REGULARITY OF NAVIER–STOKES FLOWS WITH BOUNDS FOR THE PRESSURE

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Abstract. This study derives regularity criteria for solutions of the Navier–Stokes equations. Let $\Omega(t) := \{x : |u(x,t)| > c\,\|u\|_{L^r(\mathbb{R}^3)}\}$, for some $r \geq 3$ and constant c independent of t, with measure $|\Omega|$. It is shown that if $\|p + \mathcal{P}\|_{L^{3/2}(\Omega)}$ becomes sufficiently small as $|\Omega|$ decreases, then $\|u\|_{L^{(r+6)/3}(\mathbb{R}^3)}$ decays and regularity is secured. Here p is the physical pressure and $\mathcal{P}$ is a pressure moderator of relatively broad forms. The implications of the results are discussed and regularity criteria in terms of bounds for $|p + \mathcal{P}|$ within Ω are deduced.

1. Introduction

This note is concerned with the Cauchy problem of the Navier–Stokes equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \Delta u, \qquad (1)$$
$$\nabla \cdot u = 0$$

in $\mathbb{R}^3 \times (0, \infty)$ with $u(x, 0) = u_0(x)$ divergence-free, smooth and decaying sufficiently fast at infinity. Given such an initial velocity field, it is well known that a classical solution exists up to some finite time $t = T$, which depends on $u_0(x)$. The question is whether or not the solution remains smooth (regular) beyond T, particularly up to all $t \geq T$ (global regularity).

Decades of active research on this problem since Leray's seminal work in the 1930s have resulted in a rich literature [1–33]. Yet, the prospect of a definitive answer to the above question has become increasingly remote. Early studies by Prodi [22], Serrin [24] and Ladyzhenskaya [20] found that regularity is guaranteed provided that $\int_0^T \|u\|_{L^r}^{2r/(r-3)}\, \mathrm{d}t < \infty$, for $r \in (3, \infty)$. Recently, Escauriaza, Seregin and Sverák [14] have extended this criterion to $\operatorname{esssup}_{t\in(0,T)} \|u\|_{L^3} < \infty$, for the critical case $r = 3$. Various criteria expressible in terms of the pressure p and its gradient ∇p have been derived by a number of authors [1, 4, 8, 10, 12, 15, 23, 25, 28, 30, 32]. Among these, two criteria are most relevant to the present context. One is the criterion

$$\int_0^T \|p\|_{L^r}^{2r/(2r-3)}\, \mathrm{d}\tau < \infty, \text{ for } r \in (3/2, \infty), \qquad (2)$$

which is an analogue of the Prodi–Serrin–Ladyzhanskaya result. The other is the following theorem by Seregin and Sverák [23].

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Theorem 1 (Seregin & Sverák, 2002)**.** *Let u and p solve the Navier–Stokes equations (1). Let $g(x,t): \mathbb{R}^3 \times (0,\infty) \to [0,\infty)$ be such that for any $t_0 > 0$, there exists $r = r(t_0) > 0$ such that*

$$\sup_{x_0\in\mathbb{R}^3} \sup_{t_0-r^2\le t\le t_0} \int_{B(x_0,r)} \frac{g(x,t)}{|x-x_0|}\,\mathrm{d}x < \infty.$$

If

$$\frac{|u(x,t)|^2}{2} + p(x,t) \le g(x,t) \tag{3}$$

or

$$p(x,t) \ge -2g(x,t), \tag{4}$$

for $x \in \mathbb{R}^3$ and $t \in (0,\infty)$, then $u(x,t)$ is smooth and unique.

This note derives regularity criteria in terms of bounds for an "effective" pressure. The results have some bearing on criteria (2) and (3) discussed in the preceding paragraph. Qualitatively speaking, it is proved that if an effective pressure in region(s) of high velocity is bounded by a singular function similar to $g(x,t)$, then regularity is secured. Here the effective pressure is the sum of the physical pressure and moderators, which may potentially be used to moderate the former in such region(s).

For the rest of this study, c denotes a positive constant, which may assume different values from one expression to another.

2. Results

Let $r \ge 3$ and $q := (r+6)/3$, so that $3 \le q \le r$. The evolution of $\|u\|_{L^q} := \|u\|_{L^q(\mathbb{R}^3)}$ is governed by

$$\begin{aligned}\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^q}^q &= (q-2)\int_{\mathbb{R}^3} p|u|^{q-2}\,\widehat{u}\cdot\nabla|u|\,\mathrm{d}x \\ &\quad -(q-2)\left\||u|^{(q-2)/2}\nabla|u|\right\|_{L^2}^2 - \left\||u|^{(q-2)/2}\nabla u\right\|_{L^2}^2,\end{aligned} \tag{5}$$

where $\widehat{u}$ is the unit vector in the direction of u. Our aim is to derive conditions under which the driving term in (5) becomes smaller than the corresponding dissipation terms, thereby implying a decay of $\|u\|_{L^q}$ and regularity. The following lemmas constitute an integral part of our derivation. These results are taken from Ref. [27] (see also Ref. [26]) with minor modifications.

Lemma 1. *Let*

$$\mathcal{P}(x,|u|,t) := \sum_{i=1}^{n} f_i(x,t)g_i(|u|,t), \tag{6}$$

where $u\cdot\nabla f_i(x,t) = 0$ and $g_i(\xi,t) \in C^1$, then

$$\int_{\mathbb{R}^3} \mathcal{P}\,|u|^{q-2}\,\widehat{u}\cdot\nabla|u|\,\mathrm{d}x = 0. \tag{7}$$

Proof. Let

$$h_i(|u|,t) = \frac{1}{|u|^{q-2}}\int_0^{|u|} \xi^{q-3} g_i(\xi,t)\,\mathrm{d}\xi.$$

Then by virtue of $\nabla \cdot u = 0$ and the hypothesis $u \cdot \nabla f_i = 0$, we have

$$\begin{aligned} \nabla \cdot \left(f_i h_i |u|^{q-2}\, u \right) &= f_i u \cdot \nabla \int_0^{|u|} \xi^{q-3} g_i(\xi, t)\, \mathrm{d}\xi \\ &= f_i g_i |u|^{q-2}\, \widehat{u} \cdot \nabla |u|. \end{aligned} \tag{8}$$

Summing over i and integrating the resulting equation over $\mathbb{R}^3$ proves the lemma.

The function $\mathcal{P}$ defined by (6) is called a pressure moderator for the reasons to become apparent in due course.

Lemma 2. *Let $\mathcal{P}$ be a pressure moderator satisfying*

$$\|p + \mathcal{P}\|_{L^2} \leq c_2' \|u\|_{L^4}^2 , \tag{9}$$

for some absolute constant c_2'. Then there is $c_1 > 0$, depending on c_2', $\|u_0\|_{L^2}$ and the constant c in the Sobolev inequality $\|f\|_{L^6} \leq c \|\nabla f\|_{L^2}$, such that if we define

$$\Omega(t) := \{x : |u(x,t)| > c_1 \|u\|_{L^r}\}, \tag{10}$$

then

$$\int_{\mathbb{R}^3 \backslash \Omega} |p + \mathcal{P}| |u|^{q-2}\, \widehat{u} \cdot \nabla |u| \, \mathrm{d}x \leq \frac{1}{2} \left\| |u|^{(q-2)/2} \nabla |u| \right\|_{L^2}^2 .$$

Proof. Since $|u(x,t)| \leq c_1 \|u\|_{L^r}$ for $x \in \mathbb{R}^3 \backslash \Omega$ we have

$$\begin{aligned} \int_{\mathbb{R}^3 \backslash \Omega} (p + \mathcal{P}) |u|^{q-2}\, \widehat{u} \cdot \nabla |u| \, \mathrm{d}x &\leq \left(\int_{\mathbb{R}^3 \backslash \Omega} (p + \mathcal{P})^2 |u|^{q-2} \, \mathrm{d}x \right)^{1/2} \left\| |u|^{(q-2)/2}\, \nabla |u| \right\|_{L^2} \\ &\leq c_1^{(q-2)/2} \|p + \mathcal{P}\|_{L^2} \|u\|_{L^r}^{(q-2)/2} \left\| |u|^{(q-2)/2}\, \nabla |u| \right\|_{L^2} \\ &\leq c_1^{(q-2)/2} c_2' \|u\|_{L^4}^2 \|u\|_{L^r}^{(q-2)/2} \left\| |u|^{(q-2)/2}\, \nabla |u| \right\|_{L^2} \\ &\leq c_1^{(q-2)/2} c_2' \|u\|_{L^2} \|u\|_{L^{3q}}^{q/2} \left\| |u|^{(q-2)/2}\, \nabla |u| \right\|_{L^2} , \end{aligned} \tag{11}$$

where the Hölder inequalities

$$\|u\|_{L^4} \leq \|u\|_{L^2}^{(3q-4)/(6q-4)} \|u\|_{L^{3q}}^{3q/(6q-4)}$$

and

$$\|u\|_{L^r} \leq \|u\|_{L^2}^{4/[(3q-2)(q-2)]} \|u\|_{L^{3q}}^{q(3q-8)/[(3q-2)(q-2)]}$$

have been used. The proof is completed with the following application of Sobolev inequality

$$\|u\|_{L^{3q}}^{q/2} \leq \frac{cq}{2} \left\| |u|^{(q-2)/2}\, \nabla |u| \right\|_{L^2} ,$$

together with the choice $c_1 = (cqc_2' \|u_0\|_{L^2})^{-2/(q-2)}$.

Lemma 3. *Let u and p solve the Navier–Stokes equations (1) and $\Omega(t)$ be defined by (10). Let $\mathcal{P}$ be a pressure moderator satisfying (9) and $\|p + \mathcal{P}\|_{L^{3q/2}(\Omega)} \leq c_{3q/2}' \|u\|_{L^{3q}}^2$ for some absolute constant $c_{3q/2}'$. Then there is $c_0 > 0$, depending only on $c_{3q/2}'$ and the constant c in the Sobolev inequality $\|f\|_{L^6} \leq c \|\nabla f\|_{L^2}$, such that if*

$$\|p + \mathcal{P}\|_{L^{3/2}(\Omega)} \leq c_0 \tag{12}$$

for all $t \in (0,T)$, then $\|u\|_{L^q}$ decreases on $(0,T)$.

Proof. Upon application of lemmas 1 and 2, (5) becomes

$$
\begin{aligned}
c_q \frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{L^q}^q &\le \int_\Omega (p+\mathcal{P})|u|^{q-2}\,\widehat{u}\cdot\nabla|u|\,\mathrm{d}x - \frac{1}{2}\left\||u|^{(q-2)/2}\nabla|u|\right\|_{L^2}^2 \\
&\le \left(\int_\Omega |p+\mathcal{P}|^{3/2}\,\mathrm{d}x\right)^{1/3}\left(\int_\Omega |p+\mathcal{P}|^3|u|^{3q-6}\,dx\right)^{1/6}\left\||u|^{(q-2)/2}\nabla|u|\right\|_{L^2} \\
&\quad -\frac{1}{2}\left\||u|^{(q-2)/2}\nabla|u|\right\|_{L^2}^2 \\
&\le \left(\int_\Omega |p+\mathcal{P}|^{3/2}\,\mathrm{d}x\right)^{1/3}\left(\int_\Omega |p+\mathcal{P}|^{3q/2}\,\mathrm{d}x\right)^{1/3q}\left(\int_\Omega |u|^{3q}\,\mathrm{d}x\right)^{(q-2)/6q}\left\||u|^{(q-2)/2}\nabla|u|\right\|_{L^2} \\
&\quad -\frac{1}{2}\left\||u|^{(q-2)/2}\nabla|u|\right\|_{L^2}^2 \\
&\le c'_{3q/2}\left(\int_\Omega |p+\mathcal{P}|^{3/2}\,\mathrm{d}x\right)^{1/3}\|u\|_{L^{3q}}^{q/2}\left\||u|^{(q-2)/2}\nabla|u|\right\|_{L^2} - \frac{1}{2}\left\||u|^{(q-2)/2}\nabla|u|\right\|_{L^2}^2 \\
&\le c'_{3q/2}\frac{cq}{2}\left(\int_\Omega |p+\mathcal{P}|^{3/2}\,\mathrm{d}x\right)^{1/3}\left\||u|^{(q-2)/2}\nabla|u|\right\|_{L^2}^2 - \frac{1}{2}\left\||u|^{(q-2)/2}\nabla|u|\right\|_{L^2}^2,
\end{aligned} \tag{13}
$$

where $c_q = q/(q-2)$. Now the hypothesis

$$
\|p+\mathcal{P}\|_{L^{3/2}(\Omega)} \le c_0 \tag{14}
$$

means that the driving term in (13) becomes smaller than the corresponding dissipation term if $c_0 \le (qcc'_{3q/2})^{-2}$. Hence under this condition, $\|u\|_{L^q}$ decays and regularity is secured. The lemma is proved.

Remark 1. *The condition $\|p+\mathcal{P}\|_{L^s} \le c'_s \|u\|_{L^{2s}}^2$, for $s \ge 2$, in the above lemmas trivially holds if $\mathcal{P} = c|u|^2$.*

Remark 2. *Given $\mathcal{P} = c|u|^2$ or $\mathcal{P} = 0$, (14) remains unchanged under the scalings $p(x,t) \longmapsto \lambda^2 p(\lambda x, \lambda^2 t)$ and $u(x,t) \longmapsto \lambda u(\lambda x, \lambda^2 t)$ that render (1) invariant.*

We are now in a position to state and prove the main results of this study.

Theorem 2. *Let $\phi(\xi,t) \ge 0$ be such that*

$$
\lim_{\xi \searrow 0} \phi(\xi,t) = 0.
$$

Suppose that

$$
|p(x,t) + \mathcal{P}(x,|u|,t)| \le \frac{\phi(|\Omega|,t)^{2/3}}{|\Omega|^{2/3}}, \quad \text{for } x \in \Omega, \tag{15}
$$

holds for sufficiently small $|\Omega|$, then $\|u\|_{L^q}$ remains finite and regularity follows.

Proof. Let $\delta > 0$ be such that $\phi(\xi,t) < c_0$ whenever $\xi < \delta$. We define

$$
I := \{t \in (0,T) : |\Omega(t)| < \delta\} \text{ and } J := (0,T)\backslash I.
$$

For $t \in J$, by virtue of the definition of Ω we have

$$
\|u\|_{L^r} \le \frac{(cqc'_2)^{2/(q-2)}\|u_0\|_{L^2}^{q/(q-2)}}{\delta^{1/2}}
$$

independent of t. And since $2 < q \le r$, by Hölder's inequality, $\|u\|_{L^q}$ is bounded independently of t by

$$\|u\|_{L^q} \le \frac{(cqc_2')^{6(q-2)/[q(3q-8)]} \|u_0\|_{L^2}^{1+6(q-2)/[q(3q-8)]}}{\delta^{3(q-2)^2/[2q(3q-8)]}} := U_q. \tag{16}$$

On the other hand, for $t \in I$, we have $\phi(|\Omega|, t) < c_0$ and therefore $\mathrm{d}\|u\|_{L^q}/\mathrm{d}t \le 0$ at t.

Now let $t \in I$ be arbitrary. We define $O := \cup(t_1, t_2)$, where the union is over all $0 < t_1 < t < t_2 < T$ such that $(t_1, t_2) \subseteq I$. There are two possible cases. First, O is empty. Then $t \in \overline{J}$ (the closure of J) and there exists a sequence $\{t_n\} \subseteq J$ such that $t_n \to t$. By continuity we have

$$\|u(t)\|_{L^q} = \lim_{n\to\infty} \|u(t_n)\|_{L^q} \le U_q.$$

Second, O is nonempty. Then $O = (a, b)$, for some $0 \le a < b \le T$ with $a \in \overline{J} \cup \{0\}$. As $\|u(t)\|_{L^q}$ decreases in (a, b), we have

$$\|u(t)\|_{L^q} \le \max\{\|u_0\|_{L^q}, U_q\}. \tag{17}$$

It follows that (17) holds for all $t \in (0, T)$ and regularity is secured.

Remark 3. *In essence, theorem 2 implies that no loss of regularity can occur if the effective pressure $|p + \mathcal{P}|$ in Ω grows marginally less rapidly than $|\Omega|^{-2/3}$ as $\|u\|_{L^r}$ increases.*

Remark 4. *In criterion (15), it is desirable that $\phi(\xi, t)$ tends to zero as slowly as possible. Some simple examples of $\phi(\xi, t)$ are $1/|\log \xi|$ and $1/\log|\log \xi|$.*

The bound (15) in theorem 2 is regular, being consistent with the fact that $|p+\mathcal{P}| < \infty$ whenever $|\Omega| > \delta$. It is possible to reformulate this result as is done below in terms of singular bounds, which are commonly considered in the literature.

Theorem 3. *Suppose that $\Omega(t)$ can be covered by a finite number of balls $B_i(x_i, r_i)$ in such a way that $r_i \to 0$ for each i as $|\Omega| \to 0$. Suppose further that*

$$|p + \mathcal{P}| \le c\,|\log|x - x_i||^{-\alpha_i}|x - x_i|^{-2}, \text{ for } x \in B_i \text{ and } \alpha_i > 2/3, \tag{18}$$

holds for sufficiently small $|\Omega|$. Then $\|u\|_{L^q}$ remains finite and regularity follows.

Proof. The hypothesis (18) implies

$$\begin{aligned}\int_\Omega |p + \mathcal{P}|^{3/2}\,\mathrm{d}x &\le c\sum_i \int_{B_i} |\log|x - x_i||^{-3\alpha_i/2}|x - x_i|^{-3}\,\mathrm{d}x \\ &\le c\sum_i |\log r_i|^{1-3\alpha_i/2}.\end{aligned} \tag{19}$$

Since $1-3\alpha_i/2 < 0$, the above sum vanishes in the limit $\max\{r_1, r_2, \cdots\} \to 0$. Hence for sufficiently small $|\Omega|$, necessarily ensuring sufficiently small $\max\{r_1, r_2, \cdots\}$, (14) holds and regularity follows. Thus the theorem is proved.

3. Concluding remarks

We have derived regularity criteria for solutions of the Navier–Stokes equations. It has been shown that if an effective pressure $|p+\mathcal{P}|$ in $\Omega := \{x : |u(x, t)| \ge c_1 \|u\|_{L^r}\}$, where $r \ge 3$, grows less rapidly than $|\Omega|^{-2/3}$ as $\|u\|_{L^r}$ increases, then no loss of regularity can occur. Here $\mathcal{P}$ is a pressure moderator whose effectiveness remains to be explored. This result in a sense confirms the physical pictures that $|p|$ must blow up concurrently with $|u|$ at points where the flow becomes singular.

The present results are more "localized" than the Prodi–Serrin type criterion (2). Furthermore, the pressure moderator $\mathcal{P}$ can be of broad forms, including $|u|^2/2$ of (3) as a special case. Note that $\mathcal{P}$ need not be a pointwise moderator. Rather, the moderation is in the sense of $L^{3/2}(\Omega)$-norm. It is interesting for future studies to examine whether such a pressure moderator can be constructed and to what extent the moderation can be.

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