

GENERALIZED BERNSTEIN POLYNOMIALS AND TOTAL POSITIVITY

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A thesis submitted for the degree of Doctor of Philosophy

School of Mathematical and Computational Sciences

University of St. Andrews



Scotland, U.K.

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*To my affectionate and devoted family Hursit, Şaziye, and Gülaziye Oruç
and
to my soul mate maternal uncle Bahri Kaderoğlu*

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Abstract

This thesis submitted for Ph.D. degree deals mainly with geometric properties of generalized Bernstein polynomials which replace the single Bernstein polynomial $B_n f$ by a one-parameter family of polynomials $B_n^q f$. It also provides a triangular decomposition and 1-banded factorization of the Vandermonde matrix.

We first establish the generalized Bernstein polynomials for monomials, which leads to a definition of Stirling polynomials of the second kind. These are q -analogues of Stirling numbers of the second kind. Some of the properties of the Stirling numbers are generalized to their q -analogues.

We show that the generalized Bernstein polynomials are monotonic in degree n , that is $B_n^q f \geq B_{n+1}^q f$, when the function f is convex. There is also a representation of $B_n^q f - B_{n+1}^q f$ involving second order divided differences of the function f .

Shape preserving properties of the generalized Bernstein polynomials are studied by making use of the concept of total positivity. It is proved that monotonic and convex functions produce monotonic and convex generalized Bernstein polynomials. It is also shown that the generalized Bernstein polynomials are monotonic in the parameter q for the class of convex functions. That is $B_n^q f \geq B_n^r f$, for $0 < q \leq r \leq 1$.

Finally, we look into the degree elevation and degree reduction processes on the generalized Bernstein polynomials.

Declarations

I, Halil Oruç, hereby certify that this thesis, which is approximately 19000 words in length, has been written by me, that it is the record of work carried out by me, and that it has not been submitted in any previous application for a higher degree.

H. Oruç Date ..17.. December.. 1998

I was admitted as a research student in January 1996 and as a candidate for the degree of Doctor of Philosophy in September 1996; the higher study for which this is a record was carried out in the University of St. Andrews between 1996 and 1998.

H. Oruç Date ..17.. December.. 1998

I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate for the degree of Doctor of Philosophy in the University of St. Andrews, and that the candidate is qualified to submit this thesis in application for that degree.

Dr. George M. Phillips Date ..17.. December.. 1998

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Chapter 1

Introduction

1.1 Gaussian polynomials

We begin by explaining the notation to be used. The q -integer $[r]$ is defined by

$$[r] = \begin{cases} (1 - q^r)/(1 - q), & q \neq 1, \\ r, & q = 1. \end{cases} \quad (1.1)$$

Then, in a natural way, we define the q -factorial $[r]!$ by

$$[r]! = \begin{cases} [r][r-1]\dots[1], & r = 1, 2, \dots, \\ 1, & r = 0 \end{cases} \quad (1.2)$$

and the q -binomial coefficient $\begin{bmatrix} n \\ r \end{bmatrix}$ by

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{cases} \frac{[n]!}{[r]![n-r]!}, & n \geq r \geq 0, \\ 0, & \text{otherwise.} \end{cases} \quad (1.3)$$

Some authors use the term Gaussian polynomials or Gaussian coefficients instead of q -integers. They were first studied by Gauss in connection with restricted

partitions in the following sense. Let $p(N, M, n)$ denote the number of partitions of n into at most M parts, each less than or equal to N . Then, the generating function is

$$G(N, M, q) = \left[\begin{matrix} N + M \\ M \end{matrix} \right] = \sum_{n \geq 0} p(N, M, n) q^n.$$

These q -binomial coefficients satisfy the following Pascal type identities

$$\left[\begin{matrix} n \\ r \end{matrix} \right] = q^{n-r} \left[\begin{matrix} n-1 \\ r-1 \end{matrix} \right] + \left[\begin{matrix} n-1 \\ r \end{matrix} \right] \quad (1.4)$$

and

$$\left[\begin{matrix} n \\ r \end{matrix} \right] = \left[\begin{matrix} n-1 \\ r-1 \end{matrix} \right] + q^r \left[\begin{matrix} n-1 \\ r \end{matrix} \right] \quad (1.5)$$

which are readily verified from (1.3). An induction argument using either (1.4) or (1.5) readily shows that $\left[\begin{matrix} n \\ r \end{matrix} \right]$ is a polynomial of degree $r(n-r)$ in q with positive integral coefficients, see Andrews [1]. When $q = 1$, the q -binomial coefficient reduces to the ordinary binomial coefficient. They arise due to their relationship with certain products. The following identity, which appears in the study of hypergeometric functions, is due to Cauchy. If $|q| < 1$, $|t| < 1$, then

$$1 + \sum_{n=1}^{\infty} \frac{(1-a)(1-aq)\dots(1-aq^{n-1})}{(1-q)(1-q^2)\dots(1-q^n)} t^n = \prod_{n=0}^{\infty} \frac{(1-atq^n)}{(1-tq^n)}. \quad (1.6)$$

This identity may be used to give elegant proofs (see Andrews [1]) for the Euler identity

$$(1-x)(1-qx)\dots(1-q^{n-1}x) = \sum_{r=0}^n (-1)^r q^{r(r-1)/2} \left[\begin{matrix} n \\ r \end{matrix} \right] x^r \quad (1.7)$$

and

$$\frac{1}{(1-x)(1-qx)\dots(1-q^{n-1}x)} = \sum_{r=0}^{\infty} \left[\begin{matrix} n+r-1 \\ r \end{matrix} \right] x^r. \quad (1.8)$$

We observe that the latter identities generalize the binomial expansion and binomial series respectively. One may verify them by using induction on n .

When q is a positive integer, the q -binomial coefficient $\left[\begin{matrix} n \\ r \end{matrix} \right]$ may be interpreted as counting the number of r dimensional subspaces of an n dimensional finite

vector space over the field of q elements. The proof can be seen in Andrews [1, pp. 212]. There are many other formulas related to the Gaussian polynomials (see Andrews [1]). The relation between q -binomial coefficients and complete symmetric functions will be mentioned in Section 5.1.

What follows is a generalization of forward differences. For any function f we define

$$\Delta^0 f_i = f_i$$

for $i = 0, 1, \dots, n$ and recursively,

$$\Delta^{k+1} f_i = \Delta^k f_{i+1} - q^k \Delta^k f_i \quad (1.9)$$

for $k = 0, 1, \dots, n - i - 1$, where f_i denotes $f([i]/[n])$. See Schoenberg [49], Lee and Phillips [33]. When $q = 1$, these q -differences reduce to ordinary forward differences and it is easily established by induction on k that

$$\Delta^k f_i = \sum_{r=0}^k (-1)^r q^{r(r-1)/2} \begin{bmatrix} k \\ r \end{bmatrix} f_{i+k-r}. \quad (1.10)$$

Andrews [1, pp. 121] mentions a large class of partition problems wherein the related generating function satisfies a linear homogeneous q -difference equation with polynomial coefficients. It is also emphasized that its theory has not been adequately developed and is indeed worthy of future research.

In Chapter 6, a q -divided difference operator will be discussed in connection with the generalized Bernstein polynomials and the derivatives of Bernstein polynomials.

1.2 Bernstein polynomials

There are several proofs of the Weierstrass approximation theorem, which states that the space \mathbb{P} of polynomials on $[a, b]$ is dense in $C[a, b]$. That is to say, for a continuous function f defined on $[a, b]$, given any $\epsilon > 0$ there corresponds a polynomial P such that $\|f - P\| < \epsilon$. Thus $|f(x) - P(x)| < \epsilon$ for all $x \in [a, b]$. One of the most elegant proofs was given by S.N. Bernstein in 1912. He introduced the following polynomials for a function defined on $C[0, 1]$,

$$B_n(f; x) = \sum_{i=0}^n f\left(\frac{i}{n}\right) \binom{n}{i} x^i (1-x)^{n-i}. \quad (1.11)$$

The book by Lorentz [34] is devoted to results on Bernstein polynomials (before 1951) for functions on $[0, 1]$ and \mathbb{C} , together with applications. Yet there is a more general approach to the theorem of Weierstrass involving a sequence of positive linear operators. An operator \mathcal{L} from $C[a, b]$ to $C[a, b]$ is called monotone if it maps $f \geq g$ into $\mathcal{L}f \geq \mathcal{L}g$. Let $\{\mathcal{L}_i; i = 0, 1, \dots\}$ be a sequence of linear monotone operators from $C[a, b]$ to $C[a, b]$. Then, if the sequence $\{\mathcal{L}_i f; i = 0, 1, \dots\}$ converges uniformly to f for the functions $f = 1, x, x^2$, then the sequence $\{\mathcal{L}_i f; i = 0, 1, \dots\}$ converges uniformly to f for all f in $C[a, b]$. This result is called the Bohman-Korovkin theorem after the works by Bohman [2] (1952) and Korovkin [31] (1957) (see also Cheney [6, pp. 65], DeVore and Lorentz [12, pp. 8]). It is easily verified that the operators B_i , $i = 1, 2, \dots$, are linear monotone operators on $[0, 1]$ which satisfy the conditions of the Bohman-Korovkin theorem. This justifies the uniform convergence of $B_n f$ to f for all f in $C[0, 1]$. There is an asymptotic error estimate for the Bernstein polynomials due to Voronovskaya [52] (see also Davis [9, pp. 117], DeVore and Lorentz [12, pp. 307]). That is, if f is bounded in $[0, 1]$ and $f''(x)$ exists for some $x \in [0, 1]$, then

$$\lim_{n \rightarrow \infty} n(B_n(f; x) - f(x)) = \frac{x(1-x)}{2} f''(x).$$

Variation diminishing properties of the Bernstein polynomials have been introduced by Pólya and Schoenberg [44] and by Schoenberg [48]. These works yielded some shape-preserving properties of the Bernstein polynomials and also of spline functions. For example monotonic and convex functions have monotonic and convex Bernstein polynomials respectively. Schoenberg [48] showed that if f is convex then the Bernstein polynomials are monotonic in the sense that $B_n f \geq B_{n+1} f \geq f$. The converse of the latter result is due to Kosmak [32]. That is, if $B_n f \geq B_{n+1} f$ for all $n \in \mathbb{N}$, then f is convex.

Studies on Bernstein polynomials are quite widespread including work on linear operators, variation diminishing properties, convexity, the rate of convergence and Lipschitz constants as well as multivariate Bernstein polynomials. However, their usage in geometric design, because of their simplicity, have created a new discipline in its own right.

In Computer Aided Design (CAD) and Computer Aided Geometric Design (CAGD) systems, in order to machine a shape using a computer, it is necessary to generate a computer compatible description of that shape. A collection of parametric curves and surfaces become essential tools for this purpose. A major breakthrough was the theory of Bézier curves and surfaces. Nowadays these are combined with B-splines. The theory was developed independently by P. de Casteljau at Citröen and P. Bézier at Renault. De Casteljau's development, slightly earlier (1959) than Bézier's, was never published, and so the whole theory of polynomial curves and surfaces in Bernstein form now bear Bézier's name. W. Boehm found copies of P. de Casteljau's technical reports in 1975. There he uncovered the most fundamental algorithm, which is given later in this chapter for the particular case of $q = 1$, in the field of the design of curves and surfaces. It recursively produces a parametric curve which supplies lots of information about

the shape of the curve with respect to given points. It has the form

$$p(t) = \sum_{i=0}^n a_i \binom{n}{i} t^i (1-t)^{n-i}, \quad t \in [0, 1], \quad a_i \in \mathbb{R}^2 \text{ or } \mathbb{R}^3 \quad (1.12)$$

and is known as a Bézier representation. The coefficients a_i give information about the shape of the curve p . The Bernstein-Bézier curve and surface forms have gained considerable popularity in CAGD applications in the last decade. Now there are many studies involving total positivity and the shape of curves and the Bernstein-Bézier form.

Blossoms or polar forms, introduced by Paul de Casteljau (see [11]), provides a new approach for studying parametric polynomial curves in the Bézier form. Its rigorous mathematical formulation is due to Ramshaw [45]. The basic idea is the concept of a symmetric multiaffine mapping. A mapping is called affine provided it satisfies

$$f\left(\sum_{i=1}^n a_i t_i\right) = \sum_{i=1}^n a_i f(t_i), \quad \forall a_i, t_i \in \mathbb{R} \text{ with } \sum a_i = 1.$$

A mapping is multiaffine if it is affine with respect to each variable and symmetric provided that it has the same value for all permutations of its variables. For each polynomial function $F : \mathbb{R} \rightarrow \mathbb{R}$ of degree less than or equal to n , there exists a unique function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, called the blossom of F , such that f is n -affine and symmetric, and

$$f(\underbrace{t, \dots, t}_{n \text{ fold}}) = F(t), \quad t \in \mathbb{R}.$$

Phillips [42] proposed the following generalization of the Bernstein polynomials, based on the q -integers. For each positive integer n , we define

$$B_n(f; x) = \sum_{r=0}^n f_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1 - q^s x), \quad (1.13)$$

where an empty product denotes 1 and $f_r = f([r]/[n])$. The notation requires some explanation. The function f is evaluated at ratios of the q -integers $[r]$ and

$[n]$, where q is a positive real number. Note that

$$B_n(f; 0) = f(0) \text{ and } B_n(f; 1) = f(1),$$

as for the classical Bernstein polynomial. This property is called end point interpolation.

As shown in Phillips [42], we may express the generalized Bernstein polynomial defined by (1.13) in terms of q -differences, in the form

$$B_n(f; x) = \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} \Delta^r f_0 x^r. \quad (1.14)$$

This generalizes the well known result (see, for example, Davis [9], DeVore and Lorentz [12], Farin [14], Hoschek and Lasser [24]) for the classical Bernstein polynomial. Thus it follows from (1.14) that, if f is a polynomial of degree m , then $B_n(f; x)$ is a polynomial of degree $\min(m, n)$. In particular, we need to evaluate $B_n(f; x)$ for $f = 1, x, x^2$ in order to justify applying the Bohman-Korovkin theorem on the uniform convergence of monotone operators. First we see from (1.14) that

$$B_n(1; x) = 1$$

and from (1.13) we may deduce the identity

$$\sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1 - q^s x) = 1. \quad (1.15)$$

We say that the generalized Bernstein basis forms a partition of unity. With $f(x) = x$ we have $f_0 = 0$ and

$$\Delta f_0 = f_1 - f_0 = \frac{1}{[n]}.$$

Thus we obtain from (1.14) that

$$B_n(x; x) = x.$$

From (1.13) this implies

$$\sum_{r=0}^n \frac{[r]}{[n]} \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1 - q^s x) = x. \quad (1.16)$$

This identity is useful to obtain the functional form of generalized Bernstein polynomials from its parametric form. It is also called the linear precision property.

We deduce that, for any real numbers a and b , generalized Bernstein polynomials reproduce linear functions, that is

$$B_n(ax + b; x) = ax + b. \quad (1.17)$$

We remark that the relations $B_n(1; x) = 1$ and $B_n(x; x) = x$ are significant in CAGD.

Finally, for $f(x) = x^2$ we compute $f_0 = 0$ and, $\Delta f_0 = 1/[n]^2$. Using (1.10), we have

$$\Delta^2 f_0 = \left(\frac{[2]}{[n]} \right)^2 - (1+q) \left(\frac{[1]}{[n]} \right)^2 = \frac{q(1+q)}{[n]^2}.$$

Thus, from (1.14)

$$\begin{aligned} B_n(x^2; x) &= \frac{1}{[n]} x + \frac{[n][n-1]}{(1+q)} \frac{q(1+q)}{[n]^2} x^2 \\ &= x^2 + \frac{x(1-x)}{[n]}. \end{aligned} \quad (1.18)$$

In Phillips [42], a q -integer $[r] = 1+q_n+\cdots+q_n^{r-1}$ is chosen to be dependent on the degree of the generalized Bernstein polynomials n . Then, taking a sequence $q = q_n$ such that $[n] \rightarrow \infty$ as $n \rightarrow \infty$, it follows that $B_n(x^2; x) \rightarrow x^2$. For example, we could take q_n such that $1 - \frac{1}{n} \leq q_n < 1$. Thus, by using the Bohman-Korovkin theorem, the generalized Bernstein polynomials $B_n f$ converges to f for all $f \in C[0, 1]$.

In Phillips [42] there is also a discussion on a Voronovskaya type theorem for the rate of convergence. In particular, when $q \rightarrow 1$ from below and $f''(x)$ exists

at $x \in [0, 1]$,

$$\lim_{n \rightarrow \infty} [n](B_n(f; x) - f(x)) = \frac{x(1-x)}{2} f''(x).$$

In Phillips [41], the convergence of derivatives of the generalized Bernstein polynomials is discussed. The following result is proved.

Let $f \in C^1[0, 1]$ and let the sequence q_n be chosen so that the sequence (ϵ_n) converges to zero from above faster than $(1/3^n)$, where

$$\epsilon_n = \frac{n}{1 + q_n + q_n^2 + \cdots + q_n^{n-1}} - 1.$$

Then the sequence of derivatives of the generalized Bernstein polynomials $B_n(f; x)$ converges uniformly on $[0, 1]$ to $f'(x)$.

The following de Casteljau type algorithm (see Phillips [40]) may be used for evaluating generalized Bernstein polynomials iteratively.

Generalized de Casteljau algorithm

```

for  $r = 0$  to  $n$ 
   $f_r^{[0]} := f([r]/[n])$ 
next  $r$ 
for  $m = 1$  to  $n$ 
  for  $r = 0$  to  $n - m$ 
     $f_r^{[m]} := (q^r - q^{m-1}x)f_r^{[m-1]} + xf_{r+1}^{[m-1]}$ 
  next  $r$ 
next  $m$ 
```

It is shown in Phillips [40] that, for $0 \leq m \leq n$ and $0 \leq r \leq n - m$, the iterate $f_r^{[m]}$ satisfies

$$f_r^{[m]} = \sum_{t=0}^m f_{r+t} \begin{bmatrix} m \\ t \end{bmatrix} \prod_{s=0}^{m-t-1} (q^r - q^s x)$$

and has the q -difference form

$$f_r^{[m]} = \sum_{s=0}^m q^{(m-s)r} \begin{bmatrix} m \\ s \end{bmatrix} \Delta^s f_r x^s.$$

Thus with $r = 0$ and $m = n$, we have $f_0^{[n]} = B_n(f; x)$. This generalizes the well known de Casteljau algorithm (see Farin [14], Hoschek and Lasser [24]) for evaluating the classical Bernstein polynomials. We note that the algorithm described above with $q = 1$ is probably the most fundamental one in the field of curve and surface design (see Farin [14], Hoschek and Lasser [24]).

Popoviciu [51] established an error estimate for the classical Bernstein polynomials using the modulus of continuity,

$$|B_n(f; x) - f(x)| \leq \frac{3}{2}\omega\left(\frac{1}{\sqrt{n}}\right).$$

The modulus of continuity is defined as follows.

Definition 1.1 The modulus of continuity $\omega(f, \delta) = \omega(\delta)$ of a function f on $[a, b]$ is defined by

$$\omega(\delta) = \sup_{\substack{|x-y| \leq \delta \\ x, y \in [a, b]}} |f(x) - f(y)|, \quad \delta \geq 0.$$

The modulus of continuity has the following properties:

- (i) if $0 < \delta_1 \leq \delta_2$, then $\omega(\delta_1) \leq \omega(\delta_2)$,
- (ii) $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$,
- (iii) if $\lambda > 0$, then $\omega(\lambda\delta) \leq (1 + \lambda)\omega(\delta)$,
- (iv) f is uniformly continuous on $[a, b]$ if and only if $\lim_{\delta \rightarrow 0} \omega(\delta) = 0$.

The proofs can be found in DeVore and Lorentz [12] or Rivlin [46].

Chapter 2

Stirling polynomials

Stirling polynomials, which are q -analogues of Stirling numbers, have been studied extensively. First we will find $B_n f$ when f is a monomial, which leads to a definition of Stirling polynomials of the second kind. Then we will look into some properties of Stirling polynomials, using combinatorial ideas and the concept of total positivity. Although some of the their properties are well known (see for example Carlitz [3], Gould [23], Médics and Leroux [36]) some of them, such as the equation (2.2), the identities (2.10) and (2.12), and the Theorem 2.3 on total positivity appear to be new. In the last section we introduce Stirling polynomials of the first kind as an inverse process for generating second kind Stirling polynomials.

2.1 Monomials and Stirling polynomials

We first express $[n] - [j] = q^j[n - j]$, for $0 \leq j \leq n$ and then express q -binomial coefficients as

$$\begin{bmatrix} n \\ j \end{bmatrix} = \frac{[n]^j}{[j]! q^{j(j-1)/2}} \pi_j^n, \quad 0 \leq j \leq n, \quad (2.1)$$

where

$$\pi_j^n = \prod_{r=0}^{j-1} \left(1 - \frac{[r]}{[n]}\right)$$

and an empty product denotes 1. It follows from (1.14) that $B_n(x^i; x)$ is a polynomial of degree less or equal to $\min(i, n)$. On using (1.10) with $f(x) = x^i$, we see that

$$\Delta^j f_0 = \frac{1}{[n]^i} \sum_{r=0}^j (-1)^r q^{r(r-1)/2} \begin{bmatrix} j \\ r \end{bmatrix} [j - r]^i.$$

On substituting the last expression in (1.14) and (2.1), we obtain

$$B_n(x^i; x) = \sum_{j=0}^i \pi_j^n [n]^{j-i} S_q(i, j) x^j, \quad (2.2)$$

where

$$S_q(i, j) = \frac{1}{[j]! q^{j(j-1)/2}} \sum_{r=0}^j (-1)^r q^{r(r-1)/2} \begin{bmatrix} j \\ r \end{bmatrix} [j - r]^i. \quad (2.3)$$

One may verify by induction on i using (2.3) that

$$S_q(i+1, j) = S_q(i, j-1) + [j] S_q(i, j), \quad (2.4)$$

for $i \geq 0$ and $j \geq 1$, with $S_q(0, 0) = 1$, $S_q(i, 0) = 0$ for $i > 0$ and we define $S_q(i, j) = 0$ for $j > i$. We call $S_q(i, j)$ the Stirling polynomials of the second kind since when $q = 1$ they are the Stirling numbers of the second kind. The recurrence relation (2.4) shows that, for $q > 0$, the Stirling polynomials are polynomials in q with non-negative integer coefficients and so are positive monotonic increasing functions of q . Thus $B_n(x^i; x)$ and all its derivatives are non-negative on $[0, 1]$. In particular, $B_n(x^i; x)$ is convex. This result is given in Goodman *et al.* [19].

We will verify (2.4) in another way. Tomescu [50] takes the generating function for Stirling numbers of the second kind, $S(n, m)$, as

$$x^n = \sum_{m=0}^n S(n, m)(x)_m,$$

where $(x)_0 = 1$ and $(x)_m = x(x - 1)\dots(x - m + 1)$. We replace $(x)_m$ by $X_m(x)$ and write

$$X_m(x) = x(x - [1])(x - [2])\dots(x - [m - 1]). \quad (2.5)$$

Then we can show that

$$x^n = \sum_{m=0}^n S_q(n, m)X_m(x). \quad (2.6)$$

We now multiply (2.6) by x and, on the right, put $x = (x - [m] + [m])$. On comparing coefficients of $X_m(x)$, we indeed obtain the recurrence relation

$$S_q(n + 1, m) = S_q(n, m - 1) + [m]S_q(n, m), \quad (2.7)$$

in agreement with (2.4). The values of $S_q(n, m)$ for $1 \leq m, n \leq 4$ are shown in the table below.

TABLE 1. Stirling polynomials of the second kind

$S_q(n, m)$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	1	0	0	0
$n = 2$	1	1	0	0
$n = 3$	1	$q + 2$	1	0
$n = 4$	1	$q^2 + 3q + 3$	$q^2 + 2q + 3$	1

When $q = 1$, $S_q(n, m) = S(n, m)$. An induction argument based on the recurrence relation (2.7) shows that, for $1 \leq m \leq n$, $S_q(n, m)$ is a polynomial in q of degree $(m - 1)(n - m)$.

The expressions (2.3) and (2.4) are first introduced by Carlitz [3] in connection with q -Bernoulli numbers. Then Gould [23] has defined q -Stirling numbers of the second kind, denoted by $S_2(n, m)$, in a combinatorial way as the sum of $\binom{n+m-1}{m}$ possible products of at most m factors chosen from the set $\{[1], [2], \dots, [n]\}$, where repeated factors are allowed. We remark that $S_2(n, m)$ is not the same as $S_q(n, m)$ with $q = 2$, but this should not cause confusion. The recurrence relation for $S_2(n, m)$ is

$$S_2(n, m) = S_2(n - 1, m) + [n]S_2(n, m - 1)$$

and it follows from this and (2.7) that

$$S_2(m, n - m) = S_q(n, m).$$

We observe that (2.6) is simply the Newton divided difference form for $f(x) = x^n$, interpolating at the points $[0], [1], \dots, [n]$. Thus

$$S_q(n, m) = f[[0], [1], \dots, [m]].$$

Koçak and Phillips [29] expressed the divided difference of a function f on consecutive q -integers as a multiple of a q -difference, namely

$$f[[i], [i+1], \dots, [i+n]] = q^{-\frac{1}{2}n(n+2i-1)} \frac{\Delta^n f_i}{[n]!}. \quad (2.8)$$

On applying (2.8) we obtain the simple form

$$S_q(n, m) = \frac{q^{-m(m-1)/2}}{[m]!} \Delta^m f_0, \quad (2.9)$$

where $f(x) = x^n$.

The next result explains how to derive (2.3) from an infinite series.

Theorem 2.1 *For $|x| > [m]$ the following identity holds.*

$$\sum_{n=m}^{\infty} \frac{S_q(n, m)}{x^{n+1}} = \prod_{r=0}^m \left(\frac{1}{x - [r]} \right). \quad (2.10)$$

Proof We first recall the Lagrange interpolation formula to find a partial fraction representation of $\prod_{r=0}^m \left(\frac{1}{x-[r]} \right)$,

$$p_m(x) = \sum_{r=0}^m \mathcal{L}_r(x) f(x_r),$$

where

$$\mathcal{L}_r(x) = \prod_{\substack{j=0 \\ j \neq r}}^m \left(\frac{x - x_j}{x_r - x_j} \right).$$

Let us take the function $f(x) = 1$. This implies that

$$\sum_{r=0}^m \mathcal{L}_r(x) = 1.$$

On dividing this by $\prod_{r=0}^m (x - x_r)$ we have

$$\prod_{r=0}^m \left(\frac{1}{x - x_r} \right) = \sum_{r=0}^m \frac{1}{(x - x_r) \prod_{\substack{j=0 \\ j \neq r}}^m (x_r - x_j)}.$$

Substituting $x_r = [r]$ yields

$$\prod_{r=0}^m \left(\frac{1}{x - [r]} \right) = \sum_{r=0}^m \frac{1}{(x - [r]) \prod_{\substack{j=0 \\ j \neq r}}^m ([r] - [j])}.$$

On writing

$$[r] - [j] = \begin{cases} q^j[r-j], & j \leq r, \\ -q^r[j-r], & j > r, \end{cases}$$

we obtain

$$\prod_{r=0}^m \left(\frac{1}{x - [r]} \right) = \sum_{r=0}^m (-1)^{m-r} \frac{1}{q^{r(2m-r-1)/2} [r]! [m-r]! (x - [r])}. \quad (2.11)$$

Now we expand each $\frac{1}{x-[r]}$ on the right by an infinite sum

$$\frac{1}{x - [r]} = \sum_{t=0}^{\infty} \frac{[r]^t}{x^{t+1}},$$

giving on the right of (2.11),

$$\frac{1}{[m]! q^{m(m-1)/2}} \sum_{r=0}^m (-1)^{m-r} q^{(m-r)(m-r-1)/2} \begin{bmatrix} m \\ r \end{bmatrix} \sum_{t=0}^{\infty} \frac{[r]^t}{x^{t+1}},$$

since

$$\frac{1}{[r]! [m-r]!} = \frac{1}{[m]!} \begin{bmatrix} m \\ r \end{bmatrix}.$$

We now consider the coefficient of $\frac{1}{x^{n+1}}$ in the above infinite expansion,

$$\frac{1}{[m]! q^{m(m-1)/2}} \sum_{r=0}^m (-1)^{m-r} q^{(m-r)(m-r-1)/2} \begin{bmatrix} m \\ r \end{bmatrix} [r]^n$$

and invert the order of summation, replacing r by $m-r$, giving

$$\frac{1}{[m]! q^{m(m-1)/2}} \sum_{r=0}^m (-1)^r q^{r(r-1)/2} \begin{bmatrix} m \\ r \end{bmatrix} [m-r]^n = S_q(n, m).$$

By (2.11) this implies that

$$\sum_{n=m}^{\infty} \frac{S_q(n, m)}{x^{n+1}} = \prod_{r=0}^m \left(\frac{1}{x - [r]} \right)$$

since $S_q(n, m) = 0$ when $n < m$. ■

On considering the recurrence relation (2.7), it is natural to seek to express $S_q(n+1, m)$ in terms of $S_q(r, m-1)$ for $r = m-1, \dots, n$.

Theorem 2.2 *For $n \geq 1$ and $1 \leq m \leq n+1$ we have the identity*

$$S_q(n+1, m) = \sum_{r=m-1}^n q^{r-m+1} \binom{n}{r} S_q(r, m-1). \quad (2.12)$$

Proof We use induction on n . For $n = 1, m = 1$, (2.12) gives $S_q(2, 1) = S_q(0, 0) = 1$ and for $n = 1, m = 2$, (2.12) gives $S_q(2, 2) = S_q(1, 1) = 1$. We assume

$$S_q(n, m) = \sum_{r=m-1}^{n-1} q^{r-m+1} \binom{n-1}{r} S_q(r, m-1)$$

holds for some $n \geq 2$ and all $m \leq n$. On using (2.7) we have

$$\begin{aligned} S_q(n+1, m) &= \sum_{r=m-2}^{n-1} q^{r-m+2} \binom{n-1}{r} S_q(r, m-2) \\ &\quad + [m] \sum_{r=m-1}^{n-1} q^{r-m+1} \binom{n-1}{r} S_q(r, m-1). \end{aligned}$$

Expressing $[m] = 1 + q[m - 1]$, we split the second sum above to give

$$\begin{aligned} S_q(n+1, m) &= \sum_{r=m-2}^{n-1} q^{r-m+2} \binom{n-1}{r} S_q(r, m-2) \\ &\quad + \sum_{r=m-1}^{n-1} q^{r-m+1} \binom{n-1}{r} S_q(r, m-1) \\ &\quad + [m-1] \sum_{r=m-1}^{n-1} q^{r-m+1} \binom{n-1}{r} S_q(r, m-1). \end{aligned}$$

We may write this as

$$\begin{aligned} S_q(n+1, m) &= \binom{n-1}{m-2} + \sum_{r=m-1}^{n-1} q^{r-m+2} \binom{n-1}{r} S_q(r, m-2) \\ &\quad + \binom{n-1}{m-1} + \sum_{r=m}^{n-1} q^{r-m+1} \binom{n-1}{r} S_q(r, m-1) \\ &\quad + [m-1] \sum_{r=m-1}^{n-1} q^{r-m+2} \binom{n-1}{r} S_q(r, m-1). \end{aligned}$$

It follows from (2.7) that

$$\begin{aligned} \sum_{r=m-1}^{n-1} q^{r-m+2} \binom{n-1}{r} (S_q(r, m-2) + [m-1] S_q(r, m-1)) \\ = \sum_{r=m-1}^{n-1} q^{r-m+2} \binom{n-1}{r} S_q(r+1, m-1). \end{aligned}$$

Then, on writing

$$\begin{aligned} \sum_{r=m-1}^{n-1} q^{r-m+2} \binom{n-1}{r} S_q(r+1, m-1) \\ = \sum_{r=m}^{n-1} q^{r-m+1} \binom{n-1}{r-1} S_q(r, m-1) + q^{r-m+1} S_q(n, m-1), \end{aligned}$$

we obtain

$$\begin{aligned} S_q(n+1, m) &= \sum_{r=m}^{n-1} q^{r-m+1} \left(\binom{n-1}{r} + \binom{n-1}{r-1} \right) S_q(r, m-1) \\ &\quad + \binom{n}{m-1} S_q(m-1, m-1) + q^{r-m+1} S_q(n, m-1). \end{aligned}$$

This gives

$$S_q(n+1, m) = \sum_{r=m-1}^n q^{r-m+1} \binom{n}{r} S_q(r, m-1),$$

completing the proof by induction. ■

We note that the last theorem generalizes the well known result on Stirling numbers of the second kind

$$S(n+1, m) = \sum_{r=m-1}^n \binom{n}{r} S(r, m-1).$$

In Section 4.2 we state that the power basis $\phi_i(x) = x^i$, $i = 0, 1, \dots, n$ is a totally positive basis on the interval $[0, \infty)$. We next show that there is a totally positive matrix which transforms the power basis into the basis $X_m(x)$ defined in (2.5).

Theorem 2.3 Let $\mathbf{S}^{(n)}$ be the matrix whose (i, j) entry is $S_q(i, j)$, $1 \leq i, j \leq n$. Then, for $n \geq 2$, $\mathbf{S}^{(n)}$ is given by the product of $n - 1$ matrices of order $n \times n$

$$\left[\begin{array}{c} 1 \\ \ddots \\ 1 & 1 \end{array} \right] \left[\begin{array}{c} 1 \\ \ddots \\ [2] & 1 \end{array} \right] \dots \left[\begin{array}{c} 1 \\ 1 & 1 \\ [2] \\ \ddots & \ddots \\ [n-1] & 1 \end{array} \right]$$

and is totally positive for $q \in (-1, \infty)$.

Proof We use induction on n . The result holds for $n = 2$, since $S^{(2)} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Let us denote the above product by $\mathbf{T}^{(n)}$ and assume that $\mathbf{S}^{(n)} = \mathbf{T}^{(n)}$ for some $n \geq 2$. We may express $\mathbf{T}^{(n+1)}$ as a product in block form

$$\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{B}^{n,1} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{B}^{n,2} \end{bmatrix} \cdots \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{B}^{n,n-1} \end{bmatrix} \mathbf{B}^{n+1,n},$$

where $\mathbf{0}$ is the zero (column) vector and

$$\mathbf{B}^{n,k} = (B_{i,j}^{n,k})_{i=1}^n{}_{j=1}^n = \begin{cases} 1, & i = j, \\ [k - n + i], & i \geq n - k + 1, i = j + 1, \\ 0, & \text{otherwise,} \end{cases}$$

for $k = 1, \dots, n - 1$.

We observe that $\mathbf{B}^{n,1}\mathbf{B}^{n,2}\dots\mathbf{B}^{n,n-1} = \mathbf{S}^{(n)}$ and also $T_{1,1}^{(n+1)} = 1 = S_{1,1}^{(n+1)}$.

Thus it remains only to verify the last rows of $\mathbf{S}^{(n+1)}$ and $\mathbf{T}^{(n+1)}$. We have

$$\begin{aligned} [T_{n+1,1}^{(n+1)}, \dots, T_{n+1,n+1}^{(n+1)}] &= [0, S_q(n, 1), S_q(n, 2), \dots, S_q(n, n)] \mathbf{B}^{n+1,n} \\ &= [1, S_q(n, 1) + [2]S_q(n, 2), \dots, S_q(n, n-1) + [n]S_q(n, n), 1]. \end{aligned}$$

Combining the terms in the last row using (2.4) we see that

$$[T_{n+1,1}^{(n+1)}, \dots, T_{n+1,n+1}^{(n+1)}] = [S_{n+1,1}^{(n+1)}, \dots, S_{n+1,n+1}^{(n+1)}].$$

A q -integer $[r] = 1 + q + \dots + q^{r-1}$ is non-negative in $q \in (-1, \infty)$. Thus all entries in the above product of 1-banded matrices are non-negative in this interval. As we will see from Theorem 4.1, $\mathbf{S}^{(n)}$ is totally positive in the interval $q \in (-1, \infty)$. ■

2.2 Stirling polynomials of the first kind

In this section we derive a generalization of $s(n, m)$, the Stirling numbers of the first kind. In (2.6) the monomial x^n is expressed in terms of the functions $X_m(x)$.

We now invert this process and write

$$X_n(x) = s_q(n, 0) + s_q(n, 1)x + s_q(n, 2)x^2 + \dots + s_q(n, n)x^n, \quad (2.13)$$

where the coefficients $s_q(n, m)$ are to be determined. On comparing the coefficients of x^m in the equation

$$X_{n+1}(x) = (x - [n])X_n(x),$$

we obtain the recurrence relation

$$s_q(n+1, m) = s_q(n, m-1) - [n]s_q(n, m), \quad (2.14)$$

for $n \geq 0$ and $m \geq 1$, with $s_q(0, 0) = 1$, $s_q(n, 0) = 0$ for $n > 0$ and $s_q(n, m) = 0$ for $m > n$. When $q = 1$, $s_q(n, m) = s(n, m)$. An induction argument based on the recurrence relation (2.14) shows that, for $1 \leq m \leq n$, $s_q(n, m)$ is a polynomial in q of degree $\frac{1}{2}(n-1)(n-2) - \frac{1}{2}(m-1)(m-2)$. We will call $s_q(n, m)$ a Stirling polynomial of the first kind. The values of $s_q(n, m)$ for $1 \leq m, n \leq 4$ are shown in the table below.

TABLE 2. Stirling polynomials of the first kind

$s_q(n, m)$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 1$	1	0	0	0
$n = 2$	-1	1	0	0
$n = 3$	$q + 1$	$-(q + 2)$	1	0
$n = 4$	$-(q^3 + 2q^2 + 2q + 1)$	$q^3 + 3q^2 + 4q + 3$	$-(q^2 + 2q + 3)$	1

Gould [23] has defined q -Stirling numbers of the first kind, denoted by $S_1(n, m)$, as the sum of $\binom{n}{m}$ possible products of m distinct factors chosen from the set $\{[1], [2], \dots, [n]\}$. Then he finds a corresponding recurrence relation

$$S_1(n, m) = S_1(n-1, m) + [n]S_1(n-1, m-1). \quad (2.15)$$

Thus from (2.15) and (2.14), we see that

$$s_q(n, m) = (-1)^{n-m}S_1(n-1, n-m).$$

We note that for $n \geq 1$, the first term on the right of (2.6) is zero, and the same holds for the first term on the right of (2.13). On substituting (2.13) into (2.6) we find that

$$x^n = \sum_{m=1}^n \sum_{j=1}^m S_q(n, m) s_q(m, j) x^j$$

and, on rearranging the order of the double summation, we obtain

$$x^n = \sum_{j=1}^n \sum_{m=j}^n S_q(n, m) s_q(m, j) x^j.$$

Thus, for $1 \leq j < n$,

$$\sum_{m=j}^n S_q(n, m) s_q(m, j) = 0. \quad (2.16)$$

Secondly, we can invert the order of these operations and substitute (2.6) into (2.13) to give

$$X_n(x) = \sum_{j=1}^n \sum_{m=1}^j s_q(n, j) S_q(j, m) X_m(x).$$

On rearranging the order of the double summation, we obtain

$$X_n(x) = \sum_{m=1}^n \sum_{j=m}^n s_q(n, j) S_q(j, m) X_m(x).$$

Since the functions $X_m(x)$, $1 \leq m \leq n$, are linearly independent, we have

$$\sum_{j=m}^n s_q(n, j) S_q(j, m) = 0, \quad (2.17)$$

for $1 \leq m < n$. We also note that we can extend the summations in both (2.16) and (2.17) from 1 to n . On putting $q = 1$ in (2.16) and (2.17) we recover well-known results connecting the Stirling numbers of the first and second kind. These “inner product” formulas can be viewed as a result in matrix algebra. Let **A** and **B** denote the $N \times N$ matrices whose (i, j) elements are respectively $s_q(i, j)$ and $S_q(i, j)$. Thus **A** and **B** are both lower triangular matrices with units on their main diagonals. We observe that all of the elements on the main diagonal of **AB** are units. It follows from this, (2.16) and (2.17) that **AB** = **BA** = **I**. Thus, for $1 \leq i, j \leq N$,

$$\sum_{k=1}^N s_q(i, k) S_q(k, j) = \sum_{k=1}^N S_q(i, k) s_q(k, j) = \delta_{i,j}, \quad (2.18)$$

where $\delta_{i,j}$ is the Kronecker delta function,

$$\delta_{i,j} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

Chapter 3

Convex functions and q -differences

We first give a simple observation concerning the q -differences of convex functions. Then the positivity of derivatives of the generalized Bernstein polynomials is discussed. In Section 3.2, we extend the result of Schoenberg concerning monotonicity of the classical Bernstein polynomials of convex functions. These results are given in a more concise form in Oruç and Phillips [39]. There follows a representation of $B_{n-1}f - B_n f$ in terms of second order divided differences of the function f . Analogous results concerning modulus of continuity and Bernstein polynomials are obtained in Section 3.3.

3.1 Non-negative differences

It is shown in Davis [9] that if the k th ordinary differences of f are non-negative then the k th derivative of the classical Bernstein polynomial $B_n(f; x)$ is non-

negative on $[0,1]$. We will discuss extensions of these results to the generalized Bernstein polynomials. We begin by recalling the following definition.

Definition 3.1 A function f is said to be convex on $[0, 1]$ if, for any t_0, t_1 such that $0 \leq t_0 < t_1 \leq 1$ and any $\lambda, 0 < \lambda < 1$,

$$f(\lambda t_0 + (1 - \lambda)t_1) \leq \lambda f(t_0) + (1 - \lambda)f(t_1). \quad (3.1)$$

Geometrically, this definition states that no chord of f lies below the graph of f . With $\lambda = q/(1+q)$, $t_0 = [m]/[n]$ and $t_1 = [m+2]/[n]$ in (3.1), where $0 < q \leq 1$, we see that, if f is convex,

$$f_{m+1} \leq \frac{q}{1+q}f_m + \frac{1}{1+q}f_{m+2}$$

from which we deduce that

$$f_{m+2} - (1+q)f_{m+1} + qf_m = \Delta^2 f_m \geq 0.$$

Thus the second q -differences of a convex function are non-negative, generalizing the well known result for ordinary differences (where $q = 1$).

For a fixed natural number k we now construct a set of piecewise polynomials whose k th q -differences take the value 1 at a given knot, say $([m]/[n])$, and the value 0 at all the other knots. Let $g^{k,m}$ denote such a function. For $k = 1$, we have a piecewise constant function such that

$$g^{1,m}(x) = \begin{cases} 0, & 0 \leq x \leq [m]/[n], \\ 1, & [m]/[n] < x \leq 1, \end{cases}$$

for $0 \leq m \leq n - 1$. Similarly for $k = 2$, we require

$$\Delta^2 g_j^{2,m} = \Delta^2 g^{2,m}([j]/[n]) = \begin{cases} 1, & j = m, \\ 0, & j \neq m, 0 \leq j \leq n - 2. \end{cases}$$

Since $\Delta^2 g_j^{2,m} = g_{j+2}^{2,m} - (1+q)g_{j+1}^{2,m} + qg_j^{2,m} = 1$, we can take $g_j^{2,m} = 0$, for $j = 0, \dots, m+1$ and

$$g_j^{2,m} = [j-m-1] \text{ for } j = m+2, \dots, n.$$

Thus we can take $g^{2,m}$ to be the first degree spline defined by

$$g^{2,m}(x) = \begin{cases} 0, & 0 \leq x \leq [m+1]/[n], \\ q^{-(m+1)}([n]x - [m+1]), & [m+1]/[n] < x \leq 1. \end{cases}$$

For a general value of k and $0 \leq m \leq n-k$ define

$$g^{k,m}(x) = \begin{cases} 0, & 0 \leq x \leq [m+k-1]/[n], \\ \gamma^{k,m}(x), & [m+k-1]/[n] < x \leq 1, \end{cases} \quad (3.2)$$

where

$$\gamma^{k,m}(x) = \prod_{r=m+1}^{m+k-1} \left(\frac{[n]x - [r]}{[2r-m] - [r]} \right). \quad (3.3)$$

The values of these piecewise polynomials at the knots are given by

$$g_j^{k,m} = g^{k,m}([j]/[n]) = \begin{bmatrix} j-m-1 \\ k-1 \end{bmatrix}. \quad (3.4)$$

Note that $g_j^{k,m}$ is zero for $0 \leq j \leq m+k-1$.

We now wish to evaluate $\Delta^k g_j^{k,m}$, where Δ operates on the suffix j . In the view of (3.4), we are thus concerned with the effect of applying Δ to the upper parameter of the q -binomial coefficients. We have

$$\Delta^0 \begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n \\ r \end{bmatrix},$$

and

$$\Delta^k \begin{bmatrix} n \\ r \end{bmatrix} = \Delta^{k-1} \begin{bmatrix} n+1 \\ r \end{bmatrix} - q^{k-1} \Delta^{k-1} \begin{bmatrix} n \\ r \end{bmatrix}. \quad (3.5)$$

We readily see that, for $0 \leq k \leq r$,

$$\Delta^k \begin{bmatrix} n \\ r \end{bmatrix} = q^{k(n+k-r)} \begin{bmatrix} n \\ r-k \end{bmatrix}. \quad (3.6)$$

This can be shown easily using (3.5). Thus we have

$$\begin{aligned}\Delta^k \begin{bmatrix} j-m-1 \\ k-1 \end{bmatrix} &= \Delta^{k-1} \begin{bmatrix} j-m \\ k-1 \end{bmatrix} - q^{k-1} \Delta^{k-1} \begin{bmatrix} j-m-1 \\ k-1 \end{bmatrix} \\ &= q^{(k-1)(j-m)} \left(\begin{bmatrix} j-m \\ 0 \end{bmatrix} - \begin{bmatrix} j-m-1 \\ 0 \end{bmatrix} \right).\end{aligned}$$

It follows that

$$\Delta^k g_j^{k,m} = \begin{cases} 1, & j = m, \\ 0, & \text{otherwise.} \end{cases} \quad (3.7)$$

Let $p_{k-1} \in \mathbb{P}_{k-1}$ denote the polynomial which interpolates f on the first k of these knots, $([j]/[n])$, $0 \leq j \leq k-1$, and let us write

$$\tilde{f}(x) = p_{k-1}(x) + \sum_{m=0}^{n-k} \Delta^k f_m g^{k,m}(x). \quad (3.8)$$

This is a piecewise polynomial of degree $k-1$ with respect to the knots. On the interval $[0, [k-1]/[n]]$, all of the $n-k+1$ functions $g^{k,m}(x)$ are zero and thus

$$\tilde{f}([j]/[n]) = p_{k-1}([j]/[n]) = f([j]/[n]), \quad 0 \leq j \leq k-1, \quad (3.9)$$

so that

$$\Delta^r \tilde{f}_0 = \Delta^r f_0, \quad 0 \leq r \leq k-1. \quad (3.10)$$

Also, we deduce from (3.8) and (3.7) that

$$\Delta^k \tilde{f}_m = \Delta^k f_m, \quad 0 \leq m \leq n-k,$$

and so

$$\Delta^r \tilde{f}_0 = \Delta^r f_0, \quad k \leq r \leq n. \quad (3.11)$$

Combining (3.10) and (3.11), we deduce that

$$\tilde{f}([j]/[n]) = f([j]/[n]), \quad 0 \leq j \leq n. \quad (3.12)$$

Thus the function \tilde{f} , a piecewise polynomial of degree $k-1$, takes the same values as f on all $n+1$ knots. When $k=1$, \tilde{f} is a step function which interpolates f

on all $n + 1$ knots and, when $k = 2$, the function \tilde{f} is the linear spline which interpolates f . For a general value of k , we deduce that

$$B_n(\tilde{f}; x) = B_n(f; x) \quad (3.13)$$

and thus, from (3.8) and the linearity of the Bernstein operator B_n ,

$$B_n(f; x) = B_n(p_{k-1}; x) + \sum_{m=0}^{n-k} \Delta^k f_m C_{k,m}(x) \quad (3.14)$$

say, where

$$C_{k,m}(x) = B_n(g^{k,m}; x). \quad (3.15)$$

We now state:

Theorem 3.1 *The k th derivatives of the generalized Bernstein polynomials of order n are non-negative on $[0, 1]$ for all functions f whose k th q -differences are non-negative if and only if the k th derivatives of the generalized Bernstein polynomials of the $n - k + 1$ functions $g^{k,m}(x)$, $0 \leq m \leq n - k$, are all non-negative.*

Proof This follows from (3.14) and (3.15). ■

We will find it useful to derive an alternative expression for the k th derivative of $B_n(g^{k,m}; x)$. We begin by expressing higher order q -differences (of order not less than k) in terms of the k th q -differences. For $0 \leq m \leq n - k$, we may write

$$\Delta^{m+k} f_i = \sum_{t=0}^m (-1)^t q^{t(t+2k-1)/2} \begin{bmatrix} m \\ t \end{bmatrix} \Delta^k f_{m+i-t}. \quad (3.16)$$

We may verify (3.16) by induction on m as follows. By (1.9), it is true for $m = 0$. Suppose it holds for some $m \geq 0$. We then write

$$\Delta^{m+k+1} f_i = \Delta(\Delta^{m+k} f_i) = \Delta^{m+k} f_{i+1} - q^{m+k} \Delta^{m+k} f_i.$$

Using (3.16) in the latter equation we have

$$\begin{aligned}\Delta^{m+k+1} f_i &= \sum_{t=0}^m (-1)^t q^{t(t+2k-1)/2} \begin{bmatrix} m \\ t \end{bmatrix} \Delta^k f_{i+m+1-t} \\ &\quad - q^{m+k} \sum_{t=0}^m (-1)^t q^{t(t+2k-1)/2} \begin{bmatrix} m \\ t \end{bmatrix} \Delta^k f_{i+m-t}.\end{aligned}$$

For $1 \leq t \leq m$, the coefficient of $(-1)^t q^{t(t+2k-1)/2} \Delta^k f_{i+m+1-t}$ in the above sums is

$$\left(\begin{bmatrix} m \\ t \end{bmatrix} + q^{m-t+1} \begin{bmatrix} m \\ t-1 \end{bmatrix} \right) = \begin{bmatrix} m+1 \\ t \end{bmatrix},$$

on using Pascal identity (1.4). This verifies (3.16).

We now write the q -difference form of the generalized Bernstein polynomial (1.14) as

$$B_n(f; x) = \sum_{r=0}^{k-1} \begin{bmatrix} n \\ r \end{bmatrix} \Delta^r f_0 x^r + \sum_{s=0}^{n-k} \begin{bmatrix} n \\ s+k \end{bmatrix} \Delta^{s+k} f_0 x^{s+k}.$$

Using (3.16) to replace the operator Δ^{s+k} in terms of Δ^k we have

$$B_n(f; x) = \sum_{r=0}^{k-1} \begin{bmatrix} n \\ r \end{bmatrix} \Delta^r f_0 x^r + \sum_{s=0}^{n-k} \sum_{t=0}^s (-1)^t q^{t(t+2k-1)/2} \begin{bmatrix} n \\ s+k \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \Delta^k f_{s-t} x^{s+k}.$$

Writing $m = s - t$, so that

$$\begin{bmatrix} n \\ s+k \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} n \\ m+t+k \end{bmatrix} \begin{bmatrix} m+t \\ t \end{bmatrix},$$

we obtain

$$B_n(f; x) = \sum_{r=0}^{k-1} \begin{bmatrix} n \\ r \end{bmatrix} \Delta^r f_0 x^r + \sum_{m=0}^{n-k} \Delta^k f_m D_{k,m}(x) \quad (3.17)$$

say, where

$$D_{k,m}(x) = \sum_{t=0}^{n-m-k} (-1)^t q^{t(t+2k-1)/2} \begin{bmatrix} n \\ m+t+k \end{bmatrix} \begin{bmatrix} m+t \\ t \end{bmatrix} x^{m+t+k}. \quad (3.18)$$

On comparing (3.14) and (3.17), which hold for all functions f , we deduce that

$$\frac{d^k}{dx^k} C_{k,m}(x) = \frac{d^k}{dx^k} D_{k,m}(x). \quad (3.19)$$

Thus, given that we are interested only in their k th derivatives, the sets of polynomials $C_{k,m}$ and $D_{k,m}$ are equivalent.

It is well known (see Davis [9]) that, with $q = 1$, the k th derivatives of $D_{k,m}$ are non-negative. This is easily verified from (3.18) since with $q = 1$ we have

$$\frac{d^k}{dx^k} D_{k,m}(x) = \frac{n!}{m!(n-m-k)!} x^m \sum_{t=0}^{n-m-k} (-1)^t \binom{n-m-k}{t} x^t,$$

so that, mindful of (3.19),

$$\frac{d^k}{dx^k} D_{k,m}(x) = \frac{d^k}{dx^k} C_{k,m}(x) = \frac{n!}{m!(n-m-k)!} x^m (1-x)^{n-m-k} \geq 0$$

for $0 \leq x \leq 1$. From (3.18) we can also see that, as q tends to zero from above, each q -integer tends to 1 and we have the limiting form

$$D_{k,m}(x) = x^{m+k}$$

and so its k th derivative is non-negative. We will find that the k th derivative of each $D_{k,m}$ is non-negative for $0 < q < 1$ for certain values of m which we will mention below.

We will now work with $C_{k,m}$ rather than $D_{k,m}$. From (3.15), (1.13) and (3.4) we have

$$C_{k,m}(x) = \sum_{r=m+k}^n \begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} r-m-1 \\ k-1 \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1-q^s x), \quad (3.20)$$

for $0 \leq m \leq n - k$. With $m = n - k$, we have

$$C_{k,n-k}(x) = x^n,$$

whose k th derivative is clearly non-negative on $[0, 1]$. With $m = n - k - 1$, we obtain from (3.20) that

$$C_{k,n-k-1}(x) = [n]x^{n-1}(1-x) + [k]x^n$$

and, with a little work, we find that the k th derivative of the latter polynomial is also non-negative on $[0, 1]$.

We can express $C_{k,m}(x)$ in another way, as follows. Since B_n is a linear operator, we may write

$$C_{k,m}(x) = B_n(g^{k,m}; x) = B_n(\gamma^{k,m}; x) + B_n(g^{k,m} - \gamma^{k,m}; x), \quad (3.21)$$

where $\gamma^{k,m}$ is defined in (3.3). Let

$$B_n(\gamma^{k,m}; x) = p_{k,m}(x)$$

say, where $p_{k,m}(x) \in \mathbb{P}_{k-1}$. Then we obtain from (3.21) that

$$C_{k,m}(x) = p_{k,m}(x) + q^{-(2m+k)(k-1)/2}(-1)^k S_{k,m} \quad (3.22)$$

say, where

$$S_{k,m} = \sum_{r=0}^m q^{r(k-1)} \begin{bmatrix} m+k-1-r \\ k-1 \end{bmatrix} \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-1-r} (1 - q^s x). \quad (3.23)$$

In particular, (3.22) gives

$$C_{k,0}(x) = p_{k,0}(x) + q^{-k(k-1)/2}(-1)^k \prod_{s=0}^{n-1} (1 - q^s x).$$

Since, for $0 < q < 1$, the zeros of the function $(-1)^k \prod_{s=0}^{n-1} (1 - q^s x)$ are all greater than unity, the repeated application of Rolle's theorem shows that this is true of each of its first n derivatives. Also, Euler's identity (1.7) shows that its k th derivative is positive at $x = 0$ and so is positive on $[0, 1]$. Since $p_{k,0}(x) \in \mathbb{P}_{k-1}$ it follows that k th derivative of $C_{k,0}$ is also positive on $[0, 1]$. Apart from the case $q = 1$ and the special cases discussed above, we have no proof that the k th derivative of $C_{k,m} \geq 0$ for $0 \leq m \leq n - k$.

3.2 Monotonicity for convex functions

It is well known (see Davis [9]) that, when the function f is convex on $[0, 1]$, its Bernstein polynomials are monotonic decreasing, in the sense that

$$B_{n-1}(f; x) \geq B_n(f; x), \quad n = 2, 3, \dots, \quad 0 \leq x \leq 1.$$

We now show that this result extends to the generalized Bernstein polynomials, for $0 < q \leq 1$. In Figure 3.1, which illustrates this monotonicity, the function is

concave rather than convex and thus the Bernstein polynomials are monotonic increasing. Figure 3.1 here is modelled on Fig. 6.3.1 in Davis [9], which relates to the classical Bernstein polynomials. The function is the linear spline which joins up the points $(0,0)$, $(0.2,0.6)$, $(0.6,0.8)$, $(0.9,0.7)$ and $(1,0)$ and the Bernstein polynomials are those of degrees 2, 4 and 10, with $q = 0.8$ in place of $q = 1$ in Davis [9].

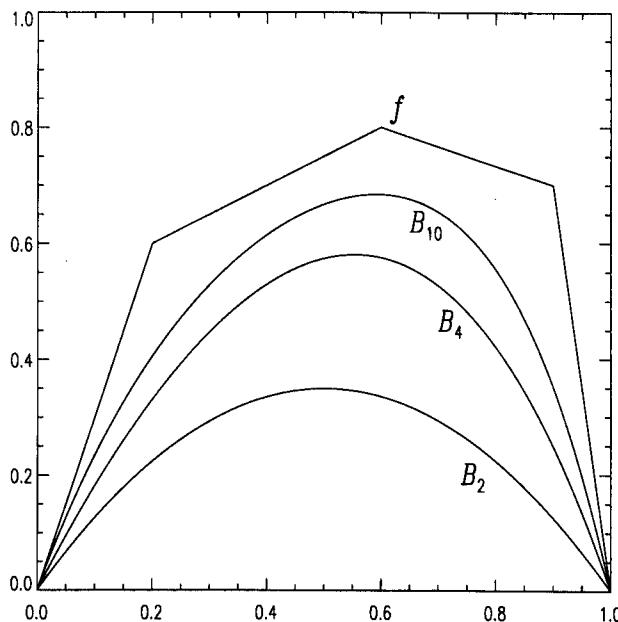


Figure 3.1: Monotonicity of generalized Bernstein polynomials for a concave function.

The polynomials are B_2, B_4 and B_{10} , with $q = 0.8$

Theorem 3.2 *Let f be convex on $[0, 1]$. Then, for $0 < q \leq 1$, $B_{n-1}(f; x) \geq B_n(f; x)$ for $0 \leq x \leq 1$ and all $n \geq 2$. If $f \in C[0, 1]$ the inequality holds strictly for $0 < x < 1$ unless f is linear in each of the intervals between consecutive knots $[r]/[n-1], 0 \leq r \leq n-1$, in which case we have the equality $B_{n-1}(f; x) = B_n(f; x)$.*

Proof The key to the proof in Davis [9] for the case $q = 1$ is to express the difference between the consecutive Bernstein polynomials in terms of powers of $x/(1 - x)$. Since the generalized Bernstein polynomials involve the product

$\prod_{s=0}^{n-r-1} (1 - q^s x)$ rather than $(1 - x)^{n-r}$ we need to modify the proof somewhat.

For $0 < q < 1$ we begin by writing

$$\begin{aligned} \prod_{s=0}^{n-1} (1 - q^s x)^{-1} (B_{n-1}(f; x) - B_n(f; x)) \\ = \sum_{r=0}^{n-1} f\left(\frac{[r]}{[n-1]}\right) \begin{bmatrix} n-1 \\ r \end{bmatrix} x^r \prod_{s=n-r-1}^{n-1} (1 - q^s x)^{-1} \\ - \sum_{r=0}^n f\left(\frac{[r]}{[n]}\right) \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=n-r}^{n-1} (1 - q^s x)^{-1}. \end{aligned}$$

We now split the first of the above summations into two, writing

$$x^r \prod_{s=n-r-1}^{n-1} (1 - q^s x)^{-1} = \psi_r(x) + q^{n-r-1} \psi_{r+1}(x),$$

where

$$\psi_r(x) = x^r \prod_{s=n-r}^{n-1} (1 - q^s x)^{-1}. \quad (3.24)$$

The resulting three summations may be combined to give

$$\prod_{s=0}^{n-1} (1 - q^s x)^{-1} (B_{n-1}(f; x) - B_n(f; x)) = \sum_{r=1}^{n-1} \begin{bmatrix} n \\ r \end{bmatrix} a_r \psi_r(x), \quad (3.25)$$

say, where

$$a_r = \frac{[n-r]}{[n]} f\left(\frac{[r]}{[n-1]}\right) + q^{n-r} \frac{[r]}{[n]} f\left(\frac{[r-1]}{[n-1]}\right) - f\left(\frac{[r]}{[n]}\right). \quad (3.26)$$

From (3.24) it is clear that each $\psi_r(x)$ is non-negative on $[0, 1]$ for $0 \leq q \leq 1$ and thus, in view of (3.25), it suffices to show that each a_r is non-negative. We return to (3.1) and put $t_0 = [r-1]/[n-1]$, $t_1 = [r]/[n-1]$ and $\lambda = q^{n-r}[r]/[n]$. Then $0 \leq t_0 < t_1 \leq 1$ and $0 < \lambda < 1$ for $1 \leq r \leq n-1$ and, comparing (3.1) and (3.26), we deduce that, for $1 \leq r \leq n-1$,

$$a_r = \lambda f(t_0) + (1 - \lambda) f(t_1) - f(\lambda t_0 + (1 - \lambda)t_1) \geq 0.$$

Thus $B_{n-1}(f; x) \geq B_n(f; x)$. Of course we have equality for $x = 0$ and $x = 1$ since all Bernstein polynomials interpolate f on these end-points. The inequality will be strict for $0 < x < 1$ unless each $a_r = 0$ which can only occur when f is linear in each of the intervals between consecutive knots $[r]/[n-1]$, $0 \leq r \leq n-1$, when we have $B_{n-1}(f; x) = B_n(f; x)$ for $0 \leq x \leq 1$. This completes the proof. ■

Theorem 3.3 For $n = 2, 3, \dots$ we have

$$\begin{aligned} B_{n-1}(f; x) - B_n(f; x) &= \frac{x(1-x)}{[n-1][n]} \sum_{r=0}^{n-2} q^{n+r-1} \binom{n-2}{r} \\ &\quad f \left[\frac{[r]}{[n-1]}, \frac{[r+1]}{[n]}, \frac{[r+1]}{[n-1]} \right] x^r \prod_{s=1}^{n-r-2} (1 - q^s x). \end{aligned} \quad (3.27)$$

Proof It follows from equations (3.25) and (3.26) that

$$B_{n-1}(f; x) - B_n(f; x) = \sum_{r=1}^{n-1} \binom{n}{r} x^r a_r \prod_{s=0}^{n-r-1} (1 - q^s x). \quad (3.28)$$

Let us evaluate the divided difference of f at the points $\frac{[r-1]}{[n-1]}$, $\frac{[r]}{[n]}$ and $\frac{[r]}{[n-1]}$. Using the symmetric form for the divided differences we obtain

$$\begin{aligned} f \left[\frac{[r-1]}{[n-1]}, \frac{[r]}{[n]}, \frac{[r]}{[n-1]} \right] &= \\ \frac{[n][n-1]^2}{q^{2r-2} [n-r]} f \left(\frac{[r-1]}{[n-1]} \right) &- \frac{[n]^2[n-1]^2}{q^{n+r-2} [r][n-r]} f \left(\frac{[r]}{[n]} \right) \\ &+ \frac{[n][n-1]^2}{q^{n+r-2} [r]} f \left(\frac{[r]}{[n-1]} \right) \end{aligned} \quad (3.29)$$

From (3.29) and (3.26) we see that

$$\binom{n}{r} a_r = \frac{q^{n+r-2}}{[n][n-1]} \binom{n-2}{r-1} f \left[\frac{[r-1]}{[n-1]}, \frac{[r]}{[n]}, \frac{[r]}{[n-1]} \right]$$

and also we obtain from (3.28)

$$\begin{aligned} B_{n-1}(f; x) - B_n(f; x) &= \frac{x(1-x)}{[n-1][n]} \sum_{r=1}^{n-1} q^{n+r-2} \binom{n-2}{r-1} \\ &\quad f \left[\frac{[r-1]}{[n-1]}, \frac{[r]}{[n]}, \frac{[r]}{[n-1]} \right] x^{r-1} \prod_{s=1}^{n-r-1} (1 - q^s x). \end{aligned}$$

Shifting the limits of the latter equation completes the proof. ■

The last theorem is a generalization of the theorem in DeVore and Lorentz [12, pp. 309].

We deduce that if the function f is convex in $[0, 1]$ then all terms in (3.27) are non-negative for $0 < q \leq 1$ and $x \in [0, 1]$ implying that

$$B_{n-1}(f; x) \geq B_n(f; x).$$

3.3 Modulus of continuity

Theorem 3.4 *If f is bounded for $0 \leq t \leq 1$ and $0 < q \leq 1$, then*

$$\| f - B_n f \|_\infty \leq \frac{3}{2} \omega \left(\frac{1}{[n]^{\frac{1}{2}}} \right).$$

Proof This generalizes the result in Rivlin [46, pp.15]. We modify Rivlin's proof as follows. We have

$$\begin{aligned} |f(t) - B_n(f; t)| &= \left| \sum_{j=0}^n (f(t) - f_j) B_j^n(t) \right| \\ &\leq \sum_{j=0}^n |f(t) - f_j| B_j^n(t) \\ &\leq \sum_{j=0}^n \omega \left(\left| t - \frac{[j]}{[n]} \right| \right) B_j^n(t), \end{aligned}$$

where $B_j^n(t)$ is a member of the generalized Bernstein basis, as defined in Section 4.2. From property (iii) of the modulus of continuity (see Section 1.2), we have

$$\omega \left(\left| t - \frac{[j]}{[n]} \right| \right) = \omega \left([n]^{\frac{1}{2}} \left| t - \frac{[j]}{[n]} \right| [n]^{-\frac{1}{2}} \right) \leq \left(1 + [n]^{\frac{1}{2}} \left| t - \frac{[j]}{[n]} \right| \right) \omega \left(\frac{1}{[n]^{\frac{1}{2}}} \right),$$

and using the fact that $B_n f$ reproduces linear polynomials we have

$$\begin{aligned} |f(t) - B_n(f; t)| &\leq \sum_{j=0}^n \left(1 + [n]^{\frac{1}{2}} \left| t - \frac{[j]}{[n]} \right| \right) \omega \left(\frac{1}{[n]^{\frac{1}{2}}} \right) B_j^n(t) \\ &\leq \omega \left(\frac{1}{[n]^{\frac{1}{2}}} \right) \left(1 + [n]^{\frac{1}{2}} \sum_{j=0}^n \left| t - \frac{[j]}{[n]} \right| \right) B_j^n(t). \quad (3.30) \end{aligned}$$

On using the Schwartz inequality in the latter sum we obtain

$$\begin{aligned}
 \sum_{j=0}^n |t - \frac{[j]}{[n]}| B_j^n(t) &\leq \left(\sum_{j=0}^n \left(t - \frac{[j]}{[n]} \right)^2 B_j^n(t) \right)^{\frac{1}{2}} \left(\sum_{j=0}^n B_j^n(t) \right)^{\frac{1}{2}} \\
 &= \left(\sum_{j=0}^n \left(t - \frac{[j]}{[n]} \right)^2 B_j^n(t) \right)^{\frac{1}{2}} \\
 &= \left(\sum_{j=0}^n \left(t^2 - 2t\frac{[j]}{[n]} + \frac{[j]^2}{[n]^2} \right) B_j^n(t) \right)^{\frac{1}{2}} \\
 &= \left(t^2 - 2t \cdot t + t^2 + \frac{t(1-t)}{[n]} \right)^{\frac{1}{2}}, \tag{3.31}
 \end{aligned}$$

on using (1.17) and (1.18). Since

$$\frac{t(1-t)}{[n]} \leq \frac{1}{4[n]}$$

for $0 \leq t \leq 1$, we deduce that

$$\sum_{j=0}^n \left(|t - \frac{[j]}{[n]}| \right) B_j^n(t) \leq \frac{1}{(4[n])^{\frac{1}{2}}}$$

and thus, from (3.30),

$$|f(t) - B_n(f; t)| \leq \omega \left(\frac{1}{[n]^{\frac{1}{2}}} \right) \left(1 + [n]^{\frac{1}{2}} \frac{1}{(4[n])^{\frac{1}{2}}} \right).$$

This completes the proof. ■

Theorem 3.4 is quoted by Phillips [42] but the proof is omitted.

Remark 3.1 For $q = 1$, in view of Theorem 3.4 and using property (iv) of the modulus of continuity, if $f(t)$ is continuous on $[0, 1]$ we see that $\omega(n^{-\frac{1}{2}}) \rightarrow 0$ as $n \rightarrow \infty$. This shows again the uniform convergence of classical Bernstein polynomials, $B_n f \rightarrow f$.

Remark 3.2 Phillips [42] shows that the error $B_n f - f$ tends to zero like $1/[n]$ for a choice of a sequence $q = q_n \rightarrow 1$ from below. Thus the rate of convergence is best for $q_n = 1$, like $1/n$, for the classical Bernstein polynomials.

Remark 3.3 *If a function satisfies*

$$|f(s) - f(t)| \leq \kappa |s - t|^\alpha$$

for $s, t \in [a, b]$, and $0 < \alpha$, then $f(x)$ is said to satisfy a Lipschitz condition of order α with constant κ and the set of such functions is denoted by $\text{Lip}_\kappa \alpha$. It is easy to see that $f(x) \in \text{Lip}_\kappa \alpha$ if and only if $\omega(\delta) \leq \kappa \delta^\alpha$. Thus, if $f(t) \in \text{Lip}_\kappa \alpha$ on $[0, 1]$

$$\|f - B_n f\|_\infty \leq \frac{3}{2} \kappa [n]^{-\frac{\alpha}{2}}.$$

Chapter 4

Total positivity

Total positivity is a powerful property that plays an important role in various domains of mechanics, mathematics, statistics and operational research. Totally positive functions figure prominently in problems involving convexity and moment spaces in approximation theory (see Karlin [28]).

It is quite useful from the point of view of design to have an approximation to a function f which mimics the shape of f . Total positivity provides a technique for discussing shape properties of approximations, due to the variation diminishing properties of totally positive functions, bases and matrices.

There is a large amount of literature concerning total positivity. We follow Karlin [28] for some basic ideas, and Goodman [18] and Carnicer and Peña [5], [4] for the applications of this concept to the shape properties of curves.

This chapter is organized as follows. We begin with some introductory material regarding total positivity. We find the transformation matrices between the generalized Bernstein basis and the power basis, and vice-versa. We also obtain

the conversion matrix, which is shown to be totally positive, from the Bernstein basis to the generalized Bernstein basis in section 4.2. We study shape properties of the generalized Bernstein polynomials in section 4.3. Basic results in these two sections are given in Goodman *et al.* [19].

4.1 Total positivity and totally positive bases

We require some preliminaries on total positivity before giving results on basis conversion, the total positivity of the generalized Bernstein basis and also the shape-preserving properties of generalized Bernstein polynomials. The following definitions and theorems can be found in Goodman [18].

Definition 4.1 For any real sequence v , finite or infinite, we denote by $S^-(v)$ the number of strict sign changes in v .

We use the same notation to denote sign changes in a function, as follows.

Definition 4.2 For a real-valued function f on an interval I , we define $S^-(f)$ to be the number of sign changes of f , that is

$$S^-(f) = \sup S^-(f(x_0), \dots, f(x_m))$$

where the supremum is taken over all increasing sequences (x_0, \dots, x_m) in I for all m .

We recall Descartes' Rule of Signs. Given any polynomial

$$p(x) = \sum_{i=0}^n a_i x^i, \quad x > 0,$$

then the number of times it changes sign on $(0, \infty)$ is bounded by the number of changes of sign in the sequence a_0, \dots, a_n .

Definition 4.3 Let

$$v_i = \sum_{k=0}^n a_{ik} u_k, \quad i = 0, \dots, m,$$

be a linear transformation, where the coefficients and variables are all real. This transformation is called variation diminishing by Pólya (see Schoenberg [47]) provided that

$$S^-(v) \leq S^-(u).$$

Definition 4.4 A matrix is said to be totally positive if all its minors are non-negative.

Explicitly this definition states that all $m \times m$ sub-matrices of \mathbf{A} of the form

$$\mathbf{B} = (a_{i_k, j_l})_{k, l=1}^m \text{ with } m \geq 1, i_1 < i_2 < \dots < i_m, j_1 < j_2 < \dots < j_m$$

have a non-negative determinant, that is $\det(\mathbf{B}) \geq 0$. We note that the process of solving a system of equations with a totally positive nonsingular \mathbf{A} by Gaussian elimination even without pivoting is numerically stable (see de Boor and Pinkus [10]).

Definition 4.5 We say that a matrix $\mathbf{A} = (a_{ij})$ is m -banded if, for some l , $a_{ij} \neq 0$ implies $|l| \leq |j - i| \leq l + m$.

In particular, 1-banded (also called bidiagonal) matrices have all their non-zero elements in two neighbouring diagonals.

Theorem 4.1 *A finite matrix is totally positive if and only if it is a product of 1-banded matrices with non-negative elements.*

Theorem 4.2 (Variation diminishing property) *If \mathbf{T} is a totally positive matrix and \mathbf{v} is any vector for which $\mathbf{T}\mathbf{v}$ is defined, then $S^-(\mathbf{T}\mathbf{v}) \leq S^-(\mathbf{v})$.*

Definition 4.6 We say that a sequence (ϕ_0, \dots, ϕ_n) of real-valued functions on an interval I is totally positive if, for any points $x_0 < \dots < x_n$ in I , the collocation matrix $(\phi_j(x_i))_{i,j=0}^n$ is totally positive.

When the totally positive functions (ϕ_0, \dots, ϕ_n) are also linearly independent we refer to them as a totally positive basis. In addition, if the basis $\Phi = \{\phi_0, \dots, \phi_n\}$ forms a partition of unity, that is

$$\sum_{i=0}^n \phi_i(x) = 1,$$

it is called normalized totally positive basis.

If Φ is a totally positive basis in an interval I we may easily deduce the following properties from the definition above.

- (i) If f is an increasing function from an interval J into I then $(\phi_0 \circ f, \dots, \phi_n \circ f)$ is totally positive on J , where $\phi_0 \circ f$ denotes the composition of ϕ_0 and f .
- (ii) If g is a positive function on I , then $(g\phi_0, \dots, g\phi_n)$ is totally positive on I .
- (iii) If \mathbf{A} is a constant $(m+1) \times (n+1)$ totally positive matrix and

$$\psi_i = \sum_{j=0}^n a_{ij} \phi_j, \quad i = 0, \dots, m,$$

then ψ_0, \dots, ψ_m is totally positive on I .

Theorem 4.3 *If (ϕ_0, \dots, ϕ_n) is totally positive on I then, for any numbers a_0, \dots, a_n ,*

$$S^-(a_0\phi_0 + \dots + a_n\phi_n) \leq S^-(a_0, \dots, a_n).$$

For the proofs of these theorems see Goodman [18].

4.2 Change of basis

Karlin [28] states that a matrix is totally positive provided all minors with consecutive columns are non-negative. Using this fact, Goodman [18] shows that the power basis (i.e. monomial basis)

$$(1, x, x^2, \dots, x^n), \quad x \geq 0,$$

whose collocation matrix is the Vandermonde matrix \mathbf{V} with $v_{i,j} = x_i^j$, $0 \leq i, j \leq n$, is totally positive on $[0, \infty)$. Thus on making the change of variable $t = x/(1-x)$, by property (i) following Definition 4.6, noting that t is an increasing function of x , we see that

$$(1, x/(1-x), x^2/(1-x)^2, \dots, x^n/(1-x)^n)$$

is totally positive on $[0, 1]$. On applying property (ii) with $g(x) = (1-x)^n$, which is non-negative on $[0, 1]$, we deduce that

$$((1-x)^n, x(1-x)^{n-1}, \dots, x^n)$$

is totally positive on $[0, 1]$. By continuity we can extend this to $[0, 1]$. Finally, by property (iii), we can multiply this basis by a $(n+1) \times (n+1)$ diagonal matrix which is totally positive and whose (i, i) element is $\binom{n}{i}$ to obtain the Bernstein basis

$$b_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad 0 \leq x \leq 1, \quad i = 0, \dots, n. \quad (4.1)$$

Since

$$\sum_{i=0}^n b_i^n(x) = 1, \quad (4.2)$$

the Bernstein basis is indeed a normalized totally positive basis. The Bernstein polynomial defined by

$$B_n(f; x) = \sum_{i=0}^n f(i/n) b_i^n(x)$$

lies in the region

$$\min_{0 \leq i \leq n} f(i/n) \leq B_n f \leq \max_{0 \leq i \leq n} f(i/n).$$

Indeed, it lies entirely inside the convex hull of its associated control polygon formed by joining the control points $(i/n, f(i/n))$, see Hoschek and Lasser [24].

We note that if (ψ_0, \dots, ψ_n) is totally positive with $\sum_{i=0}^n \psi_i > 0$ on the interval I , then defining

$$\phi_i = \frac{\psi_i}{\sum_{i=0}^n \psi_i}, \quad i = 0, \dots, n,$$

we see that (ϕ_0, \dots, ϕ_n) is a normalized totally positive basis. For example if (ψ_0, \dots, ψ_n) is a Bernstein basis, then (ϕ_0, \dots, ϕ_n) is a rational Bernstein basis and if (ψ_0, \dots, ψ_n) is a sequence of B-splines, then (ϕ_0, \dots, ϕ_n) is a sequence of rational B-splines (see Farin [14, pp. 268], Goodman [17], Carnicer and Peña [4]).

Since the power and the generalized Bernstein bases both span the space of polynomials of degree n on $[0, 1]$, each power basis function may be expressed in terms of the $n + 1$ generalized Bernstein bases functions, and vice-versa. First we note that the generalized Bernstein basis

$$B_j^n(x) = \begin{bmatrix} n \\ j \end{bmatrix} x^j \prod_{t=0}^{n-j-1} (1 - q^t x), \quad 0 \leq j \leq n, \quad 0 \leq x \leq 1,$$

forms a partition of unity. Thus we deduce that the generalized Bernstein polynomial $B_n(f; x)$ satisfies the inequalities

$$\min_{0 \leq j \leq n} f_j \leq B_n(f; x) \leq \max_{0 \leq j \leq n} f_j. \quad (4.3)$$

On using (1.7), we obtain

$$B_j^n(x) = \sum_{k=0}^{n-j} (-1)^k q^{k(k-1)/2} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} n-j \\ k \end{bmatrix} x^{j+k}.$$

Shifting the limits of the above sum and then writing

$$\begin{bmatrix} n-j \\ k-j \end{bmatrix} = \frac{\begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix}}{\begin{bmatrix} n \\ j \end{bmatrix}}, \quad (4.4)$$

we deduce that

$$B_j^n(x) = \sum_{k=0}^n (-1)^{k-j} q^{(k-j)(k-j-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} x^k. \quad (4.5)$$

Conversely, on multiplying each side of the following identity by x^j

$$\sum_{k=0}^{n-j} B_k^{n-j}(x) = 1,$$

we may obtain the power basis $\phi_j(x) = x^j$, $j = 0, \dots, n$ in terms of generalized Bernstein basis

$$x^j = \sum_{k=0}^{n-j} \begin{bmatrix} n-j \\ k \end{bmatrix} x^{j+k} \prod_{t=0}^{n-j-k-1} (1 - q^t x).$$

Thus shifting the limit of the sum and the product above and then using (4.4) we obtain

$$x^j = \sum_{k=j}^n \frac{\begin{bmatrix} k \\ j \end{bmatrix}}{\begin{bmatrix} n \\ j \end{bmatrix}} B_k^n(x), \quad j = 0, \dots, n. \quad (4.6)$$

As a consequence

$$\begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix} = \mathbf{M}^{n,q} \begin{bmatrix} B_0^n(x) \\ B_1^n(x) \\ \vdots \\ B_n^n(x) \end{bmatrix},$$

where $\mathbf{M}^{n,q}$ is an upper triangular matrix such that

$$\mathbf{M}^{n,q} = (m_{j,k}^{n,q})_{j,k=0}^n = \frac{\begin{bmatrix} k \\ j \end{bmatrix}}{\begin{bmatrix} n \\ j \end{bmatrix}} = \frac{[n-j]! [k]!}{[k-j]! [n]!}.$$

We may write $\mathbf{M}^{n,q} = \mathbf{ATB}$ such that \mathbf{A} is a diagonal matrix with $\mathbf{A} = (a_{j,j})_{j=0}^n = \frac{[n-j]!}{[n]!}$ and \mathbf{B} is a diagonal matrix with $\mathbf{B} = (b_{k,k})_{k=0}^n = [k]!$ and \mathbf{T} is a Toeplitz matrix with $\mathbf{T} = (t_{j,k})_{j,k=0}^n = \frac{1}{[k-j]!}$. We also invert the matrix $\mathbf{M}^{n,q}$ to obtain corresponding coefficients in (4.5). Thus

$$(\mathbf{M}^{n,q})^{-1} = ((m_{j,k}^{n,q})^{-1})_{j,k=0}^n = (-1)^{k-j} q^{(k-j)(k-j-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix}.$$

We note that the above formulas concerning bases conversion for the special case $q = 1$ can be found in Farouki and Rajan [16] and Goodman [18].

For $0 < q \leq 1$, $n \geq 1$, $j = 0, \dots, n$, let

$$P_j^{n,q}(x) = x^j \prod_{s=0}^{n-j-1} (1 - q^s x), \quad 0 \leq x \leq 1, \quad (4.7)$$

denote the functions which appear in the generalized Bernstein polynomials. We have seen above that

$$(P_0^{n,1}, P_1^{n,1}, \dots, P_n^{n,1})$$

is totally positive on $[0,1]$. Since the functions defined in (4.7) are a basis for the subspace of the polynomials of degree at most n then, for any q, r , $0 < q, r \leq 1$, there exists a non-singular matrix $\mathbf{T}^{n,q,r}$ such that

$$\begin{bmatrix} P_0^{n,q}(x) \\ \vdots \\ P_n^{n,q}(x) \end{bmatrix} = \mathbf{T}^{n,q,r} \begin{bmatrix} P_0^{n,r}(x) \\ \vdots \\ P_n^{n,r}(x) \end{bmatrix}.$$

Theorem 4.4 *For $0 < q \leq r$ all elements of the matrix $\mathbf{T}^{n,q,r}$ are non-negative.*

Proof We use induction on n . The result holds for $n = 1$ since $\mathbf{T}^{1,q,r}$ is the 2×2 identity matrix. Let us assume the result holds for some $n \geq 1$. Then, since

$$P_{j+1}^{n+1,q}(x) = x P_j^{n,q}(x), \quad 0 \leq j \leq n,$$

we have

$$\begin{bmatrix} P_1^{n+1,q}(x) \\ \vdots \\ P_{n+1}^{n+1,q}(x) \end{bmatrix} = \mathbf{T}^{n,q,r} \begin{bmatrix} P_1^{n+1,r}(x) \\ \vdots \\ P_{n+1}^{n+1,r}(x) \end{bmatrix}. \quad (4.8)$$

Also, we have

$$\begin{aligned} P_0^{n+1,q}(x) &= (1-x) \dots (1-q^{n-1}x)(1-q^n x) \\ &= (1-q^n x) \sum_{j=0}^n T_{0,j}^{n,q,r} P_j^{n,r}(x). \end{aligned} \quad (4.9)$$

We see that

$$\begin{aligned}(1 - q^n x) P_j^{n,r}(x) &= (1 - r^{n-j} x + (r^{n-j} - q^n)x) P_j^{n,r}(x) \\ &= P_j^{n+1,r} + (r^{n-j} - q^n) P_{j+1}^{n+1,r}(x).\end{aligned}$$

On substituting this in (4.9) we obtain

$$\begin{aligned}P_0^{n+1,q}(x) &= \sum_{j=0}^n T_{0,j}^{n,q,r} (P_j^{n+1,r} + (r^{n-j} - q^n) P_{j+1}^{n+1,r}(x)) \\ &= T_{0,0}^{n,q,r} P_0^{n+1,r}(x) + (1 - q^n) T_{0,n}^{n,q,r} P_{n+1}^{n+1,r}(x) \\ &\quad + \sum_{j=1}^n \left((r^{n+1-j} - q^n) T_{0,j-1}^{n,q,r} + T_{0,j}^{n,q,r} \right) P_j^{n+1,r}(x).\end{aligned}\quad (4.10)$$

Combining (4.8) and (4.10), we have

$$\begin{bmatrix} P_0^{n+1,q}(x) \\ P_1^{n+1,q}(x) \\ \vdots \\ P_{n+1}^{n+1,q}(x) \end{bmatrix} = \begin{bmatrix} T_{0,0}^{n,q,r} & \mathbf{v}_{n+1}^T \\ & \ddots \\ \mathbf{0} & \mathbf{T}^{n,q,r} \end{bmatrix} \begin{bmatrix} P_0^{n+1,r}(x) \\ P_1^{n+1,r}(x) \\ \vdots \\ P_{n+1}^{n+1,r}(x) \end{bmatrix}, \quad (4.11)$$

where the elements of the row vector \mathbf{v}_{n+1}^T are the coefficients of $P_1^{n+1,r}(x), \dots, P_{n+1}^{n+1,r}(x)$ given by (4.10). Putting $x = 0$ in (4.9) gives $T_{0,0}^{n,q,r} = 1$. Thus $\mathbf{T}^{n+1,q,r}$ is the matrix in block form in (4.11) which, together with (4.10), shows that all elements of $\mathbf{T}^{n+1,q,r}$ are non-negative for $0 < q \leq r$. This completes the proof. ■

Corollary 4.4.1 *For $n \geq 2$ and $0 < q \leq 1$ the matrix $\mathbf{T}^{n,q,1}$ is a totally positive matrix such that*

$$\mathbf{T}^{n,q,1} = (T_{i,j}^{n,q,1})_{i,j=0}^n = (1 - q)^{j-i} S_1(n - 1 - i, j - i),$$

where $S_1(n, k)$ is defined in Section 2.2 with $S_1(n, k) = 0$ for $k < 0$ and $k > n$, and $S_1(n, 0) = 1$ for $n \geq 0$.

Proof Substitute $r = 1$ in the proof above and use induction on n . We obtain from (4.10) that

$$\begin{aligned} P_0^{n+1,q}(x) &= S_1(n-1,0)P_0^{n+1,1}(x) + (1-q^n)S_1(n-1,n)P_{n+1}^{n+1,1}(x) \\ &\quad + \sum_{j=1}^n \left((1-q^n)(1-q)^{j-1}S_1(n-1,j-1) + S_1(n-1,j) \right) P_j^{n+1,1}(x). \end{aligned}$$

We may write this

$$P_0^{n+1,q}(x) = P_0^{n+1,1}(x) + \sum_{j=1}^n (1-q)^j ([n]S_1(n-1,j-1) + S_1(n-1,j)) P_j^{n+1,1}(x).$$

In the view of the recurrence relation given in (2.15) we obtain

$$P_0^{n+1,q}(x) = \sum_{j=0}^n (1-q)^j S_1(n,j) P_j^{n+1,1}(x).$$

Putting $r = 1$ in (4.11) shows it is true for $n + 1$. Since $S_1(n-i, j-i) = 0$ for $i > j$ the matrix $\mathbf{T}^{n+1,q,r}$ is upper triangular with its main diagonal all 1's and last column all zeros. Also, the recurrence relation (2.15) implies that all $S_1(n-i, j-i)$ are non-negative for $q \geq 0$. We will see in the next theorem concerning 1-banded factorization of $\mathbf{T}^{n+1,q,r}$ that $\mathbf{T}^{n+1,q,1}$ is totally positive matrix for $0 < q \leq 1$. ■

Corollary 4.4.2 *For $n \geq 2$ and $0 < q \leq 1$, the upper triangular totally positive matrix $\tilde{\mathbf{T}}^{n,q,1}$ with*

$$\tilde{\mathbf{T}}^{n,q,1} = (\tilde{T}_{i,j}^{n,q,1})_{i,j=0}^n = \frac{\begin{bmatrix} n \\ i \end{bmatrix}}{\begin{bmatrix} n \\ j \end{bmatrix}} (1-q)^{j-i} S_1(n-1-i, j-i)$$

transforms the classical Bernstein basis into the generalized Bernstein basis. That is

$$\begin{bmatrix} B_0^n(x) \\ B_1^n(x) \\ \vdots \\ B_n^n(x) \end{bmatrix} = \tilde{\mathbf{T}}^{n,q,1} \begin{bmatrix} b_0^n(x) \\ b_1^n(x) \\ \vdots \\ b_n^n(x) \end{bmatrix}. \quad (4.12)$$

Also $(\tilde{\mathbf{T}}^{n,q,1})^T$ is stochastic (each row has sum 1).

Proof We see that

$$(B_0^n(x), B_1^n(x), \dots, B_n^n(x))^T = C^{n,q} (P_0^{n,q}(x), P_1^{n,q}(x), \dots, P_n^{n,q}(x))^T,$$

where $C^{n,q}$ is the diagonal matrix such that $(C_{j,j}^{n,q})_{j=0}^n = \begin{bmatrix} n \\ j \end{bmatrix}$. Similarly we have

$$(b_0^n(x), b_1^n(x), \dots, b_n^n(x))^T = C^{n,1} (P_0^{n,1}(x), P_1^{n,1}(x), \dots, P_n^{n,1}(x))^T.$$

Thus

$$(B_0^n(x), B_1^n(x), \dots, B_n^n(x))^T = C^{n,q} T^{n,q,1} (C^{n,1})^{-1} (b_0^n(x), b_1^n(x), \dots, b_n^n(x))^T.$$

It can be easily verified that

$$C^{n,q} T^{n,q,1} (C^{n,1})^{-1} = \tilde{T}^{n,q,1}.$$

$\tilde{T}^{n,q,1}$ is totally positive since the product of totally positive matrices is also totally positive. The lower triangular matrix $(\tilde{T}^{n,q,1})^T$ is stochastic since both of the bases are normalized. That is

$$\sum_{i=0}^n B_i^n(x) = \sum_{j=0}^n b_j^n(x) \sum_{i=0}^n \tilde{T}_{i,j}^{n,q,1} = 1.$$

Thus

$$\sum_{i=0}^n \tilde{T}_{i,j}^{n,q,1} = 1, \quad j = 0, \dots, n,$$

which implies that each row of $(\tilde{T}^{n,q,1})^T$ has sum 1. ■

We now show that $T^{n,q,r}$ can be factorized as a product of 1-banded matrices.

First we require the following lemma.

Lemma 4.1 For $m \geq 1$ and $r, a \in \mathbb{R}$, let $\mathbf{A}(m, a)$ denote the $m \times (m + 1)$ matrix

$$\begin{bmatrix} 1 & r^m - a \\ & 1 & r^{m-1} - a \\ & & \ddots & \ddots \\ & & & 1 & r - a \end{bmatrix}.$$

Then

$$\mathbf{A}(m, a)\mathbf{A}(m+1, b) = \mathbf{A}(m, b)\mathbf{A}(m+1, a). \quad (4.13)$$

Proof We represent the elements of $\mathbf{A}(m, a)$ as

$$A_{i,j}(m, a) = \begin{cases} 1, & i = j, \\ r^{m-i} - a, & i = j - 1, \\ 0, & \text{otherwise,} \end{cases} \quad (4.14)$$

for $0 \leq i \leq m-1$. Let

$$\alpha_{i,j} = \sum_{k=0}^m A_{i,k}(m, a)A_{k,j}(m+1, b)$$

and

$$\beta_{i,j} = \sum_{k=0}^m A_{i,k}(m, b)A_{k,j}(m+1, a).$$

Then, $\alpha_{i,j}$ and $\beta_{i,j}$ are nonzero only for $j = i, j = i+1, j = i+2$. Thus we obtain from (4.14) that

$$\begin{aligned} \alpha_{i,i} &= A_{i,i}(m, a)A_{i,i}(m+1, b) + A_{i,i+1}(m, a)A_{i+1,i}(m+1, b) \\ &= 1 = \beta_{i,i} \end{aligned}$$

and

$$\begin{aligned} \alpha_{i,i+1} &= A_{i,i}(m, a)A_{i,i+1}(m+1, b) + A_{i,i+1}(m, a)A_{i+1,i+1}(m+1, b) \\ &= r^{m+1-i} - b + r^{m-i} - a = \beta_{i,i+1} \end{aligned}$$

and

$$\begin{aligned} \alpha_{i,i+2} &= A_{i,i}(m, a)A_{i,i+2}(m+1, b) + A_{i,i+1}(m, a)A_{i+1,i+2}(m+1, b) \\ &= (r^{m-i} - a)(r^{m-i} - b) = \beta_{i,i+2}. \end{aligned}$$

Thus $\mathbf{A}(m, a)\mathbf{A}(m+1, b) = \mathbf{A}(m, b)\mathbf{A}(m+1, a)$. ■

Theorem 4.5 For $n \geq 2$ and any q, r the matrix $\mathbf{T}^{n,q,r}$ is given by the product

$$\begin{bmatrix} 1 & r - q^{n-1} \\ & 1 \\ & & 1 \\ & & & \ddots \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & r^2 - q^{n-2} \\ & 1 & r - q^{n-2} \\ & & 1 \\ & & & \ddots \\ & & & & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & r^{n-1} - q \\ & 1 & r^{n-2} - q \\ & & \ddots & \ddots \\ & & & 1 & r - q \\ & & & & 1 \\ & & & & & 1 \end{bmatrix}.$$

Proof We use induction on n . The result holds for $n = 2$. Denote the above product by $\mathbf{S}^{n,q,r}$ and assume that, for some $n \geq 2$, $\mathbf{T}^{n,q,r} = \mathbf{S}^{n,q,r}$. Then we can express $\mathbf{S}^{n+1,q,r}$ as the product, in block form,

$$\mathbf{S}^{n+1,q,r} = \begin{bmatrix} 1 & \mathbf{c}_0^T \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{c}_1^T \\ 0 & \mathbf{B}_1 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{c}_2^T \\ 0 & \mathbf{B}_2 \end{bmatrix} \cdots \begin{bmatrix} 1 & \mathbf{c}_{n-1}^T \\ 0 & \mathbf{B}_{n-1} \end{bmatrix},$$

where $\mathbf{c}_0^T, \dots, \mathbf{c}_{n-1}^T$ are row vectors, $\mathbf{0}$ denotes the zero vector, \mathbf{I} the unit matrix and

$$\mathbf{B}_1 \mathbf{B}_2 \cdots \mathbf{B}_{n-1} = \mathbf{S}^{n,q,r} = \mathbf{T}^{n,q,r}.$$

Also, the first column of $\mathbf{S}^{n+1,q,r}$ has 1 in the first row and zeros below. Thus it remains only to verify that the first rows of $\mathbf{T}^{n+1,q,r}$ and $\mathbf{S}^{n+1,q,r}$ are equal. We have

$$[S_{0,0}^{n+1,q,r}, \dots, S_{0,n+1}^{n+1,q,r}] = [\mathbf{w}^T, 0],$$

where,

$$\mathbf{w}^T = \begin{bmatrix} 1 & r - q^n \end{bmatrix} \begin{bmatrix} 1 & r^2 - q^{n-1} \\ & 1 & r - q^{n-1} \end{bmatrix} \dots$$

$$\begin{bmatrix} 1 & r^n - q \\ & 1 & r^{n-1} - q \\ & & \ddots & \ddots \\ & & & 1 & r - q \end{bmatrix}.$$

In the notation defined in the lemma above,

$$\mathbf{w}^T = \mathbf{A}(1, q^n) \mathbf{A}(2, q^{n-1}) \dots \mathbf{A}(n-1, q^2) \mathbf{A}(n, q). \quad (4.15)$$

In view of the lemma, we may permute the quantities q^n, q^{n-1}, \dots, q in (4.15), leaving \mathbf{w}^T unchanged. In particular, we may write

$$\mathbf{w}^T = \mathbf{A}(1, q^{n-1}) \mathbf{A}(2, q^{n-2}) \dots \mathbf{A}(n-1, q) \mathbf{A}(n, q^n). \quad (4.16)$$

Now the product of the first $n-1$ matrices in (4.16) is simply the first row of $\mathbf{S}^{n,q,r}$ and thus

$$\mathbf{w}^T = [S_{0,0}^{n,q,r}, \dots, S_{0,n-1}^{n,q,r}] \begin{bmatrix} 1 & r^n - q^n \\ & \ddots & \ddots \\ & & 1 & r - q^n \end{bmatrix}$$

$$= [T_{0,0}^{n,q,r}, \dots, T_{0,n-1}^{n,q,r}] \begin{bmatrix} 1 & r^n - q^n \\ & \ddots & \ddots \\ & & 1 & r - q^n \end{bmatrix}.$$

This gives

$$S_{0,0}^{n+1,q,r} = T_{0,0}^{n,q,r}$$

and

$$S_{0,j}^{n+1,q,r} = (r^{n+1-j} - q^n) T_{0,j-1}^{n,q,r} + T_{0,j}^{n,q,r}, \quad j = 1, \dots, n,$$

noting that $T_{0,0}^{n,q,r} = 1$ and $T_{0,n}^{n,q,r} = 0$. Then from (4.10)

$$S_{0,j}^{n+1,q,r} = T_{0,j}^{n+1,q,r}, \quad j = 0, \dots, n,$$

and since $S_{0,n+1}^{n+1,q,r} = 0 = T_{0,n+1}^{n+1,q,r}$, the result is true for $n + 1$ and the proof is complete. ■

The following is a consequence of Theorem 4.1 and Theorem 4.5.

Theorem 4.6 *For $0 < q \leq r^{n-1}$ the matrix $\mathbf{T}^{n,q,r}$ is totally positive.*

We note that if $0 < q \leq r^{n-1}$ and

$$p = a_0^q P_0^{n,q} + \dots + a_n^q P_n^{n,q} = a_0^r P_0^{n,r} + \dots + a_n^r P_n^{n,r} \quad (4.17)$$

then Theorem 4.2 shows that

$$S^-(a_0^r, \dots, a_n^r) \leq S^-(a_0^q, \dots, a_n^q),$$

(see Goodman [18, pp. 166]). Since $(P_0^{n,1}, \dots, P_n^{n,1})$ is totally positive it follows from Theorem 4.3 that, for $0 < q \leq r^{n-1} \leq 1$ and p as in (4.17),

$$S^-(p) \leq S^-(a_0^r, \dots, a_n^r) \leq S^-(a_0^q, \dots, a_n^q). \quad (4.18)$$

Corollary 4.6.1 *For any $0 < q \leq 1$, the generalized Bernstein basis*

$$B_j^n(x) = \begin{bmatrix} n \\ j \end{bmatrix} x^j \prod_{t=0}^{n-j-1} (1 - q^t x), \quad 0 \leq j \leq n, \quad 0 \leq x \leq 1 \quad (4.19)$$

is a normalized totally positive basis.

Proof We have seen that $P_j^{n,q}(x) = x^j \prod_{t=0}^{n-j-1} (1 - q^t x)$ is a totally positive basis. Thus applying the property (iii) in Section 4.1 with a totally positive diagonal

matrix $\mathbf{A} = (A_{j,j})_{j=0}^n = \begin{bmatrix} n \\ j \end{bmatrix}$ we obtain

$$\begin{bmatrix} B_0^n(x) \\ B_1^n(x) \\ \vdots \\ B_n^n(x) \end{bmatrix} = \mathbf{A} \begin{bmatrix} P_0^{n,q}(x) \\ P_1^{n,q}(x) \\ \vdots \\ P_n^{n,q}(x) \end{bmatrix}.$$

From (1.15), the generalized Bernstein basis is normalized. ■

We deduce from this and (4.18) that for $0 < q \leq 1$

$$S^-(B_n f) \leq S^-(f(0), f([1]/[n]), \dots, f([n]/[n])) \leq S^-(f). \quad (4.20)$$

The following figures show all third degree generalized Bernstein basis polynomials $B_j^3(x)$, $j = 0, 1, 2, 3$.

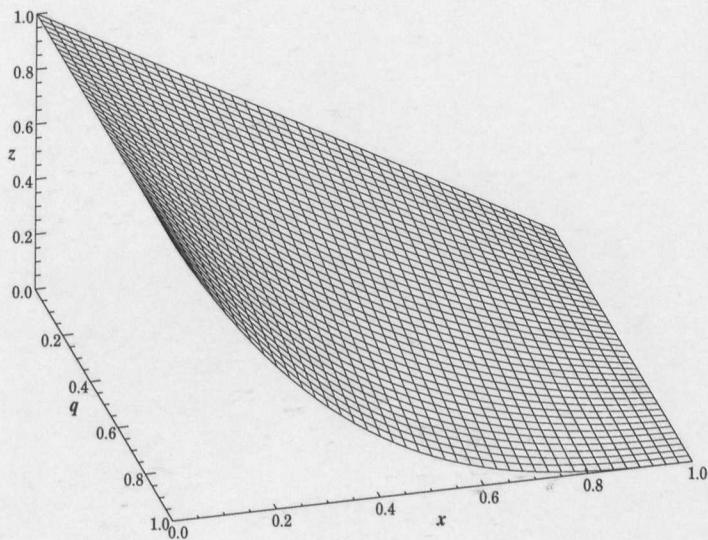


Figure 4.1: $B_0^3 = (1 - x)(1 - qx)(1 - q^2x)$ for values of q between 0 and 1

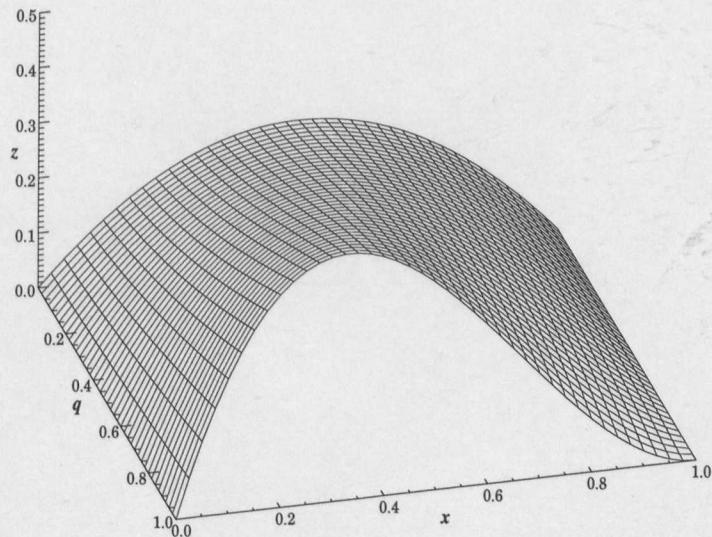


Figure 4.2: $B_1^3 = [3]x(1 - x)(1 - qx)$ for values of q between 0 and 1

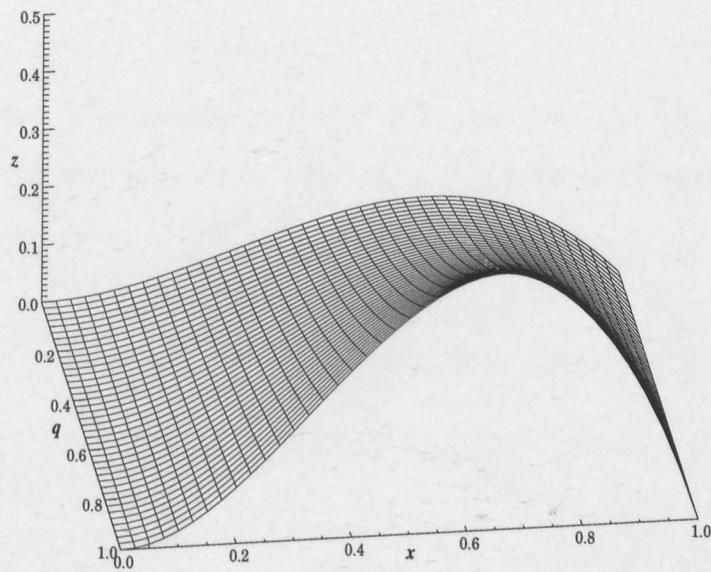


Figure 4.3: $B_2^3 = [3]x^2(1-x)$ for values of q between 0 and 1

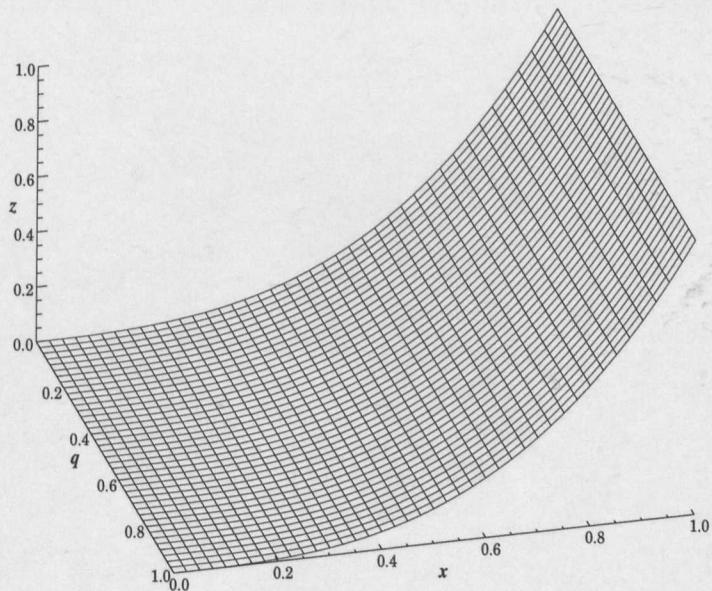


Figure 4.4: $B_3^3 = x^3$ for values of q between 0 and 1

4.3 Convexity

Since the number of sign changes of $B_n^q f$ is bounded by that of f , and also B_n^q reproduces any linear polynomial, that is $B_n^q(ax + b) = ax + b$, we have the following consequence.

Theorem 4.7 *For any function f and any linear polynomial p ,*

$$S^-(B_n^q f - p) = S^-(B_n^q(f - p)) \leq S^-(f - p),$$

for $0 < q \leq 1$.

This is illustrated by Figure 4.5. The function $f(x)$ is $\sin 2\pi x$ and the generalized Bernstein polynomials are of degree $n = 20$ with $q = 0.8$ and $q = 0.9$.

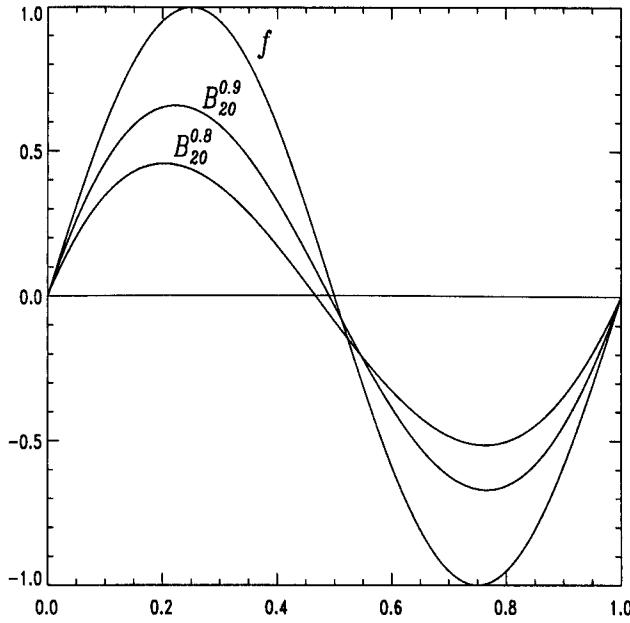


Figure 4.5: $f(x) = \sin 2\pi x$. The polynomials are $B_{20}^{0.8}f$ and $B_{20}^{0.9}f$.

The next result follows from Theorem 4.7.

Theorem 4.8 *If f is increasing (decreasing) on $[0,1]$, then $B_n^q f$ is also increasing (decreasing) on $[0,1]$, for $0 < q \leq 1$.*

Proof Let f be increasing on $[0,1]$. Then, for any constant c ,

$$S^-(B_n^q f - c) = S^-(B_n(f - c)) \leq S^-(f - c) \leq 1$$

and thus $B_n^q f$ is monotonic. Since

$$B_n^q(f; 0) = f(0) \leq f(1) = B_n^q(f; 1),$$

$B_n^q f$ is monotonic increasing. (If f is decreasing we may replace f by $-f$.) ■

We now state a result on convexity.

Theorem 4.9 *If f is convex on $[0,1]$, then $B_n^q f$ is also convex on $[0,1]$, for $0 < q \leq 1$.*

Proof Let p denote any linear polynomial. Then if f is convex we have

$$S^-(B_n^q f - p) = S^-(B_n^q(f - p)) \leq S^-(f - p) \leq 2.$$

Thus if $p(a) = B_n^q(f; a)$ and $p(b) = B_n^q(f; b)$ for $0 < a < b < 1$ then $B_n^q f - p$ cannot change sign in (a, b) . As we vary a and b , a continuity argument shows that the sign of $B_n^q f - p$ on (a, b) is the same for all a and b , $0 < a < b < 1$. From the convexity of f we see that, when $a = 0$ and $b = 1$, $0 \leq p - f$, so that

$$0 \leq B_n^q(p - f) = p - B_n^q f$$

for $0 < q \leq 1$ and thus $B_n^q f$ is convex. ■

We next show that, if f is convex, the generalized Bernstein polynomials $B_n^q f$, for n fixed, are monotonic in q . We first recall Jensen's inequality.

Theorem 4.10 (Jensen's inequality) *Let $f : I \rightarrow \mathbb{R}$ be a convex function. Let $x_0, \dots, x_n \in I$ and let $\lambda_0, \dots, \lambda_n \geq 0$ with $\lambda_0 + \dots + \lambda_n = 1$. Then*

$$f(\lambda_0 x_0 + \dots + \lambda_n x_n) \leq \lambda_0 f(x_0) + \dots + \lambda_n f(x_n).$$

For its proof see Webster [53, pp. 200].

Theorem 4.11 *For $0 < q \leq r \leq 1$ and for f convex on $[0,1]$, we have*

$$B_n^r f \leq B_n^q f.$$

Proof Let us write $\zeta_j^{n,q} = \frac{[j]}{[n]}$ and $a_j^{n,q} = \binom{n}{j}$. Then, for any function g on $[0,1]$,

$$B_n^q g = \sum_{j=0}^n g(\zeta_j^{n,q}) a_j^{n,q} P_j^{n,q}.$$

Using Theorem 4.4 we have

$$\begin{aligned} B_n^q g &= \sum_{j=0}^n \sum_{k=0}^n g(\zeta_j^{n,q}) a_j^{n,q} T_{j,k}^{n,q,r} P_k^{n,r} \\ &= \sum_{k=0}^n P_k^{n,r} \sum_{j=0}^n T_{j,k}^{n,q,r} g(\zeta_j^{n,q}) a_j^{n,q} \end{aligned} \quad (4.21)$$

since $\mathbf{T}^{n,q,r}$ is upper triangular matrix. Using the fact that generalized Bernstein polynomials reproduce linear polynomials we obtain, with $g = 1$,

$$1 = \sum_{j=0}^n a_j^{n,q} P_j^{n,q} = \sum_{k=0}^n P_k^{n,r} \sum_{j=0}^n T_{j,k}^{n,q,r} a_j^{n,q}$$

and hence

$$\sum_{j=0}^n T_{j,k}^{n,q,r} a_j^{n,q} = a_k^{n,r}, \quad k = 0, \dots, n. \quad (4.22)$$

On putting $g(x) = x$ in (4.21), we obtain

$$x = \sum_{j=0}^n \zeta_j^{n,q} a_j^{n,q} P_j^{n,q} = \sum_{k=0}^n P_k^{n,r} \sum_{j=0}^n T_{j,k}^{n,q,r} \zeta_j^{n,q} a_j^{n,q}.$$

Since

$$\sum_{j=0}^n \zeta_j^{n,r} a_j^{n,r} P_j^{n,r} = x$$

we have

$$\sum_{j=0}^n T_{j,k}^{n,q,r} \zeta_j^{n,q} a_j^{n,q} = \zeta_k^{n,r} a_k^{n,r}, \quad k = 0, \dots, n. \quad (4.23)$$

Now if f is convex, it follows from (4.22) and (4.23) and Jensen's inequality with

$$\lambda_j = (a_k^{n,r})^{-1} T_{j,k}^{n,q,r} a_j^{n,q}, \quad x_j = \zeta_j^{n,q}$$

that

$$\begin{aligned} f(\zeta_k^{n,r}) &= f\left(\sum_{j=0}^n (a_k^{n,r})^{-1} T_{j,k}^{n,q,r} \zeta_j^{n,q} a_j^{n,q}\right) \\ &\leq \sum_{j=0}^n (a_k^{n,r})^{-1} T_{j,k}^{n,q,r} a_j^{n,q} f(\zeta_j^{n,q}). \end{aligned} \quad (4.24)$$

Then (4.21) gives

$$B_n^q f = \sum_{j=0}^n f(\zeta_j^{n,q}) a_j^{n,q} P_j^{n,q} = \sum_{j=0}^n \sum_{k=0}^n f(\zeta_j^{n,q}) a_j^{n,q} T_{j,k}^{n,q,r} P_k^{n,r}.$$

Hence we see from (4.24) that

$$\begin{aligned} \sum_{k=0}^n a_k^{n,r} P_k^{n,r} \sum_{j=0}^n (a_k^{n,r})^{-1} T_{j,k}^{n,q,r} f(\zeta_j^{n,q}) a_j^{n,q} &= B_n^q f \\ \sum_{k=0}^n a_k^{n,r} P_k^{n,r} f(\zeta_k^{n,r}) &\leq B_n^q f. \end{aligned}$$

Thus

$$B_n^r f \leq B_n^q f$$

and the proof is complete. ■

Figure 4.6 illustrates the monotonicity in q of the generalized Bernstein polynomials $B_n^q(f; x)$ for the convex function $f(x) = 1 - \sin \pi x$, where $n = 10$ is fixed, $q_1 = 0.5$, $q_2 = 0.75$, and $q_3 = 1$.

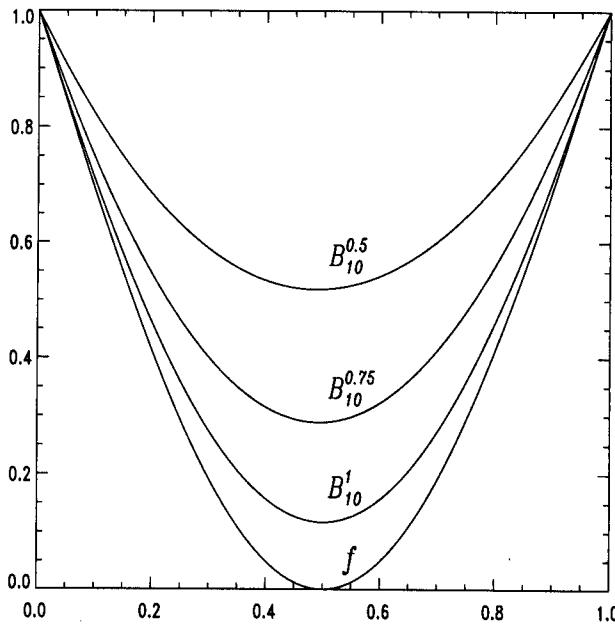


Figure 4.6: Monotonicity of generalized Bernstein polynomials in the parameter q , for $f(x) = 1 - \sin \pi x$. The polynomials are $B_{10}^{0.5}f$, $B_{10}^{0.75}f$ and B_{10}^1f .

Corollary 4.11.1 *If f is convex on $[0, 1]$ then*

$$B_n^q f \geq B_{n+1}^q f \geq f, \quad (4.25)$$

for $0 < q \leq 1$. The inequalities are strict if f is strictly convex on $[0, 1]$.

Proof The result above for $q = 1$ is first proved by Schoenberg [48]. Also see DeVore and Lorentz [12, pp. 310]. When the function f is convex, from Theorem 3.2, $B_n^q f$ is monotonic in n , that is

$$B_n^q f \geq B_{n+1}^q f.$$

From Theorem 4.11 above and the theorem of Schoenberg [48] we obtain

$$f \leq B_{n+1}^1 f \leq B_n^1 f \leq B_n^q f,$$

and

$$f \leq B_{n+1}^1 f \leq B_{n+1}^q f \leq B_n^q f.$$

Thus the approximation to a convex function by its generalized Bernstein polynomial B_n^q is not only monotonic in n and in q but it is also one-sided. ■

Consequently, the shape of $B_n^q f$ preserves the shape of the function f . For this reason we may consider B_n^q as a shape preserving operator.

Dahmen [8] includes an excellent survey on convexity and Bernstein-Bézier polynomials as well as the convexity of multivariate Bernstein polynomials.

For applications in CAGD, we are concerned with parametrically defined curves. Let us define a curve $\mathbf{P}(t)$ with the generalized Bernstein polynomials as its basis functions,

$$\mathbf{P}(t) = (\mathbf{p}_1(t), \mathbf{p}_2(t)) = \sum_{i=0}^n A_i B_i^n(t), \quad 0 \leq t \leq 1, \quad (4.26)$$

where $A_i = (x_i, y_i) \in \mathbb{R}^2$, $i = 0, \dots, n$. We will write $p(A_0, \dots, A_n)$ to denote the polygonal arc which “joins up” the points $A_i = (x_i, y_i)$ $i = 0, \dots, n$. Since the generalized Bernstein basis, which satisfies $\sum_{i=0}^n B_i^n(t) = 1$, is a normalized totally positive basis we have the following consequences.

Theorem 4.12 *The number of times any straight line l crosses the curve \mathbf{P} given by (4.26) is no more than the number of times it crosses the polygonal arc $p(A_0, \dots, A_n)$.*

Proof Consider any straight line l with the equation $ax + by + c = 0$. The number of times it crosses the curve \mathbf{P} is

$$\begin{aligned} S^-(a\mathbf{p}_1 + b\mathbf{p}_2 + c) &= S^- \left(a \sum x_i B_i^n(t) + b \sum y_i B_i^n(t) + c \sum B_i^n(t) \right) \\ &= S^- \left(\sum (ax_i + by_i + c) B_i^n(t) \right) \\ &\leq S^-(ax_0 + by_0 + c, \dots, ax_n + by_n + c) \end{aligned}$$

by Theorem 4.3. Since the latter expression is the number of times the line crosses the arc $p(A_0, \dots, A_n)$, this completes the proof. ■

Corollary 4.12.1 *If the polygon arc $p(A_0, \dots, A_n)$ is monotonic increasing or decreasing, then so is the curve P defined by (4.26).*

Proof Suppose that the arc $p(A_0, \dots, A_n)$ is monotonic increasing, that is $y_0 \leq y_1 \leq \dots \leq y_n$. Then any straight line parallel to the x -axis crosses it at most once. So from Theorem 4.12, any such line crosses the curve P at most once. Since $P(t)$ interpolates the end points of the arc $p(A_0, \dots, A_n)$, the curve P is increasing in the y -direction, that is p_2 is an increasing function. ■

Corollary 4.12.2 *If the polygon arc $p(A_0, \dots, A_n)$ is convex then so is the curve P .*

Proof If the polygon $p(A_0, \dots, A_n)$ is convex, then any straight line crosses it at most twice. Thus, by Theorem 4.12, any straight line crosses the curve P at most twice. Let l be the line interpolating the end points of the curve P . Then $l - p(A_0, \dots, A_n) \geq 0$ and since Bernstein polynomials reproduce linear polynomials we have

$$l - P = B_n l - P \geq 0.$$

Thus the curve P is convex. ■

Indeed, the latter theorem holds for any normalized totally positive basis (see Goodman [17]).

Consequently Theorem 4.12 shows that the shape of the parametric curve P closely mimics the shape of the control polygon $p(A_0, \dots, A_n)$. We can predict or manipulate the shape of the curve by making a suitable choice of the control polygon or the parameter q in the basis functions.

An affine change of variable may be used to give the corresponding generalized Bernstein basis and polynomials on $[a, b]$.

The Bernstein basis has many profound properties. We here quote some recent results on it. Goodman and Said [21] prove the following. Suppose that (ϕ_0, \dots, ϕ_n) is a basis for \mathbb{P}_n and let \mathbf{M} denote the matrix such that whenever

$$\sum_{i=0}^n C_i \phi_i(x) = \sum_{i=0}^n A_i b_i^n(x), \quad C_i, A_i \in \mathbb{R}^2,$$

then $(A_0, \dots, A_n)^T = \mathbf{M}(C_0, \dots, C_n)^T$. Then (ϕ_0, \dots, ϕ_n) is totally positive if \mathbf{M} is totally positive. This follows a conjecture based on the evidence obtained from corner cutting techniques on the polygonal arc C_0, \dots, C_n which leads to the Bézier polygon A_0, \dots, A_n , namely that the Bernstein basis has optimal shape preserving properties among all normalized totally positive bases for \mathbb{P}_n . That is, Bézier points with a Bézier polygon provide us with a better guide to the shape of a curve.

An affirmative answer to this conjecture is given by Carnicer and Peña [5] based on the following definition. A normalized totally positive basis $\Phi = (\phi_0, \dots, \phi_n)$ of \mathbb{P}_n has optimal shape preserving properties if, for any other normalized totally positive basis $\Psi = (\psi_0, \dots, \psi_n)$ of \mathbb{P}_n , there exists a stochastic totally positive \mathbf{M} (that is each row sums to 1) such that

$$(\psi_0, \dots, \psi_n) = (\phi_0, \dots, \phi_n)\mathbf{M}.$$

Then it is shown in Carnicer and Peña [5] that the Bernstein basis is the unique basis with optimal shape preserving properties among all normalized totally positive bases. For details and proofs see Carnicer and Peña [5] and Goodman and Said [21].

Yet there is another property of the Bernstein basis that influences the accuracy and reliability of various calculations on the parametric curves and surfaces. Farouki and Rajan [16] have investigated numerical stability of the polynomials in Bernstein form. For any simple root r of an arbitrary polynomial $P(x)$ on the

interior of the interval (usually $(0,1)$) we have the following advantages of using the Bernstein basis:

- The root condition number is smaller in the Bernstein basis than in the power basis.
- The root condition number decreases monotonically under Bernstein degree elevation and subdivision schemes.
- The root condition number is smaller in the Bernstein basis than in any other basis which may be expressed as a non-negative combination of the Bernstein basis in the interval.

For further details and proofs see Farouki and Rajan [16].

Farouki and Goodman [15] have shown that the Bernstein-Bézier form is optimally stable in the sense that no other non-negative basis yields systematically smaller condition numbers for the values of roots of arbitrary polynomials on the chosen interval. Indeed it is the only stable basis whose elements have no roots on the interior of that interval.

Chapter 5

Factorization of the Vandermonde matrix

This chapter is concerned with the factorization of the Vandermonde matrix into 1-banded matrices, which is preceded by decomposing it into lower and upper triangular matrices. This process uses symmetric functions. The factorization into 1-banded matrices provides an alternative way of verifying the well known result concerning the total positivity of the Vandermonde matrix. Cryer in [7] has shown that a matrix \mathbf{A} is totally positive if and only if \mathbf{A} has an LU-factorization such that \mathbf{L} and \mathbf{U} are totally positive, where \mathbf{L} is a lower triangular matrix and \mathbf{U} is an upper triangular matrix. Goodman and Sharma [22] have shown that a totally positive, symmetric, periodic, banded matrix \mathbf{A} can be factored in a symmetric manner into positive 1-banded periodic factors.

5.1 Symmetric functions

First, we require the following definitions to describe the elements of $\mathbf{L}^{(n)}$ and $\mathbf{U}^{(n)}$ in the factorization of the Vandermonde matrix.

Definition 5.1 For integers $1 \leq r \leq n$, $\sigma(n, r)$ is the r th elementary symmetric function. This is the sum of all products of r distinct real variables chosen from n variables. We set $\sigma(n, 0) = 1$, $n \geq 1$, and write,

$$\sigma(n, r) = \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} x_{i_1} \dots x_{i_r}.$$

Definition 5.2 For integers $n, r \geq 1$, $\tau(n, r)$ is the r th complete symmetric function defined by the sum of all products of n variables of order r . That is

$$\tau(n, r) = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_n \leq n} x_{i_1}^{\lambda_1} \dots x_{i_n}^{\lambda_n}, \quad \lambda_1 + \dots + \lambda_n = r$$

where $\lambda_1, \dots, \lambda_n \in \{0, 1, \dots, r\}$. We set $\tau(n, 0) = 1$, $n \geq 1$. We will also use $\tau_r(x_1, \dots, x_n)$ to denote $\tau(n, r)$.

See for example Macdonald [35].

The generating function of the elementary symmetric function is well known. We have

$$S(x) = (1 - x_1x) \dots (1 - x_nx) = \sum_{r=0}^n (-1)^r \sigma(n, r)x^r. \quad (5.1)$$

The generating function for the complete symmetric function is $\frac{1}{S(x)}$, since

$$\begin{aligned} \frac{1}{S(x)} &= \frac{1}{(1 - x_1x) \dots (1 - x_nx)} = \prod_{j=1}^n \sum_{r=0}^{\infty} x_j^r x^r \\ &= \sum_{r=0}^{\infty} \tau(n, r)x^r. \end{aligned} \quad (5.2)$$

Example 5.1

$$\begin{aligned}\sigma(3, 2) &= x_1x_2 + x_1x_3 + x_2x_3 \\ \tau(2, 2) &= \sigma(2, 2) + x_1^2 + x_2^2 \\ \tau(2, 3) &= x_1x_2^2 + x_1^2x_2 + x_1^3 + x_2^3.\end{aligned}$$

Lemma 5.1 *The complete symmetric functions satisfy the recurrence relation*

$$\tau(n, r) = \tau(n - 1, r) + x_n \tau(n, r - 1), \quad (5.3)$$

for integers $n, r \geq 1$.

Proof Consider the identity

$$\frac{1}{S(x)} = \frac{1}{(1 - x_1x)\dots(1 - x_{n-1}x)} + \frac{x_n x}{(1 - x_1x)\dots(1 - x_n x)}. \quad (5.4)$$

On using (5.2) and comparing the coefficients of x^r on both sides of (5.4), we obtain (5.3). ■

This lemma with a different (combinatorial) proof has recently appeared in Konvalin [30].

Applying a similar method as in the latter lemma, we may easily obtain a recurrence relation for the symmetric functions,

$$\sigma(n, r) = \sigma(n - 1, r) + x_n \sigma(n - 1, r - 1). \quad (5.5)$$

We now give a few identities derived from the generating functions for the symmetric and complete functions involving special cases of the variables.

Substituting $x_i = 1$, $i = 1, \dots, n$ in (5.1) and (5.2) we derive the binomial coefficients:

$$\begin{aligned}(1 - x)^n &= \sum_{r=0}^n (-1)^r \binom{n}{r} x^r, \\ \frac{1}{(1 - x)^n} &= \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r.\end{aligned}$$

Also, the recurrence relation (5.5) with $x_n = 1$ gives the corresponding recurrence relation for the binomial coefficients.

Letting $x_i = q^{i-1}$, $i = 1, \dots, n$, we obtain the Gaussian polynomials:

$$(1-x)(1-qx)\dots(1-q^{n-1}x) = \sum_{r=0}^n (-1)^r q^{r(r-1)/2} \begin{bmatrix} n \\ r \end{bmatrix} x^r,$$

$$\frac{1}{(1-x)(1-qx)\dots(1-q^{n-1}x)} = \sum_{r=0}^{\infty} \begin{bmatrix} n+r-1 \\ r \end{bmatrix} x^r.$$

Also, the recurrence relation (5.3) with $x_n = q^{n-1}$ gives the q -binomial recurrence relation (1.4).

Putting $x_i = [i]$, $i = 1, \dots, n$ we obtain Gould's q -analogues of Stirling numbers:

$$(1-x)(1-[2]x)\dots(1-[n]x) = \sum_{r=0}^n (-1)^r S_1(n, r) x^r$$

$$\frac{1}{(1-x)(1-[2]x)\dots(1-[n]x)} = \sum_{r=0}^{\infty} S_2(n, r) x^r,$$

We may replace the coefficients in the latter equations by

$$(-1)^r S_1(n, r) = s_q(n+1, n-r+1) \text{ and } S_2(n, r) = S_q(n+r, n).$$

Once again, the recurrence relations (5.5) and (5.3) give the corresponding recurrence relations for S_1 and S_2 respectively.

Lemma 5.2 *For integers $k, n \geq 1$ the following holds:*

$$\sum_{j=0}^{\min(k,n)} (-1)^j \sigma(n, j) \tau(n, k-j) = 0. \quad (5.6)$$

Proof We use the generating functions for $\sigma(n, r)$ and $\tau(n, r)$ to verify the lemma. It is easily seen that

$$S(x) \frac{1}{S(x)} = 1 = \sum_{r=0}^n (-1)^r \sigma(n, r) x^r \sum_{s=0}^{\infty} \tau(n, s) x^s = \sum_{k=0}^{\infty} a_k x^k,$$

where

$$a_k = \sum_{j=0}^{\min(k,n)} (-1)^j \sigma(n, j) \tau(n, k-j).$$

Thus the coefficient a_k is nonzero only for $k = 0$, and then $a_0 = 1$. ■

This lemma gives rise to a few identities. If $k = n$ and $x_i = 1$, $i = 1, \dots, n$ then,

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \binom{n+j-1}{j} = 0$$

and for $x_i = q^{i-1}$, $i = 1, \dots, n$

$$\sum_{j=0}^n (-1)^{n-j} q^{(n-j)(n-j-1)/2} \binom{n}{j} \binom{n+j-1}{j} = 0.$$

Setting $x_i = [i]$, $i = 1, \dots, n$, we obtain

$$\sum_{j=0}^n (-1)^j S_1(n, j) S_2(n, n-j) = 0.$$

The above lemma has appeared in Konvalin [30].

5.2 Factorization process

Since it is easier to express a triangular matrix as a product of 1-banded matrices, we first split the Vandermonde matrix into lower and upper triangular matrices. Let $\mathbf{V}^{(n)}$ be the n th order Vandermonde matrix

$$\mathbf{V}^{(n)} = \begin{bmatrix} 1 & x_0 & \dots & x_0^n \\ 1 & x_1 & \dots & x_1^n \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^n \end{bmatrix}$$

and let $\mathbf{V}^{(n)} = \mathbf{L}^{(n)} \mathbf{U}^{(n)}$ where $\mathbf{L}^{(n)}$ is a lower triangular matrix with units on its main diagonal and $\mathbf{U}^{(n)}$ is an upper triangular matrix. This factorization is unique. We apply Crout's algorithm to obtain the elements of the matrices $\mathbf{L}^{(n)}$ and $\mathbf{U}^{(n)}$.

Algorithm

do $i = 0, \dots, n$

$$l_{i,i}^{(n)} = 1 \quad (5.7)$$

enddo

do $j = 0, \dots, n$

do $i = 0, \dots, j$

$$u_{i,j}^{(n)} = v_{i,j}^{(n)} - \sum_{k=0}^{i-1} l_{i,k}^{(n)} u_{k,j}^{(n)} \quad (5.8)$$

enddo

do $i = j + 1, \dots, n$

$$l_{i,j}^{(n)} = \frac{1}{u_{j,j}^{(n)}} \left(v_{i,j}^{(n)} - \sum_{k=0}^{j-1} l_{i,k}^{(n)} u_{k,j}^{(n)} \right) \quad (5.9)$$

enddo

enddo

The algorithm evaluates the entries of upper and lower matrices as follows.

First, (5.7) sets the diagonal entries of $\mathbf{L}^{(n)}$ to 1. Then for each j , (5.8) calculates the j th column of $\mathbf{U}^{(n)}$ and then the j th column of $\mathbf{L}^{(n)}$ is calculated from (5.9). For example $j = 0$ gives

$$u_{0,0}^{(n)} = v_{0,0}^{(n)} = 1 \text{ and } l_{i,0}^{(n)} = \frac{v_{i,0}^{(n)}}{u_{0,0}^{(n)}} = 1, \text{ for } i = 1, 2, \dots, n.$$

With $j = 1$ we calculate

$$u_{0,1}^{(n)} = v_{0,1}^{(n)} = x_0 \text{ and } u_{1,1}^{(n)} = v_{1,1}^{(n)} - u_{0,1}^{(n)} = x_1 - x_0$$

and

$$\begin{aligned} l_{i,1}^{(n)} &= \frac{1}{u_{1,1}^{(n)}}(v_{i,1}^{(n)} - l_{i,0}^{(n)}u_{0,1}^{(n)}) \\ &= \frac{x_i - x_0}{x_1 - x_0}, \text{ for } i = 2, 3, \dots, n. \end{aligned}$$

Next, with $j = 2$, we compute

$$\begin{aligned} u_{0,2}^{(n)} &= v_{0,2}^{(n)} = x_0^2 \\ u_{1,2}^{(n)} &= v_{1,2}^{(n)} - l_{1,0}^{(n)}u_{0,2}^{(n)} = x_1^2 - x_0^2 \\ u_{2,2}^{(n)} &= v_{2,2}^{(n)} - (l_{2,0}^{(n)}u_{0,2}^{(n)} + l_{2,1}^{(n)}u_{1,2}^{(n)}) \\ &= x_2^2 - (x_0^2 + (x_2 - x_0)(x_1 + x_0)) = (x_2 - x_1)(x_2 - x_0) \end{aligned}$$

and

$$\begin{aligned} l_{i,2}^{(n)} &= \frac{1}{u_{2,2}^{(n)}}(v_{i,2}^{(n)} - l_{i,0}^{(n)}u_{0,2}^{(n)} - l_{i,1}^{(n)}u_{1,2}^{(n)}) \\ &= \frac{x_i^2 - x_0^2 - (x_i - x_0)(x_1 + x_0)}{(x_2 - x_1)(x_2 - x_0)} \\ &= \frac{(x_i - x_1)(x_i - x_0)}{(x_2 - x_1)(x_2 - x_0)}, \text{ for } i = 3, 4, \dots, n. \end{aligned}$$

We apply Crout's algorithm to decompose the Vandermonde matrix and from (5.8) and (5.9) we conjecture from the evidence obtained from small values of n that

$$l_{i,j}^{(n)} = \prod_{t=0}^{j-1} \frac{x_i - x_{j-t-1}}{x_j - x_{j-t-1}}, \quad 0 \leq j \leq i \leq n, \quad (5.10)$$

$$u_{i,j}^{(n)} = \tau_{j-i}(x_0, \dots, x_i) \prod_{t=0}^{i-1} (x_i - x_t), \quad 0 \leq i \leq j \leq n, \quad (5.11)$$

where an empty product denotes 1. In order to verify the formulas (5.10) and (5.11), consider

$$v_{i,j}^{(n)} = \sum_{k=0}^i l_{i,k}^{(n)} u_{k,j}^{(n)}.$$

Thus on substituting the entries of the matrices $l_{i,k}^{(n)}$ and $u_{k,j}^{(n)}$ from (5.10) and (5.11), we obtain

$$v_{i,j}^{(n)} = x_0^j + (x_i - x_0)\tau_{j-1}(x_0, x_1) + \cdots + (x_i - x_0) \dots (x_i - x_{i-1})\tau_{j-i}(x_0, \dots, x_i).$$

Now we recall the interpolating polynomial in divided difference form of a function f at the points x_0, \dots, x_i

$$p_n(x) = f[x_0] + (x - x_0)f[x_0, x_1] + \cdots + (x - x_0) \dots (x - x_{i-1})f[x_0, \dots, x_i] \quad (5.12)$$

where

$$f[x_0, \dots, x_i] = \sum_{s=0}^i \frac{f(x_s)}{\prod_{\substack{t=0 \\ t \neq s}}^i (x_s - x_t)}.$$

So, for $f(x) = x^j$ and $0 \leq i \leq j$, we have

$$f[x_0, \dots, x_i] = \sum_{s=0}^i \frac{x_s^j}{\prod_{\substack{t=0 \\ t \neq s}}^i (x_s - x_t)}. \quad (5.13)$$

We also recall the Lagrange interpolating polynomial for $f(x) = x^i$ at the points x_0, \dots, x_i , to find a partial fraction representation of the generating function of the complete symmetric functions. That is

$$x^i = \sum_{j=0}^i x_j^i \mathcal{L}_j(x),$$

where

$$\mathcal{L}_j(x) = \prod_{\substack{t=0 \\ t \neq j}}^i \left(\frac{x - x_t}{x_j - x_t} \right).$$

On making the change of variable $u = 1/x$ in the Lagrange interpolating polynomial and then cancelling u^i on each sides, we have

$$\sum_{s=0}^i x_s^i \prod_{\substack{t=0 \\ t \neq s}}^i \left(\frac{1 - x_t x}{x_s - x_t} \right) = 1.$$

On dividing this by $\prod_{t=0}^i (1 - x_t x)$ we obtain

$$\frac{1}{(1 - x_0 x)(1 - x_1 x) \dots (1 - x_i x)} = \sum_{s=0}^i \frac{x_s^i}{(1 - x_s x) \prod_{\substack{t=0 \\ t \neq s}}^i (x_s - x_t)}. \quad (5.14)$$

On the right, we expand $\frac{1}{1-x_s x}$ as an infinite series and so obtain

$$\sum_{s=0}^i \frac{x_s^i}{(1-x_s x) \prod_{\substack{t=0 \\ t \neq s}}^i (x_s - x_t)} = \sum_{s=0}^i \frac{\sum_{r=0}^{\infty} x_s^{i+r} x^r}{\prod_{\substack{t=0 \\ t \neq s}}^i (x_s - x_t)}.$$

We deduce from this and (5.14) that

$$\sum_{r=0}^{\infty} \tau_r(x_0, \dots, x_i) x^r = \sum_{s=0}^i \frac{\sum_{r=0}^{\infty} x_s^{i+r} x^r}{\prod_{\substack{t=0 \\ t \neq s}}^i (x_s - x_t)}.$$

Thus on comparing the coefficients of x^{j-i} in the above equation and using (5.13) we deduce that

$$\tau_{j-i}(x_0, \dots, x_i) = f[x_0, \dots, x_i], \text{ where } f(x) = x^j, \quad 0 \leq i \leq j. \quad (5.15)$$

Therefore we substitute $f(x) = x^j$ and $x = x_i$ in (5.12) and obtain

$$x_i^j = x_0^j + (x_i - x_0) \tau_{j-1}(x_0, x_1) + \dots + (x_i - x_0) \dots (x_i - x_{i-1}) \tau_{j-i}(x_0, \dots, x_i) = v_{i,j}^{(n)}.$$

This, together with the uniqueness of factorization, verifies the formulas (5.10) and (5.11) for the elements of $\mathbf{L}^{(n)}$ and $\mathbf{U}^{(n)}$ obtained from the decomposition of $\mathbf{V}^{(n)}$. We note that the identity (5.15) is proved in Milne-Thomson [37] and quoted in Neuman [38]. Further studies concerning complete symmetric functions through B-splines and q -binomial coefficients can also be found in Neuman [38].

Next we give an example of the factorization of the Vandermonde matrix into 1-banded matrices for $n = 3$ and then state a theorem concerning this factorization for a general value of n .

Example 5.2

$$\mathbf{V}^{(3)} = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix}$$

and $\mathbf{V}^{(3)} = \mathbf{L}^{(3)}\mathbf{U}^{(3)}$ where, as we saw above,

$$\mathbf{L}^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & \frac{x_2-x_0}{x_1-x_0} & 1 & 0 \\ 1 & \frac{x_3-x_0}{x_1-x_0} & \frac{(x_3-x_1)(x_3-x_0)}{(x_2-x_1)(x_2-x_0)} & 1 \end{bmatrix},$$

$$\mathbf{U}^{(3)} = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 0 & x_1 - x_0 & x_1^2 - x_0^2 & x_1^3 - x_0^3 \\ 0 & 0 & (x_2 - x_1)(x_2 - x_0) & (x_2 - x_1)(x_2 - x_0)(x_0 + x_1 + x_2) \\ 0 & 0 & 0 & (x_3 - x_2)(x_3 - x_1)(x_3 - x_0) \end{bmatrix}.$$

$\mathbf{L}^{(3)}$ is factorized into 1-lower banded matrices such that $\mathbf{L}^{(3)} = \mathbf{L}^{3,1}\mathbf{L}^{3,2}\mathbf{L}^{3,3}$, where

$$\mathbf{L}^{3,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{L}^{3,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \frac{x_3-x_2}{x_2-x_1} & 1 \end{bmatrix},$$

$$\mathbf{L}^{3,3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & \frac{x_2-x_1}{x_1-x_0} & 1 & 0 \\ 0 & 0 & \frac{(x_3-x_2)(x_3-x_1)}{(x_2-x_1)(x_2-x_0)} & 1 \end{bmatrix}.$$

Similarly $\mathbf{U}^{(3)}$ is factorized into 1-upper banded matrices such that $\mathbf{U}^{(3)} = \mathbf{U}^{3,3}\mathbf{U}^{3,2}\mathbf{U}^{3,1}$, where

$$\mathbf{U}^{3,3} = \begin{bmatrix} 1 & x_0 & 0 & 0 \\ 0 & x_1 - x_0 & \frac{x_1(x_1-x_0)}{x_2-x_1} & 0 \\ 0 & 0 & x_2 - x_0 & \frac{x_2(x_2-x_1)(x_2-x_0)}{(x_3-x_2)(x_3-x_1)} \\ 0 & 0 & 0 & x_3 - x_0 \end{bmatrix},$$

$$\mathbf{U}^{3,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x_0 & 0 \\ 0 & 0 & x_2 - x_1 & \frac{x_1(x_2-x_1)}{x_3-x_2} \\ 0 & 0 & 0 & x_3 - x_1 \end{bmatrix}, \quad \mathbf{U}^{3,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x_0 \\ 0 & 0 & 0 & x_3 - x_2 \end{bmatrix}$$

so that

$$\mathbf{V}^{(3)} = \mathbf{L}^{3,1}\mathbf{L}^{3,2}\mathbf{L}^{3,3}\mathbf{U}^{3,3}\mathbf{U}^{3,2}\mathbf{U}^{3,1}.$$

Theorem 5.1 For integers $n \geq 1$ and x_0, x_1, \dots, x_n distinct, the n th order real Vandermonde matrix can be factorized into $2n$ 1-banded matrices such that

$$\mathbf{V}^{(n)} = \mathbf{L}^{n,1}\mathbf{L}^{n,2}\dots\mathbf{L}^{n,n}\mathbf{U}^{n,n}\mathbf{U}^{n,n-1}\dots\mathbf{U}^{n,1} \quad (5.16)$$

where, for $1 \leq k \leq n$,

$$l_{i,j}^{n,k} = \begin{cases} 1, & i = j, \\ \prod_{t=0}^{k-n+i-2} \frac{x_i - x_{i-1-t}}{x_{i-1} - x_{i-2-t}}, & i = j+1, i \geq n-k+1, \\ 0, & \text{otherwise} \end{cases} \quad (5.17)$$

and

$$u_{i,j}^{n,k} = \begin{cases} 1, & i = j, i \leq n-k, \\ x_i - x_{n-k}, & i = j, i > n-k, \\ x_{k-n+i} \prod_{t=1}^{k-n+i} \frac{x_i - x_{i-t}}{x_{i+1} - x_{i+1-t}}, & i = j-1, i \geq n-k, \\ 0, & \text{otherwise,} \end{cases} \quad (5.18)$$

noting that an empty product denotes 1. Thus

$$\mathbf{L}^{n,1}\mathbf{L}^{n,2}\dots\mathbf{L}^{n,n} = \mathbf{L}^{(n)} \text{ and } \mathbf{U}^{n,n}\mathbf{U}^{n,n-1}\dots\mathbf{U}^{n,1} = \mathbf{U}^{(n)}$$

so that $\mathbf{V}^{(n)} = \mathbf{L}^{(n)}\mathbf{U}^{(n)}$.

Proof We use induction on n . When $n = 1$

$$\mathbf{L}^{(1)} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } \mathbf{U}^{(1)} = \begin{bmatrix} 1 & x_0 \\ 0 & x_1 - x_0 \end{bmatrix},$$

which are both 1-banded matrices, and

$$\mathbf{L}^{(1)}\mathbf{U}^{(1)} = \begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix} = \mathbf{V}^{(1)}.$$

We now split the rest of the proof into two parts, the factorization of $\mathbf{L}^{(n)}$ and the factorization of $\mathbf{U}^{(n)}$. Next we will show by induction on k , for $1 \leq k \leq n$, that

$$\mathbf{L}^{n,1}\mathbf{L}^{n,2}\dots\mathbf{L}^{n,k} = \begin{bmatrix} \mathbf{I}_{n-k} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{L}}^{n,k} \end{bmatrix}, \quad (5.19)$$

where each $\mathbf{0}$ denotes the appropriate zero matrix, \mathbf{I}_{n-k} denotes the $(n-k) \times (n-k)$ identity matrix, $\tilde{\mathbf{L}}^{n,k}$ is a $(k+1) \times (k+1)$ lower triangular matrix such that

$$\tilde{l}_{i,j}^{n,k} = \begin{cases} 1, & i = j, \\ \prod_{t=0}^{j-1} \frac{x_{n-k+i-t} - x_{n-k+j-t-1}}{x_{n-k+j-t} - x_{n-k+j-t-1}}, & 0 \leq j < i \leq k \end{cases} \quad (5.20)$$

and an empty product denotes 1.

For $k = 1$, $\tilde{\mathbf{L}}^{n,1} = \mathbf{L}^{(1)}$ and from (5.17) and (5.19) we see that

$$\begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{L}}^{n,1} \end{bmatrix} = \mathbf{L}^{n,1}.$$

We now assume that (5.19) is true for some $k \geq 1$. It is necessary to verify the following identity:

$$\begin{bmatrix} \mathbf{I}_{n-k-1} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{L}}^{n,k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{n-k} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{L}}^{n,k} \end{bmatrix} \mathbf{L}^{n,k+1}. \quad (5.21)$$

On the right, we modify $\tilde{\mathbf{L}}^{n,k}$ by adding a column and a row, defining

$$\hat{\mathbf{L}}^{n,k} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \tilde{\mathbf{L}}^{n,k} \end{bmatrix},$$

where $\mathbf{0}$ is a zero column vector. Thus

$$\hat{l}_{i,j}^{n,k} = \begin{cases} 1, & i = j, \\ \prod_{t=0}^{j-2} \frac{x_{n-k+i-1} - x_{n-k+j-t-2}}{x_{n-k+j-1} - x_{n-k+j-t-2}}, & 1 \leq j < i \leq k+1, \\ 0, & \text{otherwise.} \end{cases} \quad (5.22)$$

Also, we represent $\mathbf{L}^{n,k+1}$ in block form as

$$\mathbf{L}^{n,k+1} = \begin{bmatrix} \mathbf{I}_{n-k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{n,k+1} \end{bmatrix},$$

where each $\mathbf{0}$ is the appropriate zero matrix and $\mathbf{B}^{n,k+1}$ is the $(k+2) \times (k+2)$ 1-lower banded matrix defined by

$$b_{i,j}^{n,k+1} = \begin{cases} 1, & i = j, \\ \prod_{t=0}^{i-2} \frac{x_{n-k+i-1} - x_{n-k+i-t-2}}{x_{n-k+i-2} - x_{n-k+i-t-3}}, & i = j+1, 0 \leq j \leq k, \\ 0, & \text{otherwise.} \end{cases} \quad (5.23)$$

Thus

$$\begin{bmatrix} \mathbf{I}_{n-k-1} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{L}}^{n,k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{n-k-1} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{L}}^{n,k} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{n-k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{n,k+1} \end{bmatrix},$$

which yields

$$\tilde{\mathbf{L}}^{n,k+1} = \hat{\mathbf{L}}^{n,k} \mathbf{B}^{n,k+1}.$$

The (i,j) th element of $\hat{\mathbf{L}}^{n,k} \mathbf{B}^{n,k+1}$ is, say,

$$m_{i,j} = \sum_{s=0}^{k+1} \hat{l}_{i,s}^{n,k} b_{s,j}^{n,k+1}, \quad 0 \leq i, j \leq k+1.$$

Since $\mathbf{B}^{n,k+1}$ is 1-lower banded, its only non-zero elements are $b_{j,j}^{n,k+1}$ and $b_{j+1,j}^{n,k+1}$ so that

$$m_{i,j} = \hat{l}_{i,j}^{n,k} b_{j,j}^{n,k+1} + \hat{l}_{i,j+1}^{n,k} b_{j+1,j}^{n,k+1}.$$

Using (5.22) and (5.23) we have

$$\begin{aligned} m_{i,j} &= \prod_{t=0}^{j-2} \frac{x_{n-k+i-1} - x_{n-k+j-t-2}}{x_{n-k+j-1} - x_{n-k+j-t-2}} \\ &+ \prod_{t=0}^{j-1} \frac{x_{n-k+i-1} - x_{n-k+j-t-1}}{x_{n-k+j} - x_{n-k+j-t-1}} \prod_{t=0}^{j-1} \frac{x_{n-k+j} - x_{n-k+j-t-1}}{x_{n-k+j-1} - x_{n-k+j-t-2}}. \end{aligned}$$

It follows that

$$m_{i,j} = \frac{(x_{n-k+i-1} - x_{n-k-1}) \prod_{t=0}^{j-2} (x_{n-k+i-1} - x_{n-k+j-t-2})}{\prod_{t=0}^{j-1} (x_{n-k+j-1} - x_{n-k+j-t-2})}$$

and thus we obtain

$$m_{i,j} = \prod_{t=0}^{j-1} \frac{x_{n-k+i-1} - x_{n-k+j-t-2}}{x_{n-k+j-1} - x_{n-k+j-t-2}}, \quad 0 \leq j < i \leq k+1.$$

But we see from (5.20) that $m_{i,j} = \tilde{l}_{i,j}^{n,k+1}$. Since, when $k = n$, we have from (5.19) and (5.20)

$$\mathbf{L}^{n,1} \mathbf{L}^{n,2} \dots \mathbf{L}^{n,n} = \tilde{\mathbf{L}}^{n,n} = \mathbf{L}^{(n)},$$

this completes the proof by induction.

Next, following a similar technique we show that

$$\mathbf{U}^{n,k} \mathbf{U}^{n,k-1} \dots \mathbf{U}^{n,1} = \begin{bmatrix} \mathbf{I}_{n-k} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{U}}^{n,k} \end{bmatrix}, \quad (5.24)$$

where each $\mathbf{0}$ is the appropriate zero matrix and $\tilde{\mathbf{U}}^{n,k}$ is a $(k+1) \times (k+1)$ upper triangular matrix such that

$$\tilde{u}_{i,j}^{n,k} = \tau_{j-i}(x_0, \dots, x_i) \prod_{t=1}^i (x_{n-k+i} - x_{n-k+i-t}), \quad 0 \leq i \leq j \leq k, \quad (5.25)$$

with an empty product denoting 1. For $k = 1$,

$$\tilde{\mathbf{U}}^{n,1} = \begin{bmatrix} 1 & x_0 \\ 0 & x_n - x_{n-1} \end{bmatrix}$$

and from (5.18) we see that

$$\begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{U}}^{n,1} \end{bmatrix} = \mathbf{U}^{n,1}.$$

Now we need to verify the following:

$$\begin{bmatrix} \mathbf{I}_{n-k-1} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{U}}^{n,k+1} \end{bmatrix} = \mathbf{U}^{n,k+1} \begin{bmatrix} \mathbf{I}_{n-k} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{U}}^{n,k} \end{bmatrix}. \quad (5.26)$$

On the right, we represent $\mathbf{U}^{n,k+1}$ in block form as

$$\mathbf{U}^{n,k+1} = \begin{bmatrix} \mathbf{I}_{n-k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{n,k+1} \end{bmatrix},$$

where $\mathbf{C}^{n,k+1}$ is the $(k+2) \times (k+2)$ 1-upper banded matrix defined by

$$c_{i,j}^{n,k+1} = \begin{cases} 1, & i = j = 0, \\ x_{n-k+i-1} - x_{n-k-1}, & 1 \leq i = j \leq k+1, \\ x_i \prod_{t=1}^i \frac{x_{n-k+i-1} - x_{n-k+i-t-1}}{x_{n-k+i} - x_{n-k+i-t}}, & i = j-1, 0 \leq i \leq k+1, \\ 0, & \text{otherwise.} \end{cases} \quad (5.27)$$

We also modify $\tilde{\mathbf{U}}^{n,k}$ by adding a column and a row to give

$$\hat{\mathbf{U}}^{n,k} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \tilde{\mathbf{U}}^{n,k} \end{bmatrix},$$

where $\mathbf{0}$ is a zero column vector and

$$\hat{u}_{i,j}^{n,k} = \begin{cases} 1, & i = j = 0, \\ \tau_{j-i}(x_0, \dots, x_{i-1}) \prod_{t=1}^{i-1} (x_{n-k+i-1} - x_{n-k+i-t-1}), & 1 \leq i \leq j \leq k+1, \\ 0, & \text{otherwise.} \end{cases} \quad (5.28)$$

Thus

$$\begin{bmatrix} \mathbf{I}_{n-k-1} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{U}}^{n,k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{n-k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{n,k+1} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{n-k-1} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{U}}^{n,k} \end{bmatrix},$$

which gives

$$\tilde{\mathbf{U}}^{n,k+1} = \mathbf{C}^{n,k+1} \hat{\mathbf{U}}^{n,k}.$$

The (i, j) th element of $\mathbf{C}^{n,k+1} \hat{\mathbf{U}}^{n,k}$ is, say,

$$n_{i,j} = \sum_{s=0}^{k+1} c_{i,s}^{n,k+1} \hat{u}_{s,j}^{n,k}, \quad 0 \leq i \leq j \leq k+1.$$

Since $\mathbf{C}^{n,k+1}$ is 1-upper banded, its only non-zero entries are $c_{i,i}^{n,k+1}$ and $c_{i,i+1}^{n,k+1}$ and thus

$$n_{i,j} = c_{i,i}^{n,k+1} \hat{u}_{i,j}^{n,k} + c_{i,i+1}^{n,k+1} \hat{u}_{i+1,j}^{n,k}.$$

On using (5.27) and (5.28) we obtain, for $i \geq 1$,

$$\begin{aligned} n_{i,j} &= (x_{n-k+i-1} - x_{n-k-1}) \tau_{j-i}(x_0, \dots, x_{i-1}) \prod_{t=1}^{i-1} (x_{n-k+i-1} - x_{n-k+i-t-1}) \\ &+ x_i \tau_{j-i-1}(x_0, \dots, x_i) \prod_{t=1}^i \frac{x_{n-k+i-1} - x_{n-k+i-t-1}}{x_{n-k+i} - x_{n-k+i-t}} \prod_{t=1}^i (x_{n-k+i} - x_{n-k+i-t}). \end{aligned}$$

This gives

$$n_{i,j} = (\tau_{j-i}(x_0, \dots, x_{i-1}) + x_i \tau_{j-i-1}(x_0, \dots, x_i)) \prod_{t=1}^i (x_{n-k+i-1} - x_{n-k+i-t-1})$$

for $0 \leq i \leq j \leq k+1$. By Lemma (5.1)

$$\tau_{j-i}(x_0, \dots, x_{i-1}) + x_i \tau_{j-i-1}(x_0, \dots, x_i) = \tau_{j-i}(x_0, \dots, x_i).$$

Thus we have

$$n_{i,j} = \tau_{j-i}(x_0, \dots, x_i) \prod_{t=1}^i (x_{n-k+i-1} - x_{n-k+i-t-1}) = \tilde{u}_{i,j}^{n,k+1}.$$

Since, when $k = n$, we have from (5.24) and (5.25)

$$\mathbf{U}^{n,n} \mathbf{U}^{n,n-1} \dots \mathbf{U}^{n,1} = \tilde{\mathbf{U}}^{n,n} = \mathbf{U}^{(n)},$$

this completes the proof by induction.

Hence

$$\mathbf{L}^{n,1} \mathbf{L}^{n,2} \dots \mathbf{L}^{n,n} \mathbf{U}^{n,n} \mathbf{U}^{n,n-1} \dots \mathbf{U}^{n,1} = \mathbf{V}^{(n)}$$

and the proof of the theorem on the factorization of the Vandermonde matrix is complete. ■

Corollary 5.1.1 $\mathbf{V}^{(n)}$ is totally positive for $0 < x_0 < x_1 < \dots < x_n$.

The condition makes all elements of $\mathbf{L}^{n,k}$ and $\mathbf{U}^{n,k}$ positive for $1 \leq k \leq n$. Since each of the $2n$ matrices in the complete factorization of $\mathbf{V}^{(n)}$ is a totally positive matrix, so is $\mathbf{V}^{(n)}$. According to Cryer [7], both $\mathbf{L}^{(n)}$ and $\mathbf{U}^{(n)}$ are totally positive if and only if $\mathbf{V}^{(n)}$ is totally positive.

Example 5.3 Let us take $x_i = i$, $i = 0, \dots, n$ and calculate $\mathbf{L}^{(n)}$, $\mathbf{U}^{(n)}$ and $\mathbf{L}^{n,k}$, $\mathbf{U}^{n,k}$ for $k = 1, \dots, n$. We see from (5.10) that

$$l_{i,j}^{(n)} = \binom{i}{j}, \quad 0 \leq j \leq i \leq n$$

and from (5.11) and with $q = 1$ in (2.9)

$$u_{i,j}^{(n)} = \sum_{s=0}^i \frac{i! s^j}{(-1)^{i-s} s! (i-s)!} = i! S_q(j, i), \quad 0 \leq i \leq j \leq n.$$

We calculate from (5.17) for $1 \leq k \leq n$,

$$l_{i,j}^{n,k} = \begin{cases} 1, & i = j, \\ 1, & i = j + 1, i \geq n - k + 1, \\ 0, & \text{otherwise} \end{cases}$$

and from (5.18)

$$u_{i,j}^{n,k} = \begin{cases} 1, & i = j, i \leq n - k, \\ i + k - n, & i = j, i > n - k, \\ i + k - n, & i = j - 1, i \geq n - k, \\ 0, & \text{otherwise.} \end{cases}$$

Chapter 6

A difference operator \mathcal{D} on generalized Bernstein polynomials

In this chapter, we study some properties of a particular type of operator which is related to divided differences. The differential operator is obtained from this as a limiting case where the parameter $q \rightarrow 1$. This operator is applied to the generalized Bernstein polynomials to give results which complement those concerning derivatives of Bernstein polynomials, see Davis [9]. An inverse operator is also defined, leading to some results given in Section 7.1.

6.1 The operator \mathcal{D}

Given any function $\phi(x)$ and $q \in \mathbb{R}$ we define the operator \mathcal{D}

$$\mathcal{D}\phi(x) = \frac{\phi(qx) - \phi(x)}{qx - x} . \quad (6.1)$$

Thus $\mathcal{D}\phi(x)$ is simply a divided difference,

$$\mathcal{D}\phi(x) = \phi[x, qx].$$

The operator \mathcal{D} is introduced by Jackson [25] and its properties are studied in Exton [13]. We note that, provided $\phi'(x)$ exists,

$$\lim_{q \rightarrow 1} \mathcal{D}\phi(x) = \phi'(x).$$

As an example it is easily seen that for integer values of $r \geq 1$

$$\mathcal{D}(x^r) = [r]x^{r-1}. \quad (6.2)$$

Jackson [26] investigates the series

$$E(x) = \sum_{r=0}^{\infty} \frac{x^r}{[r]!}. \quad (6.3)$$

The series (6.3) is absolutely convergent only in $|x| < (1 - q)^{-1}$ when $|q| < 1$ whereas the exponential series is absolutely convergent for all x . However on applying \mathcal{D} term by term to $E(x)$, we see that

$$\mathcal{D}E(x) = E(x).$$

When we apply the operator \mathcal{D} repeatedly k times to an arbitrary function ϕ , we obtain

$$\mathcal{D}^k \phi(x) = \sum_{r=0}^k \frac{(-1)^r q^{r(r-1)/2} \begin{bmatrix} k \\ r \end{bmatrix} \phi(q^{k-r}x)}{q^{k(k-1)/2} (x(q-1))^k}. \quad (6.4)$$

This can be shown by induction on k . In the inductive step we write

$$\mathcal{D}^k \phi(x) = \frac{\mathcal{D}^{k-1} \phi(qx) - \mathcal{D}^{k-1} \phi(x)}{qx - x},$$

replace k by $k-1$ on the right of (6.4) and substitute in the latter equation. The coefficient of $\phi(q^{k-r}x)$ in the sums is

$$(-1)^r \frac{q^{r(r-1)/2}}{q^{k(k-1)/2} (x(q-1))^k} \left(\begin{bmatrix} k-1 \\ r \end{bmatrix} + q^{k-r} \begin{bmatrix} k-1 \\ r-1 \end{bmatrix} \right).$$

The q -binomial coefficients are combined using the Pascal identity (1.4) to give the r th term of (6.4).

It is easily verified that

$$\mathcal{D}(u(x)v(x)) = \mathcal{D}u(x).v(x) + u(qx).\mathcal{D}v(x). \quad (6.5)$$

This is quoted in Exton [13]. The repeated application of (6.5) gives the following Leibniz-type formula.

Theorem 6.1 *For integer values of $k \geq 0$,*

$$\mathcal{D}^k(u(x)v(x)) = \sum_{r=0}^k \begin{bmatrix} k \\ r \end{bmatrix} \mathcal{D}^{k-r}u(q^r x)\mathcal{D}^r v(x). \quad (6.6)$$

Proof We can verify this by induction on k . We see that (6.6) is true for $k = 0$. Let us replace k by $k - 1$ in (6.6) and assume it holds for $k \geq 1$. We then apply (6.5) and (6.1). After arranging the terms we obtain

$$\begin{aligned} \mathcal{D}^k(u(x)v(x)) &= \sum_{r=0}^{k-1} \begin{bmatrix} k-1 \\ r \end{bmatrix} \left[\left(\frac{\mathcal{D}^{k-1-r}u(q^{r+1}x) - \mathcal{D}^{k-1-r}u(q^r x)}{qx - x} \right) \mathcal{D}^r v(x) \right. \\ &\quad \left. + \mathcal{D}^{k-1-r}u(q^{r+1}x) \left(\frac{\mathcal{D}^r v(qx) - \mathcal{D}^r v(x)}{qx - x} \right) \right]. \end{aligned}$$

The latter equation can be rewritten by means of (6.1) as

$$\mathcal{D}^k(u(x)v(x)) = \sum_{r=0}^{k-1} \begin{bmatrix} k-1 \\ r \end{bmatrix} \left(q^r \mathcal{D}^{k-r}u(q^r x) \mathcal{D}^r v(x) + \mathcal{D}^{k-1-r}u(q^{r+1}x) \mathcal{D}^{r+1}v(x) \right).$$

Changing the limits of the second sum gives

$$\mathcal{D}^k(u(x)v(x)) = \sum_{r=0}^k \left(q^r \begin{bmatrix} k-1 \\ r \end{bmatrix} + \begin{bmatrix} k-1 \\ r-1 \end{bmatrix} \right) \mathcal{D}^{k-r}u(q^r x) \mathcal{D}^r v(x).$$

Combining the q -binomial coefficients using (1.5) completes the proof. ■

Let us apply the operator \mathcal{D} to the Bernstein basis polynomials. First we denote

$$B_r^{n,k}(x) = \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=k}^{n-r+k-1} (1 - q^s x). \quad (6.7)$$

Note that $B_r^{n,0}(x)$, $0 \leq r \leq n$, gives the basis polynomials of the generalized Bernstein polynomial. By direct computation using (6.1) and (6.2) we obtain

$$\mathcal{D} B_r^{n,k}(x) = \left(\begin{bmatrix} n \\ r \end{bmatrix} x^{r-1} ([r] - q^k [n]x) \prod_{s=k+1}^{n-r+k-1} (1 - q^s x) \right).$$

Replacing $([r] - q^k [n]x)$ by $([r](1 - q^{n-r+k}x) - q^k [n-r]x)$ in the last equation we obtain

$$\mathcal{D} B_r^{n,k}(x) = [n](B_{r-1}^{n-1,k+1}(x) - q^k B_r^{n-1,k+1}(x)). \quad (6.8)$$

This is a generalization of the derivative formula for the Bernstein-Bézier polynomials, which corresponds to $k = 0$, and taking the limit as $q \rightarrow 1$ in (6.8). See for example Farin [14].

6.2 Divided differences

We saw that \mathcal{D} is a divided difference operator. In this section we show that powers of \mathcal{D} behave like divided differences.

Theorem 6.2 *For a function f and non-negative integers k, m*

$$f[q^m x, q^{m+1} x, \dots, q^{m+k} x] = \frac{1}{[k]!} \mathcal{D}^k f(q^m x). \quad (6.9)$$

Proof This is true for $k = 0$. Assume (6.9) is true for some $k \geq 0$ and all $m \geq 0$. Then

$$\begin{aligned} f[q^m x, q^{m+1} x, \dots, q^{m+k} x] &= \frac{f[q^{m+1} x, \dots, q^{m+k} x] - f[q^m x, \dots, q^{m+k-1} x]}{q^{k+m} x - q^m x} \\ &= \frac{\frac{1}{[k-1]!} \mathcal{D}^{k-1} f(q^{m+1} x) - \frac{1}{[k-1]!} \mathcal{D}^{k-1} f(q^m x)}{q^m x (q^k - 1)} \\ &= \frac{1}{[k]!} \mathcal{D}^k f(q^m x), \end{aligned}$$

since

$$\mathcal{D}^k f(q^m x) = \frac{\mathcal{D}^{k-1} f(q^{m+1} x) - \mathcal{D}^{k-1} f(q^m x)}{q^{m+1} x - q^m x}.$$

This completes the proof. ■

The next result is a special case of (6.9).

Corollary 6.2.1 *For a function f and non-negative integer k*

$$f[x, qx, \dots, q^k x] = \frac{1}{[k]!} \mathcal{D}^k f(x).$$

Remark 6.1 *If f is convex on $[0,1]$ then $\mathcal{D}^2 f \geq 0$ for any $0 < q \leq 1$.*

Proof If f is convex on $[0, 1]$, then

$$f(\lambda t_0 + (1 - \lambda)t_1) \leq \lambda f(t_0) + (1 - \lambda)f(t_1) \quad (6.10)$$

for any $0 < \lambda < 1$ and any points $0 \leq t_0 \leq t_1 \leq 1$. Choosing $\lambda = \frac{1}{1+q}$, for any $0 < q < 1$ and $t_0 = q^2 x$, $t_1 = x$, we obtain from (6.10)

$$f(qx) \leq \frac{1}{1+q} f(q^2 x) + \frac{q}{1+q} f(x).$$

Thus

$$\frac{1}{(1-q)^2 x^2} f[x, qx, q^2 x] \geq 0$$

and so $\mathcal{D}^2 f \geq 0$ for any $0 < q \leq 1$. ■

6.3 Repeated applications of the operator \mathcal{D}

One of the most astonishing properties of the classical Bernstein polynomials is that not only does $B_n(f; x)$ converge uniformly to $f(x)$, but if the function

f is sufficiently differentiable, the derivatives of $B_n(f; x)$ converge uniformly to the derivatives of $f(x)$. See Davis [9], DeVore and Lorentz [12] or Lorentz [34]. Davis [9] proves the following theorem as a preliminary to proving these results concerning the derivatives of the classical Bernstein polynomials.

Theorem 6.3 *For any integer $0 \leq k \leq n$,*

$$B_n^{(k)}(f; x) = n \dots (n - k + 1) \sum_{r=0}^{n-k} \Delta^k f_r \binom{n-k}{r} x^r (1-x)^{n-k-r}, \quad (6.11)$$

where f_r denotes $f(r/n)$ and Δ is the ordinary difference operator.

We now give an analogous result involving the generalized Bernstein polynomials with the \mathcal{D} operator in place of the ordinary difference operator.

Theorem 6.4 *For any integer $0 \leq k \leq n$,*

$$\mathcal{D}^k B_n(f; x) = [n] \dots [n - k + 1] \sum_{r=0}^{n-k} \Delta^k f_r \begin{bmatrix} n-k \\ r \end{bmatrix} x^r \prod_{s=k}^{n-r-1} (1 - q^s x). \quad (6.12)$$

In this case f_r denotes $f([r]/[n])$ and Δ denotes the q -difference operator.

Proof We recall that $B_n(f; x) = \sum_{r=0}^n \binom{n}{r} \Delta^r f_0 x^r$. Then applying the operator \mathcal{D} to $B_n(f; x)$ repeatedly k times, we acquire

$$\mathcal{D}^k B_n(f; x) = \sum_{r=0}^{n-k} \frac{[n]!}{[n - k - r]! [r]!} \Delta^{k+r} f_0 x^r. \quad (6.13)$$

Now, recalling (3.16), let us express the operator Δ^{k+r} in terms of Δ^k to give

$$\mathcal{D}^k B_n(f; x) = \sum_{r=0}^{n-k} \sum_{t=0}^r (-1)^t q^{t(t+2k-1)/2} \frac{[n]!}{[n - k - r]! [r]!} \begin{bmatrix} r \\ t \end{bmatrix} \Delta^k f_{r-t} x^r.$$

Writing $m = r - t$ and putting

$$\frac{[n]!}{[n - k - m - t]! [m + t]!} \begin{bmatrix} m + t \\ t \end{bmatrix} = \frac{[n]!}{[n - k - m]! [m]!} \begin{bmatrix} n - k - m \\ t \end{bmatrix}$$

in the latter equation we obtain

$$\mathcal{D}^k B_n(f; x) = \sum_{m=0}^{n-k} \frac{[n]!}{[n-k-m]! [m]!} \Delta^k f_m x^m \sum_{t=0}^{n-k-m} (-1)^t q^{t(t+2k-1)/2} \begin{bmatrix} n-k-m \\ t \end{bmatrix} x^t.$$

Now it can be easily verified from the generalized binomial expansion (1.7), on replacing x by $q^k x$, that

$$\prod_{t=k}^{n-m-1} (1 - q^t x) = \sum_{t=0}^{n-k-m} (-1)^t q^{t(t+2k-1)/2} \begin{bmatrix} n-k-m \\ t \end{bmatrix} x^t. \quad (6.14)$$

This completes the proof. ■

Remark 6.2 From Theorem 6.4 we see that, with $0 < q \leq 1$, if $\Delta^k f_r \geq 0$ for $0 \leq r \leq n-k$ then $\mathcal{D}^k B_n(f; x) \geq 0$. If f is convex on $0 \leq x \leq 1$ then $\mathcal{D}^2 B_n(f; x) \geq 0$ for $0 < q \leq 1$. If f is increasing then $\mathcal{D} B_n(f; x) \geq 0$, for $0 < q \leq 1$.

If we express the operator \mathcal{D}^k in terms of a divided difference via (6.9), we can express (6.12) in the following form, where $B_n f[x, qx, \dots, q^k x]$ denotes the divided difference of $B_n(f; x)$ on the set of points $x, qx, \dots, q^k x$.

Corollary 6.4.1 For any integer $0 \leq k \leq n$

$$B_n f[x, qx, \dots, q^k x] = \begin{bmatrix} n \\ k \end{bmatrix} \sum_{r=0}^{n-k} \begin{bmatrix} n-k \\ r \end{bmatrix} \Delta^k f_r x^r \prod_{s=k}^{n-r-1} (1 - q^s x).$$

From the q -difference form of the generalized Bernstein polynomial (1.14) and (6.13), the last equation can be written as

$$B_n f[x, qx, \dots, q^k x] = \begin{bmatrix} n \\ k \end{bmatrix} \sum_{r=0}^{n-k} \begin{bmatrix} n-k \\ r \end{bmatrix} \Delta^{k+r} f_0 x^r. \quad (6.15)$$

We see from (6.15) that if f is a polynomial of degree m then $B_n f[x, qx, \dots, q^k x]$ is a polynomial of degree $\min(n-k, m)$.

It is well known that if $f \in C^k$

$$\lim_{x_i \rightarrow x} f[x_0, \dots, x_k] = \frac{f^k(x)}{k!},$$

where $\lim_{x_i \rightarrow x}$ denotes the limit as each $x_i \rightarrow x$, $0 \leq i \leq k$. See for example DeVore and Lorentz [12] or Phillips and Taylor [43]. Thus we have the following.

Corollary 6.4.2

$$\lim_{q \rightarrow 1} \mathcal{D}^k B_n(f; x) = B_n^{(k)}(f; x).$$

Finally, we apply the operator \mathcal{D} to $D_{k,m}(x)$ defined in (3.18). We see from (6.2) that

$$\mathcal{D}(D_{k,m}(x)) = \sum_{t=0}^{n-k-m} (-1)^t q^{t(t+2k-1)/2} \begin{bmatrix} n \\ m+t+k \end{bmatrix} \begin{bmatrix} m+t \\ t \end{bmatrix} [m+t+k] x^{m+t+k-1}.$$

Repeating the operation k times we obtain

$$\mathcal{D}^k(D_{k,m}(x)) = \frac{[n]!}{[n-m-k]! [m]!} x^m \prod_{t=k}^{n-m-1} (1 - q^t x). \quad (6.16)$$

We note that the last expression is non-negative in $0 \leq x \leq 1$, for $0 < q \leq 1$.

6.4 Inverse of \mathcal{D}

Exton [13] quotes work of Jackson (see [26] and [27]) on the inverse operation of \mathcal{D} , which Jackson calls q -integration or basic integration. When $q \rightarrow 1$ this reduces to ordinary integration. We write

$$\Phi(x) = \mathcal{D}^{-1}\phi(x)$$

if and only if

$$\phi(x) = \mathcal{D}\Phi(x),$$

and we will also write

$$[\mathcal{D}^{-1}\phi(x)]_a^b = \Phi(b) - \Phi(a).$$

Exton [13] shows that Φ exists if ϕ is Riemann integrable.

Let us now show that for any integer u

$$\mathcal{D}^{-1} x^u = \frac{x^{u+1}}{[u+1]} + c,$$

where $c \in \mathbb{R}$. We seek a function $F(x)$ such that

$$\frac{F(qx) - F(x)}{qx - x} = x^u,$$

that is

$$F(x) - F(qx) = x^{u+1}(1 - q). \quad (6.17)$$

Replacing x by qx in (6.17) and adding it to (6.17), we obtain

$$F(x) - F(q^2x) = x^{u+1}(1 - q)(1 + q^{u+1}).$$

Repeating this process, we obtain

$$F(x) - F(q^n x) = x^{u+1}(1 - q) \sum_{r=0}^{n-1} q^{r(u+1)}.$$

Letting $n \rightarrow \infty$, the infinite sum is the series expansion of $\frac{1}{1 - q^{u+1}}$. Hence we find that

$$\begin{aligned} F(x) &= x^{u+1} \frac{1 - q}{1 - q^{u+1}} + F(0) \\ &= \frac{x^{u+1}}{[u+1]} + F(0). \end{aligned} \quad (6.18)$$

Since $\mathcal{D} \left(\frac{x^{u+1}}{[u+1]} + c \right) = x^u$, for any $c \in \mathbb{R}$, we can replace $F(0)$ by c in (6.18).

From (6.5) we can deduce that

$$\mathcal{D}^{-1}(\mathcal{D}u(x)v(x)) = u(x)v(x) - \mathcal{D}^{-1}(u(qx)\mathcal{D}v(x)) \quad (6.19)$$

which is a generalization of the classical integration by parts formula.

Let us evaluate $[\mathcal{D}^{-1}B_r^{n,0}(x)]_0^1$, where $B_r^{n,0}(x)$ is defined in (6.7).

Choose

$$u(x) = -\frac{(q-x)(1-x)\dots(1-q^{n-r-1}x)}{[n-r+1]}$$

and $v(x) = x^r$. Then

$$\mathcal{D}u(x) = (1-x)\dots(1-q^{n-r-1}x)$$

and

$$\mathcal{D}v(x) = [r]x^{r-1}.$$

We then obtain from (6.19) that

$$[\mathcal{D}^{-1}(\mathcal{D}u(x).v(x))]_0^1 = \frac{q[r]}{[n-r+1]} [\mathcal{D}^{-1}(x^{r-1}(1-x)\dots(1-q^{n-r}x))]_0^1,$$

and thus

$$\begin{aligned} [\mathcal{D}^{-1}(x^r(1-x)\dots(1-q^{n-r-1}x))]_0^1 \\ = \frac{q[r]}{[n-r+1]} [\mathcal{D}^{-1}(x^{r-1}(1-x)\dots(1-q^{n-r}x))]_0^1. \end{aligned}$$

Repeating the operation r times and multiplying by $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]$, we obtain

$$[\mathcal{D}^{-1}B_r^{n,0}(x)]_0^1 = \begin{cases} \frac{q^{r+1}}{[n+1]}, & 0 \leq r \leq n-1, \\ \frac{1}{[n+1]}, & r = n. \end{cases}$$

Since \mathcal{D}^{-1} is a linear operator and

$$B_n(f; x) = \sum_{r=0}^n f_r B_r^{n,0}(x),$$

we have

$$[\mathcal{D}^{-1}B_n(f; x)]_0^1 = \frac{1}{[n+1]} \left(f_n + \sum_{r=0}^{n-1} q^{r+1} f_r \right). \quad (6.20)$$

We note that (6.20) is a generalization of the result

$$\int_0^1 B_n(f; x) dx = \frac{1}{n+1} \sum_{r=0}^n f\left(\frac{r}{n}\right)$$

for the classical Bernstein polynomials (see DeVore and Lorentz [12] and Farin [14]).

On the other hand, expanding $B_r^{n,0}$ using (1.7), we obtain

$$\int_0^1 B_r^{n,0}(x) dx = \begin{bmatrix} n \\ r \end{bmatrix} \sum_{j=0}^{n-r} (-1)^j \frac{q^{j(j-1)/2}}{(r+j+1)} \begin{bmatrix} n-r \\ j \end{bmatrix}.$$

Thus

$$\int_0^1 B_n(f; x) dx = \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} f_r \sum_{j=0}^{n-r} (-1)^j \frac{q^{j(j-1)/2}}{(r+j+1)} \begin{bmatrix} n-r \\ j \end{bmatrix}. \quad (6.21)$$

The fact that

$$\sum_{r=0}^n B_r^{n,0}(x) = 1,$$

gives rise to a special case of (6.21),

$$\int_0^1 \sum_{r=0}^n B_r^{n,0}(x) dx = 1.$$

Chapter 7

A generalization of Bézier curves

In this chapter we are concerned with the degree elevation process on the generalized Bernstein polynomials. It has some practical usage in Computer Aided Geometric Design (CAGD). Then in Section 6.2 it is shown that repeated degree elevation is indeed variation diminishing by obtaining a totally positive transformation matrix from initial set points to any degree elevated points.

7.1 Degree elevation

Long after their initial use by S.N. Bernstein in his proof of the Weierstrass theorem, Bernstein polynomials gained further celebrity when they were used in CAGD. The fundamental tool here is the use of Bézier curves and surfaces. The theory was developed independently by P. de Casteljau at Citröen and by P. Bézier a little later at the Renault automobile company. P. de Casteljau discovered an efficient algorithm which generates the Bernstein polynomials. In geometric design the Bernstein basis polynomials, together with control points,

play an important role. Recently Phillips [40] has produced a generalization of this type of algorithm in which the classical de Casteljau algorithm is recovered when the parameter q is replaced by 1.

Degree elevation is a method used in CAGD to develop the flexibility of a given polygon or surface by adding another vertex to it. This process increases the degree of the Bézier curve by one. On the other hand it does not change the curve. Farin [14] and Hoschek and Lasser [24] are good references on CAGD. We will follow Farin's notation in this chapter.

Suppose we are given vertices $\mathbf{b}_0^0, \dots, \mathbf{b}_n^0 \in \mathbb{R}^2$. The polygon P formed by $\mathbf{b}_0^0, \dots, \mathbf{b}_n^0$ is called a control polygon or Bézier polygon. Likewise the points of control polygon are called control vertices or Bézier points. Our objective is to find a new set of control vertices $\mathbf{b}_0^1, \dots, \mathbf{b}_{n+1}^1$ using the given control vertices for the same curve generated by generalized de Casteljau algorithm. We recall the basis functions for the generalized Bernstein polynomials,

$$B_j^n(t) = \begin{bmatrix} n \\ j \end{bmatrix} t^j \prod_{s=0}^{n-j-1} (1 - q^s t), \quad (7.1)$$

for $0 \leq j \leq n$. On using (1.5) in (7.1) we deduce a recurrence formula

$$B_j^n(t) = t B_{j-1}^{n-1}(t) + (q^j - q^{n-1}t) B_j^{n-1}(t). \quad (7.2)$$

Similarly on using (1.4) in (7.1) we have

$$B_j^n(t) = q^{n-j} t B_{j-1}^{n-1}(t) + (1 - q^{n-j-1}t) B_j^{n-1}(t). \quad (7.3)$$

The formulas

$$(1 - q^{n-j}t) B_j^n(t) = \frac{[n+1-j]}{[n+1]} B_j^{n+1}(t) \quad (7.4)$$

and

$$(q^{n-j}t) B_j^n(t) = \left(1 - \frac{[n-j]}{[n+1]}\right) B_{j+1}^{n+1}(t) \quad (7.5)$$

follow immediately from (7.1). We note on putting $q = 1$, that all identities above deduced from (7.1) are generalizations of formulas given in Farin [14] and Hoschek and Lasser [24]. Although all of them can be used as degree rising formulas we will use only (7.4) and (7.5).

We now develop a q -Bézier parametric curve, say $P(t)$, from the control points using the generalized de Casteljau algorithm (see section 1.1) with \mathbf{b}_i^0 in place of f_i^0 , giving

$$P(t) = \sum_{j=0}^n \mathbf{b}_j^0 B_j^n(t). \quad (7.6)$$

Using the identities (7.4) and (7.5), and writing our given curve as

$$P(t) = (1 - q^{n-j}t)P(t) + q^{n-j}tP(t)$$

we obtain

$$P(t) = \sum_{j=0}^n \frac{[n+1-j]}{[n+1]} \mathbf{b}_j^0 B_j^{n+1}(t) + \sum_{j=0}^n \left(1 - \frac{[n-j]}{[n+1]}\right) \mathbf{b}_j^0 B_{j+1}^{n+1}(t).$$

We may rewrite these two summations by shifting their limits, to give

$$P(t) = \sum_{j=0}^{n+1} \frac{[n+1-j]}{[n+1]} \mathbf{b}_j^0 B_j^{n+1}(t) + \sum_{j=0}^{n+1} \left(1 - \frac{[n+1-j]}{[n+1]}\right) \mathbf{b}_{j-1}^0 B_j^{n+1}(t)$$

where \mathbf{b}_{-1}^0 is defined as the zero vector. Comparing coefficients on both sides of the equation we obtain

$$\mathbf{b}_j^1 = \left(1 - \frac{[n+1-j]}{[n+1]}\right) \mathbf{b}_{j-1}^0 + \frac{[n+1-j]}{[n+1]} \mathbf{b}_j^0, \quad j = 0, \dots, n+1. \quad (7.7)$$

We note that degree elevation interpolates the end points, that is

$$\mathbf{b}_0^1 = \mathbf{b}_0^0 \quad \text{and} \quad \mathbf{b}_{n+1}^1 = \mathbf{b}_n^0.$$

We point out that the new set of vertices \mathbf{b}_j^1 are evaluated from the old polygon by piecewise linear interpolation at the parameter values of $[n+1-j]/[n+1]$. Let us illustrate this process with an example. We take a set of control points

(vertices of our polygon) $\mathbf{b}_0^0 = (0, 0)$, $\mathbf{b}_1^0 = (1, 2)$, $\mathbf{b}_2^0 = (3, 2)$, $\mathbf{b}_3^0 = (4, 0)$ on the real plane and the parameter value $q = 0.8$, see Figure 7.1. The polygons are generated by first and fifth degree elevations respectively. The curve in the figure is the third degree q -Bézier curve developed by the generalized de Casteljau algorithm for the points above.

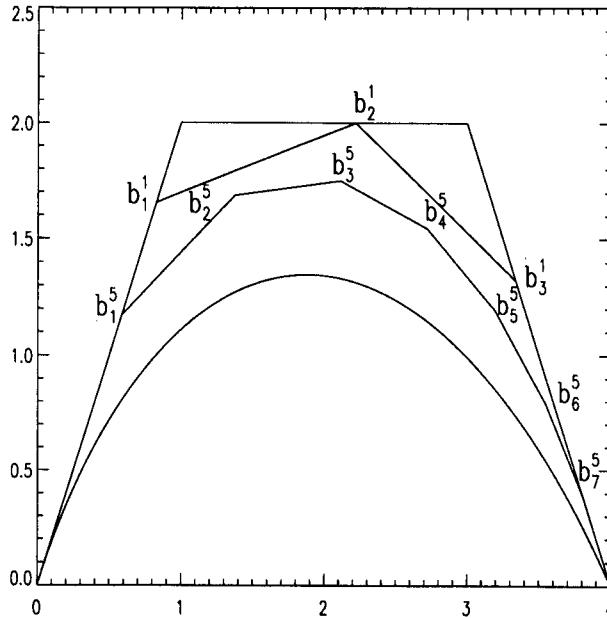


Figure 7.1: First and fifth degree elevations and a third degree parametric q -Bézier curve

The advantage of having another parameter q in degree elevation is that a change in q changes the control polygon, while keeping the end points and the degree fixed. We remark that in classical degree elevation, the control polygon is not altered provided the degree is fixed.

The degree elevation results in a new polygon \mathbf{EP} in place of \mathbf{P} . Using the same method, we repeat the degree elevation, replacing (7.7) by the recursive process

$$\mathbf{b}_i^r = \left(1 - \frac{[n+r-i]}{[n+r]}\right) \mathbf{b}_{i-1}^{r-1} + \frac{[n+r-i]}{[n+r]} \mathbf{b}_i^{r-1} \quad \begin{cases} r = 0, 1, \dots \\ i = 0, 1, \dots, n+r. \end{cases} \quad (7.8)$$

After r degree elevations we obtain a sequence of polygons $P, EP, \dots, E^r P$. The polygon $E^r P$ has vertices b_0^r, \dots, b_{n+r}^r . For every r , $E^r P$ interpolates the end points of the polygon, so that

$$b_0^0 = b_0^r \quad \text{and} \quad b_n^0 = b_{n+r}^r.$$

Using an induction argument on (7.8), we find that each b_i^r can be written explicitly in the form

$$b_i^r = \sum_{j=0}^r q^{j(n+j-i)} \frac{\begin{bmatrix} n \\ i-j \end{bmatrix} \begin{bmatrix} r \\ j \end{bmatrix}}{\begin{bmatrix} n+r \\ i \end{bmatrix}} b_{i-j}^0. \quad (7.9)$$

This sum is nonzero only for $0 \leq i - j \leq n$. Thus, replacing j by $i - j$, (7.9) can be rewritten as

$$b_i^r = \sum_{j=0}^n q^{(i-j)(n-j)} \begin{bmatrix} n \\ j \end{bmatrix} \frac{\begin{bmatrix} r \\ i-j \end{bmatrix}}{\begin{bmatrix} n+r \\ i \end{bmatrix}} b_j^0, \quad (7.10)$$

where $n \geq 3$ and $0 \leq i \leq n + r$.

We may verify (7.10) by induction on r . For $r = 0$ in (7.10) we indeed obtain b_i^0 , since the j th term in the sum is nonzero only for $j = i$. Assume (7.10) is true with $r - 1$ in place of r . We then substitute (7.10) with $r - 1$, $i - 1$ and $r - 1$, i in place of r , i into (7.8). Writing

$$\left(1 - \frac{[n+r-i]}{[n+r]}\right) / \begin{bmatrix} n+r-1 \\ i-1 \end{bmatrix} = \frac{q^{n+r-i}}{\begin{bmatrix} n+r \\ i \end{bmatrix}} \quad \text{and} \quad \frac{[n+r-i]}{[n+r]\begin{bmatrix} n+r-1 \\ i \end{bmatrix}} = \frac{1}{\begin{bmatrix} n+r \\ i \end{bmatrix}},$$

we obtain

$$b_i^r = \sum_{j=0}^n q^{(i-j)(n-j)} \begin{bmatrix} n \\ j \end{bmatrix} \frac{\left(q^{r+j-i} \begin{bmatrix} r-1 \\ i-j-1 \end{bmatrix} + \begin{bmatrix} r-1 \\ i-j \end{bmatrix}\right)}{\begin{bmatrix} n+r \\ i \end{bmatrix}} b_j^0.$$

The q -binomial coefficients in the numerator of the latter expression are combined using (1.4) to give $\begin{bmatrix} r \\ i-j \end{bmatrix}$. This verifies (7.10).

Thus the consequence of degree elevation is the following:

$$E^r P(t) = P(t) = \sum_{j=0}^{n+r} b_j^r B_j^{n+r}(t). \quad (7.11)$$

So far we have considered a parametric form of the generalized Bernstein polynomial, or q -Bézier curve, $\mathbf{P}(t) = (x(t), y(t))$ where

$$x(t) = \sum_{j=0}^n x_{j,0} B_j^n(t)$$

and

$$y(t) = \sum_{j=0}^n y_{j,0} B_j^n(t).$$

The functional form of $\mathbf{P}(t)$, which is $B_n(f; x)$, is reproduced when \mathbf{b}_j^0 is replaced by $f([j]/[n])$ in (7.6). Thus the control polygon \mathbf{P} of $B_n(f; x)$ is formed by the points $([j]/[n], f([j]/[n]))$. We consider the approximated area AP of \mathbf{P} given by a particular Riemann sum as

$$AP = \frac{1}{[n+1]} \sum_{j=0}^n q^j f_j. \quad (7.12)$$

It can be shown that AP remains unchanged under degree elevation. The approximated area of the degree elevated polygon AEP is

$$AEP = \frac{1}{[n+2]} \sum_{j=0}^{n+1} q^j y_{j,1}, \quad (7.13)$$

where $y_{j,1}$ is the ordinate of \mathbf{b}_j^1 . The points $y_{j,1}$ can be replaced by

$$y_{j,1} = \left(1 - \frac{[n+1-j]}{[n+1]}\right) f_{j-1} + \frac{[n+1-j]}{[n+1]} f_j.$$

On substituting the latter equation in (7.13) and then shifting the limits of the first sum we obtain

$$\begin{aligned} AEP &= \frac{1}{[n+2]} \sum_{j=0}^n \frac{q^j (q[n+1] + [n+1-j] - q[n-j])}{[n+1]} f_j \\ &= \frac{1}{[n+1]} \sum_{j=0}^n q^j f_j. \end{aligned}$$

Remark 7.1 When $q = 1$ and $r \rightarrow \infty$ the area AEP converges to the integral of $B_n(f; x)$, that is

$$\int_0^1 \sum_{j=0}^n f_j B_j^n(t) dt = \frac{1}{n+1} \sum_{j=0}^n f\left(\frac{j}{n}\right).$$

7.2 Degree elevation and total positivity

In this section, we will show that the process of degree elevation is variation diminishing by making use of the concept of total positivity. This implies that any given straight line does not cross the polygon $E^r P$ more than it crosses the polygon P . We recall Theorem 4.2 on totally positive matrices.

Let \mathbf{b} denote the vector such that $\mathbf{b}^T = [b_0, \dots, b_n]$ where the elements are the control vertices defined above. We also define \mathbf{b}^r as the vector whose elements are the control vertices b_i^r , $0 \leq r \leq n+r$ generated by degree elevation repeated r times. Then we have the following result.

Theorem 7.1 *For $0 < q \leq 1$ there exists a $(n+r+1) \times (n+1)$ totally positive matrix $\mathbf{T}^{r,n}$ such that $\mathbf{b}^r = \mathbf{T}^{r,n}\mathbf{b}$.*

Proof We will show, using induction on r , that $\mathbf{T}^{r,n}$ is a product of r 1-banded positive matrices. Let $\mathbf{T}^{r,n}$ denote the $(n+r+1) \times (n+1)$ matrix such that

$$T_{i,j}^{r,n} = \frac{q^{(i-j)(n-j)} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} r \\ i-j \end{bmatrix}}{\begin{bmatrix} n+r \\ i \end{bmatrix}}. \quad (7.14)$$

Thus $T_{i,j}^{r,n}$ is zero unless $0 \leq i - j \leq r$. We note that the elements $T_{i,j}^{r,n}$ are the coefficients which appear in (7.10). Now, the result holds for $r = 0$ since $\mathbf{T}^{0,n}$ is simply the $(n+1) \times (n+1)$ identity matrix. Let $\mathbf{B}^{(r)}$ denote the $(n+r+1) \times (n+r)$ 1-banded positive matrix such that

$$B_{i,j}^{(r)} = q^{(i-j)(n+r-i)} \frac{\begin{bmatrix} n+r-1 \\ j \end{bmatrix}}{\begin{bmatrix} n+r \\ i \end{bmatrix}}, \quad \text{for } 0 \leq i - j \leq 1. \quad (7.15)$$

Then $\mathbf{T}^{r,n} = \mathbf{B}^{(r)} \mathbf{B}^{(r-1)} \dots \mathbf{B}^{(1)}$. Let $\mathbf{V} = \mathbf{B}^{(r+1)} \mathbf{T}^{r,n}$. Explicitly this yields

$$V_{i,j} = \sum_{k=0}^{n+r+1} B_{i,k}^{(r+1)} T_{k,j}^{r,n}.$$

We see from (7.15) that $B_{i,k}^{(r+1)}$ is nonzero only for $k = i - 1$ and $k = i$. Thus

$$V_{i,j} = B_{i,i-1}^{(r+1)} T_{i-1,j}^{r,n} + B_{i,i}^{(r+1)} T_{i,j}^{r,n}.$$

Hence, with $r + 1$ in (7.15) and (7.14), we obtain

$$V_{i,j} = q^{(n+r-i+1)+(i-1-j)(n-j)} \frac{\begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} r \\ i-1-j \end{bmatrix}}{\begin{bmatrix} n+r+1 \\ i \end{bmatrix}} + q^{(i-j)(n-j)} \frac{\begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} r \\ i-j \end{bmatrix}}{\begin{bmatrix} n+r+1 \\ i \end{bmatrix}}.$$

After arranging the terms we have

$$V_{i,j} = q^{(i-j)(n-j)} \frac{\begin{bmatrix} n \\ j \end{bmatrix}}{\begin{bmatrix} n+r+1 \\ i \end{bmatrix}} \left(q^{r-i+j+1} \begin{bmatrix} r \\ i-1-j \end{bmatrix} + \begin{bmatrix} r \\ i-j \end{bmatrix} \right).$$

Using the Pascal identity (1.4) we obtain

$$V_{i,j} = q^{(i-j)(n-j)} \frac{\begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} r+1 \\ i-j \end{bmatrix}}{\begin{bmatrix} n+r+1 \\ i \end{bmatrix}} = T_{i,j}^{r+1,n},$$

completing the proof. ■

The following is a direct consequence of Theorem 4.2.

Theorem 7.2 (Variation Diminishing Property)

$$S^-(\mathbf{b}^r) \leq S^-(\mathbf{b}).$$

Sign changes in this section mean sign changes in the sequence of ordinates of vertices of the polygon, provided its abscissas are increasing.

Corollary 7.2.1 *If the control polygon P formed by the control vertices $(\mathbf{b}_i)_{i=0}^n = (x_{i,0}, y_{i,0})_{i=0}^n$ is increasing (decreasing) then the degree elevated polygon $E^r P$ with vertices $(\mathbf{b}_i^r)_{i=0}^{n+r} = (x_{i,r}, y_{i,r})_{i=0}^{n+r}$ is increasing (decreasing).*

Proof Let c be any real number. The number of sign changes in $P - c$ is

$$S^-(y_{0,0} - c, \dots, y_{n,0} - c) \leq 1$$

since the sequence $y_{0,0}, \dots, y_{n,0}$ is increasing. Thus, by the theorem above,

$$S^-(y_{0,r} - c, \dots, y_{n+r,r} - c) \leq S^-(y_{0,0} - c, \dots, y_{n,0} - c) \leq 1.$$

We have $y_{0,0} = y_{0,r}$, $y_{n,0} = y_{n+r,r}$ and $y_{0,0} \leq y_{n,0}$. Thus the sequence $y_{0,r}, \dots, y_{n+r,r}$ is increasing and hence $E^r P$ is increasing. If the control polygon is decreasing, its proof is very similar to the above proof. ■

Corollary 7.2.2 *If the control polygon P is convex then $E^r P$ is convex.*

Proof Let $y = l(t)$ denote the equation of a straight line. By the convexity of the polygon P , the straight line does not cross the polygon more than twice. That is $S^-(P - l(t)) \leq 2$. In other words any ordinate y of a point on this line has $S^-(y_{0,0} - y, \dots, y_{n,0} - y) \leq 2$ sign changes. By total positivity

$$S^-(y_{0,r} - y, \dots, y_{n+r,r} - y) \leq S^-(y_{0,0} - y, \dots, y_{n,0} - y) \leq 2.$$

Also by algorithm (7.8) we see that each new point is obtained as a convex combination of the two previous consecutive points. Thus $E^r P$ is convex. ■

The inverse process of degree elevation, degree reduction aims to represent a given curve of degree n as one of degree $n - 1$. In general, exact degree reduction is not possible. For example, a quadratic with a turning point cannot reasonably be replaced by a straight line. Thus the process can be viewed only as a method to approximate a given curve by one of lower degree (see Farin [14]) and note that the approximations may be quite poor.

From the q -difference form of the Bernstein polynomials (1.14) we see that a q -Bézier curve of degree n with control points $\mathbf{b}_0, \dots, \mathbf{b}_n$ in place of f_0, \dots, f_n has a degree $n - 1$ representation if and only if

$$\Delta^n \mathbf{b}_0 = 0.$$

Thus, using (1.9), we have

$$\Delta^n \mathbf{b}_0 = \sum_{i=0}^n (-1)^i q^{i(i-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} \mathbf{b}_{n-i} = 0.$$

In this case, in order to find the new points $\tilde{\mathbf{b}}_0, \dots, \tilde{\mathbf{b}}_{n-1}$ for the q -Bézier representation of degree $n - 1$ we use degree elevation formulas (7.4) and (7.5) so that

$$\sum_{i=0}^n \mathbf{b}_i B_i^n(t) = \sum_{i=0}^{n-1} \tilde{\mathbf{b}}_i \left(\frac{[n-i]}{[n]} B_i^n(t) + \left(1 - \frac{[n-1-i]}{[n]}\right) B_{i+1}^n(t) \right).$$

On comparing the coefficients of the basis functions $B_i^n(t)$, we obtain

$$\mathbf{b}_i = \frac{[n-i]}{[n]} \tilde{\mathbf{b}}_i + \left(1 - \frac{[n-i]}{[n]}\right) \tilde{\mathbf{b}}_{i-1}, \quad i = 0, 1, \dots, n-1,$$

from which we obtain

$$\tilde{\mathbf{b}}_i = \frac{[n]}{[n-i]} \mathbf{b}_i - \left(\frac{[n]}{[n-i]} - 1 \right) \tilde{\mathbf{b}}_{i-1}, \quad i = 0, 1, \dots, n-1. \quad (7.16)$$

This approximation is from the left of the control polygon, taking $\tilde{\mathbf{b}}_0 = \mathbf{b}_0$. When i is replaced by $n - i$ in (7.16) we have an approximation from the right side, with $\tilde{\mathbf{b}}_{n-1} = \mathbf{b}_n$,

$$\tilde{\mathbf{b}}_{n-i-1} = \frac{[n]}{[n]-[i]} \mathbf{b}_{n-i} - \frac{[i]}{[n]-[i]} \tilde{\mathbf{b}}_{n-i}, \quad i = 0, 1, \dots, n-1. \quad (7.17)$$

It should be noted that the formulas for reducing the degree of a q -Bézier curve, (7.16), tend to be numerically unstable (as is the Bézier curve), as the calculation of new points requires the subtraction of a previously calculated point (see Farin [14]).

Of course when we take $q = 1$ in (7.16) and (7.17) we recover the classical degree reduction formulas for Bézier curves (see Farin [14], Farouki and Rajan [16], Goodman and Said [20]).

Bibliography

- [1] G. E. ANDREWS, *The Theory of Partitions*, Cambridge University Press, 1998, chapter 3.
- [2] H. BOHMAN, *On approximation of continuous and of analytic functions*, Arkiv för Matematik, **2** (1952), pp. 43–56.
- [3] L. CARLITZ, *q -Bernoulli numbers and polynomials*, Duke Mathematical Journal, **15** (1948), pp. 987–1000.
- [4] J. M. CARNICER AND J. PEÑA, *Totally positive bases for shape preserving curve design and optimality of B-splines*, Computer Aided Geometric Design, **11** (1994), pp. 633–654.
- [5] J. M. CARNICER AND J. M. PEÑA, *Shape preserving representations and optimality of the Bernstein basis*, Advances in Computational Mathematics, **1** (1993), pp. 173–196.
- [6] E. W. CHENEY, *Introduction to Approximation Theory*, Chelsea, second ed., 1984, chapter 2.
- [7] C. W. CRYER, *Some properties of totally positive matrices*, Linear Algebra and its Applications, **15** (1976), pp. 1–25.

- [8] W. DAHMEN, *Convexity and Bernstein-Bézier polynomials*, in Curves and Surfaces, P. J. Laurent, A. L. Méhauté, and L. L. Schumaker, eds., 1991, pp. 107–134.
- [9] P. J. DAVIS, *Interpolation and Approximation*, Dover, 1975, chapter 6.
- [10] C. DE BOOR AND A. PINKUS, *Backward error analysis for totally positive systems*, Numerische Mathematik, **27** (1977), pp. 485–490.
- [11] P. DE CASTELJAU, *Shape Mathematics and CAD*, Kogan Page Ltd., 1986.
- [12] R. A. DEVORE AND G. G. LORENTZ, *Constructive Approximation*, Springer-Verlag, 1993, chapter 10.
- [13] H. EXTON, *q -Hypergeometric Functions and Applications*, Ellis Horwood, 1983, chapter 2.
- [14] G. FARIN, *Curves and Surfaces for Computer Aided Geometric Design, A Practical Guide*, Academic Press, fourth ed., 1997, chapter 3–5.
- [15] R. T. FAROUKI AND T. N. T. GOODMAN, *On the optimal stability of the Bernstein basis*, Mathematics of Computation, **65** (1996), pp. 1553–1566.
- [16] R. T. FAROUKI AND V. T. RAJAN, *On the numerical condition of polynomials in Bernstein form*, Computer Aided Geometric Design, **4** (1987), pp. 191–216.
- [17] T. N. T. GOODMAN, *Shape preserving representations*, in Mathematical Methods in CAGD, T. Lyche and L. Schumaker, eds., Academic Press, Boston, MA, 1989, pp. 333–357.
- [18] ——, *Total positivity and shape of curves*, in Total Positivity and its Applications, M. Gasca and C. A. Micchelli, eds., 1996, pp. 157–186.

- [19] T. N. T. GOODMAN, H. ORUÇ, AND G. M. PHILLIPS, *Convexity and generalized Bernstein polynomials*, Proceedings of the Edinburgh Mathematical Society, in the press.
- [20] T. N. T. GOODMAN AND H. B. SAID, *Properties of generalized Ball curves and surfaces*, Computer Aided Design, **23** (1991), pp. 554–560.
- [21] ——, *Shape preserving properties of the generalized Ball basis*, Computer Aided Geometric Design, **8** (1991), pp. 115–121.
- [22] T. N. T. GOODMAN AND A. SHARMA, *Factorization of totally positive, symmetric, periodic, banded matrices with applications*, Linear Algebra and its Applications, **178** (1993), pp. 85–107.
- [23] H. W. GOULD, *q -Stirling numbers of first and second kinds*, Duke Mathematical Journal, **28** (1961), pp. 281–289.
- [24] J. HOSCHEK AND D. LASSE, *Fundamentals of Computer Aided Geometric Design*, A K Peters, 1993, chapter 4.
- [25] F. H. JACKSON, *On q -functions and a certain difference operator*, Transactions of the Royal Society of Edinburgh, **46** (1908), pp. 253–281.
- [26] ——, *On q -definite integrals*, Quarterly Journal of Pure and Applied Mathematics, **41** (1910), pp. 193–203.
- [27] ——, *Basic integration*, Quarterly Journal of Mathematics, Oxford Series, **2** (1951), pp. 1–16.
- [28] S. KARLIN, *Total Positivity*, Stanford University Press, 1967, chapter 6.
- [29] Z. F. KOÇAK AND G. M. PHILLIPS, *An identity involving the q -factorial*, in Applications of Fibonacci numbers, Kluwer, 1996.

- [30] J. KONVALIN, *Generalized-Binomial Coefficients and the Subset-Subspace Problem*, Advances in Applied Mathematics, **21** (1998), pp. 228–240.
- [31] P. P. KOROVKIN, *On convergence of linear positive operators in the space of continuous functions*, Doklady Akademii Nauk SSSR, **114** (1957), pp. 961–964.
- [32] L. KOSMAK, *A note on Bernstein polynomials of a convex function*, Mathematica (Cluj), **2** (1960), pp. 281–282.
- [33] S. L. LEE AND G. M. PHILLIPS, *Polynomial interpolation at the points of a geometric progression*, Proceedings of the Royal Society Edinburgh, **108A** (1988), pp. 75–87.
- [34] G. G. LORENTZ, *Bernstein polynomials*, University of Toronto Press, 1953, chapter 1–2.
- [35] I. G. MACDONALD, *Symmetric Functions and Hall polynomials*, Clarendon Press Oxford, 1979, chapter 1.
- [36] A. D. MÉDICS AND P. LEROUX, *Generalized Stirling numbers, convolution formula and p, q -analogues*, Canadian Journal Mathematics, **47(3)** (1995), pp. 474–499.
- [37] L. M. MILNE-THOMSON, *Calculus of Finite Differences*, Macmillan, 1951, chapter 1.
- [38] E. NEUMAN, *On complete symmetric functions*, SIAM Journal on Mathematical Analysis, **19** (1988), pp. 736–750.
- [39] H. ORUÇ AND G. M. PHILLIPS, *A generalization of Bernstein polynomials*, Proceedings of the Edinburgh Mathematical Society, in the press.
- [40] G. M. PHILLIPS, *A de Casteljau algorithm for generalized Bernstein polynomials*, BIT, **36:1** (1996), pp. 232–236.

- [41] ——, *On generalized Bernstein polynomials*, in Numerical Analysis: A.R. Mitchell 75th Birthday Volume, D. Griffiths and G. Watson, eds., 1996, pp. 263–269.
- [42] ——, *Bernstein polynomials based on q -integers*, Annals of Numerical Mathematics, **4** (1997), pp. 511–518.
- [43] G. M. PHILLIPS AND P. J. TAYLOR, *Theory and Applications of Numerical Analysis*, Academic Press, second ed., 1996, chapter 5.
- [44] G. PÓLYA AND I. J. SCHOENBERG, *Remarks on the de la Vallée Poussin means and convex conformal maps of the circle*, Pacific Journal of Mathematics, **8** (1958), pp. 295–334.
- [45] L. RAMSHAW, *Blossoms are polar forms*, Computer Aided Geometric Design, **6** (1989), pp. 323–358.
- [46] T. J. RIVLIN, *An Introduction to the Approximation of functions*, Dover, 1969, chapter 1.
- [47] I. J. SCHOENBERG, *On smoothing operations and their generating functions*, Bulletin of the American Mathematical Society, **59** (1953), pp. 199–230.
- [48] ——, *On variation diminishing approximation methods*, in On Numerical Approximation Methods, R. E. Langer, ed., 1959, pp. 249–271.
- [49] ——, *On polynomial interpolation at the points of a geometric progression*, Proceedings of the Royal Society Edinburgh, **90A** (1981), pp. 195–207.
- [50] I. TOMESCU, *Introduction to Combinatorics*, London and Wellingborough: Collet's, 1975, chapter 2.
- [51] T. POPOVICIU, *Sur l'approximation des fonctions convexes d'ordre supérieur*, Mathematica, **10** (1935), pp. 49–54.

- [52] E. VORONOVSKAYA, *Determination de la forme asymptotique d'approximation des functions par les polynomes de M. Bernstein*, Doklady Akademii Nauk SSSR, **4** (1932), pp. 79–85.
- [53] R. WEBSTER, *Convexity*, Oxford Science Publication, 1994, chapter 5.