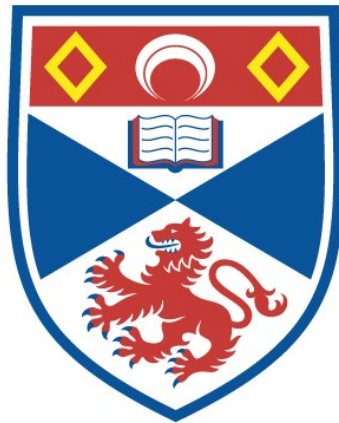


CONSTRUCTING 2-GENERATED SUBGROUPS OF THE GROUP OF HOMEOMORPHISMS OF CANTOR SPACE

James Thomas Hyde

A Thesis Submitted for the Degree of PhD
at the
University of St Andrews



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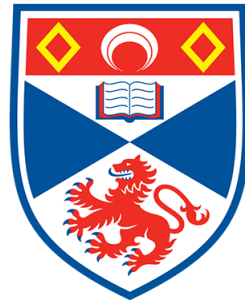
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Constructing 2-Generated Subgroups of the Group of Homeomorphisms of Cantor Space

James Thomas Hyde



University of
St Andrews

This thesis is submitted in partial fulfilment for the degree of PhD
at the University of St Andrews

April 30, 2017

Abstract

We study finite generation, 2-generation and simplicity of subgroups of $H_{\mathcal{C}}$, the group of homeomorphisms of Cantor space.

In Chapter 1 and Chapter 2 we run through foundational concepts and notation. In Chapter 3 we study vigorous subgroups of $H_{\mathcal{C}}$. A subgroup G of $H_{\mathcal{C}}$ is *vigorous* if for any non-empty clopen set A with proper non-empty clopen subsets B and C there exists $g \in G$ with $\text{supp}(g) \subseteq A$ and $Bg \subseteq C$. It is a corollary of the main theorem of Chapter 3 that all finitely generated simple vigorous subgroups of $H_{\mathcal{C}}$ are in fact 2-generated. We show the family of finitely generated, simple, vigorous subgroups of $H_{\mathcal{C}}$ is closed under several natural constructions.

In Chapter 4 we use a generalised notion of word and the tight completion construction from [13] to construct a family of subgroups of $H_{\mathcal{C}}$ which generalise Thompson's group V . We give necessary conditions for these groups to be finitely generated and simple. Of these we show which are vigorous. Finally we give some examples.

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I, James Thomas Hyde, hereby certify that this thesis, which is approximately 20000 words in length, has been written by me, and that it is the record of work carried out by me, or principally by myself in collaboration with others as acknowledged, and that it has not been submitted in any previous application for a higher degree.

I was admitted as a research student in September 2010 and as a candidate for the degree of Ph.D in September 2011; the higher study for which this is a record was carried out in the University of St Andrews between 2010 and 2014.

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Chapter 1

Introduction, Basic Notation and Terminology

1.1 Introduction

We are interested in simplicity, finite generation and 2-generation of subgroups of H_C the group of homeomorphisms of Cantor space. We study these properties firstly through their relationship to two other properties of subgroups of H_C and secondly as properties of groups in a particular family of subgroups of H_C constructed from inverse semigroups.

The thesis is organised as follows. The first two chapters give basic definitions and lay groundwork. In Chapter 3 we introduce two properties of groups of permutations (for us these will nearly always be homeomorphisms) of the Cantor set:

the property of being vigorous (a kind of transitivity condition) and the property of being flawless (a strengthening of the property of being perfect by taking into account the action of the group on the Cantor set). We use \mathcal{K} for the family of finitely generated simple vigorous groups of homeomorphisms of the Cantor set. The main objective of Chapter 3 is proving that groups in \mathcal{K} are in fact 2-generated. Towards this we show that for vigorous groups of homeomorphisms of the Cantor set being flawless is equivalent to being simple. We defer the proof of the main result of Chapter 3 till the end of the Chapter.

In the remainder of Chapter 3 we give some conditions on an inverse monoid of partial permutations of the Cantor set sufficient for its closure under compatible union to be vigorous. We also give some constructions that \mathcal{K} is closed under. Firstly the conjugate of a group in \mathcal{K} by a homeomorphism of the Cantor set is (trivially) also in \mathcal{K} . Secondly the union of two groups in \mathcal{K} generates a group in \mathcal{K} . Thirdly if G is in \mathcal{K} then under certain restrictions on an element g of G and a homeomorphism h of the Cantor set the group generated by $G \cup \{[g, h]\}$ is also in \mathcal{K} .

In Chapter 4 we investigate a construction generalising Thompson's group V . If S is a set then we write $\mathcal{I}_S^{\text{cof}}$ for the set of partial permutations of S whose domains and ranges include all but finitely many points of S . The construction takes as input a countably infinite set S , an alphabet A (which we take to be finite

of size at least 2) and an inverse submonoid I of $\mathcal{I}_S^{\text{cof}}$ and returns as output a group $\mathcal{V}(S, A, I)$ acting on A^S . First we introduce a generalised notion of strings which gives the construction an intuitive foundation. Once $\mathcal{V}(S, A, I)$ has been shown to be well defined we give some basic results. We then describe inputs which produce the group $2V$ and relate the construction to the broader context of tight completions. We then give necessary conditions on the inputs for the group to be finitely generated and simple. Finally we move onto examples of groups that may be constructed relating them to the property of being vigorous from Chapter 3.

1.2 Notation

Definition 1.2.1. For this thesis the set of natural numbers \mathbb{N} will be the set $\{1, 2, 3, \dots\}$. We emphasise we do not consider 0 to be a natural number.

Definition 1.2.2. For $n \geq 0$ we will use \mathbf{n} for the set $\{0, \dots, n - 1\}$.

Definition 1.2.3. If X is a set then the *powerset* of X is the set of subsets of X and it is denoted $P(X)$.

Definition 1.2.4. If X and Y are disjoint sets then we will write $X \sqcup Y$ for the union of X and Y . If X and Y are not disjoint then $X \sqcup Y$ is not defined. So if Z is also a set then the statement $X \sqcup Y = Z$ is equivalent to the statement $\{X, Y\}$ partitions Z .

For sets X and Y we define a *function* f from X to Y to be a subset of $X \times Y$ such that for each x in X there is a unique y in Y with the pair (x, y) in f .

Remark 1.2.5. In some works a different definition of function is used where functions are seen as triples (domain, co-domain, set of ordered pairs). We use our definition because it allows a cleaner and more transparent presentation of the contents of Chapter 4.

We will write $f : X \longrightarrow Y$ to indicate f is from X to Y . If (x, y) is in f then we will write $xf = y$. So if $f : X \rightarrow Y$ is a function from a set X to a set Y then

$$f = \{(x, xf) \mid x \in X\}.$$

A *partial function* from X to Y is a function from a subset $U \subseteq X$ to Y . We regard the partial function as equal to the underlying function and as such has the same domain and range. Since the property of being partial is not an intrinsic property of a set of pairs, it follows that the object we call a function can only be partial relative to some set.

If t is a partial function from X to Y then the *domain* of t is the set

$$\text{dom}(t) = \{x \in X \mid \text{there exists } y \in Y \text{ with } xt = y\}$$

and the *range* of t is the set

$$\text{ran}(t) = \{y \in Y \mid \exists x \in X \text{ with } y = xt\}.$$

We will act from the right and also treat partial functions, bijections and partial bijections as sets of pairs. One consequence of this is for sets U, V, X and Y and functions $f : U \rightarrow V$ and $t : X \rightarrow Y$ we will write $f \subseteq t$ if and only if $U \subseteq X$ and $uf = ut$ for all $u \in U$. We may also write functions as unions of other functions.

We will often abuse notation by allowing ourselves to apply functions not just to points in their domains but also to subsets of their domains. So if $f : R \rightarrow S$ then

$$R'f = \{rf \in S \mid r \in R'\}$$

for all $R' \subseteq R$.

We will say a subset S of a set T is *co-finite* in T if $T \setminus S$ is finite. Context will often allow us to say S is co-finite without explicitly mentioning the set T . For sets S and T we will write

- $\text{Sym}(S)$ for the set of bijections from S to S ,
- \mathcal{I}_S for the set of partial bijections from S to S ,
- \mathcal{I}_S^f for the set of partial bijections from S to S with finite domain,
- $\mathcal{I}_S^{\text{cof}}$ for the set of partial bijections from S to S with domain and image co-finite in S ,
- \mathcal{T}_S for the set of functions from S to S ,

- \mathcal{P}_S for the set of partial functions from S to S ,
- \mathcal{P}_S^f for the set of partial functions from S to S with finite domain,
- $\mathcal{P}_S^{\text{cof}}$ for the set of partial functions from S to S with domain and image co-finite in S .
- $\text{Sym}(S, T)$ for the set of bijections from S to T ,
- $\mathcal{I}_{S,T}$ for the set of partial bijections from S to T ,
- $\mathcal{I}_{S,T}^f$ for the set of partial bijections from S to T with finite domain,
- $\mathcal{I}_{S,T}^{\text{cof}}$ for the set of partial bijections from S to T with domain co-finite in S and image co-finite in T ,
- $\mathcal{T}_{S,T}$ for the set of functions from S to T ,
- $\mathcal{P}_{S,T}$ for the set of partial functions from S to T ,
- $\mathcal{P}_{S,T}^f$ for the set of partial functions from S to T with finite domain,
- $\mathcal{P}_{S,T}^{\text{cof}}$ for the set of partial functions from S to T with domain co-finite in S .

We will also write A^S for the set of functions from the set S to the set A .

If S and T are sets and f and g are partial functions from S to T and x is in S then we will say f and g *agree* at x if xf and xg are both defined and equal. We will say f and g *disagree* at x if xf and xg are both defined and not equal. We

will say f and g *disagree* if there exists a point at which they disagree. We will say f and g *agree* if there is no point x at which they disagree or equivalently, f and g agree on the intersection of their domains. We will also refer to the set that f and g agree on and the set they disagree on.

If S is a set equipped with a binary relation \leq then we will say \leq is a *partial order* on S if for all r, s and t in S we have

- $r \leq r$,
- $r \leq t$ and $t \leq r$ implies $r = t$, and
- $r \leq s$ and $s \leq t$ implies $r \leq t$.

Let S be a set equipped with a partial order and let a be an element of S . We will write a^\downarrow for the set

$$\{s \in S \mid s \leq a\},$$

and we will write a^\uparrow for the set

$$\{s \in S \mid s \geq a\}.$$

If A is a subset of S such that for all a in A the set a^\downarrow is a subset of A then A is an *order ideal* of S .

If S is a set equipped with a partial order and T is a subset of S such that there is a least element above T then that element is the *join* of T and is denoted

$\vee T$. For s and t in S if $\vee\{s, t\}$ exists then we may call it s join t or the join of s and t and denote it $s \vee t$.

1.3 Semigroups and Groups

Let S be a semigroup. If x in S is such that $x^2 = x$ then x is an *idempotent*. If y is an element of S such that for all s in S we have $sy = s$ and $ys = s$ then y is an *identity*. If z is an element of S such that for all s in S we have $sz = z$ and $zs = z$ then z is a *zero*. Clearly a semigroup may have at most one identity and at most one zero. If a semigroup has an identity then it is a *monoid*. If S is a monoid then the identity of S will be denoted 1_S and the set

$$\{s \in S \mid \text{there exists } a, b \in S \text{ such that } as = 1_S \text{ and } sb = 1_S\}$$

forms a group which we will call the *group of units* of S . If u and v in S are such that $uv = vu$ then u and v *commute*. If every pair of elements of S commute then S is *commutative*.

If X is a subset of a group G we will say X *group-generates* G if the only subgroup of G containing X is G . The definitions of what it means for a set X to *semigroup-generate* and *monoid-generate* are analogous. We may also write X generates a structure G without specifying the structure if the ambiguity is removed by context. If X generates G then we will write X is a *generating set* for G . If X is a generating set for a structure G then we write $\langle X \rangle = G$. We will

use extensions of this language (e.g. “ X is a semigroup generating set”) without comment throughout, hoping this will not introduce confusion. If S is a semigroup and X is a subset of S then we will talk of the subsemigroup of S *generated* by X meaning the intersection of all the subsemigroups of S containing X . Where the type of structure is clear from context we will use $\langle X \rangle$ for the structure generated by X .

If X is a subset of a group G then X is a group-generating set for G if and only if any element of G can be constructed from elements of X using the group operations. Analogous statements hold for semigroups and monoids.

For a set S and $x \subseteq S \times S$ we will write x^{-1} for $\{(n, m) \in S \times S \mid (m, n) \in x\}$.

Let G be a group. If $a, b \in G$ then the *commutator* of a and b is $a^{-1}b^{-1}ab$ and is denoted $[a, b]$. The *commutator subgroup* of G is the subgroup

$$G' := \langle [a, b] \mid a, b \in G \rangle$$

of G . If $G' = G$ then we say G is *perfect*.

We will say a group G *acts by bijections* on a set S if we have specified (probably implicitly) a homomorphism ϕ from G to the group $\text{Sym}(S)$ and write $sg = s(g\phi)$ where $x \in S$ and $g \in G$ (note that we are acting on the right). Also, we will allow inverse semigroups to *act by partial bijections* (where the homomorphism is to \mathcal{I}_S) and semigroups to *act by functions* (where the homomorphism is to \mathcal{T}_S). We may simply say a structure *acts* on a set if the context removes the ambiguity.

Let a group G act by bijections on a set S . For $X \subseteq S$ we define the *pointwise stabiliser* of X to be the set

$$\text{pstab}_G(X) = \{g \in G \mid xg = x \text{ for all } x \in X\}.$$

Let g be an element of G . The set

$$\text{supp}(g) = \{s \in S \mid sg \neq s\}$$

is the *support* of g . The set

$$\text{fix}(g) = \{s \in S \mid sg = s\}$$

is the set of fixed points of g . Let s be in S . The set $\{sg \in S \mid g \in G\}$ is the *orbit* of s under G . If g is in G then the set $\{sg^n \mid n \in \mathbb{N}\}$ is the *forward orbit* of s under g . We will talk about orbits in a way that is analogous for semigroups, inverse semigroup, monoids and inverse monoids. If S only has one orbit under G then G acts *transitively* on S .

Chapter 2

Fundamental Ideas

2.1 Inverse Semigroups

Definition 2.1.1. An *inverse semigroup* is a semigroup S such that for each element s of S there is a unique element called the inverse of s and denoted s^{-1} such that

$$ss^{-1}s = s \text{ and } s^{-1}ss^{-1} = s^{-1}.$$

If S is also a monoid then S is an *inverse monoid*. We will consider taking inverses as an operation.

Proposition 2.1.2. If I is an inverse semigroup and $s, t \in I$ then ss^{-1} and $s^{-1}s$ are idempotents and $(st)^{-1} = t^{-1}s^{-1}$ and $(s^{-1})^{-1} = s$. If e is an idempotent then $e^{-1} = e$.

Proof. These statements are all proved in Proposition 1 of Section 1.4 of [9]. \square

Proposition 2.1.3. *If b and d are idempotents of an inverse semigroup then $bd = db$.*

Proof. This is proved in Theorem 5.1.1 of Chapter 5 of [7]. \square

Definition 2.1.4. Let S be a set equipped with a partial order and let T be a subset of S . The set

$$\{s \in S \mid \exists t \in T : s \leq t\}$$

is the *order ideal* generated by T and is denoted T^\downarrow . If s is in S we will also write s^\downarrow for $\{s\}^\downarrow$.

We apply this notation to inverse semigroups as described in the next definition.

Definition 2.1.5. If S is an inverse semigroup then there is a natural partial order on S which we will denote \leq defined by $s \leq t$ for s and t in S if and only if there exists an idempotent $e \in S$ with $s = te$.

Proposition 2.1.6. *Let I be an inverse semigroup. Let s be an element of I and let e be an idempotent of I . Then there exists idempotents i and j of I such that $is = se$ and $sj = es$.*

Proof. This is proved in Lemma 2 of Section 1.4 of [9]. \square

Remark 2.1.7. Let s and t be elements of an inverse semigroup. It is a consequence of the above proposition that there exists an idempotent e with $se = t$

exactly if there exists an idempotent i with $is = t$. So \leq is symmetrical between right and left multiplication even though this symmetry is not evident in Definition 2.1.5.

Remark 2.1.8. Let S be a set and let I be an inverse subsemigroup of \mathcal{I}_S and let r and t be in I . Then $r \leq t$ exactly if $r \subseteq t$ that is, if r is a restriction of the map t to some subset of S .

Recall a partial permutation f of a set S is in the set $\mathcal{I}_S^{\text{cof}}$ if the sets $S \setminus \text{dom}(f)$ and $S \setminus \text{ran}(f)$ are both finite. The importance of $\mathcal{I}_S^{\text{cof}}$ to Chapter 4 justifies a proof that $\mathcal{I}_S^{\text{cof}}$ is in fact an inverse monoid.

Proposition 2.1.9. *Let S be a set. The set $\mathcal{I}_S^{\text{cof}}$ equipped with composition of partial permutations is an inverse monoid.*

Proof. First we show that $\mathcal{I}_S^{\text{cof}}$ is closed under taking inverses. Let f be in $\mathcal{I}_S^{\text{cof}}$. By the assumption that f is in $\mathcal{I}_S^{\text{cof}}$ the left hand sides of the equations

$$S \setminus \text{ran}(f) = S \setminus \text{dom}(f^{-1})$$

$$S \setminus \text{dom}(f) = S \setminus \text{ran}(f^{-1})$$

are finite. Therefore the right hand sides of the above two equations are also finite and f^{-1} is in $\mathcal{I}_S^{\text{cof}}$ as desired.

Second we show that $\mathcal{I}_S^{\text{cof}}$ is closed under composition of partial permutations. If f and t are in $\mathcal{I}_S^{\text{cof}}$ then ft is certainly a partial permutation of S . To show ft

is in $\mathcal{I}_S^{\text{cof}}$ we will show the sets $S \setminus \text{dom}(ft)$ and $S \setminus \text{ran}(ft)$ are finite.

Points which are in S but not in $\text{dom}(ft)$ are either not in $\text{dom}(f)$ or are mapped by f to a point not in $\text{dom}(t)$. Since f is in $\mathcal{I}_S^{\text{cof}}$ there are only finitely many points in $S \setminus \text{dom}(f)$ so we only need to check that the set of points mapped by f to a point in $S \setminus \text{dom}(t)$ is finite. Since $S \setminus \text{dom}(t)$ is finite and f is a partial bijection it follows that the set of points mapped by f into $S \setminus \text{dom}(t)$ is also finite. Therefore $S \setminus \text{dom}(ft)$ is finite as desired.

For similar reasons the set $S \setminus \text{ran}(ft)$ can also be seen to be finite so we may conclude that ft is in $\mathcal{I}_S^{\text{cof}}$.

The identity map from S to S is an identity for $\mathcal{I}_S^{\text{cof}}$ which is therefore an inverse monoid as desired. \square

Remark 2.1.10. For S a set we note that

- each of the sets \mathcal{T}_S , \mathcal{P}_S and \mathcal{P}_S^f are monoids when equipped with the operation of composition of partial functions,
- each of the sets \mathcal{I}_S and \mathcal{I}_S^f are inverse monoids when equipped with the operation of composition of partial bijections, and
- the set $\text{Sym}(S)$ equipped with the operation of composition of bijections is a group.

We will call \mathcal{T}_S the *transformation monoid*, \mathcal{P}_S the *monoid of partial functions*, \mathcal{I}_S the *full symmetric inverse monoid* and $\text{Sym}(S)$ the *symmetric group*.

2.2 Bicyclic Monoid

The bicyclic monoid is sufficiently important to the construction that Chapter 4 is devoted to that it warrants its own section.

Definition 2.2.1. Let b and c be the partial permutations of the natural numbers such that for each natural number n

$$nb = n + 1$$

and

$$nc = \begin{cases} n - 1 & n \neq 1 \\ \text{not defined} & n = 1 \end{cases}.$$

We define the *bicyclic monoid* to be the inverse submonoid of the monoid of $\mathcal{I}_{\mathbb{N}}$ generated by the set $\{b, c\}$ (this is the same as the inverse monoid generated by $\{b\}$). We will write B for the bicyclic monoid.

We note that the bicyclic monoid B is an inverse submonoid of $\mathcal{I}_{\mathbb{N}}^{\text{cof}}$. This is important because Chapter 4 gives a construction for Thompson's group V_2 (which is defined in Definition 2.7.2) using B and then modifies this construction by replacing B by other inverse submonoids of $\mathcal{I}_{\mathbb{N}}^{\text{cof}}$ to form new groups. It is these groups which are the focus of Chapter 4.

If i and j are natural numbers then $i \leq j$ under the *dual order on the natural numbers* exactly if $i \geq j$ under the normal order.

Definition 2.2.2. If S is a partial order and s is an element of S then the *principal order ideal* generated by s is the set s^\downarrow .

Definition 2.2.3. A partial order P is a meet *semilattice* if for any finite subset X of P there is a unique p in P with firstly $x \geq p$ for all x in X and secondly if q is such that $x \geq q$ for all x in X then $p \geq q$.

Definition 2.2.4. If S is a meet semilattice then the *Munn semigroup* of S is the set of isomorphisms between principal order ideals of S .

The bicyclic monoid is equal to the Munn semigroup of the natural numbers under the dual order. For more information on Munn semigroups see Chapter 5 of [7].

Proposition 2.2.5. *Let m and n be non-negative integers. The domain of $c^m b^n$ is $\{m+1, m+2, \dots\}$ and the range of $c^m b^n$ is $\{n+1, n+2, \dots\}$ and if i is in the domain of $c^m b^n$ then $ic^m b^n = i - m + n$.*

Proof. The partial permutation b increments its input by 1 so the partial permutation b^n increments its input by n . Therefore the domain of b^n is \mathbb{N} and the range of b^n is the set $\{n+1, n+2, \dots\}$. Since c is the inverse of b the range of c is \mathbb{N} and the domain of c^m is $\{m+1, m+2, \dots\}$. Consequently the domain of $c^m b^n$ is $\{m+1, m+2, \dots\}$ and the range of $c^m b^n$ is $\{n+1, n+2, \dots\}$. Since c^m decrements its input by m and b^n increments its input by n the partial permutation $c^m b^n$ maps i to $i - m + n$ as desired. □

Proposition 2.2.6. *Each element of B may be uniquely represented as $c^m b^n$ where m and n are non-negative integers.*

Proof. Since c and b generate B each element of B is a product over b and c . Any such product may be reduced to a product of the form $c^m b^n$ where m and n are non-negative integers by applying the relation $bc = 1_B$. This representation is unique by Proposition 2.2.5. □

2.3 The Homomorphism ϕ_S

For a countably infinite set S the inverse monoid $\mathcal{I}_S^{\text{cof}}$ is important for the construction to which Chapter 4 is devoted. In particular the homomorphism ϕ_S which is defined in Definition 2.3.6 below is very important so we will spend this section laying some groundwork related to ϕ_S which will be used extensively in Chapter 4.

Definition 2.3.1. For S a set define $\chi_S : \mathcal{I}_S^{\text{cof}} \rightarrow P(\mathcal{I}_S^{\text{cof}})$ to be the map sending each partial bijection $g \in \mathcal{I}_S^{\text{cof}}$ to the subset of $\mathcal{I}_S^{\text{cof}}$ of those partial bijections that agree with g in all but finitely many places.

Lemma 2.3.2. *If S is a set and g and h are in $\mathcal{I}_S^{\text{cof}}$ then $g \chi_S$ equals $h \chi_S$ if and only if there exists $f \in \mathcal{I}_S^{\text{cof}}$ with $fg = fh$.*

Proof. Let X be the set where g and h disagree. For $f \in \mathcal{I}_S^{\text{cof}}$ the set where fg

and fh disagree is Xf^{-1} . So fg equals fh if and only if Xf^{-1} is empty. If X is finite then f may be chosen to be the idempotent with domain $S \setminus X$ and then Xf^{-1} is empty as desired. If X is infinite then the range of any element of $\mathcal{I}_S^{\text{cof}}$ intersects X so Xf^{-1} cannot be empty and fg cannot equal fh as desired. \square

Remark 2.3.3. Note that $\mathcal{I}_S^{\text{cof}} \chi_S = \mathcal{I}_S^{\text{cof}} / R$ where R is the equivalence relation

$$R := \{(g, h) \mid \exists f \in \mathcal{I}_S^{\text{cof}} \text{ with } fg = fh\}.$$

Lemma 2.3.4. For S a set and $f, g \in \mathcal{I}_S^{\text{cof}}$ the set $(fg)\chi_S$ contains the set $\{f'g' \mid f' \in f\chi_S, g' \in g\chi_S\}$.

Proof. Let f' be in $f\chi_S$ and let g' be in $g\chi_S$. Let A be the set where f and f' disagree and let B be the set where g and g' disagree.

The set where fg and $f'g'$ disagree must be contained in the union of A and Bf^{-1} . Since these sets are both finite the set where fg and $f'g'$ disagree must also be finite so $f'g'$ is in $(fg)\chi_S$ as desired. \square

Remark 2.3.5. We note that in the bicyclic monoid cb agrees with 1_B in all but one place, however $\{xy \mid x \in c\chi_S, y \in b\chi_S\}$ does not contain 1_B since none of the partial bijections in $c\chi_S$ are defined on all of \mathbb{N} . So though $(cb)\chi_S$ contains $\{xy \mid x \in c\chi_S, y \in b\chi_S\}$ they are not equal.

Let A and B be in $\mathcal{I}_S^{\text{cof}} \chi_S$. Let a be in A and let b be in B . By Remark 2.3.3

and Lemma 2.3.4 the set

$$\{w \in \mathcal{I}_S^{\text{cof}} \mid w \text{ agrees with } ab \text{ in all but finitely many places}\}$$

is dependent on A and B but not on our choice of $a \in A$ and $b \in B$. Note that the above set is an element of $\mathcal{I}_S^{\text{cof}} \chi_S$. Consequently we may define an operation on $\mathcal{I}_S^{\text{cof}} \chi_S$ defined by the rule that the product of A and B is equal to the above set.

By definition the above operation is such that for all u and v in $\mathcal{I}_S^{\text{cof}}$ the product of $u\chi_S$ and $v\chi_S$ is equal to $(uv)\chi_S$. This is equivalent to χ_S being a homomorphism.

Remark 2.3.5 shows that the above operation is different from the operation such that for each X and Y in $\mathcal{I}_S^{\text{cof}} \chi_S$ the product of X and Y is equal to

$$XY = \{xy \mid x \in X, y \in Y\}.$$

In order to avoid ambiguity between these two operations we make the following definition.

Definition 2.3.6. For each set S fix f a bijection from $\mathcal{I}_S^{\text{cof}} \chi_S$ to some set T_S .

Now define ϕ_S to be the composition $\chi_S f$. We will denote by concatenation the binary operation on T_S with

$$(a\phi_S)(b\phi_S) = \{w \in \mathcal{I}_S^{\text{cof}} \mid w \text{ agrees with } ab \text{ in all but finitely many places}\} f$$

for each a and b in $\mathcal{I}_S^{\text{cof}}$.

2.4 Cantor Set

Definition 2.4.1. If (X, \mathcal{T}_X) is a topological space then the set of homeomorphisms from X to X is a group when equipped with the operation of composition of functions. We will use H_X for this group.

Definition 2.4.2. If (X, \mathcal{T}_X) is a topological space and U is a subset of X then U is *closed* if $X \setminus U$ is open. Subsets of topological spaces which are both open and closed are called *clopen*.

Definition 2.4.3. If $(A_s)_{s \in S}$ is a (possibly infinite) collection of sets then we define the *Cartesian product* over $(A_s)_{s \in S}$ to be the set

$$\prod_{s \in S} A_s := \left\{ f : S \longrightarrow \bigcup_{s \in S} A_s \mid sf \in A_s \text{ for all } s \in S \right\}.$$

If $S = \mathbf{n}$ for some natural number n then we may denote the Cartesian product $A_0 \times \cdots \times A_{n-1}$. If $A_s = A$ for all s and t in S then we may denote the Cartesian product A^S .

Definition 2.4.4. If S is a set and $((T_s, \mathcal{T}_s))_{s \in S}$ is a list of topological spaces indexed by S then the *product topology* on the set $\prod_{s \in S} T_s$ with respect to $(\mathcal{T}_s)_{s \in S}$ is the set of (possibly infinite) unions of Cartesian products of the form

$$\prod_{s \in S} U_s$$

with U_s in \mathcal{T}_s for each s in S , and $U_s = T_s$ for all but finitely many s in S .

Definition 2.4.5. Let A and S be sets. Equip A with the discrete topology. We will write A^S for the set of function from S to A equipped with the product topology.

Recall if S and A are sets then $\mathcal{P}_{S,A}^f$ is the set of partial functions from S to A with finite domain.

Definition 2.4.6. For A and S sets and f in $\mathcal{P}_{S,A}^f$ we write \bar{f} for the set

$$\{t \in A^S \mid t \supseteq f\}$$

which we will call a *cone* of A^S or, if context allows, a *cone*.

Definition 2.4.7. Let A be a set of size at least 2 and let f be in $\mathcal{P}_{\mathbb{N},A}^f$ such that $\text{dom}(f) = \{1, 2, \dots, n\}$ for some natural number n . We will say

$$\{t \in A^{\mathbb{N}} \mid t \supseteq f\}$$

is a *strict cone* of $A^{\mathbb{N}}$ or if context allows we will just say a *strict cone*.

Example 2.4.8. The whole of $\{0, 1\}^{\mathbb{N}}$ is a strict cone. All strict cones are also cones but the set $\{t \in \{0, 1\}^{\mathbb{N}} \mid 3t = 0\}$ is a cone but not a strict cone. The set $\{t \in \{0, 1\}^{\mathbb{N}} \mid 1t = 0 \text{ and } 2t = 0 \text{ and } 3t = 0\}$ is a strict cone.

Proposition 2.4.9. *Let A be a finite set of size at least 2. Each cone of $A^{\mathbb{N}}$ is a finite union of strict cones.*

Proof. Let C be a cone of $A^{\mathbb{N}}$. Let $f \in \mathcal{P}_{\mathbb{N},A}^f$ be such that $C = \{t \in A^{\mathbb{N}} \mid t \supseteq f\}$. Let n be the largest natural number that f is defined upon. Let M be the set of partial functions in $\mathcal{P}_{\{1,2,\dots,n\},A}^f$ which agree with f . Now the union of strict cones

$$\bigcup_{t \in M} \bar{t}$$

is equal to C as desired. □

Definition 2.4.10. A topological space (T, \mathcal{T}) is *compact* if for any set C of open sets such that $\bigcup C = T$ there exists B a finite subset of C such $\bigcup B = T$.

We now state Tychonoff's Theorem which we use below.

Theorem 2.4.11. *The product of a set of compact topological spaces is compact with respect to the product topology.*

Proof. This is proved in [8]. □

Corollary 2.4.12. *Let n be a natural number of size at least 2 then $\mathbf{n}^{\mathbb{N}}$ is compact.*

Proof. Since \mathbf{n} equipped with the discrete topology is trivially compact this follows immediately from Theorem 2.4.11. □

Proposition 2.4.13. *Let n be a natural number. The open subsets of $\mathbf{n}^{\mathbb{N}}$ are the arbitrary unions of strict cones and the clopen subsets of $\mathbf{n}^{\mathbb{N}}$ are the finite unions of strict cones.*

Proof. That the open subsets of $\mathbf{n}^{\mathbb{N}}$ are the arbitrary unions of strict cones follows immediately from Definition 2.4.4 and Proposition 2.4.9.

Before showing clopen sets are finite unions of strict cones we show finite unions of cones are clopen. On the way to this we show cones are clopen. Let f be in $\mathcal{P}_{\mathbb{N}, \mathbf{n}}^f$. The cone \bar{f} is open since it is trivially a union of cones. Let M be equal to $\mathbf{n}^{\text{dom}(f)} \setminus \{f\}$. We may write the compliment of \bar{f} as the union

$$\bigcup_{t \in M} \bar{t}$$

which is open since it is a union of cones. Since the compliment of \bar{f} is open \bar{f} is closed. Since cones are open and closed and open sets and closed sets are closed under finite union it follows that finite unions of cones are clopen.

Let C be a clopen subset of $\mathbf{n}^{\mathbb{N}}$. Since C is clopen it is open and therefore a union of strict cones. Let U be a set of strict cones such that $\bigcup U = C$. Let $V := U \cup \{\mathbf{n}^{\mathbb{N}} \setminus C\}$. Since $\bigcup V = \mathbf{n}^{\mathbb{N}}$ and $\mathbf{n}^{\mathbb{N}}$ is compact there must be a finite subset W of V such that $\bigcup W = \mathbf{n}^{\mathbb{N}}$. Now $W \setminus \{\mathbf{n}^{\mathbb{N}} \setminus C\}$ is a finite set of strict cones whose union is equal to C as desired. \square

Definition 2.4.14. We define the *Cantor set* to be the topological space $\{0, 1\}^{\mathbb{N}}$.

Definition 2.4.15. Any topological space which is homeomorphic to the Cantor set is called a *Cantor space*. We will use \mathcal{C} to denote an arbitrary Cantor space.

We will now give some propositions to further illuminate the structure of some Cantor spaces and give the reader a feel for the way in which they will be treated.

Proposition 2.4.16. *For any $n \geq 2$ any two cones of $\mathbf{n}^{\mathbb{N}}$ are homeomorphic.*

Proof. Let f and t be $\mathcal{P}_{\mathbb{N}, \mathbf{n}}^f$. We will find a homeomorphism from \bar{f} to \bar{t} . Since $\mathbb{N} \setminus \text{dom}(f)$ and $\mathbb{N} \setminus \text{dom}(t)$ are both countably infinite there exists a bijection $g : \mathbb{N} \setminus \text{dom}(t) \rightarrow \mathbb{N} \setminus \text{dom}(f)$. It is sufficient to show the map $h : \bar{f} \rightarrow \bar{t}$ sending $x \in \bar{f}$ to $gx \cup t$ is well defined, injective, surjective and a homeomorphism.

Let x be in \bar{f} . Since $\text{dom}(g) = \mathbb{N} \setminus \text{dom}(t)$ and $\text{ran}(g) = \mathbb{N} \setminus \text{dom}(f) \subseteq \mathbb{N} = \text{dom}(x)$ it follows that the $\text{dom}(gx) = \mathbb{N} \setminus \text{dom}(t)$. Therefore $gx \cup t$ is in \bar{t} so h is well defined. If y is an element of \bar{f} distinct from x and i is a natural number such that $ix \neq iy$ then $gx \cup t$ and $gy \cup t$ disagree on ig^{-1} so $gx \cup t \neq gy \cup t$. Therefore h is injective. Let z be in \bar{t} . The function $g^{-1}z \cup f$ is in \bar{f} . Since $(g^{-1}z \cup f)h = g(g^{-1}z \cup f) \cup t = gg^{-1}z \cup gf \cup t = z|_{\text{dom}(g)} \cup t = z|_{\mathbb{N} \setminus \text{dom}(t)} \cup t = z$ the injection h is also a surjection.

Let r be in $\mathcal{P}_{S,A}^f$ such that $f \subseteq r$. Now \bar{r} is a cone contained in \bar{f} . Note

$$\begin{aligned}
\bar{r}h &= \{w \in \mathbf{n}^{\mathbb{N}} \mid w \supseteq r\} h \\
&= \{wh \mid w \in \mathbf{n}^{\mathbb{N}} \text{ and } w \supseteq r\} \\
&= \{gw \cup t \mid w \in \mathbf{n}^{\mathbb{N}} \text{ and } w \supseteq r\} \\
&= \{g(g^{-1}v) \cup t \mid v \in \mathbf{n}^{\mathbb{N}} \text{ and } g^{-1}v \cup f \supseteq r\} \\
&= \{v|_{\text{dom}(g)} \cup t \mid v \in \mathbf{n}^{\mathbb{N}} \text{ and } v \supseteq gr\} \\
&= \{u \mid u \in \mathbf{n}^{\mathbb{N}} \text{ and } u \supseteq gr \cup t\}.
\end{aligned}$$

Since g maps cones to cones g must also map open sets to open sets and therefore g is a homeomorphism. \square

Since for $n \geq 2$ the set $\mathbf{n}^{\mathbb{N}}$ is a cone of itself, any cone of $\mathbf{n}^{\mathbb{N}}$ is homeomorphic to $\mathbf{n}^{\mathbb{N}}$. More generally the following proposition is well known. (See, e.g. [5])

Proposition 2.4.17. *If A and B are finite sets of size at least 2, and X and Y are countably infinite sets. Then A^X is homeomorphic to B^Y .*

Definition 2.4.18. If S and A are sets and X is a clopen subset of A^S then a *transversal* of X is a finite subset L of $\mathcal{P}_{S,A}^f$ such that for each $x \in X$ there is exactly one $l \in L$ with $l \subseteq x$.

We will now discuss transversals which are important for elements of both the groups constructed in Chapter 4 and Thompson's group V which is defined later in this chapter.

Example 2.4.19. The set of partial functions

$$\left\{ \{(3, 0), (5, 0)\}, \{(3, 0), (5, 1)\}, \{(3, 1)\} \right\} \subseteq \mathcal{P}_{\mathbb{N}, \{0,1\}}^f$$

is a transversal of $\{0, 1\}^{\mathbb{N}}$. Here $\{(3, 0), (5, 0)\}$ is the partial function sending both 3 and 5 to 0 and not defined elsewhere.

Remark 2.4.20. If S and A are sets with A infinite then A^S has only one transversal. This transversal contains only the empty function.

Remark 2.4.21. For sets S and A each partial function f from S to A with finite domain corresponds to a cone $\{t \in A^S \mid t \supseteq f\}$ of A^S . The transversals of a clopen subset X of A^S correspond exactly to the partitions of X into cones.

Lemma 2.4.22. *If S is a countably infinite set and A is a finite set of size at least 2 and X is a clopen subset of A^S then there exists a transversal of X .*

Proof. From the Proposition 2.4.13 we may fix F a finite subset of $\mathcal{P}_{S,A}^f$ with

$$X = \bigcup_{f \in F} \{x \in A^S \mid x \supseteq f\}.$$

Fix

$$T := \bigcup_{f \in F} \text{dom}(f).$$

Also fix

$$E := \{e \in A^T \mid e \supseteq f \text{ for some } f \text{ in } F\}$$

and

$$U := \bigcup_{e \in E} \{x \in A^S \mid x \supseteq e\}.$$

The above union is disjoint because if two functions in A^T are distinct then they must disagree somewhere so they can have no mutual extension. The union U is equal to X because each x in X may be restricted to some f in F which can be further restricted to some e in E . □

Remark 2.4.23. Let S be a countably infinite set and let A be a finite set. It

follows immediately from Lemma 2.4.22 that any clopen subset of A^S may be partitioned into cones.

Lemma 2.4.24. *If A is a finite set and S is an infinite set and X is a non-empty clopen subset of A^S and P is a transversal of X then there exist transversals of X of size $|P| + m(|A| - 1)$ for each natural number m .*

Proof. By induction it suffices to find a transversal of X of size $|P| + |A| - 1$. Let $p \in P$ and $s \in S \setminus \text{dom}(p)$ be given. Now the set $(P \setminus \{p\}) \cup \{p \cup (s, l) \mid l \in A\}$ is a transversal of X of the correct size. \square

Lemma 2.4.25. *Let A be a finite set and let X be a non-empty clopen subset of $A^{\mathbb{N}}$ and let P be a transversal of X such that for each p in P*

$$\text{dom}(p) = \mathbf{n} \setminus \{0\}$$

for some natural number n . Then for every natural number m there exists Q a transversal of X with $|Q| = |P| + m(|A| - 1)$ and for each $q \in Q$

$$\text{dom}(q) = \mathbf{n} \setminus \{0\}$$

for some natural number n .

Proof. This is similar to the proof of Lemma 2.4.24. The main difference is s should be the least natural number not in $\text{dom}(p)$. \square

Lemma 2.4.26. *If S is a set and A is a finite set and P and Q are transversals of X a clopen subset of A^S then $|A| - 1$ divides $|P| - |Q|$.*

Proof. Let $N := \bigcup_{p \in P \cup Q} \text{dom}(p)$. Note that P and Q are both subsets of A^N . If $p \in P$ is such that $\text{dom}(p) \subsetneq N$ then we may take $s \in N \setminus \text{dom}(p)$ and form $(P \setminus p) \cup \{p \cup (s, l) \mid l \in A\}$ a new transversal of X . We may only iterate this process a finite number of times because each transversal is a larger subset of A^N than its predecessor and A^N is finite. The only place this iteration can stop is at $\{f \in A^N \mid f \subseteq f' \text{ for some } f' \in X\}$. To complete the proof note that we have brought P and Q to the same place by finitely many steps each of which changed their orders by $|A| - 1$. \square

Recall from Definition 2.4.15 we write \mathcal{C} to denote an arbitrary Cantor space.

In a generic Cantor space there is no notion of a cone; all the structure is described by the open sets. When we work with \mathcal{C} instead of a concrete Cantor space like $\{0, 1\}^{\mathbb{N}}$ this is because the cones would be distracting without being useful.

For example in parts of Chapter 3 we work with subgroups of $H_{\mathcal{C}}$ that have no reason to interact nicely with the set of cones so it makes sense to work in \mathcal{C} . In Chapter 4 and parts of Chapter 3 we construct subgroups of $H_{\mathcal{C}}$ which interact nicely with the set of cones so we work with concrete Cantor spaces.

In Section 4.1 we describe a classical notion of word and then a generalised notion of word. These are certainly fundamental ideas to Chapter 4 so that section has a claim to belonging in this Chapter. However we reserve Section 4.1 for

Chapter 4 firstly because the reader will be better equipped for Chapter 4 with Section 4.1 fresh in their mind and secondly to avoid the possibility of a reader, familiar with the other ideas in the earlier Chapters, skipping ahead and entering Chapter 4 without the insight of Section 4.1.

2.5 Polycyclic Monoid

Recall we write $\hat{}$ to denote concatenation of words.

Definition 2.5.1. Fix $n \geq 2$. For this definition we will think of $\mathbf{n}^{\mathbb{N}}$ as infinite words over \mathbf{n} (words with a beginning but no end). For each i in \mathbf{n} let b_i and c_i be in $\mathcal{I}_{\mathbf{n}^{\mathbb{N}}}$ with for each w in $\mathbf{n}^{\mathbb{N}}$

$$wb_i = i\hat{w} \text{ and } (i\hat{w})c_i = w$$

and with $(i\hat{w})c_j$ not defined for each j in $\mathbf{n} \setminus \{i\}$. We define the *polycyclic monoid of rank n* to be the inverse submonoid of the monoid of partial bijections on $\mathbf{n}^{\mathbb{N}}$ generated by $\{b_i, c_i\}_{i \in \mathbf{n}}$. We will denote the polycyclic monoid of rank n by P_n .

Since we are acting from the right and application of an element of the polycyclic monoid modifies the beginning of a string there is an argument that our strings should be left infinite. For example $(\dots 01011)$. However we follow [2] and [19] and consider our strings to be right infinite.

Proposition 2.5.2. For $n \geq 2$ and $0 \leq i < n$ the inverse of b_i is c_i in P_n .

Proof. Let n , a natural number, be at least 2. Let w be in $\mathbf{n}^{\mathbb{N}}$ and let i be in \mathbf{n} . The product $wb_i c_i b_i$ is equal to $(i \hat{w})c_i b_i$ which is equal to wb_i . If the first letter of w is i then we may write w as $i \hat{u}$ for some u in $\mathbf{n}^{\mathbb{N}}$. Then $wc_i b_i c_i = ub_i c_i = (i \hat{u})c_i = wc_i$. If the first letter of w is not i then neither $wc_i b_i c_i$ or wc_i are defined.

Since $c_i b_i c_i = c_i$ and $b_i c_i c_i = b_i$ it follows that b_i is the inverse of c_i as desired. □

Proposition 2.5.3. *For each n we note that each non-zero element of the polycyclic monoid of rank n is equal to exactly one product of elements from c_0, \dots, c_{n-1} followed by elements from b_0, \dots, b_{n-1} .*

Proof. Let n , a natural number, be at least 2. Let w be a product over $\{b_i\}_{i \in \mathbf{n}} \cup \{c_i\}_{i \in \mathbf{n}}$. If w is not of the desired form then there must be a subproduct of the form $b_i c_j$ for some i and j in \mathbf{n} . If $i \neq j$ then w evaluates to 0. If $i = j$ then $b_i c_j$ is the identity and may be removed, shortening w . Repetitive application of this process can only terminate in zero or a product of the right form. The process must terminate because w is of finite length. □

Proposition 2.5.4. *Fix $n \geq 2$ and w_1, \dots, w_i and x_1, \dots, x_j elements of \mathbf{n} for some natural numbers i, j . If p is*

$$c_{w_1} \cdots c_{w_i} b_{x_j} \cdots b_{x_1}$$

a non-zero element of the polycyclic monoid then

1. the inverse of p is $c_{x_1} \cdots c_{x_j} b_{w_i} \cdots b_{w_1}$,
2. the domain of p is the set $\{f \in \mathbf{n}^{\mathbb{N}} \mid 1f = w_1, \dots, if = w_i\}$, and
3. the range of p is the set $\{f \in \mathbf{n}^{\mathbb{N}} \mid 1f = x_1, \dots, jf = x_j\}$.

Proof. The first point is a basic fact about inverse semigroups and the other two points follow by inspection. □

It follows that the set of domains of elements of P_n , the set of ranges of elements of P_n and the set of domains of idempotents of P_n are all equal to the union of $\{\emptyset\}$ and the set of strict cones of $\mathbf{n}^{\mathbb{N}}$.

Recall for a semigroup S with zero we denote the zero of S by 0_S .

Proposition 2.5.5. *Fix $n \geq 2$ and w_1, \dots, w_i and x_1, \dots, x_j and y_1, \dots, y_k and z_1, \dots, z_l lists over \mathbf{n} . Let p be the product of*

$$c_{w_1} \cdots c_{w_i} b_{x_j} \cdots b_{x_1}$$

and

$$c_{y_1} \cdots c_{y_k} b_{z_l} \cdots b_{z_1}$$

which are elements of the polycyclic monoid of rank n . If $j \geq k$ and

$$x_1 \cdots x_k = y_1 \cdots y_k$$

then p is equal to

$$c_{w_1} \cdots c_{w_i} b_{x_j} \cdots b_{x_{k+1}} b_{z_l} \cdots b_{z_1}.$$

If $j \leq k$ and

$$x_1 \cdots x_j = y_1 \cdots y_j$$

then p is equal to

$$b_{w_1} \cdots b_{w_i} b_{y_{j+1}} \cdots b_{y_k} c_{z_1} \cdots c_{z_1}.$$

Otherwise $p = 0_{P_n}$.

Proof. If we concatenate $c_{w_1} \cdots c_{w_i} b_{x_j} \cdots b_{x_1}$ and $c_{y_1} \cdots c_{y_k} b_{z_1} \cdots b_{z_1}$ and apply the fact that elements of the set $\{a_i b_i\}_{i \in \mathbf{n}}$ are equal to the identity and elements of the set $\{a_i b_j\}_{i, j \in \mathbf{n}}$ are equal to the zero then the result follows. \square

For the rest of this section fix a countably infinite set S , a finite alphabet A of size at least 2 and an inverse submonoid I of $\mathcal{I}_S^{\text{cof}}$. In Chapter 4 we study a construction which takes as input the triple (S, A, I) and returns an inverse submonoid $\mathcal{I}(S, A, I)$ of \mathcal{I}_{A^S} .

In the special case when S is equal to the set of natural numbers and I is the bicyclic monoid the inverse monoid $\mathcal{I}(S, A, I)$ is equal to the polycyclic monoid of rank $|A|$. This relationship between the bicyclic monoid and the polycyclic monoid motivates the use of this construction in Chapter 4. We do not describe this special case in detail here to avoid overlap with Section 4.1, Section 4.2 and Section 4.3.

2.6 Closure Under Compatible Union

In this section we will lay some ground work related to closures of inverse semigroups of partial bijections under (finite) compatible union. This will be used in Section 2.7, Section 2.8 and Chapter 4.

Definition 2.6.1. A pair of elements s, t of an inverse semigroup are *compatible* if and only if st^{-1} and $s^{-1}t$ are idempotents. We will say a subset of an inverse semigroup is *compatible* if and only if any pair of elements in it are compatible. We will talk of *compatible union* of partial bijections which is a union of pairwise compatible partial bijections.

Definition 2.6.2. Let I be an inverse semigroup. Let a and b be elements of I . The *join* of a and b denoted $a \vee b$ is the least element of I above both a and b should it exist. Let T be a subset of I . The *join* of T denoted $\bigvee T$ is the least element of I above every element of T should it exist. We will write T^\vee for the closure of T in I under finite join.

Definition 2.6.3. An inverse semigroup S is *distributive* if for all a, b and c in S with a and b compatible there exist joins $a \vee b$ and $ac \vee bc$ and the equality $(a \vee b)c = ac \vee bc$ holds.

Let I be an inverse semigroup. Recall if t is an element of I then we write t^\downarrow for the set $\{i \in I \mid i \leq t\}$. Also recall if T is a subset of I we write T^\downarrow for the set

$$\bigcup_{t \in T} t^\downarrow.$$

Definition 2.6.4. Let I be an inverse semigroup. If T is a subset of I such that $T^\downarrow = T$ then we will say T is an *order ideal* of I . If there is a finite subset A of T such that $A^\downarrow = T$ then we will write T is a *finitely generated* order ideal of I . If T is compatible then we will write T is a *compatible* order ideal of I .

Recall we say functions a and b agree if for each x in $\text{dom}(a) \cap \text{dom}(b)$ the equality $xa = xb$ holds.

Lemma 2.6.5. *Let S be a set and let X be a subset of \mathcal{I}_S . The following are equivalent*

1. X is compatible,
2. a and b agree and a^{-1} and b^{-1} agree for all a and b in X , and
3. $\bigcup X$ is a partial bijection.

Proof. X is defined to be compatible exactly if all a and b in X are compatible.

A pair a and b are defined to be compatible exactly if $ab^{-1} \subseteq 1_{\mathcal{I}_S}$ and $a^{-1}b \subseteq 1_{\mathcal{I}_S}$.

For partial bijections a and b the containment $a^{-1}b \subseteq 1_{\mathcal{I}_S}$ holds exactly if the equality $sa = sb$ holds for any $s \in \text{dom}(a) \cap \text{dom}(b)$, which is the definition of a and b agreeing. The containment $ab^{-1} \subseteq 1_{\mathcal{I}_S}$ holds exactly if for any $s \in \text{ran}(a) \cap \text{ran}(b)$ the equality $sa^{-1} = sb^{-1}$ holds, which is the definition of a^{-1} and b^{-1} agreeing. So

X is compatible exactly if for each a and b in X the pair a and b agree and the pair a^{-1} and b^{-1} agree as desired.

Since X is a set of partial bijections, the union

$$\bigcup X$$

fails to be a partial bijection exactly if there are a and b in X and s in S with $sa \neq sb$ or $sa^{-1} \neq sb^{-1}$. \square

Let S be a set and let I be an inverse submonoid of \mathcal{I}_S then the closure of I under finite compatible union is the set $\{\bigcup F \mid F \text{ is a finite compatible subset of } I\}$. Note that this set contains all finite compatible unions of elements from this set.

Lemma 2.6.6. *Let T be a distributive inverse semigroup and let S be an inverse subsemigroup of T . Then S^\vee , the closure of S under finite compatible join, is also a distributive inverse semigroup.*

Proof. We need to show four things: firstly that S^\vee is closed under join, secondly that S^\vee is closed under the operation, thirdly that S^\vee is closed under taking inverses and fourthly that S^\vee is distributive.

Let A be a finite set of finite compatible subsets of S . We wish to show that

$$\bigvee_{B \in A} \bigvee B = \bigvee_{f \in \bigcup A} f.$$

Since $\bigvee_{f \in \bigcup A} f \geq \bigvee B$ for each B in A it follows that the right hand side is greater than or equal to the left hand side. Since $\bigvee_{B \in A} \bigvee B$ is greater than or equal to each

B in A it follows that $\bigvee_{B \in A} \bigvee B$ is greater than or equal to each $s \in \bigcup A$. Therefore the left hand side is greater than or equal to the right hand side.

Let B and C be finite compatible subsets of S . We wish to show

$$\left(\bigvee B\right) \bigvee C = \bigvee_{b \in B, c \in C} bc.$$

Since T is distributive the left hand side is equal to $\bigvee_{b \in B} (b \bigvee C)$ which, again by the distributivity of T , is equal to $\bigvee_{b \in B} \bigvee_{c \in C} bc$ which simplifies to the right hand side as desired.

If F is a finite compatible subset of I then the inverse of $\bigvee F$ is $\bigvee_{f \in F} f^{-1}$. We will show that

$$\left(\bigvee F\right) \left(\bigvee_{f \in F} f^{-1}\right) \left(\bigvee F\right) = \left(\bigvee F\right)$$

As a consequence of the assumption that D is distributive the left hand side is equal to $\bigvee_{f, g, h \in F} fg^{-1}h$. The left hand side is certainly greater than or equal to the right hand side because the $\{f, g, h \in S \mid fg^{-1}h\}$ contains the set $\{f \in S \mid ff^{-1}f\}$ which is equal to F .

To see that the right hand side is greater than or equal to the left hand side note that for $f, g, h \in F$ the product fg^{-1} is an idempotent since f and g are compatible. Consequently $fg^{-1}h$ is less than or equal to h which is part of the join on the right hand side.

To see that S^\vee is distributive let a, b and c in S^\vee be such that a and b are compatible. Since S^\vee is closed under join $a \vee b$ and $ac \vee bc$ are in S^\vee . Since a, b, c

are also in T it follows that $(a \vee b)c = ac \vee bc$ as desired. \square

Definition 2.6.7. Let I be an inverse semigroup with zero. A subset S of I is *orthogonal* if $ab^{-1} = 0_I$ for all distinct a and b in S . If S is an orthogonal subset of I then the join of S is said to be an *orthogonal join*. If T is a set and R is an orthogonal subset of \mathcal{I}_T then the union $\bigcup R$ is an *orthogonal union*.

If I is an inverse semigroup of partial permutations including the empty function and S is a subset of I then S is orthogonal if and only if for all distinct a and b in S the intersections $\text{dom}(a) \cap \text{dom}(b)$ and $\text{ran}(a) \cap \text{ran}(b)$ are both empty.

Lemma 2.6.8. *Let S be a set and let I be an inverse semigroup of partial permutations of S , such that for each h in I there exists a finite set of idempotents B contained in I , such that the set*

$$\{\text{dom}(b) \mid b \in B \cup \{h\}\}$$

partitions S .

Then the closure of I under finite orthogonal join is equal to I^\vee .

Proof. If I is empty or is a group then the result is immediate so we may assume that I has an element t such that $\text{dom}(t) \neq S$. By assumption there exists a set of idempotents E of I such that $\{\text{dom}(e) \mid e \in E \cup \{t\}\}$ partitions S . Since t is not a permutation it follows that E is non-empty. Let e be an element of E . Now et is equal to the empty function therefore a subset F of I is orthogonal if and only if

for all distinct a and b in F the intersections $\text{dom}(a) \cap \text{dom}(b)$ and $\text{ran}(a) \cap \text{ran}(b)$ are both empty.

Let D be the closure of I under finite orthogonal union. Since the unions allowed in forming D are compatible D is a subset of I^\vee .

Let h be in I we will show that the set D contains the idempotent with domain the complement of the domain of h . Let B be a set of idempotents in I with

$$\{\text{dom}(b) \mid b \in B \cup \{h\}\}$$

partitions S . The idempotent $\bigcup B$ is in D and has domain the complement of the domain of h as desired.

Let g be in I^\vee . We will show g is also in D by finding g as a finite orthogonal union of elements of I . Since union distributes over multiplication we may find a natural number n and f_0, \dots, f_{n-1} pairwise compatible elements of I with $f_0 \cup \dots \cup f_{n-1} = g$. For each i in \mathbf{n} let u_i be the idempotent in D with $\text{dom}(u_i) = S \setminus \text{dom}(f_i)$. For j in \mathbf{n} let $v_j := u_0 \cdots u_{j-1}$.

Let x be a point in the domain of $f_0 \cup \dots \cup f_{n-1}$. Let j be the least element of \mathbf{n} with x in the domain of f_j . We will show x is in the domain of $v_j f_j$ and not in the domain of $v_l f_l$ for $l \neq j$.

If i in \mathbf{n} is strictly less than j then x is not in the domain of $v_i f_i$ because x is not in the domain of f_i . The point x is in the domain of $v_j f_j$ because x is in the domain of u_i for each i in \mathbf{n} strictly less than j and x is in the domain of f_j . If k

in \mathbf{n} is strictly bigger than j then x is not in the domain of $v_k f_k$ because x is not in the domain of v_k because x is not in the domain of u_j

Now $(v_0 f_0) \cup \dots \cup (v_{n-1} f_{n-1})$ equals $f_0 \cup \dots \cup f_{n-1}$ and the domains of the terms are disjoint as desired. Since the domains are disjoint and the union is a partial bijection the ranges must also be disjoint. \square

2.7 Thompson's Group V_n

In this section we will give a construction of the groups V_n which will be analogous to the construction of the group $2V$ which we will give in Section 2.8 which will roughly follow Brin's construction of $2V$ in [2].

Recall we write $H_{\mathcal{C}}$ for the group of homeomorphisms of the Cantor set and P_n for the polycyclic monoid of rank n . The relationship between the polycyclic monoids and Thompson's groups which underlies our construction of V_n in this section and our construction of $2V$ in Section 2.8 was observed in [11].

Definition 2.7.1. For each $n \geq 2$ we define D_n to be the closure of P_n under finite compatible union.

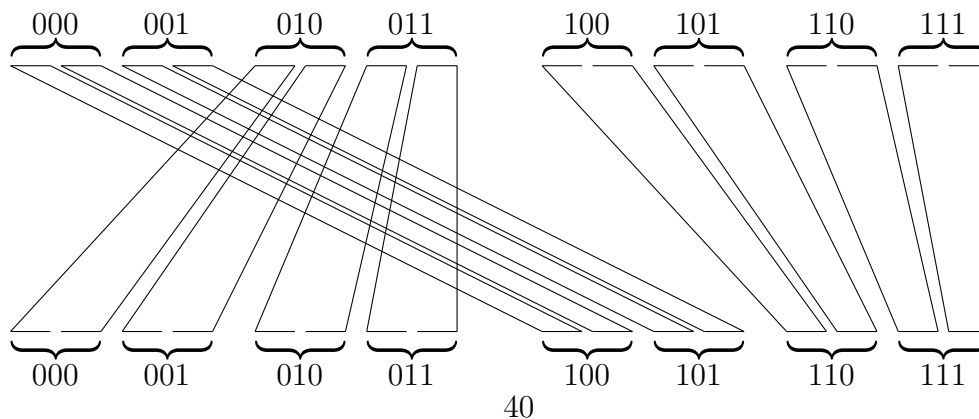
Definition 2.7.2. For each $n \geq 2$ we define V_n to be the group of units of D_n .

If F is a set of partial bijections with disjoint domains and disjoint ranges then we may call the elements of F *pieces* of $\bigcup F$.

Example 2.7.3. We will now give an example of an element of V_2 . The pieces of our element will be $c_0c_0b_0b_1$, $c_0c_1b_0$ and $c_1b_1b_1$. Note that

- $c_0c_0b_0b_1$ is defined exactly on $\{00^\wedge w \mid w \in \{0, 1\}^\mathbb{N}\}$,
- $00^\wedge wc_0c_0b_0b_1 = 10^\wedge w$ for each $w \in \{0, 1\}^\mathbb{N}$,
- $c_0c_1b_0$ is defined exactly on $\{01^\wedge w \mid w \in \{0, 1\}^\mathbb{N}\}$,
- $01^\wedge wc_0c_1b_0 = 0^\wedge w$ for each $w \in \{0, 1\}^\mathbb{N}$,
- $c_1b_1b_1$ is defined exactly on $\{1^\wedge w \mid w \in \{0, 1\}^\mathbb{N}\}$, and
- $1^\wedge wc_1b_1b_1 = 11^\wedge w$ for each $w \in \{0, 1\}^\mathbb{N}$.

The domains partition $\{0, 1\}^\mathbb{N}$ since every infinite word over $\{0, 1\}$ begins with exactly one of 00, 01 or 1 and the ranges partition $\{0, 1\}^\mathbb{N}$ since every infinite word over $\{0, 1\}$ begins with exactly one of 10, 0 or 11. We illustrate this element of V_2 with a diagram. The top copy of $\{0, 1\}^\mathbb{N}$ corresponds to the domain and the bottom copy of $\{0, 1\}^\mathbb{N}$ corresponds to the range. The labels (e.g. 000) are a prefix of elements of that part of $\{0, 1\}^\mathbb{N}$.



Lemma 2.7.4. *D_n is the closure of P_n under taking unions of pairs of partial bijections whose domains are pairwise disjoint and whose ranges are pairwise disjoint.*

Proof. It is sufficient to show that P_n satisfies the hypothesis of Lemma 2.6.8. Let h be an element of P_n . Recall the domains of elements of P_n are strict cones. Let $r \in \mathcal{P}_{\mathbf{N}, \mathbf{n}}^f$ be such that

$$\{x \in \mathbf{n}^{\mathbb{N}} \mid x \supseteq r\} = \text{dom}(h).$$

For each t in $\mathbf{n}^{\text{dom}(r)}$ let g_t be the idempotent of P_n such that

$$\text{dom}(g_t) = \{x \in \mathbf{n}^{\mathbb{N}} \mid x \supseteq t\}.$$

Fix

$$B := \{g_t \in P_n \mid t \in \mathbf{n}^{\text{dom}(r)} \text{ and } t \neq r\}.$$

Now

$$\{\text{dom}(b) \mid b \in B \cup \{h\}\}$$

partitions $\mathbf{n}^{\mathbb{N}}$ as desired. □

Remark 2.7.5. By Lemma 2.7.4 for $n \geq 2$ any element of D_n (and so also V_n) may be represented as the union of a finite set of elements of P_n with pairwise disjoint domains and pairwise disjoint ranges.

2.8 Thompson's Group $2V$

In Section 4.1 we will introduce a generalised notion of word. This is not the notion of word that we will be using in this section. In this section (as in Section 2.7) we use the traditional notion of a word thinking of $\{0, 1\}^{\mathbb{N}}$ as a set of infinite words over $\{0, 1\}$.

Definition 2.8.1. Let n, s, e, w (the initials of the compass directions) be the injections from $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ to $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ such that for each $(x, y) \in \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$

- $(x, y)n = (x, 1 \hat{\ } y)$,
- $(x, y)s = (x, 0 \hat{\ } y)$,
- $(x, y)e = (1 \hat{\ } x, y)$, and
- $(x, y)w = (0 \hat{\ } x, y)$.

We will write $2P_2$ for the inverse submonoid of $\mathcal{I}_{\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}}$ generated by these injections.

We note that our description of P_n in Definition 2.5.1 differs from our description of $2P_2$ in Definition 2.8.1 in that for P_n we give a generating set closed under taking inverses and for $2P_2$ we give a generating set not closed under taking inverses.

Definition 2.8.2. For m and n in \mathbb{N} with $n \geq 2$ we extend the definition of $2P_2$ defining mP_n to be the inverse monoid of partial bijections acting on

$$\{0, \dots, n-1\}^{\mathbb{N}} \times \dots \times \{0, \dots, n-1\}^{\mathbb{N}}$$

in a way that is analogous to to the action of $2P_2$ on $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ in Definition 2.8.1 and the action of P_n in Definition on $\{0, \dots, n-1\}^{\mathbb{N}}$ in 2.5.1.

Remark 2.8.3. If p is a non-zero element of $2P_2$ there is exactly one pair of lists a_1, \dots, a_i and b_1, \dots, b_j over $\{n, s\}$ and exactly one pair of lists c_1, \dots, c_k and d_1, \dots, d_l over $\{e, w\}$ such that

$$p = a_1^{-1} \dots a_i^{-1} c_1^{-1} \dots c_k^{-1} b_j \dots b_1 d_l \dots d_1.$$

It follows from Remark 2.8.3 that the set of domains of elements of $2P_2$, the set of ranges of elements of $2P_2$ and the set of domains of idempotents of $2P_2$ are all equal to the set

$$\{U \times V \mid U \text{ and } V \text{ are strict cones of } \{0, 1\}^{\mathbb{N}}\} \cup \{\emptyset\}.$$

Definition 2.8.4. We define $2D_2$ to be the closure of $2P_2$ under finite compatible union.

Lemma 2.8.5. $2D_2$ is the closure of $2P_2$ under taking unions of pairs of partial bijections whose domains are pairwise disjoint and whose ranges are pairwise disjoint.

Proof. It is sufficient to show that $2P_2$ satisfies the hypothesis of Lemma 2.6.8.

Let h be an element of $2P_2$. Let U and V be strict cones of $\{0, 1\}^{\mathbb{N}}$ with

$$U \times V = \text{dom}(h).$$

Let a in $\mathcal{P}_{S,A}^f$ be such that

$$\{x \in \{0, 1\}^{\mathbb{N}} \mid x \supseteq a\} = U$$

and let r in $\mathcal{P}_{S,A}^f$ be such that

$$\{x \in \{0, 1\}^{\mathbb{N}} \mid x \supseteq r\} = V$$

For each c in $\{0, 1\}^{\text{dom}(a)}$ and t in $\{0, 1\}^{\text{dom}(r)}$ let $g_{(c,t)}$ be the idempotent of $2P_2$ with domain

$$\{x \in \{0, 1\}^{\mathbb{N}} \mid x \supseteq c\} \times \{x \in \{0, 1\}^{\mathbb{N}} \mid x \supseteq t\}.$$

We can find an idempotent of $2P_2$ with domain this set because this set is the Cartesian product of two strict cones. This set is the Cartesian product of two strict cones because $\text{dom}(c) = \text{dom}(a)$ and $\text{dom}(t) = \text{dom}(r)$ and the domain of h is the Cartesian product of two strict cones. Fix

$$B := \{g_{(c,t)} \mid c \in \{0, 1\}^{\text{dom}(a)} \text{ and } t \in \{0, 1\}^{\text{dom}(r)} \text{ and } (c, t) \neq (a, r)\}$$

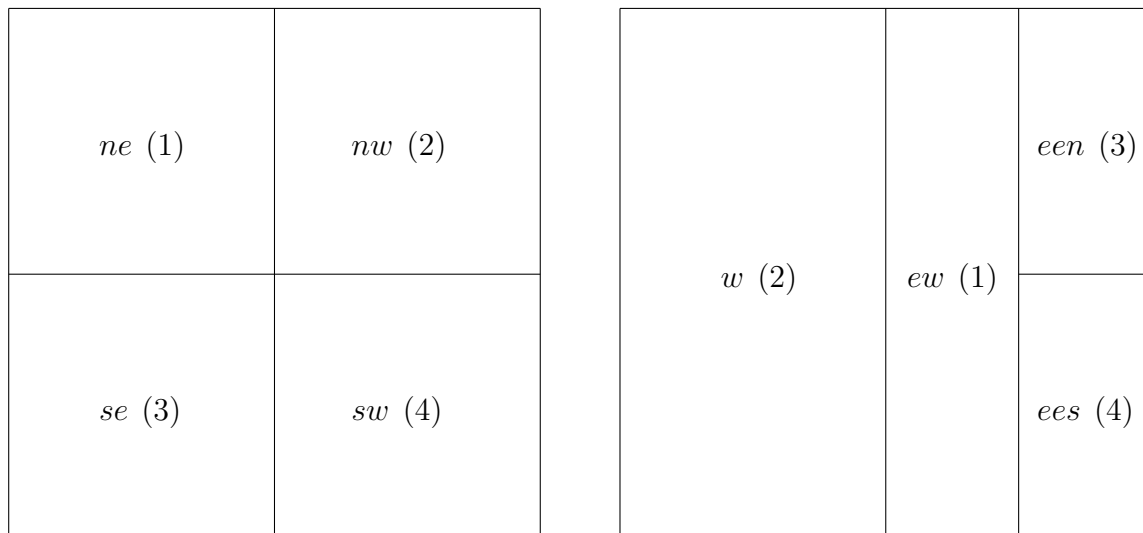
Now

$$\{\text{dom}(b) \mid b \in B \cup \{h\}\}$$

partitions $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ as desired. □

Definition 2.8.6. We define $2V$ to be the group of units of $2D_2$.

Example 2.8.7. We will now give an element of $2V$. The pieces of our element will be $n^{-1}e^{-1}ew$, $n^{-1}w^{-1}w$, $s^{-1}e^{-1}een$ and $s^{-1}w^{-1}ees$.



Each of the two large squares represents a copy of $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ the first square represents the partition of $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ into the domains of the pieces and the second large square represents the partition of $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ into the ranges of the pieces. Each rectangle is the image of the element of $2P_2$ labelling it. For each i in $\{1, 2, 3, 4\}$ the rectangle in the first large square labelled i is mapped to the rectangle in the second large square labelled i .

Remark 2.8.8. It is possible to perform an analogous construction acting on $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ instead of $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$. Similarly for each natural number m there is an analogous construction acting on the m th direct power of

$\{0, 1\}^{\mathbb{N}}$.

For each m and n in the naturals with $n > 1$ by combining the construction of the previous paragraph with the construction of Definitions 2.7.2 we may form a still more general construction acting on the m th direct power of $\{0, 1, \dots, n-1\}^{\mathbb{N}}$.

Definition 2.8.9. For m and n natural numbers with $n > 1$ we define mV_n to be the group acting on the m th direct power of $\{0, 1, \dots, n-1\}^{\mathbb{N}}$ constructed in Remark 2.8.8.

Our construction is not identical to Brin's in [2], but all of the differences are superficial, we outline them below.

The middle third Cantor set is a subset of the unit interval homeomorphic to the Cantor set which we do not define. Brin uses an identification between $\{0, 1\}^{\mathbb{N}}$ and the middle third Cantor set.

In both constructions elements of $2V_2$ are piecewise defined. Above we define the pieces first and then put them together whereas in [2] the partitions of $\{0, 1\}^{\mathbb{N}}$ that may be induced by the domains or ranges of the pieces of an element of $2V_2$ are defined and then the pieces are given as linear maps using the identification between $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ and the Cartesian product of two copies of the middle third Cantor set.

Chapter 3

Sufficient Conditions for a Group of Homeomorphisms of the Cantor Set to be 2-Generated

3.1 Main Theorem and Plan

Before laying out the structure of this chapter we will state the main theorem and its corollary, which require a definition.

Definition 3.1.1. We will say that a subset G of $\text{Sym}(\mathcal{C})$ is *vigorous* if and only if for all clopen subsets A, B, C of \mathcal{C} with B and C proper non-empty subsets of A there exists g in G with $\text{supp}(g) \subseteq A$ with $Bg \subseteq C$.

We use the word vigorous because it is evocative of thorough mixing of the Cantor set. We define vigorous for subsets of $\text{Sym}(\mathcal{C})$ but will in fact largely only be interested in vigorous subgroups of $H_{\mathcal{C}}$. (Recall $H_{\mathcal{C}}$ is the group of homeomorphisms of the Cantor set.) Thompson's group V_2 is an example of a vigorous group as Corollary 3.2.2 will show.

The main result that we establish in this chapter is the following theorem and corollary. We delay the proof of Theorem 3.1.2 till Section 3.4.

Theorem 3.1.2. *If G is a simple vigorous subgroup of $H_{\mathcal{C}}$ and E is a finitely generated subgroup of G then there exists a 2-generated group F with $E \leq F \leq G$.*

Corollary 3.1.3. *All finitely generated simple vigorous subgroups of $H_{\mathcal{C}}$ are 2-generated.*

Proof. Let G be a simple vigorous finitely generated subgroup of $H_{\mathcal{C}}$. Set $E := G$. By Theorem 3.1.2 there must exist a 2-generated group $F \leq H_{\mathcal{C}}$ with $E \leq F \leq G$. Since $E = G$ we must have $E = F = G$ so G is 2-generated as desired. \square

A group is simple exactly if the normal closure of any non-identity element is the whole group. A rough interpretation of simplicity that may be useful in this context is that conjugation is powerful enough that any element is the product of conjugates of any other non-identity element.

We spend the rest of this section on a plan of this chapter. In Section 3.2 we will prove Theorem 3.2.1 showing certain inverse monoids of partial bijections of

the Cantor set have vigorous groups of units. It will follow from Corollary 3.2.2 to Theorem 3.2.1 that Thompson's group V_2 is vigorous.

Theorem 3.1.2 applies to the the set of finitely generated simple vigorous groups of homeomorphisms of the Cantor set. In Definition 3.3.1 we name this set \mathcal{H} . In Section 3.3 we will introduce another property of subgroups of H_C and use it to discuss the set \mathcal{H} . We will show that \mathcal{H} is closed under some natural constructions. By Corollary 3.2.2 Thompson's group V_2 is in \mathcal{H} .

In Section 3.4 we will prove Theorem 3.1.2 through a series of lemmas.

We will conclude this chapter with a brief discussion of directions that Theorem 3.1.2 suggests could be explored.

We note that Epstein in [4] gives sufficient conditions for groups of homeomorphisms of a large class of topological spaces (including the Cantor set) to be simple. Theorem 3.1.2 is similar to Epstein's result in that the both theorems draw algebraic conclusions from dynamic hypotheses (though Theorem 3.1.2 also has algebraic properties).

3.2 Inverse Semigroups of Partial Bijections of the Cantor Set

In this section we give sufficient conditions for the group of units of an inverse monoid of partial bijections of the Cantor set to be vigorous and use this result to show that Thompson's group V_2 is vigorous.

Recall for each $n \geq 2$ the polycyclic monoid of rank n is denoted P_n and was defined in Definition 2.5.1. Also recall for each $n \geq 2$ the polycyclic monoid of rank n acts on $\mathbf{n}^{\mathbb{N}}$ by partial bijections.

Theorem 3.2.1. *Let I be an inverse monoid of partial bijections of the Cantor set. Assume*

1. *for each clopen subset K of \mathcal{C} , the idempotent with domain K is an element of I ,*
2. *for each clopen subset K of \mathcal{C} , there is some element f in I with $\text{dom}(f) = \mathcal{C}$ and $\text{ran}(f) \subseteq K$, and*
3. *for each f and t in I with $\text{dom}(f) \cap \text{dom}(t) = \emptyset$ and $\text{ran}(f) \cap \text{ran}(t) = \emptyset$ and with the sets $\text{dom}(f) \cup \text{dom}(t)$ and $\text{ran}(f) \cup \text{ran}(t)$ contained in proper clopen subsets of \mathcal{C} , there exists g in the group of units of I with $f \cup t \subseteq g$.*

Then the group of units of I is vigorous.

Proof. Let A , B and C be clopen sets with B and C non-empty proper clopen subsets of A . We will find g in the group of units of I with $\text{supp}(g) \subseteq A$ and $Bg \subseteq C$.

Let a and b denote the idempotents with $\text{dom}(a) = \mathcal{C} \setminus A$ and $\text{dom}(b) = B$. By Assumption 1 from the lemma a and b are in I . Using Assumption 2 from the lemma, choose f an element of I with $\text{dom}(f) = \mathcal{C}$ and $\text{ran}(f) \subseteq C$.

Since $\text{dom}(a) = \mathcal{C} \setminus A$ and $\text{dom}(bf) = B \subsetneq A$, it follows $\text{dom}(a) \cup \text{dom}(bf)$ is a proper clopen subset of \mathcal{C} and $\text{dom}(a) \cap \text{dom}(bf) = \emptyset$. Since $\text{ran}(a) = \mathcal{C} \setminus A$ and $\text{ran}(bf) \subseteq C \subsetneq A$, it follows $\text{ran}(a) \cap \text{ran}(bf) = \emptyset$ and $\text{ran}(a) \cup \text{ran}(bf)$ is contained in $\mathcal{C} \setminus (A \setminus C)$ which is a proper clopen subset of \mathcal{C} . By Assumption 3 from the lemma there exists g in the group of units of I with $a \cup bf \subseteq g$.

Since g contains a , the support of g must be a subset of A . Since $bf \subseteq g$, the image of B under g must be a subset of C . □

For each $n \geq 2$ recall firstly the closure of P_n under compatible union is D_n and secondly the group of units of D_n is V_n .

Corollary 3.2.2. *The inverse semigroup D_n satisfies the conditions of Theorem 3.2.1 for each $n \geq 2$.*

Proof. For this proof we will think of $\mathbf{n}^{\mathbb{N}}$ as the set of infinite words over \mathbf{n} as in Definition 2.5.1. Strict cones of $\mathbf{n}^{\mathbb{N}}$ take the form $\{x_1x_2 \cdots x_i \hat{\ } w \mid w \in \mathbf{n}^{\mathbb{N}}\}$ for some word $x_1x_2 \cdots x_i$ over \mathbf{n} .

1. Let a word $x_1x_2 \cdots x_i$ over \mathbf{n} be given. The idempotent

$$c_{x_1}c_{x_2} \cdots c_{x_i}b_{x_i} \cdots b_{x_2}b_{x_1}$$

in P_n has domain $\{x_1x_2 \cdots x_i \hat{\ } w \mid w \in \mathbf{n}^{\mathbb{N}}\}$. Since P_n is contained in D_n this idempotent is also in D_n .

2. Let $y_1y_2 \cdots y_j$ a word over \mathbf{n} be given. Since each clopen set contains a strict cone it is sufficient to find an element of D_n with domain equal to $\mathbf{n}^{\mathbb{N}}$ and range equal to $\{y_1 \cdots y_j \hat{\ } w \mid w \in \mathbf{n}^{\mathbb{N}}\}$. The partial bijection $b_{y_j} \cdots b_{y_2}b_{y_1}$ is as required and is in P_n which is contained in D_n .

3. Recall strict cones were defined in Definition 2.4.7. Let f and t be as in Condition 3 from Lemma 3.2.1.

Since f and t are finite unions of elements of P_n and since the domains and ranges of elements of P_n are strict cones the sets $\text{dom}(f) \cup \text{dom}(t)$ and $\text{ran}(f) \cup \text{ran}(t)$ must be clopen.

Let $U := \mathbf{n}^{\mathbb{N}} \setminus (\text{dom}(f) \cup \text{dom}(t))$ and let $V := \mathbf{n}^{\mathbb{N}} \setminus (\text{ran}(f) \cup \text{ran}(t))$. Note that U and V are also clopen. We will find s in D_n with $\text{dom}(s) = U$ and $\text{ran}(s) = V$. Finding such an s is sufficient to finish the proof because then $f \cup t \cup s$ would be a bijection containing $f \cup t$ (and in D_n since D_n is closed under compatible union) as desired.

Using Lemma 2.7.4 fix K a finite subset of P_n with the domains of elements

of K pairwise disjoint and $\bigcup K = f \cup t$.

Note $\{\text{dom}(k) \mid k \in K\}$ partitions $\text{dom}(f) \cup \text{dom}(t)$ into strict cones and $\{\text{ran}(k) \mid k \in K\}$ partitions $\text{ran}(f) \cup \text{ran}(t)$ into strict cones. Let X be a partition of $\mathbf{n}^{\mathbb{N}}$ into strict cones with $\{\text{dom}(k) \mid k \in K\} \subseteq X$ and let Y be a partition of $\mathbf{n}^{\mathbb{N}}$ into strict cones with $\{\text{ran}(k) \mid k \in K\} \subseteq Y$.

Since there is a correspondence between transversals of $\mathbf{n}^{\mathbb{N}}$ and partitions of $\mathbf{n}^{\mathbb{N}}$ into cones it follows from Lemma 2.4.26 that the difference $|X| - |Y|$ must be a multiple of $n - 1$. Since $|\{\text{dom}(k) \mid k \in K\}| = |K| = |\{\text{ran}(k) \mid k \in K\}|$ the difference $|X \setminus \{\text{dom } k \mid k \in K\}| - |Y \setminus \{\text{ran } k \mid k \in K\}|$ must also be a multiple of $n - 1$.

If $|X|$ is strictly smaller than $|Y|$ we may take one of the strict clopen sets in $X \setminus \{\text{dom}(k) \mid k \in K\}$ and replace it with n strict clopen sets preserving the fact that X is a partition of $\mathbf{n}^{\mathbb{N}}$. By repetitive application of this process (or the analogous process for $|Y| < |X|$) we may assume $|X| = |Y|$. (We cannot use Lemma 2.4.24 because that deals with clopen sets not strict clopen sets).

Let g be a bijection from $X \setminus \{\text{dom}(k) \mid k \in K\}$ to $Y \setminus \{\text{ran}(k) \mid k \in K\}$. For each x in $X \setminus \{\text{dom}(k) \mid k \in K\}$ let h_x be the element of P_n with domain x and range xg .

Now $s := \bigcup_{x \in X \setminus \{\text{dom}(k) \mid k \in K\}} h_x$ is as desired.

□

Corollary 3.2.3. *For each $n \geq 2$ Thompson's group V_n is vigorous.*

Proof. Corollary 3.2.2 states the inverse monoid D_n satisfies the conditions of Lemma 3.2.1. Since V_n is the group of units of D_n we may conclude that V_n is vigorous. \square

It is possible to prove that V_n is vigorous for each $n \geq 2$ directly without relying on Theorem 3.2.1. We sketch this proof below.

Fix $n \geq 2$. Let A be a non-empty clopen subset of $\{0, \dots, n-1\}^{\mathbb{N}}$ and let B and C be non-empty proper clopen subsets of A . Let Q be a partition of A into strict cones. Let Q_B and Q_C be partitions of A into strict cones such that Q_B and Q_C are refinements of Q and such that $|Q_B| = |Q_C|$ and such that B is a union of i strict cones in Q_B and such that C is a union of j strict cones in Q_C for some i and j natural numbers with $i \leq j$.

Recall the inverse semigroup P_n was defined in Definition 2.5.1 and the group V_n is defined to be the closure of P_n under compatible union. Also note that for any pair of strict cones of $\{0, \dots, n-1\}^{\mathbb{N}}$ there is a unique element of P_n mapping the first bijectively to the second. We may now construct an element g of V_n which firstly stabilises $\{0, \dots, n-1\}^{\mathbb{N}} \setminus A$ pointwise, secondly maps each strict cone of Q_B to a strict cone of Q_C and thirdly maps those strict cones of Q_B which are subsets of B to strict cones of Q_C which are subsets of C . The group element g has the properties we are looking for.

We will now introduce another property of subgroups of $H_{\mathcal{C}}$ and prove it is strictly stronger than vigorous to give vigorous some context.

Definition 3.2.4. We will say a subset G of $\text{Sym}(\mathcal{C})$ is *strongly vigorous* if and only if for all clopen subsets A, B and C of \mathcal{C} with B and C proper subsets of A there exists $g \in G$ with $\text{supp}(g) \subseteq A$ and $Bg = C$.

One might reasonably wonder if the property of being vigorous is equivalent to the property of being strongly vigorous for groups of homeomorphisms of the Cantor set. We give the next lemma to demonstrate the two properties are distinct. We will not discuss the property of being strongly vigorous again after the end of this section.

Lemma 3.2.5. *For $n > 2$ the group V_n is not strongly vigorous.*

Proof. For this proof we will think of $\mathbf{n}^{\mathbb{N}}$ as the set of infinite words over \mathbf{n} . Let D be the set $\{0^{\wedge}w \mid w \in \mathbf{n}^{\mathbb{N}}\}$ and let R be the set $\{0^{\wedge}w \mid w \in \mathbf{n}^{\mathbb{N}}\} \cup \{1^{\wedge}w \mid w \in \mathbf{n}^{\mathbb{N}}\}$. Let g an element of V_n be given. We will show $Dg \neq R$.

Let e in D_n be the idempotent defined exactly on D . Fix $h := eg$ and note that h is also in D_n . It is sufficient to show $\text{ran}(h) \neq R$. Using Remark 2.7.5 we may find F a finite subset of P_n with $h = \bigcup F$ and the set of domains of elements of F pairwise disjoint.

The set $\{\text{dom}(f) \mid f \in F\}$ partitions D into strict cones. Similarly the set $\{\text{ran}(f) \mid f \in F\}$ partitions $\text{ran}(h)$ into strict cones. By Lemma 2.4.26 the set

$\{\text{dom}(f) \mid f \in F\}$ has size congruent to $1 \pmod{n-1}$. But

$$|\{\text{ran}(f) \mid f \in F\}| = |F| = |\{\text{dom}(f) \mid f \in F\}| \equiv 1 \pmod{n-1}.$$

Since $R = \{0^w \mid w \in \mathbf{n}^{\mathbb{N}}\} \cup \{1^w \mid w \in \mathbf{n}^{\mathbb{N}}\}$ by Lemma 2.4.26 all partitions of R into cones have size congruent to $2 \pmod{n-1}$. Since $n > 2$ the set $\{\text{ran}(f) \mid f \in F\}$ cannot be a partition of R .

Now $\text{ran}(h)$ cannot equal R since $\{\text{ran}(f) \mid f \in F\}$ is a partition of one but not the other. □

3.3 Development of Concepts and Examples

Definition 3.3.1. We will write \mathcal{K} for the family of subgroups of H_C which are simple, vigorous and finitely generated.

Corollary 3.1.3 exactly says that all the groups in \mathcal{K} are 2-generated. In Chapter 4 we will give more examples of vigorous subgroups of H_C . In this section we will give examples of groups in \mathcal{K} by showing R Thompson's group V_2 is in \mathcal{K} (mainly by referencing other works) and describing constructions \mathcal{K} is closed under. We note that Mason showed in [17] that R Thompson's group V_2 is 2-generated.

Definition 3.3.2. We will say a subgroup G of H_C is *flawless* if and only if G is

generated by the set

$$\{[g, h] \mid g, h \in \text{pstab}_G(A) \text{ for some non-empty clopen set } A\}.$$

Both the definitions of being flawless and vigorous are original to this thesis. We use the word flawless because of its association with perfection and simplicity (as Lemmas 3.3.3 shows flawless groups are perfect and as Lemma 3.3.8 shows vigorous groups are flawless if and only if they are simple).

We emphasise that unlike the properties of being simple or perfect the properties of being vigorous and flawless are dependent not just on the groups but also on their actions on \mathcal{C} .

Lemma 3.3.3. *All flawless subgroups of $H_{\mathcal{C}}$ are perfect.*

Proof. If G is a flawless subgroup of $H_{\mathcal{C}}$ then the set

$$\{[g, h] \mid g, h \in \text{pstab}_G(A) \text{ for some non-empty clopen set } A\}$$

generates G . Since this set is contained in the set of commutators of G they must also generate G as desired. \square

We now give a construction which can produce perfect subgroups of $H_{\mathcal{C}}$ which are not flawless to show the two families are distinct.

Example 3.3.4. Let A be a finite set of size at least 2. For this example we will denote elements of $A^{\mathbb{N}}$ by infinite words over A . Let $\alpha_A : \text{Sym}(A) \longrightarrow \text{Sym}(A^{\mathbb{N}})$

be the function defined by

$$(a_1a_2a_3 \dots)(g\alpha_A) := (a_1g)(a_2g)(a_3g) \dots$$

for each $g \in \text{Sym}(A)$ and $a_1a_2a_3 \dots \in A^{\mathbb{N}}$. For each $g \in \text{Sym}(A)$ the bijection $g\alpha_A$ maps the set of cones bijectively to the set of cones so in fact α_A maps $\text{Sym}(A)$ into $H_{A^{\mathbb{N}}}$.

Let g and h be in $\text{Sym}(A)$ and let x be in $A^{\mathbb{N}}$. Since

$$(a_1a_2a_3 \dots)(g\alpha_A)(h\alpha_A) = (a_1gh)(a_2gh)(a_3gh) \dots = (a_1a_2a_3 \dots)((gh)\alpha_A)$$

the function α_A is a homomorphism. We wish to show α_A is a monomorphism.

Assume $g \neq h$ and let b be an element of A with $bg \neq bh$. For each finite word w over A let N_w be the infinite word which is just w repeated indefinitely. Now

$$N_b(g\alpha_A) = N_{bg} \neq N_{bh} = N_b(h\alpha_A)$$

so $g\alpha_A$ is not equal to $h\alpha_A$ as desired.

Assume A is such that $\text{Sym}(A)$ has a non-trivial perfect subgroup G . Since α_A is a monomorphism the image of G under α_A is isomorphic to G and therefore perfect and non-trivial. To show that $G\alpha_A$ is not flawless we will show that $\text{pstab}_{G\alpha_A}(C)$ is trivial for each non-empty clopen subset C of $A^{\mathbb{N}}$. To show this we will show the support of each non-identity element of $G\alpha_A$ intersects every strict cone of $A^{\mathbb{N}}$.

Let g be a non-identity element of G and let $a_1 \cdots a_i$ be a finite word over A . We will show that the support of g intersects the strict cone $a_1 \cdots a_i \mathcal{C}$ non-trivially. Since $\text{Sym}(A)$ acts faithfully on A and g is not the identity we may find b and c in A with $bg = c \neq b$. Now

$$(a_1 \cdots a_i N_b)(g\alpha_A) = (a_1 g) \cdots (a_i g) N_{bg} = (a_1 g) \cdots (a_i g) N_c \neq a_1 \cdots a_i N_b$$

since $c \neq b$. The infinite word $a_1 \cdots a_i N_b$ is both in the strict cone $a_1 \cdots a_i \mathcal{C}$ and the support of $g\alpha_A$ as desired.

Lemma 3.3.5. *If F and G are subgroups of $H_{\mathcal{C}}$ with F vigorous and contained in G then G is also vigorous.*

Proof. Let A , B and C be clopen subsets of $H_{\mathcal{C}}$ with B and C proper non-empty subsets of A . Since F is vigorous the set

$$F \cap \{f \in \text{pstab}_{H_{\mathcal{C}}}(A) \mid Bf \subseteq C\}$$

is non-empty. On the other hand F is contained in G so the above set is contained in

$$G \cap \{g \in \text{pstab}_{H_{\mathcal{C}}}(A) \mid Bg \subseteq C\}$$

which is therefore also non-empty so G is also vigorous. □

Lemma 3.3.6. *If G is a subgroup of $H_{\mathcal{C}}$, the subgroup generated by*

$$F := \{[g, h] \mid g, h \in \text{pstab}_G(A) \text{ for some non-empty clopen set } A\}$$

is a normal subgroup of G .

Proof. For the group generated by F to be normal it must be closed under conjugation. It is sufficient to show that F is closed under conjugation. Let A be a non-empty clopen subset of \mathcal{C} and let g, h be in $\text{pstab}_G(A)$ and let f be in G . Since A is non-empty and clopen and f is a homeomorphism the set Af is also non-empty and clopen. Since both g^f and h^f stabilise Af pointwise the conjugate $[g, h]^f = [g^f, h^f]$ is also in F as desired. \square

Lemma 3.3.7. *For each non-identity g in $H_{\mathcal{C}}$ there exists a non-empty clopen subset Y of \mathcal{C} with Yg disjoint from Y .*

Proof. For this proof we will use $\{0, 1\}^{\mathbb{N}}$ to represent the Cantor set. Since g is non-identity we may choose x in $\{0, 1\}^{\mathbb{N}}$ with $xg \neq x$. Let $n \in \mathbb{N}$ be such that $nx \neq n(xg)$. Let $A := \{y \in \{0, 1\}^{\mathbb{N}} \mid ny = nx\}$ and $B := \{y \in \{0, 1\}^{\mathbb{N}} \mid ny = n(xg)\}$. Let $Y := A \cap Bg^{-1}$. The set Y is non-empty since x is in both A and Bg^{-1} . Since A and B are disjoint it follows

$$\begin{aligned} Y \cap Yg &= (A \cap Bg^{-1}) \cap (A \cap Bg^{-1})g \\ &= A \cap Bg^{-1} \cap Ag \cap B \\ &\subseteq A \cap B \\ &= \emptyset \end{aligned}$$

as desired. \square

Lemma 3.3.8. *Let G be a vigorous subgroup of $H_{\mathcal{C}}$. Then G is flawless if and only if G is simple.*

Proof. First we assume G is simple and show G is flawless. Set

$$F := \{[g, h] \mid g, h \in \text{pstab}_G(A) \text{ for some non-empty clopen set } A\}.$$

The group G is simple and Lemma 3.3.6 shows $\langle F \rangle$ is normal so to show F generates G it is sufficient to show that F is non-trivial. To show F is non-trivial we will find a pair of non-commuting elements of G that stabilise the same non-empty clopen set pointwise.

Let A, B, C and D be non-empty disjoint clopen subsets of \mathcal{C} . Since G is vigorous we may choose p in $\text{pstab}(A \cup B)$ with $Cp \subseteq D$. Similarly we may choose q in $\text{pstab}(A \cup D)$ with $Cq \subseteq B$. Now $Cpq \subseteq D$ and $Cqp \subseteq B$ so p and q do not commute but do stabilise A pointwise as desired.

Now we assume G is flawless and show G is simple. Let N be a non-trivial normal subgroup of G . We will show that $N = G$. Let $g \in N$ be non-identity and let h be in F . We will show that h is a product of conjugates of g .

Since h is in F there must exist $u, v \in G$ with $h = [u, v]$ and the supports of u and v contained in some proper clopen subset X of \mathcal{C} . (The set X is the complement of A in the definition of F). Since g is non-identity we may use Lemma 3.3.7 to find Y a non-empty clopen subset of \mathcal{C} with $Yg \cap Y = \emptyset$. Since G is vigorous we may fix $a \in G$ with $Xa \subseteq Y$. Note $Xaga^{-1}$ does not intersect X .

Also note $u^{aga^{-1}}$ commutes with v^{-1} since their supports do not intersect. Using the equalities

$$\begin{aligned}
[[aga^{-1}, u], v] &= u^{-1}ag^{-1}a^{-1}uaga^{-1} \cdot v^{-1} \cdot ag^{-1}a^{-1}u^{-1}aga^{-1}u \cdot v \\
&= u^{-1}(ag^{-1}a^{-1}uaga^{-1})(v^{-1})ag^{-1}a^{-1}u^{-1}aga^{-1}uv \\
&= u^{-1}(v^{-1})(ag^{-1}a^{-1}uaga^{-1})ag^{-1}a^{-1}u^{-1}aga^{-1}uv \\
&= u^{-1}v^{-1}uv \\
&= h,
\end{aligned}$$

we deduce h is a product of conjugates of g so h is in N . However h was an arbitrary element of F so all of F must be contained in N . But since F generates G the normal subgroup N must be the whole of G as desired. \square

Recall \mathcal{K} is the family of finitely generated simple vigorous subgroups of H_C .

Example 3.3.9. The group V_2 is a simple vigorous and finitely generated subgroup of H_C so Corollary 3.1.3 applies and V_2 is in \mathcal{K} and 2-generated.

Proof. Lemma 3.2.3 tell us that V_2 is vigorous. That V_2 is simple and finitely generated was proved in [3] and earlier by Thompson himself in unpublished notes [20]. \square

We note that Mason showed in [17] that V_2 is 2-generated.

Having given some examples of groups in \mathcal{K} we will now show that \mathcal{K} is closed under various natural constructions to allow for more groups in \mathcal{K} to be easily constructed.

Proposition 3.3.10. *If G is in \mathcal{K} and f is a homeomorphism of the Cantor set then $G^f = \{g^f \mid g \in G\}$ is also in \mathcal{K} .*

Proof. The group G is simple and finitely generated and the map $g \mapsto g^f$ is a group isomorphism so G^f is also simple and finitely generated.

Let A, B and C be clopen subsets of \mathcal{C} with B and C non-empty proper subsets of A . Since f is a homeomorphism Af^{-1} , Bf^{-1} and Cf^{-1} are clopen sets and Bf^{-1} and Cf^{-1} are non-empty proper subsets of Af^{-1} . Since G is vigorous we may choose g with $\text{supp}(g) \subseteq Af^{-1}$ and $Bf^{-1}g \subseteq Cf^{-1}$. Now $\text{supp}(g^f) \subseteq A$ and $Bg^f \subseteq C$ so G^f is also vigorous. \square

Proposition 3.3.11. *If G and H are in \mathcal{K} then the group $\langle G \cup H \rangle$ is also in \mathcal{K} .*

Proof. Since $\langle G \cup H \rangle$ contains G which is vigorous, $\langle G \cup H \rangle$ is also vigorous by Lemma 3.3.5. Since G and H are simple and vigorous, they are also flawless by Lemma 3.3.8. Let

$$F_G := \{[g, h] \mid g, h \in \text{pstab}_G(A) \text{ for some non-empty clopen set } A\},$$

$$F_H := \{[g, h] \mid g, h \in \text{pstab}_H(A) \text{ for some non-empty clopen set } A\}$$

and

$$F_{\langle G \cup H \rangle} := \{[g, h] \mid g, h \in \text{pstab}_{\langle G \cup H \rangle}(A) \text{ for some non-empty clopen set } A\}.$$

Since F_G generates G and F_H generates H the set $F_G \cup F_H \subseteq F_{\langle G \cup H \rangle}$ must generate $\langle G \cup H \rangle$ so $\langle G \cup H \rangle$ is also flawless and therefore simple.

It remains to show G is finitely generated. If X is a finite generating set for G and Y is a finite generating set for H then $X \cup Y$ is a finite generating set for $\langle G \cup H \rangle$ as desired. \square

Proposition 3.3.12. *Let g be an element of a group $G \in \mathcal{K}$ such that the support of g is a subset of a non-empty proper clopen subset D of \mathcal{C} . Let B be a non-empty proper clopen subset of D and let h be in $H_{\mathcal{C}}$ with $\text{supp}(h)$ disjoint from B and $\text{supp}(h)g$ a subset of B . Then the group $\langle G \cup \{[g, h]\} \rangle$ is also in \mathcal{K} .*

Proof. We will write H for $\langle G \cup \{[g, h]\} \rangle$. The proof that H is finitely generated and vigorous is the same as in Proposition 3.3.11. Since H is vigorous to show that H is simple it is sufficient to show that H is flawless.

Let A be a proper clopen subset of $\mathcal{C} \setminus Bg^{-1}$ properly containing $D \setminus Bg^{-1}$. To show H is flawless it is sufficient to find u, v in H with $[u, v] = [g, h]$ and $\text{supp}(u) \cup \text{supp}(v) \subseteq A \cup Bg^{-1}$. (Note we cannot use $u := g$ and $v := h$ because h may not be in H). We may assume h is not the identity since the proposition is trivial otherwise.

Since $\text{supp}(h) \cap \text{supp}(h)g = \emptyset$ it follows that $\text{supp}(h)g$ is a subset of $\text{supp}(g) \subseteq D \subseteq A \cup Bg^{-1}$. Since h is non-trivial it also follows that $\text{supp}(h)g$ is non-empty so B is also non-empty. Since D is non-empty $A \setminus D$ is a non-empty proper clopen subset of A .

Since G is vigorous by the last paragraph we may fix $y \in G$ with $\text{supp}(y) \subseteq A$

and $By \subseteq A \setminus D$. First observe y commutes with h and second observe h^{gy} commutes with g .

$$\begin{aligned}
[g, [g, h]^y] &= g^{-1} \cdot y^{-1} h^{-1} g^{-1} h g y \cdot g \cdot y^{-1} g^{-1} h^{-1} g h y \\
&= g^{-1} (y^{-1} h^{-1}) g^{-1} h g y g y^{-1} g^{-1} h^{-1} g (h y) \\
&= g^{-1} h^{-1} (y^{-1} g^{-1} h g y \cdot g) y^{-1} g^{-1} h^{-1} g y h \\
&= g^{-1} h^{-1} g (y^{-1} g^{-1} h g y \cdot y^{-1} g^{-1} h^{-1} g y) h \\
&= g^{-1} h^{-1} g h \\
&= [g, h]
\end{aligned}$$

as desired. □

Corollary 3.3.13. *All the groups in \mathcal{K} are 2-generated and simple. This includes V_2 and any group that may be constructed from V_2 using the constructions described in Proposition 3.3.10, Proposition 3.3.11 and Proposition 3.3.12.*

Proof. This follows immediately from Corollary 3.1.3. □

We now give an example to show that \mathcal{K} is not closed under taking supergroups.

Example 3.3.14. Let $\alpha_{\{0,1\}}$ be as in Example 3.3.4. We will use G for the group

$$\langle \text{Sym}(\{0, 1\}) \alpha_{\{0,1\}} \cup V_2 \rangle.$$

The group G is a supergroup of V_2 which is in \mathcal{H} . We will show that G is not simple and is therefore not in \mathcal{H} .

We will use S_0 for the set $\{x \in \{0, 1\}^{\mathbb{N}} \mid \text{the set } 1x^{-1} \text{ is finite}\}$ and use S_1 for the set $\{x \in \{0, 1\}^{\mathbb{N}} \mid \text{the set } 0x^{-1} \text{ is finite}\}$. All elements of V_2 stabilise both S_0 and S_1 setwise. The non-identity element of $\text{Sym}(\{0, 1\})_{\alpha_{\{0,1\}}}$ maps S_0 bijectively to S_1 and maps S_1 bijectively to S_0 .

From the last paragraph it follows that all elements of G either stabilise both S_1 and S_2 setwise or switch them. From this it follows that there is a homomorphism from G to the two element group sending elements which stabilise both S_1 and S_2 setwise to the identity and sending elements which switch S_1 and S_2 to the non-identity element. Therefore G is not simple.

3.4 Proof of Theorem 3.1.2

In this section we will prove a string of lemmas which will come together to prove Theorem 3.1.2.

Lemma 3.4.1. *If $F \leq G$ are groups with X a finite generating set for F and Y a generating set for G then there exists a finite subset Y_0 of Y with $\langle Y_0 \rangle \geq F$.*

Proof. Since $X \subseteq \langle Y \rangle$ we may for each element of X fix a product over Y equal to that element of X . Since X is finite and the set of elements of Y used in any

one of the products is finite, the set Y_0 of elements of Y that are used in any of the products is finite. However $\langle Y_0 \rangle \supseteq X$ so $\langle Y_0 \rangle \geq \langle X \rangle = F$ as desired. \square

Lemma 3.4.2. *If $G \leq H_{\mathcal{C}}$ is vigorous then the set*

$$\{[\mu, \nu] \mid \mu, \nu \in \text{pstab}_G(A) \text{ for some non-empty clopen set } A\}$$

is also vigorous.

Proof. Let A , B and C be clopen subsets of \mathcal{C} with B and C non-empty proper subsets of A . First find D a non-empty proper clopen subset of A disjoint from B and not containing C . If $B \cup C = A$ then set D to be any non-empty proper clopen subset of $A \setminus B$. Otherwise set $D := A \setminus (B \cup C)$.

Since G is vigorous we may choose μ in G with $\text{supp}(\mu) \subseteq A$ and $(A \setminus D)\mu \subseteq D$ and ν in G with $\text{supp}(\nu) \subseteq A \setminus D$ with $B\nu \subseteq C \setminus D$.

Since $\text{supp}((\nu^{-1})^\mu) = \text{supp}(\nu^{-1})\mu \subseteq (A \setminus D)\mu \subseteq D$ is disjoint from B and ν maps B into C the commutator $[\mu, \nu] = (\nu^{-1})^\mu \nu$ also maps B into C as desired.

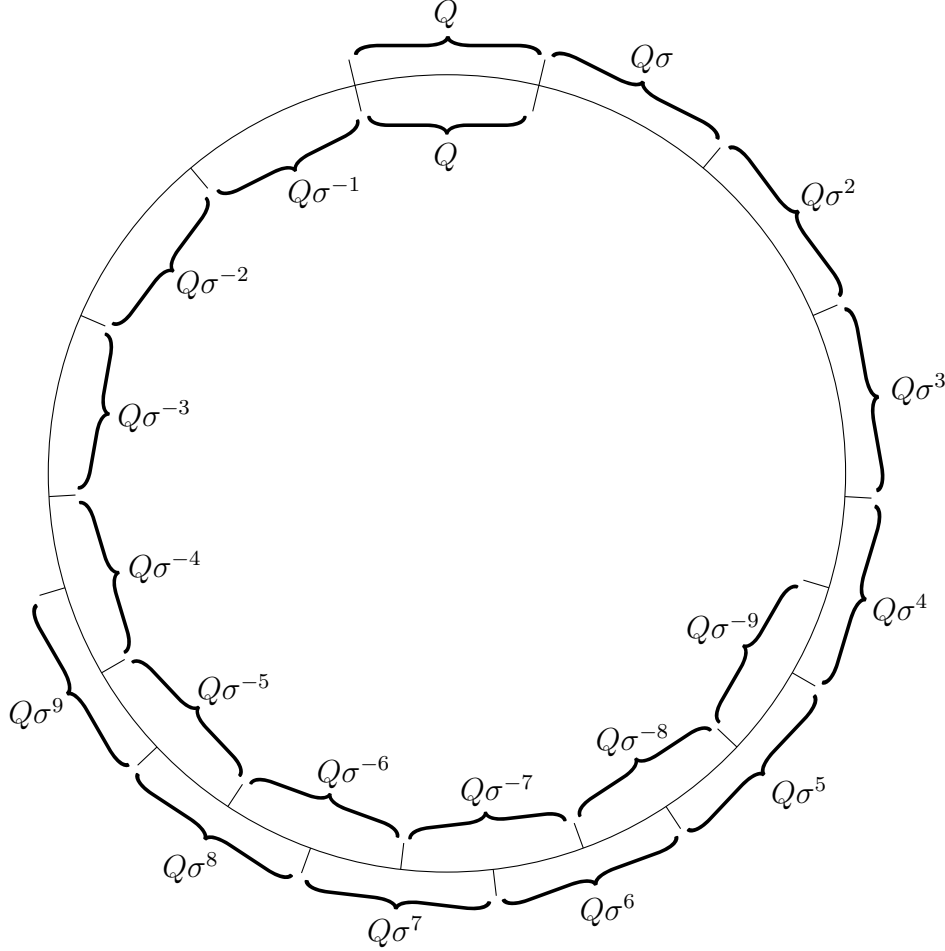
Furthermore $\text{supp}([\mu, \nu]) \subseteq \text{supp}(\mu) \cup \text{supp}(\nu) \subseteq A$. \square

Definition 3.4.3. For a natural number n , a homeomorphism σ of the Cantor set and a clopen subset Q of the Cantor set we will say *the n -step σ -orbit of Q nearly partitions \mathcal{C}* if and only if the $2n + 2$ element set of clopen sets

$$\{Q\sigma^i \mid -n \leq i \leq n\} \cup \{Q\sigma^{-(n+1)} \cap Q\sigma^{n+1}\}$$

partition \mathcal{C} .

The diagram below illustrates Definition 3.4.3 in the case when $n = 6$. The circle represents the Cantor set.



Note that substituting $n = 6$ into the set of clopen sets in Definition 3.4.3 renders

$$\{Q\sigma^i \mid -6 \leq i \leq 6\} \cup \{Q\sigma^{-7} \cap Q\sigma^7\}$$

which holds in the diagram above by inspection.

The above diagram is likely to be useful for the rest of this chapter, particularly in Lemma 3.4.5, Lemma 3.4.7, Lemma 3.4.8 and Lemma 3.4.10.

Lemma 3.4.4. *Let S be a set and let P and Q be disjoint non-empty subsets of S and let ρ be a permutation of S mapping $P \cup Q$ into P . Then for distinct integers i and j the sets $Q\rho^i$ and $Q\rho^j$ are disjoint.*

Proof. Assume $i < j$. Note $Q\rho^i$ and $Q\rho^j$ are disjoint if and only if Q and $Q\rho^{j-i}$ are disjoint. To finish observe $Q\rho^{j-i}$ is a subset of P which is disjoint from Q as desired. \square

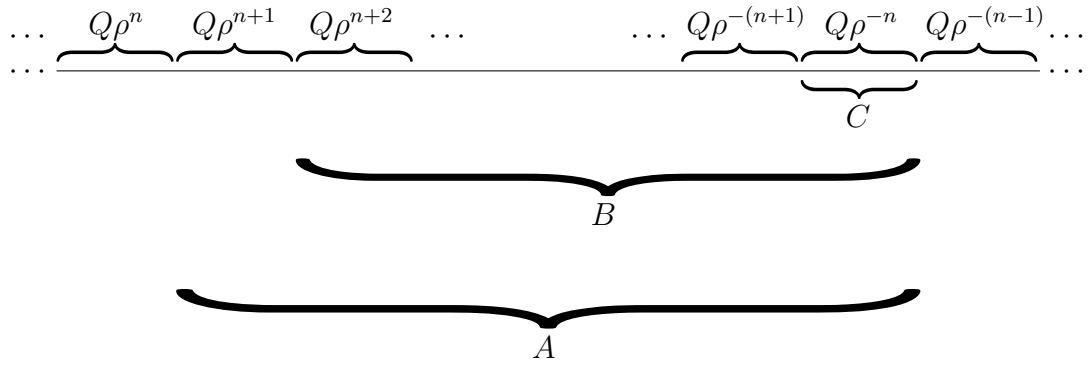
Lemma 3.4.5. *Let $G \leq H_{\mathcal{C}}$ be vigorous, let n be a natural number and let Q be a non-empty proper clopen subset of \mathcal{C} . There exists σ in G such that the n -step σ -orbit of Q nearly partitions \mathcal{C} (as in Definition 3.4.3).*

Proof. Choose a non-empty proper clopen subset P of $\mathcal{C} \setminus Q$. Since G is vigorous we may let $\rho \in G$ map $P \cup Q$ into P . By Lemma 3.4.4 for distinct integers i and j the sets $Q\rho^i$ and $Q\rho^j$ do not intersect. Again since G is vigorous we may let

$\pi \in G$ stabilise $\bigcup_{i=-(n-1)}^n Q\rho^i$ pointwise and map $\mathcal{C} \setminus \bigcup_{i=-(n-1)}^{n+1} Q\rho^i$ into $Q\rho^{-n}$.

Set $\sigma := \rho\pi$. Let $A := \mathcal{C} \setminus \bigcup_{i=-(n-1)}^n Q\rho^i$ and $B := \mathcal{C} \setminus \bigcup_{i=-(n-1)}^{n+1} Q\rho^i$ and $C := Q\rho^{-n}$.

Note $\text{supp}(\pi) \subseteq A$ and $B\pi \subseteq C$. The diagram below shows the relationship between ρ , A , B , C and n .



Note $Q\sigma^i = Q\rho^i$ for $-n \leq i \leq n$ and so $\bigcup_{i=-n}^n Q\rho^i = \bigcup_{i=-n}^n Q\sigma^i$. Also note

$$\mathcal{C} \setminus \bigcup_{i=-n}^n Q\rho^i \subseteq Q\rho^{n+1}\pi \subseteq \mathcal{C} \setminus \bigcup_{i=-(n-1)}^n Q\rho^i$$

and

$$\mathcal{C} \setminus \bigcup_{i=-n}^n Q\rho^i \subseteq Q\rho^{-n}\pi^{-1}\rho^{-1} \subseteq \mathcal{C} \setminus \bigcup_{i=-n}^{n-1} Q\rho^i.$$

But $Q\sigma^{n+1} = Q\rho^{n+1}\pi$ and $Q\sigma^{-(n+1)} = Q\rho^{-n}\pi^{-1}\rho^{-1}$ so $Q\sigma^{-(n+1)} \cap Q\sigma^{n+1}$ is contained in the intersection of the right hand sides of the above containments which is equal to $\mathcal{C} \setminus \bigcup_{i=-n}^n Q\rho^i$. Similarly $Q\sigma^{-(n+1)} \cap Q\sigma^{n+1}$ must contain the intersection of the left hand sides of the above containments which is also equal to $\mathcal{C} \setminus \bigcup_{i=-n}^n Q\rho^i$. Therefore $Q\sigma^{-(n+1)} \cap Q\sigma^{n+1}$ equals $\mathcal{C} \setminus \bigcup_{i=-n}^n Q\rho^i$. Recall $Q\sigma^i = Q\rho^i$ and $Q\sigma^j = Q\rho^j$ are disjoint for $-n \leq i < j \leq n$. Now

$$\{Q\sigma^i \mid -n \leq i \leq n\} \cup \{Q\sigma^{-(n+1)} \cap Q\sigma^{n+1}\}$$

partitions \mathcal{C} as desired. □

Lemma 3.4.6. *Let S be a set and let T be a subset of S and let g and h be in $\text{Sym}(S)$ stabilising T setwise with $\text{supp}(g) \cap \text{supp}(h) \subseteq T$. Then $\text{supp}([g, h])$ is*

contained in T .

Proof. The bijections g and h both stabilise T setwise so the restrictions of g and h to $S \setminus T$ are both permutations of $S \setminus T$. It follows the expression $[g|_{S \setminus T}, h|_{S \setminus T}]$ is a permutation of $S \setminus T$ and is in fact equal to $[g, h]|_{S \setminus T}$. By assumption $g|_{S \setminus T}$ and $h|_{S \setminus T}$ have disjoint supports and so commute therefore $[g|_{S \setminus T}, h|_{S \setminus T}] = [g, h]|_{S \setminus T}$ is the identity bijection on $S \setminus T$ as desired. \square

Lemma 3.4.7. *Let Q be a non-empty proper clopen subset of the Cantor set, let $n \geq 2$ be an integer and let σ be an element of H_C such that the n -step σ -orbit of Q nearly partitions C . Also let λ and ξ be in H_C with*

- $\text{supp}(\lambda) \subseteq Q\sigma^{-1} \cup Q \cup Q\sigma$,
- $(Q \cup Q\sigma)\lambda \subseteq Q\sigma$,
- $\xi|_{Q\sigma^{-1} \cup Q \cup Q\sigma} = \lambda|_{Q\sigma^{-1} \cup Q \cup Q\sigma}$,
- if i is odd and $2 \leq |i| < n$ then ξ stabilises $Q\sigma^i$ setwise,
- if i is even and $2 \leq |i| < n$ then ξ stabilises $Q\sigma^i$ pointwise, and
- ξ stabilises $Q\sigma^{-(n+1)} \cup Q\sigma^{-n} \cup Q\sigma^n \cup Q\sigma^{(n+1)}$ pointwise.

It is consequence of these assumptions that

1. $\text{supp}([\lambda^{-1}, (\lambda^{-1})^\sigma]) \subseteq Q\sigma^{-1} \cup Q \cup Q\sigma$,

2. $(Q \cup Q\sigma) [\lambda^{-1}, (\lambda^{-1})^\sigma] \subseteq Q\sigma$, and

3. $[\xi^{-1}, (\xi^{-1})^\sigma] = [\lambda^{-1}, (\lambda^{-1})^\sigma]$.

Proof. 1. From $(Q \cup Q\sigma)\lambda \subseteq Q\sigma$ by applying λ^{-1} it follows

$$Q \cup Q\sigma \subseteq Q\sigma\lambda^{-1}.$$

Taking the compliment of both sides of the above containment in the union $Q\sigma^{-2} \cup Q\sigma^{-1} \cup Q \cup Q\sigma$ we find

$$(Q\sigma^{-2} \cup Q\sigma^{-1} \cup Q \cup Q\sigma) \setminus (Q \cup Q\sigma) \supseteq (Q\sigma^{-2} \cup Q\sigma^{-1} \cup Q \cup Q\sigma) \setminus Q\sigma\lambda^{-1}.$$

We use this containment to get from the first line to the second line of the next string of manipulations. Since $\text{supp}(\lambda) = Q\sigma^{-1} \cup Q \cup Q\sigma$ we may deduce that $Q\sigma^{-2} \cup Q\sigma^{-1} \cup Q \cup Q\sigma = (Q\sigma^{-2} \cup Q\sigma^{-1} \cup Q \cup Q\sigma)\lambda^{-1}$, which we now use to get from the second line to the third line

$$\begin{aligned} Q\sigma^{-2} \cup Q\sigma^{-1} &= (Q\sigma^{-2} \cup Q\sigma^{-1} \cup Q \cup Q\sigma) \setminus (Q \cup Q\sigma) \\ &\supseteq (Q\sigma^{-2} \cup Q\sigma^{-1} \cup Q \cup Q\sigma) \setminus Q\sigma\lambda^{-1} \\ &\supseteq (Q\sigma^{-2} \cup Q\sigma^{-1} \cup Q \cup Q\sigma)\lambda^{-1} \setminus Q\sigma\lambda^{-1} \\ &= ((Q\sigma^{-2} \cup Q\sigma^{-1} \cup Q \cup Q\sigma) \setminus Q\sigma)\lambda^{-1} \\ &= (Q\sigma^{-2} \cup Q\sigma^{-1} \cup Q)\lambda^{-1}. \end{aligned}$$

Applying σ to both sides of the containment and using the observation

$\lambda^{-1}\sigma = \sigma(\lambda^{-1})^\sigma$ renders

$$Q\sigma^{-1} \cup Q \supseteq (Q\sigma^{-1} \cup Q \cup Q\sigma)(\lambda^{-1})^\sigma.$$

We use this when establishing the penultimate line below. We finish this part of the proof with this string of manipulations

$$\begin{aligned} \text{supp}([\lambda^{-1}, (\lambda^{-1})^\sigma]) &= \text{supp}\left(\lambda\left((\lambda^{-1})^{((\lambda^{-1})^\sigma)}\right)\right) \\ &\subseteq \text{supp}(\lambda) \cup \text{supp}\left((\lambda^{-1})^{((\lambda^{-1})^\sigma)}\right) \\ &\subseteq (Q\sigma^{-1} \cup Q \cup Q\sigma) \cup \text{supp}(\lambda^{-1})^{((\lambda^{-1})^\sigma)} \\ &\subseteq (Q\sigma^{-1} \cup Q \cup Q\sigma) \cup (Q\sigma^{-1} \cup Q \cup Q\sigma)^{((\lambda^{-1})^\sigma)} \\ &\subseteq (Q\sigma^{-1} \cup Q \cup Q\sigma) \cup (Q\sigma^{-1} \cup Q) \\ &= Q\sigma^{-1} \cup Q \cup Q\sigma. \end{aligned}$$

2. Starting with

$$(Q \cup Q\sigma)\lambda \subseteq Q\sigma$$

using the equality of $(Q\sigma^{-1} \cup Q \cup Q\sigma)\lambda$ and $Q\sigma^{-1} \cup Q \cup Q\sigma$ it follows that

$$(Q\sigma^{-1} \cup Q \cup Q\sigma)\lambda \setminus (Q \cup Q\sigma)\lambda \supseteq (Q\sigma^{-1} \cup Q \cup Q\sigma) \setminus Q\sigma.$$

Simplifying

$$((Q\sigma^{-1} \cup Q \cup Q\sigma) \setminus (Q \cup Q\sigma))\lambda \supseteq Q\sigma^{-1} \cup Q.$$

Simplifying again, switching the left hand side with the right hand side and applying λ^{-1} to both sides we find

$$(Q\sigma^{-1} \cup Q)\lambda^{-1} \subseteq Q\sigma^{-1}.$$

Since λ^{-1} stabilises $Q\sigma^{-2}$ pointwise it follows that

$$(Q\sigma^{-2} \cup Q\sigma^{-1} \cup Q)\lambda^{-1} \subseteq Q\sigma^{-2} \cup Q\sigma^{-1}.$$

Applying σ to both sides of the containment and using the observation $\lambda^{-1}\sigma = \sigma(\lambda^{-1})^\sigma$ we find

$$(Q\sigma^{-1} \cup Q \cup Q\sigma) (\lambda^{-1})^\sigma \subseteq Q\sigma^{-1} \cup Q.$$

We will use this in the next paragraph.

Since $(Q\sigma^{-1} \cup Q \cup Q\sigma) (\lambda^{-1})^\sigma \subseteq Q\sigma^{-1} \cup Q$ and $\text{supp}(\lambda^{-1}) \subseteq Q\sigma^{-1} \cup Q \cup Q\sigma$ the set $Q\sigma$ must be stabilised by $(\lambda^{-1})^{((\lambda^{-1})^\sigma)}$ pointwise. Since λ maps $(Q \cup Q\sigma)$ into $Q\sigma$ and $(\lambda^{-1})^{((\lambda^{-1})^\sigma)}$ stabilises $Q\sigma$ pointwise the commutator $[\lambda^{-1}, (\lambda^{-1})^\sigma] = \lambda(\lambda^{-1})^{((\lambda^{-1})^\sigma)}$ must also map $(Q \cup Q\sigma)$ into $Q\sigma$ as desired.

3. Note

- λ^{-1} , ξ^{-1} , $(\lambda^{-1})^\sigma$ and $(\xi^{-1})^\sigma$ all stabilise $Q\sigma^{-1} \cup Q \cup Q\sigma \cup Q\sigma^2$ setwise,
- λ^{-1} and ξ^{-1} agree on $Q\sigma^{-1} \cup Q \cup Q\sigma \cup Q\sigma^2$, and
- $(\lambda^{-1})^\sigma$ and $(\xi^{-1})^\sigma$ agree on $Q\sigma^{-1} \cup Q \cup Q\sigma \cup Q\sigma^2$

so the homeomorphisms $[\lambda^{-1}, (\lambda^{-1})^\sigma]$ and $[\xi^{-1}, (\xi^{-1})^\sigma]$ must agree on $Q\sigma^{-1} \cup Q \cup Q\sigma \cup Q\sigma^2$.

Since we showed

$$\begin{aligned} \text{supp}([\lambda^{-1}, (\lambda^{-1})^\sigma]) &\subseteq Q\sigma^{-1} \cup Q \cup Q\sigma \\ &\subseteq Q\sigma^{-1} \cup Q \cup Q\sigma \cup Q\sigma^2 \end{aligned}$$

in the first part of the proof of this lemma, it remains only to show that

$$\text{supp}([\xi^{-1}, (\xi^{-1})^\sigma]) \subseteq Q\sigma^{-1} \cup Q \cup Q\sigma \cup Q\sigma^2.$$

To show this we will use Lemma 3.4.6 with $S = \mathcal{C}$ and $T = Q\sigma^{-1} \cup Q \cup Q\sigma \cup Q\sigma^2$ and $g = \xi^{-1}$ and $h = (\xi^{-1})^\sigma$. We will show the hypotheses of Lemma 3.4.6 are satisfied.

Outside of $Q\sigma^{-1} \cup Q \cup Q\sigma \cup Q\sigma^2$ the support of ξ^{-1} is contained in the union of the clopen sets of the form $Q\sigma^i$ with $2 \leq |i| < n$ and i odd. Whereas outside of $Q\sigma^{-1} \cup Q \cup Q\sigma \cup Q\sigma^2$ the support of $(\xi^{-1})^\sigma$ is contained in the union of the clopen sets of the form $Q\sigma^i$ with $2 \leq |i| \leq n$ and i even. But these two sets are disjoint by Definition 3.4.3. So in accordance with the hypothesis of Lemma 3.4.6 the intersection of the supports of ξ^{-1} and $(\xi^{-1})^\sigma$ is a subset of $T = Q\sigma^{-1} \cup Q \cup Q\sigma \cup Q\sigma^2$.

By Lemma 3.4.6 the support of $[\xi^{-1}, (\xi^{-1})^\sigma]$ is contained in $Q\sigma^{-1} \cup Q \cup Q\sigma \cup Q\sigma^2$ as desired. \square

In Lemma 3.4.8 we will use a homeomorphism η which has the same properties as the homeomorphism λ from Lemma 3.4.7. The reason we do not use the same symbol for these two homeomorphism is that when we pull the lemmas of this section together in Theorem 3.1.2 the map η will be more akin to the map $[\lambda^{-1}, (\lambda^{-1})^\sigma]$ from the conclusion of Lemma 3.4.7 which also has the same properties as λ .

Lemma 3.4.8. *Assume Q is a non-empty proper clopen subset of the Cantor set, $n \geq 2$ is an integer and let σ in $H_{\mathcal{C}}$ be such that the n -step σ -orbit of Q nearly partitions \mathcal{C} . Also assume η is in $H_{\mathcal{C}}$ with $\text{supp}(\eta) \subseteq Q\sigma^{-1} \cup Q \cup Q\sigma$ and $(Q \cup Q\sigma)\eta \subseteq Q\sigma$. Then there exists α in the group generated by σ and η , stabilising Q pointwise with $(\mathcal{C} \setminus (Q \cup Q\sigma))\alpha \subsetneq Q\sigma^{-1}$.*

Proof. For each $i \in \mathbb{Z}$ applying σ^i to both sides of the containment $(Q \cup Q\sigma)\eta \subseteq Q\sigma$ renders the containment $(Q\sigma^i \cup Q\sigma^{i+1})\eta^{(\sigma^i)} \subseteq Q\sigma^{i+1}$. For each $i \in \mathbb{Z}$ it follows from the conclusion of last sentence and $\text{supp}(\eta^{\sigma^i}) \subseteq Q\sigma^{i-1} \cup Q\sigma^i \cup Q\sigma^{i+1}$ that if D is a subset of \mathcal{C} containing $Q\sigma^i$ and $Q\sigma^{i+1}$ and disjoint from $Q\sigma^{i-1}$ then $D\eta^{\sigma^i} \subseteq D \setminus Q\sigma^i$.

By repetitive application of the conclusion of the last sentence of the previous paragraph we achieve

$$\begin{aligned} & \mathcal{C} \setminus (Q \cup Q\sigma)\eta^{(\sigma^2)}\eta^{(\sigma^3)}\eta^{(\sigma^4)} \dots \eta^{(\sigma^{n+1})} \\ & \subseteq \mathcal{C} \setminus (Q \cup Q\sigma \cup Q\sigma^2)\eta^{(\sigma^3)}\eta^{(\sigma^4)} \dots \eta^{(\sigma^{n+1})} \end{aligned}$$

$$\begin{aligned}
&\subseteq \mathcal{C} \setminus (Q \cup Q\sigma \cup Q\sigma^2 \cup Q\sigma^3) \eta^{(\sigma^4)} \dots \eta^{(\sigma^{n+1})} \\
&\quad \vdots \qquad \qquad \qquad \vdots \\
&\subseteq \mathcal{C} \setminus (Q \cup Q\sigma \cup Q\sigma^2 \cup Q\sigma^3 \cup \dots \cup Q\sigma^{n+1}),
\end{aligned}$$

which is the first of three containments which we will bring together at the end of the proof. Similarly

$$\begin{aligned}
&(Q\sigma^{-1} \cup \dots \cup Q\sigma^{-n} \cup Q\sigma^{-(n+1)}) \eta^{(\sigma^{-(n+1)})} \eta^{(\sigma^{-n})} \dots \eta^{(\sigma^{-2})} \\
&\subseteq (Q\sigma^{-1} \cup \dots \cup Q\sigma^{-n}) \eta^{(\sigma^{-n})} \dots \eta^{(\sigma^{-2})} \\
&\quad \vdots \qquad \qquad \qquad \vdots \\
&\subseteq (Q\sigma^{-1} \cup Q\sigma^{-2}) \eta^{(\sigma^{-2})} \\
&\subseteq Q\sigma^{-1},
\end{aligned}$$

which is the second of the three containments which we will bring together at the end of the proof.

Let

$$X := (Q \cup Q\sigma \cup Q\sigma^2 \cup Q\sigma^3 \cup \dots \cup Q\sigma^{n+1})$$

and

$$Y := (Q\sigma^{-1} \cup \dots \cup Q\sigma^{-n} \cup Q\sigma^{-(n+1)}).$$

From Definition 3.4.3 the set $X \cup Y$ is the whole Cantor set. Our third containment will be $\mathcal{C} \setminus X \subsetneq Y$. To show this containment is proper we will show $Q\sigma^{-(n+1)} \cup Q\sigma^{(n+1)} \subseteq X \cap Y$.

In this paragraph we will show $Q\sigma^{-(n+1)} \subseteq X \cap Y$. Since $Q\sigma^{-(n+1)} \subseteq Y$ it is sufficient to show $Q\sigma^{-(n+1)} \subseteq X$. The set $Q\sigma^{-(n+1)}$ cannot intersect $Q\sigma^j$ for any $-n \leq j < n$ because by right multiplying by σ the set $Q\sigma^{-n}$ would then intersect $Q\sigma^{j+1}$ which contradicts Definition 3.4.3. By Definition 3.4.3 the set $Q\sigma^{-(n+1)}$ must be a subset of $Q\sigma^n \cup (Q\sigma^{-(n+1)} \cap Q\sigma^{(n+1)})$ since all of the other components of the partition have been excluded. Simplifying we find $Q\sigma^{-(n+1)} \subseteq Q\sigma^n \cup Q\sigma^{(n+1)} \subseteq X$.

We may conclude from the previous paragraph that $Q\sigma^{-(n+1)} \subseteq X \cap Y$. A symmetrical argument shows $Q\sigma^{n+1} \subseteq X \cap Y$ so $Q\sigma^{-(n+1)} \cup Q\sigma^{(n+1)} \subseteq X \cap Y$. This is enough to demonstrate that $X \cap Y$ is non-empty which is enough for the broader proof. For completeness we note that $X \cap Y = Q\sigma^{-(n+1)} \cup Q\sigma^{(n+1)}$ since $X \setminus Q\sigma^{n+1}$ is disjoint from $Y \setminus Q\sigma^{-(n+1)}$ by Definition 3.4.3.

Now fix

$$\alpha := \left(\eta^{(\sigma^2)} \eta^{(\sigma^3)} \dots \eta^{(\sigma^n)} \eta^{(\sigma^{n+1})} \right) \left(\eta^{(\sigma^{-(n+1)})} \eta^{(\sigma^{-n})} \dots \eta^{(\sigma^{-3})} \eta^{(\sigma^{-2})} \right)$$

and note that the conjugates of η used all stabilise Q pointwise so α also stabilises Q pointwise as desired.

Combining our three containments (first then third then second) we note that α maps $(\mathcal{C} \setminus (Q \cup Q\sigma))$ to a proper subset of $Q\sigma^{-1}$ is as desired. \square

Lemma 3.4.9. *Let Q be a non-empty proper clopen subset of the Cantor set and let $n \geq 2$. If σ is in $H_{\mathcal{C}}$ such that the n -step σ -orbit of Q nearly partitions \mathcal{C} and if*

η is in $H_{\mathcal{C}}$ with $\text{supp}(\eta) \subseteq Q\sigma^{-1} \cup Q \cup Q\sigma$ and $(Q \cup Q\sigma)\eta \subseteq Q\sigma$ then there exists φ in the group generated by σ and η with $(\mathcal{C} \setminus Q)\varphi \subsetneq Q$.

Proof. By Lemma 3.4.8 we may fix α in the group generated by σ and η such that α stabilises Q pointwise and $(\mathcal{C} \setminus (Q \cup Q\sigma))\alpha \subsetneq Q\sigma^{-1}$.

It follows from $(Q \cup Q\sigma)\eta \subseteq Q\sigma$ that

$$(Q\sigma^{-1} \cup Q \cup Q\sigma) \setminus ((Q \cup Q\sigma)\eta) \supseteq (Q\sigma^{-1} \cup Q \cup Q\sigma) \setminus (Q\sigma) = Q\sigma^{-1} \cup Q.$$

Since η stabilises $Q\sigma^{-1} \cup Q \cup Q\sigma$ setwise we find

$$(Q\sigma^{-1})\eta = (Q\sigma^{-1} \cup Q \cup Q\sigma)\eta \setminus ((Q \cup Q\sigma)\eta) \supseteq Q\sigma^{-1} \cup Q.$$

Applying σ renders

$$Q\eta^\sigma \supseteq Q \cup Q\sigma,$$

which we will apply between the fourth and fifth lines below.

Set

$$\varphi := \eta^\sigma \alpha \sigma.$$

To conclude the proof note

$$\begin{aligned} & (\mathcal{C} \setminus Q)\varphi \\ &= (\mathcal{C} \setminus Q)\eta^\sigma \alpha \sigma \\ &= ((\mathcal{C}\eta^\sigma) \setminus (Q\eta^\sigma))\alpha \sigma \\ &= (\mathcal{C} \setminus (Q\eta^\sigma))\alpha \sigma \end{aligned}$$

$$\begin{aligned}
&\subseteq (\mathcal{C} \setminus (Q \cup Q\sigma))\alpha\sigma \\
&\subsetneq (Q\sigma^{-1})\sigma \\
&= Q
\end{aligned}$$

as desired. □

Lemma 3.4.10. *Let Q be a non-empty proper clopen subset of the Cantor set, let m be a natural number and let σ be in $H_{\mathcal{C}}$ such that the $(2m + 2)$ -step σ -orbit of Q nearly partitions \mathcal{C} . Also let ξ be in $H_{\mathcal{C}}$ and let $\mu_1, \dots, \mu_m, \nu_1, \dots, \nu_m$ be in $\text{pstab}_{H_{\mathcal{C}}}(\mathcal{C} \setminus Q)$.*

Assume the product

$$\left(\mu_m^{\sigma^{-(2m+1)}} \cdots \mu_i^{\sigma^{-(2i+1)}} \cdots \mu_1^{\sigma^{-3}} \right) \left(\nu_1^{\sigma^3} \cdots \nu_i^{\sigma^{2i+1}} \cdots \nu_m^{\sigma^{2m+1}} \right)$$

agrees with ξ on the set $\mathcal{C} \setminus (Q\sigma^{-1} \cup Q \cup Q\sigma)$. Also assume $(Q \cup Q\sigma)\xi$ is a subset of $Q\sigma$.

Then for each $1 \leq i \leq m$ the commutator $[\mu_i, \nu_i]$ is in the group generated by σ and ξ .

Proof. Let $\eta := [\xi^{-1}, (\xi^{-1})^\sigma]$. From Lemma 3.4.7 we note $(Q \cup Q\sigma)\eta \subseteq Q\sigma$ and $\text{supp}(\eta) \subseteq Q\sigma^{-1} \cup Q \cup Q\sigma$.

Since η is in the group generated by σ and ξ we may use Lemma 3.4.8 to fix α in the group generated by σ and ξ stabilising Q pointwise with $(\mathcal{C} \setminus (Q \cup Q\sigma))\alpha \subsetneq$

$Q\sigma^{-1}$. Note that $(Q \cup Q\sigma)\alpha \supsetneq \mathcal{C} \setminus Q\sigma^{-1}$. Since α acts trivially on Q it follows that $Q\sigma\alpha \supseteq \mathcal{C} \setminus (Q\sigma^{-1} \cup Q)$. Let $1 \leq i \leq m$ be given. We will show

$$[\mu_i, \nu_i] = [\xi^{(\sigma^{2i+1}\alpha)}, \xi^{(\sigma^{-(2i+1)})}].$$

Let $\delta := \left(\mu_m^{\sigma^{-(2m+1)}} \cdots \mu_i^{\sigma^{-(2i+1)}} \cdots \mu_1^{\sigma^{-3}} \right) \left(\nu_1^{\sigma^3} \cdots \nu_i^{\sigma^{2i+1}} \cdots \nu_m^{\sigma^{2m+1}} \right)$. Note that the homeomorphisms in the product that makes δ have pairwise disjoint supports.

Since $\mu_i^{\sigma^{-(2i+1)}}$ and δ agree on $Q\sigma^{-(2i+1)}$ and δ and ξ agree on $\mathcal{C} \setminus (Q\sigma^{-1} \cup Q \cup Q\sigma) \supseteq Q\sigma^{-(2i+1)}$ it follows that ξ agrees with $\mu_i^{\sigma^{-(2i+1)}}$ on $Q\sigma^{-(2i+1)}$. From this it follows that $\xi^{\sigma^{2i+1}}$ agrees with μ_i on Q . For symmetric reasons $\xi^{\sigma^{-(2i+1)}}$ agrees with ν_i on Q . Since α stabilises Q pointwise it follows that $\xi^{(\sigma^{2i+1}\alpha)}$ also agrees with μ_i on Q . Since both parts of the commutator stabilise Q setwise we have achieved

$$[\mu_i, \nu_i]|_Q = [\xi^{(\sigma^{2i+1}\alpha)}, \xi^{(\sigma^{-(2i+1)})}]|_Q.$$

Since $\xi^{(\sigma^{2i+1})}$ acts trivially on the sets $Q\sigma$ and $Q\sigma\alpha \supseteq \mathcal{C} \setminus (Q\sigma^{-1} \cup Q)$ the homeomorphism $\xi^{(\sigma^{2i+1}\alpha)}$ acts trivially on the set $\mathcal{C} \setminus (Q\sigma^{-1} \cup Q)$. Since $\xi^{(\sigma^{-(2i+1)})}$ acts trivially on the set $Q\sigma^{-1}$ and both $\xi^{(\sigma^{-(2i+1)})}$ and $\xi^{(\sigma^{2i+1}\alpha)}$ stabilise Q setwise the support of $[\xi^{(\sigma^{2i+1}\alpha)}, \xi^{(\sigma^{-(2i+1)})}]$ must be contained in Q . To complete the proof note that the support of $[\mu_i, \nu_i]$ is also contained in Q . \square

We are now ready to prove Theorem 3.1.2 which we restate.

Theorem 3.1.2. *If G is a simple vigorous subgroup of $H_{\mathcal{C}}$ and E is a finitely generated subgroup of G then there exists a 2-generated group F with $E \leq F \leq G$.*

Proof. By Lemma 3.3.8 the group G must be flawless. Since G is flawless the set

$$\{[\psi, \omega] \mid \psi, \omega \in \text{pstab}_G(A) \text{ for some non-empty clopen set } A\}$$

generates G . By Lemma 3.4.1 we may fix a natural number m and A_1, \dots, A_m non-empty clopen subsets of \mathcal{C} and $\psi_1, \dots, \psi_m, \omega_1, \dots, \omega_m$, elements of $H_{\mathcal{C}}$ with ψ_i and ω_i stabilising A_i pointwise for each $1 \leq i \leq m$ and

$$\langle [\psi_1, \omega_1], \dots, [\psi_m, \omega_m] \rangle \geq E.$$

Let Q be a proper clopen subset of \mathcal{C} with $Q \cap A_i$ non-empty for each $1 \leq i \leq m$. By Lemma 3.4.5 we may fix $\sigma \in G$ with the $(4m + 2)$ -step σ -orbit of Q nearly partitions \mathcal{C} . The homeomorphism σ will be one of our two generators.

Let λ in G with $\text{supp}(\lambda) \subseteq Q\sigma^{-1} \cup Q \cup Q\sigma$ be such that $(Q \cup Q\sigma)\lambda \subseteq Q\sigma$. Set $\eta := [\lambda^{-1}, (\lambda^{-1})\sigma]$. By Lemma 3.4.7 the homeomorphism η has support contained in $Q\sigma^{-1} \cup Q \cup Q\sigma$ and maps $Q \cup Q\sigma$ into $Q\sigma$.

By Lemma 3.4.9 there exists $\varphi \in \langle \sigma, \eta \rangle$ with $(\mathcal{C} \setminus Q)\varphi \subsetneq Q$. Let $B := (\mathcal{C} \setminus Q)\varphi$. By Lemma 3.4.2 we may for each $1 \leq i \leq m$ choose μ_i and ν_i in G with support contained in Q with $(B)[\mu_i, \nu_i] \subseteq A_i \cap Q$.

For $1 \leq i \leq m$ set

$$\mu_{m+i} := \psi_i([\mu_i, \nu_i]^{-1}\varphi^{-1})$$

and set

$$\nu_{m+i} := \omega_i([\mu_i, \nu_i]^{-1} \varphi^{-1}).$$

Note for each $1 \leq i \leq m$

$$\begin{aligned} \text{supp}(\mu_{m+i}) &= \text{supp}\left(\psi_i([\mu_i, \nu_i]^{-1} \varphi^{-1})\right) \\ &\subseteq (\mathcal{C} \setminus (A_i \cap Q)) [\mu_i, \nu_i]^{-1} \varphi^{-1} \\ &\subseteq (\mathcal{C} \setminus B) \varphi^{-1} \\ &= \mathcal{C} \setminus (\mathcal{C} \setminus Q) \\ &= Q \end{aligned}$$

and by a similar argument $\text{supp}(\nu_{m+i}) \subseteq Q$. For each $1 \leq i \leq 2m$ observe that

$$\text{supp}\left(\mu_i^{(\sigma^{-(2i+1)})}\right) \subseteq Q\sigma^{-2i+1} \text{ and similarly } \text{supp}\left(\nu_i^{(\sigma^{(2i+1)})}\right) \subseteq Q\sigma^{2i+1}.$$

Set

$$\xi := \lambda \prod_{i=1}^{2m} \mu_i^{(\sigma^{-(2i+1)})} \nu_i^{(\sigma^{(2i+1)})}$$

noting that the product is well defined because its terms commute since they have disjoint supports. The homeomorphism ξ will be the other of our two generators.

By checking the hypotheses of Lemma 3.4.7 we will show that η is equal to $[\xi^{-1}, (\xi^{-1})^\sigma]$ and is therefore in $\langle \sigma, \xi \rangle$. The variables used in Lemma 3.4.7 take the same names as in this proof with the exception of n which we will set to be $4m+2$.

Recall the hypotheses of Lemma 3.4.7 are

- $\text{supp}(\lambda) \subseteq Q\sigma^{-1} \cup Q \cup Q\sigma$,

- $(Q \cup Q\sigma)\lambda \subseteq Q\sigma$,
- $\xi|_{Q\sigma^{-1} \cup Q \cup Q\sigma} = \lambda|_{Q\sigma^{-1} \cup Q \cup Q\sigma}$,
- if i is odd and $2 \leq |i| < n$ then ξ stabilises $Q\sigma^i$ setwise,
- if i is even and $2 \leq |i| < n$ then ξ stabilises $Q\sigma^i$ pointwise, and
- ξ stabilises $Q\sigma^{-(n+1)} \cup Q\sigma^{-n} \cup Q\sigma^n \cup Q\sigma^{(n+1)}$ pointwise.

The first two hypotheses hold by the assumptions we made when we chose λ . The third assumption holds because the support of λ is contained in $Q\sigma^{-1} \cup Q \cup Q\sigma$ and the support of the rest of the product that makes up ξ is disjoint from $Q\sigma^{-1} \cup Q \cup Q\sigma$. The last three hypotheses follow from the fact that the terms other than λ in the product that defines ξ have supports contained in one of $Q\sigma^{-(2i+1)}$ or $Q\sigma^{2i+1}$ for some $1 \leq i \leq m$.

Since $\varphi \in \langle \sigma, \eta \rangle$ we may deduce $\varphi \in \langle \sigma, \xi \rangle$. By Lemma 3.4.10 the group $\langle \sigma, \xi \rangle$ contains $[\mu_i, \nu_i]$ for each $1 \leq i \leq 2m$.

To complete the proof for each $1 \leq i \leq m$ note

$$[\psi_i, \omega_i] = [\mu_{m+i}^{\varphi[\mu_i, \nu_i]}, \nu_{m+i}^{\varphi[\mu_i, \nu_i]}] = [\mu_{m+i}, \nu_{m+i}]^{\varphi[\mu_i, \nu_i]} \in \langle \sigma, \xi \rangle$$

and recall $E \leq \langle [\psi_1, \omega_1], \dots, [\psi_m, \omega_m] \rangle$ so $E \leq \langle \sigma, \xi \rangle$ as desired. \square

3.5 Conclusion

Question 3.5.1. Does there exist a finitely presented simple group that is not 2-generated?

Question 3.5.1 is well-known and has been partially answered by the Classification of Finite Simple Groups from which it follows that if there are finitely presented simple groups which are not 2-generated then they must be infinite. There are examples of simple groups which are finitely generated but not 2-generated as shown in [6] and there are certainly examples of finitely presented non-2-generated groups and examples of finitely presented simple groups so there are no obstructions to any proper subset of the demands of the question.

Epstein in [4] provides three axioms under which a group of homeomorphisms of a space will be simple (or at least, will have a simple commutator subgroup). One of these is that there is a countable basis of open sets upon which the group acts transitively, another subsumes the property that the group is generated by elements which are wholly supported in these basic open sets. These properties seem not far from the properties we have used in this chapter to find 2 element generating sets for finitely generated groups of homeomorphisms of the Cantor set.

For this reason, it may be an interesting project for the future to try to adapt the argument of the main theorem of this chapter to the context of Epstein's Axioms, and thus investigate the question that if G is a finitely presented simple

group of homeomorphisms which satisfies Epstein's Axioms, must G actually be two-generated?

Chapter 4

On a Construction of Groups from Inverse Semigroups

Lawson and Lenz introduce the tight completion in [14]. Tight completion is a process which, when applied to an inverse semigroup, produces a distributive inverse semigroup. There is a natural homomorphism from any inverse semigroup to its tight completion which identifies pairs of elements if they are in some sense similar according to certain finite compatible subsets beneath them called tight covers. In our context it will turn out that this homomorphism is always an injection. Consequently the tight completions of the inverse semigroups of partial permutations which we construct in this chapter will turn out to be isomorphic to their closures under compatible union.

Our own interest in this process stems from the fact that Thompson's group V_2 may be constructed as the group of units of the tight completion of the polycyclic monoid of rank 2 as shown in [11]. In this chapter we generalise this construction for V_2 to find an infinite family of groups of a Higman-Thompson flavour. We are especially interested in when these groups are finitely generated and when they are simple.

The group of units of the tight completion of any (finite non-empty) direct product of polycyclic monoids is of a Higman-Thompson flavour. These direct products may be embedded in the inverse monoid of partial permutations of the Cantor set in a natural way. For these embeddings, closure under compatible union corresponds to tight completion. Our construction also relies upon a family of inverse submonoids of the monoid of partial bijections of the Cantor set for which closure under compatible union corresponds to tight completion.

We construct the polycyclic monoid of rank 2 from the bicyclic monoid (represented as an inverse submonoid of $\mathcal{I}_{\mathbb{N}}^{\text{cof}}$). We generalise this construction to apply to any inverse submonoid of $\mathcal{I}_{\mathbb{N}}^{\text{cof}}$ to produce (for each inverse submonoid of $\mathcal{I}_{\mathbb{N}}^{\text{cof}}$) an inverse monoid ripe for application of the tight completion construction. This construction relies upon a generalised notion of word (which reduces to the normal notion of word in the case of the bicyclic monoid) which we discuss in Section 4.1. For a countably infinite set S this construction applies equally to inverse

submonoids of $\mathcal{I}_S^{\text{cof}}$. We often work in $\mathcal{I}_S^{\text{cof}}$ instead of working in $\mathcal{I}_{\mathbb{N}}^{\text{cof}}$.

The construction of the title is the one taking an inverse submonoid of $\mathcal{I}_S^{\text{cof}}$ to a group via first applying the construction mentioned in the previous paragraph then the tight completion construction and then passing to the group of units. We are particularly interested in when the resulting groups are finitely generated and when they are simple. We finish the chapter with some examples of topical interest in modern research.

4.1 Classical Words and Generalised Words

In this section we will first discuss the classical notion of word (which we have been using in sections 2.5, 2.7 and 2.8) and second discuss a generalised notion of word which will be used throughout this chapter. Though this use is only implicit, the notion of generalised word described in this section will ease and enhance the reader's understanding of Chapter 4.

One can think of a word as a list of letters, and one can think of a list of letters as a function. It is this functional perspective that we will use.

Let A be a finite alphabet of size at least 2.

If i is a non-negative integer, then we think of a *classical finite word* of length i as a partial function $w : \mathbb{N} \rightarrow A$ such that $\text{dom}(w) = \{1, \dots, i\}$ (recall for us that \mathbb{N} does not include 0). The length of w is denoted $|w|$. We think of a *classical*

infinite word as a (total) function $w : \mathbb{N} \rightarrow A$. So the set of classical infinite words over A is just the set $A^{\mathbb{N}}$.

The notion of classical words comes with the notions of prefix, prefix removal, concatenation and prefix substitution, which we will now describe.

Let u be a classical finite word and let w be a classical (finite or infinite) word. In list notation if $u = a_1, a_2, \dots, a_i$ and $w = a_1, a_2, \dots, a_i, \dots$ then u is a *prefix* of w . In the function notation the word u is a prefix of the word w if u is a subset of w (recall our perspective that a function $f : D \rightarrow R$ is a subset of $D \times R$ satisfying firstly that for all $d \in D$, there is $r \in R$ such that $(d, r) \in f$ and secondly, whenever $(d, r_1), (d, r_2) \in f$ then $r_1 = r_2$). We will use \bar{u} for the set of classical infinite words which contain u as a prefix. So

$$\bar{u} = \{r \in A^{\mathbb{N}} \mid \forall x \in \text{dom}(u) : xr = xu\}.$$

We may refer to the set \bar{u} as the cone at u , or simply as a cone, if the context is clear.

Removal of the prefix u is the map from \bar{u} to $A^{\mathbb{N}}$ sending $r \in \bar{u}$ to fr where f is the map sending $n \in \mathbb{N}$ to $n + |u|$. The *concatenation* of u and w is the classical infinite word

$$u \cup tw$$

where $t : \mathbb{N} \rightarrow \mathbb{N}$ is the partial function sending $n > |u|$ to $n - |u|$. Note that the above union is disjoint. We will think of concatenation by u as a map from $A^{\mathbb{N}}$ to

\bar{u} .

If v is a classical finite word then the *classical prefix substitution* replacing u by v is the map from \bar{u} to \bar{v} formed by composing the map removal of the prefix u with concatenation by v . So if w is in \bar{u} the prefix substitution of u by v applied to w is the classical infinite word

$$v \cup gw$$

where g is the map sending $n > |v|$ to $n + |u| - |v|$. Again note that this union is disjoint. Changing the lengths of u and v changes the partial bijection g . The partial bijections arising in this way are precisely those constituting the bicyclic monoid as defined in Definition 2.2.1.

We now move onto the notion of generalised words. Fix a countably infinite set S . For generalised words the set \mathbb{N} will be replaced by the set S . Correspondingly the bicyclic monoid will be replaced by the inverse monoid $\mathcal{I}_S^{\text{cof}}$ (which was defined in Chapter 1 to be the set of partial bijections from S to S with domain and image co-finite in S).

Let $w : S \rightarrow A$ be a partial function. If $\text{dom}(w)$ is finite then we will say w is a *generalised finite word* over A and under S . If $S \setminus \text{dom}(w)$ is finite then we will say w is a *generalised co-finite word* over A and under S . If $\text{dom}(w) = S$ then we will say w is a *generalised total word* over A and under S . We may omit the over A and under S if they are clear from context.

Generalised finite words generalise classical finite words, generalised total words generalise classical infinite words. Let u be a generalised finite word and let v be a generalised total word. As in the classical case u is a *prefix* of v if u is a subset of v . Also as in the classical case we use \bar{u} for the set of generalised total words that have u as a prefix, still referring to \bar{u} as the cone at u or simply as a cone.

If u and v are generalised finite words and f is in $\mathcal{I}_S^{\text{cof}}$ such that

$$\{\text{dom}(u), \text{dom}(f)\} \text{ and } \{\text{dom}(v), \text{ran}(f)\}$$

both partition S then the *generalised prefix substitution* from \bar{u} to \bar{v} by f is the bijection $g : \bar{u} \rightarrow \bar{v}$ such that for all $w \in \bar{u}$

$$wg = v \cup f^{-1}w.$$

We note that this union is disjoint. If either of the sets $\{\text{dom}(u), \text{dom}(f)\}$ and $\{\text{dom}(v), \text{ran}(f)\}$ do not partition S then we define the generalised prefix substitution from \bar{u} to \bar{v} by f to be the empty function. In this chapter the generalised prefix substitution from \bar{u} to \bar{v} by f is thought of as a partial permutation of A^S and is denoted $(\overline{u, f, v})$. If I is an inverse submonoid of $\mathcal{I}_S^{\text{cof}}$ then we will use $\mathcal{I}(S, A, I)$ for the set of generalised prefix substitutions from cones of A^S to cones of A^S by elements of I .

Example 4.1.1. If the input of the construction of this chapter is the triple $(\mathbb{N}, \{0, 1\}, B)$ then the group constructed is equal to Thompson's group V_2 represented as an action on $\{0, 1\}^{\mathbb{N}}$.

From Proposition 2.2.6 any element of $\mathcal{I}(\mathbb{N}, \{0, 1\}, B)$ may be represented as $\left(\overline{u, c^k b^l, v}\right)$ for some non-negative integers k and l . Let u and v be in $\mathcal{P}_{\mathbb{N}, \{0, 1\}}^f$ and let k and l be non-negative integers. The partial permutation $\left(\overline{u, c^k b^l, v}\right)$ is non-empty exactly if $\{\text{dom}(u), \text{dom}(c^k b^l)\}$ and $\{\text{dom}(v), \text{ran}(c^k b^l)\}$ both partition \mathbb{N} . It follows from Proposition 2.2.5 that $\left(\overline{u, c^k b^l, v}\right)$ is non-empty exactly if both $\{\text{dom}(u), \{k+1, k+2, \dots\}\}$ and $\{\text{dom}(v), \{l+1, l+2, \dots\}\}$ partition \mathbb{N} which is the case exactly when $\text{dom}(u) = \{1, 2, \dots, k\}$ and $\text{dom}(v) = \{1, 2, \dots, l\}$. Therefore $\left(\overline{u, c^k b^l, v}\right)$ is non-empty exactly if u and v are classical finite words and $|u| = k$ and $|v| = l$. Consequently any non-empty element of $\mathcal{I}(\mathbb{N}, \{0, 1\}, B)$ may be represented as $\left(\overline{r, c^{|r|} b^{|t|}, t}\right)$ for some classical finite words r and t .

The partial permutation $\left(\overline{u, c^{|u|} b^{|v|}, v}\right)$ has domain \bar{u} and range \bar{v} and moves elements of \bar{u} by removing the prefix u and replacing it with the prefix v and shunting the rest of the classical infinite word to accommodate (f governs this shunting). We may conclude the set $\mathcal{I}(\mathbb{N}, \{0, 1\}, B)$ is equal to P_2 the polycyclic monoid of rank 2 which was defined in Definition 2.5.1.

To form the group we are interested in in this example we take the group of units of the closure of

$$\left\{ \left(\overline{u, f, v}\right) \mid u, v \in \mathcal{P}_{\mathbb{N}, \{0, 1\}}^f, f \in B \right\} = P_2$$

(along with the empty function) under compatible union. The group of units of the closure of P_2 under compatible union, represented with this action on $\{0, 1\}^{\mathbb{N}}$,

was shown to be Thompson's group V_2 in [11] as desired.

4.2 The Main Construction

For this section fix a set S , a non-empty alphabet A , which ultimately will have cardinality at least 2, and an inverse submonoid I of $\mathcal{I}_S^{\text{cof}}$ (See Proposition 2.1.9). From this data, we will construct an inverse monoid $\mathcal{I}(S, A, I)$ of generalised prefix transformations. In the special case where $S = \mathbb{N}$, $I = B$, the bicyclic monoid, and A is a 2-letter alphabet, then $\mathcal{I}(\mathbb{N}, \{0, 1\}, B)$ will be precisely the polycyclic monoid on two generators. For this reason the reader may find regarding S as \mathbb{N} to be helpful.

While our construction is not the same as that in [12], [10], [11] and [14], it nonetheless relies heavily on the notion of tight completion in [14]. Thompson's group V has been generalised in other ways. For example in [16], Thompson's group V is considered as an automorphism group of a Cantor algebra.

We note that if $f \in I$ and $v \in A^{\text{ran}(f)}$ then $fv \in A^{\text{dom}(f)}$. The following definition will be illustrated in the next diagram.

Definition 4.2.1. For $f \in I$ and $a, b \in \mathcal{P}_{S,A}^f$ define $(\overline{a, f, b}) \in \mathcal{I}_{A^S}$ by

$$(\overline{a, f, b}) := \{(u, v) \in A^S \times A^S \mid u = a \sqcup fv \text{ and } v = b \sqcup f^{-1}u\}.$$

We emphasise firstly that \sqcup represents a restriction on what is in the set and

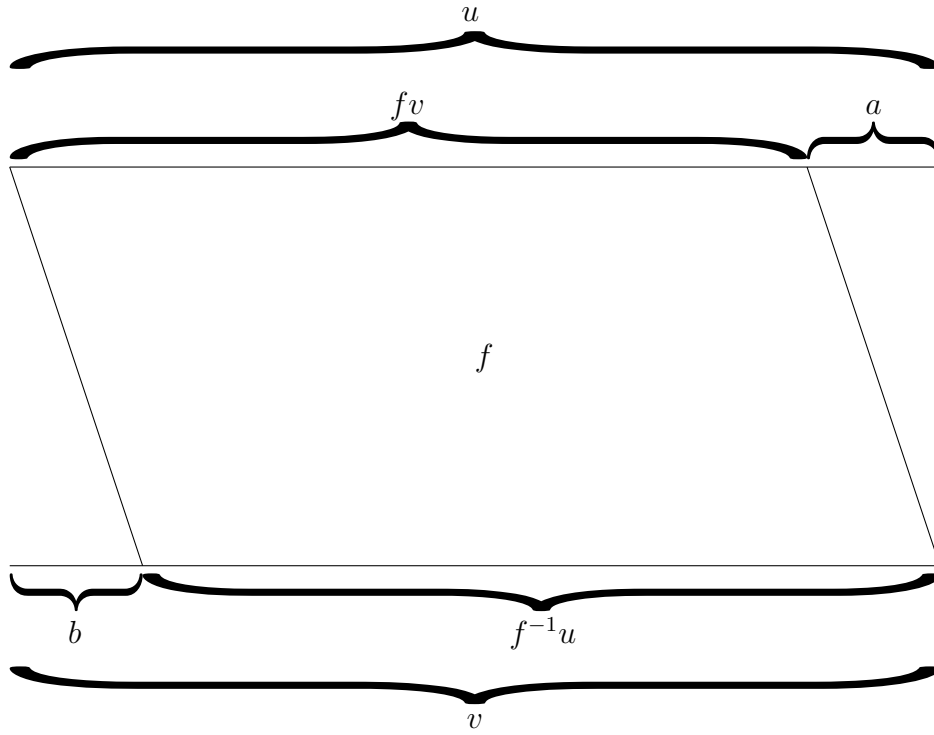
secondly that $(u, v) \in A^S \times A^S$ works with the first restriction guaranteeing that, for instance, $\text{dom}(a)$ and $\text{dom}(f)$ form a partition of S . We use the overline to distinguish between the triple (a, f, b) and the binary relation $(\overline{a, f, b})$ (which is a partial bijection on A^S with domain and range both cones). The claims in this paragraph and more will be shown in Proposition 4.2.2.

The next diagram indicates the relationships between a, b, u, v and f from the definition above (that is a, b are partial functions from S to A with finite domain, f is in I and u, v are in A^S with $u(\overline{a, f, b}) = v$).

The top horizontal line represents a copy of S labelled by A according to u . The part of the top horizontal line labelled by a represents the part of S where a is defined (u agrees with a). The part of the top horizontal line labelled by fv (that is f compose v) represents the part of S where fv is defined (u agrees with fv).

The area between the two diagonal lines represents f partially mapping the top copy of S to the bottom copy of S .

The bottom horizontal line represents a copy of S labelled by A according to v . The part of the bottom horizontal line labelled by b represents the part of S where b is defined (v agrees with b). The part of the bottom horizontal line labelled by $f^{-1}u$ (that is f^{-1} compose u) represents the part of S where $f^{-1}u$ is defined (v agrees with $f^{-1}u$).



Recall w in $\mathcal{P}_{\mathbb{N},A}^f$ is a non-empty classical finite word over A if $\text{dom}(w) = \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$. Also recall the generalised finite words over A and under S are just the elements of $\mathcal{P}_{S,A}^f$.

If a and b are non-empty classical finite words then $\text{dom}(a) \cap \text{dom}(b)$ contains at least one of $\text{dom}(a)$ and $\text{dom}(b)$ so this intersection is certainly non-empty. In contrast, if a and b are non-empty generalised finite words then $\text{dom}(a)$ and $\text{dom}(b)$ may or may not intersect. The diagram above is drawn with $\text{dom}(a)$ and $\text{dom}(b)$ disjoint to emphasise this difference.

Proposition 4.2.2. *For $f \in I$ and a, b partial functions from S to A with finite domains the set $(\overline{a, f, b}) \in A^S \times A^S$ is a partial bijection on A^S .*

Proof. Let $u_1, u_2, v_1, v_2 \in A^S$ be given. Assume $(u_1, v_1), (u_2, v_1) \in (\overline{a, f, b})$. Since $u_1 = a \sqcup fv_1 = u_2$ we have $u_1 = u_2$ as desired. Now assume $(u_1, v_1), (u_1, v_2) \in$

$(\overline{a, f, b})$. Since $v_1 = b \sqcup f^{-1}u_1 = v_2$ we have $v_1 = v_2$ as desired. \square

Proposition 4.2.3. For $f \in I$ and $a, b \in \mathcal{P}_{S,A}^f$ the following are equivalent

1. $\text{dom}(f) \sqcup \text{dom}(a) = S$ and $\text{ran}(f) \sqcup \text{dom}(b) = S$,
2. $\text{dom}((\overline{a, f, b})) = \{t \in A^S \mid t \supseteq a\}$,
3. $\text{ran}((\overline{a, f, b})) = \{t \in A^S \mid t \supseteq b\}$, and
4. $(\overline{a, f, b}) \neq \emptyset$.

Proof. $1 \Rightarrow 2$. First $\text{dom}((\overline{a, f, b}))$ is contained in $\{t \in A^S \mid t \supseteq a\}$ because

$$u(\overline{a, f, b}) = v \Rightarrow u = a \sqcup fv \Rightarrow u \supseteq a.$$

Now we show $\{t \in A^S \mid t \supseteq a\}$ is contained in $\text{dom}((\overline{a, f, b}))$. If $u \supseteq a$ then for $v := b \sqcup f^{-1}u$ it follows $u(\overline{a, f, b}) = v$ because $a \sqcup fv = a \sqcup f(b \sqcup f^{-1}u) = a \sqcup fb \sqcup ff^{-1}u = a \sqcup ff^{-1}u = u$.

$1 \Rightarrow 3$. This is similar to $1 \Rightarrow 2$.

$2 \Rightarrow 4$. This is immediate as is $3 \Rightarrow 4$.

$4 \Rightarrow 1$. Let $(u, v) \in (\overline{a, f, b})$ be given. Since $u = a \sqcup fv$ and $u, v \in A^S$ we have $\text{dom}(a) \sqcup \text{dom}(f) = S$. Similarly since $v = b \sqcup f^{-1}u$ and $u, v \in A^S$ we have $\text{dom}(b) \sqcup \text{dom}(f^{-1}) = S$. \square

In the above proof we write $u \supseteq a$. Since u is infinite and a is finite there is no way they could be equal so we could equally write $u \supsetneq a$.

Proposition 4.2.4. *Let a and b be in $\mathcal{P}_{S,A}^f$ and let f be in I . If $\text{dom}(a) \sqcup \text{dom}(f) = S$ and $\text{ran}(f) \sqcup \text{dom}(b) = S$ then for all $u \in A^S$ we have*

$$u(\overline{a, f, b}) = \begin{cases} b \sqcup f^{-1}u & : u \supseteq a \\ \text{undefined} & : u \not\supseteq a \end{cases}$$

otherwise $\overline{a, f, b}$ is the empty function.

Proof. Recall Definition 4.2.1 tells us

$$\overline{a, f, b} := \{(u, v) \in A^S \times A^S \mid u = a \sqcup fv \text{ and } v = b \sqcup f^{-1}u\}$$

If either $\text{dom}(a) \sqcup \text{dom}(f) = S$ or $\text{ran}(f) \sqcup \text{dom}(b) = S$ do not hold (because $\text{dom}(a)$ intersects $\text{dom}(f)$ for example) then the conditions of the set builder notation above cannot be satisfied for any u so the function is empty as desired.

If u contains a then $u(\overline{a, f, b}) = b \sqcup f^{-1}u$ satisfies the conditions of the set builder notation above. On the other hand if u does not contain a then there can be no $v \in A^S$ with $u = a \sqcup fv$ so u is not in the domain of $\overline{a, f, b}$. \square

Proposition 4.2.5. *The set*

$$\{\overline{a, f, b} \in \mathcal{I}_{A^S} \mid f \in I \text{ and } a, b \in \mathcal{P}_{S,A}^f\}$$

contains the zero of \mathcal{I}_{A^S} .

Proof. Let a be a non-empty function in $\mathcal{P}_{S,A}^f$. The domains of a and 1_I intersect, so by Proposition 4.2.4 the map $\overline{a, 1_I, \emptyset}$ is empty as desired \square

Proposition 4.2.6. *The map $(\overline{\emptyset, 1_I, \emptyset})$ is the identity map of H_{A^S} .*

Proof. Let $u \in A^S$ be given. Plugging in our values to the equation of Proposition 4.2.4 gives

$$u(\overline{\emptyset, 1_I, \emptyset}) = \begin{cases} \emptyset \sqcup 1_I u & : u \supseteq \emptyset \\ \text{undefined} & : u \not\supseteq \emptyset \end{cases}.$$

Since all functions contain the empty function $u(\overline{\emptyset, 1_I, \emptyset}) = \emptyset \sqcup 1_I u = u$ as desired. \square

Proposition 4.2.7. *For $a, b \in \mathcal{P}_{S,A}^f$ and $f \in I$ the inverse of $(\overline{a, f, b})$ is $(\overline{b, f^{-1}, a})$.*

Proof. Assume $\text{dom}(a) \sqcup \text{dom}(f) = S$ and $S = \text{ran}(f) \sqcup \text{dom}(b)$ since otherwise $(\overline{a, f, b}) = \emptyset = (\overline{b, f^{-1}, a})$. By Proposition 4.2.3 the images and domains are correct, so we only need check $u(\overline{a, f, b})(\overline{b, f^{-1}, a}) = u$ for $u \in \text{dom}(\overline{a, f, b}) = \{u \in A^S \mid u \supseteq a\}$. But then

$$\begin{aligned} & u(\overline{a, f, b})(\overline{b, f^{-1}, a}) \\ &= (b \sqcup f^{-1}u)(\overline{b, f^{-1}, a}) \\ &= a \sqcup f(b \sqcup f^{-1}u) \\ &= a \sqcup fb \sqcup ff^{-1}u \\ &= a \sqcup ff^{-1}u \\ &= u \end{aligned}$$

as desired. \square

Proposition 4.2.8. *Let a_1, a_2, b_1, b_2 be in $\mathcal{P}_{S,A}^f$ and let f_1, f_2 be in I . If $(\overline{a_1, f_1, b_1})$ and $(\overline{a_2, f_2, b_2})$ are non-empty and b_1 and a_2 agree then $(\overline{a_1, f_1, b_1}) (\overline{a_2, f_2, b_2})$ is equal to $(\overline{a_1 \sqcup f_1 a_2, f_1 f_2, f_2^{-1} b_1 \sqcup b_2})$. Otherwise $(\overline{a_1, f_1, b_1}) (\overline{a_2, f_2, b_2})$ is the empty function.*

Proof. By Proposition 4.2.3 the unions

- $\text{dom}(f_1) \sqcup \text{dom}(a_1)$,
- $\text{dom}(f_2) \sqcup \text{dom}(a_2)$,
- $\text{ran}(f_1) \sqcup \text{dom}(b_1)$, and
- $\text{ran}(f_2) \sqcup \text{dom}(b_2)$

are all disjoint and equal to S .

Note that for X a set and x in X and partial functions $f : X \rightarrow X$ and $g : X \rightarrow X$ the equality

$$xfg = \begin{cases} xfg & x \in \text{dom}(f) \text{ and } xf \in \text{dom}(g) \\ \text{undefined} & x \notin \text{dom}(f) \\ \text{undefined} & xf \notin \text{dom}(g) \end{cases}$$

holds. Below the first equality is an instance of the above equality, the second equality is three applications of Proposition 4.2.4, the third equality is another

application of Proposition 4.2.4 and the fourth equality is a rearrangement.

$$\begin{aligned}
u \overline{(a_1, f_1, b_1)} \overline{(a_2, f_2, b_2)} &= \begin{cases} u \overline{(a_1, f_1, b_1)} \overline{(a_2, f_2, b_2)} & u \supseteq a_1 \text{ and } u \overline{(a_1, f_1, b_1)} \supseteq a_2 \\ \text{undefined} & u \not\supseteq a_1 \\ \text{undefined} & u \overline{(a_1, f_1, b_1)} \not\supseteq a_2 \end{cases} \\
&= \begin{cases} b_2 \sqcup f_2^{-1} (u \overline{(a_1, f_1, b_1)}) & u \supseteq a_1 \text{ and } b_1 \sqcup f_1^{-1} u \supseteq a_2 \\ \text{undefined} & u \not\supseteq a_1 \\ \text{undefined} & b_1 \sqcup f_1^{-1} u \not\supseteq a_2 \end{cases} \\
&= \begin{cases} b_2 \sqcup f_2^{-1} (b_1 \sqcup f_1^{-1} u) & b_1 \sqcup f_1^{-1} u \supseteq a_2 \text{ and } u \supseteq a_1 \\ \text{undefined} & u \not\supseteq a_1 \\ \text{undefined} & b_1 \sqcup f_1^{-1} u \not\supseteq a_2 \end{cases} \\
&= \begin{cases} b_2 \sqcup f_2^{-1} b_1 \sqcup f_2^{-1} f_1^{-1} u & b_1 \sqcup f_1^{-1} u \supseteq a_2 \text{ and } u \supseteq a_1 \\ \text{undefined} & u \not\supseteq a_1 \\ \text{undefined} & b_1 \sqcup f_1^{-1} u \not\supseteq a_2 \end{cases}
\end{aligned}$$

First assume b_1 and a_2 disagree. Now for all $u \in A^S$ the function $b_1 \sqcup f_1^{-1} u$ cannot contain the function a_2 so the domain of $\overline{(a_1, f_1, b_1)} \overline{(a_2, f_2, b_2)}$ is empty as desired. Now assume b_1 and a_2 agree. Since b_1 and a_2 agree and $\text{dom}(b_1) \sqcup \text{dom}(f_1^{-1}) = S$ we obtain $b_1 \sqcup f_1^{-1} u \supseteq a_2 \Leftrightarrow f_1 f_1^{-1} u \supseteq f_1 a_2 \Leftrightarrow u \supseteq f_1 a_2$. Applying the last

sentence to where we left off we obtain

$$\begin{aligned}
u(\overline{a_1, f_1, b_1})(\overline{a_2, f_2, b_2}) &= \begin{cases} b_2 \sqcup f_2^{-1}b_1 \sqcup f_2^{-1}f_1^{-1}u & u \supseteq f_1a_2 \text{ and } u \supseteq a_1 \\ \text{undefined} & u \not\supseteq a_1 \\ \text{undefined} & u \not\supseteq f_1a_2 \end{cases} \\
&= \begin{cases} b_2 \sqcup f_2^{-1}b_1 \sqcup f_2^{-1}f_1^{-1}u & u \supseteq f_1a_2 \sqcup a_1 \\ \text{undefined} & u \not\supseteq f_1a_2 \sqcup a_1 \end{cases} \\
&= \overline{(f_1a_2 \sqcup a_1, f_1f_2, b_2 \sqcup f_2^{-1}b_1)}
\end{aligned}$$

as desired. Above the second equality is a rearrangement and the third equality is an application of Proposition 4.2.4. \square

Definition 4.2.9. Define $\mathcal{I}(S, A, I)$ to be the set

$$\{(\overline{a, f, b}) \in \mathcal{I}_{AS} \mid f \in I \text{ and } a, b \in \mathcal{P}_{S,A}^f\}.$$

Theorem 4.2.10. *The set $\mathcal{I}(S, A, I)$ is an inverse submonoid of \mathcal{I}_{AS} with zero.*

Proof. It follows from Proposition 4.2.5 that the set contains the zero of \mathcal{I}_{AS} . It follows from Proposition 4.2.6 that the set contains the identity of \mathcal{I}_{AS} . It follows from Proposition 4.2.7 that the set is closed under taking inverses in \mathcal{I}_{AS} . It follows from Proposition 4.2.8 and that zero times anything is zero that the set is closed under taking products in \mathcal{I}_{AS} . \square

Definition 4.2.11. Define $\mathcal{D}(S, A, I)$ to be the smallest inverse submonoid of \mathcal{I}_{A^S} containing $\mathcal{I}(S, A, I)$ and such that for all $f_1, f_2 \in \mathcal{D}(S, A, I)$ with $f_1 \cup f_2 \in \mathcal{I}_{A^S}$ we have $f_1 \cup f_2 \in \mathcal{D}(S, A, I)$.

Definition 4.2.12. Define $\mathcal{V}(S, A, I)$ to be the group of units of $\mathcal{D}(S, A, I)$.

The group construction $\mathcal{V}(S, A, I)$ defined above is the focus of this chapter. Later we will be interested in necessary conditions on S , A and I for $\mathcal{V}(S, A, I)$ to be finitely generated and simple.

In order to give the reader an example to hang the notation on we now give the groups $(V_n)_{n>1}$ as an example of the groups that may be constructed using the above definitions. Recall the bicyclic monoid was defined in Section 2.2.

Example 4.2.13. Let n be at least 2. Let S be equal to the set of natural numbers, let A be equal to the set $\{0, 1, \dots, n-1\}$ and let I be equal to the bicyclic monoid B . The inverse monoid $\mathcal{I}(S, A, I)$ is equal to the polycyclic monoid of rank n (why this is the case will become clear in Section 4.3). The inverse monoid $\mathcal{D}(S, A, I)$ is the closure of the polycyclic monoid under compatible union. The group $\mathcal{V}(S, A, I)$ is Thompson's group V_n .

We now give a proposition describing the identity of $\mathcal{V}(S, A, I)$.

Proposition 4.2.14. *Assume $|A| > 1$. Let g an element of $\mathcal{V}(S, A, I)$ and K a finite subset of $\mathcal{P}_{S,A}^f$ and $\{f_a\}_{a \in K}$ a finite subset of I and $\{b_a\}_{a \in K}$ a finite subset of*

$\mathcal{P}_{S,A}^f$ be such that $(\overline{a, f_a, b_a})$ is non-empty for each $a \in K$ and $g = \bigcup_{a \in K} (\overline{a, f_a, b_a})$.

Then g is the identity of $\mathcal{V}(S, A, I)$ if and only if for each $a \in K$ we have f_a is an idempotent and $b_a = a$.

Proof. First assume $a \in K$ is such that f_a is not an idempotent. Let $s \in S$ be such that sf_a is defined and not equal to s . Now g does not fix any $x \in A^S$ with $a \subseteq x$ and $sx \neq sf_ax$. Second assume $a \in K$ is such that f_a is an idempotent but $a \neq b_a$. Now g does not fix any $x \in A^S$ with $a \subseteq x$. Finally assume that for all $a \in K$ we have f_a is an idempotent and $b_a = a$. Let $x \in A^S$ be given. Let $a \in K$ be such that $a \subseteq x$. Now $xg = b_a \sqcup f_a^{-1}x = a \sqcup x|_{\text{dom}(f_a)} = x$ as desired. \square

We will spend the rest of this section on modifications which can be made to I without changing $\mathcal{D}(S, A, I)$. These modifications will therefore also not change $\mathcal{V}(S, A, I)$.

Recall $\phi_S : \mathcal{I}_S \rightarrow T_S$ is a homomorphism from \mathcal{I}_S to some set T_S which identifies partial bijections if they agree in all but finitely many places. Note that $I\phi_S\phi_S^{-1} = I$ if and only if I is closed under making finite changes.

Theorem 4.2.15. *If $\bigcap_{f \in I} \text{dom}(f) = \emptyset = \bigcap_{f \in J} \text{dom}(f)$ and $I\phi_S = J\phi_S$ then $\mathcal{D}(S, A, I) = \mathcal{D}(S, A, J)$.*

Proof. Let $f \in I$ and $a, b \in \mathcal{P}_{S,A}^f$ be given with $\text{dom}(a) \sqcup \text{dom}(f) = S = \text{ran}(f) \sqcup \text{dom}(b)$. We will show $(\overline{a, f, b})$ is in $\mathcal{D}(S, A, J)$. Let $g \in J$ be given with $g\phi_S = f\phi_S$.

Let $X \subseteq S$ be the set where f and g disagree. Since $\bigcap_{f \in J} \text{dom}(f) = \emptyset$ we may for each $x \in X$ find $h_x \in J$ with $x \notin \text{dom}(h_x)$. Let $h := \left(\prod_{x \in X} h_x h_x^{-1} \right) g \in J$ and note $h \subseteq f$. Let $Y := \text{dom}(f) \setminus \text{dom}(h)$. To complete the proof note by our choice of h that $(\overline{a, f, b}) = \bigcup_{c \in A^Y} (\overline{a \cup c, h, b \cup f^{-1}c}) \in \mathcal{D}(S, A, J)$. \square

Corollary 4.2.16. *If $\bigcap_{f \in I} \text{dom}(f) = \emptyset = \bigcap_{f \in J} \text{dom}(f)$ and $I\phi_S = J\phi_S$ then*

$$\mathcal{V}(S, A, I) = \mathcal{V}(S, A, J).$$

Proof. This follows immediately from Proposition 4.2.15. \square

Corollary 4.2.17. *If $\bigcap_{f \in I} \text{dom}(f) = \emptyset$ then $\mathcal{V}(S, A, I) = \mathcal{V}(S, A, I\phi_S\phi_S^{-1})$.*

Proof. This is a special case of Corollary 4.2.16. \square

Note that if we are interested only in the group $\mathcal{V}(S, A, I)$ and only interested in the case $\bigcap_{f \in I} \text{dom} f = \emptyset$, then Corollary 4.2.17 allows us to assume $I = I\phi_S\phi_S^{-1}$.

4.3 $2V$ in the Context of the Construction

As an example of our construction in Section 4.2, we show how Brin's group $2V$ can be constructed. Brin introduced Thompson's group $2V$ in [2] and we described $2V$ in Section 2.8. Recall $2P_2$ was defined in Definition 2.8.1. The relationship between the polycyclic monoids and Thompson groups which this section relies upon was discussed in [11]. For this section only set S to be $\mathbb{N} \cup \mathbb{N}'$ where \mathbb{N}' is a

copy of the set of natural numbers such that $\mathbb{N} \cap \mathbb{N}' = \emptyset$ and $\mathbb{N}' = \{1', 2', \dots\}$, set A to be $\{0, 1\}$, and set I to be the inverse monoid $B \times B = \langle (b, 1), (1, b), (c, 1), (1, c) \rangle$ where B is the bicyclic monoid (as in Section 2.7) acting on S such that for $x \in S$ we have

$$x(c^i b^j, c^m b^n) = \begin{cases} x + j - i & \text{if } x \in \mathbb{N} \text{ and } x + j \geq i \\ x + n' - m' & \text{if } x \in \mathbb{N}' \text{ and } x + n' \geq m' \\ \text{undefined} & \text{otherwise} \end{cases}$$

Note here and below we are using that \mathbb{N}' has the same additive and order structure as \mathbb{N} . In the function above m, n on the left hand side are natural numbers whereas m', n' on the right hand side are the corresponding \mathbb{N}' versions (e.g., if $m = 4$ then $m' = 4'$).

Theorem 4.3.1. *The group $\mathcal{V}(S, A, I)$ is isomorphic to the group $2V$.*

Proof. In order to show this it is sufficient to observe that the action of $\mathcal{I}(S, A, I)$ on $A^S = \{0, 1\}^{\mathbb{N} \cup \mathbb{N}'}$ is isomorphic to the action of $2P_2$ on $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}'}$. We will define an explicit bijection and isomorphism.

Let $\delta : \mathbb{N} \rightarrow \mathbb{N}'$ be such that $n\delta = n'$ for each $n \in \mathbb{N}$. Let $\eta : \{0, 1\}^{\mathbb{N} \cup \mathbb{N}'} \rightarrow \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}'}$ be such that for each $z \in \{0, 1\}^{\mathbb{N} \cup \mathbb{N}'}$

$$z\eta = (z|_{\mathbb{N}}, \delta z).$$

Note that η is a bijection.

Now define $\zeta : \mathcal{I}(S, A, I) \longrightarrow 2P_2$ by

$$f\zeta := \eta^{-1}f\eta.$$

for each f in $\mathcal{I}(S, A, I)$.

We will borrow the term closed downwards from the theory of partially ordered sets. We will say a subset of \mathbb{N} is *closed downwards* if it takes the form $\{i \in \mathbb{N} \mid i < n\}$ for some $n \in \mathbb{N}$. We extend this definition to subsets of \mathbb{N}' .

For a and b be in $\mathcal{P}_{S,A}^f$ and f in I recall that $(\overline{a, f, b})$ is non-empty exactly if $\{\text{dom}(a), \text{dom}(f)\}$ partitions S and $\{\text{dom}(b), \text{ran}(f)\}$ partitions S . By inspection of the domains of elements of I note that for a and b be in $\mathcal{P}_{S,A}^f$ there exists f in I with $(\overline{a, f, b})$ non-empty exactly if $\text{dom}(a) \cap \mathbb{N}$ and $\text{dom}(a) \cap \mathbb{N}'$ and $\text{dom}(b) \cap \mathbb{N}$ and $\text{dom}(b) \cap \mathbb{N}'$ are all closed downwards in which case there exists exactly one such element of I .

For each t in $\mathcal{I}(S, A, I)$ ζ there are four lists a_1, a_2, b_1 and b_2 over $\{0, 1\}$ which completely describe t . Specifically, for $(x, y) \in \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ if a_1 is not a prefix of x or a_2 is not a prefix of y then t is not defined on (x, y) , if a_1 is a prefix of x and a_2 is a prefix of y then t moves (x, y) by replacing the prefix a_1 of x with b_1 and replacing the prefix a_2 of y with b_2 treating $\{0, 1\}^{\mathbb{N}}$ as a set of infinite words. Recall this is the definition of elements of $2P_2$. □

Remark 4.3.2. Fix $m, n \in \mathbb{N}$ with $n > 1$. Let \hat{S} be m disjoint copies of $\mathbf{n}^{\mathbb{N}}$ and let $\hat{I} := mP_n$ act on \hat{S} in a similar way to the action of $I = 2P_2$ on $S = \mathbb{N} \cup \mathbb{N}'$

(which is two copies of \mathbb{N}) as defined at the beginning of this section. The group $\mathcal{V}(\hat{S}, \mathbf{n}, \hat{I})$ is isomorphic to Thompson's group mV_n . The reasons for this are similar to the reasons for Theorem 4.3.1 so we will not repeat them.

4.4 The Construction in the Context of Tight Completions

The goal of this section is Theorem 4.4.11 in which we show that for S and A sets and I an inverse subsemigroup of \mathcal{I}_S the inverse semigroup $\mathcal{D}(S, A, I)$ is isomorphic to the tight completion of $\mathcal{I}(S, A, I)$. First we will give more background on inverse semigroups and give the definition of tight completion following [13].

Recall that in Definition 2.1.5 we defined a partial order on inverse semigroups and in Definition 2.6.1 we defined what it means for elements of an inverse semigroup to be compatible with each other. Recall also that the join of two elements of a partial order is the least element above both of them, should it exist.

Recall distributive inverse semigroups were defined in Definition 2.6.3. Also recall that order ideals, finitely generated order ideals and compatible order ideals were introduced in Definition 2.6.4.

Definition 4.4.1. If I is an inverse semigroup with zero then a *tight cover* of $i \in I$ is a finite subset T of i^\downarrow such that for all $s \in i^\downarrow \setminus \{0\}$ there exists $t \in T$ with

$s^\downarrow \cap t^\downarrow \neq \{0\}$. If T is a tight cover of i then we write $T \rightarrow i$.

Definition 4.4.2. Let A be an order ideal of an inverse semigroup I . Define

$$\bar{A} := \{i \in I \mid A \text{ has a finite compatible subset which is a cover of } i\}$$

Definition 4.4.3. Let I be an inverse monoid. Define the tight completion of I to be the set

$$D_t(I) := \{\bar{A} \mid A \text{ is a finitely generated compatible order ideal of } I\}$$

equipped with the operation such that for all X and Y in $D_t(I)$ the product $X \cdot Y$ equals \overline{XY} .

Definition 4.4.4. For I an inverse semigroup we will use $\delta_I : I \rightarrow D_t(I)$ for the homomorphism sending i in I to the compatible order ideal i^\downarrow .

Recall we do not include 0 in the natural numbers.

Example 4.4.5. Let I be the set of natural numbers equipped with the operation such that for each i and j in \mathbb{N}

$$i \cdot j := \min(i, j).$$

This semigroup has a zero which is 1. The partial order on $I = \mathbb{N}$ induced by its semigroup structure coincides with the normal total order on \mathbb{N} . The non-empty finitely generated compatible order ideals of I are the sets of the form

$$\{i \in \mathbb{N} \mid i < j\}$$

for some j in \mathbb{N} . If j is in $\mathbb{N} \setminus \{1\}$ then the tight covers of j are those finite subsets of j^\downarrow that contain any natural number other than 1. If X is any finitely generated compatible order ideal other than $\{1\}$ or the empty set then $\overline{X} = \mathbb{N}$. If X is either empty or $\{1\}$ then $\overline{X} = \{1\}$.

The tight completion of I is therefore the two element inverse monoid with one zero and one identity.

Example 4.4.6. Let S be a set and let I be the set of partial bijections from I to I with domain size either 1 or 0.

The semigroup I has a zero which is the empty function. The partial order on this semigroup is such that every non-zero element is above zero and no two non-zero elements are comparable. If f is in I then the only tight covers of f are the set containing f and the set containing both f and the empty function. Distinct pairs of elements of I are compatible exactly if they have disjoint domains and disjoint ranges.

If A is a finite subset of I then A is a finitely generated compatible order ideal of I exactly if A contains the empty function and any two distinct elements of A have disjoint domains and disjoint ranges. If A is a finitely generated compatible order ideal of I then $\overline{A} = A$. So if A and B are finitely generated order ideals of I then

$$A \cdot B = \overline{AB} = AB.$$

The map from the set of finitely generated order ideals of I containing the empty function to the set of partial functions on S which sends any finitely generated order ideal A to $\bigcup A$ is an isomorphism from the tight completion of I to the set of partial bijections on S with finite domain. So in this instance the homomorphism δ_I from I to its tight completion is a monomorphism.

Lemma 4.4.7. *Let I be an inverse semigroup with zero, let x be an element of I and let A and B be finite compatible subsets of x^\downarrow such that for each a in A there exists b in B with $b \geq a$. If A is a tight cover of x then B is also a tight cover of x .*

Proof. Let y be in x^\downarrow . Let a be an element of A with $y^\downarrow \cap a^\downarrow \neq \{0\}$. Let b be an element of B with $b \geq a$ then $b^\downarrow \supseteq a^\downarrow$ so $y^\downarrow \cap b^\downarrow \supseteq y^\downarrow \cap a^\downarrow$ and so $y^\downarrow \cap b^\downarrow \neq \{0\}$ as desired. □

Lemma 4.4.8. *If I is an inverse semigroup and a and b are compatible elements of I then $a^\downarrow \cap b^\downarrow = (aa^{-1}b)^\downarrow$.*

Proof. This follows immediately from the well known inverse semigroup theory fact that the greatest element of I below both a and b is $aa^{-1}b$. □

Lemma 4.4.9. *If A is a finite compatible subset of I an inverse semigroup with zero then the set $\overline{A^\downarrow}$ is equal to the set of x in I such that $\{xx^{-1}a \mid a \in A\}$ is a tight cover of x .*

Proof. We will first show that if $\{xx^{-1}a \mid a \in A\}$ is a tight cover of x then $\overline{A^\downarrow}$ contains x . The set $\{xx^{-1}a \mid a \in A\}$ is a subset of A^\downarrow and a tight cover of x . So by definition $\overline{A^\downarrow}$ contains x as desired.

Now we will show that if x is in $\overline{A^\downarrow}$ then the set $\{xx^{-1}a \mid a \in A\}$ is a tight cover of x . Since x is in $\overline{A^\downarrow}$ there exists Y a finite compatible subset of A^\downarrow which is a tight cover of x . For each y in Y we may choose a_y in A with $y \leq a_y$. By Lemma 4.4.7 to show $\{xx^{-1}a \mid a \in A\}$ is a tight cover of x it is sufficient to show that $y \leq xx^{-1}a_y$. This follows from Lemma 4.4.8 because y is less than or equal to both x and a_y . \square

Recall the notation $a \vee b$, $\bigvee T$ and T^\vee were defined in Definition 2.6.2.

Proposition 4.4.10. *Let S be a set and let I be an inverse subsemigroup of \mathcal{I}_S with zero. Assume for all p and q in I^\vee with $\text{dom}(p) \setminus \text{dom}(q)$ non-empty there exists non-zero f in I with $\text{dom}(f) \subseteq \text{dom}(p) \setminus \text{dom}(q)$. Then the tight completion of I is isomorphic to I^\vee .*

Proof. Define $\iota : D_t(I) \rightarrow \mathcal{I}_S$ to be the map such that if A is a finite compatible subset of I then $\overline{A^\downarrow} \iota := \bigcup \overline{A^\downarrow}$. We will show

1. if A is a finite compatible subset of I then $\overline{A^\downarrow} \iota = \bigcup A$,
2. the map ι is injective, and
3. the map ι is a homomorphism.

It follows immediately from the first point that $D_t(I)\iota = I^\vee$ so these points are sufficient to show that ι is an isomorphism from $D_t(I)$ to I^\vee .

1. Let A be a finite compatible subset of I . Since $\overline{A^\downarrow}$ contains A we note $\bigcup A \subseteq \bigcup \overline{A^\downarrow} = \overline{A^\downarrow}\iota$ so it remains to show $\bigcup \overline{A^\downarrow} \subseteq \bigcup A$.

Let b be an element of $\overline{A^\downarrow}$. By Lemma 4.4.9 the set $\{bb^{-1}a \mid a \in A\}$ is a tight cover of b . Seeking a contradiction assume b is not a subset of $\bigcup A$. Since $A \cup \{b\}$ is compatible the set $\text{dom}(b) \setminus \text{dom}(\bigcup A)$ is non-empty. By the hypothesis of the theorem there exists non-zero f in I with $\text{dom}(f) \subseteq \text{dom}(b) \setminus \text{dom}(\bigcup A)$. Fix $t := ff^{-1}b$. Since $\text{dom}(t)$ is disjoint from $\text{dom}(a)$ for each a in A we find $t^\downarrow \cap \{bb^{-1}a \mid a \in A\}^\downarrow = 0$. But t is in b^\downarrow and non-zero so $\{xx^{-1}a \mid a \in A\}$ is not a tight cover of b , a contradiction.

2. Let A and B be finite compatible subsets of I with $\overline{A^\downarrow}\iota = \overline{B^\downarrow}\iota$. We will show $\overline{A^\downarrow} = \overline{B^\downarrow}$. By the first point the unions $\bigcup A$ and $\bigcup B$ are equal so it is sufficient to show $\overline{A^\downarrow} = \{i \in I \mid i \subseteq \bigcup A\}$.

We will first show $\{i \in I \mid i \subseteq \bigcup A\} \subseteq \overline{A^\downarrow}$. Let i in $\{i \in I \mid i \subseteq \bigcup A\}$ be given. By Lemma 4.4.9 it is sufficient to show $\{ii^{-1}a \mid a \in A\}$ is a tight cover of i . Let x in i^\downarrow be given. Since $\text{dom}(x) \subseteq \text{dom}(i) \subseteq \text{dom}(\bigcup A)$ we may fix a in A with $\text{dom}(x) \cap \text{dom}(a)$ non-empty. The domain of $xx^{-1}a$ is non-empty so $xx^{-1}a$ is non zero. Since $x \leq i$ the partial bijections $xx^{-1}a$ and $xx^{-1}ii^{-1}a$ are equal. By 4.4.9 since x and $ii^{-1}a$ are compatible $xx^{-1}a = xx^{-1}ii^{-1}a$ is

in $x^\downarrow \cap ii^{-1}a^\downarrow$ as desired.

We will now show $\overline{A^\downarrow} \subseteq \{i \in I \mid i \subseteq \bigcup A\}$. Let i in $\overline{A^\downarrow}$ be given. By Lemma 4.4.9 the set $\{ii^{-1}a \mid a \in A\}$ is a tight cover of i . Note $A \cup \{i\}$ is compatible. Assume seeking a contradiction that i is not a subset of $\bigcup A$. Now $\text{dom}(i) \setminus \text{dom}(\bigcup A)$ is non-empty. Using the assumption of the theorem let f in I be such that $\text{dom}(f) \subseteq \text{dom}(i) \setminus \text{dom}(\bigcup A)$. Let $t := ff^{-1}i$. Since t is in i^\downarrow and $\{ii^{-1}a \mid a \in A\}$ is a tight cover of i we may chose a non-zero x in the set $t^\downarrow \cap \{ii^{-1}a \mid a \in A\}^\downarrow$. But $\text{dom}(x) \subseteq \text{dom}(t) \cap \text{dom}(\bigcup \{ii^{-1}a \mid a \in A\}) \subseteq \text{dom}(t) \cap \text{dom}(\bigcup A)$ which is empty contradicting that t is non-zero.

3. Let P, Q in $D_t(S)$ be given. Let x and y in S be given. To finish the proof note

$$x(P\iota)(Q\iota) = z$$

$$\Leftrightarrow \text{there exists } y \text{ in } S \text{ with } x(P\iota) = y \text{ and } y(Q\iota) = z$$

$$\Leftrightarrow \text{there exists } y \text{ in } S \text{ and } u \text{ in } P \text{ and } v \text{ in } Q \text{ with } xu = y \text{ and } yv = z$$

$$\Leftrightarrow \text{there exists } y \text{ in } S \text{ and } t \text{ in } PQ \text{ with } xt = z$$

$$\Leftrightarrow x((PQ)\iota) = z.$$

as desired. □

Theorem 4.4.11. *Let S and A be sets and let I be an inverse subsemigroup of $\mathcal{I}_S^{\text{cof}}$. Then the tight completion of $\mathcal{I}(S, A, I)$ is isomorphic to $\mathcal{D}(S, A, I)$.*

Proof. It is sufficient to show the hypothesis of Proposition 4.4.10 is satisfied.

If there do not exist p and q in $\mathcal{D}(S, A, I)$ with $\text{dom}(p) \setminus \text{dom}(q)$ non-empty then the hypothesis of Proposition 4.4.10 is trivially satisfied and we are done so assume such elements exist. Let p and q in $\mathcal{D}(S, A, I)$ be given such that $\text{dom}(p) \setminus \text{dom}(q)$ is non-empty. By Proposition 4.4.10 it is sufficient to find non-zero f in $\mathcal{I}(S, A, I)$ with $\text{dom}(f) \subseteq \text{dom}(p) \setminus \text{dom}(q)$.

Let P and Q be finite compatible subsets of $\mathcal{I}(S, A, I)$ with $\bigcup P = p$ and $\bigcup Q = q$. For $u \in P$ let a_u and b_u in A and f_u in I be such that $(\overline{a_u, f_u, b_u}) = u$. Similarly for $v \in Q$ let a_v and b_v in A and f_v in I be such that $(\overline{a_v, f_v, b_v}) = v$. Let i be the product of all the elements $f_r f_r^{-1}$ of I with r running over $P \cup Q$. Note that $S \setminus \text{dom}(i)$ is finite.

Let x be in $\text{dom}(p) \setminus \text{dom}(q)$. Let t be x restricted to $S \setminus \text{dom}(i)$. Note that $\text{dom}(t) = \bigcup_{r \in P \cup Q} \text{dom}(a_r)$. Since x disagrees with a_v for every v in Q it follows that t also disagrees with a_v for every $v \in Q$. Since x contains a_u for some u in P it follows t also contains a_u for some u in P . We note the domain of $(\overline{t, i, t})$ is equal to $\{y \in A^S \mid y \supseteq t\}$ which is contained in $\text{dom}(p)$ and disjoint from $\text{dom}(q)$, as desired. \square

4.5 Some Necessary Conditions for Finite Generation and Simplicity

For this section fix S and A sets and I, J inverse submonoids of $\mathcal{I}_S^{\text{cof}}$.

In this section we will show that if $\mathcal{V}(S, A, I)$ is to be infinite, simple and finitely generated and not isomorphic to the group of units of I then the following may be assumed either because they are necessary or because they may be achieved by trivially modifying S, A or I without changing $\mathcal{V}(S, A, I)$

- $1 < |A| < |\mathbb{N}| = |S| = |I|$ (Proposition 4.5.1, Proposition 4.5.2, Proposition 4.5.3, Proposition 4.5.6 and Theorem 4.5.13),
- $\bigcap_{f \in I} \text{dom}(f) = \emptyset$ (Proposition 4.5.6),
- I is finitely generated (Theorem 4.5.13).

Proposition 4.5.1. *If $|A| < 2$ then $\mathcal{V}(S, A, I)$ is the trivial group.*

Proof. If $|A| < 2$ then $1 = |A^S|! = |\text{Sym}(A^S)| \geq |\mathcal{V}(S, A, I)|$ (note we are using $0! = 1$). □

Proposition 4.5.2. *If S is non-empty and A is infinite then $\mathcal{V}(S, A, I)$ is isomorphic to $U(I)$ the group of units of I .*

Proof. Let e be the empty function. If p and q are permutations of S and x is in A^S then $x(\overline{e, p, e})(\overline{e, q, e}) = (p^{-1}x)(\overline{e, q, e}) = q^{-1}p^{-1}x = (pq)^{-1}x = x(\overline{e, pq, e})$.

Therefore the map from the group of units of I to $\mathcal{I}(S, A, I)$ sending a unit p of I to $(\overline{e, p, e})$ is a homomorphism to the group of units of $\mathcal{I}(S, A, I)$ which is a subgroup of $\mathcal{V}(S, A, I)$.

We will show that this homomorphism is a monomorphism. Assume p and q are distinct. Let s in S be such that $sp^{-1} \neq sq^{-1}$. Let $z \in A^S$ be such that $sp^{-1}z \neq sq^{-1}z$. Now $z(\overline{e, p, e}) = p^{-1}z \neq q^{-1}z = z(\overline{e, q, e})$ as desired.

Using that A is infinite we will show that all elements of $\mathcal{V}(S, A, I)$ have the form $(\overline{e, g, e})$ for some unit g of I . This is sufficient as we will have given an isomorphism from the group of units of I to $\mathcal{V}(S, A, I)$.

Let h be in $\mathcal{V}(S, A, I)$. Let K a finite subset of $\mathcal{P}_{S,A}^f$ and $\{f_c\}_{c \in K} \subseteq I$ and $\{d_c\}_{c \in K} \subseteq \mathcal{P}_{S,A}^f$ be such that the union $\bigcup_{c \in K} (\overline{c, f_c, d_c})$ is equal to h and for each c in K the partial permutation $(\overline{c, f_c, d_c})$ is non-empty.

Since K is a finite subset of $\mathcal{P}_{S,A}^f$ and A is infinite we may find $y \in A$ not in the range of any element of K . Let $t \in A^S$ send every element of S to y . Because t is in the domain of h the map t must be in $\text{dom}((\overline{c, f_c, d_c})) = \{r \in A^S \mid r \supseteq c\}$ for some c in K . The only way this can happen is if $c = e$ so e must be an element of K .

Since the domain of $(\overline{e, f_e, d_e})$ is A^S the other elements of K are superfluous. So $g = (\overline{e, f_e, d_e})$. For h to be surjective d_e must also be the empty function and therefore f_e must be a permutation of S as desired. \square

Note that if A is infinite and I is not a monoid then $\mathcal{V}(S, A, I)$ is empty (and so not even a group).

Proposition 4.5.3. *If A and S are finite then $\mathcal{V}(S, A, I)$ is also finite.*

Proof. $|\mathcal{V}(S, A, I)| \leq |\text{Sym}(A^S)| = |A^S|!$ is finite if A and S are finite. \square

Lemma 4.5.4. *Assume A is finite. Let h be in $\mathcal{D}(S, A, I)$. Then there exists a finite subset F of $\mathcal{I}(S, A, I)$ such that F partitions h .*

Proof. By Lemma 2.6.8 it is sufficient to find a set of idempotents $B \subseteq \mathcal{I}(S, A, I)$ such that

$$\{\text{dom}(b) \mid b \in B \cup \{h\}\}$$

partitions A^S . Relying on the fact that h is in $\mathcal{D}(S, A, I)$ let K a finite subset of $\mathcal{P}_{S,A}^f$ and $\{f_a\}_{a \in K} \subseteq I$ and $\{b_a\}_{a \in K} \subseteq \mathcal{P}_{S,A}^f$ be such that $\bigcup_{a \in K} \overline{(a, f_a, b_a)} = h$. Let

$$t := \bigcap_{a \in K} f_a f_a^{-1}.$$

Note that t is in I . Let

$$C := \{c \in A^{S \setminus \text{dom}(t)} \mid c \text{ disagrees with each partial bijection in } K\}.$$

Since the set $S \setminus \text{dom}(t)$ is finite it follows that C is finite. Let B be the set $\{\overline{(c, t, c)} \mid c \in C\}$.

We now show that $\{\text{dom}(b) \mid b \in B \cup \{h\}\}$ partitions A^S . Let x be in A^S . If x disagrees with each element of K then $x|_{S \setminus \text{dom}(t)}$ is in C and x is in the domain of

$\left(\overline{x|_{S \setminus \text{dom}(t)}, t, x|_{S \setminus \text{dom}(t)}}\right)$ which is in B . If there is some $a \in K$ such that x agrees with a then x is in the domain of $\left(\overline{a, f_a, b_a}\right)$ which is a subset of the domain of h . \square

For $E \subseteq S$ we will abuse notation by writing $I|_E$ for the monoid $\{f|_E \mid f \in I\}$. We are interested in infinite finitely generated simple groups. The motivation for the following lemma is to exclude the S, A, I for which $\mathcal{V}(S, A, I)$ is only simple if it is isomorphic to $\mathcal{V}(E, A, I|_E)$ where $E = S \setminus \bigcap_{f \in I} \text{dom}(f)$.

Proposition 4.5.5. *Let $E := S \setminus \bigcap_{f \in I} \text{dom}(f)$. There is a surjective homomorphism $\psi : \mathcal{V}(S, A, I) \longrightarrow \mathcal{V}(E, A, I|_E)$.*

Proof. Define $\psi : \mathcal{V}(S, A, I) \longrightarrow \mathcal{V}(E, A, I|_E)$ by for each $g \in \mathcal{V}(S, A, I)$

$$g\psi := \bigcup_{a \in K} \left(\overline{a, f_a|_E, b_a}\right)$$

where K is a finite subset of $\mathcal{P}_{S,A}^f$ and $\{f_a\}_{a \in K} \subseteq I$ and $\{b_a\}_{a \in K} \subseteq \mathcal{P}_{S,A}^f$ are such that g is equal to the union $\bigcup_{a \in K} \left(\overline{a, f_a, b_a}\right)$.

First we show ψ is well defined. Let $g, K, \{f_a\}_{a \in K}$ and $\{b_a\}_{a \in K}$ be as in the definition of ψ . Let x be in A^E . Let m and n in $\mathcal{P}_{S,A}^f$ and l in I and a in K be such that $\left(\overline{m, l|_E, n}\right)$ and $\left(\overline{a, f_a|_E, b_a}\right)$ are defined on x . It is sufficient to show that $x \left(\overline{m, l|_E, n}\right) = x \left(\overline{a, f_a|_E, b_a}\right)$.

Let y in A^S be an extension of x . Since y is an extension of both a and m it follows that $y \left(\overline{a, f_a, b_a}\right)$ and $y \left(\overline{m, l, n}\right)$ are defined. Also $y \left(\overline{a, f_a, b_a}\right)$ and $y \left(\overline{m, l, n}\right)$

must be equal since $(\overline{a, f_a, b_a})$ and $(\overline{m, l, n})$ are both restrictions of g . In the second implication below we use that no element of I maps any point of S in or out of E .

$$\begin{aligned}
y(\overline{a, f_a, b_a}) = y(\overline{m, l, n}) &\implies f_a^{-1}y \sqcup b_a = l^{-1}y \sqcup n \\
&\implies f_a^{-1}|_E y \sqcup b_a = l^{-1}|_E y \sqcup n \\
&\implies f_a^{-1}|_E x \sqcup b_a = l^{-1}|_E x \sqcup n \\
&\implies x(\overline{a, f_a|_E, b_a}) = x(\overline{m, l|_E, n})
\end{aligned}$$

as desired.

We will now show ψ is a homomorphism. Let h in $\mathcal{V}(S, A, I)$, a finite subset J of $\mathcal{P}_{S,A}^f$, a finite subset $\{t_c\}_{c \in J}$ of I and a finite subset $\{d_c\}_{c \in J}$ of $\mathcal{P}_{S,A}^f$ be given such that h is equal to the union $\bigcup_{c \in K} (\overline{c, t_c, d_c})$. It is sufficient to show that $x(g\psi)(h\psi) = x((gh)\psi)$.

Fix $M := \{(a, c) \in K \times J \mid b_a|_{\text{dom}(c)} \subseteq c\}$. Let a_0 in K be an extension of x and let c_0 in J be an extension of $b_0 \sqcup f_{a_0}x$. Now observe

$$\begin{aligned}
x(g\psi)(h\psi) &= x\left(\bigcup_{a \in K} (\overline{a, f_a|_E, b_a})\right)(h\psi) \\
&= x(\overline{a_0, f_{a_0}|_E, b_{a_0}})(h\psi) \\
&= x(\overline{a_0, f_{a_0}|_E, b_{a_0}}) \bigcup_{c \in J} (\overline{c, t_c|_E, d_c}) \\
&= x(\overline{a_0, f_{a_0}|_E, b_{a_0}}) (\overline{c_0, t_{c_0}|_E, d_{c_0}}) \\
&= x(\overline{a_0 \sqcup f_{a_0}c_0, f_{a_0}|_E t_{c_0}|_E, t_{c_0}^{-1}b_{a_0} \sqcup d_{c_0}})
\end{aligned}$$

$$\begin{aligned}
&= x \left(\overline{a_0 \sqcup f_{a_0} c_0, f_{a_0} t_{c_0}|_E, t_{c_0}^{-1} b_{a_0} \sqcup d_{c_0}} \right) \\
&= x \bigcup_{(a,c) \in M} \left(\overline{a \sqcup f_a c, f_a t_c|_E, t_c^{-1} b_a \sqcup t_c} \right) \\
&= x((gh)\psi)
\end{aligned}$$

as desired.

Now we will show ψ is surjective. Let p be in $\mathcal{V}(E, A, I|_E)$. Let L be a finite subset of $\mathcal{P}_{D,A}^f$ and let $\{r_u\}_{u \in L}$ be a finite subset of $I|_D$ and let $\{v_u\}_{u \in L}$ be a finite subset of $\mathcal{P}_{D,A}^f$ such that $p = \bigcup_{u \in L} (\overline{u, r_u, v_u})$.

For each u in L choose w_u in I such that $w_u|_E = r_u$. Now set

$$q := \bigcup_{u \in L} (\overline{u, w_u, v_u})$$

and note $q\psi = p$, as desired. \square

Proposition 4.5.6. *If $\bigcap_{f \in I} \text{dom}(f) \neq \emptyset$ and $\mathcal{V}(S, A, I)$ is simple then $\mathcal{V}(S, A, I)$ is isomorphic to $\mathcal{V}(D, A, I|_D)$ where $D := S \setminus \bigcap_{f \in I} \text{dom}(f)$.*

Proof. This follow immediately from Proposition 4.5.5. \square

Of the groups we are interested in which are produced by the construction of this chapter some are vigorous (which was defined in Definition 3.1.1) and others respect a measure. We will use both of these properties to prove things. Since the details of measure theory are extraneous to an understanding of the construction

we will not go into the details of what measures are and instead give a rudimentary definition sufficient for our purposes and give explanatory examples.

Definition 4.5.7. If A is finite of size at least 2 then we will use λ for the unique function from the set of clopen subsets of A^S to the set of real numbers such that for each a in $\mathcal{P}_{S,A}^f$ the cone $\{x \in A^S \mid x \supseteq a\}$ is mapped by λ to $\frac{1}{|A|^{|a|}}$ and if X and Y are clopen subsets of A^S then

$$(X)\lambda + (Y)\lambda = (X \cup Y)\lambda + (X \cap Y)\lambda.$$

Example 4.5.8. If A is the set $\{a, b, c\}$ and S is the natural numbers then

- $A^S\lambda = 1,$
- $\{x \in A^S \mid 1x = a \text{ and } 2x = c\} \lambda = 1/9,$
- $\{x \in A^S \mid 1x = a \text{ or } 1x = c\} \lambda = 2/3,$ and
- $\{x \in A^S \mid 1x = a \text{ or } 2x = b\} \lambda = 5/9.$

Lemma 4.5.9. *Assume A is finite and of size at least 2. If $h \in \mathcal{D}(S, A, I)$ with $\text{dom}(h)\lambda = \text{ran}(h)\lambda$ then there exists a finite subset K of $\mathcal{P}_{S,A}^f$ and a finite set $\{f_a\}_{a \in K}$ of idempotents from I and a finite subset $\{b_a\}_{a \in K}$ of $\mathcal{P}_{S,A}^f$ such that $h \sqcup \bigcup_{a \in K} \overline{(a, f_a, b_a)} \in \mathcal{V}(S, A, I).$*

Proof. Let L a finite subset of $\mathcal{P}_{S,A}^f$ and $\{t_c\}_{c \in L}$ a finite subset of I and $\{d_c\}_{c \in K}$ a finite subset of $\mathcal{P}_{S,A}^f$ be such that $\overline{(c, t_c, d_c)}$ is non-empty for each c in L and $h = \bigcup_{c \in L} \overline{(c, t_c, d_c)}.$

Let $e := \prod_{c \in L} t_c t_c^{-2} t_c$. Note that e is an idempotent and $\text{dom}(e) = \text{ran}(e) = \bigcap_{c \in L} (\text{dom}(t_c) \cap \text{ran}(t_c))$. Let U be the set

$$\{r \in A^{S \setminus \text{dom}(e)} \mid \{x \in A^S \mid x \supseteq r\} \cap \text{dom}(h) = \emptyset\}$$

and let V be the set

$$\{r \in A^{S \setminus \text{dom}(e)} \mid \{x \in A^S \mid x \supseteq r\} \cap \text{ran}(h) = \emptyset\}.$$

Note that $\bigcup_{u \in U} \{x \in A^S \mid x \supseteq u\} = A^S \setminus \text{dom}(h)$ and $\bigcup_{v \in V} \{x \in A^S \mid x \supseteq v\} = A^S \setminus \text{ran}(h)$. It is sufficient to find an element of $\mathcal{D}(S, A, I)$ with domain equal to the set $\bigcup_{u \in U} \{x \in A^S \mid x \supseteq u\}$ and range $\bigcup_{v \in V} \{x \in A^S \mid x \supseteq v\}$. We will first show the sets U and V have the same number of elements.

Recall $\text{dom}(h)\lambda = \text{ran}(h)\lambda$. By Definition 4.5.7 if r is in $A^{S \setminus \text{dom}(e)}$ then λ maps the cone $\{x \in X \mid x \supseteq r\}$ to $|A|^{-|S \setminus \text{dom}(e)|}$. Since $S \setminus \text{dom}(h)$ is partitioned into $|U|$ many cones each mapped by λ to $|A|^{-|S \setminus \text{dom}(e)|}$ and $S \setminus \text{ran}(h)$ is partitioned into $|V|$ many cones each also mapped by λ to $|A|^{-|S \setminus \text{dom}(e)|}$ we may use the properties of λ to deduce

$$|U| \cdot |A|^{-|S \setminus \text{dom}(e)|} = (S \setminus \text{dom}(h))\lambda = (S \setminus \text{ran}(h))\lambda = |V| \cdot |A|^{-|S \setminus \text{dom}(e)|}$$

so the sets U and V have the same number of elements.

Fix a bijection $v : U \rightarrow V$. To complete the proof note

$$h \sqcup \bigsqcup_{r \in U} (\overline{r, e, r\bar{v}}) \in \mathcal{V}(S, A, I)$$

as desired. □

Definition 4.5.10. The finitary power semigroup $P_f(T)$ of a semigroup T is the set of finite subsets of T equipped with setwise multiplication. So for u, v finite subsets of T the semigroup product of u and v is

$$\{rt \mid r \in u, t \in v\}$$

where concatenation denotes the original semigroup operation on T .

Lemma 4.5.11. *If T and U are semigroups and $\mu : T \rightarrow P_f(U)$ is a function such that*

1. T is finitely generated,
2. for all $u \in U$ there is $t \in T$ with $u \in t\mu$, and
3. $(t_1\mu)(t_2\mu) \supseteq (t_1t_2)\mu$ for all $t_1, t_2 \in T$

then U is also finitely generated.

Proof. By the Point 1 we may fix X a finite generating set for T . Fix $Y := \bigcup_{t \in T} t\mu$. Since Y is finite it is sufficient to show Y is a generating set for U . Let u be in U . We will show that $u \in \langle Y \rangle$.

By Point 2 there exists t in T with $u \in t\mu$. Since X is a generating set for T there exists a natural number j and a list x_1, \dots, x_j of elements of X with $x_1 \cdots x_j = t$.

From Point 3 it follows that $(x_1\mu) \cdots (x_j\mu) \supseteq (x_1 \cdots x_j)\mu = t\mu \ni u$. Since $(x_1\mu) \cdots (x_j\mu)$ contains u we may choose y_1, \dots, y_j a list of elements of Y with y_i

in $x_i\mu$ for each $1 \leq i \leq j$ and $y_1 \cdots y_j = u$. Now u is in $\langle y_1, \dots, y_j \rangle$ which is a subset of $\langle Y \rangle$ so u is also in $\langle Y \rangle$ as desired.

□

Recall ϕ_S was defined in Definition 2.3.6.

Lemma 4.5.12. *Assume A is finite and of size at least 2. If $\bigcap_{f \in I} \text{dom}(f) = \emptyset$ then there exists a function $\mu : \mathcal{V}(S, A, I) \rightarrow P_f(I\phi_S)$ such that firstly $(g_1 g_2)\mu \subseteq (g_1\mu)(g_2\mu)$ for all $g_1, g_2 \in \mathcal{V}(S, A, I)$ and secondly for all $f \in I\phi_S$ there is $t \in \mathcal{V}(S, A, I)$ such that f is in $t\mu$*

Proof. Fix $g \in \mathcal{V}(S, A, I)$, we will define μ by constructing $g\mu$. Let a finite subset K of $\mathcal{P}_{S,A}^f$ and a finite subset $\{f_a\}_{a \in K}$ of I and a finite subset $\{b_a\}_{a \in K}$ of $\mathcal{P}_{S,A}^f$ be given such that $(\overline{a, f_a, b_a})$ is non-empty for each $k \in K$ and $g = \bigcup_{a \in K} (\overline{a, f_a, b_a})$.

Define

$$g\mu := \{f_a\phi_S\}_{a \in K} \cup \{\emptyset\}.$$

In order to show that μ is well defined we will show that $g\mu$ is independent of our choice of K , $\{f_a\}_{a \in K}$ and $\{b_a\}_{a \in K}$. Let a finite subset K' of $\mathcal{P}_{S,A}^f$ and a finite subset $\{f'_{a'}\}_{a' \in K'}$ of I and a finite subset of $\{b'_{a'}\}_{a' \in K'}$ of $\mathcal{P}_{S,A}^f$ be given such that $(\overline{a', f'_{a'}, b'_{a'}})$ is non-empty for each $k' \in K'$ and $g = \bigcup_{a' \in K'} (\overline{a', f'_{a'}, b'_{a'}})$.

Fix $x \in A^S$ and $a_0 \in K$ and $a'_0 \in K'$ with a_0 and a'_0 both restrictions of x noting that a_0 and a'_0 must agree.

It is sufficient to show $f_{a_0}\phi_S$ and $f'_{a'_0}\phi_S$ are equal. We will show the stronger statement that $f_{a_0}^{-1}$ and $f'_{a'_0}{}^{-1}$ agree. Seeking a contradiction assume s is in $\text{ran}(f_{a_0}) \cap \text{ran}(f'_{a'_0})$ with $sf_{a_0}^{-1} \neq sf'_{a'_0}{}^{-1}$. Fix $l_1, l_2 \in A$ with $l_1 \neq l_2$. Define $u \in A^S$ by

$$tu := \begin{cases} t(a_0 \cup a'_0) & \text{if } t \in \text{dom}(a_0 \cup a'_0) \\ l_1 & \text{if } t = sf_{a_0}^{-1} \\ l_2 & \text{otherwise} \end{cases}$$

for each $t \in S$. Note that

$$\begin{aligned} & s(ug) \\ &= s\left(u \bigcup_{a \in K} (\overline{a, f_a, b_a})\right) \\ &= s\left(u(\overline{a_0, f_{a_0}, b_{a_0}})\right) \\ &= sf_{a_0}^{-1}u \\ &= l_1 \\ &\neq l_2 \\ &= sf'_{a'_0}{}^{-1}u \\ &= s\left(u(\overline{a', f'_{a'}, b'_{a'}})\right) \\ &= s\left(u \bigcup_{a' \in K'} (\overline{a', f'_{a'}, b'_{a'}})\right) \\ &= s(ug) \end{aligned}$$

a contradiction so f_{a_0} and $f'_{a'_0}$ agree. So $f_{a_0}\phi_S = f'_{a'_0}\phi_S$ and μ is well defined.

Let $h \in \mathcal{V}(S, A, I)$ be given. We will now show $(gh)\mu \subseteq (g\mu)(h\mu)$. Let L a

finite subset of $\mathcal{P}_{S,A}^f$ and $\{t_c\}_{c \in L}$ a finite subset of I and $\{d_c\}_{c \in L}$ a finite subset of $\mathcal{P}_{S,A}^f$ be given such that $(\overline{c, t_c, d_c})$ is non-empty for each $c \in L$ and $h = \bigcup_{c \in L} (\overline{c, t_c, d_c})$.

Fix $M := \{(a, c) \in K \times L \mid b_a \text{ agrees with } c\}$. Observe

$$\begin{aligned}
& (gh) \mu \\
&= \left(\left(\bigcup_{a \in K} (\overline{a, f_a, b_a}) \right) \left(\bigcup_{c \in L} (\overline{c, t_c, d_c}) \right) \right) \mu \\
&= \left(\bigcup_{a \in K, c \in L} (\overline{a, f_a, b_a}) (\overline{c, t_c, d_c}) \right) \mu \\
&= \left(\bigcup_{(a,c) \in M} (\overline{a \sqcup f_a c, f_a t_c, t_c^{-1} b_a \sqcup d_c}) \right) \mu \\
&= \{(f_a t_c) \phi_S\}_{(a,c) \in M} \sqcup \{\emptyset\} \\
&\subseteq \{(f_a t_c) \phi_S\}_{(a,c) \in K \times L} \sqcup \{\emptyset\} \\
&= (\{f_a \phi_S\}_{a \in K} \sqcup \{\emptyset\}) (\{t_c \phi_S\}_{c \in L} \sqcup \{\emptyset\}) \\
&= \left(\bigcup_{a \in K} (\overline{a, f_a, b_a}) \right) \mu \left(\bigcup_{c \in L} (\overline{c, t_c, d_c}) \right) \mu \\
&= (g\mu) (h\mu)
\end{aligned}$$

as desired.

Let $i \in I$ be given. We will construct v in $\mathcal{V}(S, A, I)$ with $i\phi_S \in v\mu$. Let $q \in I$ be given with $\text{dom}(q) \neq S$. Since $qq^{-1}iqq^{-1}\phi_S = t\phi_S$ and $\text{dom}(qq^{-1}iqq^{-1}) \cup \text{ran}(qq^{-1}iqq^{-1}) \neq S$ we may assume without loss of generality that $\text{dom}(i) \cup \text{ran}(i) \neq S$.

Let $r \in \mathcal{D}(S, A, I)$ be the union of $(\overline{(S \setminus \text{dom}(i)) \times \{l_1\}, i, (S \setminus \text{ran}(i)) \times \{l_1\}})$ and $(\overline{(S \setminus \text{ran}(i)) \times \{l_2\}, i^{-1}, (S \setminus \text{dom}(i)) \times \{l_2\}})$ where $X \times \{x\}$ is the constant function from X to x . By Lemma 4.5.9 since $\text{dom}(r)\lambda = \text{ran}(r)\lambda$ we may choose v

to be an extension of r . To complete the proof note $v\mu$ contains $i\phi_S$ as desired. \square

Theorem 4.5.13. *If $\mathcal{V}(S, A, I)$ is finitely generated then $I\phi_S$ is also finitely generated and so there must exist some finitely generated $I' \in \mathcal{I}_S^{\text{cof}}$ with $I'\phi_S = I\phi_S$ and $\mathcal{V}(S, A, I') = \mathcal{V}(S, A, I)$.*

Proof. This follows immediately from Lemma 4.5.11, Lemma 4.5.12 and Corollary 4.2.16. \square

4.6 $\mathcal{V}(S, A, I)$ and the Property of Being Vigorous

For this section let S be a countably infinite set, let A be a finite set of size at least 2 and let I be an inverse submonoid of $\mathcal{I}_S^{\text{cof}}$ (note the extra assumption about cardinality not made in previous sections). First we list some conditions by number in the rest of the section.

1. The inverse monoid I is finitely generated.
2. The intersection $\bigcap_{f \in I} \text{dom}(f)$ is empty.
3. There exists f in I such that $|S \setminus \text{dom}(f)|$ is not equal to $|S \setminus \text{ran}(f)|$.
4. $|S \setminus \text{dom}(f)| = |S \setminus \text{ran}(f)|$ for all f in I .
5. The group $I\phi_S$ has no infinite commutative homomorphic images.

Definition 4.6.1. If (S, A, I) satisfies Condition 1, Condition 2, Condition 3 and fails Condition 4 then we will say (S, A, I) is of *non-area preserving type*. If (S, A, I) satisfies Condition 1, Condition 2, Condition 4, Condition 5 and fails Condition 3 then we will say (S, A, I) is of *area preserving type*.

Conjecture 4.6.2. If (S, A, I) is of non-area preserving type or of area preserving type then the commutator subgroup of $\mathcal{V}(S, A, I)$ is infinite finitely generated simple and of finite index in $\mathcal{V}(S, A, I)$.

Conjecture 4.6.3. If (S, A, I) is not of non-area preserving type or of area preserving type then $\mathcal{V}(S, A, I)$ is finite or not finitely generated or is degenerate (as in Proposition 4.5.2 or Proposition 4.5.6).

Proposition 4.6.4. *If (S, A, I) is either of non-area preserving type or of area preserving type then I is countably infinite.*

Proof. Since I is finitely generated by Condition 1 the cardinality of I can be no more than countably infinite.

From Condition 2

$$\bigcup_{f \in I} (S \setminus \text{dom}(f)) = S.$$

By an assumption in the first sentence of this section S is countably infinite and the above union is of finite sets so I must also be infinite. □

Lemma 4.6.5. *The map $\iota : \mathcal{I}_S^{\text{cof}} \longrightarrow \mathbb{Z}$ which sends $f \in \mathcal{I}_S^{\text{cof}}$ to $|S \setminus \text{dom}(f)| - |S \setminus \text{ran}(f)|$ is a homomorphism.*

Proof. For this proof we will use $\text{dom}^c(f)$ for $S \setminus \text{dom}(f)$ and $\text{ran}^c(f)$ for $S \setminus \text{ran}(f)$.

Let $f, g \in \mathcal{I}_S^{\text{cof}}$ be given. We wish to show

$$|\text{dom}^c(fg)| - |\text{ran}^c(fg)| = |\text{dom}^c(f)| + |\text{dom}^c(g)| - (|\text{ran}^c(f)| + |\text{ran}^c(g)|)$$

Since

$$\begin{aligned} |\text{dom}^c(fg)| &= |\text{dom}^c(f) \cup \text{dom}^c(g)f^{-1}| \\ &= |\text{dom}^c(f)| + |\text{dom}^c(g)| - |\text{ran}^c(f) \cap \text{dom}^c(g)| \end{aligned}$$

and similarly

$$\begin{aligned} |\text{ran}^c(fg)| &= |\text{ran}^c(g) \cup \text{ran}^c(f)g| \\ &= |\text{ran}^c(g)| + |\text{ran}^c(f)| - |\text{dom}^c(g) \cap \text{ran}^c(f)| \end{aligned}$$

we have

$$|\text{dom}^c(fg)| - |\text{ran}^c(fg)|$$

equals

$$|\text{dom}^c(f)| + |\text{dom}^c(g)| - |\text{ran}^c(f) \cap \text{dom}^c(g)|$$

minus

$$|\text{ran}^c(g)| + |\text{ran}^c(f)| - |\text{dom}^c(g) \cap \text{ran}^c(f)|$$

which reduces to

$$(|\text{dom}^c(f)| + |\text{dom}^c(g)|) - (|\text{ran}^c(g)| + |\text{ran}^c(f)|)$$

as desired. □

Definition 4.6.6. We will use $\iota_S : \mathcal{I}_S^{\text{cof}} \rightarrow \mathbb{Z}$ for the homomorphism which sends $f \in \mathcal{I}_S^{\text{cof}}$ to $|S \setminus \text{dom}(f)| - |S \setminus \text{ran}(f)|$.

Lemma 4.6.7. *Assume $I = I\phi_S\phi_S^{-1}$. If a and b are finite subsets of S and f is an element of I such that $f\iota_S = |a| - |b|$ then there exists $t \in I$ such that $\text{dom}(t) = S \setminus a$ and $\text{ran}(t) = S \setminus b$.*

Proof. We will show that there exists t in $f\phi_S\phi_S^{-1}$ with $\text{dom}(t) = S \setminus a$ and $\text{ran}(t) = S \setminus b$ which is enough to prove the lemma since $f\phi_S\phi_S^{-1}$ is a subset of $I\phi_S\phi_S^{-1} = I$. The properties t must have are being a partial bijection from $S \setminus a$ to $S \setminus b$ and agreeing with f in all but finitely many places.

By extending or restricting f construct a partial bijection r with firstly either $f \subseteq r$ or $r \subseteq f$ and secondly both $|S \setminus \text{dom}(r)| = |a|$ and $|S \setminus \text{ran}(r)| = |b|$. Note that r is in $f\phi_S\phi_S^{-1}$. Let g be an element of $\text{Sym}(S)$ with support contained in $a \cup (S \setminus \text{dom}(r))$ and $ag = S \setminus \text{dom}(r)$. Let h be an element of $\text{Sym}(S)$ with support contained in $b \cup (S \setminus \text{ran}(r))$ and $(S \setminus \text{ran}(r))h = b$.

Set $t := grh$. By inspection t has the desired properties. □

Proposition 4.6.8. *If (S, A, I) is of non-area preserving type then $I\iota_S \neq \{0\}$.*

Proof. This follows immediately from Condition 3. □

Lemma 4.6.9. *If (S, A, I) is of non-area preserving type or of area preserving type and X is a clopen subset of A^S then there exists h an idempotent in $\mathcal{D}(S, A, I)$ with $\text{dom}(h) = X = \text{ran}(h)$.*

Proof. Assume without loss of generality that $I = I\phi_S\phi_S^{-1}$. Let K be a transversal of X . For $a \in K$ let f_a be the idempotent in I with $\text{dom}(f_a) = S \setminus \text{dom}(a) = \text{ran}(f_a)$. Now $h = \bigsqcup_{a \in K} (\overline{a, f_a, a})$ is a transversal of X as desired. \square

Lemma 4.6.10. *If (S, A, I) is of non-area preserving type or of area preserving type and $p \in \mathcal{D}(S, A, I)$ with $\text{dom}(p), \text{ran}(p) \notin \{\emptyset, A^S\}$ then there exists p' in $\mathcal{D}(S, A, I)$ with $p \sqcup p' \in \mathcal{V}(S, A, I)$*

Proof. If (S, A, I) is of area preserving type then the lemma follows from Lemma 4.5.9 so assume (S, A, I) is of non-area preserving type. Assume p is as in the hypotheses of the lemma statement. Assume without loss of generality that $I = I\phi_S\phi_S^{-1}$. Let $i := \min(I\iota_S \cap \mathbb{N})$. Let $d, r \in \mathcal{P}_{S,A}^f$ be such that $\text{dom}(p) \cap \{x \in A^S \mid x \supseteq d\} = \emptyset$ and $\text{ran}(p) \cap \{x \in A^S \mid x \supseteq r\} = \emptyset$ and i divides $|d| = |r|$. Let $f_d, f_r \in I$ be such that $\text{dom}(f_d) = S = \text{dom}(f_r)$ and $\text{ran}(f_d) \sqcup d = S$ and $\text{ran}(f_r) \sqcup r = S$. Fix $g_d = (\overline{\emptyset, f_d, d}) \in \mathcal{I}(S, A, I)$ and $g_r = (\overline{\emptyset, f_r, r}) \in \mathcal{I}(S, A, I)$. Using Lemma 4.6.9 let $t_d, t_r \in \mathcal{D}(S, A, I)$ be the idempotents with $\text{dom}(t_d) = A^S \setminus (\text{ran}(g_d) \sqcup \text{dom}(p))$ and $\text{dom}(t_r) = A^S \setminus (\text{ran}(g_r) \sqcup \text{ran}(p))$.

Note $A^S = \text{ran}(p) \sqcup \text{dom}(t_r) \sqcup \text{ran}(g_r)$ and $A^S = \text{dom}(p) \sqcup \text{dom}(t_d) \sqcup \text{ran}(g_d)$.

We wish to show

$$p \sqcup g_d^{-1}t_r \sqcup g_d^{-1}p^{-1}g_r \sqcup t_d g_r \sqcup g_d^{-1}g_r^{-1}g_d g_r \in \mathcal{V}(S, A, I).$$

We will show that the ranges of the above five partial bijections form a partition

of A^S . The argument for the domains is similar.

$$\begin{aligned}
A^S &= \text{ran}(p) \sqcup \text{ran}(t_r) \sqcup \text{ran}(g_r) \\
&= \text{ran}(p) \sqcup \text{ran}(t_r) \sqcup A^S g_r \\
&= \text{ran}(p) \sqcup \text{ran}(t_r) \sqcup (\text{dom}(p) \sqcup \text{ran}(t_d) \sqcup \text{ran}(g_d))g_r \\
&= \text{ran}(p) \sqcup \text{ran}(t_r) \sqcup \text{dom}(p)g_r \sqcup \text{ran}(t_d)g_r \sqcup \text{ran}(g_d)g_r
\end{aligned}$$

As desired. □

Recall we define what it means for a group to be vigorous in Definition 3.1.1.

Theorem 4.6.11. *If (S, A, I) is of non-area preserving type then $\mathcal{V}(S, A, I)$ is vigorous.*

Proof. By Corollary 4.2.17 we may assume that $I = I\phi_S\phi_S^{-1}$. It is sufficient to show that $\mathcal{D}(S, A, I)$ satisfies the conditions of Theorem 3.2.1.

1. From Lemma 4.6.9 it follows that each clopen subset of A^S is the domain of some idempotent of $\mathcal{D}(S, A, I)$.
2. Let f in $\mathcal{P}_S^f A$ be given. We will find an element of $\mathcal{D}(S, A, I)$ with domain equal to A^S and range a subset of the cone $\{x \in A^S \mid x \supseteq f\}$.

By Lemma 4.6.5 the set $I\iota_S$ must be an inverse subsemigroup of the integers. Since (S, A, I) is of non-area preserving type this inverse subsemigroup contains a non-identity element and therefore contains some element n with $n \leq -|f|$.

Let p in I be such that $p\iota_S \leq n$. Let D be a subset of S containing $\text{dom}(f)$ such that $|D| = p\iota_S$. By Lemma 4.6.7 there exists q in I with the domain of q is S and the range of q is $S \setminus D$.

Let t in A^X be an extension of f . The partial bijection $(\overline{\emptyset, q, t})$ has domain A^S and range $\{x \in A^S \mid x \supseteq q\}$ which is a subset of $\{x \in A^S \mid x \supseteq t\}$ as desired.

3. This follows from Lemma 4.6.10 and that $\mathcal{D}(S, A, I)$ is closed under compatible union.

□

Corollary 4.6.12. *If (S, A, I) is of non-area preserving type and $\mathcal{V}(S, A, I)$ is finitely generated and simple then $\mathcal{V}(S, A, I)$ is two generated.*

Proof. This follows immediately from Theorem 3.1.2 and Theorem 4.6.11 □

Remark 4.6.13. If (S, A, I) is of area preserving type then the $\mathcal{V}(S, A, I)$ respects λ and therefore cannot be vigorous.

4.7 Area Preserving Type Examples

Denote by L the class of infinite finitely generated, transitive permutation groups that do not have any infinite commutative homomorphic images. In this section we

will reduce finding examples of triples of area preserving type to finding examples of permutation groups in L and then give some examples of these. For G a group of permutations of a set S we will use I_G for the set

$$\{ig \in \mathcal{I}_S^{\text{cof}} \mid i \text{ an idempotent of } \mathcal{I}_S^{\text{cof}} \text{ and } g \text{ in } G\}$$

which is in fact an inverse submonoid of $\mathcal{I}_S^{\text{cof}}$. The inverse monoid I_G is equal to G^\downarrow in $\mathcal{I}_S^{\text{cof}}$ but different from G^\downarrow in \mathcal{I}_S . Recall that at the beginning of Section 4.6 we listed some conditions used in the definitions of non-area preserving type and area preserving type.

Proposition 4.7.1. *If (S, A, I) is of area preserving type then there exists a group $G \leq \text{Sym}(S)$ such that G is in L and $\mathcal{V}(S, A, I_G) = \mathcal{V}(S, A, I)$.*

Proof. Let X be a generating set for I . Using Condition 4 for each x in X choose g_x a permutation of S such that g_x is an extension of x . Since $g_x\phi_S = x\phi_S$ for each x in X it follows that $\langle \{g_x\}_{x \in X} \rangle \phi_S = X\phi_S$.

Let Y be the set

$$\bigcup_{x \in X} (S \setminus \text{dom}(x)).$$

By Condition 2 each point in S is be mapped to some point in Y by some element of $\langle X \rangle$. Therefore each element of S is be mapped to an element of Y by an element of $\langle \{g_x\}_{x \in X} \rangle$. Since Y is finite the group $\langle \{g_x\}_{x \in X} \rangle$ must have finitely many orbits. Let Z be a finite set intersecting each of the orbits of $\langle \{g_x\}_{x \in X} \rangle$. Let G be the group generated by $\langle \{g_x\}_{x \in X} \rangle$ union $\text{pstab}_S(S \setminus Z)$. Note G is transitive.

The group $\text{pstab}_S(S \setminus Z)\phi_S$ is trivial so $G\phi_S = \langle \{g_x\}_{x \in X} \rangle \phi_S = I\phi_S$ and by Condition 4 the group $G\phi_S$ has no infinite commutative homomorphic images.

Since $\text{pstab}_S(S \setminus Z)$ is finite and $\langle \{g_x\}_{x \in X} \rangle$ is finitely generated G is also finitely generated. By Corollary 4.2.16 the groups $\mathcal{V}(S, A, I_G)$ and $\mathcal{V}(S, A, I)$ are equal.

□

Proposition 4.7.2. *If A is a finite set of size at least 2 and G is a group of permutations of a set S such that G is in L then the triple (S, A, I_G) is of area preserving type.*

Proof. We will go through the conditions required for a triple to be of area preserving type.

The inverse monoid I_G is generated by any generating set for G along with an idempotent whose domain misses only one point of S . Since G is in L it is finitely generated so I_G is also finitely generated.

Since I_G contains all the idempotents of $\mathcal{I}_S^{\text{cof}}$ the intersection $\bigcap_{f \in I_G} \text{dom}(f)$ is empty.

Recall ι_S was defined in Definition 4.6.6. Idempotents and permutations are both mapped to 0 by ι_S which is a homomorphism so $G\iota_S = \{0\}$.

Idempotents are mapped to the identity by ϕ_S so $I_G\phi_S = G\phi_S$. Since G does not have infinite commutative homomorphic images $G\phi_S$ also does not have infinite commutative homomorphic images.

□

By Proposition 4.7.1 and Proposition 4.7.2 to give examples of triples of area preserving type it is enough to give examples of permutation groups in L .

Example 4.7.3. Let G be an infinite finitely generated group without infinite commutative homomorphic images. When G is realised as a group of permutations of itself by the Cayley action from the right the group G is in L .

For a set A we use A^* for the set of finite classical words over A . Recall for finite classical words u and v we use $u \hat{v}$ for the concatenation of u and v . Below we define $\text{QAut}(\mathcal{T}_{2,c})$ which was discussed in [1] and [15].

Definition 4.7.4. For n in \mathbb{N} define $\text{QAut}(\mathcal{T}_{n,c})$ to be the permutation group

$$\{g \in \text{Sym}(\mathbf{n}^*) \mid \text{the set } \{(w, m) \in \mathbf{n}^* \times \mathbf{n} \mid (w \hat{m})g \neq (w)g \hat{m}\} \text{ is finite}\}.$$

Example 4.7.5. The group $\text{QAut}(\mathcal{T}_{2,c})$ with the standard action is shown in [18] to have the properties necessary for membership of L .

4.8 Non-Area Preserving Type Examples

Example 4.8.1. Let (S, A, I) be of non-area preserving type or area preserving type. Also let (T, A, J) be of non-area preserving type or area preserving type. Assume that S and T are disjoint. Let K be the embedding of $I \times J$ in $\mathcal{I}_{S \cup T}^{\text{cof}}$ where if (i, j) is in $I \times J$ and x is in $S \cup T$ then $x(i, j) = xi$ if x is in S and $x(i, j) = xj$ if x is in T . For example $2V$ may be constructed from two copies of V in this way.

If both (S, A, I) and (T, A, J) are of area preserving type then $(S \cup T, A, K)$ is of area preserving type otherwise $(S \cup T, A, K)$ is of non-area preserving type.

Example 4.8.2. Recall the bicyclic monoid $B \leq \mathcal{I}_{\mathbb{N}}^{\text{cof}}$ was defined in Definition 2.2.1. If (S, A, I) is of area preserving type (for example one of the examples of the previous section) then we may use the construction of Example 4.8.1 to construct a triple of non-area preserving type by combining I with B to get an inverse monoid acting on $S \cup \mathbb{N}$.

Example 4.8.3. For G a group let M_G be the set $G \times \mathbb{N}$ and let J_G be the inverse submonoid of $\mathcal{I}_{M_G}^{\text{cof}}$ generated by $\{b\} \cup \{f_g\}_{g \in G}$ where

$$(h, n)b = \begin{cases} (h, n+1) & \text{if } h = 1_G \\ (h, n) & \text{otherwise} \end{cases}$$

and

$$(h, n)f_g = (hg, n).$$

Let G^\vee be $\mathcal{V}(M_G, \{0, 1\}, J_G)$.

If G is a finitely generated group then G^\vee is of non-area preserving type.

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