

RUSSELL'S METAPHYSICAL ACCOUNTS OF LOGIC

Ryo Ito

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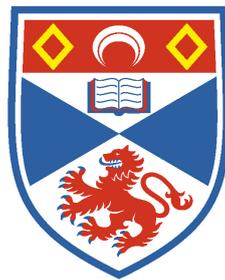
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Russell's Metaphysical Accounts of Logic

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University of
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This thesis is submitted in partial fulfilment for the degree of PhD
at the
University of St Andrews

30 June 2017

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Russell's Metaphysical Accounts of Logic

30 June 2017

Ryo Ito

Abstract: Bertrand Russell's works on logic, despite his reputation as a founder of mathematical logic, appear unnecessarily metaphysical and even naïve to contemporary logicians and philosophers. He offered several accounts of logic whilst pursuing the goal of *logicism*, the view of mathematics as reducible to logic. In their attempts to explain why those accounts look naïve nowadays, many commentators have sought one or another simple philosophical doctrine which can characterise his conception of logic. Instead of thus assuming a coherent theme underlying his works on logic, I propose to understand them as a shift from a conception of logic towards another. By looking into books, papers and manuscripts which he wrote during the period from 1898 to 1918, I argue that he inherited an antique, metaphysical conception of logic from his idealist predecessors and, through his attempts to replace some idealistic features of the conception with his realist alternatives, he became more sympathetic to—though never fully convinced of—a linguistic conception of logic, which was proposed by some of his contemporary logicians and has been widely accepted since then.

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Introduction

A historical perspective on logic

Bertrand Russell is counted as a founder of mathematical logic. But the contemporary reader of his works on logic will feel that the way in which he understood logic is not the same as the way we contemporary philosophers do. His logicist programme was set forth in *The Principles of Mathematics* (1903; henceforth, *PoM*), but it assigns a whole chapter to his metaphysical picture of reality, which appears unnecessary to those who are familiar with the set-theoretic reduction of mathematics to an axiomatic-set theory. His attempts to reduce mathematics to logic culminated in his collaborative work with Whitehead, *Principia Mathematica* (1910; henceforth, *PM*). But the voluminous book again contains a metaphysical account of judgments instead of a purely syntactical, recursive definition of the symbols and formulae of their formal system. It seems as though Russell understood logic to be something that could not be separated from and developed without metaphysics.

One may be tempted to dismiss Russell's metaphysical remarks on logic as a result of confusion. But it is, I believe, worth seeking a charitable account as to why he made those remarks, because such an account can cast light upon how we understand logic. An exegetical investigation into a past philosopher can make explicit some assumptions that we accept more or less unselfconsciously and do not even articulate. An encounter with something that does not fit our conceptual framework helps us to become aware of a totally distinct framework and to realise certain characteristics of our own framework. Russell's works on logic, which look odd to contemporary philosophers, can provide us with a better understanding of our standpoint on logic.

It is certainly reasonable to wonder whether we can legitimately speak of such a thing as *our* standpoint on logic. We have a lot of disagreements over logic. Some of us maintain that classical logic is the only logic, while others argue for a non-classical logic, some others endorsing pluralism of logics. However, whether one supports one of those logics or endorses pluralism, one will still find, I presume, Russell's works on logic bizarre. The divergence between his understanding of logic and ours seems far greater than the one between our

views of logic. It is useful to distinguish between a *view* of logic and a *conception* of logic here. A *view* of logic is an opinion as to what *logic* is which an individual or a group of individuals explicitly endorse. On the other hand, a *conception* of logic is an opinion as to what logic is which is shared by those who put forward distinct *views* of logic. A conception of logic is a presupposition upon which a whole discussion concerning various views of logic is based in the sense that it prescribes certain conditions which any opinion regarding logic should meet in order to count as a view of logic. This distinction is not a black and white one, for disagreements may come in degree. What counts as a conception of logic could become a view of logic when an opposing conception of logic becomes equally orthodox. But two opposing opinions count as conceptions of logic as long as each of them is taken for granted by a large group of philosophers and/or logicians and the two opinions are so divergent from each other that those who accept one of them cannot see the others as speaking of logic. The feeling I mentioned above suggests that contemporary philosophers subscribe to a certain conception of logic that makes Russell's works on logic look odd. In the present work, I hope to discern some characteristics of our conception of logic by casting light upon what underlies his mysterious remarks on logic.

However, the task which I wish to undertake cannot be a simple comparison between two distinct conceptions of logic, even though such attempts have indeed been made in the literature. For example, Hylton in his seminal work *Russell, Idealism, and the Emergence of Analytic Philosophy* proposes 'a view which emphasizes that a philosophical work is partly to be understood by understanding the philosophical context against which it was written' (Hylton, 1990, p.7). He thereby offers a 'historical understanding of our own philosophical position' (*ibid.*).¹ It is indeed his work that has persuaded me of the importance of a non-anachronistic approach to the history of philosophy. But most of those accounts including Hylton's vital work have sought a single notion or a philosophical doctrine in terms of which they can characterise Russell's understanding of logic. For instance some authors including Hylton attributed to Russell what van Heijenoort called the *universality* of logic (van Heijenoort, 1967b, p.324). According to van Heijenoort, Russell regarded his formal system as a universal language outside of which 'nothing can be, or has to be, said' (*ibid.*, p.326). This interpretation attracted not only Hylton but also some other prominent philosophers such as Goldfarb, Urquhart and Hintikka.² However, Proops has recently pointed out that the interpretation has various interpretational problems and fails to single out a simple theme by which we can contrast Russell's understanding of logic with our conception of logic (Proops,

¹ Hylton discusses how and to what extent we can carry out a non-anachronistic approach to the history of philosophy (Hylton, 1990, pp.3-4). For the discussion of a historical understanding of our own philosophical standpoint, see also Skinner (1969) and Rorty (1984).

² Goldfarb (1979, 2001), Urquhart (1988) and Hintikka (1988, 2010).

2007). Landini, who also opposes the universalist interpretation, tries to understand Russell's works on logic as his attempts to solve the set-theoretic paradox which bears his name as well as other paradoxes while adhering to the *doctrine of unrestricted variable* according to which the variables of a logical system must range over all the worldly items (Landini, 1998b). His interpretation has been developed by Stevens and Klement amongst others.³ However, although these authors offer cogent arguments against the universalist interpretation, their own interpretations are subject to one of the points that Proops makes: there are some works where Russell does not hesitate to give up the doctrine, readily stratifying his ontology (Proops, 2007, pp.3-7).

Russell is known to have frequently changed his own positions on various topics, and his position on logic was no exception. He offered several mutually incompatible accounts of logic during the period from 1898 to 1919—the decades that witnessed him making 'honest toil' on logic (*Introduction to Mathematical Philosophy*, p.71). It seems to me that a truly faithful interpretation of those accounts of logic can only be obtained by observing that he revised his understanding of logic so drastically that he cannot even be seen to have subscribed to a single *conception* of logic. This is why I have avoided using the phrase 'Russell's conception of logic' in favour of his 'understanding of logic.' I do not of course claim that all the commentators have neglected the changes that he made to his own position on logic. But those changes certainly merit more attention than has been paid thus far. I shall not, therefore, aim at a simple comparison between two conceptions of logic.

In light of this I should like to depict Russell's works on logic as a sort of movement from an old conception of logic towards the one that we contemporary philosophers accept. It seems to me that he inherited an antique conception of logic from his idealist predecessors and, through his attempts to replace some idealistic features of the conception with his realist alternatives, became more sympathetic to—though never fully convinced of—a linguistic conception of logic, which itself was first proposed by some of his contemporary logicians and has been widely accepted since then. To illustrate this point is the primary aim of the present work.

This aim wants a non-anachronistic approach to Russell's works on logic. When philosophers seek to get inspired by historical texts, it does not really matter whether their ways of reading the texts presuppose certain notions which were not available when they were written. By contrast, the aim of the present work cannot be reached by any anachronistic approach. If we employ concepts of mathematical logic, for example, to interpret Russell's remarks on logic, the resulting interpretation will not reveal the whole difference between his way of understudying logic and ours. There are of course some cases where we can

³ Stevens (2005) and Klement (2003, 2010).

invoke contemporary notions to interpret them without blurring the crucial difference. But in other cases it is necessary to understand his remarks *in his terms*. This is not an easy task because a straightforward way of grasping a past philosopher's idea is to articulate it with notions familiar to us. But it is, I think, also possible to understand a philosophical theory by familiarising ourselves with technical terms and with how they are employed to formulate the theory. I will sometimes introduce Russell's technical terms merely by indicating the way he uses them to present a whole theory. By so doing I hope to offer a non-anachronistic interpretation of his works on logic.

The success of the present work will not depend on whether or not it manages to discover an idea or argument that we contemporary philosophers find useful. But one might contend that unless we obtain something of philosophical importance from the historical material, it is not worth our philosophical investigation. But in which sense of the word 'philosophical'? Importantly, the word has what Rorty calls a 'honorific' use (Rorty, 1984, p.58). It is sometimes used to claim that philosophers *should* discuss some topics and *should not* be concerned with others. This is a sort of normative claim, which is, I think, often proclaimed solely on the grounds that a majority of philosophers, *i.e.*, those who are customarily regarded thus, are interested in the former rather than in the latter. As Hylton puts it, such a claim seems to 'display an unjustifiable dogmatism about what philosophy is, and about one's own right to speak for it' (Hylton, 1990, p.13). I do not claim that it is impossible to defend such a claim, even though it may be the case that if one seeks to argue for it, one may have to be engaged in a kind of historical investigation—what Rorty calls *canon formation* (Rorty, 1984). The notion that the philosophical importance of the history of philosophy lies solely in its possibility to inspire present-day philosophers may itself be, borrowing Skinner's phrase, one of 'our contingent arrangements' (Skinner, 1969, p.53). Put otherwise, the notion may be a part of our *conception of philosophy*.

Ironically enough, Russell has been influential in spreading the notion in question. In *A Critical Exposition of the Philosophy of Leibniz* (1900; henceforth *Leibniz*), he remarks as follows:

Questions concerning the influence of the times or of other philosophers, concerning the growth of a philosopher's system, and the causes which suggested his leading ideas—all these are truly historical: they require for their answer a considerable knowledge or the prevailing education, of the public to whom it was necessary to appeal, and of the scientific and political events of the period in question. But it may be doubted how far the topics dealt with in works where these elements predominate can be called properly philosophical.

(*Leibniz*, p.x)

To be precise, Russell here mainly opposes the approach which pays 'so much attention to

the *relations* of philosophies' that it does not go into the contents of their thought. This approach is not what I endorse. But he also claims that the history of philosophy can only be of philosophical importance when it provides us with 'philosophic truths' about various philosophical 'systems' as opposed historical facts about 'the actual views of the philosopher' (*Leibniz*, pp.x-xi). This claim has arguably affected many philosophers' attitudes towards the history of philosophy. As Hunter among others points out, Russell's book on Leibniz has been notably influential in shaping 'the analytic style of doing the history of philosophy' (Hunter, 1999, pp.1-2).

It is thus ironical that Russell's various notions concerning logic, which look completely odd to us, make much more sense—or so I will attempt to show—when we put them in their historical contexts and try to discern his 'actual views.' In the rest of this introduction I will indicate what I call the *idealist conception of logic* and the *linguistic conception of logic* respectively and sketch how Russell's position moved from the former towards the latter.

Two conceptions of logic

What I shall call the idealist conception of logic was widely accepted around the end of the 19th century. British Idealists assumed that logic was composed of two branches, what I shall call the *theory of logic* and the *calculus of logic*. The former was regarded as a metaphysical enquiry into the foundation of the latter. To be fair, we are familiar with a similar distinction: we sometimes use such phrases as 'philosophy of logic' or 'philosophical logic' to distinguish philosophical or foundational enquiries into logic from mathematical logic, a purely technical enterprise. What is distinctive about the idealists' standpoint on logic is rather the importance they attached to the theory of logic. It seems as though they presupposed that their calculus of logic—syllogistic logic in effect—would remain imperfect unless they supplemented it with some philosophical foundation. The theory of logic was an essential part of the whole discipline even to the extent that the calculus of logic was considered theoretically dependent on it. Furthermore, they also seem to have understood the theory of logic as a metaphysical investigation into how judgments are possible and what they are. That is, their theory of logic was the theory of judgment. The idealist conception of logic can be characterised by these two points: the indispensability of the theory of logic and the status of the theory of judgment as the theory of logic.

For example, Bradley, arguably the most prominent figure among British Idealists, devoted a voluminous book to his account of judgment as well as inference and titled it *The Principles of Logic* (1883; henceforth, *PoL*). He did so because he thought his enquiries into judgments 'were

mainly logical, and, for logic at least, must be fundamental' (*PoL*, p.iii). He viewed, as Mander puts it, logic as 'the science of knowledge,' which was 'concerned with determining how adequate various forms of thought are to express and convey truth, something determined in part by the practice of thinking itself—thought in its very nature sets its own goals and critical standards—but something determined equally by the nature of reality' (Mander, 2011, p.283). Quite a similar view of logic was endorsed by another prominent idealist Bernard Bosanquet, who offered his own theory of judgment in the book entitled *The Essentials of Logic* (1895). It would thus seem that these two prominent philosophers agreed that the theory of judgement was an integral part of logic.

Hylton points out that "'Logic," as the Idealists understood it, was in large measure the heir of Kant's Transcendental Logic, though usually with some discussion of the valid patterns of reasoning appended' (Hylton, 1990, p.75). Hylton offers a convincing account of how British Idealists came to hold what I call the idealist conception of logic:

Transcendental Logic was, among other things, an enquiry into the conditions of synthesis, and thus into the presuppositions of any synthetic judgement. Since Kant thought of Formal Logic as made up of analytic judgements, i.e. judgements not produced by synthesis and so not subject to these presuppositions, the truths of Formal Logic thus seemed to be wholly independent of Transcendental Logic. This independence, however, proved tenuous. Even the most Kantian of Kant's successors came to find it implausible that there could be any judgements at all which did not require synthesis, and thus that there could be any judgements at all which are not subject to Transcendental Logic. Formal Logic thus seemed to presuppose Transcendental Logic: before we can consider the formal, logical relations among judgements, we must first discover the necessary conditions of all judgement, for these conditions must be obeyed throughout. Given the poverty of the Formal Logic of the period, its dependence upon Transcendental Logic seemed to leave it in danger of merging into the latter entirely—if you understand the necessary conditions of judgement, then the logical relations among judgements follow more or less automatically.

(Hylton, 1990, pp.74-75)

Kant, as Hylton points out, called his theory of judgement 'Transcendental Logic' and his followers argued that it precedes Formal Logic, a study of valid patterns of inference. According to Hylton, British Idealists accepted the indispensability of the theory of logic even to the extent that it was considered capable of yielding the correct calculus of logic. Hylton refers to the following passage in Kemp Smith's well-known commentary on Kant:

Synthetic, relational factors are present in *all* knowledge, even in knowledge that may

seem, on superficial study, to be purely analytic [...]. This is the reason why, in modern logic [...] the theory of judgement receives so much more attention than the theory of reasoning.

(Kemp Smith, 1918, p.xxxviii)

It seems that British Idealists thus came to understand the theory of judgment as the theory of logic, on which the calculus of logic depended.

It is perhaps not precise to use the term ‘the idealist conception of logic,’ partly because it does not seem to have been held by Kant and his idealist predecessors, but more importantly because some of Russell’s contemporaries who we hardly count as idealists seem to have subscribed to this conception of logic. For instance, in Baldwin’s *Dictionary of Philosophy and Psychology*, Peirce first defines logic to be ‘the classification of arguments, so that all those that are bad are thrown into one division, and those which are good into another, these divisions being defined by marks recognizable even if it be not known whether arguments are good or bad’ (Peirce, 1902, p.21). Peirce calls this enterprise ‘critic’ and then remarks:

It is generally admitted that there is a doctrine which properly antecedes what we have called critic. It considers, for example, in what sense and how there can be any true proposition and false proposition, and what are general conditions to which thought or signs of any kind must conform in order to assert anything.

(Peirce, 1902, p.21)

Pierce adds that the consideration of this kind is sometimes called ‘Epistemology’ (*ibid.*). He thus accepted—and thought that his colleagues would also agree with—the indispensability of the theory of logic to the extent that it could be said to ‘properly antecede’ the calculus of logic.⁴ It would thus seem that the idealist conception of logic was more widely accepted than the name suggests.

The idealist conception of logic looks quite different from the way we understand logic. In this sense it can indeed count as a *conception* of logic, which in turn implies that our way of understanding logic is also a conception of logic. The idealist conception of logic thus indicates ‘a kaleidoscope of different ways to think about reasoning, a diversity of understandings of the very nature and purpose of “logic” itself’ (Mander, 2011, p.275). Indeed, the two characteristics of the idealistic conception of logic indicate two distinctive features of our conception of logic. Philosophers at the end of the 19th century took it for granted that the theory of logic was an integral part of the discipline while the calculus of logic was dependent upon it. By contrast, we do not believe that mathematical logic, the counterpart of

⁴ It is presumably in this sense of the word that Russell distinguishes ‘Epistemology’ from ‘Psychology’ in *Leibniz* (§98).

the calculus of logic, wants a metaphysical enquiry into judgments or into entities of any kind. We can of course develop a certain metaphysical theory to found our calculus of logic, but we do not think it indispensable. Furthermore, even when we seek a philosophical ground of mathematical logic, we will not try to obtain it by scrutinising the nature of judgment. On the contrary, if someone presents a new metaphysical account of how our judgment is possible, most of us will not count it as a new 'logic.' Hence, the status of the theory of judgment as the theory of logic has also been lost. In my view, the notion of sentence has replaced that of judgment. Mathematical logic is a study of formal languages and their models. So, in logic, we are not concerned with judgments but with sentences or more precisely with recursively defined formulae. If Kemp Smith's diagnosis is correct, one of the working assumptions behind Kant's followers' argument for the dependency of the calculus of logic upon the theory of logic was that vehicles of inferences were judgments. By contrast we seem to understand sentences as bearers of inferential powers.

I submit that our conception of logic can thus be characterised by these two features: the independence of our calculus of logic from any metaphysical enquiry and the notion of sentences as bearers of inferences. It follows that our conception of logic is *linguistic* in nature, since it is arguably based upon what I shall call the *linguistic approach to formal languages*, namely, the notion that formal languages themselves are the subject of our logical enquiry. When we examine a formal language, we are only concerned with properties of the language which follow from the recursive definition given to the language; we do not consider metaphysical properties, as it were, of the language. This explains why we do not regard mathematical logic as dependent upon any philosophical enquiry. There may be other features of our conception of logic that underlie the idealist conception of logic as well and hence cannot be detected in light of the latter. Yet, since those two features are exclusive to our conception of logic, it can be called the *linguistic* conception of logic.

A contrast similar to the one between the idealist conception of logic and the linguistic conception of logic has already been indicated by Hacking. According to him 'philosophers nowadays would say about "sentence" what Port Royal once said about "ideas"' (Hacking, 1975, p.159). 'Connections between ideas were,' he explains, 'expressed in mental discourse, and formed representations of reality in response to changes in the ego's experience and reflection' whereas '[i]n today's discussion, public discourse has replaced mental discourse' (*ibid.*). In his view, the 17th century was 'the heyday of ideas,' while from the middle of the 18th century philosophers experienced 'the heyday of meanings,' which was followed by 'the heyday of sentences' in the twentieth century. Plausible as this story is on the whole, however, it seems to have problems with its division of time into these epochs and with its classification of philosophers into the three epochs. One is that although Russell is categorised as a philosopher of meaning, he did not understand meaning in the same way as Frege—a

typical philosopher of meaning in Hacking's view—did.⁵

Yet, the idea of explicating a shift in a philosopher's position or in a generation's conception in terms of some key concepts is indeed illuminating. The idea can be applied in my view not only to the contrast between the idealist conception of logic and the linguistic conception of logic but also to the shift in Russell's understanding of logic from the former towards the latter. He replaced the notion of *judgement*—a key concept of the idealist conception of logic—with his realist alternative, the notion of *proposition*, which he was to give up in favour of that of *fact* by the time he and Whitehead finally published *PM*. The publication paved the way for the linguistic conception of logic by providing logicians, though in a somewhat sloppy manner, a fully-developed formal system—a structured set of *sentences*. I shall sketch below Russell's shift from judgments, through propositions and then facts, towards sentences. The sketch will also indicate how the following chapters are intended to contribute to the chief aim of the current work.

From Judgment to Proposition

Russell started his career as an idealist philosopher. It seems to me that even when Moore led him to turn away from British Idealism around 1898, he still accepted, at least partially, the idealist conception of logic. In Chapter 1, I will attempt to illustrate this point by showing that a certain argument which he offers in *Leibniz* against British Idealism presupposes a notion of 'judgment' that shares some features with Bradley's notion of 'judgment.' In *Leibniz*, Russell argues that Leibniz's monadism contradicts the 'traditional logic,' which Leibniz arguably endorsed. The traditional logic is in Russell's view the notion that '[e]very proposition is ultimately reducible to one which attributes a predicate to a subject' (*Leibniz*, §8). He proceeds to argue that the traditional logic implies that there can be at most one substance, contradicting Leibniz's contention that the universe is composed of numerous substances or *monads*. This argument requires us, or so I will argue, to understand what Russell calls a 'judgment' in the book as an object that corresponds to one's act of judging whether it is deemed true or false. Bradley in *PoL* tries to establish that one's judgment is 'the act which refers an ideal content (recognized as such) to a reality beyond the act' (*PoL*, §10). When thus defined, each judgment attributes its content to reality, whether the judgment itself is

⁵ As I will argue in Chapter 1 and Chapter 7, Russell understood language as essentially private, which should, in Hacking's terminology, disqualify him as a philosopher of meaning. Martinich makes the same point in his review of the work of Hacking's (Martinich, 1977). In fairness, Hacking himself expresses reservations about the classification (Hacking, 1975, pp.172-3).

said to be true or false; otherwise, the act does not even count as a 'judgment.' This notion is idealistic in that it grants a subjective, mental act of judging an objective role of shaping reality. It should be noted that either from Bradley's theory of 'judgment' or from Russell's notion of 'judgment' in *Leibniz*, it follows that one's act of judging has a corresponding part of reality regardless of the truth or falsity of the judgment. Furthermore, both Bradley in *PoL* and Russell in *Leibniz* think of their notions of 'judgment' as *logical* notions. The former understands the notion of 'judgment' as a crucial subject of his work on 'principles of logic,' whereas the latter realist employs the notion of 'judgment' to articulate what he calls the 'traditional logic.' Russell was thus under the influence of the idealist conception of logic even after he revolted against British Idealism.

In setting forth his logicist programme in *PoM*, however, Russell comes to reject one of the two characteristic notions of the idealist conception of logic. In *PoM*, he develops what I shall call the *ontology of propositions*, understanding propositions as mind-independent complexes that subsist or 'have *being*.' By so doing he intends to reject the idea that the theory of logic investigates the nature of judgment, and suggests that it should rather be concerned with propositions, the objects of judgments. He thus deprives the idealists' theory of judgment of the celebrated status of the theory of logic, granting it to his own realist ontology of propositions.

On the other hand, Russell still accepts the indispensability of the theory of logic. For he invokes the ontology of propositions so as to offer philosophical accounts of technical notions such as *class* and *propositional function*. He thus still finds the theory of logic indispensable. Hence, his rejection of the idealist conception of logic was not a sheer denial of it. Rather, it resulted from his attempts to replace various idealistic notions with their realist alternatives *within* the framework of inherited from the idealist conception of logic. I will illustrate this point in Chapter 2.

However, the indispensability of the theory of logic became less apparent to Russell than to his idealist contemporaries. For he learnt from Peano the overwhelming successfulness of predicate logic, which led him to take a more flexible position on the theoretical dependence of the calculus of logic upon the theory of logic. In the preface of *PoM*, he remarks that he assumes 'the non-existential nature of propositions (except such as happen to assert existence) and their independence of any knowing mind' (*PoM*, p.xviii). He goes on to speak of these assumptions as follows: 'the fact that they allow mathematics to be true, which most current philosophers do not, is surely a powerful argument in their favour' (*ibid.*). He thus thinks that the success of the calculus of logic can vindicate the theory of logic even though he still maintains that the former needs grounding in terms of the latter. Unlike his idealist contemporaries, he does not simply think that the correct calculus of logic will automatically follow from the correct theory of logic. Thus, for Russell, the theory of logic and the calculus

of logic are far more intertwined than his idealist predecessors once believed.

This intertwining relation of the theory of logic and the calculus of logic in Russell's perspective is, I think, the key to understanding his sinuous works between *PoM* and *PM*. For it explains the tight connection, which underlies those works, between technical considerations as to how he could avoid various paradoxes including the set-theoretic paradox and philosophical considerations as to what classes and propositional functions are at all. The connection will be examined in three chapters.

From The Principles of Mathematics to Principia Mathematica

Russell could not proceed to write the promised second volume of *PoM* immediately after the publication of the first—and last—volume of the book despite the fact that he then already invented a technically feasible solution to the set-theoretic paradox, the very first version of the theory of types. In my view, the importance attached to the theory of logic explains why. In Chapter 3, I will argue that it was not just the set-theoretic paradox that prevented him from undertaking the writing of the subsequent volume of *PoM*; rather, he understood the paradox as a *proof* that his philosophical account of classes was erroneous. He could not write the second volume because he was yet to have a correct theory of logic, which he took to be an indispensable part of logic.

From 1903 onwards, Russell, instead of seeking a plausible account of classes, made various attempts to develop a philosophical account of propositional functions as well as other kinds of functions. What he was after was a philosophically feasible account of functions that could motivate his then solution to the set-theoretic paradox and its variants. Those attempts eventually led him to discover the *theory of descriptions*, which was first presented in his celebrated paper 'On Denoting' (1905; henceforth, OD). In Chapter 4, I will discuss those attempts in order to show that he envisaged the tight connection between the theory of logic and the the calculus of logic and that one of those attempts led him to invent the theory once deemed as the 'paradigm of philosophy.' Those attempts have attracted little attention in the literature, and so, I will provide sufficient details of them to grasp how he developed one account of functions after another.

In Chapter 5, I shall first look into some manuscripts which Russell wrote subsequently to OD in preparation for the promised volume, which eventually came out as *PM*. After OD, Russell developed the *substitutional theory*, which treats function-symbols and class-symbols as *incomplete symbols*, namely, symbols which do not designate any entities on their own but can still be meaningfully employed in proper contexts. He was, however, to abandon the

theory by 1907 and it has long been one of the most contentious subjects in the literature what he then adopted and what he and Whitehead call 'propositional functions' in *PM*. In Chapter 5, I will endeavour to show that the manuscripts and *PM* can be given a coherent interpretation by focusing upon the tight connection between the theory of logic and the calculus of logic. One might expect a whole chapter or possibly more to be devoted to the discussion of the ramified theory of types in *PM*. But I shall confine myself to the problem—which I think illustrates the connection well—as to what propositional functions are considered to be in the voluminous book, putting aside many other exegetical problems concerning the work.

In *PM*, Russell and Whitehead explicitly endorse the *multiple-relation theory of judgment*, seemingly abandoning the ontology of propositions. This marks an important turning point in Russell's shift from the idealist conception of logic towards the linguistic conception of logic. But the intertwining of technical considerations and metaphysical enquiries does not, as I will argue in Chapter 5, answer the question why he abandoned the ontology of propositions, his realist alternative to the idealist theory of judgment. To see why, it is necessary to turn, as Russell did, to a purely philosophical investigation into the nature of truth.

From Proposition through Fact towards Sentence

It has been customary in the literature to think that Russell's abandonment of propositions was due to a problem internal to the ontology of propositions. In Chapter 6, I shall attempt to indicate that the existing accounts are not entirely satisfactory, before I offer my own interpretation. It seems to me that he gave up the ontology of propositions simply because of his feeling that *facts* are more preferable to propositions—the feeling which he later called 'a vivid sense of reality' in *The Philosophy of Logical Atomism* (1918; henceforth, *PLA*). I will argue that he felt that *facts* were more realistic as opposed to idealistic than propositions, and that he did so because of a commonality between his notion of proposition and Bradley's notion of judgment. As we saw above, what he calls a 'proposition' in *Leibniz* is supposed to be a part of reality, whether the judgment is considered true or false. The same applies to his ontology of propositions presented in *PoM*: whether our judgment is true or false, the object of the judgment, a proposition, is supposed to subsist in reality. Hence, even the ontology of propositions, which was meant to be a replacement of Bradley's theory of judgement, still inherited a characteristic of the latter. The step from judgment to proposition was not an entirely realistic move after all. Russell found it necessary to take a further step from proposition to fact.

When the step was indeed taken, Russell rejected the indispensability of the theory of logic

in effect. As an alternative to the notion of proposition, he introduced the notion of *form of fact* so that he could understand the theory of logic as an investigation into the forms of facts. But the adoption of the notion had a significant consequence on the status of the theory of logic. He noticed that the notion could be applied at most to the meanings of atomic sentences. For if, say, ' $\neg p$ ' were to stand for a complex fact in which the fact p occurs, it would follow that both p and $\neg p$ were facts. The notion of form of fact is thus applicable only to atomic sentences. Hence, viewing the theory of logic as an investigation into the forms of facts, he could no longer hold that the theory of logic would offer a foundation of the whole calculus of logic. At this point, he effectively rejected the indispensability of the theory of logic.

A mere coincidence or not, Russell's rejection of the indispensability of a metaphysical enquiry into the foundation of logic paralleled the rise of what I call the linguistic approach to formal systems. Some of his contemporary logicians began to prove theorems *on* formal systems including, of course, the one presented in *PM* without asking what those systems should express or seeking any metaphysical ground for treating them as *logical* systems. The separation of mathematical logic from metaphysical enquiries certainly helped then logicians to concentrate on investigating mathematical properties of formal languages. In this respect his standpoint on logic came close to the linguistic conception of logic. But he could not accept it after all. I will attempt to explain why he could not in Chapter 7.

I have sketched how Russell replaced the notion of judgment with that of proposition and then with that of fact, thereby making a shift from the idealist conception of logic towards the linguistic conception of logic. I will explicate this overall story in what follows.

1 The Idealist Notion of Judgment

1.1. Introduction

In this chapter I shall endeavour to show that Bradley and Russell once shared a certain understanding of *judgment*, which indicates their tacit acceptance of the idealist conception of logic. I will first introduce Russell's argument that what he calls the subject-predicate doctrine implies monism, namely, the claim that there is only one entity. I will then look into how he appeals to the argument to refute *monistic idealism*, that is, the view that reality as opposed to appearance is a single, all-encompassing experience called 'the Absolute.' By so doing I will point to two distinctive features of what he understands by 'judgment' in *Leibniz*. I will then turn to Bradley's theory of judgment developed in *The Principles of Logic* and in *Appearance and Reality* so that I can show that it also exhibits those features.

1.2. Russell's interpretation of Leibniz's monadism

Russell's ultimate diagnosis of Leibniz's metaphysics presented in *Leibniz* is that his monadology contradicts the assumption, which Leibniz arguably accepts, that 'every proposition has a subject and a predicate' (*Leibniz*, §4). He holds that although Leibniz endorsed a pluralism of substances maintaining that there are numerous substances or *monads*, the assumption implies that there can be at most one substance. I shall use, as Russell does, the term 'subject-predicate doctrine' to speak of this assumption. In this section I will illustrate that he attributes the doctrine to Leibniz and to Bradley amongst others and how he invokes it in his attempt to reject the latter's monistic idealism.

Let us begin by a brief introduction of the subject-predicate doctrine. Russell presents the doctrine as 'the belief that propositions must, in the last analysis, have a subject and a predicate' (*Leibniz*, §10). The doctrine thus presented wants clarifications as to what he means by 'propositions,' 'in the last analysis' and 'have a subject and a predicate' respectively. As for the first one, some of its occurrences in the book are interchangeable with 'sentence.' For example, if 'propositions must have a subject and a predicate,' Russell remarks, 'a proposition without a subject and a predicate must be no proposition, and must be *destitute of meaning*' (§10; emphasis added). He apparently sees propositions as bearers of *meaning*, though he does not explicate what he thinks meaning is. He also uses the word 'proposition' interchangeably

with 'judgment' (see, for example, §10). This may come as surprise, since in *PoM* and in subsequent works he draws a sharp distinction between judgments and propositions or between the act of making a judgment and the content of the judgment. As the distinction is also vital for our purposes, I will use the terms 'judgmental act' and 'judgmental content' to speak of the act of making a judgment and the content of it respectively. In order to tell exactly what he means by 'judgment' or 'proposition' in *Leibniz*, we need to look into what I shall call the *indistinguishability argument*, that is, his argument that the subject-predicate doctrine makes substances numerically indistinguishable from each other. But this is the task of the next section; for the moment it suffices to note that the subject-predicate doctrine is concerned with forms of judgmental contents. When Russell considers the doctrine that every judgment in fact has 'a subject and a predicate,' he has in mind the claim that every judgmental content has an entity and a property *and nothing else*: it is composed of an object and a property attached to it. And the qualification 'in the last analysis' suggests that there is a gulf between what we normally understand as the form of a judgment and its genuine form. The qualification is needed for an obvious reason that there are judgments whose contents are seemingly not composed of a single object and a property attached to it. He employs the term 'reducible' in such a way that if the subject-predicate doctrine is correct, every judgment is *reducible* to one whose content is composed of an entity and a monadic property attached to it. He attributes to Leibniz a simple way of reducing a judgment to another: for each judgment of any form, there is one of the subject-predicate form that is equivalent to the original one. However, he does not claim that this is *the* way of reducing sentences. He is aware that Bradley, as we will see in Section 1.4, appeals to another way of doing so.

Russell claims that not only Leibniz but also monistic idealists in general have taken the subject-predicate doctrine for granted:

The view that a subject and a predicate are to be found in every proposition is a very ancient and respectable doctrine; it has, moreover, by no means lost its hold on philosophy, since Mr Bradley's logic consists almost wholly of the contention that every proposition ascribes a predicate to Reality, as the only ultimate subject. [...] In the belief that propositions must, in the last analysis, have a subject and a predicate, Leibniz does not differ either from his predecessors or from his successors. Any philosophy which uses either substance or the Absolute will be found, on inspection, to depend upon this belief.

(*Leibniz*, §10)

Russell makes a plainly bold claim here that all philosophers preceding him based their metaphysics whether explicitly or implicitly upon the subject-predicate doctrine. But he does not do so without any reasons. He argues that when properly articulated, the notion of

substance, which has indeed been a central notion since ancient times, involves the doctrine in question:

Leibniz perceived [...] that the relation to subject and predicate was more fundamental than the doubtful inference to independent existence [...]. He, therefore, definitely brought his notion of substance into dependence upon this logical relation.

(Leibniz, §17)

The idea is that a self-subsistent entity is what always appears as a subject of a judgment. The definition of substance as 'that which can only be subject, not predicate' indeed presupposes the notion of subject and predicate. This does not, of course, mean that the definition also involves the subject-predicate doctrine. Yet, he moves on to argue as follows:

Change implies something which changes; it implies, that is, a subject which has preserved its identity while altering its qualities. This notion of a subject of change is, therefore, not independent of subject and predicate, but subsequent to it; it is the notion of subject and predicate applied to what is in time. It is this special form of the logical subject, combined with the doctrine that there are terms which can only be subjects and not predicates, which constitutes the notion of substance as Leibniz employs it.

(Leibniz, §17)

It is not clear what Russell means by 'qualities' here, but if we interpret them as monadic properties then he indeed understands Leibniz's account of change as involving the subject-predicate doctrine. This is why Russell defines a substance to be 'that which can only be subject, not predicate, which has many predicates, and *persists through change*' (§17; emphasis added). He thus maintains that the subject-predicate doctrine is presupposed in the notion of substance. In his view, it is impossible to reject the doctrine and at the same time invoke the notion of substance.

Russell goes on to argue that the notion of the Absolute also derives its apparent plausibility (to his contemporaries) from the subject-predicate doctrine. As we will discuss in the next section, he argues that the subject-predicate doctrine entails that there is at most one substance. According to him, the indistinguishability argument underlies monistic idealism. The idea is that if anything that counts as real has to be a substance and if the indistinguishability argument is valid then the subject-predicate doctrine implies that there is at most only one thing that is real. It is implausible that any philosopher who subscribed to the notion of the Absolute relied on the indistinguishability argument. In fact, although Russell speaks as if his argument applies to monistic idealism in general, he has mainly in mind Bradley's

view, which is, as Candlish points out, rather idiosyncratic (Candlish, 2007, pp.141-2).¹ I will employ the term ‘monistic idealism’ only when the discrepancies between Bradley’s view and other versions of monistic idealism do not matter. It is questionable, as we will see in Section 1.4, whether Bradley’s monistic idealism depends upon the subject-predicate doctrine, even though he indeed argues for the doctrine. It seems as though Russell believes that any argument other than the indistinguishability argument in favour of monistic idealism will, ‘on inspection,’ turn out to be flawed.

Russell proceeds to reject the doctrine by indicating two kinds of propositions or judgments which are not reducible to those of the subject-predicate form, *i.e.* to those whose contents are composed of an entity and a monadic property attached to it. He considers a simple notion of reducibility such that a proposition is *reducible* to another if one implies the other and *vice versa*. The ‘plainest instances’ of such propositions are in his view those which assert numbers, *e.g.* ‘There are three men.’ ‘Such propositions,’ he explains, ‘cannot be regarded as a mere sum of subject-predicate propositions, since the number only results from the singleness of the proposition, and would be absent if three propositions, asserting each the presence of one man, were juxtaposed’ (*Leibniz*, §10).²

Another sort of counterexamples to the subject-predicate doctrine comes, of course, from propositions which assert relations: ‘Again, we must admit, in some cases, relations between subjects—*e.g.* relations of position, of greater and less, of whole and part’ (*ibid.*). Russell introduces a passage where Leibniz observes that any attempt to find a monadic property which is equivalent to an asymmetric relation ends up either with a judgment asserting another asymmetric relation or with a judgment with two subjects and a monadic predicate attached to them. Leibniz dismisses the latter option as ‘contrary to the notion of accidents’ and argues that since a relation cannot be properly expressed by a judgment of the subject-predicate form, it ‘must be a mere ideal thing, the consideration of which is nevertheless useful’ (*ibid.*). According to Russell, Leibniz is right to the extent that he notes that relational expressions are not reducible to monadic predicates, though he fails to draw the conclusion that relations are real. Even in his later works, Russell mentions asymmetric relations as counterexamples to the subject-predicate doctrine (*PoM*, §215; *OKEW*, pp. 48-50). In these works he presents a more convincing argument against the possibility which Leibniz dismisses as ‘contrary to the notion of accidents.’ If, say, ‘*a* is greater than *b*’ is equivalent to ‘*a* and *b* have a such-and-such property,’ then, Russell argues, the latter should be equivalent, if not identical, to ‘*b* and *a*

¹ In fairness, the most notable divergence of Bradley’s metaphysics from others’ is not concerned with the subject-predicate doctrine. The former differs from those of Green and Bosanquet among others in that it explicitly denies, while the other two hold, that thought can fully capture reality. For discussion of the difference between Bradley’s metaphysics and Green’s, see Hylton (1990, pp.57-9, 69-70). Mander discusses commonalities and differences between Bradley’s theory of judgment and Bosanquet’s (Mander, 2011, pp.309-12).

² This observation is developed in *PoM* into a philosophical account of classes (cf. Chapter 3).

have a such-and-such property,' which in turn is equivalent to 'b is greater than a.' Thus, Russell's well-known emphasis on external relations stems from one of his objections to the subject-predicate doctrine.

Given what we have seen, the reason why Russell in *Leibniz* pays much attention to the subject-predicate doctrine is not just that he finds it incompatible with the reality of relations. Hylton seems to construe the doctrine as equivalent to the rejection of relations in explaining why Russell repudiates the subject-predicate doctrine (Hylton, 1990, p.155). But, as we have just seen, Russell in *Leibniz* thinks that relational propositions constitute only one, though important, category of propositions which are not reducible to those of subject-predicate form. He thus does not maintain that the falsity of the subject-predicate doctrine implies that relations are entities. It is true that relational propositions become the sole kind of counterexamples as he comes to endorse later the *no-classes theory*, according to which any sentence with an occurrence of a class-symbol can be translated into one without. But, in *Leibniz*, he does have two kinds of counterexamples and hence does not regard the subject-predicate doctrine as equivalent to the unreality of relations.

It seems to me that Russell in *Leibniz* finds the importance of discussing the subject-predicate doctrine in that by discerning and then rejecting it, he can develop a new pluralist metaphysics that will not suffer, as Leibniz's monadism did, from contradictions arising from the doctrine. As we will see shortly, the indistinguishability argument establishes, in his view, that the doctrine is incompatible with monadism or any pluralism. Furthermore, he thinks that unless the subject-predicate doctrine is rejected, no tenable account of space can be provided. He holds that monadism requires the reality of *space* to accommodate multiple entities but 'for a philosophy of substance, it is essential to disprove the reality of space' (*Leibniz*, §65). He argues that once the subject-predicate doctrine is granted, one can disprove, as Leibniz indeed did, the reality of space: 'The relation [...] between place and the object occupying it, is one for which the traditional logic had no room' (*ibid.*; cf. §66).³ On the other hand, Russell finds that Leibniz's argument against the monistic view of space as an attribute of the substance is also 'fairly strong' (§65). He thus concludes:

The confusions into which Leibniz falls are the penalty for taking extension as prior to space, and they reveal a fundamental objection to all monadisms. For these, since they work with substance, must deny the reality of space; but to obtain a plurality of coexistent substances, they must surreptitiously assume that reality. Spinoza, we may say, had shown that the actual world could not be explained by means of one substance; Leibniz showed that it could not be explained by means of many substances. It became

³ I will discuss another argument to the same effect in Section 2.3.

necessary, therefore, to base metaphysics on some notion other than that of substance—a task not yet accomplished.

(Leibniz, §71)

In his view, the subject-predicate doctrine not only fails to accommodate relations but also makes it impossible to offer any tenable account of space. In other words, he regards the doctrine not just as an obstacle to accepting relations but as a faulty framework within which no coherent account of space can be offered. In the next chapter we will see that he in *PoM* undertakes the ‘task’ by replacing the notion of *substance* with that of *term*.

1.3. The Indistinguishability argument

We have seen that Russell in *Leibniz* attempts to undermine monistic idealism by refuting the subject-predicate doctrine, on which he claims monistic idealism is based. Let us turn to the indistinguishability argument, which he offers to show that the subject-predicate doctrine entails monism. The argument requires, or so I will argue, that each judgmental content should not belong (only) to the judging mind but rather to the external reality and that this should be the case with ‘false’ judgments as well as true ones.

Let us begin by looking at how Russell draws the monistic conclusion from the subject-predicate doctrine. First of all, he thinks that although Leibniz offered several arguments for the identity of indiscernibles, the most cogent one was from the subject-predicate doctrine combined with the view of entity as substance.⁴ The identity of indiscernibles is here understood as such a strong claim that if there are two numerically distinct entities, there is a *monadic* property that applies to only one of those entities. According to Russell, Leibniz rightly defined a substance as ‘that which can only be subject, not predicate, which has many predicates, and persists through change’ (*Leibniz*, §17). Russell then urges that ‘if we admit that nothing can be said about a substance except to assign its predicates, it seems evident that to be a different substance is to have different predicate’ (*Leibniz*, §24). He explains:

[Suppose] A differs from B, in the sense that they are different substances; but to be thus different is to have a relation to B. This relation must have a corresponding predicate of A. But since B does not differ from itself, B cannot have the same predicate. Hence A and B will differ as to predicates, contrary to the hypothesis.

(Leibniz, §24)

⁴ I use the word ‘entity’ to speak of a self-subsistent object where an object is self-subsistent if and only if its subsistence in reality requires the subsistence of no other objects except for its parts. The term ‘object’ will be used in a broader sense to the effect that anything that is in reality is an *object* whether it is self-subsistent or not. I do not commit myself to any view on what counts as an entity or an object. I employ those terms to compare Bradley’s metaphysical picture of reality with Russell’s.

The identity of indiscernibles is thus derivable from the conjunction of the view of an entity as a substance and the subject-predicate doctrine. But according to Russell the doctrine has a further implication that there cannot be two entities:

For the numerical diversity of the substances is logically prior to their diversity as to predicates: there can be no question of their differing in respect of predicates, unless they first differ numerically. But the bare judgment of numerical diversity itself is open to all the objections which Leibniz can urge against indiscernibles. Until predicates have been assigned, the two substances remain indiscernible; but they cannot have predicates by which they cease to be indiscernible, unless they are first distinguished as numerically different.

(Leibniz, §25)

The idea is that substances must differ from one another numerically so as to have different attributes but the identity of indiscernibles makes it impossible for substances to differ without having different attributes. Russell thus thinks that if the subject-predicate doctrine holds, 'there cannot be two substances at all' (§25).

Strictly speaking, the two passages quoted above indicate two distinct arguments. If we literally understand the last quotation, his argument seems to be based on a notion of logical priority: a plurality of substances requires that substances should be numerically different *before* some attributes are attached to them. But the cogency of the argument is questionable, as it is not obvious how to articulate the relevant notion of logical priority nor why two substances having exactly the same monadic properties cannot come into existence *at the same time*. Yet, Russell does not seem worried about those points, as he presents a similar argument also in *PoM* (§428, §452). His discussion of 'A differs from B,' quoted above, suggests a different argument which does not appeal to the notion of logical priority. If we understand *difference* as a dyadic relation between entities, we obtain the following argument: if there are two distinct substances, the relation of difference must hold between the two substances; but its holding between them is not compatible with the subject-predicate doctrine. This is indeed how Russell derives monistic idealism from the doctrine in his later works. For example, in *Our Knowledge of the External World* (1914; henceforth *OKEW*), Russell remarks:

Now the traditional logic holds that every proposition ascribes a predicate to a subject, and from this it easily follows that there can be only one subject, the Absolute, for if there were two, the proposition that there were two would not ascribe a predicate to either.

(*OKEW*, p.39)

This argument appears at least to me more convincing than the one based on a notion of

logical priority. But for our purposes it is important to focus on the argument based on a notion of logical priority as it illustrates how Russell understands judgments in *Leibniz*. For this reason I use the term ‘indistinguishability argument’ exclusively to speak of the argument through the identity of indiscernibles and based on a notion of logical priority.

It is plain that although Russell presents the subject-predicate doctrine using such notions as ‘subject’ and ‘predicate,’ it is unmistakably an ontological doctrine. For it would otherwise remain unintelligible how he could draw from the doctrine such an ontological conclusion that there cannot be two substances. Every judgmental content is understood not as a mere private, internal experience but rather as a part of the objective, external reality. Unless he understands the subject-predicate doctrine as the claim that every judgmental content itself is composed of an entity and a property attached to it, he cannot infer from the doctrine that the numerical difference between two substances amounts to the difference in their monadic properties. In other words, the indistinguishability argument would fail if a judgmental content belongs merely to one’s private, internal experience. In such a case, whether a judgment content can be of a form other than the subject-predicate one would have nothing to do with whether there is an external, objective *fact* in which a binary relation holds between two substances. We are thus led to one of the two distinct features of Russell’s notion of ‘judgment’ in *Leibniz*: each judgmental content should not belong (only) to the judging mind but rather to the external reality.

Russell’s sloppiness in formulating the subject-predicate doctrine is presumably due to what I shall call, following Candlish, the *transparency thesis* (Candlish, 2007, p.106f). The thesis can be understood as the claim that when a sentence with which one makes a judgment is perfectly analysed—whatever ‘perfect’ means here—its grammatical structure is isomorphic to the ontological structure of the corresponding part of reality. Russell arguably subscribes to the transparency thesis in one form or another throughout his career.⁵ In *Leibniz*, he remarks: ‘The attributes of a substance *are* the predicates of a subject’ (*Leibniz*, §16; emphasis added). Hence, in his view, when the sentence used in a judgment is properly analysed, its content is composed of an entity and a property attached to it. The transparency thesis thus explains why the subject-predicate doctrine is presented despite its ontological nature as a doctrine concerning subjects and predicates.

Closely related to and yet distinct from the transparency thesis is what Candlish calls ‘the doctrine of real propositional constituents.’ The doctrine urges that propositions are ‘not linguistic, and their constituents include quite everyday objects’ (Candlish, 2007, p.54). As

⁵ As he develops the theory of descriptions and the notion of incomplete symbol, Russell comes to think of the language as less transparent. But what changed with the discovery of those ideas is not the transparency thesis itself but, as Candlish puts it, ‘the conception of where a language’s true grammar is to be found’ (Candlish, 2007, p.114). I will discuss the thesis in greater detail in Chapter 1 and Chapter 7.

we will discuss in the next chapter, there are some passages in *PoM* that indicates Russell's endorsement of the doctrine or something analogous. But Candlish goes further to say that this doctrine is endorsed even in *Leibniz*. He claims that the combination of this doctrine, the transparency thesis and the subject-predicate doctrine 'would, in Russell's mind, entail the conclusion that relations are not real entities,' and that Russell abandoned the subject-predicate doctrine in favour of the other two claims as well as the reality of relations (Candlish, 2007, p.133). However, Russell in *Leibniz* seems to think that a meaning of an expression may not be an *entity*, a self-subsistent object. For if he endorses the doctrine of real propositional constituents together with the transparency thesis, he cannot argue against monistic idealism without begging the question. For the conjunction of these two claims implies that a sentence, say, 'Socrates is human' expresses a complex composed of *two* entities, namely Socrates and the property of *being human* understood as a *real* object. Hence, if Russell rejects the subject-predicate doctrine in order to undermine monistic idealism, he cannot legitimately subscribe to the doctrine of real propositional constituents.⁶ Thus, Russell in *Leibniz* does not advocate the doctrine of real propositional constituents.

Russell's endorsement of the transparency thesis indicates the second feature of his notion of 'judgment' in *Leibniz*. The thesis implies that each properly analysed sentence expresses a part of reality, *whether the sentence is said to be true or false*. The transparency thesis, as it stands, does not prevent a false sentence from having a corresponding part of reality. It follows that even when a judgment is said to be false, its judgmental content belongs to the external reality. This is the second feature of his notion of 'judgment' in *Leibniz*.

One might now wonder whether Russell in *Leibniz* endorses the transparency thesis, for it is counterintuitive to say that even a false judgment describes a part of reality. One might think that what he holds is not the transparency thesis but a weaker principle according to which transparency is limited to true sentences: every sentence, when properly analysed and *when true*, describes a part of reality. However, this doctrine cannot entitle us to infer from the subject-predicate doctrine that there cannot be two substances. This version of the thesis entails that any sentence that asserts a difference between two objects is either false or not properly analysed. But this only amounts to the claim that if there are indeed two distinct objects, no properly analysed sentence will stand for the fact; but this is perfectly compatible with the claim that there are two distinct entities. One might instead attribute to Russell a more greatly modified thesis such that every true sentence describes a part of reality. This is

⁶ The doctrine of real propositional constituents seems to be, rather, a consequence of Russell's rejection of the subject-predicate doctrine and the notion of substance. The idea is roughly that if we drop the notion of substance and instead adopt his notion of *term*, or in other words, if we regard an entity as 'what can be a subject' rather than 'what can *only* be a subject,' then we can include such properties as humanity among entities. I will discuss this point in detail in the next chapter.

quite an ordinary idea. This modification may appear at first sight to be supported by some remarks in *Leibniz*, for he sometimes presents the subject-predicate doctrine in terms of the ‘validity’ of judgment. For example, after introducing the passage where Leibniz argues for the ‘ideality’ of relations, Russell remarks as follows:

It appears plainly from his discussion that he is unable to admit, *as ultimately valid*, any form of judgment other than the subject-predicate form, although, in the case he is discussing, the necessity of relational judgments is peculiarly evident.

(*Leibniz*, §10; emphasis added)

The term ‘valid’ looks interchangeable with ‘true’ here. In addition, if we combine this claim with the subject-predicate doctrine, we can still infer from the doctrine that there are no two objects: any sentence that asserts a difference between two seemingly distinct objects is false. But this deduction is, so to speak, too quick. For it makes Russell’s discussion of the identity of indiscernibles entirely redundant. The notion of indistinguishability plays no role in the above argument. It would thus seem that he endorses the transparency thesis despite of the counterintuitive implication that whether a judgment is said to be true or false its content belongs to the objective reality.

1.4. Bradley’s theory of judgment

In this section I shall attempt to show that Bradley also thinks that every judgmental content is a part of reality whether the judgment is true or false. I will do so by briefly sketching his view of judgment presented in *The Principles of Logic (PoL)* and in *Appearance and Reality* (hereafter, *AR*).⁷ It will also be seen that Bradley does not utilise the indistinguishability argument to support his monistic idealism; on the contrary, his theory of judgment is in part derived from his monistic metaphysics.

One of the central tenets of *The Principles of Logic* is that a judgment proper is ‘the act which refers an ideal content (recognized as such) to a reality beyond the act’ (*PoL*, §10).⁸ In order to see what this claim amounts to, we first need to understand his notions of ‘ideal content’ and of ‘reality.’ What he calls ‘ideal content’ or simply ‘idea’ is not a mental image, but a

⁷ The first edition of *PoL* was published in 1883, before the two editions of *AR* then appeared in 1893 and in 1897 respectively. The second edition of *PoL* came out much later in 1922. Of course, Bradley changed his position through these publications, but it is not necessary to take various modifications into consideration as far as we seek to show that he accepted a notion of judgment with the two distinctive features in question. Mander summarises how Bradley changed details of his account of judgment (Mander, 1994, p.143).

⁸ Another important theme of the book is of course *inference*. For a concise introduction of Bradley’s account of inference, see Mander (2011, pp.298-308). But Bradley more or less dismisses the notion of *term*—the last one of the traditional three subject matters of logic. He does so intentionally, since he argues that a judgment is not a combination of terms or ideas but rather an attribution of a single idea to reality. Mander accounts for Bradley’s account of ideas in connection to the notion of identity-in-difference (*ibid.*, pp.290-5).

universal which one abstracts from a particular fact given to the mind. Bradley thinks that each fact *exists* with some *content*: 'In all that is we can distinguish two sides, (i) existence and (ii) content. In other words we perceive both *that* it is and *what* it is' (*PoL*, p.3). Besides, on his view, some facts have a third element which he calls 'meaning':

But there is a class of facts which possess an other and additional third side. They have a meaning; and by a sign we understand any sort of fact which is used with a meaning. The meaning may be part of the original content, or it may have been discovered and even added by a further extension. Still this makes no difference. Take anything which can stand for anything else, and you have a sign. Besides its own private existence and content, it has this third aspect.

(*PoL*, p.3)

Hence, a meaning is a universal that one obtains by abstracting away both the particular existence of a fact and the irrelevant details of its content. 'A symbol is,' he then remarks, 'a fact which stands for something else, and by this, we may say, it both loses and gains, is degraded and exalted. In its use as a symbol it foregoes individuality, and self-existence' (*PoL*, p.3).⁹ 'A sign is,' in other words, 'any fact that has a meaning, and meaning consists of a part of the content (original or acquired), cut off, fixed by the mind, and considered apart from the existence of the sign' (*ibid.*, p.4).

If a judgment attributes an ideal content to reality, then what is reality in Bradley's view? In *PoL*, he does not explicate what he means by 'reality,' presumably because the chief aim of the book is to investigate foundational notions of logic (*PoL*, p.ix). But we may safely assume that he considers reality to be the Absolute as he does explicitly in *AR*. It is widely known that he in *AR* argues that the notion of relation leads to an infinite regress: in order for a relation to hold between terms, it requires that relations should hold between the relation and its terms, and *ad infinitum*.¹⁰ A similar argument can be found in *PoL*: 'If units have to exist together, they must stand in relation to one another; and if these relations are also units, it would seem that the second class must also stand in the relation to the first' (*PoL*, p.96). 'The real truth is,' he concludes, 'that the units on one side, and on the other side the relation existing between them, are nothing real' (*ibid.*). His rejection of the ultimate reality of relations is of course an integral part of his monistic notion that in reality there is only one entity the Absolute. It is thus plausible that he understands reality in *PoL* in the way he does in *AR*—as 'a single

⁹ It may sound strange to say that a symbol is a *fact*, since a symbol is at best an object. But Bradley does not seem to distinguish between facts and objects (in our casual sense). For example, he remarks: 'When I say "There is a wolf," the real fact is a particular wolf, not like any other, in relation to this particular environment and to my internal self, which is present in a particular condition of feeling emotion and thought' (*PoL*, p.97). This is perhaps because it is in his view ultimately not tenable to keep distinguishing an object from its background.

¹⁰ *AR*, p.21. For a detailed interpretation of the argument in *AR*, see Baxter (1996).

and all-inclusive experience, which embraces every partial diversity is concord' (AR, p.147). To be precise, Bradley in *PoL* remarks that one does not refer an ideal content to the whole reality but to 'that which I now perceive, or feel, or about some portion of it' (*PoL*, p.49). In an appendix added in the second edition of *PoL*, Bradley summarises his standpoint as follows: 'In Judgment the Reality to which in fact we refer is always something distinguished. It is Reality, as our whole world, but, at the same time and none the less, it is also *this* reality' (*PoL*, p.629). The idea is that one can only refer an ideal content to a part of reality because an individual, as a finite being, cannot experience the whole reality; yet, the part is still a part of the whole organic unity, and hence, when one refers an ideal content to the experienced part of reality, one thereby refers it also to the whole. He also remarks that 'the given reality is never consistent' due to its finiteness (AR, p.166). Hence, in his view, a part of reality which one experiences cannot be separated out from the whole reality.

Bradley's claim that a judgment is an attribution of an ideal content to reality is not contentious, when it is understood as a claim on judgmental acts. For to perform a judgmental act is to assert something to be true of reality. However, it seems that he goes further to hold that each judgmental content belongs to what we normally understand as the external reality or to a fragment of reality beyond one's private experience. For instance, he remarks:

A truth is not necessary unless in some way it is compelled to be true [...]. And compulsion is not possible without something that compels. It will hence be the real, which exerts this force, of which the judgment is asserted. We may indeed not affirm that the suggestion S-P itself is categorically true of the fact, and *that* is not our judgment. The actual judgment asserts that S-P is forced on our minds by a reality *x*. And this reality, whatever it may be, is the subject of the judgment.

(*PoL*, p.41)

Thus, on his view, what a judgment really asserts is not the meaning of the symbol used in the judgment: each judgment rather asserts that reality has the meaning as a property.¹¹ In other words, he understands a judgmental content as composed of reality and an ideal content referred to it by the judgment.¹² One might still wonder whether he really grants such an ontological status to a judgmental content, but he himself goes even further to claim that the connection between reality and an ideal content is independent of one's judging:

In the act of assertion we transfer this adjective to, and unite it with, a real substantive. And we perceive at the same time, that the relation thus set up is neither made by the act, nor merely holds within it or by right of it, but is real both independent of and

¹¹ This is not how Candlish understands Bradley's view, however. See footnote 15.

¹² In this sense, Bradley does endorse a subject-predicate doctrine. I will expand on this point shortly.

beyond it.

(*PoL*, p.10)

Bradley thus finds no problem with the idea that a judgmental content itself is a part of the objective reality. To be precise, he maintains that each judgment involves a separation of reality and an ideal content, and hence, it can never capture reality (cf. *PoL*, p.94; *AR*, Ch.XV). Hence, in his view, no judgment can perfectly correspond to reality. This is a point where his metaphysics diverges from those of other monistic idealists (cf. footnote 1). But what is important for our purposes is that he still holds that each judgmental content is a part of reality: it is an imperfect and hence falsified piece of reality.

Bradley's view of judgment thus presented may appear implausible but we can see why he endorses it if we understand his contention that each judgment does not have any specific content unless a universal used in the act of judging is attributed to reality. He first urges that each singular judgment has a very particular content whilst an ideal content itself is a universal and hence can never give us a specific content. He argues as follows¹³:

When we say "It rained last Tuesday," we mean *this* last Tuesday, and not any other; but if we keep to ideas, we do not utter our meaning. Nothing in the world that you can do to ideas, no possible torture will get out of them an assertion that is not universal. We can not escape by employing ideas of events in time, particulars as we call them. The event you describe is a single occurrence, but what you say of it will do just as well for any number of events, imaginary or real.

(*PoL*, p.63)

Bradley also urges that the only way of capturing the particularity of one's assertion is to refer to reality. One might think that 'space and time are "principles of individuation," in the sense that a temporal or spatial exclusion will confer uniqueness upon any content' (*PoL*, p.63). In other words, one might try to pick up a particular event by specifying when and where it happens, thereby pointing to its location in a series of events. But Bradley rejects this idea. 'There is nothing whatever,' he argues, 'in the idea of a series to hind that there may not be any number of series, internally all indistinguishable from the first' (*ibid.*). Time and space can only help, when we single out a series by using such indexicals as 'here' and 'now.' But this works, only if reality is given to the mind so that an ideal content can be referred to it. He thus argues that each singular judgment requires a reference to reality in order to have a specific content, because 'we find uniqueness in our contact with the real, and that

¹³ To be precise, the following discussion is about a category of singular judgments. Bradley uses the term 'synthetic judgments of sense' to speak of judgments 'which state either some fact of time or space, or again some quality of the given matter, which I do not here and now directly perceive' (*PoL*, p.49).

we do not find it anywhere else' (*PoL*, p.65; cf.pp.69-70).¹⁴ This argument from a uniqueness of a singular judgment is, of course, not applicable to universal judgements. However, each universal judgment is, Bradley argues, hypothetical, and what it affirms is 'not the actual existing behaviour of the real, but a latent quality of its disposition' (*PoL*, p.87). Therefore, on his view, every judgment, whether it is true or false, requires that its ideal content should belong to the external reality.

The view of a judgment as an attribution of an idea to reality has some important implications. Firstly, a judgment may have no grammatical subject. For regardless of the presence or absence of a grammatical subject, a judgmental content contains reality as its genuine subject. Hence, one can use a mere word 'Wolf' to assert the existence of a wolf (*PoL*, p.56). Secondly, there is no problem with judgments whose grammatical subjects are an empty name:

"A four-cornered circle is an impossibility," we are told, does not assert the actual existence of a four-cornered circle [...]. But the objection is irrelevant, unless it is maintained that in every case we affirm the reality of the *grammatical* subject. And this clearly is not what we mean to assert.

(*PoL*, p.42)

Thirdly and far more importantly for our purposes, it follows that even a judgment which we normally deem false must refer an ideal content to reality as long as it has a specific content. In other words, on Bradley's theory of judgment, the content of a false judgment must also be composed of reality and an ideal content.

It may still be questioned whether Bradley takes such a counterintuitive position on false judgments. A piece of evidence that he indeed does can be found in his discussion of the *problem of error*. Given his theory of judgment, it is natural to think that whether a judgment is true or false depends on whether reality possesses the ideal content of the judgment as a property. Bradley himself envisages this thought:

Appearance will be truth when a content, made alien to its own being, is related to some fact which accepts its qualification. The true idea is appearance in respect of its own being as fact and event, but is reality in connection with other being which it qualifies. Error, on the other hand, is content made loose from its own reality, and related to a reality with which it is discrepant.

(*AR*, pp.186-8)

¹⁴ In fairness, Bradley speaks of affirmative judgments in the discussion I am referring to. But this does not mean that he thinks that negative judgments do not attribute ideas to reality: 'What we deny is not the reference of the idea to actual fact. It is the mere idea of the fact, as so qualified, which negation excludes; it repels the suggested synthesis, not the real judgment' (*PoL*, pp. 115-6). See *PoL*, Book I, Ch.3 for greater details.

But then, since he also maintains that each judgment, regardless of its truth or falsity, must attach an ideal content to reality, it follows that every judgment is true. The problem is, in other words, that if he adopts the account of truth and falsehood given in the above quotation, he has to conclude that there is no false judgment. Hence the problem of error:

Error is without any question a dangerous subject, and the chief difficulty is as follows. We cannot, on the one hand, accept anything between non-existence and reality, while, on the other hand, error obstinately refuses to be either. [...] In false appearance there is something attributed to the real which does not belong to it. But if the appearance is not real, then it is no false appearance, because it is nothing. On the other hand, if it is false, it must therefore be true of reality, for it is something which is.

(AR, p.186)

The presence of the problem indicates that Bradley indeed thinks of a false judgmental content as a part of the external reality. For otherwise he would not be confronted by the problem.

Bradley seeks to solve the problem of error by invoking his view of truth and falsehood as a matter of degree. He maintains that the ideal contents of true judgments are *more* in accordance with reality, while those of false ones are *less* so. He then remarks:

Error *is* truth, it is partial truth, that is false only because partial and left incomplete. The Absolute *has* without subtraction all those qualities, and it has every arrangement which we seem to confer upon it by our mere mistakes.

(AR, p.192)

Hence, in his view, what we normally deem a false judgment is still a 'partial' truth. It should be remembered that he also maintains that no judgment is entirely true. As we have seen, no judgment can, in his view, fully capture reality and hence every judgment fails to be true in this sense. Thus, according to his fully developed account of truth, every judgment is neither entirely true nor entirely false: it is true to some degree depending on how much it is in accordance with the whole reality.

But it remains unclear why a 'false' ideal content can still be attributed to reality, which is considered to be perfectly consistent. Bradley offers a more detailed and complicated account of how reality can be predicated of a 'false' ideal content. First, he claims that we can dissolve a discrepancy between the subject of a judgment and the 'false' ideal content by extending the former to a bigger reality: 'Instead of one subject distracted, we get a larger subject with distinctions, and so the tension is removed' (AR, p.192). He illustrates this point by using such a notation as 'X(abcdef ...).' For example, a judgment 'X(ab) is d' is false, 'because d is not present in the subject'; however, 'the collision is resolved if we take the subject, not as mere X(a, b), but more widely as X(abcd)' (*ibid.*, p.193). It is not clear whether this solution is

applicable to every case where one makes a false judgment. Bradley himself finds a problem with this idea: even when all the complements are added to the subject, the discrepancy that the original ideal content has remains there (*ibid.*, p.194). He responds to this problem by claiming that the Absolute itself cannot be, after all, identified with a whole which we obtain by transforming the subject of a judgment into a larger one (*ibid.*, p.195). But this response leaves it utterly unclear how a false ideal content can be attributed to reality itself. Yet, in any case, what is important for our purpose is that Bradley is thus confronted by the problem of error, which makes it clear that he thinks that each judgmental content belongs to the objective reality whether the judgment is said to true or false.

Let us briefly discuss whether Russell is right in claiming that Bradley's monistic idealism is based on the subject-predicate doctrine. First, Russell's claim that Bradley advocates the doctrine can be justified to the extent that Bradley views every judgmental content as composed of an entity and a monadic property or, in his own terminology, of reality and an ideal content. Although commentators tend to dismiss Russell's claim as groundless, some of Bradley's remarks in *PoL* indeed make it excusable: 'The actual judgment affirms that S-P is in connection with *x*' (*PoL*, p.42); 'Our judgment "S is P" affirms really that Reality is such that S is P' (*ibid.*, p.630).¹⁵ Yet, Bradley's actual account of a judgment is, of course, much more complicated than the subject-predicate doctrine. For instance, he thinks, as we saw above, that the reality to which one attributes an ideal content is always a part of the whole reality. Also, he is somewhat indifferent as to whether every judgmental content is composed of an entity and a property and *nothing else*. For he remarks that what a judgment really asserts is 'S-P is forced on our minds by a reality *x*' (*PoL*, p.41). Furthermore, Russell's claim that Bradley's monistic idealism is based on the subject-predicate doctrine is much harder if not impossible to justify. For Bradley's view of a judgment as a reference of an ideal content to reality is, rather, based on his monistic idealism. Bradley does not simply assume that each judgment of any form has a corresponding judgment of the subject-predicate form. His argument for monism, which we may find in the first part of *AR*, does not depend on the doctrine and it seems as though he anticipates and presupposes the resulting monism in developing his theory of judgment in *PoL*.¹⁶ Thus, although it is possible to view Bradley as an advocate of the subject-predicate doctrine, his metaphysics does not depend upon it.

¹⁵ Those commentators include Mander (1994), Fortier (1996, p.33) and Candlish (2007, pp.131-2). But whether Bradley endorses the subject-predicate doctrine depends on whether he views reality as a part of a judgmental content, not on whether he regards an ideal content as a combination of a subject and a predicate.

¹⁶ Mander remarks that Bradley's notion that the subject of a judgment lies outside of the ideal content leads though an analogue of the indistinguishability argument to monism: 'taking different judgements, we have no way of distinguishing their referents, hence they must all be taken to refer to the same thing, reality as a whole' (Mander, 1994, p.33). But it remains unclear whether Bradley himself envisaged this argument.

1.5. Concluding remarks

It would thus seem that both Russell in *Leibniz* and Bradley in two of his major works—*PoL* and *AR*—hold that a judgmental content is 1) a part of the objective reality, 2) even when the judgment is said to be false. In Section 3, we saw that this is indeed the case with Russell in *Leibniz*, and in Section 4 with Bradley.

The notion of judgment, which we can thus find in *Leibniz* and in *PoL*, is arguably an *idealistic* notion. One may well find it implausible to think that even false judgments have corresponding parts in reality. But one of the main reasons why it does not fit our intuition seems to stem from the view that a judgment has a *corresponding* part of reality if and only if the judgment is true. This notion of truth was accepted as a satisfactory account neither by Russell in *Leibniz* nor by Bradley. In fact, idealists who reject the correspondence theory of truth can offer a plausible account as to why a judgment, whether true or false, has as its content a part of reality. For instance, if one understands, as Kant did, a judgment as a sort of experience or perception, and if one accepts a version of idealism, its judgmental content would be a part of reality, even when the judgment is said to be false in the naive sense in which it does not correspond to external fact, the sense which one can reject on the basis of idealism.

It should also be noted that both Bradley and Russell in *Leibniz* think that any theoretical inquiry into judgments counts as logic. As we saw in Section 1.4, what Bradley presents in *PoL*, his work on principles of logic, is his theory of judgment. On the other hand, Russell understands the subject-predicate principle as a (false) principle of logic:

[...] the traditional *logic*—the *logic* underlying all use of substance or of the Absolute—assumes, as I have endeavoured to show, that all propositions have a subject and a predicate.

(*Leibniz*, §65; emphasis added)

As we saw in Section 1.3, Russell thinks that the argument for the identity of indiscernibles from the subject-predicate doctrine goes further to prove that there cannot be two substances. After pointing this out, he also remarks that ‘on the principles of Leibniz’s *logic*, the identity of indiscernibles does not go far enough’ (*Leibniz*, §25; emphasis added). Furthermore, it is worth noting that when Russell published the second of edition of *Leibniz*, he did not hesitate to call the doctrine ‘subject-predicate logic.’ It would thus seem that both Bradley and Russell in *Leibniz* assume that any theoretical enquiry into the nature of judgment belongs to logic.

2 A Universe of Propositions

2.1. Introduction

From 1898 onwards, Moore and Russell developed what I call the *ontology of propositions*, the view of reality as the body of propositions, where propositions are, whether true or false, considered to be mind-independent complex entities. As we saw in the preceding chapter, Russell accepted the notion that logic was composed of the theory of logic and the calculus of logic even after his revolt against idealism. But he explicitly opposed to the idea that the idealist enquiry into *judgments* could be the theory of logic, and he proposed the ontology of *propositions* in place of it. In this chapter, I will illustrate this point by showing that Moore and Russell obtained the ontology by replacing, one by one, some idealistic notions of Bradley's theory of judgment with realist alternatives instead of rejecting the idealistic framework altogether.

In Section 2.2, I will introduce Moore's version of the ontology of propositions and then Russell's. I will then discuss two important points where their realist ontology significantly differs from Bradley's idealist theory of judgment in Section 2.3 and Section 2.4 respectively. I shall, in Section 2.5, try to show that these two points as well as a certain idea concerning proper names underlie Russell's notion that language is merely a psychological device with which one can speak of something external.

2.2. The ontology of propositions

In the first half of this section I will attempt to indicate how Moore replaces Bradley's notion of *idea* or *ideal content* with his own notion of *concept*. I will then discuss how Russell modifies the notion of *concept* and develops it into the notion of *term*.¹

2.2.1 Moore—Concepts as against ideal contents

The ontology of propositions first appears in Moore's seminal paper 'The Nature of Judgment' (hereafter NJ). He starts off NJ by claiming that "'the idea used in judgment" is not a part of the content of our ideas, nor produced by any action of our minds' (NJ, p.177). It is

¹ Moore and Russell employ 'term,' 'concept' and 'proposition' as technical terms. But to avoid a flood of italics I will not italicise these words unless otherwise there would be a risk of confusion.

plain that by thus explaining what he thinks an idea is not, he takes on Bradley's theory of judgment (cf. Section 1.4). According to Moore, although Bradley is right in maintaining that 'the idea in judgment is the universal meaning,' he still lets a psychological element sneak into his theory of judgment:

Before we can judge at all on Mr. Bradley's theory, a part of this character must have been "cut off and fixed by the mind." But my question is, whether we can thus cut off a part of the character of our ideas, and attribute that part to something else, unless we already know, in part at least, what is the character of the idea from which we are to cut off the part in question. If not, then we have already made a judgment with regard to the character of our idea.

(NJ, pp.177-8)

Moore thus argues that ideas which we use in making a judgment must be already there before we do so.² He calls ideas conceived as mind-independent entities *concepts*. According to him, concepts are not 'abstractions either from things or from ideas' but rather 'the only objects of knowledge' (NJ, p.182). He thus replaces Bradley's notion of ideal content with his own realist alternative—his notion of concept.

The adoption of the notion of concept has some consequences. First, once concepts are thus conceived as objects of judgments, they no longer have a *descriptive* nature. In Bradley's view, ideal contents are what we employ in order to describe reality or a fragment of it. On the contrary, Moore holds that concepts are themselves what we describe by using symbols. Second, concepts must somehow be able to account for the particularity of each judgment. As we saw in the last chapter, Bradley maintains that each judgmental content obtains its particularity through reference to reality. In his view, no judgmental content can be specific to a particular object and this is precisely why we have to posit something beyond them as the subject of a judgment. On the other hand, if concepts themselves are the objects of our judgments and there is nothing beyond them, it follows that each judgment gains its particular content in virtue of the particularity of the concept or the collection of concepts that is the subject of our judgment. This is perhaps related to the fact that Moore in 'Identity' comes to introduce the distinction between general concepts and their instances. We will discuss it in Section 2.4.

In Moore's view, concepts are individual entities which constitute reality. They are considered to be 'the only objects of knowledge.' It follows that every single fragment of reality of which one can think must be a concept or at least a complex of concepts. He calls complexes

² As Hylton points out, Moore's argument presupposes that abstraction is a conscious process, and Bradley indicates this point in his letter to Moore dated 10 October 1899 (Hylton, 1990, p.133fn).

composed of concepts *propositions*:

A proposition is composed not of words, nor yet of thoughts, but of concepts. Concepts are possible objects of thought; but that is no definition of them. It merely states that they may come into relation with a thinker; and in order that they may do anything, they must already be something. It is indifferent to their nature whether anybody thinks them or not.

(NJ, p.179)

Here Moore uses 'proposition' to speak of the content of a judgment rather than the act of judging. In his view, a judgmental act is merely a binary relation between the mind and a mind-independent complex entity called a proposition. I will call this view of judgment the *dyadic-relation theory of judgment*. Our act of judging does not play any substantial role in his metaphysics; it is merely a psychological act of no philosophical importance. What matters in his metaphysical picture is not a judgment but a proposition. 'A proposition is,' he remarks, 'a synthesis of concepts; and, just as concepts are themselves immutably what they are, so they stand in infinite relations to one another equally immutable' (NJ, p.180). He thus replaces Bradley's notion of *idea* with that of *concept*, and also his notion of *judgment* with that of *proposition*. This is how he obtains the realist ontology of propositions.

The dyadic-relation theory of judgment leads to another important feature of Moore's notion of concept: concepts and hence propositions may not *exist* but they all *subsist*. If our judgmental act is always a binary relation to a proposition, it follows that non-existent objects are still in reality. For otherwise we cannot make a judgment about them. According to him, when one judges that a certain thing exists, the judgment has as its object the proposition composed of the thing in question and the concept of existence. He adds that existence itself is a concept. If existence is merely among concepts, concepts themselves must have a different ontological status than existence. He thus maintains that concepts subsist but may not exist. He also notes that the view that existence is among concepts also makes it circular to define truth and falsehood of propositions in terms of existence. For the existence of an entity amounts to the truth of the proposition that the entity exists. Hence 'existence is logically subordinate to truth' (NJ, p.180).³

2.2.2 Russell—Terms as against substances

'Whatever,' Russell declares in *PoM*, 'may be an object of thought or may occur in any true or false proposition, or can be counted as *one*, I call a *term*' (*PoM*, §47). He explains this notion of *term* as 'a modification of Mr. Moore's notion of a *concept* in his article "On the Nature

³ I will discuss this idea in greater detail in Chapter 6.

of Judgment” (§47n). In this section I will indicate some commonalities and differences between these notions, before I indicate a certain way in which Russell intends to replace the old notion of substance with his notion of term.

First of all, Russell maintains that terms are the objects of our judgments. Whatever we can think of is a term and, put otherwise, anything that can be a *logical subject* of a proposition counts as a term. In effect, he holds that an entity is a logical subject of a proposition if the entity is designated by a proper name in the sentence expressing the proposition. Second, he follows Moore in maintaining that terms are ‘immutable and indestructible’ (§47). They all subsist or—using Russell’s own phraseology—*have being* permanently and timelessly in reality. Related to this, thirdly, he endorses the dyadic-relation theory of judgment, maintaining that the object of our judgment is a proposition, a mind-independent entity composed of terms. ‘*Being is,*’ he claims, ‘that which belongs to every conceivable term, to every possible object of thought—in short to everything that can possibly occur in any proposition, true or false, and to all such propositions themselves’ (§427). He also argues that ‘*terms embrace everything that can occur in a proposition*’ on pain of self-contradiction (§49; see also §52, §483). In his view, whatever we can speak of is a *term*, and hence, it is self-contradictory to hold that a certain constituent of a proposition cannot be a logical subject. The proposition ‘a such-and-such object can never be a logical subject’ itself turns the object into a logical subject. He thus concludes that ‘every constituent of every proposition must, on pain of self-contradiction, be capable of being made a logical subject’ (§52).⁴ Thus, in his view, propositions are mind-independent entities composed of other entities.

There are certain respects in which Russell’s ontology diverges from Moore’s. One is that Moore distinguishes propositions from concepts only by complexity while Russell is more wary of doing so. ‘The difference between a concept and a proposition,’ Moore claims, ‘in virtue of which the latter alone can be called true or false, would seem to lie merely in the simplicity of the former’ (NJ, p.180). On the other hand, Russell wonders whether the proposition ‘Caesar died’ can be identified with the complex entity denoted by ‘the death of Caesar’ (*PoM*, §52). His own argument for the notion of *term* suggests that these two entities must be identical, while he also believes that ‘neither truth nor falsity belongs to a mere logical subject’ (*ibid.*).⁵

Another point of difference can be found in their accounts as to how a proposition is composed of other entities. Russell maintains that each proposition contains a *relating* relation, a relation that cements, so to speak, the other constituents. Whether a relation is a relating one or not is considered to be merely a matter of two distinct kinds of occurrences (*PoM*, §54).

⁴ Russell, however, thinks that his argument has ‘the possible exception of complexes of terms of the kind denoted by *any* and connate words’ (§49). Those exceptions will be discussed in Chapter 3 and Chapter 4.

⁵ This dilemma is a part of his well-known, obscure discussion of the notion of *assertion*.

On the other hand, Moore suggests that the constituents of a proposition are all combined by a peculiar relation, the nature of which determines whether the proposition is true or false.

There has been an extended discussion on whether Russell's view that a relation occurring as a relation in a proposition combines other entities is tenable or not.⁶ Let us discuss this issue briefly before we turn to other points of difference. Bradley thinks that the idea that relations hold among entities involves an infinite regress (cf. Section 1.4). Russell, who firmly believes in relations, responds that the regress is certainly there but harmless. The regress, he argues, means nothing more than the following: 'when a relation holds between two terms, the relations of the relation to the terms, and of these relations to the relation and the terms, and on *ad infinitum*, though all implied by the proposition affirming the original relation, form no part of the *meaning* of this proposition' (*PoM*, §55; cf. §214). He distinguishes regress of *meaning* from regress of *implication*, holding that only the former is harmful and that the latter is perfectly acceptable to those who hold that 'there are no contradictions peculiar to the notion of infinity' (*ibid.*). Bradley unsurprisingly finds this response unconvincing, and many commentators side with him.⁷ The main problem with Russell's response is that he does not—and arguably cannot—account for how a relation can occur in propositions in these two different ways. He indeed admits: 'I do not know how to give a clear account of the precise nature of the distinction' between the two modes of occurrences (§54). It seems to me that whether one can be satisfied with Russell's response or not depends simply on what one takes to be an adequate philosophical account of the topic. On the one hand, Russell thinks it sufficient to provide a metaphysical account of relations which does not—at least immediately—lead to any contradiction or to any regress of meaning; on the other hand, Bradley and the commentators sympathetic to him hold that any tenable theory of relations must be not only self-consistent but also self-explanatory in the sense that it leaves no obvious questions concerning them unanswered.⁸

Connected with Bradley's regress is a second point of difference between Moore's ontology and Russell's. Moore in 'Identity' finds no problem with the notion that properties, whether monadic or relational, have instances. But Russell in *PoM* argues that relations do not have instances though unary properties may do (§55n). He asks whether the unity of a proposition, say, aRb , is brought about by a general concept of R or an instance of the relation 'particularised' to a and b (§55). First, he holds that with regard to Bradley's regress, the latter view is no better than the former one. He then goes on to argue that the latter view fails to

⁶ See, for example, Hylton (1990), Linsky (1992), Griffin (1993) and Stevens (2016). Davidson (1984), Gaskin (1995) and Peacock (2011) discuss, more broadly, what is now called the unity of the proposition.

⁷ Bradley (1910). Those commentators include Hylton (1990, pp.174-8), Linsky (1992, p.250) and Griffin (1993, pp.166-7).

⁸ Korhonen argues that 'in Russell's metaphysical framework unity must be accepted as a primitive feature of propositions' (Korhonen, 2013, pp.120-121).

explicate what instances are. For all instances of a general relation must have something in common in order to count as *instances*, while ‘the most general way in which two terms can have something in common is by both having a given relation to a given term’ (*ibid.*). But given the latter view such a relation is not a general relation but instances of some relation. Thus the latter view makes it impossible to account for instances of a relation on pain of an explanatory circle. Russell thus concludes that relations do not have instances and that it is an abstract relation itself that connects the other constituents of a proposition.⁹

Yet another point of difference is presumably the plainest to observe: Russell distinguishes *terms* into *things* and *concepts* while Moore draws no such distinction. In Russell’s view, such terms as Socrates can *only* be a logical subject in a proposition, while some other terms, like humanity, are capable of occurring in a proposition otherwise than as a logical subject. The former are called *things*, the latter *concepts*.

This distinction leads to a closely related point of divergence between these two versions of realism. It is about how ordinary objects are understood. Moore seems to maintain that ordinary objects are collections of concepts. In fact, his argument against the identity of indiscernibles, which we will examine in Section 2.2.2, seems to presuppose that what we normally count as objects are collections of concepts. On the other hand, Russell’s distinction between things and concepts suggests that he regards ordinary objects as *things* and in particular as simple, individual terms. He remarks unhesitatingly:

Points, instants, bits of matter, particular states of mind, and particular existents generally, are things in the above sense, and so are many terms which do not exist, for example, the points in a non-Euclidean space and the pseudo-existents of a novel.

(*PoM*, §48)

He goes further to include humans among ‘many terms which have no parts’ in an 1898 manuscript ‘An Analysis of Mathematical Reasoning,’ back to which we can trace the definition of term as ‘whatever can be a logical subject’ (*CP2*, p.211; p.169). It seems that he understands ordinary objects not as complexes of terms but as mere points of predication—individual objects that can be compared to what we now call *bare particulars*. Thus, although Moore regards ordinary objects as collections of concepts, Russell takes them to be individual bearers of properties. Note also that he does not thereby think that all terms are *mere* bearers of various properties. He rather maintains that concepts encapsulate ‘a special kind of diversity which may be called conceptual’ (*PoM*, §50). The idea is, it seems, that if two *things* should

⁹ In a short manuscript ‘Do Differences Differ?’ written apparently in 1898, he rejects abstract relations on the ground that they would lead us to an endless regress of a vicious kind (*CP3*, pp.55-57). In a 1901 draft of *PoM*, he still maintains that all *concepts* including relations have instances (*ibid.*, p.190). He must have changed his view on this issue by May 1902, when he completed the body of *PoM*.

be stripped of all their properties, we could no longer tell which one is one, even though we could still observe that they are numerically different; but even in such a situation, we could still differentiate two *concepts* thanks to the conceptual difference between them.

The idea that terms are bearers of properties illustrates that Russell intends to replace the notion of substance with that of term. In *PoM*, he remarks:

A term is, in fact, possessed of all the properties commonly assigned to substances or substantives. Every term, to begin with, is a logical subject: it is, for example, the subject of the proposition that itself is one. Again every term is immutable and indestructible. What a term is, it is, and no change can be conceived in it which would not destroy its identity and make it another term.

(*PoM*, §47)

As some philosophers have pointed out, there appears to be a sheer conflict between his generous inclusion of ordinary objects into terms and his unequivocal contention that terms are 'immutable and indestructible.'¹⁰ Although I cannot fully discuss various accounts given thus far by those philosophers, I find it plausible to think, as Hylton does, that terms are considered to be atemporal beings and apparent changes occurring to them are explained as various predicates attaching them with references to points of time (Hylton, 1990, pp.173-4). Hylton refers to the following passage in *PoM*:

The so-called predicates of a term are mostly derived from relations to other terms; change is due, ultimately, to the fact that many terms have relations to some parts of time which they do not have to others. But every term is eternal, timeless, and immutable; the relations it may have to parts of time are equally immutable. It is merely the fact that different terms are related to different times that makes the difference between what exists at one time and what exists at another. And though a term may cease to exist, it cannot cease to be; it is still an entity, which can be counted as one, and concerning which some propositions are true and others false.

(*PoM*, §443)

It thus seems that Russell understands terms as entities which go through changes. In this respect, his notion of term inherits a certain characteristic of the old notion of substance. As we saw in the preceding chapter, Russell in *Leibniz* characterises substance as 'that which can only be subject, not predicate, which has many predicates, and persists through change' (*Leibniz*, §17). Both terms and substances are thus considered to be bearers of properties.

¹⁰ See Hylton (1990, p.139), Makin (2000, p.181) and Candlish (2007, p.109). Peter Simons indicates a similar but still distinguishable problem 'as to how a concrete object existing in space and time like Mont Blanc can be literally part of a non-spatial, non-temporal proposition' (Simons, 1999, p.90).

But Russell thinks that his notion of term is free from the various problems which he argues the notion of substance faces. In the following two sections I will introduce two important doctrines by which he distinguishes the notion of term from that of substance, thereby freeing the former from these problems with the latter.

2.3. Reality of relations

In *Leibniz* and many other subsequent works, Russell emphasises that relations are *real* in the sense that they are not reducible to monadic properties.¹¹ The reality of relations should be distinguished from the stronger claim that relations are among individual entities; for the former is compatible with a type-theoretic distinction between relations and individual entities while the latter is not. In this section, I will first introduce various roles which the reality of relations plays in his metaphysical picture, before discussing how the reality of relations differentiates the notion of term from that of substance.

First of all, it has been pointed out that Russell takes relations as indispensable in order to account for mathematics (e.g. Hylton, 1990, pp.183-4). In *PoM*, he takes the notion of *order* or of *series* to be 'essential to any understanding of the foundations of mathematics' (§187). He goes on to argue that 'all order depends upon transitive asymmetric relations' and that unless the subject-predicate doctrine is abandoned, asymmetric relations cannot be accommodated (§208). He thus maintains that we must accept the reality of relations or else we fail to understand mathematics.

The reality of relations also underpins the mind-independence of propositions. The dyadic-relation theory urges that our judging act is a binary relation between the judging mind and a proposition. But if the reality of relations is denied, it follows that no relational sentences can be ultimately true and hence we cannot understand a judgment as a binary relation. Thus, the mind-independence of propositions relies on the reality of relations. Russell himself observes this point while discussing Lotze's view that relations are merely 'presentations in a relating consciousness' or 'beliefs in propositions asserting relations between the terms which appear related' (§425). According to Russell, this view entails the following:

The objects concerning which the beliefs are entertained are as a matter of fact wholly unrelated; indeed there cannot even be *objects*, for the plural implies diversity, and all beliefs in the relation of diversity must be erroneous. There cannot even be one object distinct from myself, since this would have to have the relation of diversity to me, which is impossible.

(*PoM*, §425)

¹¹ *PoM*, §94; *NT*, pp.500-2; *ONT*, pp.446-450; 'Mysticism and Logic,' pp.18-9; *PP*, pp.147-154; *OKEW*, p.47; *PLA*, pp.36-7.

The sharp distinction between the act of judging and the judgmental content is a characteristic of Moore and Russell's ontology of propositions, whereas Bradley says: 'what I repudiate is the separation of feeling from the felt, or of the desired from desire, or of what is thought from thinking, or the division—I might add—of anything from anything else' (*AR*, p.146). The reality of relations enables Moore and Russell to retain the sharp distinction between one's act of judging and the object of it.

Furthermore, the reality of relations underlies Russell's response to Bradley's regress. As we saw in the previous section, Russell distinguishes regress of implication from regress of meaning, claiming that only the latter is harmful. This response is based on the notion of *material implication*. He remarks that 'the assertion that q is true or p false turns out to be strictly equivalent to " p implies q "' (*PoM*, §16). A consequence of thus understanding implication is, of course, that when p implies q , the contents of p and of q may have nothing to do with each other. In other words, material implication may hold, even when the meaning of ' p ' has no relevance to that of ' q .' It is, I think, this observation that enables Russell to draw the distinction between regress of meaning and regress of implication. Now it should be observed that the notion of material implication in turn requires the reality of relations to the extent that it is considered as an external relation between propositions. Once relations are accepted as real, nothing prevents us from thinking that the relation of implication is simply an external relation that can hold between two entities regardless of the constituents of them. This point is perhaps related to the fact that he counts as an axiom of mathematics the statement that 'material implication is a relation' (*PoM*, §30).¹²

We have so far seen three important roles that the reality of relations plays in Russell's ontology of propositions. Let us turn to the consequences of the reality of relations which are more directly connected with his notion of term. The most important one is that the indistinguishability argument, which urges that there cannot be more than one substance, does not apply to the notion of term. In *Leibniz*, Russell argues that the subject-predicate doctrine leaves us no way of distinguishing two substances from each other prior to the attribution of various monadic properties to them. But no such difficulty arises with terms, as we can still differentiate two terms by means of relations owing to the reality of them. The subject-predicate doctrine merges numerical diversity into conceptual diversity, making it impossible to have two distinct bearers of conceptual diversity. On the other hand, the reality of relations allows us to distinguish numerical diversity from conceptual diversity. Once relations are considered to be real, numerical diversity becomes a simple matter of the relation of *being different from* holding between two terms.

¹² It is presumably not a coincidence that Russell, in his 1910 response to Bradley's objections to his ontology of terms, discusses the notion of material implication just after he re-introduces his solution to Bradley's regress (Russell, 1910b, pp.374-5).

This point leads to a second implication of the reality of relations for the notion of term: 'a space composed of points is not self-contradictory' (§431). In *PoM*, Russell takes on Lotze's various arguments against the view of space as composed of points. One of his arguments goes as follows:

All points are exactly alike, yet every pair have a relation peculiar to themselves; but being exactly like every other pair, the relation should be the same for all pairs.

(*PoM*, §424)

In response, Russell contends that this argument presupposes the identity of indiscernibles, which he takes to be a consequence of the subject-predicate doctrine. Once the doctrine is rejected, he continues, there is no 'logical' reason to think that two things exactly alike are in fact one and the same (cf. *Leibniz*, §66). In this way he rejects all the objections to what he calls 'the view of absolute position,' that is, the view that 'there are true propositions in which spatial relations are asserted to hold timelessly between certain terms, which may be called spatial points' (*PoM*, §423). This should come as no surprise, since, as we saw in the preceding section, Russell in *Leibniz* argues that the subject-predicate doctrine is incompatible with the reality of space. The view of absolute position is clearly what Russell in *Leibniz* calls 'the Newtonian theory of absolute space,' which he there argues is incompatible with the subject-predicate doctrine. In his view it is the subject-predicate doctrine that has prevented his preceding philosophers from developing a tenable theory of space:

The objection to Newton's theory is, that it is self-contradictory; the objection to Leibniz's, that it is plainly inconsistent with the facts, and, in the end, just as self-contradictory as Newton's. A theory free from both these defects is much to be desired, as it will be something which philosophy has not hitherto known.

(*Leibniz*, §61)

But he has the notion of term in hand now and hence thinks that there is no logical objection to the view of actual space as composed of points:

The difficulties which used to be found [...] in the nature of space [...] seem to have been derived almost exclusively from general logic. With a subject-predicate theory of judgment, space necessarily appears to involve contradictions; but when once the irreducible nature of relational propositions is admitted, all the supposed difficulties vanish like smoke.

(*PoM*, §431)

To be precise, he is careful not to claim that our universe is indeed composed of points; he only maintains that the view of space as composed of points is not self-contradictory. The

rejection of the subject-predicate doctrine, and in particular, the notion of *term* have thus freed him from the dilemma which once confronted him. This is precisely how he undertakes the 'task not yet accomplished,' namely, the task of basing 'metaphysics on some notion other than that of substance' (*Leibniz*, §71).

A third consequence of the reality of relations for the notion of term is related to the idea that points possessing exactly the same monadic properties can still be differentiated by relations. If the subject-predicate doctrine holds, we can only differentiate two entities by pointing to a monadic property which only one of the two possesses. However, once relations are counted as real objects, it is possible for one to distinguish an entity from any other entity simply in virtue of one's having an epistemic relation to it. Moore suggests this idea in 'Identity.' He argues that the identity of indiscernibles entails:

We can never say, "This red differs from that red, in virtue of having a different position"; or "in virtue of having a different spatial relation to this other thing"; or "as being the one I think of now, whereas that was the one thought of then."

(Moore, 1900-1, p.109).

He thus holds that the identity of indiscernibles makes it impossible to distinguish two instances of a simple property from each other. We will discuss in the next section how Moore thinks the identity of indiscernibles does so. What is important here is the idea, which seems to underlie the above passage, that one can distinguish an entity from others in virtue of a relation which one has to that object. Moore seemingly maintains that the reality of relations makes it possible to invoke such a relation in order to indicate a point of difference, even though he does not clarify whether one manages to identify it by *knowing that* one has a certain relation to it. In Section 2.5, I will argue that Russell also subscribes though more implicitly to the same idea. But before doing so let us turn to another important respect in which the ontology of propositions makes a contrast to Bradley's theory of judgment.

2.4. Properties as entities

In this section, I will discuss Moore and Russell's claim that if properties are among self-subsistent entities, monistic idealism breaks down. Let us begin by looking into Moore's remarks on this claim before we turn to Russell's.

In NJ, as we have seen, Moore maintains that reality is composed of concepts, where they are considered to be self-subsistent entities. At the end of the paper, he remarks:

A concept is not in any intelligible sense an 'adjective,' as if there were something substantive, more ultimate than it. For we must, if we are to be consistent, describe what appears to be most substantive as no more than a collection of such supposed

adjectives: and thus, in the end, the concept turns out to be the only substantive or subject, and no one concept either more or less an adjective than any other.

(NJ, pp.192-3)

He seems to contrast his own view of concepts with Bradley's view that our judgment always attributes something ideal and hence not real to the genuine entity. As we saw in Chapter 1, it is indeed one of Bradley's central observations that a content with which we describe the given reality is always ideal and hence incapable of fully capturing it. A point of difference between these metaphysics thus lies in the ontological status of properties.

Moore's argument in 'Identity' illustrates the importance of this point of difference in his realist ontology. He argues that if properties count as genuine entities, 'the theory that there is no difference but conceptual difference' is rejected (Moore, 1900-1, p.110). The theory in question is in our terminology the identity of indiscernibles, since he remarks that when the theory is granted, 'to talk of two things exactly alike, or with no conceptual difference, is to talk sheer nonsense—mere words' (*ibid.*, p.106). His argument against the identity of indiscernibles is complicated, as it not only tries to refute the contentious doctrine on identity but also is meant to establish that concepts—or what he calls 'predicates' in 'Identity'—have instances. He connects these two views presumably because he thinks that the identity of indiscernibles makes it hard, if not impossible, to maintain that a predicate has various instances. Indeed, it is reasonable to suppose that all the instances of a predicate, if there are such things at all, share the same monadic properties, even though they may well have different relations to various objects. Moore sets forth his argument by claiming that the identity of indiscernibles entails that when two distinct objects have a common predicate, we can always find '1) point of difference; 2) relation of predication; 3) common predicate' (Moore, 1900-1, p.108). He seems to mean by 'point of difference' a pair of distinct properties by virtue of which we can differentiate two collections of predicates. It should be remembered that he understands ordinary objects as collections of predicates (cf. p.36). He then adds that we must regard the two properties which constitute a point of difference as bearers of other properties. In his view, the identity of indiscernibles thus implies that two distinct objects contain two distinct predicates: 'We must start, on this theory, with two points of difference—two simple predicates having conceptual difference from one another, this is essential to there being two things at all' (*ibid.*). But if so, Moore continues, the following obtains:

We can never say, "The red I mean is the one surrounded by yellow, and not the one surrounded by blue." For the one surrounded by yellow is also surrounded by blue: they are not two but one, and whatever is true of that which is surrounded by yellow is also true of that which is surrounded by blue.

(Moore, 1900-1, pp.109-110)

The idea is that we cannot tell the difference between two occurrences of a simple object such as redness. For there is no 'point of difference' between such occurrences and hence they must be one and the same object. Although he remarks 'We can never *say*' in the above passage, his point is, of course, not about whether one can utter a sentence but rather about whether one can meaningfully *judge* that an occurrence of redness is distinct from another. Moore thus rejects both the identity of indiscernibles and the view that predicates do not have instances. If the former implies the latter, the above argument in effect serves as an objection to the identity of indiscernibles.

It is important to observe that Moore in arguing thus takes for granted that a property can be the subject of our judgment. This should indeed be treated as an assumption, since Bradley does not accept it. In Bradley's view, our judgment does not have the property of redness as a subject even when we judge 'The redness is surrounded by yellow.' Both Moore and Bradley arguably accept that only self-subsistent entities can be a subject of our judgement. But they disagree with each other in regard to what can be a subject of our judgment. In this respect the idea that predicates are self-subsistent entities marks a crucial difference between these two philosophers.

Hylton indicates another way in which the notion of properties as genuine entities can lead to the refutation of monistic idealism. According to him, Moore holds that the notion entails the mind-independence of propositions. Hylton points out that British Idealists such as Thomas Green and Bradley maintained that every judgment involves a relation (Hylton, 1990, pp.29-30, 154-5). He then argues that Moore follows his idealist predecessors to maintain that since every judgment is relational, judgments and relations have the same ontological status (*ibid.*, p.121). If so, propositions should be on the list of entities as long as relations are taken to be entities. However, it is not entirely clear whether Moore envisaged this argument. For, as we saw in Section 2.2, he in NJ seemingly endorses the view that the relation which combines concepts to form a proposition belongs to a peculiar category. The ontological peculiarity given to the relation is, on this view, precisely the source of its peculiar nature of combining various entities. Thus, as long as Moore endorses this account of the unity of the proposition, he is unlikely to invoke the argument in question.¹³

Let us turn to Russell. He argues that once the notion of term is accepted, every property counts as a term. Closely related to this argument is the fact that the notion of *term* is in a sense the dual notion of *substance*, which he thinks is based on the subject-predicate doctrine. He defines a substance to be 'that which can *only* be subject, not predicate, which has many predicates, and persists through change' (*Leibniz*, §17; emphasis added). By contrast, he holds that whatever *can* be a logical subject is a term. Thus, a substance is something that is always

¹³ Moore became unconvinced of the account soon after the publication of NJ, as we shall see later in Chapter 6.

a subject of our judgment, whereas a term is something that sometimes occurs as a logical subject. Importantly, when terms are thus defined, it follows that properties count as terms, since it is self-contradictory to hold that a property is not a term. Hence, he remarks:

Thus the theory that there are adjectives or attributes or ideal things, or whatever they may be called, which are in some way less substantial, less self-subsistent, less self-identical, than true substantives, appears to be wholly erroneous, and to be easily reduced to a contradiction.

(*PoM*, §49)

What he has in mind is arguably Bradley's theory of judgment, according to which an ideal content which one uses to judge is a mere 'adjectival' of reality.

The duality between term and substance leads to the opposite conclusions as to how many entities can be. If Russell's indistinguishability argument is valid, there is at most one substance (cf. Section 1.3). On the other hand, he thinks that if properties are terms, monism immediately breaks down:

Numerical identity and diversity are the source of unity and plurality; and thus the admission of many terms destroys monism. And it seems undeniable that every constituent of every proposition can be counted as one, and that no proposition contains less than two constituents.

(*PoM*, §47)

The notion that properties are among entities thus provides Russell with a direct way of arguing against Bradley's idealist metaphysics. Griffin points out that the reason why he extends the notion of being to non-existent objects lies in that 'he needs to take account of the occurrence of predicates as subjects' (Griffin, 1991, p.295). Given what we have seen, Russell presumably finds it necessary to allow properties to be subjects not just because we normally make a judgment about a property but because it will undermine monistic idealism.

There is yet another way in which Russell thinks the view of properties as genuine entities undermines monistic idealism. As we mentioned in the preceding section, Lotze holds that relations are either 'presentations in a relating consciousness' or 'internal states in the real elements which are said to stand in these relations' (*PoM*, §424). Russell responds to him by holding that it amounts to the subject-predicate doctrine—the view that 'there is only one thing, God or the Absolute, and only one type of proposition, namely that ascribing to predicates to the Absolute' (§425). He then argues that this view is self-contradictory in that the proposition 'there are predicates' must be a part of the view but not of the subject-predicate form. Thus, even if the subject-predicate doctrine is granted, monistic idealism is, in his view, untenable as long as (monadic) properties are entities.

2.5. Russell's conception of language as psychological

In the last two sections, we have discussed two important points which Moore and Russell make in their attempt to replace Bradley's theory of judgment with their ontology of propositions. This section is devoted to illustrating how these two points underlie Russell's conception of language which he hints at in *PoM*. The following discussion will also cast light on a commonality between Russell and Bradley: the former inherits from the latter a certain view on what symbols are and how they work.

Let us first take a look at what Russell says about language in *PoM*. Surprisingly or not, he pays little attention to language in that work and the following passage is presumably the only place where he explicitly states his view on this matter:

To have meaning, it seems to me, is a notion confusedly compounded of logical and psychological elements. *Words* all have meaning, in the simple sense that they are symbols which stand for something other than themselves. But a proposition, unless it happens to be linguistic, does not itself contain words: it contains the entities indicated by words. Thus meaning, in the sense in which words have meaning, is irrelevant to logic. But such concepts as a *man* have meaning in another sense: they are, so to speak, symbolic in their own logical nature, because they have the property which I call *denoting*.

(*PoM*, §51)

The relation that holds between words and their meanings is here said to be 'irrelevant to logic' on the ground that it is psychological.¹⁴

Russell does not explain why he thinks the relation between words and their meanings is psychological. It seems as though he takes for granted that symbols exist in one's mind and, therefore, that any account of the relation between symbols and their meanings necessarily involves psychological investigations into the mind. Indeed, he explains the confusion between the logical and psychological elements of meaning as follows:

The confusion is largely due, I believe, to the notion that words occur in propositions, which in turn is due to the notion that propositions are essentially mental and are to be identified with cognitions.

(*PoM*, §51)

He seems to assume that symbols are themselves mental. For without this assumption, symbols would not have to occur in propositions, even when propositions are supposed to be

¹⁴ What Russell regards as the logical notion of *meaning* is the relation called 'denoting,' which I will discuss in Chapter 4.

'essentially mental.' He thus seems to think that symbols, in the philosophically fundamental sense of the word 'symbol,' exist in one's mind. The qualification 'in the philosophically fundamental sense' is essential, as he would of course deny that a figure of ink, say, exists in one's mind. Presumably, when he understands symbols to be essentially mental, he considers symbols which one employs to make a judgment. If we call symbols of this kind *internal symbols* as opposed to *external symbols*, his assumption can be rephrased as follows: internal symbols are what philosophers have to mainly consider if they attempt a theoretical understudying of symbols in general. This fits the point that his ontology of terms is meant to be a replacement of Bradley's theory of judgment. For symbols which are important to the idealists are not those which we find outside of our experience but those by which we make a judgment.¹⁵ The idea that symbols primarily live in one's mind thus indicates a commonality between Russell and his idealist predecessors.

Russell thus seems to view a language merely as a mental device with which we can speak of *propositions*, the subject matter of his theory of logic. Let us look into how his conception of language is connected with the two important doctrines which we have seen. After presenting his view of properties as entities, he remarks:

It is interesting and not unimportant to examine very briefly the connection of the above doctrine of adjectives with certain traditional views on the nature of propositions. It is customary to regard all propositions as having a subject and a predicate, *i.e.* as having an immediate *this*, and a general concept attached to it by way of description. This is, of course, an account of the theory in question which will strike its adherents as extremely crude; but it will serve for a general indication of the view to be discussed. This doctrine develops by internal logical necessity into the theory of Mr Bradley's Logic, that all words stand for ideas having what he calls *meaning*, and that in every judgment there is a something, the true subject of the judgment, which is not an idea and does not have meaning.

(*PoM*, §51)

Russell tacitly appeals to his indistinguishability argument in saying that the view of judgment as 'having a subject and a predicate' is the same as the claim that each judgment ascribes a general concept to 'an immediate *this*.' It is plain that a judgment composed of 'an immediate *this*' and 'a general concept attached to it' is of the subject-predicate form, but, in the opposite direction, he invokes the argument, according to which the doctrine makes a substance 'wholly destitute of meaning' (*Leibniz*, §25). Yet, it is unclear why the subject-

¹⁵ If so, it is by no means a mere coincidence that Russell, from 1912 onwards, sets forth his theory of meaning in the solipsistic setting where the Cartesian doubt plays a significant role.

predicate doctrine 'develops by internal logical necessity' into the view that all words stand for what Bradley calls meanings. A key to explaining why should be found in his 'doctrine of adjectives,' namely, the idea that properties are self-substantive entities (*PoM*, §49).

As we saw in the previous section, Russell holds that monistic idealism breaks down if properties are entities. In other words, monism implies, in his view, that properties are 'less substantial, less self-subsistent, less self-identical, than true substantives' (*ibid.*). Hence, the subject-predicate doctrine, which he thinks is equivalent to monistic idealism, entails that properties are not entities. This explains why he thinks that given the doctrine, predicates such as 'red' can only stand for what Bradley calls 'meanings' or 'ideal contents,' which are considered not to be real. The notion that monism is incompatible with properties's being entities thus underlies the passage quoted above.

However, it is still unclear why the subject-predicate doctrine implies that *all* words should stand for such ideal things. In particular, it is not clear why we cannot use proper names to designate entities rather than ideal contents. A possible answer is given by Moore's argument against the identity of indiscernibles in his 'Identity.' As we saw in Section 2.3, he suggests that unless the reality of relations is granted, one cannot identify an entity simply in virtue of having a certain relation to the entity. Hence, *if* we must be able to identify an entity in order to use a proper name for it, the subject-predicate doctrine prevents us from using a proper name for an entity. On the contrary, once the reality of relations is granted, we can employ a proper name for an entity to which we have a certain relation, even if it is assumed that we must be able to differentiate the entity in order to name it. Russell thus seems to assume that one must be able to identify an entity in order to use a proper name for it. This explains why he maintains that the subject-predicate doctrine makes it impossible for us to use a proper name for any entity. Interestingly this in turn means that he has already got all the ingredients of his notion of *acquaintance*, even though he does not explicitly introduce it until 1905.¹⁶

Thus, the reality of relations and the self-subsistence of properties underlie Russell's contention that every name stands for an entity. In other words, he thinks that the subject-predicate doctrine, which is incompatible with either of these claims, prevents us from understanding our language as a transparent medium through which we can speak directly of entities.

One may well wonder whether Russell in fact presupposes that in order to designate an entity by a proper name, one must be able to identify the entity. Admittedly, he does not articulate this presupposition, much less explicitly argue for it. But Bradley does. In *PoL*, he rejects the view that a proper name designates an entity without any intermediaries:

¹⁶ Noonan (1996) and Hylton (1990, p.247) also point this out, while Amijee (2013) argues otherwise.

There is a curious illusion, now widely spread, on the subject of proper names. We find it laid down that a proper name has not got *connotation*, or, to use the more common technical term, it has no *intension*. In ordinary language, it *stands* for something but does not *mean* anything.

(PoL, p.59)

He rejects the view that a proper name '*stands* for something but does not *mean* anything' primarily for the following reason:

Originally imposed as an *arbitrary* mark, that very process, which makes it a sign and associates it firmly with the thing it signifies, must associate with it also some qualities and characters of that which it stands for. If it did not to some extent get to *mean* the thing, it never could get to *stand* for it at all. And can any one say that a proper name, if you are aware of its designation, brings *no* ideas with it, or that these ideas are mere chance conjunction?

(PoL, p.60)

Bradley seems to think that there must be something in virtue of which a symbol is tied with its designation and that a property attached to the object can only be such a thing. This suggests that he also subscribes to the assumption in question, namely, that one must be able to identify an object to use a name for it. And this indeed becomes clearer, as he goes on to argue that even when one employs a proper name to make a judgement about an object, the judgment is an attribution of a universal meaning to the reality:

The meaning of such a name is universal, and its use implies a real universality, an identity which transcends particular moments. For, unless the person were recognized as distinct, he would hardly get a name of his own, and his recognition depends on his remaining the same throughout change of context. We could not recognize anything unless it possessed an attribute, or attributes, which from time to time we are able to identify.

(PoL, p.61)

Bradley thus argues that one's ability of identifying an entity is a necessary condition for the employment of a symbol which designates it. Given that Russell does not take on this point in his attempts to refute Bradley's theory of judgment, we may say that he also accepts the idea that one must be able to identify an object in order to use a name for it.

2.6. Concluding remarks

In Section 2.2, we saw that Moore replaces Bradley's notion of *idea* with his own realist notion of *concept*, and Russell in turn develops the latter into the notion of *term*, which he takes to be a proper alternative to the defective notion of *substance*. An important way in which Russell's notion of term makes a sharp contrast to the notion of substance is, as we saw in Section 2.3, that terms can be distinguished from one another in virtue of relations between them. An important corollary of the reality of relations is that our act of judging can be understood as a binary relation between the mind and a proposition, these two being sharply distinguished from each other. What I call the dyadic-relation theory of judgment is thus underpinned by the reality of relations. Another point of difference between the ontology of propositions and Bradley's theory of judgment is whether ideas or concepts are considered as self-independent entities or merely as 'less substantive' descriptions of substantive entities. This point of difference is, as we saw in Section 2.4, closely related to Moore and Russell's contention that once properties are accepted as genuine entities, monistic idealism cannot be maintained. The reality of relations and the self-subsistence of properties thus seem to be the heart of Moore and Russell's attempts to replace the old theory of judgment by their realist ontology of propositions. But it is equally important to note that they did not simply reject Bradley's idealist theory of judgment altogether. Rather they obtained the ontology of propositions by replacing some idealist tenets with their realist doctrines one after another.

In Section 2.5, I argued that the reality of relations and the self-subsistence of properties also motivated one of Russell's central commitments, namely, the view of language as a mere psychological medium to speak of something beyond it. But, at the same time, he shared a certain view on proper names with Bradley. Russell's view of language thus seems to give us another indication that the ontology of propositions was not a sheer rejection of the whole idealistic framework but a result of his attempt to replace some idealistic notions such as the notion of ideal content with realist alternatives like his notion of term while inheriting among other things the notion that logic was composed of the theory of logic and the theory of calculus.

But we have yet to see how the ontology of propositions was meant to work as the theory of logic. This is the primary task of the following three chapters.

3 Early Theories of Classes

3.1. Introduction

Russell ‘discovered’ the set-theoretic paradox which bears his name around May or June 1901. The quotation-marks are put here because he first took the contradiction as a ‘very subtle fallacy’ on Cantor’s side and it took a while for Russell to come to see it as a genuine paradox arising from natural assumptions on classes (Grattan-Guinness, 2000, pp.310-2). Russell had thus already known of the paradox before he completed *PoM* in December 1902. It may then come as no surprise that he finished the book with the following remark:

In the case of classes, I must confess, I have failed to perceive any concept fulfilling the conditions requisite for the notion of class. And the contradiction [...] proves that something is amiss, but what this is I have hitherto failed to discover.

(PoM, pp. xv-xvi)

It should, however, be noted that Russell regards the set-theoretic paradox as a *proof* that ‘something is amiss.’ He seems to think that the genuine problem lies not in the paradox itself but rather in the fact that he has not found any tenable account of classes which meets ‘the conditions requisite for the notion of class.’

Given what we have seen thus far, all this should not be surprising. For Russell inherited the idea that any successful calculus of logic must be vindicated by the theory of logic. The present chapter is intended to provide an illustration of this claim. I will argue that he exploited the ontology of propositions—his replacement of the idealist’s theory of judgment—in his attempts to account for what classes are. It will be seen that the original theory of types presented in the appendices of *PoM* was obtained through a modification of a central doctrine in his ontology of propositions.

The early Russell’s account of classes is not a new topic in the literature. The original theory of types has attracted attention from many commentators.¹ But in my view the whole story behind the remark quoted above has not been fully uncovered. As Levine points out, the very first theory of types was a response to Frege’s objections to the theory of classes presented

¹ See for example Urquhart (1988), Cocchiarella (1980, pp.73-7) and Hylton (1990, pp.222-6).

in the body of *PoM* and any account of classes analogous to it (Levine, 2001). However, Appendix A of *PoM*, where Russell critically introduces Frege's works up to the first volume of *Grundgesetze der Arithmetik*, has scarcely been consulted in most of the literature on Russell's account of classes.

In light of this, I will devote Section 3.3 to the appendices of *PoM* and argue that the type-theory presented there could not overcome all the objections from Frege, which led Russell to make the remark quoted above. Before doing so I will introduce in Section 3.2 the theory of classes presented in the body of the book. These two sections will illustrate how he made use of the ontology of propositions in order to offer a philosophical account of classes. In Section 3.4, I will sketch how he viewed classes during the period from 1903 to 1910. It will be seen that although Russell eventually found a way of dispensing with classes, he maintained throughout the period that a correct philosophical account of classes was yet to come. In the following I will not distinguish classes from sets, though it is customary to regard the former as including the latter as well as what we call proper classes. The distinction was alien to Russell and my discussion will not depend on it.

3.2. The original theory of classes

Russell completed the final draft of *PoM* in May 1902, but after that he greatly changed his account of classes under the influence of Frege. Most of the remarks on Russell's former theory of classes can still be found in the body of the book. I will call it his *original theory of classes*. The aim of this section is to introduce it.

To begin with, it should be noted that Russell clearly envisages that classes must be *extensional* in the sense that the identity between two classes solely depends on their members. In setting forth the theory of classes in the final draft of *PoM*, he remarks:

It is essential that the classes with which we are concerned should be composed of terms, and should not be predicates or concepts, for a class must be definite when its terms are given, but in general there will be many predicates which attach to the given terms and to no others. We cannot of course attempt an intensional definition of a class as the class of predicates attaching to the terms in question and to no others, for this would involve a vicious circle.

(*PoM*, §66)

Russell is thus well aware that a class cannot be identified with a corresponding *concept*: distinct concepts may well define the same class. However, the idea that classes must be extensional leads him to a naïve-looking account of classes. This is because he takes it for granted that in order for classes to be extensional they 'should be composed of terms.' I will

discuss why he does so later. For the moment, let us see how he tries to explain classes as composed of terms. For this purpose he introduces the notion of *numerical conjunction*. So we shall first look into his theory of *denoting concepts*, to which the notion belongs.

As we saw in Section 2.2.2, Russell divides *terms* into *things* and *concepts*. Furthermore, he draws a distinction between those *concepts* which are indicated by verbs and those by adjectives (*PoM*, §48). Each concept which is designated by an adjective is called either a *predicate* or a *class-concept*. *Denoting phrases* are the expressions of the form 'any *u*,' 'all *u*'s,' 'every *u*,' 'some *u*,' 'a *u*,' or 'the *u*,' where *u* is a class-concept. Importantly, he thinks it essential to distinguish these six forms of denoting phrases sharply from one another (§58). A denoting phrase of the form 'the *u*' designates the concept which *denotes*, if any, a unique entity which satisfies *u*, while denoting phrases of the other forms designate a concept *denoting* a certain sort of combination of terms. For example, a denoting concept of the form 'all *u*'s' is considered to denote a *numerical conjunction* of terms, while a denoting concept of the form 'every *u*' denotes a *propositional conjunction* of terms. A numerical conjunction of terms occurs, say, in the proposition 'Brown and Jones are two of Miss Smith's suitors.' This proposition is about both Brown and Jones collectively, since 'this is not true of either separately' (§59). Importantly, this is not the case with such a proposition as 'Brown and Jones are paying court to Miss Smith,' in which the propositional conjunction of Brown and Jones occurs. This proposition is equivalent to the conjunction of two propositions 'Brown is paying court to Miss Smith and Jones is paying court to Miss Smith.'

One might wonder why Russell offers such a naive-looking account of quantificational expressions as the theory of denoting concepts. It is certainly not due to his poor understanding of predicate logic. For he observes that 'the explicit mention of *any*, *some*, etc., need not occur in Mathematics: formal implication will express all that is required' (*PoM*, §87). He seems to envisage that once universal quantification and implication are granted, the other logical connectives can be defined in terms of them.² He also expressly remarks: 'any *a* is a *b*' and '*x* is an *a* implies *x* is a *b*' 'do not *mean* the same thing' (§89). He develops the theory of denoting concepts not because he is yet to fully appreciate the expressive power of predicate calculus but rather because he seeks to give philosophical accounts, based on his ontology of propositions, to such technical notions as classes.

In fact, Russell proceeds to define a class to be a numerical conjunction of its members. Hence, in his view, the proposition 'Brown and Jones are two of Miss Smith's suitors' attributes a property to a class, that is, the class of Brown and Jones. But the proposition does not thereby turn Brown and Jones into a single entity; the predicate '*two* of Miss Smith's suitors' cannot be truly applied to any single entity. Hence, in Russell's view, the proposition is not about

² Russell defines negation in terms of universal quantification over propositions (*PoM*, §19).

a single logical subject composed of Brown and Jones, but about both Brown and Jones. In other words, Brown and Jones *are* logical subjects of the proposition.

This theory of classes has some consequences. One is that classes can be *denoted* by concepts of the form ‘all *u*’s.’ Every class can be determined either by a *concept* denoting the class or by enumeration, e.g. ‘Brown and James,’ where ‘and’ is the notion of numerical conjunction. Russell calls the first way of determining a class ‘intensional’ and the second ‘extensional’ (§66, §68). It should be noted that enumeration is for him not a mental or physical act of counting terms. He remarks that ‘*logically*, the extensional definition appears to be equally applicable to infinite classes’ (§71, my emphasis). This is not surprising, since the theory of logic for him is the platonist ontology of *terms*. Since we can use a concept which denotes a class, we can speak of an infinite class, and this is ‘the inmost secret of our power to deal with infinity’ (§72). Thus, in his view, we need an intensional way of determining a class in order to speak of infinite classes. But, at the same time, classes must be ‘extensional’ in the sense of being composed of their elements, for otherwise they fail, Russell assumes, to be ‘extensional’ in the sense that their identity depends solely on their members. He thus thinks that both the extensional and intensional ways of speaking of a class are essential. This is why he holds that ‘there are positions intermediate between pure intension and pure extension, and it is in these intermediate regions that Symbolic Logic has its lair’ (§66).

There are two more consequences of greater importance.³ One is that there is no such object as empty class. If a class is composed of terms and nothing else, then the empty class is simply absent, since there is nothing to be combined. Russell is aware of this consequence, while he also recognises that ‘symbolically it is quite necessary to introduce some such notion’ as the empty class (§73). His solution to this apparent conflict is to interpret a symbol for the null class as indicating the class of class-concepts which are null, that is, class-concepts which are satisfied by no entity. This solution works insofar as there subsists at least one such class-concept. And once the notion of class is defined in terms of numerical conjunction, there is no circularity involved in this ‘roundabout’ account of the null class (Frege, 1980, p.138). The other important consequence is that a class which contains only one term is identical to the term itself, since there is nothing to be added to the term in order for it to be a class (§69, §71). He thus thinks that a singleton, or in his terminology, a unit class is always identical to its sole member.⁴

It should be noted that the original theory of classes is based on the rejection of the subject-predicate doctrine. Such propositions as ‘Brown and Jones are two’ have more than one subject and an attribute attached to them, thereby constituting a counterexample to the

³ These two consequences are characteristics of what Michael Potter calls ‘fusions’ as opposed to ‘collections’ (Potter, 2004, pp. 21-3, 31).

⁴ Whether this proposal can be made to work will be discussed in Section 3.3.

doctrine. As we saw in Chapter 1, Russell in *Leibniz* expressly holds that Leibniz could not admit aggregates because he subscribed to the subject-predicate doctrine (*Leibniz*, §63).

It is also worth noting that the above account of classes is closely related to Russell's definition of natural numbers as classes of classes (*PoM*, §109-§111). In his view, it is a class to which we attribute a natural number, as is explicit in the proposition 'Brown and Jones are *two* of Miss Smith's suitors.' Russell thus thinks of numbers as 'applicable essentially to classes' and admits that his theory of numbers 'depends fundamentally upon the notion of *all*, the numerical conjunction as we agree to call it' (§109).

One may well be tempted to compare the above account of classes to Frege's introduction of *Wertverläufe* as objects which satisfy the principle of extensionality. It may be puzzling that Russell did not thus define classes, given that he was aware of the extensional nature of classes. To explain why not, Hylton introduces the distinction between *judgement-based* metaphysics and *object-based* metaphysics. According to Hylton, 'the notion of judgement is taken as basic' by Frege and '[o]ne consequence of this is what is often called the context principle: only in the context of a judgement [Satz] does a word have meaning' (Hylton, 1990, pp.223-4). On the other hand, Russell endorses an 'object-based' metaphysics, holding that 'a satisfactory explanation of classes is required to tell us what a class is in itself' (*ibid.*). His original theory of classes certainly resulted from his enquiry into what classes themselves are:

I find it hard to see what a class really is if it does not consist of objects but is nevertheless supposed to be the same for two concepts with the same extension.

(Frege, 1980, p.139)

It seems to me more appropriate to differentiate Frege's standpoint and Russell's by describing them as 'sentence-based' and 'object-based' respectively. For one thing, the term 'judgment-based metaphysics' could also be employed to depict monistic idealism. For another, what strikes me as the real divergence between these two thinkers is whether one could wholly rely on a formal *language* in introducing new technical objects. Frege does not seem to find any problems with his elucidation of *Wertverläufe* in terms of his formal language. By contrast, Russell seeks an account of classes in terms of less metaphysically-burdened objects, that is, an account as to how classes themselves are composed of other objects.

Connected with the theory of classes in the final draft of *PoM* is an important distinction between *class as many* and *class as one*. 'Taking the class,' Russell remarks, 'as equivalent simply to the numerical conjunction "A and B and C and etc.," it seems plain that it is many' (§74).⁵ On the other hand, 'it is,' he admits, 'quite necessary that we should be able to count

⁵ Classes as many can be understood as what we now call pluralities. For the development on plural logic, see, for example, Boolos (1984), McKay (2006) and Oliver & Smiley (2013).

classes as one each,' partly because 'we do habitually speak of a class' (*ibid.*). To reconcile these two points, he introduces the distinction of class as many and class as one, and claims that (almost) every class as many has a corresponding class as one.⁶ For example, *all men* are a class as many, but *the human race* is considered to be a class as one composed of all people. He finds no problem in considering a class as one to be a *whole* composed of the terms of the class (§70). Importantly, the distinction between class as many and class as one also enables him to introduce classes of classes. For if we simply combine two classes as many, say, '*A and B*' and '*C and D*,' by means of the notion of numerical conjunction, we cannot obtain a class of these two classes but only a class as many: '*A and B and C and D*.' Thus, in order to allow a class to have classes as its members, we need to turn those classes as many into classes as one. Russell illustrates this point with 'classes of all rational animals' and 'men.' The first one is considered to be the unit-class whose sole member is the class (as one) of all human beings, while the second is the class (as many) of all human beings. In this way he explains how a class can be a member of another class.

Given that the set-theoretic paradox involves a class of classes, it should come as no surprise that Russell attempts, as Levine points out, to solve it by appealing to the distinction of class as one and class as many in the final draft of *PoM* (Levine, 2001, p.224). Russell thinks that 'the source of the contradiction' lies in the assumption that 'the class as one is to be found wherever there is a class as many' (*PoM*, §104). His idea is that there are certain classes as many which do not have a corresponding class as one, or more precisely, that some propositional functions of one variable can only determine a class as many but not a class as one. He holds that propositional functions of the form $\varphi(k_\varphi)$, where φ is a variable, are the ones which do not determine any class as one. Note that if we express '*x* is an *x*' by means of a propositional function, the function will take the above form. If propositional functions of this form do not determine any classes as one, we can avoid the paradox:

Thus if *w* be the class of all classes which can be made single subjects, but as such are not members of themselves, *w* will be only many, not one, and the question whether *w* as one is a member of *w* as many is a question concerning an entity which has no being, in fact a non-entity.

(Blackwell, 1984, p.288)

This line of thought was, as we will see in the following chapter, developed in 1904, though

⁶ It should be noted that the distinction between classes as one and classes as many does not correspond to Potter's distinction between collections and fusions (cf. footnote 3). One might think that classes as one are collections and classes as many are fusions. But, as we will see in the next section, Russell takes Frege's notion of *Wertverläufe* to be 'something other than the whole composed of the terms satisfying the propositional function in question' (*PoM*, §486). It should also be seen that Russell uses the word 'collection' to mean a class as many, which is in Potter's terminology a *fusion* (*PoM*, §71).

unsuccessfully. Russell thus tries to solve the paradox by placing a certain restriction on the correspondence between classes as one and classes as many.

One might think that Russell drew the distinction in order to resolve the paradox. But this is not the case. For he had drawn it *before* he was confronted by the paradox. He ‘discovered’ the paradox in May or June 1901, while he had drawn the distinction already in a draft of *PoM* written by June 1900: ‘collections of which integers other than 1 can be asserted are essentially not wholes, since every whole is one, while collections are many’ (*CP3*, p.35). Thus, the distinction between class as one and class as many was not invented as a solution to the paradox.

Before we move on to the first theory of types presented in the appendices of *PoM*, let us point to two potential problems with the theory of classes which we have seen. One is about the ontological status of classes as many. The notion that classes as many are essentially many and hence cannot be counted as *one* entails that a class as many is not a *term*. Russell indeed remarks that a combination of terms which a denoting concept denotes is itself not a term (§59). But if classes are non-entities, what are they? Klement attributes to him the idea that ‘there are, in the end, only individuals, and that a class as many is only a way of speaking about the members of the class as one in a way that does not reduce to singular predication of each of the members separately, *or* to a predication made about the class as one’ (Klement, 2014, p.3). But it is not plausible that Russell in the final draft of *PoM* endorses such a strictly individualistic metaphysics. For he grants ‘some kind of unity’ to a class as many:

In a class as many, the component terms, though they have some kind of unity, have less than is required for a whole. They have, in fact, just so much unity as is required to make them many, and not enough to prevent them from remaining many.

(*PoM*, §70)

In the final manuscript, he also remarks:

A class also, in one sense at least, is distinct from the whole composed of its terms, for the latter is only and essentially one, while the former, where it has many terms, is, as we shall see later, the very kind of entity of which many is to be asserted.

(Blackwell, 1984, p.286)

Russell calls a class as many an ‘entity’ here. Thus, it is difficult to think that classes as many are considered to be reducible to mere ways of speaking. It seems rather that he has not fully worked out the ontological status of classes as many in the final draft. In fact, as we will see in the next section, he replaces the word ‘entity’ here by ‘object’ in the printed version. This is, of course, related to the late introduction of the notion of *object*—which is absent from the final draft—‘to cover both singular and plural, and also cases of ambiguity’ (*PoM*, §58fn;

Blackwell 1984, p.285).

Another potential problem with the original theory of classes is as to whether Russell can accept objects other than *terms* without contradicting his own argument that whatever is a logical subject of a proposition is a term. As we saw in Section 2.2.2, he argues that it is self-contradictory to claim that there is something that is not a *term*. But he thinks that in the case of classes as many, there is no such difficulty. 'A plurality of terms is,' he explains, 'not the logical subject when a number is asserted of it: such propositions have not one subject, but many subjects' (*PoM*, §70fn). The idea is that if a proposition can be about many subjects, it is possible to speak of *them* without merging them into a single logical subject. However, Russell abandons this doctrine of logical subject in the appendices of *PoM*. To see why he does is in effect to see why he invents the first theory of types.

3.3. The original theory of types

Russell completed the final draft of *PoM* in May 1902, but he greatly changed his view of classes under the influence of Frege, with whom a correspondence started in June 1902. Russell devoted Appendix A of the book to the critical discussion of Frege's philosophy and Appendix B to the exposition of the very first theory of types motivated by Frege's objections to the notion of class as many. It will be seen that the way in which Russell comes to adopt the theory of types as a reply to Frege's criticisms illustrates the role which the ontology of propositions plays as the theory of logic.

To my knowledge, only a few authors have referred to Appendix A in their accounts of the original theory of types. Bell focuses on Appendix A to illustrate how the theory of types emerged, though he does not bother to discuss Frege's argument which I shall introduce shortly (Bell, 2012). Levine points out that Frege's objection to the notion of class as many and any analogous account of classes led Russell to propose, though tentatively, the first theory of types. According to Levine, 'Frege directly challenges Russell's distinction between a class "as many" and a class "as one"' in the correspondence between them. What Levine focuses on is the following remark in Frege's letter to Russell dated 28 July 1902:

If a class name is not meaningless, then, in my opinion, it means an object. In saying something about a manifold or set, we treat it as an object. A class name can appear as the subject of a singular proposition and therefore has the character of a proper name [...].

(Frege, 1980, p.140)

According to Levine, Russell, accepting this criticism, abandons the notion of class as many and thereby his solution to the set-theoretic paradox by means of the distinction of class as

one and class as many. Levine goes on to argue that Russell instead puts forward the theory of types to resolve the paradox (Levine, 2001, pp.227-8). Russell indeed seems to accept the above criticism, as he remarks in Appendix A that ‘without a single object to represent an extension, Mathematics crumbles’ (*PoM*, §489). However, the above point concerning proper names does not seem to reveal the whole reason why Russell gives up his original theory of classes. It seems to me that he comes to endorse the theory of types through his attempt to respond to Frege’s argument against any notion of class which assimilates a singleton with its sole element.⁷ In what follows I will first introduce the argument before I discuss how Russell responds to it by means of the theory of types.

In Appendix A, Russell remarks that the original theory of classes is proven to be wrong by the following argument: ‘If a is a class of more than one term, and if a is identical with the class whose only term is a , then to be a term of a is the same thing as to be a term of the class whose only term is a , whence a is the only term of a ’ (*PoM*, §487; cf. §484, §491; Frege 1895). As we saw in the previous section, Russell in the final draft of *PoM* holds that a singleton is identical to its sole element. But, if the identity of a class depends on its members, and if any class must have the same member as its singleton, then every class contains only one element, namely, itself, which is preposterous. Accepting this argument as ‘irrefutable,’ Russell admits that ‘it is necessary to re-examine the whole doctrine of classes’ (*PoM*, §487). The argument indeed undermines his account as to how a class can be a member of another, because it relies on the notion of class as one whereas he takes it as axiomatic that ‘[a] whole composed of one term only is that one term’ (§75). On his account, it is classes as one that can be members of other classes, but, as long as a class as one is identified with a whole, it follows that any class as one is identical to its singleton. Thus, the argument forces him to hold either (a) that a class is a single entity similar to the kind of whole which he understood to be a class as one but not identical in that a class of one term is distinct from that term, or (b) that there are no such things as classes as one at all and classes are all ‘strictly and only many’ (§487). Among these two options, Russell attributes the former view to Frege and Peano but goes on to reject it on the ground that ‘it is very hard to see any entity such as Frege’s range, and the argument that there must be such an entity gives us little help’ (§488). Russell’s standpoint is indeed ‘object-based’ here: he seems to assume that unless classes are given an ontological account, they cannot be accepted.

Having thus rejected Frege’s account as well as his own one, Russell takes the option (b), which results in a new theory dealing exclusively with classes as many. To be precise, he does not thereby abandon the notion of class as one altogether: Frege’s above argument, as Russell

⁷ B. Linsky indicates that Russell accepted Frege’s argument concerning singletons (Linsky, 2004, p.136). But it is not clear whether Linsky thinks of the argument as a leading motivation for the theory of types.

understands it, only proves that 'the non-identification of the class with the class as one' is necessary (§488). In other words, classes as one can be retained if they are not assumed to observe the principle of extensionality. In fact, as we shall see shortly, Russell keeps using the term 'class as one' even in the rest of Appendix A.

Russell goes on to note that his new account of classes requires 'a modification of the logic hitherto advocated' (§488). In the final draft of *PoM*, he holds that ' ϵ cannot represent the relation between a term to its class as many' for the reason that 'since a class, except when it has one term, is essentially many, it cannot be as such represented by a single letter' (§76). In other words, 'if only single terms can be logical subjects [...], if u is a symbol standing essentially for many terms, we cannot make u a logical subject without risk of error,' since being a logical subject is equivalent to being one (§489). Levine, as we saw above, understands this as the whole problem with the original theory of classes. Yet, only after Russell confines himself to classes as many does it become necessary for him to employ a single letter to represent a class as many.

Russell solves the above problem concerning the membership sign by giving up the view, which we saw at the end of the previous section, that multiple terms do not constitute a single logical subject. To put it otherwise, he now holds: '*The* subject of a proposition may be not a single term, but essentially many terms' (§490; emphasis added). This new doctrine of logical subject leads him to the first theory of types⁸:

Although a class is many and not one, yet there is identity and diversity among classes, and thus classes can be counted as though each were a genuine unity; and in this sense we can speak of one class and of the classes which are members of a class of classes. *One* must be held, however, to be somewhat different when asserted of a class from what it is when asserted of a term; that is, there is a meaning of *one* which is applicable in speaking of *one term*, and another which is applicable in speaking of *one class*, but there is also a general meaning applicable to both cases.

(*PoM*, §490)

The idea is that a class as many can still be counted as *one* object, albeit in a higher-order sense of 'one,' and hence the membership relation can hold between an object and a class as many.⁹ This is how he is led to 'distinguish (1) terms, (2) classes, (3) classes of classes, and so on ad infinitum' (*ibid.*). He explains his new doctrine of logical subject as follows:

⁸ It should be noted that Russell in the following passage unhesitatingly introduces a type-neutral sense of *one* while also accepting type-specific senses of *one*.

⁹ In this passage, Russell thinks that every object, whether it is a term or a class, can be counted as *one* in a very broad sense. Note that he introduces the notion of *object* to cover both terms and classes.

The fundamental doctrine upon which all rests is the doctrine that the subject of a proposition may be plural, and that such plural subjects are what is meant by classes which have more than one term.

(*PoM*, §490)

The theory of types in the appendices of *PoM* is thus ‘the direct denial’ of the old view of the logical subject (§488fn).

It is plain that the theory of types resolves the problem with the membership relation and thereby accommodates classes of classes without invoking classes as one. Klement complains that the theory of types cannot accommodate the class of classes named by ‘the French and the English’ (Klement, 2014, pp.5-6). But this complaint seems to miss the very reason why Russell introduces the theory of types, though it is still fair to ask whether the theory is successful in achieving its primary goal.

The theory of types allows Russell to account for classes of classes and so on without invoking the notion of class as one. I submit that this is how he comes to endorse the theory of types in the appendices of *PoM*. The theory of types is introduced not simply as a solution to the set-theoretic paradox but as a result of a substantial ‘modification of the logic hitherto advocated,’ one that pushes apart the previously equivalent notions of being a ‘term of’ and a ‘logical subject’ of a proposition. He invented the theory of types as a philosophical account of classes, and in so doing he modified the ontology of propositions. Thus, the invention of the theory of types illustrates how he understands the relation between the ontology of propositions and his view of classes.

Confusingly, the solution to the set-theoretic paradox which Russell appeals to in the final draft is adopted also in the appendices. As we saw, he solves the paradox by denying that propositional functions of the form $\varphi(k_\varphi)$ determine any classes as one. After passing the final draft to Cambridge University Press, he named the form ‘quadratic form’ and asked the press to add some remarks on quadruple functions to the manuscript on 25 June (Blackwell, 1984, pp.287-8). Then, in a letter dated 8 August, he introduces the first theory of types to Frege (Frege, 1980, p.144). Of course, the theory of types renders the old solution redundant. However, Russell summarises his discussion leading to his theory of types as follows:

Thus the final conclusion is, that the correct theory of classes is even more extensional than that of Chapter IV; that the class as many is the only object always defined by a propositional function, and that this is adequate for formal purposes; that the class as one, or the whole composed of the terms of the class, is probably a genuine entity except where the class is defined by a quadratic function [...], but that in these cases, and in other cases possibly, the class as many is the only object uniquely defined.

(*PoM*, §492)

He thus thinks that every non-quadratic function can still have an object called 'a class as one.' He seems to think that classes as one can be retained as long as they are not employed 'for formal purposes' nor assumed to observe the principle of extensionality.

It remains to discuss why Russell found the original theory of types unsatisfactory even before publishing *PoM*.¹⁰ An important but so far overlooked problem with the theory of types is that while it enables him to escape from Frege's argument about the singleton with regard to classes as one, it recreates the same difficulty as a problem for classes as many. In his new theory a class as many will be the sole member of its singleton, but the type theory does not explain how we can distinguish a class as many from a class (as many) containing the class as its sole member. He thus thinks it necessary to accept 'Frege's range as an object distinct from its term' in the case of unit classes (§491). But this sounds ad hoc, and given his object-based metaphysics, he cannot be content with this account. This is presumably one of the reasons why he admits, in the preface of *PoM*, that he has not found any plausible account of classes. He could not find any account according to which a singleton is composed of an object and nothing else but is still distinct from that object. In addition to this problem, he seems to be worried about his account of the empty class, as he constantly mentions it in the appendices. As we saw in the preceding section, he offers the 'roundabout' account. But this means that he does not have any *straightforward* account of the empty class, and hence, it is plausible that he finds the notion of empty class still problematic. It seems as though he takes these two problems to be indications of his failure to find a correct account of classes.

Another reason why Russell could not be content with the theory of types is related to his own view of propositions as complex entities. As has been discussed in the literature, the ontology of propositions allows him to formulate an analogue of the set-theoretic paradox in terms of propositions. The propositional version of the paradox is sometimes called the Appendix B paradox or Russell-Myhill paradox. The existence of the Appendix B paradox may not pose an immediate problem, since it is possible to resolve it by erecting a hierarchy of propositions similar to the hierarchy of classes as many. The problem is rather that Russell takes it for granted that propositions must form a single category. He compares the set-theoretic paradox and the Appendix B paradox as follows:

The only solution I can suggest is, to accept the conclusion that there is no greatest number and the doctrine of types, and to deny that there are any true propositions concerning all objects or all propositions. Yet the latter, at least, seems plainly false, since all propositions are at any rate true or false, even if they had no other common properties. In this unsatisfactory state, I reluctantly leave the problem to the ingenuity

¹⁰ His letter to Frege dated 29 September 1902 reads: 'My proposal concerning logical types now seems to me incapable of doing what I had hoped it would do' (Frege, 1980, p.147).

of the reader.

(*PoM*, §349)

Russell thus finds the hierarchy of propositions implausible on the ground that there are some propositions, at least such logical truths as that every proposition is true or false, which he believes must hold of absolutely all propositions.¹¹ It is possible to hold, as he and Whitehead do later in *PM*, that there are no such things as truth and falsehood simpliciter, and that these properties are also stratified. But Russell in *PoM* finds such a hierarchy of propositions 'harsh and highly artificial' (§500). The theory of types adopted in the appendices thus leaves the Appendix B paradox unsolved.

Klement argues that the original theory of types is vulnerable to another intensional paradox and that Russell is aware of it. He explains the paradox as follows:

So long as for every plural collection of individuals there is some one thing, some one individual, that they all have in common, even if that thing is an intension rather than something extensional, we will end up positing as many individuals as classes, and Cantor's theorem will be violated.

(Klement, 2014, p.9)

If there are as many concepts as classes of individuals, and if concepts are among individuals, it would certainly contradict Cantor's theorem. Russell certainly accepts the second antecedent of this conditional, for he includes concepts among terms. According to Klement, Russell is also committed to the first antecedent, because, in endorsing the theory of types, he regards the membership relation as holding between an element of a class and a concept which denotes the members of the class. However, as we have seen, the chief reason for the introduction of the theory of types is to allow classes as many themselves to have the relation to their elements. Hence, Russell has no reason to think that the membership relation holds between an object and a concept denoting a class as many. In addition, he expressly remarks that given Cantor's theorem, 'not all classes have defining predicates' (§499). This is in fact a reason why he thinks of the type theory as 'even more extensional' than his two-fold theory of classes, another reason being that the type theory retains only classes as many which are composed of objects (§492). Thus, it does not seem plausible that Klement's intensional paradox led Russell to abandon the theory of types.

Importantly, it is only the hierarchy of propositions that Russell found 'harsh and highly artificial.' Some commentators take such a comment as a piece of evidence that he adheres

¹¹ Russell in the appendices thus has a specific reason against the stratification of propositions, whereas, as we have seen, he is happy to impose a type-stratification on classes (as many) as he has no such reason against it (cf. footnote 8).

to the so-called doctrine of the unrestricted variable, which is *prima facie* incompatible with the theory of types.¹² It is true that he in *PoM* argues that whatever can be a logical subject is a *term*, suggesting that in his view everything in the universe belongs to one and the same ontological category, over which the unrestricted variable is to range. But, as Proops points out, it is questionable that Russell in *PoM* believes that there must be only one domain of the variable (Proops, 2007, pp.3-7). In addition, Levine argues that Russell in fact rejects the notion of the unrestricted variable in earlier drafts of *PoM* (Levine, 2001, p.219). Furthermore, Russell, in the appendices of *PoM*, explicitly states that the hierarchy of classes is ‘in harmony with common sense’ (§488). Speaking of the theory of types, he also remarks¹³:

According to the view here advocated, it will be necessary, with every variable, to indicate whether its field of significance is terms, classes, classes of classes, or so on. [...] the opinion here advocated seems to adhere very closely indeed to common sense. (PoM, §492)

It would thus seem that although Russell is indeed unwilling to adopt the theory of types in *PoM*, his unwillingness does not come from the firm endorsement of the doctrine of the unrestricted variable but from a more specific problem with a hierarchy of propositions.

Among the problems which Russell leaves unsolved in *PoM*, the ones with singletons and with the empty class can be seen as directly connected to the nature of classes. For these problems are derived from his own attempt to give an extensional account of classes. This in turn explains why he did not take the set-theoretic paradox itself as a problem but rather as a *proof* that he had yet to reach a proper account of classes. He also left the Appendix B paradox unsolved. Hence, it is not surprising that he had given up his first theory of types even before *PoM* was published.

3.4. The problem of the one and the many

In this section, I will sketch how Russell dealt with classes after completing *PoM* in 1902. The chief point which I wish to make here is that he came to call the problems concerning the nature of classes ‘the old problem of the one and the many’ and that he found in these problems an advantage of his *no-classes theory*, which avoids the ontological commitment to classes by regarding class-symbols as *incomplete symbols*.

Given the problems which Russell found with the notion of class, it is not surprising that he tried to dispense with classes altogether in 1903. In his letter to Frege dated 24 May, Russell claims: ‘I have discovered that classes are entirely superfluous’ (Frege, 1980, p.158). But he

¹² For example, Cocchiarella (1980), Urquhart (1988), Hylton (1990), Landini (1998b) and Klement (2003).

¹³ Compare the following remark to Klement’s contention that Russell ‘begrudgingly and unhappily suggested’ ‘separating individuals, classes, and class of classes so on’ (Klement, 2003, p.23).

soon found this approach hopeless, and he sought to develop what is called the ‘zig-zag theory’ from then on.¹⁴ Looking back on the time when *PoM* was finished, he later in 1905 remarks as follows: ‘The only thing that induced me at that time to retain classes was the technical difficulty of stating the propositions of elementary arithmetic without them’ (*EiA*, p.193). In the letter to Frege, Russell also claims that he has solved the Appendix B paradox (Frege, 1980, pp.159-60). He then stopped speaking of this paradox and it was not until he came to pay attention to the Liar paradox that he realised that he had after all not resolved the Appendix B paradox.

In developing the zig-zag theory, Russell, as we will see in the next chapter, focused on (propositional) functions rather than classes. It seems as if he simply assumed that in order to apprehend what classes are, he first needed to have a correct account of propositional functions which tells us which propositional functions can determine classes. Indeed, he remarks in a letter to Frege dated December 1904: ‘I believe that the contradiction does not arise from the nature of a class, but from the fact that certain expressions of the form $(\phi).F(x, \phi x, \phi \xi)$ [...] do not represent functions of x ’ (Frege, 1980, p.167). As a result, the manuscripts which he wrote while struggling to develop the zig-zag theory contain only a few remarks on classes. ‘Fundamental Notions’ is a set of working notes or, using Grattan-Guinness’ term, a ‘logical diary’ which Russell wrote in 1904. At the beginning of it, Russell presents a tentative plan for the next book, which was planned to be the second volume of *PoM*. But his remark on classes is very brief: ‘Classes, somehow’ (*CP4*, p.112). He thus scarcely speaks of classes in the manuscripts or papers written during the period from 1903 to the summer of 1905.

Russell invented the theory of descriptions in the summer of 1905. He then applied to class-symbols and to function-symbols the idea that some sorts of symbols do not stand for any objects while they can still be meaningful in broader contexts. He called such symbols *incomplete symbols* and named the resulting theory the *no-classes theory*. This terminology is unfortunate, as he later develops some theories of propositional functions in which class-symbols are still treated as incomplete symbols. Hence, I will use the term ‘substitutional theory’ to speak of the theory which Russell developed from 1905 onwards, reserving the term ‘no-classes theory’ to speak of any theory which treats class-symbols as incomplete symbols.

Some earlier remarks on the substitutional theory are found in ‘On Some Difficulties in the Theory of Transfinite Numbers and Order Types’ (1905; henceforth, *OSD*). In this paper, Russell only remarks, regarding classes, that ‘if, in fact, there are classes and relations,’ the substitutional theory is ‘unnecessarily difficult and complicated’ (*CP5*, p.82). Thus, Russell remains agnostic about the ontological status of classes in this paper. But in ‘On the Sub-

¹⁴ I will discuss these approaches in detail in the next chapter.

stitutional Theory of Classes and Relations' (1906; hereafter SCR), he is convinced of the substitutional theory and remarks as follows:

Of the philosophical consequence of the [substitutional] theory I will say nothing, beyond pointing out that it affords what at least seems to be a complete solution of all the hoary difficulties about the one and the many; for, while allowing that there are many entities, it adheres with drastic pedantry to the old maxim that "whatever is, is one."

(CP5, p.261)

Russell thus comes to hold explicitly that a philosophical advantage of the substitutional theory lies in its ability to solve 'all the hoary difficulties about the one and the many.' But what are those problems?

Landini among others claims that those problems are concerned with the conflict between the so-called doctrine of the unrestricted variable and a hierarchy of types (Landini, 1998b, p.200). It is certainly true that the substitutional theory enables Russell to place a hierarchy of types on classes and propositional functions without stratifying entities: the theory does not commit him to treating classes and propositional functions as entities. But, as we will discuss in Chapter 5, it is quite questionable whether Russell took the doctrine to be a non-negotiable premiss.

It seems more natural to understand the problem of the one and the many as the difficulties which confronted Russell in his attempts to give an object-based explanation of the extensionality of classes. The view that a class as many can still be counted as one object certainly violates the maxim 'Whatever is, is one.' If this is so, the philosophical advantage of the substitutional theory which he alludes to in the above passage is that it enables him to employ classes without counting them as single objects in any sense. It would thus seem possible that what Russell in SCR means by 'the hoary difficulties about the one and the many' are the the problems concerning the nature of classes.

As we will discuss in Chapter 5, Russell considered restoring classes when he was confronted by the necessity of the Axiom of Reducibility.¹⁵ The idea is that once classes are admitted, it is possible and even natural to *reduce* any arbitrary propositional function ϕx to an extensional one, that is, $x \in \alpha$. But, in some 1906 manuscripts, he hesitates to restore classes, partly because he has yet to explain what classes are in themselves. For instance, he expressly speaks of the problems with classes in a set of working notes called 'The Paradox

¹⁵ The Axiom of Reducibility, as presented in *PM*, effectively says that every propositional function ϕx has a co-extensional *predicative* propositional function $\psi!x$. In the book, the class of objects that satisfy ϕx is contextually defined in terms of its corresponding predicative function (cf. p.112).

of the *Liar*' (CP5, 11; hereafter 'The Liar') written in September 1906.¹⁶ In the middle of the manuscript, he notes, summarising his observations, that the Liar paradox is solved by a hierarchy of propositions. But if such a hierarchy is accepted, he continues, 'it is essential to have some means of inferring that certain formulae, if they hold for all functions below a certain grade, hold for all functions absolutely' (CP5, p.332). He then remarks: 'This axiom for reducing the grade of a statement can be effected plausibly by means of *classes*' (*ibid.*). But he adds that this method invites '[t]he old difficulties as to the nature of a class' such as 'the meaning of Λ , the difference between x and $\iota'x$, etc' (*ibid.*). Note that he uses ' Λ ' for the empty set and ' $\iota'x$ ' for the singleton of x . Thus, he evidently thinks of the problems concerning the nature of classes here.

In the end, Russell and Whitehead in *PM* decide that class-symbols are incomplete symbols, while accepting the Axiom of Reducibility. One of their reasons for the acceptance is the observation that although the axiom looks ad hoc, it is a 'smaller' assumption than the subsistence of classes. In a later part of 'The Liar,' Russell remarks as follows:

As to *classes*, observe, in introducing the reducibility primitive proposition, that $x \in \alpha$ satisfies its requirements if there are classes, so that our assumption is a smaller one than the assumption that there are classes. Thus classes may be alluded to to make the primitive proposition seem plausible; but they had better not be introduced anywhere but in the talk.

(CP5, p.342)

A similar remark can indeed be found in *PM*: 'The assumption of the axiom of reducibility is [...] a smaller assumption than the assumption that there are classes' (*PM*, p.52). Another reason for favouring the hierarchy of propositional functions over the one of classes is, unsurprisingly, the difficulty in accounting for classes:

It is an old dispute whether formal logic should concern itself mainly with intensions or with extensions. [...] The facts seem to be that, while mathematical logic requires extensions, philosophical logic refuses to supply anything except intensions. Our theory of classes recognizes and reconciles these two apparently opposite facts, by showing that an extension (which is the same as a class) is an incomplete symbol, whose use always acquires its meaning through a reference to intension.

(*PM*, p.72)

It should be remembered that Russell in *PoM* views his original theory of classes as located somewhere 'intermediate between between pure intension and pure extension' (*PoM*, §66).

¹⁶ See also 'On Substitution' which was written in April and May 1906 (CP5, p.213).

There seems to be no problem in connecting this remark in *PoM* with the above one in *PM*. In fact, the authors of *PM* go on to say that ‘arguments [against classes] of more or less cogency can be elicited from the ancient problem of the One and the Many,’ where they explain the problem as follows:

If there is such an object as a class, it must be in some sense one object. Yet it is only of classes that many can be predicated. Hence, if we admit classes as objects, we must suppose that the same object can be both one and many, which seems impossible.

(*PM*, p.72fn)

Thus, Russell still thinks that the no-classes theory enables him to avoid committing himself to classes and thereby facing the problems concerning the nature of classes.

Similar remarks on classes can also be found in a short paper ‘Some Explanations in Reply to Mr. Bradley,’ which Russell wrote after the completion of *PM*. He there remarks: ‘Apart from other contradictions, the fact that a class, if there is such a thing, must be both one and many constitutes a difficulty’ (Russell, 1910b, p.376). In ‘On Appearance, Error and Contradiction,’ Bradley challenges Russell’s notion of class and attempts to indicate ‘the inherent inconsistency of “class”’ (Bradley, 1910, p.182). But now Russell is happy to concede that classes pose problems, thanks to the no-classes theory:

The theory that there are no such things as classes avoids at once the difficulties raised by Mr. Bradley and the difficulties with which I endeavour to contend in the *Principles*.

(Russell, 1910b, p.376)

Hylton points out that ‘[t]he difficulties which Russell faces over the unity of a class are by no means adventitious’ and that ‘[t]he issue of unity, or of the one and the many, is one of the major themes of the rejection of Idealism by Moore and by Russell’ (Hylton, 1990, p.225). We have indeed seen a connection between the rejection of monistic idealism and Russell’s original theory of classes in Section 3.2: it is based on his contention that judgments about a class as many constitute a counterexample to the subject-predicate doctrine, which he claims all kinds of monistic idealism rest upon.

3.5. Concluding remarks

We have seen that Russell in *PoM* exploits the ontology of propositions in his attempts to give a philosophical account of what classes are. As we saw in Section 3.2, he in the body of the book invokes the theory of denoting concepts so that he can define a class (as many) to be a numerical conjunction of its members. But the definition is vulnerable to Frege’s objections to any account of a class that assimilates a singleton with its sole element. This led Russell to

develop the very first theory of types in the appendices. The theory was thus, contra Landini, not 'a purely formal dodge' at all (Landini 1992, p.164; Landini 1998a, p.50). It was meant to be a philosophical account of what classes are. But, as we saw in Section 3.3, the theory fails to account for why a class as many can be differentiated from the class as many containing it as its sole member, and this failure underlies the remark which I quoted at the beginning of this chapter. These sections illustrate, hopefully, that Russell's ontology of propositions was meant to be a replacement of the theory of judgment, that is, the idealist foundation of logic. In Section 3.4, I endeavoured to show that he after completing *PoM* came to regard the problems concerning the nature of classes as certain signs of the conflict between the one and the many. I also argued that he adopted a no-classes theory even after abandoning the substitutional theory because it enabled him to avoid those problems.

4 Theories of Functions

4.1. Introduction

'On Denoting' is one of the most famous papers which Russell wrote. His main concern in the paper is whether there are any entities corresponding to definite descriptions. He concludes that there are not, by rejecting his old theory of *denoting concepts*. The paper thus does not seem to have much to do with his logicist programme. But, in a retrospective letter to Jourdain, Russell remarks:

In April 1904 I began working at the Contradiction again, and continued it, with few intermissions, till January 1905. I was throughout much occupied by the question of Denoting, which I thought was probably relevant, as it proved to be.

(CP4, p.xxxiii)

This remark raises two questions. First, why did Russell think that the notion of denoting 'proved to be' relevant to the set-theoretic paradox? In this letter, he explains that the theory of descriptions, which he adopted in 'On Denoting' (OD), makes it possible to define functions in general by means of propositional functions, *i.e.*, functions whose values are propositions. Hylton explains that the theory of descriptions enabled Russell to dispense with denoting concepts, arguably an exception to his view of language as a transparent medium (Hylton, 2003). In the next chapter we shall look more closely into various roles which the notion of *incomplete symbol* played within his logicist programme. The task of this chapter is to answer the other question. Why had Russell thought that the notion of denoting was 'probably relevant' before he invented the theory of descriptions?

A key to finding the connection which Russell envisaged between the set-theoretic paradox and the notion of denoting is the idea that logic has the two branches, the theory of logic and the calculus of logic.¹ In this chapter, I will attempt to show that during the period from 1903 to 1905 he constantly kept seeking a philosophical account of (propositional) functions which could vindicate his then solution to the paradox while often exploiting the notion of denoting

¹ For other possible accounts of the connection, see Urquhart's Introduction to CP4, Landini (1998a) and Kaplan (2005). I cannot discuss these accounts in detail except for some problems with Landini's interpretation.

in order to offer such a philosophical account. He seems to have taken it for granted that any solution to the paradox must be somehow based upon a philosophical account of propositions and in particular of denoting concepts. As Klement puts it, Russell's 'standards were high; he did not want a formal dodge, he wanted a philosophically, even *metaphysically*, motivated explanation for the avoidance of the contradictions' (Klement, 2003, p.15). Klement does not discuss why Russell held such 'high' standards, but we have seen why: he once accepted the idealist conception of logic, according to which a metaphysical investigation into judgments was theoretically prior to the technical development of logic (Klement, 2016). But the claim that Russell sought a philosophical account of functions to motivate his then solution to the paradox has not yet been substantiated. It takes, in my view, an orderly introduction of his theories of functions as well as a close examination of how he used them to solve the set-theoretic paradox and its analogues. This is the primary task that I wish to undertake in the following. I will introduce various theories of functions which he tentatively endorsed and then abandoned during the period from 1903 to 1905. Those theories will illustrate how he connected technical enquiries into the paradoxes with his philosophical enquiries as to what propositional functions are. It will also be seen that the notion of denoting and the paradoxes were in Russell's view more closely connected than hitherto considered in the literature: he found a decisive argument against the notion of denoting through an attempt to motivate his then solution to them.

I will begin by indicating several points which constitute Russell's objection to the notion of denoting in Section 4.2. I will then introduce the philosophical accounts of functions which he considers in *PoM*, in 1903 and in 1904 in Sections 4.3, 4.4 and 4.5 respectively. The chief aim of introducing them is to illustrate how the set-theoretic paradox and the notion of denoting are connected in his accounts of functions. I will sometimes look into details of them so that it can be seen why he replaced one account of functions with another and so on. In Section 4.6, I will argue that he discovered the problem of denoting while developing an account of why propositional functions of a certain form cannot determine a class.

4.2. The problem of denoting

Russell develops an argument against the theory of denoting concepts in the notoriously obscure passage of OD. In this section I will argue that the argument is composed of four points which I will call Nature of Denoting, No Function of Meaning, No Linguistic Bypass and No Function of Denotation respectively. I do not intend to offer a full interpretation of the passage but only to indicate that those four points are the essential components of the argument. But the following remarks are still contentious. So I will indicate where my understanding of the passage disagrees with preceding interpretations at the end of this

section.

In setting forth the argument in OD, Russell takes on the view that a denoting complex *has* a *meaning* and a *denotation*, before he combats the old theory of denoting concepts presented in *PoM*.² The new theory urges that whenever a denoting object occurs in a proposition, the proposition is not *about* the meaning of the object but *about* what the object *denotes*. The new theory is thus a variation of the original theory of denoting concepts.³ What Russell thinks is common between the two theories is the idea that whenever a denoting object, whether complex or not, occurs in a proposition, the proposition is *about* what it denotes. I will call this notion the Nature of Denoting. The notion implies that we cannot speak of a denoting complex by means of a proposition containing it as a constituent. Russell thus remarks:

When we wish to speak about the *meaning* of a denoting phrase, as opposed to its *denotation*, the natural mode of doing so is by inverted commas. Thus we say:

The centre of mass of the Solar System is a point, not a denoting complex;

“The centre of mass of the Solar System” is a denoting complex, not a point.

(OD, pp.485-6)

The idea is, in other words, that ‘when *C* occurs it is the *denotation* that we are speaking about; but when “*C*” occurs, it is the *meaning*’ (OD, p.486). Russell thus suggests using inverted commas as an expression for a function that takes a complex as an argument and returns the meaning of the complex as its value. He does not employ double quotation marks as a function which takes a linguistic item as an argument and yields an expression designating the meaning of the item. He rejects this option, because he thinks that ‘there must be a logical relation’ between meaning and denotation, ‘not merely linguistic through the phrase’ (OD, p.486). This point may be called ‘No Linguistic Bypass.’

Russell goes on to reject the possibility of speaking of a meaning by means of a function which, when a denoting complex is given as an argument, yields the meaning of the complex:

But if we speak of “the meaning of *C*,” that gives us the meaning (if any) of the denotation. “The meaning of the first line of Gray’s *Elegy*” is the same as “The meaning of ‘The curfew tolls the knell of the parting day,’” and is not the same as “The meaning of ‘the first line of Gray’s *Elegy*.’” Then in order to get the meaning we want, we must

² This remark is already contentious. Kremer thinks that the first view is the one presented in *PoM* (Kremer, 1994, pp.278-9). Salmon argues that what Russell really challenges is more broadly the view that definite descriptions are singular terms (Salmon, 2005, 1077-80). But other authors agree that the second view is the old theory of denoting concepts, though they explain the status of the first view in different ways (*e.g.*, Turnau, 1991, p.59; Pakaluk, 1993, pp.46-9). Some other authors explain the presence of these two views by arguing that Russell’s argument develops in two stages (Noonan, 1996; Makin, 2000).

³ I will discuss the new theory in detail in Section 4.6.

speaking not of "the meaning of C ," but of "the meaning of ' C ,'" which is the same as " C " by itself.

(OD, p.486)

I understand the above argument as follows: let C be a complex which has both a meaning M and a denotation D (assuming that we can thus speak of the complex and its meaning); then, in order to speak of M , one might use such an expression as 'the meaning of C '; but what the expression actually denotes is not M , but the meaning of D (assuming that D again has both a meaning and a denotation), because the symbol ' C ' in 'the meaning of C ' denotes D rather than M . We intend to use the expression 'the meaning of ...' to express an operator or a function whose values are meanings of arguments; but 'the meaning of C ' denotes the value of the function applied to D , and hence, we cannot speak of M but the meaning of D . This point becomes plain, when we consider a function expressed by 'the father of x .' The expression 'the father of the man' should denote, if it denotes at all, someone's father and not a father of a concept (if there is such a thing). Similarly, the expression 'the meaning of C ' denotes the meaning of what C denotes. Therefore, in order to speak of the meaning of C , we need such an expression as 'the meaning of " C ";' but double quotation marks are exactly what we are looking for, namely, a method with which we can denote the meaning of C . In other words, we can define a function which can be applied to meanings, only when we already have some such function. I will call this observation No Function of Meaning.

Russell thinks that No Function of Meaning leads us to 'the right phrase' such that 'the meaning has denotation and is a complex, and there is not something other than the meaning, which can be the complex, and be said to *have* both meaning and denotation' (OD, p.487). We are led to the 'right phrase' or in effect to the old theory of denoting concepts, because the above argument demonstrates that one cannot use such expressions as 'the meaning of C ' and (for a similar reason) 'the denotation of C ,' and hence, that we cannot properly state the very idea that a complex has a meaning and a denotation. But the original theory of denoting concepts is no better than the rejected one. He thinks that 'the right phrase' renders 'our difficulty in speaking of meanings more evident' (*ibid.*). He rephrases the above argument as follows: 'whenever C occurs without inverted commas, what is said is not true of the meaning, but only of the denotation'; hence we need some complex which denotes C , but ' C must not be a constituent of this complex (as it is of "the meaning of C ")' (*ibid.*). The structure of the argument is thus not affected by the adoption of the 'right phrase': we cannot define any function which we can directly apply to meanings, for if we put a meaning M in an argument place of a function f , the value of the function $f(M)$ contains M in an entity position or as a logical subject and hence denotes the value of the function applied to what M denotes, not to M itself.

One may think that even if we have no function of meanings, we can still speak of a meaning by means of a function expressed by 'that which denotes *D*' which takes an entity as its argument and returns the meaning which denotes the entity. But Russell rejects this method: 'there is no backward road from denotations to meanings, because every object can be denoted by an infinite number of different denoting phrases' (OD, p.487). This point may be called 'No Function of Denotation.'

I submit that Russell's argument in the obscure passage in OD consists of the following points, Nature of Denoting, No Linguistic Bypass, No Function of Meaning and No Function of Denotation. I will call the problem posed by those points the *problem of denoting*. These four points seem to jointly establish that we cannot expect to have a uniform method through which we can speak of a meaning. I do not claim that this exhausts the whole argument presented in the passage, though I believe that these four points constitute the very heart of the argument.

Many commentators have understood the passage in different ways. Although I cannot discuss their interpretations, I shall indicate differences between the above reading and them. For example, Searle does not observe the circularity involved in 'the meaning of "*C*"' and he instead assumes the transitivity of the relation of denoting in order to explain why the expression does not work (Searle, 1958, p.139). Blackburn and Code do not find the circularity either, though they point out that we cannot *use* a denoting phrase '*C*' to speak of the meaning of the expression (Blackburn & Code, 1978, p.72). Hylton interprets the passage in question without mentioning No Function of Meaning or any analogous point (Hylton, 1990, pp.248-54). Landini does not think that Russell accepts No Linguistic Bypass or Nature of Denoting even in OD (Landini, 1992, pp.46fn, 59). According to Pakaluk, the argument leading to No Function of Meaning rather shows that the expression 'the meaning of "*C*"' should be understood as a name, not as a description which denotes a value of a function (Pakaluk, 1993, p.51). Kremer also sees no such circularity and rather appeals to No Linguistic Bypass to explain the illegitimacy of the expression (Kremer, 1994, pp.280-1). Turnau understands what I call No Function of Meaning in connection with No Function of Denotation (Turnau 1991, p.60). Noonan also thinks that the former is partially supported by the latter, even though he explicitly points to the circularity involved in the expression 'the meaning of "*C*"' and understands the whole argument in a way analogous to the one I presented (Noonan, 1996, pp.93-5). Makin, in effect, points to all of the four points, though he understands the first view combatted by the argument differently (Makin, 2000, p.22f). Salmon also indicates the four points effectively, though he does not distinguish between Nature of Denoting and No Function of Meaning and calls them 'the Collapse' (Salmon, 2005, pp.1071, 1101, 1104). But he holds that the chief difficulty arising from the Collapse is not about systematically speaking of a meaning but about identifying which meaning we are using (*ibid.*, p.1106).

4.3. 1902 - Propositional functions in *Principles*

In order to understand Russell's various attempts to account for propositional functions, it is helpful to take a look first at his account of propositional functions in *PoM* as well as his theory of denoting concepts. In this chapter and the following one, I will use 'function' as an abbreviation for 'propositional function' as long as there is no risk of confusion.⁴ In this section, I will attempt to show that these two theories are closely tied with each other already in *PoM*. I shall begin by recalling the theory of denoting concepts.

Russell introduces the notion of *denoting* as follows: 'A concept *denotes* when, if it occurs in a proposition, the proposition is *not* about the concept, but about a term connected in a certain peculiar way with the concept' (*PoM*, §56). He thus does not, as Landini points out, claim in *PoM* that *whenever* a denoting concept occurs in a proposition, the proposition fails to be about the concept (Landini, 1998a, p.59). Hence, it should not be simply assumed that Russell envisages Nature of Denoting already in *PoM*. But the primary role of a denoting concept is certainly to enable one to speak about an entity that does not occur in a proposition. Denoting concepts are in short 'aboutness-shifting' entities (Makin, 2000, p.18). Russell also makes it clear that the relation of *denoting* is in his view not a linguistic relation between expressions and objects but an ontological one obtaining between *concepts* and *objects*. As we saw in Chapter 2, Russell unequivocally contends that 'meaning, in the sense in which words have meaning, is irrelevant to logic' (*PoM*, §51). He adds that 'concepts such as *a man*' are, by contrast, 'symbolic in their own logical nature' having the property of *denoting* (*ibid.*). He thus understands the 'logical' relation of denoting as an ontological relation between *concepts* and *objects*.⁵ The relation of *denoting* is thus a relation which, holding between a denoting concept and an object, shifts the aboutness of the proposition in question from the concept itself to what it denotes.

It is important for our purposes to observe that although denoting concepts are conceived as aboutness-shifting entities, Russell holds in *PoM* that 'it is possible to consider and make propositions about the concepts themselves' (§65). 'If we wish to speak of the concept,' he remarks, 'we have to indicate the fact by italics or inverted commas' (§56). He is thus yet to find No Function of Meaning yet.

Let us turn to propositional functions in *PoM*. First of all, they should not be confused with universals or what Russell calls *concepts*. He does not explicitly state the distinction

⁴ I will use the term 'function in general,' when I wish to clarify that he intends to speak of both kinds of functions. In some manuscripts subsequent to *PoM*, Russell distinguishes between *denoting functions* and *non-denoting functions*, where propositional functions belong to the latter.

⁵ To be precise, he accepts some cases where a denoting concept fails to denote any object. For example, the denoting concept *nothing* does not denote any term or object, though it, in his view, still denotes (§73). This explains, he seems to think, why we can employ the word 'about' even when we speak about nothing.

of functions and concepts in *PoM*; rather he takes it for granted.⁶ There are at least two indications of it. First, he formulates two self-referential paradoxes in terms of concepts and in terms of functions respectively. He sets forth the latter by considering the function ‘not- $\varphi(\varphi)$,’ which asserts that φ is not applicable to φ itself (§85). He then remarks as follows:

This contradiction can be avoided by the recognition that the functional part of a propositional function is not an independent entity. As the contradiction in question is closely analogous to the other, concerning predicates not predicable of themselves, we may hope that a similar solution will apply there also.

(*PoM*, §85)

The second sentence indicates that he distinguishes between these two paradoxes. Besides, he denies that φ in φx is ‘an independent entity’ or a *term*. This is another indication of the distinction between functions and concepts. For he includes concepts among terms. He also remarks that ‘[t]he x in φx , where φx is a propositional function, is an unrestricted variable; but the φx itself is restricted to the class which we may call φ ’ (§88).⁷ Functions in *PoM* are thus not concepts.

It is also implausible that functions are considered, in *PoM*, to be linguistic items such as open well-formed formulae. Landini claims that ‘the *Principles* does not “ontologize variables” by assuming an ontology of propositional functions and absorbing the variable into them’ (Landini, 1998a, p.49). But, as we will see below, what Russell calls ‘the true variable’ is an *object*, a certain combination of all terms. In addition, as we saw in Chapter 2, he unequivocally claims that meaning in the sense in which sentences express propositions is ‘irrelevant to logic.’ Furthermore, he can hardly be content with the view of functions as symbols, given his understanding of symbols as mental items (cf. Section 2.5). In fairness, he also calls a propositional function an ‘expression’ containing a variable (§13). But we cannot take these words at face value. For he in the same passage employs the term ‘expression’ to speak of a *proposition*, a complex entity composed of *terms*. Thus, he does not seem to understand functions as mere symbols in *PoM*.

What are functions in *PoM* then? Roughly speaking, they are considered to be complex objects in which some entities are replaced by variables. Consider the proposition ‘Socrates is a man implies Socrates is human.’⁸ ‘It is,’ Russell remarks, ‘quite certain that if we replace

⁶ His tacit distinction between functions and universals has recently been recognised in the literature. See, for example, Landini (1998b, p.127), Linsky (1999, p.21f), Klement (2003, 2004, 2010a) and Pickel (2013, p.916).

⁷ Proops cites this remark as evidence against Russell’s endorsement of the doctrine of the unrestricted variable (Proops, 2007, p.3). Russell seems to hold that given the account of functions which I shall shortly introduce, functions naturally form a hierarchy. In the next chapter, I will introduce the term ‘structured objects’ to speak of those objects which can be justifiably (in his view) assumed to form a hierarchy.

⁸ The expression is awkward but Russell views the relation expressed by ‘implies’—material implication—as

Socrates by a variable, we obtain a propositional function' (§82). But this is not his definition of function. To see why not, we need to look into his account of variables.

In *PoM*, Russell maintains that each expression of the form 'any u ' designates a concept which in turn denotes what he calls a '*variable* conjunction' of terms. Such conjunctions of terms are, as was the case with classes as many, taken as 'neither terms nor concepts, but strictly and only combinations of terms' (§59). According to him, the conjunction appears, say, in the proposition 'If you met Brown or Jones, you met a very ardent lover' (§59). The conjunction is expressed by 'or' here. This proposition is equivalent to a propositional disjunction: 'If you met Brown or if you met Jones, you met a very ardent lover.' This is in his view a characteristic of the variable conjunction. He then suggests: '*Any term* is a concept denoting the true variable; if u be a class not containing all terms, *any u* denotes a restricted variable' (§88; cf. §93). He employs the word 'variable' to speak of what the concept *any term* denotes, though he also uses it to talk about the concept itself (cf. Proops, 2007, p.6). This explains why he speaks of '*the* true variable': what *any term* denotes is the variable conjunction of all terms and hence is uniquely determined. However, he is not content with the account of the variable as what the concept *any term* denotes. This is closely related to his argument against Frege's account of functions as *unsaturated* objects.

As Klement points out, Russell in *PoM* rejects the idea that a function is what remains when we remove occurrences of an entity from a proposition (Klement, 2010, pp.639-640). Russell employs the term 'assertion' to speak of what remains when we can remove such occurrences of an entity (*PoM*, §43). But the notion of assertion is, he argues, not always available. For if we omit, say, Socrates from the proposition 'Socrates is a man implies Socrates is human,' what remains would be '... is a man implies ... is a mortal'; yet, 'no trace whatsoever appears in the would-be assertion' as to whether 'the *same* term should be substituted in the two places' (§82). This observation underlies his objection to Frege's view of functions (in general) as *unsaturated* objects: if the function ' $2x^3 + x$ ' is identified with ' $2(\quad)^3 + (\quad)$ ' then 'there is no way of distinguishing what we mean from the function involved in $2x^3 + y$ ' (§482). Russell also claims that 'it is impossible to define or isolate the constant element in a propositional function, since what remains, when a certain term, wherever it occurs, is left out of a proposition, is in general no discoverable kind of entity' (§106). He concludes that 'the term in question must be not simply omitted, but replaced by a *variable*' (*ibid.*).

The above argument explains why Russell cannot be content with the simple view that propositional functions differ from propositions only in that they contain the variable among its constituents. It should be possible for ' x ' and ' y ' in the expression ' $2x^3 + y$ ' to vary

a relation between propositions, not between sentences. I will employ single quotation marks not only to mention symbols but also to delimit expressions for propositions and for propositional functions. I will make it explicit in which way I am using them, whenever it may be ambiguous.

independently from each other. That is, ‘variables have a kind of individuality’ (§93). The individuality of variables can, he seems to think, be explained only if we understand them in relation to functions in which they occur. He is thus led to explain the variable as something more than what *any term* denotes: ‘A variable is not *any term* simply but any term as entering into a propositional function’ (*ibid.*). But then he cannot invoke the notion of variable to explain propositional functions.

Russell moves on to argue that it is necessary ‘to accept the general notion of a propositional function itself as indefinable’ (§81). He denies that a function is what remains when a term is omitted from a proposition. But he still thinks that ‘“Plato is a man implies Plato is a mortal” is, in some sense or other, the *same* function of Plato’ as ‘Socrates is a man implies Socrates is a mortal’ is of Socrates (§82). He invokes the notion of substitution to capture the ‘constancy of form’ involved here:

Given any proposition (not a propositional function), let a be one of its terms, and let us call the proposition $\varphi(a)$. Then in virtue of the primitive idea of a propositional function, if x be any term, we can consider the proposition φx , which arises from the substitution of x in place of a . We can thus arrive at the class of all propositions $\varphi(x)$.

(*PoM*, §93)

He thus thinks that we can obtain ‘a class of propositions of constant form’ through substitution.⁹ This notion of *class* used here is ‘more fundamental than the general notion of *class*, for the latter can be defined in terms of the former, but not the former in terms of the latter’ (§86). The *class* of propositions needs to be distinguished from ‘other classes,’ which we discussed in the preceding chapter (Blackwell, 1984, p.288). However, this view of a function as a *class* of propositions does not serve as a *definition*, because the operation of substitution presupposes the very notion. For it is ‘impossible, without a vicious circle, to discover any common characteristic by which the class could be defined, since the statement of any common characteristic is itself a propositional function’ (§88). Thus if one attempts to explain a function in terms of its values, one has to appeal to yet another function to characterise the range of the original function. Hence ‘what is fundamental is not particular propositional functions, but the class-concept *propositional function*’ (§91). He thus finds it necessary ‘to accept the general notion of a propositional function itself as indefinable’ (§81). ‘A propositional function is,’ he concludes, ‘the class of all propositions which arise from the variation of a single term, but this is not to be considered as a definition’ (§91).

To be precise, Russell also allows himself to use the term ‘propositional function’ to speak

⁹ Russell, as we will see, repeatedly invokes the notion of substitution to account for functions. The notion is, as we shall discuss in the next chapter, developed further into an ontological reduction of functions once he invents the notion of incomplete symbol.

of a correlation between the terms and a certain *class* of propositions. He remarks that if we mean by ‘propositional function’ a correlation between the terms and the propositions of a given *class*, it can be seen ‘as a single entity’ (§482).¹⁰ Yet, ‘the air of formal definition’ which the notion of correlation brings here is, he asserts, ‘fallacious, since propositional functions are presupposed in the class of referents and relata of a relation’ (*ibid.*). His official view in *PoM* is thus that the notion of propositional function is primitive.

4.4. 1903 - Functional Theory

After completing *PoM*, Russell tackled the set-theoretic paradox until he got involved in a political activity in January 1904. During this period, he came to understand the notion of denoting concept in terms of the distinction between *meaning* and *denotation*, and used the distinction to develop, though tentatively, two distinct accounts of functions. I will introduce those accounts and illustrate how they are, in his view, related to the set-theoretic paradox and its analogues.

4.4.1 Function as detachable meaning

The first idea which Russell tried after the completion of *PoM* in order to solve the paradox was to abandon classes altogether in favour of functions. In this subsection I will introduce this attempt and the view of functions which he presented in order to motivate it.

In the letter to Frege dated 25 May 1903, Russell remarks: ‘I believe I have discovered that classes are entirely superfluous’ (Frege, 1980, p.158). His idea is to employ propositional functions in place of classes. He distinguishes entities into individuals and functions, maintaining that two entities are identical ($‘u = v’$) either if both entities are individuals and exactly the same functions apply to them or if they are co-extensive functions. He thus includes propositional functions among entities though they are subject to the different criteria of identity.¹¹ He goes on to introduce the notion of *similarity* in the same way as Frege defines *Gleichzahligkeit* in *Die Grundlagen der Arithmetik (GLA, §72)*. Russell then defines the cardinal number of ϕ to be $‘\psi’(\phi \text{ sim } \psi)’$ (Frege, 1980, p.158). He claims that ‘[i]n this way we can do arithmetic without classes’ and that ‘this seems to me to avoid the contradiction’ (*ibid.*).¹²

It may well come as surprise that Russell does not hesitate to include functions among entities, given that he is aware of the self-referential paradox of functions already in *PoM*

¹⁰ A function thus understood can be assumed to be a one-one relation not just a many-one relation. In fact, this assumption seems to underlie the propositional paradox presented in Appendix B of *PoM*.

¹¹ In the letter to Jourdain which I partially quoted at the beginning, Russell makes it clear that in developing the theory in question, he ‘treated ϕ as an entity.’ A similar treatment of propositional functions can be found in his view of a proposition as a mode of combination, which we discuss in Section 4.5.3. This line of thought makes a sharp contrast with Frege’s exclusive distinction of objects and functions.

¹² Landini discusses the theory in question in detail. See Landini (1992).

(§85). As we will see in Section 4.4.2, Russell realised the vulnerability of the above theory to the paradox only later in 1903. He may have thought that applying the two distinct standards to individuals and to functions respectively would suffice to block the paradox.

Another problem with the above view is that it does not fit Russell's own 'object-based' metaphysics, which we saw in the previous chapter. If classes are to be avoided, it seems that he requires some contextual definitions analogous to Frege's Basic Law V so as to introduce important notions such as cardinal number. He cannot define the cardinal number of a function ϕx in terms of an equivalence class with respect to similarity, and hence, he needs something like: $C(\phi) = C(\psi) \Leftrightarrow \phi \text{ sim } \psi$. But it is questionable whether he can accept such an indirect way of introducing objects, given that he rejects Frege's notion of *Weltverläufe* by claiming that 'the argument that there must be such an entity gives us little help' (cf. p.58).

The 'superfluous' theory is thus not promising, but it is important for our purposes to observe that Russell offers a philosophical account of functions so that he could motivate the approach. A 1903 manuscript 'Functions and Objects' (CP4, 3a) illustrates how his view of functions shifted from the one presented in *PoM* to a new account that treats functions as *terms*. 'The notion of function is,' Russell first remarks, 'indefinable; but by way of explanation it may be said that a function is to be conceived as the relation of the expression X to the term x which occurs in it' (CP4, p.50). This remark suggests that he understands functions as objects of a certain sort. He goes on to remark as follows:

When any expression contains a constituent x , it is possible to conceive of x being replaced by y or z or any other term, without any other change being made in the expression. When this is done, something remains constant while the term in question is changed. This something is to be called a *function*.

(CP4, p.50)

Although he uses the word 'expression' here, what he has in mind is a sort of object; for he also says 'Whatever is not a function will be called a *term*; either a term or a function will be called an *object*, so that everything is absolutely an object' (CP4, p.51). The idea is already introduced in *PoM* that the essence of a function can be captured by substitution of terms. But he now goes further to claim that a function is something that remains when we *remove* occurrences of the variable. 'It is to be observed,' Russell indeed remarks, 'that the function itself does not contain x (unless, indeed, X contains x in more than one place, and x is not varied in all these places)' (*ibid.*). He does not, however, simply accept Frege's account of functions, as he still regards it as 'inadequate' (*ibid.*, p.50). Russell seems to believe that functions must be included among terms.

In 'Dependent Variable and Denotation' (CP4, 12), Russell develops this notion to the view of a function as a *detachable meaning*. He begins the manuscript by introducing the

distinction of *meaning* and *denotation*: 'When there are two words, such as *death* and *dies*, of which one means and the other denotes, the difference seems not psychological' (CP4, p.298). The distinction arguably corresponds to the one between denoting concepts and what they denote, and hence, what he calls a meaning is an object that has the relation of denoting to another object. But a *meaning* is also explained as what remains when we remove the subject from a proposition: 'In what is simple, the object is always a term,' whereas '[i]n what is complex one object at least is a meaning, and one at least is a term' (*ibid.*). He thus connects the distinction between denoting concepts and their denotations with the one between the two ways in which a relation occurs in propositions (cf. p.35). In this manuscript, a *meaning* is thus understood not only as what denotes an object but as an object which can form a complex entity when combined with an entity. As we will see below, these two features remain the same even when he invokes the notion of meaning in later manuscripts.

In 'Dependent Variable and Denotation,' Russell tries to explain how a meaning is *detachable* from a complex in which it occurs. According to him, in order to detach a meaning from a complex, we need to find an expression that denotes, rather than means, the meaning. He introduces the notation ' $p \frac{y}{x}$ ' to express that a meaning remains constant through substitution of variables:

From $\phi|x$ to $\phi|y$ is a case of $p \frac{y}{x}$. The process in $p \frac{y}{x}$ is this: If p is a meaning which denotes, and if x is a constituent of the meaning, then $p \frac{y}{x}$ is the meaning which results from substituting y for x .

(CP4, p.298)

In this passage he understands a meaning remaining constant through substitution as a function here. He uses '|' for function-application, presumably because he wants to emphasise that ϕ and x are separable. He thus seeks symbolism that can be employed to speak of a meaning and can indicate on its own how it can be detached from a complex.

But Russell finds a problem with the notation ' $p \frac{y}{x}$.' We cannot use it to speak of a function conceived as a meaning: "' p " ought only to denote; no single letter *means* at all' (CP4, p.298). He does not clearly state why no single letter can mean but only denote, but he is presumably influenced by Frege's view that any 'saturated' expression designates an individual entity. Russell rephrases the problem as follows:

If p is of the form in which x occurs only once, and there as subject, $\acute{x}(p)$ is got by omitting x . But it is not the *meaning* of what is so obtained, but the *denotation*, that is expressed by $\acute{x}(p)$.

(CP4, p.303)

What Russell finds problematic is thus clear: ‘We want a general symbol by which we can denote a given meaning’ (CP4, p.301). The problem here is akin, though not identical, to the problem of denoting: both problems are related to how we can speak of a meaning. But the problem here is only concerned with our employment of a single letter, while the latter problem is relevant to any attempt to systematically speak of a meaning. Although he considers several ways of denoting a meaning, none of them satisfies him and so the manuscript leaves the problem unsolved.

Russell wrote yet another manuscript ‘On the Meaning and Denotation of Phrases’ (CP4, 11a) around August 1903 (cf. CP4, p.283). This manuscript contains a more detailed exposition of the idea that a function is a detachable meaning. He attempts to solve the above problem by explicating how we can detach a function, that is, how we can find an expression which denotes a meaning. In the case of ‘Socrates died,’ we can detach what ‘died’ means by using the word ‘death.’ He generalises this procedure as follows:

We first replace X by a new complex having the same denotation, in which all the rest of X has become a proper name followed by *of*; the proper name in question then denotes what the rest of X meant.

(CP4, p.289)

If we want to speak of the meaning of ‘ X ’ containing ‘ x ,’ we need to find an expression of the form ‘ Y of x ’ which denotes the same entity as ‘ X .’ The expression ‘ Y ’ thus obtained should be a proper name and so it denotes what remains when we remove x from the meaning of ‘ X ,’ that is, the meaning in question (cf. CP4, p.292).

Thus, in his attempt to block the set-theoretic paradox by using propositional functions in place of classes, Russell offered the view of function as a detachable meaning. It was meant to offer a philosophical account of why functions, which had once been denied occurrences as terms in *PoM*, could now be treated as individual entities. But his attempt to dispense with classes was to face the problems which we saw above, and, as we will see in the next subsection, he indeed gave it up together with the account of function as a detachable entity by the end of 1903.

4.4.2 Function as a complex containing a variable

After ‘On the Meaning and Denotation of Phrases,’ Russell wrote the manuscript titled ‘On Meaning and Denotation’ (CP4, 14).¹³ We will see below that in this manuscript he maintains that a *meaning* is a complex *containing* a variable as a constituent, viewing functions as meanings. This view makes a contrast with his former account of a function as a detachable

¹³ Both manuscripts are undated, but some examples used in them suggest that Russell wrote the former and then the latter (cf. CP4, p.314).

meaning, that is, something that we obtain by *removing* a variable from a complex. It will also be seen that the new view of functions is again directly connected with his attempt to avoid the self-referential paradox of functions.

Let us first see how Russell presents the notion of denoting in 'On Meaning and Denotation.' In his view, a name *designates* an object, where not only proper names but also definite descriptions are counted as names in a broader sense (CP4, p.315). He distinguishes the relation of *designating* from that of *expressing*, the latter holding between an expression and a meaning (CP4, p.317). 'Such phrases,' he claims, 'as "Arthur Balfour," "two," "yellow," "whiteness," "good," "diversity," and single words generally, designate without expressing,' while 'when a phrase contains several words, not simply juxtaposed, but in any way combined so as to acquire unity, then the phrase, as a rule, expresses a complex meaning' (*ibid.*, p.318). *Denoting* is explained as 'the relation of the meaning expressed to the entity designated' (*ibid.*). He does not seem worried about what I call No Linguistic Bypass here, as he does not even try to offer any other explanation of the relation. It should also be noted that he now understands a meaning as a *complex* object.

One of the problems which Russell found with the account of a function as a detachable meaning was, as we saw, that no single letter could be used to speak of a meaning. In 'On Meaning and Denotation' he finds a solution to this problem. The idea is that if we want to speak of a meaning, we can appeal to another meaning which denotes the meaning in question:

[I]f we invent a proper name for the complex (as opposed to what the complex denotes), the proper name, *quâ* name, does of course designate the complex; but the idea indicated by the proper name merely expresses "the meaning of the complex so-and-so," which is a complex denoting the said complex. It is thus only through the medium of denoting that the complex can be dealt with at all as a subject.

(CP4, p.322)

Russell thinks that a symbol can only designate a denoting complex through 'a complex denoting the said complex.' It is not clear why he thinks so. But he envisages what I call Nature of Denoting here, if his reason is that wherever a denoting concept occurs as a logical subject of a proposition, the proposition is not *about* the concept. On the other hand, he does not seem to find No Function of Denotation here. For he remarks 'we can always (theoretically at least) find another idea expressing a meaning which denotes the said object' (CP4, p.321). He does not seem to wonder whether he can systematically find a denoting complex to speak of the denoting complex in question.

Russell proceeds to develop a new account of functions, according to which a function is a complex meaning containing a variable. He claims that an independent variable has a

constant meaning and a variable denotation: 'The *meaning* of the independent variable is something peculiar, and the way in which the meaning denotes is peculiar; but there is not a new single object which can be called its denotation' (*ibid.*, p.330). He then introduces a dependent variable, or a function in general, as a 'complex containing an independent variable' (*CP4*, p.331). But he regards this as a mere approximation, since it does not account for the identity between variables:

"The square of anything" is a constant meaning, but this is not the meaning of x^2 . If it were, we could deduce from the above proposition (p.40a) [i.e. 'if x is a rational number, x^2 is a rational number'] that, if x is a rational number, the square of anything is a rational number,' which is false.

(*CP4*, p.332)

Russell thus thinks that if we are to make sense of ' x^2 ' in 'if x is a rational number, x^2 is a rational number,' we need to accept that it does not express on its own a constant meaning which we can identify with the function of squaring. What this view amounts to is not clear, but it is certain that he understands a function as a complex meaning and hence includes functions among denoting complexes or meanings. That is, 'the function itself always denotes, its denotation being composed of its values' (*ibid.*, p.331). The remark that the denotation is *composed of* entities suggests that he understands, as he does in *PoM*, a denotation as a combination of entities. He also concludes from the above observation that 'an expression containing x must be treated as a whole, and must not be regarded as analyzable into bits each of which contains an independent variable' (*ibid.*, p.333). Propositional functions are thus understood as 'the most fundamental' among functions in general.

It is important to see that Russell derives a sort of type-distinction among variables from the view of a function as a variable meaning. No dependent variable can be, he argues, among the values of the individual variable since 'all the values of the individual variable are constants,' (*ibid.*). Hence, 'if we want a variable whose values are to be dependent variables containing x , we must have a new variable of a different kind' (*ibid.*, p.334). He thus introduces a type-distinction and thereby resolves the self-referential paradox of functions. Hence this account of functions is another instance of his attempts to motivate his then solution to the paradox.

In 'On Meaning and Denotation' Russell also discusses the idea that in fx , 'the f can be detached' (*ibid.*, p.337). This is in effect his former view of function as a detachable meaning. But he now finds two problems with this view. 'Such an analysis is faultless,' he first argues, 'as long as x only occurs once in the complex; but as soon as x is repeated, we find difficulties' (*ibid.*). He also points out that unless functions are somehow distinguished from individuals, 'the detachability of the function' allows self-application of functions, which gives rise to contradiction (*ibid.*, p.338). In a letter dated November 1904, Frege raises this point against

Russell's earlier attempt to use functions in place of classes. Russell responds to the letter with the following remark: 'I have known already for about a year that my attempt to make classes entirely dispensable was a failure, for essentially the same reasons as you give' (Frege, 1980, p.166). It is plausible that these flaws were the chief reasons why Russell abandoned the view of function as a detachable entity in favour of the view that a function is a complex containing a variable. The latter view of function became a basis of his attempts to solve the paradox in 1904.

4.5. 1904 - Zigzag Theory

In April 1904, Russell returned to his work on logic after several months of political activity. He was to stop working again in January 1905, after he had developed various versions of 'zig-zag theory,' on which we focus in this section.

4.5.1 The aim of the theory

Let us first see how the zig-zag theory was meant to work. It will be helpful to have an idea of what Russell was aiming at, if we are to trace the literally 'zigzag' path which he went through in developing the theory.

Russell read a paper called 'On Some Difficulties in the Theory of Transfinite Numbers and Order Types' in December 1905 in the London Mathematical Society. The paper contains an outline of the zigzag theory, though the paper was written after he had virtually abandoned the theory. He there introduces it as an attempt to avoid the set-theoretic paradox and its analogues by excluding the classes with 'a certain characteristic which we may call zigzaginess' (*EiA*, p.145). He explains the notion of zigzaginess as follows¹⁴:

If $\phi!x$ is a non-predicative function, it follows that, given any class u , there must either be members of u for which $\phi!x$ is false, or members of not- u for which $\phi!x$ is true. (For, if not, $\phi!x$ would be true when, and only when, x is a member of u ; so that $\phi!x$ would be predicative.) [...] Again, given any class u , the property $\phi!x$ belongs to either to some, but not all, of the members of u , or to some, but not all, of the members of not- u . This is the zigzag property which gives its name to the theory we are considering.

(*EiA*, p.146)

When a function is said to determine a class, all the members of the class and nothing else satisfies the function. Hence, if a propositional function does *not* determine any class, every class must have either an element which does not satisfy the function or an entity that

¹⁴ For a symbolic presentation of this idea, see Urquhart (1988, p.85). Russell himself offers another explanation of zigzaginess in 'Fundamental Notions' (*CP4*, pp.120-1). I will write ' \mathcal{x} ' for class-abstraction, though Russell originally employs the smooth-breathing symbol.

satisfies the function but is not a member of the class. The aim of zigzag theory is to divide propositional functions between those which can determine a class without leading to any contradictions and those which cannot. I shall call the former functions *legitimate* and the latter *illegitimate*. There should be multiple ways of drawing the distinction, though, as we shall see, Russell struggles to find a feasible one. He sometimes uses the term ‘predicative’ to speak of legitimate functions and employs such notations as ‘ $Fo_x\phi$ ’ to indicate that ϕ is a legitimate function of x .

In order to distinguish legitimate functions from illegitimate ones, Russell focuses upon the form of a function which we use in deriving the set-theoretic paradox. He now defines, under the influence of Frege, the class-membership relation ϵ in terms of second-order quantification (cf. *CP4*, pp.113, 250):

$$x \epsilon u := (\exists\phi). u = \hat{z}(\phi'z). \phi'x \quad \text{Df.}$$

Then, the problematic function ‘ $x \sim \epsilon x$ ’ is equivalent to ‘ $x = \hat{z}(\phi'z). \supset\phi. \sim \phi'x$.’ By generalising the class-abstraction operator to an arbitrary function f , we obtain ‘ $x = f(\phi). \supset\phi. \sim \phi'x$ ’ as a paradigm case of illegitimate functions. We cannot, however, simply abandon all functions of this form, since some of them may well be essential to mathematics. Hence, another important task that should be done in order for the zigzag theory to work is to *recover* harmless functions of the form. He indeed tried numerous sets of axioms governing which functions of the form are *reducible* to other harmless functions.¹⁵

In developing the zigzag theory, Russell allows propositional functions to occur as an argument of a function, but he does not seem to worry about the self-referential paradox of functions in the surviving manuscripts written in 1904. This appears mysterious, given that the paradox led him to abandon the view of function as a detachable entity in 1903. It seems as though he thought he could avoid it by a restriction on the function-name-forming operator. Let us use, as he does in ‘On the Nature of Functions’ (*CP4*, 9), ‘ $\hat{z}(\phi z)$ ’ as a proper name for the function ϕz , and suppose that a function ϕ can occur as an argument in a complex only when it is designated by the proper name ‘ $\hat{z}(\phi z)$.’ Then we can avoid the functional paradox by restricting the applicability of the function-name-forming operator to legitimate functions. If we have a necessary condition for a propositional function to determine a class, then we can simply apply the same condition to the function-name-forming operator. In fact, Russell, in ‘On Some Difficulties,’ clearly sees the similarity between the set-theoretic paradox and the functional paradox: ‘it is to be understood that the arguments which show that there is not always a class also show that there is not always a separable entity which is the propositional function’ (*EiA*, p.145fn; cf.also p.154fn.).

¹⁵ For instance, ‘Fundamental Notions’ contains a number of examples (cf.*CP4*, p.120, p.122, p.132, pp.133-134, p.137, p.143, p.152, p.206, p.208, pp.214-215, p.219, p.225, pp.231-232, p.253, p.258).

The zigzag theory is related to Russell's contention that Cantor's theorem should fail in the cases of 'too big' classes (CP4, p.184). The contention is ill-motivated as he wrongly connects the theorem to the Schröder-Bernstein theorem (*ibid.*, p.160). But Quine's well-known formal system NF and its (slight) modification NFU both prevent us from proving Cantor's theorem, where the latter is even proven to be consistent relative to ZF (Jensen, 1968). Hence, it is thus possible to develop the idea of the zigzag theory into a consistent system, though Russell, as we shall see in the next two subsections, did not succeed in doing so.

4.5.2 Function as an ambiguously denoting object

In 1904, Russell was mainly occupied with the technical issue as to which set of primitive propositions could provide him with an adequate resource from which to construct mathematics. But he also offered at least two accounts of functions in order to explain why his classification of functions into legitimate ones and illegitimate ones is not an *ad hoc* solution to the paradoxes but a philosophically justifiable standpoint. In this section we shall focus on one of these two accounts which is presented in some earlier parts of 'Fundamental Notions' and in 'On Functions, Classes, and Relations' (CP4, 5).¹⁶ It urges that a function is a *constant* (as opposed to *variable*) *meaning*, where a *meaning* is now considered an object that *ambiguously denotes* an object. He seems to think, in other words, that a functional expression designates a fixed object which in turn has the relation of denoting to its values in an unfixed manner—not to a combination of them in a fixed manner.

In 'On Functions, Classes, and Relations,' Russell attempts to explain both what functions in general are and why functions of the form ' $x = f(\phi) . \supset_{\phi} . \sim \phi x$ ' are illegitimate. As for the former question, he claims that '[a] *function* is any way of ambiguously denoting a set of entities, so that, by assigning a value to a variable x , called the *argument*, one and only one of the entities of the set is picked out' (CP4, p.86). Speaking of propositional functions, he remarks:

It would perhaps be better to call this kind of functions *meaning-functions*, rather than propositional functions. A function of the kind which we represent by $\phi'x$ is such that x is a constituent of $\phi'x$, and that $\phi'x$ ambiguously denotes the various values $\phi'x$. A denoting function, on the contrary, does not ambiguously denote its values, but does ambiguously denote complex meanings which denote the corresponding values.

(CP4, p.87)

¹⁶ 'Fundamental Notions' is a set of working notes which covers both accounts of functions. As Urquhart points out, Russell uses some of its discussions to write 'On Functions,' 'On the Functionality of Denoting Complexes' and 'On the Nature of Functions.' 'On Functions, Classes, and Relations' is only dated 1904, but we know that it was written no later than October 1904 (cf. CP4, p.85).

It seems that Russell understands a function (in general) as a fixed object that ambiguously denotes its values.¹⁷ He adopts the phrase ‘meaning-function’ instead of ‘propositional function’ because he tries to dispense with functions of more than one variable by replacing them with functions of functions or by *currying*¹⁸:

The notation $(\psi'\hat{x})'\hat{y}$ represents those among meaning-functions which have meaning-functions for their values. If $(\psi'\hat{x})'\hat{y}$ still contains the variable as such, we adopt the notation $\{(\chi'x)'y\}'z$; and so on. The notation is to imply that, at each stage, we are considering *meanings*. If, when we reach a stage where the variable as such is no longer a constituent, the result is a denoting complex, we are to consider the meaning, not the denotation, of the complex.

(CP4, p.87)

Russell distinguishes \hat{x} from x and calls the former ‘variable as such.’ He adopts the above view to avoid problems concerning the identity between different variables. In ‘Fundamental Notions,’ he remarks: ‘If functions of two variables are *not* of the form $(\phi'x)'y$, how do we distinguish x and y ?’ (CP4, p.248; cf.p.119).¹⁹

It should be observed that Russell introduces the circumflex as a tool for speaking of a meaning. In ‘Fundamental Notions,’ he says as follows:

When we wish to speak of the function itself, *i.e.* the constant meaning, we write $\hat{p} \supset \hat{q} . \supset . \hat{q}$. In $\hat{p} \supset \hat{q} . \supset . \hat{q}$, we do not have an undetermined term of the denotation, but that constant meaning which denotes the terms of the denotation. The circumflex has the same sort of effect as inverted commas have. E.g. we say

Any man is a biped;
“Any man” is a denoting concept.

The difference between $p \supset q . \supset . q$ and $\hat{p} \supset \hat{q} . \supset . \hat{q}$ corresponds to the difference between any man and “any man.”

(CP4, pp.128-129; cf. pp.114, 124.)

¹⁷ We can find the phrase that a concept or a meaning ‘ambiguously denotes’ its values in *PM*. It can be traced back, at least, to ‘On Meaning and Denotation,’ where Russell remarks that ‘the various values [of a dependent variable] are denoted in the same ambiguous impartial manner’ and that the concept a man ‘denotes ambiguously’ (CP4, pp.332, 342).

¹⁸ As Klement points out, the idea of currying itself can be found already in ‘Functions and Objects’ though Russell does not explicitly use it to account for the individuality of variables (Klement, 2003, p.25). Russell discusses several issues concerning *double functions*, namely, functions of two variables in ‘On Fundamentals’ and ‘On the Nature of Functions.’

¹⁹ Hylton counts as a reason for Russell’s adopting the theory of descriptions the problem concerning the identity between variables (Hylton, 2003, p.200).

This passage makes it clear that a function is now conceived as a constant meaning, as opposed to a variable meaning, and that the circumflex is granted a role similar to that of inverted commas. Russell thus employs the circumflex as a tool to speak of a function without noticing No Function of Meaning. Klement finds in the above passage ‘one obvious explanation of why Russell’s work on paradox solving brought him to explore the nature of denoting in general’ (Klement, 2010, p.644). Indeed, as Klement points out, the self-referential paradox of functions ‘requires being able to speak of a propositional function itself as opposed to its values’ (*ibid.*). But, as we have seen, even the set-theoretic paradox is, in Russell’s view, also closely connected to the notion of denoting through functions of the form ‘ $x = \hat{z}(\phi'z) . \supset_{\phi} . \sim \phi x$.’

In his attempt to explain why such functions are illegitimate, Russell develops what he calls the vicious-circle theory.²⁰ He argues that if a function of the form $(\phi).(f'x)(\phi'z)$ can itself be seen as having the form $\phi'x$, it would involve ‘a vicious circle in definition’: ‘We cannot make any statement about all statements about x , because, according to the hypothesis, such a statement would itself be one of the statements about x ’ (CP4, p.88). Note that ‘ $(\phi).(f'x)(\phi'z)$ ’ is a generalisation of ‘ $x = f(\phi) . \supset_{\phi} . \sim \phi x$.’ If we regard, as Gödel would do, the expression ‘ $x = f(\phi) . \supset_{\phi} . \sim \phi x$ ’ as merely picking up an entity which subsists independently of our use of the symbol, then there would be no circularity. But if we understand, as Russell apparently did, *definition* as an explanation of how the complex entity in question is composed of other entities, the circularity is a genuine one.²¹ It thus seems that Russell had a good reason to endorse the vicious-circle theory. But he gave up the theory at some point in the summer of 1904 for a technical reason: he found that the axiom numbered *4.2, which was motivated by the vicious-circle theory, makes the function ‘ $\hat{x} \sim \epsilon \hat{x}$ ’ legitimate (CP4, pp.125-126).

4.5.3 Function as a mode of combination

From October 1904 onwards, Russell wrote ‘On Functions’ and ‘On the Nature of Functions.’ In these manuscripts, he tentatively abandons the view of a function as a meaning in favour of the view of a function as a *mode of combination*. He develops the latter view so as to, again, give a philosophical ground to the zigzag theory.

The manuscript ‘On Functions’ was written in October 1904 and sent to Whitehead with a covering letter, which summarises the core idea of the manuscript as follows:

²⁰ One might think that the theory was named after the vicious circle principle, which Russell adopted through his dispute with Poincaré. But it was when an English translation of Poincaré’s *La science et la hypothèse* was published in 1905 that the dispute started. Yet, there remains a possibility that Russell read the original version in French in 1903 learning the idea that paradoxes arise from vicious circles.

²¹ B. Linsky makes the same point regarding propositional functions in *PM* (Linsky, 1999, pp.27-32). I will discuss his interpretation in greater detail in the next chapter.

The bare bones of the matter are these:

- (a). $\phi'x$ is not a constituent of $\phi'x$; hence
- (b). $\phi'x$ is not a function of ϕ and x ; hence
- (c). $x \in u$ is not a function of x and u , and $\hat{x} \in u$ is not a function of u . (CP4, p.96)

Russell seems to think that if (c) holds, the function ' $x \sim \epsilon x$ ' would prove illegitimate. So, if the above reasoning is correct, he only needs to establish (a) to block the set-theoretic paradox. What he submits in order to explain why (a) obtains is the view of a function as a mode of combination.

In 'On Functions,' Russell first defines a *complex* as 'a unity formed by certain constituents combined in a certain manner' (CP4, p.98). He then argues that such a manner is not a constituent of the complex: 'in a complex, the combination is a combination of *all* the constituents, and cannot therefore be itself one of the constituents' (*ibid.*). Hence, if a function $\phi'x$ is a mode of combination, it follows that $\phi'x$ is not a constituent of $\phi'x$. With this account of functions, he explains why functions of the form ' $x = f(\phi) \cdot \supset_{\phi} \cdot \sim \phi x$ ' are illegitimate. He now employs ' $\phi'x$ ' 'to denote a variable one of all the possible ways in which one entity can enter into a complex' (*ibid.*, p.98). He includes mode of combinations among entities, but he still distinguishes such variables from entity-variables on the ground that 'there are possible values of x which are not possible values of $\phi'x$ ' (*ibid.*, p.99).²² He then urges that functions which involve both kinds of occurrences of a variable are illegitimate: 'The general rule is that, if $\phi'x$ occurs in a complex, the ϕ must not be varied, because it is not a constituent of $\phi'x$ ' (CP4, p.100). But he finds it too sweeping to reject all the expressions which both quantify over functions and apply a function to an entity. Hence, he tries to 'establish the exceptions and the reasons for them' though in vain (*ibid.*).

In November 1904, Russell wrote the manuscript 'On the Nature of Functions.' A key notion in this draft is 'the difference between ϕ (in $\phi'x$) and $\hat{x}(\phi'x)$ ' (CP4, p.265). He remarks as follows:

As a rule, in places where we write $\hat{z}(\phi'z)$, the function occurs as entity; in places where we write ϕ , the function occurs as function. We may make it a principle to write $\hat{z}(\phi'z)$ for ϕ when and only when the complex in question remains significant when entities other than functions are substituted for $\hat{z}(\phi'z)$. If we adhere to this rule, "all values of ϕ " will mean, when $\hat{z}(\phi'z)$ is concerned, the same as "all values of x ," whereas when $\phi'z$ is concerned it will mean "all functions," which is narrower than "all entities." This

²² The idea that a mode of combination is an entity contrasts with his own notion of *logical form* in later writings where he emphasises that logical forms are merely *forms* and not entities (cf. TK, p.98).

seems to make a preliminary objection to such expressions as

$$x = f'\hat{z}(\phi'z) \cdot \supset_{\phi} \cdot \phi'y;$$

for “all values of ϕ ” will refer, on the left, to all *entities*, but on the right, only to all *functions*.

(CP4, pp.268-9)

Thus, Russell holds that two different kinds of varying ϕ render ' $x = f'\hat{z}(\phi'z) \supset_{\phi} \cdot \phi'y$ ' illegitimate. It is important to observe that the distinction 'between ϕ (in $\phi'x$) and $\hat{z}(\phi'z)$ ' is supported by the view of functions as modes of combination: ' $\phi'x$ designates the compound of x with other entities according to a certain mode of composition,' while ' $\hat{z}(\phi'z)$ designates that mode of composition itself' (CP4, p.265). Thus, in 'On the Nature of Functions,' Russell again appeals to the view of a function as a mode of combination in order to explain why functions of the form ' $x = f(\phi) \supset_{\phi} \cdot \phi'x$ ' are illegitimate.

But functions understood as modes of combination seem to be quite different from what we usually think of as functions. For instance, in the case of $p \supset p$, the mode of combination should be $\hat{p}\hat{R}\hat{p}$ rather than $\hat{p} \supset \hat{p}$, and if so, we cannot speak of $\hat{p} \supset \hat{p}$ as a function (cf. CP4, p.98). Russell observes this point in 'Fundamental Notions.' 'Theoretically,' he remarks, 'a complete analysis into function and arguments ought to leave no constant elements in the function, but only variable arguments in a constant manner'; yet, *practically*, each function has 'some constant element over and above the function itself' (CP4, pp.129-130). He seems to distinguish 'philosophically fundamental' functions from derivative ones or 'mathematical' ones, maintaining that only the former are modes of combination and the latter are complex entities composed of modes of combinations and other entities (*ibid.*, p.154). Such a distinction would indeed allow ordinary functions to be called *functions* though in a derivative sense. But he eventually comes to see this distinction as based merely on a confusion. In some later parts of 'Fundamental Notions,' he remarks that the 'vital distinction' between modes of combination and functions expressed by such notions as ' $f'x$ of $f'\phi$ or $f'\phi, 'x$ or $f'x, 'y$ ' has been missed (*ibid.*, p.255).²³ He now clearly observes that '[t]he form of combination in a complex, as opposed to the function or functions, is got by making *every* constituent variable' and that '[a] function on the other hand is got from a complex by varying only such constituents as occur as proper names, not as verbs' (*ibid.*). He thus comes to distinguish between functions and modes of combination and thereby abandons the view of functions as modes of combinations, though he, as we will see, returns to it in 1905.

Interestingly, Russell also tries to use the distinction between functions and modes of

²³ These parts seem to have been written after 'On the Nature of Functions,' since they come after the folios in which Russell presents a summary of 'On the Nature of Functions.'

combination to motivate the zigzag theory: he claims that ' $x = f(\phi) . \supset_{\phi} . \phi x$ ' is a mode of combination, but not a function of x . But he soon becomes suspicious about this approach: 'I am not sure but what our present distinction between function and mode-of-combination rests on oblivion of the fact that the ϕ in $\phi'x$ is not got by leaving out the x ' (CP4, p.256). He also wonders whether it is plausible to characterise a function as something 'got from a complex by varying only such constituents as occur as proper names'; for 'it won't apply to all propositional functions' (*ibid.*, p.257). Then, he is led to 'the fundamental philosophical question, upon which, the whole thing turn': 'what is a function? and what, besides the mode-of-combination, must be constant in a complex in order that a constant function may result?' (*ibid.*). As we have seen in the present section, Russell in 1904 could not find any answers to these questions: he failed to find a philosophical account of functions that would motivate his zig-zag theory and in particular would explain why functions of the form ' $x = f(\phi) . \supset_{\phi} . \phi x$ ' should be illegitimate.

4.6. 1905 - The discovery of the problem of denoting

In this section, I will focus on a 1905 manuscript 'On Fundamentals' so as to illustrate that Russell came to envisage Nature of Denoting and No Function of Meaning—arguably the central observations of the problem of denoting—through his attempt to philosophically motivate the zig-zag theory.

In April 1905, Russell resumed his work on logic and wrote some manuscripts before 'On Fundamentals.' They are lost but he seems to have thought again that the key to solving the paradoxes lies in the notion of denoting. For, at the beginning of 'On Fundamentals,' he distinguishes 'meaning-variation' from 'entity-variation,' while he employs, as we shall see shortly, 'being' or 'entity' in place of 'denotation':

It seems likely that meaning-variation must be distinguished from entity-variation, and that two variables of which one means and the other is can only be equal by accident, and can't be kept equal throughout variation.

(CP4, p.360)

As we saw in the previous section, Russell considered already in 1904 the idea that two distinct kinds of occurrences of variables explain why functions of the form ' $x = f(\phi) . \supset_{\phi} . \phi x$ ' are illegitimate. But he now connects it to the notion of denoting and develops the view of functions as having two distinct occurrences. He introduces the following claim as 'Important Principle'²⁴:

²⁴ In 'On Fundamentals,' Russell first proposes to use the symbol ' $(C\phi)x$ ' to speak of a mode of combination, suggesting that he still retains the notion of mode of combination (CP4, p.360). But, as he moves on in the

If a denoting expression denotes a meaning, and is put in a meaning-position in a complex, it will be the meaning *meant* by the denoting expression, not the meaning *denoted*, that will occur there.

(CP4, p.361)

He seems to propose that by writing ' $\phi(x)$ ' we should apply the meaning of ' ϕ ' to the denotation of ' x .' He then remarks as follows:

Corollary. If $(C\hat{x})$ is in a *meaning*-position in $(C\hat{x})\frac{x}{\hat{x}}$, and if we consider $(C\hat{x})\frac{(C\hat{x})}{\hat{x}}$, then, if we take a value of $(C\hat{x})$ which both is and denotes a meaning, it will be the meaning which it *is* that will occur on the first occurrence, and the meaning which it *denotes* that will occur on the second occurrence. Thus we do not get an instance of $p\frac{p}{\text{Var}}$ from such a case.

(CP4, p.361)

He thus thinks that the two distinct kinds of occurrences of a function forbid us from applying functions to themselves. A typical example of functions whose self-application gives rise to difficulty is, of course, of the form ' $x = f(\phi) . \supset_{\phi} . \phi x$ ':

Consider $x = (C\hat{x}) . \supset_C . \sim '(C\hat{x})$. Call this $(W\hat{x})$. Then we infer

$$(W\hat{x})(W\hat{x}) . \equiv . (W\hat{x}) = (C\hat{x}) . \supset_C . \sim '(C\hat{x})(W\hat{x}).$$

We shall say here that, though $(W\hat{x})$ and $(W\hat{x})$ are *meant*, W itself is not meant, but can only be *denoted*. In fact W is the meaning of the type equivalent to the proposition that x does not satisfy any type which it is.

(CP4, p.362)

He thus appeals to the two distinct kinds of occurrences of a meaning to explain why functions of a certain form are to be considered as illegitimate.

Importantly, after the discussion of those two kinds of occurrences of a meaning, Russell comes to hold that we cannot talk about a meaning simply by means of a proposition containing the meaning as a constituent:

It seems that if we wish to put a denoting meaning in an entity-position, and say something about the meaning itself, we can only do so by means of a denoting concept;

manuscript, he comes to use it, as in the quoted passage, to speak of a complex in general including what I call legitimate functions and illegitimate functions. Also, as the passage suggests, he employs the operation of substitution to give an account of function-application. It should be noted that since he abandoned the view of function as a detachable entity in 1903, he could no longer explain function-application simply as a connection between two objects. Klement argues that the notion of substitution thus used leads to an infinite regress (Klement, 2003, p.31).

for if, instead of a denoting concept, we put the meaning in question, then since the position is an entity-position, we shall be talking unintentionally about the denotation of the meaning instead of about the meaning.

(CP4, p.363)

He seems to think that a proposition is *about* an entity only if it occurs in an entity-position of the proposition or occurs—in *PoM*'s phraseology—as a logical subject of the proposition. But, if so, and if a meaning shifts *aboutness* to its denotation when it occurs in an entity-position, it follows that we cannot talk about the meaning by finding a proposition containing it. This is precisely what I call Nature of Denoting. The heart of the notion of denoting is, as we saw in Section 4.3, to shift *aboutness*. He thus becomes fully aware of the consequence of the nature of denoting: we cannot talk about a meaning itself by making it a constituent of a proposition. But this does not mean that he finds it impossible to speak of a meaning. He indeed thinks that inverted commas 'give a denoting concept which denotes the meaning of what is between the inverted commas' (CP4, p.363). He is yet to discover No Function of Meaning.

Discussing the above principle, Russell comes to propose the view that a complex *has* a meaning and a denotation. He considers replacing the distinction between meaning and denotation with the one between *being* and *meaning*: 'What a complex *is* is what we have called its denotation' (CP4, p.366). He then regards this two-fold nature as a characteristic of complex objects: 'An entity which is not a complex does not have the two sides, but only has being' (*ibid.*). It is important to note that when he says a complex has meaning and being, he does not mean that there are two distinct objects attached to a complex. Rather, meaning and denotation are two 'sides' of a complex, or, to put it differently, '[a] complex may occur in two ways, as meaning or as entity' (*ibid.*).²⁵

But Russell soon finds certain dilemmas with the view that a complex has meaning and denotation. First of all, he points out that 'a proposition as entity must depend upon its constituents' (CP4, p.368). He illustrates this with the proposition 'People were surprised that Scott was the author of *Waverley*.' In this, the proposition 'Scott was the author of *Waverley*' occurs as entity or as being. But if the proposition as *being* is not analysable into its constituents, it remains unclear how we can distinguish it from 'Scott is Scott' in 'People were surprised that Scott was Scott.' The *Waverley* case thus seems to show that the identity of a complex occurring as entity still depends on its constituents. However, Russell argues that the *Waverley* case also demonstrates the opposite: 'a proposition quâ entity is to be held unanalyzable' (*ibid.*, p.371). When a proposition $\phi'x$ occurs as an argument of $\psi'x$ and hence

²⁵ Pakaluk understands the view in question otherwise (Pakaluk, 1993, pp.46-9).

as an entity, the constituents of $\phi'x$ cannot be constituents of $\psi'\phi'x$. For we cannot always substitute 'Scott' in ' $\psi'\phi'$ (Scott)' with 'the author of *Waverley*' without changing the truth value of the whole proposition, as the *Waverley* case illustrates. He is thus confronted by a dilemma.²⁶ He seems to think that the dilemma and other similar difficulties all arise from the following principle:

The broad rule is that when complexes occur as *meaning*, their complexity is essential, and their constituents are constituents of any complex containing the said complexes; but when complexes occur as *entities*, their unity is what is essential, and they are not to be split into constituents.

(CP4, p.373)

He finds two more dilemmas arising from this 'broad rule.' One is that 'in "Scott is the author of *Waverley*," we must either deny that this [proposition] contains "the author of *Waverley*," quâ entity, or deny that *Waverley* is a constituent of it' (*ibid.*, p.373). The other one is that if p and q occur in a proposition ' $p \supset q$ ' as entities, their constituents are not constituents of the proposition, while the analysis of ' $\phi'x \supset_x \psi'x$ ' as obtained from ' $\phi'x \supset \psi'x$ ' seems to require that the constituents of $\phi'x$ and $\psi'x$ should also be constituents of ' $\phi'x \supset \psi'x$ ' (*ibid.*).²⁷ 'The above two dilemmas,' he remarks, 'force us to recognize a greater variety of modes of occurrence than we have yet admitted' (*ibid.*). He introduces five pairs of kinds of occurrence, which include 'occurrence as entity' and 'occurrence as meaning' (*ibid.*, pp.374-376). These distinctions of occurrence may be artificial enough to undermine the view that a complex has a meaning and a denotation, but he still does not think that the notion of denoting is rendered implausible.

After examining the five kinds of occurrences and relations between them, Russell focuses on the distinction of entity-occurrence and meaning-occurrence, and he suggests that a meaning variable is only derivative of an entity variable. Then, almost suddenly, he starts questioning how inverted commas work. First he recalls Nature of Denoting:

When a concept has meaning and denotation, if we wish to say anything about the meaning, we must put it in an *entity*-position; but if we put it itself in an entity-position, we shall be really speaking about the denotation, not the meaning, for that is always the case when a denoting complex is put in any entity-position. Thus in order to speak about the meaning, we must substitute for the meaning something which *denotes*

²⁶ Klement indicates a similar problem concerning substitution of denotation in a larger complex, though the problem presupposes the identity between such propositions as 'The King of England is a man' and 'Edward VII is a man' (Klement, 2016, pp.21-2).

²⁷ Russell also remarks that p and q should occur as entity in ' $p \supset q$,' since 'it is fallacious to use single letters' for complexes as meanings (CP4, p.373).

the meaning. Hence the meanings of denoting complexes can only be approached by means of complexes which denote these meanings. This is what complexes in inverted commas are.

(CP4, pp.381-2)

In illustrating this point Russell discovers that we cannot apply any function to a meaning unless we already have such a function:

If we say “‘any man’ is a denoting complex,’ ‘any man’ stands for ‘the meaning of the complex “any man,”” which is a denoting concept. But this is circular; for we use “any man” in explaining “any man.” And the circle is unavoidable. For if we say “the meaning of any man,” that will stand for the meaning of the denotation of any man, which is not what we want.

(CP4, p.382)

This observation is of course No Function of Meaning. This is a mere extension of Nature of Denoting to the cases where a denoting complex occurs as an argument of a function rather than as a logical subject of a proposition. But Russell discovered these two points separately and, as we saw above, he in OD treats them in different ways. He is thus confronted by the problem of denoting:

The endeavour to speak about the meanings of denoting complexes leads, if the above is correct, to the following dilemma. If we do not put the meaning in an entity-position, we merely mean it, and do not say anything about it; if, on the contrary, we put it in an entity-position, it stands for its denotation, and we get the meaning (if any) of what the complex denotes, not of what the complex means.

(CP4, p.382)

It should be noted that No Function of Meaning shows that if a function is a meaning, such an expression as ‘ $x = f(\phi) \cdot \supset_{\phi} \phi x$ ’ does not express what we intend. For if ‘ ϕ ’ is a denoting expression, ‘ $f(\phi)$ ’ does not express an application of any function f to it. This is precisely the idea, which we saw above, that two distinct occurrences of a function make it impossible to have a self-application of the function. But he did not use this idea to motivate the zig-zag theory, as he invented the theory of descriptions just after the discovery of the above problem. The theory of descriptions led him to a more drastic solution to the paradoxes, the *substitutional theory*.²⁸

²⁸ ‘On Fundamentals’ documents how he invents the theory of descriptions (CP4, pp.384f). See also Noonan (1996, pp.97-101). We will turn to the substitutional theory in the next chapter.

4.7. Concluding remarks

Russell's various accounts of functions thus provide us with the link between his attempts to solve the paradoxes and the notion of denoting. He developed various theories of function so that he could give a philosophical justification to his then solution to the set-theoretic paradox as well as the self-referential paradox of functions, while most of those theories were based upon the notion of denoting. This illustrates how he understood the relation between technical issues such as the set-theoretic paradox and philosophical issues: A technical solution wants a philosophical account of theoretical entities in question. In his letter to Russell dated April 1904, Whitehead remarks as follows: 'For the *technical* development we want reasonably general Pps which give us all we want and exclude the contradiction. I agree that Pps (except by miracle) will only turn up in a philosophical analysis of the subject—and that the better the analysis, the better the Pps' (CP4, p.xxxix). Whitehead was thus aware that his collaborator found a tight connection between a technical investigation into functions and a philosophical enquiry into them.

It is important to note that Russell did not simply follow the idealist conception of logic. In developing functional theories, he rather assumed that a technically adequate theory can come *prior* to a philosophical account underlying it. Klement remarks that 'Russell refused to accept any theory until he found a theory that was consistent with his metaphysical scruples' (Klement, 2003, p.36). But this is not entirely correct, as Russell sometimes proposed a technical solution and *then* sought a philosophical ground for endorsing it. Some passages in 'Fundamental Notions' illustrate this point well. He considers there a view that '[i]f in an expression containing ϕ and x , ϕ occurs predicatively, then " (ϕ) .the expression." is not a function of x , if the argument to ϕ contains x or ϕ ' (CP4, p.131). The idea seems, again, to be intended to be an account of why the function ' $x = \hat{z}(\phi'z) . \supset_{\phi} . \sim \phi x$ ' is illegitimate. He then remarks: 'I think this is all right; but it needs to be philosophized' (*ibid.*). When he finds a set of primitive propositions promising, he goes on to say: 'It only remains to philosophize the matter, which must be done by pointing out that $\phi'x$ does not contain ϕ as a constituent' (*ibid.*, p.146). And after introducing another set of axioms, he states: 'It becomes necessary to simplify them and to find some philosophical principle from which they follow' (*ibid.*, p.207). Of course, Russell did not accept these technical solutions after all, but his attitudes towards the relationship between the two branches of logic was certainly more flexible than Klement suggests. Russell thought that a technically successful theory concerning a certain range of theoretical entities could guide us to a correct account of what they are. This is another significant step away from the idealist conception of logic, the first step being the replacement of propositions for judgments.

5 Substitution and Construction

5.1. Introduction

The preceding chapter was devoted to examining why Russell had thought of the notion of *denoting* as ‘probably relevant’ before his invention of the theory of descriptions in the summer of 1905. In this chapter I will discuss why he thought the notion ‘proved to be’ relevant to the set-theoretic paradox. It is worth doing so because he constantly speaks of the theory of descriptions as a breakthrough in his logicist programme (for example, *MPD*, p.79; *Autobiography*, p.142).

In the literature, there are at least two accounts as to why Russell thus describes the invention of the theory of descriptions. Both accounts share the idea that what really brings progress to his programme on logic is not the theory of descriptions itself but the notion of *incomplete symbol*, under which the theory urges definite descriptions fall. A somewhat popular interpretation is that Russell adhered to the *doctrine of the unrestricted variable*. The doctrine urges that there is only one kind of variables and they range over all the objects. The doctrine is thus at its face value incompatible with type restrictions, and this is where the notion of incomplete symbol plays, on this account, a crucial role in his logicist programme: by treating class-symbols and function-symbols as incomplete symbols, he can avoid introducing type restrictions to genuine variables ranging over genuine objects. He indeed develops the *substitutional theory* from 1905 onwards, which enables him to thus treat classes and propositional functions as ‘logical fictions.’ An obvious problem with this interpretation is, however, that Russell and Whitehead seemingly quantify over propositional functions conceived as objects in *PM*, which means their abandonment of the doctrine. Those who endorse this interpretation thus construe propositional functions in *PM* as mere linguistic items so that the quantification over them can be rendered compatible with the doctrine. In Section 5.2, I will look into some papers and manuscripts on the substitutional theory in order to critically examine this interpretation.

Another interpretation is originally proposed as an interpretation of *PM* and urges that Russell and Whitehead conceive of propositions and functions as *logical constructions* in the book. According to this interpretation, the notion of incomplete symbol provides the authors with a systematic way of dealing with technical objects such as classes. In Section 5.3, I will

survey some manuscripts written in 1907 as well as *PM* and argue that although Russell certainly hits upon the notion of logical construction or at least an analogue of it then, he and Whitehead do *not* appeal to it in *PM* in their attempt to explain what propositional functions are nor to motivate the ramified theory of types. By way of explaining what the authors take to be propositional functions in *PM*, I will also argue against the idea that they are considered to be open sentences, the idea to which those who believe in Russell's firm adherence to the doctrine of the unrestricted variable appeal.

These two sections will hopefully illustrate how Russell comes to endorse the ramified theory of types presented in *PM*, and more importantly, what he subscribes to throughout his attempts to find a technically and philosophically satisfactory formal system. In the last section I will conclude that what underlies those attempts was not the doctrine of the unrestricted variable nor the notion of logical construction but rather a simple assumption that any hierarchy of technical objects needs to be motivated by a philosophical account of those objects. Given what we have seen in the present work, this is a perfectly natural assumption for Russell to hold.

5.2. Substitutional theory and the doctrine of the unrestricted variable

In this section I will first introduce some basic notions of the substitutional theory so as to explain how it treats function-symbols as incomplete symbols. I will then attempt to show that Russell developed the theory despite of what I call propositional paradoxes, before I argue that he retained the theory as a philosophical account of a hierarchy of propositional functions even after he started to seek alternative theories.¹

5.2.1 The basic notion of the substitutional theory

Once Russell invented the notion of incomplete symbol, it was presumably natural for him to apply it to function-symbols.² For, as we saw in the preceding chapter, he struggled to find a tenable account of functions before the invention. The substitutional theory enabled him to treat function-symbols as incomplete symbols, liberating him from the burden of explaining what functions are.

The core notion of the substitutional theory is to employ a pair of two entities, a proposition and an entity occurring in it, in place of a function. For example, we can replace a function '*x* is human' for a pair of the proposition 'Socrates is human' and Socrates. If we want to simulate the application of the function, say, to Plato, then we can substitute Plato for Socrates in the proposition. The resulting proposition is, of course, 'Plato is human' and this

¹ This section has benefitted from G. Moore's Introduction to *CP5* and from Proops' summary of Russell's "on-again, off-again" relationship with his so-called substitutional theory' (Proops, 2011, pp.191-4).

² I use 'function' to speak of what Russell called propositional functions.

is exactly what we obtain by applying the function 'x is human' to Plato. Russell employs the notation ' $p/a; x!q$ ' or ' $p \frac{x}{a} !q$ ' to mean, intuitively, that 'q results from p by substituting x for a in all those places (if any) where a occurs in p' (EiA, p.168). He assumes that substitution always produces a unique result (CP5, p.100). Then the expression ' $p/a; x$ ' can be treated as a definite description for the unique q such that $p/a; x!q$, and the theory of descriptions enables us to eliminate such descriptions whenever we want. We can thus regard ' $p/a; x$ ' as an alternative expression for a function application ϕx , and hence, ' p/a ' for a function ϕ itself.³ Such expressions as ' p/a ' are, importantly, not proper names, which are considered meaningful on their own, but merely part of expressions like ' $p/a; x!q$.' Russell calls such an expression as ' p/a ' a 'matrix,' maintaining that matrices do not stand for any objects on their own. The substitutional theory thus enables him to treat function-symbols as incomplete symbols.

Importantly, the substitutional theory places a type restriction on functions as part of its grammar. A matrix p/a is after all a pair of entities but not a single entity. Hence, if we are to simulate a self-application of a function p/a , we should write ' $p/a; p/a$ '; but ' $p/a; p/a!q$ ' is not a legitimate expression of the theory. In order to simulate higher-order functions, we can use multiple substitution. If we express by ' $p/a, b; x, y!q$ ' a multiple and simultaneous substitution of x for a and y for b, such an expression as ' $p/a, b$ ' can be used for a function of functions of individuals. In this way, we can obtain a hierarchy of matrices/functions: matrices of two letters stand for functions of individuals, those of three letters for functions of functions of individuals, and so on. It is worth noting that if we thus simulate higher-order functions, they will have the same 'type' as multiple relations. For instance, a matrix ' $p/a, b$ ' can be used as a substitute for ϕx , a unary function of individuals, while it can be employed to speak of a binary relation between individuals.

The substitutional theory has several advantages. First of all, it makes it possible to employ technical objects such as functions and classes without any ontological commitment to those objects, liberating Russell from the philosophical burden of explaining what they are at all. Second, it justifies type restrictions upon those objects, as it imposes these restrictions upon symbols for them as part of its syntax. It is important for our purposes to note that these advantages count as such only for those who seek philosophical grounds for their formal system. Third, insofar as it is concerned with functions and classes, the substitutional theory dispenses with type indices, leaving its variables all unrestricted. It is undeniably true that Russell includes this point among the advantages of the theory. Landini, who attributes a firm endorsement of the doctrine of the unrestricted variable to Russell, goes further to say

³ It should be remembered that once Russell invented the theory of descriptions, he also distinguished between *substitution* of a term for another and *determination* of a value of variable.

that this advantage was the *raison d'être* of the theory (Landini, 1998b, p.254). But, for my part, it is questionable whether Russell took the third advantage to be the whole point of adopting the substitutional theory. To evaluate Landini's interpretation, let us take a closer look at some manuscripts in which Russell develops the substitutional theory.

5.2.2 The development of the substitutional theory

In this section, I will sketch how Russell developed the substitutional theory after the invention of the theory of descriptions in 1905. By so doing I will make two points against the notion that Russell strictly adhered to the doctrine of the unrestricted variable. One is that even when he accepted the necessity of a hierarchy of propositions to block propositional paradoxes, he kept developing the substitutional theory until the summer of 1906. The other is that even in a paper where Russell emphasises the doctrine of the unrestricted variable, what he effectively endorses is a weaker claim that a hierarchy upon objects should be indicated by 'internal' structures of symbols for them.

In 'On Some Difficulties in the Theory of Transfinite Numbers and Order Types' (OSD), Russell makes public the substitutional theory for the first time. He sent the paper to the London Mathematical Society in November 1905. The paper contains a brief introduction to the substitutional theory and rates it as a possible solution to the set-theoretic paradox and its analogues:

The theory is safe, but drastic; and, if, in fact, there are classes and relations, it is unnecessarily difficult and complicated. For the present, therefore, it may be accepted as one way of avoiding contradictions, though not necessarily *the* way.

(EiA, p.156)

He thus presents the substitutional theory merely as a possible option here, but he soon becomes fully convinced of it. For Russell wrote 'Substitution₁' in December 1905, and this manuscript presents a set of axioms for the substitutional theory, which in turn suggests that he then attempted to develop the basic ideas of substitution into a formal system.⁴

When Russell started developing the substitutional theory, he discovered some '*oddities*' in it (CP5, p.lxxiiiif). Those *oddities* are arguably what I call propositional paradoxes, as he uses the same word to speak of them in other manuscripts. Propositional paradoxes are paradoxes which become formulable when we treat propositions as individual entities. They include the Appendix B paradox, the Liar paradox and what is sometimes called the p_0/a_0 paradox.⁵

⁴ Russell wrote two manuscripts entitled 'On Substitution.' I will use 'Substitution₁' to refer to the one written in December 1905 (CP5, 3b) and 'Substitution₂' (CP5, 5a) to speak of the one written in April and May 1906.

⁵ Russell might have understood those paradoxes as various ways of formulating the Liar paradox. For formal derivations of the Appendix B paradox and the p_0/a_0 paradox, see Landini (1998b) and Linsky (2002). As for the Liar paradox, Russell gives two distinct interpretations to the Liar sentence 'I am lying.' One is ' $(p). \phi p. \supset. \sim p$,'

As the term ‘oddities’ suggests, Russell first did not regard them as genuine contradictions. Whitehead shared this optimistic attitude towards propositional paradoxes. In his letter to Russell of 5 December 1905, Whitehead remarks:

My position over the “oddities” is thus: as a middle-class Englishman I regret that such paradoxes should be, but candour compels me to admit that they are not contradictions.
(CP5, p.lxxiii)

Whitehead also notes that ‘[w]hile they exist a contradiction may crop up any day,’ suggesting that oddities are somehow distinguished from contradictions (CP5, p.lxxiii). Russell and Whitehead did not then think of the *oddities* as genuine contradictions, presumably because they thought they could easily reject the identity conditions for propositions, which are essential in deriving the Appendix B paradox and the p_0/a_0 paradox. In fact, Whitehead, in the letter quoted above, claims that if there are no such things as ‘complex unities,’ each proposition is ‘a simple unity’ and hence there may well be more than one way of ‘enunciating’ its constituents. If the constituents of a proposition can be ‘enunciated’ in multiple ways, we can reasonably reject the assumption that if two propositions are identical, they are composed of the same constituents in the same manner.

It is important to note that Russell maintained the optimistic attitude towards the propositional paradoxes for a while, continuing to work on the substitutional theory. In a letter to Lucy Donnelly of 1 January of 1906, he remarks:

The difficulty which I came upon in 1901, and was worrying over all the time you were in Europe, has come out at last, completely and finally, so far as I can judge. It all came from considering whether the King of France is bald—a question which I decided in the same article in which I proved that George IV was [not] interested in the Law of Identity.

(Autobiography, p.174; cf. p.169)

It is plausible that he speaks of the substitutional theory here, which is indeed an application of the notion of incomplete symbol. In February 1906, he added the following note to the draft of OSD: ‘From further investigation I now feel hardly any doubt that the no-classes theory [*i.e.* the substitutional theory] affords the complete solution of all the difficulties stated in the first section of this paper’ (EiA, p.164).⁶ Russell submitted ‘On the Substitutional Theory of Classes and Relations’ (SCR) to the London Mathematical Society in April 1906. Having

which intuitively says that every proposition with the property of ϕ is false; the other is ‘ $(\exists p). \phi p . \sim p$,’ which implies a contradiction when supplemented with a premise that the proposition itself is the only proposition that satisfies ϕ . I will not distinguish between the Liar paradox and the Epimenides paradox.

⁶ I use the term ‘no-classes theory’ exclusively to speak of the idea that class-symbols are incomplete symbols.

presumably the *oddities* in mind, he remarks in SCR as follows:

The only serious danger, so far as appears, is lest some contradiction should be found to result from the assumption that propositions are entities; but I have not found any such contradiction, and it is very hard to believe that there are no such things as propositions, or to see how, if there were no propositions, any general reasoning would be possible.
(*EiA*, p.188)

In this paper he presents the substitutional theory as ‘*the* solution of the contradictions discovered by Burali-Forti and myself’ (*EiA*, p.165; emphasis added).

Russell thus kept developing the substitutional theory despite the propositional paradoxes. Far more important is the fact that he did so even after he explicitly observed that he could only avoid the paradoxes by introducing a hierarchy of propositions. After the submission of SCR, he wrote what I call ‘Substitution₂.’ This manuscript discusses the *oddities* again and, after considering some minor modifications to the substitutional theory, he asks himself:

But will so small a change avoid contradictions?

$$p_0 := (\exists p, a) : a_0 = (p \frac{b}{a} ! q) : p \frac{a_0}{a} ! r . \supset_r . \sim r \dots$$

The only way to avoid *oddities* is to decree a doctrine of types as regards *propositions*, so that in $p \frac{b}{a} ! q$, b and a must be of lower type than p and q , so that the substitution of $(p \frac{b}{a} ! q)$ for a is illegitimate.

(*CP5*, p.131)

Although he still calls propositional paradoxes *oddities* here, he also employs the word ‘contradictions’ to speak of them, suggesting that he no longer distinguishes the *oddities* from contradictions. It is also plain that he thinks here that only a hierarchy of propositions solves them. This is important to observe, as it seems to indicate, contra Landini, that the hierarchy of propositions necessitated by the propositional paradoxes was not the reason why Russell abandoned the substitutional theory. Landini claims that it is only in ‘Mathematical Logic as Based on the Theory of Types’ (henceforth, ML), which Russell wrote in July 1907, that he accepted a hierarchy of propositions in conjunction with the substitutional theory. But, as we have just seen, he observed that he needed a hierarchy of propositions to block the propositional paradoxes back in the spring of 1906 at latest. It would be simply difficult to explain why he retained the substitutional theory until July 1907, if he eventually abandoned it because, as Landini contends, the type restriction on propositional variables turned out to be ‘flatly inconsistent with the *raison d’être* of the original substitutional theory,’ namely with the doctrine of the unrestricted variable (Landini, 1998b, p.254). It seems to me that the doctrine was, for Russell, not the *raison d’être* of the theory after all. In the next section I will

explore possible reasons why he stopped developing the substitutional theory.

Russell wrote 'On "Insolubilia" and Their Solution by Symbolic Logic' (hereafter, 'Insolubilia') as a response to Poincaré's 'Les mathématiques et la logique' in June 1906. In 'Insolubilia,' Russell remains optimistic that he can resolve propositional paradoxes without giving up the advantages of the substitutional theory; and, in fact, it is in this paper that Russell emphasises the doctrine of the unrestricted variable most. He accepts Poincaré's 'vicious-circle suggestion': 'Whatever involves an apparent variable must not be among the possible values of that variable' (*EiA*, p.198). But he then remarks that 'the vicious-circle principle' does not fit the contention that 'our variables must be capable of *all* values' (*ibid.*, p.205). He then says:

Hence to reconcile the unrestricted range of the variable with the vicious-circle principle, which might seem impossible at first sight, we have to construct a theory in which every expression which contains an apparent variable (i.e. which contains such words as *all, any, some, the*) is shown to be a mere *façon de parler*, a thing with no more independent reality than belongs to (say) $\frac{x}{dx}$ or \int_a^b .

(*EiA*, p.206)

Thus, he indeed attempts to employ the notion of incomplete symbol to reconcile the doctrine of the unrestricted variable with type restrictions.

However, what Russell in 'Insolubilia' really endorse is not the doctrine of the unrestricted variable but a certain weaker claim, even though he does not distinguish these two claims from each other. Arguing against the idea of expressly restricting the range of significance, he remarks as follows:

We thus find that we are brought back after all to variables with an unrestricted range. If this is to be avoided, the range of significance must be somehow given with the variable; this can only be done by employing variables having some internal *structure* for such as are to be of some definite logical type other than individuals.

(*EiA*, p.206)

The idea presented here is simply that all type restrictions must have *internal* structures, whereby we can indicate the restrictions without stating them. This notion is weaker than the doctrine of the unrestricted variable. For the former allows us to place a type restriction upon a certain range of objects without treating them as incomplete symbols, that is, without having any systematic procedure of turning those sentences which contain expressions for the objects into those without such expressions. 'For example,' he indeed remarks, 'M. Peano's symbol " $x\exists(\phi x)$ " can only stand for a class, and no explicit statement to this effect is needed in particular cases' (*ibid.*, p.206). It thus seems that Russell does not distinguish between these two distinct notions regarding a hierarchy of objects, one being the doctrine of the unrestricted

variable, the other a much weaker claim that a hierarchy of objects only requires variables with internal structures.

It is also important to note that Russell in 'Insolubilia' does not clarify how the hierarchy of propositions can be reconciled with the doctrine of the unrestricted variable. He solves the Liar paradox or 'the paradox as to the man who says "I am lying"' by distinguishing *proposition* and *statement*. The idea is that general propositions are in reality not propositions but merely *statements* whereas we cannot legitimately speak of *all* statements.⁷ In drawing the distinction, however, he offers no syntactic procedure of eliminating *statements*. It is even unclear whether there can be any such procedure.⁸ This suggests that although he does hold that 'every expression which contains an apparent variable [...] is shown to be a mere *façon de parler*,' the notion of *façon de parler* or of incomplete symbol is tacitly extended to cases where we have no systematic way of eliminating expressions for objects of a certain range from the sentences in which they occur. This in turn suggests that he does not adhere to the doctrine of the unrestricted variable in 'Insolubilia.' For the firm endorsement of the doctrine requires that there should be a systematic syntactic procedure through which expressions for general propositions can be eliminated from the sentences in which they occur; otherwise, type restrictions placed upon those expressions reflect, as long as the transparency thesis is assumed, a hierarchy of complex objects. One may well think that Russell introduces the multiple-relation theory of judgment to this effect; but, as I will argue in the next chapter, only in 1907 or later did he come up with the theory (cf. Section 6.3.1). By contrast, the weaker claim concerning internal structures of variables is tenable even without such a syntactic procedure, if occurrences of apparent variables themselves can be regarded as indications of internal structures of *statements*. It would thus seem that what he really endorses in 'Insolubilia' is the weaker claim, though he certainly does not distinguish it from the doctrine of the unrestricted variable.

It would thus seem that in spite of his apparent emphasis on the doctrine of the unrestricted variable, Russell in 'Insolubilia' seems to endorse, in effect, the weaker claim that a hierarchy of objects requires that variables ranging over them should indicate certain 'internal' structures. He did not understand the doctrine of the unrestricted variable as a requirement but merely as an ideal in the sense that once it obtains, the weaker claim also holds. This in turn explains why he kept developing the substitutional theory even though he was aware that the propositional paradoxes would require him to abandon the alleged doctrine.

⁷ I will use the term 'general proposition' to speak of propositions containing apparent variables. Note that Russell and Whitehead use the term only for universally quantified sentences in *PM*.

⁸ Note that the multiple-relation theory of judgment, to which we will turn in the next section, does not offer any such procedure either.

5.2.3 The substitutional theory as a philosophical account of functions

In this section I will sketch how Russell treated the substitutional theory once he had started seeking alternative theories around September 1906. I will argue that he began to look for alternatives not simply because he found the theory equipped with type-indices to propositional letters incompatible with the doctrine of the unrestricted variable, but because he found the resultant theory practically too complicated. I will also attempt to indicate that he retained the core notion of the theory as a possible philosophical explanation of a hierarchy of functions even after he had given up developing a formal system of the theory.

Russell wrote 'The Paradox of the Liar' ('the Liar') in September 1906, exploring an alternative to the substitutional theory with a full stratification of propositions. In this manuscript, he considers two possible solutions to the Liar paradox: one is 'to say that there are no such things as propositions at all' and the other is 'to allow that there are propositions, but to put them in a hierarchy' (CP4, p.322). The second option is the substitutional theory equipped with a hierarchy of propositions. It is worth noting that the hierarchy which he proposes here is as fine-grained as what Landini claims Russell comes to endorse only in ML:

We shall want some such primitive proposition as

$$(\exists p_1) : p_n/a = p_1/a$$

for reducibilities.

(CP5, p.352)

On the other hand, the first option is arguably a *functional* theory, a theory quantifying only over functions and not over propositions. He thus began to explore functional theories as an alternative to the substitutional theory around September 1906, though he envisaged that the fully stratified substitutional theory could solve the propositional paradoxes.

Given this, Russell's remark in a letter to Jourdain dated 12 October 1906 wants a careful treatment. He says: 'I am engaged at present in purging it [the no-classes theory] of meta-physical elements as far as possible, with a view to getting the bare residuum on which its success depends' (Grattan-Guinness, 1977, p.93). One might think that by the phrase 'the no-classes theory' Russell is here speaking of a version of the substitutional theory. But he is more likely to talk about a functional theory here. For one thing, as we shall see in the following section, he in some 1907 manuscripts employs the phrase to speak of functional theories which define classes through functions. For another, he decided to withdraw SCR in October 1906, which suggests that he no longer thought the substitutional theory feasible. Furthermore, he in 'the Liar' introduces the p_0/a_0 paradox as 'the fallacy which led to the

abandonment of substitution before' (CP5, p.351; emphasis added).⁹

The question is then why Russell started seeking alternatives to the substitutional theory. There are three possible reasons. One is of course that Russell found a propositional hierarchy incompatible with the notion of unrestricted variable. G. Moore refers to an important letter of 7 October 1906 from Whitehead to Russell, which summarises the latter's struggle over the unrestricted variable:

The nastiness which you wanted to avoid is the Frege bugbear of propositional functions becoming unmeaning when certain terms are substituted. According to the doctrine of types we have got to put up with this. [...] The doctrine of substitution was on stronger ground here; for it did without the function entirely, and simply brought in p/a as a typographical device for pretending that we were talking of one entity when we were really talking of two.

Hence if you want the unrestricted variable, the doctrine of substitution is the true solution. But then this doctrine won't do, will it?

(CP5, p.lxxi)

This letter vindicates that Russell tried to keep the variables unrestricted through the substitutional theory. It is, however, hasty to thereby attribute a firm endorsement of the doctrine of the unrestricted variable to him. He certainly counted as an advantage of the theory its *prima facie* ability to avoid introducing type restrictions to genuine variables, but this does not mean that the whole significance of the theory lay in its apparent conformity with the doctrine of the unrestricted variable. Given what we have seen in the previous subsection, it is reasonable to think that the propositional paradoxes undermined *one* of the apparent advantages of the substitutional theory. In fact, Russell in 'the Liar' remarks:

A very great advantage of the substitutional theory is that it allows that there are propositions, though not in the same sense in which there are individuals.

(CP5, p.352)

He seems, at least here, content with the distinction between propositions and individuals. Thus, although the substitutional theory's apparent ability to avoid introducing type indices to variables was, for Russell, a strong point of the theory, he did not start exploring alternatives to it simply because it turned out that the ability was illusory.

Another possible reason why Russell began to seek alternatives is the notation of the substitutional theory, which is complicated enough by itself and becomes even more so

⁹ In a well-known letter to Ralph G. Hawtrey of January 1907, the p_0/a_0 paradox is presented as what 'pilled the substitutional theory' (CP5, p.125).

when a hierarchy is placed upon propositional variables. In ‘the Liar,’ he remarks: ‘whether technically useful or not, the idea of substitution is legitimate’ (CP5, p.349; see also pp.350, 352). He also claims that the substitutional theory has ‘some advantages even technically’ suggesting that he had generally supposed it to have no such advantages (*ibid.*, p.352). In the end, he in ‘the Liar’ finds that ‘the complications of the substitutional point of view were too great’ (*ibid.*, p.359). Whitehead shares this worry, as he remarks in the letter to Russell quoted above that he does not ‘see the necessity for its complication’ (*ibid.*, p.lxxi).

In addition to the complexity problem, Russell in ‘the Liar’ thinks that ‘if $\phi\hat{y}$ must in any case be admitted, the advantages of substitution are lost’ (*ibid.*). It is unclear why he thinks so, however. One possible reason is the presence of the p_0/a_0 paradox: he might have thought it necessary to stratify functions to block them. Back in ‘Substitution₂,’ he considers, as an alternative to a simple hierarchy of propositions, the distinction ‘between *entity-substitution* and *functional substitution*:

$$‘p \frac{x}{a} !q, p \frac{\psi \hat{x}}{\phi \hat{x}} !q, p \frac{\psi(\hat{x}, \hat{y})}{\phi(\hat{x}, \hat{y})} !q, \text{etc.}’$$

(CP5, p.183). Another possibility is that he somehow becomes unsatisfied with the notion of multiple substitution. As we saw in Section 5.2.1, although the notion can be used to define higher-order functions within the substitutional theory, it implies that a unary function of functions of individuals has the same type as that of a binary relation of individuals. Russell may have found this consequence undesirable. At any rate, he in ‘the Liar’ concludes that ‘we cannot dispense with [propositional] functions as apparent variables by means of substitution’ (*ibid.*, p.348fn). If the substitutional theory cannot thus replace functions with matrices, it loses yet another advantage of saving him explaining what functions are.

Presumably for these reasons, Russell became unsatisfied with the substitutional theory around September 1906. But, importantly, this is not to say that he abandoned the theory altogether then. On the contrary, he retained it as a possible philosophical account of functions even after he had started to develop functional theories. For example, he in ‘the Liar’ remarks: ‘whether technically useful or not, the idea of substitution is legitimate’ (CP5, p.349; see also pp.350, 352). It is not implausible to think, as he does back in *PoM*, that the essence of a propositional function can be captured by the substitution of an entity in a proposition for another to obtain a new value of the function.¹⁰ Russell seems to think that a function can be analysed, through the notion of substitution, into a pair of a proposition and an entity occurring in it. I will call this idea the *substitutional account* of functions. Indeed, in ‘On

¹⁰ Indeed, Frege also appealed to the notion of substitution to explain what a *function* is. In his view, a *function* is ‘the part that remains invariant in the expression’ through substituting a part of the expression for another (Frege, 1879, p.22). But, in fairness, what Frege thus defines to be a function is a certain part of an expression, not an object which Russell would call a function.

Types,' Russell remarks as follows:

If this form of substitution turns out feasible, perhaps it should be put into an Appendix. It is philosophically simpler than functions, but technically vastly more complicated.
(CP5, p.516)

Thus, he finds the substitutional account of functions plausible, and he maintains that once it is granted, 'philosophically all we need to assume is the hierarchy of propositions, together with individuals' (*ibid.*, p.518).¹¹

Russell retained the substitutional account even when he decided to start writing *PM* with Whitehead. He submitted a draft of *ML* to *American Journal of Mathematics* in July 1907, and this arguably marks a point where he made a decision to write *PM* using the ramified theory of functions. As Landini among others has pointed out, Russell in *MT* remarks as follows:

Although it is *possible* to replace functions by matrices, and although this procedure introduces a certain simplicity into the explanation of types, it is technically inconvenient.
(CP5, p.603)

It would thus seem that Russell even at this point believes that 'the substitutional theory is the proper explanatory starting point' (CP5, p.lxxvii).

Russell's retainment of the substitutional theory as a philosophical account suggests that he ceased to develop it mainly because of its technical inconvenience rather than of its incapability to keep variables wholly unrestricted. If so, importantly, he did not regard its ability to avoid type restrictions upon function-symbols as the *raison d'être* of the theory. If he had done so, he would have abandoned the theory altogether, when he accepted a hierarchy of propositions. He seems to have thought that the substitutional theory could offer a philosophical account of why functions would form a hierarchy.

5.3. Functional theories from 1907 to *Principia*

From September 1906 onwards Russell began to consider various functional theories as alternatives to the substitutional theory and eventually came to endorse the ramified theory of types as we can see in *PM*. Behind these theories, I will argue below, lies the notion that functions are not individual entities but *structured* objects, where I shall employ the phrase

¹¹ To be precise, Russell's position on substitution is slightly more complicated than the passage quoted here suggests. For he also holds that '[t]he point of view of substitution is perhaps unnecessarily complicated' (CP5, p.517). It is in his view 'unnecessarily' so, because he again thinks that the substitutional theory, somehow, fails to do away with function-variables (*ibid.*). Still, he claims that the substitutional account of functions can be true of functions of a certain sort, those analogous to *predicative* functions in *PM* (*ibid.*).

‘structured objects’ to speak of those objects which can, in his view, be justifiably assumed to form a hierarchy on their own.

5.3.1 Classes as constructed and functions as structured

This subsection introduces the functional theories which Russell developed from September 1906 onwards. I will attempt to show that in developing these theories he maintains that functions are structured objects while classes are not.

After ‘the Liar,’ Russell wrote a short manuscript which we call ‘Types’ (CP5, 18a). He begins the manuscript with the following remark:

A function must be an incomplete symbol. This seems to follow from the fact that $\phi!(\phi!\hat{z})$ is nonsense. The whole difficulty lies in reconciling this with the fact that a function can be an apparent variable.

(CP5, p.498)

He employs the term ‘incomplete symbol’ for a propositional function here, but importantly he does not even attempt to offer any syntactic procedure of turning sentences with functional expressions into those without them. As we saw in Section 5.2.2, he in ‘Insolubilia’ already includes general propositions among *façons de parler* without any such procedure. This sense of ‘incomplete symbol’ or ‘*façon de parler*’ is distinguishable from the one applied to definite descriptions, as he offers such a procedure in the latter’s case. It seems as though he thinks that propositional functions or general propositions can be justifiably assumed to form a hierarchy without showing, by means of such a syntactic apparatus, that symbols for them are ‘incomplete symbols’ in the strict sense in which definite descriptions are so. He seems to regard functions or general propositions as, in my terminology, *structured objects*. G. Moore, the editor of CP5, interprets the first sentence of the above passage as a ‘Fregean’ statement meaning that a function-symbol is an *unsaturated* expression (CP5, p.495). This is perhaps related to the point that I am making, for the view of functions as structured objects implies a distinctively Fregean claim that functions are in nature different from individual entities. Interestingly, Russell in ‘Types’ employs the term ‘construction’ to explain the idea that functions are structured objects¹²:

The way to explain things is as follows: (1). ϕx stands for anything containing x , and being a proposition for every value of x . (2). $f!\phi\hat{x}$ stands for anything containing values of ϕ , and containing these values with arguments which are constants or apparent variables, and containing these values only in propositional positions. (3). $F!f!\hat{\phi}\hat{x}$

¹² The passage also illustrates that he does not commit himself here to the idea that each function occurs ‘only through its values’—the idea to which we shall turn shortly.

stands for anything containing values of f in a similar way.

The point is that f and F stand for ways of construction, and do not presuppose any knowledge of what is to be put in as argument beyond what can be got from lower types alone.

(CP5, p.512)

Although he uses the word 'knowledge' here, it is plausible that this passage is about functions themselves, not about our knowledge of them. He thus seems to maintain that propositional functions are 'constructions' from objects of lower-levels thereby forming a hierarchy on their own.

However, Russell takes a different stance when it comes to classes. It seems as though he thinks that if they are to be put into a hierarchy, it is necessary to show that they are in reality *logical fictions*, namely, that class-symbols can always be eliminated through a systematic procedure from the sentences in which they occur.¹³ At the beginning of 'Types,' he seems to retain both functions and classes, and he remarks as follows:

When we consider classes, it seems evident that, if they have being at all, they have being of a different kind from that of individuals. When we have counted up all the things there are in the world, we don't get new things by taking them in groups. Thus in some sense classes must be non-entities, and statements about classes must be reducible to statements about their members.

(CP5, p.503)

He then goes on to conclude that classes are dispensable: 'It seems as if now classes too were unnecessary, provided we allow variation of ϕ in ϕx and put

$$y \in \hat{x}(\phi x) . = . \phi y \quad \text{Df'}$$

(CP5, p.506).¹⁴ It is important to note that this is how Russell introduces classes-symbols in this manuscript. Thus, in placing a type-restriction upon class-symbols, he offers a way of eliminating class-symbols from sentences in which they occur.

After 'Types' Russell seems to have written 'Fundamentals' (CP5, 19). This manuscript contains the following summary of three possible philosophical views of functions and/or classes, which is long but worth quoting:

A. *Technically.* All contradictions are avoided provided no expression containing an apparent variable is a possible value of that apparent variable. This requires two

¹³ I use the term 'logical fictions' exclusively for objects whose expressions can thus be eliminated in a systematic manner.

¹⁴ It is difficult to determine, from the context, what Russell means by 'classes too were unnecessary.'

sorts of functions, one of which can be an apparent variable while the other can't. Whether we call the two sorts ϕx and $\phi!x$, or ϕx and $x \in \alpha$, is technically indifferent. Whichever we do, we need a reducibility-primitive-proposition, so that one of the wider kind is always *equivalent* to some one of the narrower kind. [...]

B. *Philosophically.* (a). The no-classes theory, with the theory of predicative and non-predicative functions, supplies what is required, except that (α) there is a difficulty about the meaning of a function as apparent variable; (β) the distinction of predicative and non-predicative functions is obscure, and the axiom of reducibility is arbitrary.

(b). The plan of distinguishing ϕx from $x \in \hat{z}(\phi z)$, in which the former asserts the value of ϕ for the argument x , while the latter asserts that x has the property ϕ , has much to recommend it. In this plan, ϕ can only be an apparent variable when it is *explicit*, as in $x \in \hat{z}(\phi z)$, not when it is as in ϕx . This has the merit of making the reducibility-axiom obvious, since it states that " $\phi x \equiv .x$ has the property ϕ ." But it seemed to involve treating truth-functions as a type. [...] And there are grave difficulties about treating truth-functions as a type.

(c). The plan of never varying functions at all, and introducing $x \in \alpha$ as a primitive idea, has very great advantages. It is simple, it makes a very clear distinction between predicative and non-predicative functions, it allows us to use the argument that the ϕ in ϕx can't be varied because it doesn't occur in ϕx and is in fact nothing, and it makes the reducibility-axiom simply the universally-admitted axiom of classes. The objection to this plan is that it makes it hard to see why $\alpha \in \alpha$ should be meaningless. [Note that it is not strictly necessary that $\alpha \in \alpha$ is meaningless, but only that $(x).f(x \in \alpha)$ should not imply $f(\alpha \in \alpha)$.] And to get " $\text{Ex!}\phi x$ " as *not* a function of x , we still need the notion of a truth-function.

(CP5, pp.552-3)

The philosophical view labeled (a) is arguably the functional theory which Russell develops in 'Types.' The view called (b) above is, it seems, the one which he is led to endorse in this manuscript. G. Moore points out that in putting forward this view, Russell employs the symbol ' $\hat{z}(\phi z)$ ' as an expression for a *predicate* rather than for a class: 'We may decline to vary a *general* ϕ , but introduce a restricted (*predicative*) ϕ , of the form " $x \in \alpha$," where α is a *predicate*' (*ibid.*, p.554; cf.p.555). The view (b) may thus be called the theory of *predicates*. As Russell remarks, this view can give the axiom of reducibility a plausible interpretation: 'any propositional function of x is equivalent to the ascription of some predicate' (*ibid.*, p.554). The controversial axiom becomes, in other words, a simple comprehension principle for functions. However, he finds that this view also suffers from similar problems with the view (a), which

we will see in the following subsection. The view (c) is plainly a simple type theory of classes.

It is important to observe that Russell rejects the view (c) for the reason that ‘it makes it hard to see why $a \in a$ should be meaningless.’ He seems to assume that unless some form of the no-classes theory is adopted, it will be left mysterious why classes form a hierarchy. To put it differently, he assumes, again, that classes must be treated as logical fictions if they are to form a hierarchy at all. This assumption also makes sense of the following short remark: ‘Types won’t work without no-classes’ (*ibid.*, p.561). Russell maintains in ‘Insolubia’ that such class symbols as ‘ $x\epsilon(\phi x)$ ’ can be said to indicate that classes are distinct from individuals (cf. p.103). However, he does not seem to think that such symbols tell us why ‘ $x\epsilon(\phi x) \in x\epsilon(\phi x)$ ’ should be meaningless. It is equally important that, in the above passage, he does not think it necessary to treat functions as logical fictions in order to place a hierarchy upon them. In fact, the view (a) and the view (b) both impose hierarchies upon functions, but he does not even try to attach ‘no-propositional-function theory,’ so to speak, to either of them.

Thus, both in ‘Types’ and in ‘Fundamentals,’ Russell seems to assume that both classes and functions should be put into hierarchies and that only the former need to be logical fictions while the latter can be considered *structured* objects, forming a hierarchy on their own. A possible reason why Russell did not think it necessary to offer any systematic way of eliminating function-symbols is that he thought he could invoke the substitutional account of functions for that purpose. As we saw in the preceding subsection, he retained the core notion of the substitutional theory as a philosophical account of functions at least until ML.

5.3.2 Towards the contextual definition of classes

It thus seems that Russell assumed that classes must be explained as logical fictions in order for them to form a hierarchy. However, he had yet to invent a proper definition of class-symbols which would enable him to eliminate them from any possible sentence. Russell and Whitehead adopt a version of the no-classes theory in *PM*, introducing class-symbols through the following definition:

$$f\{\hat{z}(\phi z)\} . =: (\exists \psi)\phi x . \equiv_x . \psi!x : f\{\psi!\hat{x}\} \quad \text{Df.}$$

The exclamation mark or ‘shriek’ next to ψ indicates, of course, that $\psi!\hat{x}$ is a *predicative* function of x .¹⁵ The definition enables the authors to eliminate a class-symbol in question from a sentence of any form. I shall call this definition the *contextual* definition of class-symbols. Although this definition is certainly a natural application of the theory of descriptions to class-symbols, it took, as we will see below, some time for Russell to invent the definition. In this subsection, I will first indicate some problems with the simple definition of class-symbols

¹⁵ *PM* notoriously gives the notion of *predicative* function two seemingly distinct definitions. For the discussion of them, see, for example, Chihara (1973), Church (1976), Landini (1998b, p.264) and Linsky (1999, pp.83-5).

which he adopts in ‘Types’ and in ‘Fundamentals.’ I will then sketch how he invented the contextual definition of class-symbols. By so doing I will illustrate that he found it necessary to treat classes as logical fictions so as to impose a type restriction on them.

As we have seen, Russell examines a version of the no-classes theory in ‘Types,’ which introduces classes through the following simple definition:

$$y \in \hat{x}(\phi x) . = . \phi y \quad \text{Df.}$$

But this definition has some problems. An obvious one is that there are some cases where the definition is not applicable. The definition tells us nothing about cases where ‘ $\hat{x}(\phi x)$ ’ occurs without ‘ ϵ ’ occurring next to it. As we will see shortly, he envisages this problem in ‘Fundamentals.’ Another problem is that the extensionality of classes does not follow from the definition. For there is no guarantee that if two functions are co-extensional, they fall under the exactly same higher-order functions. For this reason he introduces the notion of *truth-function*. Although he gives some distinct definitions to the notion in ‘Types’ (CP5, pp.504, 510), what he intends to have through these definitions is clear: A function $f!\phi\hat{y}$ is said to be a *truth-function* of $\phi\hat{y}$, if and only if, for any function f ,

$$\phi y . \equiv_y . \psi y \quad \supset : \quad f!\phi\hat{y} . \equiv . f!\psi\hat{y}$$

(cf. *ibid.*, p.506). Truth-propositions are thus better called *extensional* functions. The proposed definition of class-symbols remains silent as to whether we can replace ‘ $\phi\hat{y}$ ’ and ‘ $\psi\hat{y}$ ’ for ‘ $\hat{y}(\phi y)$ ’ and ‘ $\hat{y}(\psi y)$ ’ respectively. But if we can, we obtain the extensionality of classes with respect to truth-functions¹⁶:

$$\phi y . \equiv_y . \psi y \quad \supset : \quad f!(\hat{y}(\phi y)) . \equiv . f!(\hat{y}(\psi y)).$$

But Russell soon finds a problem with this solution to the extensionality problem:

The only serious objection that I can see to the above is that it makes it meaningless to say (e.g.) “A is a liar,” or even “A asserts p .” For this puts p in a position which is not a truth-function.

(CP5, p.507)

He also notes that once classes are abandoned, it becomes difficult to motivate the axiom of reducibility.

In ‘Fundamentals’ Russell is confronted by more problems with the above definition of class-symbols and almost abandons the version of the no-classes theory. One is quite intrinsic to the proposed definition of class-symbols. ‘Only in the form “ $\hat{z}(\phi z)$,”’ he remarks, ‘can ϕ

¹⁶ This line of thought is developed in Introduction to the Second Edition of *Principia Mathematica* (pp.xxxix-xliiii).

become an apparent variable' (*ibid.*, p.540). Hence, he takes such an expression as ' $(\phi).\phi x$ ' to be illegitimate, as it varies ϕ without turning it into $\hat{z}(\phi z)$. 'But if ϕ is to be an apparent variable,' he then points out, ' $x \in \hat{z}(\phi z)$ ' must be a different proposition from ϕx . This rather destroys the no-classes theory' (*ibid.*, p.546). He also comes to observe the problem that the proposed definition of class-symbols does not always allow us to eliminate class-symbols:

Note that the no-classes theory is in essence abandoned by distinguishing between ϕx and $x \in \hat{z}(\phi z)$. For this requires that $\hat{z}(\phi z)$ should be a constituent of $x \in \hat{z}(\phi z)$, and therefore that $\hat{z}(\phi z)$ should be something. This difficulty seems inherent in the no-classes theory, since functions must be allowed as apparent variables. A value of an apparent variable must be something, and thus the no-classes theory won't work. It worked while we had propositions, because then they became apparent variables where a variable matrix was wanted. But if propositions are not to be apparent variables, functions must be, and therefore functions must be admitted. But then they may as well be classes.

('Fundamentals,' p.543)

G. Moore interprets 'the no-classes theory' as the substitutional theory (*ibid.*, p.lxx). But it is, I think, more plausible that Russell speaks here of the version of the no-classes theory which he has developed in 'Types.' In fact, the substitutional theory can easily do away with such expressions as ' $x \in \hat{z}(\phi z)$,' and hence, it is hard to regard them as a reason against the theory. On the contrary, as we have seen, Russell in 'Types' tries to treat class-symbols as incomplete symbols by the following definition: ' $y \in \hat{x}(\phi x) = .\phi y$.' It seems to me that what he finds problematic in the above passage is that if a class-symbol ' $\hat{x}(\phi x)$ ' can occur in a sentence without the membership symbol ϵ ' flanking it, it is in effect the abandonment of the proposed no-classes theory.

These problems with the simple no-classes theory are perhaps among the reasons why Russell comes to regard the theory of *predicates*, the view labeled (b) in the preceding subsection, as a better option than the view (a) in 'Fundamentals.' But the view (b) still faces a similar problem, as it introduces *predicates* in the same way as the simple no-classes theory deals with classes. In order to use *predicates* in place of classes, he finds it necessary to show that they are extensional in the sense in which classes are. Hence, he again invokes the notion of truth-function in 'Fundamentals': 'We want to prove that there are no other functions [than truth-functions]' (*CP5*, p.541; see also p.543).¹⁷ However, he comes to think that *predicates* must not be extensional in order for them to count as *predicates*:

¹⁷ In his attempt to prove so, he is led to consider what I will call the asymmetric theory of truth, and then, the view that '*Belief* means a certain kind of relation to several objects' (*CP5*, p.542). I take the latter view to be a rudimentary version of the multiple-relation theory of judgment (cf. Chapter 6).

It seems plain that we can truly say: “Socrates is a featherless biped” is different from “Socrates is human”. But if nothing whatever can truly be said about predicates except predications of them, featherless biped = human, and all propositions containing the one should be identical with propositions containing the other.

(CP5, p.554)

Presumably for this reason, he abandons the view (b) despite its ability to turn the Axiom of Reducibility into a plausible assumption.

However, ‘Fundamentals’ also contains some clues to the contextual definition of class-symbols, with which Russell can avoid the above problems:

The dogma also involves that $\alpha \in k$ can’t be a primitive idea, since it must be of the form $f\{\hat{x} \in \alpha\}$, so that we must put

$$\alpha \in k . = : (\exists f) : k = \hat{\beta}\{f(\hat{x} \in \beta)\} . f(\hat{x} \in \alpha).$$

So that $\hat{\beta}\{f(\hat{x} \in \beta)\}$ becomes our primitive idea; or rather,

$$F\{\hat{\beta}\{f(\hat{x} \in \beta)\}\} . = : f(\hat{x} \in \beta) . \equiv_{\beta} . g\beta : Fg \text{ roughly.}$$

(CP5, pp.553-4)

What he calls ‘the dogma’ here is the idea that ‘[p]roperties of classes are always properties of their membership’ (*ibid.*, p.553). He seems to forget to put ‘ $(\exists g)$ ’ on the right hand side of the second equation above, but he soon modifies it to get closer to the contextual definition:

$$f\{\hat{z}(\phi z)\} . = : (\exists \alpha) : \phi x . \equiv_x . x \in \alpha : f(\alpha)$$

(*ibid.*, p.556). But he still needs a variable ‘ α ’ ranging over *predicates*. In ‘Fundamentals,’ he does not, after all, reach the contextual definition but the following intermediate ones in ‘Fundamentals’:

$$x \in \hat{z}(\phi!z) . = . \phi!x \quad \text{Df}$$

$$f\{\hat{z}(\phi z)\} . = : (\exists \psi) : \phi z . \equiv_z . \psi!z : f\{\hat{z}(\psi!z)\} \quad \text{Df}$$

(*ibid.*, p.563). The second proposition does not still allow us to contextually define class-symbols due to the occurrence of ‘ $\hat{z}(\psi!z)$ ’ in the second conjunct of the right hand side: it only introduces the notation ‘ $\{\hat{z}(\phi z)\}$ ’ to all functions assuming that it is already defined for those functions which ‘!’ flanks.

Some pieces of textual evidence suggest that it was around June 1907 that Russell finally invented the contextual definition of classes. One manuscript in which he presents the contextual definition is ‘On Types,’ where he remarks:

All that is required for classes is some propositional function depending only upon the extension of its (functional) argument, and equivalent to $f(\psi\hat{z})$ in case f is already extensional. Hence we put $f\{\hat{z}\phi z\} . = : (\exists\psi)\phi x . \equiv_x . \psi!x : f\{\psi!\hat{z}\}$ Df.

(CP5, pp.518-9)

Unfortunately, we do not know when the manuscript was written. But it was certainly by June 1907 that Russell finally invented the contextual definition of class-symbols. In a letter dated 16 June 1907, Whitehead praises the contextual definition of class-symbols thus:

Your translation of intension to extension by means of

$$f\{\hat{z}(\phi z)\} . = : (\exists\psi)\phi x . \equiv_x . \psi!x : f\{\psi!\hat{x}\} \quad \text{Df.}$$

is beyond all praises. It must be right.

(CP5, p.lxxvi)

The fact that Whitehead thus praises the definition suggests that it was not very long ago that he learnt it from Russell. Given that Russell went on to submit a draft of ML in the following July, it is plausible that the contextual definition made him confident enough to start writing the voluminous book with Whitehead. This fits G. Moore's observation that 'it was not in any sense that the abandonment of the substitutional theory that was crucial' to Russell's decision to start writing up the book (CP5, p.lxxvii). The crucial thing was that he finally obtained a satisfactory way of treating classes as logical fictions. This in turn fits the claim that he regarded a contextual syntactic elimination as essential for classes but not for functions.

5.3.3 Functions in *Principia*

There has been a dispute over the metaphysical picture underlying the technical enterprise of *PM*. A point of controversy is what Russell and Whitehead take functions to be in the book. I will examine three accounts that have hitherto been proposed: the naïve realist one, the nominalist one and the logical-construction one, before I argue, contra these accounts, that the authors do not understand functions as mere open sentences nor logical constructions but as structured objects.

Given Russell's undeniable tendency towards realism, it is natural to interpret functions in *PM* as individual entities. This interpretation underlies Quine's objection to the ramified theory of types in *PM*:

Russell's method eliminates classes, but only by appeal to another realm of equally abstract or universal entities—so-called propositional functions. The phrase 'propositional function' is used ambiguously in *Principia Mathematica*; sometimes it means an open sentence and sometimes it means an attribute. Russell's no-class theory uses

propositional functions in this second sense as values of bound variables; so nothing can be claimed for the theory beyond a reduction of certain universals to others, classes to attributes.

(Quine, 1961, pp.122-3)

Quine thus argues that as long as functions are thus conceived, the no-classes theory fails to achieve its supposed goal of ontological parsimony. He also follows Ramsey and Gödel to denounce the vicious-circle principle as incompatible with Russell's own realism of functions (Ramsey, 1931, p.41; Gödel, 1944, pp.135-6; Quine, 1963). It is certain that *if* functions are considered as self-subsistent, individual entities, the principle is blatantly implausible. I will use the term 'naïve realist interpretation' to speak of the idea that functions in *PM* are such entities.

This interpretation has recently been challenged in the literature, however. Among the challenges to the interpretation I find the following particularly illuminating: Russell distinguishes universals from functions. Once these two kinds of objects are distinguished from each other, the well-known fact that he counted universals as individuals does not commit him to the naïve realist interpretation of functions at all (cf. p.75). For instance, Russell and Whitehead remark that values of a function $\varphi a, \varphi b, \varphi c$, etc, do not presuppose the function itself $\varphi \hat{x}$, whereas they think that a proposition Fa (now conceived as a 'false abstraction') presupposes the universal F and the individual a (*PM*, pp.39-40, 43). Furthermore, the notion that a function $\varphi \hat{x}$ does not occur in φa is not alien to Russell at all (cf. p.89), while he has apparently never questioned that *concepts* are constituents of propositions. Note also that since the notion of function is for Russell a new addition to logic, it is natural that he should think that the notion wants even more elucidation than that of universal. Thus, Russell's view that universals are self-subsistent entities does not reinforce the naïve realist interpretation of functions in *PM*. What we have seen in this chapter offers another objection to the interpretation: Russell himself observes that the vicious circle principle does not apply to individual entities. As we saw in the long passage quoted in Section 5.3.1, he holds that the realism of classes 'makes it hard to see why $\alpha \in \alpha$ should be meaningless' (*CP5*, p.523).

Landini and Klement, amongst others, interpret propositions and functions in *PM* as (closed and open) well-formed formulae.¹⁸ According to them, propositional-function variables in *PM* do not range over any kind of objects but await substitutional (as opposed to objectual) evaluations.¹⁹ On their accounts, the sentence ' φa ,' where φ is viewed as a free variable, has the value *true* just in case all the substitutional instances of φa are true. As Landini

¹⁸ Those who endorse this interpretation include Sainsbury (1979, pp.285-90; 1980), Landini (1998b), Hazen (2004), Stevens (2005) and Klement (2003; 2004; 2010a, pp.648-649).

¹⁹ For the substitutional account of quantification, see for example Kripke (1976).

and Klement argue, this way of evaluating the truth of a sentence fits the way Russell and Whitehead introduce the ramified hierarchy of truth (Landini 1998b, p.278; Klement 2010, p.651). They add that the nominalist interpretation makes it possible to reconcile the ramified theory of types with the doctrine of the unrestricted variable—what they take to be a non-negotiable principle for Russell. For if variables other than individual ones are evaluated in the nominalistic manner, it follows that the genuine variables of the ramified theory, that is, its individual variables are still unrestrictedly ranging over all the objects. Landini also argues that once the nominalist interpretation is granted, the Axiom of Reducibility should be regarded as the comprehension schema of *PM*'s formal system (Landini, 1998b, p.267). According to him, the nominalist interpretation can thus turn the controversial principle into a (supposedly) reasonable assumption which merely says that any open well-formed formula can be replaced by a single letter (of the corresponding type). These points are the primary merits of the nominalist interpretation.

However, there are some problems with this interpretation. First of all, if what we have seen is correct, it is questionable whether the authors of *PM* adhere to the doctrine of the unrestricted variable. Indeed, they describe the theory of types as having 'a certain consonance with with common sense' (*PM*, p.37). Landini holds that the doctrine of the unrestricted variable was the only reason for Russell's abandonment of the substitutional theory 'save the convenience of using indexed predicate variables instead of substitutional formulae' (Landini, 1998b, p.274). The technical inconvenience of the substitutional theory is, Landini claims, 'not sufficient reason' for abandoning the theory (*ibid.*). But, as we saw in Section 5.2.3, it rather seems to have been a main factor that led Russell to functional theories.

Second, the nominalist interpretation does not fit the authors' employment of the circumflex notation ' $\phi\hat{x}$ ' 'to speak of the function itself' (*PM*, p.187). If functions in *PM* are mere open formulae, the authors of the book, it would follow, speak of symbols through the notation. But Russell seems to have a good reason to deny that the ramified theory of types is concerned with symbols, since he endorses the view of symbols as psychological (cf. Section 2.5). In a 1906 review of Hugh MacColl's *Symbolic Logic and Its Applications*, Russell remarks that 'logic ought not to be concerned with forms of words, but with what such forms mean' (*CP5*, p.672). He maintains this attitude towards language even after the completion of *PM*, holding that logic is 'not concerned with forms of words' (*CP5*, p.55). In fairness, Landini and Klement make various attempts to show that the circumflex notation is not part of the formal language of the voluminous book (Landini 1998b, pp.265-7; Klement 2010, pp.651-3). But even granting this point it would remain unclear why, if functions are conceived as open formulae, the authors introduce the circumflex notation in the first place. For they sometimes employ quotation marks as a device of speaking of expressions themselves (for example, *PM*, pp.38-41). It is far more natural to think that they introduce the notation to 'distinguish the

function itself from an undetermined value of function' (*PM*, p.40; see also p.15).

Third, although there are indeed some passages in which Russell explains functions as symbols, these passages are all written *after* the first edition of the book. Graham Stevens points this out in his review of Landini's work (Stevens, 2003), and the same point should be made against Klement's appeal to various remarks which Russell makes on symbols in later works (Klement, 2010, pp.647, 653-4). In Chapter 7, I will argue that only after *PM* does Russell come to consider, though still quite reluctantly, the possibility that symbols play a central role in his account of logic. In my view, although it is indeed possible to attribute the nominalistic account of functions to *PM*, the account is better understood as something that occurred to Russell only *after* it was published.

Fourth, it is unlikely that the authors of the book understand the Axiom of Reducibility as a comprehension schema. For once the axiom is thus understood, it becomes utterly unclear why they should admit, '[t]hat the axiom of reducibility is self-evident is a proposition which can hardly be maintained' (*PM*, p.59). It should be remembered that Russell, as we saw in Section 5.3.1, attempts to find a plausible interpretation for the axiom in 'Fundamentals.' In addition, he and Whitehead attempt to render it less implausible by indicating that if a finite number of first-order functions are relevant, the disjunction of them can be seen as a predicative function (*ibid.*, p.56). But they would surely not have appealed to this argument if they had had available a much more straightforward and plausible one. He and Whitehead must have given the axiom an ontological implication at least when they wrote *PM*.

The nominalist interpretation is thus not entirely plausible. But this does not mean that we have to return to the naïve realist interpretation. On the contrary, it is possible to understand functions in *PM* as objects of a certain sort that are subject to a type restriction. Jung and Linsky have developed such an interpretation.

Jung and Linsky hold that propositions and functions in *PM* are *logical constructions* as opposed to mental constructions (Jung, 1999; Linsky, 1999). Their idea is that propositions and functions are considered to be mind-independent but not self-subsistent: they depend on propositions and functions of lower orders. As Linsky argues, once they are thus perceived, the implicit distinction between universals and functions makes perfect sense (Linsky, 1999, pp.22f). In addition, this interpretation also renders the vicious-circle principle plausible: 'It makes sense to think of propositional functions as having a unique ontological analysis which will be subject to the vicious circle principle' (Linsky, 1999, p.29). The idea is that higher-order functions are logical constructions out of lower-order ones and hence the former are ontologically dependent, as the vicious circle principle lays down, upon the latter. The logical-construction interpretation can make the principle plausible without attributing to the book the alleged nominalistic understanding of functions.

Yet, this interpretation also does not perfectly fit textual evidence. First of all, Russell

and Whitehead do not yet, at the time of *PM*, seem to envisage any clear-cut notion of logical construction. They adopt the multiple-relation theory of judgment, according to which ‘a “proposition,” in the sense in which a proposition is supposed to be *the* object of a judgment, is a false abstraction, because a judgment has several objects, not one’ (*PM*, p.44). But the multiple-relation theory of judgment fails to offer any systematic way of translating declarative sentences seemingly expressing propositions into some other expressions. As Linsky himself observes, Russell ‘does not, and indeed could not, present the construction [of propositional functions] as a formal definition’ (Linsky, 1999, p.114). The notion of logical construction thus needs to be extended to cover cases where no systematic translation is available. It is surely possible to characterise, as Linsky and Pickel do, the notion of logical construction in such a way that propositions, classes and denoting concepts can all be deemed logical constructions (Linsky 1999, Ch.7; Pickel 2013). But Russell and Whitehead do not even attempt to do so in *PM*. This suggests that they have yet to understand the book as undertaking the project of logical construction, that is, as replacing ‘some set of supposed entities with neat logical properties’ with ‘purely logical structures composed of entities which have not such neat properties’ (*CP9*, p.164). It is only in later works such as *Introduction to Mathematical Philosophy* (henceforth, *IMP*) and *Logical Atomism* that Russell clearly envisages the project of logical construction.

Furthermore, the authors of *PM* do *not* appeal to the notion of incomplete symbol to explain what functions are nor to motivate the hierarchy of them. They invoke the ‘nature’ of functions:

A [propositional] function is what ambiguously denotes some one of a certain totality, namely the values of the function; hence, this totality cannot contain any members which involve the function, since, if it did, it would contain members involving the totality, which, by the vicious-circle principle, no totality can do.

(*PM*, p.39)

The whole explanation given to functions in *PM* amounts to ‘the view that a function is an ambiguity’ (*PM*, p.40). What the authors appeal to in their attempt to motivate a hierarchy of functions are this view of functions and the fact that without any such hierarchy contradictions would arise. It seems to me that although functions in *PM* are assumed to be objects of a certain sort, they are not considered as what Jung and Linsky call *logical constructions*. They are simply explained as ambiguities and this account in turn is supposed to explain why they form a hierarchy. Linsky maintains that the multiple-relation theory of judgment is presented as the basis of constructing propositions and functions (Linsky, 1999, p.29). But the authors explain the nature of functions *before* they introduce the multiple-relation theory of judgment. Furthermore, if the whole point of the multiple-relation theory of judgment lay

in the construction of technical objects, it would follow that the authors left the construction quite unfinished, or indeed barely started. The theory does not explain how functions can be constructed. It seems as though the multiple-relation theory of judgment is introduced for a rather different reason.

Thus, although the logical-construction interpretation thus rightly points out that Russell and Whitehead consider functions to be objects other than individuals, forming a certain ontological hierarchy, yet it goes too far to say that functions are treated in the same way as denoting concepts, classes and propositions and that the authors undertake, already in *PM*, the project of logical construction, which Russell apparently comes to envisage only in later works.

It seems to me that Russell and Whitehead understand functions as structured objects, that is, as objects that can be justifiably assumed to form a hierarchy. As we have seen, they appeal to the 'nature' of functions to motivate the ramified hierarchy. They hold that a function is essentially an *ambiguity* between an open sentence and a set of objects (that are to be called the *values* of the function). This is precisely why, on their account, the function ontologically presupposes its values. They, on this basis, maintain that the account of a function as an ambiguity motivates the ramified hierarchy of types. Functions are thus considered to be structured objects.

The account of functions as ambiguities is certainly sketchy, but the authors of *PM* are content with it presumably because it offers a philosophical justification as to why functions form a hierarchy. This observation in turn explains why the authors adopt the no-classes theory despite their ontological commitment to functions. They do not do so for the ontological parsimony of classes but rather for a philosophical justification for the hierarchy placed upon classes. Klement argues that the logical construction interpretation is rendered implausible by the point made by Quine and Soames that if the authors conceive of functions as objects of any sort, it would make it mysterious why they adopt the no-classes theory to reduce classes to functions (Klement, 2010, p.647fn). If the whole point of the no-classes theory lay in the ontological parsimony, their ontological commitment to functions would cancel out the advantage. Klement's argument may appear to be applicable to my view that functions in *PM* are simply assumed to be structured objects. However, it should be remembered that Russell from 1907 onwards seems to think that classes cannot be said to form a hierarchy as long as they are counted as genuine objects whereas functions can be seen as structured on their own. The no-classes theory offers an account of why classes are subject to type restrictions by reducing them to functions, which in turn form a hierarchy thanks to their own nature.

It is not entirely clear whether the view of functions as ambiguities is meant to replace the substitutional account of functions. As we saw, Russell retains the latter in *ML*; and we can find the phrase 'matrix' even in *PM*. One plausible suggestion is that he comes up

with the ambiguity account of functions and finds it simpler than the substitutional account thereby avoiding any references to the latter in *PM*. But another is that Russell understood the substitutional account to be still at work in the background, as explaining this notion of ambiguity, since it is by the notion of substitution that we explain the range of propositions between which a function is 'ambiguous.'

5.4. Concluding remarks

In Section 5.2, I discussed how Russell developed the substitutional theory, so that I could indicate pieces of evidence that he did not regard the doctrine of the unrestricted variable as a non-negotiable premise. Among them was the point that what Russell in 'Insolubilia' effectively endorses is not the doctrine of the unrestricted variable but the claim that a hierarchy of certain objects requires that variables ranging over those objects should have *internal structures* indicating the hierarchy. In Section 5.3, I attempted to show that in considering various functional theories afterwards, Russell maintained that classes should be explained as 'logical fictions' while propositional functions could be considered structured objects. I then argued that we can reasonably interpret some remarks on propositional functions in *PM* if we think that they are regarded as structured objects.

The view of functions as structured objects is arguably in line with the notion that type-restrictions must be indicated by *internal structures* of variables. For function-symbols such as φx seem to show that functions behave quite differently from individuals and hence that the former belong to a different ontological category from that of the latter. But the textual evidence seems to leave unclear how the view of functions as *structured* objects was connected to the substitutional account of functions.

What is important for our purposes is, however, to observe that behind these three ideas concerning a hierarchy of functions lay a rather simple assumption that if a formal system imposes a type restriction upon a certain range of objects, there must be a philosophical account of them that motivates the restriction. It is—or rather *was*—perfectly natural for Russell to assume this, given what we have seen in the present work. Logic had, in his view, the two branches, the theory of logic and the calculus of logic, and hence, it was insufficient to have a technically feasible system without any support from a philosophical account of technical objects.

The notion of incomplete symbol now seems to have had various and changing meanings in Russell's logicist programme. When he invented the substitutional theory, the notion's significance lay mainly in its apparent ability to accommodate the doctrine of the unrestricted variable. But, as he realised that the ability was illusory due to the propositional paradoxes, its importance was rather to be found in its application to class-symbols. For Russell had diffi-

culty in motivating a hierarchy of classes conceived as genuine objects as well as in explaining what they were in the first place. Indeed, it was when he discovered the contextual definition of classes, a feasible way of systematically eliminating class-symbols from sentences, that he made up his mind to begin writing *PM* with his collaborator. He then came to attach another sense to the term 'incomplete symbol' when he adopted the multiple-relation theory of judgment. The theory does not provide us with any syntactic way of eliminating sentences and in this respect the notion of incomplete symbol underwent a significant extension. He eventually made some attempts to articulate what he thought incomplete symbols were, and those attempts led to the notion of logical construction. But, importantly, it was only after *PM* was completed that he came to envisage the programme of logical construction.

A question remains. The above story does not explain why the authors of *PM* adopted the multiple-relation theory of judgment. As I argued in 5.3.3, it is not plausible that the theory was introduced in order to treat propositions as structured objects. This important question is addressed in the following chapter.

6 Truths and Facts

6.1. Introduction

Russell abandoned the ontology of propositions between 1906 and 1910. In 'On the Nature of Truth and Falsehood' (1910; henceforth, ONTF), he rejects the ontology in favour the multiple-relation theory of judgment. One might think that he did so in order to accommodate a hierarchical conception of propositions within his ontological framework and thereby avoid the propositional paradoxes. However, given what we have seen in the preceding chapter, the multiple-relation theory plays a minor role, if any, in Russell and Whitehead's arguments for the ramified hierarchy in *PM*. Why then did Russell abandon the ontology of propositions?

It is reasonable to seek an answer in Russell's own arguments against the ontology of propositions presented in ONTF. He makes three points against it in this paper. First, he claims that expressions for propositions are incomplete symbols. According to him, both declarative sentences and their nominalised forms do not stand for propositions but facts conceived as complex entities, and hence, propositions should instead be expressed by that-clauses such as 'that Brown is taller than Smith' (ONTF, p.175). 'We feel,' he continues, 'that the phrase "that so-and-so" is essentially incomplete, and only acquires full significance when words are added so as to express a judgment, e.g. "I believe that so-and-so," "I deny that so-and-so," "I hope that so-and-so"' (*ibid.*, pp.175-6). The following passage contains his second and third points against propositions or what he now calls 'Objectives':

If we allow that all judgments have Objectives, we shall have to allow that there are Objectives which are false. Thus there will be in the world entities, not dependent upon the existence of judgments, which can be described as objective falsehoods. This is in itself almost incredible: we feel that there could be no falsehood if there were no minds to make mistakes. But it has the further drawback that it leaves the difference between truth and falsehood quite inexplicable.

(ONTF, p.176)

His second point is that it is 'incredible' that there are such things as objective falsehoods. He also regards the indefinability of truth and falsehood as a 'drawback,' and this constitutes his third objection to the ontology of propositions.

However, these objections to propositions simply make more mysterious why Russell abandoned propositions, because none of them is decisive. The first point amounts to a mere claim, not an argument, that propositions should be expressed by that-clauses. As for the second objection, there is certainly a sense in which ‘there could be no falsehood if there were no minds to make mistakes’; but this does not mean that there is no sense of falsehood in which a proposition is said to be false regardless of whether someone is making a mistake or not. The second objection is thus no more than the claim that a false proposition is ‘almost incredible.’ Furthermore, the indefinability of truth and falsehood is not a ‘drawback’ for those who maintain that truth and falsehood are indefinable. These three objections are thus not strong enough to explain why Russell decided to abandon propositions.

This has driven some authors to seek a ‘real’ reason for his abandonment of the metaphysics of propositions. They have focused on the view of truth and falsehood as simple, indefinable properties of propositions, which Russell endorsed in conjunction with the ontology of propositions. In what follows I will first discuss three problems with the view of truth and falsehood as primitive properties in Section 6.2. I will there argue that it is implausible that Russell abandoned propositions simply because of them. In Section 6.3, I will offer a story as to how he became unsatisfied with propositions and how those problems affected his attitude towards propositions. In closing, I will argue that the starting point of the story lies in a commonality between the ontology of propositions and the old idealistic notion of *judgment*.

6.2. Problems with the view of truth as primitive

In this section, I will discuss three kinds of problems with the view that truth and falsehood are primitive properties of propositions. It has often been argued in the literature that some of these problems were crucial in persuading Russell to reject the ontology of propositions in favour of the multiple-relation theory of judgment. However, it will be seen that he did not regard them as genuine problems.

6.2.1 Indefinability of truth and falsehood

In this subsection I will first introduce how Russell argues for the indefinability of truth and falsehood in endorsing the ontology of propositions. I will then try to indicate that although his argument poses a certain problem to his own standpoint, he does not view it as a genuine problem.

Let us begin by briefly introducing Russell’s view of truth and falsehood as indefinable properties. In *PoM*, he gives the notion of truth only a marginal treatment except for an obscure discussion on the notion of *assertion* (cf. *PoM*, §1, §84). It is in some subsequently written papers that his view of truth and falsehood enjoys an extensive discussion. He wrote ‘Meinong’s Theory of Complexes and Assumptions’ (henceforth, MT) in 1903 and

published it in three parts in 1904. It is in this paper that he states his position on truth and falsehood expressly for the first time since his endorsement of the ontology of propositions. He concludes, through a detailed discussion of Meinong's theory of *Objectives*, 'that there are, apart from and independently of judgment, true and false propositions, and that either kind may be assumed, believed, or disbelieved' (CP4, p.472). As we will see in Section 6.3.1, Russell is somewhat sceptical about false propositions but in the end he holds that 'some propositions are true and some false, just as some roses are red and some white' (*ibid.*, p.473). His comparison of truth and falsehood with redness and whiteness suggests that truth and falsehood are considered simple, indefinable properties of propositions, since he arguably counts redness and whiteness as such properties (cf. Moore, 1900-1). In fact he explicitly argues for this view of truth and falsehood in 'The Nature of Truth' (henceforth, NT), which he read to the Jewett Society, Oxford in June 1905. The impossibility of defining truth and falsehood is among the main themes of the paper: 'I have no positive doctrine to advocate, but am content to try and refute whatever positive doctrine comes into our discussion' (CP4, p.492). He thus maintains that it is impossible to define truth and falsehood. He also makes it plain that they are considered to be certain properties of propositions: 'Truth and falsehood, in fact, are properties attaching to propositions as wholes, and are not themselves, in general, parts of propositions' (*ibid.*, p.504). In endorsing the ontology of propositions, he thus holds that truth and falsehood are simple, indefinable properties of propositions.

Let us turn to Russell's argument for the indefinability of truth and falsehood. In NT, he presents the following argument against the correspondence theory of truth and effectively against any attempts to define truth and falsehood¹:

If we don't know the difference between a proposition's being true and not being true, we don't know the difference between a thing's having a property and not having it, and therefore we can't define a thing as true when it has a certain property such as correspondence with reality.

(CP4, p.494)

The idea is that every proposed definition of truth will presuppose the very notion of truth and hence fail to be a proper definition of it. The argument is arguably based on the assumption that an entity's possession of a property consists in the corresponding proposition's being true. Given this assumption, it indeed follows that any account of truth and falsehood ultimately relies upon the very notion of truth. For in order to say that an entity *a* actually possesses a property *F*, we need to check whether the proposition '*Fa*' is true or not.

It should be observed that this assumption in turn implies that even if an entity *a* is *connected*

¹ Frege offers a similar argument against the correspondence theory of truth (Frege, 1984, p.353).

with a property F to form a proposition Fa , the proposition itself remains neutral as to whether the entity *actually possesses* the property F . It will possess the property, if the proposition is true; and it will not, if the proposition is false. Russell thus effectively distinguishes between an entity's *being connected with* a property and its *actually possessing* it. Hence, the unity of a proposition requires, in his view, just one mode of combination: entities only need to be *connected with* a relation.

As Hylton, Ricketts and Korhonen point out, this argument against any definitions of truth and falsehood backfires (Hylton, 1990, p.140; Ricketts, 2001, p.105; Korhonen, 2009, p.173). If truth and falsehood are included among ordinary properties, we can generate a certain regress within the view of truth in question precisely in the same way in which Russell does against any definitions of truth and falsehood. 'The ontological primacy of propositions and of truth,' Hylton explains, 'arises from the fact that for an object to have a property (or for two objects to be related by a given relation) is taken to consist in the truth of the corresponding proposition' (Hylton, 1990, p.178). He then points out that an infinite regress arises from this account: 'For a given proposition, call it a , to have the property truth is for another proposition, that a is true, to have the property truth; for this to be so, the proposition that the proposition that a is true is true must in turn have the property truth, and so on' (*ibid.*). I shall call this regress 'the regress of predication.'

This regress indeed undermines the ontology of propositions, because it establishes that nothing in the ontology could constitute an entity's *actual possession* of a property. For the regress generates a chain of propositions in each of which a certain proposition is merely connected with—and does not actually possess—the property of being true. However long the chain may be, it never reaches a proposition which actually possesses the property of being true. The neutral mode of combination that propositions are all considered to have cannot yield any complexes where an entity actually possesses a property. The problem is, in other words, not an infinity of propositions involved in the truth of a proposition but rather the absence of what can let a proposition actually possess the property of being true.

Russell was arguably aware of the regress of predication while endorsing the ontology of propositions. Hylton, Ricketts and Korhonen point out that that the regress underlies the notoriously obscure discussion of *assertion* in the *logical sense* in *PoM*.² Although, contra their claim, Russell does not seem to speak of the regress of predication in the body of the book, he indeed tries to resolve it by appealing to the notion of *assertion* in Appendix A:

If p is a proposition, " p 's truth" is a concept which has being even if p is false, and thus " p 's truth" is not the same as p asserted. Thus no concept can be found which is

² I cannot interpret the passage in the present work.

equivalent to p asserted, and therefore assertion is not a constituent in p asserted. Yet, assertion is not a term to which p , when asserted, has an external relation; for any such relation would need to be itself asserted in order to yield what we want.

(PoM, §478)

Russell thus attempts to resolve the regress by distinguishing p 's being asserted from a concept p 's truth, holding that the former cannot be turned into a *term*.

It is, however, implausible that Russell came to abandon the ontology because of the regress of predication. On the contrary, it seems as though he remained optimistic that the regress would not do his ontology of propositions any harm. In MT, he remarks, after a discussion of false propositions, that 'there is no problem at all in truth and falsehood' (CP5, p.473). In NT, he introduces an epistemological problem caused by a regress analogous to the regress of predication and, importantly, he presents it to support his primitivist view of truth:

How then do we know whether a proposition is true or false? The first answer is, that the difficulty would be just as great if there were such a difference between true and false propositions as is desired. If all propositions having a certain property are true, and all others false, the question arises: How do we know which propositions have this property? Presumably by observing whether the proposition that they have the property has the property; but this plainly won't do.

(CP4, p.505)

Any appeal to a 'further quality' is thus subject to the same objection which he makes against the correspondence theory, and this constitutes his 'first answer.' His second answer is that the difficulty with our knowledge of truths only reinforces his claim that truth and falsehood are inexplicable. Of course, he might have changed this optimistic attitude towards the regress of predication after NT. But if he had abandoned the ontology because of it, he would certainly have mentioned it in ONTF.

It is not perfectly clear why Russell thus thought the regress of predication harmless. There are at least two possible ways to resolve it, though neither of them is entirely satisfactory. One is to hold that truth and falsehood are different from other properties and that if a proposition is connected with one of these two properties, the proposition is indeed true or false as the case may be. This idea amounts to the claim that truth and falsehood are *sui generis* objects. Korhonen indeed attributes this view to Russell (Korhonen, 2013, pp.124-6). But this idea, as Korhonen himself observes, leaves it inexplicable why Russell would then employ such a misleading expression as 'properties attaching to propositions' without any qualification.

Another solution is to accept more than one mode of combination. One might attempt, as Moore in NJ seemingly does, to distinguish between the true-proposition-constituting mode

of combination and the false-proposition-constituting one. 'A proposition is,' he remarks, 'constituted by any number of concepts, together with a specific relation between them; and according to the nature of this relation the proposition may be either true or false' (NJ, p.180).³ This distinction certainly resolves the regress of predication, as it prevents the very first step of the regress. However, this approach is flawed or at least Russell thinks so. 'If truth and falsehood were,' he argues in NT, 'respectively constituents of true and false propositions, we could tell by inspection (at least as a rule) whether a proposition is true or false' (CP4, p.504). It is reasonable for him to think that to understand a proposition is to have acquaintance with all the constituents and with the way in which they are combined with each other. But if so, and if a proposition is true or false depending on the way in which terms are combined to form it, we would be able to tell whether a proposition is true or false, simply by understanding the proposition, that is, the meaning of a sentence. Russell was presumably aware of this problem as early as in 1899 or 1900. For Moore seemingly rejected the distinction between these two modes of combination by the time he wrote his entry on truth in Baldwin's *Dictionary of Philosophy and Psychology* (Moore, 1902, pp.716-7).⁴

A plausible way of accepting more than one mode of combination is to distinguish between the proposition-constituting manner and the fact-constituting manner. The distinction resolves the regress because it enables us to think that the truth of a proposition consists in the substance of a corresponding fact, a complex in which entities are connected in the fact-constituting manner. I will call this account of predication the *dualist* account of predication. As we will see in Section 6.3, Russell in NT and some other papers makes various attempts to accommodate complexes in which entities are connected in such a manner. One might think that he does so because he tries to resolve the paradox by means of the dualist account. However, it should be remembered that it is in NT that he argues for the impossibility of defining truth and falsehood, invoking two kinds of regress, both of which are akin to the regress of predication. He would not do so if he found the regress problematic there. It seems to me that his reason for accepting complexes other than propositions stemmed from something else.

Attributing the dualist account to Russell in *PoM*, Ricketts offers an elaborate account of the abandonment of propositions. According to Ricketts, Russell in *PoM* tries to resolve the regress of predication by holding that there are two ways in which a proposition can be formed out of individual entities—'the true way or the other way.' Ricketts' interpretation

³ To be precise, Moore in NJ also understands 'truth and falsehood as properties of certain concepts, together with their relations—a whole to which we give the name of proposition' (*ibid.*, p.181).

⁴ Hylton points this out (Hylton, 1990, p.138fn). Later in *Some Main Problems of Philosophy*, Moore remarks that given the dyadic-relation theory of judgment, truth is 'a property which can only be possessed by propositions, and is only possessed by *some* among them' (Moore, 1953, p.262).

will be discussed in the following subsection.

6.2.2 Proposition and falsehood

In search for Russell's substantial reason for giving up the ontology of propositions, some philosophers have appealed to an argument that if the object of a judgment is a proposition it is impossible to make a false judgment. I will call the argument 'the impossibility argument.' In this subsection I will argue against a simple interpretation that presents the argument as the reason for his abandonment of propositions, before I turn to Ricketts' one, which combines the argument with the dualist account of predication.

Olson, Cartwright and B. Linsky have argued that Russell abandoned propositions directly because of the impossibility argument.⁵ Cartwright summarises the argument as follows:

We thus appear to have at hand a demonstration that there is no such thing as the proposition that there are no subways in Boston: if there is, then there is such a thing as the nonexistence of subways in Boston; and there is such a thing as the nonexistence of subways in Boston only if there are no subways in Boston; but there *are* subways in Boston.

(Cartwright, 1987, p.82)

The idea is that if each judgment has a proposition as its object and if the proposition is a complex constituted by the entities about which the judgment is made, then the subsistence of the proposition implies that the entities in question are *in fact* combined with each other in the way in which the judgment describes them, which in turn makes the judgment true. This is what I call the impossibility argument. It is meant to show that if the dyadic-relation theory of judgment holds, that is, if our judgment is always a binary relation between the judging mind and a proposition, then every judgment would be true, which is preposterous. One cannot avoid the argument by appealing to the distinction between *subsist* and *exist*, which Moore and Russell indeed had when they advocated the theory of truth as primitive. For the argument is concerned only with the former notion. This is perhaps why Cartwright does not pay much attention to the distinction in stating the impossibility argument as above.

The impossibility argument involves a notion that is arguably foreign to Russell, however. The crucial step in the argument is from the subsistence of the proposition *Brown is taller than Smith* to the claim that Brown is *in fact* taller than Smith. According to Cartwright, Russell had to take this step because of his account of the unity of the proposition. Russell in *PoM* holds that '[t]he verb, when used as a verb, embodies the unity of the proposition,' where 'verb' here means the relation which a verb designates (*PoM*, §54). In the case of the proposition *A*

⁵ Olson (1987), pp.80-1; Cartwright (1987), p82; Linsky (1999), pp.47-9.

differs from B, he explains that '[t]he difference which occurs in the proposition actually relates *A* and *B*' (*ibid.*). Thus, on his account, each proposition has among its constituents a relation which relates the other constituents to form a single entity. This account *appears* to entail that when the proposition 'Brown is taller than Smith' subsists, Brown has the relation of *being taller than* to Smith, that is, Brown is *in fact* taller than Smith. However, it should be observed that Russell does not accept the equivalence expressed by 'that is' here: the subsistence of a proposition is in his view not the same thing as Brown's *actually* being taller than Smith. As we saw in the preceding subsection, the subsistence of the complex 'Brown is taller than Smith' is supposed to be neutral as to whether Brown is *actually* taller than Smith. For, on his account, if the complex possesses the property of being false, Brown is not taller than Smith. The proposition itself is understood as the result of Brown and Jones *being connected with*—as opposed to *actually possessing*—the relation of *being taller than* (in the relevant order). Hence, in his view, the proposition 'Brown is taller than Smith' is neutral as to whether it is true or false, waiting for truth or falsehood to attach to it. This account of truth and falsehood is precisely why the regress of predication arises within the ontology of propositions. 'There is,' in Russell's view, 'no hint at all that insofar as it [a proposition] contains a relating relation, the proposition must be true' (Landini, 1996, p.323).

An argument presented in 'Meinong's Theory of Complexes and Assumptions' (1904; henceforth, MT) illustrates that Russell indeed thinks that entities in a proposition are only connected with each other in a way that does not automatically render a judgment about the proposition true. In this paper, he uses the notion of 'particularized relation' to discuss Meinong's idea that 'when a relation *R* is affirmed to hold between *a* and *b*, as in (say) "*a* is the father of *b*," what is really affirmed is the being or subsistence of the relation' (MT, pp.452-3). According to him, Meinong holds that the judgment '*a* is the father of *b*' has as its object 'the relation particularized as relating *a* and *b*' (*ibid.*, p.452). On the other hand, Russell maintains that a relating relation is an 'abstract' relation or a 'general concept.' He argues as follows:

But there are logical reasons for supposing that there are no such entities at all as particularized relations; most of them I have set forth elsewhere [...],⁶ but another is derived from false propositions. If what is actually meant by a relational proposition is the being of particularized relation, then, when the proposition in question is not true, it must be meaningless: for it affirms the being of what, *ex hypothesi*, does not have being, and therefore there is nothing of which it affirms the being, and therefore it affirms nothing and is meaningless.

(MT, p.453)

⁶ What Russell has in mind here is presumably *PoM*, §55.

Russell thinks that a relation R particularized as relating a and b subsists only when it is true that aRb . It should be observed that the argument presented above parallels the impossibility argument: if each judgment has a particularized relation as its object, there is no false judgment, since the subsistence of the particularized relation entails, by definition, that the corresponding judgment is true. The fact that he raises the above problem against the notion of particularized relation implies that he is aware of the impossibility argument and thinks that it does not apply to his view that an abstract relation combines entities into a proposition. He thus seems to think that a general relation connects entities into a proposition without making it true.

There are some other difficulties with the idea that Russell abandoned propositions simply because of the impossibility argument. As Proops points out, Russell does not refer to the argument, when he states his reasons against the ontology of propositions in ONTF (Proops, 2011, p.199). Proops also points out that the argument is too obvious to take Russell several years to realise and too simple to explain why Russell does not mention it in his published works (*ibid.*). In fact, if the argument itself were the reason for abandoning propositions, Russell would have said that he had a decisive argument, since the argument, if it is valid, would render the dyadic-relation theory of judgment decisively implausible. On the contrary, he in ONTF regards the dyadic-relation theory of judgment as 'not logically impossible.'⁷ The impossibility argument cannot thus be the whole story of why he abandoned the dyadic-relation theory of judgment.

Ricketts develops Cartwright's interpretation by holding that although Russell did have a way out of the impossibility of false judgments, he came to realise that it would not work and accepted the argument, abandoning propositions. Ricketts first argues that Russell was able to avoid the impossibility of false judgments by endorsing what I call the dualistic account of predication, that is, 'by separating out two notions of a relating relation' (Ricketts, 2001, p.116). One is 'that of the proposition-constituting predicative occurrence of a relation; the other is that connected to truth' (*ibid.*). If a relation can occur in a proposition either in the 'proposition-constituting' manner or in the fact-constituting manner, then the subsistence of a proposition in which a relation connects entities in the proposition-constituting way does not imply the subsistence of the complex in which the relation glues the other entities in the latter way. Ricketts goes on to argue that although Russell could thus avoid the impossibility argument, he soon found this reply undesirable. 'A Bradleyan might,' Ricketts explains, 'put the point like this: according to Russell, a relation can relate without actually relating' (Ricketts, 2001, p.115). According to Ricketts, the dualistic account of predication 'foists on Russell the burden of explaining the relationship between these two notions' (*ibid.*, p.116).

⁷ ONTF, p.176. Similar remarks can be found in *PP* (Ch.XII), *TK* (pp.105, 116, 153) and Moore (1953, p.262).

Ricketts thus holds that the dualistic account would crack the ‘transparency’ of Russell’s account of the unity of the proposition, which Russell wanted in order to avoid a Bradleyan’s criticism concerning it. Ricketts then concludes that Russell could not be content with the two notions of a relating relation and was led to accept the impossibility argument, abandoning the ontology of propositions.

However, this interpretation is not plausible. For it is unlikely that Russell found it necessary to accept the dualist account of predication because of the impossibility argument. As we have seen, he in MT does not think that the impossibility argument applies to his own account of the unity of the proposition. Hence, in his view, he does not have to draw the distinction between the proposition-constituting mode of combination and the fact-constituting one to avoid the argument in the first place. One might argue that he eventually realised that the argument would be applicable to his own account as well, and hence he drew the distinction between the two modes of combination. But this is unlikely. For, as a solution to the impossibility argument, the distinction is redundant, because it presupposes that there should be a mode of combination that does not make a resulting complex a true proposition. If he thought that the neutral mode of combination would allow him to avoid the argument, he had no reason to add another mode of combination as a response to the argument. It would thus seem difficult to think that the argument was the driving force behind his abandonment of propositions.

6.2.3 Proposition and acquaintance

Proops focuses on the notion of acquaintance and on the Principle of Sufficient Reason in explaining how Russell was led to reject the ontology of propositions. The present subsection examines his interpretation.

Proops first points out that Russell’s solution to a well-known puzzle concerning true, informative identity statements commits him to holding that we cannot be acquainted with propositions. In OD, Russell appeals to the theory of descriptions to explain why substitution of a definite description for a co-referring proper name sometimes fails to preserve the truth value. We cannot *salve veritate* replace ‘the author of *Waverley*’ by ‘Scott’ in ‘George IV wanted to know whether Scott was the author of *Waverley*.’ He explains that we cannot do so because the proposition expressed by the sentence does not contain any entity corresponding to the definite description ‘the author of *Waverley*,’ which is an incomplete symbol. Russell’s solution to the puzzle is thus to show that one of the two expressions flanking the identity sign in a true, informative statement is not a so-called *logically proper name*, or—adopting Proops’ terminology—a *genuine* name. Proops goes on to argue that given the ontological commitment to propositions, a propositional version of the George IV puzzle is formulable:

'In 1903 Russell believed that all *F*s are *G*s = $\forall x(Fx \supset Gx)$ ' (Proops, 2011, p.183).⁸ According to Proops, 'Russell would have been vividly aware of all the ingredients needed to arrive at the conclusion that some sentence-tokens are incomplete symbols' and that although 'the texts are, unfortunately, silent on this point,' 'it would be surprising if he had not done so' (*ibid.*, p.184). Proops maintains that though the propositional variations of the puzzle do not entail that Russell has to regard *all* occurrences of sentences as incomplete symbols, yet, he should have drawn a conclusion that we are not acquainted with propositions. For Russell holds that 'anything with which we are acquainted can be named,' whereas if we can name a proposition, it is possible to attach 'names to it that reflect differing degrees of discernment of the proposition's constituent structure,' thereby creating further instances of the puzzle (Proops, 2011, pp.184-5).⁹

Importantly, Proops does not simply go on to claim that Russell abandoned propositions because he had to conclude that we were not acquainted with propositions. According to Proops, when Russell realised that we cannot be acquainted with propositions on pain of the propositional variations of the George IV puzzle, he started seeking an alternative account of understanding or judgment that does not require that we should have acquaintance with propositions. The multiple-relation theory of judgment is, on Proops' view, adopted as such an alternative. But at this point, Proops continues, Russell did not have to abandon propositions altogether, and rather, he 'might have argued for the subsistence of propositions by a kind of inference to the best explanation.' The subsistence of propositions can still explain 'the apparent validity of inferences of the form: "A believes that the sun orbits the earth; so $\exists x[\text{believes}(A, x)]$ "' (Proops, 2011, pp.186-7). However, according to Proops, when Russell firmly endorsed the multiple-relation theory of judgment as an alternative account of judgment, 'Russell could no longer see his "inference to the best explanation" argument for propositions as cogent' (Proops, 2011, p.190). In Proops' view, having lost the sole ground for propositions, Russell sought and found positive arguments against propositions, which we can find in ONTF. Proops thus holds that Russell had adopted the multiple-relation theory so as to give an alternative account of judgment, *before* he concluded that there are no such things as propositions.

Proops adds that Russell came to see the impossibility argument 'having some force only after he had come to embrace the Principle of Sufficient Reason in 1910' (Proops, 2011, p.201). Proops denies that 'Russell always took the relation's occurrence "as relating" as sufficient to ensure the truth of the proposition' (*ibid.*, p.200fn). But Proops does so because he thinks it possible to distinguish two ways in which a relation combines other entities: the 'fact-

⁸ To be fair, Russell in *PoM* does not identify these two propositions with each other (cf. p.52).

⁹ It is, however, not clear whether Russell thinks that he can give significantly different names to a single entity in this way. I will discuss this point in detail shortly.

constituting' way and the 'false-proposition-constituting' way.¹⁰ Proops goes on to argue that the Principle of Sufficient Reason undermines this way of responding to the impossibility argument: 'with this principle assumed, the supposition that one and the same relation occurs in some complexes in a fact-constituting way but in others merely in a proposition-constituting way would seem implausible' (*ibid.*, p.200).

Proops' account is carefully constructed and based on illuminating observations, but there are three points where his account does not seem to be entirely satisfactory. One is that it is unlikely that Russell was 'vividly aware of all the ingredients needed to arrive at the conclusion that some sentence-tokens are incomplete symbols.' For Russell and Whitehead in *PM* do not pay attention to variations of the George IV puzzle and only considers the original one discussed in OD:

It might be thought that identity would not have much importance, since it can only hold between x and y if x and y are different symbols for the same object. This view, however, does not apply to what we shall call "descriptive phrases," *i.e.* "the so-and-so." It is in regard to such phrases that identity is important, as we shall shortly explain. A proposition such as "Scott was the author of *Waverley*" expresses an identity in which there is a descriptive phrase (namely "the author of *Waverley*"); this illustrates how, in such cases, the assertion may be important.

(*PM*, p.23)

This passage shows that Russell does not think that we can have an informative, or in his word 'important,' identity statement using two names for a single entity. If he thus envisages only the original version of the George IV puzzle, it is difficult to think that he has come to endorse the multiple-relation theory of judgment so as to solve the propositional versions of the puzzle. It should also be noted that the authors of *PM* expressly treat ordinary proper names as 'telescoped' descriptions: "'Apollo" means really "the object having such-and-such properties"' and 'The same principle applies to many uses of the proper names of existent objects, *e.g.* to all uses of proper names for objects known to the speaker only by report, and not by personal acquaintance' (*PM*, p.31). This suggests that Russell's main reason for regarding proper names as telescoped descriptions does not lie in the George IV puzzle nor its variations but rather in his observation that existential statements in which ordinary proper names occur as subjects can be meaningful and yet false.¹¹

Another point where Proops' interpretation is not entirely convincing is the idea that Russell comes to combine the impossibility argument with the Principle of Sufficient Reason to grant

¹⁰ As we saw, Russell found this distinction implausible. But Proops' interpretation does not rely on the distinction and could instead attribute the dualist account of predication to him.

¹¹ Proops also observes this point, though he thinks that this is a mere possible reason (Proops, 2011, p.172).

the argument 'some force.' Firstly, as we discussed in the preceding section, Russell thought that the argument would not apply to his own standpoint. Secondly, once the metaphysical principle is granted, the impossibility argument itself becomes redundant. In endorsing the dyadic-theory of judgment, Russell maintains that it is impossible to *explain* why a certain true proposition is true rather than false; but this is plainly incompatible with the Principle of Sufficient Reason. Thirdly, Russell, even in *TK*, may not have envisaged the connection between the principle and the view of truth and falsehood as indefinable. For he thinks that the impossibility argument 'cannot be regarded as very conclusive' (*TK*, p.110). He finds no 'logically compelling' objections to the dyadic-relation theory (*ibid.*, p.153). If he realised either the full force of the impossibility argument combined with the Principle of Sufficient Reason or the incompatibility of the principle itself with the dyadic-relation theory of judgment, he would not have said that his arguments against the dyadic-relation theory of judgment 'are not of a very definite kind' (*ibid.*).

The last point in which I do not entirely agree with Proops is his claim that, although Russell only adopts agnosticism about propositions in *PM*, he turns to eliminativism in *ONTF* (Proops, 2011, pp.188-9). In my view, Proops is entirely right in pointing out that no textual evidence indicates Russell's endorsement of eliminativism of propositions in *PM*. As Proops points out, Russell often, and certainly in *PM*, expresses his agnosticism in a way that makes him seem to endorse eliminativism. Yet, the same observation seems to apply to his remarks in *ONTF* as well. As we have seen, he first argues in *ONTF* that that-clauses seem to be incomplete symbols. But this can be seen as an argument for agnosticism rather than for eliminativism. The other two points which Russell makes against the dyadic-relation theory of judgment in *ONTF* are 1) the feeling that there are no false objectives and 2) its incapability of explaining the difference between true propositions and false ones. In the following section I will endeavour to show that Russell has been considering these two points in some papers and manuscripts written before *PM*, so that I can claim that the presence of these two points in *ONTF* does not necessarily indicate Russell's change of mind from *PM* to *ONTF*. It seems that Russell and Whitehead do not make these points in *PM* because they take the points to be too philosophical or not relevant enough to be put in there. If so, it is possible to develop an interpretation such that Russell abandoned propositions because of 'his intuitive distrust of false propositions'—the interpretation which Proops thinks 'sits awkwardly with the fact that there is, strictly speaking, no trace of eliminativism (as opposed to agnosticism) of propositions concerning propositions in *Principia*' (Proops, 2011, pp.195-6).

6.3. Towards an existential view of truth

It seems that Russell wavered between two different conceptions of fact during the period from 1904 to 1909. In advocating the dyadic-relation theory, Russell understood a fact simply as a true proposition; but from 1904 onwards he came to think that a fact should be a complex entity that subsists in the universe just in case the corresponding judgment is true. I shall use the term 'factual complex' to speak of a complex entity that subsists only if the corresponding judgment is true, reserving the term 'fact' for our general concept of facts. In my view, it was his preference for the notion of factual complex that led him to abandon the dyadic-relation theory of judgment.¹²

6.3.1 The notion of particularized relation

It is, to my knowledge, when Russell wrote 'Meinong's Theory of Complexes and Assumptions' (MT) that he started wondering whether the dyadic-relation theory should be replaced. In that paper, he remarks that 'the explanation of falsity presents grave difficulties' (MT, p.461). The task of this section is to argue that those difficulties which he finds in MT can be understood as arising from his preference to the notion of factual complex.

As we saw in Section 6.2.2, Russell in MT attributes to Meinong the view that our judgment affirms the subsistence of a particularised relation, and he rejects this view with the impossibility argument. But it seems as though Russell remains tempted to think that there are such things as particularised relations and that they are somehow connected with the truths of judgments:

[...] if we consider a relation R between a and b , we should say, when it is false that this relation holds, that there is no such thing as the relation R between a and b . But this argument has been already disposed of, by the contention that the being of this relation is not what the proposition " a has the relation R to b " really affirms. Yet, when we consider such complexes as "the difference between a and b ," which must be admitted by some door, it seems plain that, when a and b are identical, there is no difference between a and b , which seems equivalent to "the difference between a and b does not have being."

(MT, p.463)

It seems that what Russell calls particularised relations can be seen as factual complexes. Although he means by 'general concept' a relation itself, he identifies a particularized relation with a certain kind of complex in which the entities combined by the particular instance of the relation also occur. In addition, Russell himself does not hesitate to use the word 'fact'

¹² R. E. Tully suggests this interpretation but does not develop it in detail (Tully, 2003, p.348).

to speak of entities that subsist only in the case of true judgments, though this does not, of course, mean that he understands such entities as complexes (MT, p.468). Russell in MT goes on to seek an alternative to the dyadic-relation theory of judgment so that he can somehow accommodate factual complexes within his ontology.

Russell first considers the view that only true judgments have objects and false ones have no objects. I will call this view 'the asymmetric theory.' But this view is immediately confronted by a problem that if false judgments have no objects, it would follow either that there are no false judgments or that we can tell simply by virtue of making a judgment whether the judgment is true or false. To avoid these consequences, Russell argues that given the asymmetric theory, a judgment must be made of ideas or presentations of the objects of the judgment. The asymmetric theory thus suggests that a false judgment ' aRb ' is 'composed of the presentations of a and R and b suitably related, and might have no corresponding object' (MT, p.468).

The asymmetric theory appears to be the multiple-relation theory of judgment, but these two views are not the same. For the asymmetric theory explains a judgment as composed of ideas of the relevant entities, while the multiple-relation theory urges that a judgment consists of worldly entities themselves. The difference may look trivial, since in Russell's view the relation of *having a presentation of* is a dyadic relation and hence if a judgment relates, as the asymmetric theory urges, some ideas or presentations with each other, there will be a composite relation between the objects which correspond to those presentations (cf. MT, p.468). But the multiple-relation theory does not require that it should be possible to thus combine presentations to make a judgment. Furthermore, the asymmetric theory seems to take the first-person point of view in the sense that it explains a judgment as something that obtains within the mind through its manipulating presentations. On the other hand, the multiple-relation theory analyses a judgment from the third-person's point of view, including the judging mind itself among the constituents of the judgment.

One problem with the asymmetric theory is that it cannot explain how a molecular judgment is possible: a true judgment may contain, as its part, a false judgment. For instance, if a true judgment ' $\neg p$ ' has the corresponding complex $\neg p$, then we would have to admit that there is also such a complex as p , which on this view must be absent. Another problem arises from Russell's conviction that 'the presentation of a relation is not itself a relation' (MT, p.468). If the presentations ' a ,' ' R ' and ' b ' are considered to compose the judgment ' aRb ,' then it is natural to maintain, as Frege and Wittgenstein did, that the presentation ' R ' relates the others to form a single presentation ' aRb .' But Russell takes it as axiomatic that 'presentations related cannot beget a new and different presentation,' and hence, in his view, if the asymmetric view is

correct, it follows that 'we should not have presentations of complexes' (MT, p.469).¹³ Instead of pondering whether he can overcome these difficulties, he in MT chooses to give up the asymmetric theory.

What Russell considers next in MT is a theory of judgment which retains propositions and factual complexes at the same time:

The view commended by inspection would rather be the following: There is, in any case, a proposition aRb , and in this proposition, the abstract relation R occurs, not the relation particularized by its terms; but in the case where aRb is true, there is such an entity as the particularized relation, whereas, when aRb is false, there is no such entity. This entity, when it subsists, is distinct from the proposition.

(MT, p.471)

Russell thus considers what I shall call the *two-fold* theory, according to which judgments, whether true or false, have propositions as their objects, and only true ones have further entities called factual complexes. However, he finds a 'fatal' objection to this two-fold view: it is difficult 'to see what it is that is denied when the particularized relation is said not to subsist' (MT, p.471). If the judgment 'the table is black' is false, then the judgment 'there is no such thing as the blackness of the table' should be true and hence meaningful, while what the phrase 'the blackness of the table' stands for must be absent. This is precisely the impossibility argument. This problem appears to be avoidable by applying Russell's own theory of denoting concepts or the theory of descriptions, though the latter is yet to come at this point. I will discuss what happens when these theories are applied in the following subsection. In virtue of this problem, Russell in MT returns to the dyadic-relation theory of judgment in the end, even though he is still tempted to think that 'when a proposition is false, something does not subsist which would subsist if the proposition were true' (MT, p.473).

Russell thus in MT starts wondering how he can accommodate particularised relations, presumably because of Meinong's *existential* account of truth and falsehood: a judgment is true, if there exists (or, more precisely, subsists) a corresponding entity. As Simons insightfully suggests, Russell's own account of truth and falsehood is *qualitative* rather than *existential* (Simons, 1999, p.78). For Russell maintains that propositions are either true or false depending on whether the property of being true or that of being false attaches to them. It seems as though Russell comes to find the existential account more *realistic* than the qualitative one.

¹³ Note that this problem does not arise within the multiple-relation theory of judgment, as it does not require that a presentation of a relation should relate other presentations. In light of this problem, Russell considers the possibility that a presentation is always simple whether what it stands for is a complex or not. But he concludes that 'where no single word exists, as in (say) "the execution of Charles I," it seems quite evident, at first sight, that the presentation is complex' (MT, p.470).

I will discuss why he does so in Section 6.4. Before doing so I shall introduce two more attempts which he made to endorse the existential account.

6.3.2 Enquiries into the nature of truth

During the period from 1905 to 1906, Russell wrote two articles on the notion of truth: 'The Nature of Truth' (NT) and 'On the Nature of Truth' (ONT). Let us examine his views on truth in these papers.

Russell wrote NT in June 1905, presumably before he discovered the theory of descriptions.¹⁴ In this paper he again tries to reconcile the notion of factual complex with that of proposition:

It is natural to hold that when a proposition is true, there is a connected entity which has being, and that when a proposition is false there is no such connected entity. Let our proposition be of the form "A differs from B"; then when it is true, there is a difference between A and B, while when it is false there is no difference. Thus the difference between A and B is an entity when the proposition is true, but not otherwise.

(NT, p.505)

Here a proposition 'A differs from B' is distinguished from 'a connected entity,' that is, an entity that is designated by the phrase 'the difference between A and B.' What he calls a 'connected entity' here is a particularized relation, and hence, what I call a factual complex. He thus again proposes the two-fold theory.

Russell now appeals to the theory of denoting concepts, which he originally develops in *PoM*, in order to solve the problem that 'the concept "the difference between A and B" might seem to have a perfectly clear and definite meaning even when there is no difference' (NT, p.505). He claims that 'when the proposition is true this concept denotes an object, which is the difference between A and B, while when the proposition is false, the concept does not denote an object' (*ibid.*). Importantly, he in NT does not *define* truth and falsehood of propositions in terms of factual complexes; rather, he emphasises that the presence of a factual complex is merely a 'mark' or 'criterion' of truth. This is of course related to his contention in NT that it is impossible to offer any positive definitions of truth and falsehood. It should be remembered that since he presents, as we saw in Section 6.2.1, two kinds of regress analogous to the regress of predication in NT, it is unlikely that he accepts factual complexes as well as propositions as a result of his endorsement of the dualist account of predication.

In ONT, Russell considers the dyadic-relation theory and the view that a judgment 'will consist of several related ideas' (ONT, p.452). The latter theory explains a judgment as a relation of ideas, not of entities, and hence is what I call the asymmetric theory, even though it

¹⁴ NT was apparently presented in Oxford on 10 June 1905, while the lengthy manuscript 'On Fundamentals,' in the middle of which he discovered the theory, was started on 7 June 1905 (*CP4*, pp.359, 490).

is often identified with a rudimentary version of the multiple-relation theory of judgment (cf. Proops 2011; Klement 2010, p.646). Russell argues that the asymmetric theory is advantageous in that it can explain why perception, conceived as a binary relation between the mind and a factual complex, is infallible, though he adds that ‘it may well be questioned whether perception is infallible’ (ONT, pp.452-3). But he still finds a problem with the asymmetric theory. In his view, the asymmetric theory cannot explain how a true judgment can correspond to a certain factual complex, for ‘the belief that *A* and *B* have the relation *R* must be a *three-term* relation of the ideas of *A* and *B* and *R*’ (ONT, p.452fn). He seems to presuppose that the symbol ‘*R*’ cannot combine ‘*A*’ and ‘*B*’ to form ‘*ARB*.’ Thus, the asymmetric theory is, again, confronted by his conviction that ‘presentations related cannot beget a new and different presentation.’¹⁵

It is important to note that Russell calls the asymmetric theory ‘a form of the correspondence theory’ in ONT (CP4, p.451). He remarks: ‘Truth, then, we might suppose, is the quality of beliefs which have facts for their objects, and falsehood is the quality of other beliefs’ (*ibid.*). The acceptance of a correspondence theory means that Russell no longer sees as much force in the argument for indefinability of truth and falsehood as before. It might be the case that he now realises that the regress of predication is indeed problematic. But, importantly, this does not explain why he began to seek alternatives to his dyadic-relation theory of judgment. He had been doing so before he ceased to argue for the indefinability.

It should also be noted that Russell does not even mention the two-fold theory in ONT. There are some possible accounts as to why he does not. One is that he does not want to accept two kinds of complex entities, propositions and factual complexes. As Ricketts and Proops hold, he may have found it undesirable to distinguish two modes of combination. Back in MT, he indeed remarks that if particularized relations are admitted, ‘it is very difficult to see what the proposition is’ (MT, p.471). Another possible account is based on the fact that he invented the theory of descriptions between NT and ONT. As we have seen, he in NT makes use of the theory of denoting concepts to solve the problem concerning such phrases as ‘the difference between *A* and *B*.’ Plainly, once the theory is rejected, he can no longer solve the problem in this way. In OD, he simply assumes that factual complexes subsist, claiming that “‘the difference between *A* and *B*’ has a denotation when *A* and *B* differ, but not otherwise’ (OD, p.490). In his view, the theory of descriptions solves the problem with such expressions as ‘the difference between *A* and *B*.’ It can indeed be applied to these expressions as long as we have such predicates as ‘*x* is a relation *r* particularised to *a* and *b*.’ Yet, this solution has a problem. Given Russell’s subsequent uses of the notion of incomplete symbol, it implies that factual complexes are what he (ambiguously) calls ‘logical fictions,’ unless they

¹⁵ Later in *The Analysis of Mind*, Russell returns to the asymmetric theory and makes the same point (AoM, p.166).

are given another kind of expressions that are not incomplete symbols. If 'the King of France' does not stand for any entity, neither should 'the difference between A and B.' Therefore, if propositions and factual complexes are both to be retained as entities and if declarative sentences are reserved for propositions, Russell needs a different sort of expressions with which he can designate factual complexes. In OD, he does not seem to envisage this problem, but he might have realised it before ONT, giving up the two-fold theory.¹⁶ Both of those two accounts seem to me tenable, and they are not mutually incompatible.

Russell's implicit rejection of the two-fold theory in ONT has an important consequence: once the theory is rejected, his preference to factual complexes becomes equivalent to his hesitancy over false objectives. In MT and NT, as we have seen, Russell repeatedly states that if, and only if, a judgment is true, there should subsist a complex entity. Once the possibility of retaining both propositions and factual complexes is rejected, it becomes possible to express the same thought thus: there should *not* be an 'objective falsehood,' namely an entity which subsists in spite of the falsity of the corresponding judgment. In other words, once the two-fold theory is excluded, the difference between the two conceptions of *fact* is tantamount to whether there are such things as false objectives.¹⁷ His implicit rejection of the two-fold theory and his unsureness of the asymmetric theory lead him to remark as follows:

We found that two theories seem tenable, one of which regards truth as the quality of beliefs which are beliefs in facts, which are the only non-mental complexes, while the other regards truth and falsehood as both capable of belonging to non-mental complexes, which we called *propositions*, of which there are two kinds, facts, which are true, and *fictions*, which are false.

(ONT, p.454)

According to the asymmetric theory, a factual complex is an entity that subsists only when the judgment is true, while the dyadic theory of judgment urges that factual complexes are simply propositions actually possessing the property of being true. Without choosing between these two theories he closes the paper.

6.3.3 Emergence of the multiple-relation theory of judgment

The chief aim of this subsection is to illustrate how the story which I have been attributing to Russell fits his remarks in ONTF. I will start off by offering a possible account of how he

¹⁶ But once some genuine expressions (those other than incomplete symbols) are allocated to factual complexes, the original problem reappears as to why some of those expressions designate factual complexes and the others (of precisely the same form) do not. When Russell and Whitehead introduce the notation '*a-in-the-relation-R-to-b*' to speak of facts in *PM*, they seem to neglect the problem (*PM*, p.43).

¹⁷ Admittedly, Russell in MT remarks that 'the explanation of falsity presents grave difficulties,' though he envisages the two-fold theory (MT, p.461).

came up with the multiple-relation theory of judgment.

As we have seen, it is implausible to identity the asymmetric theory, which is presented in MT, NT and ONT, with the multiple-relation theory of judgment. So, when did Russell come up with the latter theory? It is, as far as I know, in 'Fundamentals' written in January 1907 that he comes to envisage a rudimentary version of it. In his attempt to 'prove that there are no other functions' than truth-functions, he naturally considers such sentences as 'I believe p ' (CP5, p.541). He thus comes to view judgments from the third-person's point of view. He goes on to consider 'the view that there is no proposition except when the proposition is true' (*ibid.*). But he is soon led to the following idea:

A proposition seems to involve union of plural and complex: their union seems to be what constitutes *truth*. A belief has several objects; when correct, it has, in a secondary sense, the complex as its object. *Belief* means a certain kind of relation to several objects.
(CP5, p.542)

The view considered here is arguably the multiple-relation theory of judgment, for he now thinks of a judgment or a belief as a relation between entities, not between presentations of them. It thus seems as though he invents the multiple-relation theory of judgment through his attempts to dispense with functions other than extensional ones.

Why then does Russell come to endorse the multiple-relation theory of judgment? As we shall discuss in the next chapter, the theory is almost destined to fail, but he still seems to have had some reasons to accept it. First, the theory retains the two advantages which he in ONT thinks the asymmetric theory has over the dyadic-relation theory. The multiple-relation theory obviously enables us to *define* truth and falsehood. In addition, it can explain why perception is infallible: if perception is to hold between the judging mind and a factual complex, what is perceived certainly holds in reality. Second, the multiple-relation theory overcomes the most serious problem with the asymmetric theory. According to the former theory, a judgment does not consist of ideas but of entities including the judging mind. Hence, the theory enables him to stick to his firm intuition that a representation of a relation is not itself a relation. Third, the multiple-relation theory can be seen as a solution to the two problems which we discussed in the last subsection. If he gave up the two-fold theory because of its commitment to the two kinds of complex objects, propositions and factual complexes, then he may have adopted the multiple-relation theory so that he could distinguish between factual complexes and propositions without invoking two distinct modes of combination. The multiple-relation theory also solves the problem with such expressions as 'the difference between A and B.' If that-clauses are expressions for propositions, we can employ declarative sentences as genuine expressions for factual complexes, thereby avoiding the use of definite

descriptions to speak of factual complexes.¹⁸ In these respects, the theory is advantageous enough for him to adopt in his attempt to accommodate the notion of factual complex within his ontology.

Let us now take a fresh look at ONTF, where Russell explicitly rejects the dyadic-relation theory of judgment in favour the multiple-relation theory. He remarks there:

We feel that when we judge truly some entity “corresponding” in some way to our judgment is to be found outside our judgment, while when we judge falsely there is no such “corresponding” entity. It is true that we cannot take as this entity simply the grammatical subject of our judgment: if we judge, e.g. “Homer did not exist,” it is obvious that Homer is not the entity which is to be found if our judgment is true, but not if it is false. Nevertheless it is difficult to abandon the view that, in some way, the truth or falsehood of a judgment depends upon the presence or absence of a “corresponding” entity of some sort.

(ONTF, p.176)

Given what we have seen in the last two sections, it is not surprising at all to see Russell remark ‘We feel that when we judge truly some entity “corresponding” in some way to our judgment is to be found outside our judgment, while when we judge falsely there is no such “corresponding” entity.’ He also observes, here again, that factual complexes cannot be identified with propositions, since ‘[t]o say that there ever was such a thing as “Charles I’s death in his bed” is merely another way of saying that Charles I died in his bed.’ In other words, unless factual complexes are somehow distinguished from propositions, the dyadic-relation theory of judgment forces us ‘either to admit objective falsehoods, or to admit that when we judge falsely there is nothing that we are judging’ (ONTF, p.177). He proceeds to raise the three objections to the dyadic-relation theory which we saw at the beginning of this chapter. His first point is that that-clauses are intuitively incomplete. This is a more or less expedient point but can still be understood as connected with his struggle to assign two distinct kinds of expressions to propositions and factual complexes respectively. The second objection is that objective falsehoods are ‘incredible.’ This is, I take, a straightforward expression of his preference of factual complexes, which do not subsist in the case of false judgments. He does not have any concrete reason for the notion of factual complex; he merely *wants* to accept it. As for the last objection, it should also come as no surprise that he counts as an advantage of the multiple-relation theory of judgment its ability to define truth and falsehood. For he has already abandoned his argument against any attempt to define truth

¹⁸ This does not, however, mean that the multiple-relation theory removes all the difficulties concerning expressions for facts (cf. footnote 16).

and falsehood, and in particular, against the correspondence theory of truth, and hence, he has no reason to think that truth and falsehood cannot be defined.

6.4. Concluding remarks

Given what we have seen, I submit that Russell came to abandon the ontology of propositions in favour of the multiple-relation theory of judgment, through his increasing attraction to a realistic notion of factual complex. Discussing Meinong's existential account of truth and falsehood, Russell in 1904 came to wonder whether it would be possible to accommodate factual complexes within his ontology of propositions.¹⁹ The point is that Russell did not have any convincing argument for factual complexes and he simply *felt* that there should be such things as factual complexes. But he was also aware that he could not simply replace propositions with factual complexes while endorsing the dyadic-relation theory of judgment: if he had done so, the impossibility argument would have forced him to conclude that there were no false judgments. This explains why Russell considers the asymmetric theory and the two-fold theory in NT. But he repudiated the latter by the time he wrote ONT, possibly because he found it undesirable or maybe impossible to give an adequate account of how factual complexes differ from propositions, and also because the theory of descriptions urged him to find expressions for factual complexes other than definite descriptions. Yet, he could not be satisfied with the asymmetric theory, for it would commit him to the claim that a symbol of a relation is also a relation, which he took to be axiomatically false. At some point between January 1907 and 1909, he eventually invented the multiple-relation theory of judgment, which could indeed avoid the above two problems with the two-fold theory as well as the difficulty with which the asymmetric theory was confronted. It would seem that these advantages together with his preference to factual complexes led him to adopt the multiple-relation theory, thereby abandoning propositions. The theory itself does not, as Proops points out, commit Russell to the eliminativism of propositions. But, given the preference to factual complexes, he did adopt the eliminativism when he came to think that propositions and factual complexes could not both be retained. Without the preference, I submit, he presumably would not have considered the various theories of truth including the multiple-relation theory.

Russell expresses his preference to factual complexes in later writings as well. In *TK*, for example, he remarks: 'There is a strong natural conviction that when a judgment is false there is not, in the world of objects, something, pointed to by the judgment, which there is when

¹⁹ Hylton, though in a different context, suggests that it is 'a reasonable speculation that it was his thinking through these issues [*i.e.* problems concerning his ontological views presented in *PoM*] in connection with Meinong which led to a shift in his own ontological views' (Hylton, 2003, p.238).

the judgment is true' (TK, p.153). In *PLA*, he claims that no one with 'a vivid sense of reality' will believe in false objectives (*PLA*, p.55). Our story fits these remarks well.

The question is why Russell thinks it more *realistic* to endorse the existential account of truth and falsehood. The ontology of propositions is, after all, quite a *realist* alternative to the idealists' theory of judgment. One possible account that can be given from our point of view is that the notion of proposition shares an important feature with that of *judgment* which we discussed in Chapter 1. A Bradleian *judgment* is such that regardless of its truth/falsity, its content is a part of reality. Similarly, a Russellian proposition subsists whether it is true or false, and hence, even a false judgment, namely a judgment whose object is a false proposition, has a corresponding part of reality. This feature shared by the notion of judgment and that of proposition may have led Russell to regard the latter notion as less realistic than it had first appeared to him. Later in *OKEW*, he indeed finds such a commonality between his old dyadic-relation theory of judgments and the preceding, idealist metaphysics:

It is therefore necessary, in analysing a belief, to look for some other logical form than a two-term relation. Failure to realise this necessity has, in my opinion, vitiated almost everything that has hitherto been written on the theory of knowledge, making the problem of error insoluble and the difference between belief and perception inexplicable.

(*OKEW*, p.58)

Russell thinks that not only Bradley's theory of judgment but also the dyadic-relation theory of judgment faced 'the problem of error' because of the mistake concerning the logical forms of judgments. It would thus seem that the abandonment of the ontology of propositions was in his view yet another significant step away from the idealist conception of logic. This explains why he abandoned it just because of the feeling that there should not be false objectives. After all, he did have a good reason to feel so.

7 Towards the Linguistic Conception of Logic

7.1. Introduction

In closing the present work, I shall undertake two tasks in this chapter. First, I will introduce the account of logic which Russell developed during the period from 1912 to 1919. He abandoned the ontology of propositions by 1910 because of his ‘vivid sense of reality’—his preference for factual complexes over propositions. But the ontology of propositions was originally intended to be his theory of logic, the philosophical foundation of the calculus of logic. What was his theory of logic once he abandoned the ontology of propositions? By answering this question I will illustrate how the subject matter of logic in his view shifted from propositions to factual complexes or *forms of facts*. The rest of this chapter will be devoted to the linguistic conception of logic that underlay the development of mathematical logic after *PM*. Those who endorsed this conception of logic regarded sentences (of a formal system) as the subject matter of logic, rather than factual complexes or any other metaphysical objects. It will be seen that Russell could not wholeheartedly take this final step towards the linguistic conception of logic.

7.2. The notion of logical form of fact

In this section I will examine Russell’s account of logic as concerned with *forms of facts*, indicating various problems with it and his responses to them.

Russell completed his ‘shilling shocker,’ *The Problems of Philosophy (PP)* in 1912. In this book, he makes some interesting points about logic. First, he holds that the logical principles are independent of us, and hence, it is misleading to call some of them ‘Laws of Thought.’ He thus seems to maintain a realistic attitude towards logic. Second, he includes the inductive principle among logical principles (*PP*, p.114). However, he in *PP* does not develop any specific account of logic nor explain what ‘logical principles’ are. It is only in subsequent works that he seeks an account of logic which does not depend on the ontology of propositions.

An attempt to develop such an account is made in a short manuscript called ‘What is logic?’, where he defines logic to be ‘the study of forms of complexes.’ He thinks that in order to

keep logic separated from psychology, logic must deal with complex entities, though he can no longer identify them with propositions:

Logic is not concerned with *judgments*, which are psychological. What are *propositions*? May be judgments, forms of words, or Meinong's Objectives. Difficulty for supposing there are objective falsehoods compels us to suppose that what *can* be *false* must be judgments or forms of words. Logic not concerned with forms of words. Hence logic not concerned with propositions.

(CP6, p.55)

Russell thus thinks that '[i]n order not to have to give being to non-entities, [we] must make logic deal with complexes' (CP6, p.55). By 'complexes' he seems to mean various entities composed of other entities, though he no longer assumes that every judgment has such a complex entity as its object. As he now adopts the multiple-relation theory of judgment, he maintains that if such a complex entity is in reality, the corresponding judgment is true. Hence, what he calls 'complexes' here are factual complexes in our terminology (cf. 137). He thus comes to understand logic, or more precisely the theory of logic, as concerned with *forms of facts*, where *facts* are understood as what we call factual complexes. It is also worth noting that he considers and quickly rejects the idea that logic is concerned with 'forms of words.'

In this manuscript, Russell also finds some difficulties in explaining what *forms of facts* are. He suggests that a form of a complex is not a constituent of the complex on pain of an endless regress analogous to Bradley's. This means that he no longer accepts the distinction between two kinds of occurrences of relations. He then tries the idea that a form is obtained through substitution of constituents of factual complexes, but he (or more probably Wittgenstein) finds some objections to this idea. The one which Russell enlarges on in the manuscript is the following:

It is difficult to see what is meant by substitution in a complex if the result fails to be a complex. You can substitute in the symbol for a complex, but not in the complex. Substitution in a complex can only have a definite meaning when the result is a complex, not when it isn't.

(CP6, p.56)

The point is that substitution of constituents of factual complexes cannot fail, if it can be done at all. But it is not entirely clear what precisely is problematic about this observation. One possible problem is that the notion of substitution is not applicable to constituents of facts as there are no cases where the 'substitution' fails. But this is not a devastating problem, since we can still *misapply*, so to speak, the notion of substitution to define a form of a fact. A more serious one is that whether we can extract the form of a factual complex in the proposed way

is entirely contingent upon whether there are other factual complexes of the same form. For instance, if the fact that Socrates is a philosopher were the sole factual complex of the form Fa , then we would not be able to thus speak of the form. In such a case, we could not execute any substitution on the complex and hence we would fail to characterise the form in question. It should be observed that propositions and sentences are not subject to those problems of substitution, since whether a proposition or a sentence of a certain form is available does not depend on whether there is a factual complex of the corresponding form. At any rate, Russell abandons the explanation of forms through substitution, and instead considers the view that the notion of form is primitive. 'It won't do,' he soon remarks, however, 'to take "being of the same form" as primitive, because that rules out non-existent complexes' (CP6, p.56). He does not explicitly tell us why non-existent complexes would be excluded by the view in question; yet, it is obvious that forms of such complexes are indispensable. We need such a formula as ' $x \neq x$ ' to have a definite meaning in order to define the empty class, whereas there should be no factual complexes of that form. Given these problems, it is not surprising that he remarks in a letter to Lady Ottoline Morrell as follows:

I can't get on with "what is logic?," the subject is hopelessly difficult, and for the present I am stuck. I feel very much inclined to leave it to Wittgenstein.

(CP6, p.54)

Russell thus gives up the manuscript, though he keeps seeking a tenable formulation of the view of logic as concerned with *forms of facts* in subsequent works, to which we are turning.

The chief aim of *The Theory of Knowledge* (TK) is to account for various epistemological notions. But it also illustrates how Russell develops the idea that the philosophical part of logic is the study of *forms of facts*. He still maintains that forms are not constituents of complexes on the ground that otherwise 'we find ourselves embarked on an endless regress' (TK, p.98). Although he does not mention the problems discussed in the previous manuscript, he suggests a range of objections to his own correspondence theory of truth and falsehood, which are also directed against the account of logic as concerned with *forms of facts*. What he takes to be the 'most serious' objection to his view of truth is that *propositions*, complex entities subsisting independently of our cognition, are essential to logic (*ibid.*, p.155). He does not expand on the objection but he seems to think thus: if factual complexes are considered to be complex entities whose internal structures are perfectly mirrored by sentences expressing them, no factual complexes can correspond to sentences containing logical constants. For instance, as Russell himself already observed in 1904, it is impossible to think that ' $\neg p$ ' expresses a factual complex rather than a proposition; for if it does, then it is natural, if not necessary, that p should also be a factual complex, since p is a constituent of the factual complex $\neg p$ (cf. p.138). Similar problems arise with *molecular facts*, namely, those factual

complexes which are expressed by such sentences as ' $p \vee q$,' ' $p \rightarrow q$ ' and ' $p \& q$ ' (TK, p.154). If there is such a factual complex as $p \vee \neg p$ and it is composed of smaller factual complexes, then we would have to accept both p and $\neg p$ as factual complexes. Hence, the notion of factual complex cannot replace that of proposition entirely and the former can be at most applied to what atomic sentences are supposed to express. These problems would all vanish if Russell were to go back to the old ontology of propositions or instead proceeds to consider forms of sentences as the subject matter of the theory of logic. A true molecular or negated proposition can be assumed to contain false propositions as its constituents, as does a true molecular or negated sentence. Thus, those problems seem to show that if one wants to understand logic as concerned with forms of complex entities of a certain kind, those complexes must be propositions rather than factual complexes. But Russell in *TK* chooses not to tackle any of these problems:

I do not profess to be able to answer all the arguments in favour of "propositions" in this sense. I can only say that, to me personally, no such entities are visible, and the admission of such entities—which must be capable of falsehood as well as truth—runs counter to the rejection of unrealities, fictions, and mere possibilities which seems to me, on general grounds, necessary and vital to all sound philosophy. Until, then, the arguments in favour of non-mental "propositions" are presented in some more unanswerable form than any now known to me, I shall continue to reject them, and to believe that the apparent reasons in their favour are fallacious, even if I cannot always detect the fallacy.

(TK, p.155)

In *TK*, Russell thus develops the account of logic as concerned with *forms of facts* only to find more problems with it.

Let us turn to *Our Knowledge of the External World as a Field for Scientific Method in Philosophy* (OKEW). In this book, Russell maintains that forms are 'the proper object of philosophical logic,' where philosophical logic is considered to be 'the beginnings' of mathematical logic (OKEW, pp.41, 43). He also remarks:

Logic, we may say, consists of two parts. The first part investigates what propositions are and what forms they may have; this part enumerates the different kinds of atomic propositions, of molecular propositions, of general propositions, and so on. The second part consists of certain supremely general propositions, which assert the truth of all propositions of certain forms. This second part merges into pure mathematics, whose propositions all turn out, on analysis, to be such general formal truths.

(OKEW, p.57)

He thus holds that logic has two branches, one corresponding to what I call the theory of logic and the other to the calculus of logic. In his view, the former branch is an investigation into forms of 'propositions':

In every proposition and in every inference there is, besides the particular subject-matter concerned, a certain form, a way in which the constituents of the proposition or inference are put together. [...] Take (say) the series of propositions, "Socrates drank the hemlock," "Coleridge drank the hemlock," "Coleridge drank opium," "Coleridge ate opium." The form remains unchanged throughout this series, but all the constituents are altered. Thus form is not another constituent, but is the way the constituents are put together.

(OKEW, pp.42-3)

He thus speaks of forms of 'propositions.' But, importantly, what he means by 'proposition' is now a sentence: 'A form of words which must be either true or false I shall call a proposition' (OKEW, p.52). One might think that he sloppily employs, as he does elsewhere, the phrase 'a form of words' to speak of a complex entity, returning to the ontology of propositions. But given his firm rejection of this ontology in *TK* and his endorsement of the multiple-relation theory of judgment in *OKEW*, it rather seems as though he now seeks to characterise the subject matter of his theory of logic in terms of sentences. This move is, it seems, related to the fact that he now minimises the role of the theory of logic in the study of inference: it 'merely enumerates forms' of sentences now (*ibid.*, p.58). He might have found that sentences are far less metaphysically burdened than propositions and than factual complexes, wanting much less discussion on what they themselves are.

Yet, Russell in *OKEW* also seems to subscribe to the account of the theory of logic as concerned with *forms of facts*. He understands philosophical logic as 'the most difficult, and philosophically the more important' part of logic. Its importance is, he explains, illustrated by the rejection of the dyadic-relation theory of judgment. According to him, the theory 'leads to absolutely insoluble difficulties in the case of error'; yet it has been endorsed by philosophers including his former self because of 'poverty in the logical inventory' of forms (*ibid.*). He seems to think that the dyadic-relation theory of judgment has unduly obtained its popularity because multiple relations other than dyadic ones have been neglected. He presupposes here that forms which he collects in the logical inventory are not those of sentences but those of propositions or those of facts. For the dyadic-relation theory of judgment is about a form of our judgmental act, and hence, about a form of an event that takes place in reality. In his framework, such an event should be understood either as a proposition or as a factual complex. He thus still tacitly invokes the notion of *form of fact*.

It seems that Russell attempts to develop this ambivalent attitude into a coherent picture in

The Philosophy of Logical Atomism (PLA). In this work he expressly holds that logic is concerned with *forms of facts*. In a frequently-cited passage, he remarks as follows:

I think one might describe philosophical logic, the philosophical portion of logic which is the portion that I am concerned with in these lectures since Christmas (1917), as an inventory, or if you like a more humble word, a “zoo” containing all the different forms that facts may have. I should prefer to say “forms of facts” rather than “forms of propositions.” To apply that to the case of molecular propositions which I dealt with last time, if one were pursuing this analysis of the forms of facts, it would be belief in a molecular proposition that one would deal with rather than the molecular proposition itself. In accordance with the sort of realistic bias that I should put into all study of metaphysics, I should always wish to be engaged in the investigation of some actual fact or set of facts, and it seems to me that that is so in logic just as much as it is in zoology. In logic you are concerned with the forms of facts, with getting hold of the different sorts of facts, different logical sorts of facts, that there are in the world.

(*PLA*, pp.47-8)

Russell thus holds that the task of the theory of logic is to collect logical *forms of facts*. To be precise, he no longer holds that what he calls ‘facts’ are complex entities in *PLA*. This is presumably related to the problem with factual complexes corresponding negated sentences. He instead holds, though speculatively, that what correspond to true negated sentences are ‘negative facts,’ suggesting that those facts do not contain any smaller facts as their parts (*PLA*, p.41). His remarks in *PLA* also suggest that he has already endorsed this idea in *OKEW*, though he does not explicitly expand on it in *OKEW*.

In *PLA*, Russell appeals to truth-tables to avoid the difficulties with molecular facts. Considering sentences, which he now calls ‘propositions,’ involving disjunction, conjunction and negation, he remarks: ‘I do not see any reason to suppose that there is a complexity in the facts corresponding to these molecular propositions’ (*ibid.*). In his view ‘the truth or falsehood of this proposition “*p* or *q*” depends upon two facts’ in the way that the truth-table of ‘or’ dictates (*ibid.*, p.39; see also p.72).

Russell proceeds to combine the realist view of logic with a nominalist understanding of propositional functions. He abandons the account of a propositional function as an ambiguity. ‘A propositional function is,’ he claims, ‘simply any expression containing an undetermined constituent, or several undetermined constituents, and becoming a proposition as soon as the undetermined constituents are determined’ (*PLA*, p.64; unitalicised). He also remarks: ‘A propositional function is nothing, but, like most of the things one wants to talk about in logic, it does not lose its importance through that fact’ (*ibid.*). The second sentence seems to show that he somehow retains propositional functions even though they are in reality ‘nothing,’ which

in turn suggests that propositional functions are considered to be 'logical constructions.' But, for my part, it is plausible that Russell does not treat propositional functions as logical constructions but rather as mere open sentences here. For he does not even attempt to offer any systematic way of removing a function-symbol from a sentence in which it occurs. It seems to me that although he comes to think of his own project as that of logical construction, he does not include propositional functions among the constructed objects. But it is hasty to take the above remarks to be evidence for the nominalist interpretation of propositional functions in *PM*, which we discussed in Chapter 5. As we have seen, he does make various attempts to 'make logic deal with complexes' in his works subsequent to *PM*.

Propositional functions are thus considered as mere linguistic items. But Russell still thinks that there are some objects that correspond to them—general facts. This explains why he remarks: 'The only thing really that you can do with a propositional function is to assert either that it is always true, or that it is sometimes true, or that it is never true' (*PLA*, p.64). As long as we employ a propositional function to assert either its universality, its satisfiability, or its unsatisfiability, there will always be a corresponding general fact. He thus speaks of factual complexes expressed by such sentences and existential statements as follows:

Those facts have got to come into the inventory of the world, and in that way propositional functions come in as involved in the study of general facts.

(*PLA*, p.71)

He seems to think that every sentence in which a propositional function is properly employed has a corresponding general fact. In this sense, he can still hold that 'propositions and propositional functions that contain only variables and nothing else at all' cover 'the whole of logic' even though he regards each proposition or propositional function as 'just a symbol' (*ibid.*, p.10). This is how he attempts to accommodate the realist view of logic as concerned with forms of facts with the nominalist account of propositional functions.

But there are some problems with this view of logic. First, it is not clear, as Russell himself wonders, whether one can reject molecular facts while retaining general propositions. For general sentences in ordinary language often involve implication (*PLA*, pp.72-3). For instance, 'All men are mortal' will be naturally translated into "' x is a man" implies " x is a mortal" whatever x may be.' Second, and more significantly, the rejection of molecular facts means that his theory of logic can no longer offer a foundation for the whole calculus of logic. He maintains that the task of the theory of logic is collect forms of atomic facts, those of general facts and those of negative facts. But there are no such things as forms of molecular facts, and hence, his theory of logic leaves the meanings of disjunction, conjunction and implication unexplained. The implication of his solution to the problems with molecular facts is far greater than he apparently envisages.

It would thus seem that, in his works written after *PM*, Russell attempts to maintain a realist standpoint on logic by regarding *forms of facts* as the subject matter of the theory of logic. There are various problems with the notion of *form of fact*, and he tries to resolve them mainly by appealing to some linguistic notions such as ‘form of a sentence.’ Yet, he does not thereby abandon the realist standpoint and rather endeavours to maintain it by preserving correspondence between sentences and facts.

7.3. The subsequent development of mathematical logic

In this section, I will contrast the linguistic approach to formal languages that underpins the development of mathematical logic from 1910 onwards with Russell’s, and attempt to explain why he could not wholeheartedly accept the former despite of its advantages. I will refer to some of his works after 1918, so that I can sketch how he responded to the linguistic approach.

From around 1910 onwards, logicians began to prove many theorems *about* formal systems. These results are obtained through what I shall call the linguistic approach to formal language. The approach is characterised by the notion that a formal system is a subject of a purely mathematical investigation, not a medium through which we can access the *logical structure of the universe*, whatever it may be. For example, Post, in his paper presenting a proof of the completeness of propositional logic, explains his standpoint as follows:

Finally a word must be said about the viewpoint that is adopted in this paper and the method that is used. We have consistently regarded the system of ‘Principia’ and the generalizations thereof as purely formal developments, and so have used whatever instruments of logic or mathematics we found useful for a study of these developments.
(Post, 1921, pp.164-5)

Post refers to C. I. Lewis’s *A Survey of Symbolic Logic* for ‘a general statement’ of his viewpoint. In that book, Lewis contrasts Russell’s ‘logico-metaphysical’ approach to mathematics with ‘the external view of mathematics’ or ‘mathematics without meaning’ (Lewis, 1918, Ch.VI, III). In making the contrast and defending the latter approach, Lewis remarks as follows:

[...] if Mr. Russell is right, the mathematician has given over the metaphysics of space and of the infinite only to be plunged into the metaphysics of classes and of functions.
(Lewis, 1918, p.356)

This is, I think, a true description of Russell’s attitude towards mathematics save that he does not commit himself to the subsistence of classes owing to the no-classes theory. On the other hand, Lewis ‘regards mathematics as dealing, not with certain denoted things—

numbers, triangles, etc.—nor with certain symbolized “concepts” or “meanings,” but solely with recognizable marks, and dealing with them in such wise that it is wholly independent of any question as to what the marks represent’ (*ibid.*, p.355). Thus, the linguistic approach treats a formal language as its subject matter and examines it using mathematical methods such as induction. Although Lewis does not name those who endorse the ‘external’ approach, Hilbert is presumably the most influential figure among them. In a paper based on lectures which he delivered in 1917, Hilbert remarks as follows:

All such questions of principle [as consistency, decidability, etc] [...] seem to me to form an important new field of research which remains to be developed. To conquer this field we must, I am persuaded, make the concept of specifically mathematical proof itself into an object of investigation, just as the astronomer considers the movement of his position, the physicist studies the theory of his apparatus, and the philosopher criticizes reason itself.

(Hilbert, 1917, p.1115 (54))

Thus, logicians, inspired by Lewis and Hilbert amongst others, began to conduct research on formal languages, applying given mathematical techniques to them. Needless to say, a great many important mathematical results were produced including Gödel’s completeness theorem and his incompleteness theorems. And, to repeat, behind those results was the linguistic approach. Since these results are about formal systems, they require that one should view a formal system itself as a subject matter, not as a mere medium through which one can speak of the logico-metaphysical structure of reality.

The linguistic approach to formal languages thus became common among logicians from 1910 onwards. In the meantime, as we saw in the preceding section, Russell was after a coherent account of what his formal language expresses. He was certainly aware of many meta-systematic theorems (though he may have failed to fully grasp them) and also of what underpinned them, that is, the linguistic approach. But he could never wholeheartedly accept it. For instance, in the second edition of *PoM* (1937), he summarises the difference between his view in 1903 and his then view as follows:

Broadly, the result is an outlook which is less Platonic, or less realist in the mediaeval sense of the word. How far it is possible to go in the direction of nominalism remains, to my mind, an unsolved question, but one which, whether completely soluble or not, can only be adequately investigated by means of mathematical logic.

(*PoM*, p.xiv)

He thinks that his view on logic has become more linguistic, because he maintains that logical constants ‘must be treated as part of the language, not as part of what the language speaks

about' (*PoM*, p.xi). Yet, it is plain that he is not perfectly convinced of the linguistic approach in logic in the above passage. There are at least two reasons why not.

One is a sort of foundationalism. If a formal system is designed to demonstrate the truth of, say, mathematical induction, it is simply beside the mark to prove theorems *about* the system using mathematical induction. His logicist programme originated in his strongly foundationalist attempt 'to arrive at a perfected mathematics which should leave no room for doubts, and bit by bit to extend the sphere of certainty from mathematics to other sciences' (*MPD*, p.36). What he desired most was 'to find some reason for supposing mathematics true' (*Autobiography*, p.57). It is true that he abandoned this strong attempt when he gave up basing mathematics upon self-evident truths. The discovery of the set-theoretic contradiction and the adoption of the axiom of reducibility among other things led him to view his programme rather as an attempt to discover premisses of mathematics which may not be self-evident at all.¹ Yet, importantly, he still counted the axioms of his formal system as 'premisses' of mathematics. In *IMP*, he remarks that although 'the ultimate logical concepts and propositions [...] are remote and unfamiliar as compared with the natural numbers,' '[i]n a synthetic, deductive treatment these fundamentals come first, and the natural numbers are only reached after a long journey' (*IMP*, p.195). It seems as though he assumes that his 'syntactic, deductive treatment' captures the ontological—as opposed to epistemological—dependence of mathematical truths upon more fundamental ones. He admits that we humans recognise the truth of the former much more easily than that of the latter. But the latter are in his view ontologically more fundamental than the former, though they may not be the *most* fundamental. As long as he employs a formal system so as to indicate the ontological dependence, this sort of foundationalism makes it pointless for him to assume the truth of a certain mathematical sentence in order to prove various theorems about the formal system. For what he aims at is to indicate how the truth of the sentence is derived from more basic ones.

Russell's foundationalism should not be confused with the alleged impossibility of a meta-linguistic point of view on a formal system within his framework. Many commentators have regarded the impossibility as a characteristic of his understanding of logic.² But, as Landini and Proops among others indicate, Russell does not think it impossible to take a meta-linguistic viewpoint on a formal system (Landini 1998b, p.32; Proops 2007, pp.18-24). Those who attribute to Russell the impossibility of a meta-systematic point of view might refer to the following passage in the second edition of *PM*:

¹ This attitude towards the axioms of his formal system is expressed especially in 'Insolubilia' (p.194), 'The Regressive Method of Discovering the Premises of Mathematics' and *PM* (p.v). See also Irvine (1989) and Hager (1994).

² See, for example, van Heijenoort (1967b), Goldfarb (1979, p.353), Hylton (1990, p.203) and Hintikka (2010).

It is obvious that, in practice, an infinite conjunction or disjunction such as the above cannot be manipulated without assumptions *ad hoc*. We can work out results for any segment of the infinite conjunction or disjunction, and we can “see” that these results hold out. But we cannot prove this, because mathematical induction is not applicable. (PM, p.xxxiv)

Russell indeed claims that mathematical induction is ‘not applicable,’ not that it is pointless to apply it to his formal system itself. But it is plausible that he only means that *in order to achieve his goal* mathematical induction cannot be employed in proving the legitimacy of infinite conjunction and disjunction. The following passage is more frequently cited as a piece of evidence that Russell found the meta-linguistic viewpoint impossible:

Some indemonstrables there must be; and some propositions, such as the syllogism, must be of the number, since no demonstration is possible without them. But with conceding others, it may be doubted whether they are indemonstrable or merely undemonstrated; and it should be observed that the method of supposing an axiom false, and deducing the consequences of this assumption, which has been found admirable in such cases as the axiom of parallels, is here not universally applicable. For all our axioms are principles of deduction; and if they are true, the consequences which appear to follow from the employment of an opposite principle will not really follow, so that arguments from the supposition of the falsity of an axiom are here subject to special fallacies.

(PoM, §17)

Yet, as Proops points out, what Russell rejects here is merely an ‘non-demonstrative’ method of checking the derivability of an axiom concerning a mode of inference from other axioms (Proops, 2007, pp.21-4). In my view, Russell rejects the method not because of his commitment to the impossibility of stepping outside of a logical system but because of his foundationalism. He seems to think that if one seeks to establish the truth of, say, the Law of Excluded Middle (at the object-language level), one cannot use it in one’s argumentation for it (even at the meta-linguistic level). He is—and in fact, we all are—in the ‘logocentric predicament’ in this sense (cf. Sheffer, 1926, p.228). We cannot expect to convince someone doubting the Law of Excluded Middle of its legitimacy by presenting a formal system which has an object-language translation of the logical law as a theorem but which requires us to employ classical logic at the meta-linguistic level. But this does not entail that we cannot take a meta-systematic point of view, of course. It is when one aims to establish the legitimacy of a certain mode of reasoning that one cannot justifiably employ the very form of inference even at the meta-linguistic level. Thus, a point of difference between Russell and those who endorsed the linguistic approach

to formal languages does not lie in whether the meta-systematic point of view is deemed possible or not. It lies simply in their distinct aims. As Proops puts it, Russell's 'aim is to show that mathematics is a branch of logic, not to investigate the logical system in which that reduction is to be carried out'; yet 'the fact that Russell does not take up these questions is not, by itself, any reason to think that he found them unintelligible or in principle resistant to mathematical treatment' (Proops, 2007, p.24).

Another reason why Russell did not ever rest content with the linguistic approach can be found in his understanding of language. He seems to have presupposed arguably throughout his career that a formal language is in essence a medium through which we can speak of something beyond it. This presupposition, it seems, underlies the following passage in Introduction to the Second Edition of *PoM*:

The formalists have forgotten that numbers are needed, not only for doing sums, but for counting. Such propositions as "There were 12 Apostles" or "London has 6,000,000 inhabitants" cannot be interpreted in their system. For the symbol "0" may be taken to mean any finite integer, without thereby making any of Hilbert's axioms false; and thus every number-symbol becomes infinitely ambiguous. The formalists are like a watchmaker who is so absorbed in making his watches look pretty that he has forgotten their purpose of telling the time, and has therefore omitted to insert any works.

(*PoM*, p.vi)

He compares with the works of a watch what the formal system expresses. He thus seems to maintain that it is essential that any language should stand for objects. Later in *MPD*, he also remarks as follows:

I have never been able to feel any sympathy with those who treat language as an autonomous province. The essential thing about language is that it has meaning—i.e. that it is related to something other than itself, which is, in general, non-linguistic.

(*MPD*, p.14)

Though we cannot always take remarks in *MPD* at face value, this passage seems to serve as a rough description of his unchanging attitude towards language. According to him, 'those who treat language as an autonomous province' maintain that 'words should never be confronted with facts but should live in a pure, autonomous world where they are compared only with other words' (*MPD*, p.148). To be fair, these remarks in *MPD* are not directed at formal language but at language in general and are made in discussion of 'the linguistic aspects of epistemology' (*ibid.*, p.14). But there seems to be no reason to think that these remarks do not apply to formal languages in his view. It would thus seem that he maintained throughout his career that a formal language itself is not the subject matter of logic but a

medium through which we can examine it. It should also be noted that the view of formal language as a medium does not entail the impossibility of a meta-linguistic point of view on such a language.³ It only means that it is of little use to take such a point of view in logic.

Russell's contention that the essence of language is to express something may well come from his tacit assumption that symbols (in the primary sense of the word) are mental items. If he thus assumed, then, given his anti-psychologism, he could not take the linguistic approach to formal languages, because it would make mental items the subject matter of our logical investigation. As we saw in Section 2.5, Russell in *PoM* subscribes to the idea that what I call *internal symbols* are more fundamental than *external* ones (cf. p.46). The former are symbols which one employs within one's mind whereas the latter are to be found outside of the mind. One might think that we learn symbols through experience and in this sense external symbols are more fundamental than internal ones. Yet, Russell's solipsistic viewpoint makes internal ones more fundamental. For if one starts off, as he does, from a solipsistic setting, one can even doubt whether one has thus learnt internal symbols. One can employ them even when one doubts the existence of external objects, on which external symbols are written. His solipsistic setting becomes apparent in *PP*, where he remarks:

Other people are represented to me by certain sense-data, such as the sight of them or the sound of their voices, and if I had no reason to believe that there were physical objects independent of my sense-data, I should have no reason to believe that other people exist except as part of my dream.

(*PP*, p.33)

It seems to me that he can maintain, by the same token, that external symbols can only be given to the mind through sense-data. The following passage from *PLA* also suggests that he understands symbols as mental and essentially representative:

I am using it in a sense to include all language of every sort and kind, so that every word is a symbol, and every sentence, and so forth. When I speak of a symbol I simply mean something that "means" something else, and as to what I mean by "meaning" I am not prepared to tell you. I will in the course of time enumerate a strictly infinite number of different things that "meaning" may mean but I shall not consider that I have exhausted the discussion by doing that. I think that the notion of meaning is always more or less psychological, and that it is not possible to get a pure logical theory of meaning, nor therefore of symbolism. I think that it is of the very essence of the explanation of what you mean by a symbol to take account of such things as knowing,

³ Landini, among others, makes this point against the so-called universalist interpretation of Russell's conception of logic (Landini, 1998b, pp.30-7).

of cognitive relations, and probably also of association. At any rate I am pretty clear that the theory of symbolism and the use of symbolism is not a thing that can be explained in pure logic without taking account of the various cognitive relations that you may have to things.

(*PLA*, pp.11-2)

He thus maintains that any account of symbolism involves 'such things as knowing' and so on. In *My Mental Development* (1944; hereafter, *MMD*), he makes it clear that his starting point in his epistemological inquiry has always been a subjective point of view. He remarks:

Theory of Knowledge, with which I have been largely concerned, has a certain essential subjectivity; it asks "how do I know what I know?" and starts inevitably from personal experience. Its date are egocentric, and so are the earlier stages of its argumentation. I have not, so far, got beyond the earlier stages, and have therefore seemed more subjective in outlook than in fact I am.

(*MMD*, p.16)

If the theory of meaning is to be applicable in the solipsistic setting, his starting point of epistemology, the theory must first deal with internal symbols rather than external ones. Hence, it is plausible that he regards private experience as a starting point of the theory of meaning and hence takes internal symbols to be fundamental. This attitude towards language was possibly brought to him by British idealists. For on their account the theory of logic is an investigation into judgments, and we often, if not always, make use of internal symbols in making a judgment.

One may well wonder how to locate, then, the formal language which Russell and Whitehead develop in *PM*. Is it supposed to be someone's private language? It seems as though Russell simply assumes that the formal language can be employed by anyone as a system of internal symbols. It becomes one's 'logically perfect language,' if one adds one's own vocabulary, and it is entirely dependent upon each person which names and predicates are to be added (cf. *PP*, pp.84-90; *PLA*, p.25). He seems to think that a formal system is a system of internal symbols which we employ but is not itself the subject matter of our logical enquiry.

Thus, Russell could not accept the linguistic approach partly because of his view of language as a psychological device to speak of something beyond it. His old-fashioned approach was, however, recognised as the standard one for a while after the publication of *PM*. For instance, Lewis, in his book we mentioned above, calls Russell's approach 'the orthodox view' and the linguistic one 'the heterodox view' (Lewis, 1918, p.354f). The temporal orthodoxy of the former also explains why Post once found it better to explicate his own approach. But the linguistic approach quickly replaced Russell's logico-metaphysical one. In his famous

criticisms of *PM*, Gödel's remarks as follows:

What strikes one as surprising in this field [*i.e.* Mathematical Logic] is Russell's pronouncedly realistic attitude, which manifests itself in many passages of his writings.

(Gödel, 1944, p.127)

It thus seems that by 1944, the linguistic approach became so dominant among logicians that Russell's approach looked 'surprising' to them.

7.4. Concluding remarks

Russell abandoned the ontology of propositions, his original theory of logic, by the time when he completed his part of *PM*. He then sought an alternative philosophical account of what the formal system of the voluminous book expressed. His idea was to understand the theory of logic as an investigation into *forms of facts*. His attempts to develop this idea culminated in *PLA*, where he indicates some solutions to various problems with the idea, even though those solutions are not entirely feasible. In the meantime, some logicians began to prove theorems about formal systems themselves, taking what I call the linguistic approach to a formal language. Although the approach led to various important mathematical results, Russell could not accept the approach because he took for granted a sort of foundationalism and also because he regarded (formal) languages as mental tools by which he could speak of the logical structure of the universe. The formal system presented in *PM* certainly played an important role in the rise of the linguistic approach, as it offered a complete formal system of logic (though its syntax was not fully specified). But the authors of the book themselves did not take the approach, failing to keep up with the subsequent development of mathematical logic.

It is important for our purposes to observe that the linguistic approach to formal languages renders the theory of logic unnecessary. For if we define a formal system through recursion and examine it using mathematical methods, there is no need to wonder what the language expresses or what the language is composed of; we are only concerned with mathematically defined features of the formal language. It is possible to consider what a formal language is, but we do not have to do so at all to achieve the chief goal of our logical enquiry, that is, to prove theorems about the language. This is, I think, why the theory of logic or what we nowadays call philosophical logic is not considered to be theoretically prior to the calculus of logic. The calculus is independent of the theory, though this does not mean, of course, that the theory has lost its place in logic.

It would thus seem that the way in which philosophers and logicians understand logic changed greatly from 1900 to 1920. The various accounts of logic which Russell offered

during this period and afterwards illustrate a transition from the old, idealist conception of logic towards the new, linguistic one, though he could not wholeheartedly accept the latter.

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