Dirac Solitons in General Relativity and Conformal Gravity

by

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Abstract

Static, spherically-symmetric particle-like solutions to the coupled Einstein-Dirac and Einstein-Dirac-Maxwell equations have been studied by Finster, Smoller and Yau (FSY). In their work, FSY left the fermion mass as a parameter set to \( \pm 1 \). This thesis generalises these equations to include the Higgs field, letting the fermion mass become a function through coupling, \( \mu \). We discuss the dynamics associated with the Higgs field and find that there exist qualitatively similar solutions to those found by FSY, with well behaved, non-divergent metric components and electrostatic potential, close to the origin, going over to the point-particle solutions for large \( r \); the Schwarzschild or Reissner-Nordström metric, and the Coulomb potential. We then go on to discuss an alternative gravity theory, conformal gravity, (CG), and look for solutions of the CG equations of motion coupled to the Dirac, Higgs and Maxwell equations. We obtain asymptotically nonvanishing, yet fully normalisable Dirac spinor components, resembling those of FSY, and, in the case where charge is included, non-divergent electrostatic potential close to the origin, matching onto the Coulomb potential for large \( r \).
The building of a universe need not happen with a bang and a whoosh, rather, the slow bringing together of things on some quiet Sunday morning until the whole is born.
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Finally, I wish to thank my parents for all the support they have given me over the years.
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3.4 Fermion binding energy, $|m| - \omega$, for EDMH and EDM solutions. $\alpha_1$ increases with increasing $m$.

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5.1 CDMH ground state solution, with $\alpha_1 = 0.125383$, $q = -0.1$, $h_0 = 0.420073$, $\omega = 1.48435$. The gauge transformation of the electrostatic potential allows a rescaling of the field to match onto the Coulomb potential for large $r$. 


Currently, one of the biggest open problems in physics is the question of how gravity behaves on the quantum scale. While many have pursued this question in many different ways, to date we are still none the wiser about how this fundamental force of nature truly interacts with nature.

One of the main obstacles of this goal, is the fact that we have two theories; one for gravity and one for the quantum world: Einstein’s general relativity (GR) and quantum mechanics (QM) respectively. These theories are generally accepted given that both are strongly supported by empirical evidence, and have stood these tests for many years. There is, however, the uncomfortable fact that they are not obviously compatible with one another and so trying to reconcile one to the other appears not to be possible.

In response to this, many new theories have been devised, coming under the blanket term quantum gravity.
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1.1 Quantum Gravity

Through QM, and by extension quantum field theory (QFT), we have quantum descriptions for three of the four fundamental forces of nature; electromagnetism, the strong force and the weak force. It then seems plausible that, in analogy, the fourth fundamental force, gravity, must also have some quantum description. The main goal of quantum gravity (QG) is to unify these four forces under a single theory.

The length scale at which quantum gravitational effects are thought to become important is called the Planck length,

\[ l_P = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-35} \text{m}, \]

which is a scale derived purely from dimensional arguments and is far smaller than we are currently able to observe. As a result, direct experimental predictions of QG theories are, at present impossible to do, and so being able to properly test QG theories is difficult. Consequently, the study of many QG theories is one of mathematical interest first, and physical interest second.

Nevertheless, the applications of a quantum theory of gravity are far reaching (such applications include early cosmology and inflation, (Coule, 2005), (Agullo et al., 2012); black holes and their evolution, (Meissner, 2004), (Modesto, 2006); singularities, cosmological and otherwise, (Brunnemann & Thiemann, 2006), (Husain & Winkler, 2005); and ‘a theory of everything’, (Lisi, 2007); among others), and the physical significance of finding such a theory will likely impact the physics community on a wide scale.

With that in mind then, is it possible to simplify the problem and work from there? Explicitly, can we glean any information by instead considering gravity as a classical field coupled to quantum fields? There are two possible reasons why such an approach may be of interest.

First is that gravity, for some reason, may not be a quantum field or that we simply cannot devise a quantum description for it, and second, we could view such an attempt as a kind of ‘pre-quantum gravity’, from which we may learn some clues about a full quantum theory. To that end, we shall pursue a semiclassical gravity theory, that of gravity as a classical field coupled to a Dirac field and the Higgs field, and see what we can discover. Before we do that
though, a note on some conventions that are to be used throughout this thesis.

1.2 Notation and Conventions

We use the 4-vector notation for spacetime vectors, denoting their components using either Greek or Latin indices, the former for temporal and spatial components, the latter restricted to purely spatial components. In compact notation

\[ x^\mu = (x^0, x^i) = (x^0, x^1, x^2, x^3). \]  

The energy-momentum 4-vector is defined as

\[ p_\mu = (E/c, p_1). \]  

We employ the Einstein summation convention wherein repeated indices imply summation

\[ x^2 = x^\mu x_\mu = x^0 x_0 + x^i x_i + \cdots. \]  

where \( x^\mu \) are the contravariant vector components, and \( x_\mu \) are the covariant vector or one-form components.

We will use the symbol \( \mathcal{L} \) to denote Lagrangian densities, but to avoid confusion, we state explicitly here that we will term them Lagrangians, as is typical in the literature.

For derivatives, we will use a comma to denote partial differentiation, and a semicolon to denote covariant differentiation, however, despite the fact that, for a scalar

\[ S_{,\mu} = S_{,\mu}, \]  

we will stick with the semicolon to denote its derivative.

1.2.1 Units

We will work in units where \( c = G = \hbar = 1 \) unless otherwise stated, though when stating equations of motion we will leave in factors of \( G \), since under this unit system, it carries units
of $L^2$, and so leaving factors in can be helpful in verifying dimensional consistency.

### 1.2.2 General Relativity Sign Conventions

In GR, there are three choices of sign convention that have no overall effect on the physics, but that can be confusing when comparing papers if the reader is unaware. In order to make clear where these sign conventions exist, we will state them here. We will name these sign choices $s_n = \pm 1$.

The first of these appears in the definition of the metric tensor

$$g_{\mu \nu} = s_1 \text{diag} (-, +, +, +),$$

(1.6)

the second instance is attached to the definition of the Riemann tensor

$$R^{\mu \nu \alpha \beta} = s_2 \left( \Gamma^{\mu \nu, \alpha} - \Gamma^{\mu \nu, \alpha} + \Gamma^{\mu \lambda \alpha} \Gamma_{\lambda \nu} - \Gamma^{\mu \lambda \beta} \Gamma_{\lambda \nu} \right),$$

(1.7)

and the third and final occurrence is attached to the contraction of the Riemann tensor

$$R_{\nu \beta} = s_3 R^{\mu}_{\nu \mu \beta}.$$  

(1.8)

When propagated through the derivation of the Einstein equations, these sign conventions affect the equations such that

$$R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = G_{\mu \nu} = s_2 s_3 8 \pi T_{\mu \nu},$$

(1.9)

and the Ricci scalar is $s_1 s_2 s_3 R$.

For simplicity, we will set these sign conventions to +1. See Appendix A for signs chosen in the literature.

### 1.3 The Dirac Equation

Dirac first derived his now well known equation in pursuit of an equation that did not allow for negative energy solutions, which the Klein-Gordon equation allows. He wanted an equation that was first order in its Hamiltonian, and also remained invariant under Lorentz
1.3. The Dirac Equation

1.3.1 The Klein-Gordon Equation

Before arriving at the Dirac equation, it is instructive to do the derivation of the Klein-Gordon equation. We start with the relativistic energy-momentum relation for free particles

\[ E^2 = p^i p_i + m^2, \]  

(1.10)

where \( E \) is the energy of the particle, \( p_i \) is the momentum, and \( m \) is the particle’s mass. If we promote the energy and momentum to their operator forms \( E \rightarrow i \partial_t \) and \( p_i \rightarrow -i \partial_i \), where \( \partial_t = \partial / \partial t \), \( \partial_i = \partial / \partial x^i \) and give them a wavefunction, \( \psi \), to operate on, we get

\[ (-\partial_t^2 + \partial_i^2 - m^2) \psi = 0, \]  

(1.11)

or, in covariant notation

\[ (\partial^\mu \partial_\mu - m^2) \psi = 0. \]  

(1.12)

This is the Klein-Gordon equation, which can be considered a relativistic form of the Schrödinger equation, describing spin-0 particles.

1.3.2 The Dirac Equation

Dirac wanted a wave equation roughly equivalent to the Klein-Gordon equation but linear in \( \partial_t \), i.e., the energy, and so as a starting point he wrote down a Hamiltonian of the form

\[ H = \alpha^i p_i + \beta m, \]  

(1.13)

where \( \alpha^i \) and \( \beta \) are the 4×4 matrices

\[ \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \]  

(1.14)

and where \( I \) is the identity 2×2 matrix, and the \( \sigma^i \)'s are the 2×2 Pauli matrices

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\( \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \) (1.15)

This gives us a wave equation

\[
(E - \alpha^i p_i - \beta m) \psi = 0, \quad (1.16)
\]

which he multiplied by

\[
E + \alpha^i p_i + \beta m, \quad (1.17)
\]

to get

\[
\left[ E^2 - (\alpha^i p_i)^2 + (\alpha^i \alpha^j + \alpha^j \alpha^i) p_i p_j + (\alpha^i \beta m + \beta m \alpha^i) - \beta^2 m^2 \right] \psi = 0, \quad (1.18)
\]

where \( i \neq j \). This reduces to the Klein-Gordon equation as long as the following anticommutation relations are true

\[
\{ \alpha^i, \alpha^j \} = 2 \delta^{ij}, \quad (1.19)
\]
\[
\{ \alpha^i, \beta \} = 0. \quad (1.20)
\]

Given these anticommutation relations, Dirac assumed his new wave equation, Equation 1.16, to be correct for the motion of free electrons.

1.3.3 Covariant Dirac Equation

Now that we have derived the Dirac equation, we want to express it in a manifestly covariant way. We multiply Equation 1.16 through by \( \beta \) and introduce new matrices

\[
\gamma^0 = \beta, \quad \gamma^i = \beta \alpha^i, \quad (1.21)
\]

and rewrite to get
\( (\gamma^0 p_0 - \gamma^i p_i - m) \psi = 0, \) \hspace{1cm} (1.22)

then we replace \( p_0 \rightarrow i \partial_t, \) \( p_i \rightarrow -i \partial_i, \) and use the more compact 4-vector notation, giving

\[ (\partial - m) \psi = 0, \] \hspace{1cm} (1.23)

where we have introduced Feynmann slash notation such that \( \partial = i\gamma^\mu \partial_\mu, \) and \( \psi \) is a 4-component vector with spin-up and spin-down, particle and antiparticle entries.

## 1.4 General Relativity

GR is arguably one of the greatest achievements of the 20th century. In his seminal work, \[ \text{Einstein} \ (1916) \] states that time and space are not distinct, but rather two aspects of some continuum, called spacetime, whose geometry is affected by the presence of energy and matter. In doing so, it demotes gravity from a fundamental force to a consequence of this change of geometry, called curvature.

Freefalling test particles in GR follow straight paths in spacetime, and these paths are affected by any curvature, thus the presence of a mass or energy affects how another mass or massless particles will move. This can be neatly summed up in the following quote by John Archibald Wheeler ‘matter tells spacetime how to curve, spacetime tells matter how to move’.

GR enjoys a high level of success in that it is not only a relatively simple theory, but that it makes predictions of the nature of the Universe. In the one hundred years since its inception, GR has stood the test of time and experimentation, with many of its predictions being directly observed, for example the motion of Mercury around the Sun \[ \text{Einstein} \ (1915), \] the existence of gravitational time dilation \[ \text{Pound & Rebka} \ (1959), \] and most recently, the existence of gravitational waves \[ \text{Abbott et al.} \ (2016). \]

### 1.4.1 Review of General Relativity

Within GR, we define spacetime as a differentiable manifold, whose local distances are given by the line element

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \] \hspace{1cm} (1.24)
where $g_{\mu\nu}$ is the metric tensor. We raise and lower indices in the usual way using the metric tensor

$$x^\mu = g^{\mu\nu}x_\nu, \quad x_\mu = g_{\mu\nu}x^\nu.$$ \hfill (1.25)

The Riemann tensor is defined in terms Christoffel symbols and their partial derivatives

$$R^\mu_{\nu\alpha\beta} = \Gamma^\mu_{\nu\beta,\alpha} - \Gamma^\mu_{\nu\alpha,\beta} + \Gamma^\mu_{\lambda\alpha} \Gamma^\lambda_{\nu\beta} - \Gamma^\mu_{\lambda\beta} \Gamma^\lambda_{\nu\alpha},$$ \hfill (1.26)

and the Ricci tensor is defined as the contraction of the first and third indices of the Riemann tensor

$$R_{\nu\beta} = R^\mu_{\nu\mu\beta}.$$ \hfill (1.27)

The Ricci scalar is defined as the contraction of Ricci tensor

$$R = R^\mu_{\mu}.$$ \hfill (1.28)

The equation of motion in GR is given by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},$$ \hfill (1.29)

where $\Lambda$ is the cosmological constant, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor, which contains information about the geometry of spacetime, and $T_{\mu\nu}$ is the energy-momentum tensor, which contains information about the distribution of matter and energy through spacetime.

### 1.5 Dirac Equation in Curved Spacetime

Since we want to discuss GR, it is natural that we will be dealing with curved spacetimes, and as such, we must promote the Dirac equation to one that applies to these curved spacetimes. Do to this, we must first discuss the concept of a vierbein, and a spin connection. We follow Yepez (2011) and Koke et al. (2016).
1.5. Dirac Equation in Curved Spacetime

1.5.1 Local Basis

In order to approximate some point, \( p \), on a manifold in GR, it is typical to define a tangent plane, \( T_p \), which is flat. The unit basis vectors that span the tangent plane depend on the coordinates at that point, and are defined as

\[
\hat{e}_\mu = \partial_\mu, \quad (1.30)
\]

so that vectors defined on \( T_p \) take the form

\[
V = V^\mu \hat{e}_\mu, \quad (1.31)
\]

with components \( V^\mu \).

The corresponding unit basis one-forms for the cotangent plane, \( T^*_p \) are defined as

\[
\hat{e}^\mu = dx^\mu = \left( \partial^0, \partial^1 \right) \quad (1.32)
\]

with one forms being defined similarly defined as vectors

\[
O = O_\mu \hat{e}^\mu. \quad (1.33)
\]

The basis vectors are inverse to the basis one-forms so that their tensor product

\[
\hat{e}^\mu \otimes \hat{e}_\nu = \delta^\mu_\nu, \quad (1.34)
\]

where \( \delta \) is the Kronecker delta.

1.5.2 Vierbeins

Since we are free to choose any set of basis vectors that span \( T_p \), we can introduce a set of coordinate-independent unit basis vectors \( \hat{e}_a \). Notice the choice of index here is Latin, which will be used to denote these non-coordinate unit basis vectors, with Greek being used for the coordinate unit basis vectors.
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The inner product of these non-coordinate basis vectors is given by

\[ \langle \hat{e}_a, \hat{e}_b \rangle = \eta_{ab}, \]  

(1.35)

where \( \eta_{ab} \) is the Minkowski metric for flat spacetime.

We can transform between the coordinate basis and the non-coordinate basis by use of

\[ \hat{e}_\mu(x) = e_\mu^a(x)\hat{e}_a \]  

(1.36)

for spacetime coordinate \( x \), spacetime index \( \mu \), and where \( e_\mu^a(x) \) is called the vierbein, a 4 \( \times \) 4 matrix with \( a = 0, 1, 2, 3 \). Explicitly, the vierbein is a transformation matrix used to convert from the non-coordinate basis to the coordinate basis. The inverse vierbein has switched indices \( e_\mu^a(x) \), and together with the vierbein, satisfies

\[ e_\mu^a(x)e_\nu^a(x) = \delta_\mu^\nu, \quad e_\mu^a(x)e_\mu^b(x) = \delta^a_b. \]  

(1.37)

Like the vierbein, the inverse vierbein is a transformation matrix for transforming from coordinate to non-coordinate unit basis vectors

\[ \hat{e}_a = e_\mu^a(x)\hat{e}_\mu(x). \]  

(1.38)

Likewise, we can convert between the flat spacetime metric and the curved one by using vierbeins according to

\[ g_{\mu\nu}(x) = e_\mu^a(x)e_\nu^b(x)\eta_{ab}. \]  

(1.39)

As the coordinate unit basis vector has a corresponding unit basis one-form, so too is there a non-coordinate equivalent, denoted by \( \hat{e}^a \), satisfying the tensor product

\[ \hat{e}^a \otimes \hat{e}_b = \delta^a_b. \]  

(1.40)

Now that we have defined the coordinate and non-coordinate basis vectors, it is clear that
we can define any vector on $T_p$ in terms of either, so that

$$V = V^\mu \hat{e}_\mu = V^a \hat{e}_a,$$  \hspace{1cm} (1.41)

and we can switch between components using

$$V^\mu = e^\mu_a V^a, \quad V^a = e^a_\mu V^\mu,$$ \hspace{1cm} (1.42)

where we have dropped the explicit statement that Greek indices denote coordinate dependence.

### 1.5.3 Spin Connections

In GR, when considering curved spacetimes, derivatives as expressed in flat spacetimes are not sufficient. One can show that a vector parallel transported in some curved spacetime is required to compare a field from point to point due to the curvature that distorts it, and so correction terms are required for derivatives. These corrections, named affine connections, modify derivatives such that they are promoted from partial derivatives to covariant derivatives. They are expressed

$$V^\mu;_\nu = V^\mu,_{\nu} + \Gamma^\mu_{\lambda\nu} V^\lambda, \hspace{1cm} (1.43)$$

$$O^\mu;_\nu = O^\mu,_{\nu} - \Gamma^\lambda_{\mu\nu} O^\lambda, \hspace{1cm} (1.44)$$

for vectors and one-forms respectively, where the $\Gamma$ is the affine connection or Christoffel symbol. In a more general sense, similar expressions apply to tensors of any rank, upper indices each having a Christoffel symbol with positive sign and lower indices each having one with negative sign.

The affine connection applies when differentiating tensors in the coordinate basis, but for the non-coordinate basis, we replace these with spin connections. The principle for these spin connections is the same as that of the affine connection, the result of which is the promotion of derivatives from flat spacetimes to the non-coordinate basis, looks like that of the coordinate basis, namely
\[ V^a_{\mu} = V^a_{\mu} + \omega^a_{\mu b} V^b, \]  
(1.45) 
\[ O^a_{\mu} = O^a_{\mu} - \omega^b_{\mu a} O_b, \]  
(1.46) 
where

\[ \omega^a_{\mu b} = \epsilon^a_{\nu} \epsilon^\lambda_{b} \Gamma^\nu_{\mu \lambda} - \epsilon^\lambda_{b} \partial^\mu \epsilon^\nu_{\lambda}, \]  
(1.47)

is the spin connection.

### 1.5.4 Dirac Operator in Curved Spacetime

Now that we have a handle on vierbeins and the spin connection, we can start to consider how the covariant derivative for the spinor must look. We might expect from our discussion of covariant derivative operations in the coordinate basis, that we ought to have a partial derivative with some correction term, which indeed we do. We write this differential operator

\[ D_\mu = \partial_\mu + \Gamma_\mu. \]  
(1.48)

We will want to contract this with the Dirac matrices, whose form we know in flat spacetime, and so we must convert using the vierbeins, so that

\[ \gamma^\mu = e^\mu_a \gamma^a, \]  
(1.49)

thus we can write a general form for the Dirac equation in curved spacetime. We have

\[ (\not{\!D} - m) \psi = 0, \]  
(1.50)

where

\[ \not{\!D} = i \gamma^\mu D_\mu = ie^\mu_a \gamma^a D_\mu. \]  
(1.51)

In order to determine the form of the connection, we must consider how Dirac spinors
1.6. Einstein-Dirac Solitons

undergo Lorentz transformations. It turns out, that for an infinitesimal Lorentz transformation of the form

$$\Lambda^a_b = \delta^a_b + \lambda^a_b,$$  \hfill (1.52)

spinors, $\psi$, and their adjoints, $\psi^+ \gamma^0$, transform like

$$\psi \rightarrow L \psi, \quad \overline{\psi} \rightarrow \overline{\psi} L^{-1},$$  \hfill (1.53)

where

$$L = 1 + \frac{1}{2} \lambda_{ab} \sigma^{ab}, \quad L^{-1} = 1 - \frac{1}{2} \lambda_{ab} \sigma^{ab},$$  \hfill (1.54)

$\lambda$ is antisymmetric under interchange of its indices, and

$$\sigma^{ab} = \frac{i}{2} [\gamma^a, \gamma^b].$$  \hfill (1.55)

By the fact the $\overline{\psi}\psi$ must transform like a scalar, and $\overline{\psi} \gamma^a \psi$ must transform like a vector, one can determine that the correction term to the spinor covariant derivative takes the form

$$\Gamma_\mu = -\frac{i}{4} \omega_{ab\mu} \sigma^{ab}.$$  \hfill (1.56)

1.6 Einstein-Dirac Solitons

Now that we know how to express the Dirac equation for curved spacetime, we can start to work with it in the context of GR. We shall now review the work of Finster, Smoller and Yau (FSY), [Finster et al. (1999c)].

FSY studied static, spherically-symmetric solutions the Einstein equations coupled to the Dirac equation, so termed the Einstein-Dirac (ED) system of equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \quad (\not{\partial} - m) \psi = 0,$$  \hfill (1.57)

where $T_{\mu\nu}$ is the energy-momentum tensor, $\not{\partial}$ is the Dirac operator for curved spacetime and
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\( m \) is the fermion mass.

For static, spherically-symmetric spacetimes, the metric is chosen so that

\[
g_{\mu\nu} = \text{diag}(-T^{-2}, A^{-1}, r^2, r^2 \sin^2 \theta),
\]

where \( T = T(r) \) and \( A = A(r) \).

1.6.1 Equations of Motion

In order to express the Dirac equations as radial equations, FSY separate out angular momentum by making an appropriate choice of spinor ansatz,

\[
\psi = e^{-i\omega t} \sqrt{\frac{T}{r}} \begin{pmatrix} \alpha \\ -i\sigma^r \beta \end{pmatrix} \Omega_{jk}(\theta, \phi) \chi_a,
\]

where \( \alpha = \alpha(r) \) and \( \beta = \beta(r) \) are the matter and antimatter components of the spinor, \( \Omega_{jk}(\theta, \phi) \) are spherical harmonics, \( \chi_a \) are 2-spinors representing spin-up and spin down states and \( \sigma^r \) is the Pauli matrix in the radial direction, found by taking the inner product of the vector formed of the Pauli matrices, and unit vector in the \( r \) direction. The Dirac equations then are

\[
\sqrt{A} \alpha' = \frac{\kappa \alpha}{2r} - \beta(m + T \omega),
\]

\[
\sqrt{A} \beta' = \alpha(T \omega - m) - \frac{\kappa \beta}{2r},
\]

and the mixed 00 and 11 components of the Einstein equations are given by

\[
rA' + A - 1 = -8\pi G|\kappa|T^2 \omega \left( \alpha^2 + \beta^2 \right),
\]

\[
\frac{2ArT'}{T} - A + 1 = -8\pi G|\kappa|T \left( T\omega \left( \alpha^2 + \beta^2 \right) - \frac{2}{r} \alpha \beta - m \left( \alpha^2 - \beta^2 \right) \right),
\]

and the 22 and 33 components, which are equal to each other, are given by

\[
- \frac{r^2 A'T'}{2T} + \frac{r A'}{2} - \frac{Ar^2 T''}{T} + \frac{2Ar^2 \left( T' \right)^2}{T^2} - \frac{Ar T'}{T} \omega = 4\pi \alpha \beta G \kappa T \left| \kappa \right| \frac{r}{\sqrt{T}},
\]
1.6. Einstein-Dirac Solitons

where the left hand sides are the components of the Einstein tensor multiplied by $r^2$, the right hand sides give the components of the energy-momentum tensor multiplied by $r^2$, and $\kappa = \pm (2j + 1)$ is the relativistic quantum number whose sign sets the parity of the system, with total angular momentum quantum number, $j = \frac{1}{2}, \frac{3}{2}, \ldots$. For the case of 2 fermions, as FSY consider, $\kappa = 2$.

The normalisation of the fermions is given by

$$\int_0^\infty 4\pi r^2 \psi^\dagger \psi \, dr = \int_0^\infty 4\pi T \alpha^2 + \beta^2 \, dr = 1. \quad (1.65)$$

Assuming solutions are regular close to the origin, FSY expand fields in powers of $r$, to find the following relations

$$\alpha = \alpha_1 r + O(r^3), \quad (1.66)$$
$$\beta = \frac{1}{3} \alpha_1 (T_0 \omega - m) r^2 + O(r^3), \quad (1.67)$$
$$T = T_0 - \frac{4}{3} \pi \alpha_1^2 G T_0^2 (2T_0 \omega - m) r^2 + O(r^3), \quad (1.68)$$
$$A = 1 - \frac{16}{3} \pi \alpha_1^2 G T_0^2 \omega r^2 + O(r^3), \quad (1.69)$$

from which they identify the parameters $\alpha_1, T_0, \omega$ and $m$. They impose the requirement that flat spacetime is to be recovered when far enough away from solutions, so that as $r \to \infty$, $T = A = 1$.

Note that the asymptotic relations stated here differ from those quoted by FSY. This is down to a typographical error in the FSY result.

1.6.2 Rescaling the Equations

Rather than directly imposing Equation 1.65 and the asymptotically flat spacetime condition, FSY instead weaken these conditions and find rescalings that they can apply to the equations of motion. This allows them to solve the equations and rescale the fields afterwards in order to find physically meaningful results. The weaker conditions they impose are
\[ \tau = \lim_{r \to \infty} T < \infty, \quad \chi^2 = \int_0^\infty T \frac{\alpha^2 + \beta^2}{\sqrt{A}} \, dr < \infty, \quad (1.70) \]

and they set \( T_0 = 1 \) and \( m = \pm 1 \). Under these new constraints, fields scale as

\[ \bar{\alpha}(\bar{r}) = \sqrt{\frac{\tau}{\chi}} \alpha(\bar{r} \chi), \quad (1.71) \]
\[ \bar{\beta}(\bar{r}) = \sqrt{\frac{\tau}{\chi}} \beta(\bar{r} \chi), \quad (1.72) \]
\[ \bar{T}(\bar{r}) = \tau^{-1} T(\bar{r} \chi), \quad (1.73) \]
\[ \bar{A}(\bar{r}) = A(\bar{r} \chi). \quad (1.74) \]

In order to satisfy the equations of motion with fields scaled in this way, the following parameter scalings must also apply

\[ \bar{m} = \chi m, \quad (1.75) \]
\[ \bar{\omega} = \chi \tau \omega. \quad (1.76) \]

### 1.6.3 Solutions

As the problem currently stands, there are 2 unassigned free parameters, \( \alpha_1 \) and \( \omega \). It turns out that for any choice of positive \( \alpha_1 \), solutions to the equations exist, and as such, \( \alpha_1 \) can be fixed to a value, reducing the problem to a 1-dimensional one in \( \omega \). Solutions can be found by implementing a binary search algorithm in \( \omega \) shooting for required boundary conditions at large \( r \).

Solving the equations, it can be shown that there exists a ground state solution, as well as excited state solutions characterised by monotonic increasing numbers of nodes in the fermion wavefunctions. In each case, \( T \) is a monotonic decreasing function that is positive, and asymptotically equal to 1, and both \( T \) and \( A \) go over exactly to the Schwarzschild metric, \( A = T^{-2} = 1 - \frac{2M}{r} \), when sufficiently far from the soliton, where \( M \) is the ADM mass of the soliton. Figure 1.1 shows a typical ground state solution with positive fermion mass, for which the fermion part of the spinor wavefunction is dominant, where Figure 1.2 shows the case for...
negative fermion mass, for which the antifermion part dominates.

In order to generate the antifermion dominated solutions, the initial conditions must be changed such that $\kappa = -2$, $m = -1$, which then switches the conditions on the fermions so that $\beta_1$ is now the parameter we are free to set. The initial conditions become

\[
\begin{align*}
\alpha &= \frac{1}{3} \beta_1 (m - T_0 \omega) r^2 + O(r^3), \\
\beta &= \beta_1 r + O(r^3), \\
T &= T_0 - \frac{4}{3} \pi \beta_1^2 G T_0^2 (2 T_0 \omega - m) r^2 + O(r^3), \\
A &= 1 - \frac{16}{3} \pi \beta_1^2 G T_0^2 \omega r^2 + O(r^3).
\end{align*}
\]

To investigate whether a particular solution is bound, one can compare the ADM mass of the soliton against twice the fermion mass, $M - 2|m|$. That is, taking the total mass of the soliton, and subtracting from it the rest mass of the fermions, leaving the binding energy.
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Figure 1.2: FSY antifermion dominated ground state solution for $\alpha_1 = 0.02$. Scaled parameter values are: fermion mass, $m = -0.75667$, $\omega = 0.63023$, and the ADM mass of the soliton, $M = 1.46824$. 

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1.7 Related Work

While we have reviewed here only the Einstein-Dirac system of equations, there is other work around this area of interest that is worth mentioning, even if only briefly.

Lisi (1995) studied the nonrelativistic Dirac-Maxwell system of equations both with and without angular dependence, and found particle-like solutions, which, when combined the the
ED problem, leads naturally into the discussion of the Einstein-Dirac-Maxwell (EDM) system studied by FSY (Finster et al. 1999b).

The Newtonian limit of the Einstein-Dirac equations, which leads to the nonrelativistic Newton-Schrödinger equations, was studied by Stuart (2010) and Harrison et al. (2003), showing that bound states exist in this regime.

An Einstein-Dirac cosmology is studied by Finster & Hainzl (2009) wherein they derive a bouncing behaviour of the space-time; a time-periodic contraction and expansion, and thus find that there is no cosmological singularity. This was further investigated in (Finster & Hainzl 2011).
Having looked at the FSY Einstein-Dirac system of equations and their solutions, we can now consider a slightly more generalised set of equations in which we involve the Higgs field. We shall call this system of equations the Einstein-Dirac-Higgs (EDH) equations.

Now that we are involving the Higgs field, the fermion mass is no longer a parameter that we can set, rather, through coupling to the Higgs field, the fermion mass will now be some function of $r$, and there will be dynamics associated with the Higgs field. This allows the fermion mass to vary, affecting how strongly the soliton binds.

As with FSY, we shall restrict our search to static, spherically symmetric-solutions.

### 2.1 Setup

We choose our metric tensor to match that of Finster, Smoller and Yau (Finster et al. 1999c), so that
\[ g_{\mu\nu} = \text{diag}(-T^{-2}, A^{-1}, r^2, r^2 \sin^2 \theta) \],
\[
\text{where } T = T(r) \text{ and } A = A(r).\]

Our action is defined as
\[ S = S_g + S_m = \int d^4x \sqrt{-g} \left( \mathcal{L}_g + \mathcal{L}_m \right), \]
with \( S_g \) and \( S_m \) denoting the gravitational and matter actions respectively, and \( g = -T^{-2}A^{-1}r^4 \sin^2 \theta \) is the determinant of \( g_{\mu\nu} \). We are using the standard Einstein-Hilbert Lagrangian, defined by
\[ \mathcal{L}_g = \frac{R - 2\Lambda}{16\pi G}, \]
and our matter Lagrangian takes the form
\[ \mathcal{L}_m = -\frac{1}{2} h^{\mu\nu} h_{\mu\nu} - \left( V(h) + \frac{\zeta}{12} R h^2 \right) + \bar{\psi} (\not\!{D} - \mu h) \psi, \]
where \( h = h(r) \) is the Higgs field, \( R \) is the Ricci scalar, \( h_m \) is the value of the Higgs field when at the stable minimum of the Mexican hat potential, \( V(h) \), with an unstable maximum at \( h = 0 \), \( m_H \) is the intrinsic Higgs mass, \( \lambda \) is the Higgs self-coupling parameter, \( \mu \) is the coupling between the fermions and Higgs field, \( \not\!{D} \equiv \gamma^\nu D_\nu \) is the Dirac operator with spacetime index \( \nu = 0, 1, 2, 3 \), \( \psi \) is the Dirac spinor, and \( \bar{\psi} = \psi^\dagger \gamma^0 = \psi^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) is the adjoint spinor.

Note that \( \zeta \) is a coupling parameter for the Higgs field and gravity. One can choose \( \zeta = 0 \), corresponding to the case where there is no explicit coupling between the two, but we will choose \( \zeta = 1 \), which corresponds to a conformal coupling between them.

### 2.2 Equations of Motion

In order to determine our equations of motion, we must apply the Euler-Lagrange (EL) equations to the total Lagrangian \( \mathcal{L}_T = \mathcal{L}_g + \mathcal{L}_m \).
2.2. Equations of Motion

\[ \frac{\partial L}{\partial F} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu F)} \right) = 0, \]  

(2.6)

with \( F \) here being a general placeholder for the field for which we want the equation of motion.

2.2.1 Dirac Equations

For the Dirac equations, we take our Dirac spinor, \( \psi \), as our field, and apply the EL equations to get

\[ (\not{D} - \mu h) \psi = 0. \]  

(2.7)

Since we are considering spherically symmetric solutions, we wish to reduce our problem to a radial one, which we can do by choosing an appropriate ansatz for the spinors. As in Pinster et al. (1999c) we choose

\[ \psi = e^{-i\omega t} \frac{\sqrt{T}}{r} \begin{pmatrix} \alpha \\ -i\sigma^r \beta \end{pmatrix} \Omega_{jk}(\theta, \phi) \chi_a, \]  

(2.8)

where \( \alpha = \alpha(r), \beta = \beta(r) \), are radial matter and antimatter components of the spinor respectively, \( \sigma^r \) is the radial Pauli matrix, \( \Omega_{jk}(\theta, \phi) \) are spherical harmonics, and \( \chi_a \) are the 2-spinors representing spin-up and spin-down states, with \( \chi_1 = (0, 1)^T \) and \( \chi_2 = (1, 0)^T \). Here, \( j = \frac{1}{2}, \frac{3}{2}, \ldots \) is the total angular momentum quantum number and \( k = -j, -j+1, \ldots j \) is the projection of the total angular momentum onto the z-axis. From this ansatz, we arrive at the radial Dirac equations in terms of matter and antimatter spinor components

\[ \sqrt{A} \alpha' = \frac{\kappa}{2r} \alpha - \beta (\mu h + \omega T), \]  

(2.9)

\[ \sqrt{A} \beta' = \alpha (\omega T - \mu h) - \frac{\kappa}{2r} \beta. \]  

(2.10)

Here, \( \kappa = \pm (2j + 1) \) is the relativistic quantum number, whose sign indicates the parity of the system.

We require that our fermions be normalisable, and do so using
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\[
\int_0^\infty \frac{4\pi r^2}{\sqrt{A}} \psi^\dagger \psi dr = \int_0^\infty \frac{4\pi T}{\sqrt{A}} (\alpha^2 + \beta^2) dr = 1. \tag{2.11}
\]

We will consider, like FSY, the singlet state of 2 fermions, and so we will set \( \kappa = 2 \).

2.2.2 Einstein Equations

Applying the EL equations to \( \mathcal{L}_T \) with respect to the metric gives the Einstein field equations, whose general form is given by

\[
G_{\mu\nu} + g_{\mu\nu} \Lambda = 8\pi GT_{\mu\nu}, \tag{2.12}
\]

where we will work with \( \Lambda = 0 \), and

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\partial (\mathcal{L}_m \sqrt{-g})}{\partial g^{\mu\nu}}, \tag{2.13}
\]

is the energy-momentum tensor. The Einstein tensor components are

\[
G_0^0 = \frac{1}{r^2} \left( A - 1 + rA' \right), \tag{2.14}
\]
\[
G_1^1 = \frac{1}{r^2} \left( -\frac{2ArT'}{T} + A - 1 \right), \tag{2.15}
\]
\[
G_2^2 = G_3^3 = \frac{1}{r^2} \left( r^2 \left( \frac{AT'}{2T} - \frac{AT''}{T^2} + 2A \left( \frac{T'}{T} \right)^2 \right) + r \left( \frac{A'}{2} - \frac{AT'}{T} \right) \right), \tag{2.16}
\]

with all non-diagonal components being equal to zero. The Ricci scalar evaluates to

\[
R = \frac{2(1-A)}{r^2} + \frac{2}{r} \left( \frac{2AT'}{T} - A' \right) + \frac{A'T'}{T} + \frac{2AT''}{T} - \frac{4A \left( \frac{T'}{T} \right)^2}{T^2}. \tag{2.17}
\]

The 00 Einstein field equation component is

\[
\left( 1 - \frac{4}{3} \pi \zeta \frac{Gh^2}{r^2} \right) (rA' + A - 1) + A r^2 = 8\pi G r^2 \left( \frac{T^2 |\kappa| \omega \left( \alpha^2 + \beta^2 \right)}{r^2} - \frac{1}{2} A \left( h' \right)^2 - \frac{1}{2} \lambda \left( h^2 - h_m^2 \right)^2 \right.
\]
\[
+ \frac{\zeta}{6r} \left( h r A'h' + 2A \left( r h' \right)^2 + h \left( r h'' + 2h' \right) \right) \right)
\]
\[
= 1. \tag{2.18}
\]
the 11 component is given by

\[
\left(1 - \frac{4}{3}\pi\zeta G h^2\right) \left(\frac{(A-1)T - 2ArT'}{T}\right) + \Lambda r^2 = 8\pi Gr^2 \left( -\frac{T}{r^3} (\alpha \beta \kappa + \mu hr (\alpha^2 - \beta^2) - r \omega T (\alpha^2 + \beta^2)) + \frac{1}{2} A(h')^2 - \lambda (h^2 - h_m^2)^2 \right.
\]
\[
- \frac{T}{3rT} \left( A\zeta hh' (rT' - 2T) \right),
\]

(2.19)

and the 22 and 33 components, equal to each other, are given by

\[
-\left(1 - \frac{4}{3}\pi\zeta G h^2\right) \left(\frac{T\alpha^2 hT' - T}{2T^2} + 2A(r r'' - 2r T')^2 + TT') \right) + \Lambda r^2 = 8\pi Gr^2 \left( \frac{\alpha \beta \kappa T |\kappa|}{2r^3} - \frac{1}{2} A(h')^2 - \lambda (h^2 - h_m^2)^2 \right.
\]
\[
+ \frac{\zeta}{6rT} \left( hr TA'h' + 2A\left(r T\left(h'\right)^2 + h\left(T\left(rh'' + h'\right) - rh'T'\right)\right) \right),
\]

(2.20)

with the terms inside the outer brackets on the right hand sides being the components of the energy-momentum tensor.

Notice, the left hand sides of the Einstein equations will flip sign when \(h^2 > \frac{3}{40\zeta}\), however, in the solutions we will discuss this condition will not be met.

### 2.2.3 Higgs Equation

For the Higgs equation, we again apply the EL equation to \(\mathcal{L}_T\), taking our Higgs field, \(h\), as our field. Doing so yields

\[
h^{\mu \nu} = \frac{dV}{dh} + \frac{\zeta}{6} R h + \mu \overline{\psi} \psi,
\]

(2.21)

where

\[
h^{\mu \nu} = \frac{1}{\sqrt{-g}} \left( \sqrt{-g} g^{\mu \nu} h_{\nu} \right)_{;\mu}
\]
\[
= Ah'' + h' \left( \frac{A'}{2} + \frac{2A}{r} - \frac{AT'}{T} \right),
\]

(2.22)
and

$$\overline{\psi} \psi = \frac{T|\kappa|}{r^2} (\alpha^2 - \beta^2).$$  \tag{2.23}$$

Explicitly written out, the Higgs equation is

$$Ah'' + h'\left(A' - A\left(\frac{T'}{T} - \frac{2}{r}\right)\right) = \frac{\mu T (\alpha^2 - \beta^2)|\kappa|}{r^2} + 4h\lambda (h^2 - h_m^2) + \frac{\zeta hR}{6}, \tag{2.24}$$

2.3 Solutions

In generating solutions, we solve Equations 2.9, 2.10, 2.18, 2.19 and 2.21. These reduce to the FSY equations if we set the Higgs field to its minimum value, $h = h_m = \frac{m_H}{\sqrt{8\lambda}}$, its derivatives to zero, $h' = h'' = 0$, and $\zeta = 0$. Notice that when we do this, the product $\mu h$ becomes a constant, which is consistent with FSY setting their fermion mass, $m$, to $\pm 1$. We can make this same choice by using appropriate values for the constants.

We want our system of equations to approach the Minkowski spacetime for large $r$, so we impose the condition that the metric components go over to the vacuum solution, i.e. the Schwarzschild metric for sufficiently large $r$

$$T^{-2} = A = 1 - \frac{2GM}{r}, \tag{2.25}$$

where $M$ is the ADM mass.

2.3.1 Scaling the Equations

When numerically determining solutions, rather than imposing the conditions given in Equations 2.11 and 2.25 we instead use the weaker conditions

$$\tau = \lim_{r \to \infty} T < \infty, \tag{2.26}$$

$$\chi^2 = \int_0^\infty \frac{T}{\sqrt{A}} (\alpha^2 + \beta^2) \, dr < \infty. \tag{2.27}$$
then set $T(0) = 1$. The problem with doing this is that the set of equations we are solving yields solutions that are unnormalised. Fortunately, there exists a rescaling of the equations that allows us to obtain unnormalised solutions, and perform the normalisation after the fact, which greatly simplifies the process. We find that, like FSY, we can scale the fields according to

$$
\tilde{\alpha}(\vec{r}) = \sqrt{\frac{\tau}{\chi}} \alpha(\vec{r} \chi),
$$

(2.28)

$$
\tilde{\beta}(\vec{r}) = \sqrt{\frac{\tau}{\chi}} \beta(\vec{r} \chi),
$$

(2.29)

$$
\tilde{T}(\vec{r}) = \tau^{-1} T(\vec{r} \chi),
$$

(2.30)

$$
\tilde{A}(\vec{r}) = A(\vec{r} \chi),
$$

(2.31)

$$
\tilde{h}(\vec{r}) = h(\vec{r} \chi).
$$

(2.32)

Substitution of these scaled functions into the equations of motion requires that $\mu$, $m_H$, $\omega$ and $\lambda$ be replaced by

$$
\tilde{\mu} = \chi \mu,
$$

(2.33)

$$
\tilde{m}_H = \chi m_H,
$$

(2.34)

$$
\tilde{\omega} = \chi \tau \omega,
$$

(2.35)

$$
\tilde{\lambda} = \chi^2 \lambda.
$$

(2.36)

As the Ricci scalar has units of inverse length squared, we expect that it must scale as such, which our analysis confirms

$$
\tilde{R} = \frac{R}{\chi^2}.
$$

(2.37)

One can do a quick check of the self consistency of these scalings, wherein we find that masses scale like $\chi$, which we expect given Equation 2.25, where the ADM mass must scale in this way. The Higgs field doesn’t scale, which also holds for its potential, and the product $\mu h$,
which is a mass term, also scales like $\chi$, as does the Higgs mass.

2.3.2 Small $r$ Asymptotic Analysis

If we assume our fields are regular at the origin, we can Taylor expand them in powers of $r$ and thus determine their behaviour at small $r$. Doing so when we set $\kappa = 2$, we find

\[
\alpha = \alpha_1 r + O(r^3), \quad (2.38)
\]

\[
\beta = -\frac{1}{3} \alpha_1 (h_0 \mu - T_0 \omega) r^2 + O(r^3), \quad (2.39)
\]

\[
T = T_0 + \left( \frac{4}{3} \pi G T_0 \left( \lambda \left( h_0^2 - h_m^2 \right)^2 + \alpha_1^2 h_0 \mu T_0 - 2 \alpha_1^2 T_0^2 \omega \right) \right. \\
- \left. \frac{4 \pi \zeta G \lambda T_0 \left( h_0^2 - h_m^2 \right) \left( h_0^2 \left( 8 \pi G h_m^2 + 3 \right) - 9 h_m^2 \right)}{9 \left( 4 \pi G h_0^2 - 3 \right)} \right) r^2 + O(r^3), \quad (2.40)
\]

\[
A = 1 + \left( \frac{8 \pi \zeta G \lambda \left( h_0^2 - h_m^2 \right) \left( h_0^2 \left( 16 \pi G h_m^2 - 3 \right) - 9 h_m^2 \right)}{9 \left( 4 \pi G h_0^2 - 3 \right)} \right. \\
- \left. \frac{8}{3} \pi G \left( \lambda \left( h_0^2 - h_m^2 \right)^2 + 2 \alpha_1^2 T_0^2 \omega \right) \right) r^2 + O(r^3), \quad (2.41)
\]

\[
h = h_0 + \left( \frac{1}{3} \left( 2 h_0 \lambda \left( h_0^2 - h_m^2 \right) + \alpha_1^2 \mu T_0 \right) - \frac{2}{9} \zeta h_0 \lambda \left( h_0^2 - h_m^2 \right) \left( 4 \pi G h_m^2 - 3 \right) \right) + O(r^3). \quad (2.42)
\]

This leaves us 7 free parameters in the system, $\omega$, $\alpha_1$, $T_0$, $\mu$, $m_H$, $\lambda$ and the initial value for the Higgs field, $h_0$. These expressions are complicated, and while it is not necessary to pass these to the solver, it is useful to at least see how each expression depends on $r$ at small scales, which gives us a useful check by eye when first beginning to find solutions, to ensure their behaviour is as expected.

On top of this, we can see that, when setting $\zeta = 0$ and $h = h_m$, these reduce to the FSY asymptotic relations.
2.3. Solutions

2.3.3 Higgs Effective Dynamics

Before trying to find numerical solutions to the EDH equations, it is worth considering how we expect solutions to behave. We follow the same rationale as Schlögel et al. (2014), who consider the Einstein equations coupled to the Higgs equation, in order to do this.

For the Dirac and Einstein equations, this is simple in that we want normalisable fermions, and so the fermionic wavefunctions must converge to 0 for large $r$ and we want to recover flat spacetime sufficiently far from the fermions, and so we expect our metric components to tend to 1 for large $r$. This leaves the Higgs field to consider.

The first case we consider is when $h = 0$. In this case, we have asymptotically curved spacetime, which comes from the fact that in the energy-momentum tensor, we have the $\lambda h^4$, which is constant for all $r$. For the case where $h = \pm h_m$, $\zeta = 0$, we recover the FSY equations, which have been shown to recover flat spacetime and normalisable fermions.

The final case to consider is the one in which $h = \pm h_m$, $\zeta = 1$. For this case, we can check that we will have asymptotically flat spacetime by simply substituting our large $r$ field values into the energy-momentum tensor, that is $T = A = 1$, $\alpha = \beta = 0$, for which we see that we do indeed recover the vacuum solution.

To further explore the dynamics, consider the general form of the Higgs equation, shown in Equation 2.21. We rewrite it so that we have

$$h^{\mu \nu} = - \frac{dV_{\text{eff}}}{dh}, \quad (2.43)$$

with

$$V_{\text{eff}} = -V(h) - \frac{\zeta}{12} Rh^2 - \mu h \bar{\psi} \psi. \quad (2.44)$$

For the case of a spatially uniform, time-dependent field, Equation 2.43 evaluates to

$$T^2 \ddot{h} = \frac{dV_{\text{eff}}}{dh}, \quad (2.45)$$

where the dot denotes the derivative with respect to time. Here we have the potential the ‘correct’ way up, Figure 2.1a, and we expect the field to roll to one of the two stable minima,
located at \( \pm h_m \). However, for the static case, as we are considering here, we end up with

\[
Ah'' + h\left(\frac{A'}{2} + \frac{2A}{r} - \frac{AT'}{T}\right) = -\frac{dV_{\text{eff}}}{dh},
\]  

(2.46)

so that our potential is now inverted, with \( \pm h_m \) being local, unstable maxima, and \( h = 0 \) a stable minimum.

Notice that when we consider the case where \( \zeta = 1 \), the coupling of the Higgs field to gravity affects the quadratic part of the potential. Furthermore, the coupling of the fermions to the Higgs field adds an extra linear term to the effective potential. Let us consider the implications of this.

First, we will consider the coupling of the Higgs field to gravity, and thus the Ricci term in the effective potential. If we look to Equation 2.17, we see that it is a complicated function of the metric components, but that when we have flat spacetime, it reduces to 0, as we would expect. This tells us that outside of the fermion source, this term must vanish from the effective potential. The fact that this term carries the same sign to the \( h^2 \) term in the intrinsic potential means these two terms will reinforce one another, affecting the depth of the quadratic part of the potential, which also affects the locations of the peaks, Figure 2.1b (Red).

Next, we consider the linear term in the potential, which comes from the coupling of the fermions to the Higgs field. This term introduces a tilt to the potential, Figure 2.1b (Black), which takes the form

\[
\bar{\mu} \psi \psi = \mu \frac{T|\kappa|}{r^2} \left(\alpha^2 - \beta^2\right),
\]  

(2.47)

from which we see that, outside of the fermion source, this term should vanish, while inside it contributes in a non-negligible way, pushing the Higgs field towards one of its peaks.

From this we see that outside of the source, we should recover the intrinsic Higgs potential, Figure 2.1b (Blue), smoothly from an evolving, dynamic potential within the source.

Now we have a sense for how the Higgs field ought to behave; we look for solutions in which the Higgs field finds its peak. This, as mentioned, is an unstable equilibrium point, and situations in which it rolls into the minimum, or rolls out to infinity are unphysical. We have seen from our asymptotic analysis that we are free to choose the initial value for the Higgs
2.3. Solutions

(a) Figure 2.1: Higgs potential plots. 2.1a is the Higgs Mexican hat potential with the conventional orientation, for the case when the Higgs field is time-dependent. 2.1b shows the inverted potential for the static, but spatially dependent Higgs field. The blue curve shows the intrinsic inverted potential, the red curve shows the effect of the Higgs coupling to gravity has on the potential, and the black curve shows the additional effect the inclusion of the fermions has on the potential.

field, \( h_0 \), but, unsurprisingly, solutions are highly sensitive to this value, thus there must exist some critical \( h_0 \) for which the forces on the Higgs field balance such that it asymptotically tends to its peak value. We find that, indeed, this is the case.

2.3.4 Solutions

Of the free parameters, \( m_H, \lambda \) will clearly have a value as determined by nature, though here we choose \( m_H = 1, \lambda = 10 \) and \( \mu \), which can be chosen depending on how strongly we want our solitons to be coupled to the Higgs field; we choose \( \mu = 10 \). Were we searching for a model for a particular particle in nature, electron, neutrino etc, we could choose \( \mu \) accordingly.

Since we are free to scale our solutions after obtaining, we set \( T_0 = 1 \) to be altered after solving. This leaves us 3 free parameters, for which we choose some positive value for \( \alpha_1 \), since like FSY, any value yields solutions, with \( \omega \) and \( h_0 \) remaining as the parameters we need to tune, leaving us with a 2-dimensional problem, which we solve by implementing a Nelder-Mead algorithm in Mathematica using its built-in NMinimize function.

We find that, as might be expected, our solutions are qualitatively similar to FSY solutions, with \( T \) decreasing monotonically from some central value, and \( A \) remaining positive over the range of the solution (note that \( A = 0 \) corresponds to the formation of a horizon). There is a ground state solution and a tower of excited states, with monotonic increasing numbers of nodes in the fermion wavefunction where the ground state has none. We also find that there
exist fermion dominated solutions, Figure 2.2, and antifermion dominated solutions, Figure
2.3, having positive and negative fermion masses respectively, but always having positive and
finite ADM mass.

The Higgs field starts at some initial value, and is naturally attracted towards the min-
imum of the potential. This attraction is offset by the tilt of potential due to the fermions,
which overcomes this and pushes the Higgs field towards its peak. By including \( h_0 \) in the mini-
imisation, we search for the value such that the Higgs field acquires kinetic energy enough to
roll up to the peak, at which point the kinetic energy is lost and so the field will sit at the peak
indefinitely.

In the case of the antifermion dominated solutions, the tilt given in Equation 2.47, while
initially dominated by \( \alpha \) thus making it positive, is, for the most part, dominated by \( \beta \), and so
it is negative, pushing the Higgs field to the negative peak in the potential.

For excited states, Figure 2.4, we find that the metric component \( T \) remains qualitatively
similar to its ground state form, while \( A \) gains extra minima. The fermion wavefunctions gain
a node, and the parametric plot comes back to the origin, not in the top right quadrant of the
plot as in the ground state, but in the bottom left. This is as FSY noticed with the ED solutions
they studied.

An interesting point with the excited state, for which we have chosen \( \alpha_1 = 0.1 \) to make the
Higgs dynamics more dramatic, is that, with the extrema appearing in \( A \), oscillations appear
in the effective potential Figure 2.4e. These oscillations appear to ‘catch’ the evolving Higgs
field and guide it toward the intrinsic peak. The potential evolves faster than the Higgs field
itself, and it can be seen that the field is actually overtaken by the potential repeatedly, either
to fall in or out, but, before it can fall away, is caught by the evolving peak to be pushed back
the other way until it finds its way to the peak.

It is interesting to note that, in principle, one can have a solution that has negative central
fermion mass, but with the Higgs field settling to a positive value so that the fermions end up
with positive mass, Figure 2.5. In these situations, the initial value for \( \alpha_1 \) is high, so that, by
Equation 2.47, the tilt in the potential is much higher. In order to counteract this, the Higgs
field must start at a smaller initial value, and this can be, and sometimes must be negative.
Such solutions are more difficult to find as the tilt is much greater initially, and so solutions
are far more sensitive to the initial value for the Higgs field.
Figure 2.2: EDH ground state solution for $\alpha_1 = 0.05$. In (a) we have the Dirac spinor components, $\alpha$ being the particle part, $\beta$ the antiparticle part. The plot of $\alpha$ vs $\beta$ is shown in (b). (c) shows the metric components $T$ and $A$ for this solution. Notice that at small $r$ they are well behaved and finite, and at large $r$ they go over exactly to the Schwarzschild metric. (d) shows the ADM mass plots, $M(r)$, as determined from the metric components, both converging to some constant value; the mass of the soliton. The plot of the Higgs field is shown in (e), with its dynamic extrema affected by the coupling to the Ricci scalar and the fermions for small $r$, eventually going over to the intrinsic values for large $r$, and the Higgs field itself rolling toward the positive peak. The vertical dashed line included in some of the plots is the cutoff radius, occurring at $r = 26.3647$. Scaled parameter values for this solution are: $h_0 = 0.082305$, scaled $\omega = 0.384081$, fermion mass, $m = 0.44095$, ADM mass, $M = 0.913948$. 
Figure 2.3: EDH antifermion dominated ground state solution for $\alpha_1 = 0.05$. This time, the Higgs field settles to the negative peak, corresponding to negative scaled fermion mass, $m = -0.626645$. However, the total mass of the soliton, the ADM mass remains positive. Cutoff radius occurring at $r = 17.6434$. Other scaled parameters: $h_0 = -0.101498$, $\omega = 0.482435$, ADM mass $M = 1.319513$. 
2.3. Solutions

Figure 2.4: EDH first excited solution for $\alpha_1 = 0.1$, chosen for its more extreme Higgs field dynamics. The fermion wavefunctions shown in 2.4a show the spinor components each have a node, where before for the ground state there were none. The plot of $\alpha$ vs $\beta$ now, rather than coming back to the origin in the top right quadrant, comes back in the bottom left, 2.4b, consistent with the observations FSY made to their ED solutions. The metric component $T$ remains qualitatively similar while $A$ gains an extra minimum, 2.4c. The ADM curves reflect the metric components; the one derived from $A$ exhibiting oscillations 2.4d. The Higgs field, 2.4e exhibits interesting dynamics, with the Higgs field ‘caught’ and guided toward the intrinsic peak by the more quickly evolving potential. Cutoff radius, $r = 17.3822$.

Scaled parameter values: $h_0 = 0.058436$, $\omega = 0.610494$, fermion mass, $m = 0.739421$, and ADM mass, $M = 1.52962$. 
Figure 2.5: EDH ground state solution for which, $\alpha_1 = 0.12$, $h_0 = -0.00544761$, so that the fermion mass is centrally negative. The Higgs field quickly leaves the negative part of the potential due to the tilt provided by the fermions, and so the fermion mass is asymptotically positive at $m = 0.532264$. Scaled parameters: $\omega = 0.321158$, and ADM mass $M = 0.99558$. The cutoff radius occurs at $r = 8.79683$. 

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2.3. Solutions

(a) Full, unprocessed plots showing the divergence of the various fields given the fact we cannot achieve infinite precision. The cutoff radius is shown as the vertical dashed line.

(b) Figure 2.6: Full, unprocessed plots showing the divergence of the various fields given the fact we cannot achieve infinite precision. The cutoff radius is shown as the vertical dashed line.

(c) Since we are not able to achieve infinite precision, we find that there is some $r$ at which the outer boundary of the solution exists and is divergent. This is defined at the point at which the Higgs field either rolls away from the peak to infinity, or into the minimum. Once this occurs, the metric component $A$ diverges from the Schwarzschild metric, and the corresponding ADM mass curve will also diverge. Since, at this point, the solution becomes unphysical, we truncate plots by defining some cutoff radius $r_c$, at the point the ADM mass curve derived from metric component $A$ is flattest, and then we extrapolate the plots outward from this point. We mark this cutoff point explicitly on plots for which this truncation has been carried out to make clear that this has happened, and, we provide an example of the full, unprocessed solution in Figure 2.6.

2.3.5 Einstein-Dirac vs Einstein-Dirac-Higgs

If we want to directly compare ED to EDH solutions, we would find that there is not a unique way to do this. For instance, we might compare $\alpha_1$, the scaling, $\chi$, or the ADM mass. In truth, all of these are perfectly reasonable, and so the choice of comparison is arbitrary, and is down to preference, though we note that not all are physical. For example, comparing $\alpha_1$ between solutions is purely mathematical.
Chapter 2. Einstein-Dirac-Higgs Solitons

Figure 2.7: Comparison plots of ED and EDH solutions matching the value of $\tau$. In order to create these plots, $\alpha_1$ was varied for the ED equations until the correct value of $\tau$ was found. The value for the EDH solution remains 0.05, while for the ED solution, $\alpha_1 = 0.0323$.

We shall compare ED to EDH using the physical value of the central redshift, or in other words, the value of $\tau$, Figure 2.7. As mentioned above, EDH solutions are qualitatively similar to ED solutions however, in order to match the central redshift, the value of $\alpha_1$ is significantly different from ED to EDH, 0.0323 and 0.05 respectively. The value for the ED $\alpha_1$ was found by varying it until the correct value of $\tau$ was reached.

2.4 EDH Spiral Plots

We have attempted to create spiral plots for the EDH equations in the same way that FSY did for their ED equations. However, we find that due to limitations in the solver, we are unable to show a large range in $\alpha_1$, since, if it becomes either too small or too large, the minimiser struggles to find satisfactory solutions. This results in solutions in which the fermions are not well converged, the ADM mass curves from the metric components don’t converge into a single value and the Higgs field doesn’t stay at the peak for long, or at all. We are limited to the range 0.0175 to 0.1.
0.0175 to 0.1.

There are several possible reasons the minimiser fails. For the large $\alpha_1$ cases, there exists the physical reason that, because of the increased tilt in the potential due to the strength of the fermions, simply finding the peak of the potential will be a much more sensitive problem in the parameters, since the Higgs field will acquire more kinetic energy, and thus, finding the exact value of $h_0$ for which the field decays such that it finds and stays at the peak, will of course require greater precision.

Other possible reasons include the fact that the cost function given to the minimiser is simply not sufficient for the low and high $\alpha_1$, whilst within range, it works very well. Indeed, for even the best cost function devised, we have found the problem to be numerically noisy, and so, the Nelder-Mead method often gets stuck in a local minimum rather than finding a global one. It may be that the precision of the minimiser is not sufficient to expand the range stated above.

Despite this, we will show the solutions we have been able to determine, plotted alongside the ED equivalents. Points in the EDH curves indicate evaluations for different values of $\alpha_1$, increasing with increasing $m$, with the connecting lines being a rough interpolation between points. The total binding energy of the soliton is shown in Figure 2.8. Again, we compare $M - 2|m|$ against the scaled fermion mass; negative values corresponding to bound solitons and positive values to unbound. Figure 2.9 shows the fermion binding energy, $|m| - \omega$, plotted against scaled fermion mass, and Figure 2.10 shows the energy of the gravitational field, $M - 2\omega$.

We note that the EDH plots vary greatly from their ED equivalents, even at small $\alpha_1$, where the expectation is that the EDH curve should begin to match onto, and thus have comparable values to the ED curve. The reasoning behind this is that, as $\alpha_1$ is reduced, the tilt in the potential from the fermions lessens, and so the Higgs field is placed closer to the intrinsic peak value at the origin, requiring less displacement in the potential to actually find the intrinsic peak. This will cause the Higgs field to remain essentially flat throughout the extent of the solution, which, as we noted above, is one requirement for the reduction of the EDH equations to the ED equations. Indeed, the ED spiral plots shown here were generated using the EDH equations reduced to the ED equations.

The inability to obtain EDH curves at the low and high $\alpha_1$ ends of the range is suspected
Figure 2.8: Binding energy of the soliton, $M - 2|\mathbf{m}|$, for ground state EDH solutions and the ED equivalent. The negative part of the curve indicates bound states, and the positive parts indicate unbound states.

to be a matter of precision, whereas the reason for the quantitative and qualitative divergence of the EDH curves from expectation is not understood.

In order to check that the issue was one inherent to the problem as opposed to limitations of using Mathematica in solving the equations, an attempt to find solutions using Python was attempted, and we found that the spiral curves from Python are in agreement with the curves from Mathematica, suggesting there is indeed some subtlety to the problem that adding the Higgs field introduces.
2.4. **EDH Spiral Plots**

**Figure 2.9:** Fermion binding energy, $|m| - \omega$, for ground state EDH and ED equivalent.

**Figure 2.10:** Energy of the gravitational field, $M - 2\omega$, for EDH and ED equivalent.
3 Einstein-Dirac-Higgs-Maxwell Solitons

An obvious extension to the EDH solutions is the inclusion of charge. This further generalises the EDH equations to what we shall call the Einstein-Dirac-Maxwell-Higgs (EDMH) equations. This was the next step that FSY took (Finster et al., 1999b), (Finster et al., 1999a), and so we will follow in the same vein.

3.1 Setup

As before, we parameterise our metric such that

\[ g_{\mu\nu} = \text{diag}(-T^{-2}, A^{-1}, r^2, r^2 \sin^2 \theta), \]  

(3.1)

with \( T = T(r) \) and \( A = A(r) \).

The general form of the action remains
Chapter 3. Einstein-Dirac-Higgs-Maxwell Solitons

\[ S = S_g + S_m = \int d^4 x \sqrt{-g} \left( \mathcal{L}_g + \mathcal{L}_m \right), \quad (3.2) \]

using again the standard Einstein-Hilbert gravitational action. Our matter Lagrangian now includes the Maxwell term, so that

\[ \mathcal{L}_m = -\frac{1}{2} h^\mu h_\mu - \left( V(h) + \frac{\zeta}{12} R h^2 \right) + \bar{\psi} (\not{D} \psi - \mu h) \psi - \frac{1}{4} F^{\mu \nu} F_{\mu \nu}, \quad (3.3) \]

\[ V(h) = \lambda \left( h^2 - h_m^2 \right)^2, \quad h_m^2 \equiv \frac{m_H^2}{8 \lambda}, \quad (3.4) \]

where \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the antisymmetric Faraday tensor, and, like FSY, we define the electromagnetic 4-potential \( A_\mu = (-\phi, 0) \), with \( \phi = \phi(r) \) being the electrostatic potential.

### 3.2 Equations of Motion

We derive our equations of motion, again by applying the EL equations to \( \mathcal{L}_T = \mathcal{L}_g + \mathcal{L}_m \), though now, since we are considering charged particles, we must include the standard minimal coupling, that is, the replacement of \( \partial_\mu \rightarrow \partial_\mu - iqA_\mu \), where \( q \) is the charge of the particle.

#### 3.2.1 Dirac Equations

We obtain Dirac equations much as in the EDH case but with the modification \( \omega \rightarrow \omega - q\phi \),

\[ \sqrt{A} \alpha' = \frac{\kappa}{2r} \alpha - \beta (\mu h + T(\omega - q\phi)), \quad (3.5) \]

\[ \sqrt{A} \beta' = \alpha (T(\omega - q\phi) - \mu h) - \frac{\kappa}{2r} \beta. \quad (3.6) \]

The normalisation condition on our fermions remains unaffected by the fact that they are now charged, and so we still have

\[ \int_0^\infty \frac{4\pi r^2}{\sqrt{A}} \bar{\psi} \psi dr = \int_0^\infty \frac{4\pi T}{\sqrt{A}} \left( \alpha^2 + \beta^2 \right) dr = 1. \quad (3.7) \]
3.2. Equations of Motion

3.2.2 Einstein Equations

We obtain our Einstein equations, 00 and 11 components respectively given by

\[
\left(1 - \frac{4}{3} \pi \zeta G h^2\right) \left(r A' + A - 1\right) + \Lambda r^2 = 8 \pi G r^2 \left(\frac{T^2 (\alpha^2 + \beta^2) |\kappa| (\omega - q \phi)}{r^2} - \frac{1}{2} A(h')^2 \right.
\]
\[
- \lambda (h^2 - h^2_m)^2 + \frac{\zeta}{6r} \left( h r A'h' + 2A \left( r (h')^2 + h(rh'' + 2h') \right) \right) - \frac{A T^2 (\phi')^2}{2 \mu_0} ,
\]

(3.8)

\[
\left(1 - \frac{4}{3} \pi \zeta G h^2\right) \frac{(A-1)T - 2Ar'T'}{T} + \Lambda r^2 = 8 \pi G r^2 \left( -\frac{T |\kappa|}{r^3} \left( \alpha \beta \kappa + \mu hr (\alpha^2 - \beta^2) \right) \right.
\]
\[
- rT (\alpha^2 + \beta^2)(\omega - q \phi) + \frac{1}{2} A(h')^2 \right.
\]
\[
- \lambda (h^2 - h^2_m)^2 - \frac{A \zeta h'h' (rT' - 2T)}{3r T} - \frac{A T^2 (\phi')^2}{2 \mu_0} ,
\]

(3.9)

and 22 and 33 components being given by

\[
- \left(1 - \frac{4}{3} \pi \zeta G h^2\right) \frac{r \left(T A' (rT' - T) + 2A \left(r T T'' - 2r (T')^2 + T T'\right)\right)}{2T^2} + \Lambda r^2
\]
\[
= 8 \pi G r^2 \left( \frac{\alpha \beta \kappa T |\kappa|}{2r^3} - \frac{1}{2} A(h')^2 - \lambda (h^2 - h^2_m)^2 \right.
\]
\[
+ \frac{\zeta}{6r T} \left( h r A'h' + 2A \left( r T (h')^2 + h(T (rh'' + h') - rh'h') \right) \right) + \frac{A T^2 (\phi')^2}{2 \mu_0} ,
\]

(3.10)

with terms on the right-hand side in the outer brackets being the energy-momentum tensor.

3.2.3 Higgs Equation

Since the Higgs boson is uncharged, its equation of motion remains unaffected by the inclusion of the Maxwell field, thus we have a Higgs equation of the form

\[
A h'' - h' \left( A \left( \frac{T'}{T} - \frac{2}{r} \right) - A' \left( \frac{2}{2} \right) \right) = -\frac{\mu T (\alpha^2 - \beta^2) |\kappa|}{r^2} - 4\lambda h (h^2 - h^2_m) - \frac{1}{6} \zeta h R . \]

(3.11)


### 3.2.4 Maxwell Equation

The general form for the Maxwell equations is found by applying the EL equations to $\mathcal{L}_T$, taking $A_\mu$ as our field, resulting in

$$F^{\mu\nu} = 4\pi q j^\mu, \quad (3.12)$$

where $q$ is the charge and $j^\mu = (\rho, j^i)$ is the current 4-vector, with charge density $\rho$, and current density $j^i$. The components of the current 4-vector are determined by

$$j^\mu = \overline{\psi} G^\mu \psi, \quad (3.13)$$

$$j^i = \frac{|\kappa| T^2}{r^2}\left(\alpha^2 + \beta^2\right), \quad (3.14)$$

$$j^t = 0. \quad (3.15)$$

The resulting form for the Maxwell equation is

$$Ar^2 \phi'' + \phi' \left(r^2 \left(\frac{A'}{2} + \frac{AT'}{T} \right) + 2Ar\right) = 4\pi q \left(\alpha^2 + \beta^2\right) |\kappa|. \quad (3.16)$$

### 3.3 Solutions

In order to obtain EDMH solutions, we solve Equations [3.5], [3.6], [3.8], [3.9], [3.11] and [3.16]. As the EDH equations reduce to the FSY Einstein-Dirac equations, so these reduce to the FSY Einstein-Dirac-Maxwell (EDM) equations when setting $h = h_m$, $h' = h'' = 0$, and $\zeta = 0$.

As before, we want to recover flat spacetime when sufficiently far from the soliton, so we require that our metric go over not to the Schwarzschild metric, which applies to the uncharged case, but the Reissner-Nordström metric, which includes this charged correction. This has the form

$$T^{-2} = A = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad (3.17)$$

where $M$ is the ADM mass and $Q$ is the enclosed charge of the solution.
3.3. Solutions

An added condition we require is that the electrostatic potential vanishes at infinity. In fact, as will be shown later, we are able to match the potential onto the Coulomb potential thus ensuring this condition holds.

3.3.1 Scaling the Equations

We find that, like FSY, our equations are invariant under the gauge transformation

\[ \phi \rightarrow \phi + C, \quad \omega \rightarrow \omega + qC, \quad (3.18) \]

with \( C \) being some real constant

\[ C = - \lim_{r \to \infty} \phi < \infty \quad (3.19) \]

This leaves us free to shift the electrostatic potential so that we can ensure it matches onto the Coulomb potential.

For scaling the fields, we again define

\[ \tau = \lim_{r \to \infty} T < \infty, \quad (3.20) \]

\[ \chi^2 = \int_0^\infty \frac{T}{\sqrt{A}} (a^2 + \beta^2) dr < \infty, \quad (3.21) \]

for which we find the following scaling relations

\[ \tilde{a} (\tilde{r}) = \sqrt{\frac{\tau}{\chi}} a (\tilde{r} \chi), \quad (3.22) \]

\[ \tilde{\beta} (\tilde{r}) = \sqrt{\frac{\tau}{\chi}} \beta (\tilde{r} \chi), \quad (3.23) \]

\[ \tilde{T} (\tilde{r}) = \tau^{-1} T (\tilde{r} \chi), \quad (3.24) \]

\[ \tilde{A}(\tilde{r}) = A(\tilde{r} \chi), \quad (3.25) \]

\[ \tilde{h} (\tilde{r}) = h (\tilde{r} \chi), \quad (3.26) \]

\[ \tilde{\phi} (\tilde{r}) = \tau (\phi (\tilde{r} \chi) + C). \quad (3.27) \]
These scaled functions satisfy the original equations provided that \( \mu, m_H, \omega, \lambda \) and \( q \) are replaced with

\[
\tilde{\mu} = \chi \mu, \quad (3.28)
\]
\[
\tilde{m}_H = \chi m_H, \quad (3.29)
\]
\[
\tilde{\omega} = \chi \tau (\omega + qC), \quad (3.30)
\]
\[
\tilde{\lambda} = \chi^2 \lambda, \quad (3.31)
\]
\[
\tilde{q} = \chi q. \quad (3.32)
\]

### 3.3.2 Small \( r \) Asymptotic Analysis

Assuming solutions are regular close to the origin, expanding fields in powers of \( r \) and setting \( \kappa = 2 \), we find

\[
\alpha = \alpha_1 r + O(r^3), \quad (3.33)
\]
\[
\beta = -\frac{1}{3} \alpha_1 (h_0 \mu + T_0 (q \phi_0 - \omega)) r^2 + O(r^3), \quad (3.34)
\]
\[
T = T_0 + \left( \frac{4}{3} \pi G \lambda (h_0^2 - h_m^2)^2 + \alpha_1^2 h_0 \mu T_0 + 2 \alpha_1^2 \tau_0^2 (q \phi_0 - \omega) \right) r^2 + O(r^3), \quad (3.35)
\]
\[
A = 1 + \left( \frac{8 \pi \zeta G \lambda (h_0^2 - h_m^2)(h_0^2(16 \pi G h_m^2 - 3) - 9 h_m^2)}{9(4 \pi G h_m^2 - 3)} \right) r^2 + O(r^3), \quad (3.36)
\]
\[
h = h_0 + \left( \frac{1}{3} (2 h_0 \lambda (h_0^2 - h_m^2) + \alpha_1^2 \mu T_0) - \frac{2}{9} \zeta h_0 \lambda (h_0^2 - h_m^2)(4 \pi G h_m^2 - 3) \right) r^2 + O(r^3), \quad (3.37)
\]
\[
\phi = \phi_0 + \frac{4}{3} \pi \alpha_1^2 q + O(r^3). \quad (3.38)
\]
These relations reveal that, unsurprisingly, we have the same free parameters as the EDH case, now with the charge included, and the freedom to choose the initial value for the electrostatic potential.

### 3.3.3 Solutions

As before, we set $m_H = 1$, $\lambda = 10$ and $\mu = 10$. Since we can rescale the equations of motion to perform the normalisation of the fermions after solving, we are free to set $T_0 = 1$, and $\phi_0 = 0$ and scale them appropriately after. This leaves us with 4 parameters, $\alpha_1$, $\omega$, $h_0$ and $q$.

We choose some value for $\alpha_1$ and $q$, leaving us, once again, with a 2-dimensional problem. We implement the Nelder-Mead method to solve the equations, finding that solutions are qualitatively similar to the EDH solutions, as might be expected.

There exist fermion dominated solutions, with positive fermion mass, Figure 3.1 and antifermion dominated solutions with negative fermion mass, Figure 3.2. In both cases, the ADM mass is positive and finite, and we can verify that the electrostatic potential matches onto the Coulomb potential for large $r$, while being well behaved for small $r$.

The metric components, $T$ and $A$ match onto the Reissner-Nordström metric exactly, ensuring that they tend to 1 for large $r$, while they are well behaved, non-diverging functions close to the origin.

### 3.4 EDMH Spiral Plots

As with the EDH equations, there are limitations on which solutions we are able to obtain for the EDMH equations; we are limited to $\alpha_1$ values 0.0175 to 0.1. Despite that, we show the spiral plots we have been able to determine. The soliton binding energy, $M - 2|m|$, is shown in Figure 3.3, the fermion binding energy, $|m| - \omega$, in Figure 3.4 and the energy of the gravitational field, $M - 2\omega$ in Figure 3.5 all plotted against their respective EDM equivalents.

Comparison with EDH shows that the EDMH spirals are qualitatively similar, and, encouragingly, the EDMH solutions are less bound than the EDH solutions, which is fully expected since the inclusion of charge to the fermions introduces a repulsive force between them that opposes the gravitational attraction.

The EDM spirals were generated by reducing the EDMH equations to the EDM equations.
Figure 3.1: EDMH ground state solution for $\alpha_1 = 0.05$ and $q = -0.0393942$. Shown in (a) are the spinor components, $\alpha$, the particle part, and $\beta$, the antiparticle part. The plot of $\alpha$ vs $\beta$ is shown in (b). The metric components, $T$ and $A$ are shown in (c) along with the Reissner-Nordström metric, which they match onto exactly for large $r$. For small $r$, these functions are well behaved. The ADM mass, $M(r)$, plots are shown in (d), both curves converging on a finite value. The plot for the electrostatic potential is shown in (e) as well as the Coulomb potential. As with the metric components, notice that for small $r$, the electrostatic potential is well behaved where the Coulomb potential diverges, and for large $r$, the potential goes over to the Coulomb potential exactly. In (f) we have the Higgs field shown along with the extrema of the potential; the Higgs field rolls up to the positive peak. The cutoff radius, occurring at $r = 31.8536$ is marked as the vertical dashed line. Parameter values for this solution are: fermion mass, $m = 0.440441$, $h_0 = 0.0823318$, scaled $\omega = 0.383479$, and ADM mass, $M = 0.913642$. 
Figure 3.2: EDMH antifermion dominated ground state solution for $\alpha_1 = 0.05$ and $q = -0.0559711$. Scaled parameter values are: fermion mass, $m = -0.625776$, $h_0 = -0.101512$, $\omega = 0.48183$, and the ADM mass of the soliton, $M = 1.31822$. The cutoff radius occurs at $r = 17.2186$. 
Figure 3.3: Binding energy of the soliton, $M - 2|m|$, for EDMH and the EDM equivalent. $\alpha_1$ increases with increasing $m$.

Figure 3.4: Fermion binding energy, $|m| - \omega$, for EDMH and EDM solutions. $\alpha_1$ increases with increasing $m$.

Figure 3.5: Energy of the gravitational field, $M - 2\omega$, for EDMH and EDM solutions. $\alpha_1$ increases with increasing $m$. 

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Conformal Gravity-Dirac-Higgs Solitons

Despite its many successes, GR still falls short when applied to distance scales far greater than that of the solar system, namely galactic and cosmological scales, for which we encounter the problems of dark matter and dark energy respectively. There have been many attempts to modify or add correction terms to the Einstein-Hilbert action, but there are infinitely many possibilities and so, while these theories may enjoy some success in addressing these issues, it is difficult to know which, if any, of these theories is correct. This is reinforced by the fact that, to date, there have been no experimental verifications to the proposed solutions. With that in mind, we shall now move away from GR and instead visit an entirely different gravity theory; conformal gravity.

4.1 Conformal Gravity

Conformal gravity (CG) is an alternative theory of gravity proposed by [Mannheim (2006)] that is coordinate independent, like GR, but which includes the extra, restrictive conformal symmetry; a local preserving of angles. Under a conformal transformation the metric is stretched
so that

$$g_{\mu\nu}(x) \rightarrow \Omega^2(x) g_{\mu\nu}(x), \quad (4.1)$$

for spacetime point, $x$, with other fields carrying their own scalings, the combinations of which leave the equations of motion unaffected. The restrictive nature of this conformal invariance leads to a unique choice for the gravitational action, namely the Weyl action

$$S_g = -\alpha_g \int d^4x \sqrt{-g} C_{\mu\nu\lambda\kappa} C^{\mu\nu\lambda\kappa}, \quad (4.2)$$

where

$$C_{\mu\nu\lambda\kappa} = R_{\mu\nu\lambda\kappa} - \frac{1}{2} (g_{\mu\lambda} R_{\nu\kappa} - g_{\mu\kappa} R_{\nu\lambda} - g_{\nu\lambda} R_{\mu\kappa} + g_{\nu\kappa} R_{\mu\lambda}) + \frac{1}{6} R (g_{\mu\lambda} g_{\nu\kappa} - g_{\mu\kappa} g_{\nu\lambda}) \quad (4.3)$$

is the Weyl tensor, the traceless part of the Riemann tensor under the Ricci decomposition, and $\alpha_g$ is a dimensionless coupling constant, which we will set to $+1$.

As well as restricting the gravitational action, conformal invariance is broken by Newton’s constant $G$ so that it is excluded from the theory. The same is true of any intrinsic particle masses, so that for instance, the intrinsic Higgs mass $m_H$ must also be prohibited.

In the same way that variation of the total Lagrangian leads to the Einstein equations in GR, CG has analogous gravitational equations of motion; the Bach equations

$$4\alpha_g W_{\mu\nu} = T_{\mu\nu}, \quad (4.4)$$

where $W_{\mu\nu}$ is the Bach tensor

$$W_{\mu\nu} = \frac{1}{3} R_{\mu\nu} - \frac{1}{3} R_{\mu\nu}^{\lambda\kappa} ;{\lambda}^{\kappa} + \frac{1}{6} g_{\mu\nu} (3R_{\alpha\beta} R^{\alpha\beta} + R_{\alpha\beta}^{\lambda\kappa} ;{\lambda}^{\kappa} + R^2) + 2R_{\mu\alpha\nu\beta} R^{\alpha\beta} - \frac{2}{3} RR_{\mu\nu}. \quad (4.5)$$

One important property of the Bach tensor is that it is traceless, which, given Equation 4.4, forces the energy-momentum tensor to also be traceless, which in turn puts restrictions on the
4.2 Setup

4.1.1 Conformal Gravitational Potential

An important way the Bach equation differs from the Einstein equation is that it is fourth order in metric derivatives, as opposed to the Einstein equation being second order, thus, when integrating the equations of motion, there are extra terms present in the gravitational potential. It can be shown that, for a vacuum, a conformal transformation can be made so that the metric can be expressed (Mannheim & Kazanas, 1989)

\[ g_{00} = w - \frac{2b}{r} + \gamma r - Kr^2, \]

where \( w, b, \gamma, \) and \( K \) are integration constants. For small \( r \), this form of the potential approximates the Schwarzschild solution provided \( w = 1 \) and \( b = M \), while for large \( r \), the linear term in the potential can potentially explain galaxy rotation curve observations (Mannheim & O’Brien, 2013), (Horne, 2016) and gravitational lensing effects (Cattani et al., 2013), (Lim & Wang, 2017). There have also been attempts at devising a quantum theory of gravity from conformal gravity (Mannheim, 2009), (Faria, 2016), and an attempt to explain the cosmological constant problem, among others (Mannheim, 2012).

4.2 Setup

We shall define our metric in a different way to FSY, instead following Mannheim (2006) and Brihaye & Verbin (2009) and parameterising it such that

\[ g_{\mu\nu} = \text{diag}(-B, B^{-1}, r^2, r^2 \sin^2 \theta), \]

where \( B = B(r) \). Our action takes the form

\[ S = S_g + S_m = \int d^4x \sqrt{-g} \left( \mathcal{L}_g + \mathcal{L}_m \right), \]

with \( S_g \) and \( S_m \) denoting gravitational and matter actions respectively and \( g = r^4 \sin^2 \theta \) is the determinant of \( g_{\mu\nu} \). Now, instead of using an Einstein-Hilbert term, our gravitational Lagrangian is given by the Weyl Lagrangian given in Equation 4.2.
The Conformal Gravity-Dirac-Higgs (CDH) matter Lagrangian is now

\[ \mathcal{L}_m = -\frac{1}{2} h^{\mu \nu} h_{\mu \nu} + \frac{1}{12} R h^4 - \lambda h^4 + \bar{\psi} (\varphi - \mu h) \psi, \] (4.9)

where \( h = h(r) \) is the Higgs field, \( \lambda \) is the Higgs self coupling constant, \( R \) is the Ricci scalar, \( \psi \) is the Dirac spinor with adjoint \( \bar{\psi} = \psi^\dagger \gamma^0 = \psi^\dagger \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ -1 \end{array} \right) \), \( \varphi \) is the Dirac operator and \( \mu \) is the coupling between the fermions and the Higgs field.

Now, our Higgs potential has only one intrinsic term; the quartic \( \lambda h^4 \), with the coupling to gravity through the Ricci scalar providing the quadratic term, and thus the Mexican hat form is dynamically generated. Of course, the Mexican hat potential can only exist provided the quadratic and quartic terms carry the opposite relative signs, and so the sign of the Ricci scalar is important in spontaneous symmetry breaking.

### 4.2.1 Conformal Symmetry

Since CG is a conformally invariant gravity theory, we need to ensure our action is also. We can check this with a straightforward calculation. The various fields in our action conformally transform as

\[ C^{\mu \nu \lambda \kappa} \rightarrow C^{\mu \nu \lambda \kappa}, \] (4.10)

\[ g^{\mu \nu} \rightarrow e^{2\eta} g^{\mu \nu}, \] (4.11)

\[ g^{\mu \nu} \rightarrow e^{-2\eta} g^{\mu \nu}, \] (4.12)

\[ h \rightarrow e^{-\eta} h, \] (4.13)

\[ R \rightarrow e^{-2\eta} R, \] (4.14)

\[ \psi \rightarrow e^{-3\eta} \psi; \] (4.15)

\[ \bar{\psi} \rightarrow e^{-3\eta} \bar{\psi}; \] (4.16)

\[ \varphi \rightarrow e^{-\eta} \varphi; \] (4.17)

\[ \sqrt{-g} \rightarrow e^{4\eta} \sqrt{-g}. \] (4.18)

For the terms in the action we see that, under these transformations, a factor of \( e^{-4\eta} \) will be picked up. The action will thus be invariant since the scaling of \( \sqrt{-g} \) cancels the scaling
4.3. Equations of Motion

Once again, deriving our equations of motion requires applying the Euler-Lagrange equations to the total Lagrangian \( \mathcal{L}_T = \mathcal{L}_g + \mathcal{L}_m \). We shall call these equations the conformal gravity-Dirac-Higgs (CDH) equations.

4.3.1 Dirac Equations

Taking the Dirac spinor as our field, we get the general form for the Dirac equations

\[
(\slashed{D} - \mu h) \psi = 0. \tag{4.19}
\]

We are again choosing to consider static, spherically symmetric solutions, for which we need to use an ansatz so that our equations reduce to radial ones. We take the FSY ansatz, and modify it to match our current metric parameterisation

\[
\psi = e^{-i\omega t} \frac{1}{B^{1/4} r} \begin{pmatrix} \alpha \\ -i\sigma^r \beta \end{pmatrix} \Omega_{jk}(\theta, \phi) \chi_a. \tag{4.20}
\]

Using this, we get our radial Dirac equations

\[
\sqrt{B} \alpha' = \frac{\kappa}{2r} \alpha - \beta \left( \frac{\omega}{\sqrt{B}} + \mu h \right), \tag{4.21}
\]

\[
\sqrt{B} \beta' = \alpha \left( \frac{\omega}{\sqrt{B}} - \mu h \right) - \frac{\kappa}{2r} \beta, \tag{4.22}
\]

with \( \kappa = \pm (2j + 1) \) being the the relativistic quantum number. The fermion normalisation condition becomes

\[
\int_0^\infty \frac{4\pi r^2}{\sqrt{B}} \bar{\psi} \psi dr = \int_0^\infty \frac{4\pi r^2}{B} \frac{(\alpha^2 + \beta^2)}{r^2} dr = 1, \tag{4.23}
\]
4.3.2 Bach Tensor

To determine the components of the Bach tensor, we apply the EL equations to the gravitational Lagrangian in Equation 4.2. Doing so, we find

\[
W_0^0 = \frac{1}{12r^4} \left( 4B'^2 - 2B''r A' + r^2 \left( rB'' - 2B' \right)^2 - 4Br \left( 2B' + r \left( B''r + 3B'' - B' \right) \right) - 4 \right),
\]

(4.24)

\[
W_1^1 = \frac{1}{12r^4} \left( -2B''r^3 \left( rB' - 2B \right) + \left( rB'' - 2B' + 2B \right)^2 - 4 \right).
\]

(4.25)

These components can be combined to end up with a surprisingly simple relation

\[
\frac{3}{B} (W_0^0 - W_1^1) = -\frac{1}{r} (rB''').
\]

(4.26)

The other components of the Bach tensor are

\[
W_3^3 = W_4^4 = \frac{1}{12r^4} \left( -4B^2 + 2B''r^4B' - r^2 \left( rB'' - 2B' \right)^2 + 2Br \left( 4B' + r \left( B''r^2 + 2B'' - 2B \right) \right) + 4 \right).
\]

(4.27)

4.3.3 Energy-Momentum Tensor

We derive the energy-momentum tensor using the relation

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\partial (\mathcal{L}_m \sqrt{-g})}{\partial g^{\mu\nu}},
\]

(4.28)

for which the 00 component has the form

\[
T_{00}^0 = -\omega \left( \alpha^2 + \beta^2 \right) |\kappa| + \frac{h^2 \left( rB' + B - 1 \right)}{6r^2} + \frac{hrB'h' + 2B \left( r \left( h' \right)^2 + h \left( rh'' + 2h' \right) \right)}{6r} - \frac{1}{2} B \left( h' \right)^2 - \lambda h^4.
\]

(4.29)
4.3. Equations of Motion

the 11 component is given by

\[ T_{11} = -\frac{|\kappa|}{B r^3} \left( \sqrt{B} \left( \alpha \beta \kappa + \mu h r (\alpha^2 - \beta^2) \right) - r \omega (\alpha^2 + \beta^2) \right) + \frac{h^2 (r B' + B - 1)}{6r^2} + \frac{hh' (r B' + 4 B)}{6r} + \frac{1}{2} B (h')^2 - \lambda h^4, \]  

(4.30)

and the 22, and 33 components are

\[ T_{22} = T_{33} \]

\[ = \frac{\alpha \beta \kappa |\kappa|}{2 \sqrt{B} r^3} + \frac{h^2 (r B'' + 2 B')}{12r} + \frac{hr B' h' + B \left( r (h')^2 + h (r h'' + h') \right)}{3r} - \frac{1}{2} B (h')^2 - \lambda h^4. \]  

(4.31)

The trace of the energy-momentum tensor vanishes when the Dirac equations and the Higgs equation are substituted in.

4.3.4 Poisson Equation

For the metric parameterised such that it depends upon only one function of \( r \), we find that it is necessary to reduce the number of equations for \( B \) from two to one. This can be done by combining the 00 and 11 Bach tensor components as in Equation 4.26 along with the 00 and 11 components of the energy-momentum tensor into a source term according to

\[ f(r) = -\frac{3}{4\alpha B} \left( T_{00}^0 - T_{11}^1 \right), \]  

(4.32)

to define a 4th-order Poisson equation

\[ \frac{1}{r} (r B)^{\prime \prime \prime} = f(r). \]  

(4.33)

Written out explicitly for the CDH equations

\[ f = -\frac{3|\kappa|}{4B^2 r^3} \left( \alpha \beta \sqrt{B} \kappa + \alpha^2 r \left( \sqrt{B} h \mu - 2 \omega \right) - \beta^2 r \left( \sqrt{B} h \mu + 2 \omega \right) \right) - \frac{hh''}{4} + \frac{(h')^2}{2} \]  

(4.34)
4.3.5 Higgs Equation

Applying the EL equations to the total Lagrangian using the Higgs field as our field gives the Higgs equation

\[ h_{\mu\nu} = \frac{1}{6}Rh + 4\lambda h^3 + \mu \bar{\psi} \psi, \]  
(4.35)

where

\[ h_{\mu\nu} = B h'' + B' h' + \frac{2B h'}{r}, \]  
(4.36)

and

\[ \bar{\psi} \psi = \frac{|\kappa|}{r^2 \sqrt{B}} (\alpha^2 - \beta^2) \]  
(4.37)

4.4 Solutions

In order to determine our CDH solutions, we solve Equations 4.21, 4.22, 4.33, 4.35, however, unlike in the case of GR, where the intrinsic Higgs mass is allowed, and thus there exists a fully Mexican hat intrinsic potential, the CG case does not allow this, and so, where before we looked to recover flat spacetime far from the soliton, now we require constant curvature so that the Ricci scalar does not vanish. To make this more explicit, let us take a look at the form of the Ricci scalar

\[ R = -B'' + \frac{4B'}{r} - \frac{2B}{r^2} + \frac{2}{r^2}, \]  
(4.38)

for which, if we set \( B \) to be equal to the right-hand side of Equation 4.6, we get

\[ R = \frac{2(1 - w)}{r^2} - \frac{6\gamma}{r} + 12K, \]  
(4.39)

with large \( r \) expression

\[ R = 12K. \]  
(4.40)
4.4. Solutions

4.4.1 Scaling the Equations

For the CDH equations we find that a scaling of the fields exists, and these take the form

\[ e^\alpha(r) = \sqrt{\chi} \alpha(\sqrt{\chi} r), \]  
\[ e^\beta(r) = \sqrt{\chi} \beta(\sqrt{\chi} r), \]  
\[ e^B(r) = B(\sqrt{\chi} r), \]  
\[ e^h(r) = \chi h(\sqrt{\chi} r), \]  

requiring parameter scaling

\[ \tilde{\omega} = \chi \omega, \]  

with all other parameters left unscaled. Now, \( \chi \) is some constant. However, these scalings are not sufficient as a means to normalise the fermions, since the normalisation integral remains unchanged. As such, we must include the normalisation as a condition during the process of solving the equations.

4.4.2 Small r Asymptotic Analysis

Under the assumption that solutions are regular close to the origin, we expand fields in powers of \( r \) and set \( \kappa = 2 \), finding

\[ \alpha = \alpha_1 r - \frac{1}{2} \alpha_1 B_1 r^2 + \frac{1}{48} \alpha_1 \left( 3 \left( 5B_1^2 - 4B_2 \right) + 8\mu^2 h_0^2 - 8\omega^2 \right) r^3 + O(r^4), \]  
\[ \beta = -\frac{1}{3} \alpha_1 (\mu h_0 - \omega) r^2 + \frac{1}{3} \alpha_1 B_1 (\mu h_0 - \omega) r^3 + O(r^4), \]  
\[ B = 1 + B_1 r + B_2 r^2 + O(r^4), \]  
\[ h = h_0 - \frac{1}{2} B_1 h_0 r + \frac{1}{3} \left( \alpha_1^2 \mu + (B_1^2 - B_2) h_0 + 2h_0^3 \lambda \right) r^2 \]  
\[ - \frac{1}{4} B_1 \left( 2\alpha_1^2 \mu + (B_1^2 - 2B_2) h_0 + 4\lambda h_0^3 \right) r^3 + O(r^4). \]
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From these relations, we see that we have 7 free parameters, $\alpha_1, h_0, B_1, B_2, \lambda, \omega,$ and $\mu$.

We note that the asymptotic analysis was performed on both the set of equations of motion using the Poisson equation, and the set containing the 00 and 11 Bach equation components, and found that these agree with one another, suggesting that these are fully complete and consistent.

4.4.3 Solutions

Asymptotic analysis of the equations reveals the parameters we are free to set at the origin. We set $B_1 = 0, B_2 = 0.3, \mu = 1$ and $\lambda = 1.27$. The choice for $B_2$ at the origin is arbitrary, but application of the scaling after solving the equations allows us to choose some value for $\chi$ that will shift the value that $B''$ goes to for large $r$. In other words, we can find solutions for any value of external curvature, $K$.

Unlike the EDH and EDMH problems, the fact that we must consider the normalisation of the fermions whilst solving the equations means that there exist solutions for particular choices of $\alpha_1$, so we pass this to the solver to find, leaving us a 3-dimensional problem in $\alpha_1, h_0,$ and $\omega$. We implement the Nelder-Mead method again, finding the addition of the extra parameter hinders it not at all when it is given Equations 4.21, 4.22, 4.33, 4.35.

Imposing time-independence means we are again working with an inverted Higgs potential, thus we want to find solutions with constant external curvature, and the Higgs field rolling up to sit at a peak so as to give finite fermion mass.

An interesting point to note is that, since the normalisation integrand is proportional to $\frac{1}{B}$, which, by Equation 4.6, we see is a growing function in $r$, in order to impose normalisation of the fermions, it is not necessary that the spinor functions be asymptotically zero, as the metric actually takes care of this for us. What becomes important then, is that the integrand is asymptotically vanishing and of course that it integrates to 1.

We find that the solutions are somewhat reminiscent of the EDH solutions, particularly in the spinor components and the Higgs field. As mentioned, the spinor components need not be asymptotically zero, which we indeed see in Figure 4.1a, but that the integrand itself is, Figure 4.1b. The metric component, $B$, is a quadratically growing function; fitting the function to a quadratic curve verifies this, Figure 4.1c.
4.4. Solutions

The Higgs field starts at some initial value, and due to the tilt in the effective potential provided by coupling to the fermions, it is pushed towards the positive peak with kinetic energy such that, as it reaches the peak, it slows and eventually stops, thus ensuring we have finite mass for the fermions, Figure 4.1d.

We find that we have an asymptotically vanishing source term, \( f \), as we might expect, since once \( h \) finds its peak, the strong inverse \( r \) dependence, both explicitly and implicitly by its relation to \( B \), will drive it towards 0, Figure 4.1e and that \( B'' \) tends to a constant value, Figure 4.1f indicating that the solution has constant external curvature. In this instance, we have scaled the solution so that the external curvature is equal to 1, which requires \( \chi = 1.8582 \). This fact that there is external curvature could of course be inferred from the fact that there are extrema in the Higgs potential.
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Figure 4.1: CDH ground state solution for $\alpha_1 = 0.125423$. 4.1a shows the spinor matter and antimatter components, $\alpha$ and $\beta$. This time, they do not approach 0 for large $r$, but they are still normalised to 1, since the integrand, 4.1b, is asymptotically 0, thanks to the metric, 4.1c, being a growing function in $r$. The Higgs field, 4.1d, starts at some initial value, is pushed towards its positive peak, then loses kinetic energy and stays for the duration of the solution. The source term, 4.1e, is asymptotically vanishing and $B''$, 4.1f, has been scaled to be asymptotically equal to 1, with $\chi = 1.8582$ required to do this. Scaled $\omega = 1.51017$, $h_0 = 0.420909$. 
As in the generalisation from EDH to EDMH, we add charge to the CDH solitons, now considering conformal gravity coupled to the Dirac, Higgs and Maxwell fields; the CDMH equations. We do this in the same way, adding the Maxwell terms to the Lagrangian, and using the minimal coupling $\omega \rightarrow \omega + q\phi$.

### 5.1 Equations of Motion

The CDMH equations of motion are
where the first two are the Dirac equations, the third is the 4th-order Poisson equation, the fourth is the Higgs equations and the fifth is the Maxwell equation. For the Poisson equation, the left hand side is same as the CDH equations, coming from the Bach tensor relation

\[ \frac{3}{B} (W_0^0 - W_1^1) = -\frac{1}{r} (rB)''' \],

and the right hand side comes from the relation

\[ f(r) = -\frac{3}{4\alpha \beta} (T_0^0 - T_1^1) \]
\[ = -\frac{3 |\kappa|}{4 B^2 r^3} (\alpha \sqrt{B} \kappa + \alpha^2 - \beta^2 r (\sqrt{B} \mu + 2q - 2\omega) - \beta^2 r (\sqrt{B} \mu - 2q + 2\omega)) \]
\[ - \frac{hh''}{4} + \frac{(h')^2}{2} \]

The fermion normalisation is as before

\[ \int_0^\infty \frac{4\pi r^2}{\sqrt{B}} \psi^\dagger \psi \, dr = \int_0^\infty \frac{4\pi}{B} (\alpha^2 + \beta^2) \, dr = 1. \]
5.2. Solutions

\[ \phi \to \phi + C, \quad \omega \to \omega + qC, \]  

where \( C \) is a real constant

\[ C = - \lim_{r \to \infty} \phi < \infty, \]  

so we are free to shift the electrostatic potential as we did in the EDMH solution.

A scaling exists for the CDMH equations of the form to the CDH equations

\[ \tilde{\alpha}(\tilde{r}) = \sqrt{\chi} \alpha(\tilde{r} \chi), \]  

\[ \tilde{\beta}(\tilde{r}) = \sqrt{\chi} \beta(\tilde{r} \chi), \]  

\[ \tilde{B}(\tilde{r}) = B(\tilde{r} \chi), \]  

\[ \tilde{h}(\tilde{r}) = \chi h(\tilde{r} \chi), \]  

\[ \tilde{\phi}(\tilde{r}) = \chi (\phi(\tilde{r} \chi) + C), \]

requiring parameter scaling

\[ \tilde{\omega} = \chi (\omega + qC), \]  

for some positive \( \chi \), and all other parameters remaining unscaled.

5.2.2 Small r Asymptotic Analysis

Taylor expanding fields near the origin yields expansions

\[ \alpha = \alpha_1 r - \frac{1}{2} \alpha_1 B_1 r^2 + \frac{1}{48} \alpha_1 \left( 15 B_1^2 - 12 B_2 + 8 \left( \mu^2 h_0^2 - (\omega - q \phi_0)^2 \right) \right) r^3 + O(r^4), \]  

\[ \beta = -\frac{1}{3} \alpha_1 \left( h_0 \mu + q \phi_0 - \omega \right) r^2 + \frac{1}{3} \alpha_1 B_1 \left( \mu h_0 + q \phi_0 - \omega \right) r^3 + O(r^4), \]  

\[ B = 1 + B_1 r + B_2 r^2 + O(r^4), \]
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\begin{equation}
 h = h_0 - \frac{1}{2}B_1 h_0 r + \frac{1}{3} \left( a_1^2 \mu + (B_1^2 - B_2) h_0 + 2h_0^3 \lambda \right) r^2 - \frac{1}{4} B_1 \left( 2a_1^2 \mu + (B_1^2 - 2B_2) h_0 + 4\lambda h_0^3 \right) r^3 + O(r^4),
\end{equation}

\begin{equation}
 \phi = \phi_0 + \frac{4}{3} \pi q a_1^2 r^2 - \frac{4}{3} \pi B_1 q a_1^2 r^3 + O(r^4),
\end{equation}

from which we get that our free parameters are \( \alpha_1, h_0, B_1, B_2, \lambda, \omega, \mu, \phi_0 \) and \( q \).

5.2.3 Solutions

We set \( B_1 = 0, B_2 = 0.3, \lambda = 1.275, \mu = 1, \phi_0 = 0, \) and \( q = -0.1 \), leaving \( \alpha_1, h_0, \) and \( \omega \) for the solver to find, again implementing the Nelder-Mead method in Mathematica.

Solutions are qualitatively similar to the CDH equivalents, with the fermions not asymptotically vanishing, Figure 5.1a, but still normalisable, with the integrand instead being asymptotically vanishing, Figure 5.1b, due to \( B \) growing quadratically with \( r \), Figure 5.1c. The Higgs field is pushed to the positive peak due to the tilt in the potential provided by coupling to the fermions, Figure 5.1d, and \( h_0 \) is found such that the kinetic energy the Higgs field picks up as it moves in the potential decays completely by the time the peak is reached, thus ensuring fermion mass is finite.

The scaling of the fields, while not being sufficient to normalise the fermions, is useful for being able to choose the value of \( B_2 \), and scaling the solution to match any value of external curvature, \( K \). Figure 5.1f shows the case in which we have scaled solutions to get external curvature of 1, requiring \( \chi = 1.8582 \).

Invariance of the equations of motion given the gauge transformation of the electrostatic potential in Equation 5.9 allows one to set the value of \( \phi_0 = 0 \) and scale after obtaining the solution in order to match it onto the Coulomb potential, thus we have that, far from the solution we have point-particle like behaviour, but non-divergent field close to the origin, Figure 5.1g.
5.2. Solutions

Figure 5.1: CDMH ground state solution, with $\alpha_1 = 0.125383$, $q = -0.1$, $h_0 = 0.420073$, $\omega = 1.48435$. The gauge transformation of the electrostatic potential allows a rescaling of the field to match onto the Coulomb potential for large $r$. 
Conclusions

Throughout this thesis, it has been shown that one can generalise the Einstein-Dirac and Einstein-Dirac-Maxwell systems of equations by including a Higgs field to let fermion mass become a function of $r$ rather than a parameter to set.

We see that, for these new systems of equations, we are able to generate localised solutions with fully normalisable fermionic wavefunctions alongside well behaved metric components and electrostatic potential. Moreover, despite the fact that these solutions are not point-particles, when far enough away, this is exactly how they look by virtue of the fact that the metric and electrostatic potential go over to their point-particle forms, that is, the Schwarzschild or Reissner-Nordström metric for uncharged and charged cases respectively, and the Coulomb potential.

Solutions have a discrete energy spectrum, with ground state and excited states, and there is the ability to model both matter and antimatter fermions. For the excited states, the Higgs dynamics show the interesting property that, as the metric function $A$ gains minima, so the
effective potential oscillates, which seems to guide the Higgs field toward the intrinsic peak, since the potential evolves faster than the field itself, it prevents it from rolling in or out, despite its crossing the peak repeatedly.

We have worked with static, spherically-symmetric solutions, for which we showed that the Higgs potential is inverted and so, rather than there being two stable minima and one stable maximum, there are instead two unstable maxima and one stable minimum. The coupling of the Higgs field to gravity introduces dynamics to the potential as well as a tilt from the coupling to the fermions, with a smooth transition from this effective potential within the source, to the intrinsic one without the source. We have discussed the dynamics of the Higgs field for such a potential, showing that in order for the fermions to have finite mass, the Higgs field must roll up to, and sit at the peak.

For this to occur, there is, for any given solution, a particular value of the Higgs field at the origin for which the kinetic energy is such that, by the time the field finds the peak, the energy has decayed away, leaving the Higgs field constant.

It has been shown that similar solutions exist for an entirely different theory of gravity, conformal gravity. While CG is still in its infancy, it has been successful in describing some of the observed astrophysical phenomena that general relativity cannot. There is a long way to go with CG, but we can see that there is, at least, a way to describe fermionic matter within the theory.

The requirement for conformal invariance in CG prohibits intrinsic mass terms, such as the Higgs mass, and so the Mexican hat Higgs potential must be generated dynamically through coupling to gravity, with the sign of the Ricci scalar being important for spontaneous symmetry breaking. We show that one can find a solution when these conditions are met, with the dynamics of the Higgs field being similar to the EDH case.

For the CG solutions, any positive value for the external curvature can be found by implementing a set of scalings for the equations, leaving us free to choose the initial value of the curvature, however, the scaling cannot be used to normalise the fermions, and so this must be handled as the equations are being solved. We have found the interesting property of the fermions wherein their asymptotic value need not be zero, since the metric grows with $r$, and so this takes care of normalisation integral.
6.1 Further Developments

We have also shown that, for charged fermions in the CG case, we can make use of a gauge transformation on the electrostatic potential in a similar way to the EDM and EDMH equations to ensure that, for large \( r \), it matches onto the Coulomb potential.

6.1 Further Developments

There still exists much scope for the development of the work done here. One could easily envision adding in strong and weak nuclear forces in order to fully model the fundamental forces in nature. Further, loosening the constraint that solutions be spherically-symmetric and static will of course allow the study of a more realistic system. One might try to recast the equations imposing cylindrical symmetry, and thus be able to model, say, a single fermion, where we have required that spins cancel.

We have restricted our study to static solutions; adding time dependence is an obvious way to further generalise the problem. Time dependence extends the Higgs field into the complex plane adding a Goldstone mode to the problem alongside the Mexican hat potential the conventional way up. There will also be implications for charged matter since moving charges generate magnetic fields, so the electromagnetic 4-potential must be generalised to reflect this.

Solitons are waves, but they exhibit particle properties; the ability to propagate indefinitely and interact with one another without change of form. It would be instructive to derive the equations for the scattering of these solutions and model their behaviour under such interactions.

Another important possible avenue of investigation is the mechanism by which states evolve into other states, for instance how excited states decay into lower states. At this point, the work done here makes no attempt to explain how this is achieved; the absorption and emission of photons is ignored entirely. Clearly then, an important addition to explain this would be a way for these radiative processes to take place.

This thesis has been concerned with a semiclassical system, classical gravity coupled to quantum fields, but if a quantum theory of gravity were found, it would be worthwhile trying to incorporate it with the Dirac, Higgs and Maxwell fields, to formulate a fully quantum theory of matter and gravity.
Appendices
In the introduction section of this thesis, the sign conventions present in GR were shown. We shall state here, for the sake of clarity, the sign conventions used by some papers in the literature. It should be noted that it is not always possible to determine all three sign conventions. Typically, one is able to determine either $s_1$, $s_2$, and $s_3$, or $s_1$ along with the product $s_2s_3$.

While the second of these two situations gives us less information than the first, it is nonetheless useful to know, and can help when comparing equations of motion from paper to paper. More immediately, it can be helpful for the reader of this thesis to understand any apparent sign discrepancies between equations stated here compared to papers in the literature.
## Appendix A. Sign Conventions in the Literature

<table>
<thead>
<tr>
<th>Source</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_2s_3$</th>
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<tbody>
<tr>
<td>Finster et al. (1999a)</td>
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<td></td>
<td></td>
<td>-1</td>
</tr>
<tr>
<td>Finster et al. (1999c)</td>
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<td></td>
<td></td>
<td>-1</td>
</tr>
<tr>
<td>Finster et al. (1999b)</td>
<td>-1</td>
<td></td>
<td></td>
<td>-1</td>
</tr>
<tr>
<td>Mannheim (2006)</td>
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<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>Mannheim (2009)</td>
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<td></td>
<td></td>
<td>-1</td>
</tr>
<tr>
<td>Mannheim (2012)</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>Brihaye &amp; Verbin. (2009)</td>
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<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>Schlögel et al. (2014)</td>
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<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Yepez (2011)</td>
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<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Throughout this thesis, many different solutions have been displayed, and with them, the parameter values. However, we shall collate this information in this appendix so that the reader need not trawl through the text and captions to find out parameter values for a certain solution. We will include, with this information, the figure to which it pertains to further simplify things.

### B.1 FSY ED

<table>
<thead>
<tr>
<th>Figure</th>
<th>$\alpha_1$</th>
<th>$\omega$</th>
<th>$M$</th>
<th>$m$</th>
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<tbody>
<tr>
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<td>0.63023</td>
<td>1.46824</td>
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</table>

### B.2 EDH

All EDH solutions have, $m_H = 1$, $\lambda = 10$, $\mu = 10$. 
Appendix B. Data for Solutions

<table>
<thead>
<tr>
<th>Figure</th>
<th>$\alpha_1$</th>
<th>$\omega$</th>
<th>$h_0$</th>
<th>$M$</th>
<th>$m$</th>
<th>Cutoff radius</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.082305</td>
<td>0.913948</td>
<td>0.44095</td>
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</tbody>
</table>

B.3 EDMH

All EDMH solutions have, $m_H = 1$, $\lambda = 10$, $\mu = 10$.

<table>
<thead>
<tr>
<th>Figure</th>
<th>$\alpha_1$</th>
<th>$\omega$</th>
<th>$h_0$</th>
<th>$M$</th>
<th>$m$</th>
<th>$q$</th>
<th>Cutoff radius</th>
</tr>
</thead>
<tbody>
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B.4 CDH

CDH solution has, $B_1 = 0$, $\lambda = 1.286$, $\mu = 1$.

<table>
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<th>Figure</th>
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</tr>
</tbody>
</table>

B.5 CDMH

CDMH solution has $B_1 = 0$, $\lambda = 1.275$, $\mu = 1$.

<table>
<thead>
<tr>
<th>Figure</th>
<th>$\alpha_1$</th>
<th>$\omega$</th>
<th>$h_0$</th>
<th>$\chi$</th>
<th>$q$</th>
</tr>
</thead>
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