Optimal Allocation with Ex-post Verification and Limited Penalties

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Several agents with privately known social values compete for a prize. The prize is allocated based on the claims of the agents, and the winner is subject to a limited penalty if he makes a false claim. If the number of agents is large, the optimal mechanism places all agents above a threshold onto a shortlist along with a fraction of agents below the threshold, and then allocates the prize to a random agent on the shortlist. When the number of agents is small, the optimal mechanism allocates the prize to the agent who makes the highest claim, but restricts the range of claims above and below.

A principal has an indivisible prize to give to one of several ex-ante identical agents. The principal's value from giving the prize to agent i is privately known by this agent. The principal asks the agents to report these values and allocates the prize based on the reports. Ex post, the principal learns the true value from allocating the prize and can penalize the winner by destroying a certain fraction of his surplus. The principal can commit to an allocation rule that determines how the prize is allocated as a function of the agents' reports and under what circumstances the prize recipient is penalized. Apart from the penalty, there are no utility transfers.

There are multiple environments that correspond to our model. For example, a development agency announces a grant competition among potential partners to deliver aid to a disaster area. Each partner organization privately knows the social value it will produce. Ex post, the agency can conduct a review of the competition winner and decide whether to debar this organization from future grant applications (or whether to try to recover some of the funds allocated to the organization).[1] For another example, a college administration has to allocate

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[1]Consider, for example, the U.S. Agency for International Development. A typical report to the Congress by the Office of the Inspector General of the Agency lists a number of organizations and

an academic scholarship or a slot in a program to one of the applicants. The students have private information about their abilities or their fit to the program. The college will be able to withdraw the remainder of the scholarship from the students with subpar performance. The last example is a firm that would like to fill a position with a fixed salary. Applicants have private information about their qualifications. The firm will eventually learn the qualification of the new hire and can choose to let him or her go.

In all these examples, the principal can punish the agent for lying about her private information by destroying a part of the prize. This penalty is limited because the agent enjoys a share of the payoff until the prize is taken away, or with some probability, the principal may fail to take the prize away because of legal or political reasons (e.g., a court might side with the worker), or imperfect monitoring. The agent has limited liability and cannot be punished beyond taking the prize away.

We characterize allocation rules that maximize the expected payoff of the principal. To understand the forces at play on the intuitive level, consider a naive rule that allocates the prize to the agent with the highest reported value. In the unique equilibrium, everyone reports the upper-bound value, and the rule de facto allocates the prize at random. This is so even if the lies are penalized ex post. An agent with a low value (values are continuously distributed) has only a slight chance of winning by truthfully reporting his value, since it is nearly certain that another agent has a higher value. Inflating the report to the upper-bound value substantially increases the probability of winning the prize, albeit at the cost of loosing a fraction of the surplus. The argument then unravels: once agents with low values inflate their reports, then agents with medium and, in turn, high values respond by inflating their reports as well.

The principal can do better than allocating the prize at random. Consider a *restricted-bid procedure* that allows the agents to submit reports within some interval between two thresholds and selects the agent with the highest report (ties are broken randomly). Ex post, the winner is penalized whenever his report is "inflated," i.e., when it is above the lower threshold and exceeds the true value. An agent's benefit from an inflated report is bounded by the increment in the probability of selection between submitting the upper threshold and the lower threshold reports. When this probability increment is small enough and does not compensate for the loss of the surplus caused by the penalty, reporting the value closest to the true value within the permitted interval is optimal. This allocation rule is superior to random allocation, as it only bunches types at the top, above the upper threshold, and at the bottom, below the lower threshold, while fully

individuals that are debarred for product substitution and inadequate performance. For instance, the Semiannual Report for the period from October 1, 2015 - March 31, 2016, states that "the implementing partner identified discrepancies in food baskets purchased for distribution in Syria and determined that the vendor fraudulently profited approximately $106,000 by manipulating the contents of more than 55,000 food baskets." The vendor was debarred. The report for the same period a year earlier describes a case of suspension of two US contractors who have built houses in Haiti using substandard materials that failed to meet safety standards.

separating types in the middle. We show that, for a small number of agents, the optimal rule has the described two-threshold structure.

The optimal allocation rule is different when the number of agents is large. It can be described as a *shortlisting procedure*. Agents report whether their values are above or below a single threshold. The former are shortlisted with certainty, while the latter are shortlisted with a probability of less than one. A winner is chosen randomly from the shortlist. If the shortlist is empty, then a winner is drawn at random from the full set. Ex post, the penalty is imposed if the winner has an above-threshold report and a below-threshold value. Note that there is no discontinuity between the restricted-bid and shortlisting procedures. As the number of agents, n, increases, the optimal thresholds of the restricted-bid procedure converge to a single threshold.

Of course, our model is a just a simplification intended to capture a relevant tradeoff in settings with ex-post verification and limited penalties. The incentive constraint bounds the ratio in probabilities of the selection of the highest and lowest types. If the low types are not promised to be selected with a sufficiently high probability, they will mimic the high types, so the principal may as well select an agent at random. The cap on the highest probability means bunching the types at the top, while the floor on the lowest probability means bunching the types at the bottom. Keeping the difference in these probabilities fixed, the principal faces the tradeoff between making the rule more competitive by selecting higher types with higher probability and reducing rents that have to be given to the low types.

In applications, bunching can take the form of categorization, quotas, or the use of irrelevant and ad hoc criteria to rule out applicants. A grant agency can sort applicants into, for example, three categories: "highly competitive," "competitive," and "non-competitive". After that, it can allot certain amounts of funding for each category and randomly allocate the appropriated funding within the categories.[2] An academic program can assign a quota for scholarships that are need-based and automatically enter every applicant who did not qualify for merit-based funding into a lottery for need-based scholarships. It can also invoke irrelevant or vague qualifying criteria such as seniority, prior allocation of scholarships, or some specific performance measure to disqualify applicants from obtaining the scholarship. As long as the application of these criteria is random and independent of merit from the perspective of the students, its effect on the incentives of the students will be equivalent to bunching.

Our analysis shows that adding agents beyond some number does not benefit the principal and that, for a large number of agents, the optimal allocation rule is a binary shortlisting procedure. There is an alternative implementation of the optimal rule for a large number of agents: The principal randomly excludes some

[2]Our model assumes a single indivisible good. This is for clarity of exposition. Extension to multiple goods is mechanical, as long as we maintain the assumption that each agent demands the same amount of good.

agents, and categorizes the remaining agents as above or below the bar. If there are agents above the bar, one of them is chosen at random. Otherwise, an agent is randomly chosen among all agents. We see similar mechanisms in practice. Job search forums are full of anecdotes of HR departments discarding every fifth application or arbitrarily dividing applications into two piles and throwing away an "unlucky" pile. If candidates apply over time, a company might keep the search open for a fixed period of time or until a certain number of candidates have applied. If the quality of the candidates does not correlate with their arrival time, the optimal rule for the company is to hire the first candidate above the bar and to hire at random from the pool of applicants if all candidates are below the bar and the search is closed.

In our model, ex-post verification coupled with limited penalty is the only incentive tool available to the principal. Ben-Porath, Dekel, and Lipman (2014) (henceforth, BDL) study a similar model. They differ in the verification technology of agents' information: verification is costly and can be done prior to the allocation decision. Thus, the principal faces a tradeoff between reducing the cost of verification and improving incentives for the agents to report their information truthfully. The optimal rule is a one-threshold mechanism. If all agents report values below the threshold, their values are not verified and the good is allocated to a "favored" agent. Otherwise, the highest report is verified. Thus, similar to the optimal rules in our paper, there is distortion and bunching at the bottom. The reason for this distortion is different: the expected value from allocating the good to the highest-value agent if all agents have low valuations does not justify paying the verification cost. In BDL, there is no distortion at the top because the agents who report high values will be verified and denied the good if they lie. The difference in the timing of verification between our models is not essential: if in our model, the principal could recover the entire good with certainty and there were verification costs, the model would become equivalent to BDL.

In our model, there are no transfers at the interim (allocation) stage and there are restricted penalties at the ex-post stage. Optimal contracts with transfers that can depend on ex-post information have been studied in, e.g., Mezzetti (2004), DeMarzo, Kremer and Skrzypacz (2005), Eraslan, Mylovanov and Yimaz (2014), Dang, Gorton and Holmström (2015), Deb and Mishra (2014), and Ekmekci, Kos and Vohra (2016). This literature is surveyed in Skrzypacz (2013).[3] Burguet, Ganuza and Hauk (2012) and Decarolis (2014) study allocation problems with transfers in which the principal has a lack of commitment and can renege on transfers ex post (e.g., because of bankruptcy). In these problems, similarly to our model, agents with low values are given rents to stop them from bidding too aggressively to win the contract.[4] For mechanism design with evidence at the interim stage see Green and Laffont (1986); Bull and Watson (2007); De-

[3]See also Glazer and Rubinstein (2004, 2006).

[4]Similar forces are at play in Mookherjee and Png (1989), who solve for the optimal penalty schedule for crimes when penalties are bounded.

neckere and Severinov (2008); Ben-Porath and Lipman (2012); Kartik and Tercieux (2012); Sher and Vohra (2015), and Koessler and Perez-Richet (2013). Finally, for the literature with costly state verification, monetary transfers, and one agent, see Townsend (1979), Gale and Hellwig (1985), Border and Sobel (1987), and Mookherjee and Png (1989).

There is a body of literature on mechanism design with partial transfers in which the agents' information is non-verifiable. In Chakravarty and Kaplan (2013) and Condorelli (2012), a benevolent principal would like to allocate an object to the agent with the highest valuation, and the agents signal their private types by exerting socially wasteful effort. Condorelli (2012) studies a general model with heterogeneous objects and agents and characterizes optimal allocation rules where a socially wasteful cost is a part of mechanism design. Chakravarty and Kaplan (2013) restrict their attention to homogeneous objects and agents, and consider environments in which a socially wasteful cost has two components: an exogenously given type and a component controlled by the principal. In particular, they demonstrate conditions under which, surprisingly, the uniform lottery is optimal.[5] Che, Gale and Kim (2013) consider the problem of efficient allocation of a resource to budget-constrained agents. They show that a random allocation with resale can outperform competitive market allocation. In an allocation problem in which the private and the social values of the agents' are private information, Condorelli (2013) characterizes the conditions under which the optimal mechanism is stochastic and does not employ payments. Bar and Gordon (2014) study an allocation problem with non-negative interim transfers (subsidies), in which the allocation might be inefficient because of incentives to save on the subsidies paid to the agents.

I. Model

A. *Preliminaries*

A principal allocates a single indivisible prize (e.g., a job, scholarship, or office space) to one of $n \geq 2$ agents. The principal's payoff from retaining the prize is normalized to 0, while her payoff from choosing an agent i is $x_i \in [a, b]$, where x_i is private information of agent i. We assume that $b > 0$ and we do not restrict a. In particular, a can be negative. The values of x_i's are i.i.d. random draws, with continuously differentiable c.d.f. F, whose density f is positive almost everywhere on $[a, b]$.

The value of the prize for every agent is $v(x_i) > 0$. Each agent i makes a statement $y_i \in [a, b]$ about his type x_i, and the principal allocates the prize to some agent, or to none of them, according to a specified rule. After an allocation has been made, the principal observes type x_i of the selected agent and, contingent

[5] See also McAfee and McMillan (1992), Hartline and Roughgarden (2008), and Yoon (2011) for environments without transfers and money burning. In addition, money burning is studied in Ambrus and Egorov (2017) in the context of a delegation model.

on this observation, can destroy a fraction $c \in (0,1)$ of the agent's payoff.[6] This assumption has multiple interpretations. For example, in the case a grant competition, the winner organization can be debarred from further grant applications after the post-implementation review. Alternatively, the grant agency can try to recover the funds in court and be successful with some probability. Finally, c can capture the expected penalty if the agency discovers the winner's true type with probability less than one.

Parameters a, b, c, and n, and functions F and v are common knowledge. In addition, we assume that $F^{n-1}(0) \leq 1-c$, so that the mass of negative agents is not too large.[7]

The principal has full commitment power and can choose any stochastic allocation rule conditional on the reports and any penalty rule conditional on the reports and the ex-post verified type of the selected agent. By the revelation principle, it is sufficient to consider allocation rules in which truthful reporting constitutes a Bayesian Nash equilibrium.

We assume that allocating the prize to agent i yields payoff x_i to the principal if the agent is not penalized and at most x_i if the agent is penalized. In other words, the penalty is never beneficial for the principal and therefore can only be used as an incentive tool.[8] The optimal penalty rule is thus trivial. Since type x_i of the selected agent is verifiable, it is optimal to penalize the agent whenever he lies, $y_i \neq x_i$, and not to penalize him otherwise.

An allocation rule p associates with every profile of statements $\bar{y} = (y_1, ..., y_n)$ a probability distribution $p(\bar{y})$ over $\{0, 1, 2, \dots, n\}$. We write $p_i(\bar{y})$ for the probability of selection of $i \in \{1, ..., n\}$ and $p_0(\bar{y})$ for the probability that the prize is not allocated conditional on report profile $\bar{y}$.

Denote by $\bar{F}$ the product c.d.f. of all n agents and by $\bar{F}_{-i}$ the product c.d.f. of all agents except i. Also denote by $\bar{x} = (x_1, ..., x_n)$ the profile of truthful reports and by $(y_i, \bar{x}_{-i})$ the same profile, except that x_i is replaced by y_i. Let $g_i(y_i)$ be the expected probability that agent i with report y_i is selected, assuming that all other agents make truthful reports,

$$g_i(y_i) = \int_{\bar{x}_{-i} \in [a,b]^{n-1}} p_i(y_i, \bar{x}_{-i}) \mathrm{d}\bar{F}_{-i}(\bar{x}_{-i}).$$

The principal would like to design an allocation rule that maximizes her expected

[6] In the Appendix, we consider an extension of this model where the penalty c is type-dependent.

[7] This assumption is useful for elegance of the exposition. We analyse a more general model in the Appendix without relying on this assumption.

[8] If the principal can benefit from penalizing agents, then she might prefer to ex-post penalize the agent whose value is negative to recover the lost payoff, even if that agent has been truthful. This is not an issue if values are nonnegative, $a \geq 0$, or if the principal faces an additional constraint that truthful reports cannot be penalized.

payoff,

$$\max_{p} \ \mathbb{E}\left[\sum\nolimits_{i=1}^{n} p_i(\bar{x})x_i\right], \tag{P$_0$}$$

subject to the incentive constraint that truthful reporting is optimal (by the revelation principle),

$$v_i(x_i)g_i(x_i) \geq \max_{y_i \in [a,b]} v_i(x_i)(1-c)g_i(y_i) \quad \forall x_i \in [a,b],\ \forall i \in \{1,...,n\}, \tag{IC$_0$}$$

and the feasibility constraint that the probabilities are nonnegative and add up to one, $(p_i(\bar{x}))_{i\in\{0,\dots,n\}} \geq 0$ and $\sum_{i=0}^{n} p_i(\bar{x}) = 1$ for all $\bar{x} \in [a,b]^n$.

B. Problem in reduced form

We will approach problem (P$_0$) by formulating and solving its reduced form. Recall that all n agents are ex-ante identical, with types distributed according to F. This assumption is important for the reduced-form approach to be applicable.

Define the *reduced-form allocation* $g : [a,b] \to \mathbb{R}_+$ by

$$g(x) = \sum_{i=1}^{n} g_i(x), \quad x \in [a,b]. \tag{1}$$

We will now formulate the principal's problem in terms of g:

$$\max_{g} \int_a^b xg(x)\mathrm{d}F(x) \tag{P}$$

subject to the incentive constraint

$$v(x)g(x) \geq v(x)(1-c) \sup_{y\in[a,b]} g(y) \ \text{ for all } x \in [a,b], \tag{IC}$$

and a generalization of the Matthews-Border feasibility criterion (Matthews 1984, Border 1991, Mierendorff 2011, Hart and Reny 2015) that guarantees the existence of an allocation rule p that induces a given g (see Lemma 1 below):

$$\int_{\{x:g(x)\geq t\}} g(x)\mathrm{d}F(x) \leq 1 - \Big(F(\{x : g(x) < t\})\Big)^n \ \text{ for all } t \in \mathbb{R}. \tag{F}$$

Variable g can be interpreted in two ways. First, $\frac{g(x)}{n}$ is the probability of an agent being chosen conditional on reporting x under a symmetric allocation rule whose reduced form is g. Second, $g(x)f(x)$ is the (improper) probability density of selection of type x from the principal's perspective. The reason for defining variable g as in (1) (rather than, for instance, $g(x) = \frac{1}{n}\sum_{i=1}^{n} g_i(x)$) is convenience: the principal's objective function (P) and the incentive constraint (IC) are independent of n.

Proposition 1 *A reduced-form allocation g is a solution of problem* (P) *if and only if there exists a solution p of problem* (P_0) *whose reduced form is g.*

As p is reducible to g by definition, the only nontrivial part of the result is the "only if" part. The feasibility condition (F) is the criterion for the existence of a (symmetric) p that implements g. This condition is due to the lemma below, which is a generalization of the Matthews-Border feasibility criterion (e.g., Border 1991, Proposition 3.1) for asymmetric mechanisms. In addition, for a symmetric p, the incentive constraints (IC_0) and (IC) are identical, even though (IC_0) is a stronger condition for a general p.

Let $(X, \mathcal{X}, \mu)$ be a measure space with measure μ. Let $\mathcal{Q}_n$ be the set of measurable functions $q : X^n \to [0,1]^n$ such that $\sum_{i=1}^n q_i \leq 1$. We say that $Q : X \to \mathbb{R}_+$ is a reduced form of $q \in \mathcal{Q}_n$ if $Q(y) = \sum_{i=1}^n \int_{X^{n-1}} q_i(y, \bar{x}_{-i}) \mathrm{d}\mu^{n-1}(\bar{x}_{-i})$ for all $y \in X$.

Lemma 1 *$Q : X \to \mathbb{R}_+$ is the reduced form of some $q \in \mathcal{Q}_n$ if and only if*

$$\int_{\{x:Q(x)\geq t\}} Q(x)\mathrm{d}\mu(x) \leq 1 - \Big(\mu(\{x : Q(x) < t\})\Big)^n \quad \textit{for all } t \in \mathbb{R}_+. \tag{2}$$

Proof. Sufficiency is due to Proposition 3.1 in Border (1991), which implies that, if Q satisfies (2), then there exists a *symmetric* q whose reduced form is Q. To prove necessity, consider $q \in \mathcal{Q}_n$ and let Q be its reduced form. For every $t \in \mathbb{R}_+$ denote $E_t = \{x \in X : Q(x) \geq t\}$. Then

$$\begin{aligned}
\int_{y\in E_t} Q(y)\mathrm{d}\mu(y) &= \int_{y\in X} \left[\sum_{i=1}^n \int_{x_{-i}\in X^{n-1}} q_i(y, x_{-i})\mathrm{d}\mu^{n-1}(\bar{x}_{-i})\right] \mathbf{1}_{\{y\in E_t\}}\mathrm{d}\mu(y) \\
&= \sum_{i=1}^n \left[\int_{(x_i,\bar{x}_{-i})\in X^n} q_i(x_i, \bar{x}_{-i})\mathbf{1}_{\{x_i\in E_t\}}\mathrm{d}\mu^n(x_i, \bar{x}_{-i})\right] \\
&\leq \sum_{i=1}^n \left[\int_{(x_i,\bar{x}_{-i})\in X^n} q_i(x_i, \bar{x}_{-i})\mathbf{1}_{\cup_j\{x_j\in E_t\}}\mathrm{d}\mu^n(x_i, \bar{x}_{-i})\right] \\
&= \int_{x\in X^n} \left(\sum_{i=1}^n q_i(x)\right) \mathbf{1}_{\cup_j\{x_j\in E_t\}}\mathrm{d}\mu^n(x) \leq \int_{x\in X^n} \mathbf{1}_{\cup_j\{x_j\in E_t\}}\mathrm{d}\mu^n(x) \\
&= 1 - \int_{x\in X^n} \mathbf{1}_{\cap_j\{x_j\in X\setminus E_t\}}\mathrm{d}\mu^n(x) = 1 - \big(\mu(X\setminus E_t)\big)^n.
\end{aligned}$$

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Proof of Proposition 1. Observe that, for every p and its reduced form g, objective functions in (P_0) and (P) are identical. We now verify that the reduced form of every solution of (P_0) is admissible for (P), and that for every solution g of (P) there is an admissible allocation p for (P_0) whose reduced form is g.

Let p be a solution of (P_0). Then its reduced form satisfies the feasibility constraint (F) by Lemma 1. The incentive constraint (IC) is satisfied as well, since (IC_0) applies separately for each i and thus, in general, is stronger than (IC).

Conversely, let g be a solution of (P). Since g satisfies (F), by Proposition 3.1 in Border (1991) there exists a symmetric p whose reduced form is g. This p will satisfy incentive constraint (IC_0), since, for symmetric mechanisms, (IC) implies (IC_0). ■

II. Optimal allocation rules

Problem (P) is interesting because of its constraints. First, the incentive constraints (IC) are global rather than local, as is often the case in mechanism design. Second, the feasibility constraint (F) is substantive and will bind at the optimum if and only if the incentive constraint (IC) slacks, which is not the case in the classical mechanism design for allocation problems. Let us now discuss the implications of these constraints on the design of optimal rules.

A. *Incentive compatibility.*

There is tension between the ability of the principal to infer the agents' information and the ability to use this information to the principal's benefit by selecting agents with higher types. Suppose that the principal selects an agent with the highest positive report and selects no one if all reports are negative. In the unique equilibrium under this rule, everybody reports the highest possible type, b.[9] Thus, communication is uninformative and the outcome of this mechanism is identical to the one where the principal disregards the agents' reports and picks an agent at random, provided $\mathbb{E}[x] \geq 0$, so that allocating the prize to a random agent is better than not allocating it at all.

The following lemma shows that without loss of generality we can consider only monotonic reduced-form allocation rules.

Lemma 2 *An optimal reduced-form allocation $g(x)$ is nondecreasing.*

Intuitively, the optimality for the principal implies the monotonicity of g, as the principal would like to select higher types with higher probability. If an allocation g is nonmonotonic, by sorting $g(F^{-1})$ in ascending order, we construct a monotonic $\tilde{g}$ that preserves the incentive and feasibility constraints but increases the principal's payoff. The proof of Lemma 2 is in the online appendix.

[9]This follows from the observation that, for low enough values of x, bidding truthfully is dominated by paying penalty c and outbidding everyone else by reporting the highest type, b, and then applying this argument inductively for other values of x.

Consider a nondecreasing reduced-form allocation rule g. By the assumption that $v(x) > 0$, the incentive constraint (IC) can be simplified as

$$g(x) \geq (1-c)g(b), \tag{3}$$

The right-hand side is the maximal payoff that agent i can obtain by lying. It is equal to the probability of selection that agent i can obtain by lying, $g(b)$, times the fraction of the retained surplus after the lie is found out, $1-c$. Unlike in the standard mechanism design problems, where typically the only binding incentive constraints are local, constraint (4) is global.

By Lemma 2 and the assumption that $v(x) > 0$, the incentive constraint (IC) becomes

$$g(x) \geq (1-c)g(b), \quad \text{for all } x \in [a,b]. \tag{4}$$

The incentive constraint (4) induces two properties of an optimal allocation rule:

1. Give a chance to low types. The right-hand side of (4) provides a uniform lower bound on g. That is, an optimal rule must select any type x, whether positive or negative, whether low or high, with a probability of at least $(1-c)g(b)$. In particular, the monotonicity of g in an optimal rule then implies *bunching at the bottom*: all agents with low enough types will be selected with the same probability.

2. Cap the odds of the best. The incentive constraint (4) tightens as the probability of selecting the highest type increases. Thus, a smaller value of g at the top decreases the probability of selecting types bunched at the bottom. An optimal rule caps g at some value below 1, leading to *bunching at the top*: all agents with high enough types will be selected with the same probability.

The incentive constraint (4) dictates a different structure of an optimal allocation than in Elchanan Ben-Porath, Eddie Dekel and Barton L. Lipman (2014) (BDL). The feature of bunching the types at the bottom is similar, but the reason behind it is not the same. In our model, the incentive constraint prevents separation at the bottom, whereas in BDL, the separation of low-valued agents is feasible but does not justify the verification cost. Unlike our model, in BDL, there is no bunching at the top because, at the optimum, the agents who report high values are verified with certainty.

B. Feasibility

By Lemma 2, optimality for the principal implies the monotonicity of g. Hence, the feasibility constraint (F) can be simplified as follows.

Lemma 3 *For every weakly increasing g, the feasibility constraint* (F) *is equiv-*

alent to

$$\int_y^b g(x)\mathrm{d}F(x) \leq 1 - F^n(y), \quad \textit{for all } y \in [a,b]. \tag{5}$$

Proof. Since g is weakly increasing, for every $t \in \mathbb{R}_+$, sets $\{x : g(x) < t\}$ and $\{x : g(x) \geq t\}$ are intervals $[a, y)$ and $[y, b]$, respectively, where $y = \inf\{x : g(x) \geq t\}$. It is then immediate that (F) is identical to (5). ■

The feasibility constraint (5) has a clear interpretation. Dividing both sides by $1 - F^n(y)$, we obtain

$$\frac{1}{1 - F^n(y)} \int_y^b g(x)\mathrm{d}F(x) \leq 1.$$

The left-hand side is a conditional probability expression. This is the probability of choosing an agent with at least type y, conditional on the highest type among all agents being at least y. Naturally, it cannot exceed 1.

There are two properties of an optimal rule that follow from (5).

3. Separation in the middle. On any interval (x', x'') where the feasibility constraint is binding, the density of the selected type, $g(x)f(x)$, must be equal to the density of the highest type, $nF^{n-1}(x)f(x)$. This implies strictly increasing $g(x) = nF^{n-1}(x)$ on (x', x''), and thus full type separation on that interval. Another implication is that, if the highest value, $\max\{x_1, \ldots, x_n\}$, is in that interval, the agent with that value must be chosen with certainty.

4. Diminishing role of the feasibility constraint for large pools of agents. As the number of agents n increases, the set of feasible reduced-form allocations satisfying (5) expands, eventually permitting all allocations as $n \to \infty$. However, the incentive constraint (IC) is independent of n, so, as we will prove later, there exists a finite $\bar{n}$ such that, for $n > \bar{n}$, the incentive constraint determines the optimal allocation, while the feasibility constraint is not binding. Intuitively, as n rises, the probability of a given low-type agent being chosen shrinks. To preserve the incentives for truthtelling, the probability of the highest type of being chosen must shrink at the same rate. Thus, a larger n does not allow for better differentiation between types. As a consequence, increasing the pool of agents over some finite size $\bar{n}$ does not confer any benefit to the principal. This contrasts to standard auction environments with independent values and monetary transfers, where the auctioneer can always benefit from more bidders, albeit at a diminishing rate.

C. *Optimal allocations*

We now describe optimal allocation rules. Assume that

$$\text{if } a < 0, \text{ then } \int_a^0 (1-c)x\mathrm{d}F(x) + \int_0^b x\mathrm{d}F(x) > 0. \tag{6}$$

Since we allow for negative types, it might be optimal for the principal to select no agent. Assumption (6) is a necessary and sufficient condition for the principal to prefer the selection of some agent over no agent. Intuitively, the least that the principal can do is to differentiate between values above and below zero. Specifically, consider an allocation rule which asks each agent to report whether his value is positive or negative, and then assigns probability $\frac{1}{n}(1-c)$ to each agent whose report is negative and probability $\frac{1}{n}$ to each agent whose report is positive. This rule is feasible and incentive compatible, and it yields a positive payoff if (6) holds. The converse argument is more involved and requires to show that, if (6) does not hold, the upper bound on what the principal can attain is nonpositive. The argument uses the upper bound result of Section III.A and thus is deferred to Section III.F.

When the number of agents is small, the optimal rule bunches the types at the top and at the bottom and separates them in the middle. It can be implemented by a *restricted-bid auction.*

Restricted-bid auction. The principal asks each agent to make a statement y_i in an interval $[\underline{x}, \overline{x}] \subset [a, b]$ and then selects an agent with the highest statement (ties are broken uniformly at random). Ex post, the chosen agent is penalized if his statement y_i is "inflated": $y_i > \underline{x}$ and $y_i > x_i$.

Informally, a restricted-bid auction categorizes the agents into three groups: "high" with types above $\bar{x}$, "middle" with types between $\underline{x}$ and $\bar{x}$, and "low" with types below $\underline{x}$. The principal then randomly chooses a candidate from the high group (*bunching at the top*). If there are no candidates in that group, the highest type among the middle group is chosen (*separation at the middle*). If there are neither high nor middle candidates, a candidate is randomly selected from the low group (*bunching at the bottom*). Provided that n is not too large, one can always find $\underline{x}$ and $\overline{x}$ that guarantees the incentive compatibility of the restricted-bid auction: the greater $\underline{x}$ and the lower $\overline{x}$ are, the less benefit there is for a low-type agent to pretend to be a high type.

However, as we noted in Section II.B, for a large enough number of agents, the feasibility constraint is nowhere binding, so optimality only requires bunching at the top and at the bottom, with the empty middle interval. This is implemented by a different mechanism called a *binary shortlisting procedure.*

Binary shortlisting procedure. The principal asks each agent to make a statement indicating whether his type is above or below some threshold $\bar{x}$. Every agent who reports $x_i \geq \bar{x}$ is shortlisted with certainty, while every agent who reports $x_i < \bar{x}$ is shortlisted with a specified probability q, which is independent of the reports. Then, an agent is chosen from the shortlist uniformly at random. In the event that the shortlist is empty, a uniformly random agent is chosen from the full list. Ex post, the chosen agent is penalized if his statement has been inflated: a type $x_i < \bar{x}$ has reported being above $\bar{x}$.

Note that there is no discontinuity between these procedures: a restricted-bid auction with $\underline{x} = \overline{x}$ is identical to the binary shortlisting procedure with the threshold $\overline{x}$ and probability parameter $q = 0$.

We say that two allocation rules p and p' are *equivalent* if their reduced forms g and g' are identical up to a measure zero.

Theorem 1 *There exists a number of agents $\bar{n}$ such that an allocation rule is optimal if and only if it is equivalent to a restricted-bid auction when $n < \bar{n}$ and to a binary shortlisting procedure when $n \geq \bar{n}$.*

We prove the theorem and solve for the parameters of the optimal allocation rule in the next section.

III. Proof of Theorem 1

We proceed with the proof of Theorem 1 as follows. First, we solve the reduced-form problem without imposing the feasibility constraint (5). The obtained solution gives an upper bound on the principal's optimal payoff, and it is optimal whenever it satisfies (5). We identify the minimum number of agents $\bar{n}$ above which (5) is not binding for this upper-bound solution, and show that this solution is a binary shortlisting procedure.

Then, we solve the problem for $n < \bar{n}$, where the feasibility constraint (5) is binding and the upper bound is unattainable. We show that the solution is a restricted-bid auction with suitably defined bounds $\underline{x}$ and $\overline{x}$. This is the most technically interesting and novel part of the analysis, where we deal with interaction of two non-standard constraints: global incentive compatibility and the Matthews-Border feasibility constraint.

A. *Upper bound on the principal's payoff*

To derive the upper bound on the principal's payoff, we solve (P) subject to the incentive constraint (4) while relaxing the feasibility constraint (5).

First, we simplify the incentive constraint.

Lemma 4 *Reduced-form allocation g satisfies the incentive constraint* (4) *if and only if there exists $r \in \mathbb{R}_+$ such that*

$$(1-c)r \leq g(x) \leq r \quad \textit{for all } x \in [a,b]. \tag{7}$$

Proof. If (4) holds, then (7) also holds with $r = \sup_{y\in[a,b]} g(y)$. Conversely, if (7) holds with some $r \in \mathbb{R}$, then it also holds with $r' = \sup_{y\in[a,b]} g(y) \leq r$, which implies (4). ■

We now state the result.

Proposition 2 *Let (z^*, r^*) be the unique solution of*

$$\int_a^{z^*} (1-c)(z^* - x)\mathrm{d}F(x) = \int_{z^*}^b (x - z^*)\mathrm{d}F(x), \tag{8}$$

$$\int_a^{z^*} (1-c)r^*\mathrm{d}F(x) + \int_{z^*}^b r^*\mathrm{d}F(x) = 1. \tag{9}$$

For any allocation rule, the principal's payoff is at most z^*. Moreover, if an allocation rule attains the payoff of z^* for the principal, then its reduced form must be almost everywhere equal to

$$g^*(x) = \begin{cases} (1-c)r^*, & x < z^*, \\ r^*, & x \geq z^*. \end{cases} \tag{10}$$

One could interpret the allocation (10) as a mechanism that gives lottery tickets to the agents. Everyone with a statement above z^* gets r^* tickets, and everyone with a statement below z^* gets $(1-c)r^*$ tickets. The probability of winning the lottery is proportional to the quantity of tickets held. Now, consider lowering z^* a little. Then, the marginal agent has a higher chance of winning. This lowers the chance of winning of all the people above z^* (weighed by 1) and all the people below (weighed by $1-c$). The first effect is good for the principal, while the second effect is bad. Since these effects are both monotone in z^*, there is a unique internal optimum given by (8). Equation (9) just says that the combined value of all lottery tickets must add up to 1.

Proof. We solve $\max_g \int_a^b xg(x)\mathrm{d}F(x)$ subject to the incentive constraint (7) and the relaxed feasibility constraint that requires the total probability of allocation not exceed the unity, $\int_a^b g(x)\mathrm{d}F(x) \leq 1$. The Lagrangian of this problem is

$$\max_g \min_z \int_a^b xg(x)\mathrm{d}F(x) + z\left(1 - \int_a^b g(x)\mathrm{d}F(x)\right), \text{ or}$$
$$\max_g \min_z \left(z + \int_a^b (x-z)g(x)\mathrm{d}F(x)\right),$$

subject to (7), where $z \geq 0$ is a Lagrange multiplier.

Observe that the incentive constraint (7) must be everywhere binding, since the objective function is linear in g. The solution is a step function that, for some constant $r \geq 0$, chooses the minimum incentive compatible value $(1-c)r$ below z and the maximum incentive compatible value r above z,

$$g(x) = \begin{cases} (1-c)r, & x < z, \\ r, & x \geq z. \end{cases}$$

Now substitute the obtained $g(x)$ into the objective function and optimize over z and r,

$$\max_{r\geq 0}\min_{z\geq 0}\left(z+\int_a^z (x-z)(1-c)r\mathrm{d}F(x)+\int_z^b (x-z)r\mathrm{d}F(x)\right). \tag{11}$$

To rule out boundary solutions, observe that, under assumption (6), this objective function is linear and strictly increasing in r at $z=0$. Hence, $z>0$ at the optimum. Furthermore, if $r=0$, then the objective function is strictly increasing in z and achieves the minimum at $z=0$, which cannot be optimal, as noted above. Hence, $r>0$ at the optimum.

Consequently, if a solution exists, it must satisfy the first-order conditions

$$\int_a^z (1-c)(x-z)\mathrm{d}F(x)+\int_z^b (x-z)\mathrm{d}F(x)=0, \tag{12}$$

$$1-\int_a^z (1-c)r\mathrm{d}F(x)-\int_z^b r\mathrm{d}F(x)=0. \tag{13}$$

Notice that these conditions are equivalent to (8) and (9).

The left-hand side of (12) is strictly decreasing in z, nonpositive at $z=b$, and, under assumption (6), positive at $z=0$, thus admitting a unique solution z^*. Moreover, $z^*\in(0,b]$. In addition, for a given $z\in(0,b]$, the left-hand side of (13) is linearly decreasing in r and positive at $r=0$, thus admitting a unique solution $r^*>0$. ■

B. Attainment of the upper bound.

The reduced-form solution g^* might not be feasible when the number of agents is small. We now derive a condition on the number of agents that ensures the feasibility of g^*.

By Lemma 3, g^* is feasible if and only if $\int_{z^*}^b g^*(x)\mathrm{d}F(x)\leq 1-F^n(z^*)$, which after substituting g^* from (10) becomes:

$$(1-F(z^*))r^*\leq 1-F^n(z^*). \tag{14}$$

Note that this is a condition on the primitives, as z^* and r^* are determined by F and c and independent of n.

Denote by $\bar{n}$ the smallest number of agents that satisfies (14). It follows that:

Corollary 1 *There exists an allocation rule that attains the upper-bound payoff of z^* if and only if $n\geq\bar{n}$.*

Condition (14) is not particularly elegant. Instead, one can use a sufficient condition, which is simple and independent of F, z^*, and r^*.

Corollary 2 *There exists an allocation rule that attains the upper-bound payoff of z^* if $c \leq \frac{n-1}{n}$.*

In other words, the principal's upper-bound payoff can be achieved when the penalty is not too large, leaving at least $\frac{1}{n}$-th of the value of the prize to the agent.

Proof. Using (9), rewrite (14) as $\frac{1-F(z^*)}{(1-c)F(z^*)+1-F(z^*)} \leq 1 - F^n(z^*)$. Solving for $1-c$ yields

$$\frac{F^{n-1}(z^*)}{1+F(z^*)+F^2(z^*)+\ldots+F^{n-1}(z^*)} \leq 1-c.$$

This inequality holds when $c \leq \frac{n-1}{n}$, because:

$$\frac{F^{n-1}(z^*)}{1+F(z^*)+\ldots+F^{n-1}(z^*)} = \frac{1}{F^{1-n}(z^*)+F^{2-n}(z^*)+\ldots+1} \leq \frac{1}{n} \leq 1-c.$$

■

C. Shortlisting procedure.

An allocation rule that implements g^* with bunching of types above and below the threshold is a binary shortlisting procedure. The threshold type is z^*, while the probability q of shortlisting low-type agents has to be calculated to give the desired probabilities, $g^*(x) = (1-c)r^*$ for $x < z^*$ and $g^*(x) = r^*$ for $x \geq z^*$.

Corollary 3 *Let $n \geq \bar{n}$. Then the binary shortlisting procedure with the threshold $\bar{x} = z^*$ and the probability parameter*

$$q = 1 - \frac{c}{s} \tag{15}$$

attains the upper bound z^*, where s is the unique solution of equation[10]

$$(1-s)s^{n-1} = \frac{1}{r^*}\left(1-\frac{1}{r^*}\right)^{n-1}, \quad s \in \left[\tfrac{n-1}{n}, 1\right]. \tag{16}$$

The proof is in the online appendix.

D. Small number of agents

When the number of agents is small, $n < \bar{n}$, attainment of the upper-bound payoff z^* is prevented by the feasibility constraint. The problem becomes more difficult, as we need to handle the interaction of the feasibility and incentive constraints.

[10] Equation (16) has two solutions on $[0,1]$. One of them, $s = 1 - \frac{1}{r^*}$, is outside the domain $[\frac{n-1}{n}, 1]$, as $n \geq \bar{n}$ implies $\frac{1}{r^*} > \frac{1}{n}$ (as shown in the proof).

Our approach is to fix a number r, find the maximal principal's payoff on the set of reduced-form allocations g with supremum r, and show that this is implemented by a restricted-bid auction. Then the optimal allocation can be determined by taking the maximum with respect to r.[11]

For $r \in \mathbb{R}_+$ denote by $\mathcal{G}_r$ the set of reduced-form allocations that are weakly increasing and satisfy the incentive constraint (7) for r. Note that $\mathcal{G}_r$ contains an optimal allocation only if[12]

$$r \in R \equiv [1, \min\{n, 1/(1-c)\}].$$

Fix $r \in R$. We would like to maximize the principal's payoff on $\mathcal{G}_r$ subject to the feasibility constraint (5),

$$\max_g \int_a^b xg(x)\mathrm{d}F(x) \tag{P$_r$}$$

$$\text{s.t.} \quad (1-c)r \le g(x) \le r, \quad x \in [a,b], \tag{17}$$

$$\int_x^b g(y)\mathrm{d}F(y) \le 1 - F^n(x), \quad x \in [a,b]. \tag{18}$$

One can interpret $g(x)f(x)$ as an improper probability density that has to satisfy $\int_a^b g(x)f(x)\mathrm{d}x \le 1$ and treat (P$_r$) as the problem of allocation of the probability mass among the types on $[a,b]$.

To solve problem (P$_r$), we allocate the probability mass among the types, starting from the highest type b and proceeding to lower types, by setting the maximum density permitted by the constraints (illustrated by Fig. 1). Formally, we solve

$$\max_g \int_a^b \left(\int_x^b g(t)\mathrm{d}F(t) \right) \mathrm{d}x \quad \text{s.t. (17) and (18).}$$

This problem is identical to (P$_r$), by integration by parts of the objective function. The solution of this problem is a pointwise maximal function that respects the constraints, $g_r(x) = r$ for all $x \ge \bar{x}_r$, and $g_r(x) = nF^{n-1}(x)$ for $x < \bar{x}_r$. The latter is derived from the constraint (18) satisfied as equality, $\int_x^b g(y)f(y)\mathrm{d}y = 1 - F^n(x)$. The threshold $\bar{x}_r$ is the point where these constraints meet (the colored areas on

[11] This approach is analogous to Elchanan Ben-Porath, Eddie Dekel and Barton L. Lipman (2014), who show that, without loss of optimality, one can restrict attention to favored-agent mechanisms parametrized by agent i and threshold v^*, and then find an optimal mechanism within this subclass.

[12] If $r > 1/(1-c)$, then every allocation in $\mathcal{G}_r$ must satisfy $g(x) \ge (1-c)r > 1$, so it violates feasibility, $\int_a^b g(x)\mathrm{d}F(x) > 1$. If $r > n$, then every allocation in $\mathcal{G}_r$ is also in $\mathcal{G}_n$, since $g(x) \le n$ by feasibility, and reducing r weakens the left-hand side of (7). Finally, if $r < 1$, then every allocation in $\mathcal{G}_r$ is inferior to the uniformly random allocation, $\int_a^b xg(x)\mathrm{d}F(x) \le \int_a^b xr\mathrm{d}F(x) < \mathbb{E}[x]$.

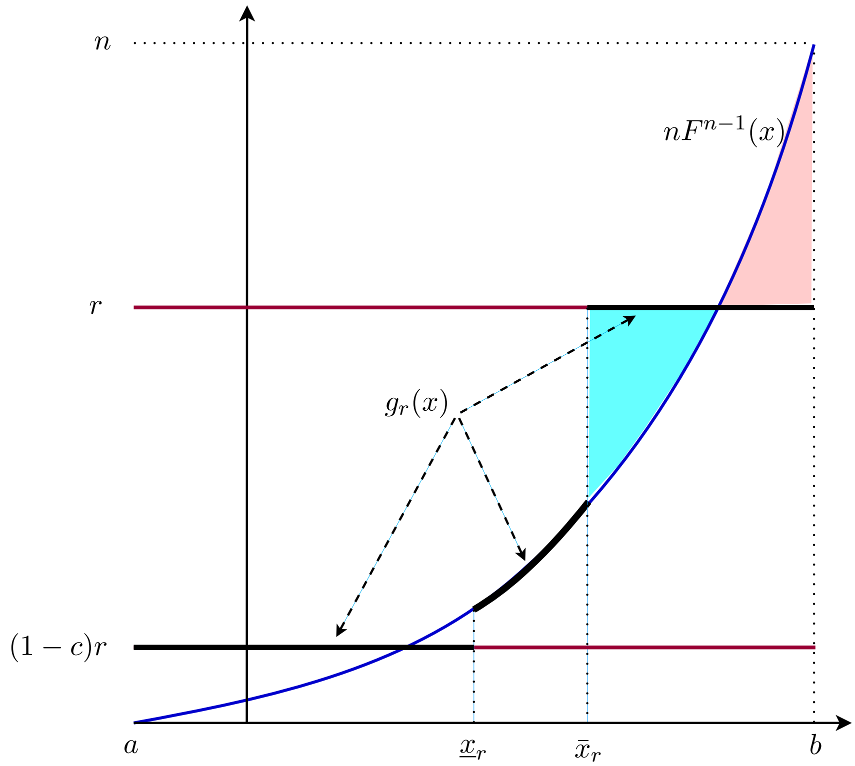

FIG. 1. A SOLUTION WITH A GIVEN SUPREMUM r.

Fig. 1 have equal size): $\int_{\bar{x}_r}^{b} r\mathrm{d}F(x) = 1 - F^n(\bar{x}_r)$, or simply[13]

$$r(1 - F(\bar{x}_r)) = 1 - F^n(\bar{x}_r). \tag{19}$$

To sum up, we allocate $g_r(x) = r$ on the interval $[\bar{x}_r, b]$ and $g_r(x) = nF^{n-1}(x)$ on the interval $[\underline{x}_r, \bar{x}_r)$. All the types below the lower threshold $\underline{x}_r$ are assigned the minimum density permitted by the incentive constraint, $(1-c)r$. The threshold $\underline{x}_r$ is the smallest number that satisfies two constraints, $\underline{x}_r \geq 0$ and the total mass not exceeding unity:

$$\int_a^{\underline{x}_r} (1-c)r\mathrm{d}F(x) + \int_{\underline{x}_r}^{\bar{x}_r} nF^{n-1}(x)\mathrm{d}F(x) + \int_{\bar{x}_r}^{b} r\mathrm{d}F(x) \leq 1.$$

The latter constraint can be simplified. Using (19) and integrating out the constant parts yields $(1-c)rF(\underline{x}_r) + (F^n(\overline{x}_r) - F^n(\underline{x}_r)) + (1 - F^n(\overline{x}_r)) \leq 1$, or,

[13]Note that there is a unique solution of (19), as $\frac{1-F^n(x)}{1-F(x)} = 1 + F(x) + ... + F^{n-1}(x) \in [1, n]$ is strictly increasing and continuous, and by assumption, $r \in R \subset [1, n]$.

equivalently,

$$F^{n-1}(\underline{x}_r) \geq (1-c)r.$$

It is apparent that either $\underline{x}_r$ solves the above as an equality or $\underline{x}_r = 0$, whichever is greater. Note that $\underline{x}_r \geq 0$, since $r \geq 1$ and, by assumption, $F^{n-1}(0) \leq 1-c$. Thus $\underline{x}_r$ is the solution of[14]

$$F^{n-1}(\underline{x}_r) = (1-c)r. \tag{20}$$

The solution of problem (P$_r$) is thus

$$g_r(x) = \begin{cases} (1-c)r, & x < \underline{x}_r, \\ nF^{n-1}(x), & \underline{x}_r \leq x < \bar{x}_r, \\ r, & x \geq \bar{x}_r, \end{cases} \tag{21}$$

where $\bar{x}_r$ and $\underline{x}_r$ are given by (19) and (20).

We have shown that g_r maximizes the principal's payoff, $\int_a^b xg(x)\mathrm{d}F(x)$, on the set of functions $\mathcal{G}_r$ for a given $r \in R$. The next proposition summarizes this result and characterizes the optimal value of r.

Proposition 3 *Let $n < \bar{n}$. Then, a reduced-form allocation g is optimal if and only if $g = g_r$, where r is the solution of*

$$(1-c)\int_a^{\underline{x}_r} (\underline{x}_r - x)\mathrm{d}F(x) = \int_{\overline{x}_r}^b (x - \overline{x}_r)\mathrm{d}F(x) \tag{22}$$

and $\overline{x}_r$ and $\underline{x}_r$ are defined by (19) *and* (20).

The optimal value of r maximizes $\int_a^b xg_r(x)\mathrm{d}F(x)$. Equation (22) is the first-order condition for this maximization problem, which turns out to have a unique solution. As in (8), the optimal thresholds equate the principal's marginal utility distortions at the top and at the bottom. The complete proof is in the online appendix.

E. Restricted-bid auction.

The reduced-form allocation g_r bunches the types above $\overline{x}_r$ and below $\underline{x}_r$ and fully separates types in the interval $[\underline{x}_r, \bar{x}_r]$. This reduced-form allocation can be implemented by the restricted-bid auction with the bid interval $[\underline{x}_r, \bar{x}_r]$. In equilibrium, an agent bids his type truthfully if it belongs to the interval $[\underline{x}_r, \bar{x}_r]$, bids $\overline{x}_r$ if his type is above $\overline{x}_r$, and bids $\underline{x}_r$ otherwise. If one or more agents have types above $\overline{x}_r$, the restricted bid auction selects one of these agents with

[14]Note that there is a unique $\underline{x}_r$ defined by (20), as $r \in R \subset [1, 1/(1-c)]$, so $(1-c)r \in [0,1]$, and $F^{n-1}(x)$ is strictly increasing and continuous.

equal probability (bunching above $\overline{x}_r$). If the highest type belongs to $[\underline{x}_r, \bar{x}_r]$, it is selected with probability one (separation). Otherwise, all bids are equal to $\underline{x}_r$ and the restricted bid auction selects one of the agents at random (bunching below $\underline{x}_r$).

By the construction of $\overline{x}_r$, as given in (19), an agent with type above $\overline{x}_r$ is selected with the probability of r/n. By (20), an agent with type below $\underline{x}_r$ is selected with the probability of at least $(1-c)r/n$, and thus has no incentive to inflate his report.

Corollary 4 *Let $n < \bar{n}$. Then, the restricted-bid auction with the bid interval $[\underline{x}_r, \bar{x}_r]$ attains the optimal payoff for the principal, where $\underline{x}_r$ and $\bar{x}_r$ are the thresholds in the optimal reduced-form allocation g_r in Proposition 3.*

Proof. The payoff of the principal from the restricted-bid auction with bid interval $[\underline{x}_r, \bar{x}_r]$ is equal to

$$\begin{aligned} V^* &= F^n(\underline{x}_r)\mathbb{E}[x|x<\underline{x}_r] + \int_{\underline{x}_r}^{\bar{x}_r} x\mathrm{d}F^n(x) + (1-F^n(\bar{x}_r))\mathbb{E}[x|x \geq \bar{x}_r] \\ &= \frac{F^n(\underline{x}_r)}{F(\underline{x}_r)}\int_a^{\underline{x}_r} x\mathrm{d}F(x) + \int_{\underline{x}_r}^{\bar{x}_r} xnF^{n-1}(x)\mathrm{d}F(x) + \frac{1-F^n(\bar{x}_r)}{1-F(\bar{x}_r)}\int_{\bar{x}_r}^b x\mathrm{d}F(x) \\ &= \int_a^{\underline{x}_r}(1-c)rx\mathrm{d}F(x) + \int_{\underline{x}_r}^{\bar{x}_r} xnF^{n-1}(x)\mathrm{d}F(x) + \int_{\bar{x}_r}^b rx\mathrm{d}F(x) = \int_a^b g_r(x)\mathrm{d}F(x), \end{aligned}$$

where in the last line we used (19), (20), and (21). ■

F. No allocation

Let us now prove that assumption (6) is necessary and sufficient for the principal to select an agent with a positive probability and to receive a positive payoff.

Proposition 4 *The optimal allocation rule chooses no agent and attains zero payoff if and only if*

$$\int_a^0 (1-c)x\mathrm{d}F(x) + \int_0^b x\mathrm{d}F(x) \leq 0. \tag{23}$$

Proof. By Proposition 2, the principal's payoff cannot exceed z^* given by the first-order condition (8). Since (8) has a unique solution, the upper-bound payoff z^* is nonpositive if

$$\int_a^z (1-c)(z-x)\mathrm{d}F(x) \geq \int_z^b (x-z)\mathrm{d}F(x) \text{ at } z=0,$$

which is identical to (23). Conversely, if (23) does not hold, then the rule

$$g(x) = \begin{cases} (1-c)r, & x < 0, \\ r, & x \geq 0. \end{cases}$$

is incentive compatible, is feasible for a small enough $r > 0$, and yields the payoff

$$r\left(\int_a^0 (1-c)x\mathrm{d}F(x) + \int_0^b x\mathrm{d}F(x)\right) > 0.$$

■

IV. Discussion and comparative statics

There are two notable features of optimal allocation when the principal must rely on reported information, which is in sharp contrast to the case of observable agent types.

First, no matter how many agents participate, low types must be chosen with a positive probability. Even agents with negative types, no matter how bad they are for the principal, must be treated the same way since the principal cannot distinguish between good and bad types and has to provide incentives for telling the truth to everyone. Moreover, the probability of choosing the very top types has to be capped to reduce the benefit of lying.

Second, in an environment with observable types, the probability of choosing a type above any given threshold is strictly increasing in the number of agents. This is not true in our model. In fact, in the restricted-bid auction, as n goes up, there is more pooling at the top: the upper threshold $\overline{x}$ decreases. Eventually, when $n \geq \bar{n}$, the optimal reduced-form allocation is a binary categorization that assigns only two values, high and low, to types above and below some threshold, respectively.

We now present comparative statics results with respect to

(a) the payoff of the principal;

(b) the size of the pooling interval of high types;

(c) the size of the separating interval in the middle for the case of a small number of agents, $n < \bar{n}$.

We denote the threshold of the high pooling interval by $\overline{x}$ and the lower threshold of the separating interval by $\underline{x}$.[15] The high pooling interval, $[\underline{x}, b]$, consists of all types above the upper quality bar $\overline{x}$ that are treated identically in the allocation mechanism. The larger the interval is, the less discriminatory the optimal mechanism will be for high types. The separating interval, $[\underline{x}, \overline{x}]$, has a positive length when $n < \bar{n}$. The size of this interval is indicative of the allocation rule's ability to discriminate the types in the middle.

[15]For $n < \bar{n}$, $\overline{x} = \overline{x}_r$ and $\underline{x} = \underline{x}_r$ as defined by (19) and (20) at the optimal r. For $n \geq \bar{n}$, $\overline{x} = z^*$ as defined by (8).

The amount of ex-post penalty affects the agents' incentives and is crucial for the structure of the optimal mechanism. As the penalty c decreases, the principal is less able to discriminate between high and low types. As c approaches zero, the gap between the probabilities assigned to high and low types vanishes, leading to the uniformly random allocation.

Proposition 5a *Suppose that the penalty c marginally increases. Then the principal is better off. The size of the high pooling interval, $[\overline{x}, b]$, decreases. Suppose in addition that $n < \bar{n}$ and that $\frac{f(x)}{F(x)}$ is decreasing.*[16] *Then, the size of the separating interval, $[\underline{x}, \overline{x}]$, increases.*

An increase in the number of applicants, n, has a non-obvious effect that we have already discussed. A larger n relaxes the feasibility constraint (F) while having no effect on the incentive constraint (IC) and the objective function (P). The principal can thus implement the allocation closer to the upper bound.

Proposition 5b *Let $n < \bar{n}$. Then, as n goes up, the principal is better off. The size of the high pooling interval, $[\overline{x}, b]$, increases, and the size of the separating interval, $[\underline{x}, \overline{x}]$, decreases. Any increase of n above $\bar{n}$ has no effect.*

While keeping the allocation ratio between high and low types fixed to ensure incentive compatibility, the principal has leeway in choosing the size of the pooling intervals for high and low types. There is a trade-off: a better differentiation of high types (smaller interval $[\bar{x}, b]$) entails worse differentiation of low types (larger interval $[a, \underline{x}]$). This tradeoff depends on the distribution of types. An f.o.s.d. improvement of the distribution increases the single optimal threshold when $n \geq \bar{n}$, and it has an ambiguous effect on the structure of the optimal mechanism when $n < \bar{n}$: both optimal thresholds can either increase or decrease.

Proposition 5c *Suppose that F is replaced by $\tilde{F}$, where $\tilde{F}$ f.o.s.d. F. Then the principal is better off. If $n \geq \bar{n}$ under F, then the size of the high pooling interval, $[\overline{x}, b]$, decreases.*

The effects of a mean-preserving spread or a rotation of the distribution (Johnson and Myatt 2006) are ambiguous. When both low and high types are less numerous, whether the principal benefits from it and whether more discrimination or more pooling of high types is optimal depends on the exact change of the distribution of types.

The proof of Propositions 5a, 5b, and 5c is in the online appendix.

V. Conclusion

In this paper, we have analyzed the problem of allocating a prize to one of several agents, where the social value of giving the prize to an agent is privately

[16] This is the well-known monotone hazard rate condition.

known by this agent. The allocation rule chooses the winner of the prize based on the agents' reports about these values. After the prize is allocated, the social value of giving the prize to the winner becomes commonly known, and the agent can be penalized for lies about the value. We have shown that, if the number of agents is low, the optimal allocation rule takes the form of a restricted-bid procedure; otherwise, it takes the form of a shortlisting procedure. In this problem, the principal faces the tradeoff between making the choice more competitive by selecting higher types with a higher probability and mainting the incentives for truthtelling by selecting low types with a positive probability. There are multiple applications that correspond to our model: a grant agency selecting an organization to fund, a college administrator awarding a scholarship, or a firm recruiting for a fixed-salary position.

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Appendix: Type-dependent penalties

Here, we consider a more general model where the penalty c depends on the agent's type. Formally, we assume that, ex post, the principal observes the selected agent's true type x_i and can impose a penalty $c(x_i) \geq 0$, which is subtracted from the agent's value $v(x_i)$. Our primary interpretation of c is the upper bound on the expected penalty that can be imposed on the agent after his type has been verified.[17] Functions v and c are bounded and almost everywhere continuous on $X \equiv [a, b]$.

As before, we formulate the principal's problem in terms of the reduced-form allocation:

$$\text{(P)} \qquad \max_g \int_{x \in X} x g(x) \mathrm{d}F(x),$$

subject to the incentive constraint,

$$\text{(IC)} \qquad v(x)g(x) \geq (v(x) - c(x)) \sup_{y \in X} g(y) \quad \text{for all } x \in X,$$

and the feasibility constraint,

$$\text{(F)} \qquad \int_{\{x: g(x) \geq t\}} g(x) \mathrm{d}F(x) \leq 1 - \Big(F(\{x : g(x) < t\})\Big)^n \quad \text{for all } t \in [0, n].$$

The idea of the solution is the same as in Section III.D. We fix a supremum value of g, denoted by r, interpret $g(x)f(x)$ as a probability density, and allocate the maximum density to high types, starting from the top, b, and proceeding

[17] The assumption that x_i is verified with certainty can be relaxed; if $\alpha(x_i)$ is the probability that x_i is verified and $L(x_i)$ is the limit on i's liability, then set $c(x_i) = \alpha(x_i)L(x_i)$.

down, subject to the constraints. However, two issues that arise because of a type-dependent incentive constraint.

The first issue is that the feasibility constraint (F) is not tractable without making more assumptions about the structure of admissible allocations g. To restore tractability, we assume that the share of the after-penalty surplus is monotonic:

Assumption 1 (Monotonicity)

$$\frac{v(x) - c(x)}{v(x)} \text{ *is weakly increasing.*}$$

That is, agents with higher types stand to lose less from lying to the principal. This is a natural assumption for the applications we consider: agents who have better values for the principal are likely to have better outside options.

Under the above assumption, using the same argument as in Lemma 2, without loss, we can consider weakly increasing allocations. By Lemma 3, for monotonic allocations the feasibility constraint (F) is equivalent to

$$(\mathrm{F}_{\max}) \qquad \int_x^b g(y)\mathrm{d}F(y) \leq 1 - F^n(x), \quad \text{for all } x \in X.$$

The second issue is that, even after simplifying the feasibility constraint, we must still handle a non-trivial interaction between feasibility and type-dependent incentive compatibility. To address this complexity, we separate the global incentive constraint (IC) into two simpler constraints. Let $r = \sup_{y \in X} g(y)$. Then, (IC) can be expressed as (c.f. Lemma 4)

$$(\mathrm{IC}_{\max}) \qquad g(x) \leq r, \qquad x \in X,$$

$$(\mathrm{IC}_{\min}) \qquad g(x) \geq h(x)r, \quad x \in X,$$

where $h(x)$ denotes the share of the after-penalty surplus truncated at zero:

$$h(x) = \max\left\{\frac{v(x) - c(x)}{v(x)}, 0\right\}, \quad x \in X.$$

For every $r \in \mathbb{R}_+$, derivation of a solution of ($\mathrm{F}_{\max}$) subject to ($\mathrm{IC}_{\max}$) and ($\mathrm{IC}_{\min}$), denoted by g_r, follows four steps.

Step 1. Existence. We identify the interval of r that ensures the existence of a feasible and incentive compatible allocation that respects $\sup g = r$. Let $\bar{r}$ be the greatest value of r that satisfies

$$\int_x^b rh(y)\mathrm{d}F(y) \leq 1 - F^n(x) \quad \text{for all } x \in X.$$

Observe that allocation $g(x) = h(x)r$, $x \in X$, is feasible and incentive compatible for all $r \in [0, \bar{r}]$. Moreover, since this is the minimal allocation that satisfies ($\mathrm{IC_{min}}$) for every given r, every incentive compatible allocation is infeasible when $r > \bar{r}$.

Step 2. Solution for negative types. The principal prefers to minimize the density assigned to the negative types. Denote by a_0 the greatest point in $[a, 0]$ that satisfies

$$\text{(A1)} \qquad \int_a^{a_0} rh(y)\mathrm{d}F(y) \geq F^n(a_0).$$

There are two possibilities. First, $a_0 = 0$ and $\int_a^0 rh(y)\mathrm{d}F(y) > F^n(a_0)$. That is, the only binding constraint for below-zero types is ($\mathrm{IC_{min}}$), so these types can be assigned the minimal incentive compatible density, $g_r(x) = h(x)r$ for all $x < 0$. Moreover, the principal prefers to allocate all available probability mass to the positive types. Thus, the total mass to types in $[0, b]$ must be fully allocated at the optimum, $\int_0^b g_r(y)\mathrm{d}F(y) = 1 - F^n(0)$.

The second possibility is $a_0 \leq 0$ and $\int_a^0 rh(y)\mathrm{d}F(y) = F^n(a_0)$. That is, the assignment of the minimal incentive compatible density $g_r(x) = h(x)r$ is feasible only for types in $[0, a_0]$. Incentive and feasibility constraints meet at a_0, and for type a_0, the feasibility constraint is binding, $\int_{a_0}^b g_r(y)\mathrm{d}F(y) = 1 - F^n(a_0)$. The feasibility constraint ($\mathrm{F_{max}}$) then implies $\int_{a_0}^b g_r(y)\mathrm{d}F(y) = 1 - F^n(a_0)$.

To sum up, in either case, we set $g_r(x) = h(x)r$ for all $x < a_0$, and the feasibility constraint must be binding at a_0,

$$\text{(A2)} \qquad \int_{a_0}^{b} g_r(y)\mathrm{d}F(y) = 1 - F^n(a_0),$$

so the total mass to types in $[a_0, b]$ must be fully allocated at the optimum. This constraint means that an agent should be selected unless all agents have types below a_0. Conditions ($\mathrm{F_{max}}$) and (A2) imply the following constraint:

$$(\mathrm{F_{min}}) \qquad \int_{a_0}^{x} g(y)\mathrm{d}F(y) \geq F^n(x) - F^n(a_0) \quad \text{for all } x \in [0, b].$$

In what follows, we disregard the types below a_0 and solve the problem on $[a_0, b]$ subject to constraint (A2).

Step 3. Concatenation of the maximal and the minimal solutions. To find an optimal allocation for the types above a_0, we consider two auxiliary problems, ($\mathrm{P_{max}}$) and ($\mathrm{P_{min}}$), whose solutions are the pointwise maximal and minimal functions subject to, respectively, ($\mathrm{IC_{max}}$)-($\mathrm{F_{max}}$) and ($\mathrm{IC_{min}}$)-($\mathrm{F_{min}}$). Allocation g_r is constructed by concatenating the two solutions.

Let $\bar{G}(x) := \int_x^b g(t)\mathrm{d}F(t)$ and consider the following problem:

$$\max_g \int_{a_0}^b \bar{G}(x)\mathrm{d}x \text{ s.t. } (\mathrm{IC}_{\max}) \text{ and } (\mathrm{F}_{\max}). \tag{$\mathrm{P}_{\max}$}$$

Similarly, let $\underline{G}(x) := \int_{a_0}^x g(y)\mathrm{d}F(y)$ and consider the following problem:

$$\min_g \int_{a_0}^b \underline{G}(x)\mathrm{d}x \text{ s.t. } (\mathrm{IC}_{\min}) \text{ and } (\mathrm{F}_{\min}). \tag{$\mathrm{P}_{\min}$}$$

Problems ($\mathrm{P}_{\max}$) and ($\mathrm{P}_{\min}$) are the same as (P), but with relaxed incentive compatibility, subject to only ($\mathrm{IC}_{\max}$) and ($\mathrm{IC}_{\min}$), respectively. Indeed, notice that the objective functions are the same up to a constant (by integration by parts). In addition, with a constant mass to be allocated, (A2), constraint ($\mathrm{F}_{\min}$) is equivalent to ($\mathrm{F}_{\max}$), but is expressed in terms of the complement sets. Thus, for any given r, ($\mathrm{P}_{\max}$) is the problem where the original incentive constraint (IC) is replaced by the constraint in which the probability of allocation to all types is capped by r. Similarly, ($\mathrm{P}_{\min}$) is the problem where the original incentive constraint (IC) is replaced by the constraint in which the probability of allocation to each type x is at least $rh(x)$.

A *concatenation* is an allocation g_r that satisfies for some $z \in (a_0, b]$:

$$g_r(x) = \begin{cases} rh(x), & x \in [a, a_0), \\ \underline{g}_r(x), & x \in [a_0, z), \\ \overline{g}_r(x), & x \in [z, b], \end{cases} \tag{A3}$$

where $\underline{g}_r(x)$ and $\overline{g}_r(x)$ denote the solutions of ($\mathrm{P}_{\min}$) and ($\mathrm{P}_{\max}$). We say that g_r is an *incentive-feasible concatenation* if it satisfies ($\mathrm{IC}_{\max}$), ($\mathrm{IC}_{\min}$), (F), and (A2).

Theorem 2 *A reduced-form allocation rule g^* is a solution of* (P) *if and only if g^* is an incentive-feasible concatenation g_r, where r solves*

$$\max_{r \in [0, \bar{r}]} \int_X x g_r(x)\mathrm{d}F(x).$$

Before proving the theorem, let us discuss what the solutions of the auxiliary problems ($\mathrm{P}_{\max}$) and ($\mathrm{P}_{\min}$) look like. The solution $\overline{g}_r$ of ($\mathrm{P}_{\max}$) is the pointwise maximal function subject to the constraints, as the following lemma shows.

Lemma 5 *For every $r \in [0, \bar{r}]$, the solution of* ($\mathrm{P}_{\max}$) *is equal to*

$$\overline{g}_r(x) = \begin{cases} nF^{n-1}(x), & x \in [a_0, \bar{x}_r), \\ r, & x \in [\bar{x}_r, b], \end{cases}$$

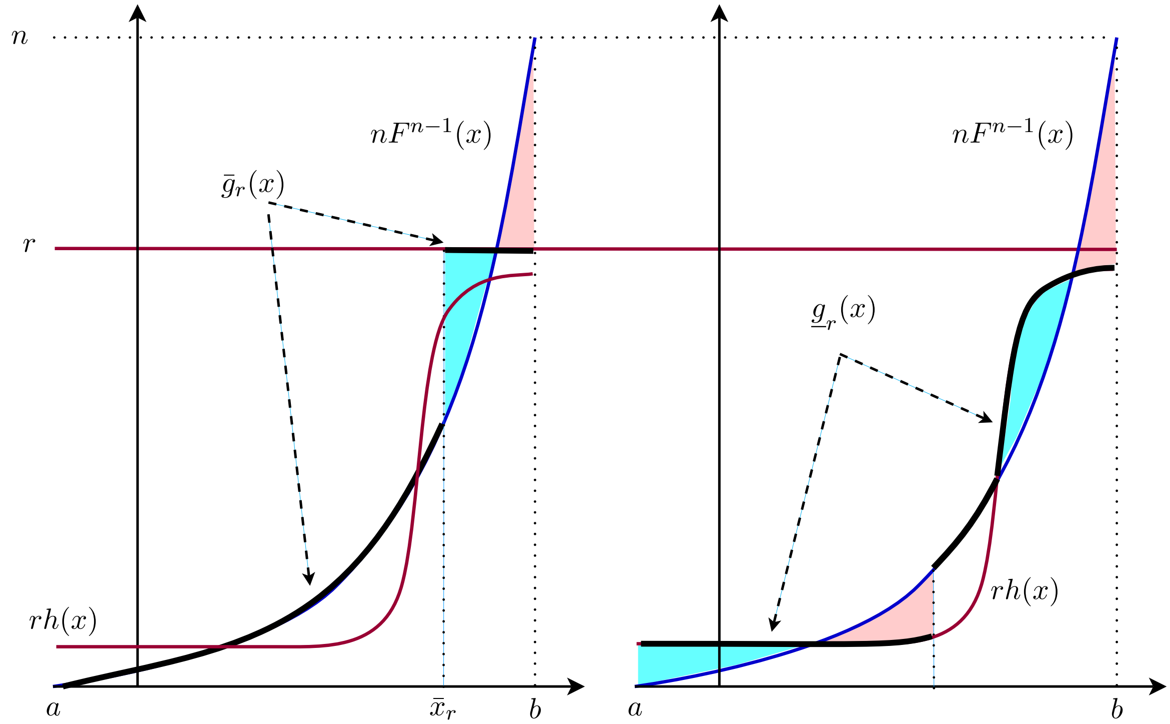

FIG. A1. EXAMPLES OF SOLUTIONS OF P_{MAX} (LEFT) AND P_{MIN} (RIGHT).

where $\bar{x}_r < b$ *is implicitly defined by*

$$\int_{\bar{x}_r}^{b} r \mathrm{d}F(x) = 1 - F^n(\bar{x}_r). \tag{A4}$$

Proof. As $r \leq \bar{r} < nF^{n-1}(b) = n$, there exists $\bar{x}_r$ such that the feasibility constraint ($\mathrm{F}_{\max}$) does not bind, while the incentive constraint ($\mathrm{IC}_{\max}$) binds for $x \geq \bar{x}_r$, and the opposite is true for $x < \bar{x}_r$. Consequently, $\overline{g}_r(x) = r$ for $x \geq \bar{x}_r$, while $\overline{g}_r(x) = nF^{n-1}(x)$ for $x < \bar{x}_r$. The value of $\bar{x}_r$ is the unique solution of (A4). i.e., the feasibility constraint binds at all $x \leq \bar{x}_r$ and slacks at all $x > \bar{x}_r$. ■

The solution $\overline{g}_r$ is illustrated by Fig. A1 (left). The blue curve is $nF^{n-1}(x)$ and the red curve is r; the black curve depicts $\overline{g}_r(x)$. Starting from the right ($x = b$), the black line follows r so long as constraint ($\mathrm{F}_{\max}$) slacks. Down from point $\bar{x}_r$ constraint ($\mathrm{F}_{\max}$) is binding, and the highest $\bar{g}_r(x)$ that satisfies this constraint is exactly $nF^{n-1}(x)$ for $x < \bar{x}_r$.

Concerning the solution $\underline{g}_r$ of ($\mathrm{P}_{\min}$), it is the pointwise minimal function subject to the constraints. It is more complex, as it involves function $h(x)$ in the constraints. Fig. A1 (right) depicts an example of $\underline{g}_r$. The blue curve is $nF^{n-1}(x)$ and the red curve is $rh(x)$; the black curve depicts $\underline{g}_r(x)$. Starting from the left ($x = a$), the black line follows $rh(x)$ up to the point where the blue area is equal to the red area (so the feasibility constraint starts binding), and then jumps to $nF^{n-1}(x)$. Then, the black curve follows $nF^{n-1}(x)$ so long as it is above $rh(x)$. After the crossing point, the incentive constraint is binding again, and the black curve again follows $rh(x)$.

A more specific result can be obtained if we make an assumption of "single-crossing" of incentive and feasibility conditions. Recall that the feasibility constraint means that the probability of choosing a type above a certain level, x, cannot exceed the probability that such a type realizes, $1 - F^n(x)$, for a given distribution F and a given number of agents n. When the incentive constraint is absent, $h = 0$, all that matters is the feasibility constraint. As we increase h uniformly for all x (constant h), in ($\mathrm{P_{min}}$) (where the constraint $g(x) \leq r$ is ignored), the incentive constraint $g(x) \geq rh(x)$ will be binding for all types below some threshold, but the feasibility constraint is still binding for all types above the threshold. The "single-crossing" assumption is a sufficient condition that yields this structure for type-dependent h. It precludes multiple alternating intervals where one of the constraints, incentive or feasibility, binds and the other slacks. Formally, for every r, there exists a threshold $\underline{x}_r$ such that, for function $g(x) = rh(x)$, the feasibility constraint ($\mathrm{F_{min}}$) is satisfied (possibly, with slack) on interval $[a_0, x]$ for any x below the threshold and is violated for any x above the threshold.

Assumption 2 (Single-crossing property) *For every $r \in \mathbb{R}_+$, there exists $\underline{x}_r \in [0, b]$ such that*

$$\text{(A5)} \qquad \int_{a_0}^{x} rh(y)\mathrm{d}F(y) \geq F^n(x) - F^n(a_0) \text{ if and only if } x \leq \underline{x}_r.$$

Assumption 2 is clearly satisfied under constant h. The concavity of $h(F^{-1}(\cdot))$ is also sufficient.

Lemma 6 *Assumption 2 holds if $h(F^{-1}(t))$ is weakly concave.*

Proof. By the concavity of $h(F^{-1}(t))$, for every $n \geq 1$, $h(F^{-1}(t)) - nt^{n-1}$ is concave. Hence, by the monotonicity of F, for all $r \geq 0$,

$$rh(y) - nF^{n-1}(y) \text{ is quasiconcave.}$$

It is immediate that the subset of $(a_0, b]$ where expression $\int_{a_0}^{x}(rh(y) - nF^{n-1}(y))\mathrm{d}F(y)$ is negative is a (possibly, empty) interval $(\underline{x}_r, b]$. If that expression is nowhere negative, then $\underline{x}_r = b$; if it is everywhere negative, then $\underline{x}_r = a_0$. Then, (A5) is immediate. ■

An example that satisfies Assumption 2 is a linear value of the prize, $v(x) = \alpha x + \beta$, and constant penalty, $c(x) = c$, $\beta \geq c \geq 0$, provided that $F^{n-1}(x)$ is weakly convex.

Lemma 7 *Let Assumption 2 hold. Then, for every $r \in [0, \bar{r}]$, the solution of problem* $(\mathrm{P_{min}})$ *is equal to*

$$\underline{g}_r(x) = \begin{cases} rh(x), & x \in [a_0, \underline{x}_r], \\ nF^{n-1}(x), & x \in (\underline{x}_r, b]. \end{cases} \tag{A6}$$

Proof. By Assumption 2, we have $(\mathrm{IC_{min}})$ binding on $[a_0, \underline{x}_r]$ and $(\mathrm{F_{min}})$ binding on $(\underline{x}_r, b]$. Consequently, $\underline{g}_r(x) = rh(x)$ on $[a_0, \underline{x}_r]$, while $\underline{g}_r(x) = nF^{n-1}(x)$ on $(\underline{x}_r, b]$. ■

Proof of Theorem 2. Because of the condition (A2), we can interpret $\frac{g(x)f(x)}{1-F^n(a_0)}$ as the probability density on $[a_0, b]$. A necessary condition for allocation g to be optimal is that

$$G(x) := \frac{1}{1 - F^n(a_0)} \int_{a_0}^{x} g(y) \mathrm{d}F(y)$$

is maximal w.r.t. the first-order stochastic dominance order (f.o.s.d.) on the set of c.d.f.s that satisfy (IC) and (F). We will prove that the set of f.o.s.d. maximal functions is the set of incentive-feasible concatenations $\{g_r\}_{r \in [0, \bar{r}]}$. Optimization on the set of these functions yields the solutions of (P).

Indeed, consider an arbitrary $\tilde{g}$ that satisfies (IC), (F), and(A2), where $r = \sup_X \tilde{g}(x)$. Let us compare

$$\tilde{G}(x) = \frac{1}{1 - F^n(a_0)} \int_{a_0}^{x} \tilde{g}(y) \mathrm{d}F(y) \quad \text{and} \quad G_r(x) = \frac{1}{1 - F^n(a_0)} \int_{a_0}^{x} \tilde{g}_r(y) \mathrm{d}F(y),$$

where g_r is an incentive-feasible concatenation (A3), where $\underline{g}_r$ and $\overline{g}_r$ are concatenated at some z. Because $\underline{g}_r$ is the solution of $(\mathrm{P_{min}})$, we have for all $x \leq z$

$$G_r(x) = \frac{1}{1 - F^n(a_0)} \int_{a_0}^{x} \underline{g}_r(y) \mathrm{d}F(y) \leq \frac{1}{1 - F^n(a_0)} \int_{a_0}^{x} \tilde{g}(y) \mathrm{d}Fy) = \tilde{G}(x).$$

Furthermore, because $\overline{g}_r$ is the solution for $(\mathrm{P_{max}})$, for all $x > z$, we have

$$1 - G_r(x) = \frac{1}{1 - F^n(a_0)} \int_{x}^{b} \overline{g}_r(t) \mathrm{d}F(t) \geq \frac{1}{1 - F^n(a_0)} \int_{x}^{b} \tilde{g}(t) \mathrm{d}F(t) = 1 - \tilde{G}(x).$$

Hence, G_r f.o.s.d. $\tilde{G}$.

It remains to show that for every $r \in [0, \bar{r}]$ there exists a unique incentive-feasible concatenation g_r. For g_r to be feasible, it must satisfy (A2) or, equivalently,

$$\text{(A7)} \qquad \int_{a_0}^{z} \underline{g}_r(x)\mathrm{d}F(x) + \int_{z}^{b} \overline{g}_r(x)\mathrm{d}F(x) = 1 - F^n(a_0).$$

Let z be the greatest solution of (A7). Such a solution exists, because the value of $\int_{a_0}^{z} \underline{g}_r(x)\mathrm{d}F(x) + \int_{z}^{b} \overline{g}_r(x)\mathrm{d}F(x)$ is continuous in z (recall that F is assumed to be continuously differentiable), and by ($\mathrm{F_{min}}$) and ($\mathrm{F_{max}}$),

$$\int_{a_0}^{b} \overline{g}_r(x)\mathrm{d}F(x) \leq 1 - F^n(a_0) \leq \int_{a_0}^{b} \underline{g}_r(x)\mathrm{d}F(x) \quad \text{for all } r \in [0, \bar{r}].$$

First, we show that $z \geq \overline{x}_r$, and consequently, $g_r(x) = r$ for all $x \geq z$ by Lemma 5. By definition, ($\mathrm{F_{max}}$) is satisfied with equality by $\overline{g}_r$ at $x = \bar{x}_r$. If ($\mathrm{F_{min}}$) is also satisfied with equality by $\underline{g}_r$ at $x = \bar{x}_r$, then (A7) is satisfied with $z = \bar{x}_r$. Hence, the greatest solution of (A7) is weakly higher than $\bar{x}_r$. If, in contrast, ($\mathrm{F_{min}}$) is satisfied with strict inequality at $x = \bar{x}_r$, then the left-hand side of (A7) is less than $1 - F^n(a_0)$ at $\bar{x}_r$, is increasing in z, and has a solution on $(\bar{x}_r, b]$. Thus, $z \geq \overline{x}_r$.

Furthermore, consider any solution z' of (A7) such that $z' < \bar{x}_r$. Then, either ($\mathrm{IC_{min}}$) is violated at some $x \geq z'$, in which case concatenation obtained at z' is not incentive compatible, or ($\mathrm{F_{min}}$) is satisfied with equality for all $x > z'$, so $\underline{g}_r(x) = nF^{n-1}(x)$ on $[z', \bar{x}_r]$. In addition, by Lemma 5, $\overline{g}_r(x) = nF^{n-1}(x)$ on $[z', \bar{x}_r]$. Hence, concatenation at any $z \in [z', \bar{x}_r]$ produces the same g_r^* and, furthermore, $z = \bar{x}_r$ is the greatest solution of (A7). Hence, an incentive-feasible concatenation is unique.

Next, we show that, for every $r \in [0, \bar{r}]$, g_r satisfies (IC), (A2), and (F). Note that g_r satisfies (A2) and (F) by construction. To prove that g_r satisfies (IC), we need to verify that $\underline{g}_r(x)$ satisfies ($\mathrm{IC_{max}}$) for $x < z$ and $\overline{g}_r(x)$ satisfies ($\mathrm{IC_{min}}$) for $x \geq z$. We have shown above that $\overline{g}_r(x) = r$ for all $x \geq z$, which trivially satisfies ($\mathrm{IC_{min}}$). To verify ($\mathrm{IC_{max}}$), observe that, for $x \leq z$, it must be that $\underline{g}_r(x) \leq r$, as otherwise z is not a solution of (A7). Assume by contradiction that $\underline{g}_r(x') > r$ for some $x' \leq z$. Since $rh(x') < r$, the constraint ($\mathrm{F_{min}}$) must be binding at x', implying $\underline{g}_r(x') = nF^{n-1}(x') \geq r$. However, we have shown above that either $z = \bar{x}_r$ or ($\mathrm{F_{min}}$) is not binding at z. We obtain the contradiction in the former case because $nF^{n-1}(x') < nF^{n-1}(\bar{x}_r) < r$, where the last inequality is by construction of $\bar{x}_r$. In the latter case, $\underline{g}_r(z) < r$, implying that $\underline{g}_r$ is decreasing somewhere on $[x', z]$, which is impossible by ($\mathrm{F_{min}}$) since ($\mathrm{F_{min}}$) is satisfied with equality at x'. ■

Online Appendix: Omitted Proofs

Proof of Lemma 2. Consider an allocation $g(x)$ that satisfies (IC) and (F). We construct a monotonic $\tilde{g}(x)$ that preserves constraints (IC) and (F), but increases the principal's payoff.

We have assumed that F has almost everywhere positive density, so F^{-1} exists. Define

$$S(t) = \left|\{y : g(F^{-1}(y)) \leq t\}\right|, \quad t \in \mathbb{R}_+.$$

Note that S is weakly increasing and satisfies $S(t) \in [0,1]$ for all t. Define

$$\tilde{g}(x) = S^{-1}(F(x))$$

for all x where $S^{-1}(F(x))$ exists, and extend $\tilde{g}$ to $[a,b]$ by right continuity. Observe that $\tilde{g}$ satisfies (F) by construction. In addition,

$$\sup_{x\in[a,b]} g(x) = \sup_{y\in[0,1]} g(F^{-1}(y)) = S^{-1}(1) = \sup_{y\in[0,1]} \tilde{g}(F^{-1}(y)) = \sup_{x\in[a,b]} \tilde{g}(x),$$

thus $\tilde{g}$ satisfies (IC). Finally, we show that $\tilde{g}$ yields a weakly greater payoff to the principal. By construction,

$$\int_a^z \tilde{g}(x)\mathrm{d}F(x) \leq \int_a^z g(x)\mathrm{d}F(x) \quad \text{for all } z \in [a,b],$$

and it holds with equality for $z = b$. Hence, using integration by parts, the expression

$$\int_a^b x(\tilde{g}(x) - g(x))\mathrm{d}F(x) = b\int_a^b (\tilde{g}(x) - g(x))\mathrm{d}F(x) \quad - \int_a^b \left(\int_a^z (\tilde{g}(x) - g(x))\mathrm{d}F(x)\right)\mathrm{d}z$$

is nonnegative. ■

Proof of Corollary 3. Let $Q = \int_a^{z^*} q\mathrm{d}F(x) + \int_{z^*}^b \mathrm{d}F(x)$ be the ex-ante probability to be short-listed, and let A and B be the expected probabilities to be chosen conditional on being shortlisted and conditional on not being short-listed, respectively:

$$A = \sum_{k=1}^{n} \frac{1}{k}\binom{n-1}{k-1} Q^{k-1}(1-Q)^{n-k} \quad \text{and} \quad B = \frac{1}{n}(1-Q)^{n-1}.$$

The associated reduced-form rule is as follows. An agent's probability $g_i(x)$ to be chosen conditional on $x_i \geq z^*$ and $x_i < z^*$ is given by A and $qA + (1-q)B$, respectively. Hence,

$$g(x) \equiv \sum\nolimits_i g_i(x) = \begin{cases} n(qA + (1-q)B), & x < z^*, \\ nA, & x \geq z^*. \end{cases} \tag{B1}$$

We now prove that g is identical to g^* whenever q satisfies (15). We have

$$
\begin{aligned}
\text{(B2)}\qquad Q &= \int_a^{z^*} q\mathrm{d}F(x) + \int_{z^*}^b \mathrm{d}F(x) = \int_a^{z^*} \left(1 - \frac{c}{s}\right)\mathrm{d}F(x) + \int_{z^*}^b \mathrm{d}F(x) \\
&= \left(\int_a^{z^*} \left(\frac{1-c}{s} - \frac{1-s}{s} \right) \mathrm{d}F(x) + \int_{z^*}^b \left(\frac{1}{s} - \frac{1-s}{s} \right) \mathrm{d}F(x) \right) \\
&= \frac{1}{s} \left(\int_a^{z^*} (1-c)\mathrm{d}F(x) + \int_{z^*}^b \mathrm{d}F(x) \right) - \frac{1-s}{s} \\
&= \frac{1/r^*}{s} - \frac{1-s}{s} = \frac{1 - r^* + r^* s}{r^* s},
\end{aligned}
$$

where we used (9). Hence, $1 - Q = \frac{r^*-1}{r^*s}$. Next,

$$
\begin{aligned}
A &= \sum_{k=1}^n \frac{1}{k} \frac{(n-1)!}{(k-1)!(n-k)!} Q^{k-1}(1-Q)^{n-k} = \frac{1}{nQ} \sum_{k=1}^n \frac{n!}{k!(n-k)!} Q^k (1-Q)^{n-k} \\
&= \frac{1}{nQ} \left(1 - (1-Q)^n\right).
\end{aligned}
$$

Substituting (B2) into the above yields

$$A = \frac{r^* s}{n(1 - r^* + r^* s)} \left(1 - \frac{(r^*-1)^n}{(r^* s)^n} \right).$$

By (16), after some algebraic transformations,

$$A = \frac{r^* s}{n(1 - r^* + r^* s)} \left(1 - \frac{(r^*-1)^n}{(r^* s)^n} \right) = \frac{r^*}{n}.$$

Also, using (B2) and (16) we obtain

$$B = \frac{1}{n}(1-Q)^{n-1} = \frac{1}{n} \frac{(r^*-1)^{n-1}}{(r^* s)^{n-1}} = \frac{(1-s)r^*}{n}.$$

Substitute A and B into (B1):

$$n(qA + (1-q)B) = \frac{(s-c)nA + cnB}{s} = \frac{(s-c)r^* + c(1-s)r^*}{s} = (1-c)r^*$$

and $nA = r^*$. Hence, $g(x) = g^*(x)$ for all $x \in X$.

It remains to show that, whenever $n \geq \bar{n}$, this shortlisting procedure is feasible and well defined, i.e., $h \geq s$ and the solution of (16) exists and is unique.

Let $n \geq \bar{n}$. Observe that $F(z^*) < 1$, as evident from (8) and the assumption that $c > 0$. Using the definition of r^*, we can rewrite (14) as

$$r^* \leq \frac{1 - F^n(z^*)}{1 - F(z^*)} = 1 + F(z^*) + F^2(z^*) + \ldots + F^{n-1}(z^*) < n.$$

In addition, $1/r^* = (1-c)F(z^*) + 1 - F(z^*) < 1$. Consequently, $\frac{1}{n} < \frac{1}{r^*} < 1$.

Observe that $(1-s)s^{n-1}$ unimodal on $[0,1]$ with zero at the endpoints and the maximum at $s = \frac{n-1}{n}$. Moreover, it is strictly decreasing on $[\frac{n-1}{n}, 1]$. Since the right-hand side of (16) is strictly between zero and the maximum, there exists a unique solution of (16) on $[\frac{n-1}{n}, 1]$.

Now we prove that $c \leq s$. It is immediate if $c \leq \frac{n-1}{n}$ (since $s \in [\frac{n-1}{n}, 1]$). Assume now that $c > \frac{n-1}{/}n$. Because $n \geq \bar{n}$, condition (14) must hold, which can be written as

$$F^{n-1}(z^*) \leq (1-c)r^*.$$

Thus, the right-hand side of (16) satisfies:

$$\frac{1}{r^*}\left(1 - \frac{1}{r^*}\right)^{n-1} = \frac{\Big(cF(z^*)\Big)^{n-1}}{r^*} \leq (1-c)c^{n-1}.$$

That is, $n \geq \bar{n}$ and (16) entail

$$(1-s)s^{n-1} = \frac{1}{r^*}\left(1 - \frac{1}{r^*}\right)^{n-1} \leq (1-c)c^{n-1}.$$

As $(1-s)s^{n-1}$ is decreasing on $[\frac{n-1}{n}, 1]$ and we have assumed $c > (n-1)/n$, it follows that $c \leq s$. ■

Proof of Proposition 3. We have already established that the solution g must satisfy (21) for some $r \in R = [1, \min\{n, 1/(1-c)\}]$. It remains to show that the optimal r is the unique solution of (22).

Let us first derive how $\overline{x}_r$ and $\underline{x}_r$ change w.r.t. r. From (19) we have

$$(1 - F(\overline{x}_r))\mathrm{d}r - rf(\overline{x}_r)\mathrm{d}\overline{x}_r = -nF^{n-1}(\overline{x}_r)f(\overline{x}_r)\mathrm{d}\overline{x}_r.$$

Hence,

$$\frac{\mathrm{d}\overline{x}_r}{\mathrm{d}r} = \frac{1 - F(\overline{x}_r)}{(r - nF^{n-1}(\overline{x}_r))f(\overline{x}_r)},$$

and thus

$$\text{(B3)} \qquad \overline{x}_r(nF^{n-1}(\overline{x}_r) - r)f(\overline{x}_r)\frac{\mathrm{d}\overline{x}_r}{\mathrm{d}r} = -\overline{x}_r(1 - F(\overline{x}_r)).$$

Next, if $\underline{x}_r = 0$, then $\frac{\mathrm{d}\underline{x}_r}{\mathrm{d}r} = 0$. Suppose that $\underline{x}_r > 0$. By (20) it satisfies

$(1-c)rF(\underline{x}_r)+1-F^n(\underline{x}_r)=1$. Hence,

$$(1-c)F(\underline{x}_r)\mathrm{d}r+(1-c)rf(\underline{x}_r)\mathrm{d}\underline{x}_r-nF^{n-1}(\underline{x}_r)f(\underline{x}_r)\mathrm{d}\underline{x}_r=0.$$

Hence,

$$\frac{\mathrm{d}\underline{x}_r}{\mathrm{d}r}=\begin{cases}\frac{F(\overline{x}_r)}{(nF^{n-1}(\underline{x}_r)-(1-c)r)f(\underline{x}_r)}, & \text{if } \underline{x}_r>0,\\ 0, & \text{if } \underline{x}_r=0.\end{cases}$$

Thus we obtain

$$\text{(B4)}\qquad \underline{x}_r((1-c)r-nF^{n-1}(\underline{x}_r))f(\underline{x}_r)\frac{\mathrm{d}\overline{x}_r}{\mathrm{d}r}=-\underline{x}_rF(\underline{x}_r).$$

Finally, with $g=g_r$, the principal's objective function is

$$W(r)=\int_a^{\underline{x}_r}x(1-c)r\mathrm{d}F(x)+\int_{\underline{x}_r}^{\overline{x}_r}xnF^{n-1}(x)\mathrm{d}F(x)+\int_{\overline{x}_r}^b xr\mathrm{d}F(x).$$

Taking the derivative w.r.t. r and using (B3) and (B4) we obtain

$$\begin{aligned}\frac{\mathrm{d}W(r)}{\mathrm{d}r}&=\int_a^{\underline{x}_r}x(1-c)\mathrm{d}F(x)+\int_{\overline{x}_r}^b x\mathrm{d}F(x)+\underline{x}_r((1-c)r-nF^{n-1}(\underline{x}_r))f(\underline{x}_r)\frac{\mathrm{d}\overline{x}_r}{\mathrm{d}r}\\&\quad+\overline{x}_r(nF^{n-1}(\overline{x}_r)-r)f(\overline{x}_r)\frac{\mathrm{d}\overline{x}_r}{\mathrm{d}r}\\&=\int_a^{\underline{x}_r}x(1-c)\mathrm{d}F(x)+\int_{\overline{x}_r}^b x\mathrm{d}F(x)-\underline{x}_rF(\underline{x}_r)-\overline{x}_r(1-F(\overline{x}_r))\\&=\int_a^{\underline{x}_r}(x-\underline{x}_r)(1-c)\mathrm{d}F(x)+\int_{\overline{x}_r}^b(x-\overline{x}_r)\mathrm{d}F(x).\end{aligned}$$

The equation $\frac{\mathrm{d}W(r)}{\mathrm{d}r}=0$ is exactly (22). To show that it has a unique solution, observe that $\frac{\mathrm{d}\underline{x}_r}{\mathrm{d}r}\geq 0$ and $\frac{\mathrm{d}\overline{x}_r}{\mathrm{d}r}>0$ (since $g_r(\underline{x}_r)=nF^{n-1}(\underline{x}_r)\geq(1-c)r$ and $g_r(\overline{x}_r)=nF^{n-1}(\overline{x}_r)\leq r$ by (IC)). Consequently, $\frac{\mathrm{d}W(r)}{\mathrm{d}r}$ is strictly decreasing in r. Moreover, for r sufficiently close to 0, we have both $\underline{x}_r$ and $\overline{x}_r$ close to a, in which case $W(r)>0$, and similarly, for $r=1/(1-c)$, we have $\overline{x}_r=\underline{x}_r=b$, in which case $W(r)<0$. ■

Proof of Propositions 5a, 5b, 5c. The points of interest are the optimal principal's payoff z^* and the structure of the optimal allocation mechanism.

First, let us deal with the optimal principal's payoff z^*.

5a: Increasing c affects only the incentive constraint (IC) by making it looser. Optimization on a larger set yields a weakly higher optimal payoff.

5b: Increasing n affects only the feasibility constraint (F) by making it looser. Optimization on a larger set yields a weakly higher optimal payoff. When $n\geq\bar{n}$, the feasibility constraint is not binding and hence has no effect on the optimal payoff.

5c: Let $\tilde{F}(x) \leq F(x)$ for all x. This affects the feasibility constraint (F) by making it looser for all x. Optimization on a larger set yields a weakly higher optimal payoff.

Next, we deal with the structure of the optimal allocation mechanism: threshold $\bar{x}$ of the high pooling interval and threshold $\underline{x}$ of the low pooling interval for the case of $n < \bar{n}$. The interval $[\underline{x}, \overline{x}]$ is the separating interval. There are three cases to consider.

Case 1: $n \geq \bar{n}$. By Proposition 2, the optimal allocation has to satisfy the equation

$$(1-c)\int_a^{z^*}(z^*-x)\mathrm{d}F(x) = \int_{z^*}^b (x-z^*)\mathrm{d}F(x).$$

Integrating by parts, we obtain

$$(1-c)\int_a^{z^*} F(x)\mathrm{d}x = \int_{z^*}^b (1-F(x))\mathrm{d}x. \tag{B5}$$

In this case, the threshold of the high pooling interval and the principal's payoff are the same, $\bar{x} = z^*$. The separating interval is empty.

5a: From (B5) it is immediate that $\frac{\mathrm{d}z^*}{\mathrm{d}c} > 0$. That is, the size of the high pooling interval is decreasing in c.

5b: Equation (B5) is independent of n, so a change in n has no effect (so long as $n \geq \bar{n}$).

5c: Let $\tilde{F}(x) \leq F(x)$ for all x. From (B5) it is immediate that replacing F with $\tilde{F}$ yields a greater solution z^*. That is, the high pooling interval shrinks.

Case 2: $n < \bar{n}$ and $\underline{x} = 0$. By Proposition 3, the optimal allocation has to satisfy equation (22) where we use $\underline{x} = 0$:

$$\int_a^0 (-x)(1-c)\mathrm{d}F(x) = \int_{\overline{x}}^b (x-\overline{x})\mathrm{d}F(x).$$

Integrating by parts, we obtain

$$(1-c)\int_a^0 F(x)\mathrm{d}x = \int_{\overline{x}}^b (1-F(x))\mathrm{d}x. \tag{B6}$$

Note that (19) is satisfied, as it has a free variable r that does not appear in (B6).

Assuming that variations of the parameters are marginal and $\underline{x}$ remains equal to zero, the value of interest is the threshold $\overline{x}$ of the high pooling interval. The change in the length of the separating interval $t = \overline{x} - \underline{x}$ is the same as the change in $\overline{x}$.

5a: From (B6) it is immediate that $\frac{\mathrm{d}\overline{x}}{\mathrm{d}c} > 0$. That is, the high pooling interval is decreasing and the separating interval is increasing in c.

5b: Equation (B6) is independent of n. Hence, a change in n has no effect, so long as $\underline{x} = 0$.

5c: Let $\tilde{F}(x) \leq F(x)$ for all x. From (B6) it is immediate that replacing F by $\tilde{F}$ yields a greater solution $\overline{x}$. That is, the high pooling interval shrinks and the separating interval expands.

Case 3: $n < \bar{n}$ and $\underline{x} > 0$. By Proposition 3, the optimal allocation is described by thee variables, $\overline{x}$, $\underline{x}$, and r, that must satisfy (19), (20), and (22). Combining (19) and (20) to eliminate r, we obtain

$$(1-c)\frac{1-F^n(\bar{x})}{1-F(\bar{x})} = F^{n-1}(\underline{x}). \tag{B7}$$

Also, integrating (22) by parts, we obtain

$$(1-c)\int_a^{\underline{x}} F(x)\mathrm{d}x = \int_{\bar{x}}^b (1-F(x))\mathrm{d}x. \tag{B8}$$

Thus, the structure of the optimal allocation is characterized by $\bar{x}$ and $\underline{x}$ that satisfy (B7) and (B8).

Let us now evaluate $\frac{\mathrm{d}\bar{x}}{\mathrm{d}n}$, $\frac{\mathrm{d}\underline{x}}{\mathrm{d}n}$, $\frac{\mathrm{d}\bar{x}}{\mathrm{d}c}$, and $\frac{\mathrm{d}(\bar{x}-\underline{x})}{\mathrm{d}c}$. After taking the full differential of (B7) and (B8) w.r.t. $\bar{x}$, $\underline{x}$, c, and n, we obtain

$$\begin{aligned} 0 &= L_{\bar{x}}\mathrm{d}\bar{x} - L_{\underline{x}}\mathrm{d}\underline{x} - L_c\mathrm{d}c + L_n\mathrm{d}n, \\ 0 &= M_{\bar{x}}\mathrm{d}\bar{x} + M_{\underline{x}}\mathrm{d}\underline{x} - M_c\mathrm{d}c, \end{aligned} \tag{B9}$$

where

$$\begin{aligned} L_{\bar{x}} &= (1-c)\frac{\mathrm{d}}{\mathrm{d}\bar{x}}(1+F(\bar{x})+F^2(\bar{x})+...+F^{n-1}(\bar{x})) > 0, \\ L_{\underline{x}} &= \frac{\mathrm{d}}{\mathrm{d}\underline{x}}F^{n-1}(\underline{x}) > 0, \\ L_c &= 1+F(\bar{x})+F^2(\bar{x})+...+F^{n-1}(\bar{x}) > 0, \\ L_n &= -\left((1-c)\frac{F^n(\bar{x})}{1-F(\bar{x})}\ln F(\bar{x}) + F^{n-1}(\underline{x})\ln F(\underline{x})\right) > 0, \\ M_{\bar{x}} &= 1-F(\bar{x}) > 0, \\ M_{\underline{x}} &= (1-c)F(\underline{x}) > 0, \\ M_c &= \int_a^{\underline{x}} F(x)\mathrm{d}x > 0, \end{aligned}$$

where we used $c > 0$, $\underline{x} > a$ and $\bar{x} < b$ (i.e., the best payoff is better than random allocation) and that $f(x)$ is everywhere positive.

To evaluate $\frac{\mathrm{d}\bar{x}}{\mathrm{d}n}$ and $\frac{\mathrm{d}\underline{x}}{\mathrm{d}n}$, we set $\mathrm{d}c = 0$ and solve the system of equations (B9),

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}n} = -\frac{L_n M_{\underline{x}}}{L_{\bar{x}} M_{\underline{x}} + L_{\underline{x}} M_{\bar{x}}} < 0,$$
$$\frac{\mathrm{d}\underline{x}}{\mathrm{d}n} = \frac{L_n M_{\bar{x}}}{L_{\bar{x}} M_{\underline{x}} + L_{\underline{x}} M_{\bar{x}}} > 0,$$

and hence $\frac{\mathrm{d}(\bar{x}-\underline{x})}{\mathrm{d}n} < 0$.

To evaluate $\frac{\mathrm{d}\bar{x}}{\mathrm{d}c}$ and $\frac{\mathrm{d}\underline{x}}{\mathrm{d}c}$, we set $\mathrm{d}n = 0$ and solve the system of equations (B9),

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}c} = \frac{L_{\underline{x}} M_c + L_c M_{\underline{x}}}{L_{\bar{x}} M_{\underline{x}} + L_{\underline{x}} M_{\bar{x}}} > 0,$$
$$\frac{\mathrm{d}\underline{x}}{\mathrm{d}c} = \frac{L_{\bar{x}} M_c - L_c M_{\bar{x}}}{L_{\bar{x}} M_{\underline{x}} + L_{\underline{x}} M_{\bar{x}}}.$$

To prove $\frac{\mathrm{d}(\bar{x}-\underline{x})}{\mathrm{d}c} > 0$, it is sufficient to check that $\frac{L\underline{x} - L_{\bar{x}}}{(1-c)L_c} > 0$. By (B7) we have

$$L_c = 1 + F(\bar{x}) + F^2(\bar{x}) + ... + F^{n-1}(\bar{x}) = \frac{1}{1-c} F^{n-1}(\underline{x}).$$

Thus,

$$\begin{aligned}
\frac{L\underline{x} - L_{\bar{x}}}{(1-c)L_c} &= \frac{\frac{\mathrm{d}}{\mathrm{d}\bar{x}} F^{n-1}(\underline{x})}{F^{n-1}(\underline{x})} - \frac{\frac{\mathrm{d}}{\mathrm{d}\bar{x}}(1 + F(\bar{x}) + F^2(\bar{x}) + ... + F^{n-1}(\bar{x}))}{1 + F(\bar{x}) + F^2(\bar{x}) + ... + F^{n-1}(\bar{x})} \\
&= \frac{(n-1)f(\underline{x})}{F(\underline{x})} - \frac{(1 + 2F(\bar{x}) + ... + (n-1)F^{n-2}(\bar{x}))f(\bar{x})}{1 + F(\bar{x}) + F^2(\bar{x}) + ... + F^{n-1}(\bar{x})} \\
&> (n-1)\left(\frac{f(\underline{x})}{F(\underline{x})} - \frac{f(\bar{x})}{F(\bar{x})}\right) \geq 0,
\end{aligned}$$

where we use

$$\frac{(1 + 2x + 3x^2 ... + (n-1)x^{n-2})}{1 + x + x^2 + ... + x^{n-1}} < \frac{n-1}{x}, \quad x \in (0,1),$$

and the hazard rate condition, $F(x)/f(x)$ is increasing.

Lastly, we cannot conclude anything from (B7)-(B8) about how the thresholds change if F is f.o.s.d. improved.

To summarize:

5a: The high pooling interval decreases and, under the hazard rate condition, the separating interval increases in c;

5b: The high pooling interval increases and the separating interval decreases in n.

5c: The result is ambiguous. If $\tilde{F}(x) \leq F(x)$ for all x, we are unable to make any conclusions about how thresholds $\bar{x}$ and $\underline{x}$ change if F is replaced by $\tilde{F}$. ■