A STUDY OF THE MATHEMATICS OF SUPERSYMMETRY THEORIES

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'A Study of the Mathematics of Supersymmetry Theories'

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STATEMENT

I was admitted under Ordnance General No. 12 as a research student on the 9th. of June 1980, and as a candidate for the degree of Doctor of Philosophy under Resolution of the University Court, 1967, No.1 on 7th. of October 1981.

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DECLARATIONS

I declare this Thesis to be of my own composition, to be a record of my own work, and not previously to have been presented in application for a higher degree

D. Williams

I declare that the conditions of the Resolution and the Regulations appertaining to the degree of Doctor of Philosophy in the University of St. Andrews have been fulfilled.

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ABSTRACT

This Thesis consists of three parts. In the first part a theory of integration is constructed for supermanifolds and supergroups. With this theory expressions for the invariant integral on several Lie supergroups are obtained incuding the super Poincaré group and superspace. The unitary irreducible representations of the super Poincaré group are examined by considering the unitary irreducible representations of a certain set of Lie groups equivalent to the super Poincaré group. These irreducible representations contain, at most, particles of a single spin.

In the second part a detailed examination of the massive representations of the super Poincaré algebra is undertaken. Supermultiplets of second quantized fields are constructed for each of the massive representations, which allows an understanding of the auxiliary fields of supersymmetry theories.

In the third part super Poincaré invariant superfields on superspace are constructed from the supermultiplets of the second part. This enables a connection between the the representations of part one and those of part two to be established. An examination of action integrals on superspace is made enabling the relationship between the integration theory constructed in part one and the Berezin integral to be established.

CONTENTS

INTRODUCTION

In this Thesis we will examine the formulation of supersymmetry theories on Minkowski space and on superspace and determine the precise mathematical relationship between these different formulations.

A supersymmetry, in its broadest sense, is a symmetry that relates particles which have different spins. Thus the 'non-relativistic' SU(6) theories of the 1960's could be considered as an early attempt in this direction. This was doomed to failure since it could not describe high energy dynamics. These failures motivated a study by Coleman and Mandula that resulted in the famous theorem [1] which states that, under quite broad conditions, the only group of symmetries in relativistic field theory is the direct product of the Poincaré group with an internal symmetry group that must be the product of a compact semisimple group and U(1) groups. The direct product structure indicates that no Lie group can be used to connect particles of different spin in a quantum field theory.

It was not until 1971 that a way of avoiding the Coleman-Mandula theorem was discovered by Gol'fand and Likhtman [2]. They achieved this by adding additional generators to the Poincaré Lie algebra such that the law of composition of the new generators was by an anticommutator amongst themselves and by a commutator with the Poincaré generators.

This work went unnoticed until after its rediscovery by Wess and Zumino [3] in 1974. The new generators introduced by these authors had the property of turning Bosons into Fermions and vice-versa. They are what is now known as supersymmetry generators. Using the algebra they had constructed they built a supermultiplet the component fields of which transformed into each other under the action of the supersymmetry generators. They were also able to construct a Lagrangian for the supermultiplet that was invariant under supersymmetry transformations.

It was not long before it was noticed that the supersymmetry algebra extending the Poincaré group was a class of mathematical constructs called Lie superalgebras. These had appeared previously in the context of string theories [4], [5] and had in fact been studied by Berezin and G.I.Kac [6] in 1970 which even predates the work of Gol'fand and Likhtman. The classification of the simple Lie superalgebras was announced in 1975 by V.G.Kac [7].

The paper of Wess and Zumino spawned a vast amount of papers on supersymmetry. The next important steps were by Salam and Strathdee who described irreducible representations of the supersymmetry algebra [8] and introduced the concept of a superfield [9]. In the superfield approach it was realized that it was possible to construct a 'supergroup' from a superalgebra using 'odd' and 'even' parameters and to define superspace as a coset space of four even and four odd dimensions on which could be defined superfields. These superfields, when expanded as a Taylor series, mimicked the structure of the known supermultiplets.

In fact supergroups had been considered previously by Berezin and G.I.Kac [6] for very different reasons. It had been noted by Berezin [10] in 1961 that there was a remarkable coincidence of the fundamental formulae of the operator calculus for Fermi and Bose fields, the difference being that the Bose fields required the use of (normal) commuting variables and Fermi fields required the use of anticommuting variables. Anticommuting variables had in fact been studied before then by Martin [11] in 1959 who is credited (by Berezin) as being the first person to study the subject now known as 'supermathematics'.

In the course of his study of second quantization Berezin defined what has come to be known as the 'Berezin integral' in his own words it is 'a type of linear functional on a Grassman algebra' [12]. This led to a study of analysis on Grassman algebras and the study of supermanifolds and supergroups.

The concept of supermanifold has been refined over the years reaching what, in our opinion, is the most useful definition in the work of Rogers [13], [14]. Meanwhile physicists have continued to use superfields as a tool for constructing supersymmetry and supergravity theories without worrying unduly about the underlying mathematics (cf. [15], [16], [17], [18], [19]), but knowing that the same results can be obtained using the component fields on Minkowski space.

This gap between the mathematics of superspace and the component formulation of supersymmetry theories is what we intend to fill with this thesis. The work involved in doing this divides naturally into three areas covered in Parts I, II and III of this Thesis.

In Part I 'Supermanifolds and Integration Theory' we first review the supermathematics that we will need in this Thesis. Then in the second chapter we study linear Lie supergroups and are able to construct matrix representations for some of the supergroups corresponding to the simple Lie superalgebras. In the third chapter, using the fact that corresponding to each Lie supergroup there is an ordinary Lie group which we call 'the equivalent Lie group', we construct a theory of integration based on normal measure-theoretical methods. With this theory we are able to obtain invariant integrals on several Lie supergroups and on the super Poincaré group and superspace. In the fourth and final chapter of Part I we examine the unitary irreducible representations of the equivalent Lie group of a general Lie supergroup. We are able to show that for any linear Lie supergroup all its unitary irreducible representations can be obtained. We obtain some of the representations of the super Poincaré group and are able to demonstrate that each irreducible representation obtained in this way describes particles of a single spin value. This result is rather disconcerting and it is not until we study superfields in Part III that it can be understood.

In Part II 'Super Poincaré Invariant Theories on Space-Time' we make a detailed examination of the irreducible representations first described by Salam and Strathdee [8] . We are able to construct from these representations, in the massive case, sets of second quantized fields that form supermultiplets labeled by an index $j = 0, \frac{1}{2}, 1, \cdots$. The $j = 0$ supermultiplet is the well known Wess-Zumino multiplet [3] . These results have never previously been obtained in an explicit form even though it is generally assumed to be possible. This construction is based on the series of papers by Weinberg [20] and allows a complete understanding of the auxiliary fields that occur in supersymmetry theories.

In Part III 'Superfields and the Irreducible Representations of the Super Poincaré Algebra' we are able to reconcile the theory of Part I with that of Part II. We begin by using second quantized fields that transform according to some representation of the Lorentz group to construct second quantized fields on superspace that transform according to the same representation of the super Lorentz group. We find that the field required for this decomposes into a reducible representation of the equivalent Lie group of the super Lorentz group. This representation is not fully reducible but consists of lower triangular matrices with irreducible representations occuring along the diagonal. These irreducible representations can be identified with certain of the unitary irreducible representations constructed in Part I. We then extend this in the third chapter to construct fields on superspace from the chiral supermultiplets of Part II. The component fields of these chiral superfields are then a 'Grassman extension' of the component fields of the supermultiplet defined on Minkowski space. In the fourth chapter we show how a general superfield can be decomposed to reveal its component superfields and thus determine the irreducible supermultiplets that it contains. In the fifth and final chapter we consider how, using the theory we have developed, a Lagrangian function for a supermultiplet can be extracted from a superfield. We show how a Lagrangian function can be constructed for a scalar superfield and how the component Lagrangian mass and kinetic energy terms arise as one dimensional representation spaces of the equivalent Lie group of the super Poincaré group. This gives an understanding of the 'Berezin integral' in terms of our integration theory as an operator that projects out the required representation spaces.

At the end of the Thesis is an appendix which details our conventions, and contains several identities relating to the Pauli and Dirac matrices that we use.

PART I

SUPERMANIFOLDS AND INTEGRATION THEORY.

CHAPTER 1

PRELIMINARIES.

1.1 Introduction.

In this Part we examine the theory of supermanifolds and supergroups with a view to constructing the irreducible unitary representations of supergroups and relating these to the irreducible representations of superalgebras.

The mathematical theory of supermanifolds and supergroups has been developed by many authors following two approaches

(1) The sheaf of algebras of C^{∞} functions over a differentiable manifold locally homeomorphic to $\mathbb{R}^n$ is extended to a $\mathbb{Z}_2$-graded sheaf of algebras such that the algebra contains both commuting and anticommuting functions. This line of argument has been pursued by Berezin and Kac [6], Berezin and Leites [21] and Kostant [22]. Berezin has in fact considered this and (2).

(2) Commuting and anticommuting coordinates are introduced in the base manifold by making the coordinates copies of either the even or odd parts of a Grassman algebra. There are many variations in this development, mainly in respect of the topology endowed on the coordinates.

We shall follow the latest development in this second approach that was initiated by Rogers [13], [14]. (See these articles for references to earlier work and the relationship between the various approaches. See also the review by Berezin [6].) This approach endows superspace with a much finer topology than any of the previous approaches.

In the remainder of this chapter we give the definitions in 'supermathematics' that we use. They are, in the main, as used by Rogers for the basic superspace and differential geometry and as given by Leites [23] for the supermatrix analysis.

In Chapter 2 we examine linear Lie supergroups and obtain some new results, which enable matrix representations of certain Lie supergroups to be easily constructed and differential operators to be constructed for superalgebra generators.

In Chapter 3 we construct a theory of integration based on normal measure theoretical arguments using the topology endowed on supermanifolds by Rogers [13] . That is, as for complex and quaternionic manifolds, we view the manifold as a differentiable manifold locally isomorphic to $\mathbb{R}^n$ with additional Grassman analytic structure. The work in this chapter has been published in the article 'The Haar Integral for Lie Supergroups' [24] . This contrasts with the integration scheme previously used by physicists which has followed the heuristic scheme proposed by Berezin [25] in the context of path integral quantization. This has been taken to be the correct method of integration on supermanifolds constructed as in scheme (2) above and is widely used in the physics literature (see the review by Nieuwenhuizen [26]). We conclude this chapter with a discussion of the Berezin integral and various other later developments in superintegration theory.

In Chapter 4 we apply the integration theory constructed in Chapter 3 to the construction of unitary irreducible representations of supergroups. We are able to show that it is possible (in principle) to construct all the unitary irreducible representations of linear Lie supergroups (defined later). We are also able to show that each of the irreducible representations of the super Poincaré group contains particles of a single spin value so that they cannot be directly related to the irreducible representations of the super Poincaré algebra.

1.2 Basic Definitions.

1.2.1 Grassman Algebra and Superspace.

The definitions given in this section follow those as given by Rogers [13] , [14] except for the norm on E_L. For this we use a Euclidean norm (hence E_L) rather than a Banach norm (vis. Rogers B_L). We note that Boyer and Gitler [27] also use a Eucidean norm.

Let L be a positive integer and denote the basis elements of $\mathbb{R}^L$ by ε_i, $i=1,2,\cdots,L$. Let E_L denote the Grassman algebra over $\mathbb{R}^L$ with antisymmetric product given by $\wedge$. Then E_L has basis $\varepsilon_0, \varepsilon_i, \varepsilon_{i\wedge j}, \cdots, \varepsilon_{i\wedge j\wedge k}, \cdots \varepsilon_{i\wedge j\wedge k\wedge\cdots\wedge l}$ where ε_0 is the unit element of E_L. For notational convenience we have defined $\varepsilon_{i\wedge j}=\varepsilon_i\wedge\varepsilon_j$ etc. so that:

$$\varepsilon_{i\wedge j\wedge\cdots\wedge k\wedge l\wedge\cdots\wedge m} = -\varepsilon_{i\wedge j\wedge\cdots\wedge l\wedge k\wedge\cdots\wedge m},$$

(of course $\varepsilon_0\wedge\varepsilon_i=\varepsilon_i$ etc.).

E_L has a natural $\mathbb{Z}$ grading in which (for $l=1,2,\cdots,L$) the homogeneous part $E_L^{(l)}$ consists of all real linear combinations of basis elements $\varepsilon_{i\wedge j\wedge\cdots\wedge k}$ involving l indices, with $E_L^{0} = \{\mathbb{R}\varepsilon_0\}$. Elements of $E_L^{(l)}$ will be said to be of level l, denoted by $l(x)=l$ if $x\in E_L^{(l)}$. This induces a $\mathbb{Z}_2$ grading $E_L=E_{L0}\oplus E_{L1}$, with $E_{L0}=\{E_L^{(l)}, l \text{ even}\}$ and $E_{L1}=\{E_L^{(l)}, l \text{ odd}\}$. Then E_{L0}, E_{L1}, E_L are vector spaces over $\mathbb{R}$ with $\dim E_{L0} = \dim E_{L1} = 2^{L-1} = \mathcal{N}$ and $\dim E_L = 2\mathcal{N}$.

The degree or parity of a homogeneous $\mathbb{Z}_2$-graded element $x\in E_{L\alpha}$ is defined by $|x|=\alpha$, $\alpha\in\{0,1\}$. It will be said that x is 'even' if $|x|=0$, and is 'odd' if $|x|=1$.

It is convenient to denote the basis elements of E_{L0} by e_i, $i=0,1,\cdots,\mathcal{N}-1$ and correspondingly the basis elements of E_{L1} by f_j, $j=1,2,\cdots\mathcal{N}$ with the assignment $e_0=\varepsilon_0$, and the restriction that if $i<k$ then $l(e_i)\leqslant l(e_k)$

and $\ell(f_i) \leqslant \ell(f_k)$. (Where a specific basis is needed it is better to revert to the $\underline{\varepsilon}$ basis rather than have complicated index assignments.)

A matrix representation of E_L can be constructed by assigning for each $a \in E_L$ the matrix $(ad(a))_{mm'}$ defined by

$$a \wedge (\underline{e}_i, \underline{f}_j)_m = (\underline{e}_i, \underline{f}_j)_{m'} (ad(a))_{m'm}$$

with $(\underline{e}_i, \underline{f}_j)_m = (\underline{e}_0, \underline{e}_1, \cdots, \underline{e}_{\mathcal{N}-1}, \underline{f}_1, \underline{f}_2, \cdots, \underline{f}_{\mathcal{N}})_m$.

As an example the matrices of $ad(E_2)$ are given by

$$ad(\varepsilon_0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad ad(\varepsilon_1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$ad(\varepsilon_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{and } ad(\varepsilon_{1\wedge 2}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where we have put $(\underline{e}_i, \underline{f}_j)_m = (\varepsilon_0, \varepsilon_{1\wedge 2}, \varepsilon_1, \varepsilon_2)_m$. The matrices $ad(\underline{\varepsilon})$ are then supermatrices as defined in section 1.2.5. To demonstrate the nilpotent nature of the Grassman algebra we can choose a different order ie. $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_{1\wedge 2})_m$ that is, ordered by level to obtain

$$ad'(\varepsilon_0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad ad'(\varepsilon_1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$ad'(\varepsilon_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \text{and } ad'(\varepsilon_{1\wedge 2}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The representation is now by lower triangular matrices.

A typical element of E_{L0} will be denoted by $x = x_i e_i$ (summation implied) $i = 0,1,\cdots,\mathcal{N}-1$, $x_i \in \mathbb{R}$, and a typical element of E_{L1} will be denoted by $\theta = \theta_j f_j$ (summation implied) $j = 1,2,\cdots,\mathcal{N}$, $\theta_j \in \mathbb{R}$. We will write an element of E_L as $x+\theta$.

We define a norm on E_L by $\|x+\theta\| = \left(\sum_{i=0}^{\mathcal{N}-1}(x_i)^2 + \sum_{j=1}^{\mathcal{N}}(\theta_j)^2\right)^{\frac{1}{2}}$ and the corresponding distance function as $d(x+\theta, x'+\theta') = \|x-x'+\theta-\theta'\|$. E_L is then a topological vector space with a Hausdorff topology. The basis vectors in E_L (e_i, f_j or equivalently $\underline{e}_{i\wedge j\wedge\cdots\wedge k}$) are defined to be orthonormal. The complex Grassman space is defined by $\mathbb{C}E_L = E_L \oplus iE_L$.

Superspace is defined as $E_L^{m,n} = E_{L0}^m \times E_{L1}^n = E_{L0}\times\cdots\times E_{L0}\times E_{L1}\times\cdots\times E_{L1}$ ie. the Cartesian product of m copies of E_{L0} and n copies of E_{L1}. A typical element of $E_L^{m,n}$ is then of the form $(x^1, x^2,\cdots,x^m, \theta^1,\cdots\theta^n)$, where $x^\mu = \sum_{i=0}^{\mathcal{N}-1} x_i^\mu e_i$, $x_i^\mu \in \mathbb{R}$, for each $\mu = 1,2,\cdots,m$ and $\theta^\nu = \sum_{j=1}^{\mathcal{N}} \theta_j^\nu f_j$, $\theta_j^\nu \in \mathbb{R}$, for each $\nu = 1,2,\cdots,n$. This will be written (x,θ) unless we wish to consider explicitly the real variables x_i^μ and θ_j^ν in which case we will write (x_i, θ_j). That is the Greek indices which appear as superscripts distinguish the various copies of E_{L0} or E_{L1}, while the Latin indices which appear as subscripts indicate the component 'within' E_{L0} or E_{L1}. Summation is assumed over all repeated indices including those that are suppressed eg.

$$a_\mu x^\mu = \sum_{\mu=1}^{m} a_\mu x_i^\mu e_i = \sum_{\mu=1}^{m}\sum_{i=0}^{\mathcal{N}-1} a_\mu x_i^\mu e_i .$$

1.2.2 Superanalysis.

(a) Differentiation.

We define differentiation in the standard way, as a Frechet derivative, following Rogers [13]. That is if U is an open set in $E_L^{m,n}$ and $f: U \to E_L$ then f is said to be once differentiable on U if there exist $m+n$ functions $\frac{\partial f}{\partial x^\mu}, \frac{\partial f}{\partial \theta^\nu} : U \to E_L$ for $\mu = 1,2,\cdots,m;\ \nu = 1,2,\cdots,n$

and a function $\eta : E_L^{m,n} \longrightarrow E_L$ such that if $(x,\theta),(x+h,\theta+k) \in \mathcal{U}$ the functions $\frac{\partial f}{\partial x^\mu}, \frac{\partial f}{\partial \theta^\nu}$ satisfy

$$f(x+h,\theta+k) = f(x+\theta) + h_\mu \frac{\partial f}{\partial x^\mu} + k_\nu \frac{\partial f}{\partial \theta^\nu} + \|(h,k)\| \eta(h,k) \cdots (1)$$

and $\|\eta(h,k)\| \longrightarrow 0$ as $\|(h,k)\| \longrightarrow 0$.

A full discussion of this is given by Boyer and Gitler [27] . Differentiation with respect to the components x_i^μ and θ_j^ν is also defined in the same way, we denote this by $\frac{\partial}{\partial x_i^\mu}$ and $\frac{\partial}{\partial \theta_j^\nu}$. There is an analogue of the Cauchy-Reiman equations ie.

$$e_i \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x_i^\mu} \quad \text{and} \quad f_j \frac{\partial}{\partial \theta^\nu} = \frac{\partial}{\partial \theta_j^\nu}$$

which was proved by Boyer and Gitler [27] . The normal rule for differentiating a product and function of a function are valid if one notes that $|\frac{\partial}{\partial \theta}| = 1$. So that

$$\begin{aligned}\frac{\partial}{\partial \theta}(fg) &= \left(\frac{\partial}{\partial \theta} f\right) g + (-1)^{|\frac{\partial}{\partial \theta}||f|} f \left(\frac{\partial}{\partial \theta} g\right), \\ &= \left(\frac{\partial}{\partial \theta} f\right) g + (-1)^{|f|} f \left(\frac{\partial}{\partial \theta} g\right).\end{aligned}$$

The parity of a function f is defined by $|f| = 0$ if $f : E_L^{m,n} \rightarrow E_{L0}$ and $|f| = 1$ if $f : E_L^{m,n} \rightarrow E_{L1}$. To differentiate a product of two functions when the first is not homogeneous we decompose it into its homogeneous components.

There are essentially three types of superfunction (i) not differentiable, (ii) once differentiable denoted as G^1 and (iii) infinitely differentiable denoted as G^∞. This is proved in reference [27] where examples can be found.

(b) Superadjoint Operations.

Two 'adjoint' operations have been proposed in the literature initially by Rittenberg and Scheunert [28] and later with more detail by Rittenberg [29] and Ne'eman [30] . The aim being to construct analogues of hermitian conjugation for supermatrices. This works and it is possible to construct supergroups using these operations analagous to the unitary groups, but such groups are not Lie supergroups (see later). The first of these operations is also used in the book by DeWitt [31] in defining a real Grassman algebra (see later).

Definition 1.1

An 'S' adjoint operation in $\mathbb{C}E_L$ is a mapping $\alpha \to \alpha^{\#}$ which satisfies

(i) $E^{\#}_{Lk} \subset E_{Lk}$, for $k = 0, 1$,

(ii) $(\alpha b)^{\#} = b^{\#}\alpha^{\#}$, for $\alpha, b \in \mathbb{C}E_L$

and (iii) $\alpha^{\#\#} = \alpha$, for $\alpha \in \mathbb{C}E_L$.

An operation satisfying these constraints complex conjugates and reverses the order of the Grassman indices ie. for

$$(a + ib)\varepsilon_{1\wedge 2\wedge 3\wedge 4} \in \mathbb{C}E_L$$

we have

$$((a + ib)\varepsilon_{1\wedge 2\wedge 3\wedge 4})^{\#} = (a - ib)\varepsilon_{4\wedge 3\wedge 2\wedge 1}.$$

In particular we note that

$$\varepsilon_0^{\#} = \varepsilon_0, \quad \varepsilon_1^{\#} = \varepsilon_1, \quad \varepsilon_{1\wedge 2}^{\#} = -\varepsilon_{1\wedge 2}, \quad \varepsilon_{1\wedge 2\wedge 3}^{\#} = -\varepsilon_{1\wedge 2\wedge 3} \text{ and}$$
$$\varepsilon_{1\wedge 2\wedge 3\wedge 4}^{\#} = \varepsilon_{1\wedge 2\wedge 3\wedge 4}.$$

In general we can see that if

$\ell(e_i) = 4n,\ n = 0, 1, 2, \cdots$ then $e_i^{\#} = e_i$,

$\ell(f_j) = 4n+1,\ n = 0, 1, 2, \cdots$ then $f_j^{\#} = f_j$,

$\ell(e_i) = 4n+2,\ n = 0, 1, 2, \cdots$ then $e_i^{\#} = -e_i$,

and if $\ell(f_j) = 4n+3,\ n = 0, 1, 2, \cdots$ then $f_j^{\#} = -f_j$.

The function $\# : x \to x^{\#}$ or $\# : \theta \to \theta^{\#}$ is clearly not differentiable.

This operation is used by DeWitt [31] as follows. He first defines a complex Grassman algebra D_L and then specifies its real subalgebra $\mathbb{R}D_L$ by the prescription $\alpha \in \mathbb{R}D_L$ if $\alpha = \alpha^{\#}$. This clearly gives a subalgebra of D_L but is a very unwieldy procedure.

Definition 1.2

A superadjoint operation in $\mathbb{C}E_L$ is a map $\alpha \to \alpha^{\S}$ which satisfies

(i) $E^{\S}_{Lk} \subset E_{Lk}$ for $k = 0, 1$,

(ii) $(\alpha\beta)^{\S} = \alpha^{\S}\beta^{\S}$ for $\alpha, \beta \in \mathbb{C}E_L$

and (iii) $\alpha^{\S} = (-1)^k \alpha$ for $\alpha \in \mathbb{C}E_{Lk}$, $k = 0, 1$.

In order to constuct an operation satisfying these requirements it is necessary to 'pair off' the generators of the Grassman algebra so that it can only be constructed if L is even. The following assignments for $L=2$

$$\varepsilon_0^{\S} = \varepsilon_0 \,,\ \varepsilon_1^{\S} = -\varepsilon_2 \,,\ \varepsilon_2^{\S} = \varepsilon_1 \text{ and } \varepsilon_{1\wedge 2}^{\S} = \varepsilon_{1\wedge 2}$$

satisfy the requirements of Definition 1.2, note that this operation affects real and imaginary parts of the algebra in the same way. The operation is clearly not differentiable.

1.2.3 Lie Superalgebras.

The theory of Lie superalgebras has been the subject of many research papers. The most comprehensive description of the basic mathematical theory is the article by Kac [32] , the book by Scheunert [33] is also useful. Here we give only the basic definitions.

A Lie superalgebra of dimension $(m|n)$ over a field F is a $\mathbb{Z}_2$- graded vector space $L = L_0 \oplus L_1$ with $\dim L_0 = m$ and $\dim L_1 = n$ equipped with a bracket operation $[\ ,\ ]$ as product defined for every $\gamma, \gamma', \gamma'' \in L$ such that

(a) $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$,

(b) $[a\gamma + b\gamma', \gamma''] = a[\gamma, \gamma''] + b[\gamma', \gamma'']$ for all $a, b \in F$,

(c) $[\gamma, \gamma'] = -(-1)^{|\gamma||\gamma'|}[\gamma', \gamma]$

and (d) $[[\gamma, \gamma'], \gamma''] = [\gamma, [\gamma', \gamma'']] - (-1)^{|\gamma||\gamma'|}[\gamma', [\gamma, \gamma'']]$.

The identity in (d) is known as the 'Graded Jacobi Identity' it is in fact valid for <u>any</u> algebra if we replace the bracket operation with the appropriate product. It is given in many forms in the literature we choose the version with the least number of commutation factors, all other quoted versions can be obtained from this by repeated use of (c) above.

If we have an associative algebra $S = S_0 \oplus S_1$ and define a new multiplication on S by

$$[s, s'] = ss' - (-1)^{|s||s'|} s's \quad \text{for all} \quad s, s' \in S$$

then we have a Lie superalgebra and conditions (b), (c) and (d) are automatically satisfied. In particular if S is a matrix algebra with a $\mathbb{Z}_2$-gradation then with multiplication defined as above we have a Lie superalgebra.

1.2.4 Super Differential Geometry.

There has been a considerable amount of work carried out in super differential geometry in the literature. We in fact only need a limited amount of this as constructed by Rogers [13], [14]. Most of this work concentrates on the construction of fibre bundles [34] on superspace and the use of these to construct gauge theories in superspace [35]. In our opinion it is first neeessary to establish precicely what superspace is and its relationship to Minkowski space before theories constructed in this way can be given a true meaning.

We denote supermanifolds over $E_L^{m,n}$ by large latin letters eg. M and the corresponding manifolds over $\mathbb{R}^{\mathcal{N}(m+n)}$ by script letters $\mathcal{M}_L$, the subscript being a reminder that we are in fact considering a whole family of manifolds (one for each L). The dimension of M is denoted by $(m|n)$. The space of vector fields on M is denoted by $D'(M)$ and admits a grading $D'(M) = D'(M)_0 \oplus D'(M)_1$ (see Rogers [13]). The 'tangent' space at a point p of M is denoted by $ST_p(M)$, and again is graded; the even part, which 'corresponds' to the tangent space of $\mathcal{M}_L$, being denoted by $T_p(M)$ so that

$$T_p(M) \cong T_p(\mathcal{M}_L).$$

Lie supergroups are denoted by large latin letters, eg. G and we denote the equivalent Lie group of G by $\mathcal{G}_L$. The formal definition of a Lie supergroup is as follows.

Definition 1.3

An $(m|n)$ dimensional Lie supergroup is a set G such that:

(a) the set G is an abstract group,

(b) the set G is an $(m|n)$ dimensional analytic supermanifold

and (c) the mapping $(g_1, g_2) \rightarrow g_1 g_2^{-1}$ of the product supermanifold $G \times G \rightarrow G$ is analytic.

The equivalent Lie group $\mathcal{G}_L$ then has dimension $\mathcal{N}(m+n)$ when considered as a vector space over $\mathbb{R}$ or $\mathbb{C}$ (as appropriate).

The 'tangent' space at the identity e of G is denoted by $\mathcal{L}(G)$ so that $\mathcal{L}(G) = ST_e(G)$ and forms a 'Lie super module' (or 'Lie left B_L-module in the terminology of Rogers [13]). That is if $\gamma, \gamma' \in \mathcal{L}(G)$ and $a \in E_L$ then

$$a[\gamma, \gamma'] = [a\gamma, \gamma'] = (-1)^{|a||\gamma|}[\gamma, a\gamma']$$
$$= a(\gamma\gamma' - (-1)^{|\gamma||\gamma'|}\gamma'\gamma) \in \mathcal{L}(G).$$

The even part of $\mathcal{L}(G)$ is denoted by $\mathcal{L}_0(G)$ and called a 'Lie module'. The basis of the Lie supermodule of a Lie supergroup of dimension $(m|n)$ is denoted by $\{\alpha_\mu, \beta_\nu\}$ with $|\alpha_\mu| = 0, |\beta_\nu| = 1, \mu = 1, 2, \cdots, m$ and $\nu = 1, 2, \cdots, n$. We note that $\beta_\nu \in \mathcal{L}(G)$ but $\beta_\nu \notin \mathcal{L}_0(G)$. It can, however, still be considered as a basis element of $\mathcal{L}_0(G)$ since this consists of all vectors of the form

$$x^\mu \alpha_\mu + \theta^\nu \beta_\nu \text{ with } x \in E_{L0} \text{ and } \theta \in E_{L1}.$$

A basis of the Lie algebra of $\mathcal{G}_L$ is then given by $\{\underline{e}_i \alpha_\mu, \underline{f}_j \beta_\nu\}$ and is denoted by $\mathcal{L}(\mathcal{G}_L)$

1.2.5 Supermatrix Algebra.

(a) General

For an extensive review of this subject see Leites [23]. Although it is possible to construct a very general theory of supermatrices that are rectangular, in the work we have done, we find that it is only necessary to consider square matrices that are partitioned in block form as $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and row or column vectors. Supermatrix multiplication is defined as for normal matrix multiplication.

The set $M(p|q;E_L)$, $p \geq 1$, $q \geq 0$ is defined to be the set $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ such that A is $p \times p$ and D is $q \times q$ with entries from E_{L0}, while B and C have dimensions $p \times q$ and $q \times p$ and have entries from E_{L1}. The set $\bar{M}(p|q;E_L)$, $p \geq 1$, $q \geq 0$ is defined to be such that A is $p \times p$ and D is $q \times q$ with entries from E_{L1}, while B and C have dimensions $p \times q$ and $q \times p$ and have entries from E_{L0}. We define

$$\tilde{M}(p|q;E_L) = M(p|q;E_L) \oplus \bar{M}(p|q;E_L).$$

Parity of supermatrices is defined by $|M| = 0$ if $M \in M(p|q;E_L)$ and $|M| = 1$ if $M \in \bar{M}(p|q;E_L)$.

Row vectors z are of the form $z = (x, \theta) = (x^1, x^2, \cdots, \theta^1, \theta^2, \cdots)$, with $x \in E_{L0}$ and $\theta \in E_{L1}$ if $|z| = 0$ and of the form $z = (\theta, x)$ if $|z| = 1$. Column vectors are of the form $z = \begin{bmatrix} x \\ \theta \end{bmatrix}$ if $|z| = 0$ and of the form $z = \begin{bmatrix} \theta \\ x \end{bmatrix}$ if $|z| = 1$.

To multiply a matrix $M \in M(p|q;E_L)$ by a scalar $\alpha \in E_L$ we associate to the scalar α the matrix $\begin{bmatrix} \alpha I_p & 0 \\ 0 & (-1)^{|\alpha|}\alpha I_q \end{bmatrix}$ with I_p, I_q unit matrices of dimension $p \times p$ and $q \times q$ respectively.

Then $\alpha M = \begin{bmatrix} \alpha I_p & 0 \\ 0 & (-1)^{|\alpha|}\alpha I_q \end{bmatrix} \begin{bmatrix} A & C \\ B & D \end{bmatrix}$,

this then satisfies $\alpha M = (-1)^{|\alpha||M|} M\alpha$.

This rule is also valid for multiplying vectors by scalars ie. for $|\alpha| = 1$

$$a\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} aI_p & 0 \\ 0 & -aI_q \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} ax \\ -a\theta \end{bmatrix} = \begin{bmatrix} xa \\ \theta a \end{bmatrix} = \begin{bmatrix} x \\ \theta \end{bmatrix} a.$$

Also it is applicable when we differentiate a matrix function. Recall that $|\frac{\partial}{\partial x}| = 0$ and $|\frac{\partial}{\partial\theta}| = 1$ so that

$$\frac{\partial}{\partial\theta}\begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} I_p\frac{\partial}{\partial\theta} & 0 \\ 0 & -I_q\frac{\partial}{\partial\theta} \end{bmatrix}\begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix},$$

$$= \begin{bmatrix} \frac{\partial}{\partial\theta}A(\theta) & \frac{\partial}{\partial\theta}B(\theta) \\ -\frac{\partial}{\partial\theta}C(\theta) & -\frac{\partial}{\partial\theta}D(\theta) \end{bmatrix}.$$

This is particularly important when we differentiate a column vector eg.

$$\frac{\partial}{\partial\theta}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

but for a scalar function $\frac{\partial}{\partial\theta}(\theta) = 1$.

(b) Operations Defined on Supermatrices.

We detail all the operations we use in this Thesis. We do not provide proofs since the results are 'well known'.

(i) Transpose.

This is defined in the normal way. Its importance is when applied to the set $\tilde{M}(p|0;E_L)$. Then we have for $X, Y \in \tilde{M}(p|0;E_L)$

$$(XY)^t = (-1)^{|X||Y|}Y^tX^t.$$

Note that this result is <u>not</u> true for general supermatrices. It is used in proving many of the following results.

(ii) Supertranspose.

This is defined by

$$M^{st} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{st} = \begin{bmatrix} A^t & (-1)^{|M|}C^t \\ (-1)^{|M|+1}B^t & D^t \end{bmatrix} \text{ for } M \in \tilde{M}(p|q;E_L).$$

This is as defined by Leites [23]. Some authors interchange the sign factors $(-1)^{M+1}$ and $(-1)^{M}$ eg. Rittenberg and Scheunert [28]. This has no effect on the properties of supertranspose. It satisfies the following Lemma.

Lemma 1.4

For all $X, Y \in \tilde{M}(p|q; E_L)$

(a) $(st)^4 = id$, ie $((((X)^{st})^{st})^{st})^{st} = X$,

(b) $(XY)^{st} = (-1)^{|X||Y|} Y^{st} X^{st}$,

(c) $\alpha (X)^{st} = (\alpha X)^{st}$

and (d) $[X, Y]^{st} = -[X^{st}\ Y^{st}]$.

The action of supertranspose on even row or column supervectors is given by $\begin{bmatrix} x \\ \theta \end{bmatrix}^{st} = (x, \theta)$ but $(x, \theta)^{st} = \begin{bmatrix} x \\ -\theta \end{bmatrix}$.

(ii) Superdeterminant.

This is defined for all <u>invertible</u> $X \in \tilde{M}(p|q; E_L)$, so that $X \in M(p|q; E_L)$. There are three alternative definitions, which are equivalent to each other.

$$\begin{aligned} \mathrm{Sdet}(X) = \mathrm{Sdet}\begin{bmatrix} A & C \\ B & D \end{bmatrix} & \\ = \det A \det D' &, \\ = \det A \det^{-1}(D - CA^{-1}B) &, \\ = \det(A - BD^{-1}C) \det^{-1} D &, \end{aligned}$$

with $X^{-1} = \begin{bmatrix} A' & C' \\ B' & D' \end{bmatrix}$ and $\det^{-1} D = (\det D)^{-1}$.

Lemma 1.5

For all invertible $A, X, Y \in \tilde{M}(p|q; E_L)$

(a) $\mathrm{Sdet}(XY) = \mathrm{Sdet}(X)\,\mathrm{Sdet}(Y)$,

(b) $\mathrm{Sdet} : M(p|q; E_L) \longrightarrow E_{L0}$,

(c) $\mathrm{Sdet}(X^{st}) = \mathrm{Sdet}(X)$,

(d) $\mathrm{Sdet}(X^{-1}) = (\mathrm{Sdet}(X))^{-1}$

and (e) $\mathrm{Sdet}(AXA^{-1}) = \mathrm{Sdet}(X)$.

We note that $X \in M(p|q; E_L)$ is invertible if and only if

$\mathrm{Sdet}(X) = a \in E_{L0}$ and $a_0 \neq 0$ (ie. the component in the $\underline{\varepsilon}_0 = \underline{e}_0$ direction).

(iii) p-transpose.

This is used in the definition of the supergroup $B(n; E_L)$ (see Chapter 2, section 2.2.5) it was given by Scheunert [23] for real valued supermatrices in the context of superalgebras and has also been given by Rittenberg [29] and Ne'eman [30]. Kac [32] avoids this in his definition of the superalgebra P(n) (in our terminology $\mathfrak{b}(n; E_L)$). It is defined by

$$X^p = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^p = \begin{bmatrix} D^t & (-1)^{|X|+1} B^t \\ (-1)^{|X|} C^t & A^t \end{bmatrix}$$

and satisfies the following Lemma.

Lemma 1.6

For all $X, Y \in \tilde{M}(p|q; E_L)$,

(a) $(X^p)^p = X$,

(b) $(XY)^p = (-1)^{|X||Y|} Y^p X^p$,

(c) $(\alpha X)^p = (-1)^{|\alpha|} \alpha (X)^p$

and (d) $[X, Y]^p = -(-1)^{|X||Y|} [X^p, Y^p]$.

(iv) Supertrace.

This is defined by

$$\mathrm{Str}(X) = \mathrm{Str}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \mathrm{tr}(A) - (-1)^{|X|}\mathrm{tr}(D).$$

Here $\mathrm{tr}(A)$ means the trace of A in the normal (matrix) sense, ie. the sum of the diagonal elements.

Lemma 1.7

For all $X, Y \in \tilde{M}(p|q; E_L)$, $\alpha \in E_L$

(a) $\mathrm{Str}(\alpha X) = \alpha\, \mathrm{Str}(X)$,

(b) $\mathrm{Str}(XY) = (-1)^{|X||Y|} \mathrm{Str}(YX)$,

(c) $\mathrm{Str}(XYX^{-1}) = \mathrm{Str}(Y)$, for X invertible,

(d) $\mathrm{Str}(X^{st}) = \mathrm{Str}(X)$

and (e) $\mathrm{Str}(X+Y) = \mathrm{Str}(X) + \mathrm{Str}(Y)$.

(v) Sadjoint.

This makes use of the adjoint operation $^{\#}$ defined in section 1.2.2. It was originally given by Rittenberg and Scheunert [28] and has been repeated by Rittenberg [29] and Ne'eman [30] . We use it in an example in Chapter 2 of a supergroup that is continuously parametrized but is <u>not</u> a Lie supergroup. It is defined by

$$X^{SA} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{SA} = \begin{bmatrix} A^{t\#} & C^{t\#} \\ B^{t\#} & D^{t\#} \end{bmatrix}$$

<u>Lemma 1.8</u>

For all $X, Y \in \tilde{M}(p|q; E_L)$ and $\alpha \in E_L$,

(a) $(X^{SA})^{SA} = X$,

(b) $(XY)^{SA} = Y^{SA} X^{SA}$,

(c) $(\alpha X)^{SA} = (-1)^{|\alpha||X|} \alpha^{\#} X^{SA}$

and (d) $[X, Y]^{SA} = [Y^{SA}, X^{SA}]$.

(vi) Superadjoint.

This makes use of the superadjoint operation § defined in section 1.2.2 of this chapter. We obtained it from the same sources as the Sadjoint operation. It has the same drawbacks when used to construct supergroups. It is defined by

$$X^{\ddagger} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{\ddagger} = \begin{bmatrix} A^{\S} & B^{\S} \\ C^{\S} & D^{\S} \end{bmatrix}^{st}.$$

Here § acts on the individual entries.

<u>Lemma 1.9</u>

For all $X, Y \in \tilde{M}(p|q; E_L)$ and $\alpha \in E_L$

(a) $(((X^{\ddagger})^{\ddagger})^{\ddagger})^{\ddagger} = X$,

(b) $(XY)^{\ddagger} = Y^{\ddagger} X^{\ddagger}$,

(c) $(\alpha X)^{\ddagger} = \alpha^{\S} X^{\ddagger}$

and (d) $[X, Y]^{\ddagger} = -[X^{\ddagger}, Y^{\ddagger}]$.

(vii) The Exponential Function.

This is defined as for normal matrices by its series expansion ie.

$$\exp(X) = e^{X} = I + X + \frac{1}{2!}(X)^2 + \frac{1}{3!}(X)^3 + \cdots$$

for all $X \in \tilde{M}(p|q;E_L)$. In practice we only ever exponentiate even matrices in which case $\exp : M(p|q;E_L) \longrightarrow M(p|q;E_L)$.

Lemma 1.10

For all $X, Y \in M(p|q;E_L)$

(a) $\mathrm{Sdet}(\exp X) = \exp(\mathrm{Str}\, X)$,

(b) $\exp(X^{\#}) = (\exp X)^{\#}$,

(c) $\exp(X^{\S}) = (\exp X)^{\S}$,

(d) $\exp(X^{st}) = (\exp X)^{st}$,

(e) $\exp(X^{SA}) = (\exp X)^{SA}$,

(f) $\exp(X^{\ddagger}) = (\exp X)^{\ddagger}$,

(g) $\exp(XYX^{-1}) = X(\exp Y)X^{-1}$ if X is invertible ,

(h) $\exp(X^{P}) = (\exp X)^{P}$,

(j) $\exp(XY) = \exp X \exp Y$ if and only if $[X, Y] = 0$,

(k) $(\exp X)^{-1} = \exp - X$

and (l) $(\exp X)^{-1}$ exists for all $X \in M(p|q;E_L)$.

We note that it is possible to define the logarithm of supermatrices as the inverse function of $\exp$.

(c) The Super Jacobian Matrix.

Consider a change of coordinate system in $E_L^{m,n}$ written in the form

$$\begin{bmatrix} y \\ \phi \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix}$$

where $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M(p|q;E_L)$ and is invertible.

Then $(y, \phi) = (x, \theta) \begin{bmatrix} A^t & C^t \\ -B^t & D^t \end{bmatrix}$.

So that $\dfrac{\partial y^\mu}{\partial x^\sigma} = (A^t)_{\mu\sigma}$, $\dfrac{\partial y^\mu}{\partial \theta^\rho} = -(B^t)_{\mu\rho}$,

$\dfrac{\partial \phi^\nu}{\partial x^\sigma} = (C^t)_{\nu\sigma}$ and $\dfrac{\partial \phi^\nu}{\partial \theta^\rho} = (D^t)_{\nu\rho}$.

We can then construct the matrix of these coefficients which we call the super Jacobian matrix and denote by $\dfrac{\partial(y,\phi)}{\partial(x,\theta)}$. Then

$$\frac{\partial(y,\phi)}{\partial(x,\theta)} = \begin{bmatrix} \frac{\partial y^\mu}{\partial x^\sigma} & \frac{\partial \phi^\nu}{\partial x^\sigma} \\ \frac{\partial y^\mu}{\partial \theta^\rho} & \frac{\partial \phi^\nu}{\partial \theta^\rho} \end{bmatrix} = \begin{bmatrix} A^t & C^t \\ -B^t & D^t \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{st}.$$

Now the rule for differentiating a composite function is given by Rogers [13]. In our terminology we have

$$\frac{\partial}{\partial x^\sigma} f(y(x,\theta), \phi(x,\theta)) = \frac{\partial y^\mu}{\partial x^\sigma}\frac{\partial f}{\partial y^\mu} + \frac{\partial \phi^\nu}{\partial x^\sigma}\frac{\partial f}{\partial \phi^\nu}$$

and $$\frac{\partial}{\partial \theta^\rho} f(y(x,\theta), \phi(x,\theta)) = \frac{\partial y^\mu}{\partial \theta^\rho}\frac{\partial f}{\partial y^\mu} + \frac{\partial \phi^\nu}{\partial \theta^\rho}\frac{\partial f}{\partial \phi^\nu}.$$

These equations are more conveniently written as a matrix equation

$$\begin{bmatrix} \frac{\partial f}{\partial x^\sigma} \\ \frac{\partial f}{\partial \theta^\rho} \end{bmatrix} = \begin{bmatrix} \frac{\partial y^\mu}{\partial x^\sigma} & \frac{\partial \phi^\nu}{\partial x^\sigma} \\ \frac{\partial y^\mu}{\partial \theta^\rho} & \frac{\partial \phi^\nu}{\partial \theta^\rho} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial y^\mu} \\ \frac{\partial f}{\partial \phi^\nu} \end{bmatrix}.$$

So that if (y,ϕ) and (x,θ) are related as above we have

$$\begin{bmatrix} \frac{\partial f}{\partial x^\sigma} \\ \frac{\partial f}{\partial \theta^\rho} \end{bmatrix} = \begin{bmatrix} A^t & C^t \\ -B^t & D^t \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial y^\mu} \\ \frac{\partial f}{\partial \phi^\nu} \end{bmatrix}$$

Inverting this we obtain

$$\begin{bmatrix} \frac{\partial f}{\partial y^\mu} \\ \frac{\partial f}{\partial \phi^\nu} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{st^{-1}} \begin{bmatrix} \frac{\partial f}{\partial x^\sigma} \\ \frac{\partial f}{\partial \theta^\rho} \end{bmatrix}$$

We note that if $\Gamma(G)$ is a representation of a supergroup then $(\Gamma(G))^{st^{-1}}$ is also a representation of the supergroup. Since

$$\Gamma(g)^{st^{-1}}\Gamma(g')^{st^{-1}} = (\Gamma(g')^{st}\Gamma(g)^{st})^{-1} = (\Gamma(g)\Gamma(g'))^{st^{-1}}$$
$$= \Gamma(gg')^{st^{-1}}.$$

CHAPTER 2

LINEAR LIE SUPERGROUPS.

2.1 Generalizations.

In this chapter we examine linear Lie supergroups, that is Lie supergroups for which we have a matrix realization. Much of what we say is used implicitly in the physics literature but formal definitions seem to be absent. Matrix supergroups were first considered by Rittenberg and Scheunert [28] as purely algebraic constructions from Lie superalgebras, they have been given by Rogers [36] as examples of Lie supergroups and are the reason for the study of supermatrices as given by Leites [23]. The book by DeWitt [31] mentions matrix representations of 'super Lie groups' and implies that these representations exist for all 'super Lie groups'. Since there are Lie groups for which no matrix representations exist (cf. [37]) and we can obtain a supergroup from these simply by extending the base field $\mathbb{R}$ to the ring E_{L0} it follows that there are Lie supergroups for which no matrix representations exist. Thus we see the need for the following formal definitions.

Definition 2.1

A matrix supergroup G is a subset of $\tilde{M}(p|q;\mathbb{C}E_L)$ such that for every $u,v \in G$ (i) $u^{-1} \in G$,

(ii) $uv \in G$

and (iii) the identity element $e \in G$ is given by

$$I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} = e .$$

We note that as for matrix groups we do not need to specify associativity, this is implied by the definition of multiplication of supermatrices. An immediate consequence of this definition is that all elements of a matrix supergroup are elements of $M(p|q;E_L)$, since they must have an inverse, ie. they are even supermatrices.

Definition 2.2

An $(m|n)$ dimensional real (complex) linear Lie supergroup G is a Lie supergroup with a faithful representation by matrices from $M(p|q;\mathbb{C}E_L)$ depending on m even and n odd Grassman real (complex) parameters.

Since $x \in E_{L0}$ $(\theta \in E_{L1})$ can be written $x = x_i e_i$ $(\theta = \theta_j f_j)$ and the e_i's $(f_j$'s$)$ can be considered as matrices (see Chapter 1, section 1.2.1) we can deduce that if G is a linear Lie supergroup then G_L is a linear Lie group for each L. One can therfore consider the matrix representation of G as a 'coded' matrix representation of G_L.

As with linear Lie groups we do not insist that a linear Lie supergroup requires only one coordinate chart we simply insist that if several charts are required then each chart can be expressed as a set of supermatrices.

Given a Lie supergroup we can construct a basis for $ST_e(G)$ by differentiation with respect to each of the parameters in turn and then evaluating the result at the identity of the supergroup. Then, in general, we have to check that we have sufficient linearly independent matrices and if necessary use the commutators to make good any deficiency. All the parametrizations we use will be such that e the identity of G is obtained by setting $x = \theta = 0$ in the chart containing the identity. The basis we obtain is then a set of m even supermatrices and n odd supermatrices, they need not have real or complex numbers as entries, to illustrate this point consider the following example

Example 2.3

Consider the linear Lie supergroup defined by

$$G = \begin{bmatrix} 1 + \varepsilon_1\theta & \varepsilon_2 x - \varepsilon_1\varepsilon_2\theta x \\ 0 & 1 + \varepsilon_1\theta \end{bmatrix}.$$

So that $\left(\frac{\partial G}{\partial x}\right)_e = \begin{bmatrix} 0 & \varepsilon_2 \\ 0 & 0 \end{bmatrix}$ and $\left(\frac{\partial G}{\partial \theta}\right)_e = \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & -\varepsilon_1 \end{bmatrix}$.

Then the matrices $\left(\frac{\partial G}{\partial x}\right)_e, \left(\frac{\partial G}{\partial \theta}\right)_e$ form the basis of a Lie superalgebra with $\left|\left(\frac{\partial G}{\partial x}\right)_e\right| = 0$ and $\left|\left(\frac{\partial G}{\partial \theta}\right)_e\right| = 1$. This can be demonstrated by evaluating the commutators

$$[\left(\tfrac{\partial G}{\partial x}\right)_e, \left(\tfrac{\partial G}{\partial x}\right)_e] = 0,$$

$$[\left(\tfrac{\partial G}{\partial x}\right)_e, \left(\tfrac{\partial G}{\partial \theta}\right)_e] = \left[\begin{bmatrix} 0 & \underline{\varepsilon}_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \underline{\varepsilon}_1 & 0 \\ 0 & -\underline{\varepsilon}_1 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 0 & -\underline{\varepsilon}_2\underline{\varepsilon}_1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \underline{\varepsilon}_1\underline{\varepsilon}_2 \\ 0 & 0 \end{bmatrix} = 0,$$

and
$$[\left(\tfrac{\partial G}{\partial \theta}\right)_e, \left(\tfrac{\partial G}{\partial \theta}\right)_e] = \left[\begin{bmatrix} \underline{\varepsilon}_1 & 0 \\ 0 & -\underline{\varepsilon}_1 \end{bmatrix} \begin{bmatrix} \underline{\varepsilon}_1 & 0 \\ 0 & -\underline{\varepsilon}_1 \end{bmatrix}\right] = 0.$$

Thus we have exhibited a Lie superalgebra that has a supermatrix representation with entries from E_L. Clearly many examples can be constructed, and they each form the basis of a Lie module. Hence one should not assume that a matrix representation of a Lie superalgebra is in terms of complex valued supermatrices.

We can of course reverse the above process and starting with a Lie superalgebra $L = L_0 \oplus L_1$ we can construct the corresponding Lie super module

$$\begin{aligned} \mathcal{L} &= E_L \otimes L \qquad \qquad \cdots (1) \\ &= E_L \otimes L_0 \oplus E_L \otimes L_1 \\ &= E_{L0} \otimes L_0 \oplus E_{L1} \otimes L_1 \oplus E_{L1} \otimes L_0 \oplus E_{L0} \otimes L_1 . \end{aligned}$$

Now we take the even part of this - the Lie module

$$\mathcal{L}_0 = E_{L0} \otimes L_0 \oplus E_{L1} \otimes L_1 \qquad \cdots (2)$$

and exponentiate to obtain the corresponding Lie supergroup. The use of the exponential function can be justified by recalling that we can express the Grassman algebra as a set of matrices (see Chapter 1 section 1.2.1).

The simple Lie superalgebras have been classified by Kac [7] , [32] and the real forms of these algebras determined by Kac [32] and Parker [38] who gives precise details for the construction of the real forms. Since we can always construct faithful matrix representations for the superalgebras which obey the commutator $[X, Y] = XY - (-1)^{|X||Y|}YX$ (noting that in some cases it is necessary to use the adjoint representation) we can define simple Lie supergroups as follows, knowing that they must be linear Lie supergroups.

Definition 2.4

(a) A simple Lie supergroup is a Lie supergroup with a simple Lie superalgebra.

(b) A semisimple Lie supergroup is a Lie supergroup with a semisimple Lie superalgebra.

For terminology for the superalgebras we follow Scheunert [33], so that we can consistently use small Latin letters to denote the algebras and Lie modules, and large Latin letters for the corresponding supergroups. Our terminology is such that $spl(n|m;\mathbb{R})$ is a real form of the complex superalgebra $spl(n|m)$. The corresponding Lie module is denoted by $spl(n|m;E_L)$ and this is the Lie module of the linear Lie supergroup $SPL(n|m;E_L)$.

In the sequel we need to use the adjoint representation of a linear Lie supergroup G and know the relationship between this and the adjoint representation of its Lie module. The definition of the adjoint operator in the context of Lie superalgebras has been given precisely by Kac [32] and Scheunert [33]. The definition of the adjoint representation of a supergroup has been given by DeWitt [31] in a form that leaves the action of the matrix realization of the adjoint representation on the Lie superalgebra undefined. Our definitions enable the adjoint representation of a Lie module to be determined as supermatrices with the action precisely determined.

Definition 2.5

Let L be a real Lie superalgebra of dimension $(n|m)$ with basis $\{\alpha_\mu, \beta_\nu\}$. Let $\mathcal{L} = E_L \otimes L$ be the Lie supermodule constructed from L. For every $\gamma \in \mathcal{L}$ define the $(n|m)$ dimensional supermatrix $(ad(\gamma))_{ab}$ by

$$[\gamma, (\alpha, \beta)_b] = (\alpha, \beta)_a (ad(\gamma))_{ab}. \quad \cdots\cdots(3)$$

Here $(\alpha, \beta)_b = (\alpha_1, \alpha_2, \cdots, \alpha_n, \beta_1, \beta_2, \cdots, \beta_m)_b$ which is thus an even supervector.

Proposition 2.6

The set of supermatrices $\{(ad(\gamma))_{ab}, \gamma \in \mathcal{L}\}$ form an $(n|m)$ dimensional representation of $\mathcal{L}$ which will be called the 'adjoint representation'.

Proof

Let $\gamma, \gamma' \in \mathcal{L}$ and $g, g' \in E_L$ then

$$\begin{aligned}
&[(g\gamma + g'\gamma'), (\alpha, \beta)_b] \\
&\quad = (\alpha, \beta)_a (ad(g\gamma + g'\gamma'))_{ab}, \\
&\quad = (\alpha, \beta)_a (ad(g\gamma))_{ab} + (\alpha, \beta)_a (ad(g'\gamma'))_{ab}, \\
&\quad = (\alpha, \beta)_a \begin{bmatrix} gI_n & 0 \\ 0 & (-1)^{|g|} gI_m \end{bmatrix}_{aa'} (ad(\gamma))_{a'b} \\
&\qquad + (\alpha, \beta)_a \begin{bmatrix} g'I_n & 0 \\ 0 & (-1)^{|g'|} g'I_m \end{bmatrix}_{aa'} (ad(\gamma'))_{a'b}, \\
&\quad = g(\alpha, \beta)_a (ad(\gamma))_{ab} + g'(\alpha, \beta)_a (ad(\gamma'))_{ab}. \quad \cdots\cdots(4)
\end{aligned}$$

Now consider

$$[[\gamma, \gamma'], (\alpha, \beta)_a] = (\alpha, \beta)_b (ad([\gamma, \gamma']))_{ba}. \quad \cdots\cdots(5)$$

But using the Jacobi identity of section 1.2.3 of Chapter 1 we have

$$\begin{aligned}
&[[\gamma, \gamma'], (\alpha, \beta)_a] \\
&\quad = [\gamma, [\gamma', (\alpha, \beta)_a]] - (-1)^{|\gamma||\gamma'|} [\gamma', [\gamma, (\alpha, \beta)_a]], \\
&\quad = [\gamma, (\alpha, \beta)_b (ad(\gamma'))_{ba}] - (-1)^{|\gamma||\gamma'|} [\gamma', (\alpha, \beta)_b (ad(\gamma))_{ba}], \\
&\quad = (\alpha, \beta)_c (ad(\gamma))_{cb} (ad(\gamma'))_{ba} \\
&\qquad - (-1)^{|\gamma||\gamma'|} (\alpha, \beta)_c (ad(\gamma'))_{cb} (ad(\gamma))_{ba},
\end{aligned}$$

$$= (\alpha, \beta)_b \, [\mathrm{ad}(\gamma), \mathrm{ad}(\gamma')]_{ba}.$$

Combining this result with equation (5) we see that

$$\mathrm{ad}([\gamma, \gamma'])_{ba} = [\mathrm{ad}(\gamma), \mathrm{ad}(\gamma)]_{ba}. \qquad \ldots (6)$$

Thus $\mathrm{ad}(\gamma)$ provides a representation of $\mathcal{L}$ in terms of $(n|m)$ dimensional supermatrices.

Note that we make no claims that this representation is faithful. The crucial point in the above definition is in choosing to write $(\alpha, \beta)_a (\mathrm{ad}(\gamma))_{ab}$ ie. the order in which they are multiplied as supermatrices. If we had chosen to follow the customary practice used for Lie algebras ie. by writing $(\mathrm{ad}(\gamma))_{ab} (\alpha, \beta)_a$ we would have had difficulty in deciding how to define multiplication by anticommuting scalars.

Now we want to construct the adjoint representation of G we observe first that since G can be considered as a coded representation of $\mathcal{G}_L$ and $\mathcal{L}_0$ is isomorphic to the Lie algebra of $\mathcal{G}_L$ the proof of the following Lemma follows from the theory of Lie groups and Lie algebras.

Lemma 2.7

Let G be a linear Lie supergroup and $\mathcal{L}_0$ its Lie module. Then for any $g \in G$ and $\gamma \in \mathcal{L}_0$

$$g \gamma g^{-1} \in \mathcal{L}_0. \qquad \ldots (7)$$

We now give the definition of the adjoint representation of G. Again we insist that the supermatrices are written in the appropriate order for matrix multiplication.

Definition 2.8

Let G be a linear Lie supergroup of dimension $(n|m)$ and let $\{\alpha_\mu, \beta_\nu\}$ be a basis of its Lie superalgebra L. For each $g \in G$ let $\mathrm{Ad}(g)$ be the matrix defined by

$$g(\alpha,\beta)_a g^{-1} = (\alpha,\beta)_b (Ad(g))_{ba} \qquad \cdots(8)$$

for $a = 1, 2, \cdots, m+n$.

Lemma 2.7 demonstrates that $Ad(g)$ is well defined.

Theorem 2.9

(a) The set of matrices $Ad(g)$ form an $(n|m)$ dimensional representation of G, called the adjoint representation of G.

(b) The associated representation of $\mathcal{L}_0$ defined by equation (3) is the adjoint representation of $\mathcal{L}_0$. That is

$$ad(\alpha) = \left(\tfrac{\partial}{\partial x} Ad(\exp x\alpha)\right)_{x=0}, \quad x \in E_{L0} \qquad \cdots(9)$$

and
$$ad(\beta) = \left(\tfrac{\partial}{\partial \theta} Ad(\exp \theta\beta)\right)_{\theta=0}, \quad \theta \in E_{L1}. \qquad \cdots(10)$$

Proof

(a) We have to show that $Ad(g)Ad(g') = Ad(gg')$.

Consider
$$\begin{aligned}(\alpha,\beta)_b (Ad(gg'))_{ba} &= (gg')(\alpha,\beta)_a (gg')^{-1} \\ &= g\,(g')(\alpha,\beta)_a g'^{-1} g^{-1} \\ &= g\,(\alpha,\beta)_b (Ad(g'))_{ba} g^{-1} \\ &= g\,(\alpha,\beta)_b g^{-1} (Ad(g'))_{ba}\end{aligned}$$

since g^{-1} is even. Thus

$$(\alpha,\beta)_b (Ad(gg'))_{ba} = (\alpha,\beta)_b (Ad(g))_{bc} (Ad(g'))_{ca}.$$

The matrices $Ad(g)$, $g \in G$ are clearly of dimension $(n|m)$.

(b) Let $g = \exp x\alpha'$ then from the definition of $Ad(g)$ we have

$$\begin{aligned}&(\alpha,\beta)_b (Ad(\exp x\alpha'))_{ba} \\ &\quad = \exp(x\alpha')(\alpha,\beta)_a \exp-(x\alpha'), \\ &\quad = \left(1 + x\alpha' + \tfrac{(x\alpha')^2}{2} + \cdots\right)(\alpha,\beta)_a \left(1 - x\alpha' + \tfrac{(x\alpha')^2}{2} + \cdots\right), \\ &\quad = 1 + [x\alpha', (\alpha,\beta)_a] + \text{terms in } x^2, \\ &\quad = 1 + x(\alpha,\beta)_b (ad(\alpha'))_{ba} + \text{terms in } x^2,\end{aligned}$$

Thus differentiating with respect to x we obtain equation (9). Equation (10) is obtained in a similar way.

2.2 Examples of Linear Lie Supergroups.

2.2.1 The General Linear Lie Supergroup.

We define this by

$$PL(p|q;E_L) = \{u \in M(p|q;E_L) \text{ and } u^{-1} \in M(p|q;E_L)\}, \qquad \ldots (11)$$

for $p \geqslant 1$, $q \geqslant 0$. We note that if $q = 0$ we have $PL(p|0;E_L) = GL(p;E_L)$ (see section 2.2.2). If we suppose that $u = \exp X$, then the only constraint on X is $X \in M(p|q;E_L)$, so that the Lie module of $PL(p|q;E_L)$ is given by

$$p\ell(p|q;E_L) = \{X \in M(p|q;E_L)\}. \qquad \ldots (12)$$

But if $u = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M(p|q;E_L)$ and is invertible then A^{-1} and D^{-1} exist and

$$u = \begin{bmatrix} I_p & BD^{-1} \\ 0 & I_q \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I_p & 0 \\ D^{-1}C & I_q \end{bmatrix} \qquad \ldots (13)$$

with
$$u^{-1} = \begin{bmatrix} I_p & 0 \\ -D^{-1}C & I_q \end{bmatrix} \begin{bmatrix} (A-BD^{-1}C)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} I_p & -BD^{-1} \\ 0 & I_q \end{bmatrix}. \qquad \ldots (14)$$

We can reparametrize these expressions to give

$$u = \begin{bmatrix} I_p & B' \\ 0 & I_q \end{bmatrix} \begin{bmatrix} A' & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I_p & 0 \\ C' & I_q \end{bmatrix} \qquad \ldots (15)$$

and
$$u^{-1} = \begin{bmatrix} I_p & 0 \\ -C' & I_q \end{bmatrix} \begin{bmatrix} A'^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} I_p & -B' \\ 0 & I_q \end{bmatrix}. \qquad \ldots (16)$$

Here the parametrisation can be chosen such that A', D, B and C' depend on different sets of parameters with the parameters of A' and D even and the parameters of B' and C' odd. Thus $PL(p|q;E_L)$ can be decomposed into three subsupergroups. We can identify the Lie modules of these subgroups with Lie submodules of the Lie module $p\ell(p|q;E_L)$ as follows

$$X = X_{-1} \oplus X_0 \oplus X_1,$$
$$= \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \qquad \ldots (17)$$

with X_i, $i = -1, 0, 1$ a submodule of $p\ell(p|q;E_L)$ and

$$\exp(X_{-1})=\begin{bmatrix} I_p & B' \\ 0 & I_q \end{bmatrix},\ \exp(X_0)=\begin{bmatrix} A' & 0 \\ 0 & D \end{bmatrix} \text{ and } \exp(X_1)=\begin{bmatrix} I_p & 0 \\ C' & I_q \end{bmatrix}$$

with $A' = GL(p;E_L)$ and $D = GL(q;E_L)$.

We note that the important point here is that the Lie module can be decomposed as in equation (6), that is we have a $\mathbb{Z}$ grading as well as the $\mathbb{Z}_2$ grading, and this $\mathbb{Z}$ gradation gives a decomposition into submodules.

The dimension of $PL(p|q;E_L)$ is $(p^2+q^2|2pq)$. We can construct $PL(p|q;\mathbb{C}E_L)$ in a similar way.

We note that $PL(p|q;E_L)$ is a four component supergroup if $q \neq 0$ and two component if $q = 0$. This is because if $u \in PL(p|q;E_L)$ then $\mathrm{Sdet}(u) = a \in E_{L0}$ with $a_0 \neq 0$ (ie. the component in the ε_0 direction) but a_0 can be either positive or negative so that the supergroup splits into two sections. If $q \neq 0$ then we have four cases (i) $\det(A')$ and $\det(D')$ (see section 1.2.5 (ii) of chapter 1 for the definition of D') are both positive, (ii) $\det(A')$ and $\det(D')$ are both negative, (iii) $\det(A')$ is positive and $\det(D')$ is negative and (iv) $\det(A')$ is negative and $\det(D')$ is positive. Thus we have four disjoint components. If $q = 0$ there are only two components.

2.2.2 Supergroups Obtained from Lie Groups.

Given any real linear Lie group parametrized by $x^\mu \in \mathbb{R}$, $\mu = 1, 2, \cdots$ we can allow the parameters to take values from E_{L0} and thus obtain a Lie supergroup. We denote this such that $SL(n;E_L)$ is obtained from $SL(n;\mathbb{R})$.

2.2.3 Real Lie Supergroups Obtained from $sp\ell(n|m)$.

The Lie superalgebra $sp\ell(n|m)$ is defined by

$$sp\ell(p|q) = \{ X \text{ is a complex valued supermatrix}, \ Str(X) = 0 \} \quad \ldots(18)$$

for $p, q \geqslant 1$. Note that $sp\ell(1|1)$ is not simple. The real forms are given by Parker [38] and can be classified by their Lie subalgebras. We list them in the following table with our designation.

Designation	Lie Subalgebra L_0	Restrictions
$sp\ell(p\|q;\mathbb{R})$	$s\ell(p;\mathbb{R}) \oplus s\ell(q;\mathbb{R}) \oplus \mathbb{R}$	$p \geqslant 1, q \geqslant 2$
$su^*(p\|q;\mathbb{R})$	$su^*(p) \oplus su^*(q) \oplus \mathbb{R}$	p, q even
$su(m, p-m\|n, q-n;\mathbb{R})$	$su(m, p-m) \oplus su(n, p-n) \oplus i\mathbb{R}$	$p, q \geqslant 1$

...(19)

If $p = q$ none of these algebras is simple, they have a one dimensional ideal that is a multiple of the identity. Parker identifies the case separately for $sp\ell(p|p;\mathbb{R})$ but the others do exist as can be established by evaluating the Killing form (see Scheunert [33] for a definition of this) in individual cases and obtaining different values. Each of these algebras has dimension $(p^2 + q^2 - 1 \,|\, 2pq)$ with the even dimension reduced by one if the ideal is factored out.

To construct the corresponding Lie module we form the tensor product $E_L \otimes L$, and then take the even part $E_{L0} \otimes L_0 \oplus E_{L1} \otimes L_1$. To construct a basis for the real forms we choose a set of matrices as basis for $sp\ell(p|q)$ considered as a real Lie superalgebra ie. if the set $\{X\}$ is a basis we consider the set $\{X, iX\}$. Then we determine the possible ways of imposing a complex structure on this vector space.

Case (i) $SPL(n|m;E_L)$.

The superalgebra $sp\ell(n|m;\mathbb{R})$ is obtained simply by restricting the domain to $\mathbb{R}$. The definition of the corresponding Lie module is

$$sp\ell(p|q;E_L) = \{ X \in p\ell(p|q;E_L), \ str\, X = 0 \}. \qquad \ldots(20)$$

This admits the $\mathbb{Z}$ grading such that

$$X = X_{-1} \oplus X_0 \oplus X_1 \;,$$
$$= \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}. \qquad \cdots\cdots(21)$$

Clearly each X_i, $i = -1, 0, 1$ is a submodule exactly as for $pl(p|q;E_L)$.

Now recall from section 1.2.5 of Chapter 1 that $\mathrm{Sdet}(\exp X) = \exp(\mathrm{Str}X)$, so that if $\mathrm{Str}\,X = 0$ then $\mathrm{Sdet}(\exp X) = 1$ thus we can define the linear Lie supergroup $SPL(p|q;E_L)$ by

$$SPL(p|q;E_L) = \{ U \in PL(p|q;E_L) ,\ \mathrm{Sdet}(U) = 1 \}. \qquad \cdots\cdots(22)$$

This has been given previosly by Rittenberg and Scheunert [28] as a purely algebraic supergroup. It can be decomposed as

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$
$$= U_{-1} U_0 U_1 \;,$$
$$= \begin{bmatrix} I_p & BD^{-1} \\ 0 & I_q \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I_p & 0 \\ D^{-1}C & I_q \end{bmatrix},$$
$$= \begin{bmatrix} I_p & B' \\ 0 & I_q \end{bmatrix} \begin{bmatrix} A' & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I_p & 0 \\ D' & I_q \end{bmatrix}. \qquad \cdots\cdots(23)$$

Here the last line is simply a re-parametrization. We can now identify the subsupergroups U_1, U_0 and U_{-1} with the submodules X_1, X_0 and X_{-1} respectivly and so construct the supergroup in a very straightforward way. We note that X_1 and X_{-1} are abelian so that

$$\exp(X_i) = 1 + X_i = U_i \text{ for } i = 1, -1.$$

It is well known (cf. Scheunert [33]) that the submodule X_0 has an ideal with generator $\begin{bmatrix} \frac{1}{p} I_p & 0 \\ 0 & \frac{1}{q} I_q \end{bmatrix}$,

so that $X_0 = \begin{bmatrix} a' & 0 \\ 0 & d' \end{bmatrix} \oplus \begin{bmatrix} y/p\, I_p & 0 \\ 0 & y/q\, I_q \end{bmatrix}$ with $y \in E_{L0}$,

$a' \in sl(p;E_L)$ and $d' \in sl(q;E_L)$. It is convenient to factor out this ideal to give.

$$u = \begin{bmatrix} I_p & B' \\ 0 & I_q \end{bmatrix} \begin{bmatrix} A'' & 0 \\ 0 & D'' \end{bmatrix} \begin{bmatrix} \exp y/p\, I_p & 0 \\ 0 & \exp \frac{y}{q} I_q \end{bmatrix} \begin{bmatrix} I_p & 0 \\ C' & I_q \end{bmatrix}, \quad \ldots (29)$$

then $A'' \in SL(p;E_L)$ and $D'' \in SL(q;E_L)$ and each of the matrix blocks A'', B', C' and D'' depend on disjoint sets of parameters.

If $p=q$ then $\begin{bmatrix} \exp y/p\, I_p & 0 \\ 0 & \exp y/q\, I_q \end{bmatrix}$ is a multiple of the identity and so is an invariant subgroup of $SPL(p|p;E_L)$. To obtain a simple Lie supergroup in this case we need to factor it out. This limits the representations of the supergroup (and its Lie Module) that have straightforward matrix representations (the adjoint representation is faithful). To obtain a Haar integral (see Chapter 3), in this case, it is better to leave in the ideal and construct the required supergroup as a coset space.

Case (ii) $su^*(p|q;E_L)$.

In this case we set $CX = MX^*M^{-1}$, for $X \in spl(p|q;\mathbb{C}E_L)$ and obtain a complex structure by taking the set $\{Y = X + CX\}$ with

$$M = \begin{bmatrix} 0 & -I_r & 0 & 0 \\ I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_s \\ 0 & 0 & I_s & 0 \end{bmatrix} \quad \text{and } 2r = p,\ 2s = q.$$

A basis of $su^*(p|q;E_L)$ is then obtained by evaluating Y for each basis element X and iX. Now if $X \in spl(p|q;E_L)$ so is X^*. Thus

$$Y = X + MX^*M^{-1}$$

and

$$Y^* = X^* + MXM^{-1}$$

are elements of $su^*(p|q;E_L)$. Combining these two equations gives

$$\begin{aligned} -Y + M^{-1}Y^*M &= M^{-1}X^*M - MX^*M^{-1}, \\ &= M^{-1}(X^*M - MMX^*M^{-1}), \\ &= M^{-1}(X^*MM - MMX^*)M^{-1}. \end{aligned}$$

But $MM = -I_{2r+2s}$ so that

$$M^{-1}Y^*M = Y.$$

Thus the definition of $su^*(p|q;E_L)$ is given by

$$su^*(p|q;E_L) = \{X \in spl(p|q;\mathbb{C}E_L),\ M^{-1}X^*M = X\}. \qquad \cdots(25)$$

The definition of the superalgebra $su^*(p|q;\mathbb{R})$ is

$$su^*(p|q;\mathbb{R}) = \{X \in spl(p|q),\ M^{-1}X^*M = X\}. \qquad \cdots(26)$$

Now if $X, Y \in su^*(p|q;E_L)$ then

$$\begin{aligned} M^{-1}[X,Y]^*M &= M^{-1}(XY-(-1)^{|X||Y|}YX)^*M, \\ &= M^{-1}X^*Y^*M-(-1)^{|X||Y|}M^{-1}Y^*X^*M, \\ &= XY-(-1)^{|X||Y|}YX, \\ &= [X,Y]. \end{aligned}$$

So that the Lie module is a closed algebraic system as required.

Now if we put $U = \exp X$ we have

$$\begin{aligned} U = \exp X &= \exp(M^{-1}X^*M), \\ &= M^{-1}(\exp X^*)M, \\ &= M^{-1}(\exp X)^*M, \\ &= M^{-1}U^*M. \end{aligned}$$

Thus we can define the supergroup $SU^*(p|q;E_L)$ by

$$SU^*(p|q;E_L) = \{U \in SPL(p|q;\mathbb{C}E_L),\ U^* = MUM^{-1}\}. \cdots(27)$$

Now consider a general element of $spl(p|q;E_L)$ given by $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ which of course admits the $\mathbb{Z}$ gradation

$$\begin{aligned} X &= X_{-1} \oplus X_0 \oplus X_1, \\ &= \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}, \end{aligned}$$

and suppose that it is also an element of $su^*(p|q;E_L)$ so that it must satisfy the constraint given in equation (25). Then

$$\begin{bmatrix} 0 & -I_r & 0 & 0 \\ I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_s \\ 0 & 0 & I_s & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & I_r & 0 & 0 \\ -I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & I_s \\ 0 & 0 & -I_s & 0 \end{bmatrix} = \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix}.$$

Thus

$$\begin{bmatrix} \begin{bmatrix} 0 & -I_r \\ I_r & 0 \end{bmatrix} a \begin{bmatrix} 0 & I_r \\ -I_r & 0 \end{bmatrix} & \begin{bmatrix} 0 & -I_r \\ I_r & 0 \end{bmatrix} b \begin{bmatrix} 0 & I_s \\ -I_s & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & -I_s \\ I_s & 0 \end{bmatrix} c \begin{bmatrix} 0 & I_r \\ -I_r & 0 \end{bmatrix} & \begin{bmatrix} 0 & -I_s \\ I_s & 0 \end{bmatrix} d \begin{bmatrix} 0 & I_s \\ -I_s & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix}$$

which demonstrates that we have preserved the $\mathbb{Z}$ gradation of $spl(p|q;\mathbb{C}E_L)$. We can thus use the procedures of Case (i) to construct group elements.

Example 2.10 $su^*(2|2;E_L)$

Using the above procedures we find a basis of the Lie superalgebra $su^*(2|2;\mathbb{R})$ to be

Basis of L_0

$$\alpha_1 = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\alpha_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}, \quad \alpha_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}, \quad \alpha_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\alpha_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

Basis of

$$\beta_1=\begin{bmatrix}0&0&0&1\\0&0&-1&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad \beta_2=\begin{bmatrix}0&0&0&i\\0&0&i&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad \beta_3=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&-1&0&0\\1&0&0&0\end{bmatrix},$$

$$\beta_4=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&i&0&0\\i&0&0&0\end{bmatrix},\quad \beta_5=\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad \beta_6=\begin{bmatrix}0&0&i&0\\0&0&0&-i\\0&0&0&0\\0&0&0&0\end{bmatrix},$$

$$\beta_7=\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{bmatrix},\quad \beta_8=\begin{bmatrix}0&0&0&0\\0&0&0&0\\i&0&0&0\\0&-i&0&0\end{bmatrix}.$$

We note that the sets $\{\beta_1,\beta_2,\beta_5,\beta_6\}$ and $\{\beta_2,\beta_3,\beta_7,\beta_8\}$ form abelian subgroups which we can identify with X_{-1} and X_1 respectivly. The general group element can be written, with the parameters $x^\mu \in E_{L0}$, $\mu=1,2,\cdots,7$ and $\theta^\nu \in E_{L1}$, $\nu=1,2,\cdots,8$ as

$$u=\begin{bmatrix}I_2&B\\0&I_2\end{bmatrix}\begin{bmatrix}A&0\\0&D\end{bmatrix}\begin{bmatrix}yI_4\end{bmatrix}\begin{bmatrix}I_2&0\\C&I_2\end{bmatrix},$$

with

$$A=\begin{bmatrix}\cos\frac{x^1}{2}\exp\frac{i}{2}(x^2+x^3) & \sin\frac{x^1}{2}\exp\frac{i}{2}(x^2-x^3)\\ -\sin\frac{x^1}{2}\exp-\frac{i}{2}(x^2-x^3) & \cos\frac{x^1}{2}\exp-\frac{i}{2}(x^2+x^3)\end{bmatrix},$$

$$B=\begin{bmatrix}\theta^5+i\theta^6 & \theta^1+i\theta^2\\ -\theta^1+i\theta^2 & \theta^5-i\theta^6\end{bmatrix},\qquad C=\begin{bmatrix}\theta^7+i\theta^8 & \theta^3+i\theta^4\\ -\theta^3+i\theta^4 & \theta^7-i\theta^8\end{bmatrix},$$

$$D=\begin{bmatrix}\cos\frac{x^4}{2}\exp\frac{i}{2}(x^5-x^6) & \sin\frac{x^4}{2}\exp\frac{i}{2}(x^5-x^6)\\ -\sin\frac{x^4}{2}\exp-\frac{i}{2}(x^5-x^6) & \cos\frac{x^4}{2}\exp-\frac{i}{2}(x^5-x^6)\end{bmatrix}$$

and $\quad y=\exp x^7.$

Here we have noted that $su^*(2)=su(2)$ and used a standard paremetrization as given by Cornwell [39] Chapter 3. We note that since we have $p=q$ the ideal of L_0 given by the generator α_7 is an ideal of the whole algebra so that we do not have a simple Lie supergroup.

Case (iii) $su(m, p-m | n, q-n ; E_L)$

In this case we have to use a different operation for odd and even elements of the basis of $spl(p|q ; E_L)$. We have

$$c_0 X = -N X^{\dagger} N \quad \text{for even elements}$$

and

$$c_1 X = i N X^{\dagger} N \quad \text{for odd elements,}$$

with $X^{\dagger} = (X^t)^*$ and $N = \begin{bmatrix} -I_m & 0 & 0 & 0 \\ 0 & I_{p-m} & 0 & 0 \\ 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & I_{q-n} \end{bmatrix}.$

A basis of $su(m, p-m | n, q-n ; \mathbb{R})$ is then given by

$$Y = X - N X^{\dagger} N \quad \text{for even elements}$$

and

$$Y = X + i N X^{\dagger} N \quad \text{for odd elements,}$$

these can be combined to give

$$Y = X + (i)^{|X|+2} N X^{\dagger} N.$$

It is straightforward to characterize the Lie algebra and Lie module following the procedures of the previous case. We obtain

$$su(m, p-m | n, q-m ; E_L) = \{ X \in spl(p|q ; \mathbb{C}E_L), \quad X = (i)^{|X|+2} N X^{\dagger} N \}. \qquad \cdots (29)$$

We can find no straightforward way of characterizing a supergroup element. Neither can we find a way of giving a $\mathbb{Z}$ gradation, with a decomposition into submodules, of the Lie Module. We give two examples of the Lie superalgebra generators.

<u>Example 2.11</u> $su(1,1|1 ; \mathbb{R})$

$$\alpha_1 = \begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \alpha_4 = \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 2i \end{bmatrix},$$

$$\beta_1=\begin{bmatrix}0&0&-i\\0&0&0\\1&0&0\end{bmatrix},\ \beta_2=\begin{bmatrix}0&0&-1\\0&0&0\\i&0&0\end{bmatrix},\ \beta_3=\begin{bmatrix}0&0&0\\0&0&i\\0&1&0\end{bmatrix},\ \beta_4=\begin{bmatrix}0&0&0\\0&0&1\\0&i&0\end{bmatrix}.$$

<u>Example 2.12</u> $su(2|1;\mathbb{R})$.

$$\alpha_1=\begin{bmatrix}i&0&0\\0&-i&0\\0&0&0\end{bmatrix},\ \alpha_2=\begin{bmatrix}0&i&0\\i&0&0\\0&0&0\end{bmatrix},\ \alpha_3=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix},\ \alpha_4=\begin{bmatrix}i&0&0\\0&i&0\\0&0&2i\end{bmatrix},$$

$$\beta_1=\begin{bmatrix}0&0&i\\0&0&0\\1&0&0\end{bmatrix},\ \beta_2=\begin{bmatrix}0&0&1\\0&0&0\\i&0&0\end{bmatrix},\ \beta_3=\begin{bmatrix}0&0&0\\0&0&i\\0&1&0\end{bmatrix},\ \beta_4=\begin{bmatrix}0&0&0\\0&0&1\\0&i&0\end{bmatrix}.$$

It is worthwhile noting that there are many papers in the physics literature [40] that deal with a 'supergroup' denoted by '$SU(N/M)$'. In our notation this is $spl(N|M;\mathbb{C}E_L)$, so that the fact that the Lie superalgebra '$su(N/M)$' decomposes into three subalgebras (which is implicit in some of these papers), is not in dispute here.

2.2.4 The Real Orthosymplectic Group $OSP(n|2r;E_L)$.

We do not consider all of the real forms of the superalgebra $osp(n|2r)$ only the simplest case obtained by restricting the domain to $\mathbb{R}$.

It is possible to decompose the superalgebras $osp(2|2r;\mathbb{R})$ into three subsets each of which is a subalgebra. In the general case, which we treat, there is a decomposition into five subsets as $X=X_{-2}\oplus X_{-1}\oplus X_0\oplus X_1\oplus X_2$ such that $\{X_{-2}\oplus X_{-1}\}$, $\{X_0\}$ and $\{X_1\oplus X_2\}$ are subalgebras. Both of these decompositions, for the superalgebra, are given by Kac [32].

Our definition of $osp(n|2r;\mathbb{R})$ is given by

$$osp(n|2r;\mathbb{R})=\{X\in spl(n|2r;\mathbb{R}),\ X^{st}h+(-1)^{|X|}hX=0\}\cdots(30)$$

with X^{st} as defined in Chapter 1 section 1.2.4 and

$$h = \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_r \\ 0 & -I_r & 0 \end{bmatrix}.$$

The superalgebra has dimension $\left(\frac{n^2-n}{2}+2r^2-r \mid 2nr\right)$. The corresponding Lie module is then defined by

$$osp(n|2r;E_L) = \{X \in spl(n|2r;E_L),\ X^{st}h + hX = 0\} \cdots (31)$$

noting that every element of a Lie module is even.

With these conventions a general element of $osp(n|2r;E_L)$ can be written in the form (see Kac [32])

$$X = \begin{bmatrix} a & y & z \\ -z^t & b & c \\ y^t & d & -b^t \end{bmatrix}$$

with $a \in so(n;E_L)$, $b \in gl(r;E_L)$, c and d symmetric with entries from E_{L0}, and y and z having entries from E_{L1}. We then have the decomposition

$$X = X_{-2} \oplus X_{-1} \oplus X_0 \oplus X_1 \oplus X_2$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & d & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & y & 0 \\ 0 & 0 & 0 \\ y^t & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -b^t \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & z \\ -z^t & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix},$$

with the sets $\{X_{-2} \oplus X_{-1}\}$, $\{X_0\}$ and $\{X_1 \oplus X_2\}$ as submodules. We can exponentiate each of these submodules separately and construct a supergroup element as

$$u = exp(X_{-2} \oplus X_{-1})\, exp(X_0)\, exp(X_1 \oplus X_2)$$

$$= \begin{bmatrix} I_n & Y & 0 \\ 0 & I_r & 0 \\ Y^t & D+\frac{Y^tY}{2} & I_r \end{bmatrix} \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & (B)^{t^{-1}} \end{bmatrix} \begin{bmatrix} I_n & 0 & Z \\ -Z^t & I_r & C-\frac{Z^tZ}{2} \\ 0 & 0 & I_r \end{bmatrix} \cdots (32)$$

We note that

$$u^{-1} = \begin{bmatrix} I_n & 0 & -Z \\ Z^t & I_r & -C-\frac{Z^tZ}{2} \\ 0 & 0 & I_r \end{bmatrix} \begin{bmatrix} A^{-1} & 0 & 0 \\ 0 & B^{-1} & 0 \\ 0 & 0 & B^t \end{bmatrix} \begin{bmatrix} I_n & -Y & 0 \\ 0 & I_r & 0 \\ -Y^t & -D+\frac{Y^tY}{2} & I_r \end{bmatrix} \quad \ldots(33)$$

Here $A \in SO(n;E_L)$, $B \in GL_e(r;E_L)$ the component of $GL(r;E_L)$ connected to the identity, Z and Y are conveniently parametrized by a different odd parameter (θ^ν) in each position and C and D are conveniently parametrized by a different even parameter (x^μ) at each position.

This construction certainly meets the condition normally given for the orthosymplectic group (see equation (35) below) and in fact meets the more restrictive condition

$$SOSP(n|2r;E_L) = \{u \in SPL(n|2r;E_L),\ u^{st}h\,u = h\}. \quad \ldots(34)$$

This definition can be obtained from equation (31) using the properties of exp and st given in section 1.2.5 of Chapter 1 as follows

$$X^{st}h = -hX \ ,$$

$$\exp(X^{st}) = \exp(-hXh^{-1}) \ ,$$

$$(\exp X)^{st} = h(\exp -X)h^{-1}$$

thus
$$h = (\exp X)^{st}h \exp X.$$

So that with $u = \exp X$ we have $u^{st}h\,u = h$ as required.
It is clear that the only element in common between the three sub-supergroups of equation (33) is the identity so that the representation is faithful and that the product of any two matrices of this form can also be decomposed in this way. So that we can deduce that this construction does give the complete group, provided we choose a suitable parametrization for A and D.

The definition often given [28], [29], [30] for the orthosymplectic groups is

$$OSP(n|2r;E_L) = \{u \in PL(n|2r;E_L),\ u^{st}h\,u = h\}. \quad \ldots(35)$$

It is clearly obtained from equation (34) by replacing $SO(n;E_L)$ by $O(n;E_L)$ and $GL_e(r;E_L)$ by $GL(n;E_L)$. Since these are both two component supergroups $OSP(n|2r;E_L)$ is a four component supergroup.

2.2.5 Real Lie Supergroups Obtained from $b(n)$.

The Lie superalgebra $b(n)$ is defined by Scheunert [33] (see also Kac [32] who denotes it P(n)) as

$$b(n) = \{X \in spl(n|n),\ X^P = -(-1)^{|X|}X\} \qquad \cdots(36)$$

with X^P, the 'p-transpose' of X, defined in section 1.2.5 of Chapter 1. The definition restricts the matrix X to be of the form $\begin{bmatrix} a & b \\ c & -a^t \end{bmatrix}$ with $b^t = b$ and $c^t = -c$, so that there is a $\mathbb{Z}$ gradation of the algebra such that

$$\begin{aligned} X &= X_{-1} \oplus X_0 \oplus X_1\,, \\ &= \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} a & 0 \\ 0 & a^t \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}, \end{aligned} \qquad \cdots(37)$$

and each X_i, $i = -1, 0, 1$ is a subalgebra.

The dimension of the superalgebra is $(n^2-1\,|\,n^2)$.

There are two real forms of this superalgebra which are

$$b(n;\mathbb{R}) = \{X \in spl(n|n;\mathbb{R}),\ X^P = -(-1)^{|X|}X\} \qquad \cdots(38)$$

and $$b^*(n;\mathbb{R}) = \{X \in su^*(n|n;\mathbb{R}),\ X^P = -(-1)^{|X|}X\}. \qquad \cdots(39)$$

Both of these algebras admit the decomposition of equation (37) and both satisfy the constraints $b^t = b$ and $c^t = -c$. From these we can construct the corresponding Lie modules as

$$b(n;E_L) = \{X \in spl(n|n;E_L),\ X^P = -X\} \qquad \cdots(40)$$

and $$b^*(n;E_L) = \{X \in su^*(n|n;E_L),\ X^P = -X\}. \qquad \cdots(41)$$

Then since, as stated in section 1.2.5 of Chapter 1, $\exp(X^P) = (\exp X)^P$ we can define the supergroups $B(n;E_L)$ and $B^*(n;E_L)$ as

$$B(n;E_L) = \{U \in SPL(n|n;E_L),\ UU^P = 1\} \qquad \cdots(42)$$

and $$B^*(n;E_L) = \{U \in SU^*(n|n;E_L),\ UU^P = 1\}. \qquad \cdots(43)$$

The supergroup $B(n;E_L)$ has been given previously by Rittenberg [29] and Ne'eman [30] (who call it P(n−1)).

The decomposition into subgroups of these supergroups is inherited from the decomposition of $SPL(n|n;E_L)$ and $SU^*(n|n;E_L)$ respectively.

2.2.6 The Supergroup $USPL(1|1)$.

This is an example of a matrix supergroup that is not a Lie supergroup, even though it is parametrized by continuous parameters. All that we say applies immediately to $USPL(n|m)$, $n,m=1,2,\cdots$. This supergroup was originally given by Rittenberg and Scheunert [29] as an analogue of the unitary groups, its definition is

$$USPL(1|1) = \{ U \in SPL(1|1;\mathbb{C}E_L),\ U^{SA} = U^{-1} \}. \qquad \cdots(44)$$

Its Lie module is

$$usp\ell(1|1) = \{ X \in sp\ell(1|1;\mathbb{C}E_L),\ X^{SA} = -X \}. \qquad \cdots(45)$$

We can easily check that the supergroup is closed using Lemma 1.8 of Chapter 1. Consider $U, V \in USPL(1|1)$ then

$$(UV)^{SA}UV = V^{SA}U^{SA}UV = 1.$$

Also the 'Lie module' is closed since for $X, Y \in usp\ell(1|1)$

$$\begin{aligned}[X,Y]^{SA} &= [Y^{SA}, X^{SA}] \\ &= [-Y, -X] \\ &= [Y, X] \\ &= -[X, Y] \text{ since } |X| = |Y| = 0.\end{aligned}$$

Now consider a general even element of the 'Lie module'. This must satisfy

$$\begin{bmatrix} a+ib & 0 \\ 0 & c+id \end{bmatrix}^{SA} = -\begin{bmatrix} a+ib & 0 \\ 0 & c+id \end{bmatrix} \quad a,b,c,d \in E_{L0}$$

so that $$\begin{bmatrix} a^{\#}-ib^{\#} & 0 \\ 0 & c^{\#}-id^{\#} \end{bmatrix} = \begin{bmatrix} -a-ib & 0 \\ 0 & -c-id \end{bmatrix}.$$

Now recall the effect of $^{\#}$ from section 1.2.2(b) of Chapter 1. We see that the real part can only have elements of level $4n+2, n=0,1,2,\cdots$

and the imaginary part can only have elements of level $4n+1$.

Consider also a general odd element. This must satisfy

$$\begin{bmatrix} 0 & \phi+i\psi \\ \chi+i\theta & 0 \end{bmatrix}^{SA} = -\begin{bmatrix} 0 & \phi+i\psi \\ \chi+i\theta & 0 \end{bmatrix} \quad \phi,\psi,\chi,\theta \in E_{L1}$$

$$\begin{bmatrix} 0 & \chi^{\#}-i\theta^{\#} \\ \phi^{\#}-i\psi^{\#} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\phi-i\psi \\ -\chi-i\theta & 0 \end{bmatrix}.$$

Thus $\phi^{\#}=-\chi$ and $\theta=\psi^{\#}$ with no other restrictions on the entries.

A general odd matrix then has the form $\begin{bmatrix} 0 & \phi+i\psi \\ -\phi^{\#}+i\psi^{\#} & 0 \end{bmatrix}$.

The result of these considerations is that we cannot obtain a basis for the Lie module $usp\ell(1|1)$ of the form $E_{L0}\otimes\alpha_\mu \oplus E_{L1}\otimes\beta_\nu$, or $usp\ell(1|1)$ is not a submodule of $sp\ell(1|1;\mathbb{C}E_L)$ but is in fact a ring embedded in $sp\ell(1|1;\mathbb{C}E_L)$.

Now consider a general even element of $USPL(1|1)$. This must satisfy

$$\begin{bmatrix} \exp(x+iy) & 0 \\ 0 & \exp(x+iy) \end{bmatrix}^{SA} \begin{bmatrix} \exp(x+iy) & 0 \\ 0 & \exp(x+iy) \end{bmatrix} = I_2 .$$

So that $\quad \exp(x^{\#}-iy^{\#})\exp(x+iy) = 1,$

ie. $\quad x^{\#}-iy^{\#}+x+iy = 0.$

Thus $x^{\#}=-x$ and $y^{\#}=y$ exactly as for the Lie module. But x, y are not then differentiable elements of E_{L0} so that this group is not a Lie supergroup.

These arguments apply to the supergroups $SUPL(n|m)$ and $UOSP(n|2r)$ constructed using the superadjoint operation detailed in section 1.2.5(vi) of Chapter 1. These were given by Rittenberg and Scheunert [28] and have been repeated by Rittenberg [29] and Ne'eman [30] .

2.3 The Linear Differential Operators Corresponding to the Lie Superalgebra Generators.

Linear differential operators have been given previously for the generators of the super Poincaré group (cf. Fayet and Ferrara [41]). The general case of any linear Lie group acting on some carrier space has never previously been determined.

Let M be an $(m|n)$ dimensional supermanifold on which the action of a linear Lie supergroup G is defined. Let (U, ψ) be a coordinate chart on M with coordinate functions (x^μ, θ^ν). Let α, β be, respectivly, even and odd matrix representations of elements of the Lie superalgebra of G. (α, β may of course be part of a reducible, irreducible etc. representation depending how we wish the group action defined).

Let $f \in G^\infty(M)$ with $\psi \circ f = f(x^\mu, \theta^\nu)$. We define the group action on $G^\infty(M)$ by extending the definition for left translation of functions defined on G. That is for $g, g' \in G$ and $f \in G^\infty(G)$ we have

$$g\, f(g') = f(g^{-1} g). \qquad \dots (46)$$

So that if $\hat{g}$ is the linear operator corresponding to g we define

$$\hat{g} f(x^\mu, \theta^\nu) = f(g^{-1}(x^\mu, \theta^\nu)). \qquad \dots (47)$$

Proposition 2.13

With the above definitions the linear differential operator $\hat{\gamma}$ corresponding to an element γ of the Lie superalgebra of G is given by

$$\hat{\gamma} = -(x^\mu, \theta^\nu)\, \gamma^{st} \begin{bmatrix} \frac{\partial}{\partial x^\sigma} \\ \frac{\partial}{\partial \theta^\rho} \end{bmatrix}.$$

Proof

First recall Taylors theorem (cf. Rogers [13]) which for our purposes is best written

$$f(x^\mu + y^\mu, \theta^\nu + \phi^\nu) = f(x^\mu, \theta^\nu) + y^\mu \left(\frac{\partial f}{\partial y^\mu}\right)_{\substack{y=0\\ \phi=0}} + \phi^\nu \left(\frac{\partial f}{\partial \phi^\nu}\right)_{\substack{y=0\\ \phi=0}} + \text{other terms}. \qquad \dots (48)$$

We are able to make the identifications

$$\left(\frac{\partial f}{\partial y^\mu}(x+y,\theta+\phi)\right)_{\substack{y=0\\ \phi=0}} = \frac{\partial f}{\partial x^\mu}(x,\theta)$$

and

$$\left(\frac{\partial f}{\partial \phi^\nu}(x+y,\theta+\phi)\right)_{\substack{y=0\\ \phi=0}} = \frac{\partial f}{\partial \theta^\nu}(x,\theta).$$

So that equation (48) can be written

$$f(x^\mu + y^\mu, \theta^\nu + \phi^\nu) = f(x^\mu, \theta^\nu) + y^\mu \frac{\partial f(x,\theta)}{\partial x^\mu} + \phi^\nu \frac{\partial f}{\partial \theta^\nu}(x,\theta) + \text{other terms.} \quad \cdots(49)$$

Now put $\begin{bmatrix} y \\ \phi \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = X \begin{bmatrix} x \\ \theta \end{bmatrix}$ with $|X| = 0$.

Equation (49) then becomes

$$f\left((1+X)\begin{pmatrix} x \\ \theta \end{pmatrix}\right) = f(x,\theta) + \left((A^\mu_\sigma x^\mu + B^\mu_\rho \theta^\rho)^t, (C^\nu_\sigma x^\sigma + D^\nu_\rho \theta^\rho)^t\right) \begin{bmatrix} \frac{\partial f}{\partial x^\mu} \\ \frac{\partial f}{\partial \theta^\nu} \end{bmatrix} + \text{other terms}$$

$$= f(x,\theta) + (x^\sigma, \theta^\rho) \begin{bmatrix} (A^\mu_\sigma)^t & (C^\nu_\sigma)^t \\ -(B^\mu_\rho)^t & (D^\nu_\rho)^t \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x^\mu} \\ \frac{\partial f}{\partial \theta^\nu} \end{bmatrix} + \text{other terms}$$

$$= f(x,\theta) + (x^\sigma, \theta^\rho) X^{st} \begin{bmatrix} \frac{\partial f}{\partial x^\mu} \\ \frac{\partial f}{\partial \theta^\nu} \end{bmatrix} + \text{other terms.} \quad \cdots(50)$$

Now consider an even supergroup element in a small neighbourhood of the identity such that

$$g = \exp(t\alpha) \simeq 1 + t\alpha \quad , \quad t \in E_{L0}.$$

Since g is a linear Lie group this must apply to $\hat{g}$ so that

$$\hat{g} \simeq \hat{I} + (\widehat{t\alpha}), \quad \cdots(51)$$

with $\hat{I}$ the identity operator and $(\widehat{t\alpha})$ the operator corresponding to $t\alpha$.

If we combine equations (47), (50) and (51) we obtain

$$(\hat{I} + (\widehat{t\alpha}))f(x,\theta) = -(x,\theta)(t\alpha)^{st} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial \theta} \end{bmatrix} + f(x,\theta).$$

So that

$$(\widehat{t\alpha}) f(x,\theta) = -t\,(x,\theta)\,\alpha^{st}\begin{bmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial \theta}\end{bmatrix}.$$

Thus with the identification $(\widehat{t\alpha}) = t(\hat{\alpha})$ we have

$$\hat{\alpha}\, f(x,\theta) = -(x,\theta)\,\alpha^{st}\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial \theta}\end{bmatrix} f(x,\theta).$$

This argument can be repeated for a group element $g = \exp(\tau\beta),\ \tau \in E_{L1}$ to obtain

$$\hat{\beta}\, f(x,\theta) = -(x,\theta)\,\beta^{st}\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial \theta}\end{bmatrix} f(x,\theta)$$

provided we remember that $|(x,\theta)| = 0$ so that $\tau(x,\theta) = (x,\theta)\tau$ as was shown in section 1.2.5 of Chapter 1.

These arguments can readily be extended to multi-component functions, that is functions that transform under some $(p|q)$ dimensional representation of the supergroup G ie.

$$\psi_a = (\Gamma(G))_a{}^b\,\psi_b, \quad \text{for } a = 1,2,\dots,r.$$

Here $\Gamma(G)$ is a $(p|q)$ dimensional matrix representation of the Lie supergroup G acting on a vector space $\psi_a,\ a = 1,2,\cdots p+q$ with $|\psi_a| = d$ for $a = 1,2,\cdots,p$ and $|\psi_a| = d+1$ for $a = p+1,\cdots,p+q$; and $d \in \{0,1\}$.

The result is

$$\hat{\gamma} = (\Gamma(\gamma))_a{}^b - (x,\theta)\,\gamma^{st}\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial \theta}\end{bmatrix}. \qquad \cdots\cdot (52)$$

<u>Example 2.14</u>

We use the matrix representation of $su(2|1;\mathbb{R})$ as given in Example 2.12 considered to be acting on a carrier space $(x,y,\theta)^t$ with $x,y \in E_{L0},\ \theta \in E_{L1}$. Then

$$\hat{\alpha}_1 = -(x,y,\theta)\begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix}^{st}\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial \theta} \end{bmatrix} = -ix\frac{\partial}{\partial x} + iy\frac{\partial}{\partial y}.$$

Similarly

$$\hat{\alpha}_2 = -iy\frac{\partial}{\partial x} - ix\frac{\partial}{\partial y} ,$$
$$\hat{\alpha}_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} ,$$
$$\hat{\alpha}_4 = -ix\frac{\partial}{\partial x} - iy\frac{\partial}{\partial y} - 2i\theta\frac{\partial}{\partial \theta} ,$$
$$\hat{\beta}_1 = x\frac{\partial}{\partial \theta} - i\theta\frac{\partial}{\partial x} ,$$
$$\hat{\beta}_2 = ix\frac{\partial}{\partial \theta} - \theta\frac{\partial}{\partial x} ,$$
$$\hat{\beta}_3 = -i\theta\frac{\partial}{\partial y} + y\frac{\partial}{\partial \theta}$$

and

$$\hat{\beta}_4 = -\theta\frac{\partial}{\partial y} + iy\frac{\partial}{\partial \theta} .$$

These operators then satisfy the same commutation relations as the matrices eg,

$$[\hat{\beta}_1 , \hat{\beta}_1] = 2\left(-ix\frac{\partial}{\partial x} + ix\theta\frac{\partial^2}{\partial\theta\partial x} - i\theta x\frac{\partial^2}{\partial x\partial\theta} - i\theta\frac{\partial}{\partial\theta}\right)$$
$$= 2\left(-ix\frac{\partial}{\partial x} - i\theta\frac{\partial}{\partial\theta}\right)$$
$$= \hat{\alpha}_1 + \hat{\alpha}_4 .$$

2.4 A Review of Super Poincaré Groups.

Super Poincaré Groups can be constructed in any number of dimensions. They are real supergroups <u>only</u> if it is possible to define Majorana spinors in the chosen dimension. For example with <u>one</u> time dimension this is possible in (2,3,4,8,9,10)mod 8 dimensions but <u>not</u> for (5,6,7)mod 8 dimensions. The generators of the superalgebra of the super Poincaré group are

(i) $L_{\lambda\mu} = -L_{\mu\lambda}$, the Lorentz generators,

(ii) K_σ, the translation generators,

and (iii) Q_α the supersymmetry generators.

Their commutation relations are

$$[L_{\lambda\mu} , L_{\sigma\rho}] = g_{\lambda\sigma}L_{\mu\rho} - g_{\lambda\rho}L_{\mu\sigma} - g_{\mu\sigma}L_{\lambda\rho} + g_{\mu\rho}L_{\lambda\sigma} ; \quad (53a)$$

$$[L_{\lambda\mu}, K_\sigma] = g_{\lambda\sigma}K_\mu - g_{\mu\sigma}K_\lambda \quad , \qquad \cdots\cdots(53b)$$

$$[K_\sigma, K_\rho] = 0 \quad , \qquad \cdots\cdots(53c)$$

$$[L_{\lambda\mu}, Q_\alpha] = \tfrac{1}{2}(\gamma_\lambda\gamma_\mu)_{\alpha\beta}Q_\beta \quad , \qquad \cdots\cdots(53d)$$

$$[K_\sigma, Q_\alpha] = 0 \qquad \cdots\cdots(53e)$$

and

$$[Q_\alpha, Q_\beta] = (\gamma^\sigma C)_{\alpha\beta}K_\sigma . \qquad \cdots\cdots(53f)$$

Here if d is the number of dimensions $\lambda, \mu, \sigma, \rho = 1, 2, \cdots, d$ and $\alpha, \beta = 1, 2, \cdots, 2^\xi$ with ξ given by $\xi = \frac{1}{2}(d-1)$ for d odd and $\xi = \frac{1}{2}d$ if d is even; $g_{\lambda\lambda} = -1$ for a spacelike dimension and $g_{\lambda\lambda} = 1$ for a timelike dimension with $g_{\lambda\mu} = 0$ if $\lambda \neq \mu$. The Dirac (γ) matrices are as appropriate to the number of dimensions and are taken to be purely imaginary for a Majorana representation, likewise C is taken to be purely imaginary in this case. Our conventions for the Dirac matrices with $d = 4$ and one time dimension are given in the appendix. Note that we choose $g_{\lambda\mu} = \mathrm{diag}(-1, -1, -1, 1)$ ie. we choose to order the dimensions as spacelike followed by timelike. This superalgebra as specified above will be called $siso(d-t, t; \mathbb{R})$ where t is the number of time dimensions and we have assumed that d and t are such that we can construct a real superalgebra.

A supermatrix representation of the superalgebra can be constructed in the block form

$$\begin{bmatrix} L_{\lambda\mu} & K_\sigma & u(Q_\alpha) \\ 0 & 0 & 0 \\ 0 & V(Q_\alpha) & \Gamma(L)_{\lambda\mu} \end{bmatrix} \qquad \cdots\cdots(54)$$

with

$$(L_{\lambda\mu})_{ba} = -\delta_{b\lambda}g_{a\mu} + \delta_{b\mu}g_{a\lambda} \quad , \qquad \cdots\cdots(55)$$

$$(K_\sigma)_a = \delta_{a\sigma} \quad , \qquad \cdots\cdots(56)$$

$$\Gamma(L_{\lambda\mu})_{ba} = \tfrac{1}{2}((\gamma_\lambda\gamma_\mu)^t)_{ba} \quad , \qquad \cdots\cdots(57)$$

$$V(Q_\alpha)_a = \delta_{\alpha a} \qquad \cdots\cdots(58)$$

and

$$(u(Q_\alpha))_{ab} = \tfrac{1}{2}(\gamma^a C)_{\alpha b} . \qquad \cdots\cdots(59)$$

This representation has been written in the standard block form $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$

(see chapter 1 section 1.2.5) and acts on the space $(x^{\mu}, 1, \theta^{\nu})^t$ with $\mu = 1, 2, \cdots d$ and $\nu = 1, 2, \cdots 2^{\xi}$, ie the extended Minkowski space (superspace) of supersymmetry theories.

We note the choice of representation $\Gamma(L_{\lambda\mu})$ used (and correspondingly the $[L, Q]$ commutator), this is the <u>negative transpose</u> of what one might expect, and is chosen to conform to standard usage in the physics literature.

The algebra, of course, admits the semi-direct structure

$$\begin{bmatrix} L_{\lambda\mu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Gamma(L_{\lambda\mu}) \end{bmatrix} \circledS \begin{bmatrix} 0 & K_{\sigma} & U(Q_{\alpha}) \\ 0 & 0 & 0 \\ 0 & V(Q_{\alpha}) & 0 \end{bmatrix} \qquad \cdots\cdots(60)$$

of the Lorentz algebra $so(d-t, t; \mathbb{R})$ generated by $\{L_{\lambda\mu}\}$ and the supersymmetry algebra $st(d-t, t; \mathbb{R})$ generated by $\{K_{\sigma}, Q_{\alpha}\}$. The translations generated by $\{K_{\sigma}\}$ also form a subalgebra, but the supersymmetry generators on their own do not form a subalgebra. The subalgebra with generators $\{L_{\mu\nu}, K_{\sigma}\}$ is the algebra of the Poincaré group and will be denoted by $iso(d-t, t; \mathbb{R})$.

Corresponding to the decomposition of the algebra we can construct the super Poincaré group elements as

$$g = \begin{bmatrix} I & t & T(-\tau) \\ 0 & 1 & 0 \\ 0 & -\tau & I \end{bmatrix} \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \Gamma(\Lambda) \end{bmatrix}, \qquad \cdots\cdots(61)$$

$$= \begin{bmatrix} I & 0 & T(-\tau) \\ 0 & 1 & 0 \\ 0 & -\tau & I \end{bmatrix} \begin{bmatrix} I & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \Gamma(\Lambda) \end{bmatrix},$$

$$= \begin{bmatrix} \Lambda & t & T(-\tau)\Gamma(\Lambda) \\ 0 & 1 & 0 \\ 0 & -\tau & \Gamma(\Lambda) \end{bmatrix}.$$

We note that the inverse is

$$g^{-1} = \begin{bmatrix} \Lambda^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \Gamma(\Lambda^{-1}) \end{bmatrix} \begin{bmatrix} I & -t & T(\tau) \\ 0 & 0 & 0 \\ 0 & \tau & I \end{bmatrix}, \qquad \cdots\cdot(62)$$

$$= \begin{bmatrix} \Lambda^{-1} & -\Lambda^{-1}t & -\Lambda^{-1}T(\tau) \\ 0 & 1 & 0 \\ 0 & -\Gamma(\Lambda^{-1})\tau & \Gamma(\Lambda^{-1}) \end{bmatrix}.$$

In these expressions Λ is a Grassman valued Lorentz transformation in the appropriate number of dimensions, $\Gamma(\Lambda)$ is the representation correspoding to $\Gamma(L_{\mu\nu})$, $t = t^{\sigma} \in E_{LO}$ a column vector correspoding to the 'translations', $\tau = \tau^{\alpha} \in E_{LO}$ a column vector corresponding to the 'supertranslations' and

$$(T(-\tau))_{\alpha b} = -\tfrac{1}{2}\tau^{\alpha}(\gamma^{a}C)_{\alpha b}. \qquad \cdots\cdot(63)$$

Note the negative sign in the last expression which arises from the rule for multiplying supermatrices by scalar quantities. We also note that

$$T(-\tau)\tau = -\frac{\tau^{\alpha}}{2}(\gamma^{a}C)_{\alpha b}\tau^{b} = 0 \qquad \cdots\cdot(64)$$

since $\gamma^{a}C$ is symmetric and $\tau^{\alpha}\tau^{b}$ is antisymmetric.

Consider now the subgroup obtained by setting $t = 0$ and $\tau = 0$, and its action on the even coordinates only ie. Λ. This will, in general, be a four component supergroup corresponding to the four components of the Lorentz group. These will be linked by the operators corresponding to Space Inversion P, Time Inversion T and Total Inversion PT. We identify the subgroups

(i) $SO_0(d-t, t; E_L)$, consisting of the component connected to the identity and called the proper orthochronous super Lorentz group,

(ii) $SO(d-t,t;E_L) = \{h \in SO_0(d-t,t;E_L), PTh\}$,

consisting of two components and called the proper homogeneous super Lorentz group

and (iii) $O(d-t,t;E_L) = \{h \in SO_0(d-t,t;E_L), Ph, Th, PTh\}$

called the homogeneous super Lorentz group. $\Lambda \in O(d-t,t;E_L)$ satisfies $\Lambda^t g \Lambda = g$.

The group action defined on superspace $(x^\mu, 1, \theta^\nu)^t$ does not give a representation of these super groups because of the presence of the representation $\Gamma(\Lambda)$ which is obtained from

$$\Gamma(L_{\mu\nu}) = S \begin{bmatrix} \Gamma^{0,\frac{1}{2}}(L_{\mu\nu}) & 0 \\ 0 & \Gamma^{\frac{1}{2},0}(L_{\mu\nu}) \end{bmatrix} S^{-1} \qquad \cdots (65)$$

for some matrix S. When exponentiated this gives a representation of a covering group of the Lorentz group. We note that the definition of a covering group remains unchanged and that all its properties are the same in the case of the super Lorentz groups. We indicate the covering group by a bar ie. the covering group of $O(n;E_L)$ is denoted by $\bar{O}(n;E_L)$.

It follows from the above discussion that there are three Poincaré supergroups

(i) the proper orthochronous Poincaré supergroup denoted by $ISO_0(d-t,t;E_L)$,

(ii) the proper homogeneous Poincaré supergroup denoted by $ISO(d-t,t;E_L)$ with two components,

and (iii) the homogeneous Poincaré supergroup denoted by $IO(d-t,t;E_L)$ with four components.

It also follows that there are three super Poincaré (super)groups denoted by

(i) $SISO_0(d-t,t;E_L)$ the proper orthochronous super Poincare group,

(ii) $SISO(d-t, t; E_L)$ the proper homogeneous super Poincaré group with two components

and (iii) $SIO(d-t, t; E_L)$ the homogeneous super Poincaré group with four components.

Each of these supergroups will have its own set of covering groups. We concern ourselves mainly with the proper orthochronous group in which case the covering group can be taken to be isomorphic to

$$SL(2, \mathbb{C}E_L) \circledS ST(3, 1; E_L) \qquad \cdots (66)$$

for $d = 4$ and $t = 1$.

In the sequel it is convenient to denote an element g of the super Poincare group by

$$g = \{\Lambda \mid t \mid \tau\} \qquad \cdots (67)$$

where Λ, t and τ refer to the matrix blocks of equation (61) and, respectivly, denote super Lorentz transformation, translation and 'super' translation. We use the same terminology to denote an element g of the equivalent Lie group to the super Poincaré group. Similarly an element of the covering group of the super Poincare group will be denoted by

$$g = [\Lambda \mid t \mid \tau]. \qquad \cdots (68)$$

Multiplication of group elements in this notation is deduced from the matrix representation of equation (61) and is such that

$$[\Lambda \mid t \mid \tau][\Lambda' \mid t' \mid \tau']$$

$$= [\Lambda\Lambda' \mid \Lambda t' + t + T(\tau)\Gamma(\Lambda)\tau' \mid \Gamma(\Lambda)\tau' + \tau] \qquad \cdots (69)$$

and $$[\Lambda \mid t \mid \tau]^{-1} = [\Lambda^{-1} \mid -\Lambda^{-1} t \mid -\Gamma(\Lambda^{-1})\tau]. \qquad \cdots (70)$$

In the sequel we will need the adjoint representation of the super Poincaré group. If we choose the order of the generators to be such that for $\gamma \in siso(d-t, t; E_L)$ the supermatrix $ad(\gamma)$ is given by

$$[\gamma, (L_{\eta\zeta}, K_\rho, Q_\beta)_a] = (L_{\epsilon\kappa}, K_\xi, Q_\gamma)_b \, (ad(\gamma))_{ba}$$

then it can be determined from the commutation relations of equation (53) to be the $(\frac{1}{2}(d^2+d)|\xi)$ dimensional supermatrix written in block form as

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}.$$

Here a is $\frac{1}{2}(d^2-d)\times\frac{1}{2}(d^2-d)$ with the double index $\eta\zeta$ labeling its rows and $\varepsilon\kappa$ labeling its columns and each pair taking the values $12, 13, 14, 23, 24, 34$; e is $d\times d$ with ρ labeling its rows and ξ labeling its columns and j is $\xi\times\xi$ with β labeling its rows and γ its columns. The dimensions of the other blocks follow from these. We find that the matrices of the adjoint representation are

$$\mathrm{ad}(K_\sigma) = \begin{bmatrix} 0 & 0 & 0 \\ (g_{\zeta\sigma}\delta^{\xi}_{\eta} - g_{\eta\sigma}\delta^{\xi}_{\zeta}) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (71)$$

$$\mathrm{ad}(L_{\lambda\mu}) = \begin{bmatrix} \begin{matrix}(g_{\lambda\eta}\delta^{\varepsilon}_{\mu}\delta^{\kappa}_{\zeta} - g_{\lambda\zeta}\delta^{\varepsilon}_{\mu}\delta^{\kappa}_{\eta} \\ -g_{\mu\eta}\delta^{\varepsilon}_{\lambda}\delta^{\kappa}_{\zeta} + g_{\mu\zeta}\delta^{\varepsilon}_{\lambda}\delta^{\kappa}_{\eta})\end{matrix} & 0 & 0 \\ 0 \quad 0 & (g_{\lambda\rho}\delta^{\xi}_{\mu} - g_{\mu\rho}\delta^{\xi}_{\lambda}) & 0 \\ 0 \quad 0 & 0 & (-\frac{1}{2}(\gamma_\lambda\gamma_\mu)^t)_{\gamma\beta} \end{bmatrix}, \quad (72)$$

and

$$\mathrm{ad}(Q_\alpha) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & (\gamma^{\xi}C)_{\alpha\beta} \\ (\frac{1}{2}(\gamma_\eta\gamma_\zeta)^t)_{\gamma\alpha} & 0 & 0 \end{bmatrix}. \quad (73)$$

As before we can obtain a matrix representation of the supergroup by exponentiating the subgroups generated by $\{L_{\lambda\mu}\}$ and $\{K_\sigma, Q_\alpha\}$. This representation is then the adjoint representation of the supergroup as given in Definition 2.8. We obtain

$$\mathrm{Ad}(g) = \mathrm{Ad}(\{\Lambda|t|\tau\})$$
$$= \begin{bmatrix} I & 0 & 0 \\ V+Tu & I & 2T \\ u & 0 & I \end{bmatrix} \begin{bmatrix} \mathrm{Ad}(\Lambda) & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \Gamma(\Lambda) \end{bmatrix},$$

$$= \begin{bmatrix} Ad(\Lambda) & 0 & 0 \\ (V+TU)Ad(\Lambda) & \Lambda & 2T\Gamma(\Lambda) \\ U\,Ad(\Lambda) & 0 & \Gamma(\Lambda) \end{bmatrix}. \qquad \cdots(74)$$

Here Λ, $\Gamma(\Lambda)$ are the matrix representatives of the covering group of the super Lorentz group as used before in this section, T is the matrix defined by equation (63), $Ad(\Lambda)$ is the adjoint representation of the super Lorentz group, the matrix V is given by

$$V_{\eta\zeta}{}^{\xi} = t^{\sigma}(g_{\zeta\sigma}\,\delta^{\xi}_{\eta} - g_{\eta\sigma}\,\delta^{\xi}_{\zeta}) \qquad \cdots(75)$$

and the matrix U is given by

$$U_{\eta\zeta,\gamma} = \frac{\tau^{\alpha}}{2}((\gamma_{\eta}\gamma_{\zeta})^{t})_{\gamma\alpha}. \qquad \cdots(76)$$

We note that the matrix TU can be rewritten as follows

$$\begin{aligned} (TU)_{\eta\zeta}{}^{\xi} &= -\tau^{\alpha}(\gamma^{\xi}C)_{\alpha\beta}\,\frac{\tau^{\delta}}{2}((\gamma_{\eta}\gamma_{\zeta})^{t})_{\beta\delta}\,, \\ &= -\frac{\tau^{\alpha}\tau^{\delta}}{2}(\gamma^{\xi}C)_{\alpha\beta}(C^{-1}\gamma_{\zeta}\gamma_{\eta}C)_{\beta\delta}\,, \\ &= -\frac{\tau^{\alpha}\tau^{\delta}}{2}(\gamma^{\xi}\gamma_{\zeta}\gamma_{\eta}C)_{\alpha\delta} \\ &= -\frac{\tau^{\alpha}\tau^{\delta}}{2}\,i\,\varepsilon^{\xi}{}_{\zeta\eta\varepsilon}(\gamma^{\varepsilon}C)_{\alpha\delta}. \end{aligned} \qquad \cdots(77)$$

Since only the antisymmetric part of the matrix $(\gamma^{\xi}\gamma_{\zeta}\gamma_{\eta}C)$ gives a non zero contribution.

The differential operators corresponding to the generators of the super Poincaré group can be constructed using Proposition 2.13. they are

$$\hat{L}_{\lambda\mu} = x_{\lambda}\frac{\partial}{\partial x^{\mu}} - x_{\mu}\frac{\partial}{\partial x^{\lambda}} + \frac{1}{2}\,\theta^{\alpha}(\gamma_{\lambda}\gamma_{\mu})_{\alpha\beta}\frac{\partial}{\partial\theta^{\beta}}\,, \qquad \cdots(78a)$$

$$K_{\sigma} = -\frac{\partial}{\partial x^{\sigma}} \qquad \cdots(78b)$$

and $$\hat{Q}_{\alpha} = \frac{1}{2}(\gamma^{\sigma}C)_{\alpha\beta}\,\theta^{\beta}\frac{\partial}{\partial x^{\sigma}} - \frac{\partial}{\partial\theta^{\alpha}}. \qquad \cdots(78c)$$

We do not make use of these in this Thesis. They are presented for comparison with the corresponding operators for second quantized fields which are constructed in Chapter 5 of Part III.

In Chapter 5 of Part III we also need a matrix representation on a vector consisting of the derivatives $(\frac{\partial}{\partial x^{\mu}}, \frac{\partial}{\partial\theta^{\nu}})$. This is obtained by constructing the Jacobian matrix as given in section 1.2.5(c) of

Chapter 1. We obtain

$$\Gamma(g)\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial\theta}\end{bmatrix} = \begin{bmatrix} I & 0\\ T(\tau)^t & I\end{bmatrix}\begin{bmatrix}(\Lambda^t)^{-1} & 0\\ 0 & \Gamma(\Lambda^t)^{-1}\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial\theta}\end{bmatrix}. \qquad \cdots\cdot(79)$$

CHAPTER 3

INTEGRATION ON SUPERMANIFOLDS AND SUPERGROUPS.

3.1 Measure and Integration on Superspace and Supermanifolds.

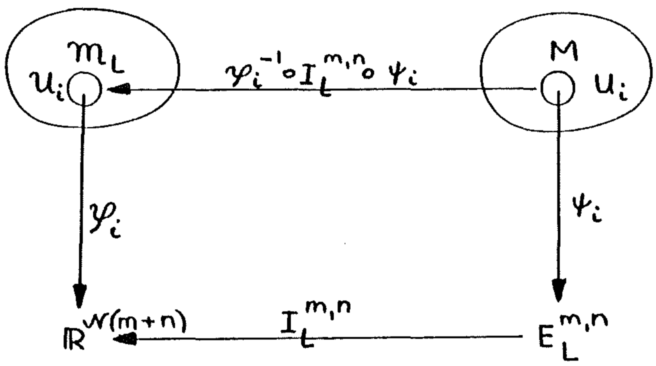
Using the topology endowed on superspace $E_L^{m,n}$ and consequently on supermanifolds, Rogers [13] was able to establish the fact that to each real supermanifold M and given value for L there is a topologically equivalent real differentiable manifold $\mathcal{M}_L$. It is convenient therefore to define the topological homeomorphisms $I_L^{m,n}$ as follows

Definition 3.1

(a) For each L let I_L be the topological homeomorphism $I_L : E_L \rightarrow \mathbb{R}^{2^N}$.

(b) For each L let $I_L^{m,n}$ be the topological homeomorphism $I_L^{m,n} : E_L^{m,n} \rightarrow \mathbb{R}^{N(m+n)}$. Clearly we can define the inverse $(I_L^{m,n})^{-1}$.

The mappings $I_L^{m,n}$ are displayed below.



In this figure M is any real supermanifold, $\mathcal{M}_L$ the equivalent real manifold (for fixed L), $\mathcal{U}_i$ is a coordinate neighbourhood in $\mathcal{M}_L$ corresponding to the 'Grassman' coordinate neighbourhood U_i with ψ_i, φ_i respectively their coordinate charts.

To avoid cumbersome notation we will write simply I_L instead of $I_L^{m,n}$ where it is clear to do so, and also I_L instead of $\varphi_i^{-1} \circ I_L^{m,n} \circ \psi_i$. We can then regard I_L as the map $I_L : M \rightarrow \mathcal{M}_L$, for which the inverse I_L^{-1} is defined (since $\mathcal{M}_L$ was constructed from M).

Now to each open subset $A \subset E_L^{m,n}$ there corresponds an open subset

$\mathcal{A} = I_L(A) \subset \mathbb{R}^{\mathcal{N}(m+n)}$. In terms of measure theory (cf. Cohn [42]) the open subsets of $\mathbb{R}^{\mathcal{N}(m+n)}$ generate the Borel sets of $\mathbb{R}^{\mathcal{N}(m+n)}$, so that we can construct an equivalence relation between the Borel sets of $E_L^{m,n}$ and the Borel sets of $\mathbb{R}^{\mathcal{N}(m+n)}$. It is then natural to define the measure of any subset of $E_L^{m,n}$ as the Lebesgue measure of the corresponding subset of $\mathbb{R}^{\mathcal{N}(m+n)}$. For every function $f: E_L^{m,n} \to E_L$ we have the functions $f \circ I_L^{-1}: \mathbb{R}^{\mathcal{N}(m+n)} \to E_L$, $I_L \circ f: E_L^{m,n} \to \mathbb{R}^{\mathcal{N}}$ and $I_L \circ f \circ I_L^{-1}: \mathbb{R}^{\mathcal{N}(m+n)} \to \mathbb{R}^{\mathcal{N}}$, which we can regard as alternative descriptions of the original function f. It is natural, therefore, to define the integral of f to be the integal of the components of $I_L \circ f \circ I_L^{-1}$ rewritten as an element of E_L. Formally the definitions are as follows.

<u>Definition 3.2</u>

(a) The measure μ of a set $A \subset E_L^{m,n}$ is defined to be the Lebesgue measure μ_L of the set $I_L(A) \subset \mathbb{R}^{\mathcal{N}(m+n)}$ ie.

$$\mu(A) = \mu_L(I_L(A)) \in \mathbb{R}. \quad \cdots(1)$$

(b) The integral of a function $f: E_L^{m,n} \to \mathbb{R}$ is defined to be the real number obtained in evaluating the integral of $f \circ I_L^{-1}: \mathbb{R}^{\mathcal{N}(m+n)} \to \mathbb{R}$. That is

$$\int f \, d\mu = \int f \circ I_L^{-1} \, d\mu_L \in \mathbb{R}. \quad (2)$$

(c) Let p_i, q_j be the projection functions,

$$p_i: E_L \to \mathbb{R}, \quad q_j: E_L \to \mathbb{R}, \quad i = 0, 1, \cdots, \mathcal{N}-1, \quad j = 1, 2, \cdots, \mathcal{N}$$

such that p_i (respectively q_j) projects the component in the direction $\underline{e}_i$ (respectively $\underline{f}_j$), so that $p_K(x + \theta) = p_K(x_i \underline{e}_i + \theta_j \underline{f}_j) = x_K$ and $q_\ell(x + \theta) = q_\ell(x_i \underline{e}_i + \theta_j \underline{f}_j) = \theta_\ell$. Then the integral of a function $g: E_L^{m,n} \to E_L$ is defined to be

$$\int g \, d\mu = \sum_{i=0}^{\mathcal{N}-1} \underline{e}_i \int p_i \circ g \, d\mu + \sum_{j=1}^{\mathcal{N}} \underline{f}_j \int q_j \circ g \, d\mu, \quad \cdots(3)$$

so that

$$\int g \, d\mu = \sum_{i=0}^{\mathcal{N}-1} \underline{e}_i \int p_i \circ g \circ I_L^{-1} \, d\mu_L + \sum_{j=1}^{\mathcal{N}} \underline{f}_j \int q_j \circ g \circ I_L^{-1} \, d\mu_L. \quad \cdots(4)$$

The definition of measure and integration for functions $h: E_L^{m,n} \to E_L^{p,q}$

is a straightforward extension of the definition above and need not be given separately. In keeping with the standard coordinate representation on $\mathbb{R}^n$ in which for $x^i \in \mathbb{R}$, $i = 1, 2, \cdots, n$ we have $d\mu = \prod_{i=1}^{n} dx^i$, we will write $\hat{d}x = d\mu$ or $\hat{d}\theta = d\mu$ for the 'line' element in E_{L0} and E_{L1} respectively, the volume element in $E_L^{m,n}$ can then be written

$$\hat{d}^m x\, \hat{d}^n\theta = \prod_{\mu=1}^{m} \hat{d}x^\mu \prod_{\nu=1}^{n} \hat{d}\theta^\nu = \prod_{\mu=1}^{m}\prod_{i=0}^{\mathcal{N}-1} dx_i^\mu \prod_{\nu=1}^{n}\prod_{j=1}^{\mathcal{N}} d\theta_j^\nu . \qquad \cdots(5)$$

We see that the integral, as defined, provides a positive integral on $E_L^{m,n}$ (ie $f \geqslant 0$ implies that $\int f \geqslant 0$) and that the set of integrable functions on $E_L^{m,n}$ can be constructed from the set of Lebesgue integrable functions on $\mathbb{R}^{\mathcal{N}(m+n)}$. We will denote this set by $\mathcal{L}^1(E_L^{m,n}, E_L)$. It is clear that we can construct the spaces $\mathcal{L}^p(E_L^{m,n}, E_L)$ for $p \geqslant 1$ modelled on the L^p spaces of normal integration theory. We now give the definition of measure and integration on supermanifolds.

<u>Definition 3.3</u>

Let M be a G^∞ supermanifold over $E_L^{m,n}$, let $\mathcal{M}_L = I_L(M)$ be the related manifold over $\mathbb{R}^{\mathcal{N}(m+n)}$ and let $d\mu_L(\mathcal{M}_L)$ be a volume form on $\mathcal{M}_L$. Then:

(a) The integral of a function $f : M \to \mathbb{R}$ is given by

$$\int f\, d\mu(M) = \int f \circ I_L^{-1}\, d\mu_L(\mathcal{M}_L)$$

for all $f \circ I_L^{-1} \in \mathcal{L}^1(\mathcal{M}_L, \mathbb{R})$. If $f \circ I_L^{-1} \in \mathcal{L}^1(\mathcal{M}_L, \mathbb{R})$ then $f \in \mathcal{L}^1(M, \mathbb{R})$, where $\mathcal{L}^1(\mathcal{M}_L, \mathbb{R})$ is the set of integrable functions defined on $\mathcal{M}_L$.

(b) The integral of a function $g : M \to E_L$ is given by

$$\int g\, d\mu(M) = \sum_{i=1}^{\mathcal{N}-1} e_i \int p_i \circ g \circ I_L^{-1}\, d\mu_L(\mathcal{M}_L) + \sum_{j=1}^{\mathcal{N}} f_j \int q_j \circ g \circ I_L^{-1}\, d\mu_L(\mathcal{M}_L). \qquad \cdots(6)$$

Now if $\{x^\mu, \theta^\nu\}$ are coordinate functions on an open set $U_k \subset M$ then $\{x_i^\mu, \theta_j^\nu\}$ are coordinate functions of $I_L(U_k) = \mathcal{U}_k \subset \mathcal{M}_L$ and the volume form can be written

$$d\mu_L(\mathcal{M}_L) = h(x_i, \theta_j) \prod_{\mu, i} dx_i^\mu \prod_{\nu, j} d\theta_j^\nu , \qquad \cdots(7)$$

where $h(x_i, \theta_j) : I_L \circ \mathcal{U} \longrightarrow \mathbb{R}$.

This is conveniently abbreviated to give

$$d\mu(M) = h(x, \theta) \hat{d}^m x \, \hat{d}^n \theta , \qquad \cdots(8)$$

with $h(x, \theta) : U \longrightarrow \mathbb{R}$,

and such that $h(x, \theta) = h(x_i, \theta_j)$. $\qquad \cdots(9)$

Note that h must be a real valued function since we have defined a real valued measure.

Given a set of one forms

$$\psi_s^\alpha = \sum_{i=0}^{\mathcal{N}-1} \sum_{\mu=1}^{m} f_{s\mu}^{\alpha i}(x_k, \theta_l) dx_i^\mu + \sum_{i=\mathcal{N}}^{2\mathcal{N}-1} \sum_{\mu=m+1}^{m+n} f_{s\mu}^{\alpha i}(x_k, \theta_l) d\theta_{i-(\mathcal{N}-1)}^{\mu-(m+1)} \cdots(10)$$

for $\alpha = 1, 2, \cdots, m+n$; $\quad s = 0, 1, \cdots, 2\mathcal{N}-1$

and $f_{s\mu}^{\alpha i} : \mathbb{R}^{\mathcal{N}(m+n)} \longrightarrow \mathbb{R}$ on an open subset $\mathcal{U}_\alpha$ of $\mathcal{M}_L$ that are linearly independent, a real valued volume form on U_α can be constructed as

$$|\det[f_{s\mu}^{\alpha i}(x_k, \theta_l)]| \prod_{\mu, i} dx_i^\mu \prod_{\nu, j} d\theta_j^\nu$$

$$= |\det[f_{s\mu}^{\alpha i}(x_k, \theta_l)]| \, \hat{d}^m x \, \hat{d}^n \theta , \qquad \cdots(11)$$

where $f_{s\mu}^{\alpha i}$ is in the (α, s) row and (μ, i) column of the $(m+n)\mathcal{N} \times (m+n)\mathcal{N}$ matrix of coefficients of the one forms. The function $|\det[f_{s\mu}^{\alpha i}(x_k, \theta_l)]|$ will be called the weight function on $\mathcal{U}_\alpha$ (or U_α).

We now have to consider what happens when we need several coordinate charts to cover M. As in normal manifold theory we need simply to insist that the Jacobian of the coordinate change on each $\mathcal{U}_i \cap \mathcal{U}_j$ is equal to 1 so that there is no 'change of scale' over the manifold. The Jacobian here, being the one for the charts of $\mathcal{M}_L$. It is not the 'Super Jacobian' for M. We assume that we are dealing with orientable manifolds. This means that we define a supermanifold M to be orientable if $\mathcal{M}_L$ is orientable. All Lie supergroups are therefore orientable together with super coset spaces, which are defined in section 3.3.

3.2 Invariant Integrals for Lie Supergroups.

The theory of invariant (Haar) integrals for topological groups is well known. (See Nachbin [43] or Hewitt and Ross [44]). Since every Lie supergroup G is equivalent to a Lie group $\mathcal{G}_L$ (for given L) we are thus able, following section 3.1, to define the invariant integral over a Lie supergroup G as equal to that over its related Lie group $\mathcal{G}_L$ (for given L). In this way the existence of an invariant integral for each supergroup is guaranteed.

Our problem, then, is to learn how to evaluate invariant integrals for this class of Lie group. We will see that for the simple Lie supergroups $SPL(n|m;E_L)$, $OSP(n|2r;E_L)$ and $B(n;E_L)$ this can be expressed in a very convenient way in terms of the (known) Haar integrals over the simple Lie groups. In this section we examine techniques for evaluating Haar integrals over Lie supergroups in general.

Definition 3.4

If G is a Lie supergroup, $f \in \mathcal{L}^1(G, \mathbb{R})$ and $U, V \in G$ the left translation $Uf \in \mathcal{L}^1(G, \mathbb{R})$ is defined by $Uf(V) = f(U^{-1}V)$. Similarly the right translation is defined by $f(V)U = f(VU)$. Clearly if e is the identity of G then $ef = f = fe$. Also $U(Vf) = UV(f)$, $(fU)V = (f)UV$ and $(Uf)V = U(fV)$ for each $U, V \in G$.

Definition 3.5

(a) A positive integral on a Lie supergroup G is said to be left invariant if for every $f \in \mathcal{L}^1(G, \mathbb{R})$ and $U, V \in G$ we have

$$\int f(U^{-1}V)\,d\mu(V) = \int f(V)\,d\mu(V). \qquad \cdots\cdots(12)$$

(b) Similarly a positive integral is right invariant if

$$\int f(VU)\,d\mu(V) = \int f(V)\,d\mu(V)\,. \qquad \ldots(13)$$

(c) A supergroup for which the left and right invariant integrals are equal is said to be unimodular.

In particular we note that if a subset $A \subset G$ is measurable and $\mu(A)$ is left and right invariant then

$$\mu(UA) = \mu(A) = \mu(AV) \in \mathbb{R}_+ \qquad \ldots(14)$$

for all $U, V \in G$.

Proposition 3.6

The left and right invariant integrals over a semisimple Lie supergroup are identical for each L.

Proof

If a Lie supergroup G is semisimple it is equal to its closed commutator subgroup. It follows that this is true for each related Lie group $\mathcal{G}_L$. Then by proposition 15 p.83 of Nachbin [43] $\mathcal{G}_L$ is unimodular. Hence G is unimodular.

We can also see that any connected Lie supergroup considered as a Lie group is the semi-direct product of two components one of which corresponds to the Lie subalgebra of the Lie superalgebra the other being a connected nilpotent group. By proposition 1.4 of p.366 of Helgason [45] the nilpotent part is always unimodular. Now a semi-direct product of two groups is unimodular if both groups are separately unimodular. (See Hewitt and Ross [44] p.210 section (15.29). (The fact that the functional $\delta(h)$ is equal to 1 is most easily seen from the statement in section (15.23) of the same book.)) Thus a Lie supergroup is unimodular if the Lie group corresponding to the Lie subalgebra of its Lie superalgebra is unimodular. In particular, since the Poincaré group is unimodular, the extended Poincaré group is unimodular. (It also coincides with its closed commutator subgroup.)

We note that since the Lebesgue integral is a Haar integral for the additive group $\mathbb{R}^n$ (see Nachbin [43]) we already have one example of an invariant integral for a supergroup ie. the additive group $E_L^{m,n}$.

We now restrict our attention to linear Lie supergroups which were defined in Chapter 2 and to left invariant integrals. It is possible to carry through the following procedure for right invariant integrals but all the groups of interest are unimodular so that this is unnecessary.

In section 3.1 we stated how the weight function is obtained from a linearly independent set of 1- forms on a manifold. There is a well known prescription for finding left invariant 1-forms (sometimes called Maurer-Cartan 1-forms) on a linear Lie group (cf. Chevalley [46]), which is conveniently expressed in the following Lemma.

Lemma 3.7

Let $\mathcal{U}$ be a matrix representation of a chart containing the identity e of a linear Lie group $\mathcal{G}$ parametrized by $\{x^\mu\}$, $x^\mu \in \mathbb{R}$. Let $\alpha^\sigma = \left(\frac{\partial \mathcal{U}}{\partial x^\sigma}\right)_e$. Evaluate the matrix elements A^μ_σ from the equations

$$\mathcal{U}^{-1} \frac{\partial \mathcal{U}}{\partial x^\mu} = A^\mu_\sigma \alpha^\sigma, \quad \mu = 1, 2, \cdots . \qquad \cdots (15)$$

The left invariant 1-forms w^μ are then given by $w^\mu = A^\mu_\sigma dx^\sigma$ in the chart $\mathcal{U}$, and a left invariant weight function is given by $|\det A^\mu_\sigma|$.

Theorem 3.8

Let G be an $(m|n)$ dimensional linear Lie supergroup and let $\mathcal{U}$ be a matrix representation of a chart containing the identity $e \in G$ expressed in terms of the parameters $\{x^\mu, \theta^\nu\}$. Let $\alpha_\mu = \left(\frac{\partial \mathcal{U}}{\partial x^\mu}\right)_e$ and $\beta_\nu = \left(\frac{\partial \mathcal{U}}{\partial \theta^\nu}\right)_e$. Suppose further that if $\mathcal{U}_\alpha$, $\alpha = 1, 2, \cdots$ is an atlas for G the

Jacobian of coordinate change on each $U_\alpha \cap U_\beta$ is equal to one. The left invariant weight function ω_G can then be obtained from the equations

$$u^{-1}\frac{\partial u}{\partial x^\mu} = a_\mu^\sigma \alpha_\sigma + b_\mu^\rho \beta_\rho \qquad \cdots(16a)$$

and

$$u^{-1}\frac{\partial u}{\partial \theta^\nu} = c_\nu^\sigma \alpha_\sigma + d_\nu^\rho \beta_\rho \qquad \cdots(16b)$$

as

$$\omega_G = |(\det[p_0(a_\mu^\sigma(x_0^\mu,0))]\det[p_0(d_\nu^\rho(x_0^\mu,0))])^{\mathcal{N}}|. \cdots(17)$$

Proof

Consider first the equation

$$u^{-1}\frac{\partial u}{\partial x^\mu} = a_\mu^\sigma \alpha_\sigma + b_\mu^\rho \beta_\rho .$$

Multiply by e_i on the left then recall from Chapter 1 section 1.2.5 that $aX = (-1)^{|a||X|}Xa$ for $a \in E_L$, $X \in \tilde{M}(p|q;E_L)$ to obtain for each i

$$u^{-1}e_i\frac{\partial u}{\partial x^\mu} = a_\mu^\sigma e_i \alpha_\sigma + b_\mu^\rho e_i \beta_\rho$$

Now expand a_μ^σ and b_μ^ρ in terms of the projection functions p_K and q_ℓ (see Definition 3.2(c)) noting that by construction $|a| = 0$ and $|b| = 1$, and that $e_i\frac{\partial u}{\partial x^\mu} = \frac{\partial u}{\partial x_i^\mu}$ (see section 1.2.2 of Chapter 1) to obtain

$$u^{-1}\frac{\partial u}{\partial x_i^\mu} = p_0(a_\mu^\sigma)e_i\alpha_\sigma + p_K(a_\mu^\sigma)e_K e_i \alpha_\sigma + q_j(b_\mu^\rho) f_j e_i \beta_\rho ,$$

with $K = 1,2,\cdots,\mathcal{N}-1$ and $j = 1,2,\cdots,\mathcal{N}$.

If we now consider the Taylor expansion of a_μ^σ in all the real variables x_i^η, θ_j^ξ, $i = 0$ it is clear that $p_0(a_\mu^\sigma(x,\theta)) = p_0(a_\mu^\sigma(x_0,0))$ so that

$$u^{-1}\frac{\partial u}{\partial x_i^\mu} = p_0(a_\mu^\sigma(x_0,0))e_i\alpha_\sigma + p_K(a_\mu^\sigma)e_K e_i\alpha_\sigma + q_j(b_\mu^\rho)f_j e_i \beta_\rho . \qquad \cdots(18)$$

By a similar calculation we obtain

$$u^{-1}\frac{\partial u}{\partial \theta_j^\nu} = -q_\ell(c_\nu^\sigma) f_\ell f_j \alpha_\sigma + p_0(d_\nu^\rho(x_0,0))f_j\beta_\rho + p_K(d_\nu^\rho)e_K f_j \beta_\rho . \qquad \cdots(19)$$

Suppose the left invariant 1-forms on $\mathcal{G}_L$ written as a matrix equation take the form

$$\begin{bmatrix} w^{\mu}_{i} \\ \omega^{\nu}_{j} \end{bmatrix} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \begin{bmatrix} dx^{\sigma}_{k} \\ d\theta^{\varrho}_{l} \end{bmatrix}$$

with the indices ordered such that

$$(dx^{\sigma}_{k}, d\theta^{\varrho}_{l})^{t}$$
$$= (dx^{1}_{0}, \cdots, dx^{m}_{0}, dx^{1}_{1}, \cdots, dx^{m}_{1}, \cdots, dx^{m}_{N-1}, d\theta^{1}_{1}, \cdots d\theta^{n}_{1}, d\theta^{1}_{2}, \cdots, d\theta^{n}_{N})^{t}$$

and the indices of the matrix $(w^{\mu}_{i}, \omega^{\nu}_{j})^{t}$ having the same ordering. A comparison of Lemma 3.7 with equations (18) and (19) (assuming the same index ordering) then shows that the left invariant weight function for $\mathcal{G}$ can be written as

$$W_{\mathcal{G}} = \left| \det \left[\begin{array}{cccccc|cccccc}
p_{11} & p_{12} & \cdots & & & & q_{11} & q_{12} & \cdots & & & \\
0 & p_{22} & p_{23} & \cdots & & & 0 & q_{22} & \cdots & & & \\
0 & 0 & p_{33} & \cdots & p_{ij} & & 0 & 0 & q_{33} & \cdots & q_{ij} & \\
 & & 0 & \ddots & & & & & 0 & \ddots & & \\
 & 0 & & \ddots & \ddots & & & 0 & & \ddots & \ddots & \\
 & & & 0 & p_{N-1,N-1} & & & & & 0 & q_{N-1,N-1} & \\
 & & & 0 & 0 & p_{NN} & & & & & 0 & q_{NN} \\
\hline
0 & r_{12} & \cdots & & & & s_{11} & s_{12} & & & & \\
0 & 0 & r_{23} & \cdots & & & 0 & s_{22} & s_{23} & & s_{ij} & \\
 & & 0 & \ddots & r_{ij} & & & 0 & s_{33} & & & \\
 & & & \ddots & \ddots & & & & 0 & \ddots & & \\
 & 0 & & & 0 & \vdots & & 0 & & \ddots & \ddots & \vdots \\
 & & & & 0 & r_{N-1,N} & & & & & 0 & s_{N-1,N-1} \\
 & & & & 0 & 0 & & & & & 0 & s_{NN}
\end{array} \right] \right|$$

where p_{ij}, q_{ij}, r_{ij} and s_{ij} are $m\times m$, $m\times n$, $n\times m$ and $n\times n$ sub-matrices such that

$$(p_{ii})^{\sigma}_{\mu} = P_{0}(a^{\sigma}_{\mu}(x_{0}, 0)), \quad i = 0, 1, \cdots N-1; \ \sigma, \mu = 1, 2, \cdots, m;$$

$$(s_{ii})^{\varrho}_{\nu} = p_o(\alpha^{\varrho}_{\mu}(x_o, 0)), \quad i = 1, 2, \cdots, \mathcal{N}; \quad \varrho, \mu = 1, 2, \cdots, n;$$

and where p_{ij}, q_{ij}, s_{ij} are zero for $i > j$ and r_{ij} is zero for $i \geqslant j$. The values of the elements of the other sub-matrices are irrelevant, since with this structure

$$w_G = |\left(\prod_{i=0}^{\mathcal{N}-1} \det p_{ii}\right)\left(\prod_{j=1}^{\mathcal{N}} \det s_{ii}\right)|.$$

Thus

$$w_G = |\left(\det[p_o(\alpha^{\sigma}_{\mu}(x_o, 0))] \det[p_o(\alpha^{\varrho}_{\nu}(x_o, 0))]\right)^{\mathcal{N}}|.$$

To extend this to a Lie supergroup G containing several charts there are two cases to consider (i) G is one connected component, (ii) G has several components.

In the first case the answer for real manifolds is to choose the Jacobian determinant on the overlapping regions to have value 1. We can acheive this in the case of a Lie supergroup by insisting that $\det\left[p_o\left(\frac{\partial(x^{\mu}, \theta^{\nu})}{\partial(y^{\sigma}, \phi^{\varrho})}\right)\right] = 1$ on each $U_{\alpha} \cap U_{\beta}$ where $\frac{\partial(x^{\mu}, \theta^{\nu})}{\partial(y^{\sigma}, \phi^{\varrho})}$ denotes the super Jacobian matrix as constructed in section 1.2.5(c) of Chapter 1. The weight function is then the same in each chart, that is, for each U_{α} $w_{U_{\alpha}} = w_G((x^{\mu}, \theta^{\nu})_{\alpha})$.

This requirement is different to requiring that the superdeterminant of the coordinate change is equal to one. Since suppose the super Jacobian matrix is given by

$$M = \begin{bmatrix} A & C \\ B & D \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ 0 & D_o \end{bmatrix} \begin{bmatrix} I + A_n & C_n \\ B_n & I + D_n \end{bmatrix}$$

with the entries of A_o and D_o elements of $\mathbb{R}$ and the elements of A_n, B_n, C_n and D_n nilpotent. Then our requirement is that

$$\det(A_o) \det(D_o) = 1.$$

Whereas requiring the superdeterminant to be one gives

$$\det(A_o) \det^{-1}(D_o) = 1.$$

Thus the requirements are the same if and only if

$$\det(A_0) = \det(D_0) = 1.$$

In the second case it is clear that each component of a many component supergroup is a topological copy of the component connected to the identity. So that we just need to use this fact to obtain an integral over the whole supergroup.

3.3 Invariant Integration on Homogeneous Spaces.

Having obtained an invariant integral on a supergroup we now consider how to construct invariant integration on a homogeneous space G/H with G, H Lie supergroups and H a subgroup of G. A homogeneous space is defined by:

Definition 3.9

Let G be a Lie supergroup and H a Lie sub-supergroup of G. The set $gH = \{ y \in G \,,\, y = gh \,,\, h \in H \}$ is called a left coset of H. These cosets then partition G into a set of mutually disjoint subsets. The set of left cosets is then called a left coset space of G by H and denoted G/H.

Suppose in addition that the space G/H inherits a topology from the projection map $\pi : G \to G/H$ then the space is called a left homogeneous space.

Right cosets, coset spaces and homogeneous spaces are defined in a similar way.

Here we use the same terminology for coset spaces and homogeneous spaces ie. G/H, since our spaces are assumed to have the inherited topology. We consider only left homogeneous spaces though everything we say applies equally to right homogeneous spaces.

We refer to the discussion in Hewitt and Ross [44] to see that there are three possibilities (i) there is a G invariant integral on G/H, (ii) there is a relativly G invariant integral on G/H or (iii) there is a quasi G invariant integral on G/H. The definition of relativly invariant is as follows.

Definition 3.10

Let G be a Lie supergroup and H a Lie sub-supergroup of G. Then an integral is said to be relativly left invariant on G/H if

(i) $0 \leq |\int f \, d(G/H)| < \infty$,

(ii) $\int (f + f') \, d(G/H) = \int f \, d(G/H) + \int f' \, d(G/H)$,

(iii) $\int \alpha f \, d(G/H) = \alpha \int f \, d(G/H)$

and (iv) $\int g f \, d(G/H) = \kappa(g) \int f \, d(G/H)$

for all $f, f' \in \mathcal{L}^1(G/H, \mathbb{R})$, $g \in G$, $\alpha \in \mathbb{R}$.

In a similar way relativly right invariant integrals can be defined. Conditions (i),(ii) and (iii) are self explanatory, but condition (iv) requires some comment. Left translations of functions defined on G/H are defined by analogy with those on the group ie.

$$gf(\rho) = f(g^{-1}\rho), \quad g \in G, \quad \rho \in G/H. \qquad \cdots\cdots(20)$$

The group action on G/H is defined by

$$g^{-1} : \rho H \longrightarrow g^{-1}\rho H, \quad \rho \in G/H, \qquad \cdots\cdots(21)$$

ie. we carry out the group multiplication as usual, then factor out H. Now since $(gg')f = g(g'f)$ it follows that the function κ must satisfy

$$\kappa(gg') = \kappa(g)\kappa(g'),$$

and also that $\kappa(G) \subset [0, \infty]$ and that κ is a continuous function of G (see Hewitt and Ross [44]).

Definition 3.11

If $\kappa(g) = 1$ for all $g \in G$ then the integral of Definition 3.10 is said to be G-invariant.

Both of these cases occur in practice, there are also important cases in the theory of Lie groups where the integral is not even relativly invariant (eg. $SL(2,\mathbb{R}) \Big/ \begin{bmatrix} x & y \\ 0 & \frac{1}{x} \end{bmatrix}$, see Hewitt and Ross [44] ,p213).

To cover all possible cases a quasi-invariant integral is defined, this and a relativly invariant integral can be taken to act on any space on which the group action is defined. (eg. some Hilbert space on which we want to represent the supergroup action.) The definition of a quasi-invariant integral is then as follows.

Definition 3.12

Let G be a Lie supergroup and H a Lie sub-supergroup of G. Then an integral is said to be quasi-invariant if

(i) $0 \leq |\int f \, d(G/H)| < \infty$,

(ii) $\int (f + f') \, d(G/H) = \int f \, d(G/H) + \int f' \, d(G/H)$,

(iii) $\int a f \, d(G/H) = a \int f \, d(G/H)$

and (iv) $\int f \, d(G/H) = 0$ implies that $\int g f \, d(G/H) = 0$

for all $f, f' \in \mathcal{L}^1(G/H, \mathbb{R})$, $g \in G$, $a \in \mathbb{R}$.

That is we require the measure defined on G/H to be such that the action of $g \in G$ leaves the null sets of G/H invariant. This can always be done. In fact if we choose our measure to be a volume form on the manifold $\mathcal{G}_L/\mathcal{H}_L$ then the null sets are those sets which are of smaller dimension than the manifold $\mathcal{G}_L/\mathcal{H}_L$. The action of G on a subset of G/H cannot change the dimension of that subset so that our chosen volume form is a quasi-invariant measure on G/H.

There are two important cases to note.

(i) If $G/H = K$ with K a subgroup of G then a G invariant integral exists and can be taken to be the Haar integral on K. (see Hewitt and Ross p206 [44]).

(ii) If G/H is not a subgroup of G then the integral on G/H is invariant if the modular functions of G and H are the same. Thus if G and H are both unimodular (ie. their left and right invariant integrals are the same) then there is a G-invariant integral on G/H. In particular if G and H are semisimple there is a G-invariant integral on G/H.

3.4 Examples of Left Invariant Integrals.

3.4.1 Supergroups Obtained from Lie Groups.

We consider Lie supergroups constructed as in section 2.2.2 of Chapter 2. Let $G(x, \mathbb{R})$ denote a Lie group and $G(x, E_L)$ denote the Lie supergroup obtained by extending the domain of the parameters to E_{L0}. Now if $\omega_{G(x,\mathbb{R})}$ is the weight function of $G(x, E_L)$ the weight function of $G(x, E_L)$ is given by Theorem 3.8 as

$$\omega_{G(x,E_L)} = \left(\omega_{G(x,\mathbb{R})}\right)^{\mathcal{N}} \qquad \cdots (23)$$

for each L.

3.4.2 The Supergroup $SPL(p|q; E_L)$.

Recall from Chapter 2 section 2.2.3 that a general element u of the supergroup can be written

$$u = \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} e^{y/p} I_p & 0 \\ 0 & e^{y/q} I_q \end{bmatrix} \begin{bmatrix} I & 0 \\ C & I \end{bmatrix},$$

with

$$u^{-1} = \begin{bmatrix} I & 0 \\ -C & I \end{bmatrix} \begin{bmatrix} e^{-y/p} I_p & 0 \\ 0 & e^{-y/q} I_q \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} I & -B \\ 0 & 0 \end{bmatrix}.$$

Denote the parameters of the matrix blocks A, B, C and D by x^α, θ^β, θ^γ and x^δ respectively. Consider first a parameter from the set $\{x^\alpha\}$, then

$$u^{-1}\frac{\partial u}{\partial x^\alpha} = \begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}\begin{bmatrix} e^{-y/p}I_p & 0 \\ 0 & e^{y/q}I_q \end{bmatrix}\begin{bmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}\begin{bmatrix} I & -B \\ 0 & 0 \end{bmatrix}$$

$$\times\begin{bmatrix} I & B \\ 0 & I \end{bmatrix}\begin{bmatrix} \frac{\partial A}{\partial x^\alpha} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} e^{y/p}I_p & 0 \\ 0 & e^{y/q}I_q \end{bmatrix}\begin{bmatrix} I & 0 \\ C & I \end{bmatrix},$$

$$= \begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}\begin{bmatrix} e^{-y/p}I_p & 0 \\ 0 & e^{y/q}I_q \end{bmatrix}\begin{bmatrix} A^{-1}\frac{\partial A}{\partial x^\alpha} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\times\begin{bmatrix} e^{y/p}I_p & 0 \\ 0 & e^{y/q}I_q \end{bmatrix}\begin{bmatrix} I & 0 \\ C & I \end{bmatrix},$$

$$= \begin{bmatrix} A^{-1}\frac{\partial A}{\partial x^\alpha} & 0 \\ -C A^{-1}\frac{\partial A}{\partial x^\alpha} & 0 \end{bmatrix}.$$

By similar calculations we obtain

$$u^{-1}\frac{\partial u}{\partial \theta^\beta} = \begin{bmatrix} A^{-1}\frac{\partial B}{\partial \theta^\beta} DC & A^{-1}\frac{\partial B}{\partial \theta^\beta} D \\ -CA^{-1}\frac{\partial B}{\partial \theta^\beta} DC & -CA^{-1}\frac{\partial B}{\partial \theta^\beta} D \end{bmatrix},$$

$$u^{-1}\frac{\partial u}{\partial \theta^\gamma} = \begin{bmatrix} 0 & 0 \\ -\frac{\partial C}{\partial \theta^\gamma} & 0 \end{bmatrix},$$

$$u^{-1}\frac{\partial u}{\partial x^\delta} = \begin{bmatrix} 0 & 0 \\ D^{-1}\frac{\partial D}{\partial x^\delta} C & D^{-1}\frac{\partial D}{\partial x^\delta} \end{bmatrix},$$

$$u^{-1}\frac{\partial u}{\partial y} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}.$$

Now applying Theorem 3.8 the weight function of $G = SPL(p|q; E_L)$ is given by

$$\begin{aligned} w_G &= \left(\left|\det[\{A^{-1}\tfrac{\partial A}{\partial x^\alpha}\}(x_0^\alpha, 0)]\det[\{D^{-1}\tfrac{\partial D}{\partial x^\delta}\}(x_0^\delta, 0)]\right.\right. \\ &\quad \left.\left.\times\det[\{\tfrac{\partial C}{\partial \theta^\gamma}\}(0,0)]\det[\{A^{-1}\tfrac{\partial B}{\partial \theta^\beta} D\}(x_0^\alpha, x_0^\delta, 0)]\right|\right)^{w}, \\ &= w_A w_D w_C \left(\left|\det[\{A^{-1}\tfrac{\partial B}{\partial \theta^\beta} D\}(x_0^\alpha, x_0^\delta, 0)]\right|\right)^{w} \end{aligned}$$

where we have written $w_A = w_{\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}}$ etc. and the expression $\{A^{-1}\frac{\partial A}{\partial x^\alpha}\}$ means the matrix of coefficients obtained when $A^{-1}\frac{\partial A}{\partial x^\alpha}$ is expanded in terms of the superalgebra of A etc.

Clearly we can choose a parametrization such that $w_C = 1$ and w_B and

ω_B are the weight functions for $SL(p;E_L)$ and $SL(q;E_L)$ respectively. Now B is an arbitrary matrix with entries from E_{L1} so that we can choose its parametrization to be such that $B_{ij} = \theta_{ij}$, for $i = 1,2,\cdots,p$ $j = 1,2,\cdots,q$ and $\theta_{ij} \in E_{L1}$. Then $\frac{\partial B}{\partial \theta_{ij}} = e_{ij}$ ie. the matrix with one in the ij position and zero's elsewhere. So that

$$\left(A^{-1}\frac{\partial B}{\partial \theta_{ij}} D\right)_{af} = (A^{-1})_{ab}\,(e_{ij})_{bc}\,(D)_{cf} ,$$
$$= A^{-1}_{ai}\, D_{jf}$$

and
$$\left\{A^{-1}\frac{\partial B}{\partial \theta_{ij}} D\right\}_{if,aj} = A^{-1}_{ai}\, D_{jf} ,$$

$$= \begin{bmatrix} (A^{-1})_{11} D_{jf} & (A^{-1})_{21} D_{jf} & (A^{-1})_{31} D_{jf} & \cdots \\ (A^{-1})_{12} D_{jf} & (A^{-1})_{22} D_{jf} & & \vdots \\ (A^{-1})_{13} D_{jf} & \vdots & & \\ \vdots & & & \end{bmatrix}_{ia} ,$$

$$= (A^{-1})_{ia} \otimes (D)_{jf} .$$

Therefore
$$\det\left\{A^{-1}\frac{\partial B}{\partial \theta_{ij}} D\right\} = \det(A^{-1})^p \det(D)^q . \qquad \cdots(24)$$

The weight function for $SPL(p|q;E_L)$ can then be written
$$\omega_{SPL(p|q;E_L)} = \left(\omega_{SL(p;\mathbb{R})}\,\omega_{SL(q;\mathbb{R})}\right)^{\mathcal{N}} . \qquad \cdots(25)$$

The decomposition of $SU^*(p|q;E_L)$ and $PL(p|q;E_L)$ is identical to that for $SPL(p|q;E_L)$, so that the analysis of this section can be repeated for these supergroups. The results are

$$\omega_{SU^*(p|q;E_L)} = \left(\omega_{SU^*(p;\mathbb{R})}\,\omega_{SU^*(q;\mathbb{R})}\right)^{\mathcal{N}} \qquad \cdots(26)$$

and
$$\omega_{PL(p|q;E_L)} = \left(\omega_{GL(p;\mathbb{R})}\,\omega_{GL(q;\mathbb{R})}\right)^{\mathcal{N}} . \qquad \cdots(27)$$

3.4.3 The Supergroup $OSP(n|2r;E_L)$.

The procedure we follow is identical to that of the prvious section. We denote the parameters of the matrix blocks A, B, C, D, Y and Z of equation (32) of section 2.2.4 of Chapter 2 by x^α, x^β, x^γ, x^δ, θ^ξ and

θ^ξ respectively to obtain

$$U^{-1}\frac{\partial U}{\partial x^\alpha} = \begin{bmatrix} A^{-1}\frac{\partial A}{\partial x^\alpha} & 0 & A^{-1}\frac{\partial A}{\partial x^\alpha} z \\ -z^t A^{-1}\frac{\partial A}{\partial x^\alpha} & 0 & -z^t A^{-1}\frac{\partial A}{\partial x^\alpha} z \\ 0 & 0 & 0 \end{bmatrix},$$

$$U^{-1}\frac{\partial U}{\partial x^\beta} = \begin{bmatrix} 0 & 0 & -z B^t \frac{\partial (B^t)^{-1}}{\partial x^\beta} \\ B^{-1}\frac{\partial B}{\partial x^\beta} z^t & B^{-1}\frac{\partial B}{\partial x^\beta} & \begin{array}{c} B^{-1}\frac{\partial B}{\partial x^\beta}\left(C + \frac{z^t z}{2}\right) \\ +\left(-C+\frac{z^t z}{2}\right) B^t \frac{\partial (B^t)^{-1}}{\partial x^\beta} \end{array} \\ 0 & 0 & B^t \frac{\partial (B^t)^{-1}}{\partial x^\beta} \end{bmatrix},$$

$$U^{-1}\frac{\partial U}{\partial x^\gamma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial C}{\partial x^\gamma} \\ 0 & 0 & 0 \end{bmatrix},$$

$$U^{-1}\frac{\partial U}{\partial x^\delta} = \begin{bmatrix} -z B^t \frac{\partial D}{\partial x^\delta} B z^t & -z B^t \frac{\partial D}{\partial x^\delta} B & -z B^t \frac{\partial D}{\partial x^\delta} B\left(C + \frac{z^t z}{2}\right) \\ \left(-C+\frac{z^t z}{2}\right) B^t \frac{\partial D}{\partial x^\delta} B z^t & \left(-C+\frac{z^t z}{2}\right) B^t \frac{\partial D}{\partial x^\delta} B & \begin{array}{c} \left(-C+\frac{z^t z}{2}\right) B^t \frac{\partial D}{\partial x^\delta} \\ \times B\left(C+\frac{z^t z}{2}\right) \end{array} \\ B^t \frac{\partial D}{\partial x^\delta} B z^t & B^t \frac{\partial D}{\partial x^\delta} B & B^t \frac{\partial D}{\partial x^\delta} (B^t)^{-1} \end{bmatrix},$$

$$U^{-1}\frac{\partial U}{\partial \theta^\xi} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

with
$$a_{11} = -z B^t \frac{\partial Y^t}{\partial \theta^\xi} A + A^{-1}\frac{\partial Y}{\partial \theta^\xi} A z + z B^t\left(\frac{Y^t}{2}\frac{\partial Y}{\partial \theta^\xi} - \frac{\partial Y^t}{\partial \theta^\xi}\cdot\frac{Y}{2}\right) B z^t,$$

$$a_{12} = A^{-1}\frac{\partial Y}{\partial \theta^\xi} B - z B^t \left(\frac{Y^t}{2}\frac{\partial Y}{\partial \theta^\xi} - \frac{\partial Y^t}{\partial \theta^\xi}\frac{Y}{2}\right) B,$$

$$a_{13} = -z B^t \frac{\partial Y^t}{\partial \theta^\xi} A z + A^{-1}\frac{\partial Y}{\partial \theta^\xi} B\left(C + \frac{z^t z}{2}\right) - z B^t\left(\frac{Y^t}{2}\frac{\partial Y}{\partial \theta^\xi} - \frac{\partial Y^t}{\partial \theta^\xi}\frac{Y}{2}\right) B\left(C + \frac{z^t z}{2}\right),$$

$$a_{21} = -\left(-C+\frac{z^t z}{2}\right) B^t \frac{\partial Y^t}{\partial \theta^\xi} A - z^t A^{-1}\frac{\partial Y}{\partial \theta^\xi} B z^t + \left(-C+\frac{z^t z}{2}\right) B^t\left(\frac{Y^t}{2}\frac{\partial Y}{\partial \theta^\xi} - \frac{\partial Y^t}{\partial \theta^\xi}\frac{Y}{2}\right) B z^t,$$

$$a_{22} = -z^t A^{-1}\frac{\partial Y}{\partial \theta^\xi} B + \left(-C+\frac{z^t z}{2}\right) B^t\left(\frac{Y^t}{2}\frac{\partial Y}{\partial \theta^\xi} - \frac{\partial Y^t}{\partial \theta^\xi}\frac{Y}{2}\right) B,$$

$$a_{23} = -\left(-C+\frac{z^t z}{2}\right) B^t \frac{\partial Y^t}{\partial \theta^\xi} A z - z^t A^{-1}\frac{\partial Y}{\partial \theta^\xi} B\left(C+\frac{z^t z}{2}\right) + \left(-C+\frac{z^t z}{2}\right) B^t\left(\frac{Y^t}{2}\frac{\partial Y}{\partial \theta^\xi} - \frac{\partial Y^t}{\partial \theta^\xi}\frac{Y}{2}\right) B\left(C+\frac{z^t z}{2}\right),$$

$$a_{31} = -B^t \frac{\partial Y^t}{\partial \theta^\xi} A + B^t\left(\frac{Y^t}{2}\frac{\partial Y}{\partial \theta^\xi} - \frac{\partial Y^t}{\partial \theta^\xi}\frac{Y}{2}\right) B z^t,$$

$$a_{32} = B^t\left(\frac{Y^t}{2}\frac{\partial Y}{\partial \theta^\xi} - \frac{\partial Y^t}{\partial \theta^\xi}\frac{Y}{2}\right) B$$

and $\quad a_{33} = -B^t \frac{\partial Y^t}{\partial \theta^\xi} A Z + B^t\left(\frac{Y^t}{2}\frac{\partial Y}{\partial \theta^\xi} - \frac{\partial Y}{\partial \theta^\xi}\frac{Y}{2}\right)B\left(C + \frac{Z^t Z}{2}\right)$,

and

$$U^{-1}\frac{\partial U}{\partial \theta^\xi}\begin{bmatrix} 0 & 0 & \frac{\partial Z}{\partial \theta^\xi} \\ \frac{\partial Z^t}{\partial \theta^\xi} & 0 & \frac{Z^t}{2}\frac{\partial Z}{\partial \theta^\xi} + \frac{\partial Z^t}{\partial \theta^\xi}\frac{Z}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, as for $SPL(p|q;E_L)$ we can use Theorem 3.8 to obtain for $G = SOSP(n|2r;E_L)$,

$$\begin{aligned} \omega_G = \Big(|\det[\{A^{-1}\tfrac{\partial A}{\partial x^\alpha}\}(x^\alpha_0, 0)]\det[\{B^{-1}\tfrac{\partial B}{\partial x^\beta}\}(x^\beta_0, 0)] \\ \times \det[\{B^t \tfrac{\partial D}{\partial x^\delta} B\}(x^\beta_0, x^\delta_0, 0)]\det[\{A^{-1}\tfrac{\partial Y}{\partial \theta^\xi} B\}(x^\alpha_0, x^\beta_0, 0)]|\Big)^{w}, \end{aligned}$$

$$\begin{aligned} = \omega_A \omega_B \Big(|\det[\{B^t \tfrac{\partial D}{\partial x^\delta} B\}(x^\beta_0, x^\delta_0, 0)] \\ \times \det[\{A^{-1}\tfrac{\partial Y}{\partial \theta^\xi} B\}(x^\alpha_0, x^\beta_0, 0)]|\Big)^{w}. \end{aligned}$$

Here ω_A is the weight function for $SO(n;E_L)$ and ω_B is the weight function for $GL(r;E_L)$. Now if we choose a parametrization for Y such that it is parametrized by a different odd parameter at each position we can use equation (24) to obtain

$$\begin{aligned} &\det[\{A^{-1}\tfrac{\partial Y}{\partial \theta^\xi} B\}(x^\alpha_0, x^\beta_0, 0)] \\ &\quad = \det[A^{-1}(x^\alpha_0)]^n \det[B(x^\beta_0)]^r, \\ &\quad = (\det[B(x^\beta_0)])^r. \end{aligned}$$

By a similar calculation (noting that D is symmetric) we find that

$$\begin{aligned} &\det[\{B^t \tfrac{\partial D}{\partial x^\delta} B\}(x^\beta_0, x^\delta_0, 0)] \\ &\quad = (\det[B(x^\beta_0)])^{r+1}. \end{aligned}$$

The weight function can now be written

$$\omega_G = \omega_A \omega_B (|\det[B(x^\beta_0)]|)^{w(2r+1)}. \qquad \ldots(28)$$

A parametrization that is often used for $B \in GL(r,\mathbb{R})$ is by a different real number at each position, with the constraint that $\det[B] \neq 0$. The weight function is then $|(\det(B))|^{-r}$. This parametrization is clearly suitable in this application with the additional constraint $\det[B] \geq 0$. The weight function now becomes

$$\omega_G = \omega_A (|\det[B(x^\beta_0)]|)^{w(r+1)}. \qquad \ldots(29)$$

3.4.4 The Supergroups $B(n;E_L)$ and $B^*(n;E_L)$.

We recall from section 2.2.5 of Chapter 2 that the supergroups $B(n;E_L)$ and $B^*(n;E_L)$ inherit the decomposition into subgroups from $SPL(n|n;E_L)$. Determination of the weight function follows the same procedure and we arrive at (essentially) the same result ie. the weight function depends only on the weight function of $SL(n;E_L)$ or $SU^*(n;E_L)$. The weight functions are

$$\omega_{B(n;E_L)} = (\omega_{SL(n;\mathbb{R})})^{\mathcal{N}}$$

and
$$\omega_{B^*(n;E_L)} = (\omega_{SU^*(n;\mathbb{R})})^{\mathcal{N}}.$$

3.4.5 The Super Poincaré Group and Superspace.

In this section we construct the left invariant integral for the covering group $SL(2,\mathbb{C}E_L) \circledS ST(3,1;E_L)$ of the super Poincaré group as given by equation (66) of section 2.4 of Chapter 2. We do this by constructing the left invariant integrals for $SL(2,\mathbb{C}E_L)$ and $ST(3,1;E_L)$ and then combining the results.

First consider the supergroup $SL(2,\mathbb{C}E_L)$. We choose as parametrization for $g \in SL(2,\mathbb{C}E_L)$

$$g = \begin{bmatrix} e^{z^1} & z^3 e^{z^1} \\ z^3 e^{z^1} & z^2 z^3 e^{z^1} + e^{-z^1} \end{bmatrix},$$

with $z^j = x^j + i y^j$, $x^j, y^j \in E_{L0}$ for $j = 1, 2, 3$.

The inverse g^{-1} is then given by

$$g^{-1} = \begin{bmatrix} z^2 z^3 e^{z^1} + e^{-z^1} & -z^2 e^{z^1} \\ -z^3 e^{z^1} & e^{z^1} \end{bmatrix},$$

and a basis of the Lie algebra of $SL(2,\mathbb{C}E_L)$ (considered as a six parameter Lie group) is

$$\alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \alpha_4 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix},$$

$$\alpha_5 = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \alpha_6 = \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix}.$$

Now using the prescription given in Theorem 3.8 and noting that $e^{2z'} = e^{2x'}(\cos 2y' + i \sin 2y')$ we find that

$$
\begin{aligned}
g^{-1}\frac{\partial g}{\partial x^1} &= \alpha_1 \ , \\
g^{-1}\frac{\partial g}{\partial x^2} &= \alpha_2 \ , \\
g^{-1}\frac{\partial g}{\partial x^3} &= (y^2 \sin 2y' - x^2 \cos 2y')e^{2x'}\alpha_1 \\
&\quad -(((x^2)^2 - (y^2)^2)\cos 2y' - 2x^2y^2 \sin 2y')e^{2x'}\alpha_2 \\
&\quad + e^{2x'}\sin 2y'\alpha_3 - (x^2 \sin 2y' + y^2 \cos 2y')e^{2x'}\alpha_4 \\
&\quad - (((x^2)^2 - (y^2)^2)\sin 2y' + 2x^2y^2\cos 2y')e^{2x'}\alpha_5 \\
&\quad + e^{2x'}\cos 2y'\alpha_6 \ , \\
g^{-1}\frac{\partial g}{\partial y^1} &= \alpha_4 \ , \\
g^{-1}\frac{\partial g}{\partial y^2} &= \alpha_5 \ ,
\end{aligned}
$$

and

$$
\begin{aligned}
g^{-1}\frac{\partial g}{\partial y^3} &= (y^2 \cos 2y' + x^2 \sin 2y')e^{2x'}\alpha_1 \\
&\quad + (2x^2y^2\cos 2y' + ((x^2)^2 + (y^2)^2)\sin 2y')e^{2x'}\alpha_2 \\
&\quad - e^{2x'}\sin 2y'\alpha_3 + (y^2 \sin 2y' - x^2\cos 2y')e^{2x'}\alpha_4 \\
&\quad - (2x^2y^2\sin 2y' + ((x^2)^2 + (y^2)^2)\cos 2y')e^{2x'}\alpha_5 \\
&\quad + e^{2x'}\cos 2y'\alpha_6 .
\end{aligned}
$$

Thus using equation (17) we find the weight function of $SL(2, \mathbb{C}E_L)$ to be

$$
\begin{aligned}
w_{SL(2,\mathbb{C}E_L)} &= (\sin 4y'_0 \, e^{4x'_0})^{w} \\
&= \sin^{w} 4y'_0 \, e^{4wx'_0} \qquad \ldots\text{(32)}
\end{aligned}
$$

Now consider the group $ST(3,1;E_L)$. Equations (61) and (63) of Chapter 2 tell us that a representation of $h \in ST(3,1;E_L)$ is given by

$$h = \begin{bmatrix} I_4 & t^e & -\tfrac{1}{2}\tau^\alpha(\gamma^e C)_{\alpha\mu} \\ 0 & 1 & 0 \\ 0 & -\tau^\alpha & I_4 \end{bmatrix},$$

and equations (63) and(64) tell us that the inverse is

$$h^{-1} = \begin{bmatrix} I_4 & -t^{\rho} & +\frac{1}{2}\tau^{\alpha}(\gamma^{\rho}C)_{\alpha\mu} \\ 0 & 1 & 0 \\ 0 & \tau^{\alpha} & I_4 \end{bmatrix}.$$

Then using the matrix representation of the superalgebra given by equation (54) of Chapter 2 we find that

$$h^{-1}\frac{\partial h}{\partial t^{\sigma}} = K_{\sigma}$$

and $$h^{-1}\frac{\partial h}{\partial \tau^{\beta}} = Q_{\beta} + \frac{1}{2}\tau^{\alpha}(\gamma^{\rho}C)_{\alpha\mu}\delta^{\mu}_{\beta}K_{\rho}.$$

Thus the weight function is given by

$$\omega_{ST(3,1;E_L)} = 1. \qquad \cdots(33)$$

Now using Propositions 28 and 29 of Nachbin [43] we can see that the weight function of the covering group of the super Poincaré group is equal to the product of the weight function of the super translation group $ST(3,1;E_L)$ and the covering group of the super Lorentz group $SL(2,\mathbb{C}E_L)$ if each of the three groups is separately unimodular. $I_L(ST(3,1;E_L))$ is a connected nilpotent group so that it is unimodular. (See comments after Proposition 3.6 on p.67 of this Thesis). Proposition 3.6 tells us that $SL(2,\mathbb{C}E_L)$ is unimodular since it is a simple Lie supergroup. The covering group of the super Poincaré group is unimodular as noted on p.67 of this Thesis. Thus the weight function of the covering group of the super Poincare group is given by

$$\begin{aligned}\omega_{\overline{SISO}_0(3,1;E_L)} &= \omega_{SL(2,\mathbb{C}E_L)}\,\omega_{ST(3,1;E_L)} \\ &= (\sin 4y'_0)^{\mathcal{N}} e^{4\mathcal{N}x'_0}. \end{aligned} \qquad \cdots(34)$$

Now superspace is defined to be the homogeneous space

$$\begin{aligned} SISO(3,1;E_L)/SO(3,1;E_L) \\ = ST(3,1;E_L). \end{aligned} \qquad \cdots(35)$$

Since this is itself a group it fits the conditions of special case (i) of section 3.3 so that a super Poincaré integral on superspace is just the Haar integral on superspace itself.

3.5 A Critique of Alternative Integration Theories on Supermanifolds and of Superforms.

3.5.1 An Overview.

The theory we have developed has been based, in a natural way, on the topology of $E_L^{m,n}$. This is closely analogeous to integration on complex manifolds. This gives results very different from the theory normally used in the physics literature ie.

(i) Integration is not the 'inverse' of differentiation (or even the same as differentiation!). Of course, neither is the Lebesgue integral.

(ii) The Jacobian used for transformations is that for the components (x_i^μ, θ_j^ν) not the 'Super Jacobian' or 'Berezin function'.

(iii) The result of integration is an element of E_L ie. $\int : f \to E_L$ for $f \in \mathcal{L}^1(M, E_L)$ not a real number.

Most authors who write about supermanifolds accept the Berezin theory of integration [25] unchallenged (eg. [15] , [16] , [19] , [30] , [47] and [48]) and many others use the, related, notion of a superform as if it possesses the same properties as a differential form on a real manifold (eg. [15] , [16] , [19] and [49]). We can find only three other authors (Rogers [50] , DeWitt [31] and Rabin [51]) who have questioned the validity of the Berezin integral, and then they have simply tried to modify it to conform to their particular criteria. Rabin [51] has done little but repeat what has been said by DeWitt [31] .

At this stage our theory bears little relationship to the theory of Berezin (or Rogers or DeWitt). The fact that the Berezin theory can be understood in terms of our integration theory must wait until Chapter 5 of Part III. We content ourselves, here, with noting what these other authors have done.

3.5.2 The Berezin Integral.

The Berezin integral was originally defined in the context of path integral quantization [25] . It is applicable to 'odd' variables and is defined such that

$$\int^{B} d\theta = 0 \,, \quad \int^{B} \theta \, d\theta = 1 .$$

It has passed through several stages of development in its adaptation to supermanifolds the latest that we are aware of is given by Leites [23] . This is formulated in terms of the 'sheaf' definition of a supermanifold. Consider a superdomain which is a region R of $\mathbb{R}^n$ parametrized by $\{u^\mu, \mu = 1, 2, \cdots, n\}$, a set of generators of a Grassman algebra $\{\xi_i, i = 1, 2, \cdots, L\}$ and the set of functions

$$H = \{ f \mid f(u^\mu, \xi_i) = f_0(u^\mu) + \sum_{i=1}^{L} f_i(u^\mu)\xi_i + \sum_{i,j=1}^{L} f_{ij}(u^\mu)\xi_{i\wedge j} + \cdots + f_{12\cdots L}(u^\mu)\xi_{1\wedge 2\wedge\cdots\wedge L}\},$$

with each $f_{ij\cdots L}(u^\mu) \in C^\infty(R)$.

A superdomain is denoted by $S^{n,L}$ and is said to have coordinate system (u, ξ). A supermanifold is then (roughly speaking) a manifold together with a sheaf of commuting and anticommuting functions that admits an 'atlas' consisting of superdomains. Note that the definition used by Berezin and his co-workers for differentiation is such that $f \in H$ is a differentiable function. The class of differentiable functions is then considerably larger than the one we use.

The Berezin integral of $f \in H$ is then defined by

$$\int^{B} f = (-1)^{\left(-\frac{L}{2}(L-1) + Ln\right)} \int f_{12\cdots L}(u^\mu)\, d^n u . \qquad \cdots(36)$$

We note that it is the sign factor on the right hand side of this expression that has been the subject of the changes in definition mentioned above.

It is easy to see that $\int^{B} : H \to \mathbb{R}$. The transformation of coordinate factor is given by the superdeterminant of the super Jacobian matrix.

It was shown by Rogers [13] that a G^{∞}-supermanifold can be constructed from a sheaf-supermanifold, but unfortunately the Berezin integral cannot be transferred to give a notion of measure on a G^{∞}-supermanifold.

It is clear that Berezin integration is a linear functional defined on H, that is similar to a projection map. It is stated in reference [12] that invariant integrals can be constructed with the Berezin integral and that the result for $SPL(p|q; E_L)$ (called there $U(p,q)$) is zero.

The Berezin integal satisfies the requirements of physicists in that it can be used to pick out a Lagrangian function for the component fields from some quadratic form of a superfield. Provided, of course, one chooses the correct quadratic form.

3.5.3 The DeWitt Scheme of Integration.

DeWitt [31] defines integration on the even variables as a contour integral taken on the 'body' of the supermanifold ie. the normal manifold contained within the supermanifold - all DeWitt's manifolds are complex. For integration on the odd variables he defines a 'volume' element by

$$d^n\theta = i^{n(n-1)/2} d\theta^1 d\theta^2 \cdots d\theta^n$$

and then uses Berezin integration. The most appropriate comment we can make on this theory is given in a quote from his book [31] p.10 "Measure theoretical notions play no role here. Integration over $R_\alpha(\mathbb{C}E_L)$ becomes a purely formal procedure, the utility of which rests ultimately on the naturalness with which it can be used to encode certain algebraic information". We need say no more.

3.5.4 The Rogers Scheme of Integration.

The scheme given by Rogers [50] is a modification of the DeWitt scheme. Essentially she notes that by a suitable change of variable one can convert certain manifolds over $E_L^{m,n} = E_{L0}^m \times E_{L1}^n$ to manifolds over $\mathbb{R}^m \times E_{L1}^n$. She then defines integration as for the DeWitt scheme. This is claimed to remove inconsistencies that appear when trying to integrate in different even coordinate systems, effectively by using only one special coordinate system and a restricted class of manifold.

3.5.5 Differential forms on Supermanifolds.

As mentioned in section 3.5.1 many authors make use of differential forms on supermanifolds as if they were precise analogues of differential forms on real manifolds. These were originally proposed by Berezin [52] in order to construct his theory of integration by analogy with the theory for real manifolds. All other authors seem to follow this without question.

Here we consider only 1-forms on an $(m|n)$ dimensional manifold M. A 1-form on M is then defined as a map

$$\phi : D^1(M) \to G^\infty(M)$$

and it is assumed that $\phi \in D_1(M)$ the dual space to $D^1(M)$. Thus at any point $p \in M$ we require that

$$\phi : T_p(M) \to E_L$$

and $\phi_p \in T_p^*(M)$ the dual space to $T_p(M)$. But $T_p(M)$ is isomorphic to $T_p(\mathcal{M}_L)$ the tangent space to the equivalent manifold $\mathcal{M}_L$ so that we would expect $T_p^*(M)$ to be isomorphic to $T_p^*(\mathcal{M}_L)$. Then we must have

$$\phi_p : T_p(M) \to \mathbb{R}$$

and therefore

$$\phi_p : D^1(M) \to R(M)$$

the set of real valued functions on M, which (appart from the constant functions) are not differentiable. Thus the definition above is inadequate and needs to be modified in some way. We do not pursue this since we have no need to use differential forms in this Thesis.

CHAPTER 4

UNITARY IRREDUCIBLE REPRESENTATIONS OF LIE SUPERGROUPS.

4.1 Preliminary Discussion.

In the previous chapter we were able to obtain a theory of integration on superspace and Lie supermanifolds by considering integration to be equivalent to that on the real vector space underlying a supermanifold. With this theory, we were able to obtain expressions for the weight functions of several important Lie supergroups.

In this chapter we are going to study the unitary irreducible representations of Lie supergroups. We will do this by constructing the unitary irreducible representations for the equivalent Lie group.

The method we are going to use is the theory of induced representations. This concept was discovered by Frobenius [53] over eighty years ago in his study of finite groups and used by Wigner [54] to construct the unitary irreducible representations of the Poincaré group. Inducing was revealed as an indispensible tool for constructing representations of non compact groups by the work of Mackay [55]. It has even been suggested that it is the only method that has been used so far to obtain non-trivial representations of such groups [56]. We are able to use this theory because the equivalent Lie group can be expressed as a sequence of semi-direct products each of which contributes representations (see section 2).

Firstly we need to define what we mean by a unitary representation of a group $\mathcal{G}$. We suppose that we have a Hilbert space $\mathcal{H}$ with scalar product $x \cdot y$ for $x, y \in \mathcal{H}$ and that the action of the group elements on this Hilbert space is by operators $U(g)$, $g \in \mathcal{G}$ such that

$$(U(g)y)\cdot(U(g)x) = y \cdot x \qquad \ldots(1)$$

for each $g \in \mathcal{G}$, $x, y \in \mathcal{H}$.

Now consider an arbitrary group $\mathcal{G}$, and the set of complex valued square integrable functions on $\mathcal{G}$, $\mathcal{L}^2(\mathcal{G}, \mathbb{C})$. This set does not form a Hilbert space but if we define the set of functions $\mathcal{E} = \{ f \in \mathcal{L}^2(\mathcal{G}, \mathbb{C}),\ \int f f^* d\mu(\mathcal{G}) = 0 \}$, then the set $\mathcal{H}(\mathcal{G}) = \mathcal{L}^2(\mathcal{G}, \mathbb{C})/\mathcal{E}$ is a Hilbert space. Now the group action on $f \in \mathcal{H}(\mathcal{G})$ is given by Definition 3.4 of Chapter 3 as $gf(g') = f(g^{-1}g')$ for $g, g' \in \mathcal{G}$ then $f(g^{-1}g') \in \mathcal{H}(\mathcal{G})$ and equation (1) is satisfied. This then gives the archetype of a unitary representation. Thus the existence of a Haar integral on $\mathcal{G}$ guarantees the existence of unitary representations.

Consider now a locally compact (but not compact) abelian group $\mathcal{G}$, which is then isomorphic to $\mathbb{R}^n$ for some n, and consider the set of functions $\mathcal{H}(\mathcal{G})$ as defined above. This forms what is known as the regular representation of $\mathcal{G}$.

But all of the irreducible representations of an abelian group are one dimensional, so that we would expect $\mathcal{H}(\mathcal{G})$ to decompose into a sum of one dimensional subspaces. This does in fact happen and a convenient basis for these one dimensional subspaces is given by the characters of $\mathcal{G}$ defined as follows.

<u>Definition 4.1</u>

(a) A character of an arbitrary locally compact group $\mathcal{G}$ is a continuous function $\chi : \mathcal{G} \to \mathbb{C}$ such that

$$|\chi(g)| = 1$$

and $$\chi(g)\chi(g') = \chi(gg') \quad \text{for each } g, g' \in \mathcal{G}.$$

(b) If $\mathcal{G}$ is in addition an abelian group isomorphic to $\mathbb{R}^n$ then for $x = (x^1, x^2, \cdots, x^n) \in \mathcal{G}$ and each $p \in \mathbb{R}^n$ the function

$$\chi_p = \exp i\, p.x \qquad \cdots(2)$$

is a character of $\mathcal{G}$. The set $\{p\} \cong \mathbb{R}^n$ is called the dual group of $\mathcal{G}$ and denoted by $\hat{\mathcal{G}}$.

Now each χ_p is not an element of $\mathcal{L}^2(\mathcal{G},\mathbb{C})$ since

$$\int \chi_p \chi_p^* dx = \int e^{ip.x} e^{-ip.x} dx = \infty .$$

But we can write any $f \in \mathcal{L}^2(\mathcal{G},\mathbb{C})$ in the form

$$f(x) = \int g(p)\, e^{ip.x} dp \qquad \cdots(3)$$

with

$$g(p) = \int f(x)\, e^{-ip.x} dx . \qquad \cdots(4)$$

That is each element of the regular representation can be expressed as an integral that can be thought of as a weighted sum of the irreducible representations. This is called a direct integral decomposition. If we had chosen to consider a compact abelian group then $p \in \{0, \pm\frac{1}{2}, \pm 1, \cdots\}$ and $0 \leq x^\mu < 2\pi$ so that $\int \chi_p \chi_p^* dx = 2\pi$ and we would instead have had a decomposition into a direct sum of irreducible representations. This decomposition into irreducible representations given by the characters is not unique, for instance we can alter the characters by adding any element of the set $\mathcal{E}$ without changing the substance of what we say. But, whatever we do, the representations remain equivalent to the one we have constructed. This then typifies a Type I representation. There are other types of representation called Type II and Type III that do not possess this property. Fortunately groups with Type II and III representations do not seem to occur in particle physics. Examples do exist in other branches of physics eg. the 'magnetic translation group' which has Type II representations and has been discussed by Brown [57] and Zak [58] .

The following classes of Lie group possess only Type I representations (1) compact, (2) locally compact abelian, (3) semisimple, (4) nilpotent, (5) Eucidean group and (6) Poincaré group.

Now $E_L^{m,n}$ is a locally compact Lie supergroup and is topologically isomorphic to $\mathbb{R}^{\mathcal{N}(m+n)}$, so that the above arguments apply. If we recall the definition of inner product as given in section 1.2.1 of

Chapter 1 we can see that the characters of the supergroup $G = E_L^{m,n}$ can be labeled by $(p,\phi) \in E_L^{m,n}$ and written

$$\chi_{(p,\phi)} = \exp i(p,\phi)\cdot(x,\theta) \quad \text{for} \quad (x,\theta) \in G.$$

These characters then serve as a basis for the one (complex not Grassman) dimensional representations of the supergroup $E_L^{m,n}$ and any complex valued function f on $E_L^{m,n}$ can be written as

$$f(x,\theta) = \int \hat{d}^m p\, \hat{d}^n \phi \{ g(p,\phi)\, e^{ip\cdot x} e^{ip\cdot\phi} \}, \qquad \cdots(6)$$

for some function $g : E_L^{m,n} \longrightarrow \mathbb{C}$.

Of course a function taking values in $\mathbb{C}E_L$ must be written as a sum of integrals of this type. So that if $f' : E_L^{m,n} \longrightarrow \mathbb{C}E_L$ with component functions $f'_{e_i} : E_L^{m,n} \longrightarrow \mathbb{C}$ and $f'_{f_j} : E_L^{m,n} \longrightarrow \mathbb{C}$ then

$$\begin{aligned} f'(x,\theta) &= e_i f'_{e_i}(x,\theta) + f_j f'_{f_j}(x,\theta) \\ &= e_i \int \hat{d}^m p\, \hat{d}^n \phi \{ g_{e_i}(p,\phi)\, e^{ip\cdot x} e^{i\phi\cdot\theta} \}, \\ &\quad + f_j \int \hat{d}^m p\, \hat{d}^n \phi \{ g_{f_j}(p,\phi)\, e^{ip\cdot x} e^{i\phi\cdot\theta} \}, \end{aligned} \qquad \cdots(7)$$

for some choice of the functions g_{e_i} and g_{f_j}, which can be determined from the inverse Fourier transforms

$$g_{e_i}(p,\phi) = \int \hat{d}^m x\, \hat{d}^n \theta \{ f_{e_i}(x,\theta)\, e^{-ip\cdot x} e^{-i\phi\cdot\theta} \} \qquad \cdots(8)$$

and

$$g_{f_j}(p,\phi) = \int \hat{d}^m x\, \hat{d}^n \theta \{ f_{f_j}(x,\theta)\, e^{-ip\cdot x} e^{-i\phi\cdot\theta} \}. \qquad \cdots(9)$$

Equations (6), (7), (8) and (9) give us the foundations of the theory of Fourier analysis on superspace, which we do not pursue.

4.2 Non Abelian Lie Supergroups.

Consider any $(m|n)$ dimensional Lie supergroup G with Lie superalgebra generators $\{\alpha_\mu, \beta_\nu\}$. The equivalent Lie group $\mathcal{G}_A$ then has generators $\{e_i\alpha_\mu, f_j\beta_\nu\}$, $i = 0,1,\cdots,2^{A-1}-1$; $j = 1,2,\cdots,2^{A-1}$ and the equivalent Lie group $\mathcal{G}_{A+1}$ has generators $\{e_i\alpha_\mu, \varepsilon_{A+1}f_j\alpha_\mu, f_j\beta_\nu, \varepsilon_{A+1}e_i\beta_\nu\}$.

The additional generators in the step $A \rightarrow A+1$ ie. $\{\varepsilon_{A+1} f_j \alpha_\mu, \varepsilon_{A+1} e_i \beta_\nu\}$ form the basis of an abelian Lie algebra so that the coset space $\mathcal{G}_{A+1}/\mathcal{G}_A$ is an abelian subgroup of $\mathcal{G}_{A+1}$ and we have the decomposition

$$\mathcal{G}_{A+1} = \mathcal{G}_A \circledS \mathcal{G}_{A+1}/\mathcal{G}_A. \qquad \cdots\cdot(10)$$

Here the symbol $\circledS$ signifies semidirect product and we adopt the convention that $H \circledS K$ satisfies $HH \subset H$, $KK \subset K$ and $HK \subset K$.

Clearly then we can construct $\mathcal{G}_L$ for any L as a sequence of semidirect products as follows

$$\mathcal{G}_L = ((\cdots((\cdots((\mathcal{G}_0 \circledS \mathcal{G}_1/\mathcal{G}_0) \circledS \mathcal{G}_2/\mathcal{G}_1 \cdots) \circledS \mathcal{G}_{A+1}/\mathcal{G}_A) \circledS \cdots) \circledS \mathcal{G}_L/\mathcal{G}_{L-1}), \quad \cdots(11)$$

with $\mathcal{G}_0$ the Lie group with generators $\{\varepsilon_0 \alpha_\mu\}$.

Now consider any group $\mathcal{G}$ that admits the semidirect product structure $\mathcal{G} = \mathcal{H} \circledS \mathcal{N}$ with $\mathcal{N}$ an abelian subgroup of $\mathcal{G}$. Let $\alpha_h(n)$ denote the automorphism of $\mathcal{N}$ by $\mathcal{H}$ given by

$$\alpha_h(n) = hnh^{-1}, \qquad \cdots\cdot(12)$$

for each $n \in \mathcal{N}$ and fixed $h \in \mathcal{H}$.

For each $y \in \hat{\mathcal{N}}$ we define the transform of the character χ_y by h as

$$\chi_y(n)h = \chi_y(hnh^{-1}) = \chi_{y'}(n), \qquad \cdots\cdot(13)$$

and define the orbit of χ_y to be the set

$$\{\chi_y h, \ h \in \mathcal{H}\}. \qquad \cdots\cdot(14)$$

The group

$$\mathcal{H}_{\chi_y} = \{y \in \mathcal{H}, \ \chi_y h = \chi_y\} \qquad \cdots\cdot(15)$$

is called the stability group of χ_y.

With these definitions we can state the two main theorems on induced representations that we will be using. These are as given by Mackey[55].

Theorem 4.2

For each $\chi_y, y \in \hat{n}$ choose an irreducible representation $\Delta_{\mathcal{H}_{\chi_y}}$ of $\mathcal{H}_{\chi_y}$ and consider the subgroup $S_y = \mathcal{H}_{\chi_y} \circledS \mathcal{N}$ consisting of all $hn \in \mathcal{H} \circledS \mathcal{N}$ with $h \in \mathcal{H}_{\chi_y}$. Then $\Delta_{S_y} = \Delta_{\mathcal{H}_{\chi_y}} \chi_y(n)$ is an irreducible representation of S_y and the induced representation $(\Delta_{S_y})\uparrow\mathcal{G}$ is an irreducible representation of $\mathcal{G}$.

If χ_y and $\chi_{y'}$ lie in the same orbit then every $(\Delta_{\mathcal{H}_{\chi_y}} \chi_y(n))\uparrow\mathcal{G}$ is equivalent to some $(\Delta_{\mathcal{H}_{\chi_{y'}}} \chi_{y'}(n))\uparrow\mathcal{G}$. Thus it suffices to choose just one χ_y from each orbit.

Let C be the set of characters which includes just one member of each orbit. Then as χ_y varies over C and Δ varies over the set of irreducible representations of $\mathcal{H}_{\chi_y}$ the irreducible representations $(\Delta\,\chi_y)\uparrow\mathcal{G}$ are mutually inequivalent.

Theorem 4.3

Suppose it is possible to choose the set C of χ_y such that C is a Borel set. Then every irreducible representation of $\mathcal{G}$ is equivalent to some $(\Delta\,\chi_y)\uparrow\mathcal{G}$.

Since the representations we are discussing here are on a Hilbert space we can replace 'irreducible representation' by 'unitary irreducible representation' in the statement of these theorems.

Now consider equation (11) in the light of these theorems. We have the following results.

Theorem 4.4

Let G be a linear Lie supergroup. Then given a value for L we can proceed through the sequence $\mathcal{G}_0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \cdots \mathcal{G}_L$ and obtain all the unitary irreducible representations of $\mathcal{G}_L$ provided we can ascertain the representations of $\mathcal{G}_0$.

Proof

Suppose we have all the unitary irreducible representations of $\mathcal{G}_0$. The stability group for each character of $\mathcal{G}_1/\mathcal{G}_0$ is a subgroup of $\mathcal{G}_0$ for which we can obtain all the unitary representations by restricting the domain of the representations of $\mathcal{G}_0$. Then using Theorem 4.2 we can obtain all the representations of $\mathcal{G}_1$. We can repeat this procedure for each $A = 2, 3, \cdots, L$.

Now since $\mathcal{G}_L$ is a linear Lie supergroup the orbit of an element g of $\mathcal{G}_{A+1}/\mathcal{G}_A$ is a closed subset of $\mathcal{G}_{A+1}/\mathcal{G}_A$ and every element of $\mathcal{G}_{A+1}/\mathcal{G}_A$ lies in one and only one such orbit. The set C is obtained by choosing one element from each of these closed subsets and must clearly be a Borel set.

We thus satisfy the requirements of Theorem 4.3 this completes the proof.

Theorem 4.5

If $\mathcal{G}_0$ has only Type I representations then $\mathcal{G}$ has only Type I representations.

Proof

If $\mathcal{G}_0$ has only Type I representations then every subgroup of $\mathcal{G}_0$ has only Type I representations, since if a subgroup of $\mathcal{G}_0$ had Type II or Type III representations it would induce to give Type II or Type III representations for $\mathcal{G}_0$. Now the stability group of $n \in \mathcal{G}_1/\mathcal{G}_0$ is one of these subgroups so that $\mathcal{G}_1$ has only Type I representations. Then proceding through the sequence $A = 2, 3, \cdots, L$ we can see that $\mathcal{G}$ can only have Type I representations.

Theorem 4.6

At each step $\mathcal{G}_A \rightarrow \mathcal{G}_{A+1}$ the stability group of the character $\chi = \exp i(0,0,\cdots).(x,\theta)$ ie. $\chi = 1$ for all $g \in \mathcal{G}_{A+1}/\mathcal{G}_A$ is $\mathcal{G}_A$ so that we retain all the representations so far obtained. If we denote the unitary irreducible representations of $\mathcal{G}_A$ by $R(\mathcal{G}_A)$ then

$$R(\mathcal{G}_L) \supset R(\mathcal{G}_{L-1}) \supset \cdots \supset R(\mathcal{G}_{A+1}) \supset R(\mathcal{G}_A) \supset \cdots \supset R(\mathcal{G}_0).$$

Proof

This is an immediate consequence of Theorem 4.4.

Thus it is clear that to obtain all the representations of a linear Lie supergroup we construct the representations obtained in each step $\mathcal{G}_A \rightarrow \mathcal{G}_{A+1}$ for $A = 0, 1, \cdots, L$. To construct the representations at each step we need to determine the set C and the stability group for each $y \in \hat{n}$. In practice we cannot do this in two seperate steps.

This analysis then tells us which representations we need to induce to representations of $\mathcal{G}_L$. To carry out the induction process we need a further theorem. For this we refer to Cornwell [39] Chapter 5 section 7 from which we obtain the following theorem (rewritten in our notation).

Theorem 4.7

Let V_Δ be a carrier space for the representation Δ of $S_y = \mathcal{H}_{\chi_y} n$ and let $\Phi_\Delta(s)$ be a set of operators defined for all $s \in S_y$ to act on V_Δ in such a way that

$$\Phi_\Delta(s)\psi_n = \Delta(s)_{n'n}\psi_{n'} \qquad \cdots(16)$$

where $\psi_1, \psi_2, \cdots, \psi_d$ are a basis for V_Δ. Let V be the vector space of all mappings $\phi(S\tau_j)$ into V_Δ, of right cosets $S\tau_1, S\tau_2, \cdots$ of $\mathcal{G}$ with respect to S. For each $g \in \mathcal{G}$ define the operator $\Phi(g)$ by

$$\Phi(g)\phi(S\tau_K) = \Phi_\Delta(\tau_K g \tau_j^{-1})\,\phi(S\tau_j), \qquad \cdots(17)$$

where $\phi \in V$ and τ_j is the coset representative such that

$$\tau_K g \in S\tau_j. \qquad \cdots(18)$$

Then

(a) for any $g, g' \in \mathcal{G}$

$$\Phi(gg') = \Phi(g)\Phi(g') \qquad \cdots(19)$$

and

(b) there exists a basis ϕ_{kt} of V $(k = 1, 2, \cdots, m; t = 1, 2, \cdots, d)$ such that for all $g \in \mathcal{G}$ and $j = 1, 2, \cdots m; r = 1, 2, \cdots, d$

$$\Phi(g)\phi_{jr} = \Gamma(g)_{kt,jr}\,\phi_{kt} \qquad \cdots(20)$$

where Γ is the induced representation of $\mathcal{G}$ defined in equation (17). Thus V is a carrier space for the induced representation Γ of $\mathcal{G}$ and the $\Phi(g)$ are the corresponding operators acting on V.

Now suppose we choose $\hat{\chi}$ as a representative character for a given orbit then the set of characters in this orbit are given by

$$\{\chi = \hat{\chi}g \,,\; g \in \mathcal{G}\}.$$

But g admits the decomposition $g = s\Upsilon_i$ for some coset Υ_i and $s \in S_{\hat{\chi}}$. That is $\chi = \hat{\chi}s\Upsilon_i = \hat{\chi}\Upsilon_i$ and this relationship must be one:one so that we can put

$$\Upsilon_i = \Upsilon(\chi, \hat{\chi}) \qquad \cdots(21)$$

and label the cosets by our representative character and the characters in the corresponding orbit. Thus equation (17) can be rewritten as

$$\Phi(g)\,\phi(s\Upsilon(\chi, \hat{\chi})) = \Phi_\Delta(\Upsilon(\chi,\hat{\chi})\,g\,\Upsilon^{-1}(\chi',\hat{\chi}))\,\phi(s\Upsilon(\chi',\hat{\chi})) \qquad \cdots(22)$$

Now equation (18) imposes the condition that

$$\Upsilon(\chi,\hat{\chi})\,g\,\Upsilon^{-1}(\chi',\hat{\chi}) \in S,$$

so that

$$\hat{\chi}\,\Upsilon(\chi,\hat{\chi})\,g\,\Upsilon^{-1}(\chi',\hat{\chi}) = \hat{\chi}s = \hat{\chi}.$$

Thus

$$\chi\,g\,\Upsilon^{-1}(\chi',\hat{\chi}) = \hat{\chi} = \chi'\Upsilon^{-1}(\chi',\hat{\chi}).$$

Therefore χ' must satisfy

$$\chi' = \chi g \qquad \cdots(23)$$

or $$\chi = \chi' g^{-1}. \qquad \cdots(24)$$

Also since the coset representatives are labeled by χ we can put

$$\phi(s\tau(\chi,\hat{\chi})) = \phi(\chi).$$

Equation (22) can now be written

$$\Phi(g)\phi(\chi) = \Phi_\Delta(\tau(\chi' g^{-1},\hat{\chi})g\tau^{-1}(\chi',\hat{\chi}))\phi(\chi g). \qquad \cdots(25)$$

Now define $$\phi_{\chi,n}(\chi') = \begin{cases} \psi_n & \text{if } \chi = \chi' \\ 0 & \text{otherwise.} \end{cases} \qquad \cdots(26)$$

So that $$\phi_{\chi' g^{-1},n}(\chi) = \begin{cases} \psi_n & \text{if } \chi' = \chi g \\ 0 & \text{otherwise,} \end{cases} \qquad \cdots(27)$$

and $$\phi_{\chi',n}(\chi g) = \begin{cases} \psi_n & \text{if } \chi' = \chi g \\ 0 & \text{otherwise.} \end{cases} \qquad \cdots(28)$$

If we consider the action of $\Phi(g)$ on these mappings we have, from equation (22) using equation (28)

$$\begin{aligned}\Phi(g)\phi_{\chi',n}(\chi) &= \Phi_\Delta(\tau(\chi' g,\hat{\chi})g\tau^{-1}(\chi',\hat{\chi}))\phi_{\chi',n}(\chi g), \\ &= \begin{cases} \Delta(\tau(\chi' g,\hat{\chi})g\tau^{-1}(\chi',\hat{\chi}))_{mn}\psi_m & \text{if } \chi' = \chi \\ 0 & \text{otherwise} \end{cases}, \\ &= \Delta(\tau(\chi' g^{-1},\hat{\chi})g\tau^{-1}(\chi',\hat{\chi}))_{mn}\phi_{\chi' g^{-1},m}(\chi); \end{aligned} \qquad \cdots(29)$$

where we have used equation (27). Equation (29) then satisfies condition (b) of Theorem 4.3 so that the mappings defined by equation (26) are indeed a basis of the induced representation. We can delete the argument χ of the mappings and replace χ' by χ to obtain

$$\Phi(g)\phi_{\chi,n} = \Delta(\tau(\chi g^{-1},\hat{\chi})g\tau^{-1}(\chi,\hat{\chi}))_{mn}\phi_{\chi g^{-1},m}. \qquad \cdots(30)$$

Now the characters χ are in one to one correspondence to the elements $y \in \hat{\mathcal{N}}$ so that the above equation could equally well have been written in terms of y. In fact this is the most convenient way to express equation (30). To do this we first have to determine how the group $\mathcal{H}$ acts on $\hat{\mathcal{N}}$ in some convenient way. In our case $\mathcal{N} = \mathcal{G}_{A+1}/\mathcal{G}_A$ and $\mathcal{H} = \mathcal{G}_A$. Let $g_n \in \mathcal{N}$ and $g \in \mathcal{G}_A$. Now since $\mathcal{N}$ is abelian all its elements can be written in the form

$$g_n = I + \alpha_\mu x^\mu + \beta_\nu \theta^\nu, \qquad \cdots(31)$$

with $x^\mu = x^\mu_j f_j \varepsilon_{A+1}$ and $\theta^\nu = \theta^\nu_i \varepsilon_i \varepsilon_{A+1}$.

The action of the automorphism $a_h(n)$ of equation (12) is then given by

$$\begin{aligned} a_g(g_n) &= g g_n g^{-1} \\ &= g(I + \alpha_\mu x^\mu + \beta_\nu \theta^\nu) g^{-1} \\ &= I + g(\alpha, \beta)_a \begin{bmatrix} x \\ \theta \end{bmatrix}_a g^{-1} \\ &= I + g(\alpha, \beta)_a g^{-1} \begin{bmatrix} x \\ \theta \end{bmatrix}_a . \end{aligned} \qquad \cdots(32)$$

Now using the definition of $Ad(g)$ given by equation (8) of Chapter 2 we have

$$a_g(g_n) = I + (\alpha, \beta)_b (Ad(g))_{ba} \begin{bmatrix} x \\ \theta \end{bmatrix}_a . \qquad \cdots(33)$$

Now the characters of $n = \mathcal{G}_{A+1}/\mathcal{G}_A$ are given by

$$\chi_{(y,\phi)}(g_n) = \exp i\{(y, \phi).(x, \theta)\}$$

and combining equation (13) with equation (33) we have

$$\begin{aligned} \chi_{(y,\phi)}(g_n) g &= \exp i\left\{(y, \phi). \left((Ad(g))_{ba} \begin{bmatrix} x \\ \theta \end{bmatrix}_a\right)\right\}, \\ &= \exp i\{((y, \phi)_b (Ad(g))_{ba}).(x, \theta)\}. \end{aligned} \qquad \cdots(34)$$

That is we can view the automorphism a_g in three equivalent ways

(i) $$a_g : \begin{bmatrix} x \\ \theta \end{bmatrix} \longrightarrow (Ad(g))_{ba} \begin{bmatrix} x \\ \theta \end{bmatrix}_a \qquad \cdots(35)$$

where $(x, \theta)^t$ parametrize n,

(ii) $$a_g : (y, \phi)_b \longrightarrow (y, \phi)_a (Ad(g))_{ab} \qquad \cdots(36)$$

where (y, ϕ) parametrize $\hat{n}$

or (iii) $$a_g : \chi_{(y,\phi)} \longrightarrow \chi_{((y,\phi)Ad(g))}. \qquad \cdots(37)$$

Thus rewritting equation (30) in terms of (y, ϕ) the parameters of $\hat{n}$ we obtain

$$\Phi^{\hat{g},\hat{\phi},\Delta_{\mathcal{H}}}(g)\,\phi_{(y,\phi),n} \qquad \cdots(38)$$
$$= \Delta^{\hat{g},\hat{\phi},\Delta_{\mathcal{H}}}(\Upsilon((y,\phi)\mathrm{Ad}(g^{-1}),(\hat{g},\hat{\phi}))g\Upsilon^{-1}((y,\phi),(\hat{g},\hat{\phi})))_{mn}\phi_{(y,\phi)\mathrm{Ad}(g^{-1}),m}.$$

Here $(\hat{g},\phi)$ are the parameters corresponding to the choice of a character $\hat{\chi}$ in a given orbit and we have written $\Delta^{\hat{g},\hat{\phi},\Delta_{\mathcal{H}}}$ and $\Phi^{\hat{g},\hat{\phi},\Delta_{\mathcal{H}}}$ to indicate the dependence of the representation on the choice of irreducible representation of $\mathcal{H}_{\chi_{\hat{g},\hat{\phi}}}$.

We note that equation (34) above defines the orbit of a character and can also be used to obtain the stability group but in practice it is easier to make use of the Lie algebra of $\mathcal{G}_A$ acting on the Lie algebra of $\mathcal{G}_{A+1}/\mathcal{G}_A$ for this step. To see this we observe that there is a one to one correspondence between the elements of the Lie algebra of $\mathcal{G}_{A+1}/\mathcal{G}_A$ and the elements of $\hat{\mathcal{N}}$, so that requiring that h is an element of the stability group of an element (y,ϕ) is equivalent to demanding that h leaves invariant the corresponding element of the Lie algebra of $\mathcal{G}_{A+1}/\mathcal{G}_A$. Now h must lie in some one parameter subgroup of $\mathcal{G}_A$ so that we can put $h = \exp \alpha\gamma,\ \alpha \in \mathbb{R},\ \gamma \in \mathcal{L}(\mathcal{G}_A)$. Then our requirement becomes

$$h\,\delta\,h^{-1} = \delta$$

where δ is in the Lie algebra element corresponding to the character ie.

$$\exp(\alpha\gamma)\,\delta\,\exp(-\alpha\gamma) = \delta$$
$$(1+\alpha\gamma+\tfrac{1}{2}\alpha^2\gamma^2+\cdots)\,\delta\,(1-\alpha\gamma+\tfrac{1}{2}\alpha^2\gamma^2+\cdots) = \delta$$
$$\delta + \alpha[\gamma,\delta]+\cdots = \delta$$

so that we require $\quad [\gamma,\delta] = 0.$

4.3 The Unitary Irreducible Representations of $\mathcal{G} = \overline{SISO_0}(3,1;E_L)$.

4.3.1 Preliminaries.

In this section we will construct some of the unitary irreducible representations of the covering group of the proper orthochronous super Poincare group. Extension to the homogeneous super Poincare group can be carried out by introducing unitary operators corresponding to P, T and PT.

A review of the super Poincaré groups in any number of dimensions was presented in section 2.4 of Chapter 2, here we put $d = 4$ and $t = 1$. The Dirac matrices then become the four dimensional ones as specified in the appendix.

We are looking for representations that we can associate with the well known representations of the Lie superalgebra $siso(3,1;E_L)$ as originally described by Salam and Strathdee [8] . These are known to be labeled by a 'superspin' index $j = 0, \frac{1}{2}, 1, \cdots$ and a 'mass parameter' M. We deal with these representations in Part II of this Thesis in some detail. These representations consist of the direct sum of four Poincaré type representations (except for $j = 0$ which has only three such representations) with spins of j, $j+\frac{1}{2}$, $j-\frac{1}{2}$, j (the $j-\frac{1}{2}$ representation does not exist for $j = 0$) together with the supersymmetry generators which link the representations.

We will show, by a series of examples, that the unitary irreducible representations we can construct act on state vectors which have at most one index that we can associate with spin and that this is such that an irreducible representation acts on state vectors with a single fixed spin. The connection between the representations we construct here and the Salam-Strathdee representations is the subject of Part III of this Thesis.

We consider group elements of the form $g = [\Lambda | t | \tau]$ for $g \in \mathcal{G}_L$ parametrized by $\tau^{\alpha} \in E_{L1}$, $\alpha = 1,2,3,4$ for the supertranslations, $t^{\sigma} \in E_{L0}$, $\sigma = 1,2,3,4$ for the translations and for the Lorentz transformations we parametrize by $y^{\mu\nu}$; $\mu, \nu = 1, 2, 3, 4$; $\mu \neq \nu$ and $y^{\mu\nu} = -y^{\nu\mu} \in E_{L0}$ so that there are six independent parameters which we take to correspond to the Lie algebra generators $L_{\mu\nu}$ ie. $\Lambda = \exp(y^{\mu\nu} L_{\mu\nu})$. We will need to write the parameters of Λ as a vector in which case we choose them to be ordered as

$$y = (y^{12},\ y^{13},\ y^{14},\ y^{23},\ y^{24},\ y^{34}). \qquad \cdots(39)$$

We also use another order but this is best specified later.

The elements of $\mathcal{G}_{A+1}/\mathcal{G}_A = n$ are then parametrized by

$$\{ y^{\mu\nu}_{j\wedge(A+1)} \underline{f}_j \underline{\varepsilon}_{A+1},\ t^{\sigma}_{j\wedge(A+1)} \underline{f}_j \underline{\varepsilon}_{A+1},\ \tau^{\alpha}_{i\wedge(A+1)} \underline{e}_i \underline{\varepsilon}_{A+1} \} \qquad \cdots(40)$$

with $\underline{e}_i$ a basis of E_{A0} so that $i = 0, 1, \cdots, 2^{A-1}-1$ and $\underline{f}_j$ a basis of E_{A1} so that $j = 1, 2, \cdots, 2^{A-1}$.

Correspondingly the elements of $\hat{n}$ are in one to one correspondence to the vector

$$\begin{aligned}(\ell,\ k,\ \phi) &= (\ell_{\mu\nu},\ k_\sigma,\ \phi_\alpha), \qquad \cdots(41)\\ &= (\ell^{j\wedge(A+1)}_{\mu\nu},\ k^{j\wedge(A+1)}_{\sigma},\ \phi^{i\wedge(A+1)}_{\alpha}),\end{aligned}$$

with $\mu, \nu, \sigma, \alpha, j, i$ taking the values specified above. The characters of n are then given by

$$\chi_{(\ell,k,\phi)} = \exp i\{(\ell, k, \phi).(u, x, \theta)\}. \qquad \cdots(42)$$

The action of a group element $g \in \mathcal{G}_A$ on a character is then specified by equation (37) to be

$$\chi_{(\ell,k,\phi)}\, g = \chi_{((\ell,k,\phi)Ad(g))} \qquad \cdots(43)$$

with $Ad(g)$ given by equation (74) of Chapter 2. We note that the 'translation' element of $\hat{n}$ is often denoted by p (we do in Parts II and III .) this corresponds to the Hermitian generator of the Poincare group

$$P_\sigma = \frac{i}{\hbar} K_\sigma . \qquad \cdots(44)$$

It is related to our parameter k by

$$\frac{1}{\hbar} P^{o}_{\sigma} = k^{o}_{\sigma} . \qquad \cdots(45)$$

The action of $g = [\Lambda | t | \tau] \in \mathcal{G}_A$ on $\hat{n}$ is given by

$$(\ell, k, \phi)\begin{bmatrix} Ad(\Lambda) & 0 & 0 \\ (V+TU)Ad(\Lambda) & \Lambda & 2T\Gamma(\Lambda) \\ U\,Ad(\Lambda) & 0 & \Gamma(\Lambda) \end{bmatrix} \quad \cdots(46)$$

$$= (\ell Ad(\Lambda) + k(V+TU)Ad(\Lambda) + \phi U Ad(\Lambda),\ \rho\Lambda,\ \phi\Gamma(\Lambda) + 2\rho T\Gamma(\Lambda))$$

with the matrices V, T and U specified by equations (75), (63) and (76) respectively of Chapter 2. The representation $\Gamma(\Lambda)$ is given by

$$\Gamma(\Lambda) = S^{*}\begin{bmatrix} \Gamma^{0,\frac{1}{2}}(\Lambda^{-1})^{t} & 0 \\ 0 & \Gamma^{\frac{1}{2},0}(\bar{\Lambda}^{-1})^{t} \end{bmatrix}(S^{-1})^{*}, \quad \cdots(47)$$

$$= \exp \tfrac{1}{2}\, y^{\mu\nu}(\gamma^{M}_{\mu}\gamma^{M}_{\nu})^{t}$$

with the similarity transformation S defined by equation (36) of the Appendix and the Dirac matrices being those defined by equation (35) of the appendix.

The matrix $Ad(\Lambda)$ is equivalent to the representation $\begin{bmatrix} \Gamma^{0,1}(\Lambda) & 0 \\ 0 & \Gamma^{1,0}(\Lambda) \end{bmatrix}$. It is convenient to have an explicit representation in this decomposed form and to have the corresponding characters for this decomposition. We find that a suitable matrix representation of $ad(L_{\mu\nu})$ is given by

$$ad(L_{12}) = \begin{bmatrix} \begin{matrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & O \\ O & \begin{matrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \end{bmatrix}, \quad ad(L_{13}) = \begin{bmatrix} \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{matrix} & O \\ O & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{matrix} \end{bmatrix},$$

$$ad(L_{23}) = \begin{bmatrix} \begin{matrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix} & O \\ O & \begin{matrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix} \end{bmatrix}, \quad ad(L_{14}) = \begin{bmatrix} \begin{matrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{matrix} & O \\ O & \begin{matrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{matrix} \end{bmatrix},$$

$$ad(L_{24}) = \begin{bmatrix} \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{matrix} & O \\ O & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{matrix} \end{bmatrix}, \quad ad(L_{34}) = \begin{bmatrix} \begin{matrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & O \\ O & \begin{matrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \end{bmatrix}. \quad \cdots(48)$$

Corresponding to this representation the vector ℓ takes the form

$$\left(\tfrac{1}{\sqrt{2}}(\ell_{13}+i\ell_{24}),\ \tfrac{1}{\sqrt{2}}(\ell_{23}-i\ell_{14}),\ \tfrac{1}{\sqrt{2}}(\ell_{12}-i\ell_{34}),\ \tfrac{1}{\sqrt{2}}(\ell_{13}-i\ell_{24}),\ \tfrac{1}{\sqrt{2}}(\ell_{23}+i\ell_{14}),\ \tfrac{1}{\sqrt{2}}(\ell_{12}+i\ell_{34})\right)$$
$$= (r_1, r_2, r_3, r_1^*, r_2^*, r_3^*). \qquad \cdots\cdots(49)$$

4.3.2 The Representations of $\mathcal{G}_L$ Obtained from the Representations of $\mathcal{G}_0 = SISO_0(3,1;E_0) = ISO_0(3,1;\mathbb{R})$.

The representations of $\mathcal{G}_0$ are the well known representations of the Poincaré group. In our terminology we consider elements of the form $[\Lambda | t | 0]$ which admit the semidirect product decomposition

$$[\Lambda|t|0] = [I|t|0][\Lambda|0|0]. \qquad \cdots\cdots(50)$$

The elements of $\hat{n}$ are the four vector $k^0 = (k_1^0, k_2^0, k_3^0, k_4^0)$ and the action of Λ on the elements of $\hat{n}$ is given by

$$k'^0_\nu = k^0_\mu (\Lambda)_{\mu\nu}. \qquad \cdots\cdots(51)$$

The action of Λ on k^0 leaves invariant the quadratic form

$$k^0_\mu k^0_\nu g^{\mu\nu} = -(k_1^0)^2 - (k_2^0)^2 - (k_3^0)^2 + (k_4^0)^2 = \left(\frac{Mc}{\hbar}\right)^2, \qquad \cdots\cdots(52)$$

where we have written the constant on the right hand side in its standard form with M being the 'mass', c the velocity of light and $\hbar$ Plank's constant.

We are able to see that there are six distinct types of orbit, for which the set of representative characters forms a Borel set so that we obtain every representation. These are:

(i) The orbit consists of four vectors k^0 for which $\left(\frac{Mc}{\hbar}\right)^2 > 0$ and $k_4^0 > 0$. A convenient choice of representative is $\hat{k}^0 = (0, 0, 0, \hat{k}_4^0)$. The stability group is isomorphic to a covering group of $SO(3;\mathbb{R})$ which has finite dimensional unitary representations labeled by $j = 0, \frac{1}{2}, 1, \cdots$. The labels M, j thus serve to label all these representations. The vectors of the carrier space of these representations are interpreted as being particles of mass M and spin j.

(ii) The orbit consists of four vectors k° for which $\left(\frac{Mc}{\hbar}\right)^2 > 0$ and $k_4^{\circ} < 0$. A convenient choice of representative is $\hat{k}^{\circ} = (0, 0, 0, \hat{k}_4^{\circ})$. The stability group is again isomorphic to a covering group of $SO(3, \mathbb{R})$. The representations are labeled by M, j but we require M to be negative. The particles corresponding to these representations are thus considered to be 'non physical'.

(iii) The orbit consists of four vectors k° for which $\left(\frac{Mc}{\hbar}\right)^2 < 0$. A convenient choice of representative character is $\hat{k}^{\circ} = (\hat{k}_1, 0, 0, 0)$. The stability group is isomorphic to a covering group of $SO(2,1;\mathbb{R})$. The condition $\left(\frac{Mc}{\hbar}\right)^2 < 0$ implies that M is imaginary so that these representations are again considered to be 'non physical'.

(iv) The orbit consists of four vectors k° for which $\left(\frac{Mc}{\hbar}\right)^2 = 0$ but $k_4^{\circ} > 0$ for which a convenient choice of representative is $\hat{k}^{\circ} = (0, 0, \hat{k}_4^{\circ}, \hat{k}_4^{\circ})$. The stability group in this case is isomorphic to a covering group of $SO(2;\mathbb{R}) \circledS T(2,\mathbb{R})$. If we induce up from some representation of $T(2,\mathbb{R})$ we obtain representations that are infinite dimensional and labeled by two real numbers (apart from the 'mass' which has been set to zero). These representations are again discarded as 'non physical'. The unitary representations of $\overline{SO}(2,\mathbb{R})$ are all one dimensional and are labeled by $\lambda = 0, \pm\frac{1}{2}, \pm 1, \cdots$. After inducing to representations of $\mathcal{G}_0$ they are still one dimensional and are interpreted as massless particles of spin λ.

(v) The orbit consists of four vectors k° for which $\left(\frac{Mc}{\hbar}\right)^2 = 0$ but The analysis is the same as for case (iv) but these representations are for particles of negative momentum, so are discarded on physical grounds.

(vi) The orbit consists of the four vector $k^{\circ} = (0, 0, 0, 0)$. The resulting representations are then induced from the infinite dimensional unitary representations of $\overline{SO}_0(3,1;\mathbb{R})$. These are interpreted as representing space-time symmetries and not particles, since any resulting particle would have zero mass and zero momentum. We note that these representations require two half - integer labels corresponding to the two Casimir operators of $SO_0(3,1;\mathbb{R})$.

We note that the representations above that are of physical interest ((i) and (iv)), in that they serve to provide representations of physical particles, can be labeled by a number that takes half-integer values (j or λ) and a real positive number (M). These labels are insufficient to classify all of the representations. The same is true if we try to classify the representations above by the eigenvalues of the two independent Casimir operators $K^2 = \hbar^2 K_\mu K_\nu g^{\mu\nu}$ and $W^2 = \hbar^4 g_{\alpha\beta}\, \varepsilon^{\alpha\lambda\mu\varepsilon} L_{\lambda\mu} K_\varepsilon\, \varepsilon^{\beta\eta\zeta\xi} L_{\eta\zeta} K_\xi$ of the Poincaré group.

In Parts II and III of this Thesis we make use of the representations of massive particles as described in (i) above. We quote the result as given by Cornwell [39] in terms of the four vector k^o that we are using here and also in terms of the alternative four vector p^o.

$$\Phi^{\hat{k}^o, j}([\Lambda|t|0])\,\phi_{k^o_i m} = \Phi^{\hat{p}^o j}([\Lambda|t|0])\,\phi_{p^o_i m}$$

$$= \exp i\{(k^o\Lambda^{-1})_\alpha t^\alpha_o\}\, D^j([B(k^o\Lambda^{-1}, \hat{k}^o)^{-1}\Lambda B(k^o, \hat{k}^o)|0|0|0])_{m'm}\, \phi_{k^o\Lambda^{-1}, m'} \quad \cdots(53)$$

$$= \exp \tfrac{i}{\hbar}\{(\Lambda p^o)_\alpha t^\alpha_o\}\, D^j([B(\Lambda p^o, \hat{p}^o)^{-1}\Lambda B(p^o, \hat{p}^o)|0|0|0])_{m'm}\, \phi_{\Lambda p^o, m'} \cdots(54)$$

Here D^j is the $(2j+1)\times(2j+1)$ dimensional representation of $SU(2)$ the covering group of $SO(3;\mathbb{R})$. The coset representative $B(k^o, \hat{k}^o)$ is the <u>inverse</u> of the coset representative $\Upsilon(k^o, \hat{k}^o)$ and is known as the 'Lorentz boost' from $\hat{k}^o$ to k^o. This may be chosen so that

$$B(k^o, \hat{k}^o)_{\lambda\mu} = \delta_{\lambda\mu} + \frac{(k^o_\lambda k^o_\mu)}{|k^o|^2}(\cosh\theta - 1)\ , \text{ for } \lambda, \mu = 1, 2, 3\ ;\cdots(55a)$$

$$B(k^o, \hat{k}^o)_{\lambda 4} = B(k^o, \hat{k}^o)_{4\lambda} = \frac{(k^o_\lambda)}{|k^o|}\sinh\theta\ , \text{ for } \lambda = 1, 2, 3 \cdots(55b)$$

and $$B(k^o, \hat{k}^o)_{44} = \cosh\theta \cdots(55c)$$

with $|k^o| = ((k^o_1)^2 + (k^o_2)^2 + (k^o_3)^2)^{\frac{1}{2}}$, $\sinh\theta = \dfrac{|k^o|}{\hat{k}^o_4}$ and $\cosh\theta = \dfrac{(|k^o|^2 + (k^o_4)^2)^{\frac{1}{2}}}{\hat{k}^o_4}$.

We also note that $(k^{o}\Lambda^{-1})_{\alpha} = (\Lambda k^{o})_{\alpha} = \frac{1}{\hbar}(\Lambda p^{o})_{\alpha}$, this was used to obtain equation (56).

Now we want to construct a representation of $\mathcal{G}_L$ from any representation of $\mathcal{G}_0$. Consider first inducing to a representation of $\mathcal{G}_1 = \mathcal{G}_0 Ⓢ \mathcal{G}_1/\mathcal{G}_0$ we choose the character of $\mathcal{G}_1/\mathcal{G}_0$ defined by $\hat{\phi}' = (0, 0, 0, 0)$ and the representation of $\mathcal{G}_0$ given by (say) $\Phi_{\mathcal{G}_0}([\Lambda|t|0])$. The orbit consists of a single element and the coset representative is thus $\Upsilon(\phi', \hat{\phi}') = I$ so that the operators of the induced representation are given by

$$\Phi_{\mathcal{G}_1}([\Lambda|t|\tau]) = \Phi_{\mathcal{G}_0}([\Lambda|t|0]).$$

This argument can be repeated at each step of the sequence

$$\mathcal{G}_0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \cdots \mathcal{G}_L$$

so that the operators $\Phi_{\mathcal{G}_L}([\Lambda|t|\tau])$ of a representation of $\mathcal{G}_L$ obtained by inducing from a representation $\Phi_{\mathcal{G}_0}([\Lambda|t|0])$ of $\mathcal{G}_0$ are given by

$$\Phi_{\mathcal{G}_L}([\Lambda|t|\tau]) = \Phi_{\mathcal{G}_0}([\Lambda|t|0]). \qquad \cdots(56)$$

A consequence of this is that any operator corresponding to a group transformation of $\mathcal{G}_L/\mathcal{G}_0$ leaves the state vectors unchanged. Thus the eigenvalues of the operators $\varepsilon_i K_\sigma$, $\varepsilon_i L_{\mu\nu}$ and $f_j Q_\alpha$ for $i = 1, 2, \cdots, \mathcal{N}-1$ and $j = 1, 2, \cdots, \mathcal{N}$ are zero. In particular for the representation given by equation (56) we have

$$\left.\begin{aligned} \varepsilon_i K_\sigma \, \phi_{k,m} &= 0 \qquad \cdots(57) \\ \varepsilon_i L_{\mu\nu} \, \phi_{k,m} &= 0 \qquad \cdots(58) \end{aligned}\right\} \text{for } i = 1, 2, \cdots, \mathcal{N}-1$$

and

$$f_j \, Q_\alpha \, \phi_{k,m} = 0 \quad \text{for } j = 1, 2, \cdots, \mathcal{N} \qquad \cdots(59)$$

for each $\sigma, \mu, \nu, \alpha = 1, 2, 3, 4$.

4.3.3 The Representations Obtained in the Step $\mathcal{G}_0 \to \mathcal{G}_1$.

A basis of the Lie algebra of $\mathcal{G}_0$ is

$$\{\varepsilon_0 L_{\mu\nu}, \varepsilon_0 K_\sigma\} \qquad \cdots(60)$$

and a basis of the Lie algebra of $\mathcal{G}_1/\mathcal{G}_0$ is

$$\{\varepsilon_1 Q_\alpha\} \qquad \cdots\cdots(61)$$

The action of the Lie algebra of $\mathcal{G}_0$ on the Lie algebra of $\mathcal{G}_1/\mathcal{G}_0$ is given by

$$[\varepsilon_0 L_{\mu\nu}, \varepsilon_1 Q_\alpha] = -\varepsilon_1 \tfrac{1}{2}(\gamma_\mu \gamma_\nu)_{\alpha\beta} Q_\beta \qquad \cdots\cdots(62)$$

and

$$[\varepsilon_0 K_\sigma, \varepsilon_1 Q_\alpha] = 0. \qquad \cdots\cdots(63)$$

We can deduce immediately that the translation subgroup generated by $\{\varepsilon_0 K_\sigma\}$ is part of the stability group for any character of $\mathcal{G}_1/\mathcal{G}_0$. Now consider equation (62) with $\alpha = 1$. Using the Majorana representation of the Dirac matrices given by equation (35) of the appendix we find

$$[\varepsilon_0 L_{12}, \varepsilon_1 Q_1] = -\tfrac{1}{2}\varepsilon_1 Q_4 \quad , \qquad \cdots\cdots(64a)$$

$$[\varepsilon_0 L_{13}, \varepsilon_1 Q_1] = \tfrac{1}{2}\varepsilon_1 Q_2 \quad , \qquad \cdots\cdots(64b)$$

$$[\varepsilon_0 L_{14}, \varepsilon_1 Q_1] = \tfrac{1}{2}\varepsilon_1 Q_4 \quad , \qquad \cdots\cdots(64c)$$

$$[\varepsilon_0 L_{23}, \varepsilon_1 Q_1] = -\tfrac{1}{2}\varepsilon_1 Q_3 \quad , \qquad \cdots\cdots(64d)$$

$$[\varepsilon_0 L_{24}, \varepsilon_1 Q_1] = \tfrac{1}{2}\varepsilon_1 Q_1 \qquad \cdots\cdots(64e)$$

and

$$[\varepsilon_0 L_{34}, \varepsilon_1 Q_1] = -\tfrac{1}{2}\varepsilon_1 Q_3 . \qquad \cdots\cdots(64f)$$

Thus

$$[(\varepsilon_0 L_{12} + \varepsilon_0 L_{14}), \varepsilon_1 Q_1] = 0 \qquad \cdots\cdots(65a)$$

and

$$[(\varepsilon_0 L_{23} - \varepsilon_0 L_{34}), \varepsilon_1 Q_1] = 0. \qquad \cdots\cdots(65b)$$

So that the group generated by

$$\{(\varepsilon_0 L_{12} + \varepsilon_0 L_{14}), (\varepsilon_0 L_{23} - \varepsilon_0 L_{34})\} \qquad \cdots\cdots(66)$$

leaves the character $\chi(0;0;\hat{\phi}'_1,0,0,0)$ of $\mathcal{G}_1/\mathcal{G}_0$ invariant. It is also clear from equations (68) that the orbit of the character corresponding to the choice of any non zero $\phi'_1 \in \mathbb{R}$ is the complete set of characters of $\mathcal{G}_1/\mathcal{G}_0$.

Thus if we set $\hat{\phi}'_1 = 1$ the representations of $\mathcal{G}_1$ are given by

$$(\Delta_{\mathcal{H}} \exp i\,\hat{\phi}'_1 . \theta) \uparrow \mathcal{G}_1 \qquad \cdots\cdots(67)$$

with $\Delta_{\mathcal{H}}$ some unitary irreducible representation of the group $\mathcal{H}$ generated by

$$\{(\varepsilon_o L_{12} + \varepsilon_o L_{14}),\ (\varepsilon_o L_{23} - \varepsilon_o L_{34}),\ \varepsilon_o K_\sigma\}. \qquad \cdots(68)$$

Thus before we can induce we need to determine the representations of $\mathcal{H}$.

The Lie algebra of $\mathcal{H}$ is given by

$$[(\varepsilon_o L_{12} + \varepsilon_o L_{14}),\ (\varepsilon_o L_{23} - \varepsilon_o L_{34})] = 0 \quad , \qquad \cdots(69a)$$

$$[(\varepsilon_o L_{12} + \varepsilon_o L_{14}),\ \varepsilon_o K_\sigma] = g_{1\sigma}\varepsilon_o K_2 - g_{2\sigma}\varepsilon_o K_1 + g_{1\sigma}\varepsilon_o K_4 - g_{4\sigma}\varepsilon_o K_1 \cdots(69b)$$

$$[(\varepsilon_o L_{23} - \varepsilon_o L_{34}),\ \varepsilon_o K_\sigma] = g_{2\sigma}\varepsilon_o K_3 - g_{3\sigma}\varepsilon_o K_2 - g_{3\sigma}\varepsilon_o K_4 + g_{4\sigma}\varepsilon_o K_3 \cdots(69c)$$

and $\quad [\varepsilon_o K_\sigma\ ,\ \varepsilon_o K_\rho] = 0.$

This algebra possesses the obvious semi-direct structure which we can write symbolically as $L \circledS K$ with $\{\varepsilon_o L \quad\}$ and $\{\varepsilon_o K_\sigma\}$ both generating abelian groups which we will call respectively $\mathcal{H}'$ and $\mathcal{N}'$. These will both have one dimensional representations given by their characters. To determine the nature of the group $\mathcal{H}'$ we exponentiate using the matrix expressions for $L_{\mu\nu}$ as given in equation (55) of Chapter 2. Thus $h \in \mathcal{H}'$ is given by

$$h = \exp a(\varepsilon_o L_{12} + \varepsilon_o L_{14}) \exp b(\varepsilon_o L_{23} - \varepsilon_o L_{34})$$

$$= \exp\begin{bmatrix} 0 & a & 0 & -a \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a & 0 & 0 & 0 \end{bmatrix} \exp\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & -b & 0 & b \\ 0 & 0 & b & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & a & 0 & -a \\ -a & 1-\frac{a^2}{2} & 0 & \frac{a^2}{2} \\ 0 & 0 & 1 & 0 \\ -a & -\frac{a^2}{2} & 0 & 1+\frac{a^2}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\frac{b^2}{2} & b & \frac{b^2}{2} \\ 0 & -b & 1 & b \\ 0 & -b^2 & b & 1+\frac{b^2}{2} \end{bmatrix}.$$

So that $\mathcal{H}'$ is isomorphic to $\mathbb{R}^2$. It thus has characters labeled by continuous parameters.

The two generators $(\varepsilon_o L_{12} + \varepsilon_o L_{14})$ and $(\varepsilon_o L_{23} - \varepsilon_o L_{34})$ allow us to construct four possible stability groups - each of which we need. It is convenient to deal with these in descending size order. We first note that the commutators of equations (69b) and (69c) can be rewritten as

$$[(\varepsilon_oL_{12}+\varepsilon_oL_{14}), \begin{bmatrix}\varepsilon_oK_2+\varepsilon_oK_4\\ \varepsilon_oK_3\\ \varepsilon_oK_1\\ \varepsilon_oK_2-\varepsilon_oK_4\end{bmatrix}_\sigma] = \begin{bmatrix}0\\ 0\\ -(\varepsilon_oK_2+\varepsilon_oK_4)\\ 2\varepsilon_oK_1\end{bmatrix}_\sigma \quad \cdots(70)$$

and

$$[(\varepsilon_oL_{23}-\varepsilon_oL_{34}), \begin{bmatrix}\varepsilon_oK_2+\varepsilon_oK_4\\ \varepsilon_oK_3\\ \varepsilon_oK_1\\ \varepsilon_oK_2-\varepsilon_oK_4\end{bmatrix}_\sigma] \begin{bmatrix}0\\ \varepsilon_oK_2+\varepsilon_oK_4\\ 0\\ -2\varepsilon_oK_3\end{bmatrix}_\sigma . \quad \cdots(71)$$

Case (i) Stability group $= \mathcal{H}'$.

Since both generators of $\mathcal{H}'$ commute with $(\varepsilon_oK_2+\varepsilon_oK_4)$ we can identify the character of $\mathcal{N}'$ corresponding to this choice of stability group as

$$\exp i\,\hat{k}^o.t = \exp i(0,\hat{k}^o_2,0,\hat{k}^o_2).t \quad \cdots(72)$$

for any $k^o_2 \in \mathbb{R}$.

The representations of $\mathcal{H}'$ are given by

$$\exp i\,\hat{\ell}^o.y$$
$$= \exp i(\ell^o_{12},0,c^o_{12},c^o_{23},0,-c^o_{23}).(y^{12}_o,y^{13}_o,y^{14}_o,y^{23}_o,y^{24}_o,y^{34}_o) \cdots(73)$$

for $\ell^o_{12}, \ell^o_{23} \in \mathbb{R}$, $c^o_{12}, c^o_{23} \neq 0$.

Thus we obtain the one dimensional representations of $\mathcal{H}$ given by

$$\Delta_{\mathcal{H}} = \exp i\,\hat{\ell}^o.y \ \exp i\hat{k}^o.t \quad \cdots(74)$$

which are labeled by three real numbers. The corresponding representations of $\mathcal{G}_1$ are given by equation (67) as

$$(\exp i\,\hat{\ell}^o.y \ \exp i\hat{k}^o.t \ \exp i\,\hat{\phi}'.\tau)\uparrow \mathcal{G}_1 \quad \cdots(75)$$

and we can use equation (38) to obtain these explicitly. Since these do not seem to have any physical significance we do not carry out the induction.

Case (ii) Stability group generated by $(\varepsilon_oL_{12}+\varepsilon_oL_{14})$.

From equation (70) we can see that this generator commutes with

$\{\varepsilon_0 K_3, (\varepsilon_0 K_2 + \varepsilon_0 K_4)\}$, and from equation (71) we see that

$$[(\varepsilon_0 L_{23} - \varepsilon_0 L_{34}), \varepsilon_0 K_3] = \varepsilon_0 K_2 + \varepsilon_0 K_4.$$

Thus the orbit of any fixed element of $\hat{n}$, $\hat{k}^o = (0, 0, \hat{k}^o_3, 0)$ consists of all elements of the form $(0, a, \hat{k}^o_3, a)$ for $a \in \mathbb{R}$. To obtain all possible characters we need to allow $\hat{k}^o_3$ to range over all non zero values ($\hat{k}^o_3 = 0$ is covered by Case (i) above).

The representations of the stability subgroup of $\mathcal{H}$ are also given by characters of the form

$$\exp i\,\hat{\ell}^o . y \quad \text{with} \quad \hat{\ell}^o = (\hat{\ell}^o_{12}, 0, \hat{\ell}^o_{12}, 0, 0, 0), \hat{\ell}^o_{12} \in \mathbb{R}.$$

The representations of $\mathcal{H}$ can now be obtained by induction ie.

$$\Delta_{\mathcal{H}} = (\exp i\,\hat{\ell}^o . y \, \exp i\,\hat{k}^o . t) \uparrow \mathcal{H}. \qquad \ldots(76)$$

This will then be an infinite dimensional representation of $\mathcal{H}$ labeled by two real numbers, one of which must be non zero.

Case (iii) Stability group generated by $(\varepsilon_0 L_{23} - \varepsilon_0 L_{34})$.

From equation (71) we see that this generator commutes with $\{\varepsilon_0 K_1, \varepsilon_0 K_2 + \varepsilon_0 K_4\}$ and from equation (70) we see that

$$[(\varepsilon_0 L_{12} + \varepsilon_0 L_{14}), \varepsilon_0 K_1] = -(\varepsilon_0 K_2 + \varepsilon_0 K_4).$$

Thus the orbit of any fixed element of $\hat{n}'$, $\hat{k}^o = (k^o_1, 0, 0, 0)$ consists of all elements of the form $(\hat{k}'_0, a, 0, a)$, for $a \in \mathbb{R}$. To obtain all possible characters we need to allow $\hat{k}'_0$ to range over all non zero values ($k'_0 = 0$ is covered by Case (i) above). Then, as in Case (ii), we find that to obtain the representations of $\mathcal{H}$ we have to induce, to obtain

$$\Delta_{\mathcal{H}} = (\exp i\,\hat{\ell}^o . y \, \exp i\,\hat{k}^o . t) \uparrow \mathcal{H} \qquad \ldots(77)$$

for $\hat{\ell}^o = (0, 0, 0, \hat{\ell}^o_{23}, 0, \hat{\ell}^o_{23})$, $\hat{k}^o = (k^o_1, 0, 0, 0)$ and $\ell^o_{23}, k^o_1 \in \mathbb{R}$ $\hat{k}^o_1 \neq 0$. That is the representations are labeled by two real numbers.

Case (iv) Stability group is the identity of $\mathcal{H}'$.

This case includes all the characters we have not covered in Cases (i) -

(iii). We can identify the following possibilities (a) $\hat{k}^o = (\hat{k}_1^o, 0, \hat{k}_3^o, 0)$, (b) $\hat{k}^o = (\hat{k}_1^o, \hat{k}_2^o, 0, -\hat{k}_2^o)$, (c) $\hat{k}^o = (0, \hat{k}_2^o, \hat{k}_3^o, -\hat{k}_2^o)$, (d) $\hat{k}^o = (0, \hat{k}_2^o, 0, \hat{k}_4^o)$, (e) $\hat{k}^o = (0, \hat{k}_2^o, 0, -\hat{k}_2^o)$ or (f) $\hat{k}^o = (\hat{k}_1^o, \hat{k}_2^o, \hat{k}_3^o, -\hat{k}_2^o)$.
Here $\hat{k}_1^o, \hat{k}_2^o, \hat{k}_3^o$ and $\hat{k}_4^o$ are real numbers with all of them non zero.

For each possibility we obtain a one dimensional representation of $\mathcal{N}'$ given by the character

$$\exp i \hat{k}^o . x .$$

The representations of $\mathcal{H}$ are then obtained by induction

$$\Delta_{\mathcal{H}} = (\exp i \hat{k}^o . x) \uparrow \mathcal{H} \qquad \cdots (78)$$

and each representation is labeled by (at most) three real non zero numbers.

What is clear from this analysis is that any representation obtained in the step $\mathcal{G}_0 \to \mathcal{G}_1$ is labeled by a set of continuous parameters. Non of these representations has an integer label that we can associate with particle spin.

4.3.4 The Representations Obtained in the Step $\mathcal{G}_1 \to \mathcal{G}_2$

A basis of the Lie algebra of $\mathcal{G}_1$ is

$$\{\varepsilon_0 L_{\mu\nu}, \varepsilon_0 K_\sigma, \varepsilon_1 Q_\alpha\} \qquad \cdots (79)$$

and a basis of the Lie algebra of $\mathcal{G}_2/\mathcal{G}_1$ is

$$\{\varepsilon_{1\wedge 2} L_{\mu\nu}, \varepsilon_{1\wedge 2} K_\sigma, \varepsilon_2 Q_\alpha\} . \qquad \cdots (80)$$

The action of the Lie algebra of $\mathcal{G}_1$ on the Lie algebra of $\mathcal{G}_2/\mathcal{G}_1$ is given by

$$[\varepsilon_0 L_{\lambda\mu}, \varepsilon_{1\wedge 2} L_{\sigma\rho}] = g_{\lambda\sigma} \varepsilon_{1\wedge 2} L_{\mu\rho} - g_{\lambda\rho} \varepsilon_{1\wedge 2} L_{\mu\sigma} - g_{\mu\sigma} \varepsilon_{1\wedge 2} L_{\lambda\rho} + g_{\mu\rho} \varepsilon_{1\wedge 2} L_{\lambda\sigma}, \qquad \cdots (81a)$$

$$[\varepsilon_0 K_\sigma, \varepsilon_{1\wedge 2} L_{\lambda\mu}] = g_{\mu\sigma} \varepsilon_{1\wedge 2} K_\lambda - g_{\lambda\sigma} \varepsilon_{1\wedge 2} K_\mu, \qquad \cdots (81b)$$

$$[\varepsilon_1 Q_\alpha, \varepsilon_{1\wedge 2} L_{\lambda\mu}] = 0 \quad , \qquad \cdots (81c)$$

$$[\varepsilon_0 L_{\lambda\mu}\,,\ \varepsilon_{1\wedge 2} K_\sigma] = g_{\lambda\sigma}\varepsilon_{1\wedge 2} K_\mu - g_{\mu\sigma}\varepsilon_{1\wedge 2} K_\lambda\,, \quad \text{.....(81d)}$$

$$[\varepsilon_0 K_\sigma\,,\ \varepsilon_{1\wedge 2} K_\rho] = 0\,, \quad \text{.....(81e)}$$

$$[\varepsilon_1 Q_\alpha\,,\ \varepsilon_{1\wedge 2} K_\sigma] = 0\,, \quad \text{.....(81f)}$$

$$[\varepsilon_0 L_{\lambda\mu}\,,\ \varepsilon_2 Q_\alpha] = -\tfrac{1}{2}(\gamma_\lambda\gamma_\mu)_{\alpha\beta}\,\varepsilon_2 Q_\beta\,, \quad \text{.....(81g)}$$

$$[\varepsilon_0 K_\sigma\,,\ \varepsilon_2 Q_\alpha] = 0\,, \quad \text{.....(81h)}$$

and

$$[\varepsilon_1 Q_\alpha\,,\ \varepsilon_2 Q_\beta] = -(\gamma^\sigma C)_{\alpha\beta}\,\varepsilon_{1\wedge 2} K_\sigma. \quad \text{.....(81j)}$$

The characters of $\mathcal{G}_2/\mathcal{G}_1$ are in one to one correspondence to the vector $(\ell^{1\wedge 2}_{\mu\nu}, k^{1\wedge 2}_\sigma, \phi^2_\alpha)$. It is convenient, first, to deal with the three cases such that the representative character in a given orbit is chosen to be of the form

(i) $\ell^{1\wedge 2}_{\mu\nu} = 0\,,\ \phi^2_\alpha = 0$ for each μ, ν, α ,

(ii) $\ell^{1\wedge 2}_{\mu\nu} = 0\,,\ k^{1\wedge 2}_\sigma = 0$ for each μ, ν, σ ,

and (iii) $k^{1\wedge 2}_\sigma = 0\,,\ \phi^2_\alpha = 0$ for each σ, α .

Case (i) Representative character : $\chi_{(0,\hat{k},0)}$.

We can see from equations (81e) and (81f) that the stability subgroup generators must include $\{\varepsilon_1 Q_\alpha, \varepsilon_0 K_\sigma\}$ and that, depending on our choice of character, it will also contain some subgroup of the Lorentz group generated by $\{\varepsilon_0 L_{\lambda\mu}\}$. In fact we can recognise that the possibilities are the same as for the representations given in section 4.3.2 for $\mathcal{G}_0$. That is we have five possible types of orbit for which we can choose the representative characters (i) $\hat{k}^{1\wedge 2} = (0, 0, 0, \hat{k}^{1\wedge 2}_4)$, with $\hat{k}^{1\wedge 2}_4 > 0$, (ii) $\hat{k}^{1\wedge 2} = (0, 0, 0, \hat{k}^{1\wedge 2}_4)$ with $\hat{k}^{1\wedge 2}_4 < 0$, (iii) $\hat{k}^{1\wedge 2} = (\hat{k}^{1\wedge 2}_1, 0, 0, 0)$ with $\hat{k}^{1\wedge 2} \in \mathbb{R} - \{0\}$, (iv) $\hat{k}^{1\wedge 2} = (0, 0, \hat{k}^{1\wedge 2}_4, \hat{k}^{1\wedge 2}_4)$ with $\hat{k}^{1\wedge 2}_4 > 0$ or (v) $\hat{k}^{1\wedge 2} = (0, 0, \hat{k}^{1\wedge 2}_4, \hat{k}^{1\wedge 2}_4)$ with $\hat{k}^{1\wedge 2}_4 < 0$. We do not consider the possibility $\hat{k}^{1\wedge 2} \equiv 0$ since this gives us the representations of $\mathcal{G}_1$ that we already have.

For each of these five possibilities we have to consider representations of the appropriate stability group. This again is a semi-direct product of the form (Some subgroup of the Lorentz group)Ⓢ(The group generated by $\{\varepsilon_1 Q_\alpha, \varepsilon_0 K_\sigma\}$).

Now if we require that any representation we obtain is a candidate for the representation of a physical particle, we need the representation to be labeled by a number that takes integer or half integer values, which we can associate with the spin of the particle. This implies that the stability group must contain a representation of a covering group of a rotation group in two or three dimensions. The obvious candidates then are type (i) and type (ii) above.

Type (i)

The stability group is generated by

$$\{\varepsilon_0 L_{ij}, \varepsilon_0 K_\sigma, \varepsilon_1 Q_\alpha\} \text{ with } i, j = 1, 2, 3. \qquad \cdots(82)$$

This admits the semi-direct structure $\mathcal{H}' \circledS \mathcal{N}'$ with $\mathcal{H}'$ generated by $\{\varepsilon_0 L_{ij}\}$ and $\mathcal{N}'$ generated by $\{\varepsilon_0 K_\sigma, \varepsilon_1 Q_\alpha\}$.

Now suppose we choose some character of $\mathcal{N}'$ such that $\hat{\phi}' \neq 0$ then the stability subgroup of $\mathcal{H}'$ will be generated by the intersection of $\{\varepsilon_0 L_{ij}\}$ and the set $\{(\varepsilon_0 L_{12} + \varepsilon_0 L_{14}), (\varepsilon_0 L_{23} - \varepsilon_0 L_{34})\}$. Thus for this character choice the stability subgroup of $\mathcal{H}'$ is just the identity. The resulting induced representations will not be labeled by any integer or half-integer valued parameter. Thus we conclude that we require $\hat{\phi}' \equiv 0$.

Now consider the choice of a character such that $\hat{k}^0 \neq 0$. The obvious choice is such that we do not alter the stability group. So that we put

$$\hat{k}^0 = (0, 0, 0, \hat{k}^0_4) \text{ with } \hat{k}^0_4 > 0.$$

The representations of $\mathcal{H}$ are then given by equation (57), which we can then induce to a representation of $\mathcal{G}_2$. Alternatively we can recognise that the result can be obtained in one step by considering the character $\chi_{(0,\hat{k},0)}$ with $\hat{k} = (0,0,0,\hat{k}_4^0; 0,0,0,\hat{k}_4^{1\wedge 2})$. The result is then the same as equations (57) and (59) with k^0 and $\hat{k}^0$ replaced by $k = k^0\varepsilon_0 + k^{1\wedge 2}\varepsilon_{1\wedge 2}$ and $\hat{k} = (0,0,0,k^0\varepsilon_0 + k^{1\wedge 2}\varepsilon_{1\wedge 2})$ respectively, $\Lambda \in \overline{SO}_0(3,1;E_2)$ and $\phi_{k^0,m}$ replaced by $\phi_{k,m}$. Specifically we have

$$\Phi^{\hat{k},j}([\Lambda|t|\tau])\,\phi_{k,m}$$
$$= \exp i\{(k\Lambda^{-1})_\alpha t_0^\alpha\}\, D^j([B(k\Lambda^{-1},\hat{k})^{-1}\Lambda B(k,\hat{k})|0|0])_{m'm}\,\phi_{k\Lambda^{-1},m'} . \qquad \cdots(83)$$

We note that this formula is valid for all $\hat{k}$ such that $\hat{k}_4^0 \geqslant 0$ and $\hat{k}_4^{1\wedge 2} \in \mathbb{R}$ with at least one of them non zero, it then subsumes the representations given by equation (53). The representation is complex valued even though it is written in terms of Grassman parameters, we have 'encoded' $\mathcal{G}_2$ in Grassman form. The eigenvalues of the translation generators are

$$\varepsilon_0 K_\sigma\, \phi_{k,m} = i\, k^0_\sigma\, \phi_{k,m} \qquad \cdots(84a)$$

and

$$\varepsilon_{1\wedge 2} K_\sigma\, \phi_{k,m} = i\, k^{1\wedge 2}_\sigma\, \phi_{k,m} . \qquad \cdots(84b)$$

This representation can now be induced to a representation of $\mathcal{G}_L$, in which case in addition to equations (84a) and (84b) we have

$$\varepsilon_i K_\sigma\, \phi_{k,m} = 0 \quad \text{for} \quad i = 1,2,\cdots,N-1,\ \varepsilon_i \neq \varepsilon_{1\wedge 2} \cdots(84c)$$

We observe that it is also a representation of $ISO(3,1;E_L)$.

Type (iv)

The arguments given above for type (i) can be repeated, we obtain 'massless' type representations with a spin index. We do not consider these.

Any other choice of $\hat{k}^0$, for either of these types of representation, will result in a reduction of the 'Lorentz' part of the stability group of $\hat{k}^{1\wedge 2}$ We do not consider these.

Case (ii) Representative character $\chi_{(0,0,q)}$.
We can see from equation (81h) that the generators of the stability group contain $\{\varepsilon_0 K_\sigma\}$ and from equation (81j) we see that $\{\varepsilon_1 Q_\alpha\}$ are excluded as generators of the stability group. The stability group generators contain some subset of $\{\varepsilon_0 L_{\mu\nu}\}$. In fact we can see that we have a repeat of the arguments of section 4.3.3. The representations we obtain will be a copy of those cited there. Since these are labeled by a set of continuous parameters non of them are suitable for representing particles with spin.

Case (iii) Representative character $\chi_{(\ell,0,0)}$.
We can see from equation (81c)that the stability group generators must include $\{\varepsilon_1 Q_\alpha\}$ and equation (81b) excludes $\{\varepsilon_0 K_\sigma\}$ from being generators of the stability group.

The stability group is thus generated by some subset of the generators $\{\varepsilon_0 L_{\mu\nu}\}$ and by $\{\varepsilon_1 Q_\alpha\}$. If we choose a representation of the stability group to be such that $\hat{q}' \not\equiv 0$ then any representation we obtain will be labeled by continuous parameters. We thus choose $\hat{q}' \equiv 0$ and examine the possible subsets of $\{\varepsilon_0 L_{\mu\nu}\}$ that can generate stability groups.

Recall that the action of the Lorentz group on the characters $\{\ell^{1\wedge 2}_{\mu\nu}\}$ is given by the adjoint representation. A matrix representation of $\mathrm{ad}(L_{\mu\nu})$ is given by equation (48) and the corresponding six-vector constructed from $\ell^{1\wedge 2}_{\mu\nu}$ is given by equation (49). We clearly need only consider one of the 3×3 sub-representations of the adjoint representation. We choose to use the top left hand one acting on the vector $(r_1^{1\wedge 2}, r_2^{1\wedge 2}, r_3^{1\wedge 2}) = r^{1\wedge 2}$ with $r_i^{1\wedge 2} \in \mathbb{C}$ for $i = 1,2,3$. This representation is isomorphic to $SO(3,\mathbb{C})$ considered as a real (six parameter) Lie group. This leaves invariant the quadratic form

$$z^2 = (r_1^{1\wedge 2})^2 + (r_2^{1\wedge 2})^2 + (r_3^{1\wedge 2})^2, \quad z \in \mathbb{C}. \qquad \cdots(85)$$

The parameter z then serves as a label for the representations (except for the case when $r_1^{1\wedge 2} = r_2^{1\wedge 2} = r_3^{1\wedge 2} = 0$ which we ignore since it just gives us the representations of $\mathcal{G}_1$). We can identify two types of orbit.

Type (i) $z = 0$.

We choose as representative character $r = (0, i, 1)$. The stability group for this character is then the abelian group generated by $\{(\xi_0 L_{12} - \xi_0 L_{14}), (\xi_0 L_{23} + \xi_0 L_{34})\}$. We note that in the representation defined by equation (48)

$$(\xi_0 L_{12} - \xi_0 L_{14}) = i(\xi_0 L_{23} + \xi_0 L_{34}).$$

To examine the structure of the stability group we exponentiate

$$\exp a(\xi_0 L_{12} - \xi_0 L_{14})$$

$$= \begin{bmatrix} 1 & a & ia \\ -a & 1 - \frac{a^2}{2} & \frac{ia^2}{2} \\ ia & \frac{ia^2}{2} & 1 + \frac{a^2}{2} \end{bmatrix}, \text{ with } a \in \mathbb{C}.$$

So that the stability group is isomorphic to $\mathbb{C}$ and has representations labeled by continuous parameters.

Type (ii) $z \in \mathbb{C}, z \neq 0$.

There are two possibilities for a representative character $\chi_{(r,0,0)}$ either $r = (0, 0, z)$ or $r = (0, 0, -z)$ in either case the stability group is isomorphic to $SO(2, \mathbb{C})$. So that, in this case, we do obtain a character label that takes integer values, ie. the parameter corresponding to the subgroup $SO(2, \mathbb{R})$.

When, in Part III, we come to consider superfields, we will be looking for fields that can be expanded in terms of the plane waves on superspace. Representations of the type described here are invariant on superspace.

These three cases cover a large number of the possible orbits in the step $\mathcal{G}_1 \to \mathcal{G}_2$ but not all of them. The others can be obtained by choosing character representatives of the form $\chi(\hat{\ell}, \hat{k}, \hat{\phi})$ with at least two of $\{\hat{\ell}, \hat{k}, \hat{\phi}\}$ non zero. The stability group in each of these cases will be the intersection of the stability groups of the appropriate characters of case (i), (ii) or (iii). What is clear is that very few of these representations will have a discrete label. For this reason we do not investigate them.

4.3.5 Final Remarks.

We have considered many of the representations of $\mathcal{G}_0, \mathcal{G}_1$ and $\mathcal{G}_2$. If we continued to examine $\mathcal{G}_3, \mathcal{G}_4, \cdots$ we would obtain a larger number of representations at each step, many of which would be similar to the ones we have already obtained and the majority would be parametrized by continuous parameters.

It is clear that any irreducible representation we obtain that can be associated with an elementary particle - ie. it has a spin index, will have only a single spin value. Thus non of these representations can be related directly to the Salam - Strathdee [8] representations.

The representations constructed here act on state vectors in a complex Hilbert space, but we can anticipate that any representation on a superspace will have some sort of Grassman structure, which is absent here. This suggests that we have to combine these representations in some way to give such a structure. It is not clear how to do this at the moment. This is why in the next part of this Thesis we approach the problem from the opposite side and construct the irreducible representations of the super Poincaré algebra. In Part III we are finally able to obtain the connection.

PART II

SUPER POINCARE INVARIANT THEORIES ON SPACE-TIME.

CHAPTER 1.

INTRODUCTION AND REVIEW.

1.1 Introduction.

In this part we examine some of the representations of the super Poincaré algebra. These have been described by Salam and Strathdee [8], [9] and this general description is given in most review articles on the subject (eg. Fayet and Ferrara [41]).

The method for producing the representations is based on the Wigner [54] method of induced representation, as was used in Chapter 4 of Part I. To make use of the theory in this instance it is re-formulated in terms of Lie algebras and then assumed to be applicable to Lie superalgebras. This certainly produces a set of representations, which have labels that can be associated with mass and spin, but we cannot be sure of obtaining all possible representations. It is also possible to construct Casimir operators which again can be used to classify the representations so obtained. As with the Poincaré group these do not give a unique classification but do give sufficient labels to differentiate the representations of physical interest.

In Chapter 2 we examine the representations of the super Poincaré algebra, primarily in terms of the 'little algebra', for the case of a massive particle. This extends all previous work in that although the mass/spin content of the representations is well known general expressions for the particle states in each representation have never been given.

In Chapter 3 we construct, using the theory of Chapter 2, second quantized fields. Our methodology in this chapter is based on the series of articles by Weinberg [20]. This approach has a number of advantages, the most important of which are:

(i) There is a precise meaning assigned to the effect of a group (or algebra) transformation on the fields. For instance a second quantized field transforming as a representation $\Gamma(\Lambda)$ of the Lorentz group has the transformation property

$$U([\Lambda|t])\,\psi_m(x)\,U([\Lambda|t])^{-1} = \Gamma(\Lambda^{-1})_{mm'}\,\psi_{m'}(\Lambda x + t)\,,$$

for each $[\Lambda|t]$ in the covering group of the Poincaré group. Whereas a classical field, as discussed by many authors, transforms as

$$([\Lambda|t])\,\psi_m(x) = \Gamma(\Lambda)_{mm'}\,\psi'_{m'}(\Lambda^{-1}x + t)\,,$$

with ψ' some other field.

(ii) We are able to construct supermultiplets of fields such that each field has $(2j+1)$ components, here j is the (integer or half integer) number specifying the spin of the field. These fields then form the building blocks for constructing the fields more usually encountered in supersymmetry theories.

(iii) We do not, at this stage, need to concern ourselves with a Lagrangian since fields constructed in this way oboy only the Klein-Gordon equation.

This approach has never previously been taken for supersymmetry theories.

In Chapter 4 we examine methods of constructing combinations of these chiral supermultiplets in such a way that the fields obey equations other than the Klein-Gordon equation. We need to do this in order to have fields that transform in a simple way under the action of the parity operator P. Thus we introduce additional field components in such a way that the required equation is satisfied. These additional field components are constructed from the same creation and annihilation operators as our original fields. In keeping with standard terminology we will call these auxiliary fields.

The need to introduce additional field components is not unique to supersymmetry as often claimed (cf. West [59]). It happens in many particle theories. An example is the two component photon field in quantum electrodynamics described by the vector field A_μ which has four degrees of freedom (components) consisting of (i) two physical degrees of freedom, (ii) one gauge degree of freedom and (iii) one 'auxiliary field' of spin zero normally eliminated by requiring $\partial^\mu A_\mu = 0$.

This has nothing to do with the particle in question being 'on mass shell' or 'off mass shell' since in all cases the field components satisfy the Klein-Gordon equation. The term 'off mass shell' implies that we do not satisfy the Klein-Gordon equation.
In Chapter 4 we also consider the construction of Lagrangian densities for supermultiplets that satisfy non-trivial wave equations. We end Chapter 4 by constructing a combination of two chiral supermultiplets that we will need in Chapter 4 of Part III. We complete this chapter with a review of the super Poincaré algebra in the form we need it in this part of the Thesis. This is additional to the review given in section 2.5 of Chapter 2 of Part I.

1.2 A Review of the Super Poincaré Algebra.

We gave a review of the super Poincaré algebras and the super Poincaré groups in Chapter 2 of Part I. Here we consider the four dimensional real super Poincaré algebra written in terms of the six Hermitian generators $M_{\lambda\mu} = \frac{\hbar}{i} L_{\lambda\mu}$, $\lambda, \mu = 1, 2, 3, 4$, $M_{\lambda\mu} = -M_{\mu\lambda}$; the four Hermitian generators $P_\sigma = \frac{\hbar}{i} K_\sigma$, $\sigma = 1, 2, 3, 4$ and the generators Q_α, $\alpha = 1, 2, 3, 4$. Note that it is the set $\{L_{\lambda\mu}, K_\sigma, Q_\alpha\}$ that generate the real Lie superalgebra. The commutators between $\{M_{\lambda\mu}, P_\sigma\}$ are

$$[M_{\lambda\mu}, M_{\alpha\beta}] = \frac{\hbar}{i}\{g_{\lambda\alpha} M_{\mu\beta} - g_{\lambda\beta} M_{\mu\alpha} - g_{\mu\alpha} M_{\lambda\beta} + g_{\mu\beta} M_{\lambda\alpha}\}, \quad \cdots(1)$$

$$[M_{\lambda\mu}, P_\alpha] = \frac{\hbar}{i}\{g_{\lambda\alpha} P_\mu - g_{\mu\alpha} P_\lambda\} \quad \cdots(2)$$

and $$[P_\alpha, P_\beta] = 0. \quad \cdots(3)$$

The set $\{Q_\alpha\}$ must form a representation space for a representation of L_0 which must, in terms of $M_{\lambda\mu}$ and P_σ, be purely imaginary. Thus

$$[M_{\lambda\mu}, Q_\alpha] = \Gamma(M_{\lambda\mu})_{\beta\alpha} Q_\beta \qquad \cdots(4)$$

and $$[P_\sigma, Q_\alpha] = \Gamma(P_\sigma)_{\beta\alpha} Q_\beta . \qquad \cdots(5)$$

Since $\{P_\sigma\}$ form an abelian Lie algebra $\Gamma(P_\sigma) = 0$. We are free to choose the representation $\Gamma(M_{\lambda\mu})$. Our choice is the four dimensional representation given by

$$\Gamma(M_{\lambda\mu})_{\alpha\beta} = \begin{cases} \frac{\hbar}{2i}(\gamma_\mu^{M^t}\gamma_\lambda^{M^t})_{\alpha\beta} & \text{if } \lambda \neq \mu \\ 0 & \text{if } \lambda = \mu . \end{cases} \qquad \cdots(6)$$

Here γ_μ^M are the Dirac matrices in the Majorana representation as given in equation (35) of the appendix. Then combining equations (4) and (6) we have

$$\begin{aligned} [M_{\lambda\mu}, Q_\alpha^M] &= \frac{\hbar}{2i}(\gamma_\mu^{M^t}\gamma_\lambda^{M^t})_{\beta\alpha} Q_\beta^M , \\ &= \frac{\hbar}{2i}(\gamma_\lambda^M \gamma_\mu^M)_{\alpha\beta} Q_\beta^M . \end{aligned} \qquad \cdots(7)$$

Here the superscript M on Q indicates that we are considering the Majorana representation. Note that in defining $\Gamma(M_{\lambda\mu})$ we have used the negative transpose of the representation generated by $\sigma_{\lambda\mu} = \frac{-1}{2}\gamma_\lambda\gamma_\mu$, this is so that our algebra agrees with that normally quoted in the literature (eg. [9]).

The anticommutator $[Q^M, Q^M]$ is determined by the graded Jacobi identity as given in section 1.2.3 of Chapter 1 of Part I as

$$[Q_\alpha^M, Q_\beta^M] = k\frac{i}{\hbar}(\gamma^{M\sigma}C^M)_{\alpha\beta} P_\sigma .$$

We choose the constant $k = 1$ to give

$$[Q_\alpha^M, Q_\beta^M] = \frac{i}{\hbar}(\gamma^{M\sigma}C^M)_{\alpha\beta} P_\sigma . \qquad \cdots(8)$$

This choice of k is determined by the fact that we will be dealing with second quantized fields, where we will require

$$[[Q_\alpha, Q_\beta], \chi] = (\gamma^\sigma C)_{\alpha\beta} \frac{\partial}{\partial x^\sigma} \chi$$

(see equations (67) and (68) of Chapter 3). If we had chosen to work with classical fields we would have been obliged to choose $k = -1$.

In the sequel we will need this algebra in two component form and also we have need of the adjoint of the operator Q_α in two component form. We put $Q^M_\alpha = e^{ia\pi} S_\alpha$ with $S_\alpha = S^\dagger_\alpha$. Then

$$
\begin{aligned}
([Q^M_\alpha , Q^M_\alpha])^\dagger &= 2e^{-i2a\pi} S_\alpha S_\alpha , \\
&= -\frac{i}{\hbar} P^\dagger_4 , \\
&= -\frac{i}{\hbar} P_4 , \\
&= -[Q_\alpha , Q_\alpha] , \\
&= -2e^{i2a\pi} S_\alpha S_\alpha .
\end{aligned}
$$

Thus $$a = -\frac{1}{4} ,$$

so that $$Q^{M\dagger}_\alpha = i Q^M_\alpha . \qquad \cdots(9)$$

Now if $(S^c_M)_{\alpha\beta}$ is the similarity transformation, given by equation (36) of the appendix, that links the chiral and Majorana representations of the Dirac matrices ie. it satisfies $S^c_M \gamma^M_\mu (S^c_M)^{-1} = \gamma^c_\mu$ then it is easy to verify from equation (8) that

$$Q^c_\alpha = (S^c_M)_{\alpha\beta} Q^M_\beta , \qquad \cdots(10)$$

where the superscript c indicates that the Q^c_α are the operators corresponding to the chiral Dirac matrices. Thus

$$
\begin{aligned}
Q^{c\dagger}_\alpha = (S^c_M)^*_{\alpha\beta} Q^{M\dagger}_\beta &= i (S^c_M)^*_{\alpha\beta} Q^M_\beta \\
&= i (S^c_M)^*_{\alpha\beta} ((S^c_M)^{-1})_{\beta\gamma} Q^c_\gamma .
\end{aligned}
$$

So that

$$Q^{c\dagger}_\alpha = i \begin{bmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{bmatrix}_{\alpha\gamma} Q^c_\gamma . \qquad \cdots(11)$$

Now we define

$$
\left.
\begin{aligned}
Q_{L\frac{1}{2}} &= Q^c_1 , \\
Q_{L-\frac{1}{2}} &= Q^c_2 , \\
Q_{R\frac{1}{2}} &= Q^c_3 \\
\text{and} \quad Q_{R-\frac{1}{2}} &= Q^c_4 .
\end{aligned}
\right\} \qquad \cdots(12)
$$

Here our notation reflects the fact that in the chiral representation of the Dirac matrices $\{Q_{L\frac{1}{2}} , Q_{L-\frac{1}{2}}\}$ form a basis of the irreducible representation $\Gamma^{0,\frac{1}{2}}$ of the covering group of the Lorentz group (the 'left handed' representation) and $\{Q_{R\frac{1}{2}}, Q_{R-\frac{1}{2}}\}$ form a basis of the irreducible representation $\Gamma^{\frac{1}{2},0}$ of the covering group of the Lorentz group (the 'right handed' representation). Thus

$$Q^{\dagger}_{Ln} = (i\sigma_2)_{nn'} Q_{Rn'} \quad \text{for} \quad n = \tfrac{1}{2}, -\tfrac{1}{2} \qquad \cdots(13)$$

and

$$Q^{\dagger}_{Rn} = -(i\sigma_2)_{nn'} Q_{Ln'} \quad \text{for} \quad n = \tfrac{1}{2}, +\tfrac{1}{2}. \qquad \cdots(14)$$

There are only four linearly independent operators in the set $\{Q_{Ln}, Q^{\dagger}_{Ln}, Q_{Rn}, Q^{\dagger}_{Rn}, n = \frac{1}{2}, -\frac{1}{2}\}$ but it proves convenient in the sequel to make use of all of them to cast our formulae in convenient forms.

Thus with these conventions we have the following two alternatives for equations (7) and (8):

(i) The left handed set

$$[M_{\lambda\mu}, Q_{Ln}] = \tfrac{\hbar}{2i}(\sigma^L_\lambda \sigma^R_\mu)_{nn'} Q_{Ln'} \quad , \qquad \cdots(15)$$

$$[M_{\lambda\mu}, Q^{\dagger}_{Ln}] = \tfrac{\hbar}{2i}(\sigma^R_2 \sigma^R_\lambda \sigma^L_\mu \sigma^L_2)_{nn'} Q^{\dagger}_{Ln'} \quad , \qquad \cdots(16)$$

$$[Q_{Ln}, Q_{Ln'}] = 0 \quad , \qquad \cdots(17)$$

$$[Q^{\dagger}_{Ln}, Q^{\dagger}_{Ln'}] = 0 \qquad \cdots(18)$$

and

$$[Q_{Ln}, Q^{\dagger}_{Ln'}] = \tfrac{1}{\hbar}(\sigma^L_\mu)_{nn'} P^\mu. \qquad \cdots(19)$$

(ii) The right handed set

$$[M_{\lambda\mu}, Q_{Rn}] = \tfrac{\hbar}{2i}(\sigma^R_\lambda \sigma^L_\mu)_{nn'} Q_{Rn'} \quad , \qquad \cdots(20)$$

$$[M_{\lambda\mu}, Q^{\dagger}_{Rn}] = \tfrac{\hbar}{2i}(\sigma^L_2 \sigma^L_\lambda \sigma^R_\mu \sigma^R_2)_{nn'} Q^{\dagger}_{Rn'} \quad , \qquad \cdots(21)$$

$$[Q_{Rn}, Q_{Rn'}] = 0 \quad , \qquad \cdots(22)$$

$$[Q^{\dagger}_{Rn}, Q^{\dagger}_{Rn'}] = 0 \qquad \cdots(23)$$

and

$$[Q_{Rn}, Q^{\dagger}_{Rn'}] = \tfrac{1}{\hbar}(\sigma^R_\mu)_{nn'} P^\mu. \qquad \cdots(24)$$

It is also convenient to have equations (13) and (18) written as

$$[Q_{Ln}, Q_{Rn'}] = -\tfrac{i}{\hbar}(\sigma^R_\mu \sigma^R_2)_{nn'} P_\mu \quad , \qquad \cdots(25)$$

$$= -\tfrac{i}{\hbar}(\sigma^L_\mu \sigma^L_2)_{n'n} P_\mu \quad . \qquad \cdots(26)$$

In these equations σ^L_μ and σ^R_μ are as defined by equations (3) and (4) respectively of the appendix. We note the symmetry in exchange of L and R that these equations possess.

In Chapter 2 we will find it convenient to use the operators J_+ , J_- and J_3 defined as follows

$$J_+ = M_{23} + i M_{31} , \qquad \cdots(27)$$

$$J_- = M_{23} - i M_{31} \qquad \cdots(28)$$

and

$$J_3 = M_{12} . \qquad \cdots(29)$$

Equation (15) can now be written

$$[J_+ , Q_{L\frac{1}{2}}] = -\hbar Q_{L-\frac{1}{2}} , \qquad \cdots(30a)$$

$$[J_+ , Q_{L-\frac{1}{2}}] = 0 , \qquad \cdots(30b)$$

$$[J_- , Q_{L\frac{1}{2}}] = 0 , \qquad \cdots(30c)$$

$$[J_- , Q_{L-\frac{1}{2}}] = -\hbar Q_{L\frac{1}{2}} , \qquad \cdots(30d)$$

$$[J_3 , Q_{L\frac{1}{2}}] = -\frac{\hbar}{2} Q_{L\frac{1}{2}} \qquad \cdots(30e)$$

and

$$[J_3 , Q_{L-\frac{1}{2}}] = \frac{\hbar}{2} Q_{L-\frac{1}{2}} . \qquad \cdots(30f)$$

We note that equations (30e) and (30f) can be written as

$$[J_3 , Q_{Ln}] = -\frac{\hbar}{2} (\sigma_3^L)_{nn'} Q_{Ln'} \qquad \cdots(31)$$

Equation (16) can now be written

$$[J_+ , Q^\dagger_{L\frac{1}{2}}] = 0 , \qquad \cdots(32a)$$

$$[J_+ , Q^\dagger_{L-\frac{1}{2}}] = \hbar Q^\dagger_{L\frac{1}{2}} , \qquad \cdots(32b)$$

$$[J_- , Q^\dagger_{L\frac{1}{2}}] = \hbar Q^\dagger_{L-\frac{1}{2}} , \qquad \cdots(32c)$$

$$[J_- , Q^\dagger_{L-\frac{1}{2}}] = 0 , \qquad \cdots(32d)$$

$$[J_3 , Q^\dagger_{L\frac{1}{2}}] = \frac{\hbar}{2} Q^\dagger_{L\frac{1}{2}} \qquad \cdots(32e)$$

and

$$[J_3 , Q^\dagger_{L-\frac{1}{2}}] = -\frac{\hbar}{2} Q^\dagger_{L-\frac{1}{2}} . \qquad \cdots(32f)$$

We note that equations (32e) and (32f) can be written as

$$[J_3 , Q^\dagger_{Ln}] = \frac{\hbar}{2} (\sigma_3^L)_{nn'} Q^\dagger_{Ln'} . \qquad \cdots(33)$$

CHAPTER 2

THE IRREDUCIBLE REPRESENTATIONS OF THE SUPER POINCARE ALGEBRA (THE LITTLE ALGEBRA).

In this chapter we construct the representations of the super Poincaré algebra using the method of induced representations formulated in terms of the superalgebra. This is different from Chapter 4 of Part I where the representations were in terms of the Lie group equivalent to the Lie supergroup. We have to do this since on space-time alone we do not have the concept of a supergroup. A review of this procedure for the Poincaré Lie algebra is given by Weinberg [60] .

Firstly we observe that the operator $P_\sigma P^\sigma$ commutes with every generator of the superalgebra so that its eigenvalues serve as a label for the representations we obtain. As usual we denote this eigenvalue by M^2c^2, where c is the velocity of light and M is interpreted as the rest mass of the particle. Here we consider only the possibility $M > 0$.

We now choose a particle state and look for the 'little superalgebra' that leaves this state invariant. As with the Poincaré group it is convenient to choose the rest state ie. a particle with four-momentum $(0, 0, 0, Mc)$. The 'little superalgebra' is then spanned by

$$\{M_{ij}\ ;\ i,j = 1,2,3\ ;\ Q_{Ln}\ ,\ Q^\dagger_{Ln}\ ;\ n = \tfrac{1}{2}, -\tfrac{1}{2}\}, \qquad \cdots(1)$$

This corresponds to the stability subgroup of Chapter 4 of Part I. We note that it is not a superalgebra in its own right since the commutator of the Q's gives a P. This suggests that there are in fact other representations, but we do not pursue this point.

Now let $|Mc, j, m\rangle$ with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$ and $m = j, j-1, \cdots, -j+1, -j$, be an eigenvector of J_3, P_σ and J^2 such that

$$J^2 |Mc, j, m\rangle = \hbar^2 j(j+1) |Mc, j, m\rangle, \qquad \cdots(2)$$

$$J_3 |Mc, j, m\rangle = \hbar m |Mc, j, m\rangle, \qquad \cdots(3)$$

$$P_i |Mc, j, m\rangle = 0 \quad \text{for } i = 1, 2, 3 \qquad \cdots(4)$$

and

$$P_4 |Mc, j, m\rangle = Mc |Mc, j, m\rangle. \qquad \cdots(5)$$

Here $J^2 = M_{12}M_{12} + M_{13}M_{13} + M_{23}M_{23}$,

$$= J_3J_3 + \hbar J_3 + J_-J_+ \; . \qquad \text{....(6)}$$

That is we are insisting that our representation includes a particle of spin j at rest. Our conventions for angular momentum operators are as given by Cornwell [39] Chapter 12. We also suppose that

$$Q_{Ln}|Mc, j, m\rangle = 0. \qquad \text{....(7)}$$

Lemma 2.1

If the representation incudes a particle of spin j then it also includes a particle of spin $j+\frac{1}{2}$, and if $j \geqslant \frac{1}{2}$ a particle of spin $j-\frac{1}{2}$.

Proof

We consider the particle state $Q^{\dagger}_{Ln}|Mc, j, m\rangle$, which we assume is a basis state of the representation. Then

$$\begin{aligned} &J_3 Q^{\dagger}_{Ln}|Mc, j, m\rangle \\ &\quad = ([J_3, Q^{\dagger}_{Ln}] + Q^{\dagger}_{Ln}J_3)|Mc, j, m\rangle \; , \\ &\quad = (\tfrac{\hbar}{2}(\sigma_3)_{nn'} + \hbar m\,\delta_{nn'})|Mc, j, m\rangle \; , \qquad \text{.....(8)} \end{aligned}$$

where we have used equation (33) of Chapter 1,

$$\begin{aligned} &J_+ Q^{\dagger}_{Ln}|Mc, j, m\rangle \\ &\quad = ([J_+, Q^{\dagger}_{Ln}] + Q^{\dagger}_{Ln}J_+)|Mc, j, m\rangle \; , \\ &\quad = \hbar\{(j-m)(j+m+1)\}^{\frac{1}{2}} Q_{Ln}{}^{\dagger}|Mc, j, m+1\rangle \\ &\qquad\qquad + [J_+, Q^{\dagger}_{Ln}]|Mc, j, m\rangle \qquad \text{....(9)} \end{aligned}$$

and

$$\begin{aligned} &J_- Q^{\dagger}_{Ln}|Mc, j, m\rangle \\ &\quad = ([J_-, Q^{\dagger}_{Ln}] + Q^{\dagger}_{Ln}J_-)|Mc, j, m\rangle \; , \\ &\quad = \hbar\{(j+m)(j-m+1)\}^{\frac{1}{2}} Q^{\dagger}_{Ln}|Mc, j, m-1\rangle \\ &\qquad\qquad + [J_-, Q^{\dagger}_{Ln}]|Mc, j, m\rangle. \qquad \text{....(10)} \end{aligned}$$

In particular consider the state $Q^{\dagger}_{L\frac{1}{2}}|M, j, j\rangle$ for which

$$J_3 Q^{\dagger}_{L\frac{1}{2}}|Mc, j, j\rangle = \hbar(m+\tfrac{1}{2})Q^{\dagger}_{L\frac{1}{2}}|Mc, j, j\rangle \qquad \text{....(11)}$$

and

$$\begin{aligned} J^2 Q^{\dagger}_{L\frac{1}{2}}|Mc, j, j\rangle &= (J_3J_3 + \hbar J_3 + J_-J_+)Q^{\dagger}_{L\frac{1}{2}}|Mc, j, j\rangle, \\ &= \hbar^2\{(j+\tfrac{1}{2})+1\}\{j+\tfrac{1}{2}\}Q^{\dagger}_{L\frac{1}{2}}|Mc, j, j\rangle \qquad \text{....(12)} \end{aligned}$$

Now if we apply the operator J_- successively to $Q^{\dagger}_{L\frac{1}{2}}|M, j, j\rangle$ we can conclude that these represent the various states of a particle of spin $j+\frac{1}{2}$.

Now consider the state $Q^{+}_{L-\frac{1}{2}}|Mc,j,j\rangle$

$$J_3 Q^{+}_{L-\frac{1}{2}}|Mc,j,j\rangle = \hbar(j-\tfrac{1}{2})Q^{+}_{L-\frac{1}{2}}|Mc,j,j\rangle, \qquad \cdots(13)$$

but this state is not an eigenvector or J^2 unless $j=0$. Consider also the state $Q^{+}_{L\frac{1}{2}}|Mc,j,j-1\rangle$

$$J_3 Q^{+}_{L\frac{1}{2}}|Mc,j,j-1\rangle = \hbar(j-\tfrac{1}{2})Q^{+}_{L\frac{1}{2}}|Mc,j,j\rangle, \qquad \cdots(14)$$

which again is not an eigenvector of J^2 unless $j=0$, but we can consider the state

$$\alpha Q^{+}_{L-\frac{1}{2}}|Mc,j,j\rangle + \beta Q^{+}_{L\frac{1}{2}}|Mc,j,j-1\rangle, \qquad \cdots(15)$$

with α and β two complex numbers chosen so that

$$J_+(\alpha Q^{+}_{L-\frac{1}{2}}|Mc,j,j\rangle + \beta Q^{+}_{L\frac{1}{2}}|Mc,j,j-1\rangle) = 0. \qquad \cdots(16)$$

Now

$$\begin{aligned} &J^2(\alpha Q^{+}_{L-\frac{1}{2}}|Mc,j,j\rangle + \beta Q^{+}_{L\frac{1}{2}}|Mc,j,j-1\rangle) \\ &= (J_3J_3 + \hbar J_3)(\alpha Q^{+}_{L-\frac{1}{2}}|Mc,j,j\rangle + \beta Q^{+}_{L\frac{1}{2}}|Mc,j,j-1\rangle), \\ &= \hbar^2((j-\tfrac{1}{2})+1)(j-\tfrac{1}{2})(\alpha Q^{+}_{L-\frac{1}{2}}|Mc,j,j\rangle + \beta Q^{+}_{L\frac{1}{2}}|Mc,j,j-1\rangle). \end{aligned} \qquad \cdots(17)$$

So that we can conclude that this state represents a state of a particle of spin $j-\frac{1}{2}$ and by successive application of J_- we can obtain the other states.

Lemma 2.2

The representation includes a second particle of spin j.

Proof

Consider the particle state $Q^{+}_{L\frac{1}{2}}Q^{+}_{L-\frac{1}{2}}|Mc,j,j\rangle$.

Then

$$J_3 Q^{+}_{L\frac{1}{2}}Q^{+}_{L-\frac{1}{2}}|Mc,j,j\rangle = \hbar Q^{+}_{L\frac{1}{2}}Q^{+}_{L-\frac{1}{2}}|Mc,j,j\rangle$$

and

$$J^2 Q^{+}_{L\frac{1}{2}}Q^{+}_{L-\frac{1}{2}}|Mc,j,j\rangle = \hbar^2 j(j+1)Q^{+}_{L\frac{1}{2}}Q^{+}_{L-\frac{1}{2}}|Mc,j,j\rangle.$$

Thus this state can be used to construct the various states of a particle of spin j. But we must be sure that these states are independent of our original states $|Mc, j, j\rangle$. To see this we need only observe that

$$Q_{Ln}|Mc, j, j\rangle = 0 \text{ from equation (6)},$$

but

$$Q_{L\frac{1}{2}} Q^{\dagger}_{L\frac{1}{2}} Q^{\dagger}_{L-\frac{1}{2}} |Mc, j, j\rangle = Q^{\dagger}_{L-\frac{1}{2}} |Mc, j, j\rangle.$$

So that we can conclude that our representation contains: two particles of spin j, one particle of spin $j+\frac{1}{2}$ and if $j \geqslant \frac{1}{2}$ a particle of spin $j-\frac{1}{2}$. This is in fact all of the particles contained in a representation, we have not demonstrated this since it is an obvious deduction that we can make at the end of the chapter. Our next step is to obtain precise relationships between these various states. This result has never previously been given in the literature.

Theorem 2.3

Suppose the rest states of the particles in a representation are denoted by $|Mc, j, m\rangle$, $|Mc, j+\frac{1}{2}, m'\rangle$, $|Mc, j-\frac{1}{2}, m''\rangle$ and $|Mc, j, m\rangle\rangle$ with spin values j, $j+\frac{1}{2}$, $j-\frac{1}{2}$ and j respectively, and $m = j, j-1, \cdots, -j+1, j$; $m' = j+\frac{1}{2}, j-\frac{1}{2}, \cdots, -(j+\frac{1}{2})+1, -(j+\frac{1}{2})$ and $m'' = j-\frac{1}{2}, j-\frac{3}{2}, \cdots, -(j-\frac{1}{2})$.

Then we can choose the relative phases to be such that

$$\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{Ln}|Mc, j, m\rangle = \left(\begin{matrix}\frac{1}{2} & j \\ n & m\end{matrix}\middle|\begin{matrix}j+\frac{1}{2} \\ m+n\end{matrix}\right)|Mc, j+\tfrac{1}{2}, m+n\rangle + \left(\begin{matrix}\frac{1}{2} & j \\ n & m\end{matrix}\middle|\begin{matrix}j-\frac{1}{2} \\ m+n\end{matrix}\right)|Mc, j-\tfrac{1}{2}, m+n\rangle, \quad \cdots(18)$$

$$\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{Ln}|Mc, j+\tfrac{1}{2}, m+\tfrac{1}{2}\rangle = 2n\left(\begin{matrix}\frac{1}{2} & j \\ -n & m+\frac{1}{2}+n\end{matrix}\middle|\begin{matrix}j+\frac{1}{2} \\ m+\frac{1}{2}\end{matrix}\right)|Mc, j, m+\tfrac{1}{2}+n\rangle\rangle, \quad \cdots(19)$$

$$\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{Ln}|Mc, j-\tfrac{1}{2}, m+\tfrac{1}{2}\rangle = 2n\left(\begin{matrix}\frac{1}{2} & j \\ -n & m+\frac{1}{2}+n\end{matrix}\middle|\begin{matrix}j-\frac{1}{2} \\ m+\frac{1}{2}\end{matrix}\right)|Mc, j, m+\tfrac{1}{2}+n\rangle\rangle, \quad \cdots(20)$$

$$\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{Ln}|Mc, j, m\rangle\rangle = 0, \quad \cdots(21)$$

$$\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{Ln}|Mc, j, m\rangle = 0, \quad \cdots(22)$$

$$\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} Q_{Ln}|Mc, j+\tfrac{1}{2}, m+\tfrac{1}{2}\rangle = \left(\begin{matrix}\frac{1}{2} & j \\ n & m+\frac{1}{2}-n\end{matrix}\middle|\begin{matrix}j+\frac{1}{2} \\ m+\frac{1}{2}\end{matrix}\right)|Mc, j, m+\tfrac{1}{2}-n\rangle, \quad \cdots(23)$$

$$\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} Q_{Ln}|Mc, j-\tfrac{1}{2}, m+\tfrac{1}{2}\rangle = \left(\begin{matrix}\frac{1}{2} & j \\ n & m+\frac{1}{2}-n\end{matrix}\middle|\begin{matrix}j-\frac{1}{2} \\ m+\frac{1}{2}\end{matrix}\right)|Mc, j, m+\tfrac{1}{2}-n\rangle \quad \cdots(24)$$

and

$$\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} Q_{Ln}|Mc, j, m\rangle\rangle = 2n\left\{\left(\begin{matrix}\frac{1}{2} & j \\ -n & m\end{matrix}\middle|\begin{matrix}j+\frac{1}{2} \\ m+n\end{matrix}\right)|Mc, j+\tfrac{1}{2}, m-n\rangle + \left(\begin{matrix}\frac{1}{2} & j \\ -n & m\end{matrix}\middle|\begin{matrix}j-\frac{1}{2} \\ m+n\end{matrix}\right)|Mc, j-\tfrac{1}{2}, m-n\rangle\right\}. \quad \cdots(25)$$

Here $\begin{pmatrix} \frac{1}{2} & j & \Big| & j\pm\frac{1}{2} \\ n & m & \Big| & m+n \end{pmatrix}$ are Clebsh-Gordan coefficients of $SU(2)$. If $j=0$ then the $j-\frac{1}{2}$ representation does not exist and the representation consists just of the vectors $|Mc, j, m\rangle$, $|Mc, j+\frac{1}{2}, m'\rangle$ and $|Mc, j, m\rangle\rangle$.

Proof

We carry this out in several stages.

(i) $\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{Ln}|Mc, j, m\rangle$

As noted in eqations (11) and (12) $Q^{\dagger}_{L\frac{1}{2}}|Mc, j, j\rangle$ is an eigenvector of J_3 with eigenvalue $\hbar(j+\frac{1}{2})$ and of J^2 with eigenvalue $\hbar^2((j+\frac{1}{2})+1)(j+\frac{1}{2})$. The factor $\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}$ is inserted for latter convenience. We choose the relative phase of $|Mc, j+\frac{1}{2}, j+\frac{1}{2}\rangle$ to be such that

$$|Mc, j+\tfrac{1}{2}, j+\tfrac{1}{2}\rangle = \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{L\frac{1}{2}}|Mc, j, j\rangle \qquad \cdots(26)$$

and let the phases of $|Mc, j+\frac{1}{2}, m\rangle$, for $m = j+\frac{1}{2}-1, \cdots, -(j+\frac{1}{2})$, be determined by repeated application of J_-, with the convention that

$$J_-|Mc, j+\tfrac{1}{2}, m\rangle = \hbar\mu^{j+\frac{1}{2}}_{m}|Mc, j+\tfrac{1}{2}, m-1\rangle \qquad \cdots(27)$$

and

$$\mu^{j+\frac{1}{2}}_{m} = \{((j+\tfrac{1}{2})+m)((j+\tfrac{1}{2})-m+1)\}^{\frac{1}{2}}. \qquad \cdots(28)$$

(cf. Cornwell [39], Chapter 12).

Now we rewrite equation (10) for $n=\frac{1}{2}$ making use of equation (32c) of Chapter 1 to give

$$J_- Q^{\dagger}_{L\frac{1}{2}}|Mc, j, m\rangle = \hbar Q^{\dagger}_{L-\frac{1}{2}}|Mc, j, m\rangle + \hbar\mu^{j}_{m} Q^{\dagger}_{L\frac{1}{2}}|Mc, j, m-1\rangle \cdots(29)$$

Similarly for $n=-\frac{1}{2}$ we have

$$J_- Q^{\dagger}_{L-\frac{1}{2}}|Mc, j, m\rangle = \hbar\mu^{j}_{m} Q^{\dagger}_{L-\frac{1}{2}}|Mc, j, m-1\rangle \qquad \cdots(30)$$

Then combining equation (26) with equation (29) we have

$$J_-|Mc, j+\tfrac{1}{2}, j+\tfrac{1}{2}\rangle = \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}\{\hbar Q^{\dagger}_{L-\frac{1}{2}}|Mc, j, j\rangle + \hbar\mu^{j}_{j} Q^{\dagger}_{L\frac{1}{2}}|Mc, j, j-1\rangle\}$$

and using equation (27) gives

$$|Mc, j+\tfrac{1}{2}, j-\tfrac{1}{2}\rangle = \left(\frac{\hbar}{Mc}\right)^{2} \frac{(Q^{\dagger}_{L-\frac{1}{2}}|Mc, j, j\rangle + \mu^{j}_{j} Q^{\dagger}_{L\frac{1}{2}}|Mc, j, j-1\rangle)}{\mu^{j+\frac{1}{2}}_{j+\frac{1}{2}}},$$

$$= \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} \left(\frac{1}{\mu^{j+\frac{1}{2}}_{j+\frac{1}{2}}} Q^{\dagger}_{L-\frac{1}{2}} |Mc, j, j\rangle + \frac{\mu^{j}_{j}}{\mu^{j+\frac{1}{2}}_{j+\frac{1}{2}}} Q^{\dagger}_{L\frac{1}{2}} |Mc, j, j-1\rangle \right). \quad \cdots(31)$$

Now let ψ^{j}_{m} be a vector of the $2j+1$ dimensional representation of $SU(2)$ such that

$$J_3 \psi^{j}_{m} = \hbar m \psi^{j}_{m} \quad \cdots(32)$$

and

$$J^2 \psi^{j}_{m} = \hbar^2 (j+1) j \psi^{j}_{m}. \quad \cdots(33)$$

We can write

$$\psi^{j+\frac{1}{2}}_{j+\frac{1}{2}} = \psi^{\frac{1}{2}}_{\frac{1}{2}} \otimes \psi^{j}_{j}$$

and apply J_- to obtain

$$J_- \psi^{j+\frac{1}{2}}_{j+\frac{1}{2}} = (J_- \psi^{\frac{1}{2}}_{\frac{1}{2}}) \otimes \psi^{j}_{j} + \psi^{j}_{j} \otimes (J_- \psi^{j}_{j}).$$

Then using the standard definition of J_- as in equations (27) and (28) we have

$$\hbar \mu^{j+\frac{1}{2}}_{j+\frac{1}{2}} \psi^{j+\frac{1}{2}}_{j-\frac{1}{2}} = \hbar \mu^{\frac{1}{2}}_{\frac{1}{2}} (\psi^{\frac{1}{2}}_{-\frac{1}{2}} \otimes \psi^{j}_{j}) + \hbar \mu^{j}_{j} (\psi^{\frac{1}{2}}_{\frac{1}{2}} \otimes \psi^{j}_{j-1}). \quad \cdots(34)$$

But
$$\psi^{j+\frac{1}{2}}_{j-\frac{1}{2}} = \left(\begin{matrix} \frac{1}{2} & j \\ -\frac{1}{2} & j \end{matrix} \middle| \begin{matrix} j+\frac{1}{2} \\ j-\frac{1}{2} \end{matrix} \right) \psi^{\frac{1}{2}}_{-\frac{1}{2}} \otimes \psi^{j}_{j} + \left(\begin{matrix} \frac{1}{2} & j \\ \frac{1}{2} & j-1 \end{matrix} \middle| \begin{matrix} j+\frac{1}{2} \\ j-\frac{1}{2} \end{matrix} \right) \psi^{\frac{1}{2}}_{\frac{1}{2}} \otimes \psi^{j}_{j-1}.$$

So that
$$\frac{\mu^{\frac{1}{2}}_{\frac{1}{2}}}{\mu^{j+\frac{1}{2}}_{j+\frac{1}{2}}} = \left(\begin{matrix} \frac{1}{2} & j \\ -\frac{1}{2} & j \end{matrix} \middle| \begin{matrix} j+\frac{1}{2} \\ j-\frac{1}{2} \end{matrix} \right) = \frac{1}{\mu^{j+\frac{1}{2}}_{j+\frac{1}{2}}} \quad \text{since } \mu^{\frac{1}{2}}_{\frac{1}{2}} = 1 \quad \cdots(35)$$

and
$$\frac{\mu^{j}_{j}}{\mu^{j+\frac{1}{2}}_{j+\frac{1}{2}}} = \left(\begin{matrix} \frac{1}{2} & j \\ \frac{1}{2} & j-1 \end{matrix} \middle| \begin{matrix} j+\frac{1}{2} \\ j-\frac{1}{2} \end{matrix} \right). \quad \cdots(36)$$

Inserting equations (35) and (36) into equation (31) we obtain

$$|Mc, j+\tfrac{1}{2}, j-\tfrac{1}{2}\rangle$$
$$= \left(\begin{matrix} \frac{1}{2} & j \\ -\frac{1}{2} & j \end{matrix} \middle| \begin{matrix} j+\frac{1}{2} \\ j-\frac{1}{2} \end{matrix} \right) \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{L-\frac{1}{2}} |Mc, j, j\rangle$$
$$+ \left(\begin{matrix} \frac{1}{2} & j \\ \frac{1}{2} & j-1 \end{matrix} \middle| \begin{matrix} j+\frac{1}{2} \\ j-\frac{1}{2} \end{matrix} \right) \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{L\frac{1}{2}} |Mc, j, j-1\rangle. \quad \cdots(37)$$

Similarly applying J_- to equation (30) and repeating this analysis gives

$$|Mc, j+\tfrac{1}{2}, j-\tfrac{3}{2}\rangle$$
$$= \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} \frac{\left(2\mu^{j}_{j} Q^{\dagger}_{L-\frac{1}{2}} |Mc, j, j-1\rangle + \mu^{j}_{j} \mu^{j}_{j-1} Q^{\dagger}_{L\frac{1}{2}} |Mc, j, j-2\rangle\right)}{\mu^{j+\frac{1}{2}}_{j+\frac{1}{2}} \, \mu^{j+\frac{1}{2}}_{j-\frac{1}{2}}}, \quad \cdots(38)$$

whereas applying J_- to equation (34) gives

$$\mu^{j+\frac{1}{2}}_{j+\frac{1}{2}}\,\mu^{j+\frac{1}{2}}_{j-\frac{1}{2}}\,\psi^{j+\frac{1}{2}}_{j-3/2} = \mu^{\frac{1}{2}}_{\frac{1}{2}}\{0 + \psi^{\frac{1}{2}}_{-\frac{1}{2}} \otimes (\mu^{j}_{j}\,\psi^{j}_{j-1})\}$$
$$+\mu^{j}_{j}\{\mu^{\frac{1}{2}}_{\frac{1}{2}}\,\psi^{\frac{1}{2}}_{-\frac{1}{2}} \otimes \psi^{j}_{j-1} + \mu^{j}_{j-1}\,\psi^{\frac{1}{2}}_{\frac{1}{2}} \otimes \psi^{j}_{j-2}\}.$$

Thus

$$\psi^{j+\frac{1}{2}}_{j-3/2} = \frac{2\mu^{\frac{1}{2}}_{\frac{1}{2}}\mu^{j}_{j}(\psi^{\frac{1}{2}}_{\frac{1}{2}} \otimes \psi^{j}_{j-1}) + \mu^{j}_{j}\mu^{j}_{j-1}(\psi^{\frac{1}{2}}_{\frac{1}{2}} \otimes \psi^{j}_{j-2})}{\mu^{j+\frac{1}{2}}_{j+\frac{1}{2}}\,\mu^{j+\frac{1}{2}}_{j-\frac{1}{2}}}$$

and we have

$$\frac{2\mu^{\frac{1}{2}}_{\frac{1}{2}}\mu^{j}_{j}}{\mu^{j+\frac{1}{2}}_{j+\frac{1}{2}}\,\mu^{j+\frac{1}{2}}_{j-\frac{1}{2}}} = \left(\begin{matrix}\frac{1}{2} & j \\ -\frac{1}{2} & j-1\end{matrix}\,\middle|\,\begin{matrix}j+\frac{1}{2} \\ j-3/2\end{matrix}\right) \qquad \cdots(39)$$

and

$$\frac{\mu^{j}_{j}\mu^{j}_{j-1}}{\mu^{j+\frac{1}{2}}_{j+\frac{1}{2}}\,\mu^{j+\frac{1}{2}}_{j-\frac{1}{2}}} = \left(\begin{matrix}\frac{1}{2} & j \\ \frac{1}{2} & j-2\end{matrix}\,\middle|\,\begin{matrix}j+\frac{1}{2} \\ j-3/2\end{matrix}\right). \qquad \cdots(40)$$

Then substituting equations (39) and (40) into equation (38) we obtain

$$|Mc, j+\tfrac{1}{2}, m\rangle = \left(\begin{matrix}\frac{1}{2} & j \\ -\frac{1}{2} & m+\frac{1}{2}\end{matrix}\,\middle|\,\begin{matrix}j+\frac{1}{2} \\ m\end{matrix}\right)\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{L\frac{1}{2}}|Mc, j, m+\tfrac{1}{2}\rangle$$
$$+\left(\begin{matrix}\frac{1}{2} & j \\ \frac{1}{2} & m-\frac{1}{2}\end{matrix}\,\middle|\,\begin{matrix}j+\frac{1}{2} \\ m\end{matrix}\right)\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{L\frac{1}{2}}|Mc, j, m-\tfrac{1}{2}\rangle. \qquad \cdots(42)$$

Now recall equations (15), (16) and (17) which imply that we can write

$$|Mc, j-\tfrac{1}{2}, j-\tfrac{1}{2}\rangle = \alpha\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{L\frac{1}{2}}|Mc, j, j-1\rangle$$
$$+\beta\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{L-\frac{1}{2}}|Mc, j, j\rangle \qquad \cdots(43)$$

with α, β such that

$$J_+|Mc, j-\tfrac{1}{2}, j-\tfrac{1}{2}\rangle = 0. \qquad \cdots(44)$$

With ξ^{j}_{m} defined by

$$J_+|Mc, j-\tfrac{1}{2}, m\rangle = \hbar\,\xi^{j-\frac{1}{2}}_{m}|Mc, j-\tfrac{1}{2}, m+1\rangle \qquad \cdots(45)$$

and
$$\xi^{j}_{m} = \{(j-m)(j+m+1)\}^{\frac{1}{2}}. \qquad \cdots(46)$$

equation (9) can be rewritten as

$$J_+ Q^{\dagger}_{L\frac{1}{2}}|Mc, j, m\rangle = \hbar\,\xi^{j}_{m}\, Q^{\dagger}_{L\frac{1}{2}}|Mc, j, m+1\rangle \qquad \cdots(47)$$

and
$$J_+ Q^{\dagger}_{L-\frac{1}{2}}|Mc, j, m\rangle = \hbar\, Q^{\dagger}_{L\frac{1}{2}}|Mc, j, m\rangle$$
$$+\hbar\,\xi^{j}_{m}\, Q^{\dagger}_{L-\frac{1}{2}}|Mc, j, m+1\rangle. \qquad \cdots(48)$$

Then, since $\xi^{j}_{j} = 0$, if we apply J_+ to equation (43) we obtain

$$0 = \alpha \xi^{j}_{j-1} Q^{\dagger}_{L\frac{1}{2}} |Mc, j, m\rangle + \beta\{Q^{\dagger}_{L\frac{1}{2}} |Mc, j, j\rangle + 0\}$$

hence

$$\beta = -\alpha \xi^{j}_{j-1}. \qquad \cdots(49)$$

But consider

$$\psi^{j-\frac{1}{2}}_{j-\frac{1}{2}} = \alpha' \psi^{\frac{1}{2}}_{\frac{1}{2}} \otimes \psi^{j}_{j-1} + \beta' \psi^{\frac{1}{2}}_{-\frac{1}{2}} \otimes \psi^{j}_{j}$$

so that

$$\begin{aligned} J_+ \psi^{j-\frac{1}{2}}_{j-\frac{1}{2}} = 0 &= \alpha'\{(J_+ \psi^{\frac{1}{2}}_{\frac{1}{2}}) \otimes \psi^{j}_{j-1} + \psi^{\frac{1}{2}}_{\frac{1}{2}} \otimes (J_+ \psi^{j}_{j-1})\} \\ &\quad + \beta'\{(J_+ \psi^{\frac{1}{2}}_{-\frac{1}{2}}) \otimes \psi^{j}_{j} + \psi^{\frac{1}{2}}_{-\frac{1}{2}} \otimes (J_+ \psi^{j}_{j})\} \\ &= \alpha' \psi^{\frac{1}{2}}_{\frac{1}{2}} \otimes \xi^{j}_{j-1} \psi^{j}_{j} + \beta' \xi^{\frac{1}{2}}_{-\frac{1}{2}} \psi^{\frac{1}{2}}_{\frac{1}{2}} \otimes \psi^{j}_{j}. \end{aligned}$$

Now $\xi^{\frac{1}{2}}_{-\frac{1}{2}} = 1$ so that

$$\beta' = -\alpha' \xi^{j}_{j-1}. \qquad \cdots(50)$$

But α', β' are Clebsh-Gordan coefficients so that we can compare equations (49) and (50) to obtain for equation (43)

$$\begin{aligned} |Mc, j-\tfrac{1}{2}, j-\tfrac{1}{2}\rangle &= \left(\begin{smallmatrix} \frac{1}{2} & j \\ -\frac{1}{2} & j \end{smallmatrix} \middle| \begin{smallmatrix} j-\frac{1}{2} \\ j-\frac{1}{2} \end{smallmatrix}\right) \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{L-\frac{1}{2}} |M, j, j\rangle \\ &\quad + \left(\begin{smallmatrix} \frac{1}{2} & j \\ \frac{1}{2} & j-1 \end{smallmatrix} \middle| \begin{smallmatrix} j-\frac{1}{2} \\ j-\frac{1}{2} \end{smallmatrix}\right) \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{L\frac{1}{2}} |Mc, j, j-1\rangle. \qquad \cdots(51) \end{aligned}$$

Now, by repeated application of J_-, repeating the above argument we obtain

$$\begin{aligned} |Mc, j-\tfrac{1}{2}, m\rangle &= \left(\begin{smallmatrix} \frac{1}{2} & j \\ -\frac{1}{2} & m+\frac{1}{2} \end{smallmatrix} \middle| \begin{smallmatrix} j-\frac{1}{2} \\ m \end{smallmatrix}\right) \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{L-\frac{1}{2}} |Mc, j, m+\tfrac{1}{2}\rangle \\ &\quad + \left(\begin{smallmatrix} \frac{1}{2} & j \\ \frac{1}{2} & m-\frac{1}{2} \end{smallmatrix} \middle| \begin{smallmatrix} j-\frac{1}{2} \\ m \end{smallmatrix}\right) \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q_{L\frac{1}{2}} |Mc, j, m-\tfrac{1}{2}\rangle. \qquad \cdots(52) \end{aligned}$$

Let $C_{nm,r}$ be the $2(2j+1) \times 2(2j+1)$ matrix of Clebsh-Gordan coefficients for $SU(2)$ defined by

$$C_{nm,r} \begin{bmatrix} D^{j+\frac{1}{2}}(a) & 0 \\ 0 & D^{j-\frac{1}{2}}(a) \end{bmatrix}_{rr'} (C^{-1})_{r',n'm'}$$
$$= D^{\frac{1}{2}}(a)_{nn'} \otimes D^{j}(a)_{mm'} \qquad \cdots(53)$$

with $a \in SU(2)$ and $D^{k}(a)$ the irreducible representation of $SU(2)$ in $(2k+1) \times (2k+1)$ matrix form. Then equations (41) and (52) can be combined to give

$$(|Mc, j+\tfrac{1}{2}, a\rangle, |Mc, j-\tfrac{1}{2}, a'\rangle)_r = (C^{-1})_{r,nm}\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{Ln}|Mc, j, m\rangle.$$

Then inverting this we have

$$\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{Ln}|Mc, j, m\rangle = C_{nm,r}(|Mc, j+\tfrac{1}{2}, a\rangle, |Mc, j-\tfrac{1}{2}, a'\rangle)_r \quad \cdots(54)$$

which is the same as equation (18) ie. the first of our results.

(ii) $Q^{\dagger}_{Ln} Q^{\dagger}_{Ln'} |Mc, j, m\rangle$

We define

$$|Mc, j, m\rangle\rangle = \left(\tfrac{\hbar}{Mc}\right) Q^{\dagger}_{L\frac{1}{2}} Q^{\dagger}_{L-\frac{1}{2}} |Mc, j, m\rangle. \quad \cdots(55)$$

Then noting equation (18) of Chapter 1 and using equation (54)

$$\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{L-\frac{1}{2}} |Mc, j+\tfrac{1}{2}, j+\tfrac{1}{2}\rangle = -|Mc, j, j\rangle\rangle$$

and $$J_-\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{L\frac{1}{2}} |Mc, j+\tfrac{1}{2}, j+\tfrac{1}{2}\rangle = -J_-|Mc, j, j\rangle\rangle.$$

Now using equation (27) we have

$$\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} \mu^{j+\frac{1}{2}}_{j+\frac{1}{2}} Q^{\dagger}_{L-\frac{1}{2}} |Mc, j+\tfrac{1}{2}, j-\tfrac{1}{2}\rangle = -\mu^{j}_{j}|Mc, j, j-1\rangle\rangle,$$

thus

$$\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{L-\frac{1}{2}} |Mc, j+\tfrac{1}{2}, j-\tfrac{1}{2}\rangle = \frac{-\mu^{j}_{j}}{\mu^{j+\frac{1}{2}}_{j+\frac{1}{2}}} |Mc, j, j-1\rangle\rangle.$$

Clearly the argument proceeds as in (i), by applying J_- successively.

The general result is

$$\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{L-\frac{1}{2}} |Mc, j+\tfrac{1}{2}, m+\tfrac{1}{2}\rangle = -\left(\begin{matrix}\frac{1}{2} & j \\ \frac{1}{2} & m\end{matrix}\middle|\begin{matrix} j+\frac{1}{2} \\ m+\frac{1}{2}\end{matrix}\right)|Mc, j, m\rangle\rangle. \quad \cdots(56)$$

Again from equation (55) using equation (54) and applying J_+ we obtain

$$\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} J_+ Q^{\dagger}_{L\frac{1}{2}} |Mc, j+\tfrac{1}{2}, -(j+\tfrac{1}{2})\rangle = J_+|Mc, j, -j\rangle\rangle. \quad \cdots(57)$$

The argument proceeds as before using the coefficients ξ^{j}_{m}. The general result we obtain is

$$\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{\dagger}_{L\frac{1}{2}} |Mc, j+\tfrac{1}{2}, m+\tfrac{1}{2}\rangle = \left(\begin{matrix}\frac{1}{2} & j \\ -\frac{1}{2} & m+1\end{matrix}\middle|\begin{matrix} j+\frac{1}{2} \\ m+\frac{1}{2}\end{matrix}\right)|Mc, j, m+1\rangle\rangle \cdots(58)$$

Equations (56) and (58) can now be combined to obtain equation (19), the second of our required results.

Now if we combine equations (54) and (55) we obtain

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{+}_{L-\frac{1}{2}} \left\{ \left(\begin{matrix} \frac{1}{2} & j \\ \frac{1}{2} & m \end{matrix} \middle| \begin{matrix} j+\frac{1}{2} \\ m+\frac{1}{2} \end{matrix}\right) |Mc, j+\tfrac{1}{2}, m+\tfrac{1}{2}\rangle \right.$$

$$\left. + \left(\begin{matrix} \frac{1}{2} & j \\ \frac{1}{2} & m \end{matrix} \middle| \begin{matrix} j-\frac{1}{2} \\ m+\frac{1}{2} \end{matrix}\right) |Mc, j-\tfrac{1}{2}, m+\tfrac{1}{2}\rangle \right\} = -|Mc, j, m\rangle\rangle.$$

Then using equation (58) we have

$$\left\{ -\left(\begin{matrix} \frac{1}{2} & j \\ \frac{1}{2} & m \end{matrix} \middle| \begin{matrix} j+\frac{1}{2} \\ m+\frac{1}{2} \end{matrix}\right)\left(\begin{matrix} \frac{1}{2} & j \\ \frac{1}{2} & m \end{matrix} \middle| \begin{matrix} j+\frac{1}{2} \\ m+\frac{1}{2} \end{matrix}\right) |Mc, j, m\rangle\rangle \right.$$

$$\left. + \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} \left(\begin{matrix} \frac{1}{2} & j \\ \frac{1}{2} & m \end{matrix} \middle| \begin{matrix} j-\frac{1}{2} \\ m+\frac{1}{2} \end{matrix}\right) Q^{+}_{L-\frac{1}{2}} |Mc, j-\tfrac{1}{2}, m+\tfrac{1}{2}\rangle \right\} = -|Mc, j, m\rangle\rangle,$$

thus $$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{+}_{L-\frac{1}{2}} |Mc, j-\tfrac{1}{2}, m+\tfrac{1}{2}\rangle = \frac{\left(\begin{matrix} \frac{1}{2} & j \\ \frac{1}{2} & m \end{matrix} \middle| \begin{matrix} j+\frac{1}{2} \\ m+\frac{1}{2} \end{matrix}\right)^2 - 1}{\left(\begin{matrix} \frac{1}{2} & j \\ \frac{1}{2} & m \end{matrix} \middle| \begin{matrix} j-\frac{1}{2} \\ m+\frac{1}{2} \end{matrix}\right)} |Mc, j, m\rangle\rangle.$$

Now the Clebsh-Gordan coefficients must satisfy

$$\left(\begin{matrix} \frac{1}{2} & j \\ \frac{1}{2} & m \end{matrix} \middle| \begin{matrix} j+\frac{1}{2} \\ m+\frac{1}{2} \end{matrix}\right)^2 + \left(\begin{matrix} \frac{1}{2} & j \\ \frac{1}{2} & m \end{matrix} \middle| \begin{matrix} j-\frac{1}{2} \\ m+\frac{1}{2} \end{matrix}\right)^2 = 1.$$

So that

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{+}_{L-\frac{1}{2}} |Mc, j-\tfrac{1}{2}, m+\tfrac{1}{2}\rangle = -\left(\begin{matrix} \frac{1}{2} & j \\ \frac{1}{2} & m \end{matrix} \middle| \begin{matrix} j-\frac{1}{2} \\ m+\frac{1}{2} \end{matrix}\right) |Mc, j, m\rangle\rangle. \quad \cdots(59)$$

By a similar procedure we find that

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{+}_{L\frac{1}{2}} |Mc, j-\tfrac{1}{2}, m-\tfrac{1}{2}\rangle = \left(\begin{matrix} \frac{1}{2} & j \\ -\frac{1}{2} & m \end{matrix} \middle| \begin{matrix} j-\frac{1}{2} \\ m-\frac{1}{2} \end{matrix}\right) |Mc, j, m\rangle\rangle. \quad \cdots(60)$$

Now combining equations (59) and (60) we obtain equation (20).

Lastly apply the operator $\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{+}_{L\frac{1}{2}}$ to equation (55) to give

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{+}_{L\frac{1}{2}} |Mc, j, m\rangle\rangle = \left(\frac{\hbar}{Mc}\right)^{\frac{3}{2}} Q^{+}_{L\frac{1}{2}} Q^{+}_{L\frac{1}{2}} Q^{+}_{L-\frac{1}{2}} |Mc, j, m\rangle$$

$$= \left(\frac{\hbar}{Mc}\right)^{\frac{3}{2}} \tfrac{1}{2} [Q^{+}_{L\frac{1}{2}}, Q^{+}_{L\frac{1}{2}}] Q^{+}_{L-\frac{1}{2}} |Mc, j, m\rangle$$

$$= 0$$

similarly $$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{+}_{L-\frac{1}{2}} |Mc, j, m\rangle\rangle = 0.$$

So that we have proved equation (21).

Finally we note that if $j = 0$ then the $j - \frac{1}{2}$ representation does not exist but the formulae given are still correct.

(iii) The action of the operators Q_{Ln}.

Firstly we note that equation (22) is true by definition.

Recall equation (19) of Chapter 1 to obtain

$$[Q_{Ln}, Q^{\dagger}_{Ln'}] = \frac{Mc}{\hbar}\delta_{nn'}$$

when acting on the particle rest states, so that

$$Q_{Ln}Q^{\dagger}_{Ln'} = \frac{Mc}{\hbar}\delta_{nn'} - Q^{\dagger}_{Ln'}Q_{Ln}. \quad \cdots(61)$$

Now apply $\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L\frac{1}{2}}$ to equation (18) with $n=\frac{1}{2}$ to obtain

$$|Mc,j,m\rangle = \left(\begin{matrix}\frac{1}{2} & j\\ \frac{1}{2} & m\end{matrix}\middle|\begin{matrix}j+\frac{1}{2}\\ m+\frac{1}{2}\end{matrix}\right)\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L\frac{1}{2}}|Mc,j+\tfrac{1}{2},m+\tfrac{1}{2}\rangle$$
$$+\left(\begin{matrix}\frac{1}{2} & j\\ \frac{1}{2} & m\end{matrix}\middle|\begin{matrix}j-\frac{1}{2}\\ m+\frac{1}{2}\end{matrix}\right)\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L\frac{1}{2}}|Mc,j-\tfrac{1}{2},m+\tfrac{1}{2}\rangle \quad \cdots(62)$$

and to equation (18) with $n=-\frac{1}{2}$ to obtain

$$0 = \left(\begin{matrix}\frac{1}{2} & j\\ -\frac{1}{2} & m+1\end{matrix}\middle|\begin{matrix}j+\frac{1}{2}\\ m+\frac{1}{2}\end{matrix}\right)\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L\frac{1}{2}}|Mc,j+\tfrac{1}{2},m+\tfrac{1}{2}\rangle$$
$$+\left(\begin{matrix}\frac{1}{2} & j\\ -\frac{1}{2} & m+1\end{matrix}\middle|\begin{matrix}j-\frac{1}{2}\\ m+\frac{1}{2}\end{matrix}\right)\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L\frac{1}{2}}|Mc,j-\tfrac{1}{2},m+\tfrac{1}{2}\rangle. \quad \cdots(63)$$

Now consider equation (62) with $m=j$, ie.

$$|Mc,j,j\rangle = \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L\frac{1}{2}}|M,j+\tfrac{1}{2},m+\tfrac{1}{2}\rangle$$

then apply J_-, noting equation (30c) of Chapter 1, to obtain

$$\frac{\mu^{j}_{j}}{\mu^{j+\frac{1}{2}}_{j+\frac{1}{2}}}|Mc,j,j-1\rangle = \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L\frac{1}{2}}|Mc,j+\tfrac{1}{2},j-\tfrac{1}{2}\rangle.$$

That is using equation (36) we have

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L\frac{1}{2}}|Mc,j+\tfrac{1}{2},j-\tfrac{1}{2}\rangle = \left(\begin{matrix}\frac{1}{2} & j\\ \frac{1}{2} & j-1\end{matrix}\middle|\begin{matrix}j+\frac{1}{2}\\ j-\frac{1}{2}\end{matrix}\right)|M,j,j-1\rangle.$$

Clearly the analysis proceeds as before, in general we have

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L\frac{1}{2}}|Mc,j+\tfrac{1}{2},m+\tfrac{1}{2}\rangle = \left(\begin{matrix}\frac{1}{2} & j\\ \frac{1}{2} & m\end{matrix}\middle|\begin{matrix}j+\frac{1}{2}\\ m+\frac{1}{2}\end{matrix}\right)|Mc,j,m\rangle. \cdots(64)$$

Now substitute equation (64) into equation (62) to obtain

$$\left(1-\begin{pmatrix}\frac{1}{2} & j \\ \frac{1}{2} & m\end{pmatrix}\Big|\begin{matrix}j+\frac{1}{2}\\ m+\frac{1}{2}\end{matrix}\Big)^2\right)|Mc,j,m\rangle$$

$$=\begin{pmatrix}\frac{1}{2} & j \\ \frac{1}{2} & m\end{pmatrix}\Big|\begin{matrix}j-\frac{1}{2}\\ m+\frac{1}{2}\end{matrix}\Big)\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L\frac{1}{2}}|Mc,j-\tfrac{1}{2},m+\tfrac{1}{2}\rangle.$$

So that

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L\frac{1}{2}}|Mc,j-\tfrac{1}{2},m+\tfrac{1}{2}\rangle=\begin{pmatrix}\frac{1}{2} & j \\ \frac{1}{2} & m\end{pmatrix}\Big|\begin{matrix}j-\frac{1}{2}\\ m+\frac{1}{2}\end{matrix}\Big)|Mc,j,m\rangle. \quad \cdots(65)$$

Now put $n=-\frac{1}{2}$ in equation (18) and act on it with $\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L-\frac{1}{2}}$ to give

$$|Mc,j,m\rangle=\begin{pmatrix}\frac{1}{2} & j \\ -\frac{1}{2} & m\end{pmatrix}\Big|\begin{matrix}j+\frac{1}{2}\\ m-\frac{1}{2}\end{matrix}\Big)\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L-\frac{1}{2}}|Mc,j+\tfrac{1}{2},m-\tfrac{1}{2}\rangle$$

$$+\begin{pmatrix}\frac{1}{2} & j \\ -\frac{1}{2} & m\end{pmatrix}\Big|\begin{matrix}j-\frac{1}{2}\\ m-\frac{1}{2}\end{matrix}\Big)\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L-\frac{1}{2}}|Mc,j-\tfrac{1}{2},m-\tfrac{1}{2}\rangle \cdots(66)$$

First we put $m=j$ and then act with J_+, following the previous argument. We obtain the general result

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L-\frac{1}{2}}|Mc,j+\tfrac{1}{2},m+\tfrac{1}{2}\rangle=\begin{pmatrix}\frac{1}{2} & j \\ -\frac{1}{2} & m+1\end{pmatrix}\Big|\begin{matrix}j+\frac{1}{2}\\ m+\frac{1}{2}\end{matrix}\Big)|Mc,j,m+1\rangle. \cdots(67)$$

Substituting this into equation (66) we obtain

$$|Mc,j,m\rangle=\begin{pmatrix}\frac{1}{2} & j \\ -\frac{1}{2} & m\end{pmatrix}\Big|\begin{matrix}j+\frac{1}{2}\\ m-\frac{1}{2}\end{matrix}\Big)\begin{pmatrix}\frac{1}{2} & j \\ -\frac{1}{2} & m\end{pmatrix}\Big|\begin{matrix}j+\frac{1}{2}\\ m-\frac{1}{2}\end{matrix}\Big)|Mc,j,m\rangle$$

$$+\begin{pmatrix}\frac{1}{2} & j \\ -\frac{1}{2} & m\end{pmatrix}\Big|\begin{matrix}j-\frac{1}{2}\\ m-\frac{1}{2}\end{matrix}\Big)\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L-\frac{1}{2}}|Mc,j-\tfrac{1}{2},m-\tfrac{1}{2}\rangle.$$

Thus

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L-\frac{1}{2}}|Mc,j-\tfrac{1}{2},m-\tfrac{1}{2}\rangle=\begin{pmatrix}\frac{1}{2} & j \\ -\frac{1}{2} & m\end{pmatrix}\Big|\begin{matrix}j-\frac{1}{2}\\ m-\frac{1}{2}\end{matrix}\Big)|Mc,j,m\rangle. \quad \cdots(68)$$

So that combining equations (64) and (67) gives equation (23) and combining equations (65) and (68) gives equation (24).

Now we apply the operator $\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L\frac{1}{2}}$ to equation (55) to give

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L\frac{1}{2}}|Mc,j,m\rangle\rangle=\left(\frac{\hbar}{Mc}\right)^{3/2}Q_{L\frac{1}{2}}Q^{\dagger}_{L\frac{1}{2}}Q^{\dagger}_{L-\frac{1}{2}}|Mc,j,m\rangle,$$

then using equation (61) we have

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{L\frac{1}{2}}|Mc,j,m\rangle\rangle=\left(\frac{\hbar}{Mc}\right)^{3/2}\left(\left(\frac{Mc}{\hbar}\right)-Q^{\dagger}_{L\frac{1}{2}}Q_{L\frac{1}{2}}\right)Q^{\dagger}_{L-\frac{1}{2}}|Mc,j,m\rangle,$$

$$=\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q^{\dagger}_{L-\frac{1}{2}}|M,j,m\rangle+Q^{\dagger}_{L\frac{1}{2}}Q^{\dagger}_{L-\frac{1}{2}}Q_{L\frac{1}{2}}|Mc,j,m\rangle,$$

$$=\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q^{\dagger}_{L-\frac{1}{2}}|M,j,m\rangle,$$

by equation (7).

Now using equation (18) we have

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q_{L\frac{1}{2}} |Mc, j, m\rangle\rangle = \begin{pmatrix} \frac{1}{2} & j \\ -\frac{1}{2} & m \end{pmatrix}\!\left|\begin{matrix} j+\frac{1}{2} \\ m-\frac{1}{2} \end{matrix}\right) |Mc, j+\tfrac{1}{2}, m-\tfrac{1}{2}\rangle$$

$$+ \begin{pmatrix} \frac{1}{2} & j \\ -\frac{1}{2} & m \end{pmatrix}\!\left|\begin{matrix} j-\frac{1}{2} \\ m-\frac{1}{2} \end{matrix}\right) |Mc, j-\tfrac{1}{2}, m-\tfrac{1}{2}\rangle. \quad \ldots(69)$$

Similarly applying $\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q_{L-\frac{1}{2}}$ to equation (55) we obtain

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q_{L-\frac{1}{2}} |Mc, j, m\rangle = -\begin{pmatrix} \frac{1}{2} & j \\ \frac{1}{2} & m \end{pmatrix}\!\left|\begin{matrix} j+\frac{1}{2} \\ m+\frac{1}{2} \end{matrix}\right) |Mc, j+\tfrac{1}{2}, m+\tfrac{1}{2}\rangle$$

$$- \begin{pmatrix} \frac{1}{2} & j \\ \frac{1}{2} & m \end{pmatrix}\!\left|\begin{matrix} j-\frac{1}{2} \\ m+\frac{1}{2} \end{matrix}\right) |Mc, j-\tfrac{1}{2}, m+\tfrac{1}{2}\rangle. \quad \ldots(70)$$

Combining equations (69) and (70) we obtain equation (25).

This completes the proof of the theorem. It is convenient in the next chapter to have, in addition, some of these formulae expressed in terms of the operators Q_{Rn} and Q^{+}_{Rn}. We have, respectively, from equations (19), (20) and (25)

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q_{Rn} |Mc, j+\tfrac{1}{2}, m+n\rangle = \begin{pmatrix} \frac{1}{2} & j \\ n & m \end{pmatrix}\!\left|\begin{matrix} j+\frac{1}{2} \\ m+n \end{matrix}\right) |Mc, j, m\rangle\rangle, \quad \ldots(71)$$

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q_{Rn} |Mc, j-\tfrac{1}{2}, m+n\rangle = \begin{pmatrix} \frac{1}{2} & j \\ n & m \end{pmatrix}\!\left|\begin{matrix} j-\frac{1}{2} \\ m+n \end{matrix}\right) |Mc, j, m\rangle\rangle \quad \ldots(72)$$

and

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} Q^{+}_{Rn} |Mc, j, m\rangle\rangle = \begin{pmatrix} \frac{1}{2} & j \\ n & m \end{pmatrix}\!\left|\begin{matrix} j+\frac{1}{2} \\ m+n \end{matrix}\right) |Mc, j+\tfrac{1}{2}, m+n\rangle$$

$$+ \begin{pmatrix} \frac{1}{2} & j \\ n & m \end{pmatrix}\!\left|\begin{matrix} j-\frac{1}{2} \\ m+n \end{matrix}\right) |Mc, j-\tfrac{1}{2}, m+n\rangle. \quad \ldots(73)$$

We note that each of the vectors $|Mc, j, m\rangle$, $|Mc, j+\frac{1}{2}, m'\rangle$, $|Mc, j-\frac{1}{2}, m''\rangle$ and $|Mc, j, m\rangle\rangle$ is needed in the representation derived above and no other vector is needed. We thus have the following result.

Corollary 2.4

The set of vectors $\{|Mc, j, m\rangle, |Mc, j+\frac{1}{2}, m'\rangle, |Mc, j-\frac{1}{2}, m''\rangle, |Mc, j, m\rangle\rangle\}$ provide a basis of an irreducible representation of the rest state algebra of the super Poincaré algebra.

We could now construct the 'boosted' states from the rest states but it is more convenient to leave this until the next chapter.

CHAPTER 3

THE CONSTRUCTION OF CHIRAL SUPERMULTIPLETS.

3.1 Preliminaries.

In the previous chapter we obtained expressions for the action of the supersymmetry generators on the rest states. From this we can recognise that a supersymmetry representation consists of a reducible representation of the Poincaré group consisting of two particles of spin j, one of spin $j+\frac{1}{2}$ and one of spin $j-\frac{1}{2}$ and the action of the supersymmetry generators is to transform the 'fields' between these representations.

Firstly we must consider how to construct operator fields that are Lorentz invariant from these Poincaré representations. Then we need to determine the action of the supersymmetry operators on these fields. A method for constructing Lorentz invariant fields has been given by Weinberg [20] as mentioned in the introduction. In this section we review this procedure and note an identity that we need in the subsequent sections.

As noted in section 4.3.2 of Chapter 4 of Part I the unitary irreducible representations of the Poincaré group corresponding to a particle of mass M and spin j with $\hat{p} = (0, 0, 0, Mc)$ is given by

$$\Phi^{\hat{p},j}([\Lambda|t])\,\phi_{p,m} = \exp \tfrac{i}{\hbar}\{(\Lambda p)_\sigma t^\sigma\}\, D^j([B(\Lambda p,\hat{p})^{-1}\Lambda B(p,\hat{p})|0])_{m'm}\,\phi_{\Lambda p,m'} \quad \ldots(1)$$

for $m = -j, -j+1, \cdots, j-1, j$.

Here we have simplified the notation of Chapter 4 of Part I by suppressing the indices on t and p which we have no need of in this part of the Thesis. We have also put $[\Lambda|t] = [\Lambda|t|0]$. This is because we only consider Minkowski space- time in this part and do not consider how (or even if!) it is embedded in superspace. The indices will re-appear in Part III when we consider the embedding of Minkowski space in superspace and how these fields are represented as superfields.

Now define the one particle Fock state $|p,j,m\rangle$ and identify it with $\left(\frac{p_4}{Mc}\right)^{\frac{1}{2}}\phi_{p,m}$. The 'rest' Fock states can then be identified with the states used in Chapter 2. Now by comparison with equation (1) we define the unitary operator $u([\Lambda|t])$ by

$$u([\Lambda|t])|p,j,m\rangle = \left(\frac{(\Lambda p)_4}{p_4}\right)^{\frac{1}{2}} \exp\frac{i}{\hbar}\{(\Lambda p)_\sigma t^\sigma\}\, D^j([B(\Lambda p,\hat{p})^{-1}\Lambda B(p,\hat{p})|0])_{m'm}|\Lambda p,j,m'\rangle. \qquad \dots(2)$$

Next we introduce the single particle creation operators $a^+_{p,j,m}$ and the vacuum state $|0\rangle$ by

$$|p,j,m\rangle = a^+_{p,j,m}|0\rangle, \qquad \dots(3)$$

and suppose that

$$u([\Lambda|t])|0\rangle = |0\rangle \text{ for all } [\Lambda|t]. \qquad \dots(4)$$

Equation (2) can now be written

$$u([\Lambda|t])\, a^+_{p,j,m}\, u([\Lambda|t])^{-1} = \left(\frac{(\Lambda p)_4}{p_4}\right)^{\frac{1}{2}} \exp\frac{i}{\hbar}\{(\Lambda p)_\sigma t^\sigma\}\, D^j([B(\Lambda p,\hat{p})^{-1}\Lambda B(p,\hat{p})|0])_{m'm}\, a^+_{\Lambda p,j,m'}. \qquad \dots(5)$$

In particular we note that

$$a^+_{p,j,m} = \left(\frac{Mc}{p_4}\right)^{\frac{1}{2}} u([B(p,\hat{p})|0])\, a^+_{\hat{p},j,m}\, u^{-1}([B(p,\hat{p})|0]). \qquad \dots(6)$$

Now we define the corresponding annihilation operators $a_{p,j,m}$ by

$$a_{p,j,m} = (a^+_{p,j,m})^+ \qquad \dots(7)$$

and take the adjoint of equation (5) to obtain

$$u([\Lambda|t])\, a_{p,j,m}\, u([\Lambda|t])^{-1} = \left(\frac{(\Lambda p)_4}{p_4}\right)^{\frac{1}{2}} \exp -\frac{i}{\hbar}\{(\Lambda p)_\sigma t^\sigma\}\, D^j([B(\Lambda p,\hat{p})^{-1}\Lambda B(p,\hat{p})|0])^*_{m'm}\, a_{\Lambda p,j,m'}.$$

Since the matrices D^j are unitary we can rewrite this expression as

$$u([\Lambda|t])\, a_{p,j,m}\, u([\Lambda|t])^{-1} = \left(\frac{(\Lambda p)_4}{p_4}\right)^{\frac{1}{2}} \exp -\frac{i}{\hbar}\{(\Lambda p)_\sigma t^\sigma\}\, D^j([B(\Lambda p,\hat{p})^{-1}\Lambda B(p,\hat{p})|0])_{mm'}\, a_{\Lambda p,j,m'}. \qquad \dots(8)$$

It is convenient to rewrite equation (5) so that the indices are in the same order as those of equation (8). We recall that since the representations D^j of $SU(2)$ are real or pseudo-real (cf. Cornwell [39] Chapter 5) there exists a $(2j+1)\times(2j+1)$ matrix Z_j such that

$$D^j([R|0])^* = Z_j^{-1} D^j([R|0]) Z_j \qquad \cdots(9)$$

Here $[R|0]$ denotes an element of $SU(2)$, the covering group of $SO(3,\mathbb{R})$, that maps onto the pure rotation R in the 2:1 homeomorphic mapping of $SU(2)$ onto $SO(3,\mathbb{R})$. We note that the matrix Z_j can be chosen to satisfy

$$Z_j^{-1} = Z_j^{\dagger} \qquad \cdots(10)$$

and

$$Z_j^* Z_j = (-1)^{2j} \qquad \cdots(11)$$

For $j=\frac{1}{2}$ we put $Z_{\frac{1}{2}} = \sigma_2$.

Equation (5) can now be written

$$u([\Lambda|t])\, a^{\dagger}_{p,j,m}\, u([\Lambda|t])^{-1} = \left(\frac{(\Lambda p)_4}{p_4}\right)^{\frac{1}{2}} \exp\frac{i}{\hbar}\{(\Lambda p)_\sigma t^\sigma\}\left(Z_j^{-1} D^j([B(p,\hat{p})^{-1}\Lambda^{-1}B(\Lambda p,\hat{p})|0]\, Z_j\right)_{mm'} a^{\dagger}_{\Lambda p,j,m'}. \qquad \cdots(12)$$

Next we assume that if the particle has a corresponding antiparticle its creation and annihilation operators $b^{\dagger}_{p,j,m}$ and $b_{p,j,m}$ have the same transformation properties as $a^{\dagger}_{p,j,m}$ and $a_{p,j,m}$ respectively.

We can now construct Lorentz invariant fields defined on space-time using these operators, but it is advantageous to use an intermediate step in the construction. To this end we let $\Gamma^{(j)}([\Lambda|0])$ be a $(2j+1)\times(2j+1)$ dimensional representation of the covering group of the orthochronous Lorentz group that coincides with D^j when Λ is a rotation. Thus $\Gamma^{(j)}$ can be either $\Gamma^{j,0}$ the 'right handed' representation or $\Gamma^{0,j}$ the 'left handed' representation. We note that $\Gamma^{(j)}$ is <u>not</u> a unitary representation. Now we define the ancilliary operators $\alpha_{p,j,m}$ and $\beta_{p,j,m}$ by

$$\alpha_{p,j,m} = \left(\frac{2p_4}{Mc}\right)^{\frac{1}{2}} \Gamma^{(j)}([B(p,\hat{p})|0])_{mm'}\, a_{p,j,m'} \qquad \cdots(13)$$

and

$$\beta_{p,j,m} = \left(\frac{2p_4}{Mc}\right)^{\frac{1}{2}} \left(\Gamma^{(j)}([B(p,\hat{p})|0]) Z_j\right)_{mm'}\, b^{\dagger}_{p,j,m'}. \qquad \cdots(14)$$

The transformation properties of these operators are

$$u([\Lambda|t])\,\alpha_{p,j,m}\,u([\Lambda|t])^{-1}$$
$$= \exp -\tfrac{i}{\hbar}\{(\Lambda p)_\sigma t^\sigma\}\,\Gamma^{(j)}([\Lambda^{-1}|0])_{mm'}\,\alpha_{\Lambda p,j,m'} \qquad \cdots(15)$$

and
$$u([\Lambda|t])\,\beta_{p,j,m}\,u([\Lambda|t])^{-1}$$
$$= \exp \tfrac{i}{\hbar}\{(\Lambda p)_\sigma t^\sigma\}\,\Gamma^{(j)}([\Lambda^{-1}|0])_{mm'}\,\beta_{\Lambda p,j,m'}. \qquad \cdots(16)$$

We note that if the particle under consideration is its own antiparticle then $\alpha^{+}_{p,j,m} = \beta^{+}_{p,j,m}$, but $\alpha_{p,j,m} \neq \beta_{p,j,m}$ unless $j = 0$.

Finally to obtain the field for a particle of spin j transforming as the $(2j+1)$ dimensional representation $\Gamma^{(j)}$ of the orthochronous Lorentz group we define

$$\chi_{j,m}(x)$$
$$= \left(\frac{1}{2\pi\hbar}\right)^3 \int d^3p \left(\frac{Mc}{2p_4}\right)\left\{\xi\,\alpha_{p,j,m}\,e^{-\frac{i}{\hbar}p.x} + \eta\,\beta_{p,j,m}\,e^{\frac{i}{\hbar}p.x}\right\} \qquad \cdots(17)$$

with ξ, η complex numbers such that $|\xi| = |\eta| = 1$. We note that the field $\chi_{j,m}(x)$ is Lorentz invariant for any value of ξ and η but if we require crossing symmetry and causality then we need this restriction. This field then has the transformation property

$$u([\Lambda|t])\,\chi_{j,m}(x)\,u([\Lambda|t])^{-1}$$
$$= \Gamma^{(j)}([\Lambda^{-1}|0])_{mm'}\,\chi_{j,m'}(\Lambda x + t). \qquad \cdots(18)$$

We complete this section with the proof of a result we will need later.

Lemma 3.1

The Clebsh-Gordan matrix $C_{nm,r}$ defined by

$$C_{nm,r}\begin{bmatrix} D^{j+\frac{1}{2}}(R) & 0 \\ 0 & D^{j-\frac{1}{2}}(R) \end{bmatrix}_{rr'}(C^{-1})_{r',n'm'} = D^{\frac{1}{2}}(R)_{nn'} \otimes D^{j}(R)_{mm'} \cdots(19)$$

(cf. equation (53) of Chapter 2) and the matrices $Z_{\frac{1}{2}}$, Z_j, $Z_{j+\frac{1}{2}}$ and $Z_{j-\frac{1}{2}}$ defined by eqations (9), (10) and (11) are related by

$$(C^{-1})_{r,nm}\,(Z_{\frac{1}{2}})_{nn'} \otimes (Z_j)_{mm'}\,C_{n'm',r'}\,(Z^{-1}_{j+\frac{1}{2}} \oplus Z^{-1}_{j-\frac{1}{2}})_{r'r''} = I_{rr''} \quad \cdots(20)$$

Proof

Take the complex conjugate of equation (18) and note that $C_{nm,r}$ is by convention (cf. Cornwell [39]) real, to give

$$c_{nm,r}\begin{bmatrix} D^{j+\frac{1}{2}}(R)^{*} & 0 \\ 0 & D^{j-\frac{1}{2}*}(R)\end{bmatrix}_{rr'}(c^{-1})_{r',n'm'} = D^{\frac{1}{2}}(R)^{*}_{nn'}\otimes D^{j}(R)^{*}_{mm'}\,.$$

Then using equation (9) we have

$$c_{nm,r}\begin{bmatrix} z^{-1}_{j+\frac{1}{2}}D^{j+\frac{1}{2}}(R)z_{j+\frac{1}{2}} & 0 \\ 0 & z^{-1}_{j-\frac{1}{2}}D^{j-\frac{1}{2}}(R)z_{j-\frac{1}{2}}\end{bmatrix}_{rr'}(c^{-1})_{r',n'm'}$$

$$= (z^{-1}_{\frac{1}{2}}D^{\frac{1}{2}}(R)z_{\frac{1}{2}})_{nn'}\otimes(z^{-1}_{j}D^{j}(R)z_{j})_{mm'}\,,$$

$$= ((z^{-1}_{\frac{1}{2}}\otimes z^{-1}_{j})(D^{\frac{1}{2}}(R)\otimes D^{j}(R))(z_{\frac{1}{2}}\otimes z_{j}))_{nm,n'm'}\,,$$

$$= ((z^{-1}_{\frac{1}{2}}\otimes z^{-1}_{j})\,c\begin{bmatrix} D^{j+\frac{1}{2}}(R) & 0 \\ 0 & D^{j-\frac{1}{2}}(R)\end{bmatrix}c^{-1}(z_{\frac{1}{2}}\otimes z_{j}))_{nm,n'm'}\,.$$

Thus the matrix

$$(c^{-1})_{r,nm}(z_{\frac{1}{2}})_{nn'}\otimes(z_{j})_{mm'}\,c_{n'm',r'}\begin{bmatrix} z^{-1}_{j+\frac{1}{2}} & 0 \\ 0 & z^{-1}_{j-\frac{1}{2}}\end{bmatrix}_{r'r''}$$

commutes with $\begin{bmatrix} D^{j+\frac{1}{2}}(R) & 0 \\ 0 & D^{j-\frac{1}{2}}(R)\end{bmatrix}$ for all R.

So that Shur's Lemma implies that it must be of the form $\begin{bmatrix} \eta I_{2j+2} & 0 \\ 0 & \eta' I_{2j}\end{bmatrix}$. But $c_{nm,r}$ and z_j are chosen such that their determinant is one. The result follows.

3.2 The Left Handed Supermultiplet.

We assume that $Q_\alpha|0\rangle = 0$ so that we can express the action of Q_α on our rest state creation operators as a commutator

$$\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q_\alpha\,,\,a^{+}_{\hat{p},k,m}] = M(Q_\alpha)_{m'k',mk}\,a^{+}_{\hat{p},k',m'}\,. \qquad \ldots(21)$$

Here $M(Q_\alpha)_{m'k',mk}$ is a matrix whose coefficients can be determined from Theorem 2.3 of Chapter 2. We order the operators such that

$$a^{+}_{\hat{p},k,m} = \begin{bmatrix} a^{+}_{\hat{p},j,m} \\ a^{+}_{\hat{p},j+\frac{1}{2},m} \\ a^{+}_{\hat{p},j-\frac{1}{2},m} \\ \tilde{a}^{+}_{\hat{p},j,m}\end{bmatrix}_{k},\quad k = j,\ j+\tfrac{1}{2},\ j-\tfrac{1}{2},\ j \qquad (22)$$

$$\text{with} \qquad a^{+}_{\hat{p},j,m}|0\rangle = |\hat{p},j,m\rangle = |Mc,j,m\rangle \,, \qquad \cdots(23)$$

$$a^{+}_{\hat{p},j+\frac{1}{2},m}|0\rangle = |\hat{p},j+\tfrac{1}{2},m\rangle = |Mc,j+\tfrac{1}{2},m\rangle \,, \qquad \cdots(24)$$

$$a^{+}_{\hat{p},j-\frac{1}{2},m}|0\rangle = |\hat{p},j-\tfrac{1}{2},m\rangle = |Mc,j-\tfrac{1}{2},m\rangle \qquad \cdots(25)$$

$$\text{and} \qquad \underset{\sim}{a}^{+}_{\hat{p},j,m}|0\rangle = |\hat{p},j,m\rangle\rangle = |Mc,j,m\rangle\rangle \,. \qquad \cdots(26)$$

Here the vectors $|Mc,j,m\rangle$ etc. are as used in Chapter 2. Note that the operators $a^{+}_{\hat{p},j,m}$ and $\underset{\sim}{a}^{+}_{\hat{p},j,m}$ create different particles and are distinguished by the tilde ie. $\underset{\sim}{a}^{+}$ for the operator that creates the particle with two braces!

Then the matrices $M(Q_\alpha)$ can be written

$$M(Q^{+}_{Ln})_{m''k',m'k} =$$

$$\left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \left(\begin{array}{cc|c} \frac{1}{2} & j & j+\frac{1}{2} \\ n & m & m+n \end{array}\right) & 0 & 0 & 0 \\ \left(\begin{array}{cc|c} \frac{1}{2} & j & j-\frac{1}{2} \\ n & m & m+n \end{array}\right) & 0 & 0 & 0 \\ 0 & 2n\left(\begin{array}{cc|c} \frac{1}{2} & j & j+\frac{1}{2} \\ -n & m+n+\frac{1}{2} & m+\frac{1}{2} \end{array}\right) & 2n\left(\begin{array}{cc|c} \frac{1}{2} & j & j-\frac{1}{2} \\ -n & m+n+\frac{1}{2} & m+\frac{1}{2} \end{array}\right) & 0 \end{array}\right]_{m''k',m'k} \qquad \cdots(27)$$

and $M(Q_{Ln})_{m''k',m'k} =$

$$\left[\begin{array}{cccc} 0 & \left(\begin{array}{cc|c} \frac{1}{2} & j & j+\frac{1}{2} \\ n & m+\frac{1}{2}+n & m+\frac{1}{2} \end{array}\right) & \left(\begin{array}{cc|c} \frac{1}{2} & j & j-\frac{1}{2} \\ n & m-n-\frac{1}{2} & m-\frac{1}{2} \end{array}\right) & 0 \\ 2n\left(\begin{array}{cc|c} \frac{1}{2} & j & j+\frac{1}{2} \\ -n & m & m-n \end{array}\right) & 0 & 0 & 0 \\ 2n\left(\begin{array}{cc|c} \frac{1}{2} & j & j-\frac{1}{2} \\ -n & m & m-n \end{array}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]_{m''k',m'k} . \qquad \cdots(28)$$

The parity of the creation and annihilation operators is defined by

$$|a^{+}_{p,j,m}| = |a_{p,j,m}| = |b_{p,j,m}| = |b^{+}_{p,j,m}|$$
$$= \begin{cases} 0 & \text{if } j \text{ is an integer,} \\ 1 & \text{if } j \text{ is a half-integer.} \end{cases}$$

This is conveniently expressed as

$$|a^{+}_{p,j,m}| = (-1)^{2j} \text{ etc.} \qquad \cdots(29)$$

The order of the indices of the matrices $M(Q_\alpha)$ is chosen so that the matrices obey the same commutation rules as the operators Q_α, in the same way as the matrices $ad(\gamma)$ as defined in Chapter 2 of Part I.

Now we combine equation (20) with equation (6) to obtain the action of the Q_α on a boosted state. Thus

$$
\begin{aligned}
&\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q_\alpha\,,\, a^{+}_{p,k,m}] \\
&= \left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q_\alpha\,,\, \left(\tfrac{Mc}{p_4}\right)^{\frac{1}{2}} U([B(p,\hat{p})|0])\, a^{+}_{\hat{p},k,m}\, U([B(p,\hat{p})|0])^{-1}]\,, \\
&= \left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}\left(\tfrac{Mc}{p_4}\right)^{\frac{1}{2}}\Big\{Q_\alpha U([B(p,\hat{p})|0])\, a^{+}_{\hat{p},k,m}\, U([B(p,\hat{p})|0])^{-1} \\
&\qquad - (-1)^{2j} U([B(p,\hat{p})|0])\, a^{+}_{\hat{p},k,m}\, U([B(p,\hat{p})|0])^{-1} Q_\alpha\Big\}\,, \\
&= \left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}\left(\tfrac{Mc}{p_4}\right)^{\frac{1}{2}} U([B(p,\hat{p})|0])\Big\{U([B(p,\hat{p})|0])^{-1} Q_\alpha U([B(p,\hat{p})|0])\, a^{+}_{\hat{p},k,m} \\
&\qquad - (-1)^{2j} a^{+}_{\hat{p},k,m}\, U([B(p,\hat{p})|0])^{-1} Q_\alpha U([B(p,\hat{p})|0])\Big\} U([B(p,\hat{p})|0])^{-1}\,, \\
&= \left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}\left(\tfrac{Mc}{p_4}\right)^{\frac{1}{2}} U([B(p,\hat{p})|0]) \\
&\qquad \times [U([B(p,\hat{p})|0])^{-1} Q_\alpha U([B(p,\hat{p})|0])\,,\, a^{+}_{\hat{p},k,m}]\, U([B(p,\hat{p})|0])^{-1} \quad \cdots(30)
\end{aligned}
$$

To proceed we need to determine the action of the operator corresponding to a boost on the supersymmetry generators Q_α. Since, later, we also need to work in two-component form it is convenient to do this at the same time. Recall equations (15)and (20) of Chapter 1 which are

$$[M_{\lambda\mu}\,,\, Q_{Ln}] = \tfrac{\hbar}{2i}(\sigma^L_\lambda \sigma^R_\mu)_{nn'} Q_{Ln'} \qquad \cdots(31)$$

and

$$[M_{\lambda\mu}\,,\, Q_{Rn}] = \tfrac{\hbar}{2i}(\sigma^R_\lambda \sigma^L_\mu)_{nn'} Q_{Rn'}\,. \qquad \cdots(32)$$

These equations imply that

$$U([\Lambda|0])Q_{Ln}U([\Lambda|0])^{-1} = \Gamma^{0,\frac{1}{2}}([\Lambda|0])^{-1}_{nn'} Q_{Ln'} \qquad \cdots(33)$$

and

$$U([\Lambda|0])Q_{Rn}U([\Lambda|0])^{-1} = \Gamma^{\frac{1}{2},0}([\Lambda|0])^{-1}_{nn'} Q_{Rn'}\,, \qquad \cdots(34)$$

or alternatively

$$U([\Lambda|0])^{-1} Q_{Ln}U([\Lambda|0]) = \Gamma^{0,\frac{1}{2}}([\Lambda|0])_{nn'} Q_{Ln'} \qquad \cdots(35)$$

and

$$U([\Lambda|0])^{-1} Q_{Rn}U([\Lambda|0]) = \Gamma^{\frac{1}{2},0}([\Lambda|0])_{nn'} Q_{Rn'}\,. \qquad \cdots(36)$$

Now combining equations (30) and (35) we obtain

$$
\begin{aligned}
&\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q_{Ln}\,,\, a^{+}_{p,k,m}] \\
&= \left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}\left(\tfrac{Mc}{p_4}\right)^{\frac{1}{2}} U([B(p,\hat{p})|0])[Q_{Ln'}\,,\, a^{+}_{\hat{p},k,m}]\, U([B(p,\hat{p})|0])^{-1} \\
&\qquad\qquad \times \Gamma^{0,\frac{1}{2}}([B(p,\hat{p})|0])_{nn'}
\end{aligned}
$$

$$= \left(\frac{Mc}{P_4}\right) U([B(p,\hat{p})|0]) a^+_{\hat{p},k',m'} U([B(p,\hat{p})|0])^{-1}$$
$$\times \Gamma^{0,\frac{1}{2}}([B(p,\hat{p})|0])_{nn'} M(Q_{Ln'})_{m'k',mk}$$

$$= \Gamma^{0,\frac{1}{2}}([B(p,\hat{p})|0])_{nn'} M(Q_{Ln'})_{m'k',mk}\, a^+_{p,k',m'}.$$

So that

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} [Q_{Ln}, a^+_{p,k,m}]$$
$$= \Gamma^{0,\frac{1}{2}}([B(p,\hat{p})|0])_{nn'} M(Q_{Ln'})_{m'k',mk}\, a^+_{p,k',m'}. \qquad \ldots(37)$$

Similarly

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} [Q_{Rn}, a^+_{p,k,m}]$$
$$= \Gamma^{\frac{1}{2},0}([B(p,\hat{p})|0])_{nn'} M(Q_{Rn'})_{m'k',mk}\, a^+_{p,k',m'}. \qquad \ldots(38)$$

Now if we put

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} [Q_\alpha, a_{\hat{p},k,m}] = N(Q_\alpha)_{m'k',mk}\, a_{\hat{p},k',m'}. \qquad \ldots(39)$$

We have

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} [Q_{Ln}, a_{\hat{p},k,m}]$$
$$= \Gamma^{0,\frac{1}{2}}([B(p,\hat{p})|0])_{nn'} N(Q_{Ln'})_{m'k',mk}\, a_{p,k',m'} \qquad \ldots(40)$$

and

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} [Q_{Rn}, a_{\hat{p},k,m}]$$
$$= \Gamma^{\frac{1}{2},0}([B(p,\hat{p})|0])_{nn'} N(Q_{Rn'})_{m'k',mk}\, a_{p,k',m'}. \qquad \ldots(41)$$

The transformation properties of the antiparticle creation and annihilation operators $b_{p,j,m}$ and $b^+_{p,j,m}$ are obtained simply by replacing a with b in equations (37), (38), (40) and (41).

To determine the coefficients of the matrices $N(Q_{Ln'})$ and $N(Q_{Rn'})$ we consider equation (21) and take the adjoint as follows

$$[Q_\alpha, a^+_{\hat{p},k,m}] = M(Q_\alpha)_{m'k',mk}\, a^+_{\hat{p},k',m'} ,$$
$$[a_{\hat{p},k,m}, Q^+_\alpha] = a_{\hat{p},k',m'}\, M(Q_\alpha)_{m'k',mk} ,$$
$$(-1)^{2j}[Q^+_\alpha, a_{\hat{p},k,m}] = M(Q_\alpha)_{m'k',mk}\, a_{\hat{p},k',m'} . \qquad \ldots(42)$$

The next step is to define the ancilliary operators following equations (13) and (14), which we choose to all be left handed. Thus

$$\alpha^L_{p,j,m} = \left(\frac{2P_4}{Mc}\right)^{\frac{1}{2}} \Gamma^{0,j}([B(p,\hat{p})|0])_{mm'}\, a_{p,j,m'}, \qquad \ldots(43a)$$

$$\alpha^L_{p,j+\frac{1}{2},m} = \left(\frac{2p_4}{Mc}\right)^{\frac{1}{2}}\Gamma^{0,j+\frac{1}{2}}([B(p,\hat{p})|0])_{mm'}\, a_{p,j+\frac{1}{2},m'}\ , \qquad \ldots(43b)$$

$$\alpha^L_{p,j-\frac{1}{2},m} = \left(\frac{2p_4}{Mc}\right)^{\frac{1}{2}}\Gamma^{0,j-\frac{1}{2}}([B(p,\hat{p})|0])_{mm'}\, a_{p,j-\frac{1}{2},m'} \qquad \ldots(43c)$$

and $$\underset{\sim}{\alpha}^L_{p,j,m} = \left(\frac{2p_4}{Mc}\right)^{\frac{1}{2}}\Gamma^{0,j}([B(p,\hat{p})|0])_{mm'}\, \underset{\sim}{a}_{p,j,m'} \qquad \ldots(43d)$$

and

$$\beta^L_{p,j,m} = \left(\frac{2p_4}{Mc}\right)^{\frac{1}{2}}(\Gamma^{0,j}([B(p,\hat{p})|0])z_j)_{mm'}\, b^{+}_{p,j,m'}\ , \qquad \ldots(44a)$$

$$\beta^L_{p,j+\frac{1}{2},m} = \left(\frac{2p_4}{Mc}\right)^{\frac{1}{2}}(\Gamma^{0,j+\frac{1}{2}}([B(p,\hat{p})|0])z_{j+\frac{1}{2}})_{mm'}\, b^{+}_{p,j+\frac{1}{2},m'}\ , \qquad \ldots(44b)$$

$$\beta^L_{p,j-\frac{1}{2},m} = \left(\frac{2p_4}{Mc}\right)^{\frac{1}{2}}(\Gamma^{0,j-\frac{1}{2}}([B(p,\hat{p})|0])z_{j-\frac{1}{2}})_{mm'}\, b^{+}_{p,j-\frac{1}{2},m'} \qquad \ldots(44c)$$

and $$\underset{\sim}{\beta}^L_{p,j,m} = \left(\frac{2p_4}{Mc}\right)^{\frac{1}{2}}(\Gamma^{0,j}([B(p,\hat{p})|0])z_j)_{mm'}\, \underset{\sim}{b}^{+}_{p,j,m'}\ . \qquad \ldots(44d)$$

To proceed we rewrite equations (37), (38), (40) and (41) in terms of the ancilliary operators and reduce the resulting equations. The analysis is very similar in each case so that we only detail one of the calculations then quote the results.

Consider equation (38) for the field $\beta^L_{p,j,m}$

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q_{Rn}\ ,\ \beta^L_{p,j,m}]$$
$$= \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[-i(\sigma_2 Q^{+}_L)_n\ ,\ \beta^L_{p,j,m}]\ ,$$
$$= (\Gamma^{\frac{1}{2},0}([B(p,\hat{p})|0]))_{nn'}(\Gamma^{0,j}([B(p,\hat{p})|0])z_j)_{mm'}\begin{pmatrix}\frac{1}{2} & j & \Big| & j+\frac{1}{2}\\ n' & m' & \Big| & \underbrace{m'+n'}_{=m''}\end{pmatrix}$$
$$\times\,(\Gamma^{0,j+\frac{1}{2}}([B(p,\hat{p})|0])z_{j+\frac{1}{2}})^{-1}_{m''m'''}\,\beta^L_{p,j+\frac{1}{2},m'''}$$
$$+(\Gamma^{\frac{1}{2},0}([B(p,\hat{p})|0]))_{nn'}(\Gamma^{0,j}([B(p,\hat{p})|0])z_j)_{mm'}\begin{pmatrix}\frac{1}{2} & j & \Big| & j-\frac{1}{2}\\ n' & m' & \Big| & \underbrace{m'+n'}_{=m''}\end{pmatrix}$$
$$\times\,(\Gamma^{0,j-\frac{1}{2}}([B(p,\hat{p})|0])z_{j-\frac{1}{2}})^{-1}_{m''m'''}\,\beta^L_{p,j-\frac{1}{2},m'''}\ ,$$
$$= (\Gamma^{\frac{1}{2},0}([B(p,\hat{p})|0]))_{nn'}\otimes(\Gamma^{0,j}([B(p,\hat{p})|0]))_{mm'}$$
$$\times\,((z_{\frac{1}{2}}\otimes z_j)C(z^{-1}_{j+\frac{1}{2}}\oplus z^{-1}_{j-\frac{1}{2}}))_{m'm''}$$
$$\times((\Gamma^{0,j+\frac{1}{2}}([B(p,\hat{p})|0]))\oplus(\Gamma^{0,j-\frac{1}{2}}([B(p,\hat{p})|0])))^{-1}_{m''m'''}\begin{bmatrix}\beta^L_{p,j+\frac{1}{2},r}\\ \beta^L_{p,j-\frac{1}{2},r}\end{bmatrix}_{m'''}.$$

Here we have recognised that

$$\left(\begin{pmatrix}\frac{1}{2} & j & \Big| & j+\frac{1}{2}\\ n' & m' & \Big| & m'+n'\end{pmatrix}\oplus\begin{pmatrix}\frac{1}{2} & j & \Big| & j-\frac{1}{2}\\ n' & m' & \Big| & m'+n'\end{pmatrix}\right) = C_{n'm',\,m'+n'}$$

is the Clebsh-Gordan matrix as defined by equation (53) of Chapter 2. Now using equation (20) we obtain

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q_{Rn}, \beta^{L}_{p,j,m}]$$

$$= (\Gamma^{\frac{1}{2},0}([B(p,\hat{p})|0]))_{nn'} \otimes (\Gamma^{0,j}([B(p,\hat{p})|0]))_{mm'} C_{n'm',m''}$$

$$\times (\Gamma^{0,j+\frac{1}{2}}([B(p,\hat{p})|0]) \oplus \Gamma^{0,j-\frac{1}{2}}([B(p,\hat{p})|0]))^{-1}_{m''m'''} \begin{bmatrix} \beta^{L}_{p,j+\frac{1}{2},r} \\ \beta^{L}_{p,j-\frac{1}{2},r} \end{bmatrix}_{m'''} ,$$

$$= (\Gamma^{\frac{1}{2},0}([B(p,\hat{p})|0])_{nn'} \otimes (\Gamma^{0,j}([B(p,\hat{p})|0]))_{mm'} (\Gamma^{0,\frac{1}{2}}([B(p,\hat{p})|0])^{-1}_{n'n''}$$

$$\otimes \Gamma^{0,j}([B(p,\hat{p})|0])^{-1}_{m'm''} C_{n''m'',m'''} \begin{bmatrix} \beta^{L}_{p,j+\frac{1}{2},r} \\ \beta^{L}_{p,j-\frac{1}{2},r} \end{bmatrix}_{m'''} ,$$

using equation (53) of Chapter 2. Then using equations (46) and (1) of the appendix we obtain

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q^{+}_{Ln}, \beta^{L}_{p,j,m}]$$

$$= \frac{1}{Mc}(\sigma_2 \sigma_\mu p_\mu)_{nn'} \left\{ \begin{pmatrix} \frac{1}{2} & j & j+\frac{1}{2} \\ n' & m & m+n' \end{pmatrix} \beta^{L}_{p,j+\frac{1}{2},m+n'} \right.$$

$$\left. + \begin{pmatrix} \frac{1}{2} & j & j-\frac{1}{2} \\ n' & m & m+n' \end{pmatrix} \beta^{L}_{p,j-\frac{1}{2},m+n'} \right\} \qquad \cdots (45)$$

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q^{+}_{Ln}, \beta^{L}_{p,j+\frac{1}{2},m+\frac{1}{2}}]$$

$$= \frac{-i}{Mc}(\sigma_2 \sigma_\mu \sigma_2 p_\mu)_{nn'} \begin{pmatrix} \frac{1}{2} & j & j+\frac{1}{2} \\ n' & m+\frac{1}{2}+n' & m+\frac{1}{2} \end{pmatrix} \mathcal{R}^{L}_{p,j,m+\frac{1}{2}-n'} , \qquad \cdots (46)$$

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q^{+}_{Ln}, \beta^{L}_{p,j-\frac{1}{2},m-\frac{1}{2}}]$$

$$= \frac{-i}{Mc}(\sigma_2 \sigma_\mu \sigma_2 p_\mu)_{nn'} \begin{pmatrix} \frac{1}{2} & j & j-\frac{1}{2} \\ n' & m-\frac{1}{2}+n' & m-\frac{1}{2} \end{pmatrix} \mathcal{R}^{L}_{p,j,m-\frac{1}{2}-n'} , \qquad \cdots (47)$$

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q^{+}_{Ln}, \mathcal{R}^{L}_{p,j,m}] = 0 \qquad , \cdots (48)$$

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q_{Ln}, \beta^{L}_{p,j,m}] = 0 \qquad , \cdots (49)$$

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q_{Ln}, \beta^{L}_{p,j+\frac{1}{2},m+\frac{1}{2}}]$$

$$= -(\sigma_2)_{nn'} \begin{pmatrix} \frac{1}{2} & j & j+\frac{1}{2} \\ n' & m+\frac{1}{2}-n' & m+\frac{1}{2} \end{pmatrix} \beta^{L}_{p,j,m+\frac{1}{2}-n'} \qquad , \cdots (50)$$

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q_{Ln}, \beta^{L}_{p,j-\frac{1}{2},m-\frac{1}{2}}]$$

$$= -(\sigma_2)_{nn'} \begin{pmatrix} \frac{1}{2} & j & j-\frac{1}{2} \\ n' & m-\frac{1}{2}-n' & m+\frac{1}{2} \end{pmatrix} \beta^{L}_{p,j,m-\frac{1}{2}-n'} \qquad \cdots (51)$$

and
$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q_{Ln}, \mathcal{R}^{L}_{p,j,m}]$$

$$= i\begin{pmatrix} \frac{1}{2} & j & j+\frac{1}{2} \\ n & m & m+n \end{pmatrix} \beta^{L}_{p,j+\frac{1}{2},m+n} + i\begin{pmatrix} \frac{1}{2} & j & j-\frac{1}{2} \\ n & m & m+n \end{pmatrix} \beta^{L}_{p,j-\frac{1}{2},m+n} ; \cdots (52)$$

Then, for the annihilation fields, we obtain

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q_{Ln}, \alpha^L_{p,j,m}] = (-1)^{2j}\left(\begin{matrix}\frac{1}{2} & j\\ n & m\end{matrix}\middle|\begin{matrix}j+\frac{1}{2}\\ m+n\end{matrix}\right)\alpha^L_{p,j+\frac{1}{2},m+n} + (-1)^{2j}\left(\begin{matrix}\frac{1}{2} & j\\ n & m\end{matrix}\middle|\begin{matrix}j-\frac{1}{2}\\ m+n\end{matrix}\right)\alpha^L_{p,j-\frac{1}{2},m+n}, \qquad \cdots(53)$$

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q_{Ln}, \alpha^L_{p,j+\frac{1}{2},m+n'}] = (-1)^{2j+1}(i\sigma_2)_{nn'}\left(\begin{matrix}\frac{1}{2} & j\\ n' & m\end{matrix}\middle|\begin{matrix}j+\frac{1}{2}\\ m+n'\end{matrix}\right)\tilde{\alpha}^L_{p,j,m}, \qquad \cdots(54)$$

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q_{Ln}, \alpha^L_{p,j-\frac{1}{2},m+n'}] = (-1)^{2j+1}(i\sigma_2)_{nn'}\left(\begin{matrix}\frac{1}{2} & j\\ n' & m\end{matrix}\middle|\begin{matrix}j-\frac{1}{2}\\ m+n'\end{matrix}\right)\tilde{\alpha}^L_{p,j,m}, \qquad \cdots(55)$$

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q_{Ln}, \tilde{\alpha}^L_{p,j,m}] = 0, \qquad \cdots(56)$$

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q^{+}_{Ln}, \alpha^L_{p,j,m}] = 0, \qquad \cdots(57)$$

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q^{+}_{Ln}, \alpha^L_{p,j+\frac{1}{2},m}] = (-1)^{2j+1}\frac{1}{Mc}(\sigma_2\sigma_\mu\sigma_2 p_\mu)_{nn'}\left(\begin{matrix}\frac{1}{2} & j\\ n' & m+\frac{1}{2}-n'\end{matrix}\middle|\begin{matrix}j+\frac{1}{2}\\ m+\frac{1}{2}\end{matrix}\right)\alpha^L_{p,j,m+\frac{1}{2}-n'}, \qquad \cdots(58)$$

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q^{+}_{Ln}, \alpha^L_{p,j-\frac{1}{2},m}] = (-1)^{2j+1}\frac{1}{Mc}(\sigma_2\sigma_\mu\sigma_2 p_\mu)_{nn'}\left(\begin{matrix}\frac{1}{2} & j\\ n' & m-\frac{1}{2}-n'\end{matrix}\middle|\begin{matrix}j-\frac{1}{2}\\ m-\frac{1}{2}\end{matrix}\right)\alpha^L_{p,j,m-\frac{1}{2}-n'} \qquad \cdots(59)$$

and

$$\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q^{+}_{Ln}, \tilde{\alpha}^L_{p,j,m}] = \frac{i}{Mc}(\sigma_2\sigma_\mu p_\mu)_{nn'}\left\{\left(\begin{matrix}\frac{1}{2} & j\\ n' & m\end{matrix}\middle|\begin{matrix}j+\frac{1}{2}\\ m+n'\end{matrix}\right)\alpha^L_{p,j+\frac{1}{2},m+n'} + \left(\begin{matrix}\frac{1}{2} & j\\ n' & m\end{matrix}\middle|\begin{matrix}j-\frac{1}{2}\\ m+n'\end{matrix}\right)\alpha^L_{p,j-\frac{1}{2},m+n'}\right\}. \qquad \cdots(60)$$

We can now construct the fields that form the left handed supermultiplet. The phase factors η and ξ of equation (17) have been chosen with some foreknowledge. We define

$$\chi^L_{j,m}(x) = \left(\frac{1}{2\pi\hbar}\right)^3\int d^3p\left(\frac{Mc}{2p_4}\right)\left\{(-1)^{2j}\alpha^L_{p,j,m}e^{-\frac{i}{\hbar}p\cdot x} - i\tilde{\beta}^L_{p,j,m}e^{\frac{i}{\hbar}p\cdot x}\right\}, \qquad \cdots(61a)$$

$$\chi^L_{j+\frac{1}{2},m}(x) = \left(\frac{1}{2\pi\hbar}\right)^3\int d^3p\left(\frac{Mc}{2p_4}\right)\left\{\alpha^L_{p,j+\frac{1}{2},m}e^{-\frac{i}{\hbar}p\cdot x} + \beta^L_{p,j+\frac{1}{2},m}e^{\frac{i}{\hbar}p\cdot x}\right\}, \qquad \cdots(61b)$$

$$\chi^L_{j-\frac{1}{2},m}(x) = \left(\frac{1}{2\pi\hbar}\right)^3\int d^3p\left(\frac{Mc}{2p_4}\right)\left\{\alpha^L_{p,j-\frac{1}{2},m}e^{-\frac{i}{\hbar}p\cdot x} + \beta^L_{p,j-\frac{1}{2},m}e^{\frac{i}{\hbar}p\cdot x}\right\} \qquad \cdots(61c)$$

and $$\underset{\sim}{\chi}^{L}_{j,m}(x) = \left(\frac{1}{2\pi\hbar}\right)^{3}\int d^{3}p\left(\frac{Mc}{2p_4}\right)\left\{i(-1)^{2j}\underset{\sim}{\alpha}_{p,j,m}e^{-\frac{i}{\hbar}p\cdot x}+\beta^{L}_{p,j,m}e^{\frac{i}{\hbar}p\cdot x}\right\}. \quad \cdots(61d)$$

So that

$$U([\Lambda|t])\chi^{L}_{j,m}(x)U([\Lambda|t])^{-1} = \Gamma^{0,j}([\Lambda^{-1}|0])_{mm'}\chi^{L}_{j,m'}(\Lambda x+t); \cdots(62a)$$

$$U([\Lambda|t])\chi^{L}_{j+\frac{1}{2},m}(x)U([\Lambda|t])^{-1} = \Gamma^{0,j+\frac{1}{2}}([\Lambda^{-1}|0])_{mm'}\chi^{L}_{j+\frac{1}{2},m'}(\Lambda x+t); \cdots(62b)$$

$$U([\Lambda|t])\chi^{L}_{j-\frac{1}{2},m}(x)U([\Lambda|t])^{-1} = \Gamma^{0,j-\frac{1}{2}}([\Lambda^{-1}|0])_{mm'}\chi^{L}_{j-\frac{1}{2},m'}(\Lambda x+t) \cdots(62c)$$

and $$U([\Lambda|t])\underset{\sim}{\chi}^{L}_{j,m}(x)U([\Lambda|t])^{-1} = \Gamma^{0,j}([\Lambda^{-1}|0])_{mm'}\chi^{L}_{j,m'}(\Lambda x+t); \cdots(62d)$$

The action of the supersymmetry generators on these fields can now be evaluated as follows.

$$\begin{aligned}
&\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q^{\dagger}_{Ln}, \underset{\sim}{\chi}^{L}_{j,m}] \\
&= \left(\frac{1}{2\pi\hbar}\right)^{3}\int d^{3}p\left(\frac{Mc}{2p_4}\right)\left\{\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q^{\dagger}_{Ln}, i(-1)^{2j}\underset{\sim}{\alpha}^{L}_{p,j,m}]e^{-\frac{i}{\hbar}p\cdot x} + \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}[Q^{\dagger}_{Ln}, \beta^{L}_{p,j,m}]e^{\frac{i}{\hbar}p\cdot x}\right\}, \\
&= \left(\frac{1}{2\pi\hbar}\right)^{3}\int d^{3}p\left(\frac{Mc}{2p_4}\right)\frac{1}{Mc}(\sigma_2\sigma_\mu)_{nn'}\left\{\left(\begin{matrix}\frac{1}{2} & j\\ n' & m\end{matrix}\middle|\begin{matrix}j+\frac{1}{2}\\ m+n'\end{matrix}\right)\left\{-p_\mu\alpha^{L}_{p,j+\frac{1}{2},m+n'}e^{-\frac{i}{\hbar}p\cdot x}\right.\right. \\
&\quad \left.+p_\mu\beta^{L}_{p,j+\frac{1}{2},m+n'}e^{\frac{i}{\hbar}p\cdot x}\right\} + \left(\begin{matrix}\frac{1}{2} & j\\ n' & m\end{matrix}\middle|\begin{matrix}j-\frac{1}{2}\\ m+n'\end{matrix}\right)\left\{-p_\mu\alpha^{L}_{p,j-\frac{1}{2},m+n'}e^{-\frac{i}{\hbar}p\cdot x}\right. \\
&\quad \left.\left.+p_\mu\beta^{L}_{p,j-\frac{1}{2},m+n'}e^{\frac{i}{\hbar}p\cdot x}\right\}\right\}, \\
&= -\frac{i\hbar}{Mc}\left(\sigma_2\sigma_\mu\frac{\partial}{\partial x^\mu}\right)_{nn'}\left\{\left(\begin{matrix}\frac{1}{2} & j\\ n' & m\end{matrix}\middle|\begin{matrix}j+\frac{1}{2}\\ m+n'\end{matrix}\right)\chi^{L}_{j+\frac{1}{2},m+n'} + \left(\begin{matrix}\frac{1}{2} & j\\ n' & m\end{matrix}\middle|\begin{matrix}j-\frac{1}{2}\\ m+n'\end{matrix}\right)\chi^{L}_{j-\frac{1}{2},m+n'}\right\}.
\end{aligned}$$

So that

$$[Q^{\dagger}_{Ln}, \underset{\sim}{\chi}^{L}_{j,m}] = -i\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}\left(\sigma_2\sigma_\mu\frac{\partial}{\partial x^\mu}\right)_{nn'}\left\{\left(\begin{matrix}\frac{1}{2} & j\\ n' & m\end{matrix}\middle|\begin{matrix}j+\frac{1}{2}\\ m+n'\end{matrix}\right)\chi^{L}_{j+\frac{1}{2},m+n'} + \left(\begin{matrix}\frac{1}{2} & j\\ n' & m\end{matrix}\middle|\begin{matrix}j-\frac{1}{2}\\ m+n'\end{matrix}\right)\chi^{L}_{j-\frac{1}{2},m+n'}\right\}. \quad \cdots(63a)$$

Similarly

$$[Q^{\dagger}_{Ln}, \chi^{L}_{j+\frac{1}{2},m+\frac{1}{2}}] = -i\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}\left(\sigma_2\sigma_\mu\sigma_2\frac{\partial}{\partial x^\mu}\right)_{nn'}\left(\begin{matrix}\frac{1}{2} & j\\ n' & m+\frac{1}{2}-n'\end{matrix}\middle|\begin{matrix}j+\frac{1}{2}\\ m+\frac{1}{2}\end{matrix}\right)\chi^{L}_{j,m+\frac{1}{2}-n'}, \quad \cdots(63b)$$

$$[Q^{\dagger}_{Ln}, \chi^{L}_{j-\frac{1}{2},m-\frac{1}{2}}] = -i\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}\left(\sigma_2\sigma_\mu\sigma_2\frac{\partial}{\partial x^\mu}\right)_{nn'}\left(\begin{matrix}\frac{1}{2} & j\\ n' & m-\frac{1}{2}-n'\end{matrix}\middle|\begin{matrix}j-\frac{1}{2}\\ m-\frac{1}{2}\end{matrix}\right)\chi^{L}_{j,m-\frac{1}{2}-n'}, \quad \cdots(63c)$$

$$[Q^{\dagger}_{Ln}, \chi^{L}_{j,m}] = 0 \quad ; \cdots(63d)$$

$$[Q_{Ln}, \chi^L_{j,m}] = \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\begin{pmatrix}\frac{1}{2} & j \\ n & m\end{pmatrix}\left|\begin{matrix}j+\frac{1}{2}\\ m+n\end{matrix}\right) \chi^L_{j+\frac{1}{2},m+n}$$
$$+\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\begin{pmatrix}\frac{1}{2} & j \\ n & m\end{pmatrix}\left|\begin{matrix}j-\frac{1}{2}\\ m+n\end{matrix}\right) \chi^L_{j-\frac{1}{2},m+n} \quad , \qquad \cdots(63e)$$

$$[Q_{Ln}, \chi^L_{j+\frac{1}{2},m+\frac{1}{2}}] = -\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}(\sigma_2)_{nn'}\begin{pmatrix}\frac{1}{2} & j \\ n' & m+\frac{1}{2}-n'\end{pmatrix}\left|\begin{matrix}j+\frac{1}{2}\\ m+\frac{1}{2}\end{matrix}\right) \underset{\sim}{\chi}^L_{j,m+\frac{1}{2}-n'} \quad , \qquad \cdots(63f)$$

$$[Q_{Ln}, \chi^L_{j-\frac{1}{2},m-\frac{1}{2}}] = -\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}(\sigma_2)_{nn'}\begin{pmatrix}\frac{1}{2} & j \\ n' & m-\frac{1}{2}-n'\end{pmatrix}\left|\begin{matrix}j-\frac{1}{2}\\ m-\frac{1}{2}\end{matrix}\right) \underset{\sim}{\chi}^L_{j,m-\frac{1}{2}-n'} \quad , \qquad \cdots(63g)$$

and $$[Q_{Ln}, \underset{\sim}{\chi}^L_{j,m}] = 0. \qquad \cdots(63h)$$

This is not the most convenient way of expressing these relationships in the sequel. We define the field $\chi^L_{\frac{1}{2},n;j,m}$ by

$$\chi^L_{\frac{1}{2},n;j,m} = c_{nm,r}\begin{bmatrix}\chi^L_{j+\frac{1}{2},m'}\\ \chi^L_{j-\frac{1}{2},m''}\end{bmatrix}_r \quad , \qquad \cdots(64)$$

so that

$$\mathcal{U}([\Lambda|t])\,\chi^L_{\frac{1}{2},n;j,m}(x)\,\mathcal{U}([\Lambda|t])^{-1}$$
$$= \Gamma^{0,\frac{1}{2}}([\Lambda^{-1}|0])_{nn'}\Gamma^{0,j}([\Lambda^{-1}|0])_{mm'}\,\chi^L_{\frac{1}{2},n';j,m'}(\Lambda x+t). \qquad \cdots(65)$$

The final result of this section can now be expressed as a theorem.

Theorem 3.2

Let the fields $\chi^L_{j,m}$, $\chi^L_{\frac{1}{2},n;j,m}$ and $\underset{\sim}{\chi}^L_{j,m}$ be as defined by equations (61) and (64). Then the action of the supersymmetry generators Q_{Ln}, Q_{Rn} on these fields is given by

$$[Q_{Ln}, \chi^L_{j,m}] = \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} \chi_{\frac{1}{2},n;j,m} \quad , \qquad \cdots(66a)$$

$$[Q_{Ln}, \chi^L_{\frac{1}{2},r;j,m}] = -\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}(\sigma^L_2)_{nr}\,\underset{\sim}{\chi}^L_{j,m} \quad , \qquad \cdots(66b)$$

$$[Q_{Ln}, \underset{\sim}{\chi}^L_{j,m}] = 0 \quad , \qquad \cdots(66c)$$

$$[Q_{Rn}, \chi^L_{j,m}] = 0 \quad , \qquad \cdots(66d)$$

$$[Q_{Rn}, \chi^L_{\frac{1}{2},r;j,m}] = -\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}\left(\sigma^L_\mu\sigma^L_2\frac{\partial}{\partial x^\mu}\right)_{nr}\chi^L_{j,m} \qquad \cdots(66e)$$

and $$[Q_{Rn}, \underset{\sim}{\chi}^L_{j,m}] = -\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}\left(\sigma^L_\mu\frac{\partial}{\partial x^\mu}\right)_{nr}\chi^L_{\frac{1}{2},r;j,m}. \qquad \cdots(66f)$$

With $n, r = \frac{1}{2}, -\frac{1}{2}$; $m = j, j-1, \cdots, -j+1, -j$ and j taking integer or half-integer values. We call these the left handed chiral supermultiplets.

Proof

From equation (63a) we have

$$[Q^{+}_{Ln}\,,\,\underset{\sim}{\chi}^{L}_{j,m}] = -i\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma_2\sigma_\mu\frac{\partial}{\partial x^\mu})_{nn'}\,\chi^{L}_{\frac{1}{2},n';j,m}\,,$$
$$= [\,i(\sigma_2 Q_R)_n\,,\,\underset{\sim}{\chi}^{L}_{j,m}]\,,$$

using equation (13) of Chapter 1. So that

$$[Q_{Rn}\,,\,\underset{\sim}{\chi}^{L}_{j,m}] = -\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma_\mu\frac{\partial}{\partial x^\mu})_{nn'}\,\chi^{L}_{\frac{1}{2},n';j,m}\,.$$

From equations (63b) and (63c) we have

$$[Q^{+}_{Ln}\,,\,(\chi^{L}_{j+\frac{1}{2},m+\frac{1}{2}}\,,\,\chi^{L}_{j-\frac{1}{2},m-\frac{1}{2}})_r]$$
$$= -\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} i(\sigma_2\sigma_\mu\sigma_2)_{nn'}(C)^{-1}_{r,n'm''}\,\frac{\partial}{\partial x^\mu}\,\chi^{L}_{j,m''}.$$

So that

$$[Q^{+}_{Ln}\,,\,\chi^{L}_{\frac{1}{2},m;j,m'}]$$
$$= -i\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} C_{mm',r}\,(C)^{-1}_{r,n'm''}\,(\sigma_2\sigma_\mu\sigma_2)_{nn'}\,\frac{\partial}{\partial x^\mu}\,\chi^{L}_{j,m''}$$

thus

$$[Q_{Rn}\,,\,\chi^{L}_{\frac{1}{2},m;j,m'}] = -\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma_\mu\sigma_2\frac{\partial}{\partial x^\mu})_{nm}\,\chi^{L}_{j,m'}\,.$$

The other equations are derived in the same way. It remains to note that $\sigma_\mu = \sigma^{L}_\mu$ as defined in the appendix.

A check of the Jacobi identities in the form

$$[[Q_{Ln}\,,\,Q_{Rn'}]\,,\,\chi] = [Q_{Ln}\,,\,[Q_{Rn'}\,,\,\chi]]$$
$$+\,[Q_{Rn'}\,,[Q_{Ln}\,,\,\chi]]$$

gives the result

$$[[Q_{Ln}\,,\,Q_{Rn'}]\,,\,\chi] = -(\sigma^{L}_\mu\,\sigma^{L}_2)_{n'n}\,\frac{\partial}{\partial x^\mu}\,\chi \qquad \cdots(67)$$

for each of the three fields in the above theorem. This is as required by equation (25) for second quantized fields, since

$$[\tfrac{i}{\hbar}P_\sigma\,,\,\chi] = \frac{\partial}{\partial x^\sigma}\,\chi\,. \qquad \cdots(68)$$

Which is proved as follows

$$U([I\,|\,t])\,\chi(x)\,U([I\,|\,-t])$$
$$= \left(\frac{1}{2\pi\hbar}\right)^{3}\int d^3p\left(\frac{Mc}{2p_4}\right)\left\{\alpha_p\,e^{-\frac{i}{\hbar}p\cdot(x+t)} + \beta_p\,e^{\frac{i}{\hbar}p\cdot(x+t)}\right\},$$
$$= \left(\frac{1}{2\pi\hbar}\right)^{3}\int d^3p\left(\frac{Mc}{2p_4}\right)\left\{\alpha_p\,e^{-\frac{i}{\hbar}p\cdot x}(1-\tfrac{i}{\hbar}p\cdot t + \cdots)\right.$$
$$\left. +\,\beta_p\,e^{\frac{i}{\hbar}p\cdot x}(1+\tfrac{i}{\hbar}p\cdot t + \cdots)\right\}. \qquad \cdots(69)$$

But $\quad u([I\mid t]) = \exp \frac{i}{\hbar} t^\sigma P_\sigma = 1 + \frac{i}{\hbar} t^\sigma P_\sigma + \cdots$

so that

$$u([I\mid t])\, \chi\, u([I\mid -t]) = \chi + t^\sigma [\tfrac{i}{\hbar} P_\sigma, \chi] + \cdots \qquad \cdots(70)$$

Then comparing equations (69) and (70) the result follows. This should be contrasted with the corresponding result for classical fields, which needs a negative sign inserting.

We observe that with $j = 0$ in Theorem 3.2 we obtain the two component form of the Wess-Zumino model [3].

3.3 Construction of Right Handed Supermultiplets from Left Handed Supermultiplets.

We could repeat the analysis of section 2 and thus construct a set of right handed fields. It is much easier to construct one from the other.

We first observe that if $\{A_i^{(j)}, iA_i^{(j)}, i = 1,2,3\}$ are the generators of $\Gamma^{0,j}(\Lambda)$ with the $A_i^{(j)}$ antihermitian, generating rotations and $iA_i^{(j)}$ generating boosts, then $\{A_i^{(j)}, -iA_i^{(j)}\}$ are the corresponding generators of $\Gamma^{j,0}(\Lambda)$.

Now consider the action of the matrix Z_j as defined by equation (9) on a Lorentz boost

$$\begin{aligned} Z_j^{-1}\,\Gamma^{0,j}(B(p,\hat{p}))\,Z_j &= \exp(p\cdot Z_j^{-1}(iA^{(j)})Z_j) \quad \text{for some } p, \\ &= \exp(p\cdot iA^{(j)*}) \quad \text{since the } A^{(j)} \text{ generate } D^j(R), \\ &= \exp(p\cdot(-iA^{(j)})^*), \\ &= \Gamma^{j,0}(B(p,\hat{p}))^*. \end{aligned}$$

Then since for any rotation R

$$\Gamma^{0,j}(R) = \Gamma^{j,0}(R),$$

we have

$$Z_j^{-1}\,\Gamma^{0,j}(\Lambda)\,Z_j = \Gamma^{j,0}(\Lambda)^*. \qquad \cdots(71)$$

Similarly

$$Z^{-1}\,\Gamma^{j,0}(\Lambda)\,Z_j \;=\; \Gamma^{0,j}(\Lambda)^{*}. \qquad \cdots\cdots(72)$$

Now let $\psi^{L}_{\alpha}(x)$ be a second quantized field that transforms as

$$U([\Lambda|t])\,\psi^{L}_{\alpha}(x)\,U([\Lambda|t])^{-1} = \Gamma^{0,j}([\Lambda^{-1}|0])_{\alpha\alpha'}\psi^{L}_{\alpha'}(\Lambda x+t) \qquad \cdots(73)$$

ie. as a left handed field, and consider the field

$$\psi^{R}_{\alpha}(x) \;=\; (Z_j)_{\alpha\alpha'}\,(\psi^{L}_{\alpha'}(x))^{*}. \qquad \cdots(74)$$

Thus

$$\begin{aligned}
\psi^{R}_{\alpha}(x) &= (Z_j)_{\alpha\alpha'}\,(\psi^{L}_{\alpha'}(x))^{*} \\
&= \left(\tfrac{1}{2\pi\hbar}\right)^{3}\int d^{3}p\left(\tfrac{Mc}{2p_4}\right)^{\frac{1}{2}}\Big\{\eta^{*}(Z_j)_{\alpha\alpha'}\Gamma^{0,j}(B(p,\hat p))_{\alpha'\alpha''}\,a^{\dagger}_{p,\alpha''}\,e^{\frac{i}{\hbar}p\cdot x} \\
&\qquad\qquad + \xi^{*}(Z_j)_{\alpha\alpha'}(\Gamma^{0,j}(B(p,\hat p))Z_j)^{*}_{\alpha'\alpha''}\,b_{p,\alpha''}\,e^{-\frac{i}{\hbar}p\cdot x}\Big\}, \\
&= \left(\tfrac{1}{2\pi\hbar}\right)^{3}\int d^{3}p\left(\tfrac{Mc}{2p_4}\right)^{\frac{1}{2}}\Big\{\eta^{*}(Z_j Z_j^{-1}\Gamma^{j,0}(B(p,\hat p))Z_j)_{\alpha\alpha'}\,a^{\dagger}_{p,\alpha'}\,e^{\frac{i}{\hbar}p\cdot x} \\
&\qquad\qquad + \xi^{*}(Z_j Z_j^{-1}\Gamma^{j,0}(B(p,\hat p)Z_j Z_j^{*})_{\alpha\alpha'}\,b_{p,\alpha'}\,e^{-\frac{i}{\hbar}p\cdot x}\Big\}, \\
&= \left(\tfrac{1}{2\pi\hbar}\right)^{3}\int d^{3}p\left(\tfrac{Mc}{2p_4}\right)^{\frac{1}{2}}\Big\{\xi^{*}(-1)^{2j}\Gamma^{j,0}(B(p,\hat p))_{\alpha\alpha'}\,b_{p,\alpha'}\,e^{-\frac{i}{\hbar}p\cdot x} \\
&\qquad\qquad + \eta^{*}(\Gamma^{j,0}(B(p,\hat p))Z_j)_{\alpha\alpha'}\,a^{\dagger}_{p,\alpha'}\,e^{\frac{i}{\hbar}p\cdot x}\Big\}. \qquad \cdots(75)
\end{aligned}$$

In the last line we have made use of equation (11). This clearly transforms in the required manner ie.

$$U([\Lambda|t])\,\psi^{R}_{\alpha}(x)\,U([\Lambda|t])^{-1} = \Gamma^{j,0}([\Lambda^{-1}|0])_{\alpha\alpha'}\,\psi^{R}_{\alpha'}(\Lambda x+t)\ .$$

We note that $\psi^{R}_{\alpha}(x)$ is constructed from the adjoints of the operators used for $\psi^{L}_{\alpha}(x)$ so that it is the antiparticle field of $\psi^{L}_{\alpha}(x)$. Also

$$\begin{aligned}
(Z_j)_{\alpha\alpha'}(\psi^{R}_{\alpha'}(x))^{*} &= (Z_j Z_j^{*})_{\alpha\alpha'}\,\psi^{L}_{\alpha}(x)\,, \\
&= (-1)^{2j}\,\psi^{L}_{\alpha}(x)\,,
\end{aligned}$$

so that applying the transformation twice does not return us precicely to the starting point but includes an overall phase factor.

Now using equation (74) we can construct a set of right handed fields from the left handed fields in equations (61). We obtain

$$\chi'^{R}_{j,m}(x) = \left(\frac{1}{2\pi\hbar}\right)^3 \int d^3p \left(\frac{Mc}{2p_4}\right)\left\{-i(-1)^{2j}\alpha^R_{p,j,m}\, e^{-\frac{i}{\hbar}p\cdot x} + (-1)^{2j}\underset{\sim}{\beta}^R_{p,j,m}\, e^{\frac{i}{\hbar}p\cdot x}\right\}, \quad \cdots(76a)$$

$$\chi^{R}_{j+\frac{1}{2},m}(x) = \left(\frac{1}{2\pi\hbar}\right)^3 \int d^3p \left(\frac{Mc}{2p_4}\right)\left\{-(-1)^{2j}\alpha^R_{p,j+\frac{1}{2},m}\, e^{-\frac{i}{\hbar}p\cdot x} + \beta^R_{p,j+\frac{1}{2},m}\, e^{\frac{i}{\hbar}p\cdot x}\right\}, \quad \cdots(76b)$$

$$\chi^{R}_{j-\frac{1}{2},m}(x) = \left(\frac{1}{2\pi\hbar}\right)^3 \int d^3p \left(\frac{Mc}{2p_4}\right)\left\{-(-1)^{2j}\alpha^R_{p,j-\frac{1}{2},m}\, e^{-\frac{i}{\hbar}p\cdot x} + \beta^R_{p,j-\frac{1}{2},m}\, e^{\frac{i}{\hbar}p\cdot x}\right\} \quad \cdots(76c)$$

and
$$\underset{\sim}{\chi}'^{R}_{j,m}(x) = \left(\frac{1}{2\pi\hbar}\right)^3 \int d^3p \left(\frac{Mc}{2p_4}\right)\left\{(-1)^{2j}\underset{\sim}{\alpha}^R_{p,j,m}\, e^{-\frac{i}{\hbar}p\cdot x} + i(-1)^{2j}\beta^R_{p,j,m}\, e^{\frac{i}{\hbar}p\cdot x}\right\}, \quad \cdots(76d)$$

with

$$\alpha^R_{p,k,m} = \left(\frac{2p_4}{Mc}\right)^{\frac{1}{2}} \Gamma^{0,j}([B(p,\hat{p})\|0])_{mm'}\, b_{p,k,m} \quad \cdots(77a)$$

and
$$\beta^R_{p,k,m} = \left(\frac{2p_4}{Mc}\right)^{\frac{1}{2}} (\Gamma^{0,j}([B(p,\hat{p})\|0])Z_j)_{mm'}\, a^{\dagger}_{p,k,m} \quad \cdots(77b)$$

for $k = j,\ j+\frac{1}{2},\ j-\frac{1}{2}$ and of course similar definitions for $\underset{\sim}{\alpha}^R_{p,j,m}$ and $\underset{\sim}{\beta}^R_{p,j,m}$.

It is in fact convenient to work with a slightly different set of fields.

We define
$$\chi^{R}_{j,m}(x) = (-1)^{2j}\chi'^{R}_{j,m}(x) \quad \cdots(78a)$$

and
$$\underset{\sim}{\chi}^{R}_{j,m}(x) = (-1)^{2j}\underset{\sim}{\chi}'^{R}_{j,m}(x). \quad \cdots(78b)$$

The action of the supersymmetry operators on a field is obtained as follows

$$\begin{aligned}[Q_{Ln}, \chi^R_{j,m}] &= [Q_{Ln}, (Z_j)_{mm'}(\chi^L_{j,m'})^*] \\ &= (Z_j)_{mm'}[Q_{Ln}, \chi^{L*}_{j,m}] \\ &= (Z_j)_{mm'}([\chi^L_{j,m'}, Q^{\dagger}_{Ln}])^*\end{aligned}$$

So that

$$[Q_{Ln}, \chi^R_{j,m}] = (Z_j)_{mm'}(-1)^{2j}([Q^{\dagger}_{Ln}, \chi^L_{j,m}])^* \quad \cdots(79)$$

and similarly

$$[Q^{\dagger}_{Ln}, \chi^R_{j,m}] = (Z_j)_{mm'}(-1)^{2j}([Q^{\dagger}_{Ln}, \chi^L_{j,m}])^* \quad \cdots(80)$$

Thus for the field $\underset{\sim}{\chi}^R_{j,m}(x)$ as defined above

$$[Q_{Ln}, \underset{\sim}{\chi}^R_{j,m}] = (z_j)_{mm'}([Q^+_{Ln}, \underset{\sim}{\chi}^L_{j,m'}])^*$$

$$= (z_j)_{mm'}\left\{-i\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma_2^L\sigma_\mu^L\frac{\partial}{\partial x^\mu})_{nn'}\left\{\begin{pmatrix}\frac{1}{2} & j \\ n' & m'\end{pmatrix}\bigg|\begin{matrix}j+\frac{1}{2}\\ m'+n'\end{matrix}\right)\chi^L_{j+\frac{1}{2},m'+n'} + \begin{pmatrix}\frac{1}{2} & j \\ n' & m'\end{pmatrix}\bigg|\begin{matrix}j+\frac{1}{2}\\ m'+n'\end{matrix}\right)\chi^L_{j-\frac{1}{2},m'+n'}\right\}\right\}^*,$$

$$= i\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}(z_j)_{mm'}(\sigma_2^{L*}\sigma_\mu^{L*}\frac{\partial}{\partial x^\mu})_{nn'}\left\{\begin{pmatrix}\frac{1}{2} & j \\ n' & m'\end{pmatrix}\bigg|\begin{matrix}j+\frac{1}{2}\\ m'+n'\end{matrix}\right)\chi^{L*}_{j+\frac{1}{2},m'+n'} + \begin{pmatrix}\frac{1}{2} & j \\ n' & m'\end{pmatrix}\bigg|\begin{matrix}j+\frac{1}{2}\\ m'+n'\end{matrix}\right)\chi^{L*}_{j-\frac{1}{2},m'+n'}\right\},$$

$$= \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}\left(\sigma_2^L(i\sigma_\mu^{L*})\sigma_2^L\frac{\partial}{\partial x^\mu}\right)_{nn''}\left((z_{\frac{1}{2}})_{n''n'}\otimes(z_j)_{mm'}\right)C_{n'm',m''} \times\left(z^{-1}_{j+\frac{1}{2}}\oplus z^{-1}_{j-\frac{1}{2}}\right)_{m''m'''}\begin{bmatrix}\chi^R_{j+\frac{1}{2},r}\\ \chi^R_{j-\frac{1}{2},r}\end{bmatrix}_{m'''},$$

Where we have used the fact that $z_{\frac{1}{2}} = \sigma_2^L = (\sigma_2^L)^{-1}$ and used equation (11). Now if we use equation (20) and equations (13) and (4) of the appendix we obtain

$$[Q_{Ln}, \underset{\sim}{\chi}^R_{j,m}] = \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma_\mu^R\frac{\partial}{\partial x^\mu})_{nn'}\left\{\begin{pmatrix}\frac{1}{2} & j \\ n' & m\end{pmatrix}\bigg|\begin{matrix}j+\frac{1}{2}\\ m+n'\end{matrix}\right)\chi^R_{j+\frac{1}{2},m+n'} + \begin{pmatrix}\frac{1}{2} & j \\ n' & m\end{pmatrix}\bigg|\begin{matrix}j-\frac{1}{2}\\ m+n'\end{matrix}\right)\chi^R_{j-\frac{1}{2},m+n'}\right\} \quad \cdots(81a)$$

Similarly

$$[Q_{Ln}, \chi^R_{j+\frac{1}{2},m+\frac{1}{2}}] = i\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma_\mu^R\sigma_2^R\frac{\partial}{\partial x^\mu})_{nn'}\begin{pmatrix}\frac{1}{2} & j \\ n' & m+\frac{1}{2}-n'\end{pmatrix}\bigg|\begin{matrix}j+\frac{1}{2}\\ m+\frac{1}{2}\end{matrix}\right)\chi^R_{j,m+\frac{1}{2}-n'}, \quad \cdots(81b)$$

$$[Q_{Ln}, \chi^R_{j-\frac{1}{2},m+\frac{1}{2}}] = i\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma_\mu^R\sigma_2^R\frac{\partial}{\partial x^\mu})_{nn'}\begin{pmatrix}\frac{1}{2} & j \\ n' & m+\frac{1}{2}-n'\end{pmatrix}\bigg|\begin{matrix}j-\frac{1}{2}\\ m+\frac{1}{2}\end{matrix}\right)\chi^R_{j,m+\frac{1}{2}-n'}, \quad \cdots(81c)$$

$$[Q_{Ln}, \chi^R_{j,m}] = 0 \quad \cdots(81d)$$

$$[Q^+_{Ln}, \chi^R_{j,m}] = \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}(\sigma_2^R)_{nn'}\left\{\begin{pmatrix}\frac{1}{2} & j \\ n' & m\end{pmatrix}\bigg|\begin{matrix}j+\frac{1}{2}\\ n'+m\end{matrix}\right)\chi^R_{j+\frac{1}{2},n'+m} + \begin{pmatrix}\frac{1}{2} & j \\ n' & m\end{pmatrix}\bigg|\begin{matrix}j-\frac{1}{2}\\ n'+m'\end{matrix}\right)\chi^R_{j-\frac{1}{2},n'+m}\right\}, \quad \cdots(81e)$$

$$[Q^+_{Ln}, \chi^R_{j+\frac{1}{2},m+\frac{1}{2}}] = \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\begin{pmatrix}\frac{1}{2} & j \\ n & m+\frac{1}{2}-n\end{pmatrix}\bigg|\begin{matrix}j+\frac{1}{2}\\ m+\frac{1}{2}\end{matrix}\right)\underset{\sim}{\chi}^R_{j,m+\frac{1}{2}-n}, \quad \cdots(81f)$$

$$[Q^+_{Ln}, \chi^R_{j-\frac{1}{2},m+\frac{1}{2}}] = \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\begin{pmatrix}\frac{1}{2} & j \\ n & m+\frac{1}{2}-n\end{pmatrix}\bigg|\begin{matrix}j-\frac{1}{2}\\ m+\frac{1}{2}\end{matrix}\right)\underset{\sim}{\chi}^R_{j,m+\frac{1}{2}-n} \quad \cdots(81g)$$

and $$[Q^+_{Ln}, \underset{\sim}{\chi}^R_{j,m}] = 0 . \quad \cdots(81h)$$

As with the left handed fields, it is convenient to construct a single field from the $j+\frac{1}{2}$ and $j-\frac{1}{2}$ fields ie.

$$\chi^R_{\frac{1}{2},n;j,m} = C_{nm,r}\begin{bmatrix}\chi^R_{j+\frac{1}{2},m'}\\ \chi^R_{j-\frac{1}{2},m''}\end{bmatrix}_r \qquad \cdots(82)$$

The final result of this section can now be given as a theorem.

Theorem 3.3

Let the fields $\chi^R_{j,m}(x)$, $\chi^R_{\frac{1}{2},n;j,m}(x)$ and $\underset{\sim}{\chi}^R_{j,m}(x)$ be as defined by equations (76) and (82). Then the action of the supersymmetry generators Q_{Ln} and Q_{Rn} on these fields is given by

$$[Q_{Ln}, \underset{\sim}{\chi}^R_{j,m}(x)] = i\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma^R_\mu \tfrac{\partial}{\partial x^\mu})_{nr}\chi^R_{\frac{1}{2},r;j,m}(x) \qquad , \qquad \cdots(83a)$$

$$[Q_{Ln}, \chi^R_{\frac{1}{2},r;j,m}(x)] = i\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma^R_\mu\sigma^R_2 \tfrac{\partial}{\partial x^\mu})_{nr}\chi^R_{j,m}(x) \qquad , \qquad \cdots(83b)$$

$$[Q_{Ln}, \chi^R_{j,m}(x)] = 0 \qquad , \qquad \cdots(83c)$$

$$[Q_{Rn}, \chi^R_{j,m}(x)] = i\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\chi^R_{\frac{1}{2},n;j,m}(x) \qquad , \qquad \cdots(83d)$$

$$[Q_{Rn}, \chi^R_{\frac{1}{2},r;j,m}(x)] = -i\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}(\sigma^R_2)_{nr}\underset{\sim}{\chi}^R_{j,m}(x) \qquad \cdots(83e)$$

and $$[Q_{Rn}, \underset{\sim}{\chi}^R_{j,m}(x)] = 0.$$

With $n, r = \frac{1}{2}, -\frac{1}{2}$; $m = j, j-1, \cdots, -j+1, -j$ and j taking integer or half integer values. We call these the right handed chiral supermultiplets.

Proof

This follows the procedures of the proof of Theorem 3.2.

We note that a check of the Jacobi identities on these right handed fields gives the result stated in equation (67).

3.4 Supermultiplets that are Symmetric under the Interchange of L and R.

While the fields used in the previous two sections are perfectly adequate as they stand it is more convenient, and more pleasing, to redefine them so that the commutators of Theorems 3.2 and 3.3 become symmetric under the interchange of L and R. We acheive this by altering the relative phases of the fields, this leaves an overall phase factor undetermined.

To reduce this choice we demand that the differential operator linking the fields is also symmetric in the interchange of L and R. This still does not give a unique choice. Our choice is such that the Dirac equation takes its standard form.

We define
$$\underset{\sim}{\chi}'^{R}_{j,m} = e^{-ib\pi}\underset{\sim}{\chi}^{R}_{j,m} \quad , \qquad \cdots(84a)$$
$$\underset{\sim}{\chi}'^{L}_{j,m} = e^{ib\pi}\underset{\sim}{\chi}^{L}_{j,m} \quad , \qquad \cdots(84b)$$
$$\chi'^{R}_{j,m} = e^{-ic\pi}\chi^{R}_{j,m} \quad , \qquad \cdots(84c)$$
$$\chi'^{L}_{j,m} = e^{ic\pi}\chi^{L}_{j,m} \quad , \qquad \cdots(84d)$$
$$\chi'^{R}_{\frac{1}{2},n;j,m} = e^{-ia\pi}\chi^{R}_{\frac{1}{2},n;j,m} \qquad \cdots(84e)$$
and
$$\chi'^{L}_{\frac{1}{2},n;j,m} = e^{ia\pi}\chi^{L}_{\frac{1}{2},n;j,m} \qquad \cdots(84f)$$
with $a, b, c \in \mathbb{R}$.

Now from equation (66a)
$$[Q_{Ln}, e^{-ic\pi}\chi'^{L}_{j,m}] = \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} e^{-ia\pi}\chi'^{L}_{\frac{1}{2},n;j,m}$$
and from equation (83d)
$$[Q_{Rn}, e^{ic\pi}\chi'^{R}_{j,m}] = i\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} e^{ia\pi}\chi'^{R}_{\frac{1}{2},n;j,m} .$$
If we require these equations to be symmetric in the interchange of L and R then
$$e^{-ia\pi}e^{ic\pi} = e^{i\frac{\pi}{2}}e^{ia\pi}e^{-ic\pi} ,$$
$$-a\pi + c\pi = \frac{\pi}{2} + a\pi - c\pi .$$
Thus
$$c = a + \tfrac{1}{4} . \qquad \cdots(85)$$
Repeating this procedure on the rest of equations (66) and (83) we obtain the additional restriction
$$b = a - \tfrac{3}{4} . \qquad \cdots(86)$$
We are thus free to choose (say) $0 \leq a < 2\pi$, the parameters b and c are then fixed.

To restrict this choice we require that if $\chi^{L}_{\frac{1}{2},n;j,m}$ and $\chi^{R}_{\frac{1}{2},n;j,m}$ are self conjugate for $j = 0$, that is in terms of the creation and annihilation operators we put $a_{p,j,m} = b_{p,j,m}$, then the differential operators relating the fields are also symmetric under the interchange of L and R. Since for $j \neq 0$ the differential operator acting on the index m can be constructed from the $j = \frac{1}{2}$ by a sieries of tensor products this will be true for all j.

By its definition in equations (61b), (61c) and (64) we can write

$$\chi^{L}_{\frac{1}{2},n;0,0}(x) = \left(\frac{1}{2\pi\hbar}\right)^{3}\int d^{3}p\left(\frac{Mc}{2p_4}\right)^{\frac{1}{2}}\Big\{\Gamma^{0,\frac{1}{2}}([B(p,\hat{p})|0])_{nn'}\,a_{p,n',0}\,e^{-\frac{i}{\hbar}p.x} + (\Gamma^{0,\frac{1}{2}}([B(p,\hat{p})|0])\sigma_2)_{nn'}\,a^{+}_{p,n',0}\,e^{\frac{i}{\hbar}p.x}\Big\} \quad \cdots(87)$$

and from equation (75)

$$\begin{aligned}\chi^{R}_{\frac{1}{2},n;0,0}(x) &= (\sigma_2)_{nn'}\,\chi^{L*}_{\frac{1}{2},n';0,0}(x)\\ &= \left(\frac{1}{2\pi\hbar}\right)^{3}\int d^{3}p\left(\frac{Mc}{2p_4}\right)^{\frac{1}{2}}\Big\{-\Gamma^{\frac{1}{2},0}([B(p,\hat{p})|0])_{nn'}\,a_{p,n',0}\,e^{-\frac{i}{\hbar}p.x}\\ &\qquad + (\Gamma^{\frac{1}{2},0}([B(p,\hat{p})|0])\sigma_2^{L})_{nn'}\,a^{+}_{p,n',0}\,e^{\frac{i}{\hbar}p.x}\Big\}\end{aligned} \quad \cdots(88)$$

Now we act on equation (87) with the Dirac operator

$$\begin{aligned}&(\sigma^{L}_{\mu}\tfrac{\partial}{\partial x^{\mu}})_{nn'}\,\chi^{L}_{\frac{1}{2},n';0,0}(x)\\ &= \left(\frac{1}{2\pi\hbar}\right)^{3}\int d^{3}p\left(\frac{Mc}{2p_4}\right)^{\frac{1}{2}}\Big\{-\frac{i}{\hbar}(\sigma^{L}_{\mu}p_{\mu})_{nn'}\,\Gamma^{0,\frac{1}{2}}([B(p,\hat{p})|0])_{n'n''}\,a_{p,n'',0}\,e^{-\frac{i}{\hbar}p.x}\\ &\qquad + \frac{i}{\hbar}(\sigma^{L}_{\mu}p_{\mu})_{nn'}(\Gamma^{0,\frac{1}{2}}([B(p,\hat{p})|0])\sigma_2)_{n'n''}\,a^{+}_{p,n'',0}\,e^{\frac{i}{\hbar}p.x}\Big\},\end{aligned}$$

and use equation ( ) of the appendix to obtain

$$\begin{aligned}&(\sigma^{L}_{\mu}\tfrac{\partial}{\partial x})_{nn'}\,\chi^{L}_{\frac{1}{2},n';0,0}(x)\\ &= \left(\frac{1}{2\pi\hbar}\right)^{3}\int d^{3}p\left(\frac{Mc}{2p_4}\right)^{\frac{1}{2}}\Big\{-\frac{iMc}{\hbar}\Gamma^{\frac{1}{2},0}([B(p,\hat{p})|0])_{nn'}\,a_{p,n',0}\,e^{-\frac{i}{\hbar}p.x}\\ &\qquad + \frac{iMc}{\hbar}(\Gamma^{\frac{1}{2},0}([B(p,\hat{p})|0])\sigma_2)_{nn'}\,a^{+}_{p,n'}\,e^{\frac{i}{\hbar}p.x}\Big\}.\end{aligned}$$

So that

$$(\sigma^{L}_{\mu}\tfrac{\partial}{\partial x^{\mu}})_{nn'}\,\chi^{L}_{\frac{1}{2},n';0,0}(x) = i\frac{Mc}{\hbar}\chi^{R}_{\frac{1}{2},n;0,0}(x). \quad \cdots(89)$$

Similarly

$$(\sigma^{R}_{\mu}\tfrac{\partial}{\partial x^{\mu}})_{nn'}\,\chi^{R}_{\frac{1}{2},n';0,0}(x) = \frac{iMc}{\hbar}\chi^{L}_{\frac{1}{2},n;0,0}(x). \quad \cdots(90)$$

The requirement that equations (89) and (90) are symmetric in the interchange of L and R gives the restriction that

$$\alpha = n\frac{\pi}{2}\ , \quad n = 0, \pm 1, \pm 2, \cdots$$

ie. $\pi a = 0, \pm \frac{\pi}{2}, \pm\pi, \cdots$.

Our choice is $a = 0,$ ....(91)

so that $b = \frac{5}{4}$ ....(92)

and $c = \frac{1}{4}$ . ....(93)

Theorem 3.4

With the choices above the action of the supersymmetry generators on the fields $\{\underset{\sim}{\chi}'^{R}_{j,m}(x),\ \chi'^{R}_{\frac{1}{2},n';j,m}(x),\ \chi'^{R}_{j,m}(x)\}$ and $\{\underset{\sim}{\chi}'^{L}_{j,m}(x),\ \chi'^{L}_{\frac{1}{2},n';j,m}(x),\ \chi'^{L}_{j,m}(x)\}$ of the chiral supermultiplets can be written as

$$[Q_{Ln}, \underset{\sim}{\chi}'^{R}_{j,m}] = e^{i\frac{3\pi}{4}}\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma_\mu^R \tfrac{\partial}{\partial x^\mu})_{nn'}\,\chi'^{R}_{\frac{1}{2},n';j,m}\ , \quad \text{....(94a)}$$

$$[Q_{Ln}, \chi'^{R}_{\frac{1}{2},n';j,m}] = e^{i\frac{3\pi}{4}}\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma_\mu^R\sigma_2^R \tfrac{\partial}{\partial x^\mu})_{nn'}\,\chi'^{R}_{j,m}\ , \quad \text{....(94b)}$$

$$[Q_{Ln}, \chi'^{R}_{j,m}] = 0\ , \quad \text{....(94c)}$$

$$[Q_{Rn}, \chi'^{R}_{j,m}] = e^{i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\chi'^{R}_{\frac{1}{2},n;j,m}\ , \quad \text{....(94d)}$$

$$[Q_{Rn}, \chi'^{R}_{\frac{1}{2},n';j,m}] = e^{-i\frac{3\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}(\sigma_2^R)_{nn'}\,\underset{\sim}{\chi}'^{R}_{j,m} \quad \text{....(94e)}$$

and $$[Q_{Rn}, \underset{\sim}{\chi}'^{R}_{j,m}] = 0\ ; \quad \text{....(94f)}$$

and $$[Q_{Rn}, \underset{\sim}{\chi}'^{L}_{j,m}] = e^{i\frac{3\pi}{4}}\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma_\mu^L \tfrac{\partial}{\partial x^\mu})_{nn'}\,\chi'^{L}_{\frac{1}{2},n';j,m}\ , \quad \text{....(95a)}$$

$$[Q_{Rn}, \chi'^{L}_{\frac{1}{2},n';j,m}] = e^{i\frac{3\pi}{4}}\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma_\mu^L\sigma_2^L \tfrac{\partial}{\partial x^\mu})_{nn'}\,\chi'^{L}_{j,m}\ , \quad \text{....(95b)}$$

$$[Q_{Rn}, \chi'^{L}_{j,m}] = 0\ , \quad \text{....(95c)}$$

$$[Q_{Ln}, \chi'^{L}_{j,m}] = e^{i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\chi'^{L}_{\frac{1}{2},n;j,m}\ , \quad \text{....(95d)}$$

$$[Q_{Ln}, \chi'^{L}_{\frac{1}{2},n';j,m}] = e^{-i\frac{3\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}(\sigma_2^L)_{nn'}\,\underset{\sim}{\chi}'^{L}_{j,m} \quad \text{....(95e)}$$

and $$[Q_{Ln}, \underset{\sim}{\chi}'^{L}_{j,m}] = 0\ . \quad \text{....(95f)}$$

These sets of equations are then symmetric in the interchange of L and R. Also the differential operators relating the fields $\chi'^{L}_{\frac{1}{2},n;j,m}(x)$ and $\chi'^{R}_{\frac{1}{2},n;j,m}(x)$ for $j=0$ (if they are self conjugate) are symmetric in the interchange of L and R. ie.

$$\frac{\hbar}{Mc}(\sigma_\mu^L \tfrac{\partial}{\partial x^\mu})_{nn'}\,\chi'^{L}_{\frac{1}{2},n';0,0} = i\,\chi'^{R}_{\frac{1}{2},n;0,0} \quad \text{....(96a)}$$

and $$\frac{\hbar}{Mc}(\sigma_\mu^R \tfrac{\partial}{\partial x^\mu})_{nn'}\,\chi'^{R}_{\frac{1}{2},n';0,0} = i\,\chi'^{L}_{\frac{1}{2},n;0,0}\ . \quad \text{....(96b)}$$

This is the form we will use for these expressions from now on.

CHAPTER 4

NON-CHIRAL SUPERMULTIPLETS.

4.1 Introduction.

In Chapter 3 we detailed the construction of the chiral supermultiplets for both left handed and right handed fields. By choice we constrcted the right handed fields as the antiparticle fields of the left handed set, but we could have chosen to construct these from the same set of creation and annihilation operators. In all of these chiral supermultiplets the number of independent field components was the same as the number of independent creation (or annihilation) operators. That is each field had $2k+1,\ k=0, \frac{1}{2}, 1, \cdots$ components with $k=j, j+\frac{1}{2}, j-\frac{1}{2}$ as appropriate. Such fields describe particles of a single spin value and obey only the Klein-Gordon equation.

In this chapter we want to examine supermultiplets of fields that do obey field equations other than the Klein-Gordon equation. This implies that not all the field components are linearly independent, conversely we can say that if our set of fields do obey some field equation in addition to the Klein-Gordon equation then not all the field components are linearly independent.

In the remainder of this section we will examine fields that do obey field equations following the methods of Weinberg [20] . In the next section we discuss the Wess-Zumino model [3] ie. the supermultiplet constructed from both left and right handed supermultiplets with $j=0$. In the third section we extend this to the case for general j. In the fourth section we consider invariants that can be constructed from these supermultiplets. Lastly in the fifth section we consider a combination of the $j=k+\frac{1}{2}$ and $j=k-\frac{1}{2}$ supermultiplets that we will need in Part III of this Thesis.

The reason for introducing additional field components according to

Weinberg [20] is that if we require a theory to transform simply under P (parity), C (charge conjugation) and T (time reversal) this cannot be acheived with a $2k+1$ component field. The $2k+1$ transform simply under T and CP but not under C or P.

To obtain a theory that transforms simply under C and P it is convenient to use $2(2k+1)$ component fields that transform as $\Gamma^{0,k}(\Lambda)\oplus\Gamma^{k,0}(\Lambda)$ and are constructed from the $2k+1$ creation and annihilation operators. The antiparticle field (if it exists) is given by the adjoint of this so that in total we have $4(2k+1)$ field components constructed from the $2(2k+1)$ creation and annihilation operators. These can alternatively be combined into a $2(2k+1)$ component complex field, eg. the Dirac field.

We note that fields constructed in this way consist of particles of a single spin value. Many other types of field are considered in the physics literature (eg. the field A_μ transforming as $\Lambda = S\Gamma^{\frac{1}{2},\frac{1}{2}}(\Lambda)S^{-1}$ for some similarity transformation S). These consist of several spin values unless constrained in some way to remove unwanted components (eg. $\partial/\partial x^\mu A_\mu = 0$ removes the 'spin 0' component of A_μ).

Given the left handed field $\psi^L_{k,m}$ we now want to show how to construct a right handed field $\psi^R_{k,m}$ so that the field

$$\psi_{k,\sigma} = \begin{bmatrix} \psi^L_{k,m} \\ \psi^R_{k,m} \end{bmatrix} \qquad \cdots(1)$$

satisfies a differential equation (in addition to the Klein-Gordon equation). In so doing we construct the differential operator connecting the left and right handed components of the field. First consider the spin zero field. We have

$$\psi^L_0(x) = \left(\frac{1}{2\pi\hbar}\right)^3\int d^3p\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\left\{a_p\, e^{-\frac{i}{\hbar}p.x} + a^+_p\, e^{\frac{i}{\hbar}p.x}\right\} \qquad \cdots(2)$$

$$= \psi_0^R(x)$$

since in this case the particle is its own antiparticle. Thus the equation relating the two fields is

$$\psi_0^L(x) = \psi_0^R(x). \qquad \cdots(3)$$

We have already determined the result for the spin $\frac{1}{2}$ field in Chapter 3. This is given by equations (96) of that Chapter as

$$\frac{i\hbar}{Mc}\left(\sigma_\mu^L \frac{\partial}{\partial x^\mu}\right)_{nn'} \psi^L_{\frac{1}{2},n'} = \psi^R_{\frac{1}{2},n} \qquad \cdots(4)$$

and

$$\frac{i\hbar}{Mc}\left(\sigma_\mu^R \frac{\partial}{\partial x^\mu}\right)_{nn'} \psi^R_{\frac{1}{2},n'} = \psi^L_{\frac{1}{2},n}. \qquad \cdots(5)$$

The combined result is then

$$\frac{i\hbar}{Mc}\begin{bmatrix} 0 & \left(\sigma_\mu^R \frac{\partial}{\partial x^\mu}\right) \\ \left(\sigma_\mu^L \frac{\partial}{\partial x^\mu}\right) & 0 \end{bmatrix}_{\sigma\sigma'} \psi_{\frac{1}{2},\sigma'} = \psi_{\frac{1}{2},\sigma} \qquad \cdots(6)$$

which we can recognise as the Dirac equation.

In general the differential equation will take the form of equation (6). We define the operators $\prod_j^L(\partial)$ and $\prod_j^R(\partial)$ to be such that the field $\psi_{j,\sigma}$ satisfies

$$\begin{bmatrix} 0 & \prod_j^R(\partial) \\ \prod_j^L(\partial) & 0 \end{bmatrix}_{\sigma\sigma'} \psi_{j,\sigma'} = \psi_{j,\sigma}. \qquad \cdots(7)$$

The following proposition then enables us to evaluate $\prod_j^R(\partial)$ and $\prod_j^L(\partial)$ successively for each $j = 1, \frac{3}{2}, \cdots$

<u>Proposition 4.1</u>

Suppose $\psi^A_{\frac{1}{2},\sigma}$, $\psi^A_{j-\frac{1}{2},\sigma'}$, $\psi^A_{j,\sigma''}$ and $\psi^A_{j+\frac{1}{2},\sigma'''}$ are a set of fields related by

$$C_{nm,r}\left(\psi^A_{j+\frac{1}{2},\sigma'''}, \psi^A_{j-\frac{1}{2},\sigma'}\right)_r$$

$$= \left(\begin{matrix}\frac{1}{2} & j \\ n & m\end{matrix}\middle|\begin{matrix}j+\frac{1}{2} \\ \sigma'''\end{matrix}\right)\psi^A_{j+\frac{1}{2},\sigma'''} + \left(\begin{matrix}\frac{1}{2} & j \\ n & m\end{matrix}\middle|\begin{matrix}j-\frac{1}{2} \\ \sigma'\end{matrix}\right)\psi^A_{j-\frac{1}{2},\sigma'} \qquad \cdots(8)$$

$$= \psi^A_{\frac{1}{2},n} \otimes \psi^A_{j,m}.$$

for $A = L$ or R and $C_{nm,r}$ as defined by equation (53) of Chapter 1. Further suppose that we know the differential operators $\prod_k^A(\partial)$ for

$A = L$ or R and $k = \frac{1}{2}, j$. The differential operators relating the left and right $j+\frac{1}{2}$ and $j-\frac{1}{2}$ fields are then given by

$$\left(\overline{\Pi}^{A}_{j+\frac{1}{2}}(\partial)\right)_{\sigma\rho} = \begin{pmatrix} \frac{1}{2} & j \\ n & m \end{pmatrix}\Big| \begin{matrix} j+\frac{1}{2} \\ \sigma \end{matrix}\Big)\left(\overline{\Pi}^{A}_{\frac{1}{2}}(\partial)\right)_{nn'} \otimes \left(\overline{\Pi}^{A}_{j}(\partial)\right)_{mm'}\Big(\begin{matrix} \frac{1}{2} & j \\ n' & m' \end{matrix}\Big| \begin{matrix} j+\frac{1}{2} \\ \rho \end{matrix}\Big) \quad \ldots.(9)$$

$$\left(\overline{\Pi}^{A}_{j-\frac{1}{2}}(\partial)\right)_{\sigma\rho} = \Big(\begin{matrix} \frac{1}{2} & j \\ n & m \end{matrix}\Big| \begin{matrix} j-\frac{1}{2} \\ \sigma \end{matrix}\Big)\left(\overline{\Pi}^{A}_{\frac{1}{2}}(\partial)\right)_{nn'} \otimes \left(\overline{\Pi}^{A}_{j}(\partial)\right)_{mm'}\Big(\begin{matrix} \frac{1}{2} & j \\ n' & m' \end{matrix}\Big| \begin{matrix} j-\frac{1}{2} \\ \rho \end{matrix}\Big) \quad \ldots.(10)$$

Proof

Consider equation (8) for $A = L$ so that

$$\begin{aligned}
C_{nm,r}&\left(\psi^{L}_{j+\frac{1}{2},\sigma'''},\ \psi^{L}_{j-\frac{1}{2},\sigma'}\right)_r \\
&= \psi^{L}_{\frac{1}{2},n} \otimes \psi^{L}_{j,m} \quad, \\
&= \left(\overline{\Pi}^{R}_{\frac{1}{2}}(\partial)\right)_{nn'} \psi^{R}_{\frac{1}{2},n'} \otimes \left(\overline{\Pi}^{R}_{j}(\partial)\right)_{mm'} \psi^{R}_{j,m'} \quad, \\
&= \left(\overline{\Pi}^{R}_{\frac{1}{2}}(\partial)\right)_{nn'} \otimes \left(\overline{\Pi}^{R}_{j}(\partial)\right)_{mm'} \ \psi^{R}_{\frac{1}{2},n'} \otimes \psi^{R}_{j,m'} \quad, \\
&= \left(\overline{\Pi}^{R}_{\frac{1}{2}}(\partial)\right)_{nn'} \otimes \left(\overline{\Pi}^{R}_{j}(\partial)\right)_{mm'} C_{n'm',r'}\left(\psi^{R}_{j+\frac{1}{2},\rho'''},\ \psi^{R}_{j-\frac{1}{2},\rho'}\right)_{r'} .
\end{aligned}$$

So that

$$\begin{aligned}
&\left(\psi^{L}_{j+\frac{1}{2},\sigma'''},\ \psi^{L}_{j-\frac{1}{2},\sigma'}\right)_r \\
&= (C^{-1})_{r,nm}\left(\overline{\Pi}^{R}_{\frac{1}{2}}(\partial)\right)_{nn'} \otimes \left(\overline{\Pi}^{R}_{j}(\partial)\right)_{mm'} C_{n'm',r'}\left(\psi^{R}_{j+\frac{1}{2},\rho'''},\psi^{R}_{j-\frac{1}{2}}\right)_{r'}
\end{aligned}$$

But $C_{nm,r}$ is the Clebsh-Gordon matrix so that we have

$$\begin{aligned}
\psi^{L}_{j+\frac{1}{2},\sigma} &= \left(\overline{\Pi}^{R}_{j+\frac{1}{2}}(\partial)\right)_{\sigma\sigma'} \psi^{R}_{j+\frac{1}{2},\sigma'} \\
&= \Big(\begin{matrix} \frac{1}{2} & j \\ n & m \end{matrix}\Big| \begin{matrix} j+\frac{1}{2} \\ \sigma \end{matrix}\Big)\left(\overline{\Pi}^{R}_{\frac{1}{2}}(\partial)\right)_{nn'} \otimes \left(\overline{\Pi}^{R}_{j}(\partial)\right)_{mm'}\Big(\begin{matrix} \frac{1}{2} & j \\ n' & m' \end{matrix}\Big| \begin{matrix} j+\frac{1}{2} \\ \rho \end{matrix}\Big)\psi^{R}_{j+\frac{1}{2},\rho}
\end{aligned}$$

and

$$\begin{aligned}
\psi^{L}_{j-\frac{1}{2},\sigma} &= \left(\overline{\Pi}^{R}_{j-\frac{1}{2}}(\partial)\right)_{\sigma\sigma'} \psi^{R}_{j-\frac{1}{2},\sigma'} \\
&= \Big(\begin{matrix} \frac{1}{2} & j \\ n & m \end{matrix}\Big| \begin{matrix} j-\frac{1}{2} \\ \sigma \end{matrix}\Big)\left(\overline{\Pi}^{R}_{\frac{1}{2}}(\partial)\right)_{nn'} \otimes \left(\overline{\Pi}^{R}_{j}(\partial)\right)_{mm'}\Big(\begin{matrix} \frac{1}{2} & j \\ n' & m' \end{matrix}\Big| \begin{matrix} j-\frac{1}{2} \\ \rho \end{matrix}\Big)\psi^{R}_{j-\frac{1}{2},\rho} .
\end{aligned}$$

These results are still true if we replace R by L so that equations (9) and (10) follow.

We note that the degree of the differential operator $\overline{\Pi}_j(\partial)$ is given by $2j$

so that we only obtain a first order equation if $j = \frac{1}{2}$.

4.2 The Wess-Zumino Model.

This is the simplest supersymmetric model and was first given by Wess and Zumino [3]. It is obtained by putting $j = 0$ in equations (94) and (95) of Chapter 3 and then combining these into a single supermultiplet, noting that the $j - \frac{1}{2}$ does not exist.

We recall equation (12) of Chapter 1

$$Q^c_\alpha = \begin{bmatrix} Q_{Ln} \\ Q_{Rn'} \end{bmatrix}_\alpha \qquad \cdots(11)$$

and define

$$\chi_\beta = \begin{bmatrix} \chi^L_{\frac{1}{2},n;0,0} \\ \chi^R_{\frac{1}{2},n';0,0} \end{bmatrix} \qquad \cdots(12)$$

similar to the definition of equation (1). Equations (94b), (94e), (95b) and (95e) can now be combined to give

$$[Q^c_\alpha , \chi_\beta]$$
$$= \begin{bmatrix} e^{-i\frac{3\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}(\sigma_2^L)_{nn'}\underset{\sim}{\chi}^L_{0,0} & e^{i\frac{3\pi}{4}}\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma_\mu^R\sigma_2^R\frac{\partial}{\partial x^\mu})_{nn'}\chi^R_{0,0} \\ e^{i\frac{3\pi}{4}}\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma_\mu^L\sigma_2^L\frac{\partial}{\partial x^\mu})_{nn'}\chi^L_{0,0} & e^{-i\frac{3\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}(\sigma_2^R)_{nn'}\underset{\sim}{\chi}^R_{0,0} \end{bmatrix}_{\alpha\beta}$$
$$= e^{-i\frac{3\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\begin{bmatrix} (\sigma_2^L)_{nn'}\underset{\sim}{\chi}^L_{0,0} & \\ 0 & (\sigma_2^R)_{nn'}\underset{\sim}{\chi}^R_{0,0} \end{bmatrix}_{\alpha\beta}$$
$$+ e^{i\frac{3\pi}{4}}\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}\begin{bmatrix} 0 & (\sigma_\mu^R\sigma_2^R\frac{\partial}{\partial x^\mu})_{nn'}\chi^R_{0,0} \\ (\sigma_\mu^L\sigma_2^L\frac{\partial}{\partial x^\mu})_{nn'}\chi^L_{0,0} & 0 \end{bmatrix}_{\alpha\beta} \qquad \cdots(13)$$

It is convenient to define new scalar fields A, B, F and G by

$$\underset{\sim}{\chi}^L_{0,0} = \frac{1}{\sqrt{2}}(F + iG) \quad , \qquad \cdots(14a)$$
$$\underset{\sim}{\chi}^R_{0,0} = \frac{1}{\sqrt{2}}(F - iG) \quad , \qquad \cdots(14b)$$
$$\chi^L_{0,0} = \frac{1}{\sqrt{2}}(A + iB) \qquad \cdots(14c)$$

and

$$\chi^R_{0,0} = \frac{1}{\sqrt{2}}(A - iB) \quad . \qquad \cdots(14d)$$

Now using these definitions and noting the definitions of the chiral Dirac matrices in terms of the left and right handed Pauli matrices as given by equations (38) of the appendix, we can rewrite equation (13) as

$$[Q^c_\alpha , \chi_\beta] = \frac{1}{\sqrt{2}} e^{-i\frac{3\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}(C^c)_{\alpha\beta} F + \frac{1}{\sqrt{2}} e^{-i\frac{3\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}(\gamma^c_5 C^c)_{\alpha\beta} iG$$
$$- \frac{1}{\sqrt{2}} e^{i\frac{3\pi}{4}}\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}(\gamma^{c\mu} C^c)_{\alpha\beta} \frac{\partial}{\partial x^\mu} A + \frac{1}{\sqrt{2}} e^{i\frac{3\pi}{4}}\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}(\gamma^c_5 \gamma^{c\mu} C^c)_{\alpha\beta} \frac{\partial}{\partial x^\mu} iB. \quad \cdots(15a)$$

Similarly, using the above definitions we obtain the following commutators from equations (94) and (95) of Chapter 3.

$$[Q^c_\alpha , A] = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} \chi_\alpha \quad , \quad \cdots(15b)$$

$$[Q^c_\alpha , iB] = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} (\gamma^c_5)_{\alpha\beta} \chi_\beta \quad , \quad \cdots(15c)$$

$$[Q^c_\alpha , F] = \frac{-1}{\sqrt{2}} e^{i\frac{3\pi}{4}}\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma^{c\mu})_{\alpha\beta} \frac{\partial}{\partial x^\mu} \chi_\beta \quad \cdots(15d)$$

and $$[Q^c_\alpha , iG] = \frac{1}{\sqrt{2}} e^{i\frac{3\pi}{4}}\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma^c_5 \gamma^{c\mu})_{\alpha\beta} \frac{\partial}{\partial x^\mu} \chi_\beta . \quad \cdots(15e)$$

These equations were obtained using the chiral representation of the Dirac matrices. We can transform from these to any other representation so that we can delete the superscript ^{c} and consider them as valid for any representation.

The supermultiplet defined by equations (15) can be demonstrated to be closed if the Jacobi identities are evaluated on each of the fields. We include the calculation since it demonstrates that we do need <u>all</u> of the fields (we have not yet required the spinor field χ_β to satisfy the Dirac equation), and shows the use of the Fierz identity as given in the appendix. This check is usefully carried out at every possible stage of these calculations as a check on the algebra.

The Jacobi identity takes the form

$$[[Q_\alpha , Q_\beta], X] = [Q_\alpha, [Q_\beta, X]] + [Q_\beta , [Q_\alpha , X]] \quad \cdots(16)$$

with X any field of either parity. Thus for the field A we have

$$[[Q_\alpha, Q_\beta], A] = [Q_\alpha , \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\chi_\beta]$$
$$+ [Q_\beta , \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} \chi_\alpha]$$

$$= \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} \Big\{ e^{-i\frac{3\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} (C)_{\alpha\beta} F + e^{-i\frac{3\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} (\gamma_5 C)_{\alpha\beta} iG$$
$$- e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma^\mu C)_{\alpha\beta} \frac{\partial}{\partial x^\mu} A + e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma_5 \gamma^\mu C)_{\alpha\beta} \frac{\partial}{\partial x^\mu} iB \Big\}$$
$$+ \{\alpha \Longleftrightarrow \beta\}.$$

Here $\{\alpha \Longleftrightarrow \beta\}$ indicates the same expression with the indices α, β interchanged. Now using the symmetry properties of the matrices C, $\gamma_5 C$ $\gamma^\mu C$ and $\gamma_5 \gamma^\mu C$ as given in the appendix we have

$$[[Q_\alpha, Q_\beta], A] = (\gamma^\mu C)_{\alpha\beta} \frac{\partial}{\partial x^\mu} A$$

as required. Similarly we obtain the corresponding result for the fields B, F and G.

Now consider the field χ_γ.

$$[[Q_\alpha, Q_\beta], \chi_\gamma] = [Q_\alpha, [Q_\beta, \chi_\gamma]] + [Q_\beta, [Q_\alpha, \chi_\gamma]]$$

$$= [Q_\alpha, \frac{1}{\sqrt{2}} e^{-i\frac{3\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} (C)_{\beta\gamma} F + \frac{1}{\sqrt{2}} e^{-\frac{3\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} (\gamma_5 C)_{\beta\gamma} iG$$
$$- \frac{1}{\sqrt{2}} e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma^\mu C)_{\beta\gamma} \frac{\partial}{\partial x} A + \frac{1}{\sqrt{2}} e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma_5 \gamma^\mu C)_{\beta\gamma} \frac{\partial}{\partial x^\mu} iB],$$
$$+ \{\alpha \Longleftrightarrow \beta\}$$

$$= -\frac{1}{2} e^{-i\frac{3\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} (C)_{\beta\gamma} e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma^\mu)_{\alpha\delta} \frac{\partial}{\partial x^\mu} \chi_\delta$$
$$+ \frac{1}{2} e^{-i\frac{3\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} (\gamma_5 C)_{\beta\gamma} e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma_5 \gamma^\mu)_{\alpha\delta} \frac{\partial}{\partial x^\mu} \chi_\delta$$
$$- \frac{1}{2} e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma^\mu C)_{\beta\gamma} \frac{\partial}{\partial x^\mu} e^{i\frac{\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} \chi_\alpha$$
$$+ \frac{1}{2} e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma_5 \gamma^\mu C) \frac{\partial}{\partial x^\mu} e^{i\frac{\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} (\gamma_5)_{\alpha\delta} \chi_\delta$$
$$+ \{\alpha \Longleftrightarrow \beta\},$$

$$= -\frac{1}{2} (C)_{\beta\gamma} (\gamma^\mu \frac{\partial}{\partial x^\mu} \chi)_\alpha + \frac{1}{2} (\gamma_5 C)_{\beta\gamma} (\gamma_5 \gamma^\mu \frac{\partial}{\partial x^\mu} \chi)_\alpha$$
$$+ \frac{1}{2} (\gamma^\mu C)_{\beta\gamma} \frac{\partial}{\partial x^\mu} \chi_\alpha - \frac{1}{2} (\gamma_5 \gamma^\mu C)_{\beta\gamma} (\gamma_5 \frac{\partial}{\partial x^\mu} \chi)_\alpha$$
$$+ \{\alpha \Longleftrightarrow \beta\}$$

Now we make use of the Fierz identity as given by equation (42) of the appendix to rearrange the indices. We obtain

$$[[Q_\alpha, Q_\beta], \chi_\beta]$$
$$= \frac{1}{8} (\Gamma^A C)_{\alpha\beta} (\Gamma_A \gamma^\mu \frac{\partial}{\partial x^\mu} \chi)_\gamma - \frac{1}{8} (\Gamma^B \gamma_5 C)_{\alpha\beta} (\Gamma_B \gamma_5 \gamma^\mu \frac{\partial}{\partial x^\mu} \chi)_\gamma$$
$$+ \frac{1}{8} (\Gamma^C \gamma^\mu C)_{\alpha\beta} (\Gamma_C \frac{\partial}{\partial x^\mu} \chi)_\gamma + \frac{1}{8} (\Gamma^D \gamma_5 \gamma^\mu C)_{\alpha\beta} (\Gamma_D \gamma_5 \frac{\partial}{\partial x^\mu} \chi)_\gamma$$
$$+ \{\alpha \Longleftrightarrow \beta\}$$

Now only terms with the matrices $(\ )_{\alpha\beta}$ symmetric will contribute so that this becomes

$$
\begin{aligned}
&[[Q_\alpha , Q_\beta], \chi_\gamma] \\
&\quad = \tfrac{1}{4}(\Gamma^A C)_{\alpha\beta}(\Gamma_A \gamma^\mu \tfrac{\partial}{\partial x^\mu}\chi)_\gamma - \tfrac{1}{4}(\Gamma^B\gamma_5 C)_{\alpha\beta}(\Gamma_B\gamma_5\gamma^\mu\tfrac{\partial}{\partial x^\mu}\chi)_\gamma \\
&\quad + \tfrac{1}{4}(\Gamma^C\gamma^\mu C)_{\alpha\beta}(\Gamma_C\tfrac{\partial}{\partial x^\mu}\chi)_\gamma + \tfrac{1}{4}(\Gamma^D\gamma_5\gamma^\mu C)_{\alpha\beta}(\Gamma_D\gamma_5\tfrac{\partial}{\partial x^\mu}\chi)_\gamma .
\end{aligned}
$$

With these limitations we can replace the matrices $\Gamma_A, \dots$ as follows.

If $\Gamma^A = \gamma^a$ then $\Gamma_A = \gamma_a$,

if $\Gamma^A = \gamma^a\gamma^b$ then $\Gamma_A = \gamma_b\gamma_a$,

if $\Gamma^B\gamma_5 = \gamma^a$ then $\Gamma^B = \gamma^a\gamma_5$ and $\Gamma_B = \gamma_5\gamma_a$,

if $\Gamma^B\gamma_5 = \gamma^a\gamma^b$ then $\Gamma^B = \gamma^a\gamma^b\gamma_5$ and $\Gamma_B = \gamma_5\gamma_b\gamma_a$,

if $\Gamma^C\gamma^\mu = \gamma^a$ then $\Gamma^C = \gamma^a\gamma_\mu$ and $\Gamma_C = \gamma^\mu\gamma_a$,

if $\Gamma^C\gamma^\mu = \gamma^a\gamma^b$ then $\Gamma^C = \gamma^a\gamma^b\gamma_\mu$ and $\Gamma_C = \gamma^\mu\gamma_b\gamma_a$,

if $\Gamma^D\gamma_5\gamma^\mu = \gamma^a$ then $\Gamma^D = \gamma^a\gamma_\mu\gamma_5$ and $\Gamma_D = \gamma_5\gamma^\mu\gamma_a$

and if $\Gamma^D\gamma_5\gamma^\mu = \gamma^a\gamma^b$ then $\Gamma^D = \gamma^a\gamma^b\gamma_\mu\gamma_5$ and $\Gamma_D = \gamma_5\gamma^\mu\gamma_b\gamma_a$.

Now

$$
\begin{aligned}
&[[Q_\alpha , Q_\beta], \chi_\gamma] \\
&= \tfrac{1}{4}(\gamma^a C)_{\alpha\beta}(\gamma_a\gamma^\mu\tfrac{\partial}{\partial x^\mu}\chi)_\gamma + \tfrac{1}{4}(\gamma^a\gamma^b C)_{\alpha\beta}(\gamma_b\gamma_a\gamma^\mu\tfrac{\partial}{\partial x^\mu}\chi)_\gamma \\
&- \tfrac{1}{4}(\gamma^a C)_{\alpha\beta}(\gamma_5\gamma_a\gamma_5\gamma^\mu\tfrac{\partial}{\partial x^\mu}\chi)_\gamma - \tfrac{1}{4}(\gamma^a\gamma^b C)_{\alpha\beta}(\gamma_5\gamma_b\gamma_a\gamma_5\gamma^\mu\tfrac{\partial}{\partial x^\mu}\chi)_\gamma \\
&+ \tfrac{1}{4}(\gamma^a C)_{\alpha\beta}(\gamma^\mu\gamma_a\tfrac{\partial}{\partial x^\mu}\chi)_\gamma + \tfrac{1}{4}(\gamma^a\gamma^b C)_{\alpha\beta}(\gamma^\mu\gamma_b\gamma_a\tfrac{\partial}{\partial x^\mu}\chi)_\gamma \\
&+ \tfrac{1}{4}(\gamma^a C)_{\alpha\beta}(\gamma_5\gamma^\mu\gamma_a\gamma_5\tfrac{\partial}{\partial x^\mu}\chi)_\gamma + \tfrac{1}{4}(\gamma^a\gamma^b C)_{\alpha\beta}(\gamma_5\gamma^\mu\gamma_b\gamma_a\gamma_5\tfrac{\partial}{\partial x^\mu}\chi)_\gamma ,
\end{aligned}
$$

$$
= \tfrac{1}{2}(\gamma^a C)_{\alpha\beta}(\gamma_a\gamma^\mu\tfrac{\partial}{\partial x^\mu}\chi)_\gamma + \tfrac{1}{2}(\gamma_a C)_{\alpha\beta}(\gamma^\mu\gamma_a\tfrac{\partial}{\partial x^\mu}\chi)_\gamma ,
$$

$$
\begin{aligned}
&= \tfrac{1}{2}(\gamma^a C)_{\alpha\beta}(\gamma_a\gamma^\mu\tfrac{\partial}{\partial x^\mu}\chi)_\gamma - \tfrac{1}{2}(\gamma^a C)_{\alpha\beta}(\gamma_a\gamma^\mu\tfrac{\partial}{\partial x^\mu}\chi)_\gamma \\
&\qquad + (\gamma^\mu C)_{\alpha\beta}\tfrac{\partial}{\partial x^\mu}\chi_\gamma .
\end{aligned}
$$

So that

$$
[[Q_\alpha , Q_\beta], \chi_\gamma] = (\gamma^\mu C)_{\alpha\beta}\tfrac{\partial}{\partial x^\mu}\chi_\gamma .
$$

Now we want to examine the consequences of requiring the spinor χ_β to satisfy the Dirac equation. We can combine equations (96) of Chapter 3 to give

$$(\gamma^\mu \tfrac{\partial}{\partial x^\mu})_{\alpha\beta}\, \chi_\beta = -i \tfrac{Mc}{\hbar} \chi_\beta \qquad \cdots(17)$$

Then acting on equation (15a) with this operator we obtain

$$(\gamma^\mu \tfrac{\partial}{\partial x^\mu})_{\gamma\beta} [Q_\alpha, \chi_\beta] = [Q_\alpha, -i\tfrac{Mc}{\hbar} \chi_\gamma],$$

$$= \tfrac{1}{\sqrt{2}} (\gamma^\mu \tfrac{\partial}{\partial x^\mu})_{\gamma\beta} \Big\{ e^{-i\frac{3\pi}{4}} \big(\tfrac{Mc}{\hbar}\big)^{\frac{1}{2}} (C)_{\alpha\beta} F + e^{-i\frac{3\pi}{4}} \big(\tfrac{Mc}{\hbar}\big)^{\frac{1}{2}} (\gamma_5 C)_{\alpha\beta}\, iG$$
$$- e^{i\frac{3\pi}{4}} \big(\tfrac{\hbar}{Mc}\big)^{\frac{1}{2}} (\gamma^\rho C)_{\alpha\beta} \tfrac{\partial}{\partial x^\rho} A + e^{i\frac{3\pi}{4}} \big(\tfrac{\hbar}{Mc}\big)^{\frac{1}{2}} (\gamma_5 \gamma^\rho C)_{\alpha\beta} \tfrac{\partial}{\partial x^\rho}\, iB \Big\},$$

$$= -\tfrac{1}{\sqrt{2}} e^{-i\frac{3\pi}{4}} \big(\tfrac{Mc}{\hbar}\big)^{\frac{1}{2}} (\gamma^\mu C)_{\gamma\alpha} \tfrac{\partial}{\partial x^\mu} F - \tfrac{1}{\sqrt{2}} e^{-i\frac{3\pi}{4}} \big(\tfrac{Mc}{\hbar}\big)^{\frac{1}{2}} (\gamma^\mu \gamma_5 C)_{\gamma\alpha} \tfrac{\partial}{\partial x^\mu}\, iG$$
$$- \tfrac{1}{\sqrt{2}} e^{i\frac{3\pi}{4}} \big(\tfrac{\hbar}{Mc}\big)^{\frac{1}{2}} (\gamma^\mu \gamma^\rho C)_{\gamma\alpha} \tfrac{\partial}{\partial x^\mu} \tfrac{\partial}{\partial x^\rho} A$$
$$- \tfrac{1}{\sqrt{2}} e^{i\frac{3\pi}{4}} \big(\tfrac{\hbar}{Mc}\big)^{\frac{1}{2}} (\gamma^\mu \gamma_5 \gamma^\rho C)_{\gamma\alpha} \tfrac{\partial}{\partial x^\mu} \tfrac{\partial}{\partial x^\rho}\, iB.$$

Thus

$$[Q_\alpha, \chi_\gamma]$$
$$= \tfrac{1}{\sqrt{2}} e^{i\frac{3\pi}{4}} \big(\tfrac{\hbar}{Mc}\big)^{\frac{1}{2}} (\gamma^\mu C)_{\alpha\gamma} \tfrac{\partial}{\partial x^\mu} F - \tfrac{1}{\sqrt{2}} e^{i\frac{3\pi}{4}} \big(\tfrac{\hbar}{Mc}\big)^{\frac{1}{2}} (\gamma^\mu \gamma_5 C)_{\alpha\gamma} \tfrac{\partial}{\partial x^\mu}\, iG$$
$$+ \tfrac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \big(\tfrac{\hbar}{Mc}\big)^{3/2} (\gamma^\mu \gamma^\rho C)_{\gamma\alpha} \tfrac{\partial}{\partial x^\mu} \tfrac{\partial}{\partial x^\rho} A$$
$$- \tfrac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \big(\tfrac{\hbar}{Mc}\big)^{3/2} (\gamma^\mu \gamma^\rho \gamma_5 C)_{\gamma\alpha} \tfrac{\partial}{\partial x^\mu} \tfrac{\partial}{\partial x^\rho}\, iB. \qquad \cdots(18)$$

Now

$$(\gamma^\mu \gamma^\rho \tfrac{\partial}{\partial x^\mu} \tfrac{\partial}{\partial x^\rho})_{\alpha\beta} = (\gamma^\mu \gamma_\rho \tfrac{\partial}{\partial x^\mu} \tfrac{\partial}{\partial x_\rho})_{\alpha\beta}$$
$$= \delta_{\alpha\beta} \tfrac{\partial}{\partial x^\mu} \tfrac{\partial}{\partial x_\mu}. \qquad \cdots(19)$$

Since if $\mu \neq \rho$ the terms cancel. Also all the fields must satisfy the Klein-Gordon equation. So that

$$\tfrac{\partial}{\partial x^\mu} \tfrac{\partial}{\partial x_\mu} A = -\big(\tfrac{Mc}{\hbar}\big)^2 A \qquad \cdots(20)$$

and

$$\tfrac{\partial}{\partial x^\mu} \tfrac{\partial}{\partial x_\mu} B = -\big(\tfrac{Mc}{\hbar}\big)^2 B. \qquad \cdots(21)$$

Thus, using equations (19), (20) and (21) equation (18) can be written

$$[Q_\alpha, \chi_\gamma]$$
$$= \tfrac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \big(\tfrac{\hbar}{Mc}\big)^{\frac{1}{2}} (\gamma^\mu C)_{\alpha\gamma} \tfrac{\partial}{\partial x^\mu} F + \tfrac{1}{\sqrt{2}} e^{i\frac{3\pi}{4}} \big(\tfrac{\hbar}{Mc}\big)^{\frac{1}{2}} (\gamma_5 \gamma^\mu C)_{\alpha\gamma} \tfrac{\partial}{\partial x^\mu}\, iG$$
$$+ \tfrac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \big(\tfrac{Mc}{\hbar}\big)^{\frac{1}{2}} (C)_{\alpha\gamma} A - \tfrac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} (\gamma_5 C)_{\alpha\gamma} \big(\tfrac{Mc}{\hbar}\big)^{\frac{1}{2}} iB. \qquad \cdots(22)$$

Now if we compare this with equation (15a) we can see that if χ_β satisfies the Dirac equation then

$$A = -F$$

and

$$B = G.$$

Alternatively this may be stated as

$$\underset{\sim}{\chi}^{L}_{0,0} = -\chi^{R}_{0,0}$$

and
$$\underset{\sim}{\chi}^{R}_{0,0} = -\chi^{L}_{0,0}$$

Now consider equations (15b) and (15d) with $A = -F$, these imply that

$$\tfrac{1}{2} e^{i\frac{\pi}{4}} \left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}} \chi_{\alpha} = \tfrac{1}{2} e^{i\frac{3\pi}{4}} \left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma^{\mu})_{\alpha\beta} \tfrac{\partial}{\partial x^{\mu}} \chi_{\beta}$$

ie.
$$(\gamma^{\mu} \tfrac{\partial}{\partial x^{\mu}})_{\alpha\beta} = -i \tfrac{Mc}{\hbar} \chi_{\alpha} .$$

Thus χ_{β} must satisfy the Dirac equation and this then implies that

$$B = G .$$

Similarly consider equations (15c) and (15e) with $B = G$, these again imply that χ_{β} must satisfy the Dirac equation, which implies that $A=-F$.

We thus reach the important conclusion that given the three constraints

$$A = -F \quad , \qquad \cdots(27a)$$

$$B = G \qquad \cdots(27b)$$

and
$$(\gamma^{\mu} \tfrac{\partial}{\partial x^{\mu}})_{\alpha\beta} \chi_{\beta} = -i \tfrac{Mc}{\hbar} \chi_{\alpha} \,, \qquad \cdots(27c)$$

any one implies the other two.

4.3 Supermultiplets for $j \neq 0$ Satisfying Wave Equations.

We observe that the spin j index m is left unchanged by the action of the supersymmetry generators in equations (94) and (95) of Chapter 3. So that using the differential operators $\prod^{L}_{j}(\partial)$ or $\prod^{R}_{j}(\partial)$ as defined in equation (7) we can construct multiplets obeying the commutators of equations (94) or (95) of Chapter 3, that are either left or right handed with respect to the spin j index. It follows that we can construct sets of fields obeying equations (15) that are either left or right handed with respect to the spin j index. That is we can obtain the following set of commutators

$$[Q_{\alpha}, A^{A}_{j,m}] = \tfrac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}} \chi^{A}_{\alpha;j,m} \quad , \qquad \cdots(28a)$$

$$[Q_{\alpha}, iB^{A}_{j,m}] = \tfrac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}} (\gamma_5)_{\alpha\beta} \chi^{A}_{\beta;j,m} \quad , \qquad \cdots(28b)$$

$$[Q_{\alpha}, F^{A}_{j,m}] = -\tfrac{1}{\sqrt{2}} e^{i\frac{3\pi}{4}} \left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma^{\mu})_{\alpha\beta} \tfrac{\partial}{\partial x^{\mu}} \chi^{A}_{\beta;j,m} \,, \qquad \cdots(28c)$$

$$[Q_{\alpha}, iG^{A}_{j,m}] = \tfrac{1}{\sqrt{2}} e^{i\frac{3\pi}{4}} \left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma_5 \gamma^{\mu})_{\alpha\beta} \tfrac{\partial}{\partial x^{\mu}} \chi^{A}_{\beta;j,m} \qquad \cdots(28d)$$

and $$[Q_\alpha\, ,\, \chi^A_{\beta;j,m}] = \tfrac{1}{\sqrt{2}}\, e^{-i\frac{3\pi}{4}} \left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}} (C)_{\alpha\beta} F^A_{j,m}$$
$$+\tfrac{1}{\sqrt{2}}\, e^{-i\frac{3\pi}{4}} \left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}} (\gamma_5 C)_{\alpha\beta}\, i G^A_{j,m} - \tfrac{1}{\sqrt{2}}\, e^{i\frac{3\pi}{4}} \left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma^\mu C)_{\alpha\beta} \tfrac{\partial}{\partial x^\mu} A^A_{j,m}$$
$$+\tfrac{1}{\sqrt{2}}\, e^{i\frac{3\pi}{4}} \left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma_5 \gamma_\mu C)_{\alpha\beta} \tfrac{\partial}{\partial x^\mu}\, i B^A_{j,m} \qquad 28e)$$

for $A = L$ or $A = R$.

The analysis of the previous section can be repeated to obtain the three constraints

$$A^A_{j,m} = -F^A_{j,m}\ , \qquad \cdots(29a)$$
$$B^A_{j,m} = G^A_{j,m} \qquad \cdots(29b)$$

and $$(\gamma^\mu)_{\alpha\beta} \tfrac{\partial}{\partial x^\mu} \chi_{\beta;j,m} = -i \tfrac{Mc}{\hbar}\, \chi_{\alpha;j,m} \qquad \cdots(29c)$$

for $A = L$ or $A = R$. As before any one of these constraints implies the other two.

Now suppose we require a combination of supermultiplets given by equation (28) such that each field obeys some field equation other than the Klein-Gordon equation. The obvious choice is to combine a left handed supermultiplet with a right handed one and require that the fields are related as in Proposition 4.1. We then have the fields

$$A_{j,\beta} = \begin{bmatrix} A^L_{j,m} \\ A^R_{j,m'} \end{bmatrix}_\beta \ , \qquad \cdots(30a)$$

$$B_{j,\beta} = \begin{bmatrix} B^L_{j,m} \\ B^R_{j,m'} \end{bmatrix}_\beta \ , \qquad \cdots(30b)$$

$$F_{j,\beta} = \begin{bmatrix} F^L_{j,m} \\ F^L_{j,m'} \end{bmatrix}_\beta \ , \qquad \cdots(30c)$$

$$G_{j,\beta} = \begin{bmatrix} G^L_{j,m} \\ G^L_{j,m'} \end{bmatrix}_\beta \qquad \cdots(30d)$$

and $$\chi_{\alpha;j,\beta} = \begin{bmatrix} \chi^L_{\alpha;j,m} \\ \chi^R_{\alpha;j,m} \end{bmatrix}_\beta \qquad \cdots(30e)$$

with $\beta = 1, 2, \cdots, 2(2j+1)$.

These fields satisfy the commutators of equations (28) and we can impose any one of the constraints

$$A_{j,\beta} = -F_{j,\beta} , \qquad \cdots(31a)$$

$$B_{j,\beta} = G_{j,\beta} \qquad \cdots(31b)$$

or

$$(\gamma^\mu)_{\alpha\beta} \frac{\partial}{\partial x^\mu} \chi_{\beta;j,\gamma} = -\frac{iMc}{\hbar} \chi_{\alpha;j,\gamma} , \qquad \cdots(31c)$$

which then implies the other two. By construction they obey the field equations

$$\begin{bmatrix} 0 & \Pi^R_j(\partial) \\ \Pi^L_j(\partial) & 0 \end{bmatrix}_{\beta\beta'} A_{j,\beta'} = A_{j,\beta} , \qquad \cdots(32a)$$

$$\begin{bmatrix} 0 & \Pi^R_j(\partial) \\ \Pi^L_j(\partial) & 0 \end{bmatrix}_{\beta\beta'} B_{j,\beta'} = B_{j,\beta} \qquad \cdots(32b)$$

and

$$\begin{bmatrix} 0 & \Pi^R_j(\partial) \\ \Pi^L_j(\partial) & 0 \end{bmatrix}_{\beta\beta'} (\gamma^\mu \frac{\partial}{\partial x^\mu})_{\alpha\alpha'} \chi_{\alpha';j,\beta'} \qquad \cdots(32c)$$

$$= -i\frac{Mc}{\hbar} \chi_{\alpha;j,\beta} ,$$

with the equations of motion of $F_{j,\beta}$ and $G_{j,\beta}$ being obtained from equations (32a)and (32b) by using equations (31a) and (31b) respectively.

4.4 Lagrangian Densities for Supermultiplets.

We consider first the Wess-Zumino supermultiplet as constructed in section 2, in four component form, so that we have a non trivial Lagrangian. The standard Lagrangian density kinetic energy term for a spinor field is given by

$$\mathcal{L}^{KE}_\chi = i\hbar c \, \bar{\chi} \gamma^\mu \frac{\partial}{\partial x^\mu} \chi . \qquad \cdots(33)$$

But χ is a Majorana spinor so that

$$\bar{\chi} = \chi^t (C^t)^{-1}. \qquad \cdots(34)$$

Hence

$$\mathcal{L}^{KE}_\chi = i\hbar c \, \chi^t_\beta ((C^{t^{-1}}) \gamma^\mu \frac{\partial}{\partial x^\mu})_{\beta\gamma} \chi_\gamma . \qquad \cdots(35)$$

Now the charge conjugation matrix C is defined by (see equation (29) of the appendix)

$$C^{-1} \gamma^\mu C = -(\gamma^\mu)^t.$$

So that

$$C^{-1}\gamma^{\mu} = -(\gamma^{\mu})^{t}C^{-1} \quad ,$$

$$(C^{-1}\gamma^{\mu})^{t} = -((\gamma^{\mu})^{t}C^{-1})^{t} \quad ,$$

thus

$$((C^{-1})^{t}\gamma^{\mu})_{\beta\alpha} = -(C^{-1}\gamma^{\mu})_{\alpha\beta} . \qquad \cdots(36)$$

Then combining equations (35) and (36) we have

$$\mathcal{L}_{\chi}^{KE} = -i\hbar c\, \chi_{\beta}(C^{-1}\gamma^{\mu})_{\gamma\beta}\frac{\partial}{\partial x^{\mu}}\chi_{\gamma} . \qquad \cdots(37)$$

Now consider the action of the supersymmetry generators Q_{α} on $\mathcal{L}_{\chi}^{KE}$.

$$\begin{aligned}
&[Q_{\alpha}, \mathcal{L}_{\chi}^{KE}] \\
&= [Q_{\alpha}, -i\hbar c\chi_{\beta}(C^{-1}\gamma^{\mu})_{\gamma\beta}\frac{\partial}{\partial x^{\mu}}\chi_{\gamma}] \quad , \\
&= -i\hbar c[Q_{\alpha}, \chi_{\beta}](C^{-1}\gamma^{\mu})_{\gamma\beta}\frac{\partial}{\partial x^{\mu}}\chi_{\gamma} \\
&\qquad\qquad + i\hbar c\chi_{\beta}(C^{-1}\gamma^{\mu})_{\gamma\beta}\frac{\partial}{\partial x^{\mu}}[Q_{\alpha}, \chi_{\gamma}] \quad , \\
&= -\frac{i\hbar c}{\sqrt{2}}\Big\{e^{-i\frac{3\pi}{4}}\Big(\frac{Mc}{\hbar}\Big)^{\frac{1}{2}}(C)_{\alpha\beta}F + e^{-i\frac{3\pi}{4}}\Big(\frac{Mc}{\hbar}\Big)^{\frac{1}{2}}(\gamma_{5}C)_{\alpha\beta}iG \\
&- e^{i\frac{3\pi}{4}}\Big(\frac{\hbar}{Mc}\Big)^{\frac{1}{2}}(\gamma^{\rho}C)_{\alpha\beta}\frac{\partial}{\partial x^{\rho}}A + e^{i\frac{3\pi}{4}}\Big(\frac{\hbar}{Mc}\Big)^{\frac{1}{2}}(\gamma_{5}\gamma^{\rho}C)_{\alpha\beta}\frac{\partial}{\partial x^{\rho}}iB\Big\}(C^{-1}\gamma^{\mu})_{\alpha\beta}\frac{\partial}{\partial x^{\mu}}\chi_{\gamma} \\
&+ \frac{i\hbar c}{\sqrt{2}}\chi_{\beta}(C^{-1}\gamma^{\rho})_{\gamma\beta}\frac{\partial}{\partial x^{\rho}}\Big\{e^{-i\frac{3\pi}{4}}\Big(\frac{Mc}{\hbar}\Big)^{\frac{1}{2}}(C)_{\alpha\gamma}F + e^{-i\frac{3\pi}{4}}\Big(\frac{Mc}{\hbar}\Big)^{\frac{1}{2}}(\gamma_{5}C)_{\alpha\gamma}iG \\
&- e^{i\frac{3\pi}{4}}\Big(\frac{\hbar}{Mc}\Big)^{\frac{1}{2}}(\gamma^{\mu}C)_{\alpha\gamma}\frac{\partial}{\partial x^{\mu}}A + e^{i\frac{3\pi}{4}}\Big(\frac{\hbar}{Mc}\Big)^{\frac{1}{2}}(\gamma_{5}\gamma^{\mu}C)_{\alpha\gamma}\frac{\partial}{\partial x^{\mu}}iB\Big\} , \\
&= -\frac{i\hbar c}{\sqrt{2}}\Big\{-e^{-i\frac{3\pi}{4}}\Big(\frac{Mc}{\hbar}\Big)^{\frac{1}{2}}(C^{-1}\gamma^{\mu}C)_{\gamma\alpha}F\frac{\partial}{\partial x^{\mu}}\chi_{\gamma} \\
&- e^{-i\frac{3\pi}{4}}\Big(\frac{Mc}{\hbar}\Big)^{\frac{1}{2}}(C^{-1}\gamma^{\mu}\gamma_{5}C)_{\gamma\alpha}iG\frac{\partial}{\partial x^{\mu}}\chi_{\gamma} \\
&- e^{i\frac{3\pi}{4}}\Big(\frac{\hbar}{Mc}\Big)^{\frac{1}{2}}(C^{-1}\gamma^{\mu}\gamma^{\rho}C)_{\gamma\beta}\Big(\frac{\partial}{\partial x^{\rho}}A\Big)\frac{\partial}{\partial x^{\mu}}\chi_{\gamma} \\
&- e^{i\frac{3\pi}{4}}\Big(\frac{\hbar}{Mc}\Big)^{\frac{1}{2}}(C^{-1}\gamma^{\mu}\gamma_{5}\gamma^{\rho}C)_{\gamma\alpha}\Big(\frac{\partial}{\partial x^{\rho}}iB\Big)\frac{\partial}{\partial x^{\mu}}\chi_{\gamma}\Big\} \\
&+ \frac{i\hbar c}{\sqrt{2}}\chi_{\beta}\Big\{e^{-i\frac{3\pi}{4}}\Big(\frac{Mc}{\hbar}\Big)^{\frac{1}{2}}(\gamma^{\rho})_{\alpha\beta}\frac{\partial}{\partial x^{\rho}}F + e^{-i\frac{3\pi}{4}}\Big(\frac{Mc}{\hbar}\Big)^{\frac{1}{2}}(\gamma_{5}\gamma^{\rho})_{\alpha\beta}\frac{\partial}{\partial x^{\rho}}iG \\
&- e^{i\frac{3\pi}{4}}\Big(\frac{\hbar}{Mc}\Big)^{\frac{1}{2}}(\gamma^{\mu}\gamma^{\rho})_{\alpha\beta}\frac{\partial}{\partial x^{\rho}}\frac{\partial}{\partial x^{\mu}}A \\
&\quad + e^{i\frac{3\pi}{4}}\Big(\frac{\hbar}{Mc}\Big)^{\frac{1}{2}}(\gamma_{5}\gamma^{\mu}\gamma^{\rho})_{\alpha\beta}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x^{\rho}}iB\Big\} . \qquad \cdots(38)
\end{aligned}$$

Now from the appendix we have the following identities

$$(C^{-1}\gamma^{\mu}C)_{\gamma\alpha} = -(\gamma^{\mu t})_{\gamma\alpha} = -(\gamma^{\mu})_{\alpha\gamma} \quad , \qquad \cdots(39a)$$

$$(C^{-1}\gamma^{\mu}\gamma_{5}C)_{\gamma\alpha} = -(\gamma^{\mu t}\gamma_{5}^{t})_{\gamma\alpha} = -(\gamma_{5}\gamma^{\mu})_{\alpha\gamma} = (\gamma^{\mu}\gamma_{5})_{\alpha\gamma} , \qquad \cdots(39b)$$

$$(C^{-1}\gamma^\mu\gamma^\rho C)_{\gamma\alpha} = (\gamma^{\mu t}\gamma^{\rho t})_{\gamma\alpha} = (\gamma^\rho\gamma^\mu)_{\alpha\gamma} \qquad \cdots(39c)$$

and
$$(C^{-1}\gamma^\mu\gamma_5\gamma^\rho C)_{\gamma\alpha} = (\gamma^{\mu t}\gamma_5^t\gamma^{\rho t})_{\gamma\alpha} = (\gamma^\rho\gamma_5\gamma^\mu)_{\alpha\gamma}. \qquad \cdots(39d)$$

Also as noted in equation (19)

$$(\gamma^\mu\gamma^\rho)_{\alpha\beta}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\rho} = \delta_{\alpha\beta}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}. \qquad \cdots(40)$$

Inserting equations (39) and (40) into equation (38) we obtain

$$\begin{aligned}
&[Q_\alpha, \mathcal{L}_\chi^{KE}] \\
&= -\frac{i\hbar c}{\sqrt{2}}\Big\{e^{-i\frac{3\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac12}(\gamma^\mu)_{\alpha\gamma}F\left(\tfrac{\partial}{\partial x^\mu}\chi_\gamma\right) - e^{i\frac{3\pi}{4}}\left(\tfrac{\hbar}{Mc}\right)^{\frac12}\left(\tfrac{\partial}{\partial x^\mu}A\right)\left(\tfrac{\partial}{\partial x_\mu}\chi_\alpha\right) \\
&\quad - e^{-i\frac{3\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac12}(\gamma^\mu\gamma_5)_{\alpha\gamma}\, iG\left(\tfrac{\partial}{\partial x^\mu}\chi_\gamma\right) + e^{i\frac{3\pi}{4}}\left(\tfrac{\hbar}{Mc}\right)^{\frac12}(\gamma_5)_{\alpha\gamma}\left(\tfrac{\partial}{\partial x^\mu}iB\right)\left(\tfrac{\partial}{\partial x_\mu}\chi_\gamma\right)\Big\} \\
&\quad + \frac{i\hbar c}{\sqrt{2}}\Big\{e^{-i\frac{3\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac12}(\gamma^\rho)_{\alpha\beta}\chi_\beta\left(\tfrac{\partial}{\partial x^\rho}F\right) - e^{i\frac{3\pi}{4}}\left(\tfrac{\hbar}{Mc}\right)\chi_\alpha\left(\tfrac{\partial}{\partial x^\mu}\tfrac{\partial}{\partial x_\mu}A\right) \\
&\quad + e^{-i\frac{3\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac12}(\gamma_5\gamma^\rho)_{\alpha\beta}\chi_\beta\left(\tfrac{\partial}{\partial x^\rho}iG\right) + e^{i\frac{3\pi}{4}}\left(\tfrac{\hbar}{Mc}\right)^{\frac12}(\gamma_5)_{\alpha\beta}\chi_\beta\left(\tfrac{\partial}{\partial x^\mu}\tfrac{\partial}{\partial x_\mu}iB\right)\Big\}. \qquad \cdots(41)
\end{aligned}$$

Now consider the possible kinetic energy Lagrangian term for A

$$\mathcal{L}_A^{KE} = \frac{\hbar^2}{2M}\left(\frac{\partial}{\partial x^\mu}A\right)\left(\frac{\partial}{\partial x_\mu}A\right) \qquad \cdots(42)$$

The action of the supersymmetry generators Q_α on $\mathcal{L}_A^{KE}$ is given by

$$\begin{aligned}
&[Q_\alpha, \mathcal{L}_A^{KE}] \\
&= \frac{\hbar^2}{2m}\Big\{[Q_\alpha, \tfrac{\partial}{\partial x^\mu}A]\tfrac{\partial}{\partial x_\mu}A + \tfrac{\partial}{\partial x^\mu}A\,[Q_\alpha, \tfrac{\partial}{\partial x^\mu}A]\Big\}, \\
&= \frac{\hbar^2}{M}\left(\frac{\partial}{\partial x^\mu}\frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac12}\chi_\alpha\right)\frac{\partial}{\partial x_\mu}A, \\
&= \frac{\hbar^2}{\sqrt{2}M}e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac12}\left(\frac{\partial}{\partial x^\mu}\chi_\alpha\right)\left(\frac{\partial}{\partial x_\mu}A\right). \qquad \cdots(43)
\end{aligned}$$

Consider the possible term for B

$$\mathcal{L}_B^{KE} = \frac{\hbar^2}{2M}\left(\frac{\partial}{\partial x^\mu}B\right)\left(\frac{\partial}{\partial x_\mu}B\right), \qquad \cdots(44)$$

ie.
$$\mathcal{L}_{iB}^{KE} = \frac{\hbar^2}{2M}\left(\frac{\partial}{\partial x^\mu}iB\right)\left(\frac{\partial}{\partial x_\mu}iB\right) = -\mathcal{L}_B^{KE}.$$

So that

$$\begin{aligned}
&[Q_\alpha, \mathcal{L}_{iB}^{KE}] \\
&= 2\,[Q_\alpha, \tfrac{\hbar^2}{2M}\tfrac{\partial}{\partial x^\mu}iB]\,\tfrac{\partial}{\partial x_\mu}iB, \\
&= \frac{\hbar^2}{\sqrt{2}M}e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac12}\left(\frac{\partial}{\partial x^\mu}iB\right)\left(\gamma_5\frac{\partial}{\partial x_\mu}\chi\right)_\alpha.
\end{aligned}$$

Thus

$$[Q_\alpha, \mathcal{L}_B^{KE}] = \frac{-\hbar}{\sqrt{2}M}e^{i\frac{\pi}{4}}\left(\frac{\partial}{\partial x^\mu}iB\right)\left(\gamma_5\frac{\partial}{\partial x_\mu}\chi\right)_\alpha. \qquad \cdots(45)$$

Consider the possible term for F given by

$$\mathcal{L}_F^{KE} = F^2 . \qquad \cdots(46)$$

So that

$$\begin{aligned}[Q_\alpha , \mathcal{L}_F^{KE}] &= 2F[Q_\alpha , F] , \\ &= -\sqrt{2}\, F\, e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma^\mu)_{\alpha\beta} \frac{\partial}{\partial x^\mu} \chi_\beta . \end{aligned} \qquad \cdots(47)$$

Lastly consider the possible term for G given by

$$\mathcal{L}_G^{KE} = G^2 = -(iG)^2 . \qquad \cdots(48)$$

So that

$$\begin{aligned}&[Q_\alpha , \mathcal{L}_G^{KE}] \\ &= -i\sqrt{2}\, G\, e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma_5 \gamma^\mu)_{\alpha\beta} \frac{\partial}{\partial x^\mu} \chi_\beta . \end{aligned} \qquad \cdots(49)$$

Now we construct the following candidate for the kinetic energy term for the supermultiplet

$$\mathcal{L}^{KE} = \mathcal{L}_\chi^{KE} + a\mathcal{L}_A^{KE} + b\mathcal{L}_B^{KE} + f\mathcal{L}_F^{KE} + g\mathcal{L}_G^{KE} \qquad \cdots(50)$$

with the constants $a, b, c, g \in \mathbb{C}$ to be determined below. It is convenient to check the invariance in four seperate steps. We consider seperately the terms in $A\chi$, $B\chi$, $F\chi$ and $G\chi$.

Consider first the terms in $A\chi$. We have

$$\begin{aligned}&[Q_\alpha , \text{Terms in } A\chi] \\ &= \frac{i\hbar c}{\sqrt{2}} e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} \left(\frac{\partial}{\partial x^\mu} A\right)\left(\frac{\partial}{\partial x_\mu}\chi_\alpha\right) - \frac{i\hbar c}{\sqrt{2}} e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} \chi_\alpha \left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} A\right) \\ &\quad + \frac{a\hbar^2}{\sqrt{2}M} e^{i\frac{\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} \left(\frac{\partial}{\partial x^\mu}\chi_\alpha\right)\left(\frac{\partial}{\partial x_\mu} A\right) , \\ &= \frac{1}{\sqrt{2}} e^{i\frac{5\pi}{4}} \frac{\hbar^{3/2} c^{\frac{1}{2}}}{M^{\frac{1}{2}}} \left(\frac{\partial}{\partial x^\mu} A\right)\left(\frac{\partial}{\partial x^\mu}\chi_\alpha\right) - \frac{\hbar c}{\sqrt{2}} e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} \chi_\alpha \left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} A\right) \\ &\quad + \frac{a}{\sqrt{2}} e^{i\frac{\pi}{4}} \frac{\hbar^{3/2} c^{\frac{1}{2}}}{M^{\frac{1}{2}}} \left(\frac{\partial}{\partial x^\mu} A\right)\left(\frac{\partial}{\partial x^\mu}\chi_\alpha\right) . \end{aligned} \qquad \cdots(51)$$

Now if we set $a = 2$ this is a total divergence, so that it does not contribute to the action integral $\int \mathcal{L}^{KE}$.

Now consider the terms in $B\chi$. We have

$$
\begin{aligned}
&[Q_\alpha\,,\ \text{Terms in } B\chi] \\
&= -\frac{i\hbar c}{\sqrt{2}}\, e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma_5)_{\alpha\gamma} \left(\frac{\partial}{\partial x^\mu} iB\right)\left(\frac{\partial}{\partial x_\mu}\chi_\gamma\right) \\
&\quad + \frac{i\hbar c}{\sqrt{2}}\, e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma_5)_{\alpha\gamma}\, \chi_\gamma \left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} iB\right) \\
&\quad - \frac{b}{\sqrt{2}}\, \frac{\hbar^2}{M}\, e^{i\frac{\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} \left(\frac{\partial}{\partial x^\mu} iB\right)\left(\gamma_5 \frac{\partial}{\partial x_\mu}\chi_\gamma\right)\ , \\
&= \frac{1}{\sqrt{2}}\, e^{i\frac{\pi}{4}}\, \frac{\hbar^{3/2} c^{\frac{1}{2}}}{M^{\frac{1}{2}}} (\gamma_5)_{\alpha\gamma} \left(\frac{\partial}{\partial x^\mu} iB\right)\left(\frac{\partial}{\partial x_\mu}\chi_\gamma\right) \\
&\quad + \frac{i}{\sqrt{2}}\, \hbar c\, e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma_5)_{\alpha\beta}\, \chi_\beta \left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} iB\right) \\
&\quad - \frac{b}{\sqrt{2}}\, e^{i\frac{\pi}{4}}\, \frac{\hbar^{3/2} c^{\frac{1}{2}}}{M^{\frac{1}{2}}} \left(\frac{\partial}{\partial x^\mu} iB\right)\left(\gamma_5 \frac{\partial}{\partial x_\mu}\chi_\gamma\right) . \qquad \cdots(52)
\end{aligned}
$$

If we set $b = 2$ this is again a total divergence, so that it does not contribute to the action.

Now consider the terms in $F\chi$. We have

$$
\begin{aligned}
&[Q_\alpha\,,\ \text{Terms in } F\chi] \\
&= -\frac{i\hbar c}{\sqrt{2}}\, e^{-i\frac{3\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} (\gamma^\mu)_{\alpha\gamma}\, F \left(\frac{\partial}{\partial x^\mu}\chi_\gamma\right) \\
&\quad + \frac{i\hbar c}{\sqrt{2}}\, e^{-i\frac{3\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} (\gamma^\rho)_{\alpha\gamma}\, \chi_\gamma \left(\frac{\partial}{\partial x^\rho} F\right) \\
&\quad - \sqrt{2} f\, e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} F (\gamma^\mu)_{\alpha\gamma} \left(\frac{\partial}{\partial x^\mu}\chi_\gamma\right)\ ,
\end{aligned}
$$

and with $f = Mc$ this becomes

$$
\begin{aligned}
&[Q_\alpha\,,\ \text{Terms in } F\chi] \\
&= -\frac{1}{\sqrt{2}}\, e^{i\frac{3\pi}{4}} M^{\frac{1}{2}} c^{3/2} \hbar^{\frac{1}{2}} (\gamma^\mu)_{\alpha\gamma} \left\{ F\left(\frac{\partial}{\partial x^\mu}\chi_\gamma\right) + \left(\frac{\partial}{\partial x^\mu} F\right)\chi_\gamma \right\}\ , \\
&= -\frac{1}{\sqrt{2}}\, e^{i\frac{3\pi}{4}} (Mc^3\hbar)^{\frac{1}{2}} (\gamma^\mu)_{\alpha\gamma} \frac{\partial}{\partial x^\mu} (F\chi_\gamma) . \qquad \cdots(53)
\end{aligned}
$$

We can identify this as a total divergence, so that it provides no contribution to the action.

Lastly we consider terms in $G\chi$. We obtain

$$
\begin{aligned}
&[Q_\alpha\,,\ \text{Terms in } G\chi] \\
&= \frac{i\hbar c}{\hbar}\, e^{-i\frac{3\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} (\gamma^\mu\gamma_5)_{\alpha\gamma}\, iG \left(\frac{\partial}{\partial x^\mu}\chi_\gamma\right) \\
&\quad + \frac{i\hbar c}{\sqrt{2}}\, e^{-i\frac{3\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} (\gamma_5\gamma^\rho)_{\alpha\gamma}\, \chi_\beta \left(\frac{\partial}{\partial x^\mu} iG\right) \\
&\quad - ig\sqrt{2}\, e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma_5\gamma^\rho)_{\alpha\gamma}\, G \left(\frac{\partial}{\partial x^\mu}\chi_\beta\right)\ ,
\end{aligned}
$$

and with $g = Mc$ this becomes

$$[Q_\gamma , \text{Terms in } G\chi]$$
$$= \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} (\hbar c^3 M)^{\frac{1}{2}} (\gamma_5 \gamma^\mu)_{\alpha\gamma} \frac{\partial}{\partial x^\mu} (\chi_\gamma G). \qquad \cdots(54)$$

This, again, is a total divergence so that it provides no contribution to the action.

Combining these results we can see that the following kinetic energy term

$$\mathcal{L}^{KE} = i\hbar c\, \bar{\chi} \gamma^\mu \frac{\partial}{\partial x^\mu} \chi + \frac{\hbar^2}{M} \left\{ \frac{\partial}{\partial x^\mu} A \frac{\partial}{\partial x_\mu} A + \frac{\partial}{\partial x^\mu} B \frac{\partial}{\partial x_\mu} B \right\}$$
$$+ Mc^2 \{ F^2 + G^2 \} \qquad \cdots(55)$$

is such that the action integral

$$\mathcal{A}_{KE} = \int \mathcal{L}^{KE} d^4x \qquad \cdots(56)$$

satisfies

$$[Q_\alpha , \mathcal{A}_{KE}] = 0. \qquad \cdots(57)$$

Now consider the possible mass term for the field χ_α

$$\mathcal{L}^M_\chi = -Mc^2 \bar{\chi} \chi \qquad \cdots(58)$$

The action of the supersymmetry generators on this term is given by

$$[Q_\alpha , \mathcal{L}^M_\chi] = [Q_\alpha , -Mc^2 \bar{\chi}\chi]$$
$$= [Q_\alpha , -Mc^2 \chi_\beta (C^{-1})_{\gamma\beta} \chi_\gamma]$$
$$= Mc^2 \chi_\beta (C^{-1})_{\gamma\beta} [Q_\alpha , \chi_\gamma] - [Q_\alpha , \chi_\beta] Mc^2 (C^{-1})_{\gamma\beta} \chi_\gamma$$
$$= 2Mc^2 \chi_\beta (C^{-1})_{\gamma\beta} [Q_\alpha , \gamma_\gamma]$$
$$= \sqrt{2} Mc^2 \left\{ \chi_\alpha F e^{-i\frac{3\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} + \left(\frac{Mc}{\hbar}\right) e^{-i\frac{3\pi}{4}} (\gamma_5)_{\alpha\beta} \chi_\beta \, iG \right. \qquad \cdots(59)$$
$$\left. - e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma^\mu)_{\alpha\beta} \chi_\beta \frac{\partial}{\partial x^\mu} A + e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma_5 \gamma^\mu)_{\alpha\beta} \chi_\beta \frac{\partial}{\partial x^\mu} iB \right\}.$$

Now consider

$$\mathcal{L}^M_{AF} = Mc^2 AF. \qquad \cdots(60)$$

So that

$$[Q_\alpha , \mathcal{L}^M_{AF}]$$
$$= -\frac{1}{\sqrt{2}} Mc^2 e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} (\gamma^\mu)_{\alpha\beta} A \frac{\partial}{\partial x^\mu} \chi_\beta$$
$$- \frac{1}{\sqrt{2}} Mc^2 e^{-i\frac{3\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{2} \chi_\alpha F . \qquad \cdots(61)$$

Lastly consider

$$\mathcal{L}^{M}_{BG} = Mc^2 BG \qquad \cdots(62)$$

So that

$$\begin{aligned}[Q_\alpha\,, Mc^2 BG] \\ = \frac{Mc^2}{\sqrt{2}}\Big\{ -i\, e^{i\frac{3\pi}{4}}\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}(\gamma_5\gamma^\mu)_{\alpha\beta}\, B\, \tfrac{\partial}{\partial x^\mu}\chi_\beta \\ + i\, e^{-i\frac{3\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}(\gamma_5\chi)_\alpha G \Big\}\end{aligned} \qquad \cdots(63)$$

Combining the mass terms of equations (58), (60) and (62) we have

$$\begin{aligned}\mathcal{L}^M &= \mathcal{L}^M_\chi + \mathcal{L}^M_{AF} + \mathcal{L}^M_{BG}\,, \\ &= -Mc^2\bar{\chi}\chi = 2Mc^2\{AF + BG\}.\end{aligned} \qquad (64)$$

with

$$\begin{aligned}&[Q_\alpha\,, \mathcal{L}^M] \\ &= -\sqrt{2}\,Mc^2 e^{i\frac{3\pi}{4}}\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}\Big\{(\gamma^\mu)_{\alpha\beta}\chi_\beta\left(\tfrac{\partial}{\partial x^\mu}A\right) + (\gamma^\mu)_{\alpha\beta}A\left(\tfrac{\partial}{\partial x^\mu}\chi_\beta\right) \\ &\qquad -(\gamma_5\gamma^\mu)_{\alpha\beta}\chi_\beta\left(\tfrac{\partial}{\partial x^\mu} iB\right) - (\gamma_5\gamma^\mu)_{\alpha\beta}B\left(\tfrac{\partial}{\partial x^\mu}\chi_\beta\right)\Big\}\,, \\ &= -\sqrt{2}\,Mc^2 e^{i\frac{3\pi}{4}}\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}\tfrac{\partial}{\partial x^\mu}\left((\gamma^\mu\chi)_\alpha A\right) \\ &\quad +\sqrt{2}\,Mc^2 e^{i\frac{3\pi}{4}}\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}\tfrac{\partial}{\partial x^\mu}\left((\gamma_5\gamma^\mu\chi)_\alpha B\right).\end{aligned}$$

This is a total divergence, so that it provides no contribution to the ac action integal

$$\mathcal{A}_M = \int \mathcal{L}^M\, d^4x\,. \qquad (66)$$

The Lagrangian density in full is

$$\begin{aligned}\mathcal{L} = i\hbar c\,\bar{\chi}\gamma^\mu\tfrac{\partial}{\partial x^\mu}\chi + \tfrac{\hbar^2}{M}\Big\{\tfrac{\partial}{\partial x^\mu}A\tfrac{\partial}{\partial x_\mu}A + \tfrac{\partial}{\partial x^\mu}B\tfrac{\partial}{\partial x_\mu}B\Big\} \\ + Mc^2\{F^2+G^2\} - Mc^2\bar{\chi}\chi + 2Mc^2\{AF - BG\}.\end{aligned} \qquad \cdots(67)$$

We can obtain the equations of motion of the fields by using the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial}{\partial x^\mu}\phi\right)}\right) = 0 \qquad \cdots(68)$$

for $\phi \in \{\chi, A, B, G, F\}$. We obtain the following equations

$$i\hbar c\,\gamma^\mu\tfrac{\partial}{\partial x^\mu}\chi - Mc^2\chi = 0 \quad , \qquad \cdots(69a)$$

$$-i\hbar c\,\tfrac{\partial}{\partial x^\mu}\bar{\chi}\gamma^\mu - Mc^2\bar{\chi} = 0 \quad , \qquad \cdots\cdots(69b)$$

$$-\frac{\hbar^2}{M}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}A + Mc^2F = 0 \quad , \qquad \cdots(69c)$$

$$-\frac{\hbar^2}{M}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}B + Mc^2G = 0 \quad , \qquad \cdots(69d)$$

$$Mc^2(F + A) = 0 \qquad \cdots(69e)$$

and

$$Mc^2(G - B) = 0 \quad . \qquad \cdots(69f)$$

Equation (69e) implies that

$$A = -F \qquad \cdots(70a)$$

and equation (69f) implies that

$$B = G \qquad \cdots(70b)$$

in complete agreement with the conclusions of section 4.2.

Substituting these expressions into equations (69c) and (69d) we obtain

$$\left(\frac{\hbar}{Mc}\right)^2\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}A + A = 0 \qquad \cdots(71a)$$

and

$$\left(\frac{\hbar}{Mc}\right)^2\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}B + B = 0 \quad . \qquad \cdots(71b)$$

Then using equations (70a) and (70b) we can obtain the equations of motion of the fields F and G. They are

$$\left(\frac{\hbar}{Mc}\right)^2\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}F + F = 0 \qquad \cdots(72a)$$

and

$$\left(\frac{\hbar}{Mc}\right)^2\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}G + G = 0 \quad . \qquad \cdots(72b)$$

Now if we consider the analysis of section 3 it is clear that we can construct a set of fields $\{A^A_{j,m}\ ,\ B^B_{j,m}\ ,\ \chi^A_{\alpha j\, j,m}\ ,\ F^A_{j,m}\ ,\ G^A_{j,m}\}$ for $A = L$ or $A = R$ which have the property that the function

$$\begin{aligned} \mathcal{F}^A_{j,mm'} = {} & i\hbar c\, \bar{\chi}^A_{j,m}\gamma^\mu\frac{\partial}{\partial x^\mu}\chi^A_{j,m'} \\ & + \frac{\hbar}{M}\left\{\frac{\partial}{\partial x^\mu}A^A_{j,m}\frac{\partial}{\partial x_\mu}A^A_{j,m'} + \frac{\partial}{\partial x^\mu}B^A_{j,m}\frac{\partial}{\partial x_\mu}B^A_{j,m'}\right\} \\ & + Mc\left\{F^A_{j,m}F^A_{j,m'} + G^A_{j,m}G^A_{j,m'}\right\} - Mc^2\,\bar{\chi}^A_{j,m}\chi^A_{j,m'} \qquad \cdots(73) \\ & + Mc^2\left\{A^A_{j,m}F^A_{j,m'} + A^A_{j,m'}F^A_{j,m} - B^A_{j,m}G^A_{j,m'} - B^A_{j,m'}G^A_{j,m}\right\} \end{aligned}$$

satisfies

$$[\,Q_\alpha\ ,\ \int \mathcal{F}^A_{j,mm'}\, d^4x\,] = 0 \qquad \cdots(74)$$

for $A = L$ or $A = R$.

In order to construct a Lagrangian density we require a scalar function built from the fields, that contains at most two derivative operators in each term (otherwise we violate causality cf. Ramond [61]). This rules out the use of the operators $\prod_j(\partial)$ as constructed in section 1 except for $j=0$ and $j=\frac{1}{2}$.

The action of the Poincaré group on $\mathcal{F}^L_{j,mm'}$ is given by

$$U([\Lambda|t])\,\mathcal{F}^L_{j,mm'}(x)\,U([\Lambda|t])^{-1}$$
$$= \Gamma^{0,j}(\Lambda^{-1})_{mn}\,\Gamma^{0,j}(\Lambda^{-1})_{m'n'}\,\mathcal{F}^L_{j,nn'}(\Lambda x+t) \qquad \cdots(75)$$

with a similar expression for $\mathcal{F}^R_{j,mm'}(x)$.

Now consider $(Z_j^{-1})_{mm'}\,\mathcal{F}^L_{j,mm'}(x)$, with Z_j the matrix defined in equation (9) of Chapter 3. Then

$$U([\Lambda|t])\,(Z_j^{-1})_{mm'}\,\mathcal{F}^L_{j,mm'}(x)\,U([\Lambda|t])^{-1}$$
$$= (Z_j^{-1})_{mm'}\,\Gamma^{0,j}(\Lambda^{-1})_{mn}\Gamma^{0,j}(\Lambda^{-1})_{m'n'}\,\mathcal{F}^L_{j,nn'}(\Lambda x+t)\ ,$$
$$= (Z_j^{-1})_{mm'}(\Gamma^{0,j}(\Lambda^{-1})^{*\dagger})_{nm}\,\Gamma^{0,j}(\Lambda^{-1})_{m'n'}\,\mathcal{F}^L_{j,nn'}(\Lambda x+t)\ ,$$
$$= (Z_j^{-1}\Gamma^{j,0}(\Lambda^{-1})Z_j)^{\dagger}_{nm}(Z_j^{-1})_{mm'}\,\Gamma^{0,j}(\Lambda^{-1})_{m'n'}\,\mathcal{F}^L_{j,nn'}(\Lambda x+t)\ ,$$
$$= (Z_j^{-1}\Gamma^{j,0}(\Lambda^{-1})^{\dagger}Z_j)_{nm}(Z_j^{-1})_{mm'}\Gamma^{0,j}(\Lambda^{-1})_{m'n'}\,\mathcal{F}^L_{j,nn'}(\Lambda x+t)\ ,$$
$$= (Z_j^{-1})_{nm}(\Gamma^{j,0}(\Lambda^{-1})^{\dagger}\Gamma^{0,j}(\Lambda^{-1}))_{mn'}\mathcal{F}^L_{j,nn'}(\Lambda x+t)\ ,$$
$$= (Z_j^{-1})_{mm'}\,\mathcal{F}^L_{j,mm'}(\Lambda x+t)\ .$$

So that

$$\int (Z_j^{-1})_{mm'}\,\mathcal{F}^L_{j,mm'}(x)\,d^4x \qquad \cdots(76)$$

is a Poincaré invariant, which also satisfies

$$[Q_\alpha,\ \int (Z_j^{-1})_{mm'}\,\mathcal{F}^L_{j,mm'}(x)\,d^4x\,] = 0 \qquad \cdots(77)$$

with similar expressions being valid for $(Z_j^{-1})_{mm'}\,\mathcal{F}^R_{j,mm'}(x)$.

4.5 The Combination of the $j=k+\frac{1}{2}$ and $j=k-\frac{1}{2}$ Supermultiplets.

In this section we will combine the $j=k+\frac{1}{2}$ and $j=k-\frac{1}{2}$ supermultiplets in a way that proves useful in Chapter 4 of Part III, when we construct the general scalar superfield.

Consider two supermultiplets as defined by either equation (94) or (95) of Chapter 3 with $j=k+\frac{1}{2}$ and $j=k-\frac{1}{2}$ respectively. Then consider the following combination of these fields with $A=L$ or $A=R$

$$\chi^{AA}_{\frac{1}{2},n;k,m} = \begin{pmatrix} \frac{1}{2} & k \\ n & m \end{pmatrix}\begin{vmatrix} k+\frac{1}{2} \\ n+m \end{pmatrix} \chi^{A}_{k+\frac{1}{2},n+m} + \begin{pmatrix} \frac{1}{2} & k \\ n & m \end{pmatrix}\begin{vmatrix} k-\frac{1}{2} \\ n+m \end{pmatrix} \chi^{A}_{k-\frac{1}{2},n+m} , \quad \cdots(78a)$$

$$\underset{\sim}{\chi}^{AA}_{\frac{1}{2},n;k,m} = \begin{pmatrix} \frac{1}{2} & k \\ n & m \end{pmatrix}\begin{vmatrix} k+\frac{1}{2} \\ n+m \end{pmatrix} \underset{\sim}{\chi}^{A}_{k+\frac{1}{2},n+m} + \begin{pmatrix} \frac{1}{2} & k \\ n & m \end{pmatrix}\begin{vmatrix} k-\frac{1}{2} \\ n+m \end{pmatrix} \underset{\sim}{\chi}^{A}_{k-\frac{1}{2},n+m} \quad \cdots(78b)$$

and $$\chi^{AAA}_{\frac{1}{2},n';\frac{1}{2},n;k,m} = \begin{pmatrix} \frac{1}{2} & k \\ n & m \end{pmatrix}\begin{vmatrix} k+\frac{1}{2} \\ n+m \end{pmatrix} \chi^{A}_{\frac{1}{2},n';k+\frac{1}{2},n+m} + \begin{pmatrix} \frac{1}{2} & k \\ n & m \end{pmatrix}\begin{vmatrix} k-\frac{1}{2} \\ n+m \end{pmatrix} \chi^{A}_{\frac{1}{2},n';k-\frac{1}{2},n+m} . \quad \cdots(78c)$$

We adopt the convention, here, that the superscripts indicate the chirality of the various subscripts and are written in the same order as the subscripts. The commutators of these fields are identical to those of equation (95) of Chapter 3 if $A=L$ and to those of equation (94) if $A=R$, with the $\frac{1}{2},n'$ and k,m indices remaining unaffected by the supersymmetry transformations. It is also easy to see that we can alter the transformation properties of these fields by acting on the k,m or $\frac{1}{2},n'$ indices of the fields with the differential operators $\prod_j(\partial)$ of section 4.1 to convert them from right to left handed or vice versa.

Thus from equations (78) and equations (94) and (95) of Chapter 3 we can write down the following set of commutators

$$[Q_{\bar{C}n}, \underset{\sim}{\chi}^{AB}_{\frac{1}{2},n'';k,m}] = e^{i\frac{3\pi}{4}}\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma_\mu^C \tfrac{\partial}{\partial x^\mu})_{nn'}\, \chi^{ACB}_{\frac{1}{2},n'';\frac{1}{2},n';k,m} , \quad \cdots(79a)$$

$$[Q_{\bar{C}n}, \chi^{ACB}_{\frac{1}{2},n'';\frac{1}{2},n';k,m}] = e^{i\frac{3\pi}{4}}\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma_\mu^C \sigma_2^C \tfrac{\partial}{\partial x^\mu})_{nn'}\, \chi^{AB}_{\frac{1}{2},n'';k,m} , \quad \cdots(79b)$$

$$[Q_{\bar{C}n}, \chi^{AB}_{\frac{1}{2},n'';k,m}] = 0 , \quad \cdots(79c)$$

$$[Q_{Cn}, \chi^{AB}_{\frac{1}{2},n'';k,m}] = e^{i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} \chi^{ACB}_{\frac{1}{2},n'';\frac{1}{2},n;k,m} , \quad \cdots(79d)$$

$$[Q_{Cn}, \chi^{ACB}_{\frac{1}{2},n'';\frac{1}{2},n';k,m}] = e^{-i\frac{3\pi}{4}}\left(\frac{Mc}{\hbar}\right)(\sigma_2^C)_{nn'}\, \underset{\sim}{\chi}^{AB}_{\frac{1}{2},n'';k,m} \quad \cdots(79e)$$

and $$[Q_{Cn}, \underset{\sim}{\chi}^{AB}_{\frac{1}{2},n'';k,m}] = 0 .$$

These equations are then valid for $A=L$ or $A=R$, $B=L$ or $B=R$ and $C=L$ or $C=R$ with $\bar{C}=L$ if $C=R$ or $\bar{C}=R$ if $C=L$ and $k=0,\frac{1}{2},1,\cdots$. If $k=0$ this supermultiplet contains only the $k=j+\frac{1}{2}$ chiral supermultiplet of equations (94) and (95) of Chapter 3. In the sequel we set $A=R$, $B=L$ and $C=L$.

We now want to use the two spin $\frac{1}{2}$ indices on the field $\chi^{RLL}_{\frac{1}{2},n'';\frac{1}{2},n';k,m}$ to construct a field that transforms as the tensor product of a vector (transforming as Λ) and a spin k field.

Lemma 4.2

Suppose the field P^{RL}_{rn} transforms as

$$U([\Lambda|t])\, P^{RL}_{rn}(x)\, U([\Lambda|t])^{-1}$$
$$= \Gamma^{0,\frac{1}{2}}(\Lambda^{-1})_{nn'}\,\Gamma^{\frac{1}{2},0}(\Lambda^{-1})_{rr'}\, P^{RL}_{r'n'}(\Lambda x+t).$$

Then the field $(\sigma_2^L\sigma_\mu^L)_{nr}P^{RL}_{rn}(x)$ transforms as

$$U([\Lambda|t])\,(\sigma_2^L\sigma_\mu^L)_{nr}\,P^{RL}_{rn}(x)\,U([\Lambda|t])^{-1}$$
$$= (\Lambda^{-1})_{\mu\nu}(\sigma_2^L\sigma_\nu^L)_{nr}\,P^{RL}_{rn}(\Lambda x+t).$$

Proof

From equation (43) of the appendix we have

$$\Gamma^{0,\frac{1}{2}}(\Lambda)\, x^\mu\sigma_\mu\,(\Gamma^{0,\frac{1}{2}}(\Lambda))^\dagger = (\Lambda x)^\mu\sigma_\mu\ ,$$
$$= \Lambda^\mu{}_\nu\, x^\nu\sigma_\mu\ ,$$
$$= x^\mu\,\Lambda^\nu{}_\mu\,\sigma_\nu\ .$$

So that

$$\Gamma^{0,\frac{1}{2}}(\Lambda)\,\sigma_\mu\,(\Gamma^{0,\frac{1}{2}}(\Lambda))^\dagger = \Lambda_\mu{}^\nu\sigma_\nu = (\Lambda^{-1})_{\mu\nu}\sigma_\nu\ . \qquad \cdots(80)$$

Now

$$U([\Lambda|t])\,(\sigma_2^L\sigma_\mu^L)_{nr}\,P^{RL}_{rn}(x)\,U([\Lambda|t])^{-1}$$
$$= \Gamma^{0,\frac{1}{2}}(\Lambda^{-1})_{nn'}\,(\sigma_2^L\sigma_\mu^L)_{nr}\,\Gamma^{\frac{1}{2},0}(\Lambda^{-1})_{rr'}\,P^{RL}_{r'n'}(\Lambda x+t)\ ,$$
$$= (\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})^*)^\dagger_{n'n}\,(\sigma_2^L\sigma_\mu^L)_{nr}\,\Gamma^{\frac{1}{2},0}(\Lambda^{-1})_{rr'}\,P^{RL}_{r'n'}(\Lambda x+t)\ ,$$
$$= (\sigma_2^L\,\Gamma^{\frac{1}{2},0}(\Lambda^{-1})\,\sigma_2^L)^\dagger_{n'n}\,(\sigma_2^L\sigma_\mu^L)_{nr}\,\Gamma^{\frac{1}{2},0}(\Lambda^{-1})_{rr'}\,P^{RL}_{r'n'}(\Lambda x+t)\ ,$$
$$= (\sigma_2^L)_{n'n''}\,\Gamma^{\frac{1}{2},0}(\Lambda^{-1})^\dagger_{n''n}\,(\sigma_\mu^L)_{nr}\,\Gamma^{\frac{1}{2},0}(\Lambda^{-1})_{rr'}\,P^{RL}_{r'n'}(\Lambda x+t)\ ,$$
$$= (\sigma_2^L)_{n'n''}\,\Gamma^{0,\frac{1}{2}}(\Lambda)_{n''n}\,(\sigma_\mu^L)_{nr}\,\Gamma^{0,\frac{1}{2}}(\Lambda)^\dagger_{rr'}\,P^{RL}_{r'n'}(\Lambda x+t)\ ,$$
$$= (\sigma_2^L)_{n'n}\,(\Lambda^{-1})_{\mu\nu}\,(\sigma_\nu^L)_{nr'}\,P^{RL}_{r'n'}(\Lambda x+t)\ .$$

In this calculation we have used equation (71) of Chapter 3 and equation (4) of the appendix and in the last line we have used equation (80). We now have

$$\mathcal{U}([\Lambda|t])(\sigma_2^L\sigma_\mu^L)_{nr}P^{RL}_{rr}(x)\,\mathcal{U}([\Lambda|t])^{-1}$$
$$= (\Lambda^{-1})_{\mu\nu}(\sigma_2^L\sigma_\nu^L)_{nr}P^{RL}_{rn}(\Lambda x+t)\,,$$

as required.

We define

$$(\sigma_2^L\sigma_\mu^L)_{nr}\chi^{RLL}_{\frac12,r;\frac12,n;k,m} = v^L_{\mu;k,m}\,, \qquad \cdots(81)$$

so that

$$\mathcal{U}([\Lambda|t])\,v^L_{\mu;k,m}(x)\,\mathcal{U}([\Lambda|t])^{-1}$$
$$= (\Lambda^{-1})_{\mu\nu}\Gamma^{0,k}(\Lambda^{-1})_{mm'}v^L_{\nu;k,m'}(\Lambda x+t). \qquad \cdots(82)$$

We also need the 'inverse' of equation (81), to obtain this we have to expand in terms of the components as follows

$$\sigma_2\sigma_1=\begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix},\ \sigma_2\sigma_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ \sigma_2\sigma_3=\begin{bmatrix}0 & i\\ i & 0\end{bmatrix},\ \sigma_2\sigma_4=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}.$$

So that

$$v^L_{1;j,m} = -i\chi^{RLL}_{\frac12,\frac12;\frac12,\frac12;j,m} + i\chi^{RLL}_{\frac12,-\frac12;\frac12,-\frac12;j,m}\,,$$
$$v^L_{2;j,m} = \chi^{RLL}_{\frac12,\frac12;\frac12,\frac12;j,m} + \chi^{RLL}_{\frac12,-\frac12;\frac12,-\frac12;j,m}\,,$$
$$v^L_{3;j,m} = i\chi^{RLL}_{\frac12,-\frac12;\frac12,\frac12;j,m} + i\chi^{RLL}_{\frac12,\frac12;\frac12,-\frac12;j,m}$$

and
$$v^L_{4;j,m} = -i\chi^{RLL}_{\frac12,-\frac12;\frac12,\frac12;j,m} + i\chi^{RLL}_{\frac12,\frac12;\frac12,-\frac12;j,m}\,.$$

Thus

$$\chi^{RLL}_{\frac12,-\frac12;\frac12,-\frac12;j,m} = \tfrac12(v^L_{2;j,m} - i v^L_{1;j,m})\,,$$
$$\chi^{RLL}_{\frac12,\frac12;\frac12,\frac12;j,m} = \tfrac12(v^L_{2;j,m} + i v^L_{1;j,m})\,,$$
$$\chi^{RLL}_{\frac12,\frac12;\frac12,-\frac12;j,m} = \tfrac12(v^L_{3;j,m} + v_{4;j,m})$$

and
$$\chi^{RLL}_{\frac12,-\frac12;\frac12,\frac12;j,m} = \tfrac{i}{2}(v^L_{4;j,m} - v_{3;j,m})\,.$$

Combining these expressions into a matrix we find that

$$\chi^{RLL}_{\frac12,n;\frac12,r;j,m} = \tfrac12(\sigma_\mu^L\sigma_2^L)_{nr}v^L_{\mu;j,m}\,. \qquad \cdots(83)$$

Now we put $A=R$ and $B=L$ in equations (79) and use equations (81) and (83) to obtain the following set of commutators.

$$[Q_{Rn},\underset{\sim}{\chi}{}^{RL}_{\frac12,n';k,m}] = \tfrac12 e^{i\frac{3\pi}{4}}\left(\tfrac{\hbar}{Mc}\right)^{\frac12}(\sigma_\mu^L\sigma_\rho^R\sigma_2^R)_{nn'}\frac{\partial}{\partial x_\mu}v^L_{\rho;k,m}\,; \cdots(84a)$$
$$[Q_{Rn},v^L_{\rho;k,m}] = e^{i\frac{3\pi}{4}}\left(\tfrac{\hbar}{Mc}\right)^{\frac12}(\sigma_\mu^L\sigma_\rho^L)_{nn'}\frac{\partial}{\partial x_\mu}\chi^{RL}_{\frac12,n';k,m}\,, \cdots(84b)$$
$$[Q_{Rn},\chi^{RL}_{\frac12,n';k,m}] = 0 \qquad , \cdots(84c)$$

$$[Q_{Ln}, \chi^{RL}_{\frac{1}{2},n';k,m}] = \tfrac{1}{2} e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}(\sigma_\mu^R \sigma_2^R)_{nn'} \, v^L_{\mu;k,m} \quad , \qquad \cdots(84d)$$

$$[Q_{Ln}, v^L_{\mu;k,m}] = (\sigma_\mu^L)_{nn'} \, e^{-i\frac{3\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}} \underset{\sim}{\chi}^{RL}_{\frac{1}{2},n';k,m} \qquad \cdots(84e)$$

and $\quad [Q_{Ln}, \underset{\sim}{\chi}^{RL}_{\frac{1}{2},n';k,m}] = 0.$

A check of the Jacobi identities shows this to be a closed supermultiplet. The field $v^L_{\mu;k,m}$ is a four component vector (ignoring the spin j index) so that it contains both spin 1 and spin 0 components. It is convenient to seperate these, but first we want to construct a field from $\underset{\sim}{\chi}^{RL}$ that is left handed in both indices. We define

$$\underset{\sim}{\chi}^{LL}_{\frac{1}{2},n;k,m} = \tfrac{i\hbar}{Mc}\left(\sigma_\mu^R \tfrac{\partial}{\partial x^\mu}\right)_{nn'} \underset{\sim}{\chi}^{RL}_{\frac{1}{2},n';k,m} \qquad \cdots(85)$$

Now from equation (84a) we have

$$[Q_{Rn}, \underset{\sim}{\chi}^{LL}_{\frac{1}{2},n';k,m}] = \tfrac{1}{2}\tfrac{i\hbar}{Mc}\left(\sigma_\nu^R \tfrac{\partial}{\partial x^\nu}\right)_{n'r} e^{i\frac{3\pi}{4}}\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma_\mu^L \sigma_\varrho^R \sigma_2^R)_{nr} \times \tfrac{\partial}{\partial x^\mu} v^L_{\varrho;k,m},$$

$$= \tfrac{1}{2} e^{i\frac{5\pi}{4}}(\sigma_\nu^R \sigma_\varrho^L \sigma_\mu^R \sigma_2^L)_{n'n} \tfrac{\partial}{\partial x^\nu}\tfrac{\partial}{\partial x^\mu} v^L_{\varrho;k,m}\left(\tfrac{\hbar}{Mc}\right)^{3/2},$$

$$= \tfrac{1}{2} e^{i\frac{5\pi}{4}}\left(\tfrac{\hbar}{Mc}\right)^{3/2}((2g_{\nu\varrho} - \sigma_\varrho^R \sigma_\nu^L)\sigma_\mu^R \sigma_2^L)_{n'n} \tfrac{\partial}{\partial x^\nu}\tfrac{\partial}{\partial x^\mu} v^L_{\varrho;k,m} .$$

Here we have used equation (9) of the appendix. Now

$$[Q_{Rn}, \underset{\sim}{\chi}^{LL}_{\frac{1}{2},n';k,m}]$$
$$= e^{i\frac{5\pi}{4}}(\sigma_\mu^R \sigma_2^L)_{n'n} \tfrac{\partial}{\partial x^\mu}\tfrac{\partial}{\partial x_\varrho} v^L_{\varrho;k,m}\left(\tfrac{\hbar}{Mc}\right)^{3/2}$$
$$- \tfrac{1}{2} e^{i\frac{5\pi}{4}}(\sigma_\varrho^R \sigma_2^L)_{n'n} \tfrac{\partial}{\partial x_\mu}\tfrac{\partial}{\partial x^\mu} v^L_{\varrho;k,m}\left(\tfrac{\hbar}{Mc}\right)^{3/2}.$$

So that using equation (14) of the appendix and the Klein-Gordon equation we have

$$[Q_{Rn}, \underset{\sim}{\chi}^{LL}_{\frac{1}{2},n';k,m}] \qquad \cdots(86)$$
$$= e^{i\frac{\pi}{4}}(\sigma_\mu^L \sigma_2^L)_{nn'}\left\{\left(\tfrac{\hbar}{Mc}\right)^{3/2}\tfrac{\partial}{\partial x^\mu}\tfrac{\partial}{\partial x_\varrho} v^L_{\varrho,j,m} + \tfrac{1}{2}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}} v^L_{\mu;k,m}\right\}.$$

Also from equation (84e) we find that

$$[Q_{Ln}, v^L_{\mu;k,m}] = e^{-i\frac{\pi}{4}}\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma_\mu^L \sigma_\varrho^L)_{nr} \tfrac{\partial}{\partial x_\varrho} \underset{\sim}{\chi}^{LL}_{\frac{1}{2},r;k,m}. \qquad \cdots(87)$$

We now have a supermultiplet consisting of the fields $\chi^{RL}_{\frac{1}{2},n;k,m}$, $v_{\mu;k,m}$ and $\chi^{LL}_{\frac{1}{2},n;k,m}$ and defined by the commutators of equations (84b), (84c), (84d), (84f), (86) and (87). A check of the Jacobi identities demonstrates that we still have a closed supermultiplet with

$V^{L}_{\mu;k,m}$ an unconstrained vector (ignoring the spin j index). To separate off the 'spin 0' field we carry out the following field redefinition. We define the field $J^{L}_{\mu;k,m}$ by

$$J^{L}_{\mu;k,m} = V^{L}_{\mu;j,m} + \left(\frac{\hbar}{Mc}\right)^2 \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_e} V^{L}_{e;k,m}. \qquad \cdots(88)$$

So that

$$J^{L}_{\mu;k,m} - \left(\frac{\hbar}{Mc}\right)^2 \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_e} J^{L}_{e;k,m} = V^{L}_{\mu;k,m} \qquad \cdots(89)$$

and
$$J^{L}_{\mu;k,m} + \left(\frac{\hbar}{Mc}\right)^2 \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_e} J^{L}_{e;k,m} = V^{L}_{\mu;k,m} + 2\left(\frac{\hbar}{Mc}\right)^2 \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_e} V^{L}_{e;k,m} \cdots(90)$$

Now

$$\begin{aligned}\frac{\partial}{\partial x_\mu} J^{L}_{\mu;k,m} &= \frac{\partial}{\partial x_\mu} V^{L}_{\mu;k,m} + \left(\frac{\hbar}{Mc}\right)^2 \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\mu} V^{L}_{e;k,m}\,, \\ &= \frac{\partial}{\partial x_\mu} V^{L}_{\mu;k,m} - \frac{\partial}{\partial x_\mu} V^{L}_{e;k,m}\,, \\ &= 0.\end{aligned}$$

Here we have used the fact that $V^{L}_{e;k,m}$ must satisfy the Klein-Gordon equation. Thus $J^{L}_{\mu;j,m}$ is a constrained four vector, the 'spin zero' component has been set to zero. This implies that we need a 'spin zero' field in addition to $J^{L}_{\mu;j,m}$. We define

$$D^{L}_{j,m} = \left(\frac{\hbar}{Mc}\right) \frac{\partial}{\partial x_e} V^{L}_{e;k,m}. \qquad \cdots(91)$$

Now in terms of these fields we have

$$[Q_{Rn}\,,\, \underset{\sim}{\chi}^{LL}_{\frac{1}{2},n';k,m}] = \frac{1}{2} e^{i\frac{\pi}{4}} (\sigma^{L}_{\mu}\sigma^{L}_{2})_{nn'} \left\{ \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} J^{L}_{\mu;k,m} + \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} \frac{\partial}{\partial x_\mu} D^{L}_{j,m}\right\}, \cdots(92a)$$

$$[Q_{Rn}\,,\, J^{L}_{\mu;k,m}] = e^{-i\frac{\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} ((\sigma^{R}_{e}\sigma^{R}_{\mu}) - \delta_{\mu e})_{nr} \frac{\partial}{\partial x_e} \chi^{RL}_{\frac{1}{2},r;k,m}\,, \qquad \cdots(92b)$$

$$[Q_{Rn}\,,\, D^{L}_{k,m}] = e^{-i\frac{\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} \chi^{RL}_{\frac{1}{2},r;k,m}\,, \qquad \cdots(92c)$$

$$[Q_{Rn}\,,\, \chi^{RL}_{\frac{1}{2},n';k,m}] = 0\,, \qquad \cdots(92d)$$

$$[Q_{Ln}\,,\, \underset{\sim}{\chi}^{LL}_{\frac{1}{2},n';k,m}] = 0\,, \qquad \cdots(92e)$$

$$[Q_{Ln}\,,\, J^{L}_{\mu;k,m}] = e^{-i\frac{\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} ((\sigma^{L}_{\mu}\sigma^{L}_{e}) - \delta_{\mu e})_{nr} \frac{\partial}{\partial x_e} \underset{\sim}{\chi}^{LL}_{\frac{1}{2},r;k,m}\,, \cdots(92f)$$

$$[Q_{Ln}\,,\, D^{L}_{k,m}] = e^{i\frac{3\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} \underset{\sim}{\chi}^{LL}_{\frac{1}{2},r}\,, \qquad \cdots(92g)$$

and
$$[Q_{Ln}\,,\, \chi^{RL}_{\frac{1}{2},n';k,m}] = \frac{1}{2} e^{i\frac{\pi}{4}} (\sigma^{R}_{\mu}\sigma^{R}_{2})_{nn'} \left\{ \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} J^{L}_{\mu;k,m} - \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} \frac{\partial}{\partial x_\mu} D \right\} \cdots(92h)$$

This then specifies the field structure we need in Part III of this Thesis.

We note that by construction these equations must be symmetric in the interchange of L and R. If we set $k=0$ we obtain the commutation relations of the so called 'Yang Mills' supermultiplet (cf. Sohnius [62]) which can also be written in terms of the field $F_{\mu\nu}$ defined by

$$F_{\mu\nu} = \tfrac{1}{2}\left(\frac{\partial}{\partial x^{\mu}} J^{L}_{\nu;0,0} - \frac{\partial}{\partial x^{\nu}} J^{L}_{\mu;0,0}\right).$$

PART III

SUPERFIELDS AND THE IRREDUCIBLE REPRESENTATIONS OF THE SUPER POINCARE ALGEBRA.

CHAPTER 1

INTRODUCTION

In Part I we were able to define integration on supermanifolds and construct some of the unitary irreducible representations of the Lie group corresponding to the super Poincaré group. Each irreducible representation acted on state vectors corresponding to particles with a single spin value.

In Part II we constructed sets of second quantized fields on space-time that transformed into each other under the action of the irreducible representations of the super Poincaré algebra. Only now, when we construct fields on superspace do we find the connection between these different representation theories.

We take space-time (M_4) to be embedded in superspace ($M_L^{4,4}$) by (space-time) = p_0(superspace), with p_0 the projection operator as given in Definition 3.2 of Chapter 3 of Part I, so that if $\{x^\mu, \theta^\nu\}$ are the coordinates of superspace then $\{x_0^\mu\}$ are the coordinates of space-time. In our opinion, superspace is simply an artifact which enables certain manipulations on supermultiplets to be carried out in a simple, straightforward way. We do not agree with the viewpoint that we actually live in a supermanifold (cf. Leites [23]).

In Chapter 2 we examine how to construct super Lorentz invariant fields on $M_L^{4,4}$ from the second quantized fields constructed in Part II. We find that the fields on superspace are constructed from the fields on space-time by a 'Grassman extension' of the space-time fields. This is similar to the 'z-expansion' of Rogers [13] , which is used to construct superdifferentiable functions from C^∞ functions. The representation of the super Lorentz group acting on these superfields is such that the representation of the Lie group equivalent to the super Lorentz group on these fields is reducible, but not fully reducible. We are able to identify the irreducible blocks that occur along the diagonal of this representation with certain of the representations constructed in Chapter 4 of Part I. The fact that the representation is by triangular matrices also tells us that the representation on the real vector space corresponding to superspace is not unitary. This implies that the representation on superspace cannot be unitary. This is at variance with the standard physics interpretation, (cf. Fayet and Ferrara [41]), in which it is assumed that the supergroup operators constructed from the supersymmetry generators are unitary.

In Chapter 3 we examine the construction of superfields from the chiral supermultiplets of Chapter 3 of Part II. We find the result to be as we might have anticipated, in that we obtain chiral superfields. This demonstrates that we can construct a superfield representation from any given supermultiplet defined on space-time.

In Chapter 4 we carry out the reverse process by taking a general superfield and decomposing it into its irreducible components. We find that a superfield, transforming as a left or right handed irreducible representation of the super Lorentz group, decomposes into the sum of four supermultiplets (unless it is a scalar superfield, in which case it decomposes into the sum of three supermultiplets), consisting of (i) a chiral left handed supermultiplet, (ii) a chiral right handed supermultiplet, and (iii) an 'intermediate' supermultiplet corresponding to the supermultiplet constructed in section 4.5 of Part II, which can be expressed in terms of two chiral supermultiplets. The fact that a superfield of this type can be decomposed in this way is known, the result was obtained by Sokatchev [63] , using projection operators constructed from the 'superspin' Casimir operator, but the result has not been given explicitly.

In Chapter 5 we consider the construction of action integrals on superspace. We are able to deduce the well known results and determine the reasons for the known choice of terms. This enables us to understand the Berezin integral as a tool for extracting component Lagrangians from superfields.

CHAPTER 2

SUPER LORENTZ INVARIANT SECOND QUANTIZED FIELDS ON SUPERSPACE.

2.1 Representations of $I_L(ISO_0(3,1;E_L))$ Obtained by Extending the base Field from $\mathbb{R}$ to the Ring E_{L0}.

In this section we will examine how to construct a representation of the Lie group $I_L(ISO(3,1;E_L))$ from a given representation of $ISO(3,1;\mathbb{R})$. We consider only massive representations as given by equation (54) of Chapter 4 of Part I. That is the representation of a particle of mass M and spin j with $\hat{p}^0 = (0,0,0,Mc)$ is given by

$$\Phi^{\hat{p}^0,j}([\Lambda_0|t_0|0])\phi_{p^0,m} \qquad \cdots(1)$$
$$= \exp\frac{i}{\hbar}\{(\Lambda_0 p^0)_\sigma t_0^\sigma\} D^j([B(\Lambda_0 p^0,\hat{p}^0)^{-1}\Lambda_0 B(p^0,\hat{p}^0)|0|0])_{m'm}\phi_{\Lambda_0 p^0,m'},$$

for $m = -j, -j+1, \cdots, j-1, j$. Here we have written Λ_0 to indicate that it is non-nilpotent ie. $\Lambda_0 \in SO_0(3,1;\mathbb{R})$, then later we will take $\Lambda \in SO_0(3,1;E_L)$. This equation was also given as equation (1) of Chapter 3 of Part II. But there, since we were only interested in representations on space-time, the index $_0$ on p, t and Λ was suppressed.

In Chapter 4 of Part I we were able to induce this representation to a representation of $I_L(SISO(3,1;E_L))$ and obtain a unitary representation. In this part of this Thesis we take a different approach, we observe that if $\{\phi_{p^0,m}\}$ are a basis of a representation of $ISO_0(3,1;\mathbb{R})$, then $\{\varepsilon_i\phi_{p^0,m}\}$ provide a basis of a representation of $I_L(ISO_0(3,1;E_L))$. We will now construct the operators corresponding to this representation, and show that this gives a reducible representation of $I_L(ISO(3,1;E_L))$.

The action of the Lie algebra generators $P_\sigma = \frac{\hbar}{i}K_\sigma$ on the basis set $\{\phi_{p^0,m}\}$ is well known and is given by

$$P_\sigma\,\phi_{p_0,m} = p^0_\sigma\,\phi_{p^0,m}. \qquad \cdots(2)$$

Thus the action of the generators $\varepsilon_i P_\sigma$ of $I_L(ISO(3,1;E_L))$ on the basis set $\{\varepsilon_i\phi_{p^0,m}\}$ is given by

$$\varepsilon_i P_\sigma\,\varepsilon_K\,\phi_{p^0,m} = p_0\,(\varepsilon_i\varepsilon_K\,\phi_{p_0,m}). \qquad \cdots(3)$$

In principle we can obtain an expansion for the action of the generators

$M_{\mu\nu} = \frac{\hbar}{i} L_{\mu\nu}$ from equation (1) but it would be very complicated and we never need it explicitly, so that we assume the existance of some matrix $M(M_{\mu\nu})$ such that

$$M_{\mu\nu}\,\phi_{p^o,m} = M(M_{\mu\nu})_{p'^o,m';p^o,m}\,\phi_{p'^o,m'}. \qquad \cdots(4)$$

Then

$$e_i M_{\mu\nu} e_K \phi_{p^o,m} = M(M_{\mu\nu})_{p'^o,m';p^o,m}\, e_i e_K \phi_{p'^o,m'} \qquad \cdots(5)$$

defines the action of the generators $e_i M_{\mu\nu}$ of $I_L(ISO(3,1;E_L))$ on the basis $\{e_K, \phi_{p^o,m}\}$.

Now we introduce the Fock space state $|e_i, p^o, m\rangle$ and identify it with $\left(\frac{p_4^o}{Mc}\right)^{\frac{1}{2}} e_i \phi_{p^o,m}$ and define the corresponding operator $O([\Lambda|t|0])$ by

$$\begin{aligned} &O([\Lambda|t|0]) \\ &\quad = O([I|t_n|0])\,U([I|t_o|0])\,U([\Lambda_o|0|0])\,O([\Lambda_n|0|0]) \qquad \cdots(6)\end{aligned}$$

with t_n defining a 'nilpotent' translation and Λ_n defining a 'nilpotent' Lorentz transformation. So that

$$\begin{aligned} &O([\Lambda|t|0]) \\ &\quad = \exp\tfrac{i}{\hbar}\{e_i P_\sigma t_i^\sigma\}\exp\tfrac{i}{\hbar}\{e_o P_\sigma t_o^\sigma\}\,U([\Lambda_o|0|0])\exp\tfrac{i}{\hbar}\{y_i^{\mu\nu} e_i M_{\mu\nu}\} \qquad \cdots(7)\end{aligned}$$

with $i = 1, 2, \cdots, \mathcal{N}-1$ that is $i \neq 0$ as in our previous conventions.

Then

$$\begin{aligned} &O([I|t_n|0])\,|e_K, p^o, m\rangle \\ &\quad = \exp\tfrac{i}{\hbar} P_\sigma e_i t_i^\sigma\, |e_K, p^o, m\rangle\,, \\ &\quad = (1 + \tfrac{i}{\hbar} P_\sigma e_i t_i^\sigma)\,|e_K, p^o, m\rangle\,, \\ &\quad = |e_i, p^o, m\rangle + \tfrac{i}{\hbar} p_\sigma^o t_i^\sigma\, |e_i e_K, p^o, m\rangle\,; \qquad \cdots(8) \\ &U([I|t_o|0])\,U([\Lambda_o|0|0])\,|e_K, p^o, m\rangle \\ &\quad = \left(\frac{(\Lambda_o p^o)_4}{p_4^o}\right)^{\frac{1}{2}} \exp\tfrac{i}{\hbar}\{(\Lambda_o p^o)_\sigma t_o^\sigma\}\, D^j([B(\Lambda_o p^o, \hat{p})^{-1}\Lambda_o B(p^o, \hat{p}^o)|0|0])_{m'm} \\ &\qquad\qquad \times |e_K, \Lambda_o p^o, m'\rangle \qquad \cdots(9)\end{aligned}$$

and finally with $y^{\mu\nu} \in \mathbb{R}$ so that the transformation is in some fixed nilpotent direction

$$O([\Lambda_n|0|0])\,|\varepsilon_K, p^o, m\rangle$$
$$= \exp \tfrac{i}{\hbar}\{y^{\mu\nu}\varepsilon_i M_{\mu\nu}\}\,|\varepsilon_K, p^o, m\rangle,$$
$$= (1 + \tfrac{i}{\hbar}\, y^{\mu\nu}\varepsilon_i M_{\mu\nu})\,|\varepsilon_K, p^o, m\rangle,$$
$$= |\varepsilon_K, p^o, m\rangle + \tfrac{i}{\hbar}\, y^{\mu\nu} M(M_{\mu\nu})_{p'^o, m'; p^o, m}\,|\varepsilon_i\varepsilon_K, p'^o, m'\rangle. \quad \ldots(10)$$

Now we introduce the single particle creation operators $a^{+}_{\varepsilon_i, p^o, m}$ and the vacuum state $|0\rangle$ by

$$|\varepsilon_i, p^o, m\rangle = a^{+}_{\varepsilon_i, p^o, m}\,|0\rangle \quad \ldots(11)$$

and suppose that

$$O([\Lambda|t|\tau])|0\rangle = |0\rangle, \quad \ldots(12)$$

for all $[\Lambda|t|\tau] \in \overline{SISO}_0(3,1;E_L)$.

The action of the Grassman algebra E_{Lo} is defined by its action on the vectors $\varepsilon_i\,\phi_{p^o, m}$ by

$$\varepsilon_i\,|\varepsilon_K, p^o, m\rangle = |\varepsilon_i\varepsilon_K, p^o, m\rangle. \quad \ldots(13)$$

So that

$$\varepsilon_i\, a^{+}_{\varepsilon_K, p^o, m} = a^{+}_{\varepsilon_i\varepsilon_K, p^o, m}. \quad \ldots(14)$$

Equations (8), (9) and (10) can now be rewritten in terms of the creation operators as

$$O([I|t_n|0])\, a^{+}_{\varepsilon_i, p^o, m}\, O([I|t_n|0])^{-1}$$
$$= a^{+}_{\varepsilon_i, p^o, m} + \tfrac{i}{\hbar}\, p^o_\sigma\, t^\sigma_K\, a^{+}_{\varepsilon_K\varepsilon_i, p^o, m} \quad \ldots(15)$$

with $k = 1, 2, \cdots, \mathcal{N}-1$;

$$U([\Lambda_o|t_o|0])\, a^{+}_{\varepsilon_i, p^o, m}\, U([\Lambda_o|t_o|0])^{-1}$$
$$= \left(\frac{(\Lambda p^o)_4}{p^o_4}\right)^{\frac{1}{2}} \exp \tfrac{i}{\hbar}\{(\Lambda_o p^o)_\sigma t^\sigma_o\}$$
$$\times D^j([B(\Lambda_o p^o, \hat{p}^o)^{-1}\Lambda_o B(p^o, \hat{p}^o)|0|0])_{m'm}\, a^{+}_{\varepsilon_i, \Lambda_o p^o, m'}, \quad \ldots(16)$$

for which it is convenient to define the matrix $U_{p'^o, m'; p^o, m}$ by

$$U([\Lambda_o|t_o|0])\, a^{+}_{\varepsilon_i, p^o, m}\, U([\Lambda_o|t_o|0])^{-1}$$
$$= U_{p'^o, m'; p^o, m}\, a^{+}_{\varepsilon_i, p'^o, m'} \quad \ldots(17)$$

with $p'^o = \Lambda_o p^o$.

and

$$\begin{aligned}
&O([\Lambda_n|0|0])\,a^{\dagger}_{e_k,p^o,m}\;O([\Lambda_n|0|0])^{-1}\\
&\quad= \left(1+\tfrac{i}{\hbar}\,y^{\mu\nu}e_i M_{\mu\nu}\right)a^{\dagger}_{e_k,p^o,m}\left(1-\tfrac{i}{\hbar}\,y^{\mu\nu}e_i M_{\mu\nu}\right),\\
&\quad= a^{\dagger}_{e_k,p^o,m} + \tfrac{i}{\hbar}\,y^{\mu\nu}[e_i M_{\mu\nu},\,a^{\dagger}_{e_k,p^o,m}],\\
&\quad= a^{\dagger}_{e_k,p^o,m} + \tfrac{i}{\hbar}\,y^{\mu\nu}M(M_{\mu\nu})_{p'^o,m';\,p^o,m}\,a^{\dagger}_{e_ie_k,p'^o,m'}. \qquad \cdots(18)
\end{aligned}$$

These three equations specify the representation we have obtained. To examine the structure of this representation we construct its matrices. Thus

$$O([I|t_n|0])\left(a^{\dagger}_{\varepsilon_0,p^o,m};\,a^{\dagger}_{\varepsilon_{1n2},p^o,m};\,\cdots;\,a^{\dagger}_{e_{N-1},p^o,m}\right)_k O([I|t_n|0])^{-1}$$

$$=\left(a^{\dagger}_{\varepsilon_0,p^o,m},\,\cdots\,a^{\dagger}_{e_{N-1},p^o,m}\right)_{k'}\begin{bmatrix} I & & & & & O \\ \frac{i}{\hbar}p^o_\sigma t^\sigma_{1n2} I & I & & & & \\ \vdots & \vdots & I & & & \\ \vdots & \text{Non zero off Diagonal entries in this region only.} & & \ddots & & \\ \vdots & & & & I & \\ \frac{i}{\hbar}p^o_\sigma t^\sigma_{e_{N-1}} & \cdots & & & \cdots & I \end{bmatrix}_{k'k} \cdots(19)$$

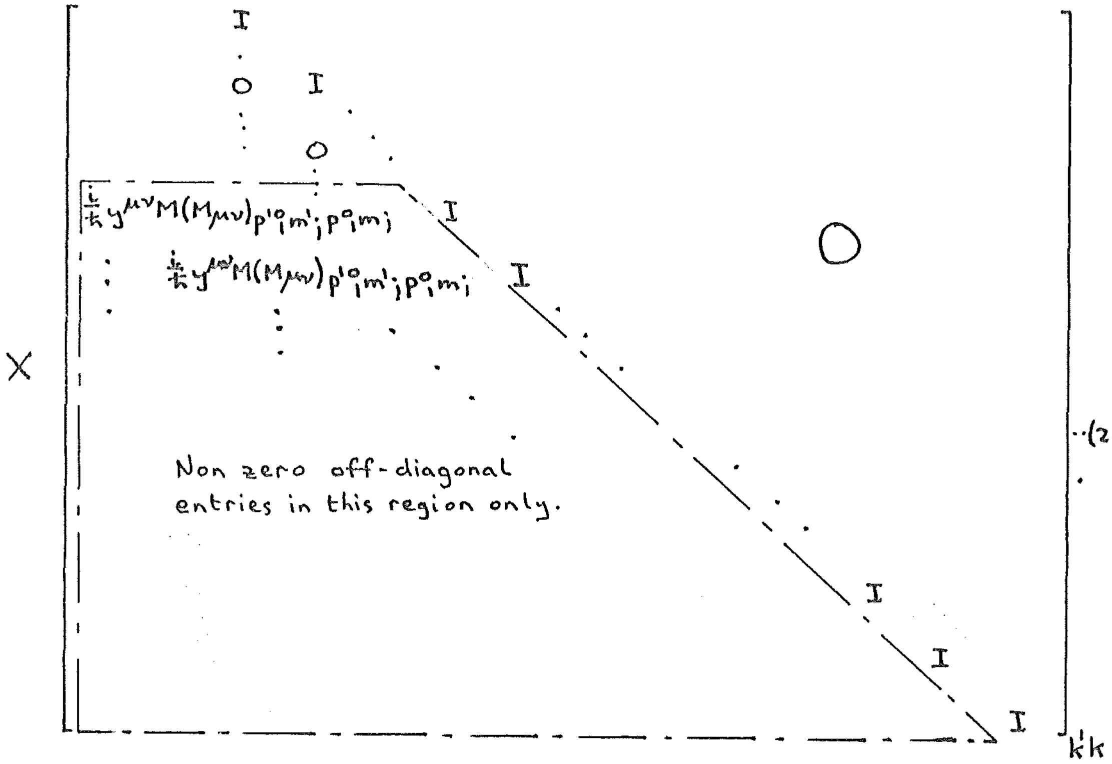
$$U([\Lambda_o|t_o|0])\left(a^{\dagger}_{\varepsilon_0,p^o,m};\,a^{\dagger}_{\varepsilon_{1n2},p^o,m};\,\cdots;\,a^{\dagger}_{e_{N-1},p^o,m}\right)_k U([\Lambda_o|t_o|0])^{-1}$$

$$=\left(a^{\dagger}_{\varepsilon_0,p'^o,m'};\,\cdots\,a^{\dagger}_{e_{N-1},p'^o,m'}\right)_{k'}\begin{bmatrix} U_{p'^o,m';p^o,m} & & & O \\ & U_{p'^o,m';p^o,m} & & \\ & & \ddots & \\ O & & & U_{p'^o,m';p^o,m} \end{bmatrix}_{k'k} \cdots(20)$$

and

$$O([\Lambda_n|0|0|0])(a^{+}_{\xi_0,p^0,m};a^{+}_{\xi_{1\wedge 2},p^0,m};\cdots;a^{+}_{\xi_{N-1},p^0,m})_k\,O([\Lambda_n|0|0|0])^{-1}$$

$$=(a^{+}_{\xi_0,p'^0,m'};\cdots a^{+}_{\xi_i,p'^0,m'};a^{+}_{\xi_i\xi_{1\wedge 2},p'^0,m'};\cdots;a^{+}_{\xi_{N-1},p^0,m})_{k'}$$



$\cdots(21)$

In these expressions we have assumed that the basis vectors are ordered by level. We cannot be precise about where the non-zero entries are, for instance $\xi_a\xi_\Lambda a^{+}_{\xi_b\xi_\Lambda,p^0,m} = a^{+}_{\xi_a\xi_\Lambda\xi_b\xi_\Lambda,p^0,m} = 0$ for all $a,b=0,1,\cdots,N-1$ and any $\Lambda = i\wedge j\wedge\cdots\wedge k$. So that there will be zero's in the regions of non-zero entries that we have indicated in equations (19) and (21). All that matters is that the matrices of equations (19) and (21) are lower triangular and the matrix of equation (20) has entries only in the blocks along the diagonal. So that the matrices of the representation are reducible but not fully reducible. Further reduction of the off-diagonal blocks is possible but this serves no useful purpose. For our purposes it is sufficient to note that the blocks occuring along the diagonal must again be representations of $I_L(ISO(3,1;E_L))$, with bases $\{a^{+}_{\xi_i,p^0,m}\}$ for $i=0,1,2,\cdots,N-1$.

The action of the group operators of these representations is then

$$u'([\Lambda_o|t_o|0])\, a^{+}_{\varrho_i,p^o,m}\, u'([\Lambda_o|t_o|0])^{-1}$$
$$= u_{p'^o,m';p^o,m^o}\ a^{+}_{\varrho_i,p'^o,m'} \qquad \cdots(22)$$

and

$$o'([\Lambda_n|t_n|0])\, a^{+}_{\varrho_i,p^o,m}\, o'([\Lambda_n|t_n|0])^{-1}$$
$$= a^{+}_{\varrho_i,p^o,m}\,. \qquad \cdots(23)$$

We can identify these representations as being identical to the unitary irreducible representations of $I_L(SISO(3,1;E_L))$ constructed in Chapter 4 of Part I from a massive representation of $ISO(3,1;\mathbb{R})$ (note that the unitary irreducible representations of this type are unitary irreducible representations of both $I_L(SISO(3,1;E_L))$ and $I_L(ISO(3,1;E_L))$) and given, there, by equation (56).

We note from the structure of the matrices in equations (19) and (20) that the representation obtained by this procedure is not unitary since the matrices are clearly not unitary.

Before completing this section we define the annihilation operators $a_{\varrho_i,p^o,m}$ by

$$(a^{+}_{\varrho_i,p,m})^{+} = a_{\varrho_i,p,m}\,. \qquad \cdots(24)$$

Their transformation properties are obtained by taking the adjoint of equations (15), (16), (17) and (18). We note that this gives the implied definition (from equation (14)) that

$$(\varrho_i)^{+} = \varrho_i \qquad \cdots(25)$$

which is in complete agreement with the definitions given in Chapter 1 of Part I.

2.2 Quantized Fields on Superspace Obtained from a Representation of $ISO(3,1;\mathbb{R})$.

We recall from equation (35) of Chapter 3 of Part I that superspace is defined as the homogeneous space $SISO_o(3,1;E_L)/SO_o(3,1;E_L) = M_L^{4,4}$.

This space has dimension $(4|4)$ and is isomorphic to $E_L^{4,4}$, it will be parametrized by $\{x^\mu, \theta^\nu\}$ with $\mu = 1,2,3,4$ and $\nu = 1,2,3,4$. Corresponding to this space we have the real space $I_L(M_L^{4,4})$ parametrized by $\{x_i^\mu, \theta_j^\nu\}$ of real dimension $8\mathcal{N}$, which is the space we are really considering.

To construct a field on this space we proceed by analogy with the normal Minkowski space procedure. We recall from section 4.1 of Chapter 4 of Part I that the set of plane waves on $I_L(M_L^{4,4})$ are given by

$$e^{\frac{i}{\hbar} p^i_\mu x^\mu_i} e^{\frac{i}{\hbar} \phi^j_\nu \theta^\nu_j} \qquad \text{....(26)}$$

for $i = 0,1,\cdots,\mathcal{N}-1$; $j = 1,2,\cdots,\mathcal{N}$; which we can write succinctly as

$$e^{\frac{i}{\hbar} p\cdot x} e^{\frac{i}{\hbar}\phi\cdot\theta} \qquad \text{....(27)}$$

with $p \in E_{L0}$ and $\phi \in E_{L1}$. Then p_μ, ϕ_ν span a superspace of dimension $(4|4)$ isomorphic to $M_L^{4,4}$.

Now any complex valued function on $M_L^{4,4}$ can be written as

$$f(x,\theta) = \int \hat{d}^4 p\, \hat{d}^4\phi \left\{ g(p,\phi)\, e^{\frac{i}{\hbar} p\cdot x} e^{\frac{i}{\hbar}\phi\cdot\theta} \right\}. \qquad \text{....(28)}$$

But we require this function to be an element of the representation space of the unitary irreducible representation defined by equation (56) of Chapter 4 of Part I. That is it can only be constructed from the plane waves restricted by the orbit conditions

$$M^2c^2 = (p_4^0)^2 - (p_1^0)^2 - (p_2^0)^2 - (p_3^0)^2 \quad \text{and} \quad p_4^0 > 0. \qquad \text{....(29)}$$

Thus equation (28) takes the form

$$f(x,\theta) = \int \hat{d}^4 p\, \hat{d}^4\phi \left\{ e^{\frac{i}{\hbar} p\cdot x} e^{\frac{i}{\hbar}\phi\cdot\theta}\, \Theta(p_4^0)\, \delta\big((p_4^0)^2 - (p_1^0)^2 - (p_2^0)^2 - (p_3^0)^2 - M^2c^2\big) \right\} \qquad \text{....(30)}$$

with $\Theta(p_4^0) = \begin{cases} 0 & \text{if } p_4^0 \leqslant 0, \\ 1 & \text{if } p_4^0 > 0 \end{cases}$

and $\delta(\alpha) = \begin{cases} 0 & \text{if } \alpha \neq 0, \\ 1 & \text{if } \alpha = 0. \end{cases}$

Thus in this case ....(31)

$$f(x,\theta) = \int d^4p^o\{g(p^o)\, e^{\frac{i}{\hbar}p^o.x_o}\,\Theta(p_4^o)\,\delta((p_4^o)^2-(p_1^o)^2-(p_2^o)^2-(p_3^o)^2-M^2c^2)\}.$$

So that $f(x,\theta)$ is independent of θ and the x dependence is restricted to the 'real' x values x_o^{μ}.

This 'four dimensional' integral can be further reduced to a 'three dimensional' integral if we note that equation (29) implies only three independent parameters for the orbit. Conventionally we choose these to be the three 'space' dimensions and eliminate the 'time' dimension. To do this we make use of the identity

$$\delta(a^2-b^2) = \frac{1}{2|b|}(\delta(a-b)+\delta(a+b)). \qquad ....(32)$$

Equation (31) can then be written

$$f(x,\theta) = \int d^3p^o\left\{\frac{1}{2p_4^o}\,g(p^o)\,e^{\frac{i}{\hbar}p^o.x_o}\right\}. \qquad ....(33)$$

To obtain the equivalent expression for the operator field using creation operators we simply replace $g(p^o)$ with $g'(p^o)a^+_{p^o}$. It is also convenient to retain a record of the representative character of the orbit, we do this by writing $g'(p^o) = g'(p^o,\hat{p}^o)$ where $\hat{p}^o$ is the representative element for the orbit. Thus a creation field is given by

$$\phi^c(x) = \int\frac{d^3p^o}{2p_4^o}\{g'(p^o,\hat{p}^o)\,a^+_{p^o}\,e^{\frac{i}{\hbar}p^o.x_o}\}. \qquad ....(34)$$

To obtain the equivalent expression for an annihilation field we can simply take the adjoint of equation (34) to obtain

$$\phi^A(x) = \int\frac{d^3p^o}{2p_4^o}\{g'^{+}(p^o,\hat{p}^o)\,a_{p^o}\,e^{-\frac{i}{\hbar}p^o.x_o}\}. \qquad ....(35)$$

In general we have several creation and annihilation operators so that our field will be a multi-component field. There need not, of course, be the same number of field components as creation operators except that the number of field components must exceed the number of creation operators and annihilation operators. Conventionally we include both the creation and annihilation operators in the construction of the field. Thus in general we will have

$$\phi_m(x) = \int \frac{d^3 p^o}{2p_4^o} \{ g(p^o, \hat{p}^o)_{mn} a^{+}_{p^o, n} e^{\frac{i}{\hbar} p.x} + g'(p^o, \hat{p}^o)_{mn} a_{p^o, n} e^{-\frac{i}{\hbar} p.x} \} \cdots (35)$$

with $m = 1, 2, \cdots$ indexing the field components. The functions $g(p^o, \hat{p}^o)$ and $g'(p^o, \hat{p}^o)$ are then chosen to obtain the required transformation properties of the field.

2.3 Super Lorentz Invariant Second Quantized Fields on Superspace.

In section 1 we showed how a representation of $I_L(ISO(3,1; E_L))$ could be constructed by extending the base field $\mathbb{R}$ to the ring E_{Lo}. The set of operators which form a basis of this representation are $a^{+}_{\varepsilon_i, p^o, m}$ with $i = 0, 1, 2, \cdots, \mathcal{N}-1$; $p^o = (p_1^o, p_2^o, p_3^o, p_4^o)$ subject to $p_4^o > 0$ and $(p_4^o)^2 - (p_1^o)^2 - (p_2^o)^2 - (p_3^o)^2 = M^2c^2$, and $m = j, j-1, \cdots, -j+1, -j$. The subsets of operators $a^{+}_{\varepsilon_i, p^o, m}$, for each i, were shown to form the basis of irreducible unitary representations as constructed in section 4.3.2 of Chapter 4 of Part I. In section 2 we showed how a field could be constructed on superspace from these representations. Now we will show how to construct a super Lorentz invariant field using these operators. The procedure, in part, is analogous to the procedure in Minkowski space as detailed in Chapter 3 of Part II.

We define the ancilliary operators $\alpha_{\varepsilon_i, p, m}$ and $\beta_{\varepsilon_i, p, m}$ by

$$\alpha_{\varepsilon_i, p, m} = \begin{cases} \left(\frac{2p_4^o}{Mc}\right)^{\frac{1}{2}} \Gamma^{(j)}([B(p^o, \hat{p}^o)|0|0|0])_{mm'} a_{\varepsilon_i, p^o, m'} & \text{if } p = (p^o, 0), \\ 0 & \text{otherwise} \end{cases} \quad \cdots (37)$$

$$\beta_{\varepsilon_i, p, m} = \begin{cases} \left(\frac{2p_4^o}{Mc}\right)^{\frac{1}{2}} (\Gamma^{(j)}([B(p^o, \hat{p}^o)|0|0|0])_{mm'} a^{+}_{\varepsilon_i, p^o, m} & \text{if } p = (p^o, 0) \\ 0 & \text{otherwise.} \end{cases} \quad \cdots (38)$$

for each $i = 0, 1, 2, \cdots, \mathcal{N}-1$ and each $m = j, j-1, \cdots, -j+1, -j$. Here $\Gamma^{(j)}$ is the representation as specified for equations (13) and (14) of Chapter 3 of Part II and $p = (p_1^o, p_2^o, p_3^o, p_4^o, p_1^{1\wedge 2}, p_2^{1\wedge 2}, \cdots, p_4^{1\wedge 2\wedge \cdots \wedge L})$.

Now we construct the set of fields $\chi^{\varepsilon_i}_m(x)$ on superspace $M^{4,4}_L$ for each $i = 0, 1, 2, \cdots, \mathcal{N}-1$ and $m = j, j-1, \cdots, -j+1, -j$ as

$$\chi^{\varepsilon_i}_m(x) = \left(\frac{1}{2\pi\hbar}\right)^3 \int \hat{d}^4 p \, \hat{d}^4 \phi \left\{ \xi_i \alpha_{\varepsilon_i, p, m} e^{-\frac{i}{\hbar} p.x} + \eta_i \beta_{\varepsilon_i, p, m} e^{\frac{i}{\hbar} p.x} \right\} \cdots (39)$$

Here ξ_i and η_i are phase factors such that $\xi_i, \eta_i \in \mathbb{C}, |\xi_i| = |\eta_i| = 1$. Then it follows from section 2 that

$$\chi^{\varepsilon_i}_m(x) = \left(\frac{1}{2\pi\hbar}\right)^3 \int d p^0 \left(\frac{M_c}{2 p^0_4}\right) \left\{ \xi \alpha_{\varepsilon_i, p^0, m} e^{-\frac{i}{\hbar} p.x} + \eta \beta_{\varepsilon_i, p^0, m} e^{\frac{i}{\hbar} p.x} \right\} \cdots (40)$$

and that $\chi^{\varepsilon_i}_m(x)$ depends only on x_0, ie. it is independent of the nilpotent components x_n of x.

We can write down the transformation properties of the fields $\chi^{\varepsilon_i}_m(x)$ under the action of group elements of the form $[\Lambda_0 | t_0 | 0]$ by reference to Chapter 3 of Part II as

$$\begin{aligned} & \mathcal{U}([\Lambda_0 | t_0 | 0]) \chi^{\varepsilon_i}_m(x) \, \mathcal{U}([\Lambda_0 | t_0 | 0])^{-1} \\ &= \mathcal{U}([\Lambda_0 | t_0 | 0]) \chi^{\varepsilon_i}_m(x_0) \, \mathcal{U}([\Lambda_0 | t_0 | 0])^{-1}, \\ &= \Gamma(\Lambda_0^{-1})_{mm'} \chi^{\varepsilon_i}_{m'}(\Lambda_0 x_0 + t_0), \\ &= \Gamma(\Lambda_0^{-1})_{mm'} \chi^{\varepsilon_i}_{m'}(\Lambda_0 x + t_0). \end{aligned} \qquad \cdots (41)$$

We note that these fields are Lorentz invariant, on superspace, only because they are independent of x_n.

To proceed we need explicit expressions for the action of $\varepsilon_i P_\sigma$ and $\varepsilon_i M_{\mu\nu}$ on the fields. The result for P_σ was evaluated in Chapter 3 of Part II and given as equation (68) there. We repeat it, modified to account for our changes in terminology as

$$\left[\frac{i}{\hbar} \varepsilon_0 P_\sigma, \chi^{\varepsilon_i}_m(x)\right] = \frac{\partial}{\partial x_0^\sigma} \chi^{\varepsilon_i}_m(x).$$

Thus

$$\begin{aligned} & \varepsilon_K \left[\frac{i}{\hbar} \varepsilon_0 P_\sigma, \chi^{\varepsilon_i}_m(x)\right] \\ &= \left[\frac{i}{\hbar} \varepsilon_K P_\sigma, \chi^{\varepsilon_i}_m(x)\right] = \varepsilon_K \frac{\partial}{\partial x_0^\sigma} \chi^{\varepsilon_i}_m(x) \\ &\qquad\qquad\qquad\qquad = \frac{\partial}{\partial x_0^\sigma} \chi^{\varepsilon_K \varepsilon_i}_m(x). \end{aligned} \qquad \cdots (42)$$

To evaluate the effect of $\underline{e}_i M_{\mu\nu}$ on the fields we first consider

$$\Lambda_0 = \exp \tfrac{i}{\hbar} y M_{\mu\nu} ,$$

with $y \in \mathbb{R}$. So that

$$U([\exp \tfrac{i}{\hbar} y M_{\mu\nu} | 0 | 0 | 0]) = I + \tfrac{i}{\hbar} y M_{\mu\nu} + \cdots .$$

Then

$$U([\exp \tfrac{i}{\hbar} y M_{\mu\nu} | 0 | 0 | 0])\, \chi^{\underline{e}_i}_m(x)\, U([\exp - \tfrac{i}{\hbar} y M_{\mu\nu} | 0 | 0 | 0])$$

$$= \chi^{\underline{e}_i}_m(x) + \tfrac{i}{\hbar} y [M_{\mu\nu}, \chi^{\underline{e}_i}_m(x)] + \cdots ,$$

$$= \left(\tfrac{1}{2\pi\hbar}\right)^3 \int d^3p^0 \left(\tfrac{Mc}{2p^0_4}\right) \Gamma^{(j)}(1 - y \tfrac{i}{\hbar} M_{\mu\nu})_{mm'} \Big\{ \alpha_{\underline{e}_i, p^0, m'} e^{-\frac{i}{\hbar} p^0 . ((1 + y \frac{i}{\hbar} M_{\mu\nu}) x_0)} + \beta_{\underline{e}_i, p^0, m'} e^{\frac{i}{\hbar} p^0 . ((1 + y \frac{i}{\hbar} M_{\mu\nu}) x_0)} \Big\},$$

$$= \left(\tfrac{1}{2\pi\hbar}\right)^3 \int d^3p^0 \left(\tfrac{Mc}{2p^0_4}\right) \Gamma^{(j)}(1 - y \tfrac{i}{\hbar} M_{\mu\nu})_{mm'} \Big\{ \alpha_{\underline{e}_i, p^0, m'} e^{-\frac{i}{\hbar} p^0 . x_0} (1 - \tfrac{i}{\hbar} p^0 \cdot y M_{\mu\nu} x + \cdots) + \beta_{\underline{e}_i, p^0, m'} e^{\frac{i}{\hbar} p . x} (1 + \tfrac{i}{\hbar} p^0 \cdot y \tfrac{i}{\hbar} M_{\mu\nu} x + \cdots) \Big\} .$$

So that

$$\tfrac{i}{\hbar} y [M_{\mu\nu}, \chi^{\underline{e}_i}_m(x)]$$

$$= \left(\tfrac{1}{2\pi\hbar}\right)^3 \int d^3p^0 \left(\tfrac{Mc}{2p^0_4}\right) \Big\{ - y \tfrac{i}{\hbar} p^0 \cdot \tfrac{i}{\hbar} M_{\mu\nu} x_0 \, \alpha_{\underline{e}_i, p^0, m} e^{-\frac{i}{\hbar} p^0 . x_0}$$
$$- \Gamma^{(j)} (y \tfrac{i}{\hbar} M_{\mu\nu})_{mm'} \alpha_{\underline{e}_i, p^0, m'} e^{-\frac{i}{\hbar} p^0 . x_0}$$
$$+ y \tfrac{i}{\hbar} p^0 \cdot \tfrac{i}{\hbar} M_{\mu\nu} x_0 \, \beta_{\underline{e}_i, p^0, m} e^{\frac{i}{\hbar} p^0 . x_0}$$
$$- \Gamma (y \tfrac{i}{\hbar} M_{\mu\nu})_{mm'} \beta_{\underline{e}_i, p^0, m'} e^{\frac{i}{\hbar} p^0 . x_0} \Big\} ,$$

$$= - y \, \Gamma(\tfrac{i}{\hbar} M_{\mu\nu})_{mm'} \chi^{\underline{e}_i}_m(x) + \left(\tfrac{1}{2\pi\hbar}\right)^3 \int d^3p^0 \left(\tfrac{Mc}{2p^0_4}\right) y \Big\{ - \tfrac{i}{\hbar} p^0 \cdot \tfrac{i}{\hbar} M_{\mu\nu} x_0 \, \alpha_{\underline{e}_i, p^0, m} e^{-\frac{i}{\hbar} p^0 . x_0} + \tfrac{i}{\hbar} p^0 \cdot \tfrac{i}{\hbar} M_{\mu\nu} x_0 \, \beta_{\underline{e}_i, p^0, m} e^{\frac{i}{\hbar} p^0 . x_0} \Big\} . \quad \ldots (43)$$

Now
$$\begin{aligned} p^0 . (\tfrac{i}{\hbar} M_{\mu\nu} x_0) &= p^0 . (L_{\mu\nu} x_0) , \\ &= p^{0\sigma} g_{\sigma\rho} (L_{\mu\nu})^{\rho}{}_{\xi} x_0^{\xi} , \\ &= p^{0\sigma} g_{\sigma\rho} (- g_{\xi\nu} \delta^{\rho}_{\mu} + g_{\xi\mu} \delta^{\rho}_{\nu}) x_0^{\xi} , \\ &= - p^0_{\mu} x_{0\nu} + p^0_{\nu} x_{0\mu} . \end{aligned} \quad \ldots (44)$$

Here we have used the expression for the 4×4 representation of $L_{\mu\nu}$ given by equation (55) of Chapter 2 of Part I.

Thus inserting equation (44) into equation (43) we obtain

$$[\tfrac{i}{\hbar}M_{\mu\nu}\,,\chi^{\varrho_i}_{m}(x)] = -\Gamma^{(j)}(\tfrac{i}{\hbar}M_{\mu\nu})_{mm'}\chi^{\varrho_i}_{m'}(x) + (x_{o\mu}\tfrac{\partial}{\partial x^\nu_o} - x_{o\nu}\tfrac{\partial}{\partial x^\mu_o})\chi^{\varrho_i}_{m}(x). \quad \text{....(45)}$$

We can thus deduce the action of $\frac{i}{\hbar}\varrho_i M_{\mu\nu}$ to be

$$[\tfrac{i}{\hbar}\varrho_i M_{\mu\nu}\,,\chi^{\varrho_k}_{m}(x)] = -\Gamma^{(j)}(\tfrac{i}{\hbar}M_{\mu\nu})_{mm'}\chi^{\varrho_i\varrho_k}_{m'}(x) + (x_{o\mu}\tfrac{\partial}{\partial x^\nu_o} - x_{o\nu}\tfrac{\partial}{\partial x^\mu_o})\chi^{\varrho_i\varrho_k}_{m}(x). \quad \text{....(46)}$$

Thus the action of a 'nilpotent' translation can be determined from equation (42) to be

$$O([I|t_n|0])\,\chi^{\varrho_k}_{m}(x)\,O([I|t_n|0])^{-1} = \chi^{\varrho_k}_{m}(x) + t^\sigma_i\tfrac{\partial}{\partial x^\sigma_o}\chi^{\varrho_i\varrho_k}_{m}(x). \quad \text{....(47)}$$

The corresponding action of a 'nilpotent' Lorentz transformation Λ_n can be determined, in a limited way, from equation (46) to be

$$\begin{aligned}O([\Lambda_n|0|0])\,\chi^{\varrho_k}_{m}(x)\,O([\Lambda_n|0|0])^{-1} &= \chi^{\varrho_k}_{m}(x) - y^{\mu\nu}\Gamma(\tfrac{i}{\hbar}M_{\mu\nu})_{mm'}\chi^{\varrho_i\varrho_k}_{m'}(x)\\ &\quad + y^{\mu\nu}(x_{o\mu}\tfrac{\partial}{\partial x^\nu_o} - x_{o\nu}\tfrac{\partial}{\partial x^\mu_o})\chi^{\varrho_i\varrho_k}_{m}(x) \quad \text{....(48)}\end{aligned}$$

with $\Lambda_n = \exp\frac{i}{\hbar}y^{\mu\nu}\varrho_i M_{\mu\nu} = 1 + \frac{i}{\hbar}\varrho_i M_{\mu\nu}$, ie. Λ_n depends only on the components in a <u>fixed</u> nilpotent direction.

Equations (47) and (48) suggest how we can define a super Lorentz invariant field. It does not consist of the direct sum $\sum_i \chi^{\varrho_i}_{m}(x)$ as one might have expected but it must contain derivatives, with respect to the coordinates x^μ_o, of the fields $\chi^{\varrho_i}_{m}(x)$ for $i \neq 0$, in fact we need to use a series . We state the main result of this chapter as a theorem.

<u>Theorem 2.1</u>

Define the field $\psi_m(x)$ by

$$\begin{aligned}\psi_m(x) &= \chi^{\varrho_o}_{m}(x) + \sum_{j=1}^{N-1} x^\mu_j\tfrac{\partial}{\partial x^\mu_o}\chi^{\varrho_j}_{m}(x) \qquad \text{....(49)}\\ &\quad + \tfrac{1}{2!}\sum_{j,k=1}^{N-1} x^\mu_j x^\sigma_k\tfrac{\partial}{\partial x^\mu_o}\tfrac{\partial}{\partial x^\sigma_o}\chi^{\varrho_j\varrho_k}_{m}(x) + \cdots\,,\end{aligned}$$

where the field $\chi^{\epsilon_0}_m(x)$ transforms under the Lorentz group as

$$U([\Lambda_0|t_0|0])\,\chi^{\epsilon_0}_m(x)\,U([\Lambda_0|t_0|0])^{-1}$$
$$= \Gamma(\Lambda_0^{-1})_{mm'}\,\chi^{\epsilon_0}_{m'}(\Lambda_0 x + t_0).$$

Then for all $[\Lambda|t|0]$ of the covering group of $ISO_0(3,1;E_L)$

$$O([\Lambda|t|0])\,\psi_m(x)\,O([\Lambda|t|0])^{-1}$$
$$= \Gamma(\Lambda^{-1})_{mm'}\,\psi_{m'}(\Lambda x + t)\,. \qquad \cdots(50)$$

Proof

We note that $O([\Lambda|t|0]) = O([I|t|0])\,O([\Lambda|0|0])$ so that it is sufficient to demonstrate the result for $[I|t|0]$ and $[\Lambda|0|0]$ seperately. We deal with $[I|t|0]$ first, and note that

$$[I|t|0][I|t'|0] = [I|t+t'|0]$$

so that, since the result is clearly true for $t_0^\sigma = t_0^\sigma\epsilon_0$, it is only necessary to consider $t_n^\sigma = t^\sigma\epsilon_\ell$, $\ell = 1,2,\cdots,\mathcal{N}-1$. Then

$$O([I|t_n|0])\,\psi_m(x)\,O([I|-t_n|0])$$
$$= \psi_m(x) + t^\sigma[\tfrac{i}{\hbar}\epsilon_\ell P_\sigma, \psi_m(x)]$$
$$= \chi^{\epsilon_0}_m(x) + \sum_{j=1}^{\mathcal{N}-1} x^\mu_j \frac{\partial}{\partial x^\mu_0}\chi^{\epsilon_j}_m(x) + \frac{1}{2!}\sum_{j,k=1}^{\mathcal{N}-1} x^\mu_j x^\sigma_k \frac{\partial}{\partial x^\mu_0}\frac{\partial}{\partial x^\sigma_0}\chi^{\epsilon_j\epsilon_k}_m(x) + \cdots$$
$$+ t^\sigma\frac{\partial}{\partial x^\sigma_0}\chi^{\epsilon_\ell}_m(x) + t^\sigma\sum_{j=1}^{\mathcal{N}-1} x^\mu_j\frac{\partial}{\partial x^\mu_0}\frac{\partial}{\partial x^\sigma_0}\chi^{\epsilon_\ell\epsilon_j}_m(x) + \cdots\quad)$$
$$= \chi^{\epsilon_0}_m(x) + \sum_{j=1}^{\mathcal{N}-1}(x+t_n)^\mu_j \frac{\partial}{\partial x^\mu_0}\chi^{\epsilon_j}_m(x)$$
$$+ \frac{1}{2!}\sum_{j,k=1}^{\mathcal{N}-1}(x+t_n)^\mu_j (x+t_n)^\sigma_k \frac{\partial}{\partial x^\mu_0}\frac{\partial}{\partial x^\sigma_0}\chi^{\epsilon_j\epsilon_k}_m(x) + \cdots\quad)$$
$$= \psi_m(x+t_n).$$

So that

$$O([I|t_n|0])\,\psi_m(x)\,O([I|t_n|0])^{-1} = \psi_m(x+t_n)$$

for any nilpotent t_n. Combining this with equation (41) we have

$$O([I|t|0])\,\psi_m(x)\,O([I|t|0])^{-1} = \psi_m(x+t)$$

for all $t \in E_{L0}$.

To show that the result is true for operators of the form $O([\Lambda|0|0])$ we observe that the result is clearly true for $L=0$ and $L=1$. So that if we can show that it is true for $L=A+1$, given that it is true for $L=A$ we have proved the result.

Let Λ_A be an element of $SO_0(3,1;E_A)$ and define the fields

$$\psi_{Am}(x_A) = \psi_m(x) \text{ with } x \in E_{A0}.$$

Then by supposition

$$O([\Lambda_A|0|0|0])\psi_{Am}(x_A)\,O([\Lambda_A|0|0|0])^{-1}$$
$$= \Gamma(\Lambda_A^{-1})_{mm'}\psi_{Am'}(\Lambda_A x_A). \qquad \cdots(52)$$

Let the basis of E_A be denoted by $\{e_i, f_j\}$ for $i = 0,1,2,\cdots,2^{A-1}-1$ and $j = 1,2,\cdots,2^{A-1}$. Then a basis of E_{A+1} is given by $\{e_i, \varepsilon_{A+1}f_j, f_j, \varepsilon_{A+1}e_i\}$, and

$$\psi_{(A+1)m}(x_{A+1}) = \psi_{Am}(x_A) + x^{\mu}_{(A+1)\wedge j}\frac{\partial}{\partial x_0^{\mu}}\chi_m^{\varepsilon_{A+1}f_j}(x_0),$$
$$+ x^{\mu}_{(A+1)\wedge j}x_k^{\sigma}\frac{\partial}{\partial x_0^{\mu}}\frac{\partial}{\partial x_0^{\sigma}}\chi_m^{\varepsilon_{A+1}f_je_k}(x_0) + \cdots,$$
$$= \left(1 + x^{\mu}_{(A+1)\wedge j}\frac{\partial}{\partial x_0^{\mu}}f_j\right)\psi_{Am}(x_A). \qquad \cdots(53)$$

Now with a suitable choice of parametrization we have

$$O([\Lambda_{A+1}|0|0|0]) = \exp\frac{i}{4}\{y^{\mu\nu}_{(A+1)\wedge j}\varepsilon_{A+1}f_jM_{\mu\nu}\}\,O([\Lambda_A|0|0|0])$$
$$= \left(1 + \frac{i}{4}y^{\mu\nu}_{(A+1)\wedge j}\varepsilon_{A+1}f_jM_{\mu\nu}\right)O([\Lambda_A|0|0|0]). \qquad \cdots(54)$$

Then using equation (53) we have

$$O([\Lambda_{A+1}|0|0|0])\psi_{(A+1)m}(x_{A+1})\,O([\Lambda_{A+1}|0|0|0])^{-1}$$
$$= O([\Lambda_{A+1}|0|0|0])\left(\psi_{Am}(x_A) + x^{\mu}_{(A+1)\wedge j}\frac{\partial}{\partial x_0^{\mu}}\varepsilon_{A+1}f_j\psi_{Am}(x_A)\right)$$
$$\times O([\Lambda_{A+1}|0|0|0])^{-1},$$
$$= O([\Lambda_{A+1}|0|0|0])\psi_{Am}(x_A)\,O([\Lambda_{A+1}|0|0|0])^{-1}$$
$$+ O([\Lambda_{A+1}|0|0|0])x^{\mu}_{(A+1)\wedge j}\frac{\partial}{\partial x_0^{\mu}}\varepsilon_{A+1}f_j\psi_{Am}(x_A)\,O([\Lambda_{A+1}|0|0|0])^{-1}.$$

Now using equations (52) and (54) we obtain

$$O([\Lambda_{A+1}|0|0|0])\psi_{(A+1)m}(x_{A+1})\,O([\Lambda_{A+1}|0|0|0])^{-1}$$
$$= \left(1 + \frac{i}{4}y^{\mu\nu}_{(A+1)\wedge j}\varepsilon_{A+1}f_jM_{\mu\nu}\right)\Gamma(\Lambda_A^{-1})_{mm'}\psi_{Am'}(\Lambda_A x_A)\left(1 - \frac{i}{4}y^{\mu\nu}_{(A+1)\wedge j}\varepsilon_{A+1}f_jM_{\mu\nu}\right)$$
$$+ x^{\mu}_{(A+1)\wedge j}\frac{\partial}{\partial x_0^{\mu}}\varepsilon_{A+1}f_j\Gamma(\Lambda_A^{-1})_{mm'}\psi_{Am'}(\Lambda_A x_A).$$

Here we have noted that the operator $(1+\frac{i}{\hbar} y^{\mu\nu}_{(A+1)\wedge j}\, \varepsilon_{A+1} f_j M_{\mu\nu})$ leaves the second term invariant. Now using equation (48) we obtain

$$O([\Lambda_{A+1}|010])\psi_{(A+1)m}(x_{A+1})\; O([\Lambda_{A+1}|010])^{-1}$$
$$= \Gamma(\Lambda_A^{-1})_{mm'}\psi_{Am'}(\Lambda_A x_A) + \Gamma(\Lambda_A^{-1})_{mm''}\Gamma(-y^{\mu\nu}_{(A+1)\wedge j}\varepsilon_{A+1}f_j\tfrac{i}{\hbar}M_{\mu\nu})_{m''m'}\psi_{Am'}(\Lambda_A x_A)$$
$$+ \Gamma(\Lambda_A^{-1})y^{\mu\nu}_{(A+1)\wedge j}\varepsilon_{A+1}f_j(x'_{o\mu}\tfrac{\partial}{\partial x'^{\nu}_o} - x'_{o\nu}\tfrac{\partial}{\partial x'^{\mu}_o})\psi_{Am'}(\Lambda_A x_A)$$
$$+ x^{\mu}_{(A+1)\wedge j}\tfrac{\partial}{\partial x'^{\mu}_o}\varepsilon_{A+1}f_j\,\Gamma(\Lambda_A^{-1})_{mm'}\psi_{Am'}(\Lambda_A x_A) \quad ,$$

with $x'^{\mu} = (\Lambda_A x_A)^{\mu}$ so that $x'^{\mu}_o = (\Lambda_o x_o)^{\mu}$.

Thus

$$O([\Lambda_{A+1}|010])\psi_{(A+1)m}(x_{A+1})\; O([\Lambda_{A+1}|0])^{-1}$$
$$= \Gamma(\Lambda_A^{-1})_{mm'}\{\psi_{Am'}(\Lambda_A x_A) + \Gamma(-y^{\mu\nu}_{(A+1)\wedge j}\varepsilon_{A+1}f_j M_{\mu\nu})_{m'm''}\psi_{Am''}(\Lambda_A x_A)$$
$$+ y^{\mu\nu}_{(A+1)\wedge j}\varepsilon_{A+1}f_j(x'_{o\mu}\tfrac{\partial}{\partial x'^{\nu}_o} - x'_{o\nu}\tfrac{\partial}{\partial x'^{\mu}_o})\psi_{Am'}(\Lambda_A x_A)$$
$$+ x^{\mu}_{(A+1)\wedge j}\tfrac{\partial}{\partial x'^{\mu}_o}\varepsilon_{A+1}f_j\psi_{Am'}(\Lambda_A x_A)\} \;,$$

$$= \Gamma(\Lambda_A^{-1})_{mm'}\Gamma(1 - y^{\mu\nu}_{(A+1)\wedge j}\varepsilon_{A+1}f_j\tfrac{i}{\hbar}M_{\mu\nu})_{m'm''}\{1 + x^{\mu}_{(A+1)\wedge j}\tfrac{\partial}{\partial x'^{\mu}_o}\varepsilon_{A+1}f_j$$
$$+ y^{\mu\nu}_{(A+1)\wedge j}\varepsilon_{A+1}f_j(x'_{o\mu}\tfrac{\partial}{\partial x'^{\nu}_o} - x'_{o\nu}\tfrac{\partial}{\partial x'^{\mu}_o})\}\psi_{Am''}(\Lambda_A x_A) \;,$$

$$= \Gamma(\Lambda_{A+1}^{-1})_{mm'}\{1 + y^{\mu\nu}_{(A+1)\wedge j}\varepsilon_{A+1}f_j(x'_{o\mu}\tfrac{\partial}{\partial x'^{\nu}_o} - x'_{o\nu}\tfrac{\partial}{\partial x'^{\mu}_o})$$
$$+ x^{\mu}_{(A+1)\wedge j}\tfrac{\partial}{\partial x^{\mu}_o}\varepsilon_{A+1}f_j\}\psi_{Am'}(\Lambda_A x_A) \quad ,$$

$$= \Gamma(\Lambda_{A+1}^{-1})_{mm'}\left(1 + y^{\mu\nu}_{(A+1)\wedge j}\varepsilon_{A+1}f_j(g_{\sigma\mu}\delta^{\rho}_{\nu} - g_{\sigma\nu}\delta^{\rho}_{\mu})x'^{\sigma}_o\tfrac{\partial}{\partial x'^{\rho}_o}\right)$$
$$\times\left(1 + x^{\xi}_{(A+1)\wedge j}\tfrac{\partial}{\partial x^{\xi}_o}\varepsilon_{A+1}f_j\right)\psi_{Am'}(\Lambda_A x_A) \quad ,$$

where we have used the nilpotency of the terms to factorize. Thus using equation (53) and noting that we have to transform the x variable in the second bracket we obtain

$$O([\Lambda_{A+1}|010])\psi_{A+1m}(x_{A+1})\; O([\Lambda_{A+1}|010])^{-1}$$
$$= \Gamma(\Lambda_{A+1}^{-1})_{mm'}\psi_{A+1m'}(\Lambda_{A+1}x_{A+1}) \; .$$

Thus for Λ any element of the covering group of $SO_o(3,1;E_L)$ we have

$$O([\Lambda|010])\psi_m(x)\; O([\Lambda|010])^{-1}$$
$$= \Gamma(\Lambda^{-1})_{mm'}\psi_{m'}(\Lambda x) \; . \qquad \cdots(55)$$

Then combining equations (55) and(51) we have the required result.

We note that this result applies to any field $\varepsilon_i \psi_m(x)$ for $i = 0,1,\cdots,\mathcal{N}-1$ or $f_j \psi_m(x)$ for $j = 1,2,\cdots,\mathcal{N}$. The series we have used in the definition of $\psi_m(x)$ is similar in structure to the 'z-expansion' given by Rogers [13] for constructing super-differentiable functions on $E_L^{m,n}$ from C^∞ functions defined on $\mathbb{R}^m$. An examination of the above proof shows that it could easily be modified to cover the case of any finite dimensional representation of a Lie supergroup obtained from a Lie group by extending the base field $\mathbb{R}$ to the ring E_{L0}.

The fact that our functions $\psi_m(x)$ on superspace are constructed as a 'z-expansion' has an interesting consequence. That is, if $\chi_m^{\varepsilon_0}(x_0)$ satisfies some differential equation, then $\psi_m(x)$ will also satisfy the same differential equation. In particular, since $\chi_m^{\varepsilon_0}(x_0)$ satisfies the Klein-Gordon equation

$$\frac{\partial}{\partial x_0^\mu}\frac{\partial}{\partial x_{0\mu}}\chi_m^{\varepsilon_0}(x_0) = -\left(\frac{Mc}{\hbar}\right)^2 \chi_m(x_0),$$

then

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\psi_m(x) = -\left(\frac{Mc}{\hbar}\right)^2 \psi_m(x). \qquad \cdots(56)$$

This fact is used extensively in Chapter 4.

CHAPTER 3

THE CHIRAL SUPERFIELDS.

In the previous Chapter we constructed fields that were super Lorentz invariant from our second quantized fields. We now use the chiral set of fields to construct the chiral superfield. That is we take the set of fields on Minkowski space given by equation (61) of Chapter 3 of Part II, construct from each of these a super Lorentz invariant field and then combine these into a superfield that is chiral in the sense that it contains only left handed or right handed fields.

We choose to work in the chiral representation for the Dirac matrices so that the matrix representation of the super Poincaré group as given by equation (61) of Chapter 2 of Part I takes the form given below when acting on the superspace $M_L^{4,4}$ parametrized by $(x^\mu, \theta^{Ln}, \theta^{Rn'})$,

$$\begin{bmatrix} \Lambda^\rho{}_\mu & t^\rho & \begin{array}{c} 0 \\ -\frac{1}{2}\tau^{Rm}(\sigma^L_\mu \sigma^L_2)_{mr}(\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})^t)_{rn} \end{array} & \begin{array}{c} -\frac{1}{2}\tau^{Lm'}(\sigma^R_\mu \sigma^R_2)_{m'r}(\Gamma^{\frac{1}{2},0}(\Lambda^{-1})^t)_{rn'} \\ 0 \end{array} \\ 0 & 1 & 0 & 0 \\ 0 & -\tau^{Lm} & (\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})^t)^m{}_n & 0 \\ 0 & -\tau^{Rm'} & 0 & (\Gamma^{\frac{1}{2},0}(\Lambda^{-1})^t)^{m'}{}_{n'} \end{bmatrix} \begin{bmatrix} x^\mu \\ 1 \\ \theta^{Ln} \\ \theta^{Rn'} \end{bmatrix}$$

$$= \begin{bmatrix} \Lambda^\rho{}_\mu x^\mu + t^\rho - \frac{1}{2}\tau^{Rm}(\sigma^L_\mu \sigma^L_2)_{mr}(\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})^t)_{rn}\theta^{Ln} - \frac{1}{2}\tau^{Lm'}(\sigma^R_\mu \sigma^R_2)_{m'r}(\Gamma^{\frac{1}{2},0}(\Lambda^{-1})^t)_{rn'}\theta^{Rn'} \\ 1 \\ \theta^{Ln}\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})_n{}^m - \tau^{Lm} \\ \theta^{Rn'}\Gamma^{\frac{1}{2},0}(\Lambda^{-1})_{n'}{}^{m'} - \tau^{Lm'} \end{bmatrix},$$

$$= \begin{bmatrix} \Lambda^\rho{}_\mu x^\mu + t^\rho + \frac{1}{2}\theta^{Ln}\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})_{nr}(\sigma^R_\mu \sigma^R_2)_{rm}\tau^{Rm} + \frac{1}{2}\theta^{Rn'}\Gamma^{\frac{1}{2},0}(\Lambda^{-1})_{n'r}(\sigma^L_\mu \sigma^L_2)_{rm'}\tau^{Lm'} \\ 1 \\ \theta^{Ln}\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})_{n'}{}^m - \tau^{Lm} \\ \theta^{Rn'}\Gamma^{\frac{1}{2},0}(\Lambda^{-1})_{n'}{}^{m'} - \tau^{Lm'} \end{bmatrix}. \quad \cdots(1)$$

The fields we are going to use to construct the chiral superfields are constructed as in Chapter 2 and are

$$\begin{aligned} \psi^L_{j,m}(y) &= \chi^{\prime L\underline{e}_0}_{j,m}(y_0) + y^\mu_k \frac{\partial}{\partial y^\mu_0} \chi^{\prime L\underline{e}_k}_{j,m}(y_0) + \cdots, \\ &= \chi^{\prime L}_{j,m}(y_0) + y^\mu_k \underline{e}_k \frac{\partial}{\partial y^\mu_0} \chi^{\prime L}_{j,m}(y_0) + \cdots, \end{aligned} \quad \cdots(2)$$

$$\underset{\sim}{\psi}{}^{L}_{j,m}(y) = \underset{\sim}{\chi}{}'^{L\varepsilon_0}_{j,m}(y_0) + y^{\mu}_{k}\frac{\partial}{\partial y^{\mu}_{0}}\underset{\sim}{\chi}{}'^{L\varepsilon_k}_{j,m}(y_0) + \cdots,$$

$$= \underset{\sim}{\chi}{}'^{L}_{j,m}(y_0) + y^{\mu}_{k}\frac{\partial}{\partial y^{\mu}_{0}}\varepsilon_k\,\chi'^{L}_{j,m}(y_0) + \cdots, \qquad \cdots\cdots(3)$$

and
$$\psi^{L}_{\frac{1}{2},n;j,m}(y) = \chi'^{L\varepsilon_0}_{\frac{1}{2},n;j,m}(y_0) + y^{\mu}_{k}\frac{\partial}{\partial y^{\mu}_{0}}\chi'^{L\varepsilon_k}_{\frac{1}{2},n;j,m}(y_0) + \cdots,$$

$$= \chi'^{L}_{\frac{1}{2},n;j,m}(y_0) + y^{\mu}_{k}\frac{\partial}{\partial y^{\mu}_{0}}\varepsilon_k\,\chi'^{L}_{\frac{1}{2},n;j,m}(y_0) + \cdots. \qquad \cdots\cdots(4)$$

Here $k = 1, 2, \cdots, \mathcal{N}-1$. Note that we choose to use the variable y not x. This is because the even parameter of the chiral superfields will turn out to be different to the real (Grassman) parameter x.

The transformation properties of these fields are

$$O([\Lambda|t])\,\psi^{L}_{j,m}(y)\,O([\Lambda|t])^{-1} = \Gamma^{0,j}(\Lambda^{-1})_{mm'}\,\psi^{L}_{j,m'}(\Lambda y + t), \cdots(5)$$

$$O([\Lambda|t])\,\underset{\sim}{\psi}{}^{L}_{j,m}(y)\,O([\Lambda|t])^{-1} = \Gamma^{0,j}(\Lambda^{-1})_{mm'}\,\underset{\sim}{\psi}{}^{L}_{j,m'}(\Lambda y + t), \cdots(6)$$

and
$$O([\Lambda|t])\,\psi^{L}_{\frac{1}{2},n;j,m}(y)\,O([\Lambda|t])^{-1}$$

$$= \Gamma^{0,j}(\Lambda^{-1})_{mm'}\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})_{nn'}\,\psi^{L}_{\frac{1}{2},n';j,m'}(\Lambda y + t). \cdots\cdots(7)$$

Now consider the field defined by

$$\theta^{Ln}_{k}\,f_k\,\psi^{L}_{\frac{1}{2},n;j,m}(y) \qquad \cdots\cdots(8)$$

Its transformation properties are

$$O([\Lambda|t])\,\theta^{Ln}_{k}f_k\,\psi^{L}_{\frac{1}{2},n;j,m}(y)\,O([\Lambda|t])^{-1}$$

$$= \theta^{Ln}_{k}f_k\,O([\Lambda|t])\,\psi^{L}_{\frac{1}{2},n;j,m}(y)\,O([\Lambda|t])^{-1},$$

$$= \theta^{Ln}_{k}f_k\,\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})_{nn'}\Gamma^{0,j}(\Lambda^{-1})_{mm'}\,\psi^{L}_{\frac{1}{2},n';j,m'}(\Lambda y + t),$$

$$= (\theta^{Ln}\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})_{nn'})\,\Gamma^{0,j}(\Lambda^{-1})_{mm'}\,\psi^{L}_{\frac{1}{2},n';j,m'}(\Lambda y + t). \qquad \cdots(9)$$

Also consider the field defined by

$$\theta^{Ln}_{k}\,\theta^{Ln'}_{\ell}\,(\sigma_2^{L})_{nn'}\,f_k f_\ell\,\underset{\sim}{\psi}{}^{L}_{j,m}(y). \qquad \cdots(10)$$

Its transformation properties are

$$O([\Lambda|t])\,\theta^{Ln}_{k}\theta^{Ln'}_{\ell}(\sigma_2^{L})_{nn'}f_k f_\ell\,\underset{\sim}{\psi}{}^{L}_{j,m}(y)\,O([\Lambda|t])^{-1},$$

$$= \theta^{Ln}_{k}\theta^{Ln'}_{\ell}(\sigma_2^{L})_{nn'}f_k f_\ell\,\Gamma^{0,j}(\Lambda^{-1})_{mm'}\,\underset{\sim}{\psi}{}^{L}_{j,m}(\Lambda y + t). \qquad \cdots(11)$$

But
$$\theta^{Ln}_{k}\,\theta^{Lm}_{\ell}\,(\sigma_2^{L})_{n'm'}\,\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})_{n}{}^{n'}\,\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})_{m}{}^{m'}f_k f_\ell,$$

$$= \theta^{Ln}_{k}\theta^{Lm}_{\ell}\;\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})_{n}{}^{n'}(\sigma_2^{L})_{n'm'}\,(\Gamma^{0,\frac{1}{2}}(\Lambda^{-1}))^{t\,m'}{}_{m}\,f_k f_\ell,$$

$$= \theta^{Ln}_{k}\theta^{Lm}_{\ell}\;\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})_{n}{}^{n'}(\Gamma^{\frac{1}{2},0}(\Lambda^{-1}))^{\dagger}_{n'}{}^{m'}(\sigma_2^{L})_{m'm}\,f_k f_\ell,$$

$$= \theta^{Ln}_{k}\,\theta^{Lm}_{\ell}\,(\sigma_2^{L})_{nm}\,f_k f_\ell. \qquad \cdots(12)$$

So that combining equation (12) with equation (11) we have

$$\begin{aligned} &O([\Lambda|t])\,\theta^{Ln}\theta^{Lm}(\sigma_2^L)_{nm}\,\underset{\sim}{\Psi}{}^L_{j,m''}(y) \\ &= (\theta^{Ln}\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})_{nn'})(\theta^{Lm}\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})_{mm'})\,(\sigma_2^L)_{n'm'}\,\Gamma^{0,j}(\Lambda^{-1})_{m''r}\,\Psi^L_{j,r}(\Lambda y+t). \end{aligned} \quad \cdots\cdots (13)$$

Now we construct the field $\Psi^L_{j,m}(y,\theta^L,\theta^R)$ which we anticipate will be our chiral superfield by

$$\begin{aligned} \Psi_{j,m}(y,\theta^L,\theta^R) &= \Psi^L_{j,m}(y) + a\,\theta^{Ln}\Psi^L_{\frac{1}{2},n;j,m}(y) \\ &\quad + \frac{b}{2!}\,\theta^{Ln}\theta^{Ln'}(\sigma_2^L)_{nn'}\,\underset{\sim}{\Psi}{}^L_{j,m}(y) \end{aligned} \quad \cdots (14)$$

with a, b $\in \mathbb{C}$ being constants yet to be determined. Then using equations (5), (9) and (13) we find that the transformation property of $\Psi_{j,m}(y,\theta^L,\theta^R)$ to be

$$\begin{aligned} &O([\Lambda|t])\,\Psi^L_{j,m}(y,\theta^L,\theta^R)\;O([\Lambda|t])^{-1} \\ &= \Gamma^{0,j}(\Lambda^{-1})_{mm'}\,\Psi^L_{j,m}(\Lambda y+t,\ \theta^L\Gamma^{0,\frac{1}{2}}(\Lambda^{-1}),\ \theta^R). \end{aligned} \quad \cdots\cdots (15)$$

Now we want to consider the action of the supersymmetry part of the supergroup $I_L(SISO_0(3,1;E_L))$. The operators corresponding to this are given by

$$\exp \tau^\alpha Q_\alpha\ ,\quad \tau^\alpha \in E_{L1} \quad \cdots\cdots (16)$$

but

$$\exp \tau^\alpha Q_\alpha = 1 + \tau^\alpha Q_\alpha + \frac{1}{2!}(\tau^\alpha Q_\alpha)^2 + \frac{1}{3!}(\tau^\alpha Q_\alpha)^3 + \frac{1}{4!}(\tau^\alpha Q_\alpha)^4 \cdots (17)$$

This is an unweildy expression so we choose a parametrization such that

$$\exp \tau^\alpha Q_\alpha = \exp \tau_1^\alpha \varepsilon_1 Q_\alpha \exp \tau_2^\alpha \varepsilon_2 Q_\alpha \cdots \exp \tau^\alpha_{1\wedge 2\wedge\ldots}\varepsilon_{1\wedge 2\wedge\ldots} Q_\alpha . \quad \cdots\cdots (18)$$

Then for each $\underline{f}_j$ we have

$$\exp \tau^\alpha \underline{f}_j Q_\alpha = 1 + \tau^\alpha \underline{f}_j Q_\alpha\ ,\quad \text{with}\quad \tau^\alpha \in \mathbb{R}. \quad \cdots\cdots (19)$$

We put

$$O([I|0|\tau]) = \exp \tau^\alpha Q_\alpha\ ,\quad \tau^\alpha \in E_{L1} \quad \cdots\cdots (20)$$

and define a supersymmetry group operation following the decomposition of equation (61) of Chapter 2 of Part I by

$$O([\Lambda|t|\tau]) = O([I|0|\tau])\,O([\Lambda|t]). \quad \cdots\cdots (21)$$

Before evaluating the effect of a supersymmetry transformation on the field $\Psi^{L}_{j,m}(y,\theta^{L})$ it is convenient to evaluate two identities we need in the calculation.

(i) $$(\theta^{Ln}_{k} - \delta^{i}_{k}\tau^{Ln})f_k(1-(\sigma_\mu\sigma_2\tfrac{\partial}{\partial y^\mu})_{rr'}\tau^{Rr}\theta^{Lr'}_{\ell}f_i f_\ell)\psi^{L}_{\frac{1}{2},n;j,m}(y)$$
$$=(\theta^{Ln}_{k}f_k - f_i\tau^{Ln}+\tfrac{1}{2}(\sigma_2)_{rr'}\theta^{Lr}_{k}\theta^{Lr'}_{\ell}(\sigma_\mu\tfrac{\partial}{\partial y^\mu})_{nn'}\tau^{Rn'}f_i f_k f_\ell)\psi^{L}_{\frac{1}{2},n;j,m}(y). \quad \cdots(22)$$

Proof

$$(\theta^{Ln}_{k} - \delta^{i}_{k}\tau^{Ln})f_k(1-(\sigma_\mu\sigma_2\tfrac{\partial}{\partial y^\mu})_{rr'}\tau^{Rr}\theta^{Lr'}_{\ell}f_i f_\ell)$$

$$= \theta^{Ln}_{k}f_k - f_i\tau^{Ln} - \theta^{Ln}_{k}f_k(\sigma_\mu\sigma_2\tfrac{\partial}{\partial y^\mu})_{rr'}\tau^{Rr}\theta^{Lr'}_{\ell}f_i f_\ell + \tau^{Ln}f_i(\sigma_\mu\sigma_2\tfrac{\partial}{\partial y^\mu})_{rr'}\tau^{Rr}\theta^{Lr'}_{\ell}f_i f_\ell \quad ,$$

$$= \theta^{Ln}_{k}f_k - f_i\tau^{Ln} + \theta^{Ln}_{k}(\sigma_\mu\sigma_2\tfrac{\partial}{\partial y^\mu})_{rr'}\tau^{Rr}\theta^{Lr'}_{\ell}f_i f_k f_\ell \quad ,$$

$$= \theta^{Ln}_{k}f_k - f_i\tau^{Ln} + (\sigma_\mu\sigma_2\tfrac{\partial}{\partial y^\mu})_{rr'}\theta^{Lp}_{k}\tau^{Rq}\theta^{Lr'}_{\ell}f_i f_k f_\ell\,\delta^{n}_{p}\delta^{r}_{q} \quad ,$$

Now we use the Fierz rearrangement formula as given in equation (16) of the appendix to obtain

$$(\theta^{Ln}_{k} - \delta^{i}_{k}\tau^{Ln})f_k(1-(\sigma_\mu\sigma_2\tfrac{\partial}{\partial y^\mu})_{rr'}\tau^{Rr}\theta^{Lr'}_{\ell}f_i f_\ell)$$

$$= \theta^{Ln}_{k}f_k - f_i\tau^{Ln} + \tfrac{1}{2}(\sigma_\mu\sigma_2)_{rr'}\tfrac{\partial}{\partial y^\mu}\theta^{Lp}_{k}\tau^{Rq}\theta^{Lr'}_{\ell}f_i f_k f_\ell(\Gamma^A)^{n}{}_{q}(\Gamma_A)^{r}{}_{q} \quad ,$$

$$= \theta^{Ln}_{k}f_k - f_i\tau^{Ln} + \tfrac{1}{2}(\Gamma_A\sigma_\mu\sigma_2)_{pr'}\theta^{Lp}_{k}(\Gamma^A\tau^R)^n\tfrac{\partial}{\partial y^\mu}\theta^{Lr'}_{\ell}f_i f_k f_\ell \,.$$

Now this expression is zero unless $\Gamma_A\sigma_\mu\sigma_2 = \sigma_2$, so that $\Gamma^A = \sigma_\mu$.

Thus

$$(\theta^{Ln}_{k} - \delta^{i}_{k}\tau^{Ln})f_k(1-(\sigma_\mu\sigma_2\tfrac{\partial}{\partial y^\mu})_{rr'}\tau^{Rr}\theta^{Lr'}_{\ell}f_i f_\ell)$$
$$= \theta^{Ln}_{k}f_k - f_i\tau^{Ln} + \tfrac{1}{2}(\sigma_2)_{rr'}\theta^{Lr}_{k}\theta^{Lr'}_{\ell}(\sigma_\mu\tfrac{\partial}{\partial y^\mu})_{nn'}\tau^{Rn'}f_i f_k f_\ell \,. \quad \cdots(23)$$

(ii) $$(\theta^{L}_{k} - \delta^{i}_{k}\tau^{L})^n(\theta^{L}_{\ell} - \delta^{i}_{\ell}\tau^{L})^{n'}(\sigma_2)_{nn'}f_k f_\ell$$
$$= (\theta^{Ln}_{k}\theta^{Ln'}_{\ell}(\sigma_2)_{nn'}f_k f_\ell - 2\tau^{Ln}\theta^{Ln'}_{\ell}(\sigma_2)_{nn'}f_i f_\ell \,.$$

Proof

$$\tfrac{1}{2}(\theta^{L}_{k} - \delta^{i}_{k}\tau^{L})^n(\theta^{L}_{\ell} - \delta^{i}_{\ell}\tau^{L})^{n'}(\sigma_2)_{nn'}f_k f_\ell$$

$$= \tfrac{1}{2}(\theta^{Ln}_{k}\theta^{Ln'}_{\ell}(\sigma_2)_{nn'}f_k f_\ell - \theta^{Ln}_{k}\delta^{i}_{\ell}\tau^{Ln'}(\sigma_2)_{nn'}f_k f_\ell + \delta^{i}_{k}\tau^{Ln}\theta^{Ln'}_{\ell}(\sigma_2)_{nn'}f_k f_\ell + \delta^{i}_{k}\tau^{Ln}\delta^{i}_{\ell}\tau^{Ln'}(\sigma_2)_{nn'}f_k f_\ell) \quad ,$$

$$= \tfrac{1}{2}(\theta_k^{Ln}\theta_\ell^{Ln'}(\sigma_2)_{nn'}f_kf_\ell - 2\tau^{Ln}\theta_\ell^{Ln'}(\sigma_2)_{nn'}f_if_\ell).$$

Now consider the action of $O([I|0|\tau]) = \exp \tau^\alpha f_i Q_\alpha$ on the field $\Psi^L_{j,m}(y,\theta^L)$.

$$O([I|0|\tau])\,\Psi^L_{j,m}(y,\theta^L)\,O([I|0|\tau])^{-1}$$

$$= \Psi^L_{j,m}(y,\theta^L) + \tau^\alpha f_i[Q_\alpha, \Psi^L_{j,m}(y,\theta^L)]$$

$$= \Psi^L_{j,m}(y,\theta^L)$$
$$+ \tau^\alpha f_i[Q_\alpha, \psi^L_{j,m}(y) + a\theta^{Ln}\psi^L_{\frac{1}{2},n;j,m}(y) + \tfrac{b}{2!}\theta^{Ln}\theta^{Ln'}(\sigma_2)_{nn'}\underset{\sim}{\psi}^L_{j,m}(y)],$$

$$= \Psi^L_{j,m}(y,\theta^L) + \tau^\alpha f_i[Q_\alpha, \psi^L_{j,m}(y)] - a\tau^\alpha f_i\theta^{Ln}[Q_\alpha, \psi^L_{\frac{1}{2},n;j,m}(y)]$$
$$+ \tfrac{b}{2!}\tau^\alpha f_i\theta^{Ln}\theta^{Ln'}(\sigma_2)_{nn'}[Q_\alpha, \underset{\sim}{\psi}^L_{j,m}(y)] \quad ,$$

$$= \psi^L_{j,m}(y) + a\theta_k^{Ln}f_k\psi^L_{\frac{1}{2},n;j,m}(y) + \tfrac{b}{2!}\theta_k^{Ln}\theta_\ell^{Ln'}(\sigma_2)_{nn'}f_kf_\ell\underset{\sim}{\psi}^L_{j,m}(y)$$
$$+ \tau^{Ln}f_i e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\psi^L_{\frac{1}{2},n;j,m}(y) - a\tau^{Ln}f_i\theta_k^{Ln}f_k e^{-i\frac{3\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}(\sigma_2^L)_{nn'}\underset{\sim}{\psi}^L_{j,m}(y)$$
$$- a\tau^{Rn}f_i\theta_k^{Ln'}f_k e^{i\frac{3\pi}{4}}\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma_\mu\sigma_2\tfrac{\partial}{\partial y^\mu})_{nn'}\psi^L_{j,m}(y)$$
$$+ \tfrac{b}{2!}e^{i\frac{3\pi}{4}}\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}\tau^{Rn}f_i\theta_k^{Ln'}\theta_\ell^{Lm'}(\sigma_2)_{n'm'}f_kf_\ell(\sigma_\mu\tfrac{\partial}{\partial x^\mu})_{nr}\psi^L_{\frac{1}{2},r;j,m}(y),$$

$$= \psi^L_{j,m}(y) - a\tau^{Rn}f_i\theta_k^{Ln'}f_k e^{i\frac{3\pi}{4}}\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}(\sigma_\mu\sigma_2)_{nn'}\tfrac{i}{\hbar}[P_\mu, \psi^L_{j,m}(y)]$$
$$+ a\theta_k^{Ln}f_k\psi^L_{\frac{1}{2},n;j,m}(y) + \tau^{Ln}f_i e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\psi^L_{\frac{1}{2},n;j,m}(y)$$
$$+ \tfrac{b}{2!}e^{i\frac{3\pi}{4}}\left(\tfrac{\hbar}{Mc}\right)^{\frac{1}{2}}\tau^{Rn}f_i\theta_k^{Ln'}\theta_\ell^{Lm'}(\sigma_2)_{n'm'}f_kf_\ell(\sigma_\mu)_{nr}\tfrac{i}{\hbar}[P_\mu, \psi^L_{\frac{1}{2},r;j,m}(y)]$$
$$+ \tfrac{b}{2!}\theta_k^{Ln}\theta_\ell^{Ln'}(\sigma_2)_{nn'}f_kf_\ell\underset{\sim}{\psi}^L_{j,m}(y) - a\tau^{Ln}f_i\theta_k^{Ln'}f_k e^{-i\frac{3\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}(\sigma_2)_{nn'}\underset{\sim}{\psi}^L_{j,m}(y).$$

Now we set

$$a = -e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}} \qquad \cdots(24)$$

and

$$b = -e^{-i\frac{\pi}{2}}\left(\tfrac{Mc}{\hbar}\right). \qquad \cdots(25)$$

Then using equations (22) and (23) we obtain

$$O([I|0|\tau])\,\Psi^L_{j,m}(y,\theta^L)\,O([I|0|\tau])^{-1}$$
$$= \psi^L_{j,m}(y + e^{i\pi}\tau^{Rn}f_i\theta_k^{Ln'}f_k(\sigma_\mu^L\sigma_2^L)_{nn'})$$
$$+ e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}(\theta_k^{Ln}f_k - \tau^{Ln}\delta_i^k f_k)(1 - \tau^{Rn''}f_i\theta_\ell^{Ln'}(\sigma_\mu^L\sigma_2^L)_{n''n'}\tfrac{\partial}{\partial x^\mu}\psi^L_{\frac{1}{2},n;j,m}(y))$$
$$+ \tfrac{1}{\sqrt{2}}e^{-i\frac{\pi}{2}}\left(\tfrac{Mc}{\hbar}\right)(\theta_k^{Ln}f_k - \tau^{Ln}\delta_i^k f_k)(\theta_\ell^{Ln'}f_\ell - \tau^{Ln'}\delta_i^\ell f_\ell)(\sigma_2^L)_{nn'}\underset{\sim}{\psi}^L_{j,m}(y).$$

Then since

$$(\theta^L_k f_k - \tau^{Ln} S^k_i f_k)(\theta^{Ln'}_\ell f_\ell - \tau^{Ln'} S^\ell_i f_\ell)(\sigma_2)_{nn'}$$
$$= (\theta^L_k f_k - \tau^{Ln} S^k_i f_k)(\theta^{Ln'}_\ell f_\ell - \tau^{Ln'} S^\ell_i f_\ell)(1 - \tau^{Rn''} f_i \theta^{Ln'}_\ell f_\ell (\sigma^L_\mu \sigma^L_2)_{n''n'})$$

we have

$$O([I|0|\tau]) \Psi^L_{j,m}(y, \theta^L)\, O([I|0|\tau])^{-1}$$
$$= \Psi^L_{j,m}(y^\mu - \tau^{Rn} f_i \theta^{Ln'} (\sigma^L_\mu \sigma^L_2)_{nn'} \;,\; \theta^{Ln} - \tau^{Ln} f_i) . \qquad \cdots (26)$$

But this is true for each f_i so that with the parametrization of equation (18) we have

$$O([I|0|\tau]) \Psi^L_{j,m}(y, \theta^L)\, O([I|0|\tau])^{-1}$$
$$= \Psi^L_{j,m}(y^\mu - \tau^{Rn} \theta^{Ln'} (\sigma^L_\mu \sigma^L_2)_{nn'} \,,\, \theta^{Ln} - \tau^{Ln}) \quad , \qquad \cdots (27)$$

with

$$\Psi^L_{j,m}(y, \theta^L) = \psi^L_{j,m}(y) - e^{i\frac{\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} \theta^{Ln} \psi^L_{\frac{1}{2},n;j,m}(y)$$
$$- \frac{1}{2} e^{-i\frac{\pi}{2}} \left(\frac{Mc}{\hbar}\right) \theta^{Ln} \theta^{Ln'} (\sigma^L_2)_{nn'} \underset{\sim}{\psi}^L_{j,m}(y) . \qquad \cdots (28)$$

Combining this result with equations (15) and (21) we obtain

$$O([\Lambda|t|\tau]) \Psi^L_{j,m}(y, \theta^L)\, O([\Lambda|t|\tau])^{-1}$$
$$= \Gamma^{0,j}(\Lambda^{-1})_{mm'} \Psi^L_{j,m'}((\Lambda y)^\mu + t + \theta^{Ln} \Gamma^{0,\frac{1}{2}}(\Lambda^{-1})_{nr} (\sigma^R_\mu \sigma^R_2)_{rr'} \tau^{Rr'} ,$$
$$\theta^{Ln'} \Gamma^{0,\frac{1}{2}}(\Lambda^{-1})_{n'}{}^{n} - \tau^{Ln}) . \qquad \cdots (29)$$

This result was obtained in a form that is symmetric in the interchange of L and R, so that we might naively anticipate that we can obtain the result for right handed fields just by exchanging L and R. But the action of the supersymmetry transformation on the even coordinates is not the same as that given in equation (1) for x^μ, thus, as we will see, we need to attach a superscript L to y of equation (29) in order to obtain symmetry in L, R interchange.

From equation (27) we can see that the action of a supersymmetry transformation on the coordinate y is given by

$$y'^\mu = y^\mu - \tau^{Rn} (\sigma^L_\mu \sigma^L_2)_{nn'} \theta^{Ln'} . \qquad \cdots (30)$$

Suppose that

$$y^{\mu} = x^{\mu} + f(\theta^{L}, \theta^{R}) \,,$$

with $f(\theta^{L}, \theta^{R})$ some function of θ^{L} and θ^{R}, to be determined. Then using equation (31) in equation (30) we obtain

$$\begin{aligned} y'^{\mu} &= x^{\mu} + f(\theta^{L}, \theta^{R}) - \tau^{Rn}(\sigma^{L}_{\mu}\sigma^{L}_{2})_{nn'}\theta^{Ln'} \,, \\ &= x'^{\mu} + f'(\theta^{L}, \theta^{R}) \,, \\ &= x^{\mu} - \tfrac{1}{2}\tau^{Rn}(\sigma^{L}_{\mu}\sigma^{L}_{2})_{nn'}\theta^{Ln'} - \tfrac{1}{2}\tau^{Ln}(\sigma^{R}_{\mu}\sigma^{R}_{2})_{nn'}\theta^{Rn'} \\ &\quad + f(\theta^{L}+\tau^{L}, \theta^{R}+\tau^{R}) \,. \end{aligned}$$

This implies that $f(\theta^{L}, \theta^{R})$ must satisfy

$$\begin{aligned} f(\theta^{L}, \theta^{R}) &= \tfrac{1}{2}\tau^{Rn}(\sigma^{L}_{\mu}\sigma^{L}_{2})_{nn'}\theta^{Ln'} - \tfrac{1}{2}\tau^{Ln}(\sigma^{R}_{\mu}\sigma^{R}_{2})_{nn'}\theta^{Rn'} \\ &\quad + f(\theta^{L}+\tau^{L}, \theta^{R}+\tau^{R}) \,. \end{aligned} \qquad \cdots(32)$$

A solution of this equation is given by

$$f(\theta^{L}, \theta^{R}) = -\tfrac{1}{2}\theta^{Rn}(\sigma^{L}_{\mu}\sigma^{L}_{2})_{nn'}\theta^{Ln'} \,, \qquad \cdots(33)$$

which is demonstrated as follows, starting with the right hand side of equation (32)

$$\begin{aligned} &\tfrac{1}{2}\tau^{Rn}(\sigma^{L}_{\mu}\sigma^{L}_{2})_{nn'}\theta^{Ln'} - \tfrac{1}{2}\tau^{Ln}(\sigma^{R}_{\mu}\sigma^{R}_{2})_{nn'}\theta^{Rn'} + f(\theta^{L}+\tau^{R}, \theta^{R}+\tau^{R}) \\ &= \tfrac{1}{2}\tau^{Rn}(\sigma^{L}_{\mu}\sigma^{L}_{2})_{nn'}\theta^{Ln'} - \tfrac{1}{2}\tau^{Ln}(\sigma^{R}_{\mu}\sigma^{R}_{2})_{nn'}\theta^{Rn'} - \tfrac{1}{2}(\theta^{R}+\tau^{R})^{n}(\sigma^{L}_{\mu}\sigma^{L}_{2})_{nn'}(\theta^{L}+\tau^{L})^{n'} \,, \\ &= -\tfrac{1}{2}\theta^{Rn}(\sigma^{L}_{\mu}\sigma^{L}_{2})_{nn'}\theta^{Ln'} \,, \\ &= f(\theta^{L}, \theta^{R}) \,, \end{aligned}$$

as required. In this calculation we have made use of equation (14) of the appendix.

Thus

$$y^{\mu} = x^{\mu} - \theta^{Rn}(\sigma^{L}_{\mu}\sigma^{L}_{2})_{nn'}\theta^{Ln'}$$

We note that y^{μ} is <u>complex valued</u> in its nilpotent part. We can now rewrite equation (29) in its final form, which is symmetric in the interchange of L and R. This result is conveniently expressed as a theorem.

<u>Theorem 3.1</u>

The chiral superfields $\Phi^{A}_{j,m}(y^{A}, \theta)$ defined by

$$\Phi^{A}_{j,m}(y^{A},\theta^{A}) = \psi^{A}_{j,m}(y) - e^{i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{An}\,\psi^{A}_{\frac{1}{2},n;j,m}(y)$$
$$-\frac{1}{2}e^{-i\frac{\pi}{2}}\left(\frac{Mc}{\hbar}\right)\theta^{An}\theta^{An'}(\sigma_2{}^{A})_{nn'}\,\underset{\sim}{\psi}^{A}_{j,m}(y) \qquad \cdots(35)$$

for $A = L$ or $A = R$, $j = 0, \frac{1}{2}, 1, \cdots$ and $m = j, j-1, \cdots, -j+1, -j$;

transforms under the super Poincaré group as

$$O([\Lambda|t|\tau])\,\Phi^{A}_{j,m}(y^{A},\theta^{A})\,O([\Lambda|t|\tau])^{-1}$$
$$= \Gamma^{Aj}(\Lambda^{-1})_{mm'}\,\Phi^{A}_{j,m'}(\Lambda y^{A} + t + \theta^{An}\Gamma^{A\frac{1}{2}}(\Lambda^{-1})_{nr}(\sigma_\mu^{\bar{A}}\sigma_2^{\bar{A}})_{rr'}\,\tau^{\bar{A}r'},$$
$$\theta^{An'}\Gamma^{A\frac{1}{2}}(\Lambda^{-1})_{n'}{}^{n} - \tau^{An}) \qquad \cdots(36)$$

with

$$\bar{A} = \begin{cases} L \text{ if } A = R \\ R \text{ if } A = L \end{cases}, \qquad \cdots(37)$$

$$\Gamma^{Aj} = \begin{cases} \Gamma^{0,j} & \text{if } A = L \\ \Gamma^{j,0} & \text{if } A = R \end{cases} \qquad \cdots(38)$$

and the even Grassman parameter defined by

$$y^{A\mu} = x^{\mu} - \frac{1}{2}\theta^{\bar{A}n}(\sigma_\mu^{A}\sigma_2^{A})_{nn'}\,\theta^{An'}. \qquad \cdots(39)$$

Here x^{μ} is as given in equation (1) so that it is real.

It is interesting to note that the parameters $y^{L\mu}$ and $y^{R\mu}$ are the complex conjugates of each other. To show this we observe that since θ^{Ln} and $\theta^{Rn'}$ form the components of a spinor in the chiral representation of the Dirac matrices that is constructed from a four component (real) Majorana spinor then they must satisfy

$$\theta^{Ln*} = (\sigma_2)^{n}{}_{n'}\,\theta^{Rn'}$$

and

$$\theta^{Rn*} = -(\sigma_2)^{n}{}_{n'}\,\theta^{Ln'}.$$

Then

$$\begin{aligned} y^{L\mu*} &= x^{\mu} - \frac{1}{2}\left(\theta^{Rn}(\sigma_\mu^{L}\sigma_2^{L})_{nn'}\,\theta^{Ln'}\right)^{*}, \\ &= x^{\mu} - \frac{1}{2}(-\sigma_2)^{n}{}_{n'}\,\theta^{Ln'}(\sigma_2\sigma_\mu^{R}\sigma_2\,\sigma_2^{R})_{nn''}(\sigma_2)^{n''}{}_{n'''}\,\theta^{Rn'''} \\ &= x^{\mu} - \frac{1}{2}\theta^{Ln'}(\sigma_2\sigma_2\,\sigma_\mu^{R}\sigma_2^{R}\sigma_2\,\sigma_2)_{n'n''}\,\theta^{Rn''}, \\ &= y^{\mu} - \frac{1}{2}\,\theta^{Ln}(\sigma_\mu^{R}\sigma_2^{R})_{nn'}\,\theta^{Rn'}. \end{aligned}$$

So that

$$(y^{L\mu})^{*} = y^{R\mu}. \qquad \cdots(40)$$

For later use we need the chiral superfields expressed in terms of the real parameter x^μ. We give the result as a corollary.

Corollary 3.2

The chiral superfields of Theorem 3.1 can be expressed in terms of the real (Grassman) parameters as

$$\begin{aligned}
&\Psi^A_{j,m}(x,\theta^L,\theta^R)\\
&= \psi^A_{j,m}(x) - \tfrac{1}{2}\theta^{\bar{A}r}(\sigma_\mu^A\sigma_2^A)_{rr'}\theta^{Ar}\tfrac{\partial}{\partial x^\mu}\psi^A_{j,m}(x)\\
&-\tfrac{1}{16}\left(\tfrac{Mc}{\hbar}\right)^2\theta^{An}(\sigma_2^A)_{nn'}\theta^{An'}\theta^{\bar{A}r}(\sigma_2^{\bar{A}})_{rr'}\theta^{\bar{A}r'}\psi^A_{j,m}(x)\\
&- e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{An}\psi^A_{\frac{1}{2},n;j,m}(x)\\
&-\tfrac{1}{4}e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{An}(\sigma_2^A)_{nn'}\theta^{An'}\theta^{\bar{A}r}(\sigma_\mu^A\sigma_2^A)_{rr'}\tfrac{\partial}{\partial x^\mu}\psi^A_{\frac{1}{2},n;j,m}(x)\\
&-\tfrac{1}{2}e^{-i\frac{\pi}{2}}\left(\tfrac{Mc}{\hbar}\right)\theta^{An}(\sigma_2^A)_{nn'}\theta^{An'}\underset{\sim}{\psi}^A_{j,m}(x) \qquad \cdots(41)
\end{aligned}$$

with $A,\bar{A}$ as defined by equation (37). Then with Γ^{Aj} defined by equation (38) $\Psi^A_{j,m}(x,\theta^L,\theta^R)$ transforms under the super Poincaré group as

$$\begin{aligned}
&O([\Lambda|t|\tau])\,\Psi^A_{j,m}(x,\theta^L,\theta^R)\,O([\Lambda|t|\tau])^{-1}\\
&=\Gamma^{Aj}(\Lambda^{-1})_{mm'}\Psi^A_{j,m}\big((\Lambda x)^\mu + t^\mu + \tfrac{1}{2}\theta^{Ln}\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})_{nr}(\sigma_\mu^R\sigma_2^R)_{rm}\tau^{Lm}\\
&+\tfrac{1}{2}\theta^{Rn'}\Gamma^{\frac{1}{2},0}(\Lambda^{-1})_{n'r}(\sigma_\mu^L\sigma_2^L)_{rm'}\tau^{Lm'},\ \theta^{Ln'}\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})_{n'}{}^{n} - \tau^{Ln},\\
&\qquad \theta^{Rn'}\Gamma^{\frac{1}{2},0}(\Lambda^{-1})_{n'}{}^{n''} - \tau^{Rn''}\big) \qquad \cdots(42)
\end{aligned}$$

Proof

We first expand each term of equation (35) as a Taylor sieries in the variable $\frac{1}{2}\theta^{\bar{A}r}(\sigma_\mu^A\sigma_2^A)_{rr'}\theta^{Ar'}$.

$$\begin{aligned}
&\Psi^A_{j,m}(x,\theta^L,\theta^R)\\
&=\psi^A_{j,m}(x) - \tfrac{1}{2}\theta^{\bar{A}n}(\sigma_\mu^A\sigma_2^A)_{nn'}\theta^{An'}\tfrac{\partial}{\partial x^\mu}\psi^A_{j,m}(x)\\
&+\tfrac{1}{8}\theta^{\bar{A}n}(\sigma_\mu^A\sigma_2^A)_{nn'}\theta^{An'}\theta^{\bar{A}r}(\sigma_\rho^A\sigma_2^A)_{rr'}\theta^{Ar'}\tfrac{\partial}{\partial x^\mu}\tfrac{\partial}{\partial x^\rho}\psi^A_{j,m}(x)\\
&- e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{An}\psi^A_{\frac{1}{2},n;j,m}(x)\\
&+\tfrac{1}{2}e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{An}\theta^{\bar{A}r}(\sigma_\mu^A\sigma_2^A)_{rr'}\theta^{Ar'}\tfrac{\partial}{\partial x^\mu}\psi^A_{\frac{1}{2},n;j,m}(x)\\
&-\tfrac{1}{2}e^{-i\frac{\pi}{2}}\left(\tfrac{Mc}{\hbar}\right)\theta^{An}\theta^{An'}(\sigma_2^A)_{nn'}\underset{\sim}{\psi}^A_{j,m}(x).
\end{aligned}$$

All other terms vanish. We need to modify only the third and fifth terms to obtain the desired result. In both cases we use the Fierz rearrangement formula as given by equation (16) of the appendix. First consider the third term.

$$\tfrac{1}{8}\,\theta^{\bar{A}n}(\sigma_\mu^A\sigma_2^A)_{nn'}\,\theta^{An'}\theta^{\bar{A}r}(\sigma_\rho^A\sigma_2^A)_{rr'}\,\theta^{Ar'}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\rho}\psi^A_{j,m}(x)$$
$$=\tfrac{-1}{8}\,\theta^{\bar{A}n}\delta^a_n(\sigma_\mu^A\sigma_2^A)_{an'}\,\theta^{An'}\theta^{Ar}\delta^b_r(\sigma_\rho^{\bar{A}}\sigma_2^{\bar{A}})_{rr'}\,\theta^{\bar{A}r'}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\rho}\psi^A_{j,m}(x),$$
$$=\tfrac{-1}{16}\,\theta^{\bar{A}n}(\Gamma^B\sigma_\mu^A\sigma_2^A)_{rn'}\,\theta^{An'}\theta^{Ar}(\Gamma_B\sigma_\rho^{\bar{A}}\sigma_2^{\bar{A}})_{nr'}\,\theta^{\bar{A}r'}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\rho}\psi^A_{j,m}(x),$$
$$=\tfrac{1}{16}\,\theta^{Ar}(\Gamma^B\sigma_\mu^A\sigma_2^A)_{rn'}\,\theta^{An'}\theta^{\bar{A}n}(\Gamma_B\sigma_\rho^{\bar{A}}\sigma_2^{\bar{A}})_{nr'}\,\theta^{\bar{A}r'}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\rho}\psi^A_{j,m}(x).$$

Now this expression is zero unless $\Gamma^B = \sigma_\mu^A$ so that $\Gamma_B = \sigma_\mu^A$. Also

$$\sigma_\mu^A\,\sigma_\rho^{\bar{A}}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\rho} = g^{\mu\rho}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\rho}\,.$$

Thus the third term becomes

$$\tfrac{1}{2}\,\theta^{Ar}(\sigma_2^A)_{rr'}\,\theta^{Ar'}\theta^{\bar{A}n}(\sigma_2^{\bar{A}})_{nn'}\,\theta^{\bar{A}n'}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\,.$$

Now consider the fifth term.

$$\tfrac{1}{2}\,e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{An}\theta^{\bar{A}r}(\sigma_\mu^A\sigma_2^A)_{rr'}\,\theta^{Ar'}\frac{\partial}{\partial x^\mu}\psi^A_{\frac{1}{2},r;j,m}(x)$$
$$=\tfrac{1}{2}\,e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{An}\theta^{\bar{A}r}\delta^a_r(\sigma_\mu^A\sigma_2^A)_{ar'}\,\theta^{Ar'}\frac{\partial}{\partial x^\mu}\,\delta^b_n\,\psi^A_{\frac{1}{2},b;j,m}(x),$$
$$=\tfrac{1}{4}\,e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{An}\theta^{\bar{A}r}(\Gamma^B\sigma_\mu^A\sigma_2^A)_{nr'}\,\theta^{Ar'}\frac{\partial}{\partial x^\mu}\,(\Gamma_B)_{nb}\,\psi^A_{\frac{1}{2},b;j,m}(x),$$
$$=-\tfrac{1}{4}\,e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{An}(\sigma_2^A)_{nn'}\,\theta^{An'}\theta^{\bar{A}r}(\sigma_\mu^A)_{rr'}\frac{\partial}{\partial x^\mu}\psi^A_{\frac{1}{2},r';j,m}(x).$$

The chiral superfield can now be written

$$\begin{aligned}
&\Psi^A_{j,m}(x,\theta)\\
&= \psi^A_{j,m}(x) - \tfrac{1}{2}\,\theta^{\bar{A}n}(\sigma_\mu^A\sigma_2^A)_{nn'}\,\theta^{\bar{A}n'}\frac{\partial}{\partial x^\mu}\psi^A_{j,m}(x)\\
&\quad+\tfrac{1}{16}\,\theta^{Ar}(\sigma_2^A)_{rr'}\,\theta^{Ar'}\theta^{\bar{A}n}(\sigma_2^{\bar{A}})_{nn'}\,\theta^{\bar{A}n'}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\psi^A_{j,m}(x)\\
&\quad- e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{An}\psi^A_{\frac{1}{2},n;j,m}(x)\\
&\quad-\tfrac{1}{4}\,e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{An}(\sigma_2^A)_{nn'}\,\theta^{An'}\theta^{\bar{A}r}(\sigma_\mu^A)_{rr'}\frac{\partial}{\partial x^\mu}\psi^A_{\frac{1}{2},r';j,m}(x)\\
&\quad-\tfrac{1}{2}\,e^{-i\frac{\pi}{2}}\left(\tfrac{Mc}{\hbar}\right)\theta^{An}\theta^{An'}(\sigma_2^A)_{nn'}\,\underset{\sim}{\psi}{}^A_{j,m}(x)\,.
\end{aligned}\qquad\cdots(43)$$

To complete this part of the proof we note that $\psi^L_{j,m}(x)$ must satisfy the Klein-Gordon equation so that

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\psi^A_{j,m} = -\left(\tfrac{Mc}{\hbar}\right)^2\psi^A_{j,m}\,.$$

The transformation properties of the chiral superfields follow from equations (1) and (36).

CHAPTER 4

THE GENERAL SUPERFIELD.

4.1 The General Scalar Superfield.

In this Chapter we are going to consider fields on superspace that transform as the extended Lorentz group. That is fields that have the transformation property

$$O([\Lambda|t|\tau])\,\Phi_m(x,\theta)\,O([\Lambda|t|\tau])^{-1} =$$
$$= \Gamma(\Lambda^{-1})_{mm'}\,\Phi_{m'}\left((\Lambda x)^\mu + t^\mu - \tfrac{1}{2}\theta^\beta \Gamma^M(\Lambda^{-1})_{\beta\beta'}(\gamma^\mu C)_{\beta'\alpha}\tau^\alpha,\ \theta^\beta \Gamma^M(\Lambda^{-1})_\beta{}^\alpha - \tau^\alpha\right) \quad \cdots\cdots(1)$$

with $\Gamma(\Lambda)$ some representation of the covering group of the extended Lorentz group and $\Gamma^M(\Lambda)$ the four dimensional spinor representation of the extended Lorentz group as specified in equation (61) of Chapter 2 of Part I. In this section we consider the general scalar superfield ie. we choose $\Gamma(\Lambda)=1$ and consider the single component field $\Phi(x,\theta)$. We consider a real scalar superfield that then admits the Taylor expansion

$$\Phi(x,\theta) = A(x) + \theta^\alpha B_\alpha(x) + \tfrac{1}{2!}\theta^\alpha\theta^\beta C_{\alpha\beta}(x) + \tfrac{1}{3!}\theta^\alpha\theta^\beta\theta^\gamma D_{\alpha\beta\gamma}(x)$$
$$+ \tfrac{1}{4!}\theta^\alpha\theta^\beta\theta^\gamma\theta^\delta E_{\alpha\beta\gamma\delta}(x)\,, \quad \cdots\cdots(2)$$

in which we can identify 16 linearly independent components. We can recognize that the chiral superfields of Chapter 3 can be obtained from equation (2) by specifying relationships between these components or setting some to zero.

Consider the action of a supersymmetry transformation $O([1|0|\tau])$ on equation (2) with $\tau^\alpha = \alpha^\alpha f_j$, $\alpha^\alpha \in \mathbb{R}$ so that all second order and higher order terms vanish. Then

$$O([1|0|\tau])\,\Phi(x,\theta)\,O([1|0|\tau])^{-1}$$
$$= \Phi(x,\theta) + \tau^\alpha[Q_\alpha, \Phi(x,\theta)]$$
$$= \Phi(x^\mu + \tfrac{1}{2}\tau^\alpha(\gamma^\mu C)_{\alpha\beta}\theta^\beta,\ \theta - \tau). \quad \cdots\cdots(3)$$

Now we can expand each term of equation (2) as a Taylor sieries.

$$A(x^\mu + \tfrac{1}{2}\tau^\alpha(\gamma^\mu C)_{\alpha\beta}\theta^\beta) = A(x) + \theta^\gamma\left(\tfrac{\partial}{\partial\theta^\gamma} A(x^\mu + \tfrac{1}{2}\tau^\alpha(\gamma^\mu C)_{\alpha\beta}\theta^\beta)\right)_{\theta=0}$$
$$+ \tfrac{1}{2!}\theta^\gamma\theta^\delta\left(\tfrac{\partial}{\partial\theta^\delta}\tfrac{\partial}{\partial\theta^\gamma} A(x^\mu + \tfrac{1}{2}\tau^\alpha(\gamma^\mu C)_{\alpha\beta}\theta^\beta)\right)_{\theta=0} + \cdots\,,$$

$$= A(x) + \theta^{\gamma}\Big(\frac{\partial}{\partial\theta^{\gamma}}(x^{\mu}+\tfrac{1}{2}\tau^{\alpha}(\gamma^{\mu}C)_{\alpha\beta}\theta^{\beta})\frac{\partial A(x^{\mu}+\frac{1}{2}\tau^{\alpha}(\gamma^{\mu}C)_{\alpha\beta}\theta^{\beta})}{\partial(x^{\mu}+\frac{1}{2}\tau^{\alpha}(\gamma^{\mu}C)_{\alpha\beta}\theta^{\beta})}\Big)_{\theta=0}$$

$$+\frac{1}{2!}\theta^{\gamma}\theta^{\delta}\Big(\frac{\partial}{\partial\theta^{\gamma}}\Big(\frac{\partial}{\partial\theta^{\delta}}(x^{\mu}+\tfrac{1}{2}\tau^{\alpha}(\gamma^{\mu}C)_{\alpha\beta}\theta^{\beta})\frac{\partial A(x^{\mu}+\frac{1}{2}\tau^{\alpha}(\gamma^{\mu}C)_{\alpha\beta}\theta^{\beta})}{\partial(x^{\mu}+\frac{1}{2}\tau^{\alpha}(\gamma^{\mu}C)_{\alpha\beta}\theta^{\beta})}\Big)\Big)_{\theta=0}$$

$$+ \cdots$$

$$= A(x) - \tfrac{1}{2}\theta^{\gamma}\tau^{\alpha}(\gamma^{\mu}C)_{\alpha\beta}\delta^{\beta}_{\gamma}\frac{\partial}{\partial x^{\mu}}A(x)$$

$$+\frac{1}{2!}\theta^{\gamma}\theta^{\delta}\Big(\frac{\partial}{\partial\theta^{\gamma}}\Big(\tfrac{1}{2}\tau^{\alpha}(\gamma^{\mu}C)_{\alpha\beta}\delta^{\beta}_{\gamma}\frac{\partial A(x^{\mu}+\frac{1}{2}\tau^{\alpha}(\gamma^{\mu}C)_{\alpha\beta}\theta^{\beta})}{\partial(x^{\mu}+\frac{1}{2}(\gamma^{\mu}C)_{\alpha\beta}\theta^{\beta})}\Big)\Big)_{\theta=0}$$

$$+ \cdots$$

$$= A(x) + \tfrac{1}{2}\tau^{\alpha}(\gamma^{\mu}C)_{\alpha\beta}\theta^{\beta}\frac{\partial}{\partial x^{\mu}}A(x)$$

since all second order and higher terms vanish because they contain $(\tau^{\alpha})^2$.

Thus

$$A(x^{\mu}+\tfrac{1}{2}\tau^{\alpha}(\gamma^{\mu}C)_{\alpha\beta}\theta^{\beta}) = A(x) + \tfrac{1}{2}\tau^{\alpha}(\gamma^{\mu}C)_{\alpha\beta}\theta^{\beta}\frac{\partial}{\partial x^{\mu}}A(x). \qquad \cdots(4)$$

We can now use this result to write down the expansion of the other terms.

$$(\theta-\tau)^{\alpha}B_{\alpha}(x^{\mu}+\tfrac{1}{2}\tau^{\alpha}(\gamma^{\mu}C)_{\alpha\beta}\theta^{\beta})$$
$$= \theta^{\alpha}B_{\alpha}(x) - \tau^{\alpha}B_{\alpha}(x) + \tfrac{1}{2}\theta^{\alpha}\tau^{\gamma}(\gamma^{\mu}C)_{\gamma\beta}\theta^{\beta}\frac{\partial}{\partial x^{\mu}}B_{\alpha}(x), \qquad \cdots(5)$$

$$(\theta-\tau)^{\alpha}(\theta-\tau)^{\beta}C_{\alpha\beta}(x^{\mu}+\tfrac{1}{2}\tau^{\alpha}(\gamma^{\mu}C)_{\alpha\beta}\theta^{\beta})$$
$$= \theta^{\alpha}\theta^{\beta}C_{\alpha\beta}(x) - 2\tau^{\alpha}\theta^{\beta}C_{\alpha\beta}(x) + \tfrac{1}{2}\theta^{\alpha}\theta^{\beta}\tau^{\gamma}(\gamma^{\mu}C)_{\gamma\delta}\theta^{\delta}\frac{\partial}{\partial x^{\mu}}C_{\alpha\beta}(x), \qquad \cdots(6)$$

$$(\theta-\tau)^{\alpha}(\theta-\tau)^{\beta}(\theta-\tau)^{\gamma}D_{\alpha\beta\gamma}(x^{\mu}+\tfrac{1}{2}\tau^{\alpha}(\gamma^{\mu}C)_{\alpha\beta}\theta^{\beta})$$
$$= \theta^{\alpha}\theta^{\beta}\theta^{\gamma}D_{\alpha\beta\gamma}(x) - 3\tau^{\alpha}\theta^{\beta}\theta^{\gamma}D_{\alpha\beta\gamma}(x) + \tfrac{1}{2}\theta^{\alpha}\theta^{\beta}\theta^{\gamma}\tau^{\delta}(\gamma^{\mu}C)_{\delta\varepsilon}\theta^{\varepsilon}D_{\alpha\beta\gamma}(x) \qquad \cdots(7)$$

and

$$(\theta-\tau)^{\alpha}(\theta-\tau)^{\beta}(\theta-\tau)^{\gamma}(\theta-\tau)^{\delta}E_{\alpha\beta\gamma\delta}(x^{\mu}+\tfrac{1}{2}\tau^{\xi}(\gamma^{\mu}C)_{\xi\eta}\theta^{\eta})$$
$$= \theta^{\alpha}\theta^{\beta}\theta^{\gamma}\theta^{\delta}E_{\alpha\beta\gamma\delta}(x) - 4\tau^{\alpha}\theta^{\beta}\theta^{\gamma}\theta^{\delta}E_{\alpha\beta\gamma\delta}(x) \qquad \cdots(8)$$

Now we use the expansions in equations (4), (5), (6), (7) and (8) in equation (3) to obtain

$$\tau^{\alpha}[Q_{\alpha}, A(x) + \theta^{\beta}B_{\beta}(x) + \tfrac{1}{2!}\theta^{\beta}\theta^{\gamma}C_{\beta\gamma}(x) + \tfrac{1}{3!}\theta^{\beta}\theta^{\gamma}\theta^{\delta}D_{\beta\gamma\delta}(x) + \tfrac{1}{4!}\theta^{\beta}\theta^{\gamma}\theta^{\delta}\theta^{\varepsilon}E_{\beta\gamma\delta\varepsilon}(x)] \qquad \cdots(9)$$

$$= \tfrac{1}{2}\tau^{\alpha}(\gamma^{\mu}C)_{\alpha\beta}\theta^{\beta}\frac{\partial}{\partial x^{\mu}}A(x) - \tau^{\alpha}B_{\alpha}(x) - \tfrac{1}{2}\tau^{\alpha}\theta^{\beta}\theta^{\gamma}(\gamma^{\mu}C)_{\alpha\gamma}\frac{\partial}{\partial x^{\mu}}B_{\beta}(x)$$
$$- \tau^{\alpha}\theta^{\beta}C_{\alpha\beta}(x) + \tfrac{1}{4}\tau^{\alpha}\theta^{\beta}\theta^{\gamma}\theta^{\delta}(\gamma^{\mu}C)_{\alpha\delta}\frac{\partial}{\partial x^{\mu}}C_{\beta\gamma}(x) - \tfrac{1}{2}\tau^{\alpha}\theta^{\beta}\theta^{\gamma}D_{\alpha\beta\gamma}(x)$$
$$-\tfrac{1}{12}\tau^{\alpha}\theta^{\beta}\theta^{\gamma}\theta^{\delta}\theta^{\varepsilon}(\gamma^{\mu}C)_{\alpha\varepsilon}\frac{\partial}{\partial x^{\mu}}D_{\beta\gamma\delta}(x) - \tfrac{1}{6}\tau^{\alpha}\theta^{\beta}\theta^{\gamma}\theta^{\delta}E_{\alpha\beta\gamma\delta}(x).$$

To obtain the action of the supersymmetry generators on the component fields we equate coeficients of θ.

The terms with no θ give

$$\tau^\alpha [Q_\alpha , A] = -\tau^\alpha B_\alpha .$$

So that

$$[Q_\alpha , A] = -B_\alpha . \qquad \cdots (10a)$$

The terms linear in θ give

$$\tau^\alpha [Q_\alpha , \theta^\beta B_\beta] = -\tau^\alpha \theta^\beta [Q_\alpha , B_\beta]$$
$$= \tfrac{1}{2} \tau^\alpha (\gamma^\mu C)_{\alpha\beta} \frac{\partial}{\partial x^\mu} A - \tau^\alpha \theta^\beta C_{\alpha\beta} .$$

So that

$$[Q_\alpha , B_\beta] = -\tfrac{1}{2} (\gamma^\mu C)_{\alpha\beta} \frac{\partial}{\partial x^\mu} A + C_{\alpha\beta} . \qquad \cdots (10b)$$

The terms quadratic in θ give

$$\tau^\alpha [Q_\alpha , \tfrac{1}{2} \theta^\beta \theta^\gamma C_{\beta\gamma}] = -\tfrac{1}{2} \tau^\alpha \theta^\beta \theta^\gamma (\gamma^\mu C)_{\alpha\gamma} \frac{\partial}{\partial x^\mu} B_\beta$$
$$-\tfrac{1}{2} \tau^\alpha \theta^\beta \theta^\gamma D_{\alpha\beta\gamma} .$$

So that

$$[Q_\alpha , C_{\beta\gamma}] = -(\gamma^\mu C)_{\alpha\gamma} \frac{\partial}{\partial x^\mu} B_\beta - D_{\alpha\beta\gamma} . \qquad \cdots (10c)$$

The terms cubic in θ give

$$\tau^\alpha [Q_\alpha , \tfrac{1}{6} \theta^\beta \theta^\gamma \theta^\delta D_{\beta\gamma\delta}] = \tfrac{1}{4} \tau^\alpha \theta^\beta \theta^\gamma \theta^\delta (\gamma^\mu C)_{\alpha\delta} \frac{\partial}{\partial x^\mu} C_{\beta\gamma}$$
$$-\tfrac{1}{6} \tau^\alpha \theta^\beta \theta^\gamma \theta^\delta E_{\alpha\beta\gamma\delta} .$$

So that

$$[Q_\alpha , D_{\beta\gamma\delta}] = -\tfrac{3}{2} (\gamma^\mu C)_{\alpha\delta} \frac{\partial}{\partial x^\mu} C_{\beta\gamma} + E_{\alpha\beta\gamma\delta} . \qquad \cdots (10d)$$

The terms quartic in θ give

$$\tau^\alpha [Q_\alpha , \tfrac{1}{24} \theta^\beta \theta^\gamma \theta^\delta \theta^\varepsilon E_{\beta\gamma\delta\varepsilon}] = -\tfrac{1}{12} \tau^\alpha \theta^\beta \theta^\gamma \theta^\delta \theta^\varepsilon (\gamma^\mu C)_{\alpha\varepsilon} \frac{\partial}{\partial x^\mu} D_{\beta\gamma\delta} .$$

So that

$$[Q_\alpha , E_{\beta\gamma\delta\varepsilon}] = -2 (\gamma^\mu C)_{\alpha\varepsilon} \frac{\partial}{\partial x^\mu} D_{\beta\gamma\delta} . \qquad \cdots (10e)$$

A check of the Jacobi identities now shows that these fields do form a closed supermultiplet (or at least the direct sum of several supermultiplets). But in order to satisfy these identities we need to make use of the symmetry properties of the component fields (ie. $C_{\alpha\beta} = -C_{\beta\alpha}$).

To overcome this we define the scalar fields $M(x)$, $N(x)$ and $G(x)$, the spinor fields $B'_\alpha(x)$ and $F_\alpha(x)$ and the vector field $V_\mu(x)$ by

$$B'_\alpha(x) = -B_\alpha(x), \qquad \cdots\cdots(11)$$

$$C_{\alpha\beta}(x) = \frac{i}{4}(C)_{\alpha\beta}M(x) - \frac{1}{4}(\gamma_5 C)_{\alpha\beta}N(x) - \frac{i}{2}(\gamma^\mu\gamma_5 C)_{\alpha\beta}V_\mu(x) \qquad \cdots\cdots(12)$$

$$D_{\alpha\beta\gamma}(x) = -\frac{i}{2}\left((C)_{\alpha\beta}F_\gamma(x) + (C)_{\beta\gamma}F_\alpha(x) + (C)_{\gamma\alpha}F_\beta(x)\right) \qquad \cdots\cdots(13)$$

and $$E_{\alpha\beta\gamma\delta}(x) = \frac{3}{4}\left((C)_{\alpha\beta}(C)_{\gamma\delta} + (C)_{\beta\gamma}(C)_{\delta\alpha}\right)G(x). \qquad \cdots\cdots(14)$$

Here the matrix $(C)_{\alpha\beta}$ is the charge conjugation matrix and the imaginary factors are chosen so that the coefficients are real in the Majorana representation as given in the Appendix. Also the coefficients of M, N and V_μ are chosen by use of the Jacobi identities (see later) and are not independent of each other.

Proposition 4.1.

The component fields A, B'_α, M, N, V_μ, F_α and G of the general scalar field defined by

$$\begin{aligned}\Phi(x,\theta) = {} & A(x) - \theta^\alpha B'_\alpha(x) + \frac{i}{8}(C)_{\alpha\beta}\theta^\alpha\theta^\beta M(x) - \frac{1}{8}(\gamma^5 C)_{\alpha\beta}\theta^\alpha\theta^\beta N(x) \\ & -\frac{i}{4}(\gamma^\mu\gamma_5 C)_{\alpha\beta}\theta^\alpha\theta^\beta V_\mu(x) \\ & +\frac{i}{12}\theta^\alpha\theta^\beta\theta^\gamma\left((C)_{\alpha\beta}F_\gamma(x) + (C)_{\beta\gamma}F_\alpha(x) + (C)_{\gamma\alpha}F_\beta(x)\right) \\ & +\frac{3}{96}\theta^\alpha\theta^\beta\theta^\gamma\theta^\delta\left((C)_{\alpha\beta}(C)_{\gamma\delta} + (C)_{\beta\gamma}(C)_{\delta\alpha}\right)G(x),\end{aligned} \qquad \cdots\cdots(15)$$

satisfy the commutators

$$[Q_\alpha, A] = B'_\alpha, \qquad \cdots\cdots(16a)$$

$$[Q_\alpha, B'_\beta] = \frac{1}{2}(\gamma^\mu C)_{\alpha\beta}\frac{\partial}{\partial x^\mu}A - \frac{i}{4}(C)_{\alpha\beta}M + \frac{1}{4}(\gamma_5 C)_{\alpha\beta}N + \frac{i}{2}(\gamma^\mu\gamma_5 C)_{\alpha\beta}V_\mu, \qquad \cdots\cdots(16b)$$

$$[Q_\alpha, M] = F_\alpha - i(\gamma^\mu)_{\alpha\beta}\frac{\partial}{\partial x^\mu}B'_\beta, \qquad \cdots\cdots(16c)$$

$$[Q_\alpha, N] = i(\gamma_5)_{\alpha\beta}F_\beta - (\gamma^\mu\gamma_5)_{\alpha\beta}\frac{\partial}{\partial x^\mu}B'_\beta, \qquad \cdots\cdots(16d)$$

$$[Q_\alpha, V_\mu] = -\frac{1}{2}(\gamma_\mu\gamma_5)_{\alpha\beta}F_\beta - \frac{i}{2}(\gamma^\rho\gamma_\mu\gamma_5)_{\alpha\beta}\frac{\partial}{\partial x^\rho}B'_\beta, \qquad \cdots\cdots(16e)$$

$$[Q_\alpha, F_\beta] = \frac{1}{4}(\gamma^\mu C)_{\alpha\beta}\frac{\partial}{\partial x^\mu}M - \frac{i}{4}(\gamma^\mu\gamma_5 C)_{\alpha\beta}\frac{\partial}{\partial x^\mu}N + \frac{1}{2}(\gamma^\rho\gamma^\mu\gamma_5 C)_{\alpha\beta}\frac{\partial}{\partial x^\rho}V_\mu + \frac{i}{2}(C)_{\alpha\beta}G \qquad \cdots\cdots(16f)$$

and $$[Q_\alpha, G] = i(\gamma^\mu)_{\alpha\beta}\frac{\partial}{\partial x^\mu}F_\beta \qquad \cdots\cdots(16g)$$

and form a closed supermultiplet.

Proof

We derive the commutators from those of equation (10).

Equation (10a) gives (16a) immediatly.

Equation (10b) with equations (11) and (12) gives equation (15b).

Now we rewrite equation (10c) with the definitions of equations (11) and (12) to give

$$[Q_\alpha, \tfrac{i}{4}(C)_{\beta\gamma}M - \tfrac{1}{4}(\gamma_5 C)_{\beta\gamma}N - \tfrac{i}{2}(\gamma^\mu\gamma_5 C)_{\beta\gamma}V_\mu]$$
$$= (\gamma^\mu C)_{\alpha\gamma}\frac{\partial}{\partial x^\mu}B'_\beta + \tfrac{i}{2}((C)_{\alpha\beta}F_\gamma + (C)_{\beta\gamma}F_\alpha + (C)_{\gamma\alpha}F_\beta),$$
$$= \tfrac{1}{4}(\Gamma^A\gamma^\mu C)_{\beta\gamma}\frac{\partial}{\partial x^\mu}(\Gamma_A B')_\alpha + \tfrac{i}{2}(\tfrac{1}{4}(\Gamma^B C)_{\gamma\beta}(\Gamma_B F)_\alpha$$
$$+ (C)_{\beta\gamma}F_\alpha - \tfrac{1}{4}(\Gamma^C C)_{\beta\gamma}(\Gamma_C F)_\alpha)$$

after performing a Fiertz rearrangement. Then

$$[Q_\alpha, \tfrac{i}{2}(C)_{\beta\gamma}M - \tfrac{1}{4}(\gamma_5 C)_{\beta\gamma}N - \tfrac{i}{2}(\gamma^\mu\gamma_5 C)_{\beta\gamma}V_\mu]$$
$$= \tfrac{1}{4}(\Gamma^A\gamma^\mu C)_{\beta\gamma}\frac{\partial}{\partial x^\mu}(\Gamma_A B)_\alpha + \tfrac{i}{2}(-\tfrac{1}{2}(\Gamma'^B C)_{\beta\gamma}(\Gamma'_B F)_\alpha$$
$$+ (C)_{\beta\gamma}F_\alpha)$$

with $(\Gamma'^B C)$ an antisymmetric matrix. Now we expand the matrices $(\Gamma^A\gamma^\mu C)$ and $(\Gamma'^B C)$ in terms of the 16 components $\{C, \gamma_5 C, \gamma^\mu C, \gamma^\mu\gamma_5 C, \gamma^\mu\gamma^\sigma C\}$ which form a basis of the set of 4×4 matrices and compare coefficients to obtain

$$[Q_\alpha, M] = -i(\gamma^\mu)_{\alpha\beta}\frac{\partial}{\partial x^\mu}B'_\beta + F_\alpha \qquad ,$$
$$[Q_\alpha, N] = -(\gamma^\mu\gamma_5)_{\alpha\beta}\frac{\partial}{\partial x^\mu}B'_\beta + i(\gamma_5)_{\alpha\beta}F_\beta \qquad ,$$

and $$[Q_\alpha, V_\mu] = -\tfrac{i}{2}(\gamma^\rho\gamma_\mu\gamma_5)_{\alpha\beta}\frac{\partial}{\partial x^\rho}B'_\beta - \tfrac{1}{2}(\gamma_\mu\gamma_5)_{\alpha\beta}F_\beta .$$

These are then equations (16c), (16d) and (16e).

Now we rewrite equation (10d) with the definitions given in equations (12), (13) and (14) to give

$$[Q_\alpha, -\tfrac{i}{2}((C)_{\beta\gamma}F_\delta + (C)_{\gamma\delta}F_\beta + (C)_{\delta\beta}F_\gamma)]$$
$$= -\tfrac{3}{2}(\gamma^\mu C)_{\alpha\delta}\frac{\partial}{\partial x^\mu}(\tfrac{i}{4}(C)_{\alpha\beta}M - \tfrac{1}{4}(\gamma_5 C)_{\alpha\beta}N - \tfrac{i}{2}(\gamma^\mu\gamma_5 C)_{\alpha\beta}V_\mu)$$
$$+ \tfrac{3}{4}((C)_{\alpha\beta}(C)_{\gamma\delta} + (C)_{\beta\gamma}(C)_{\delta\alpha})G$$

and by performing Fierz rearrangements on both sides we obtain equation (16f).

Similarly we obtain equation (16g).

A check of the Jacobi identities demonstrates that we have a closed supermultiplet.

The supermultiplet defined by equations (16) is the subject of many papers and was first given by Wess and Zumino [3]. It takes many forms depending on authors conventions and possible field redefinitions. Its form as a superfield was first given by Salam and Strathdee [9].

4.2 The General Scalar Superfield in Two Component Form.

In this section we take the general scalar superfild as given in Proposition 4.1 and decompose it into its irreducible supermultiplets. To do this it is convenient to work in the chiral representation of the Dirac matrices as given in the appendix. We also note the specification of these matrices in terms of the left handed and right handed Pauli matrices as given by equations (38) of the appendix. The scalar and vector fields remain as previously defined but the two spinor fields now decompose into left handed and right handed components ie.

$$B'_\alpha = \begin{bmatrix} B'_{Ln} \\ B'_{Rn'} \end{bmatrix} \qquad \cdots(17a)$$

and

$$F_\alpha = \begin{bmatrix} F_{Ln} \\ F_{Ln'} \end{bmatrix} \qquad \cdots(17b)$$

Proposition 4.2

In two component form the general scalar superfield takes the form

$$\begin{aligned}
&\Phi(x, \theta) \\
&= A(x) - \theta^{Ln} B'_{Ln}(x) - \theta^{Rn} B'_{Rn}(x) + \tfrac{i}{8}\, \theta^{Ln}\theta^{Ln'}(\sigma_2^L)_{nn'} M(x) \\
&\quad + \tfrac{i}{8}\, \theta^{Rn}\theta^{Rn'}(\sigma_2^R)_{nn'} M(x) - \tfrac{1}{8}\, \theta^{Ln}\theta^{Ln'}(\sigma_2^L)_{nn'} N(x) \\
&\quad + \tfrac{1}{8}\, \theta^{Rn}\theta^{Rn'}(\sigma_2^R)_{nn'} N(x) - \tfrac{i}{2}\, \theta^{Ln}(\sigma_\mu^R \sigma_2^R)_{nn'}\, \theta^{Rn'} V_\mu(x) \\
&\quad + \tfrac{i}{4}\, \theta^{Ln}\theta^{Ln'}(\sigma_2^L)_{nn'}\, \theta^{Rr} F_{Rr}(x) + \tfrac{i}{4}\, \theta^{Rn}\theta^{Rn'}(\sigma_2^R)_{nn'}\, \theta^{Lr} F_{Lr}(x)
\end{aligned}$$

$$+\frac{1}{16}\,\theta^{Ln}\theta^{Ln'}(\sigma_2^L)_{nn'}\,\theta^{Rr}\theta^{Rr'}(\sigma_2^R)_{rr'}\,G(x)\quad, \qquad \cdots(18)$$

and its component fields satisfy the following commutators

$$[Q_{Ln}, A] = B'_{Ln} \quad, \qquad \cdots(19a)$$

$$[Q_{Rn}, A] = B'_{Rn} \quad, \qquad \cdots(19b)$$

$$[Q_{Ln}, B'_{Ln'}] = \tfrac{1}{4}(\sigma_2^L)_{nn'}(N - iM) \quad, \qquad \cdots(19c)$$

$$[Q_{Ln}, B'_{Rn'}] = -\tfrac{1}{2}(\sigma_\mu^R\sigma_2^R)_{nn'}\frac{\partial}{\partial x^\mu}A + \tfrac{1}{2}i(\sigma_\mu^R\sigma_2^R)_{nn'}V_\mu \quad, \qquad \cdots(19d)$$

$$[Q_{Rn}, B'_{Ln'}] = -\tfrac{1}{2}(\sigma_\mu^L\sigma_2^L)_{nn'}\frac{\partial}{\partial x^\mu}A - \tfrac{1}{2}i(\sigma_\mu^L\sigma_2^L)_{nn'}V_\mu \quad, \qquad \cdots(19e)$$

$$[Q_{Rn}, B'_{Rn'}] = -\tfrac{1}{4}(\sigma_2^R)_{nn'}(N + iM) \quad, \qquad \cdots(19f)$$

$$[Q_{Ln}, M] = F_{Ln} + i(\sigma_\mu^R)_{nn'}\frac{\partial}{\partial x^\mu}B_{Rn'} \quad, \qquad \cdots(19g)$$

$$[Q_{Rn}, M] = F_{Rn} + i(\sigma_\mu^L)_{nn'}\frac{\partial}{\partial x^\mu}B_{Ln'} \quad, \qquad \cdots(19h)$$

$$[Q_{Ln}, N] = iF_{Ln} - (\sigma_\mu^R)_{nn'}\frac{\partial}{\partial x^\mu}B_{Rn'} \quad, \qquad \cdots(19j)$$

$$[Q_{Rn}, N] = -iF_{Rn} + (\sigma_\mu^L)_{nn'}\frac{\partial}{\partial x^\mu}B_{Ln'} \quad, \qquad \cdots(19k)$$

$$[Q_{Ln}, V_\mu] = -\tfrac{1}{2}(\sigma_\mu^L)_{nn'}F_{Rn'} - \tfrac{i}{2}(\sigma_\varrho^R\sigma_\mu^R)_{nn'}\frac{\partial}{\partial x^\varrho}B_{Ln'} \quad, \qquad \cdots(19l)$$

$$[Q_{Rn}, V_\mu] = \tfrac{1}{2}(\sigma_\mu^R)_{nn'}F_{Ln'} + \tfrac{i}{2}(\sigma_\varrho^L\sigma_\mu^L)_{nn'}\frac{\partial}{\partial x^\varrho}B_{Rn'} \quad, \qquad \cdots(19m)$$

$$[Q_{Ln}, F_{Ln'}] = \tfrac{1}{2}(\sigma_\mu^R\sigma_\varrho^L\sigma_2^L)_{nn'}\frac{\partial}{\partial x^\mu}V_\varrho + \tfrac{i}{2}(\sigma_\mu^L)_{nn'}G \quad, \qquad \cdots(19n)$$

$$[Q_{Ln}, F_{Rn'}] = -\tfrac{i}{4}(\sigma_\mu^R\sigma_2^R)_{nn'}\frac{\partial}{\partial x^\mu}(N - iM) \quad, \qquad \cdots(19p)$$

$$[Q_{Rn}, F_{Ln'}] = \tfrac{i}{4}(\sigma_\mu^L\sigma_2^L)_{nn'}\frac{\partial}{\partial x^\mu}(N + iM) \quad, \qquad \cdots(19q)$$

$$[Q_{Rn}, F_{Rn'}] = -\tfrac{1}{2}(\sigma_\mu^L\sigma_\varrho^R\sigma_2^R)_{nn'}\frac{\partial}{\partial x^\mu}V_\varrho + \tfrac{i}{2}(\sigma_2^R)_{nn'}G \quad, \qquad \cdots(19r)$$

$$[Q_{Ln}, G] = -i(\sigma_\mu^R)_{nn'}\frac{\partial}{\partial x^\mu}F_{Rn'} \qquad \cdots(19s)$$

and

$$[Q_{Rn}, G] = -i(\sigma_\mu^L)_{nn'}\frac{\partial}{\partial x^\mu}F_{Ln'}\,. \qquad \cdots(19t)$$

<u>Proof</u>

If we note the chiral Dirac matrices written in terms of the left and right handed Pauli matrices we can write down equation (18) immediately from equation (15).

Equations (19a) and (19b) also follow immediately from equation (16a). Now, from equation (16b) and the appendix we have

$$[Q_\alpha, B'_\beta]$$

$$= \tfrac{1}{2}\begin{bmatrix} 0 & -\sigma_\mu^R\sigma_2^R \\ -\sigma_\mu^L\sigma_2^L & 0 \end{bmatrix}_{\alpha\beta}\frac{\partial}{\partial x^\mu}A - \tfrac{i}{4}\begin{bmatrix} \sigma_2^L & 0 \\ 0 & \sigma_2^R \end{bmatrix}_{\alpha\beta}M$$

$$+\begin{bmatrix} \sigma_2^L & 0 \\ 0 & -\sigma_2^R \end{bmatrix}_{\alpha\beta} N + \frac{1}{2} i \begin{bmatrix} 0 & \sigma_\mu^R \sigma_2^R \\ -\sigma_\mu^L \sigma_2^L & 0 \end{bmatrix}_{\alpha\beta} \mathcal{V}_\mu .$$

Which gives equations (19c), (19d), (19e) and (19f). The remaining commutators are obtained in a similar way.

Proposition 4.3

The set of fields defined by

$$\overline{P} = \frac{1}{2}(N + iM) \quad , \qquad \cdots(20a)$$

$$\zeta_{Rr} = B'_{Rr} + i\left(\frac{\hbar}{Mc}\right)^2 (\sigma_e^L)_{rr'} \frac{\partial}{\partial x_e} F_{Lr'} \quad , \qquad \cdots(20b)$$

and

$$\overline{R} = -A + \left(\frac{\hbar}{Mc}\right)^2 G - 2i \frac{\partial}{\partial x_\mu} \mathcal{V}_\mu \left(\frac{\hbar}{Mc}\right)^2 \quad , \qquad \cdots(20c)$$

form an irreducible supermultiplet of the general scalar superfield.

Their commutators are

$$[Q_{Ln}, \overline{R}] = 0 \quad , \qquad \cdots(21a)$$

$$[Q_{Rn}, -\tfrac{1}{2}\overline{R}] = \zeta_{Rn} \quad , \qquad \cdots(21b)$$

$$[Q_{Ln}, \zeta_{Rr}] = (\sigma_\mu^R \sigma_2^R)_{nr} \frac{\partial}{\partial x_\mu} \left(\tfrac{1}{2}\overline{R}\right) \quad , \qquad \cdots(21c)$$

$$[Q_{Rn}, \zeta_{Rr}] = -(\sigma_2^R)_{nr} \overline{P} \quad , \qquad \cdots(21d)$$

$$[Q_{Ln}, -\overline{P}] = (\sigma_\mu^R)_{nr} \frac{\partial}{\partial x_\mu} \zeta_{Rr} \qquad \cdots(21e)$$

and

$$[Q_{Rn}, \overline{P}] = 0 \quad . \qquad \cdots(21f)$$

Proof

We first note that

$$\begin{aligned} &(\sigma_\mu^R)_{nr} \frac{\partial}{\partial x_\mu} \zeta_{Rr} \\ &= (\sigma_\mu^R)_{nr} \frac{\partial}{\partial x_\mu} \left(B_{Rr'} + i\left(\frac{\hbar}{Mc}\right)^2 (\sigma_e^L)_{rr'} \frac{\partial}{\partial x_e} F_{Lr'} \right) \quad , \\ &= (\sigma_\mu^R)_{nr} \frac{\partial}{\partial x_\mu} B'_{Rr} + i\left(\frac{\hbar}{Mc}\right)^2 (\sigma_\mu^R \sigma_e^L)_{nr} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_e} F_{Lr} \quad , \end{aligned}$$

so that using the fact that

$$(\sigma_\mu^R \sigma_e^L)_{nr} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_e} = \delta_{nr} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\mu}$$

and the Klein-Gordon equation we have

$$(\sigma_\mu^R)_{nr} \frac{\partial}{\partial x_\mu} \zeta_{Rr} = (\sigma_\mu^R)_{nr} \frac{\partial}{\partial x_\mu} B'_{Rr} - iF_{Lr} \quad . \qquad \cdots(22)$$

Now from equations (19h) and (19k) we have

$$[Q_{Rn}, \overline{P}] = 0 \quad ,$$

and from equations (19g) and (19j) we have, using equation (22)

$$[Q_{Ln}, \bar{P}] = iF_{Ln} - (\sigma_\mu^R)_{nr} \frac{\partial}{\partial x^\mu} B'_{Rr} \quad ,$$
$$= -(\sigma_\mu^R)_{nr} \frac{\partial}{\partial x^\mu} \zeta_{Rr} \,.$$

From equations (19f) and (19q) we have

$$[Q_{Rn}, \zeta_{Rr}]$$
$$= [Q_{Rn}\,,\ B'_{Rr} + i\left(\frac{\hbar}{Mc}\right)^2 (\sigma_\ell^L)_{rr'} \frac{\partial}{\partial x^\mu} F_{Lr'}] \quad ,$$
$$= -\frac{1}{4}(\sigma_\mu^R)_{nr}(N+iM) + i\left(\frac{\hbar}{Mc}\right)^2 (\sigma_\ell^L)_{ra} \frac{\partial}{\partial x_\ell} \frac{i}{4}(\sigma_\mu^L \sigma_2^L)_{na} \frac{\partial}{\partial x^\mu}(N+iM),$$
$$= -\frac{1}{2}(\sigma_2^R)_{nr}\bar{P} - \frac{1}{2}\left(\frac{\hbar}{Mc}\right)^2 (\sigma_\ell^L \sigma_\mu^R \sigma_2^R)_{rn} \frac{\partial}{\partial x_\ell} \frac{\partial}{\partial x^\mu} \bar{P}$$

where use has been made of equation (14) of the appendix. So that

$$[Q_{Rn}, \zeta_{Rr}] = (\sigma_2^R)_{nr}\bar{P} \,.$$

Now using equations (19d) and (19n) we have

$$[Q_{Ln}, \zeta_{Rr}]$$
$$= [Q_{Ln}, B'_{Rr} + i\left(\frac{\hbar}{Mc}\right)^2 (\sigma_\ell^L)_{rr'} \frac{\partial}{\partial x_\ell} F_{Lr'}] \,,$$
$$= -\frac{1}{2}(\sigma_\mu^R \sigma_2^R) \frac{\partial}{\partial x^\mu} A + \frac{1}{2}(\sigma_\mu^R \sigma_2^R)\, V_\mu$$
$$+ i\left(\frac{\hbar}{Mc}\right)^2 (\sigma_\ell^L)_{rn} \frac{\partial}{\partial x_\ell} \left\{ \frac{1}{2}(\sigma_\mu^R \sigma_\nu^L \sigma_2^L)_{na} \frac{\partial}{\partial x^\mu} V_\nu + \frac{i}{2}(\sigma_2^L)_{na} G \right\} ,$$
$$= -\frac{1}{2}(\sigma_\mu^R \sigma_2^R)_{nr} \frac{\partial}{\partial x^\mu} A + \frac{1}{2} i (\sigma_\mu^R \sigma_2^R)_{nr} V_\mu$$
$$+ \frac{1}{2}(\sigma_\ell^L \sigma_2^L)_{rn} \frac{\partial}{\partial x_\ell} G \left(\frac{\hbar}{Mc}\right)^2 + \frac{1}{2} i \left(\frac{\hbar}{Mc}\right)^2 (\sigma_\ell^L \sigma_\nu^R \sigma_\mu^L \sigma_2^R)_{rn} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\ell} V_\nu ,$$
$$= \frac{1}{2}(\sigma_\mu^R \sigma_2^R)_{nr} \left\{ -\frac{\partial}{\partial x^\mu} A + \left(\frac{\hbar}{Mc}\right)^2 \frac{\partial}{\partial x^\mu} G \right\} + \frac{1}{2} i (\sigma_\mu^R \sigma_2^R)_{nr} V_\mu$$
$$+ \frac{1}{2} i \left(\frac{\hbar}{Mc}\right)^2 ((\sigma_\ell^L (2g_{\nu\ell} - \sigma_\mu^R \sigma_\nu^L)\, \sigma_2^R)_{rn} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\ell} V_\nu \quad .$$

Here use has been made of equations (9) and (15) of the appendix. Now using the Klein-Gordon equation and equation (14) of the appendix we obtain

$$[Q_{Ln}, \zeta_{Rr}]$$
$$= \frac{1}{2}(\sigma_\mu^R \sigma_2^R)_{nr} \frac{\partial}{\partial x^\mu} \left\{ -A + \left(\frac{\hbar}{Mc}\right)^2 G - 2i\left(\frac{\hbar}{Mc}\right)^2 \frac{\partial}{\partial x_\ell} V_\ell \right\} ,$$

from which equation (19c) follows.

Now using equations (19a), (19ℓ) and (19s) we obtain

$$[Q_{Ln}, \bar{R}]$$
$$= [Q_{Ln}, (-A + \left(\frac{\hbar}{Mc}\right)^2 G - 2i\left(\frac{\hbar}{Mc}\right)^2 \frac{\partial}{\partial x_\ell} V_\ell)]$$

$$= -B'_{Ln} - \left(\frac{\hbar}{Mc}\right)^2 i(\sigma_\mu^R)_{nr}\frac{\partial}{\partial x^\mu}F_{Rr}$$
$$-2i\left(\frac{\hbar}{Mc}\right)^2\frac{\partial}{\partial x_\rho}g^{\rho\nu}\left\{-\tfrac{1}{2}(\sigma_\nu^L)_{nr}F_{Rr} - \tfrac{i}{2}(\sigma_\mu^R\sigma_\rho^R)_{nr}\frac{\partial}{\partial x^\mu}B'_{Lr}\right\} \quad ,$$
$$= -B'_{Ln} - i\left(\frac{\hbar}{Mc}\right)^2(\sigma_\mu^R)_{nr}\frac{\partial}{\partial x^\mu}F_{Rr} + i\left(\frac{\hbar}{Mc}\right)^2(\sigma_\mu^R)_{nr}\frac{\partial}{\partial x^\mu}F_{Rr} + B'_{Ln} \, ,$$
$$= 0 \, .$$

Here use has been made of the Klein-Gordon equation.

Lastly using equations (19b), (19m) and (19t) we obtain

$$[Q_{Rn}, \bar{R}]$$
$$= [Q_{Rn}, (-A + \left(\frac{\hbar}{Mc}\right)^2 G - 2i\frac{\partial}{\partial x_\rho}g^{\rho\nu}V_\nu\left(\frac{\hbar}{Mc}\right)^2)] \quad ,$$
$$= -B'_{Rn} - i\left(\frac{\hbar}{Mc}\right)^2(\sigma_\mu^L)_{nr}\frac{\partial}{\partial x^\mu}F_{Lr}$$
$$-2i\left(\frac{\hbar}{Mc}\right)^2\frac{\partial}{\partial x_\rho}g^{\rho\nu}\left\{\tfrac{1}{2}(\sigma_\rho^R)_{nr}F_{Lr} + \tfrac{i}{2}(\sigma_\mu^L\sigma_\rho^L)_{nr}\frac{\partial}{\partial x_\rho}B'_{Rr}\right\} \quad ,$$
$$= -2\left(B'_{Rr} + i\left(\frac{\hbar}{Mc}\right)^2(\sigma_\mu^L)_{nr}\frac{\partial}{\partial x^\mu}F_{Lr}\right) \, .$$

We can recognize the supermultiplet of Proposition 4.3 as a right handed $j = 0$ chiral supermultiplet. The fields are related to the fields of the chiral supermultiplet by choosing

$$\tfrac{1}{2}\bar{R} = \phi^R \quad . \qquad \cdots(23a)$$

Then
$$\zeta_{Rr} = e^{-i\frac{3\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\phi^R_{\frac{1}{2},r} \qquad \cdots(23b)$$

and
$$\bar{P} = -e^{i\frac{\pi}{2}}\left(\frac{Mc}{\hbar}\right)\underset{\sim}{\phi}^R \qquad \cdots(23c)$$

Thus we have the following relationships

$$\phi^R = \tfrac{1}{2}\left(-A + \left(\frac{\hbar}{Mc}\right)^2 G - 2i\frac{\partial}{\partial x_\rho}V_\rho\left(\frac{\hbar}{Mc}\right)^2\right) \quad , \qquad \cdots(24a)$$
$$\phi^R_{\frac{1}{2},r} = e^{i\frac{3\pi}{4}}\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}\left(B'_{Rr} + i\left(\frac{\hbar}{Mc}\right)^2(\sigma_\rho^L)_{rr'}\frac{\partial}{\partial x_\rho}F_{Lr'}\right) \qquad \cdots(24b)$$

and
$$\underset{\sim}{\phi}^R = \tfrac{1}{2}e^{i\frac{\pi}{2}}\left(\frac{\hbar}{Mc}\right)(N + iM) \quad . \qquad \cdots(24c)$$

Proposition 4.4

The set of fields defined by

$$P = \tfrac{1}{2}(N - iM) \quad , \qquad \cdots(25a)$$
$$\zeta_{Lr} = B'_{Lr} + i\left(\frac{\hbar}{Mc}\right)^2(\sigma_\mu^R)_{rr'}\frac{\partial}{\partial x^\mu}F_{Rr'} \qquad \cdots(25b)$$

and
$$R = -A + \left(\frac{\hbar}{Mc}\right)^2 G + 2i\left(\frac{\hbar}{Mc}\right)^2\frac{\partial}{\partial x_\rho}V_\rho \quad . \qquad \cdots(25c)$$

form an irreducible supermultiplet of the general scalar superfield.

Their commutation relations are

$$[Q_{Ln}, \tfrac{1}{2}R] = -\zeta_{Ln} \quad , \qquad \cdots(26a)$$

$$[Q_{Rn}, \tfrac{1}{2}R] = 0 \quad , \qquad \cdots(26b)$$

$$[Q_{Ln}, \zeta_{Lr}] = (\sigma_2^L)_{nr} P \quad , \qquad \cdots(26c)$$

$$[Q_{Rn}, \zeta_{Lr}] = (\sigma_\mu^L \sigma_2^L)_{nr} \frac{\partial}{\partial x_\mu} \tfrac{1}{2}R \quad , \qquad \cdots(26d)$$

$$[Q_{Ln}, P] = 0 \qquad \cdots(26e)$$

and

$$[Q_{Rn}, P] = (\sigma_e^L)_{nr} \frac{\partial}{\partial x_e} \zeta_{Lr} \quad . \qquad \cdots(26f)$$

Proof

This is essentially the same as the proof of Proposition 4.3, so we do not include it.

We can recognize the supermultiplet of Proposition 4.4 as a left handed chiral supermultiplet with $j = 0$. The fields here are related to the fields of the chiral supermultiplet by choosing

$$\tfrac{1}{2} R = \psi^L \quad . \qquad \cdots(27a)$$

Then

$$\zeta_{Lr} = e^{-i\frac{3\pi}{4}} \left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} \psi^L_{\frac{1}{2},r} \qquad \cdots(27b)$$

and

$$P = e^{i\frac{\pi}{2}} \left(\frac{Mc}{\hbar}\right) \underset{\sim}{\psi}^L \quad . \qquad \cdots(27c)$$

Thus we have the following relationships

$$\psi^L = \tfrac{1}{2}\left(-A + \left(\frac{\hbar}{Mc}\right)^2 G + 2i\left(\frac{\hbar}{Mc}\right)^2 \frac{\partial}{\partial x_e} V_e\right) \quad , \qquad \cdots(28a)$$

$$\psi^L_{\frac{1}{2},r} = e^{i\frac{3\pi}{4}} \left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}} \left(B'_{Lr} + i\left(\frac{\hbar}{Mc}\right)^2 (\sigma_e^R)_{rr'} \frac{\partial}{\partial x_e} F_{Rr'}\right) \qquad \cdots(28b)$$

and

$$\underset{\sim}{\psi}^L = \tfrac{1}{2} e^{i\frac{\pi}{2}} \left(\frac{\hbar}{Mc}\right)(iM - N) \quad . \qquad \cdots(28c)$$

Proposition 4.5

The set of fields defined by

$$\phi_{Ln} = B'_{Ln} - i\left(\frac{\hbar}{Mc}\right)^2 (\sigma_\mu^R)_{nr} \frac{\partial}{\partial x_\mu} F_{Rr} \quad , \qquad \cdots(29a)$$

$$\phi_{Rn} = B'_{Rn} - i\left(\frac{\hbar}{Mc}\right)^2 (\sigma_\mu^L)_{nr} \frac{\partial}{\partial x_\mu} F_{Lr} \quad , \qquad \cdots(29b)$$

$$A_\mu = V_\mu + \left(\frac{\hbar}{Mc}\right)^2 \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_e} V_e \qquad \cdots(29c)$$

and

$$K = A + \left(\frac{\hbar}{Mc}\right)^2 G \quad . \qquad \cdots(29d)$$

form an irreducible supermultiplet of the general scalar superfield.

Their commutation relations are

$$[Q_{Ln}, \phi_{Lr}] = 0 \quad , \qquad \cdots(30a)$$

$$[Q_{Rn}, \phi_{Lr}] = -\tfrac{1}{2}(\sigma_\mu^L \sigma_2^L)_{nr}\{\tfrac{\partial}{\partial x^\mu}K + 2iA_\mu\} \quad , \qquad \cdots(30b)$$

$$[Q_{Ln}, \phi_{Rr}] = -\tfrac{1}{2}(\sigma_\mu^R \sigma_2^R)_{nr}\{\tfrac{\partial}{\partial x^\mu}K + 2iA_\mu\} \quad , \qquad \cdots(30c)$$

$$[Q_{Rn}, \phi_{Rr}] = 0 \quad , \qquad \cdots(30d)$$

$$[Q_{Ln}, A_\mu] = -\tfrac{i}{2}\tfrac{\partial}{\partial x^\mu}\phi_{Ln} + \tfrac{i}{2}(\sigma_\mu^L \sigma_e^L)_{nr}\tfrac{\partial}{\partial x_e}\phi_{Lr} \quad , \qquad \cdots(30e)$$

$$[Q_{Rn}, A_\mu] = -\tfrac{i}{2}\tfrac{\partial}{\partial x^\mu}\phi_{Rn} + \tfrac{i}{2}(\sigma_e^L \sigma_\mu^L)_{nr}\tfrac{\partial}{\partial x_e}\phi_{Rr} \quad , \qquad \cdots(30f)$$

$$[Q_{Ln}, K] = \phi_{Ln} \qquad \cdots(30g)$$

and

$$[Q_{Rn}, K] = \phi_{Rn} \quad . \qquad \cdots(30h)$$

Proof

Using equations (19c) and (19p) we find

$$
\begin{aligned}
&[Q_{Ln}, \phi_{Lr}] \\
&= [Q_{Ln}, B'_{Lr} - i\left(\tfrac{\hbar}{Mc}\right)^2(\sigma_\mu^R)_{rr'}\tfrac{\partial}{\partial x^\mu}F_{Rr'}] \quad , \\
&= \tfrac{1}{4}(\sigma_2^L)_{nr}(N-iM) + i\left(\tfrac{\hbar}{Mc}\right)^2(\sigma_\mu^R)_{ra}\tfrac{\partial}{\partial x^\mu}\tfrac{i}{4}(\sigma_e^R\sigma_2^R)_{na}\tfrac{\partial}{\partial x_e}(N-iM) \quad , \\
&= \tfrac{1}{4}(\sigma_2^L)_{nr}(N-iM) - \tfrac{1}{4}(\sigma_\mu^R\sigma_e^L\sigma_2^L)_{rn}\left(\tfrac{\hbar}{Mc}\right)^2\tfrac{\partial}{\partial x^\mu}\tfrac{\partial}{\partial x_e}(N-iM) \quad ,
\end{aligned}
$$

where we have made use of equation (14) of the appendix. Now making use of the Klein-Gordon equation we find that

$$[Q_{Ln}, \phi_{Lr}] = 0 \, .$$

Now using equations (19e) and (19r) we find

$$
\begin{aligned}
&[Q_{Rn}, \phi_{Lr}] \\
&= [Q_{Rn}, B'_{Lr} - i\left(\tfrac{\hbar}{Mc}\right)^2(\sigma_\mu^R)_{rr'}\tfrac{\partial}{\partial x^\mu}F_{Rr'} \quad , \\
&= -\tfrac{1}{2}(\sigma_\mu^L\sigma_2^L)_{nr}\tfrac{\partial}{\partial x^\mu}A - \tfrac{1}{2}i(\sigma_e^L\sigma_2^L)V_e \\
&\quad - i\left(\tfrac{\hbar}{Mc}\right)^2(\sigma_\mu^R)_{ra}\tfrac{\partial}{\partial x^\mu}\{-\tfrac{1}{2}(\sigma_\nu^L\sigma_e^R\sigma_2^R)_{na}\tfrac{\partial}{\partial x^\nu}V_e + \tfrac{i}{2}(\sigma_2^R)_{na}G\} \quad , \\
&= -\tfrac{1}{2}(\sigma_\mu^L\sigma_2^L)_{nr}\tfrac{\partial}{\partial x^\mu}A - \tfrac{1}{2}i(\sigma_\mu^L\sigma_2^L)_{nr}V_\mu - \tfrac{1}{2}\left(\tfrac{\hbar}{Mc}\right)^2(\sigma_\mu^R\sigma_2^R)_{rn}G \\
&\quad + \tfrac{i}{2}\left(\tfrac{\hbar}{Mc}\right)^2(\sigma_\mu^R\sigma_e^L\sigma_\nu^R\sigma_2^L)_{rn}\tfrac{\partial}{\partial x^\mu}\tfrac{\partial}{\partial x^\nu}V_e \quad , \\
&= -\tfrac{1}{2}(\sigma_\mu^L\sigma_2^L)_{nr}\tfrac{\partial}{\partial y^\mu}\{A + \left(\tfrac{\hbar}{Mc}\right)^2 G\} - \tfrac{1}{2}i(\sigma_\mu^L\sigma_2^L)_{nr}V_\mu \\
&\quad - \tfrac{i}{2}\left(\tfrac{\hbar}{Mc}\right)^2((\sigma_\mu^R)(2g_{e\nu} - \sigma_\nu^L\sigma_e^R)(\sigma_2^R))_{rn}\tfrac{\partial}{\partial x^\mu}\tfrac{\partial}{\partial x^\nu}V_e \, .
\end{aligned}
$$

Here we have made use of equations (9) and (15) of the appendix. Then making use of the Klein-Gordon equation we have

$$[Q_{Rn}, \phi_{Lr}]$$
$$= -\tfrac{1}{2}(\sigma_\mu^L \sigma_2^L)_{nr}\left\{\frac{\partial}{\partial x_\mu}\left(A + \left(\frac{\hbar}{Mc}\right)^2 G + 2i\left(\frac{\hbar}{Mc}\right)^2 \frac{\partial}{\partial x_e} V_e + 2i V_\mu\right\}$$

from which equation (30b) follows.

Now using equations (19f) and (19q) we have

$$[Q_{Rn}, \phi_{Rr}]$$
$$= [Q_{Rn}, B'_{Rr} - i\left(\frac{\hbar}{Mc}\right)^2 (\sigma_e^L)_{rr'} \frac{\partial}{\partial x_e} F_{Lr'}] ,$$
$$= -\tfrac{1}{4}(\sigma_2^R)(N - iM) + \tfrac{1}{4}(\sigma_2^R)_{nr}(N - iM) .$$

So that

$$[Q_{Rn}, \phi_{Rr}] = 0 .$$

Now using equations (19d) and (19n) we have

$$[Q_{Ln}, \phi_{Rr}] =$$
$$= [Q_{Ln}, B'_{Rr} - i\left(\frac{\hbar}{Mc}\right)^2 (\sigma_e^L)_{rr'} \frac{\partial}{\partial x_e} F_{Lr'}] ,$$
$$= -\tfrac{1}{2}(\sigma_\mu^R \sigma_2^R)_{nr} \frac{\partial}{\partial x_\mu} A + \tfrac{1}{2} i (\sigma_\mu^R \sigma_2^R)_{nr} V_\mu$$
$$- i\left(\frac{\hbar}{Mc}\right)^2 (\sigma_e^L)_{ra} \frac{\partial}{\partial x_e}\left\{\tfrac{1}{2}(\sigma_\mu^R \sigma_\nu^L \sigma_2^L)_{na} \frac{\partial}{\partial x_\mu} V_\nu + \tfrac{i}{2}(\sigma_2^L)_{na} G\right\} .$$

Then using equations (9) and (15) of the appendix and the Klein-Gordon equation we have

$$[Q_{Ln}, \phi_{Rr}]$$
$$= -\tfrac{1}{2}(\sigma_\mu^R \sigma_2^R)_{nr}\left\{\frac{\partial}{\partial x_\mu}\left\{A + \left(\frac{\hbar}{Mc}\right)^2 G - 2i\left(\frac{\hbar}{Mc}\right)^2 \frac{\partial}{\partial x_e} V_e - 2i V_\mu\right\}$$

from which equation (30c) follows.

Now using equations (19a) and (19s) we have

$$[Q_{Ln}, \pi]$$
$$= [Q_{Ln}, A + \left(\frac{\hbar}{Mc}\right)^2 G]$$
$$= B'_{Ln} - i\left(\frac{\hbar}{Mc}\right)^2 (\sigma_e^R)_{nn'} \frac{\partial}{\partial x_e} F_{Rn'}$$

which gives equation (30g).

Now using equations (19b) and (19t) we have

$$
\begin{aligned}
&[Q_{Rn}, \mathcal{K}] \\
&\quad = [Q_{Rn}, A + (\tfrac{\hbar}{Mc})^2 G] \\
&\quad = B'_{Rn} - i(\tfrac{\hbar}{Mc})^2 (\sigma_e^L)_{nn'} \tfrac{\partial}{\partial x_e} F_{Rn'} \quad ,
\end{aligned}
$$

which gives equation (30h).

Finally using equation (191) we have

$$
\begin{aligned}
&[Q_{Ln}, A_\mu] \\
&\quad = [Q_{Ln}, V_\mu + (\tfrac{\hbar}{Mc})^2 \tfrac{\partial}{\partial x_\mu}\tfrac{\partial}{\partial x_e} V_e] \\
&\quad = -\tfrac{1}{2}(\sigma_\mu^L)_{nr} F_{Rr} - \tfrac{i}{2}(\sigma_e^R \sigma_\mu^R)_{nr} \tfrac{\partial}{\partial x_e} B'_{Lr} \\
&\quad + (\tfrac{\hbar}{Mc})^2 \tfrac{\partial}{\partial x_\mu}\tfrac{\partial}{\partial x_e} \{ -\tfrac{1}{2}(\sigma_e^L)_{nr} F_{Rr} - \tfrac{i}{2}(\sigma_\xi^R \sigma_e^R)\tfrac{\partial}{\partial x_\xi} B'_{Lr} \} \quad , \\
&\quad = -\tfrac{1}{2}(\sigma_\mu^L)_{nr} F_{Rr} - \tfrac{i}{2}(\sigma_e^R \sigma_\mu^R)_{nr} \tfrac{\partial}{\partial x_e} B'_{Lr} \\
&\qquad -\tfrac{1}{2}(\tfrac{\hbar}{Mc})^2 (\sigma_e^R)_{nr} \tfrac{\partial}{\partial x_\mu}\tfrac{\partial}{\partial x_e} F_{Rr} + \tfrac{i}{2}\tfrac{\partial}{\partial x_\mu} B'_{Ln} \quad .
\end{aligned}
$$

Here use has been nade of the Klein-Gordon equation. Now making use of equation (8) of the appendix we find that

$$
\begin{aligned}
&[Q_{Ln}, A_\mu] \\
&\quad = -\tfrac{i}{2}\tfrac{\partial}{\partial x_\mu}(B'_{Ln} - i(\tfrac{\hbar}{Mc})^2 (\sigma_e^R)_{nr} \tfrac{\partial}{\partial x_e} F_{Rr}) \\
&\qquad -\tfrac{1}{2}((\sigma_\mu^L)_{nr} F_{Rr} - i(\sigma_\mu^R \sigma_e^R)_{nr} \tfrac{\partial}{\partial y_e} B'_{Lr}), \\
&\quad = -\tfrac{i}{2}\tfrac{\partial}{\partial x_\mu}\phi_{Ln} + \tfrac{i}{2}(\sigma_\mu^L)_{na}(i F_{Rn} + (\sigma_e^R)_{ab} \tfrac{\partial}{\partial x_e} B'_{Lb}), \\
&\quad = -\tfrac{i}{2}\tfrac{\partial}{\partial x_\mu}\phi_{Ln} + \tfrac{i}{2}(\sigma_\mu^L \sigma_e^L)_{na} \tfrac{\partial}{\partial x_e}\phi_{La} \quad ,
\end{aligned}
$$

which is equation (30e).

Equation (30f) is derived in a similar way.

We can identify the supermultiplet defined by Proposition 4.5 with the supermultiplet constructed in section 4.5 of Chapter 4 of Part II and given in equation (92), with $k = 0$. But we cannot equate the fields, we must first construct super Lorentz invariant fields from the fields of Part II. Thus, by reference to equations (49) and (50) of Chapter 2, we define the fields $\psi^{RL}_{\frac{1}{2},n;k,m}(x)$, $\underset{\sim}{\psi}^{LL}_{\frac{1}{2},n;k,m}(x)$, $\mathcal{D}^L_{k,m}(x)$ and $\mathcal{J}^L_{\mu;j,m}(x)$ by

$$
\psi^{RL}_{\frac{1}{2},n;k,m}(x) = \chi^{RL\,\xi_0}_{\frac{1}{2},n;k,m}(x_0) + \sum_{j=1}^{N-1} x_j^\mu \frac{\partial}{\partial x_0^\mu}\chi^{RL\,\xi_j}_{\frac{1}{2},n;k,m}(x_0) + \cdots \quad , \qquad \cdots(31a)
$$

$$\underset{\sim}{\psi}^{LL}_{\frac{1}{2},n;k,m}(x) = \underset{\sim}{\chi}^{LL\varepsilon_0}_{\frac{1}{2},n;k,m}(x_0) + \sum_{j=1}^{N-1} x^\mu_j \frac{\partial}{\partial x^\mu_0} \chi^{RL\varepsilon_j}_{\frac{1}{2},n;k,m}(x_0) + \cdots, \quad \cdots(31b)$$

$$\mathcal{D}^L_{k,m}(x) = D^{L\varepsilon_0}_{k,m}(x_0) + \sum_{j=1}^{N-1} x^\mu_j \frac{\partial}{\partial x^\mu_0} D^{L\varepsilon_j}_{k,m}(x_0) + \cdots \quad \cdots(31c)$$

and $$\mathcal{J}^L_{\mu;k,m}(x) = J^{L\varepsilon_0}_{\mu;k,m}(x_0) + \sum_{j=1}^{N-1} x^\mu_j \frac{\partial}{\partial x^\mu_0} J^{L\varepsilon_j}_{\mu;k,m}(x_0) + \cdots. \quad \cdots(31d)$$

Since we are considering the general scalar superfield we set $k = 0$. These fields then obey the same algebra as the fields of equation (92) of Chapter 4 of Part II.

Now the explicit relationship between the fields is obtained by setting

$$\phi_{Ln} = e^{i\frac{3\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} \underset{\sim}{\psi}^{LL}_{\frac{1}{2},n;0,0} \quad \cdots(32a)$$

Then
$$\phi_{Rn} = e^{-i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}} \psi^{RL}_{\frac{1}{2},n;0,0}, \quad \cdots(32b)$$

$$\kappa = \mathcal{D}^L_{0,0} \quad \cdots(32c)$$

and
$$A_\mu = \frac{1}{2}\left(\frac{Mc}{\hbar}\right) e^{-i\frac{\pi}{2}} \mathcal{J}^L_{\mu;0,0} \quad \cdots(32d)$$

Thus in terms of our original fields we have

$$\underset{\sim}{\psi}^{LL}_{\frac{1}{2},r;0,0} = e^{-i\frac{3\pi}{4}}\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}\left(B'_{Lr} - i\left(\frac{\hbar}{Mc}\right)^2(\sigma^R_\mu)_{rr'}\frac{\partial}{\partial x^\mu}F_{Rr'}\right) \quad \cdots(33a)$$

$$\psi^{RL}_{\frac{1}{2},r;0,0} = e^{i\frac{\pi}{4}}\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}\left(B'_{Rr} - i\left(\frac{\hbar}{Mc}\right)^2(\sigma^L_\mu)_{rr'}\frac{\partial}{\partial x^\mu}F_{Lr'}\right) \quad \cdots(33b)$$

$$\mathcal{D}^L_{0,0} = A + \left(\frac{\hbar}{Mc}\right)^2 G \quad \cdots(33c)$$

and
$$\mathcal{J}^L_{\mu;0,0} = 2\left(\frac{\hbar}{Mc}\right)e^{i\frac{\pi}{2}}\left(v_\mu - \frac{i}{2}\frac{\partial}{\partial x^\mu}(\psi^L - \psi^R)\right) \quad \cdots(33d)$$

Here we have noted that from equations (24a) and (28a)

$$\frac{\partial}{\partial x_\rho} v_\rho = -\frac{i}{2}\left(\frac{Mc}{\hbar}\right)^2(\psi^L - \psi^R)$$

Now consider equations (24), (28) and (33) and invert these expressions to obtain the fields of the general scalar superfield in terms of the component superfields. We obtain

$$A = \frac{1}{2}\left(\mathcal{D} - (\psi^L + \psi^R)\right), \quad \cdots(34a)$$

$$B'_{Lr} = \frac{1}{2}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\left(e^{-i\frac{3\pi}{4}} \psi^L_{\frac{1}{2},r} + e^{i\frac{3\pi}{4}} \underset{\sim}{\psi}^{LL}_{\frac{1}{2},r}\right), \quad \cdots(34b)$$

$$B'_{Rr} = \frac{1}{2}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\left(e^{-i\frac{3\pi}{4}} \psi^R_{\frac{1}{2},r} + e^{-i\frac{\pi}{4}} \psi^{RL}_{\frac{1}{2},r}\right), \quad \cdots(34c)$$

$$v_\mu = \frac{1}{2}\left(\left(\frac{Mc}{\hbar}\right)e^{-i\frac{\pi}{2}}\mathcal{J}_\mu - \frac{\partial}{\partial x^\mu}e^{-i\frac{\pi}{2}}(\psi^L - \psi^R)\right), \quad \cdots(34d)$$

$$M = e^{i\pi}\left(\frac{Mc}{\hbar}\right)\left(\underset{\sim}{\psi}^L + \underset{\sim}{\psi}^R\right), \quad \cdots(34e)$$

$$N = e^{i\frac{\pi}{2}}\left(\frac{Mc}{\hbar}\right)\left(\underset{\sim}{\psi}^{L} - \underset{\sim}{\psi}^{R}\right) \quad , \qquad \cdots(34f)$$

$$F_{Lr} = \frac{1}{2}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}(\sigma_{\ell}^{R})_{rn}\frac{\partial}{\partial x_{\ell}}\left(e^{-i\frac{\pi}{4}}\psi^{R}_{\frac{1}{2},n} + e^{i\frac{\pi}{4}}\psi^{RL}_{\frac{1}{2},n}\right) \quad , \qquad \cdots(34g)$$

$$F_{Rr} = \frac{1}{2}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}(\sigma_{\ell}^{L})_{rn}\frac{\partial}{\partial x_{\ell}}\left(e^{-i\frac{\pi}{4}}\psi^{L}_{\frac{1}{2},n} + e^{i\frac{\pi}{4}}\underset{\sim}{\psi}^{LL}_{\frac{1}{2},n}\right) \qquad \cdots(34h)$$

and
$$G = \frac{1}{2}\left(\frac{Mc}{\hbar}\right)^{2}\left(\mathcal{D} + (\psi^{L} + \psi^{R})\right) . \qquad \cdots(34j)$$

We can now rewrite the general scalar superfield of equation (18) in term terms of these fields and then decompose it into its three constituent supermultiplets. Thus

$$\begin{aligned}
&\Phi(x, \theta^{L}, \theta^{R}) \\
&= \tfrac{1}{2}\mathcal{D} - \tfrac{1}{2}\psi^{L} - \tfrac{1}{2}\psi^{R} - \tfrac{1}{2}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{Ln}\left(e^{-i\frac{3\pi}{4}}\psi^{L}_{\frac{1}{2},n} + e^{i\frac{3\pi}{4}}\underset{\sim}{\psi}^{LL}_{\frac{1}{2},n}\right) \\
&\quad - \tfrac{1}{2}\theta^{Rn}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\left(e^{-i\frac{3\pi}{4}}\psi^{R}_{\frac{1}{2},n} + e^{-i\frac{\pi}{4}}\psi^{RL}_{\frac{1}{2},n}\right) + \tfrac{i}{8}\theta^{Ln}\theta^{Ln'}(\sigma_2^{L})_{nn'}e^{i\pi}\left(\tfrac{Mc}{\hbar}\right)\left(\underset{\sim}{\psi}^{L} + \underset{\sim}{\psi}^{R}\right) \\
&\quad + \tfrac{i}{8}\theta^{Rn}\theta^{Rn'}(\sigma_2^{R})_{nn'}e^{i\pi}\left(\tfrac{Mc}{\hbar}\right)\left(\underset{\sim}{\psi}^{L} + \underset{\sim}{\psi}^{R}\right) - \tfrac{1}{8}\theta^{Ln}\theta^{Ln'}(\sigma_2^{L})_{nn'}e^{i\frac{\pi}{2}}\left(\tfrac{Mc}{\hbar}\right)\left(\underset{\sim}{\psi}^{L} - \underset{\sim}{\psi}^{R}\right) \\
&\quad + \tfrac{1}{8}\theta^{Rn}\theta^{Rn'}(\sigma_2^{R})_{nn'}e^{i\frac{\pi}{2}}\left(\tfrac{Mc}{\hbar}\right)\left(\underset{\sim}{\psi}^{L} - \underset{\sim}{\psi}^{R}\right) \\
&\quad - \tfrac{i}{2}\theta^{Ln}(\sigma_\mu^{R}\sigma_2^{R})_{nn'}\theta^{Rn'}\left(\tfrac{1}{2}\left(\tfrac{Mc}{\hbar}\right)e^{-i\frac{\pi}{2}}J_\mu - \tfrac{1}{2}\tfrac{\partial}{\partial x_\mu}e^{-i\frac{\pi}{2}}(\psi^{L} - \psi^{R})\right) \\
&\quad + \tfrac{i}{4}\theta^{Ln}(\sigma_2^{L})_{nn'}\theta^{Ln'}\theta^{Rr}\tfrac{1}{2}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}(\sigma_\ell^{L})_{rn''}\tfrac{\partial}{\partial x_\ell}\left(e^{-i\frac{\pi}{4}}\psi^{L}_{\frac{1}{2},n''} + e^{i\frac{\pi}{4}}\underset{\sim}{\psi}^{LL}_{\frac{1}{2},n''}\right) \\
&\quad + \tfrac{i}{4}\theta^{Rn}(\sigma_2^{R})_{nn'}\theta^{Rn'}\theta^{Lr}\tfrac{1}{2}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}(\sigma_\ell^{R})_{rn''}\tfrac{\partial}{\partial x_\ell}\left(e^{-i\frac{\pi}{4}}\psi^{R}_{\frac{1}{2},n''} + e^{-i\frac{\pi}{4}}\psi^{RL}_{\frac{1}{2},n''}\right) \\
&\quad + \tfrac{1}{16}\theta^{Ln}\theta^{Ln'}(\sigma_2^{L})_{nn'}\theta^{Rr}\theta^{Rr'}(\sigma_2^{R})_{rr'}\tfrac{1}{2}\left(\tfrac{Mc}{\hbar}\right)\left(D + (\psi^{L} + \psi^{R})\right) \quad ,
\end{aligned}$$

$$\begin{aligned}
&= -\tfrac{1}{2}\Big\{\psi^{L} - \tfrac{1}{2}\theta^{Rn}(\sigma_\mu^{L}\sigma_2^{L})_{nn'}\theta^{Ln'}\tfrac{\partial}{\partial x_\mu}\psi^{L} \\
&\quad - \tfrac{1}{16}\theta^{Ln}(\sigma_2^{L})_{nn'}\theta^{Ln}\theta^{Rr}(\sigma_2^{R})_{rr'}\theta^{Rr'}\left(\tfrac{Mc}{\hbar}\right)^{2}\psi^{L} - \theta^{Ln}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}e^{i\frac{\pi}{4}}\psi^{L}_{\frac{1}{2},n} \\
&\quad - \tfrac{1}{4}e^{i\frac{\pi}{4}}\theta^{Ln}(\sigma_2^{L})_{nn'}\theta^{Ln'}\theta^{Rr}(\sigma_\ell^{L})_{rr'}\tfrac{\partial}{\partial x_\ell}\psi^{L}_{\frac{1}{2},r'}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}} \\
&\quad - \tfrac{1}{2}e^{-i\frac{\pi}{2}}\theta^{Ln}(\sigma_2^{L})_{nn'}\theta^{Ln'}\left(\tfrac{Mc}{\hbar}\right)\underset{\sim}{\psi}^{L}\Big\} \\
&\quad -\tfrac{1}{2}\Big\{\psi^{R} - \tfrac{1}{2}\theta^{Ln}(\sigma_\mu^{R}\sigma_2^{R})_{nn'}\theta^{Rn'}\tfrac{\partial}{\partial x_\mu}\psi^{R} \\
&\quad - \tfrac{1}{16}\theta^{Rn}(\sigma_2^{R})_{nn'}\theta^{Rn'}\theta^{Lr}(\sigma_2^{L})_{rr'}\theta^{Lr'}\left(\tfrac{Mc}{\hbar}\right)^{2}\psi^{L} - \theta^{Rn}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}e^{i\frac{\pi}{4}}\psi^{R}_{\frac{1}{2},n} \\
&\quad - \tfrac{1}{4}e^{i\frac{\pi}{4}}\theta^{Rn}(\sigma_2^{R})_{nn'}\theta^{Rn'}\theta^{Lr}(\sigma_\ell^{R})_{rr'}\tfrac{\partial}{\partial x_\ell}\psi^{R}_{\frac{1}{2},r'}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}} \\
&\quad - \tfrac{1}{2}e^{-i\frac{\pi}{2}}\theta^{Rn}(\sigma_2^{R})_{nn'}\theta^{Rn'}\left(\tfrac{Mc}{\hbar}\right)\underset{\sim}{\psi}^{R}\Big\} \\
&\quad + \tfrac{1}{2}\Big\{\mathcal{D} + \tfrac{1}{16}\theta^{Ln}(\sigma_2^{L})_{nn'}\theta^{Ln'}\theta^{Rr}(\sigma_2^{R})_{rr'}\theta^{Rr'}\left(\tfrac{Mc}{\hbar}\right)\mathcal{D} \\
&\quad - \theta^{Ln}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}e^{i\frac{3\pi}{4}}\underset{\sim}{\psi}^{LL}_{\frac{1}{2},n} + \tfrac{1}{4}e^{i\frac{3\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{Ln}(\sigma_2^{L})_{nn'}\theta^{Ln'}\theta^{Rr}(\sigma_\ell^{L})_{rr'}\tfrac{\partial}{\partial x_\ell}\underset{\sim}{\psi}^{LL}_{\frac{1}{2},r'} \\
&\quad - \theta^{Rn}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}e^{-i\frac{\pi}{4}}\psi^{RL}_{\frac{1}{2},n} + \tfrac{1}{4}e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{Rn}(\sigma_2^{R})_{nn'}\theta^{Rn'}\theta^{Lr}(\sigma_\ell^{R})_{rr'}\tfrac{\partial}{\partial x_\ell}\psi^{RL}_{\frac{1}{2},r'} \\
&\quad - \tfrac{1}{2}\theta^{Ln}(\sigma_\mu^{R}\sigma_2^{R})_{nn'}\theta^{Rn'}\left(\tfrac{Mc}{\hbar}\right)J_\mu\Big\} . \qquad \cdots(35)
\end{aligned}$$

The first term in this expansion can be identified with a left handed scalar chiral supermultiplet as constructed in Chapter 3, and given in equation (41). The second term can be recognized as a right handed chiral scalar supermultiplet. The third term is different from either of the chiral supermultiplets, since this is a scalar superfield it can be constructed from a $j=\frac{1}{2}$ chiral supermultiplet. We will call this superfield an intermediate superfield.

We can deduce from this that the general scalar superfield consists of three irreducible supermultiplets ie. one left handed chiral $j=0$ supermultiplet, one right handed chiral $j=0$ supermultiplet and one $j=\frac{1}{2}$ chiral supermultiplet which can be taken to be left handed or right handed. Thus there are $(4+4+8)=16$ linearly independent field components underlying the superfield corresponding to the 16 real (Grassman) components of the superfield.

4.3 The General Superfield.

4.3.1 Left Handed and Right Handed superfields.

We call a superfield as defined by equation (1) left handed if it transforms as

$$\begin{aligned} &O([\Lambda|t|\tau])\,\Phi^{L}_{j,m}(x,\theta)\,O([\Lambda|t|\tau])^{-1} \\ &\quad = \Gamma^{0,j}(\Lambda^{-1})_{mm'}\,\Phi^{L}_{j,m}\big(\Lambda x + t + \tfrac{1}{2}\tau^{\alpha}(\gamma^{\mu}C)_{\alpha\beta}\theta^{\beta}, \\ &\qquad\qquad \theta^{\alpha'}\Gamma^{M}(\Lambda^{-1})_{\alpha'}{}^{\alpha} - \tau^{\alpha}\big) \end{aligned} \qquad \cdots(36)$$

and we call it right handed if $\Gamma^{0,j}$ is replaced by $\Gamma^{j,0}$. So that we can have left or right handed chiral (left or right handed!) superfields.

To evaluate the field content, and the decomposition, of these superfields we do not need to do any additional work. We just need to note that all that happens is the addition of an index j,m to each of the field components. In this case since $j\neq 0$ the general superfield

$\Phi^{\Lambda}_{j,m}(x,\theta)$ consists of four irreducible supermultiplets with superspin j, j, $j+\frac{1}{2}$ and $j-\frac{1}{2}$, the last two being combined into the fields that comprise the intermediate superfield of equation (35).

4.3.2 Fields Transforming as a General Representation of the Super Lorentz Group.

Any general superfield can be decomposed into the direct sum of superfields that transform as $\Gamma^{j,j}(\Lambda^{-1})$ for $j, j' = 0, \frac{1}{2}, 1, \cdots$ so that it is sufficient to consider fields of this type. We observe that

$$\Gamma^{j,j'}(\Lambda^{-1}) = \Gamma^{j,0}(\Lambda^{-1}) \otimes \Gamma^{0,j'}(\Lambda^{-1}),$$

so that we can consider a superfield transforming by this representation to decompose in a similar way ie.

$$\begin{aligned} &\Phi^{RL}_{j,m;\,j',m'}(x,\theta) \\ &= \Phi^{R}_{j,m}(x,\theta)\,\Phi^{L}_{j',m'}(x,\theta)\,. \end{aligned}$$

Thus we can deduce the number of independent supermultiplets that comprise such a superfield, but to find precise relationships we would need to carry out a case by case study.

CHAPTER 5

LAGRANGIAN FUNCTIONS FOR SUPERFIELDS.

5.1 Super Poincare Invariant Functions on Superspace.

The aim of this chapter is to discover how to extract a Lagrangian function for the component fields from a superspace representation. The result for the $j=0$ supermultiplet is of course known. The Lagrangian for this was given in section 4.4 of Chapter 4 of Part II. The Lagrangian can be extracted from the superfield by the normal prescription used by Physicists and claimed to be integration on superspace. But this is an ad-hoc procedure since there is no justification for taking the $\theta\theta$ term of $(\Phi^L\Phi^L + \Phi^R\Phi^R)$ as the 'mass' term and the $\theta\theta\theta\theta$ term of $\Phi^L\Phi^R$ for the 'kinetic energy' term, other than they supply the expected result.

We want to use the theory we have developed so far in this Thesis to deduce, so far as is possible, the result. In this section we consider action integrals on superspace in general.

We assume that the Lagrangian density is constructed from our superfield in such a way that it is a scalar function

$$\mathcal{L}_{SS} : E_L^{4,4} \to E_L \qquad \cdots(1)$$

and define the action integral $\mathcal{A}_{SS}$ by

$$\mathcal{A}_{SS} = \int_V \hat{d}^4x\, \hat{d}^4\theta \{\mathcal{L}_{SS}(x,\theta)\} . \qquad \cdots(2)$$

Here V is some volume in superspace such that $\mu(V) \neq 0$ ie. it is not a null set. In this chapter we use a different terminology to that of Chapter 3 of Part I. There all integrals were over all of superspace, here we want to consider the function $\mathcal{L}_{SS}$ defined over all superspace and the action as the integral over some arbitrary volume in the space. To write equation (2) in the terminology of Chapter 3 of Part I we would define

$$\mathcal{L}'_{SS}(x,\theta) = \begin{cases} \mathcal{L}_{SS}(x,\theta) & \text{if } (x,\theta) \in V , \\ 0 & \text{otherwise} \end{cases}$$

then

$$\int_V \hat{d}^4x\, \hat{d}^4\theta\, \{\mathcal{L}_{ss}\} = \int \hat{d}^4x\, \hat{d}^4\theta\, \{\mathcal{L}'_{ss}\}\,.$$

We observe that since $\mathcal{L}_{SS}$ is a scalar function $\mathcal{A}_{SS}$ is invariant.

Now $\mathcal{A}_{SS}$ will take values in E_L and is evaluated using the prescription given by equations (3) and (4) of Chapter 3 of Part I. It thus can be considered, under the action of the Lie group G_L equivalent to G = (super Poincaré group), to be a vector with many components, which form the basis of a representation of G_L. One can think of the action as on a vector with $2\mathcal{N}$ components corresponding to the projections, as defined by equations (3) and (4) of Chapter 3 of Part I. But it is advantageous to decompose in a different way, which gives considerably more components. These components then form the representation space for a representation of G_L, which will in general be reducible. We can anticipate that it will reduce into a direct sum of reducible, but not completely reducible representation spaces.

Any candidate for the action integral on Minkowski space must be a one dimensional space, so that we are looking for one dimensional representation spaces of G_L contained in $\mathcal{L}_{SS}$, or equivalently $\mathcal{A}_{SS}$.

Now $\mathcal{L}_{SS}$ must be expressible as a scalar superfield so that it will admit a decomposition into the sum of a chiral left handed superfield, a chiral right handed superfield and an intermediate superfield, as described in Chapter 4. It is thus sufficient to determine the one dimensional representation spaces that can be constructed for each of these possibilities. We choose to use the real Grassman basis (x, θ) throughout this chapter. The left handed (y^L, θ) and right handed (y^R, θ) bases are not related to the real basis by a group transformation, so that we avoid them. They are related by a transformation from the complexification of the super Poincaré group which is the reason that they supply the 'correct' result.

Now let

$$\mathcal{L}_{SS}(x,\theta) = A + \theta^{\alpha}B_{\alpha} + \frac{1}{2!}\theta^{\alpha}\theta^{\beta}C_{\alpha\beta} + \frac{1}{3!}\theta^{\alpha}\theta^{\beta}\theta^{\gamma}D_{\alpha\beta\gamma} + \frac{1}{4!}\theta^{\alpha}\theta^{\beta}\theta^{\gamma}\theta^{\delta}E_{\alpha\beta\gamma\delta} . \quad \cdots(3)$$

That is $\mathcal{L}_{SS}$ has the same form as the superfield defined by equation (2) of Chapter 4, and the action of the supersymmetry generators Q_{α} on the component fields of $\mathcal{L}_{SS}$ is as given by equation (10) of Chapter 4.

To proceed with the analysis we need the action of the 'nilpotent' generators of $\mathcal{G}_L$ expressed as differential operators, we already have an expression for P_{σ} given by equation (68) of Chapter 3 of Part II from which we can construct the differential operators of $\mathcal{G}_L$ as follows. Since

$$[P_{\sigma}, \Phi] = \frac{\partial}{\partial x^{\sigma}}\Phi ,$$

we have

$$[e_i P_{\sigma}, \Phi] = e_i\frac{\partial}{\partial x^{\sigma}}\Phi = \frac{\partial}{\partial x^{\sigma}_i}\Phi . \quad \cdots(4)$$

Now from equation (1) of Chapter 4 we have

$$1 + y^{\mu\nu}[e_i M_{\mu\nu}, \Phi(x,\theta)]$$
$$= \Phi(\Lambda x, \theta\Gamma^{M}(\Lambda^{-1}))$$
$$= \left(1 + y^{\mu\nu}e_i\left(x_{\mu}\frac{\partial}{\partial x^{\nu}} - x_{\nu}\frac{\partial}{\partial x^{\mu}}\right)\right)\left(1 + y^{\mu\nu}e_i\frac{1}{2}\theta^{\alpha}((\gamma_{\mu}\gamma_{\nu})^{t})_{\alpha}{}^{\beta}\frac{\partial}{\partial\theta^{\beta}}\Phi(x,\theta)\right),$$

where we have used equation (48) of Chapter 2 and a Taylor expansion in θ. So that

$$[e_i M_{\mu\nu}, \Phi(x,\theta)]$$
$$= \left(e_i\left(x_{\mu}\frac{\partial}{\partial x^{\nu}} - x_{\nu}\frac{\partial}{\partial x^{\mu}}\right) + \frac{1}{2}e_i(\gamma_{\mu}\gamma_{\nu})_{\alpha\beta}\theta^{\beta}\frac{\partial}{\partial\theta^{\alpha}}\right)\Phi(x,\theta) ,$$
$$= \left(e_i x_{k\mu} e_k\frac{\partial}{\partial x^{\nu}} - e_i x_{k\nu} e_k\frac{\partial}{\partial x^{\mu}} + \frac{1}{2}(\gamma_{\mu}\gamma_{\nu})_{\alpha\beta}e_i\theta^{\beta}_j f_j\frac{\partial}{\partial\theta^{\alpha}}\right)\Phi(x,\theta) ,$$
$$= \left(x_{k\mu}\frac{\partial}{\partial x^{\nu}_{i\wedge k}} - x_{k\nu}\frac{\partial}{\partial x^{\mu}_{i\wedge k}} + \frac{1}{2}(\gamma_{\mu}\gamma_{\nu})_{\alpha\beta}\theta^{\beta}_j\frac{\partial}{\partial\theta^{\alpha}_{i\wedge j}}\right)\Phi(x,\theta) . \quad \cdots(5)$$

Similarly

$$[f_j Q_\alpha \,,\, \Phi(x,\theta)]$$
$$= \left(\tfrac{1}{2}(\gamma^\mu C)_{\alpha\beta}\, \theta^\beta_\ell \frac{\partial}{\partial x^\mu_{j\wedge\ell}} - \frac{\partial}{\partial\theta^\alpha_j}\right)\Phi(x,\theta) \,. \qquad \cdots(6)$$

Now, motivated by the standard approach, we consider the term in $\mathcal{L}_{SS}$ given by $\theta^1\theta^2\theta^3\theta^4 E(x)$, where we have removed the indices for convenience. We expand the field $E(x)$ in terms of the nilpotent parameters as

$$E(x) = E(x_0) + \sum_{i=1}^{N-1} \varepsilon_i x^\mu_i \frac{\partial}{\partial x^\mu_0} E(x_0)$$
$$+ \frac{1}{2!}\sum_{i,j=1}^{N-1} \varepsilon_i\varepsilon_j\, x^\mu_i x^\sigma_j \frac{\partial}{\partial x^\mu_0}\frac{\partial}{\partial x^\sigma_0} E(x_0) + \cdots \qquad \cdots(7)$$

Then

$$\theta^1\theta^2\theta^3\theta^4 E(x)$$
$$= \theta^1\theta^2\theta^3\theta^4\{E(x_0) + \sum_{i=1}^{N-1} \varepsilon_i x^\mu_i \frac{\partial}{\partial x^\mu_0} E(x_0) + \cdots\}$$
$$= \theta^1_j\theta^2_k\theta^3_\ell\theta^4_m f_j f_k f_\ell f_m E(x_0)$$
$$+ \theta^1_j\theta^2_k\theta^3_\ell\theta^4_m f_j f_k f_\ell f_m \sum_{i=1}^{N-1} \varepsilon_i x^\mu_i \frac{\partial}{\partial x^\mu_0} E(x_0) + \cdots \qquad \cdots(8)$$

Clearly the term of interest is $\theta^1_j\theta^2_k\theta^3_\ell\theta^4_m f_j f_k f_\ell f_m E(x_0)$ and examination of the operators defined by equations (4), (5) and (6) show that

$$\int_V \hat{d}^4x\, \hat{d}^4\theta\, \{\theta^1_j\theta^2_k\theta^3_\ell\theta^4_m E(x_0)\} \qquad \cdots(9)$$

is invariant under transformations of the Lie group $\mathcal{G}_L$ for each j, k, ℓ and m. That is it provides us with a one dimensional representation space for the Lie group $\mathcal{G}_L$.

Now this result is valid for any type of superfield (general, chiral or intermediate), but if $\mathcal{L}_{SS}$ is a chiral superfield it does not lead to any useful solution, since, from equation (43) of Chapter 3, the last term can then be written as

$$\theta^1\theta^2\theta^3\theta^4 E(x) = \theta^1\theta^2\theta^3\theta^4 \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} A(x)\left(\frac{\hbar}{Mc}\right)^2 , \qquad \cdots(10)$$

so that we are led to the term

$$\theta^1_j\theta^2_k\theta^3_\ell\theta^4_m f_j f_k f_\ell f_m \frac{\partial}{\partial x^\mu_0}\frac{\partial}{\partial x_{0\mu}} A(x_0) \,. \qquad \cdots(11)$$

Then

$$\int_V \hat{d}^4x \, \hat{d}^4\theta \, \{ \theta^1_j \theta^2_k \theta^3_l \theta^4_m \frac{\partial}{\partial x^\mu_0} \frac{\partial}{\partial x_{0\mu}} A(x_0) \}$$

$$= \int_V \prod_{i=1}^{N-1} d^4x_i \, \hat{d}^4\theta \left\{ \int d^4x_0 \left\{ \theta^1_j \theta^2_k \theta^3_l \theta^4_m \frac{\partial}{\partial x^\mu_0} \frac{\partial}{\partial x_{0\mu}} A(x_0) \right\}\right\},$$

$$= 0 .$$

If we consider the term constructed, by this procedure from the $\theta\theta\theta$ term , we find that it is the integral of a total derivative, so that it cannot contribute any useful terms either. We are thus led to consider the term with two θ's of the chiral superfield.

If we refer to equation (41) of Chapter 3 we see that there are two terms to be considered. Consider first the term of the form

$$\theta^{Rn} (\sigma_\mu^L \sigma_2^L)_{nn'} \theta^{Ln'} \frac{\partial}{\partial x^\mu} C(x) \qquad \cdots(12)$$

where $C(x)$ is some scalar field. Following the above procedure we are led to

$$\int_V \hat{d}^4x \, \hat{d}^4\theta \, \{ \theta^{Rn}_j (\sigma_\mu^L \sigma_2^L)_{nn'} \theta^{Ln'} \frac{\partial}{\partial x^\mu_0} C(x_0) \}$$

$$= \int \prod_{i=1}^{N-1} d^4x_i \, \hat{d}^4\theta \left\{ \int d^4x_0 \left\{ \theta^{Rn}_j (\sigma_\mu^L \sigma_2^L)_{nn'} \theta^{Ln'} \frac{\partial}{\partial x^\mu_0} C(x_0) \right\}\right\} ,$$

$$= 0 ,$$

since we are integrating a total derivative.

Now consider the term of the form (for a left handed chiral superfield)

$$\theta^{Ln} (\sigma_2^L)_{nn'} \theta^{Ln'} C(x) ,$$

where $C'(x)$ is some scalar field. Following the above procedure we are led to the terms

$$\int_V \hat{d}^4x \, \hat{d}^4\theta \, \{ \theta^{Ln}_j (\sigma_2^L)_{nn'} \theta^{Ln'}_k \; C'(x_0)$$

which are invariant for each j, k. We note that in terms of the parameter y this would have been the last term.

Thus, given a candidate for a Lagrangian density, we can find certain one dimensional representation spaces of the Lie group G_L that provide candidates for a component Lagrangian density.

Before completing this section we need to consider the units of the infinitesimals $d\theta^{\nu}_{j}$ and dx^{μ}_{i}. It is clear that we require the combinations $x^{\mu}P_{\mu}$ and $\theta^{\alpha}Q_{\alpha}$ to be dimensionless, and that the combinations $\left(\frac{\hbar}{Mc}\right)P_{\mu}$ and $\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{\alpha}$ are dimensionless. This implies that x^{μ} should be measured in units of $\left(\frac{\hbar}{Mc}\right)$ and θ^{α} in units of $\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}$. But

$$x^{\mu} = x^{\mu}_{0}\xi_{0} + x^{\mu}_{i\wedge j}\xi_{i\wedge j} + x^{\mu}_{i\wedge j\wedge k}\xi_{i\wedge j\wedge k} + \cdots$$

with $i<j<k<\cdots$. So that each 'level' in the Grassman algebra must be measured in the same units. This suggests that the infinitessimal $\prod_{i=0}^{\mathcal{N}-1} dx_{i} = \hat{d}x$ should be in units of $\left(\frac{\hbar}{Mc}\right)^{\mathcal{N}}$ and by a similar argument $\prod_{j=1}^{\mathcal{N}} d\theta_{i} = \hat{d}\theta$ should be in units of $\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}\mathcal{N}}$. This is, of course, unsatisfactory. What we can recognise is that if we had chosen to work with the operators $\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}Q_{\alpha}$, $\left(\frac{\hbar}{Mc}\right)P_{\mu}$ and $\left(\frac{\hbar}{Mc}\right)M_{\mu\nu}$ the coordinates in superspace could have been made dimensionless and we would have had no problem.

To overcome this problem we take $\hat{d}x$ to be measured in units of $\left(\frac{\hbar}{Mc}\right)$ and $\hat{d}\theta$ to be measured in units of $\left(\frac{\hbar}{Mc}\right)^{\frac{1}{2}}$.

5.2 The construction of $\mathcal{L}_{SS}$ for a Real Scalar Superfield.

In this section we will consider the most general function that can be constructed for $\mathcal{L}_{SS}$ with the following assumptions:

(i) $\mathcal{L}_{SS}$ is a scalar under the action of the super Poincaré group.

(ii) $\mathcal{L}_{SS}$ is a quadratic form containing at most two derivatives.

Let $\Phi(x,\theta)$ be a real scalar superfield and define the $(5|4)$ dimensional supervector $\underset{\sim}{\Phi}$ by

$$\underset{\sim}{\Phi}_a = \begin{bmatrix} \frac{\partial}{\partial x^\mu}\Phi(x,\theta) \\ \Phi(x,\theta) \\ \frac{\partial}{\partial\theta^\nu}\Phi(x,\theta) \end{bmatrix}_a . \qquad \cdots(14)$$

Then the most general function we can construct for $\mathcal{L}_{SS}$ is given by

$$\mathcal{L}_{SS} = \underset{\sim}{\Phi}^{st}_b \, \Omega_{ba} \, \Phi_a \qquad \cdots(15)$$

with Ω some $(5|4)\times(5|4)$ dimensional supermatrix, determined by the invariance requirement (i) above.

For $g \in G =$ (super Poincaré group) suppose the transformation of $\underset{\sim}{\Phi}$ is given by

$$\underset{\sim}{\Phi}'_a = \Gamma(g)_{ab}\underset{\sim}{\Phi}_b \ . \qquad \cdots(16)$$

So that

$$\underset{\sim}{\Phi}'^{st}_a = \underset{\sim}{\Phi}^{st}_b \, \Gamma(g)^{st}_{ba} \qquad \cdots(17)$$

The requirement of invariance then demands that

$$\underset{\sim}{\Phi}'^{st}_a \Omega_{ab} \underset{\sim}{\Phi}'_b = \underset{\sim}{\Phi}_a \Omega_{ab} \underset{\sim}{\Phi}_b$$

ie.

$$\underset{\sim}{\Phi}'_a \Gamma(g)^{st}_{aa'} \Omega_{a'b'} \Gamma(g)_{b'b} \underset{\sim}{\Phi}_b = \underset{\sim}{\Phi}^{st}_a \Omega_{ab} \Phi_b$$

So that Ω must satisfy

$$\Gamma(g)^{st}_{aa'} \Omega_{a'b'} \Gamma(g)_{b'b} = \Omega_{ab} \qquad \cdots(18)$$

The representation of the super Poincaré group on the vector $\underset{\sim}{\Phi}_a$ is given by equation (79) of Chapter 2 of Part I, noting that Φ transforms as a scalar, as

$$\Gamma(g) = \begin{bmatrix} I & 0 & 0 \\ 0 & 1 & 0 \\ T(\tau)^t & 0 & I \end{bmatrix} \begin{bmatrix} (\Lambda^t)^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Gamma(\Lambda)^{t^{-1}} \end{bmatrix} \qquad \cdots(19)$$

Now let

$$\Omega = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & J \end{bmatrix}$$

with the matrix blocks corresponding to those of $\Gamma(g)$. The structure of Ω is then determined by

$$\begin{bmatrix} \Lambda^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \Gamma(\Lambda)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 & T \\ 0 & 1 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & J \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 1 & 0 \\ T^t & 0 & I \end{bmatrix}$$

$$\times \begin{bmatrix} (\Lambda^{-1})^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \Gamma(\Lambda^{-1})^t \end{bmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & J \end{bmatrix}.$$

$$\begin{bmatrix} \Lambda^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \Gamma(\Lambda)^{-1} \end{bmatrix} \begin{bmatrix} A+TG+CT^t+TJT^t & B+TH & C+TJ \\ D+FT^t & E & F \\ G-JT^t & H & J \end{bmatrix}$$

$$\times \begin{bmatrix} (\Lambda^{-1})^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \Gamma(\Lambda^{-1})^t \end{bmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & J \end{bmatrix}.$$

Since this must be true for each $g \in G$ we can deduce that

$$G = 0, \quad C = 0 \text{ and } J = 0.$$

Now if we set $T = 0$ the invariance requirement becomes

$$\begin{bmatrix} \Lambda^{-1} A \Lambda^{-1t} & \Lambda^{-1} B & \Lambda^{-1} C\, \Gamma(\Lambda^{-1})^t \\ D\Lambda^{-1t} & E & 0 \\ \Gamma(\Lambda^{-1}) G \Lambda^{-1t} & 0 & 0 \end{bmatrix} = \begin{bmatrix} A & B & C \\ D & E & 0 \\ G & 0 & 0 \end{bmatrix}.$$

So that we can deduce that

$$D = 0, \quad B = 0, \quad C = 0 \text{ and } G = 0.$$

Also the matrix A must satisfy

$$\Lambda^{-1} A (\Lambda^{-1})^t = A$$

so that $A_{\lambda\mu} = k g_{\lambda\mu}$, $k \in E_L$, and $E = k' \in E_L$.

Thus

$$\Omega = \begin{bmatrix} k g_{\lambda\mu} & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad \ldots(20)$$

The most general function we can construct for $\mathcal{L}_{SS}$ is thus given by

$$\mathcal{L}_{SS} = k'\Phi(x,\theta)\Phi(x,\theta) + k\left(\frac{\partial}{\partial x^\mu}\Phi(x,\theta)\right)\left(\frac{\partial}{\partial x_\mu}\Phi(x,\theta)\right) \quad \cdots(21)$$

with $k, k' \in E_{Lo}$. We note that since k or k' can be zero each term is an invariant.

5.3 The $j = 0$ Chiral Superfield - an Example.

It is easy to see that if we construct a Lagrangian from just a left handed or right handed superfield we will obtain just a Klein-Gordon term for each of the component fields. This is precisely what we would expect, and it is certainly not obtained by the normal procedures used by physicists. But we want to construct a non trivial Lagrangian. We thus consider the superfield $(\Phi^L + \Phi^R)$, with both fields constructed from the same set of creation and annihilation operators. $(\Phi_L + \Phi_R)$ is then a real superfild. The most general form of $\mathcal{L}_{SS}$ is thus given by equation (21) as

$$\begin{aligned}\mathcal{L}_{SS} &= k'(\Phi^L + \Phi^R)(\Phi^L + \Phi^R) + k'\left(\frac{\partial}{\partial x^\mu}(\Phi^L + \Phi^R)\right)\left(\frac{\partial}{\partial x_\mu}(\Phi^L + \Phi^R)\right)\\ &= k'(\Phi^L\Phi^L + \Phi^R\Phi^R + 2\Phi^L\Phi^R) \qquad \cdots(22)\\ &\quad + k\left(\left(\frac{\partial}{\partial x^\mu}\Phi^L\right)\left(\frac{\partial}{\partial x_\mu}\Phi^L\right) + \left(\frac{\partial}{\partial x^\mu}\Phi^R\right)\left(\frac{\partial}{\partial x_\mu}\Phi^R\right) + 2\left(\frac{\partial}{\partial x^\mu}\Phi^L\right)\left(\frac{\partial}{\partial x_\mu}\Phi^R\right)\right).\end{aligned}$$

Here we have noted that $|\Phi^L| = |\Phi^R| = 0$. Now using equation (43) of Chapter 3 we have

$$\begin{aligned}&\Phi^L(x,\theta)\,\Phi^L(x,\theta)\\ &= \psi^L\psi^L - \theta^{Rr}(\sigma_\mu^L\sigma_2^L)_{rr'}\theta^{Lr'}\psi^L\frac{\partial}{\partial x^\mu}\psi^L\\ &\quad + \frac{1}{8}\theta^{Ln}(\sigma_2^L)_{nn'}\theta^{Ln'}\theta^{Rr}(\sigma_2^R)_{rr'}\theta^{Rr'}\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\psi^L\right)\psi^L\\ &\quad + \frac{1}{4}\theta^{Rr}(\sigma_\mu^L\sigma_2^L)_{rr'}\theta^{Lr}\theta^{Rn}(\sigma_\rho^L\sigma_2^L)_{nn'}\theta^{Ln'}\left(\frac{\partial}{\partial x^\mu}\psi^L\right)\left(\frac{\partial}{\partial x_\rho}\psi^L\right)\\ &\quad - 2e^{i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\psi^L\theta^{Ln}\psi^L_{\frac{1}{2},n} + \theta^{Rr}(\sigma_\mu^L\sigma_2^L)_{rr'}\theta^{Lr'}\left(\frac{\partial}{\partial x}\psi^L\right)\theta^{Ln}\psi^L_{\frac{1}{2},n}e^{i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\\ &\quad + e^{i\frac{\pi}{2}}\left(\frac{Mc}{\hbar}\right)\theta^{Ln}\psi^L_{\frac{1}{2},n}\theta^{Ln'}\psi^L_{\frac{1}{2},n'} - \frac{1}{2}e^{i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{Ln}(\sigma_2^L)_{nn'}\theta^{Ln'}\theta^{Rr}(\sigma_\mu^L)_{rr'}\left(\frac{\partial}{\partial x^\mu}\psi^L_{\frac{1}{2},r}\right)\psi^L\\ &\quad - e^{-i\frac{\pi}{2}}\left(\frac{Mc}{\hbar}\right)\theta^{Ln}\theta^{Ln'}(\sigma_2^L)_{nn'}\psi^L\underset{\sim}{\psi}^L\;.\end{aligned}$$

It is convenient to rearrange this expression so that the terms have, as far as possible, common factors in the θ variables. Consider first the terms in $\theta\theta$. We have

$$-\theta^{Rr}(\sigma_\mu^L\sigma_2^L)_{rr'}\psi^L\frac{\partial}{\partial x^\mu}\psi^L + e^{i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)\theta^{Ln}\psi^L_{\frac{1}{2},n}\theta^{Ln'}\psi^L_{\frac{1}{2},n'}$$
$$-e^{-i\frac{\pi}{2}}\left(\frac{Mc}{\hbar}\right)\theta^{Ln}\theta^{Ln'}(\sigma_2^L)_{nn'}\psi^L\underset{\sim}{\psi}^L$$
$$=-\frac{1}{2}\theta^{Rr}(\sigma_\mu^L\sigma_2^L)_{rr'}\frac{\partial}{\partial x^\mu}(\psi^L\psi^L) + e^{i\frac{\pi}{2}}\left(-\theta^{L1}\theta^{L2}\left(\psi^L_{\frac{1}{2},1}\psi^L_{\frac{1}{2},2}-\psi^L_{\frac{1}{2},2}\psi^L_{\frac{1}{2},1}\right)\right)$$
$$-e^{-i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)\theta^{Ln}(\sigma_2^L)_{nn'}\theta^{Ln'}\psi^L\underset{\sim}{\psi}^L \quad ,$$
$$=-\frac{1}{2}\theta^{Rr}(\sigma_\mu^L\sigma_2^L)_{rr'}\frac{\partial}{\partial x^\mu}(\psi^L\psi^L) - e^{-i\frac{\pi}{2}}\left(\frac{Mc}{\hbar}\right)\theta^{Ln}(\sigma_2^L)_{nn'}\theta^{Ln'}\psi^L\underset{\sim}{\psi}^L$$
$$+e^{i\frac{\pi}{2}}\left(\frac{Mc}{\hbar}\right)-\frac{i}{2}\theta^{Ln}(\sigma_2^L)_{nn'}\theta^{Ln'}i\,\psi^L_{\frac{1}{2},r}(\sigma_2^L)_{rr'}\psi^L_{\frac{1}{2},r'} \quad ,$$
$$=-\frac{1}{2}\theta^{Rr}(\sigma_\mu^L\sigma_2^L)_{rr'}\theta^{Lr'}\frac{\partial}{\partial x^\mu}(\psi^L\psi^L)$$
$$-\frac{1}{c\hbar}\theta^{L1}\theta^{L2}\left\{-Mc^2\psi^L_{\frac{1}{2},r}(\sigma_2^L)_{rr'}\psi^L_{\frac{1}{2},r'} - 2Mc^2\psi^L\underset{\sim}{\psi}^L\right\}. \quad \cdots(23)$$

Consider now the terms in $\theta\theta\theta$. We have

$$\theta^{Rr}(\sigma_\mu^L\sigma_2^L)_{rr'}\theta^{Lr'}\left(\frac{\partial}{\partial x^\mu}\psi^L\right)e^{i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{Ln}\psi^L_{\frac{1}{2},n}$$
$$-\frac{1}{2}e^{i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{Ln}(\sigma_2^L)_{nn'}\theta^{Ln'}\theta^{Rr}(\sigma_\mu^L)_{rr'}\left(\frac{\partial}{\partial x^\mu}\psi^L_{\frac{1}{2},r'}\right)\psi^L \quad ,$$
$$=\frac{1}{2}\theta^{Rr}(\Gamma^A\sigma_\mu^L\sigma_2^L)_{nr'}\theta^{Lr'}\theta^{Ln}(\Gamma_A)_{rn'}\psi^L_{\frac{1}{2},n'}\left(\frac{\partial}{\partial x^\mu}\psi^L\right)$$
$$-\frac{1}{2}e^{i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{Ln}(\sigma_2^L)_{nn'}\theta^{Ln'}\theta^{Rr}(\sigma_\mu^L)_{rr'}\left(\frac{\partial}{\partial x^\mu}\psi^L_{\frac{1}{2},r}\right)\psi^L \quad ,$$
$$=-\frac{1}{2}\theta^{Ln}(\sigma_2^L)_{nn'}\theta^{Ln'}\theta^{Rr'}e^{i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\left\{\left(\frac{\partial}{\partial x^\mu}\psi^L\right)(\sigma_\mu)_{rr'}\psi^L_{\frac{1}{2},r}\right.$$
$$\left.+\psi^L\left(\frac{\partial}{\partial x^\mu}(\sigma_\mu^L)_{rr'}\psi^L_{\frac{1}{2},r}\right)\right\} \quad ,$$
$$=-\frac{1}{2}\theta^{Ln}(\sigma_\mu^L)_{nn'}\theta^{Ln'}\theta^{Rr}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\frac{\partial}{\partial x}(\sigma_\mu)_{rr'}\left(\psi^L\psi^L_{\frac{1}{2},r'}\right) \quad . \quad \cdots(24)$$

Consider now the terms in $\theta\theta\theta\theta$. We have

$$\frac{1}{8}\theta^{Ln}(\sigma_2^L)_{nn'}\theta^{Ln'}\theta^{Rr}(\sigma_2^R)_{rr'}\theta^{Rr'}\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\psi^L\right)\psi^L$$
$$+\frac{1}{4}\theta^{Rr}(\sigma_\mu^L\sigma_2^L)_{rr'}\theta^{Lr}\theta^{Rn}(\sigma_\rho^L\sigma_2^L)_{nn'}\theta^{Ln'}\left(\frac{\partial}{\partial x^\mu}\psi^L\right)\left(\frac{\partial}{\partial x_\rho}\psi^L\right)$$
$$=\frac{1}{16}\theta^{Ln}(\sigma_2^L)_{nn'}\theta^{Ln'}\theta^{Rr}(\sigma_2^R)_{rr'}\theta^{Rr'}\left\{\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\right)\psi^L\psi^L-2\left(\frac{\partial}{\partial x^\mu}\psi^L\right)\left(\frac{\partial}{\partial x_\mu}\psi^L\right)\right\}$$
$$-\frac{1}{8}\theta^{Rr}(\sigma_2^L)_{nr'}\theta^{Lr'}\theta^{Ln}(\sigma_2^R)_{rn'}\theta^{Rn'}\left(\frac{\partial}{\partial x^\mu}\psi^L\right)\left(\frac{\partial}{\partial x_\mu}\psi^L\right) \quad ,$$
$$=\frac{1}{16}\theta^{Ln}(\sigma_2^L)_{nn'}\theta^{Ln'}\theta^{Rr}(\sigma_2^R)_{rr'}\theta^{Rr'}\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\right)\psi^L\psi^L \quad ,$$
$$=\frac{1}{4}\theta^{L1}\theta^{L2}\theta^{R1}\theta^{R2}\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\right)\psi^L\psi^L . \quad \cdots(25)$$

Combining these results we have

$$\begin{aligned}
&\Psi^L(x,\theta)\,\Psi(x,\theta)\\
&\quad= \psi^L\psi^L - 2e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{Ln}\psi^L\psi^L_{\frac{1}{2},n} - \tfrac{1}{2}\,\theta^{Rr}(\sigma_\mu^L\sigma_2)_{rr'}\,\theta^{Lr'}\tfrac{\partial}{\partial x^\mu}(\psi^L\psi^L)\\
&\qquad - \tfrac{1}{c\hbar}\,\theta^{L1}\theta^{L2}\left\{-Mc^2\psi^L_{\frac{1}{2},r}(\sigma_2^L)_{rr'}\psi^L_{\frac{1}{2},r} - 2Mc^2\psi^L\underline{\psi}^L\right\}\\
&\qquad - \tfrac{1}{2}\,\theta^{Ln}(\sigma_2^L)_{nn'}\,\theta^{Ln'}\theta^{Rr}e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\tfrac{\partial}{\partial x^\mu}(\sigma_\mu^L)_{rr'}(\psi^L\psi^L_{\frac{1}{2},r'})\\
&\qquad + \tfrac{1}{4}\,\theta^{L1}\theta^{L2}\theta^{L3}\theta^{L4}\left(\tfrac{\partial}{\partial x^\mu}\tfrac{\partial}{\partial x_\mu}\right)\psi^L\psi^L\ . \qquad \cdots(26)
\end{aligned}$$

We can write down the term $\Psi^R\Psi^R$ using the fact that we have symmetry in the interchange of L and R, as

$$\begin{aligned}
&\Psi^R(x,\theta)\,\Psi^R(x,\theta)\\
&\quad= \psi^R\psi^R - 2e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{Rn}\psi^R\psi^R_{\frac{1}{2},n} - \tfrac{1}{2}\,\theta^{Ln}(\sigma_\mu^R\sigma_2^R)_{nn'}\,\theta^{Rn'}\tfrac{\partial}{\partial x^\mu}(\psi^R\psi^R)\\
&\qquad - \tfrac{1}{c\hbar}\,\theta^{R1}\theta^{R2}\left\{-Mc^2\psi^R_{\frac{1}{2},r}(\sigma_2^R)\psi^R_{\frac{1}{2},r} - 2Mc^2\psi^R\underline{\psi}^R\right\}\\
&\qquad - \tfrac{1}{2}\,\theta^{Rn}(\sigma_2^R)_{nn'}\,\theta^{Rn'}\theta^{Lr}e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\tfrac{\partial}{\partial x^\mu}(\sigma_\mu^R)_{rr'}(\psi^R\psi^R_{\frac{1}{2},r'})\\
&\qquad + \tfrac{1}{4}\,\theta^{L1}\theta^{L2}\theta^{R1}\theta^{R2}\left(\tfrac{\partial}{\partial x^\mu}\tfrac{\partial}{\partial x_\mu}\right)\psi^R\psi^R\ . \qquad \cdots(27)
\end{aligned}$$

Now using the same procedure for $\Psi^L\Psi^R$ and making use of the identity

$$\theta^{Rr}\theta^{Ln}\psi^L_n\psi^R_r = \tfrac{1}{2}\,\theta^{Rr}(\sigma_\mu^L\sigma_2^L)_{rn}\,\theta^{Ln}(\sigma_2^L\sigma_\mu^L)_{n'r'}\psi^L_{\frac{1}{2},n'}\psi^R_{\frac{1}{2},r'}$$

which is proved using equations (81) and (83) of Chapter 4 of Part II, we obtain

$$\begin{aligned}
&\Psi^L(x,\theta)\,\Psi^R(x,\theta)\\
&\quad= \psi^L\psi^R - e^{i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{\frac{1}{2}}\left\{\psi^L\theta^{Rn}\psi^R_{\frac{1}{2},n} + \psi^R\theta^{Ln}\psi^L_{\frac{1}{2},n}\right\}\\
&\qquad - \tfrac{1}{2}\,\theta^{Rr}(\sigma_\mu^L\sigma_2^L)_{rr'}\,\theta^{Lr'}\left\{\left(\tfrac{\partial}{\partial x^\mu}\psi^L\right)\psi^R - \psi^L\left(\tfrac{\partial}{\partial x^\mu}\psi^R\right)\right\}\\
&\qquad + e^{i\frac{\pi}{2}}\left(\tfrac{Mc}{\hbar}\right)\tfrac{1}{2}(\sigma_\mu^L\sigma_2^L)_{rr'}\,\theta^{Rr}\theta^{Lr'}(\sigma_2^L\sigma_\mu^L)_{nn'}\psi^L_{\frac{1}{2},n}\psi^R_{\frac{1}{2},n'}\\
&\qquad - \tfrac{1}{2}\,e^{-i\frac{\pi}{2}}\left(\tfrac{Mc}{\hbar}\right)\left\{\psi^L\theta^{Rn}(\sigma_2^R)_{nn'}\theta^{Rn'}\underline{\psi}^R + \psi^R\theta^{Ln}(\sigma_2^L)_{nn'}\theta^{Ln'}\underline{\psi}^L\right\}\\
&\qquad + \tfrac{1}{2}\,\theta^{Rn}(\sigma_\mu^L\sigma_2^L)_{nn'}\,\theta^{Ln'}\left\{\left(\tfrac{\partial}{\partial x^\mu}\psi^L\right)\theta^{Rr}\psi^R_{\frac{1}{2},r} - \left(\tfrac{\partial}{\partial x^\mu}\psi^R\right)\theta^{Lr}\psi^L_{\frac{1}{2},r}\right\}\\
&\qquad + \tfrac{1}{2}\,e^{-i\frac{\pi}{4}}\left(\tfrac{Mc}{\hbar}\right)^{3/2}\left\{\theta^{Ln}\psi^L_{\frac{1}{2},n}\theta^{Rr}(\sigma_2^R)_{rr'}\theta^{Rr'}\underline{\psi}^R + \theta^{Rn}\psi^R_{\frac{1}{2},n}\theta^{Lr}(\sigma_2^L)_{rr'}\theta^{Lr'}\underline{\psi}^L\right\}\\
&\qquad + \tfrac{M}{\hbar^2}\,\theta^{L1}\theta^{L2}\theta^{R1}\theta^{R2}\left\{\tfrac{\hbar^2}{4M}\left(\tfrac{\partial}{\partial x^\mu}\tfrac{\partial}{\partial x_\mu}\right)\psi^L\psi^R - Mc^2\underline{\psi}^L\underline{\psi}^R\right.\\
&\qquad\qquad \left. - \tfrac{\hbar^2}{M}\left(\tfrac{\partial}{\partial x^\mu}\psi^L\right)\left(\tfrac{\partial}{\partial x_\mu}\psi^R\right) - \tfrac{i}{2}\hbar c\,\overline{\psi}\,\gamma^\mu\tfrac{\partial}{\partial x^\mu}\psi\right\}. \qquad \cdots(28)
\end{aligned}$$

We note that this equation transforms into itself under interchange of L and R.

In a similar way we obtain

$$
\begin{aligned}
&\left(\frac{\partial}{\partial x^\mu}\Phi^L(x,\theta)\right)\left(\frac{\partial}{\partial x_\mu}\Phi^L(x,\theta)\right) \\
&= \left(\frac{\partial}{\partial x^\mu}\psi^L\right)\left(\frac{\partial}{\partial x_\mu}\psi^L\right) - 2e^{i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\theta^{Ln}\left(\frac{\partial}{\partial x^\mu}\psi^L\right)\left(\frac{\partial}{\partial x_\mu}\psi^L_{\frac{1}{2},n}\right) \\
&\quad -\tfrac{1}{2}\theta^{Rr}(\sigma_\mu^L\sigma_2^L)_{rr'}\theta^{Lr'}\frac{\partial}{\partial x^\mu}\left(\left(\frac{\partial}{\partial x^\ell}\psi^L\right)\left(\frac{\partial}{\partial x_\ell}\psi^L\right)\right) \\
&\quad -\frac{1}{c\hbar}\theta^{L1}\theta^{L2}\Big\{-Mc^2\left(\frac{\partial}{\partial x^\mu}\psi^L_{\frac{1}{2},r}\right)(\sigma_2^L)_{rr'}\left(\frac{\partial}{\partial x_\mu}\psi^L_{\frac{1}{2},r}\right) \\
&\qquad\qquad -2Mc\left(\frac{\partial}{\partial x^\mu}\psi^L\right)\left(\frac{\partial}{\partial x_\mu}\underline{\psi}^L\right)\Big\} \\
&\quad -\tfrac{1}{2}\theta^{Ln}(\sigma_2^L)_{nn'}\theta^{Ln'}\theta^{Lr}e^{i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\frac{\partial}{\partial x^\mu}(\sigma_\mu^L)_{rr'}\left(\left(\frac{\partial}{\partial x^\ell}\psi^L\right)\left(\frac{\partial}{\partial x_\ell}\psi^L_{\frac{1}{2},r'}\right)\right) \\
&\quad +\tfrac{1}{4}\theta^{L1}\theta^{L2}\theta^{R1}\theta^{R2}\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\right)\left(\left(\frac{\partial}{\partial x^\ell}\psi^L\right)\left(\frac{\partial}{\partial x_\ell}\psi^L\right)\right) \quad , \qquad \ldots(29)
\end{aligned}
$$

the term $\left(\frac{\partial}{\partial x^\mu}\Phi^R\right)\left(\frac{\partial}{\partial x_\mu}\Phi^R\right)$ is obtained by interchanging L and R in this expression, and

$$
\begin{aligned}
&\left(\frac{\partial}{\partial x^\mu}\Phi^L(x,\theta)\right)\left(\frac{\partial}{\partial x_\mu}\Phi^R(x,\theta)\right) \\
&= \left(\frac{\partial}{\partial x^\mu}\psi^L\right)\left(\frac{\partial}{\partial x_\mu}\psi^R\right) - e^{i\frac{\pi}{4}}\left(\frac{Mc}{\hbar}\right)^{\frac{1}{2}}\Big\{\left(\frac{\partial}{\partial x^\mu}\psi^L\right)\theta^{Rn}\left(\frac{\partial}{\partial x_\mu}\psi^R_{\frac{1}{2},n}\right) \\
&\qquad\qquad +\left(\frac{\partial}{\partial x^\mu}\psi^R\right)\theta^{Ln}\left(\frac{\partial}{\partial x_\mu}\psi^L_{\frac{1}{2},n}\right)\Big\} \\
&\quad -\tfrac{1}{2}\theta^{Rr}(\sigma_\mu^L\sigma_2^L)_{rr'}\theta^{Lr'}\Big\{\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\ell}\psi^L\right)\left(\frac{\partial}{\partial x_\ell}\psi^R\right)-\left(\frac{\partial}{\partial x_\ell}\psi^L\right)\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\ell}\psi^R\right)\Big\} \\
&\quad +\tfrac{1}{2}e^{i\frac{\pi}{2}}\left(\frac{Mc}{\hbar}\right)\theta^{Rr}(\sigma_\mu^L\sigma_2^L)_{rr'}\theta^{Lr'}(\sigma_2^L\sigma_\mu^L)_{nn'}\left(\frac{\partial}{\partial x^\ell}\psi^L_{\frac{1}{2},n}\right)\left(\frac{\partial}{\partial x_\ell}\psi^R_{\frac{1}{2},n}\right) \\
&\quad -\tfrac{1}{2}e^{-i\frac{\pi}{2}}\left(\frac{Mc}{\hbar}\right)\Big\{\left(\frac{\partial}{\partial x^\ell}\psi^L\right)\theta^{Rn}(\sigma_2^R)_{nn'}\theta^{Rn'}\left(\frac{\partial}{\partial x_\ell}\underline{\psi}^R\right) \\
&\qquad\qquad +\left(\frac{\partial}{\partial x^\ell}\psi^R\right)\theta^{Ln}(\sigma_2^L)_{nn'}\theta^{Ln'}\left(\frac{\partial}{\partial x_\ell}\underline{\psi}^L\right)\Big\} \\
&\quad +\tfrac{1}{2}\theta^{Rn}(\sigma_\mu^L\sigma_2^L)_{nn'}\theta^{Ln'}\Big\{\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\ell}\psi^L\right)\theta^{Rr}\left(\frac{\partial}{\partial x_\ell}\psi^R_{\frac{1}{2},r}\right) \\
&\qquad\qquad -\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\ell}\psi^R\right)\theta^{Lr}\left(\frac{\partial}{\partial x_\ell}\psi^L_{\frac{1}{2},r}\right)\Big\} \\
&\quad +\frac{M}{\hbar}\theta^{L1}\theta^{L2}\theta^{R1}\theta^{R2}\Big\{\frac{\hbar}{Mc}\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\right)\left(\left(\frac{\partial}{\partial x^\ell}\psi^L\right)\left(\frac{\partial}{\partial x_\ell}\psi^R\right)\right) \\
&\quad -Mc^2\left(\frac{\partial}{\partial x^\ell}\underline{\psi}^L\right)\left(\frac{\partial}{\partial x_\ell}\underline{\psi}^R\right)-\frac{\hbar}{M}\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\ell}\psi^L\right)\left(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\ell}\psi^R\right) \\
&\qquad\qquad -\frac{i}{2}\hbar c\left(\frac{\partial\bar{\Psi}}{\partial x^\ell}\right)\gamma^\mu\frac{\partial}{\partial x^\mu}\left(\frac{\partial}{\partial x_\ell}\psi\right)\Big\} \,. \qquad \ldots(30)
\end{aligned}
$$

Now comparison of equations (26) to (30) with equation (35) of Chapter 4 and equation (43) of Chapter 3 shows that $\Phi^L\Phi^L$, $\Phi^R\Phi^R$, $\left(\frac{\partial}{\partial x^\mu}\Phi^L\right)\left(\frac{\partial}{\partial x_\mu}\Phi^L\right)$ and $\left(\frac{\partial}{\partial x^\mu}\Phi^R\right)\left(\frac{\partial}{\partial x_\mu}\Phi^R\right)$ are scalar chiral superfields, and $\Phi^L\Phi^R$ and $\left(\frac{\partial}{\partial x^\mu}\Phi^L\right)\left(\frac{\partial}{\partial x_\mu}\Phi^R\right)$ are general scalar superfields, which do not decompose in the same way as the general superfield of Chapter 4 but have the structure

(Intermediate superfield) + $\theta^{Rn}(\sigma_2^R)_{nn'}\theta^{Rn'}$ (Left handed chiral superfield)+ $\theta^{Ln}(\sigma_2^L)_{nn'}\theta^{Ln'}$(Right handed chiral superfield).

Thus in this case we can only identify the last term with a one dimensional subspace. The full list of one dimensional representation spaces is given below. In each case we have noted that the Minkowski space field used to construct a field $\psi(x)$ is given by $\chi(x_0)$ as specified by equations (2), (3) and (4) of Chapter 3.

(i) From equation (26), using equation (13) we obtain

$$\mathcal{A}_{SS}^{LM} = k'\int_V \hat{d}^4x\, \hat{d}^4\theta \left\{ \frac{1}{c\hbar}\, \theta_j^{L1}\theta_k^{L2} \left\{ -Mc^2 \chi^L_{\frac{1}{2},r}(\sigma_2^L)_{rr'}\chi^L_{\frac{1}{2},r'} - 2Mc^2 \chi^L \underset{\sim}{\chi}^L \right\}\right\} \quad \cdots(31)$$

for each $j,k = 1,2,\cdots,\mathcal{N}$.

(ii) From equation (27), using equation (13) we obtain

$$\mathcal{A}_{SS}^{RM} = k'\int_V \hat{d}^4x\, \hat{d}^4\theta \left\{ \frac{1}{c\hbar}\, \theta_j^{R1}\theta_k^{R2} \left\{ -Mc^2 \chi^R_{\frac{1}{2},r}(\sigma_2^R)_{rr'}\chi^R_{\frac{1}{2},r'} - 2Mc^2 \chi^R \underset{\sim}{\chi}^R \right\}\right\} \quad \cdots(32)$$

for each $j,k = 1,2,\cdots,\mathcal{N}$.

(iii) From equation (28), using equation (9) we obtain

$$\begin{aligned}\mathcal{A}_{SS}^{KE} = 2k'\int_V \hat{d}^4x\, \hat{d}^4\theta \Big\{ \frac{M}{\hbar^2}\, \theta_j^{L1}\theta_k^{L2}\theta_l^{R1}\theta_m^{R2} \Big\{ &\frac{\hbar^2}{4m}\left(\frac{\partial}{\partial x_0^\mu}\frac{\partial}{\partial x_{0\mu}}\right)\chi^L\chi^R \\ &- Mc^2 \underset{\sim}{\chi}^L \underset{\sim}{\chi}^R - \frac{\hbar}{M}\left(\frac{\partial}{\partial x_0^\mu}\chi^L\right)\left(\frac{\partial}{\partial x_{0\mu}}\chi^R\right) - \frac{i\hbar c}{2}\,\bar{\chi}\,\gamma^{c\mu}\frac{\partial}{\partial x_0^\mu}\chi \Big\}\Big\} \quad \cdots(33)\end{aligned}$$

for each $j,k,l,m = 1,2,\cdots,\mathcal{N}$.

(iv) From equation (29) using equation (13) we obtain

$$\begin{aligned}\mathcal{A}'^{LM}_{SS} = k\int_V \hat{d}^4x\, \hat{d}^4\theta \Big\{ \frac{1}{c\hbar}\, \theta_j^{L1}\theta_k^{L2} \Big\{ &-Mc^2\left(\frac{\partial}{\partial x_0^\mu}\chi^L_{\frac{1}{2},r}\right)(\sigma_2^L)_{rr'}\left(\frac{\partial}{\partial x_{0\mu}}\chi^L_{\frac{1}{2},r'}\right) \\ &-2Mc^2\left(\frac{\partial}{\partial x_0^\mu}\chi^L\right)\left(\frac{\partial}{\partial x_{0\mu}}\underset{\sim}{\chi}^L\right)\Big\} \quad \cdots(34)\end{aligned}$$

for each $j,k = 1,2,\cdots,\mathcal{N}$.

(v) From equation (29) with L exchanged with R, using equation (13) we obtain

$$\begin{aligned}\mathcal{A}'^{RM}_{SS} = k\int_V \hat{d}^4x\, \hat{d}^4\theta \Big\{ \frac{1}{c\hbar}\, \theta_j^{R1}\theta_k^{R2} \Big\{ &-Mc^2\left(\frac{\partial}{\partial x_0^\mu}\chi^R_{\frac{1}{2},r}\right)(\sigma_2^R)_{rr'}\left(\frac{\partial}{\partial x_{0\mu}}\chi^R_{\frac{1}{2},r'}\right) \\ &-2Mc^2\left(\frac{\partial}{\partial x_0^\mu}\chi^R\right)\left(\frac{\partial}{\partial x_{0\mu}}\underset{\sim}{\chi}^R\right)\Big\} \quad \cdots(35)\end{aligned}$$

for each $j,k = 1,2,\cdots,\mathcal{N}$.

(vi) From equation (30) using equation (9) we obtain

$$\mathcal{A}_{SS}^{\prime RKG} = 2k\int_V \hat{d}^4x\,\hat{d}^4\theta\Big\{\frac{M}{\hbar^2}\,\theta_j^{L1}\theta_k^{L2}\,\theta_\ell^{R1}\theta_m^{R2}\Big\{-Mc^2\Big(\frac{\partial}{\partial x_{o\varrho}}\underset{\sim}{\chi}^L\Big)\Big(\frac{\partial}{\partial x_o^\varrho}\underset{\sim}{\chi}^R\Big)$$
$$+\frac{\hbar^2}{4M}\Big(\frac{\partial}{\partial x_o^\mu}\frac{\partial}{\partial x_{o\mu}}\Big)\Big(\Big(\frac{\partial}{\partial x_o^\varrho}\chi^L\Big)\Big(\frac{\partial}{\partial x_{o\varrho}}\chi^R\Big)\Big) \qquad \cdots(36)$$
$$-\frac{\hbar}{M}\Big(\frac{\partial}{\partial x_o^\mu}\frac{\partial}{\partial x_o^\varrho}\chi^L\Big)\Big(\frac{\partial}{\partial x_{o\mu}}\frac{\partial}{\partial x_{o\varrho}}\chi^R\Big)-\frac{i\hbar c}{2}\Big(\frac{\partial}{\partial x_o^\varrho}\bar{\chi}\Big)\gamma^{c\mu}\frac{\partial}{\partial x_o^\mu}\Big(\frac{\partial}{\partial x_{o\mu}}\chi\Big)\Big\}\Big\}$$

for each $j,k,\ell,m = 1,2,\cdots,\mathcal{N}$.

It is convenient at this point to combine these integrals in such a way that we can convert to the Majorana representation for the Dirac matrices. Thus from equations (31) and (32), and using the representation of the Dirac matrices given in the appendix we obtain

$$\mathcal{A}_{SS}^{LM} + \mathcal{A}_{SS}^{RM}$$
$$= k'\int_V \hat{d}^4x\,\hat{d}^4\theta\Big\{\frac{1}{2c\hbar}\,\theta_j^\alpha(C^M)_{\alpha\beta}\,\theta_k^\beta\Big\{-Mc^2\bar{\chi}\chi - 2Mc^2\chi^L\underset{\sim}{\chi}^L$$
$$-2Mc^2\chi^R\underset{\sim}{\chi}^R\Big\}\Big\}$$
$$= k'\int_V \hat{d}^4x\,\hat{d}^4\theta\Big\{\frac{1}{2c\hbar}(\theta_j^1\theta_k^4+\theta_k^1\theta_j^4-\theta_j^2\theta_k^3-\theta_k^2\theta^3\Big\{-2Mc^2\chi^R\underset{\sim}{\chi}^R$$
$$-2Mc^2\chi^L\underset{\sim}{\chi}^L - Mc^2\bar{\chi}\chi\Big\}\Big\}$$

Now if we choose some volume in superspace $V = V_oV_n$, we can integrate over the nilpotent part V_n to obtain

$$\mathcal{A}_{SS}^{LM}+\mathcal{A}_{SS}^{RM} = \frac{k'a}{Mc^2}\int_{V_o} d^4x_o\,\Big\{-Mc^2\bar{\chi}\chi - 2Mc^2\chi^L\underset{\sim}{\chi}^L$$
$$-2Mc^2\chi^R\underset{\sim}{\chi}^R\Big\} \qquad \cdots(37)$$

where a is some real number.

To change equation (33) to the Majorana representation we simply change labels on the θ's and the Dirac matrices to obtain

$$\mathcal{A}_{SS}^{KG} = 2k'\int \hat{d}^4x\,\hat{d}^4\theta\Big\{\frac{M}{\hbar^2}\,\theta_j^1\theta_k^2\theta_\ell^3\theta_m^4\Big\{-Mc^2\underset{\sim}{\chi}^L\underset{\sim}{\chi}^R$$
$$-\frac{\hbar}{M}\Big(\frac{\partial}{\partial x_o^\mu}\chi^L\Big)\Big(\frac{\partial}{\partial x_{o\mu}}\chi^R\Big)-\frac{i\hbar c}{2}\bar{\chi}\gamma^{M\mu}\frac{\partial}{\partial x_o^\mu}\chi\Big\}\Big\}.$$

Here we have noted that the first term does not contribute to the action integral. We observe that this term arises as the last term of the 'chiral' superfields mentioned above that have the form $\theta\theta$(Chiral superfield). Now integrating we obtain

$$\mathcal{A}^{KE}_{ss} = \frac{2k'\alpha'}{Mc^2}\int_{V_0} d^4x_0 \left\{-Mc^2 \underset{\sim}{\chi}^L \underset{\sim}{\chi}^R - \frac{\hbar}{M}\left(\frac{\partial}{\partial x^\mu_0}\chi^L\right)\left(\frac{\partial}{\partial x_{0\mu}}\chi^R\right) - \frac{i}{2}\hbar c\, \bar{\chi}\gamma^{M\mu}\frac{\partial}{\partial x^\mu_0}\chi\right\} \quad \cdots(38)$$

where α' is some real number, which will in general be different from α.

Similarly we obtain from equations (35) and (36)

$$\mathcal{A}'^{LM}_{ss} + \mathcal{A}'^{RM}_{ss} = \frac{k\alpha}{Mc^2}\int_{V_0} d^4x_0 \left\{-Mc^2\left(\frac{\partial}{\partial x^\mu_0}\bar{\chi}\right)\left(\frac{\partial}{\partial x_{0\mu}}\chi\right) - 2Mc^2\left(\frac{\partial}{\partial x^\mu_0}\chi^L\right)\left(\frac{\partial}{\partial x_{0\mu}}\underset{\sim}{\chi}^L\right) - 2Mc^2\left(\frac{\partial}{\partial x^\mu_0}\chi^R\right)\left(\frac{\partial}{\partial x_{0\mu}}\underset{\sim}{\chi}^R\right)\right\} \quad \cdots(39)$$

and

$$\mathcal{A}'^{KE}_{ss} = \frac{2k\alpha'}{Mc^2}\int_{V_0} d^4x_0 \left\{-Mc^2\left(\frac{\partial}{\partial x^\mu_0}\underset{\sim}{\chi}^L\right)\left(\frac{\partial}{\partial x_{0\mu}}\underset{\sim}{\chi}^R\right) - \frac{\hbar}{M}\left(\frac{\partial}{\partial x^\rho_0}\frac{\partial}{\partial x^\mu_0}\chi^L\right)\left(\frac{\partial}{\partial x_{0\rho}}\frac{\partial}{\partial x_{0\mu}}\chi^R\right) - \frac{i\hbar c}{2}\left(\frac{\partial}{\partial x^\rho_0}\bar{\chi}\right)\left(\gamma^{M\mu}\frac{\partial}{\partial x^\mu_0}\frac{\partial}{\partial x^\rho_0}\chi\right)\right\} \cdots(40)$$

Now consider any term in the Lagrangian of the form $\left(\frac{\partial}{\partial x^\mu_0}A\right)\left(\frac{\partial}{\partial x_{0\mu}}B\right)$ as in equations (40) and (41). Since

$$\frac{\partial}{\partial x^\mu_0}\frac{\partial}{\partial x_{0\mu}}(AB) = \left(\frac{\partial}{\partial x^\mu_0}\frac{\partial}{\partial x_{0\mu}}A\right)B + 2\left(\frac{\partial}{\partial x^\mu_0}A\right)\left(\frac{\partial}{\partial x_{0\mu}}B\right) + A\left(\frac{\partial}{\partial x^\mu_0}\frac{\partial}{\partial x_{0\mu}}B\right),$$

we have

$$\left(\frac{\partial}{\partial x^\mu_0}A\right)\left(\frac{\partial}{\partial x_{0\mu}}B\right) = \frac{1}{2}\left(\frac{\partial}{\partial x^\mu_0}\frac{\partial}{\partial x_{0\mu}}\right)AB - \frac{1}{2}\left(\frac{\partial}{\partial x^\mu_0}\frac{\partial}{\partial x_{0\mu}}A\right)B - \frac{1}{2}A\left(\frac{\partial}{\partial x^\mu_0}\frac{\partial}{\partial x_{0\mu}}B\right).$$

But each of these fields must satisfy the Klein-Gordon equation, also the first term on the right hand side does not contribute to the action integral. So that, we have, essentially

$$\left(\frac{\partial}{\partial x^\mu_0}A\right)\left(\frac{\partial}{\partial x_{0\mu}}B\right) = \left(\frac{Mc}{\hbar}\right)^2 AB. \quad \cdots(41)$$

Thus if we put $k = k'\left(\frac{\hbar}{Mc}\right)^2$ in equations (39) and (40) they are, respectively, identical to equations (37) and (38).

Now if we put

$$\chi^L = \frac{1}{\sqrt{2}}(A + iB) \quad , \qquad \cdots(42a)$$
$$\chi^R = \frac{1}{\sqrt{2}}(A - iB) \quad , \qquad \cdots(42b)$$
$$\underset{\sim}{\chi}^L = \frac{1}{\sqrt{2}}(F - iG) \qquad \cdots(42c)$$

and
$$\underset{\sim}{\chi}^R = \frac{1}{\sqrt{2}}(F - iG) \quad , \qquad \cdots(42d)$$

as in equation (14) of Chapter 4 of Part II, we obtain the mass and kinetic energy terms for the action given by equations (38) and (39) in complete aggrement with those constructed in section 4.4 of Chapter 4 of Part II.

Thus we are able to deduce the mass and kinetic energy terms for the component fields from the superspace representation. We cannot deduce the relationship between the two since they are independent one dimensional representation spaces. In addition the relationship will depend on the volume in superspace over which we integrate.

5.4 Concluding Remarks.

In this chapter we have been able to construct a procedure for determining the Lagrangian for the component fields from a superfield, and shown that this procedure does, in fact, give the required result when applied to a chial superfield.

This procedure consists of three steps (i) construct a suitable super Poincaré invariant quadratic form for the superfield, (ii) identify, within this quadratic form the one dimensional representation spaces of the Lie group equivalent to the super Poincare group, and (iii) integrate over the nilpotent variables to obtain the component Lagrangian. We can see that the important steps are (i) and (ii). The fact that we integrate in step (iii) is really irrelevant, we can just define a projection operator for this step.

In fact, if we choose the projection operator $p_{1\wedge 2\wedge 3\wedge 4}$, as given in Definition 3.2 of Chapter 3 of Part I this will sufice. We observe that choosing the projection onto the last component of the Grassman algebra does not, in general, give the required result. Since, if L is odd the last component is odd. (eg. If $L = 5$ the highest level element of the algebra is of the form $a\,\xi_{1\wedge 2\wedge 3\wedge 4\wedge 5}$, $a \in \mathbb{R}$.)

We identify the Berezin integral as a projection operator similar to this in the following way. We first choose the superfield component containing our one dimensional represention space, say $\theta\theta C(x)$ or $\theta\theta\theta\theta E(x)$, then $\int^B$ performs the following operations

$$\int^B d^2\theta : \begin{cases} \theta\theta\, C(x) \rightarrow C(x_0) \;, \\ \text{all other terms} \rightarrow 0 \end{cases}$$

and

$$\int^B d^4\theta : \begin{cases} \theta\theta\theta\theta\, E(x) \rightarrow E(x_0) \;, \\ \text{all other terms} \rightarrow 0 \;. \end{cases}$$

This is clearly what is required, but is only effective if we have first correctly chosen the representation space. Its action involves rather more than straightforward integration, if we interpret it in the context of our theory as used in the previous section. This is why the attempts of DeWitt [31] and Rogers [50] appear rather artificial, since they try to interpret $\int^B$ purely as a theory of integration.

APPENDIX

1. General Conventions.

We use the standard convention that all repeated indices are summed, the range of values is usually clear from the context. If this is not so we include the summation explicitly.

We prefer to write all derivatives in full ie. $\frac{\partial}{\partial x^\mu}$, $\frac{\partial}{\partial \theta^\nu_j}$ rather than use the abbreviation ∂_μ.

Our metric takes the form, in Minkowski space M_4

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{\mu\nu}$$

with $\mu, \nu = 1, 2, 3, 4$.

We use the metric for raising and lowering indices on both Minkowski space and superspace variables ie. $g^{\mu\nu} x_\nu = x^\mu$, $g_{\mu\nu} x^\nu_i = x_{i\mu}$.

2. The Pauli Matrices.

2.1 Basic Definitions.

We use the definition of the Pauli matrices as given by Cornwell [39] .

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \qquad \cdots(1)$$

and in addition

$$\sigma_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \qquad \cdots(2)$$

A Latin subscript on a σ matrix, eg. σ_i , is assumed to take the values 1, 2 or 3 and a Greek subscript is assumed to take the values $1, 2, 3$ or 4 corresponding to the coordinates of space-time.

We define

$$\sigma_\mu^L = \sigma_\mu \quad \text{for} \quad \mu = 1, 2, 3, 4 , \qquad \cdots(3)$$

and

$$\sigma_\mu^R = \begin{cases} -\sigma_i & \text{for} \quad \mu = i = 1, 2, 3 , \\ \sigma_4 & \text{for} \quad \mu = 4 . \end{cases} \qquad \cdots(4)$$

Note that $(\sigma_\mu^L)^2 = (\sigma_\mu^R)^2 = 1 \quad \text{for } \mu = 1, 2, 3, 4$ ....(5)

and $\sigma_\mu^L g^{\mu\nu} = \sigma_\mu^R .$ ....(6)

2.2 Useful Identities.

We make use of the following identities in the body of this Thesis.

(i) $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} .$ ....(7)

Proof

This is proved by direct calculation.

(ii) $\sigma_\mu^L \sigma_\rho^L + \sigma_\rho^R \sigma_\mu^R = 2\delta_{\rho\mu}$ ....(8)

Proof

$$\sigma_\mu^L \sigma_\rho^L + \sigma_\rho^R \sigma_\mu^R$$

$$= \begin{cases} \sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} & \text{for } \mu, \rho = i, j = 1, 2, 3 , \\ \sigma_i \sigma_4 - \sigma_4 \sigma_i = 0 & \text{for } \mu = i = 1, 2, 3 , \\ \sigma_4 \sigma_j - \sigma_j \sigma_4 = 0 & \text{for } \rho = j = 1, 2, 3 , \\ \sigma_4 \sigma_4 + \sigma_4 \sigma_4 = 2 & \text{for } \rho = \mu = 4 . \end{cases}$$

$$= 2\delta_{\rho\mu} .$$

(iii) $\sigma_\mu^L \sigma_\rho^R + \sigma_\rho^L \sigma_\mu^R = 2g_{\mu\rho} .$ ....(9)

Proof

$$\sigma_\mu^L \sigma_\rho^R + \sigma_\rho^L \sigma_\mu^R$$

$$= \begin{cases} -\sigma_i \sigma_j - \sigma_j \sigma_i = -2\delta_{ij} & \text{for } j, i = 1, 2, 3 , \\ \sigma_i \sigma_4 - \sigma_4 \sigma_i = 0 & \text{for } i = 1, 2, 3 , \\ -\sigma_4 \sigma_j + \sigma_j \sigma_4 = 0 & \text{for } j = 1, 2, 3 , \\ \sigma_4 \sigma_4 + \sigma_4 \sigma_4 = 2 . \end{cases}$$

$$= 2g_{\mu\rho} .$$

(iv) $\sigma_\rho^L \sigma_\mu^R \sigma_\nu^L \sigma_\rho^L \frac{\partial}{\partial x^\mu} = 4\frac{\partial}{\partial x^\nu} .$ ....(10)

Proof

$$\begin{aligned} \sigma_\rho^L \sigma_\mu^R \sigma_\nu^L \sigma_\rho^L \frac{\partial}{\partial x^\mu} &= (2g_{\rho\mu} - \sigma_\mu^L \sigma_\rho^R)(2g_{\nu\rho} - \sigma_\rho^R \sigma_\nu^L)\frac{\partial}{\partial x^\mu} , \\ &= 4\frac{\partial}{\partial x^\nu} - 2\sigma_\mu^L \sigma_\nu^L - 2\sigma_\mu^L \sigma_\nu^L + 4\sigma_\mu^L \sigma_\nu^L , \\ &= 4\frac{\partial}{\partial x^\nu} . \end{aligned}$$

(v) $\sigma_\mu^L \sigma_\nu^L \sigma_\rho^L \sigma_\mu^L \frac{\partial}{\partial x_\rho} = 4 \frac{\partial}{\partial x_\nu}$ . ....(11)

Proof

$$\sigma_\mu^L \sigma_\nu^L \sigma_\rho^L \sigma_\mu^L \frac{\partial}{\partial x_\rho}$$
$$= (2\delta_{\mu\nu} - \sigma_\nu^R \sigma_\mu^R)(2\delta_{\rho\mu} - \sigma_\mu^R \sigma_\rho^R)\frac{\partial}{\partial x_\rho} ,$$
$$= 4\frac{\partial}{\partial x_\nu} - 2\sigma_\nu^R \sigma_\rho^R \frac{\partial}{\partial x_\rho} - 2\sigma_\nu^R \sigma_\rho^R \frac{\partial}{\partial x_\rho} + 4\sigma_\nu^R \sigma_\rho^R \frac{\partial}{\partial x_\rho} ,$$
$$= 4\frac{\partial}{\partial x_\nu} .$$

(vi) $\sigma_2^L \sigma_\mu^L \sigma_2^L = (\sigma_\mu^R)^t$ . ....(12)

Proof

This is by direct calculation.

(vii) $\sigma_2^L \sigma_\mu^L \sigma_2^L = (\sigma_\mu^R)^*$. ....(13)

Proof

This is by direct calculation.

(viii) $(\sigma_\mu^L \sigma_2^L)_{nr} = (\sigma_\mu^R \sigma_2^R)_{rn}$ . ....(14)

Proof

From equation (13)

$$(\sigma_2^L \sigma_\mu^L \sigma_2^L)_{nr} = (\sigma_\mu^R)_{rn}$$
$$(\sigma_\mu^L \sigma_2^L)_{nr} = (\sigma_2^L)_{nn'} (\sigma_\mu^R)_{rn'}$$
$$= (\sigma_\mu^R \sigma_2^R)_{rn} .$$

(ix) $(\sigma_\mu^A \sigma_\rho^B \cdots \sigma_\xi^C \sigma_\nu^D \sigma_2^E)_{nr} = (\sigma_\nu^{\bar{D}} \sigma_\xi^{\bar{C}} \cdots \sigma_\rho^{\bar{B}} \sigma_\mu^{\bar{A}} \sigma_2^{\bar{E}})_{rn}$ ...(15)

for $A, B, C, D, E, \cdots = L$ or R,

and $\bar{A}, \bar{B}, \cdots = \begin{cases} L & \text{if } A, B, \cdots = R , \\ R & \text{if } A, B, \cdots = L . \end{cases}$

That is we reverse the order of the Greek subscripts and repace L by R and vice versa.

Proof

This is by repeated use of equation (14).

2.3 Fierz Rearrangement.

The four Pauli matrices $\{\sigma_\mu\}$ form a basis for the four dimensional vector space of 2×2 complex matrices. The set $\{\sigma_\mu \sigma_2\}$ form an alternative basis with $\{\sigma_i\sigma_2\}$ symmetric, and $\sigma_4\sigma_2$ antisymmetric.

If $\{\Gamma^A, A=1,2,3,4\}$ is a basis for the set of complex 2×2 matrices and $\{\Gamma_A, A=1,2,3,4\}$ is a second basis such that for each $A=1,2,3,4$

$$(\Gamma^A)_{nr}(\Gamma_A)_{rn'} = (I_2)_{nn'}$$

then the following formula is valid

$$\delta^\alpha_a \, \delta^\beta_b = \sum_{A=1}^{4} \tfrac{1}{2}\,(\Gamma^A)^\alpha_b\,(\Gamma_A)^\beta_a \,. \qquad \cdots(16)$$

Of course if we set $\Gamma^A = \sigma_A$ then $\Gamma_A = \sigma_A$ as well.

3. The Dirac Matrices.

3.1 Basic Definitions.

Our conventions follow Cornwell [39] Chapter 17.

The four contravariant Dirac matrices γ^μ satisfy

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I_4 \,. \qquad \cdots(17)$$

They are chosen to be such that

$$(\gamma^i)^\dagger = -\gamma^i \quad \text{for } i=1,2,3 \qquad \cdots(18)$$

and
$$(\gamma^4)^\dagger = \gamma^4 \,. \qquad \cdots(19)$$

The matrix γ^5 is defined by

$$\gamma^5 = i\gamma^1\gamma^2\gamma^3\gamma^4 \qquad \cdots(20)$$

so that
$$(\gamma^5)^\dagger = \gamma^5 . \qquad \cdots(21)$$

Equation (17) implies that

$$(\gamma^i)^2 = -I_4 \quad \text{for } i=1,2,3 \qquad \cdots(22)$$

and
$$(\gamma^4)^2 = (\gamma^5)^2 = I_4 \,. \qquad \cdots(23)$$

Equations (18) and (19) together with equation (23) imply that

$$(\gamma^\mu)^\dagger = \gamma^4\gamma^\mu\gamma^4 . \qquad \cdots(24)$$

The four covariant matrices are defined by

$$\gamma_\mu = \gamma^\nu g_{\nu\mu} \quad , \qquad \cdots(25)$$

also
$$\gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_4 \,. \qquad \cdots(26)$$

Thus
$$\gamma_j = -\gamma^j \quad \text{for } j=1,2,3 \text{ and } 5 \qquad \cdots(27)$$

but
$$\gamma_4 = \gamma^4 . \qquad \cdots(28)$$

The charge conjugation matrix is defined by

$$C^{-1}\gamma_\mu C = -(\gamma_\mu)^t \quad , \qquad \cdots(29)$$

it is necessarily antisymmetric ie.

$$C^t = -C \qquad \cdots(30)$$

and is chosen to be unitary ie.

$$C^{+}C = I_4 \qquad \cdots(31)$$

For each representation of the Dirac matrices C is then unique, except for a multiplicative constant of the form $e^{i\eta}, \eta \in \mathbb{R}$.

We note that if two sets of Dirac matrices γ^μ and γ'^μ are related by the similarity transformation S ie.

$$S\gamma^\mu S^{-1} = \gamma'^\mu \quad , \qquad \cdots(32)$$

then it follows from equation (28) that the corresponding charge conjugation matrices C and C' are related by

$$SC(S)^t = C' \qquad \cdots(33)$$

modulo a multiplication factor of the form $e^{i\eta}, \eta \in \mathbb{R}$.

The six matrices

$$\begin{aligned}\sigma_{\lambda\mu} &= -\tfrac{1}{2}\gamma_\lambda\gamma_\mu \quad , \quad \lambda \neq \mu \\ &= -\tfrac{1}{4}[\gamma_\lambda , \gamma_\mu] \; , \end{aligned} \qquad \cdots(34)$$

provide a basis of a representation of the Lie algebra of the Lorentz group $SO(3,1;\mathbb{R})$. This is a reducible representation that can be transformed by a similarity transformation to the representation $\Gamma^{0,\frac{1}{2}} \oplus \Gamma^{\frac{1}{2},0}$.

3.2 Explicit Representations.

It is convenient to use two explicit representations for the Dirac matrices.

(i) Majorana Representation.

This is constructed so that $\gamma_\mu , \mu = 1,2,3,4$ is purely imaginary, it then follows that the six matrices $\sigma_{\mu\nu}$ of equation (33) are real. We take a matrix representation to be

$$\gamma_1^M = \begin{bmatrix} i\sigma_3 & \\ 0 & -i\sigma_3 \end{bmatrix}, \quad \gamma_2^M = \begin{bmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{bmatrix}, \quad \gamma_3^M = \begin{bmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{bmatrix}$$
$$\gamma_4^M = \begin{bmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{bmatrix}, \quad \gamma_5^M = \begin{bmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix}, \quad C^M = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix}. \quad \cdots(35)$$

(ii) Chiral Representation.

This is related to the Majorana representation above by the similarity transformation

$$S_M^c = (S_M^c)^{-1} = (S_M^c)^{\dagger} = \frac{1}{2}\begin{bmatrix} I-\sigma_2 & I+\sigma_2 \\ I+\sigma_2 & -I+\sigma_2 \end{bmatrix} \quad \cdots(36)$$

It is important because the representation of the Lie algebra of the Lorentz group is in the form of a direct sum $\Gamma^{0,\frac{1}{2}} \oplus \Gamma^{\frac{1}{2},0}$. The matrix representation is given by

$$\gamma_j^c = \begin{bmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{bmatrix} \text{ for } j=1,2,3, \quad \gamma_4^c = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$
$$\gamma_5^c = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C^c = \begin{bmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix}. \quad \cdots(37)$$

This is conveniently written in terms of the left and right handed Pauli matrices as

$$\gamma_\mu^c = \begin{bmatrix} 0 & -\sigma_\mu^L \\ -\sigma_\mu^R & 0 \end{bmatrix}, \qquad \sigma_{\lambda\mu}^c = -\frac{1}{2}\begin{bmatrix} \sigma_\lambda^L\sigma_\mu^R & 0 \\ 0 & \sigma_\lambda^R\sigma_\mu^L \end{bmatrix},$$
$$\gamma_5^c = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \qquad C^c = \begin{bmatrix} \sigma_2^L & 0 \\ 0 & \sigma_2^R \end{bmatrix}. \quad \cdots(38)$$

3.3 Dirac Matrices as a Basis for the 4×4 Complex Matrices.

The space of 4×4 complex matrices is a 16 dimensional vector space. We can choose various combinations of the Dirac matrices as a basis for this space. The two most important are as follows.

antihermitian and generating the rotations, then the set $\{A_i^{(j)}, -iA_i^{(j)}\}$ generate $\Gamma^{j,0}(\Lambda)$. It follows that

$$(\Gamma^{0,j}(\Lambda^{-1}))^{\dagger} = \Gamma^{j,0}(\Lambda) , \qquad \cdots(44)$$

and in particular

$$(\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})^{\dagger}) = \Gamma^{\frac{1}{2},0}(\Lambda) . \qquad \cdots(45)$$

Now if we choose $x^{\mu} = (0, 0, 0, Mc)$ and set $(\Lambda x)^{\mu} = p^{\mu}$, we can use equations (43) and (45) to obtain

$$\Gamma^{0,\frac{1}{2}}(\Lambda)_{nr'}\Gamma^{\frac{1}{2},0}(\Lambda^{-1})_{r'r} = \frac{1}{Mc}(p^{\mu}\sigma_{\mu}^{L})_{nr} \qquad \cdots(46)$$

Also since

$$\left(\frac{1}{Mc}(p^{\mu}\sigma_{\mu}^{L})\right)^{-1} = \frac{1}{Mc}(p^{\mu}\sigma_{\mu}^{R}) , \qquad \cdots(47)$$

as can be proved by direct calculation, we have

$$\Gamma^{\frac{1}{2},0}(\Lambda)_{nr'}\Gamma^{0,\frac{1}{2}}(\Lambda^{-1})_{r'r} = \frac{1}{Mc}(p^{\mu}\sigma_{\mu}^{R})_{nr} . \qquad \cdots(48)$$

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