seems illegitimate within this context. Quine, unlike Boolos then, is in a position to offer definite answer to the question of why we may want to take classical quantificational logic to be a significant and solid unity.

Chapter 3: Higher-Order Logic as the Logic of Plurals?

With his two papers "To Be is to Be a Value of a Variable (or to Be Some Values of Some variables)" (1984) and "Nominalist Platonism" (1985), Boolos returns to consider the logical status of second-order logic.[1] Instead of arguing that second-order logic has some claim to the honorific "logic" because its set-theoretic commitments are only to some limited amount of set theory, in these papers, he argues that at least monadic second-order logic need not be committed to any set-like entities at all when it is translated into a theory of plural logic. In this chapter, we will examine Boolos's plural interpretation of monadic second-order logic (MSOL) along with some criticisms of it found in the work of Resnik, Parsons, and Linnebo.[2] In light of these criticisms, I will then conclude by returning to Quine speculating what significance he would have attributed to Boolos's final attempts at placing second-order logic firmly within the bounds of logic.[3]

I

To motivate his position, Boolos begins his "To Be is to Be a Value of a Variable" by asking whether the quantifier-variable notation along with the usual logical connectives and identity of first-order logic are sufficient to represent quantification and cross-reference in English. He suspects that most philosophers and logicians think it is. These philosophers and logicians would certainly include in their logic truth-functional logic, but they would also readily

[1] George Boolos, "To Be is to Be a Value of a Variable (or to Be Some Values of Some Variables)," in *Logic, Logic, and Logic*, pp. 54-72; "Nominalist Platonism," in ibid., pp. 73-87.
[2] Michael D. Resnik, "Second-Order Logic Still Wild," *The Journal of Philosophy* 85 (1988), pp. 75-87; Charles Parsons, "The Structuralist View of Mathematical Objects," *Synthese* 84 (1990), pp. 303-46; Øystein Linnebo, "Plural Quantification Exposed," *Noûs* 37:1 (2003), pp. 71-92.
[3] As I mentioned in the introduction, Quine wrote very little of Boolos's plural interpretation of MSOL. The one brief remark I have been able to find occurs in his late essay "Structure and Nature," *Journal of Philosophy* 89:1 (1992), p. 6. The remark is made in the context of a discussion of David Lewis's *Parts of Classes* (New York: Blackwell, 1991).

agree that this is not all there is to logic because a vast number of inferences of natural language, dependent upon the quantificational words 'all' and 'some' and cross-referential works like 'it', 'who', and 'that', are not captured by truth-functional logic alone. To do this work, logic requires that quantification theory be added to express these other sorts of inferences. Also, essential to cross-reference is identity; and as identity lacks distinctive content and pervades all of natural language, it too may be uncontroversially included as a logical notion. These logicians and philosophers conclude, according to Boolos, that once logic has all of these resources at hand, there are very few inferences of natural language having to do with cross-reference, quantification, and generalization that first-order quantification theory with identity cannot express. And furthermore, that the variety of inferences not captured by first-order quantification theory with identity is nowhere near as great or as interesting as those inferences that are. Boolos, himself, has doubts about this conclusion. In particular, he questions whether we actually do know how much or how little within the province of logic first-order quantification theory with identity cannot treat. What he will go on to show is that it may be surprisingly more than the logicians and philosophers who hold to the view described above suppose.[4]

Boolos spends much of the rest of his paper presenting examples aimed at leading his readers to the view that logic must include resources beyond first-order quantification theory with identity. In what follows, I will not discuss all of his examples, but I will present some of them and in particular, those that seem most relevant to the debate over the logical status of second-order logic. Boolos begins with some examples of well-known constructions, sentences, and inferences that are non-first-orderizable, those that comprise numerical quantifiers such as "more," "most," and "as many." The inference

[4] Boolos, "To Be is to Be a Value of a Variable," pp. 54-5.

Most democrats are left-of-center.
Most democrats dislike Reagan.
Therefore, some who are left-of-center dislike Reagan.

is one such. Another, that he has given a fair amount of attention to, is "For every *A* there is a *B*," which cannot be expressed in first-order logic as it can be understood as the synonymous "There are at least as many *B*s as *A*s." One may think that this sentence could be represented by '$(\forall x)(Ax \supset (\exists y)By)$', but this is equivalent to '$(\exists x)Ax \supset (\exists y)By$'.[5] We can however express such numerical statements as "There is at least one *A*," "There are at least two *A*s", "There are at least three *A*s," etc. in first-order notation, respectively, as '$(\exists x)Fx$', '$(\exists x)(\exists y)(Fx \,.\, Fy \,.\, x \neq y)$', '$(\exists x)(\exists y)(\exists z)(Fx \,.\, Fy \,.\, Fz \,.\, x \neq y \,.\, x \neq z \,.\, y \neq z)$', etc. The same can also be done in first-order notation for "at most" and "exactly". Likewise, one may have thought that the examples Boolos provides here would also go over into first-order notation.

Another non-first-orderizable sentence of a different sort, and perhaps the most familiar of them, is the Geach-Kaplan sentence

(1) Some critics admire only one another,

cited also by Quine, attributing it to Peter Geach and David Kaplan.[6] Boolos explains that this sentence is supposed to mean that there is a collection of critics, each of whose members admires no one not in that collection, and none of whose members admires himself. In second-order logic, this sentence can be rendered as

(2) $(\exists X)((\exists x)Xx \,.\, (\forall x)(\forall y)[Xx \,.\, Axy \supset x \neq y \,.\, Xy])$,

where 'A' is the dyadic predicate '① admires ②'. The Geach-Kaplan sentence may sound simple enough that one would think it should be representable in first-order logic. A proof given by Kaplan shows however that this is not the case for the second-order version of the Geach-

[5] Ibid., p. 55; also Boolos, "For every *A* there is a *B*," *Linguistic Inquiry* 12 (1981), pp. 465-66.
[6] Quine, *Methods* (1982), p. 293; *Roots of Reference*, p. 111.

Kaplan sentence is not equivalent to any first-order sentence. His technique for showing non-first-orderizability is to substitute '$(x = 0 \vee x = y = 1)$' for 'Axy' which yields

(3) $(\exists X)((\exists x)Xx \,.\, (\forall x)(\forall y)[Xx \,.\, (x = 0 \vee x = y = 1) \supset x \neq y \,.\, Xy])$,

a sentence true in all nonstandard models of arithmetic and false in the standard model.[7]

It is sentences like the Geach-Kaplan sentence whose non-first-orderizability Boolos finds truly surprising because unlike sentences involving numerical quantifier words, these sentences "*look* as if they 'ought to be' symbolizable in first-order logic."[8] What appears to account for the non-first-orderizability of sentences like Geach-Kaplan is their use of plural forms. To illustrate this he gives the example:

(4) There is a horse that is faster than Zev and also faster than the sire of any horse that is slower than it.

Taking the universe of discourse to be the set of horses, and using '0', 's', '>', and '<' for 'Zev', '① the sire of ②', '① is faster than ②', and '① is slower than ②' respectively, (4) can be rendered into first-order notation as

(5) $(\exists x)(x > 0 \,.\, (\forall y)[y < x \supset x > s(y)])$

But now making some adjustments so that the sentence expresses a plural form, we get

(6) There are some horses that are faster than Zev and also faster than the sire of any horse that is slower than them,

the content of which Boolos thinks is made more explicit by

(7) There are some horses that are all faster than Zev and also faster than the sire of any horse that is slower than all of them.

Finally, Boolos takes as an acceptable paraphrase of both (6) and (7)

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[7] Boolos, "To Be is to Be a Value of a Variable," pp. 56-7.
[8] Ibid., p. 57.

(8) There is a nonempty collection (class, totality) X of horses, such that all members of X are faster than Zev and such that, whenever any horse is slower than all members of X, then all members of X are faster than the sire of that horse.

And this paraphrase is easily represented in second-order logic (with the same domain and interpretation of predicates as in (5)) as

(9) $(\exists X)((\exists x)Xx \,.\, (\forall x)(Xx \supset x > 0) \,.\, (\forall y)[(\forall x)(Xx \supset y < x) \supset (\forall x)(Xx \supset x > s(y))])$.

This second-order sentence, like the Geach-Kaplan sentence, is non-first-orderizable as it is false in the standard model of arithmetic when the domain is the set of natural numbers and we reinterpret '0' as 'zero', '>' as '① is greater than ②', '<' as '① is less than ②' and 's' as the successor function. It is true in any non-standard model of arithmetic as the set of non-standard elements of the model will always be a suitable value for 'X'.[9]

Boolos's interest in providing these examples of non-first-orderizable sentences containing plural forms is not so much an interest in the application of formal methods to the analysis of natural language as it is a return to the question of the relationship between second-order logic and set theory. His first move in this direction is to consider the sentence:

(10) There are some sets that are such that no one of them is a member of itself and also such that every set that is not a member of itself is one of them. (Otherwise put, there are some sets, no one of which is a member of itself, and of which every set that is not a member of itself is one.)

Taking the universe of discourse to be sets and '$\in$' to be 'is a member of', this sentence can be rendered into the second-order sentence:

(11) $(\exists X)((\exists x)Xx \,.\, (\forall x)[Xx \supset \neg x \in x] \,.\, (\forall x)[\neg x \in x \supset Xx])$

which is equivalent to

(12) $(\exists X)((\exists x)Xx \,.\, (\forall x)[Xx \equiv \neg x \in x])$.

[9] Ibid., pp. 57-8.

He next observes that '$(\exists x)\neg x \in x$' follows directly from (12), and conversely, when '$(\exists x)\neg x \in x$' holds, then there is some set in the totality X of sets that is not self-membered. X witnesses the truth of (12), and so (12) comes out equivalent to '$(\exists x)\neg x \in x$', what Boolos describes as "an obvious truth concerning sets."[10]

He does not find it surprising in light of (10)'s near self-evidence and the validity of the second-order comprehension schema, '$(\exists X)(\forall x)(Xx \equiv A(x))$', which includes as an instance of it

(13) $(\exists X)(\forall x)(Xx \equiv \neg x \in x)$,

that (12) should be equivalent to the first-order '$(\exists x)\neg x \in x$'. However, he takes the second-order (12) to more accurately reflect the semantic structure and meaning of (10) than its first-order counterpart. Still there is an important question for Boolos to raise here dating back to at least his "On Second-Order Logic;" can we use second-order logic to make assertions about all sets? Towards answering this question, he suggests we consider (13) as it is somewhat simpler than both (11) and (12). What (13) appears to express is the existence of a totality or collection containing all and only those sets x which are not members of themselves. Acceptance of the validity of (13) then, with its quantifiers taken as ranging over all sets and '$\in$' interpreted as membership, commits us to the Russell set, the set of all sets that are not self-membered, and this we know results in paradox.[11]

Avoidance of this conclusion can be blocked in a number of ways as we have seen. The method Boolos suggests here is the one he described in his 1975 paper, that it is illegitimate to use second-order formulas when the objects over which the individual variables in the formula range do not themselves constitute a set. This strategy leaves all instances of the second-order comprehension schema as logical truths and allows one to read all formulas of the form 'Xx' as

[10] Ibid., p. 64.
[11] Ibid., pp. 64-5.

'x is a member of the set X'. He now rejects this way of avoiding the paradox in that it has the following undesirable feature:

> The principal drawback of this way out is that there are certain assertions about sets that we wish to make, which certainly cannot be made by means of a first-order formula—perhaps to claim that there is a "totality" or "collection" containing all and only sets that do not contain themselves is to attempt to make one of these assertions—but which, it appears, could be expressed by means of a second-order formula if only it were permissible so to express them. To declare it illegitimate to use second-order formulas in discourse about all sets deprives second-order logic of its utility in an area in which it might have been expected to be of considerable value.[12]

In his 1971 article "The Iterative Conception of Set," Boolos observed that some axioms of set theory cannot be fully expressed by way of first-order logic because they cannot fully express the principles from which the whole of ZF can be generated.[13] For some axioms of set theory, e.g., separation and replacement, first-order logic provides an axiom schema that stands in for some infinite number of instances of that schematic form. Consider the first-order version of the separation axiom

(14) $(\forall z)(\exists y)(\forall x)(x \in y \equiv x \in z \,.\, Fx)$,

where 'y' is not free in F. This axiom schema, on Boolos's view, fails to express a single principle that says any monadic predicate put in place of 'F' will specify a set; the separation schema asserts instead an infinite number of instances of this particular schematic form. In this way, he thinks that the separation principle cries out for the second-order formulation

(15) $(\forall X)(\forall z)(\exists y)(\forall x)(x \in y \equiv x \in z \,.\, Xx)$.

His complaint is that

> [w]hatever our reasons for adopting Zermelo-Fraenkel set theory in its usual [first-order] formulation may be, we accept this theory because we accept a stronger theory consisting of a *finite* number of principles, among them some for whose complete expression second-order formulas are required. We ought to be able to formulate a theory that reflects our beliefs.[14]

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[12] Ibid., p. 65.

[13] George Boolos, "The Iterative Conception of Set," in *Logic, Logic, and Logic*, pp. 22, 25.

[14] Boolos, "To Be is to Be the Value of a Variable," p.65. (Boolos's italics)

Herein lies the heart of Boolos's project, how can second-order logic be so understood as to legitimize it for use in asserting claims about all sets?

What further complicates a solution is that he also wants to maintain the truth of second-order comprehension principles such as '$(\exists X)(\forall x)(Xx \equiv \neg x \in x)$' if he is to use second-order logic in discourse about all sets. One standard way of doing this is to allow other set-like objects, called 'ultimate (or 'proper') classes' as values of the second-order variables. Ultimate classes have members but are not themselves members of sets on account of their being "too big" to be sets. Boolos rejects this way out though on the grounds that "[s]et theory is supposed to be a theory about *all* set-like objects" and ultimate classes themselves are in essentials "set-like non-sets."[15]

Boolos's proposed solution to these difficulties is to give up the idea that the use of plural forms are to be understood as committing us to the existence of sets (or any other set-like entities such as classes, collections, or totalities) of the entities over which the individual variables range. Instead, we are to render second-order formulas into ordinary English plural forms thus eliminating the obstacles the standard interpretation of second-order logic presents for representing discourse about all sets. For example,

(16) $\neg(\exists X)((\exists x)Xx \,.\, (\forall x)[Xx \supset (x \in x \vee (\exists y)[y \in x \,.\, Xy \,.\, y \neq x])])$,

which is equivalent to the second-order statement of the set-theoretic induction axiom:

(17) $(\forall X)((\exists x)Xx \supset (\exists x)[Xx \,.\, (\forall y)(y \in x \supset \neg Xy)])$,

can be understood to mean

> (18) It is not the case that there are some sets each of which either contains itself or contains at least one of the others.

[15] Ibid., p. 66. (Boolos's italics)

Similarly, this can be done for the second-order version of the axiom of separation, (15) above, reading it as

> (19) It is not the case that there are some sets that are such that it is not the case that for any set z there is a set y such that for any set x, x is a member of y if and only if x is a member of z and also one of them.

In the interest of perspicuity, Boolos then re-writes (19) as

> (20) $\neg$ there are some sets such that $\neg(\forall z)(\exists y)(\forall x)[x \in y \equiv (x \in z \,.\, x$ is one of them)].

Though he notes that more properly (19) and (20) have the meaning:

> (21) $\neg(\exists X)((\exists x)Xx \,.\, \neg(\forall z)(\exists y)(\forall x)[x \in y \equiv (x \in z \,.\, Xx)])$

which includes the non-emptiness clause '$(\exists x)Xx$' unlike the full separation axiom. Separation can be fully expressed by adding 'and there is a set with no members' to (19) and '$(\exists y)(\forall x)\neg x \in y$' to (20).[16]

In relation to these last couple remarks, Boolos notes two features of his use of plurals as a method for interpreting second-order logic. First, he considers the case where there is exactly one Cheerio in the bowl in front of him. In ordinary English, he agrees with the intuitive view that would count the statement "There are some Cheerios in the bowl" as false, but for his present purposes, this does not really matter. He instead makes "the customary logician's assumption, which eliminates needless verbiage, that the use of plural forms does not commit one to the existence of two or more things of the kind in question."[17] In contrast, and in favor to a more literal understanding of plurals in ordinary English, he does assume that phrases such 'some critics' are committed to the existence of at least one critic, or in other words, to the non-emptiness of the class of critics. It is this feature that explains the additional clause '$(\exists x)Xx$' in

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[16] Ibid., pp. 66-7.
[17] Ibid., p. 67.

the above formulas.[18] Having said this, Boolos is now in a position to set out his translation of second-order logic into English plural talk.

First of all, he observes that translation from any logical notation into English will be complicated by a general lack of resources in natural languages for expressing cross-reference, so it will be necessary to augment English somehow with such resources. Boolos proposes subscripted pronouns, for example 'it$_x$', 'that$_x$', 'it$_y$', etc., and 'them$_X$', 'that$_X$', 'them$_Y$', etc. Such indexing, he observes, is very much continuous with the ordinary usage of 'former' and 'latter' in unaugmented English. The translation then proceeds as follows: translate Vv as 'it$_v$ is one of them$_V$', '$v \in v'$' as 'it$_v$ is a member of it$_{v'}$', $v = v'$ as 'it$_v$ is identical with it$_{v'}$', . as 'and', $\neg$ as 'not', and, where F^* is the translation of F, translate $(\exists v)F$ as 'there is a set that$_v$ is such that F^*'. The translation for the second-order existential is not quite what may be expected because of the non-emptiness condition we saw above. To accommodate this feature of discourse about plurals, Boolos translates $(\exists V)F$ as follows: let F^* be the translation of F, and let F^{**} be the translation of the result of substituting an occurrence of $\neg v = v$ for each occurrence of Vv in F. $(\exists V)F$ then translates as 'either there are some sets that$_V$ are such that F^*, or F^{**}'.[19]

Much of the recent literature on Boolos's plural interpretation of second-order logic clarifies the translation by construing it into a language called plural first-order (PFO).[20] This language has in addition to the usual first-order individual variables 'x_i' (for every natural number i), plural first-order variables 'xx_i' (for every natural number i), and a two-place logical predicate '$\propto$', which takes in its first argument place, individual variables and in its second argument place, plural variables. The plural existential quantifier '$\exists xx_i$' is interpreted as 'there

[18] Ibid., p. 67.
[19] Ibid., pp. 67-8.
[20] See for example the appendix to Agustín Rayo, "Word and Objects," *Noûs* 36:3 (2002), pp. 436-463; and also Linnebo, "Plural Quantification Exposed," pp. 73-4.

are some objects$_i$ such that', and '$x_i \propto xx_j$' as 'it$_i$ is one of them$_j$'. The interpretation of '$\exists xx_i$' allows for the case where the objects$_i$ is just a single object. Indeed all pluralities are counted as non-empty, so $(\forall vv)(\exists v)v \propto vv$ is taken as an axiom. Boolos's plural interpretation then translates into PFO as follows:

$Tr(Vv) = v \propto vv$
$Tr(\neg F) = \neg Tr(F)$
$Tr(F \,.\, G) = Tr(F) \,.\, Tr(G)$
$Tr((\exists v)F = (\exists v)Tr(F)$
$Tr((\exists V)F = (\exists vv)Tr(F) \vee Tr(F^*)$, where F^* is the result of substituting $\neg v = v$ everywhere for Vv.

There is also a translation of the second-order comprehension schema into:

$$(\exists vv)(\forall v)(v \propto vv \equiv \varphi) \vee (\forall v)(v \neq v \equiv \varphi)$$

which can be rendered equivalently but more perspicuously as

$$(\exists v)\varphi \supset (\exists vv)(\forall v)(v \propto vv \equiv \varphi)$$

where φ is a formula of PFO and does not contain vv free. Finally, to complete the theory of PFO, we add to the usual natural deduction rules for first-order logic, rules for the plural quantifiers and also for identity.

As an illustration of his translation of second-order logic into talk of plurals, Boolos presents as an example '$(Xx \equiv \neg x \in x)$' which he renders into augmented English as 'It$_x$ is one of them$_X$ if and only if it$_x$ is not a member of itself', or in PFO, '$(x \propto xx \equiv \neg x \in x)$'. Now, with the quantifiers ranging over sets, he renders '$(\forall x)(Xx \equiv \neg x \in x)$' as 'Every set is such that it is one of them$_X$ if and only if it is not a member of itself', or again in PFO, '$(\forall x)(x \propto xx \equiv \neg x \in x)$'; and finally, he renders '$(\exists X)(\forall x)(Xx \equiv \neg x \in x)$' as 'Either there are some sets that are such that every set is one of them if and only if it is not a member of itself or every set is a member of itself', or once more in PFO, '$(\exists xx)(\forall x)(x \propto xx \equiv \neg x \in x)$'. It should be noted here that Boolos's translation is only for monadic second-order logic (MSOL), but given a pairing function, which he says will be available in many of the most important applications of second-order logic, the monadic second-order variables will be sufficient for doing the work of all second-order variables. For example, the axiom of replacement can be symbolized in MSOL as:

(22) $(\forall X)((\forall x)(\forall y)(\forall z)[X\langle x, y\rangle \,.\, X\langle x, z\rangle \supset y = z] \supset (\forall u)(\exists v)(\forall y)[y \in v \equiv (\exists x)(x \in u \,.\, X\langle x, y\rangle)])$.

Shapiro, however, has observed that handling the second-order variables in this way results in the arbitrary posit of a pairing function. Furthermore, this pairing function will amount to an axiom of infinity when the domain has at least two elements because no such function exists on finite domains.[21]

Putting concerns about the assumption of a pairing function aside, Boolos's translation of MSOL into augmented English plural talk, a language we already understand,[22] appears to provide him with a perfectly acceptable way of applying second-order logic to discourse about *all* sets while at the same time, precluding Russell's Paradox. In addition, he claims to avoid ontological commitments over and above the ontological commits of the first-order variables, in this case, all sets; the application of second-order logic to set theory does not force any commitments to set-like non-sets, i.e. (ultimate) classes.

Still, he recognizes that someone may object that we only understand our use of plural forms in natural language because we have some prior understanding of statements about collections, totalities, or sets, and this prior understanding should be made explicit by analyzing these statements as claims about the existence of certain collections, totalities, or sets. This objection, Boolos thinks, may arise from the idea that any precise and adequate semantics for natural language must be interpretable in terms of set theory. However, he sees this as confusing

[21] Shapiro, *Foundations without Foundationalism*, p. 63.
[22] Boolos, "To Be is to Be a Value of a Variable," p. 69.

two questions—whether talk of plurals in our language is intelligible with whether a semantic theory can be specified for those sentences that contain plural forms. Given Tarski's work on truth theories for formal languages, Boolos explains, we should not accept as self-evident that an adequate semantics for a natural language, such as English, can be formulated within that language. We may indeed have to look to set theory, but this has no bearing on our prior understanding of English plural forms.[23]

In his "Nominalist Platonism," Boolos goes a step further in dispelling this objection by developing a truth theory for second-order set theory that in fact does not analyze plural forms in terms of set-like non-sets. The second-order variables of this theory take as their values the same sorts of entities as the first-order variables, namely all sets. Boolos formulates this truth theory in the second-order language of set theory with a new satisfaction predicate that contains two first-order variables 's' and 'F' and one second-order variable 'R' such that R and the sequence s satisfy the formula F. A sequence is a function from the set of first-order variables where the first-order variables have as values all sets. The theory is then as follows:

> If F is $u \in v$, then R and s satisfy F iff $s(u) \in s(v)$;
> if F is $u = v$, then R and s satisfy F iff $s(u) = s(v)$;
> if F is Vv, then R and s satisfy F iff $R\langle V, s(v)\rangle$;
> if F is $\neg G$, then R and s satisfy F iff $\neg(R$ and s satisfy $G)$;
> if F is $(G \,.\, H)$, then R and s satisfy F iff (R and s satisfy G . R and s satisfy H);
> if F is $(\exists v)G$, then R and s satisfy F iff $(\exists x)(\exists t)$(t is a sequence . $t(v) = x$. $(\forall u)$(u is a first-order variable . $u \neq v \supset t(u) = s(u)$) . R and t satisfy G);
> if F is $(\exists V)G$, then R and s satisfy F iff $(\exists X)(\exists T)((\forall x)(Xx \equiv T\langle V, x\rangle)$. $(\forall U)$(U is a second-order variable . $U \neq V \supset (\forall x)(T\langle U, x\rangle \equiv R\langle U, x\rangle)$) . T and s satisfy G);

where '$\langle\, ,\, \rangle$' is the ordered-pair function sign.[24] So here we have a truth theory for the second-order language of set theory that has second-order variables ranging over the same entities as the first-order variables, namely sets; there is no need to posit additional values for the second-order

[23] Ibid., p. 70.
[24] Boolos, "Nominalist Platonism," pp. 80-1.

variables, such as classes, ultimate or otherwise. The second-order variable '*R*' in the new satisfaction predicate takes on the role of 'one of them' in our augmented English plural forms.

Boolos offers one final consideration against the idea that our ordinary talk of plurals commits us to sets, or collections, presenting the example,

(23) There are some sets of which every set that is not a member of itself is one,

claimed to be false on the basis that this sentence entails the existence of a set that is "too big", specifically the set of all sets that are not members of themselves. He finds such a claim unlikely, comparing with it the claim that there is a set of all trucks and observing that this sentence does not appear to follow from the true sentence, "There are some trucks of which every truck is one." But an even stronger consideration against the falsity of (23), he thinks, is his strong intuition that English sentences of the form "There are some *A*s of which every *B* is one," are synonymous with sentences of the form "There are some A*s* and every *B* is an *A*." If we grant Boolos his intuition, then (23) means the same as the trivial truth

(24) There are some sets and every set that is not a member of itself is a set,

and so does not entail the existence of a "too big" set.[25]

From these considerations, Boolos rests with his conclusion that second-order formulas that include individual variables ranging over all sets can be interpreted in terms of English plural forms, in terms of a language that we already understand. Second-order quantification does not force us to recognize further entities that have members and would be sets if it were not for their unwieldy size.[26]

Boolos concludes by remarking on an additional benefit obtained by the interpretation of second-order logic in terms of English plural forms. It is not just when we apply second-order

[25] Boolos, "To Be is to Be a Value of a Variable," pp. 70-1.
[26] Ibid., p. 71.

logic to set theory that the plural interpretation avoids multiplying entities beyond necessity. This advantage carries over to our ordinary plural talk in that we need not take ourselves to be committed to the existence of sets when we use plural forms, as he has already alluded to in his truck example mentioned two paragraphs prior. Now, he considers a second bowl of Cheerios. In this bowl there are well over two hundred Cheerios, but it seems highly counterintuitive to think that in addition to the Cheerios, there is also a set of Cheerios in the bowl, and then perhaps also all subsets of the Cheerios in the bowl. We would no doubt feel a similar awkwardness in talking about a set of critics in addition to the critics mentioned in the sentence, "Some critics admire only one another."[27] As he explains,

> The lesson to be drawn from the foregoing reflections on plurals and second-order logic is that neither the use of plurals nor the employment of second-order logic commits us to the existence of extra items beyond those to which we are already committed. We need not construe second-order quantifiers as ranging over anything other than the objects over which our first-order quantifiers range, and, in the absence of other reasons for thinking so, we need not think that there are collections of (say) Cheerios, in addition to the Cheerios. Ontological commitment is carried by our *first*-order quantifiers; a second-order quantifier needn't be taken to be a kind of first-order quantifier in disguise, having items of a special kind, collections, in its range. It is not as though there were two sorts of things in the world, individuals, and collections of them, which our first- and second-order variables, respectively, range over and which our singular and plural forms, respectively, denote.[28]

Let us turn now to some critical response to Boolos's plural interpretation of second-order logic.

II

Probably the first fully developed criticism of Boolos's position comes from Michael Resnik in his "Second-Order Logic Still Wild" (1988). Resnik's overriding criticism is that his logical/linguistic intuitions concerning sentences that use plural forms differ significantly from those of Boolos. In fact, his intuitions are precisely those that Boolos thinks get second-order

[27] Ibid., p. 72; Boolos, "To Be is to Be a Value of a Variable," p. 74, 77-8.
[28] Boolos, "To Be is to Be a Value of a Variable," p. 72. (Boolos's italics)

logic wrong in supposing that second-order variables take classes as their values. Resnik urges, in light of Boolos's work on second-order logic, a reconsideration of the Geach-Kaplan sentence:

(1) Some critics admire only one another.

He admits that while this sentence appears to say nothing of classes, he understands it as saying,

(2) There is a non-empty collection of critics each member of which admires no one but another member.

And this, he says holds for many sentences that include plural quantification; he cannot process them without understanding them as referring to collections. For consider again Boolos's translation of the second-order formula:

(3) $(\exists X)((\exists x)Xx \,.\, (\forall x)(\forall y)[Xx \,.\, Axy \supset x \neq y \,.\, Xy])$,

as

(4) There are some critics such that any *one of them* admires another critic only if the latter is *one of them* distinct from the former.

Resnik sees this translation as explicitly referring to collections in its 'one of them' with the referent of 'one' being a member of it. Though he will grant that (3) and (4) can be read as (1) thus doing away with the problematic 'one of them', this move just brings us back to trying to work out the ontological commitments of (1) and discerning what these commitments are was supposed to be the reason for moving to (3) and then to (4). Returning to (1) is unhelpful at best in settling this ontological dispute.[29]

There is something appealing in both Resnik and Boolos's understanding of plural forms. It is not difficult to agree with Boolos that when we eat some Cheerios we are not also eating a set of Cheerios, and certainly not all the subsets of Cheerios. However, when we eat one of the Cheerios, it seems equally difficult not to agree with Resnik that we are eating *one of them*, one

[29] Resnik, "Second-Order Logic Still Wild," pp. 77-8. (Resnik's italics)

member of a collection. Parsons, in his "The Structuralist View of Mathematical Objects," (1990) has suggested a way to make sense of these two opposing intuitions. That is to think of pluralities in terms of Russell's notion of 'a class as many', which allows for discourse about collections without forcing a commitment to a class as a whole, as a single object, i.e. the notion of class or set found ordinarily in set theory.[30]

Further support for this way of understanding plural terms appears in Linnebo's "Plural Quantification Exposed" (2003) which discusses a slight extension of the language PFO to PFO+ containing predicates and relations, in addition to the logical relation '$\propto$', that take plural expressions as arguments where the predication is non-distributive. The plural predication '$F(xx)$' is distributive just in the case that it is equivalent to '$(\forall x)(x \propto xx \equiv Fx)$'. As examples of distributive and non-distributive predication, he provides the respective examples, 'The boys ran across the field' and 'The boys lifted the piano'. Once we allow plural expressions to occur as subjects of predications, Linnebo observes, it is hard not to see them as standing for some sort of entities (and in fact the satisfaction relation Boolos defines for MSOL uses a non-distributive plural predication with the relation 'R' in its first argument place). Furthermore, he remarks, that given the language of PFO+, we can define an identity predicate holding between pluralities as '$xx = yy \equiv (\forall u)(u \propto xx \equiv u \propto yy)$', and this ability to state identity conditions has often been taken as a indication of objecthood.[31]

Perhaps we do not want to say that these plural "entities" are sets, but it is hard to avoid the conclusion that they are some form of collection, perhaps Russell's sets as many or some other set-like non-sets. This conclusion, of course, will not be acceptable to Boolos as he takes

[30] Parsons, "Structuralist View of Mathematical Objects," p. 326; Russell, *Principles of Mathematics*, §§ 70, 74, 104.
[31] Linnebo, "Plural Quantification Exposed," pp. 79, 90 fn. 20.

set theory "to be a theory about *all* set-like objects."[32] The very point of his plural interpretation for MSOL is that it does not result in additional ontological commitments beyond what the first-order variables were already committed to. Allowing, as further objects, plural entities, in addition to sets and perhaps also ultimate classes, is just the opposite result of what he intended. And if sets and ultimate classes are enough to do the work of plural entities why not just stick with them so as to avoid ontological excesses? Perhaps if our interest is in capturing our intuitions about English plural forms, plural entities are better then ultimate classes, but either way, Boolos would be forced to acknowledge additional set-like objects. This looks to be a difficult position for him. Linnebo, though, points out one central assumption required for this to be decisive against Boolos's plural interpretation; that is it requires a full account of what it is to be an object in the first place and neither Boolos, nor Resnik, nor Parsons, nor Linnebo provides this. At best, this dispute remains a standoff between intuitions about the ontological status of English plural talk.[33]

Both Resnik and Parsons raise a related objection to the alternate semantics Boolos proposes for MSOL. Boolos's new semantic theory was supposed to show that the second-order variables need not assume classes as their values. So does his theory actually do this? As all of the other clauses of his definition of the satisfaction predicate are the usual ones, Parsons focuses on the alternate clause for second-order existential quantification:

(5) if F is $(\exists V)G$, then R and s satisfy F iff $(\exists X)(\exists T)((\forall x)(Xx \equiv T\langle V, x\rangle) \,.\, (\forall U)(U$ is a second-order variable $.\ U \neq V \supset (\forall x)(T\langle U, x\rangle \equiv R\langle U, x\rangle))\ .\ T$ and s satisfy $G)$

suggesting that we look at this clause "platonistically." Then R codes an assignment of second-order entities to the variables of the language in that the variable V is assigned the abstract $\lambda x R\langle V, x\rangle$. This clause then reads "There is a (second-order entity) X and an assignment T such

[32] Boolos, "To Be is to Be a Value of a Variable," p. 66. (Boolos's italics)
[33] Linnebo, "Plural Quantification Exposed," pp. 79-80.

that T assigns X to V and agrees with R in what it assigns to other variables, such that T and s satisfy G."[34] Once this move is made, Parsons finds it hard to see why we should think that the second-order variables are not assigned values in a way analogous to the assignment of values to the first-order variables. He explains,

> The values of the individual variables, Boolos says, are just the terms of the sequences. Why should we not say similarly, that X is a value of a second-order variable if it is $\lambda x R\langle V, x\rangle$ for some R and V, that is if $(\exists R)(\exists V)(\forall x)(Xx \equiv R\langle V, x\rangle)$? The difference between the treatment of first- and second-order variables seems to lie just in the facts that in the second-order case functions are coded by predicates and that a function from individuals to n-argument second-order entities can be coded by an $n + 1$-argument second-order entity.[35]

So much like the competing intuitions over the ontological status of English plural forms, Boolos's alternate semantics for MSOL fails to be decisive in determining the ontological commitments of his plural interpretation MSOL.

Linnebo, however, develops a further criticism of Boolos's interpretation that does not rely on intuitions about the ontological commitments of English plural forms and that does look to illustrate a serious point of instability in Boolos's position. He begins by reviewing the reasons Boolos claims that set theory needs second-order logic. First, there are claims about sets, such as

(6) There are some sets that are all and only the non-self-membered sets,

that look to be meaningful, and even true, but cannot be paraphrased in the usual set-theoretic way in first-order logic without introducing a Russell set and the accompanying paradox. Second, there are results in set theory that are provable regardless of what set-theoretic predicate is substituted for some schematic predicate letter. Linnebo observes, for example, that whenever 'F' is a set-theoretic predicate, the union of the ordinal numbers satisfying 'F' is well-ordered by the membership relation. To express this kind of result, Boolos sees a need for allowing

[34] Parsons, "The Structuralist View of Mathematical Objects," pp. 327.
[35] Ibid., p. 328; Resnik, "Second-Order Logic Still Wild," pp. 80-3 expresses similar worries, though less succinctly.

quantification over predicate letters, and this can be accomplished by adding a theory of second-order logic to set theory. Finally, and Linnebo remarks, most importantly, Boolos thinks that second-order logic is required to fully express the intended meanings of the axioms of replacement and separation. Here, a problem arises for Boolos. On the standard understanding of second-order logic, the second-order variables require there to be entities for them to range over, usually in the form of either sets or classes. In order to bring out the difficulties these options pose for Boolos, Linnebo appeals to an argument advanced by Parsons to the effect that adding to set theory this apparently necessary theory of second-order logic always leads in a natural way to a universe of sets larger than the one with which we began.

The process begins by adding a weak theory of classes to ZFC. By a weak theory of classes is meant a theory of classes with a predicative class comprehension schema:

(7) $(\exists F)(\forall x)(x \in F \equiv \varphi)$

where 'φ' does not contain 'F' free and its quantifiers are restricted to sets. The result is a conservative extension of ZFC known as Neumann-Bernays-Gödel set theory (NBG). Parsons thinks that the addition of this weak theory of classes can be justified merely by considerations of our use of predication in language. However, this predicative theory does not allow for the expression of instances of replacement or separation that contain bound class variables. To gain the resources needed to express all instances of replacement and comprehension, including those that do contain bound class variables, we must add the full impredicative class comprehension schema,

(7′) $(\exists F)(\forall x)(x \in F \equiv \varphi)$

where 'φ' does not contain 'F' free but its quantifiers are not restricted to sets; they may also range over classes. This yields Morse-Kelley set theory (MK), a set theory significantly stronger than NBG both in its logical strength and ontological commitments. MK's increase in power though no longer leaves open the linguistic justification for quantification over class variables used in the expansion of ZFC to NBG. Instead, the impredicative class comprehension schema relies on the notion of an arbitrary subset of the domain which depends on the sorts of combinatorial intuitions contained in set theory itself, namely the notion of an arbitrary subset of the domain. Once we appeal to these kinds of combinatorial intuitions though—the kind of combinatorial intuitions that give rise to our understanding of the notion of set in the first place—Parsons explains, we have all the resources needed to treat the classes of MK themselves as sets so allowing us to increase the size of our original set-theoretic universe. Furthermore, this natural expansion of our original set theory can be repeated again and again resulting in larger and larger universes of sets. There is no natural stopping place.[36]

What Linnebo finds surprising here, assuming Parsons is correct, is that "[i]t is impossible ever to quantify over absolutely all sets. For whenever a domain is specified for the first-order quantifiers of our set theory, we can carry out this three-step extension procedure, which will lead us to accept an even larger domain of sets."[37] He summarizes the results of Parsons's considerations in two claims—that of ontological proliferation and of inexhaustibility. Ontological proliferation claims there is more than one ontological category of set-like entities; there are also classes. And inexhaustibility claims the impossibility of every quantifying over absolutely all sets. Ontological proliferation occurs already in the first step as soon as we allow for the addition of the weak theory of classes, and inexhaustibility occurs only in the final step.[38]

[36] Charles Parsons, "Sets and Classes," *Noûs* 8:1 (1974); pp. 1-12, especially pp. 5-11; see also Linnebo, "Plural Quantification Exposed," pp. 81-3.
[37] Linnebo, "Plural Quantification Exposed," p. 83.
[38] Ibid., p. 83.

Linnebo began his account of Parson's argument by stating three reasons that lead Boolos to recognize a need for the applicability of second-order logic to set theory. Boolos also thinks that set theory is a theory of all set-like entities and that it should be possible to quantify over absolutely all sets. He does not want to recognize set-like non-sets in the form of classes nor does he want to be forced by the threat of inconsistency to restrict second-order quantification to domains that themselves constitute sets. His plural interpretation of MSOL is supposed to allow him to satisfy both these aims simultaneously.

In light of this discussion, Linnebo is now in a place to consider the logicality of Boolos's translation of MSOL into English plural forms. In the interest of perspicuity, he considers instead the translation of MSOL into the language PFO described above, but nothing of any particular philosophical significance hangs on this move. To this end, he first provides three criteria to serve as a partial analysis of what it is for a theory to count as pure logic. The first is ontological innocence, that PFO is not committed to any entities beyond those of the first-order domain; the second is universal applicability, that PFO can be applied to any universe of discourse regardless of subject matter (he notes that this would distinguish PFO from both set theory and second-order logic with the standard set-theoretic semantics. Both set theory and second-order logic with standard semantics are applicable to any universe of discourse so long as it is set-sized); and the third is cognitive primacy, that PFO presupposes no extra-logical knowledge in order to be understood. Lying behind all three of these criteria is the idea that logic be unconditioned and presuppositionless.[39] I think that Boolos would not object to any of the criteria Linnebo proposes as something like these are just the sorts of features he praises in his plural interpretation and a lack of which he finds problematic for standard second-order logic.

[39] Ibid., pp. 75-6.

Linnebo is willing to grant that much of PFO can uncontroversially be counted as lying within the domain of logic, e.g. the tautologies, the non-emptiness axiom, and the natural deduction rules; where he sees a worry about logicality arising, if anywhere, is in the plural comprehension axioms. What he intends to show now is that the considerations that allow for the addition of a theory of plural quantification to set theory will be strong enough to yield iterated extensions for plural quantification of the sort Parsons described in the case where we add the usual theory of second-order quantification to set theory.[40] Linnebo reminds us that because we want to add a theory of plural quantification in order to fully express the axioms of replacement and separation, adopting the predicative plural comprehension axioms will not be enough. We will need to add the full impredicative plural comprehension axioms. This means that we cannot take the plural comprehension axioms

$$(\exists v)\varphi \supset (\exists vv)(\forall v)(v \propto vv \equiv \varphi)$$

to determine pluralities only where φ contains no bound plural variables; we must also allow φ to contain bound plural variables. And this requires that we understand what these bound plural variables range over. Such understanding entails our understanding of the notion of a determinate range of arbitrary sub-pluralities of the original domain.[41]

This notion of a determinate range of arbitrary sub-pluralities, Linnebo thinks is far from primitive and unanalyzable, so we will require an account of the kinds of considerations that give it content. As he explains,

> The need for an account of this notion becomes particularly acute when we want to apply the plural comprehension axioms to the domain of higher set theory. For applied to this domain, the notion of a determinate range of arbitrary sub-pluralities becomes extremely complicated and abstract. In fact, the notion of an arbitrary sub*set*, which is closely related to but weaker than that of an arbitrary sub-*plurality*, is one of the most difficult and problematic concepts of all set theory. . . . For instance, if we understand the

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[40] Ibid., p. 75, 84.
[41] Ibid., p. 85.

> notion of an arbitrary subset, we understand all the concepts that are needed to express the Continuum Hypothesis. So it would be highly unreasonable to regard this notion as unanalyzable.[42]

The considerations that Linnebo thinks do give content to this notion of a determinate range of arbitrary sub-pluralities belong to combinatorics and set theory. Such considerations provide a good understanding of the range of arbitrary sub-pluralities of small finite collections, the sorts of collections our ordinary English talk of plurals is most often concerned with. He explains that this understanding most likely comes from the operations that we can perform on such collections, such as going through the items of the collection one at a time and applying a process of acceptance or rejection to these items. Our understanding of an arbitrary sub-plurality in the case of infinite collections is then gained by extrapolating from our understanding of these small, finite cases.[43]

Here, Linnebo finds a serious difficulty in Boolos's claim the theory of plural quantification is pure logic for our understanding of the notion of a determinate range of arbitrary sub-pluralities appears to depend upon a prior understanding of combinatorics, and so possibly also of set theory. Once these combinatorial ideas are in play, he argues, there is no reason that we cannot collect pluralities into pluralities of a higher level, so pluralities of pluralities, and then of course, we can continue to repeat this process obtaining higher and higher levels of pluralities. The result of quantifying over pluralities parallels that of quantifying over classes; the combinatory considerations used to justify impredicative plural comprehension lead

[42] Ibid., p. 85. (Linnebo's italics)

[43] Ibid., pp. 85-6; this idea comes from Paul Bernays, "Platonism in Mathematics," (1935) in *Philosophy of Mathematics: Selected Readings*, 2nd ed., Paul Benacerraf and Hilary Putnam, eds. (New York: Cambridge University Press, 1983) pp. 258-71. Parsons, in "Sets and Classes," appeals to just this argument of Bernays in arguing, above, that our understanding of the notion of set itself depends upon combinatorial rather than linguistic considerations, pp. 9-10.

us to theories of far greater strength than PFO.[44] Hence, Linnebo concludes, adding PFO to set theory yields even stronger extensions of set theory:

> There is no conception of plural quantification that allows us to add impredicative PFO to ZFC set theory without naturally leading to further extensions as well. This means that if Boolos wants to apply the theory of plural quantification to set theory, he will have to accept higher plural quantification as well. This leads to a stratified theory of higher pluralities. When we develop this theory up to the level of some ordinal number α, the resulting theory will be isomorphic with impredicative simple type theory of order α, in the sense of being equi-interpretable with it.[45]

Boolos's plural quantification suffers from the same phenomenon of inexhaustibility Parsons describes in the case of quantification over classes. In neither case will it ever be possible to quantify over all sets. Just as with classes we could "singularize" each new level of pluralities, i.e., treat them as sets, but each time we carried out this process would yield a larger domain of sets than the one we began with. Even if we did not singularize each new level of pluralities, he explains, we would still end up with the phenomenon of inexhaustibility in that there will be no way to quantify over all pluralities for there will always be higher levels. The reason for this in both cases, that of classes and of pluralities, are the kinds of combinatory considerations that give content to the notions of arbitrary sub-collection or sub-plurality used to justify the respective impredicative comprehension schemata.[46]

Linnebo's primary aim here is not to show that Boolos's plural interpretation of MSOL fails to deny the inexhaustibility phenomenon Parsons described for set theory where the second-order variables ranged over classes. Rather his point is to show that the impredicative plural comprehension axioms rely too heavily on combinatorial and set-theoretic ideas for us to consider PFO a purely logical theory; PFO does not share the cognitive primacy of our ordinary first-order quantification theory. That Boolos's translation of MSOL into PFO is also subject to inexhaustibility is just a manifestation of combinatorial ideas built into the theory of PFO. There

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[44] Linnebo, "Plural Quantification Exposed," pp. 86.
[45] Ibid., p. 88.
[46] Ibid., p. 88.

is also a further conclusion concerning ontological innocence that Linnebo suggests can be drawn from these considerations—that the truth of the plural comprehension axioms "consists in much the same as does the truth of the corresponding *set theoretic* comprehension axioms. The present suggestion therefore implies that *both* sorts of comprehension axioms carry ontological commitments."[47] The proponents of PFO could attempt to deny this conjecture, but then they would be forced to recognize that two logically equivalent theories, the inexhaustible set theory with classes and the set theory with an inexhaustible hierarchy of plural quantification, could have, as Linnebo describes it, "radically different ontological commitments." And this view, he thinks, could not help but raise the question of why the "notion of ontological commitment should be so important to the philosophy of mathematics in the first place."[48]

III

In conclusion, I turn now to consider what Quine's view of this dispute over the logical status of plural quantification might have been. Perhaps the best case Boolos makes for accepting a theory of plural logic as pure logic goes by way of his appeal to our ordinary understanding of English plural forms. After all, these claims do seem to be relatively straightforward and so well within the realm of what we may be willing to think of as self-evident, trivial, or obvious. Furthermore, to claim that simple sentences, such as "Some critics admire only one another" or "There are some Cheerios in the bowl" are committed to the powerful mathematical theory of sets looks highly counterintuitive, particularly when we start asking questions about whether we might be eating in addition to the Cheerios, a set of Cheerios.

[47] Ibid., p. 89. (Linnebo's italics)
[48] Ibid., p. 89.

But Quine, I think, would not feel the force of such arguments in favor of plural quantification for remember why he urged the regimentation of ordinary language into the notation of first-order logic to begin with: it was to provide a reasonable standard for judging the ontological commitments of a theory where ordinary language provided only unclarity and confusion. As he explains,

> The common man's ontology is vague and untidy in two ways. It takes in many purported objects that are vaguely or inadequately defined. But also, what is more significant, it is vague in its scope; we cannot even tell in general which of these vague things to ascribe to a man's ontology at all, which things to count him as assuming. Should we regard grammar as decisive? Surely not; the nominalizing of verbs is often a mere stylistic variation. But where can we draw the line?
>
> It is a wrong question; there is no line to draw. Bodies are assumed, yes; they are the things, first and foremost. Beyond them there is a succession of dwindling analogies....
>
> My point is not that ordinary language is slipshod, slipshod though it be. We must recognize this grading off for what it is, and recognize that a fenced ontology is just not implicit in ordinary language. The idea of a boundary between being and nonbeing is a philosophical idea, an idea of technical science in a broad sense. Scientists and philosophers seek a comprehensive system of the world, and one that is oriented to reference even more squarely and utterly than ordinary language. Ontological concern is not a correction of a lay thought and practice; it is foreign to the lay culture, though an outgrowth of it.[49]

Like much of science, ontology does not rely on our intuitions about natural language. Indeed, this was much the reason for his talk of *paraphrase* in regimenting ordinary language into canonical notation; the aim was not synonymy but rather the resolution of ambiguity. And such is the aim of science generally. For Quine, science is an artificial construct for the organizing of our experience. As Peter Hylton explains, commenting on the above quoted passage,

> To those who complain that Quine's ideas about ontology distort common sense, his answer is that any attempt at ontology is bound to do so, for common sense does not contain an answer to the ontological question, not even implicitly. To those who complain of artificiality, the answer is that the very question is a product of artifice, as are all advances in our knowledge.[50]

Here there is considerable divergence between Quine and Boolos over the importance of the relationship between synonymy and logical analysis. Quine, as we have seen, sees a desire

[49] W.V. Quine, "Things in Their Place in Theories," in *Theories and Things* (Cambridge: Harvard University Press, 1981), p. 9.
[50] Peter Hylton, "Quine of Reference and Ontology," in *The Cambridge Companion to Quine*, Roger Gibson, ed. (New York: Cambridge University Press, 2004), p. 128. Full consideration of Quine's views on ontology, though very apt, would take me very far from the present concern of higher-order logic. Hylton's article is an excellent overview of this topic in Quine's philosophy.

for synonymy as wholly misplaced as a criterion of correct analysis; the ordered pair is his paradigm. In fact, he does not think we can make any rigorous sense out of the notion of synonymy at all. Boolos, in contrast, frequently appeals to synonymy as a standard for measuring the success of a correct logical analysis. "There are some Cheerios in the bowl" should not be analyzed in terms of set theory because Boolos thinks it absurd to suggest that in eating some Cheerios, we are also eating a set of Cheerios. For this reason, he thinks his plural interpretation of MSOL captures the meaning of the English sentence more accurately than does an analysis in set theory. He does not however provide an account of what counts as a synonymous analysis or why this should be our criterion for a correct analysis. Given his sympathy with Quine's view that an extensional account of second-order logic is far more desirable that an intensional one, it is unlikely that Boolos would want to analyze synonymy in terms of intensional entities, i.e., meanings.

What Boolos is more likely to have in mind as a criterion of synonymy is something like sameness of truth conditions or logical equivalence, but it seems that we are under no obligation to say that either of these criteria captures what we ordinarily mean in saying that one sentence is synonymous with some other sentence. Both are artificial standards that may or may not help us to arrive at a definitive view of when a sentence is synonymous with another. Much of the appeal and force in Boolos's suggestion that MSOL should be understood in terms of plural quantification is that the plural interpretation is more natural, more intuitive, for expressing ordinary English plural forms. However, once we admit that the criterion of synonymy employed in judging the accuracy of our analysis into MSOL is artificial, we have less reason to think that synonymy in its more intuitive sense is of primary importance to such an analysis. We would then appear to be at liberty to employ other equally artificial devices in our analyses of ordinary English plural talk, for example, we may analyze it in terms of set theory. We may not be eating a set of Cheerios in addition to the Cheerios themselves, but so long as we only aim to paraphrase when we give a logical analysis, as Quine does, such oddities are of little concern.[51]

By similar reasoning though, Quine himself cannot appeal to his own logical or linguistic intuitions as justifying his view that English plural forms are more properly paraphrased in terms of classes or sets. This may seem to present a problem for him by making his claim that a sentence such as "Some critics admire only one another" commits one to the existence of classes now look arbitrary. Why should we understand talk of plurals as committing us to sets rather than to just a plurality of the objects that we refer to singularly? The beginning of a response would be the same as above. Quine's aim is not synonymy but resolution of ambiguity. Of course, the problem remains in that his proposed paraphrase in terms of classes appears prejudiced in favor of his taking English plural forms to commit us to classes and so outstripping the bounds of what we may reasonably consider to be pure logic.

However, I do not think Quine's paraphrase in terms of classes is as arbitrary as all this in light of Linnebo's considerations against taking the language of PFO to be pure logic. We saw there that the language of PFO relies on significant combinatorial, and perhaps even set-theoretic, ideas violating what Linnebo calls logic's cognitive primacy. In addition, he argued that set theory with quantification over classes and set theory with plural quantification are logically equivalent theories. Quine's move to paraphrase English plural talk in terms of classes now seems far from arbitrary. Indeed, Linnebo's view on this matter is, I think, very much in

[51] A similar point could be made on the basis of Boolos's acceptance of the view that use of a plural form does not commit one to the existence of two or more objects; the domain need only be non-empty. 'There are some Cheerios' will be true then if there is only one Cheerio. He admits that this is artificial but of no matter for the development of his logical theory. However, much of the appeal in his theory of plural quantification is the natural way that it regiments English plural forms. Once we start introducing such artificialities (a technique that Quine fully endorses), there seems little reason to complain about the use of set theory to regiment our ordinary plural talk. See above p. 90.

line with what would have been Quine's own response had he ever given substantial consideration to Boolos's plural interpretation of MSOL.

For consider again Quine's discussion of Henkin's branching quantifiers presented at the end of chapter I. There, Quine focused on the lack of a complete proof procedure for Henkin's theory indicating that the addition of branching quantifiers went beyond ordinary quantification theory in assuming the power of extra-logical mathematics. A theory of branching quantifiers could not be viewed as obvious throughout, a criterion of logicality that parallels Linnebo's condition of cognitive primacy. Quine concluded then that the theory of branching quantifiers was fairly represented by the extra-logical mathematical theory of functions instead. Indeed, by appeal to this earlier line of thought, Quine does not even need to count on the correctness of Parsons's argument as Linnebo does.[52] For regardless of the success of Linnebo's line of thought, Quine can again make his argument based on the completeness of ordinary quantification theory. No matter how intuitive Boolos's translation of MSOL into ordinary plural talk may seem, this theory will still lack in completeness; there will still be some truths of MSOL that are not obvious or capable of being made so by a finite number of individually obvious steps.

It is not entirely clear how much importance Quine stakes in the objects of a theory in any case as he explains, "Structure is what matters to a theory, and not the choice of its objects.... I extend the doctrine to objects generally, for I see all objects as theoretical."[53] We have already seen some hint of this view in our earliest discussions of Quine's criticisms of second-order logic. Recall that he described both Frege and Russell in their extension of the theory of quantification to predicate letters as quantifying over attributes, yet he ultimately

[52] Boolos indeed does reject much of the Parsons's argument with regard to sets; see his "Reply to Charles Parsons' 'Sets and Classes'," in *Logic, Logic, and Logic*, pp. 30-6.
[53] W.V. Quine, "Things and Their Place in Theories," p. 20.

describes their attempted reduction of mathematics to logic as a reduction to a theory of classes or sets, not to a theory of attributes. All that distinguishes classes from attributes is that classes, given their extensionality, have a readily available criterion of identity. In the interest of clarity then, Quine finds the mathematical content of Frege and Russell's logic more accurately represented in terms of classes. Similarly, it seems that he would view a choice between class theory and plural logic for the most part as both logically and ontologically insignificant in light of Linnebo's conclusion that PFO and simple type theory are logically equivalent theories. Quine's preference for paraphrasing sentences such as Geach-Kaplan in terms of classes then is not so arbitrary for class theory makes explicit that our ordinary talk of plurals leads to a mathematical theory of far greater strength than ordinary predicate logic. Such was also what motivated him in his preference for representing Henkin's branching quantifiers in terms of quantification over functions.

Remember though that he also concluded that his considerations over branching quantifiers proved nothing, and the same can be said for these considerations of Boolos's theory of plural quantification. The response I have suggested here as Quine's also proves nothing. Quine has merely offered some considerations for carving out a subclass of truths to which we might reasonably apply the label, honorific or not, of "logic".

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