FLATNESS, EXTENSION AND AMALGAMATION IN MONOIDS, SEMIGROUPS AND RINGS

JAMES HENRY RENSHAW

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Department of Pure Mathematics

University of St Andrews

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DECLARATION

I declare that the accompanying thesis has been composed by myself and that it is a record of my own work. No part of the thesis has been accepted in any previous application for a higher degree.

JAMES HENRY RENSHAW

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JOHN M HOWIE

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ABSTRACT

We begin our study of amalgamations by examining some ideas which are well-known for the category of R-modules. In particular we look at such notions as direct limits, pushouts, pullbacks, tensor products and flatness in the category of S-sets.

Chapter II introduces the important concept of free extensions and uses this to describe the amalgamated free product.

In Chapter III we define the extension property and the notion of purity. We show that many of the important notions in semigroup amalgams are intimately connected to these. In Section 2 we deduce that 'the extension property implies amalgamation' and more surprisingly that a semigroup U is an amalgamation base if and only if it has the extension property in every containing semigroup.

Chapter IV revisits the idea of flatness and after some technical results we prove a result, similar to one for rings, on flat amalgams.

In Chapter V we show that the results of Hall and Howie on perfect amalgams can be proved using the same techniques as those used in Chapters III and IV.

We conclude the thesis with a look at the case of rings. We show that almost all of the results for semigroup amalgams examined in the previous chapters, also hold for ring amalgams.

CONTENTS

Introduction

Let $\underset{\sim}{K}$ be a class of algebras all of the same type. The definition of an amalgam will be considered more carefully in Section 1 but for the moment we may think of it as a family $(B_i)_{i \in I}$ of algebras in $\underset{\sim}{K}$ intersecting in a common sub-algebra A, called the core of the amalgam. It is clear that $\bigcup_{i \in I} B_i$ need not be a member of $\underset{\sim}{K}$. The main question is: can we embed the 'partial algebra' $\bigcup_{i \in I} B_i$ in an algebra $C \in \underset{\sim}{K}$? If we can we say that the amalgam is weakly embedded (or embeddable) in C. If in addition this embedding can take place 'without collapse', that is to say the intersection of the algebras B_i in C is isomorphic to A, then we say that the embedding is strong. If every amalgam from $\underset{\sim}{K}$ can be (weakly, strongly) embedded then $\underset{\sim}{K}$ has the (weak, strong) amalgamation property. A recent paper by Kiss, Márki, Pröhle and Tholen gives a comprehensive description of a wide class of algebras with or without the amalgamation property. For example, the classes

(1) groups,

(2) abelian groups,

(3) finite groups,

(4) R-modules,

(5) S-sets,

(6) lattices,

(7) Boolean algebras,

all have the strong amalgamation property, while the classes,

(8) fields,

(9) distributive lattices,

(10) Banach spaces,

have the weak but not the strong amalgamation property.

There are however some classes which do not have even the weak amalgamation property, notably rings and semigroups. For example let $U = \{u,v,w,0\}$ be a four element null semigroup. Let $S = U \cup \{a\}$, with $au = ua = v$ and all other products equal to 0. Let $T = U \cup \{b\}$, with $bv = vb = w$ and all other products equal to 0. Then S and T are semigroups with a common sub-semigroup U. Suppose that this amalgam could be embedded in a semigroup P, say. Then in P we have

$$w = bv = b(ua) = (bu)a = 0.a = 0$$

and so we have a contradiction. We can of course confine our attention to a subclass of the class of all semigroups. A great deal of work in this area has been done by, for example T E Hall and G Clarke. In fact Clarke [6] has effectively managed to reduce the problem of determining which varieties of semigroup have the weak (strong) amalgamation property, to a group theoretic one.

The question thus arises: under what circumstances is a semigroup/ring amalgam embeddable in a semigroup/ring?

The problem for rings was first tackled by P M Cohn in 1959. His results and techniques were of a homological nature, using R-modules and tensor products of R-modules. He proved among other things, that if $(S_i)_{i \in I}$ is a collection of rings with a common subring R, and if each S_i/R is flat as an R-module then the amalgam is strongly embeddable. Probably one of the most important parts of this work was an extremely useful description of the amalgamated free product in terms of tensor products of modules.

The amalgamated free product, which had earlier been used by Schreier in 1927 to prove that the class of groups had the strong amalgamation property, was also the main tool of J M Howie in his early work on semigroup amalgams. Howie extended Schreier's result by proving that any semigroup amalgam with an almost unitary core is strongly embeddable (among the almost unitary subsemigroups are the subgroups). An alternative proof of one of Howie's results on unitary amalgams was given in 1976 by G B Preston who introduced to the theory of semigroup amalgams the techniques of representations of semigroups. This work was taken up by T E Hall in 1978 and later recast in terms of S-sets by Howie.

Our approach to the problem will be a homological one. We aim to carry on where Cohn, Hall and Howie left off and hope to show that the techniques and results involved in the study of ring and semigroup amalgams are very closely linked.

After preliminaries, we begin our investigation by looking at various constructions which will be of use in later chapters. In particular the notions of pushout, pullback and tensor product will play a central role in most of our work. The results in Chapters I.3 and I.4 are probably well-known in other categories. However, there does not appear to be any concise reference available for the category of S-sets and so we prove most of the results in detail. Chapter I.5 is the first of two chapters on flatness and we confine ourself at this stage to study results of a more fundamental nature, for example we study connections between flatness and notions such as direct limits, injectivity and dominions.

Chapter II introduces the important concept of free-extensions and uses this to describe, in a similar manner to that for rings, the amalgamated free product. Some necessary and/or sufficient conditions for embeddability of an amalgam are then deduced.

In Chapter III we define the extension property and the notion of purity, first introduced for rings by P M Cohn. We show that many of the important notions in semigroup amalgams either imply the extension property or are intimately connected with it. In Section 2 we prove that an amalgam of semigroups S_i in which the core U has the extension property in each S_i, is strongly embeddable. Many of the principal results on amalgamation can be deduced from this. Even more surprisingly we show that a semigroup U is an amalgamation base if and only if it has the extension property in every containing semigroup.

Chapter IV revisits the idea of flatness and after some technical results we prove a result similar to Cohn's on flat amalgams.

In Chapter V we show that the results of Hall and Howie on perfect amalgams can be proved using the same techniques as those used in Chapters II and IV.

Finally, we examine the case of rings in Chapter VI. Almost all of the results for semigroup amalgams examined in the previous chapters, hold for ring amalgams. In particular the notion of the extension property is just as important for rings as for semigroups. One of the more surprising results in this chapter is that the ring theoretic version of the <u>perfect amalgams</u> of Hall and Howie are precisely the <u>flat</u> amalgams of Cohn. Although the theories of ring amalgams and of semigroup amalgams have developed independently, it would seem that their paths have converged.

CHAPTER I

1. Preliminaries

Let $\underset{\sim}{K}$ be a class of algebras of some fixed type. An amalgam in $\underset{\sim}{K}$ consists of an algebra A, called the core of the amalgam, a family of algebras $\{B_i : i \in I\}$ and a family of monomorphisms $\{\varphi_i : A \to B_i : i \in I\}$. The amalgam is denoted by $[A;B_i,\varphi_i(i \in I)]$ or simply $[A;B_i]$. We shall say that the amalgam $[A;B_i,\varphi_i(i \in I)]$ is weakly embeddable (in an algebra $C \in \underset{\sim}{K}$) if there exists monomorphisms $\vartheta_i : B_i \to C$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi_i"] \arrow[d, "\varphi_j"'] & B_i \arrow[d, "\vartheta_i"] \\ B_j \arrow[r, "\vartheta_j"'] & C \end{tikzcd}$$

commutes for all $i \neq j$ in I. If, in addition, we have that for all $i \neq j$ in I,

$$\vartheta_i(B_i) \cap \vartheta_j(B_j) = (\vartheta_i \circ \varphi_i)(A),$$

then we say that the amalgam is strongly embeddable (in C).

We shall be dealing with the case when $\underset{\sim}{K}$ is the class of all semigroups or the class of all monoids. In the final chapter we shall look at the case for rings.

Given a semigroup amalgam $[U; S_i, \varphi_i]$, is there a natural candidate in which to embed the amalgam? The answer is of course yes, the amalgamated free product which we shall now describe.

Let $\{S_i : i \in I\}$ be a family of disjoint semigroups. If $a \in \bigcup_{i \in I} S_i$, then there is a unique k in I such that $a \in S_k$. Following Howie [22], we shall refer to this k as the index of a and write $k = \sigma(a)$. Consider the collection of all finite 'words'

$$(a_1, a_2, \ldots, a_m),$$

where $m \geq 1$, $a_r \in \bigcup_{i \in I} S_i$ and $\sigma(a_r) \neq \sigma(a_{r+1})$. Define a binary operation on this collection of words by the rule that

$$(a_1,\ldots,a_m)(b_1,\ldots,b_n) = \begin{cases} (a_1,\ldots,a_m,b_1,\ldots,b_n) & : \sigma(a_m) \neq \sigma(b_1) \\ (a_1,\ldots,a_m b_1,\ldots,b_n) & : \sigma(a_m) = \sigma(b_1). \end{cases}$$

Then this family of words together with this binary operation form a semigroup, called the free product of the semigroup S_i, and denoted by $\pi^*\{S_i : i \in I\}$. It is clear that the maps $\gamma_i : S_i \to \pi^*\{S_i : i \in I\}$ defined by

$$\gamma_i(s_i) = (s_i), \qquad i \in I,$$

are monomorphisms.

Now let $[U; S_i, \varphi_i]$ be an amalgam of semigroups and let ρ be the congruence on $\pi^*\{S_i : i \in I\}$ generated by the relation

$$R = \{(\gamma_i\varphi_i(u),\ \gamma_j\varphi_j(u)) : u \in U,\ i,j \in I\}.$$

We shall denote the quotient $\pi^*\{S_i : i \in I\}/\rho$ by $\pi^*_U\{S_i : i \in I\}$ or $\pi^*_U S_i$ (or simply $S_1 * S_2 * S_3 * \ldots * S_r$ if $|I| = r \in \mathbb{N}$) and call it the free product of the amalgam $[U; S_i, \varphi_i]$ or the amalgamated free product. It comes equipped with natural mappings $\vartheta_i : S_i \to \pi^*_U S_i$ given by

$$\vartheta_i = \rho^{\natural} \circ \gamma_i, \qquad i \in I,$$

and it is clear that the diagram

$$\begin{tikzcd} U \arrow[r, "\varphi_i"] \arrow[d, "\varphi_j"'] & S_i \arrow[d, "\vartheta_i"] \\ S_j \arrow[r, "\vartheta_j"'] & \pi^*_U S_i \end{tikzcd}$$

commutes for all $i \neq j$ in I.

LEMMA 1.1 [Howie, 22, Proposition VII.1.2]. *Let* $\pi^* S_i$ *be the free product of a family of semigroups* S_i. *Then* $\pi^* S_i$ *is the coproduct in the category of semigroups of the family* $\{S_i : i \in I\}$. *That is to say, if* T *is a semigroup for which homomorphisms* $\psi_i : S_i \to T$ *exist, then there exists a unique homomorphism* $\Phi : \pi^* S_i \to T$ *such that* $\Phi \circ \gamma_i = \psi_i$ $(i \in I)$.

LEMMA 1.2 [Howie, 22, Proposition VII.1.10]. *If* $[U; S_i, \varphi_i]$ *is an amalgam, then* $\pi^*_U S_i$ *is the pushout in the category of semigroups of the diagram* $\{U \to S_i\}_{i \in I}$. *That is to say, if* Q *is a semigroup for which homomorphisms* $\tau_i : S_i \to Q_i$ $(i \in I)$ *exist such that* $\tau_i \circ \varphi_i = \tau_j \circ \varphi_j$, $(i \neq j \text{ in } I)$, *then there exists a unique homomorphism* $\delta : \pi^*_U S_i \to Q$ *such that* $\delta \circ \vartheta_i = \tau_i$ $(i \in I)$.

THEOREM 1.3 [Howie, 22, Theorem VII.1.11]. *The amalgam* $[U; S_i, \varphi_i]$ *is embeddable in a semigroup if and only if it is embeddable in* $\pi^*_U S_i$.

Associativity of free product with amalgamation is provided by

THEOREM 1.4 [Howie, 20, Theorem 1.3]. Let $[U; S_i, \varphi_i : i \in I]$ be an amalgam. Suppose that the index set I is partitioned into disjoint subsets J_k $(k \in K)$ and that the amalgam $[U; S_i, \varphi_j : j \in J_k]$ is embeddable for each k. Let $P_k = \pi^*_U\{S_j : j \in J_k\}$. Let ϑ_k $(= \vartheta_j \circ \varphi_j$ for every j in $J_k)$ be the natural monomorphism from U into P_k and suppose that the amalgam

$$[U; P_k, \vartheta_k : k \in K]$$

is embeddable. The the amalgam

$$[U; S_i, \varphi_i, i \in I]$$

is embeddable and

$$\pi^*_U\{S_i : i \in I\} \simeq \pi^*_U\{\pi^*_U\{S_j : j \in J_k\} : k \in K\}.$$

THEOREM 1.5 [Howie, 20, Theorem 1.4]. If U is a semigroup such that every amalgam $[U; S,T]$ of two semigroups with U as core is embeddable then every amalgam $[U; S_i]$ of arbitrarily many semigroups with U as core is embeddable.

A semigroup U satisfying the conditions of Theorem 1.5 will be called an amalgamation base.

A slight modification of the proof of the above theorem allows us to deduce:

THEOREM 1.6 *Let* U *be a semigroup and let* $\underset{\sim}{K}$ *be a class of semigroups which contain* U *as a subsemigroup and suppose that* $\underset{\sim}{K}$ *satisfies*

(1) *for all* $S,T \in K$, *the amalgam* $[U; S,T]$ *is embeddable, and*

(2) *for all* $S,T \in \underset{\sim}{K}$, $S *_U T \in \underset{\sim}{K}$.

If $\{S_i : i \in I\}$ *is an arbitrary collection of semigroups in* $\underset{\sim}{K}$, *then the amalgam* $[U; S_i]$ *is embeddable*.

Informally we have: Suppose that P is a property that a semigroup U may have in some of its containing semigroups and suppose that whenever U has property P in semigroups S and T then the amalgam $[U; S,T]$ is embeddable and U has property P in $S *_U T$. If $\{S_i : i \in I\}$ is an arbitrary collection of semigroups such that U has property P in each S_i then the amalgam $[U; S_i]$ is embeddable.

Let U be a semigroup. We shall denote by ${}^1U$, the monoid obtained from U by adjoining an identity 1, whether or not U already has one.

The following easily proved results will be of use later.

THEOREM 1.7 *Let* $[U; S,T]$ *be an amalgam*. *Then* ${}^1S *_{{}^1U} {}^1T = {}^1(S *_U T)$, *where* $S *_U T$ *is the* SEMIGROUP *free product of the amalgam* $[U; S,T]$ *and* ${}^1S *_{{}^1U} {}^1T$ *is the* MONOID *free product of the amalgam* $[{}^1U; {}^1S, {}^1T]$.

THEOREM 1.8 *The amalgam* $[U;\ S,\ T]$ *is embeddable if and only if the amalgam* $[{}^1U;\ {}^1S,\ {}^1T]$ *is embeddable.*

Let U be a subsemigroup of a semigroup S. We say that U *dominates* an element d of S if for all semigroups T and for all homomorphisms $\beta,\gamma : S \to T$,

$$[(\forall u \in U)\ \beta(u) = \gamma(u)] \text{ implies } \beta(d) = \gamma(d).$$

The set of elements dominated by U is called the *dominion* of U in S and is written $\mathrm{Dom}_S(U)$. If $\mathrm{Dom}_S(U) = U$ we say that U is *closed in* S and if U is closed in every containing semigroup we say that U is *absolutely closed*.

THEOREM 1.9 [Howie, 22, VII.2.3]. *Let* U *be a subsemigroup of a semigroup* S. *Let* S' *be a semigroup disjoint from* S *and let* $\alpha : S \to S'$ *be an isomorphism. Let* μ,μ' *be the natural maps from* S, S', *respectively into the free product of the amalgam* $[U;S,S',i,\alpha|U]$. *Then*

$$\mu(S) \cap \mu'(S') = \mu(\mathrm{Dom}_S(U)).$$

It can be shown that the maps μ, μ' above are always 1-1 and so we see that U is closed in S if and only if the amalgam $[U;\ S,S';\ i,\alpha|U]$ is strongly embeddable. An alternative and more useful description of the dominion will be given later.

Let U be a monoid with identity 1. A set X together with a map $f : X \times U \to X$ is called a *right* U-*set* if

(i) $f(x,1) = x$, for all x in X, and

(ii) $f(x,uv) = f(f(x,u),v)$, for all x in X, u,v in U.

As is usual we shall denote $f(x,u)$ by xu and simply refer to X as the U-set.

If X and Y are right U-sets and if $\alpha:X \to Y$ is a map, then we say that α is a (right) U-map if for all x in X, u in U,

$$f(xu) = f(x)u.$$

The collection of right U-sets and right U-maps forms a category which we shall denote by ENS-$\underline{U}$. Notice that ENS-$\{1\}$ is naturally equivalent to ENS, the category of sets. The dual notions of left U-sets and left U-maps are obvious and the category of left U-sets will be denoted by $\underline{U}$-ENS. If X is a right U-set and also a left S-set and if in addition

$$s(xu) = (sx)u, \text{ for all } x \text{ in } X,\ u \text{ in } U,\ s \text{ in } S,$$

then we say that X is an (S,U)-biset. The category of (S,U)-bisets will be denoted by $\underline{S}$-ENS-$\underline{U}$ and its maps called (S,U)-maps.

Let $X \in$ ENS-$\underline{U}$ and let σ be an equivalence on X. We say that σ is a (right) U-congruence on X if

$$(x,y) \in \sigma,\ u \in U \text{ implies } (xu,yu) \in \sigma.$$

It is clear that the quotient X/σ becomes a right U-set if we define

$$(x\sigma).u = (xu)\sigma, \text{ for all } x \text{ in } X,\ u \text{ in } U.$$

EXAMPLE 1.10. Let $f:X \to Y$ be a right U-map. Then $\ker f = \{(a,b) \in X \times X : f(a) = f(b)\}$, the kernel of the map f, is a right U-congruence on X and $\operatorname{im} f \simeq X/\ker f$.

EXAMPLE 1.11. Let $f:X \to Y$ be a right U-monomorphism. Define ρ_f on Y by

$$\rho_f = (\mathrm{im} f \times \mathrm{im} f) \cup 1_Y.$$

Then ρ_f is a right U-congruence on Y. We shall normally denote the quotient Y/ρ_f by Y/X and an element $y\rho_f$ by $\overline{y}$.

The following is easy to prove.

LEMMA 1.12. *Let* $f:X \to Y$ *and* $g:Y \to Z$ *be right* U-*monomorphisms. Then there exists a right* U-*monomorphism* $h : Y/X \to Z/X$ *and* $Z/Y \simeq (Z/X)/(Y/X)$.

We have mentioned the term monomorphism to mean 1-1 map. An obvious, related question is "are the epimorphisms in ENS-$\underline{U}$ onto?" The answer is not surprisingly, yes.

LEMMA 1.13. *Let* $f:X$ Y *be a right* U-*epimorphism. Then* f *is onto*.

Proof. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, shift left=1.5, "g"] \arrow[r, shift right=1.5, "h"'] & Y/\mathrm{im} f \end{tikzcd}$$

where $g(y) = \overline{y}$ and $h(y) = \overline{f(x)}$, for some x in X. Then it is clear that $h \circ f = g \circ f$ and so $h = g$. Hence $Y \simeq \mathrm{im} f$ and f is onto.

A useful result concerning U-monomorphisms is:

THEOREM 1.14. *Let* $f:A \to B$ and $g:B \to C$ *be right* U-*maps. Then* $g \circ f$ *is* 1-1 *if and only if* f *is* 1-1 *and* $\ker g \cap \rho_f = 1_B$.

Proof. Suppose that $g \circ f$ is 1-1. Then clearly f is 1-1 and $1_B \subseteq \ker g \cap \rho_f$. Suppose then that $(x,y) \in \ker g \cap \rho_f$. Then either $x = y$, giving us the required result, or $x = f(a)$, $y = f(a')$ for some a,a' in A. Hence $gf(a) = gf(a')$ and so $a = a'$, since $g \circ f$ is 1-1.

Conversely, suppose that $(g \circ f)(a) = (g \circ f)(a')$. Then $(f(a),f(a')) \in \ker g \cap \rho_f = 1_B$. Hence $a = a'$, since f is 1-1.

We end this section by mentioning a few of the main results on amalgamations to date.

Let U be a subsemigroup of a semigroup S. We say that U is unitary in S if for all u in U, s in S

$$us \in U \text{ or } su \in U \text{ implies } s \in U.$$

A related concept is that at almost unitary subsemigroups. A subsemigroup U of a semigroup S is said to be almost unitary [Howie, 22], if there exist mappings $\lambda : S \to S$, $\rho : S \to S$ such that

(1) $\lambda^2 = \lambda$, $\rho^2 = \rho$

(2) $\lambda(st) = (\lambda s)t$, $(st)\rho = s(t\rho)$, for all $s,t \in S$

(3) $\lambda(s\rho) = (\lambda s)\rho$, for all $s \in S$

(4) $s(\lambda t) = (s\rho)t$, for all $s,t \in S$

(5) $\lambda|_U = \rho|_U = 1_U$

(6) U is unitary in $\lambda S \rho$.

Notice that for notational purposes we have written λ on the left and ρ on the right. It is easy to see that 'unitary' implies 'almost unitary' (take $\lambda = \rho = 1_U$). It is also easy to show that if

U is a group with identity e, then U is almost unitary in every containing semigroup (take $\lambda(s) = es$, $s\rho = se$, $s \in S$).

THEOREM 1.15. [Howie, 22, VII.3.11]. The amalgam $[U;S_i]$ is embeddable if U is almost unitary in each S_i.

In particular we see that every group is an amalgamation base in the class of all semigroups.

Say that a subsemigroup U of a semigroup S is relatively unitary if for all u in U, s in S

(1) $us \in U$ implies $us \in uU \cup \{u\}$,

(2) $su \in U$ implies $su \in Uu \cup \{u\}$.

It is easy to see that if U is almost unitary in S then U is relatively unitary in S.

If U is a subsemigroup of a semigroup S, then we say that the pair (U,S) is a (weak) amalgamation pair if every amalgam of the form $[U; S,T]$ is (weakly) embeddable.

THEOREM 1.16 [Howie, 15, Theorem 4.3]. If (U,S) is a weak amalgamation pair then U is relatively unitary in S.

The following definition is due to T E Hall [13], the notation and terminology being due to Howie [23]. Let U be a submonoid of a monoid S. Say that U is right perfect in S if for all $X \in \text{ENS-}\underline{S}$, all $Y \in \text{ENS-}\underline{U}$ and all right U-monomorphisms $f:X \to Y$, there exists $Z \in \text{ENS-}\underline{S}$, a U-monomorphism $g:Y \to Z$ and an S-monomorphism $h:X \to Z$ such that

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

commutes.

THEOREM 1.17 [Hall, 13; Howie, 23]. *Let $[U;S_i]$ be an amalgam such that U is right perfect in each S_i. Then the amalgam is strongly embeddable.*

It is known, Hall [13], Howie [23], that if U is an *inverse monoid* i.e. a monoid such that every principal left and right ideal is generated by a unique idempotent, then U is right perfect in every containing monoid. Hence

THEOREM 1.18 [Howie, 21, 23; Hall, 13]. *Every inverse semigroup is an amalgamation base in the class of all semigroups.*

Say that a monoid U has the *right extension property* in a containing monoid S if for all $X \in \text{ENS-}\underline{U}$, there exists $Y \in \text{ENS-}\underline{S}$ and a U-monomorphism $f:X \to Y$. It can be shown (see Hall [13, Theorem 3]) that if U is right perfect in S then U has the right extension property in S.

THEOREM 1.19 [Hall, 13, Theorem 7]. *If (U,S) is a weak amalgamation pair then U has the right extension property in S.*

2. Indecomposable U-sets

Let U be a monoid and let $X \in \text{ENS-}\underline{U}$. We say that X is decomposable if there exists non-empty sub. U-sets of X, X_1 and X_2 say, such that $X = X_1 \dot{\cup} X_2$. Otherwise we say that X is indecomposable. We say that X is cyclic if $X = xU$, for some x in X.

LEMMA 2.1 [Knauer, 28, Lemma 2.1]. *If X is a cyclic right U-set then X is indecomposable.*

LEMMA 2.2 [Knauer, 28, Lemma 2.2]. *Let $(X_i)_{i \in I}$ be a family of indecomposable U-sets with $\bigcap_{i \in I} X_i \neq \varphi$. Then $\bigcup_{i \in I} X_i$ is indecomposable.*

LEMMA 2.3 [Knauer, 28, Lemma 2.3]. *A monoid U has the property that every indecomposable U-set is cyclic if and only if U is a group.*

Let $X \in \text{ENS-}\underline{U}$. We say that $x, y \in X$ are connected and write $x \sim y$ if there exists $u_1, \ldots, u_n, v_1, \ldots, v_n$ in U, $x_1, \ldots, x_n$ in X such that

$$
\begin{gathered}
x = x_1 u_1, \\
x_1 v_1 = x_2 u_2, \\
\ldots\ldots \\
x_n v_n = y.
\end{gathered}
$$

It is easy to verify that $\sim$ is an equivalence relation on X. We say that X is connected if $x \sim y$ for all x, y in X.

LEMMA 2.4 <u>Let</u> $X \in$ ENS-$\underline{U}$. <u>Then</u> X <u>is indecomposable if and only if</u> X <u>is connected</u>. <u>Moreover every</u> U-set <u>can be decomposed, in a unique way, into a disjoint union of indecomposable sub</u> U-<u>sets</u>.

<u>Proof</u>. Suppose that X is connected and suppose that $X = A \dot{\cup} B$ for some $A,B \in$ ENS-$\underline{U}$. Let $a \in A$, $b \in B$. Then since X is connected there exists a set of equations in X

$$\begin{aligned} a &= x_1u_1, \\ x_1v_1 &= x_2u_2, \\ &\cdots \\ x_nv_n &= b. \end{aligned}$$

Since A and B are disjoint we see that $x_1 \in A$. Similarly $x_2,\ldots,x_n \in A$ and so $b = x_nv_n \in A$ giving us the required contradiction.

Conversely, suppose that X is indecomposable but not connected. Define a U-congruence ρ on X by

$$(x,y) \in \rho \text{ if and only if } x \sim y.$$

It is clear that for each x in X, the congruence class $x\rho$ is a sub U-set of X and that X is the disjoint union of these U-sets. But if X is not connected, then there are at <u>least</u> two U-sets in this union, contradicting the fact that X is indecomposable.

Now let X be any U-set. Define ρ as above and notice that $X = \dot{\bigcup} x\rho$ and each $x\rho$ is connected and hence indecomposable. Suppose there exists a family $(A_i)_{i \in I}$ of indecomposable U-sets such that $X = \dot{\bigcup}_{i \in I} A_i$. Let $i \in I$. Then A_i is connected and so for all $x \neq y$

in A_i we have that $(x,y) \in \rho$. Hence we see that $A_i \subseteq x\rho$. Suppose that $A_i \neq x\rho$. Then $x\rho \backslash A_i$ is a U-set since if $z \in x\rho\backslash A_i$ then we must have $z \in A_j$ for some $j \neq i$ in I. Hence for all u in U we see that $zu \in A_j$. But $zu \in x\rho$ and so $zu \in x\rho\backslash A_i$ since $A_j \cap A_i = \varphi$. Consequently $x\rho = A_i \dot{\cup} x\rho\backslash A_i$ contradicting the fact that $x\rho$ is indecomposable. Hence $A_i = x\rho$ and the decomposition is unique.

The following concept will prove useful later. Let U be a semigroup. Say that U is <u>left reversible</u> if any two principal right ideals of U intersect. The definition of <u>right reversible</u> is dual. Let U be a monoid and let $X \in$ ENS-$\underline{U}$. Say that X is <u>reversible</u> if any two cyclic sub U-sets of X intersect.

The following is easy to prove.

LEMMA 2.5 <u>Let</u> U <u>be a monoid and let</u> $X \in$ ENS-$\underline{U}$ <u>be reversible</u>. <u>Then</u> X <u>is connected</u>.

LEMMA 2.6 [Bulman-Fleming and McDowell, $\underset{\sim}{4}$, Lemma 2.4]. <u>The following are equivalent</u>

(1) U <u>is left reversible</u>,

(2) <u>every connected right</u> U-<u>set is reversible</u>,

(3) <u>every sub</u> U-<u>set of a connected right</u> U-<u>set is connected.</u>

COROLLARY 2.7 <u>Let</u> U <u>be a left reversible monoid. Let</u> $\lambda : A \to B$ <u>be a right</u> U-<u>monomorphism and suppose that there exists</u> a, a' <u>in</u> A <u>such that</u> $\lambda(a) \sim \lambda(a')$ <u>in</u> B. <u>Then</u> $a \sim a'$ <u>in</u> A.

<u>Proof</u>. Let B_o be the equivalence class modulo $\sim$ containing $\lambda(a)$ and $\lambda(a')$. Then B_o is a connected sub U-set of B. From Lemma 2.6 (3) we see that $B_o \cap \operatorname{im} \lambda$ is a connected sub U-set of B_o. It

is also clear that "connectedness" is preserved under isomorphisms and so $\lambda^{-1}(B_o \cap \text{im } \lambda)$ is a connected sub U-set of A containing a and a'.

From Lemmas 2.4, 2.5 and 2.6 we deduce

COROLLARY 2.8 Let U be a left reversible monoid and let $X \in$ ENS-$\underline{U}$. Then X is indecomposable if and only if X is reversible.

We can also deduce

COROLLARY 2.9 Let U be left reversible and let $X \in$ ENS-$\underline{U}$. Then X is indecomposable if and only if there exists a set I such that $X = \bigcup_{a \in I} aU$ and $aU \cap bU \neq \varphi$ for all $a,b \in I$.

Proof Suppose that X is indecomposable. Then $X = \bigcup_{x \in X} xU$ and by Corollary 2.8 we see that $xU \cap yU \neq \varphi$ for all $x,y \in X$.

Conversely, suppose that $X = \bigcup_{a \in I} aU$ and that $aU \cap bU \neq \varphi$ for all $a,b \in I$. Let $x,y \in X$. Then $x = au$ and $y = bv$ for some $a,b \in I$ and some $u,v \in U$. But there exists $u_1,u_2 \in U$ such that $au_1 = bu_2$ and so we have

$$x = au$$

$$au_1 = bu_2$$

$$bv = y.$$

Hence $x \sim y$ and X is indecomposable by Lemma 2.4.

3. Direct Limits

The concept of direct limit in the category S-ENS-T is identical to that for R-modules. Since, however, there does not appear to be any concise reference for the following results, we prove them in detail.

Let I be a quasi-ordered set (i.e. a set with a relation $\leq$ which is reflexive and transitive). A direct system is a collection of (S,T)-bisets $(X_i)_{i \in I}$ together with (S,T)-maps $\varphi_j^i : X_i \to X_j$ for all $i \leq j \in I$ such that

(1) $\varphi_i^i = 1_{X_i}$, $(i \in I)$,

(2) $\varphi_k^j \circ \varphi_j^i = \varphi_k^i$ whenever $i \leq j \leq k$.

The direct limit of the system (X_i, φ_j^i) is an (S,T)-biset X together with (S,T)-maps $\alpha_i : X_i \to X$ such that

(3) $\alpha_j \circ \varphi_j^i = \alpha_i$, whenever $i \leq j$,

(4) If $Y \in$ S-ENS-T and $\beta_i : X_i \to Y$ are (S,T)-maps such that $\beta_j \circ \varphi_j^i = \beta_i$ whenever $i \leq j$, then there exists a unique (S,T)-map $\psi : X \to Y$ such that the diagram

$$\begin{tikzcd} X_i \arrow[r, "\alpha_i"] \arrow[d, "\beta_i"'] & X \arrow[dl, "\psi"] \\ Y & \end{tikzcd}$$

commutes for all i in I.

Direct limits, if they exist, are obviously unique up to isomorphism. That they do indeed exist follows from

THEOREM 3.1 <u>Direct limits exists in $\underline{S}$-ENS-$\underline{T}$.</u>

<u>Proof</u> Let (X_i, φ_j^i) be a direct system in $\underline{S}$-ENS-$\underline{T}$. Let σ be the (S,T)-congruence on $\dot{\bigcup}_{i \in I} X_i$ generated by the relation

$$R = \{(\lambda_j \varphi_j^i(x_i),\ \lambda_i(x_i)) : x_i \in X_i,\ i,j \in I\},$$

where $\lambda_k : X_k \to \dot{\bigcup}_{i \in I} X_i$ are the natural inclusion maps.

Let $X = \dot{\bigcup}_{i \in I} X_i/\sigma$ and define $\alpha_i : X_i \to X$ by $\alpha_i = \sigma^{\natural} \circ \lambda_i$. Then it is clear that α_i is a well-defined (S,T)-map. Let $i \leq j \in I$ and suppose that $x_i \in X_i$. Then we have

$$\begin{aligned} \alpha_j \varphi_j^i(x_i) &= (\lambda_j \varphi_j^i(x_i))\sigma, \\ &= (\lambda_i(x_i))\sigma, \text{ by definition of } \sigma, \\ &= \alpha_i(x_i). \end{aligned}$$

Hence $\alpha_j \circ \varphi_j^i = \alpha_i$ whenever $i \leq j$.

Now there exists $Y \in \underline{S}$-ENS-$\underline{T}$ and (S,T)-maps $\beta_i : X_i \to Y$ such that $\beta_j \circ \varphi_j^i = \beta_i$ if $i \leq j$. Define $\psi : X \to Y$ by

$$\psi(\lambda_k(x_k)\sigma) = \beta_k(x_k), \quad (k \in I,\ x_k \in X_k).$$

Then ψ is well-defined. For suppose that $(\lambda_k(x_k))\sigma = (\lambda_i(x_i))\sigma$ for some $i,k \in I$. Then $(\lambda_k(x_k),\ \lambda_i(x_i)) \in \sigma$ and so we can find a sequence of elementary R-transitions

$$\lambda_k(x_k) = a_1 \to b_1 = a_2 \to b_2 = \ldots \to b_n = \lambda_i(x_i). \qquad (*)$$

Now we see that for each $1 \leq j \leq n$, we have either

(5) $(a_j,b_j) \in R$, or

(6) $(b_j,a_j) \in R$.

Suppose that $(\lambda_m(x_m),\ \lambda_n(x_n)) \in R$ for some $n,m \in I$. Then we see that $x_m = \varphi_m^n(x_n)$ and consequently, $\beta_m(x_m) = \beta_n(x_n)$. Applying this idea a finite number of times to the sequence (*), tells us that $\beta_k(x_k) = \beta_i(x_i)$, as required.

It is clear that ψ is an (S,T)-map and that $\psi \circ \alpha_i = \beta_i$ for all $i \in I$. Lastly, it is easy to verify that ψ is unique with this property.

EXAMPLE 3.2 Let I be a set with quasi-order given by $i \leq j$ if and only if $i = j$. If (X_i,φ_j^i) is a direct system with index set I, then the direct limit, usually called the <u>coproduct</u>, is simply $\dot{\bigcup\limits_{i \in I}} X_i$.

EXAMPLE 3.3 Let I be a set with a 'special' element o. Let the quasi-order $\leq$ be given by $i \leq j$ if and only if either $i = j$ or $i = o$. If (X_i,φ_j^i) is a direct system with index set I, then the direct limit, usually called the <u>pushout</u>, is isomorphic to $\dot{\bigcup\limits_{i \in I\setminus\{o\}}} X_i/\rho$, where ρ is the (S,T)-congruence generated by

$$\{(\varphi_i^o(x_o),\varphi_j^o(x_o)) : x_o \in X_o,\ i,j \in I\setminus\{o\}\}.$$

(This is not the construction given in Theorem 3.1.)

In the above example, we shall almost always be dealing with the case $|I| = 3$, in which case we see that the pushout of the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \\ C & \end{tikzcd}$$

is isomorphic to $(B \dot{\cup} C)/\rho$ where ρ is generated by

$$\{(\alpha(a),\beta(a)) : a \in A\}.$$

EXAMPLE 3.4 Let (X_i,φ_j^i) be a direct system in $\underline{S}$-ENS-$\underline{T}$. We see from Theorem 3.1 that the direct limit, X, of this system is given by, $X = \dot{\bigcup}_{i \in I} X_i/\rho$, where ρ is the (S,T)-congruence generated by

$$R = \{(\lambda_j\varphi_j^i(x_i),\lambda_i(x_i)) : i \leq j \in I,\ x_i \in X_i\}.$$

It is reasonably clear that ρ is in fact the <u>equivalence</u> generated by R. Hence the direct limit in $\underline{S}$-ENS-$\underline{T}$ of the system (X_i,φ_j^i), is infact the direct limit in ENS.

As a particular consequence of this example we have

COROLLARY 3.5 *Let U be a submonoid of a monoid S. Let (X_i,φ_j^i) be a direct system in ENS-$\underline{S}$. Then the direct limit of the system (X_i,φ_j^i) in ENS-$\underline{S}$ is the direct limit of (X_i,φ_j^i) in ENS-$\underline{U}$.*

LEMMA 3.6 *Let U be a monoid and consider the following pushout diagram in ENS-$\underline{U}$.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\alpha"] \\ C \arrow[r, "\beta"'] & P \end{tikzcd}$$

(1) <u>If</u> f <u>is</u> 1-1 <u>then so is</u> β.

(2) <u>If</u> g <u>is onto then so is</u> α.

<u>Proof</u> Suppose that f is 1-1. We see that $P \simeq (B \dot{\cup} C)/\sigma$ where σ is generated by

$$R = \{(f(a),g(a)) : a \in A\}.$$

Suppose then that $\beta(c) = \beta(c')$, i.e. that $(c,c') \in \sigma$. Then either $c = c'$ in $B \dot{\cup} C$ and hence in C, or there exists a sequence of R-transitions

$$c = y_1 \rightarrow x_1 = y_2 \rightarrow x_2 = \dots \rightarrow x_n = c', \quad (n \geq 1).$$

We can assume that the above sequence is of minimal length. Now, we have either $(y_1,x_1) \in R$ or $(x_1,y_1) \in R$. But since $y_1 \in C$, we see that $(x_1,y_1) \in R$, i.e. that

$$y_1 = g(a) \quad \text{and} \quad x_1 = f(a), \quad \text{for some } a \in A.$$

Similarly, we see that

$$y_2 = f(a') \quad \text{and} \quad x_2 = g(a'), \quad \text{for some } a' \in A.$$

But f is 1-1 and so $a = a'$. Hence $c = y_1 = x_2$ and we have a sequence

$$c = x_2 = y_2 \to x_3 = \ldots \to x_n = c',$$

contradicting the minimality of the length of the original sequence.

Suppose now that g is onto and let $p \in P$. It is clear that either

(1) $p = \alpha(b)$ for some $b \in B$, or

(2) $p = \beta(c)$ for some $c \in C$.

In case (2) we have that $p = \beta g(a) = \alpha f(a)$, for some a in A. Hence in either case $p \in \operatorname{im}\alpha$ and so α is onto.

The following description of pushouts shall prove useful later.

LEMMA 3.7 Let

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \alpha} \\ C & \xrightarrow[\beta]{} & P \end{array}$$

be a pushout diagram in ENS-U and suppose that g is onto and that f is 1-1. Then $P \simeq B/\rho$ where $\rho = R \cup 1_B$ and

$$R = \{(f(a),f(a')) : (a,a') \in \ker g\}.$$

Proof It is straightforward, though tedious, to show that the relation R above is transitive and that B/ρ acts as a pushout for the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

LEMMA 3.8 *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\alpha"] \\ C \arrow[r, "\beta"'] & D \end{tikzcd}$$

be a pushout diagram in ENS-*U*. *If* $\alpha(b) = \beta(c)$ *for some* b *in* B *and* c *in* C, *then there exists* a, a' *in* A *(not necessarily unique) such that* $b = f(a)$, $c = g(a')$.

Proof The proof is a straightforward consequence of Example 3.3.

LEMMA 3.9 *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\alpha"] \\ C \arrow[r, "\beta"'] & D \end{tikzcd}$$

be a pushout diagram in ENS-*U* *and suppose that* f *and* g *are* 1-1. *If* $\alpha(b) = \beta(c)$ *for some* b *in* B *and* c *in* C, *then there exists a unique* a *in* A *such that* $b = f(a)$, $c = g(a)$.

Proof This follows from Lemma 3.8 and Lemma 3.6 (1).

We can of course consider the notion dual to that of direct limit, namely that of the *inverse limit*. We shall, however, have cause to consider only one kind of inverse limit, the *pullback*. We say that the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

in ENS-U, is a *pullback* if for all $Q \in$ ENS-U and all U-maps $\varphi : Q \to B$, $\vartheta : Q \to C$ such that $f \circ \varphi = g \circ \vartheta$, there exists a unique U-map $\psi : Q \to A$ such that

$$\begin{tikzcd} Q \arrow[dr, "\psi"] \arrow[drr, bend left, "\varphi"] \arrow[ddr, bend right, "\vartheta"'] & & \\ & A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "f"] \\ & C \arrow[r, "g"'] & D \end{tikzcd}$$

commutes.

The following is straightforward to prove.

LEMMA 3.10 *Let*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

be a pullback diagram in ENS-U. *Then*

$$A \simeq \{(b,c) \in B \times C : f(b) = g(c)\}$$

and the maps $\alpha : A \to B$ *and* $\beta : A \to C$ *are given by*

$$\alpha(b,c) = b, \quad \beta(b,c) = c.$$

The following is now easy to prove.

LEMMA 3.11 *Let*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & D \end{array}$$

be a pullback diagram in ENS-*U*.

(1) *If* f *is* 1-1 *then so is* β,

(2) *if* g *is onto then so is* α.

LEMMA 3.12 *The commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & D \end{array}$$

is a pullback if and only if whenever $f(b) = g(c)$ *for some* b *in* B, c *in* C *then there exists a unique* a *in* A *such that* $b = \alpha(a)$, $c = \beta(a)$.

Proof The proof is an easy consequence of Lemma 3.10.

We can now see from Lemmas 3.12 and 3.9 that if

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \alpha} \\ C & \xrightarrow[\beta]{} & P \end{array}$$

is a pushout in ENS-U and if f and g are both 1-1, then it is also a pullback.

LEMMA 3.13 Let

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\gamma"'] & B \arrow[d, "\alpha"] \arrow[ddr, "\varepsilon"] & \\
C \arrow[r, "\beta"'] \arrow[drr, "\psi"'] & P & \\
 & & D
\end{tikzcd}$$

be a commutative diagram of U-sets and U-monomorphisms and suppose that P is the pushout of

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\gamma"'] & B \\
C &
\end{tikzcd}$$

We know there exists a unique map $\delta : P \to D$ such that $\delta\alpha = \varepsilon$ and $\delta\beta = \psi$. Then δ is 1-1 if and only if the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\gamma"'] & B \arrow[d, "\varepsilon"] \\
C \arrow[r, "\psi"'] & D
\end{tikzcd}$$

is a pullback.

Proof. Suppose that $\delta : P \to D$ is 1-1, and suppose that $\varepsilon(b) = \psi(c)$ for some b in B, c in C. Then we see that $\delta\alpha(b) = \delta\beta(c)$ and so $\alpha(b) = \beta(c)$. From Lemma 3.9, there exists a unique a in A such that $b = f(a)$, $c = \gamma(a)$. Hence from Lemma 3.12,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \gamma \downarrow & & \downarrow \varepsilon \\ C & \xrightarrow[\psi]{} & D \end{array}$$

is a pullback.

Conversely, suppose that $\delta(p) = \delta(p')$. We see from Example 3.3 that there are three cases to consider:

(1) $p = \alpha(b)$, $p' = \alpha(b')$, b,b' in B,

(2) $p = \beta(c)$, $p' = \beta(c')$, c,c' in C,

(3) $p = \alpha(b)$, $p' = \beta(c)$, b in B, c in C.

In case (1) we see that $\varepsilon(b) = \varepsilon(b')$ and so, $b = b'$ since ε is 1-1. Hence $p = p'$ as required. Case (2) is similar to case (1). In case (3) we have $\varepsilon(b) = \psi(c)$ and so by Lemma 3.12, there exists a unique a in A such that $b = f(a)$, $c = \gamma(a)$. Hence $p = \alpha(b) = \alpha f(a) = \beta\gamma(a) = \beta(c) = p'$ as required.

Recall that if $f : X \to Y$ is a U-monomorphism then

$$\rho_f = \mathrm{im}f \times \mathrm{im}f \cup 1_Y$$

is a U-congruence on Y and we write Y/ρ_f as Y/X.

THEOREM 3.14 <u>Consider the following commutative diagram in</u> ENS-$\underline{U}$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\gamma"'] & B \arrow[d, "\varepsilon"] \\ C \arrow[r, "\psi"'] \arrow[d, "\varphi"'] & D \arrow[d, "\alpha"] \\ E \arrow[r, "\beta"'] & F \end{tikzcd}$$

where the top square is a pullback and the bottom square is a pushout. Suppose also that φ is onto and that ψ is 1-1. Then the following are equivalent:

(1) $\alpha\varepsilon$ is 1-1,

(2) ε is 1-1 and $\operatorname{Ker}\alpha \cap \rho_\varepsilon = 1_B$,

(3) ε is 1-1 and $\operatorname{Ker}\varphi \cap \rho_\gamma = 1_C$,

(4) ε and $\varphi\gamma$ are both 1-1.

Proof We see from Theorem 1.14 that (1) and (2) are equivalent.

(2) implies (3). Suppose that $(x,y) \in \operatorname{Ker}\varphi \cap \rho_\gamma$. Then we have either $x = y$ as required, or

$$x = \gamma(a),\ y = \gamma(a') \quad \text{and} \quad \varphi(x) = \varphi(y), \quad a,a' \in A.$$

Hence $\alpha\psi(x) = \beta\varphi(x) = \beta\varphi(y) = \alpha\psi(y)$ and so $(\psi(x),\psi(y)) \in \operatorname{Ker}\alpha$. But $\psi(x) = \psi\gamma(a) = \varepsilon f(a)$ and $\psi(y) = \psi\gamma(a') = \varepsilon f(a')$. Hence $(\psi(x),\psi(y)) \in \rho_\varepsilon$ and so $\psi(x) = \psi(y)$. But ψ is 1-1 and so $x = y$.

(3) implies (4). From Lemma 3.11 (1) we see that γ is 1-1 and so from Theorem 1.14, $\varphi\gamma$ is 1-1.

(4) *implies* (1). Suppose that $\alpha\varepsilon(b) = \alpha\varepsilon(b')$. We see from Lemma 3.7 that either $\varepsilon(b) = \varepsilon(b')$ and so $b = b'$ as required, or there exists $(c,c') \in \operatorname{Ker}\varphi$ such that $\varepsilon(b) = \psi(c)$ and $\varepsilon(b') = \psi(c')$. Hence from Lemma 3.12 there exists a unique a in A and a unique a' in A such that

$$f(a) = b, \quad \gamma(a) = c,$$

and

$$f(a') = b', \quad \gamma(a') = c',$$

consequently, $\varphi\gamma(a) = \varphi(c) = \varphi(c') = \varphi\gamma(a')$ and so $a = a'$ since $\varphi\gamma$ is 1-1. Hence $b = b'$ as required.

The following rather technical lemma will help to simplify some of the later arguments.

LEMMA 3.15 *Suppose we have a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\gamma"'] & B \arrow[d, "\alpha"] \arrow[ddr, "\varepsilon"] & \\ C \arrow[r, "\beta"'] \arrow[drr, "\psi"'] & P \arrow[dr, "\delta"] & \\ & & D \end{tikzcd}$$

of U-sets and U-monomorphisms where P *is the pushout of*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\gamma"'] & B \\ C & \end{tikzcd}$$

Suppose also that there exists a U-set E and a U-epimorphism $\varphi : D \to E$ *such that*

(1) $\operatorname{Ker} \varphi \subseteq \rho_\psi$

(2) $\operatorname{im} \varphi\psi \subseteq \operatorname{im} \varphi\varepsilon$, *and*

(3) $\varphi\varepsilon$ *is* 1-1.

Then $E/B \simeq D/P$.

Proof Define $\Phi : D/P \to E/B$ by $\Phi(d\rho_\delta) = \varphi(d)\rho_{\varphi\varepsilon}$. Suppose that $d\rho_\delta = d'\rho_\delta$ in D/P. Then we have two possibilities

(i) $d = d'$, in which case $\varphi(d)\rho_{\varphi\varepsilon} = \varphi(d)\rho_{\varphi\varepsilon}$, or

(ii) $d = \delta(p)$, $d' = \delta(p')$ for some $p,p' \in P$. Now if $p \in \operatorname{im} \alpha$ we see that $\varphi(d) \subseteq \operatorname{im} \varphi\delta\alpha = \operatorname{im} \varphi\varepsilon$, while if $p \in \operatorname{im} \beta$, then $\varphi(d) \in \operatorname{im} \varphi\delta\beta = \operatorname{im} \varphi\psi \subseteq \operatorname{im} \varphi\varepsilon$, by (2). Hence we see that $(\varphi(d),\varphi(d')) \in \rho_{\varphi\varepsilon}$ and so Φ is well-defined.

It is clear that Φ is onto and is a U-map. To show that Φ is 1-1 we suppose that $(\varphi(d),\varphi(d')) \in \rho_{\varphi\varepsilon}$. We have two cases to consider:

(iii) $\varphi(d) = \varphi(d')$, or

(iv) $\varphi(d),\varphi(d') \in \operatorname{im} \varphi\varepsilon$.

In case (iii) we see that $(d,d') \in \operatorname{Ker} \varphi \subseteq \rho_\psi$ by (1). But $\psi = \delta\beta$ and so $\operatorname{im} \psi \subseteq \operatorname{im} \delta$. Hence $\rho_\psi \subseteq \rho_\delta$ and so $(d,d') \in \rho_\delta$, as required. In case (iv) we have $\varphi(d) = \varphi\varepsilon(b)$ and $\varphi(d') = \varphi\varepsilon(b')$ for some b,b' in B. Hence $(d,\varepsilon(b))$, $(d',\varepsilon(b')) \in \operatorname{Ker} \varphi \subseteq \rho_\delta$. But $\varepsilon = \delta\alpha$ and so $\rho_\varepsilon \subseteq \rho_\delta$. We deduce that $(\varepsilon(b),\varepsilon(b')) \in \rho_\delta$ and so by transitivity of ρ_δ we have $(d,d') \in \rho_\delta$.

A useful corollary to this result is:

COROLLARY 3.16 *Consider the following pullback diagram in* ENS-$\underline{U}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\gamma"'] & B \arrow[d, "\varepsilon"] \\ C \arrow[r, "\psi"'] & D \end{tikzcd}$$

where ε *and* ψ *are* 1-1. *Suppose that the map* $\gamma : A \to C$ *splits i.e. suppose that there exists a map* $\vartheta : C \to A$ *such that* $\vartheta \circ \gamma = 1_A$. *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\gamma"'] & B \arrow[d, "\alpha"] \\ C \arrow[r, "\beta"'] & P \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[r, "\psi"] \arrow[d, "\vartheta"'] & D \arrow[d, "\varphi"] \\ A \arrow[r, "\tau"'] & E \end{tikzcd}$$

be pushout diagrams. Then the map $\varphi \circ \varepsilon : B \to E$ *is* 1-1 *and* $E/B \simeq D/P$.

Proof That $\varphi \circ \varepsilon$ is 1-1 follows from Theorem 3.14. It is readily seen that ϑ is onto and so from Lemma 3.7 we see that if $(d,d') \in \text{Ker } \varphi$, then $d,d' \in \text{im } \psi$. Hence $\text{Ker } \varphi \subseteq \rho_\psi$. Also it is easy to see that if $c \in C$ then $(c,\gamma\vartheta(c)) \in \text{Ker } \vartheta$. Hence, by Lemma 3.7, we see that $\varphi\psi(c) = \varphi\psi(\gamma\vartheta(c))$. But $\psi\gamma = \varepsilon f$ and so $\varphi\psi(c) = \varphi\varepsilon f\vartheta(c)$. Hence we have shown that $\text{im } \varphi\psi \subseteq \text{im } \varphi\varepsilon$. The result now follows from Lemma 3.15.

Let I be a quasi-ordered set. Say that I is *directed* if for all i,j in I, there exists k in I with $k \geq i$, $k \geq j$.

We can show

THEOREM 3.17 *Let* (X_i,φ_j^i) *be a direct system in* $\underline{S}$-ENS-$\underline{T}$ *with directed index set and let* (X,α_i) *be the direct limit of this system.*

Then $\alpha_i(x_i) = \alpha_j(x_j)$ *in* X *if and only if there exists* $K \geq i,j$ *such that* $\varphi_K^i(x_i) = \varphi_K^j(x_j)$.

Proof Suppose that there exists $K \geq i,j$ such that $\varphi_K^i(x_i) = \varphi_K^j(x_j)$. Then $\alpha_K\varphi_K^i(x_i) = \alpha_K\varphi_K^j(x_j)$ and so $\alpha_i(x_i) = \alpha_j(x_j)$.

Conversely, suppose that $\alpha_i(x_i) = \alpha_j(x_j)$. Consider the (S,T)-biset $B = \dot{\bigcup}_{i \in I} X_i/\rho$ where ρ is defined by the rule that $(\lambda_i(x_i), \lambda_j(x_j)) \in \rho$ if and only if there exists $K \geq i,j$ with $\varphi_K^i(x_i) = \varphi_K^j(x_j)$. Define $\beta_i : X_i \to B$ by $\beta_i = \rho^{\natural} \circ \lambda_i$. Then β_i are well-defined (S,T)-maps and $\beta_j\varphi_j^i = \beta_i$ whenever $i \leq j$. Hence there exists a unique (S,T)-map $\psi : X \to B$ such that $\psi \circ \alpha_i = \beta_i$, ($i$ in I). (It can be shown that ψ is infact an isomorphism, but we will not require this.) We now see that $\beta_i(x_i) = \psi\alpha_i(x_i) = \psi\alpha_j(x_j) = \beta_j(x_j)$. Hence $(\lambda_i(x_i),\lambda_j(x_j)) \in \rho$ and the result follows.

The following corollary is now immediate.

COROLLARY 3.18 *Let* (X_i,φ_j^i) *be a direct system in* S-ENS-T *with directed index set and let* (X,α_i) *be the direct limit. Then for all* i *in* I, *the map* $\alpha_i : X_i \to X$ *is* 1-1 *if and only if the maps* $\varphi_K^i : X_i \to X_K$ *are* 1-1 *for all* $K \geq i$.

We shall need the following result later.

THEOREM 3.19 *Let* (X_i,φ_j^i) *be a direct system in* S-ENS-T *with directed index set and let* (X,α_i) *be the direct limit. For each* i *in* I *let* σ_i *be an* (S,T)-*congruence on* X_i *and suppose that there exists an* (S,T)-*map* $\vartheta_j^i : X_i/\sigma_i \to X_j/\sigma_j$ *whenever* $i \leq j$, *such that the diagrams*

$$\begin{tikzcd}
X_i \arrow[r, "\varphi_j^i"] \arrow[d, "\sigma_i^\natural"'] & X_j \arrow[d, "\sigma_j^\natural"] \\
X_i/\sigma_i \arrow[r, "\vartheta_j^i"'] & X_j/\sigma_j .
\end{tikzcd}$$

commute for all $i \leq j$. Then there exists an (S,T)-congruence σ on X and (S,T)-maps $\beta_i : X_i/\sigma_i \to X/\sigma$ such that $(X/\sigma, \beta_i)$ is the direct limit in $\underline{S}$-ENS-$\underline{T}$ of the system $(X_i/\sigma_i, \vartheta_j^i)$.

Proof We see from the construction of the direct limit that if $x \in X$, then $x = \alpha_i(x_i)$ for some i in I, x_i in X_i. Define σ on X by $(\alpha_i(x_i), \alpha_j(x_j)) \in \sigma$ if and only if there exists $K \geq i,j$ such that $(\varphi_K^i(x_i),\ \varphi_K^j(x_j)) \in \sigma_K$. It is easy to check that σ is an (S,T)-congruence on X. The only point we would stress is that σ is well-defined. To see this, first notice that ϑ_j^i is given by

$$\vartheta_j^i(x_i\sigma_i) = (\varphi_j^i(x_i))\sigma_j, \quad \text{whenever} \quad i \leq j.$$

Suppose then that $\alpha_i(x_i) = \alpha_m(x_m)$, $\alpha_j(x_j) = \alpha_n(x_n)$ and that $(\alpha_i(x_i), \alpha_j(x_j)) \in \sigma$. From Theorem 3.17 we see that there exists $p \geq i,m$ such that $\varphi_p^i(x_i) = \varphi_p^m(x_m)$ and $s \geq j,n$ such that $\varphi_s^j(x_j) = \varphi_s^n(x_n)$. We also know that there exists $K \geq i,j$ such that $(\varphi_K^i(x_i), \varphi_K^j(x_j)) \in \sigma_K$. Since I is directed, there exists $r \geq K,p,s$ and we deduce

$$\varphi_r^i(x_i) = \varphi_r^p\varphi_p^i(x_i) = \varphi_r^p\varphi_p^m(x_m) = \varphi_r^m(x_m).$$

Similarly, $\varphi_r^j(x_j) = \varphi_r^n(x_n)$.

Now $(\varphi_K^i(x_i),\ \varphi_K^j(x_j)) \in \sigma_K$ and so on applying the map ϑ_r^K we see that $(\varphi_r^i(x_i),\ \varphi_r^j(x_j)) \in \sigma_r$. Hence since σ_r is a well-defined congruence we see that $(\varphi_r^m(x_m),\ \varphi_r^n(x_n)) \in \sigma_r$ and so $(\alpha_m(x_m),\ \alpha_n(x_n)) \in \sigma$.

Now define $\beta_i : X_i/\sigma_i \to X/\sigma$ by $\beta_i(x_i\sigma_i) = (\alpha_i(x_i))\sigma$. Then β_i is a well-defined (S,T)-map and it is clear that if $i \leq j$ then $\beta_j \circ \vartheta_j^i = \beta_i$. Suppose that Q is an (S,T)-biset and that $f_i : X_i/\sigma_i \to Q$ are (S,T)-maps such that $f_j \circ \vartheta_j^i = f_i$ whenever $i \leq j$. Then we have a commutative diagram

$$\begin{tikzcd}
X_i \arrow[rr, "\varphi_j^i"] \arrow[d, "\sigma_i^\natural"'] & & X_j \arrow[d, "\sigma_j^\natural"] \\
X_i/\sigma_i \arrow[rr, "\vartheta_j^i"'] \arrow[dr, "f_i"'] & & X_j/\sigma_j \arrow[dl, "f_j"] \\
& Q &
\end{tikzcd}$$

whenever $i \leq j$. Since X is the direct limit of the X_i then there exists a unique (S,T)-map $\psi : X \to Q$ such that $\psi \circ \alpha_i = f_i \circ \sigma_i^\natural$. Define $\Phi : X/\sigma \to Q$ by $\Phi(x\sigma) = \psi(x)$. Then it is straightforward to show that Φ is a well-defined (S,T)-map, that $\Phi \circ \beta_i = f_i$ for all i and that Φ is unique with this property.

We end this section with a result that will prove useful in Section 5.

LEMMA 3.20 Let I be a directed quasi-ordered set. Let (X_i, φ_j^i) and (Y_i, ϑ_j^i) be directed systems in ENS-U (sharing the same index set) and suppose that there are monomorphisms $f_i : X_i \to Y_i$ such that

$$\begin{tikzcd} X_i \arrow[r, "\varphi^i_j"] \arrow[d, "f_i"'] & X_j \arrow[d, "f_j"] \\ Y_i \arrow[r, "\vartheta^i_j"'] & Y_j \end{tikzcd}$$

commutes whenever $i \leq j$. Suppose also that (X,α_i) and (Y,β_i) are the direct limits of these systems. Then there exists a U-monomorphism $f : X \to Y$ such that

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "\alpha_i"'] & Y_i \arrow[d, "\beta_i"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes for every i.

Proof We have a commutative diagram

$$\begin{tikzcd} & X & \\ X_i \arrow[ur, "\alpha_i"] \arrow[r, "\varphi^i_j"'] \arrow[d, "f_i"'] & & X_j \arrow[ul, "\alpha_j"'] \arrow[d, "f_j"] \\ Y_i \arrow[rr, "\vartheta^i_j"] \arrow[dr, "\beta_i"'] & & Y_j \arrow[dl, "\beta_j"] \\ & Y & \end{tikzcd}$$

Let $x = \alpha_i(x_i) \in X$. Define $f : X \to Y$ by $f(x) = \beta_i f_i(x_i)$. We need to show that f is a well-defined U-monomorphism. Suppose then that

$\alpha_i(x_i) = \alpha_j(x_j)$ for some $i,j \in I$. From Theorem 3.17 we see that there exists $K \geq i,j$ such that $\varphi_K^i(x_i) = \varphi_K^j(x_j)$. Hence

$$\beta_i f_i(x_i) = \beta_K f_K \varphi_K^i(x_i) = \beta_K f_K \varphi_K^j(x_j) = \beta_j f_j(x_j), \text{ as required.}$$

Suppose then that $\beta_i f_i(x_i) = \beta_j f_j(x_j)$. From Theorem 3.17 again we see that there exists $K \geq i,j$ such that

$$\vartheta_K^i f_i(x_i) = \vartheta_K^j f_j(x_j).$$

But $\vartheta_m^n f_n = f_m \varphi_m^n$ for all $n \leq m$ and so we see that

$$f_K \varphi_K^i(x_i) = f_K \varphi_K^j(x_j).$$

Since f_K is 1-1 we deduce that $\varphi_K^i(x_i) = \varphi_K^j(x_j)$ and so by Theorem 3.17, $\alpha_i(x_i) = \alpha_j(x_j)$ and f is 1-1.

4. Tensor Products

We shall now describe what is essentially a non-additive version of the classical tensor product construction in modules, see for example Rotman [35]. This construction has been used by various authors, including Stenstrom [39], Howie [23] and Bulman-Fleming and McDowell [4], and, as we shall see, is intimately connected with amalgamated free products of semigroups.

Let $X \in$ ENS-$\mathbf{U}$ and $Y \in \mathbf{U}$-ENS. The tensor product of X and Y over U is a set T together with a map $f : X \times Y \to T$ with the properties

(1) $f(xu,y) = f(x,uy)$, x in X, y in Y, u in U,

(2) If G is a set and $g : X \times Y \to G$ a map such that $g(xu,y) = g(x,uy)$ for all x in X, y in Y, u in U, then there exists a unique map $\psi : T \to G$ such that $\psi \circ f = g$.

Being a universal construction, the tensor product, if it exists, is essentially unique. To see that it does indeed exist, consider the equivalence relation τ on $X \times Y$ generated by the relation

$$\{((xu,y),(x,uy)) : x \in X,\ u \in U,\ y \in Y\},$$

and the map $\tau^\natural : X \times Y \to (X \times Y)/\tau$. Then it is easy to check that the pair $((X \times Y)/\tau, \tau^\natural)$ is a tensor product of X and Y over U. We usually denote the tensor product by $X \otimes_U Y$, or simply $X \otimes Y$ and denote an element $(x,y)\tau$ of $X \otimes_U Y$, by $x \otimes y$.

The following are easy to prove and will be used later without reference.

LEMMA 4.1 *Let* U *be a monoid and* $X \in$ ENS-$\underline{U}$. *Then* $X \otimes_U U \simeq X$.

LEMMA 4.2 *Let* $f : A \to B$ *be a right* U-*map and* $g : C \to D$ *a left* U-*map*. *Then there exists a map* $f \otimes g : A \otimes_U C \to B \otimes_U D$ *given by* $(f \otimes g)(a \otimes c) = f(a) \otimes g(c)$.

The tensor product of X and Y is normally only a set. However, if $X \in \underline{S}$-ENS-$\underline{U}$ and $Y \in \underline{U}$-ENS-$\underline{T}$ then $X \otimes_U Y$ becomes an (S,T)-biset if we define

$$s(x \otimes y) = sx \otimes y \quad \text{and} \quad (x \otimes y).t = x \otimes yt.$$

THEOREM 4.3 *Let* U *and* S *be monoids*. *Let* $A \in$ ENS-$\underline{U}$, $B \in \underline{U}$-ENS-$\underline{S}$ *and* $C \in \underline{S}$-ENS. *Then* $(A \otimes_U B) \otimes_S C \simeq A \otimes_U (B \otimes_S C)$.

Proof The proof is essentially the same as that for R-modules. See for example Rotman [35], Excercise 1.10.

The question naturally arises: When are two elements in a tensor product equal? The following result will prove useful.

LEMMA 4.4 [Bulman-Fleming and McDowell, 4, Lemma 1.2]. *Let* U *be a monoid*, $A \in$ ENS-$\underline{U}$, $a, a' \in A$, $B \in \underline{U}$-ENS *and* $b, b' \in B$. *Then* $a \otimes b = a' \otimes b'$ *in* $A \otimes_U B$ *if and only if there exists* $a_1, \ldots, a_n$ *in* A, $b_2, \ldots, b_n$ *in* B, $u_1, \ldots, u_n, v_1, \ldots, v_n$ *in* U *such that*

$$a = a_1 u_1,$$

$$a_1 v_1 = a_2 u_2, \quad u_1 b = v_1 b_2,$$

$$\cdots\cdots$$

$$a_n v_n = a', \quad u_n b_n = v_n b'.$$

A set of equations of the above form will be called a (U-) Scheme (of length n) over A and B joining (a,b) to (a',b').

Recall that if $X \in$ ENS-U and $x,x' \in X$, then x is connected to x', and we write $x \sim x'$ if there exists a set of equations

$$x = x_1u_1,$$
$$x_1v_1 = x_2u_2,$$
$$\cdots\cdots$$
$$x_nv_n = x'.$$

The following is now clear.

LEMMA 4.5 Let $X \in$ ENS-U, $Y \in$ U-ENS and suppose that $x \otimes y = x' \otimes y'$ in $X \otimes_U Y$. Then $x \sim x'$ in X and $y \sim y'$ in Y.

In order to study connections between tensor products and direct limits, the following theorem, which is certainly well-known in other categories, will be useful.

THEOREM 4.6 (Adjoint Isomorphism). If $A \in$ ENS-U, $B \in$ U-ENS-S and $C \in$ ENS-S then there is a bijection

$$f : \mathrm{Hom}_S(A \otimes_U B,\ C) \longrightarrow \mathrm{Hom}_U(A,\ \mathrm{Hom}_S(B,C)).$$

If $A \in$ U-ENS, $B \in$ S-ENS-U and $C \in$ S-ENS then there is a bijection

$$g : \mathrm{Hom}_S(B \otimes_U A,\ C) \longrightarrow \mathrm{Hom}_U(A,\ \mathrm{Hom}_S(B,C)).$$

Proof We shall discuss only the first isomorphism. The second can be treated similarly.

Notice that $\mathrm{Hom}_S(B,C) \in \mathrm{ENS}\text{-}\underline{U}$ if we define

$$(ku)(b) = k(ub),\ k \in \mathrm{Hom}_S(B,C),\ u \in U,\ b \in B.$$

Now define the map f by

$$f(k)(a)(b) = k(a \otimes b),\ k \in \mathrm{Hom}_S(A \otimes_U B,\ C),\ a \in A,\ b \in B.$$

Then it is straightforward to check that f is a well-defined map and that $f(k) \in \mathrm{Hom}_U(A,\ \mathrm{Hom}_S(B,C))$. To show that f is a bijection we construct an inverse $f' : \mathrm{Hom}_U(A,\ \mathrm{Hom}_S(B,C)) \to \mathrm{Hom}_S(A \otimes_U B,\ C)$, by

$$f'(h)(a \otimes b) = h(a)(b),\ h \in \mathrm{Hom}_U(A,\ \mathrm{Hom}_S(B,C)),\ a \otimes b \in A \otimes_U B.$$

It is again straightforward to check that f' is a well-defined map and that $f'(h) \in \mathrm{Hom}_S(A \otimes_U B,\ C)$. It is clear that f and f' are mutually inverse bijections.

Notice that if $S = \{1\}$ the theorem reduces to the existence of a bijection $f : \mathrm{Hom}(A \otimes_U B,\ C) \to \mathrm{Hom}_U(A,\ \mathrm{Hom}(B,C))$, for all $A \in \mathrm{ENS}\text{-}\underline{U}$, $B \in \underline{U}\text{-}\mathrm{ENS}$, $C \in \mathrm{ENS}$.

THEOREM 4.7 <u>Let</u> U <u>be a monoid and let</u> (X_i, φ^i_j) <u>be a direct system in</u> ENS-$\underline{U}$ <u>with direct limit</u> (X, α_i). <u>Let</u> $B \in \underline{U}$-ENS. <u>Then</u> $(X \otimes_U B,\ \alpha_i \otimes 1)$ <u>is the direct limit in</u> ENS <u>of the direct system</u> $(X_i \otimes_U B,\ \varphi^i_j \otimes 1)$.

<u>Proof</u> It is clear that for all $i \leq j$ in I, the diagram

$$\begin{tikzcd} X_i \otimes_U B \arrow[rr, "\varphi_j^i \otimes 1"] \arrow[dr, "\alpha_i \otimes 1"'] & & X_j \otimes_U B \arrow[dl, "\alpha_j \otimes 1"] \\ & X \otimes_U B & \end{tikzcd}$$

commutes. Suppose then there exists $Y \in ENS$ and $\beta_i : X_i \otimes_U B \to Y$ such that the diagram

$$\begin{tikzcd} X_i \otimes_U B \arrow[rr, "\varphi_j^i \otimes 1"] \arrow[dr, "\beta_i"'] & & X_j \otimes_U B \arrow[dl, "\beta_j"] \\ & Y & \end{tikzcd}$$

commutes whenever $i \leq j$. Then $\beta_i \in \mathrm{Hom}(X_i \otimes_U B, Y)$ and so we see from Theorem 4.6 that there exists $\overline{\beta}_i \in \mathrm{Hom}_U(X_i, \mathrm{Hom}(B,Y))$ given by

$$\overline{\beta}_i(x_i)(b) = \beta_i(x_i \otimes b), \quad x_i \in X_i,\ b \in B.$$

Now if $i \leq j$ then we have

$$\begin{aligned} (\overline{\beta}_j \circ \varphi_j^i)(x_i)(b) &= \beta_j(\varphi_j^i(x_i) \otimes b) \\ &= \beta_i(x_i \otimes b), \\ &= \overline{\beta}_i(x_i)(b), \end{aligned}$$

for all $x_i \in X_i$, $b \in B$. Hence $\overline{\beta}_j \circ \varphi^i_j = \overline{\beta}_i$. Consequently, there exists a unique U-map $\overline{\psi} : X \to \mathrm{Hom}(B,Y)$ such that

$$\begin{tikzcd} X_i \arrow[r, "\alpha_i"] \arrow[d, "\overline{\beta}_i"'] & X \arrow[dl, "\overline{\psi}"] \\ \mathrm{Hom}(B,Y) & \end{tikzcd}$$

commutes for all i in I. From Theorem 4.6 we see that there exists a map $\psi \in \mathrm{Hom}(X \otimes_U B,\ Y)$ such that

$$\psi(x \otimes b) = \overline{\psi}(x)(b).$$

Now the diagram

$$\begin{tikzcd} X_i \otimes_U B \arrow[r, "\beta_i"] \arrow[d, "\alpha_i \otimes 1"'] & Y \\ X \otimes_U B \arrow[ur, "\psi"'] & \end{tikzcd}$$

commutes for all i in I, since

$$\begin{aligned}(\psi \circ (\alpha_i \otimes 1))(x_i \otimes b) &= \overline{\psi}(\alpha_i(x_i))(b), \\ &= \overline{\beta}_i(x_i)(b), \\ &= \beta_i(x_i \otimes b).\end{aligned}$$

Lastly, it is easy to check that ψ is unique with this property.

From Example 3.2 we can thus deduce

LEMMA 4.8 <u>Let</u> U <u>be a monoid, let</u> $A \in U\text{-ENS}$ <u>and let</u> $X_i \in \text{ENS-}\underline{U}$, $i \in I$. <u>Then</u> $(\dot{\bigcup}_{i \in I} X_i) \otimes_U A \simeq \dot{\bigcup}_{i \in I} (X_i \otimes_U A)$.

From Theorem 4.7 and its dual, we see that the following Corollary holds.

COROLLARY 4.9 <u>Let</u> (X_i, φ_j^i) <u>be a direct system in</u> <u>S</u>-ENS-<u>U</u> <u>with direct limit</u> (X, α_i). <u>Let</u> $A \in$ ENS-<u>S</u> <u>and</u> $B \in$ <u>U</u>-ENS. <u>Then</u> $(A \otimes_S X \otimes_U B,\ 1 \otimes \alpha_i \otimes 1)$ <u>is the direct limit in</u> ENS <u>of the system</u> $(A \otimes_S X_i \otimes_U B,\ 1 \otimes \varphi_j^i \otimes 1)$.

Recall that if $f : X \to Y$ is a right U-monomorphism then $Y/X = Y/\rho_f$ where

$$\rho_f = \operatorname{im} f \times \operatorname{im} f \cup 1_Y.$$

LEMMA 4.10 <u>Let</u> $f : X \to Y$ <u>be a right</u> U-<u>monomorphism and let</u> A <u>be a left</u> U-<u>set</u>. <u>Then</u> $\bar{y} \otimes a = \bar{y}' \otimes a'$ <u>in</u> $(Y/X) \otimes_U A$ <u>if and only if either</u> $y \otimes a = y' \otimes a'$ <u>in</u> $Y \otimes_U A$ <u>or there exists</u> x_1, x_1' <u>in</u> X, $a_1 \sim a_1'$, <u>in</u> A <u>such that</u>

$$y \otimes a = f(x_1) \otimes a_1, \quad y' \otimes a' = f(x_1') \otimes a_1' \quad \underline{\text{in}} \quad Y \otimes_U A.$$

<u>Proof</u> Suppose that $\bar{y} \otimes a = \bar{y}' \otimes a'$ in $(Y/X) \otimes_U A$. From Lemma 4.4 we have a set of equations

$$\begin{aligned} \bar{y} &= \overline{y_1 u_1}, & & \\ \overline{y_1 v_1} &= \overline{y_2 u_2}, & u_1 a &= v_1 a_2 \\ & \cdots\cdots & & \\ \overline{y_n v_n} &= \bar{y}', & u_n a_n &= v_n a'. \end{aligned}$$

For each equation $\overline{y_i v_i} = \overline{y_{i+1} u_{i+1}}$ we have two possibilities:

(i) $y_i v_i = y_{i+1} u_{i+1}$, or

(ii) $y_i v_i,\ y_{i+1} u_{i+1} \in \operatorname{im} f$.

If case (i) holds for all i, then $y \otimes a = y' \otimes a'$ in $Y \otimes_U A$, otherwise, suppose that j is the smallest value of i such that (ii) holds. Then $y \otimes a = y_1 u_1 \otimes a = y_1 \otimes u_1 a = \ldots = y_j v_j \otimes a_{j+1} = f(x_1) \otimes a_{j+1}$, for some $x_1 \in X$. A similar result holds for $y' \otimes a'$. Thus, changing the notation, $y \otimes a = f(x_1) \otimes a_1$ and $y' \otimes a' = f(x_1') \otimes a_1'$ for some $x_1, x_1' \in X$, $a_1, a_1' \in A$. Also since $a \sim a'$ then we see that $a_1 \sim a_1'$.

Conversely, if $y \otimes a = y' \otimes a'$ in $Y \otimes_U A$, then it is clear that $\overline{y} \otimes a = \overline{y}' \otimes a'$ in $(Y/X) \otimes_U A$. Suppose then that there exists $x_1, x_1' \in X$, and $a_1 \sim a_1'$ in A such that $y \otimes a = f(x_1) \otimes a_1$ and $y' \otimes a' = f(x_1') \otimes a_1'$. Then we have a set of equations

$$a_1 = u_2 a_2,\ v_2 a_3 = u_3 a_3, \ldots, v_n a_n = a_1', \text{ where } u_i, v_i \in U,\ a_i \in A.$$

Hence,
$$\begin{aligned}\overline{y} \otimes a = \overline{f(x_1)} \otimes a &= \overline{f(x_1 u_2)} \otimes a_2 \\ &= \overline{f(x_1' v_2)} \otimes a_2, \\ &= \ldots \\ &= \overline{f(x_1')} \otimes a_1' = \overline{y'} \otimes a'.\end{aligned}$$

The following Corollary is now immediate.

COROLLARY 4.11 <u>Let</u> $f : X \to Y$ <u>be a right</u> U-<u>monomorphism and let</u> $A \in$ <u>U</u>-ENS. <u>Then</u> $\overline{f(x)} \otimes a = \overline{f(x')} \otimes a'$ <u>in</u> $(Y/X) \otimes_U A$ <u>if and only if</u> $a \sim a'$ <u>in</u> A.

COROLLARY 4.12 <u>Let</u> U <u>be a submonoid of a monoid</u> <u>S</u> <u>and let</u> $A \in$ U-ENS. <u>Then</u> $\overline{1} \otimes a = \overline{1} \otimes a'$ <u>in</u> $(S/U) \otimes_U A$ <u>if and only if</u> $a \sim a'$ <u>in</u> A.

COROLLARY 4.13 Let $Y \in$ ENS-U, $A \in U$-ENS and suppose that U is a submonoid of a monoid S. If $y \otimes \bar{s} \otimes a = y' \otimes \bar{1} \otimes a'$ in $Y \otimes_U (S/U) \otimes_U A$ then there exists $y_1 \in Y$, $a_1 \in A$ such that

$$y \otimes s \otimes a = y_1 \otimes 1 \otimes a_1 \quad \text{in} \quad Y \otimes_U S \otimes_U A.$$

Proof From Lemma 4.4 we deduce that there exists equations

$$y = y_1 u_1$$

$$y_1 v_1 = y_2 u_2, \quad \overline{u_1 s} \otimes a = \overline{v_1 s_2} \otimes a_2$$

$$\cdots\cdots$$

$$y_n v_n = y', \qquad \overline{u_n s_n} \otimes a_n = \overline{v_n} \otimes a'.$$

For each equation $\overline{u_i s_i} \otimes a_i = \overline{v_i s_{i+1}} \otimes a_{i+1}$, we see from Lemma 4.10 that we have two cases: either

(i) $u_i s_i \otimes a_i = v_i s_{i+1} \otimes a_{i+1}$;

or (ii) $u_i s_i \otimes a_i = 1 \otimes a''$, $v_i s_{i+1} \otimes a_{i+1} = 1 \otimes a'''$, for some $a'', a''' \in A$.

If case (i) holds for all i then we have $y \otimes s \otimes a = y' \otimes 1 \otimes a'$, otherwise the existence of a smallest i such that case (ii) holds gives us $y \otimes s \otimes a = y_1 \otimes 1 \otimes a_1$ for some $y_1 \in Y$, $a_1 \in A$ in a similar manner to the proof of Lemma 4.10. Hence the result.

LEMMA 4.14 Let $f : X \to Y$ be a right U-monomorphism and B a left U-set. If $y \otimes b = f(x) \otimes b'$ in $Y \otimes_U B$ then $\bar{y} \otimes b = \overline{f(x)} \otimes b$ in $(Y/X) \otimes_U B$.

Proof From Corollary 4.11 we see that $\overline{f(x)} \otimes b = \overline{f(x)} \otimes b'$. Hence $\overline{y} \otimes b = \overline{f(x)} \otimes b' = \overline{f(x)} \otimes b$.

From Corollary 4.13 we can deduce

LEMMA 4.15 *Let* U *be a submonoid of a monoid* S. *Let* $Y \in$ ENS-U *and* $B \in U$-ENS. *If* $y \otimes s \otimes b = y' \otimes 1 \otimes b'$ *in* $Y \otimes_U S \otimes_U B$, *then* $y \otimes \overline{s} \otimes b = y' \otimes \overline{1} \otimes b$ *in* $Y \otimes_U (S/U) \otimes_U B$.

Proof Since $b \sim b'$ in B we see from Corollary 4.12 that $\overline{1} \otimes b = \overline{1} \otimes b'$ in $(S/U) \otimes_U B$. Hence

$$y \otimes \overline{s} \otimes b = y' \otimes \overline{1} \otimes b' = y' \otimes \overline{1} \otimes b \text{ in } Y \otimes_U (S/U) \otimes_U B.$$

We end this section by mentioning an alternative description of the dominion. Recall that if U is a subsemigroup of a semigroup S then U is said to *dominate* an element d of S if for all semigroups T and for all homomorphisms $\beta, \gamma : S \to T$,

$$[(\forall u \in U)\ \beta(u) = \gamma(u)] \text{ implies } \beta(d) = \gamma(d).$$

The set of elements dominated by U is called the *dominion* of U in S, and is written $\mathrm{Dom}_S(U)$.

The following was first proved by Stenström [39].

THEOREM 4.16 [Howie, 22, Theorem VII.2.5]. *If* U *is a subsemigroup of a semigroup* S *and if* $d \in S$, *then* $d \in \mathrm{Dom}_S(U)$ *if and only if* $d \otimes 1 = 1 \otimes d$ *in* ${}^1S \otimes_{{}^1U} {}^1S$.

5. Flat U-Sets

Let U be a monoid and let $X \in$ ENS-U. We shall say that X is (right) flat if for all $A,B \in$ U-ENS and all U-monomorphisms $f : A \to B$, the induced map $1 \otimes f : X \otimes_U A \to X \otimes_U B$ is 1-1. Left flat U-sets are defined dually.

We shall have occasion to make use of the following result.

THEOREM 5.1 [Bulman-Fleming and McDowell, 4, Lemma 2.2]. *Let* U *be a monoid and let* $X \in$ ENS-U. *Then* X *is flat if and only if for all* $B \in$ U-ENS *and all* $b,b' \in B$ *the map*

$$X \otimes (Ub \cup Ub') \to X \otimes B$$

is 1-1.

LEMMA 5.2 [Bulman-Fleming and McDowell, 4, Lemma 2.4]. *Let* U *be a monoid. Then* U *is left reversible if and only if the one element* U-*set* $Z = \{z\}$ *is left flat.*

The following easily proved result will also prove useful.

LEMMA 5.3 *Let* S,T *be monoids and suppose that* $X \in$ ENS-S, $Y \in$ S-ENS-T *are such that* X *is flat as a right* S-*set and* Y *is flat as a right* T-*set. Then* $X \otimes_S Y$ *is flat as a right* T-*set.*

From Lemma 4.8 we can deduce

LEMMA 5.4 *Let* U *be a monoid and let* $X_i \in$ ENS-U *for* $i \in I$. *Then* $\dot{\bigcup}_{i \in I} X_i$ *is flat if and only if each* X_i *is flat.*

Proof Let $f : A \to B$ be a left U-monomorphism. Then by Lemma 4.8 we see that

$$\left(\dot{\bigcup_{i \in I}} X_i\right) \otimes_U A \simeq \dot{\bigcup_{i \in I}} \left(X_i \otimes_U A\right) \quad \text{and} \quad \left(\dot{\bigcup_{i \in I}} X_i\right) \otimes_U B \simeq \dot{\bigcup_{i \in I}} \left(X_i \otimes_U B\right).$$

The result is now clear.

Let U be a monoid and let $X \in \underline{U}\text{-ENS}$. We say that X is *(left) injective* if for all left U-monomorphisms $f : A \to B$ and all left U-maps $\vartheta : A \to X$, there exists a left U-map $\varphi : B \to X$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\vartheta"'] & B \arrow[dl, "\varphi"] \\ X & \end{tikzcd}$$

commutes.

THEOREM 5.5 *Let U be a monoid and $X \in \underline{U}\text{-ENS}$. Then X is injective if and only if the following two conditions hold:*

(1) *X contains a fixed element i.e. an element x such that $ux = x$ for all u in U,*

(2) *for all U-monomorphisms $f : A \to B$ with B cyclic and all U-maps $\vartheta : A \to X$, there exists a U-map $\varphi : B \to X$ such that $\vartheta = \varphi \circ f$.*

Proof Suppose that X is injective. Then it is clear that condition (2) holds. Let {z} be the one element U-set and consider the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "1_X"'] & X \dot{\cup} \{z\} \\ X & \end{tikzcd}$$

Since X is injective we see that there exists a U-map $\varphi : X \dot{\cup} \{z\} \to X$ such that $\varphi|_X = 1_X$. Hence we see that

$$u\varphi(z) = \varphi(uz) = \varphi(z), \quad u \in U,$$

and so $\varphi(z)$ is a fixed element in X.

Conversely, let $f : A \to B$ be any U-monomorphism and let $\vartheta : A \to X$ be any U-map. To simplify notation we shall consider f as an inclusion map.

Consider all pairs (A_i, φ_i) with the properties

(1) $A \subseteq A_i \subseteq B$, and

(2) $\varphi_i : A_i \to X$ and $\varphi_i | A = \vartheta$.

Then the collection of these pairs is non-empty since (A, ϑ) is such a pair. Order this collection by

$$(A_i, \varphi_i) \leq (A_j, \varphi_j) \text{ if and only if } A_i \subseteq A_j \text{ and } \varphi_j|_{A_i} = \varphi_i.$$

By Zorn's lemma there is a maximal such pair (A_0, φ_0), say. If $A_0 = B$ the theorem is proved. Otherwise let $b \in B | A_0$ and consider the set

$$I = \{u \in U : ub \in A_0\}.$$

Then I is either empty or is a left ideal of U.

If $I = \varphi$, then consider the diagram

$$\begin{tikzcd} A \subseteq A_o \arrow[d, phantom] & \subseteq A_o \dot{\cup} Ub \\ X \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, "\vartheta"'] \subseteq A_o \arrow[d, "\varphi_o"] \subseteq A_o \dot{\cup} Ub \arrow[dl, "\varphi"] \\ X \end{tikzcd}$$

where $\varphi : A_o \dot{\cup} Ub \to X$ is given by $\varphi(a_o) = \varphi_o(a_o)$ and $\varphi(ub) = x_o$ where x_o is a fixed element of X chosen once for all. This contradicts the maximality of (A_o,φ_o) and so $A_o = B$.

On the other hand if $I \neq \varphi$ then consider the well-defined U-map $\psi : Ib \to X$ given by $\psi(vb) = \varphi_o(vb)$ $(v \in I)$. By property (2) there exists a U-map $\varepsilon : Ub \to X$ such that

$$\begin{tikzcd} Ib \arrow[r] \arrow[d, "\psi"'] & Ub \arrow[dl, "\varepsilon"] \\ X \end{tikzcd}$$

commutes. Now define $\varphi : A_o \cup Ub \to X$ by $\varphi(a_o) = \varphi_o(a_o)$ and $\varphi(ub) = \varepsilon(ub)$ $(u \in U)$. Then φ is well-defined since if $a_o = ub$ for some $a_o \in A$, $u \in U$ then clearly $u \in I$. Hence

$$\varphi(a_o) = \varphi_o(a_o) = \varphi_o(ub) = \psi(ub) = \varepsilon(ub) = \varphi(ub).$$

Since $Ub \not\subseteq A_o$ we again contradict the maximality of (A_o,φ_o) and so the theorem is proved.

We shall now proceed to deduce a connection between flatness and injectivity similar to that for R-modules.

Let $X \in$ ENS-$\underline{U}$ and let I be any set. Let $X^* = \mathrm{Hom}(X,I)$, the collection of all maps from X to I. Then X^* is a left U-set if if we define $(uf)(x) = f(xu)$.

Suppose $\alpha : X \to Y$ is a right U-map. Then α induces a left U-map $\alpha^* : Y^* \to X^*$ given by $\alpha^*(f) = f \circ \alpha$.

From now on we will assume that $|I| \geq 2$.

LEMMA 5.6 Suppose that $\alpha : X \to Y$ is a right U-map. Then

(1) α is 1-1 if and only if α^* is onto,

(2) α is onto if and only if α^* is 1-1.

Proof (1) Suppose that α is 1-1. Let $g \in X^*$ and define $f \in Y^*$ by

$$f(y) = \begin{cases} g(x), & \text{if } \ y = \alpha(x), \\ i_o, & \text{if } \ y \notin \operatorname{im}\alpha. \end{cases}$$

(Here $i_o \in I$ is chosen once for all.) Since α is 1-1 then f is well-defined and $\alpha^*(f) = f \circ \alpha = g$. Hence α^* is onto.

Conversely, suppose that α^* is onto. Let $\alpha(x) = \alpha(x')$ and assume that $x \neq x'$. Then since $|I| \geq 2$ we can find $g \in X^*$ such that $g(x) \neq g(x')$. Hence, since α^* is onto, there exists $f \in Y^*$ such that $\alpha^*(f) = g$. But $\alpha(x) = \alpha(x')$ implies $(f \circ \alpha)(x) = (f \circ \alpha)(x')$, i.e. $g(x) = g(x')$ giving the required contradiction. Hence α is 1-1.

(2) Suppose that α is onto and let $\alpha^*(f) = \alpha^*(f')$. Then $f \circ \alpha = f' \circ \alpha$ and so $f = f'$ since α is onto. Hence α^* is 1-1.

Conversely, suppose that α^* is 1-1 and suppose that α is not onto. Then there exists $y' \in Y$ with $y' \notin \operatorname{im}\alpha$. Let $i \neq j \in I$ and define $f, f' \in Y^*$ by

$$f(y) = \begin{cases} i : y \in \operatorname{im}\alpha \cup \{y'\} \\ j : \text{otherwise} \end{cases}$$

$$f'(y) = \begin{cases} i : y \in \operatorname{im}\alpha \\ j : \text{otherwise.} \end{cases}$$

Then $f \neq f'$ since $f(y') = i \neq j = f'(y')$. But $f \circ \alpha = f' \circ \alpha$ i.e. $\alpha^*(f) = \alpha^*(f')$. This contradicts the fact that α^* is 1-1 and so α is onto as required.

LEMMA 5.7 Let Y be a flat right U-set. Then Y^* is an injective left U-set.

Proof Let $f : A \to B$ be a left U-monomorphism and let $g : A \to Y^*$ be a left U-map. Since $g \in \mathrm{Hom}_U(A, \mathrm{Hom}(Y,I))$. We see from Theorem 4.6 that there exists $\overline{g} \in \mathrm{Hom}(Y \otimes_U A, I)$ given by $\overline{g}(y \otimes a) = g(a)(y)$. So we have a diagram

$$\begin{array}{ccc} Y \otimes_U A & \xrightarrow{1 \otimes f} & Y \otimes_U B \\ {\scriptstyle \overline{g}} \downarrow & & \\ I & & \end{array}$$

where $1 \otimes f$ is 1-1 by flatness of Y. Define $h : Y \otimes_U B \to I$ by

$$h(y \otimes b) = \begin{cases} \overline{g}(y' \otimes a) & : \text{if } y \otimes b = y' \otimes f(a) \\ i_o & : \text{otherwise} \end{cases}$$

where $i_o \in I$ is chosen once for all. Then h is well-defined, since $1 \otimes f$ is 1-1 and it is clear that $h \circ (1 \otimes f) = \overline{g}$. By Theorem 4.6 we see that $h \in \mathrm{Hom}(Y \otimes_U B, I)$ induces a map $h' \in \mathrm{Hom}_U(B, \mathrm{Hom}(Y,I))$ given by $h'(b)(y) = h(y \otimes b)$. It is easy to check that $h' \circ f = g$ and so Y^* is injective as required.

As a corollary we have

COROLLARY 5.8 Let Y be a flat right U-set such that $|Y| \geq 2$. Then $T(Y)$, the full transformation semigroup over Y, is an injective left U-set.

Proof Simply take $I = Y$ in Lemma 5.7.

As one would expect, the converse of Lemma 5.7 is also true.

THEOREM 5.9 Let $Y \in$ ENS-$\underline{U}$. Then Y is flat if and only if Y^* is injective in $\underline{U}$-ENS.

Proof Suppose that Y^* is injective in $\underline{U}$-ENS and let $f : A \to B$ be a left U-monomorphism. Then the map $f^* : \mathrm{Hom}_U(B,Y^*) \to \mathrm{Hom}_U(A,Y^*)$ given by $f^*(g) = g \circ f$, is a U-epimorphism. To see this, suppose that $h \in \mathrm{Hom}_S(A,Y^*)$. Then we have a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \\ Y^* & \end{tikzcd}$$

Since Y^* is injective we see that there exists $g \in \mathrm{Hom}_U(B,Y^*)$ such that $g \circ f = h$, i.e. such that $f^*(g) = h$. We now have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_U(B,Y^*) \arrow[r, "f^*"] \arrow[d] & \mathrm{Hom}_U(A,Y^*) \arrow[d] \\ \mathrm{Hom}(Y \otimes B, I) \arrow[r, "(1\otimes f)^*"'] & \mathrm{Hom}(Y\otimes A, I) \end{tikzcd}$$

where the vertical maps are the isomorphisms of Theorem 4.6. Hence $(1 \otimes f)^*$ is onto and so $1 \otimes f$ is 1-1 by Lemma 5.6.

A rather interesting corollary is

COROLLARY 5.10 Let $Y \in$ ENS-$\underline{U}$ and suppose that $|Y| \geq 2$. Then Y is flat if and only if $T(Y)$ is injective in $\underline{U}$-ENS.

If $|Y| = 1$ then the above corollary fails, since in this case $|T(Y)| = 1$ and hence $T(Y)$ is injective while Y is flat only if U is right reversible (Lemma 5.2).

THEOREM 5.11 Let $Y \in$ ENS-$\underline{U}$. Then Y is flat if and only if for all left U-sets A, all cyclic left U-sets B and all U-monomorphisms $f : A \to B$, the induced map

$$1 \otimes f \; : \; Y \otimes A \longrightarrow Y \otimes B,$$

is 1-1.

Proof One way round is obvious. Suppose then that $f : A \to B$ is any left U-monomorphism with B cyclic and let $g : A \to Y^*$ be a left U-map. We use Theorem 5.5 to show that Y^* is injective. First notice that Y^* contains fixed elements e.g. the maps $[i] : Y \to I$ given by $[i](y) = i$ for all $y \in Y$. Since $g \in \mathrm{Hom}_U(A, \mathrm{Hom}(Y,I))$, we see from Theorem 4.6 that there exists $\bar{g} \in \mathrm{Hom}(Y \otimes A, I)$ given by $\bar{g}(y \otimes a) = g(a)(y)$. So we have a diagram

$$\begin{tikzcd} Y \otimes_U A \arrow[r, "1 \otimes f"] \arrow[d, "\bar{g}"'] & Y \otimes_U B \\ I & \end{tikzcd}$$

where $1 \otimes f$ is 1-1 by assumption. Define $h : Y \otimes B \to I$ by

$$h(y \otimes b) = \begin{cases} \bar{g}(y' \otimes a) & : \text{if } y \otimes b = y' \otimes f(a) \\ i_o & : \text{otherwise} \end{cases}$$

where $i_o \in I$ is chosen once for all. Then h is well-defined since $1 \otimes f$ is 1-1 and clearly $h \circ (1 \otimes f) = \bar{g}$. By Theorem 4.6 there exists $h' \in \mathrm{Hom}_U(B, \mathrm{Hom}(Y, I))$ such that $h'(b)(y) = h(y \otimes b)$. Hence $h' \circ f = g$ and so Y^* is injective by Theorem 5.5. From Theorem 5.9, Y is flat.

The following results will play an important role when we examine the connections between flatness and amalgamations.

LEMMA 5.12 <u>Let</u> I <u>be a directed quasi-ordered set</u>. <u>Let</u> (X_i, φ_j^i) <u>be a direct system in</u> ENS-<u>U</u> <u>with direct limit</u> (X, α_i) <u>and let</u> $f : A \to B$ <u>be a left</u> U-<u>monomorphism</u>. <u>If the maps</u> $1 \otimes f : X_i \otimes_U A \to X_i \otimes_U B$ <u>are</u> 1-1 <u>for all</u> i <u>in</u> I, <u>then the map</u> $1 \otimes f : X \otimes_U A \to X \otimes_U B$ <u>is</u> 1-1.

<u>Proof</u> Let $A_i = X_i \otimes A$ and $B_i = X \otimes B$. Then $X \otimes A$ and $X \otimes B$ are the direct limits of the systems $(A_i, \varphi_j^i \otimes 1)$, $(B_i, \varphi_j^i \otimes 1)$ respectively. The result now follows from Lemma 3.20.

The following concept will prove useful later. A right U-set X is said to be <u>(right) quasi-flat</u> if for all left U-sets A, all <u>flat</u> left U-sets B and all U-monomorphisms $f : A \to B$, the induced map $1 \otimes f : X \otimes A \to X \otimes B$ is 1-1. Clearly flat implies quasi-flat. It is not known whether the converse is true or false.

From Lemma 5.12 we have

THEOREM 5.13 <u>Let</u> U <u>be a monoid and</u> I <u>a directed quasi-ordered set</u>. <u>Let</u> (X_i, φ_j^i) <u>be a direct system in</u> ENS-<u>U</u> <u>with direct limit</u> (X, α_i). <u>If each</u> X_i <u>is [quasi-] flat then so is</u> X.

We now proceed to show that I needs to be directed in the above theorem. First we need a few lemmas.

LEMMA 5.14 <u>Let</u> U <u>be a submonoid of an abelian group</u> S. <u>Then</u> S <u>is flat as a</u> U-<u>set</u>.

<u>Proof</u> We use Theorem 5.1. Let $A \in$ ENS-<u>U</u> and suppose that $a \otimes s = a' \otimes s'$ in $A \otimes_U S$. We have a U-scheme

$$a = a_1 u_1$$

$$a_1 v_1 = a_2 u_2, \quad u_1 s = v_1 s_2,$$

$$\dots$$

$$a_n v_n = a', \quad u_n s_n = v_n s',$$

over A and S joining (a,s) to (a',s'). We need to find a U-scheme over $(aU \bigcup a'U)$ and S joining (a,s) to (a',s').

<u>NOTATION</u> Define

$$x_0 = 1, \quad x_1 = v_1, \quad x_{i+1} = x_i v_{i+1}, \quad i = 1,2,\ldots,n-1,$$

$$y_0 = 1, \quad y_1 = u_1, \quad y_{i+1} = y_i u_{i+1}, \quad i = 1,2,\ldots,n-1,$$

$$z_1 = x_1, \quad z_i = x_i x_{i-2}^{-1},$$

$$w_1 = y_1, \quad w_i = y_i y_{i-2}^{-1}, \quad i = 2,3,\ldots,n.$$

Observe first that

(1) $x_{i-2} z_i = x_i = x_{i-1} v_i$,

(2) $v_i x_i^{-1} = x_{i-1}^{-1} = z_{i+1} x_{i+1}^{-1}$,

(3) $y_{i-1} u_i = y_{i-2} w_i$,

(4) $w_i y_i^{-1} = u_{i-1} y_{i-1}^{-1}$.

It is also clear that $x_i, y_i \in U$ and since $z_i = v_i v_{i-1}$ while $w_i = u_i u_{i-1}$ as is easily checked, we deduce that $z_i, w_i \in U$. Also, we have

$$\begin{aligned} ax_n &= a_1 u_1 v_1 v_2 \cdots v_n, \\ &= a_2 u_1 u_2 v_2 \cdots v_n, \\ &= \cdots \\ &= a_n u_1 u_2 \cdots u_n v_n, \\ &= a' u_1 u_1 \cdots u_{n-1} u_n, \\ &= \cdots \\ &= a' y_n. \end{aligned}$$

Consider now the following equations

$$
\begin{array}{rlrl}
a &= a.x_o & x_o s &= z_1(sx_1^{-1}) \\
a.z_1 &= (ax_o).v_1 & v_1(sx_1^{-1}) &= z_2(sx_2^{-1}) \\
(ax_o)\, z_2 &= (ax_1)v_2 & v_2(sx_2^{-1}) &= z_3(sx_3^{-1}) \\
& & \cdots\cdots & \\
(ax_{i-2})z_i &= (ax_{i-1})v_i & v_i(sx_i^{-1}) &= z_{i+1}(sx_{i+1}^{-1}) \\
& & \cdots\cdots & \\
(ax_{n-3})z_{n-1} &= (ax_{n-2})v_{n-1} & v_{n-1}(sx_{n-1}^{-1}) &= z_n(sx_n^{-1}) \\
(ax_{n-2})z_n &= (a'y_{n-2})w_n & w_n(sx_n^{-1}) &= u_{n-1}(s'y_{n-1}^{-1}) \quad * \\
(a'y_{n-2})u_{n-1} &= (a'y_{n-3})\, w_{n-1} & w_{n-1}(s'y_{n-1}^{-1}) &= u_{n-2}(s'y_{n-2}^{-1}) \\
& & \cdots\cdots & \\
(a'y_{i-1})u_i &= (a'y_{i-2})w_i & w_i(s'y_i^{-1}) &= u_{i-1}(s'y_{i-1}^{-1}) \\
& & \cdots\cdots & \\
(a'y_o)u_1 &= a'w_1 & w_1(s'y_1^{-1}) &= y_o s' \\
a'y_o &= a' & &
\end{array}
$$

It is straightforward to check that these equations give us the required scheme. The only point we would like to stress is the validity of the equations marked (*). Notice that $x_{n-2}z_n = x_n$ by (1) above and that $y_{n-2}w_n = y_{n-1}u_n = y_n$ by (3). But $ax_n = a'y_n$ as already noted above. A similar procedure will hold for the equation $w_n(sx_n^{-1}) = u_{n-1}(s'y_{n-1}^{-1})$.

We now record a fairly straightforward result about dominions (see Sections 1 and 4).

LEMMA 5.15 *Let* U *be a submonoid of a group* S. *Then* $\mathrm{Dom}_S\, U$ *is the subgroup generated by* U.

THEOREM 5.16 *Let* U *be a submonoid of a monoid* S, *and consider the pushout diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & S \arrow[d, "\alpha"] \\ S \arrow[r, "\beta"'] & P \end{tikzcd}$$

in ENS-*U*. *If* P *is flat then* U *is closed in* S.

Proof Suppose that $d \otimes 1 = 1 \otimes d$ in ${}^1S \otimes_{{}^1U} \otimes {}^1S$ and suppose that $d \in S$. It is easy to check that if e is the identity of U (and hence of S) then $d \otimes e = e \otimes d$ in $S \otimes_U S$. Hence we have

$$\begin{aligned} \alpha(d) \otimes e &= \alpha(e) \otimes d \\ &= \beta(e) \otimes d \\ &= \beta(d) \otimes e \text{ in } P \otimes_U S. \end{aligned}$$

But, P is flat and so $P \to P \otimes_U S$ is 1-1 and so we see that $\alpha(d) = \beta(d)$ in P. By Lemma 3.9 $d \in U$, and U is closed in S.

We now see that if U is a submonoid of an abelian group S but is *not* a subgroup of S, then S is flat as a U-set but the pushout of the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & S \\ S & \end{tikzcd}$$

is *not* flat. This shows that Theorem 5.13 cannot be generalised to the case when I is not directed. Notice also that if S is flat as a U-set then U need not be closed in S. However, there are flatness conditions associated with closure.

LEMMA 5.17 *Let* U *be a submonoid of a monoid* S, *and let* $f : X \to Y$ *be a right* U-*monomorphism*. *Suppose that* $Y/X \otimes_U U \to Y/X \otimes_U S$ *is* 1-1. *If* $y \otimes 1 = f(x) \otimes s$ *in* $Y \otimes_U S$, *then* $y \in \operatorname{im} f$.

Proof We see from Lemma 4.14 that $\overline{y} \otimes 1 = \overline{f(x)} \otimes 1$ in $Y/X \otimes S$ and hence in $Y/X \otimes U$. It now follows that $y \in \operatorname{im} f$.

COROLLARY 5.18 *Let* U *be a submonoid of a monoid* S *and let* S/U *be (right,left) flat as a* U-*set*. *Then* U *is closed in* S. *Alternatively if* S *is flat and* S/U *is quasi-flat then* U *is closed in* S.

Proof In Lemma 5.17, take $X = U$, $Y = S$ and $f = i : U \to S$. If $s \otimes 1 = 1 \otimes s$ in $S \otimes_U S$ then by Lemma 5.17 we see that $s \in U$. Hence $\operatorname{Dom}_S U = U$ and U is closed in S.

LEMMA 5.19 *Let* $f : X \to Y$ *be a left* U-*monomorphism and suppose that* $X \neq \varphi$. *If* Y/X *is quasi-flat, then* U *is left reversible*.

Proof Let $u, v \in U$, $x \in X$. We have

$$u \otimes \overline{f(x)} = 1 \otimes \overline{f(ux)} = 1 \otimes \overline{f(vx)} = v \otimes \overline{f(x)} \text{ in } U \otimes_U (Y/X).$$

Since Y/X is quasi-flat, then $u \otimes \overline{f(x)} = v \otimes \overline{f(x)}$ in $(uU \cup vU) \otimes_U (Y/X)$.

Hence from Lemma 4.5 we see that $u \sim v$ in $uU \cup vU$ from which we easily deduce that $uU \cap vU \neq \varphi$.

The following corollary will be used later without reference.

COROLLARY 5.20 *Let* U *be a submonoid of a monoid* S. *If* S/U *is flat then* U *is reversible*.

6. Absolutely flat semigroups

Let U be a monoid. Then U is said to be *right absolutely flat* if every right U-set is flat. *Left absolutely flat* monoids are defined dually, while U is said to be *absolutely* flat if U is both right and left absolutely flat. A semigroup U is said to be *(right,left) absolutely flat* if the monoid ^1U is (right,left) absolutely flat.

LEMMA 6.1 [Bulman-Fleming and McDowell, 4] *A monoid* U *is absolutely flat as a semigroup if and only if it is absolutely flat as a monoid*.

From Corollary 5.20 we see

LEMMA 6.2 [Bulman-Fleming and McDowell, 4, Lemma 2.4] *If* U *is left absolutely flat then* U *is left reversible*.

Kilp proved the following.

LEMMA 6.3 [See Bulman-Fleming and McDowell, 4, Proposition 2.5] *If all cyclic left* U-*sets over a monoid* U *are flat, then* U *is regular*.

THEOREM 6.4 [Bulman-Fleming and McDowell, 4, Theorem 4.2] *Inverse semigroups are absolutely flat*.

Bulman-Fleming and McDowell [4] have given examples of absolutely flat semigroups which are not inverse and regular semigroups which are not absolutely flat.

LEMMA 6.5 [Bulman-Fleming and McDowell, 4, Lemma 1.1] *Let* U *be a monoid. Let* $A \in$ ENS-U *and let* $u, v \in U$, $a, a' \in A$. *Denote by* $\vartheta(u,v)$ *the smallest left* U-*congruence on* U *which identifies* u *and* v. *Then* $a \otimes T = a' \otimes T$ *in* $A \otimes_U (U/\vartheta(u,v))$ *if and only if either* $a = a'$ *or there exists* $a_1, \ldots, a_n \in A$, $x_1, \ldots, x_n$, $y_1, \ldots, y_n \in U$ *where* $\{x_i, y_i\} = \{u, v\}$ *for* $i = 1, 2, \ldots, n$ *such that*

$$a = a_1x_1,\ a_1y_1 = a_2x_2, \ldots, a_ny_n = a'.$$

Let U be a semigroup. Denote by R (respectively L) the collection of all non-empty right (left) ideals of ${}^1U$. Say that U is *left* R-*reductive* if for all I in R and for all a, b in I,

$$[xa = xb \text{ for all } x \text{ in } I] \text{ implies } a = b.$$

This is equivalent to saying that every right ideal of ${}^1U$ is a left-reductive semigroup.

THEOREM 6.6 *If a semigroup* U *is left absolutely flat, then* U *is left* R-*reductive*.

Proof Let us assume that U is left absolutely flat but *not* left R-reductive. Then there is a right ideal of ${}^1U$, I say, and a, b in I such that $a \neq b$, but $xa = xb$ for all x in I. Now the left ${}^1U$-set ${}^1U/\vartheta(a,b)$ is flat, by assumption and so the map

$$I \otimes_{{}^1U} {}^1U/\vartheta(a,b) \longrightarrow {}^1U \otimes_{{}^1U} {}^1U/\vartheta(a,b)$$

is 1-1. We have

$$a \otimes \overline{1} = 1 \otimes \overline{a} = 1 \otimes \overline{b} = b \otimes \overline{1}$$

in ${}^1U \otimes_{{}^1U} {}^1U/\vartheta(a,b)$ and hence in $I \otimes_{{}^1U} {}^1U/\vartheta(a,b)$. By Lemma 6.5 we see that either $a = b$, giving a contradiction, or there exists $c_1,\ldots,c_n \in I$, $x_1,\ldots,x_n$, $y_1,\ldots,y_n \in {}^1U$ with $\{x_i,y_i\} = \{a,b\}$ for $i = 1,\ldots,n$ such that

$$a = c_1x_1,\ c_1y_1 = c_2x_2,\ldots,\ c_ny_n = b.$$

But since $\{x_i,y_i\} = \{a,b\}$ and since $ca = cb$ for all $c \in I$ it follows that $c_ix_i = c_iy_i$ for $i = 1,\ldots,n$ and so $a = b$, giving the required contradiction.

COROLLARY 6.7 *If U is right (left) absolutely flat then U is right (left) reductive.*

The converse to Theorem 6.6 is false. In fact Bulman-Fleming and McDowell [4, Corollary 5.3] have shown that the semigroup $M^o[G;I,\Lambda;P]$ where $G = \{e,x\}$ is the group C_2, $I = \Lambda = \{1,2\}$ and $P = \begin{bmatrix} e & x \\ e & e \end{bmatrix}$, is neither left nor right absolutely flat. It is not too difficult however to show that this semigroup is both right L-reductive and left R-reductive.

Finally, we summarise the properties of absolutely flat semigroups in a theorem.

THEOREM 6.8 *Let a semigroup U be left absolutely flat. Then U is*

(1) *Regular*

(2) *left reversible*

(3) *left R-reductive*

(4) *absolutely closed* (Corollary 5.18).

CHAPTER II FREE PRODUCTS AND AMALGAMATION

In [7], P M Cohn constructed the free product of a ring amalgam using direct limits and tensor products of R-modules. In Section 2 we make the analogous construction for semigroups. First we introduce a concept which will prove extremely valuable in later sections. Except where otherwise indicated, all tensor products will be over U.

1. Free extensions

Let U be a submonoid of a monoid S. Let $X \in$ ENS-$\underline{S}$, $Y \in$ ENS-$\underline{U}$ and let $f : X \to Y$ be a right U-map. The free S-extension of X and Y is a right S-set $F(S;X,Y)$ together with an S-map $h : X \to F(S;X,Y)$ and a U-map $g : Y \to F(S;X,Y)$ such that:

(1) $g \circ f = h$;

(2) Whenever there is an S-set Z, an S-map $\beta : X \to Z$ and a U-map $\alpha : Y \to Z$ such that $\alpha \circ f = \beta$, there exists a unique S-map $\psi : F(S;X,Y) \to Z$ such that

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "h"] \arrow[ddr, bend right, "\beta"'] & & Y \arrow[dl, "g"'] \arrow[ddl, bend left, "\alpha"] \\
& F(S;X,Y) \arrow[d, "\psi"] & \\
& Z &
\end{tikzcd}$$

commutes.

As with all universal constructions the free-extension if it exists, is essentially unique.

THEOREM 1.1 Free S-extensions exists.

Proof Let $f : X \to Y$ be a right U-map, where $X \in$ ENS-$\underline{S}$, $Y \in$ ENS-$\underline{U}$. Let σ be the equivalence on $Y \otimes_U S$ generated by the relation

$$R = \{(f(x) \otimes s,\ f(x') \otimes s') : xs = x's',\ x,x' \in X,\ s,s' \in S\}.$$

Then σ is a right S-congruence on $Y \otimes_U S$. Define $g : Y \to (Y \otimes_U S)/\sigma$ by $g(y) = (y \otimes 1)\sigma$. Then g is a U-map and $h = g \circ f : X \to (Y \otimes_U S)/\sigma$ is an S-map. It is now routine to verify property (2) above.

NOTE If $X \in \underline{T}$-ENS-$\underline{S}$, $Y \in \underline{T}$-ENS-$\underline{U}$ and f is a (T,U)-map then $F(S;X,Y) \in \underline{T}$-ENS-$\underline{S}$.

It is possible to characterise the free S-extension in another way. Consider the map $\varphi : X \otimes_U S \to X$ given by $\varphi(x \otimes s) = xs$. Then φ is an S-epimorphism. Also the map $f \otimes 1 : X \otimes_U S \to Y \otimes_U S$ is an S-map.

THEOREM 1.2 Let $X \in$ ENS-$\underline{S}$, $Y \in$ ENS-$\underline{U}$ and let $f : X \to Y$ be a map. Then the free S-extension, $F(S;X,Y)$ of X and Y is the pushout in ENS-$\underline{S}$ (and hence in ENS-$\underline{U}$) of the diagram

$$\begin{tikzcd} X \otimes_U S \arrow[r, "f\otimes 1"] \arrow[d, "\varphi"'] & Y \otimes_U S \\ X & \end{tikzcd}$$

Proof Consider the map $\vartheta : Y \otimes_U S \to F(S;X,Y)$ given by $\vartheta(y \otimes s) = g(y).s$. Then ϑ is a well-defined S-map and the diagram

$$\begin{tikzcd} X \otimes_U S \arrow[r, "f\otimes 1"] \arrow[d, "\varphi"'] & Y \otimes_U S \arrow[d, "\vartheta"] \\ X \arrow[r, "h"'] & F(S;X,Y) \end{tikzcd}$$

commutes, where g and h are as in the definition of $F(S;X,Y)$. Suppose now that there exists an S-set Z and S-maps $\alpha : Y \otimes_U S \to Z$, $\beta : X \to Z$ such that $\alpha \circ (f \otimes 1) = \beta \circ \varphi$. Then we have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\beta"'] & & Y \arrow[dl, "\gamma"] \\ & Z & \end{tikzcd}$$

where $\gamma(y) = \alpha(y \otimes 1)$. Hence there exists a unique S-map $\psi : F(S;X,Y) \to Z$ such that $\psi \circ g = \gamma$ and $\psi \circ h = \beta$. It is now routine to verify that

$$
\begin{tikzcd}
X \otimes_U S \arrow[r, "f\otimes 1"] \arrow[d, "\varphi"'] & Y \otimes_U S \arrow[d, "\vartheta"] \arrow[ddr, "\alpha"] & \\
X \arrow[r, "h"] \arrow[drr, "\beta"'] & F(S;X,Y) \arrow[dr, "\psi"] & \\
 & & Z
\end{tikzcd}
$$

commutes.

THEOREM 1.3 *Let* U *be a submonoid of a monoid* S. *Let* I *be a quasi-ordered set and let* (X_i,φ_j^i) *and* (Y_i,ϑ_j^i) *be direct systems in* ENS-*S* *and* ENS-*U* *respectively with direct limits* (X,α_i) *and* (Y,β_i). *Suppose that there exist* U-*maps* $f_i : X_i \to Y_i$ *such that whenever* $i \leq j$ *in* I, $f_j\varphi_j^i = \vartheta_j^i \circ f_i$, *and let* $Z_i = F(S;X_i,Y_i)$. *Then there exists* S-*maps* $\psi_j^i : Z_i \to Z_j$ $(i \leq j)$, *a* U-*map* $f : X \to Y$ *and* S-*maps* $\psi_j^i : Z_i \to F(S;X,Y)$ *such that* $(F(S;X,Y),\gamma_i)$ *is the direct limit in* ENS-*S* *of the system* (Z_i,ψ_j^i).

Proof The result is straightforward enough to prove directly, but it follows almost immediately from Theorem 1.2 and Rotman [35, Theorem 2.21], which states that any two direct limits, perhaps with different index sets, commute.

Since disjoint unions (coproducts) are direct limits, we can deduce,

COROLLARY 1.4 *Let* U *be a submonoid of a monoid* S. *Let* $A,B \in$ ENS-*S*, $C,D \in$ ENS-*U* *and let* $f : A \to C$ *and* $g : B \to D$ *be* U-*maps*. *Then* $F(S;A\dot{\cup}B,C\dot{\cup}D) \simeq F(S;A,C)\ \dot{\cup}\ F(S;B,D)$.

The following easily proved result will prove useful later, and will be used without reference.

LEMMA 1.5 Let U be a submonoid of a monoid S. Let $A \in ENS\text{-}S$ and $B \in ENS\text{-}U$. Then $F(S;A,A \dot{\cup} B) \simeq A \dot{\cup} (B \otimes_U S)$.

2. Amalgamated free products

We now proceed to construct the free product of a semigroup amalgam $[U;S_1,S_2]$, as a direct limit of U-sets. All tensor products will be over U. First of all, recall the definition of the free product. The free product of the amalgam $[U;S_1,S_2]$ is the semigroup $S_1 *_U S_2 = (S_1 * S_2)/\rho$ where ρ is the congruence on the free product $S_1 * S_2$, generated by

$$\{(\alpha_1(u),\alpha_2(u)) : u \in U\},$$

where $\alpha_i : S_i \to S_1 *_U S_2$ $(i = 1,2)$ are the natural monomorphisms. A typical element of $S_1 *_U S_2$ will be written as

$$(s_1,\ldots,s_n), \qquad s_i \in S_1 \cup S_2.$$

This is not the standard notation (see Howie [22]) but will prove more useful here.

Let $[U;S_1,S_2]$ be a monoid amalgam. Let $W_1 = S_1$, $W_2 = S_1 \otimes S_2$ and define $f_1 : W_1 \to W_2$ by $f_1(s_1) = s_1 \otimes 1$. Then f_1 is a well-defined (S_1,U)-map. Suppose we have constructed a sequence $W_1 \xrightarrow{f_1} W_2 \xrightarrow{f_2} \ldots \xrightarrow{f_{n-2}} W_{n-1}$, and suppose that W_k is an (S_1,S_i)-biset, where

$i \equiv k \pmod 2$ and f_k is an (S_1,U)-map. Let $i \equiv n \pmod 2$ and define $W_n = F(S_i;W_{n-2},W_{n-1})$. From Theorem 1.1 we see that $W_n = (W_{n-1} \otimes S_i)/\sigma_{n-2}$, where σ_{n-2} is generated by

$$R_{n-2} = \{(f_{n-2}(w_{n-2}) \otimes s_i,\ f_{n-2}(w'_{n-2}) \otimes s'_i) : w_{n-2}s_i = w'_{n-2}s'_i\}$$

and we have an (S_1,U)-map $f_{n-1} : W_{n-1} \to W_n$ given by $f_{n-1}(w_{n-1}) = (w_{n-1} \otimes 1)\sigma_{n-2}$.

We show

THEOREM 2.1 *Let* $[U;S_1,S_2]$ *be a monoid amalgam. Then* $S_1 *_U S_2$ *is the direct limit in* $\underline{U}$-ENS-$\underline{U}$ *of the system* $(W_n,f_n)_{n \geq 1}$.

Proof First of all we have monoid homomorphisms $\vartheta_i : S_i \to S_1 *_U S_2$ $(i = 1,2)$. Define $\varphi_n : W_n \to S_1 *_U S_2$ inductively as follows. Let $\varphi_1 = \vartheta_1$ and put $\varphi_2(s_1 \otimes s_2) = \varphi_1(s_1)\vartheta_2(s_2)$. Then it is easy to see that φ_1 and φ_2 are well-defined, that φ_1 is an (S_1,S_1)-map, that φ_2 is an (S_1,S_2)-map and that $\varphi_2 \circ f_1 = \varphi_1$. Suppose we have defined $\varphi_k : W_k \to S_1 *_U S_2$ $(k = 1,\ldots,n-1)$ such that

(1) φ_k is an (S_1,S_i)-map, $i \equiv k \pmod 2$,

and (2) $\varphi_k \circ f_{k-1} = \varphi_{k-1}$, $k = 2,\ldots,n-1$.

Then we have a commutative diagram

$$\begin{tikzcd}
W_{n-2} \arrow[rr, "f_{n-2}"] \arrow[dr] \arrow[ddr, "\varphi_{n-2}"'] & & W_{n-1} \arrow[dl, "f_{n-1}"] \arrow[ddl, "\varphi_{n-1}"] \\
& W_n & \\
& S_1 *_U S_2 &
\end{tikzcd}$$

Hence since $W_n = F(S_i;W_{n-2},W_{n-1})$ we have a unique (S_1,S_i)-map $\varphi_n : W_n \to S_1 *_U S_2$, $i \equiv n \pmod 2$, such that $\varphi_n \circ f_{n-1} = \varphi_{n-1}$. Hence by induction we have a commutative diagram

$$\begin{tikzcd}
W_1 \arrow[r, "f_1"] \arrow[drr, "\varphi_1"'] & W_2 \arrow[r, "f_2"] \arrow[dr, "\varphi_2"] & W_3 \arrow[r, "f_3"] \arrow[d, "\varphi_3"] & \cdots \arrow[r, "f_{n-1}"] & W_n \arrow[r, "f_n"] \arrow[dll, "\varphi_n"] & {} \\
 & & S_1 *_U S_2 & & &
\end{tikzcd}$$

Now let Q be a (U,U)-biset and suppose that there exists $\varepsilon_n : W_n \to Q$ for each n, such that $\varepsilon_n \circ f_{n-1} = \varepsilon_{n-1}$ $(n \geq 2)$. We need to find a unique (U,U)-map $\psi : S_1 *_U S_2 \to Q$ such that $\psi \circ \varphi_n = \varepsilon_n$ $(n \geq 1)$. We see that W_n has the rather complicated structure

$$W_n = (\ldots((S_1 \otimes S_2 \otimes S_1)/\sigma_1 \otimes \ldots \otimes S_i)/\sigma_{n-2}.$$

To simplify notation we shall write a typical element of W_n as $[s_1,\ldots,s_n]$, ($s_i \in S_1$ for i odd, $s_i \in S_2$ for i even). We see that

$$[s_1,\ldots,s_iu,s_{i+1},\ldots,s_n] = [s_1,\ldots,s_i,us_{i+1},\ldots,s_n],$$

$$[s_1,\ldots,s_n] = ([s_1,\ldots,s_{n-1}] \otimes s_n)\sigma_{n-2},$$

and

$$\varphi_n[s_1,\ldots,s_n] = (s_1,\ldots,s_n)\rho.$$

LEMMA 2.2 <u>For all</u> $i \geq 2$

$$[s_1,\ldots,s_{i-1},1,s_{i+1}] = [s_1,\ldots,s_{i-1}s_{i+1},1,1].$$

<u>Proof</u> Let $w_{i-1} = [s_1,\ldots,s_{i-1}]$. Then we have

$$\begin{aligned}[s_1,\ldots,s_{i-1},1,s_{i+1}] &= ((w_{i-1}\otimes 1)\sigma_{i-2}\otimes s_{i+1})\sigma_{i-1},\\ &= (f_{i-1}(w_{i-1})\otimes s_{i+1})\sigma_{i-1},\\ &= (f_{i-1}(w_{i-1}s_{i+1})\otimes 1)\sigma_{i-1},\\ &= ((w_{i-1}s_{i+1}\otimes 1)\sigma_{i-2}\otimes 1)\sigma_{i-1},\\ &= [s_1,\ldots,s_{i-1}s_{i+1},1,1].\end{aligned}$$

A simple inductive argument then gives us

COROLLARY 2.3 <u>For all</u> $i \geq 2$

$$[s_1,\ldots,s_{i-1},1,s_{i+1},\ldots,s_n] = [s_1,\ldots,s_{i-1}s_{i+1},\ldots,s_n,1,1].$$

Now define $\psi : S_1 *_U S_2 \to Q$ by

$$\psi((s_1,\ldots,s_n)\rho) = \begin{cases}\varepsilon_n[s_1,\ldots,s_n], & s_1 \in S_1\\ \varepsilon_{n+1}[1,s_1,\ldots,s_n], & s_1 \in S_2.\end{cases}$$

Assume for the moment that ψ is well-defined. Then it is clearly a (U,U)-map and the diagrams

$$\begin{tikzcd} W_n \arrow[r, "\varphi_n"] \arrow[d, "\varepsilon_n"'] & S_1 *_U S_2 \arrow[dl, "\psi"] \\ Q & \end{tikzcd}$$

commute, since

$$\psi \circ \varphi_n[s_1,\ldots,s_n] = \psi(s_1,\ldots,s_n)\rho$$
$$= \varepsilon_n[s_1,\ldots,s_n] \text{ since } s_1 \in S_1.$$

Lastly it is clear that ψ is unique with this property.

Hence to prove Theorem 2.1 we require to prove

LEMMA 2.4 The map ψ given above is well-defined.

Proof In [15], Howie showed that $w\rho = w'\rho$ in $S_1 *_U S_2$ if and only if w can be connected to w' by a finite sequence of E-, S- and M-steps. We explain these terms in turn and show in each case that if $w \to w'$ by a single step then $\psi(w\rho) = \psi(w'\rho)$.

First we say that w is connected to w' by an S-step if

$$w = (s_1,\ldots,s_{i-1},u,s_{i+1},\ldots,s_n), \quad w' = (s_1,\ldots,s_{i-1}us_{i+1},\ldots,s_n).$$

Hence if $s_1 \in S_1$ then

$$\begin{aligned}
\psi(w\rho) &= \varepsilon_n[s_1,\ldots,s_{i-1},u,s_{i+1},\ldots,s_n] \\
&= \varepsilon_n[s_1,\ldots,s_{i-1}u,1,s_{i+1},\ldots,s_n], \\
&= \varepsilon_n[s_1,\ldots,s_{i-1}us_{i+1},\ldots,s_n,1,1] \text{ by Corollary 2.3,} \\
&= \varepsilon_n \circ f_{n-1} \circ f_{n-2}[s_1,\ldots,s_{i-1}us_{i+1},\ldots,s_n] \\
&= \varepsilon_{n-2}[s_1,\ldots,s_{i-1}us_{i+1},\ldots,s_n] \\
&= \psi(w'\rho).
\end{aligned}$$

A similar conclusion holds if $s_1 \in S_2$.

Since M-steps are the reverse of S-steps, the same conclusion applies to this case.

For E-steps it is convenient to list six cases separately:

(a) $w = (s_1,\dots,s_i u,s_{i+1},\dots,s_n)$, $w' = (s_1,\dots,s_i,us_{i+1},\dots,s_n)$, $1 \le i < n$;

(b) $w = (s_1,\dots,s_i,us_{i+1},\dots,s_n)$, $w' = (s_1,\dots,s_i u,s_{i+1},\dots,s_n)$, $1 \le i < n$;

(c) $w = (s_1,\dots,s_n u)$, $w' = (s_1,\dots,s_n,u)$;

(d) $w = (s_1,\dots,s_n,u)$, $w' = (s_1,\dots,s_n u)$;

(e) $w = (us_1,\dots,s_n)$, $w' = (u,s_1,\dots,s_n)$;

(f) $w = (u,s_1,\dots,s_n)$, $w' = (us_1,\dots,s_n)$;

cases (a) and (b) are trivial. As for case (c), if $s_1 \in S_1$ then

$$\begin{aligned}
\psi(w\rho) &= \varepsilon_n[s_1,\dots,s_n u],\\
&= \varepsilon_{n+1} \circ f_n[s_1,\dots,s_n u],\\
&= \varepsilon_{n+1}[s_1,\dots,s_n u,1],\\
&= \varepsilon_{n+1}[s_1,\dots,s_n,u],\\
&= \psi(w'\rho).
\end{aligned}$$

If $s_1 \in S_2$ the procedure is similar. Case (d) is similar to case (c). In case (e), if $s_1 \in S_1$,

$$\begin{aligned}
\psi(w\rho) &= \varepsilon_n[us_1,\dots,s_n],\\
&= \varepsilon_{n+2} \circ f_{n+1} \circ f_n[us_1,\dots,s_n],\\
&= \varepsilon_{n+2}[1,1,us_1,\dots,s_n], \text{ by Corollary 2.3}\\
&= \varepsilon_{n+2}[1,u,s_1,\dots,s_n],\\
&= \psi(w'\rho),
\end{aligned}$$

while if $s_1 \in S_2$, we have

$$\begin{aligned}\psi(w\rho) &= \varepsilon_{n+1}[1, us_1, \ldots, s_n],\\ &= \varepsilon_{n+1}[u, s_1, \ldots, s_n],\\ &= \psi(w'\rho).\end{aligned}$$

Case (f) is similar to case (e). It is now clear that if w is connected to w' by a finite sequence of E-, S- and M-steps then $\psi(w\rho) = \psi(w'\rho)$. Thus ψ is well-defined and so the proof of Theorem 2.1 is complete.

In order to make use of Theorem 2.1 later, we require some further observations.

Denote by $f^{(n-1)}$ the map $(f_{n-1} \circ f_{n-2} \circ \ldots \circ f_1) : W_1 \to W_n$, by $g^{(1)}$ the map $S_2 \to W_2$ given by $g^{(1)}(s_2) = 1 \otimes s_2$ and by $g^{(n-1)}$ $(n \geq 2)$ the map $(f_{n-1} \circ \ldots \circ f_2 \circ g^{(1)}) : S_2 \to W_n$. We have

THEOREM 2.5 *The amalgam* $[U;S_1,S_2]$ *is weakly embeddable if and only if for all* $n \geq 1$ *the maps* $f^{(n)}$ *and* $g^{(n)}$ *are* 1-1.

Proof From Theorem I.1.3 the amalgam is weakly embeddable if and only if the maps $\vartheta_i : S_i \to S_1 *_U S_2$ are 1-1. By Theorem I.3.17 and Corollary I.3.18, these maps are 1-1 if and only if $f^{(n)}$ and $g^{(n)}$ are 1-1 for all $n \geq 1$.

Notice that Theorem 2.5 is saying that the *monoid* amalgam $[U;S_1,S_2]$ is weakly embeddable in $S_1 *_U S_2$ if and only if the U-*set* amalgam $[U;S_1,S_2]$ is weakly embeddable in each W_n $(n \geq 2)$ (with respect to the maps $f^{(n)}$ and $g^{(n)}$ above).

We can in fact show

THEOREM 2.6 *The monoid amalgam* $[U;S_1,S_2]$ *is strongly embeddable if and only if the U-set amalgam* $[U;S_1,S_2]$ *is strongly embeddable in each* W_n $(n \geq 2)$ *(with respect to the maps* $f^{(n)}$ *and* $g^{(n)}$ *defined above).*

Proof Suppose that the monoid amalgam is strongly embeddable. From Theorem 2.5 we see that the maps $f^{(n)}$ and $g^{(n)}$ are 1-1 $(n \geq 1)$. Suppose then that $f^{(n)}(s_1) = g^{(n)}(s_2)$ for some s_1 in S_1 and s_2 in S_2. Then $\varphi_{n+1} \circ f^{(n)}(s_1) = \varphi_{n+1} \circ g^{(n)}(s_2)$ in $S_1 *_U S_2$, i.e. $\varphi_1(s_1) = \varphi_2 \circ g^{(1)}(s_2)$. But $\varphi_1 = \vartheta_1$ and $\varphi_2 \circ g^{(1)} = \vartheta_2$ and so $\vartheta_1(s_1) = \vartheta_2(s_2)$. Hence $s_1 = s_2 \in U$ and the U-*set* amalgam is strongly embeddable.

Conversely suppose that the U-*set* amalgam is strongly embeddable in each W_n $(n \geq 2)$. Then $f^{(n)}$ and $g^{(n)}$ are 1-1 $(n \geq 1)$, and so the *monoid* amalgam is weakly embeddable from Theorem 2.5. Suppose now that $\vartheta_1(s_1) = \vartheta_2(s_2)$ in $S_1 *_U S_2$. Then $\varphi_1(s_1) = \varphi_2 \circ g^{(1)}(s_2)$ and so from Theorem I.3.17 there exists $k \geq 2$ such that

$$f_k \circ f_{k-1} \circ \dots \circ f_1(s_1) = f_k \circ \dots \circ f_2 \circ g^{(1)}(s_2),$$

in W_{k+1}. Hence $s_1 = s_2 \in U$ as required.

The following lemma will be of use later.

LEMMA 2.7 *Suppose that the monoid amalgam* $[U;S_1,S_2]$ *is weakly embeddable and suppose also that the map* $\varphi_2 : W_2 \to S_1 *_U S_2$ *is* 1-1. *Then the amalgam is strongly embeddable if and only if* $s_1 \otimes 1 = 1 \otimes s_2$ *in* $S_1 \otimes_U S_2$ *implies* $s_1 = s_2 \in U$.

<u>Proof</u> The direct half follows from Theorem 2.6. Conversely, suppose that $\vartheta_1(s_1) = \vartheta_2(s_2)$ in $S_1 *_U S_2$. Then we have $\varphi_2 \circ f_1(s_1) = \varphi_2 \circ g^{(1)}(s_2)$, since $\varphi_2 \circ f_1 = \varphi_1 = \vartheta_1$ and $\varphi_2 \circ g^{(1)} = \vartheta_2$. But φ_2 is 1-1 and so $f_1(s_1) = g^{(1)}(s_2)$, i.e. $s_1 \otimes 1 = 1 \otimes s_2$ in $W_2 = S_1 \otimes S_2$. Hence $s_1 = s_2 \in U$ and the amalgam is strongly embeddable.

CHAPTER III EXTENSIONS AND AMALGAMATIONS

1. The extension properties and pure sub U-sets

In 1978 T E Hall [13] introduced four extension properties for U-sets (the representation extension property, the free representation extension property, the strong representation extension property and the orbit preserving representation extension property) which are intimately connected with amalgamation. We introduce another in this section and provide a connection not only with the above extension properties but also with the almost unitary property of Howie.

Let U be a submonoid of a monoid S. We say that U has the right extension property in S, if for all $X \in$ ENS-$\underline{U}$ the map $X \to X \otimes_U S$, given by $x \mapsto x \otimes 1$ is 1-1. The left extension property is defined dually. We shall say that U has the extension property in S if for all $X \in$ ENS-$\underline{U}$ and all $Y \in \underline{U}$-ENS, the map $X \otimes_U Y \to X \otimes_U S \otimes_U Y$, given by $x \otimes y \mapsto x \otimes 1 \otimes y$, is 1-1. We shall say that a monoid U is (right,left) absolutely extendable if U has the (right, left) extension property in every containing monoid.

THEOREM 1.1 *Let* U *be a submonoid of a monoid* S. *If* U *has the extension property in* S *then* U *has both the right and left extension properties in* S.

Proof Take $Y = U$ for the right extension property and $X = U$ for the left.

LEMMA 1.2 *Let U be a submonoid of a monoid S, let S be a submonoid of a monoid T and let S have the extension property in T. Then U has the extension property in S if and only if U has the extension property in T.*

Proof Using Theorem I.4.3 we see that if $X \in \text{ENS-}\underline{U}$ and $Y \in \underline{U}\text{-ENS}$, then

$$X \otimes_U S \otimes_U Y \simeq X \otimes_U (S \otimes_S S) \otimes_U Y \simeq (X \otimes_U S) \otimes_S (S \otimes_U Y),$$

and

$$X \otimes_U T \otimes_U Y \simeq X \otimes_U (S \otimes_S T \otimes_S S) \otimes_U Y \simeq (X \otimes_U S) \otimes_S T \otimes_S (S \otimes_U Y).$$

Since S has the extension property in T, then we see that $X \otimes_U S \otimes_U Y \to X \otimes_U T \otimes_U Y$ is 1-1. Now consider the commutative diagram

$$\begin{tikzcd}
X \otimes_U Y \arrow[r] \arrow[d] & X \otimes_U S \otimes_U Y \arrow[dl] \\
X \otimes_U T \otimes_U Y &
\end{tikzcd}$$

It is now clear that $X \otimes_U Y \to X \otimes_U S \otimes_U Y$ is 1-1 if and only if $X \otimes_U Y \to X \otimes_U T \otimes_U Y$ is 1-1. Hence the result.

The following result is fairly immediate.

THEOREM 1.3 *Let* U *be a monoid*. *If* U *is absolutely flat then* U *is absolutely extendable*.

Proof Let $X \in$ ENS-$\underline{U}$, $Y \in \underline{U}$-ENS and suppose that U is absolutely flat. Then for every monoid S containing U as submonoid the map $U \otimes_U Y \to S \otimes_U Y$ is 1-1, since Y is left flat. Since X is right flat, the map $X \otimes_U U \otimes_U Y \to X \otimes_U S \otimes_U Y$ is 1-1. But $X \otimes_U U \otimes_U Y \cong X \otimes_U Y$ and so U is absolutely extendable.

Let U be a submonoid of a monoid S. Recall that U is said to be *right perfect* in S, (Howie [23]) if for all $X \in$ ENS-$\underline{S}$, $Y \in$ ENS-$\underline{U}$ and all U-monomorphisms $f : X \to Y$ there exists $Z \in$ ENS-$\underline{S}$, a U-monomorphism $g : Y \to Z$ and an S-monomorphism $h : X \to Z$ such that

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

commutes.

Notice that U is right perfect in S if and only if whenever $X \in$ ENS-$\underline{S}$, $Y \in$ ENS-$\underline{U}$ and $f : X \to Y$ is a U-monomorphism the map $g : Y \to F(S;X,Y)$ is 1-1.

Say that U is *(right,left) absolutely perfect* if U is (right, left) perfect in every containing monoid. From [13] Theorem 3, we have:

THEOREM 1.4 *Let* U *be a submonoid of a monoid* S. *Then* U *is right perfect in* S *if and only if* U *has the right extension property in* S *and* S *is left flat*.

We also have:

LEMMA 1.5 [5, Proposition 1.1] <u>A monoid</u> U <u>is (left,right) absolutely flat if and only if every containing monoid of</u> U <u>is (left, right) flat</u>.

THEOREM 1.6 (1) <u>If</u> U <u>is right absolutely perfect then</u> U <u>is left absolutely flat</u>.

(2) U <u>is absolutely perfect if and only if</u> U <u>is absolutely flat</u>.

(3) <u>If</u> U <u>is (left,right) absolutely perfect then</u> U <u>is absolutely extendable</u>.

<u>Proof</u> (1) This follows from Theorem 1.4 and Lemma 1.5.

(2) The 'only if' follows from (1) and its dual. Suppose then that U is absolutely flat. From Theorems 1.1 and 1.3 we see that U is both right and left absolutely extendable, and hence absolutely perfect from Theorem 1.4.

(3) Suppose that U is right absolutely perfect. Let $X \in \text{ENS-}\underline{U}$ and $Y \in \underline{U}\text{-ENS}$. The map $X \to X \otimes_U S$ given by $x \mapsto x \otimes 1$ is 1-1 by Theorem 1.4. Hence, since Y is left flat, $X \otimes Y \to X \otimes S \otimes Y$ is 1-1.

NOTE The converses of (1) and (3) are not true. First, by [13, Theorem 25 (iii)] and Lemma 1.5 the three element right zero semigroup is left absolutely flat, but is not right absolutely perfect by [13, Theorem 25 (ii)]. Also, from [13, Theorem 20] any finite cyclic semigroup is an amalgamation base and so from

Theorem 2.14 below is absolutely extendable. But no cyclic semigroup which is not a group is right absolutely perfect [13, Theorem 24].

Reinterpreting a definition of Hall [13], we say that U has the (right) orbit-preserving extension property in S if for all $X \in \text{ENS-}\underline{U}$ there exists $Z \in \text{ENS-}\underline{S}$ and a U-monomorphism $f : X \to Z$ such that $yu \in Z\backslash \text{im}f$ whenever $y \in Z\backslash \text{im}f$ and $u \in U$. It is easy to prove that if A is a U-orbit of X, then f(A) is a U-orbit of Z. (A is a U-orbit of X if A is a minimal (w.r.t. $\subseteq$) subset of X with the property that for all u in U, x in X, $x \in A$ if and only if $xu \in A$.)

THEOREM 1.7 Let U be a submonoid of a monoid S. If U has the orbit preserving extension property in S then U has the extension property in S.

Proof Let $X \in \text{ENS-}\underline{U}$ and $Y \in \underline{U}\text{-ENS}$ and suppose that U has the orbit-preserving extension property in S. By assumption there exists $Z \in \text{ENS-}\underline{S}$ and a U-monomorphism $f : X \to Z$. Now, since $Z\backslash \text{im}f$ is a sub-U-set, the U-set Z may be identified with $X \dot{\cup} Z\backslash \text{im}f$ and so $Z \otimes Y = X \otimes Y \dot{\cup} [(Z\backslash \text{im}f) \otimes Y]$ from Lemma I.4.8. Hence the map $f \otimes 1 : X \otimes Y \to Z \otimes Y$ is 1-1. Now define the map $\varphi : X \otimes S \otimes Y \to Z \otimes Y$ by $\varphi(x \otimes s \otimes y) = f(x).s \otimes y$, and check that $\varphi(x \otimes 1 \otimes y) = f(x) \otimes y$. Thus $X \otimes Y \to X \otimes S \otimes Y$ is 1-1 as required.

Let U be a subsemigroup of a semigroup S. Say that U is quasi-unitary in S if there exists a $({}^1U, {}^1U)$-map $\varphi : {}^1S \to {}^1S$ such that

(1) $\varphi^2 = \varphi$, $\varphi(1) = 1$;

(2) for all u in U, $\varphi(s) \in U$ whenever $su \in U$ or $us \in U$.

It is easy to see that $\varphi(u) = u$ for all $u \in U$.

It is of interest to compare this with Howie's definition [15, 22] of almost unitary. Recall that a subsemigroup U of a semigroup S is almost unitary if there exist mappings $\lambda : S \to S$ (written on the left), $\rho : S \to S$ (written on the right) such that

(1) $\lambda^2 = \lambda, \quad \rho^2 = \rho;$

(2) $\lambda(st) = \lambda(s)t; \quad (st)\rho = s(t\rho);$

(3) $\lambda(s\rho) = (\lambda s)\rho;$

(4) $s(\lambda t) = (s\rho)t;$

(5) $\lambda|U = \rho|U = 1_U;$

(6) for all $u \in U$, $\lambda s\rho \in U$ whenever $us \in U$ or $su \in U$.

Then we have

LEMMA 1.8 If U is an almost unitary subsemigroup of a semigroup S then U is quasi-unitary.

Proof Define $\varphi : {}^1S \to {}^1S$ by $\varphi(1) = 1$ and $\varphi(s) = \lambda s\rho$ for all $s \in S$. It is now easy to check that φ has the required properties; the only point we would make is that φ is indeed a ${}^1U$-map since

$$\begin{aligned}
\varphi(us) = \lambda(us)\rho &= (\lambda(us))\rho; && \text{by (3)}\\
&= (\lambda(u)s)\rho; && \text{by (2)}\\
&= (us)\rho; && \text{by (5)}\\
&= u.(s\rho); && \text{by (2)}\\
&= (u\rho)(s\rho); && \text{by (5)}\\
&= u.\lambda(s\rho); && \text{by (4)}\\
&= u.\varphi(s),
\end{aligned}$$

and similarly $\varphi(su) = \varphi(s).u$.

The converse of Lemma 1.8 is not true. This follows from an example in [17, Section 3].

LEMMA 1.9 *Let* U *be a quasi-unitary subsemigroup of a semigroup* S. *Let* $X \in$ ENS-U *and suppose that* $x \otimes 1 = x' \otimes su$ *in* $X \otimes_{{}^1U} {}^1S$. *Then* $\varphi(s) \in {}^1U$ *and* $x = x'\varphi(s)u$.

Proof In the notation of Lemma I.4.4 we have

$$x = x_1u_1,$$

$$x_1v_1 = x_2u_2, \quad u_11 = v_1s_2,$$

$$\cdots\cdots$$

$$x_nv_n = x', \quad u_ns_n = v_n(su).$$

We see that $\varphi(s_2) \in {}^1U$ and $u_1 = \varphi(u_1) = v_1\varphi(s_2)$. Also, $u_2s_2 = v_2s_3$ implies $u_2\varphi(s_2) = v_2\varphi(s_3)$ and so $\varphi(s_3) \in {}^1U$. Continuing in this fashion we see that $\varphi(s_i) \in {}^1U$, $\varphi(s) \in {}^1U$ and we have

$$x = x_1u_1 = x_1v_1\varphi(s_2) = x_2u_2\varphi(s_2) = \ldots$$

$$\ldots = x_nu_n\varphi(s_n) = x_nv_n\varphi(s)u = x'\varphi(s)u.$$

THEOREM 1.10 *Let* U *be a quasi-unitary subsemigroup of a semigroup* S. *Then* ${}^1U$ *has the extension property in* ${}^1S$.

Proof Let $X \in$ ENS-${}^1U$ and $Y \in {}^1U$-ENS and suppose that $x \otimes 1 \otimes y = x' \otimes 1 \otimes y'$ in $X \otimes {}^1S \otimes Y$. From Lemma I.4.4 we have a set of equations

$$x \otimes 1 = x_1 \otimes s_1 u_1,$$

$$x_1 \otimes s_1 v_1 = x_2 \otimes s_2 u_2, \quad u_1 y = v_1 y_2,$$

$$\cdots\cdots$$

$$x_n \otimes s_n v_n = x' \otimes 1, \qquad u_n y_n = v_n y'.$$

From Lemma 1.9, $\varphi(s_1) \in {}^1U$ and $x = x_1\varphi(s_1)u_1$. So applying $1 \otimes \varphi$ to both sides of equation $x_1 \otimes s_1 v_1 = x_2 \otimes s_2 u_2$ we obtain $x_1 \otimes \varphi(s_1)v_1 = x_2 \otimes \varphi(s_2)u_2$, i.e. $x_1\varphi(s_1)v_1 \otimes 1 = x_2 \otimes \varphi(s_2)u_2$. Lemma 1.9 again gives us $\varphi(s_2) \in {}^1U$ and $x_1\varphi(s_1)v_1 = x_2\varphi(s_2)u_2$. Continuing in this way we see that $\varphi(s_i) \in {}^1U$ and that $x_i\varphi(s_i)v_i = x_{i+1}\varphi(s_{i+1})u_{i+1}$. We thus obtain a set of equations

$$x = x_1\varphi(s_1)u_1,$$

$$x_1\varphi(s_1)v_1 = x_2\varphi(s_2)u_2, \quad u_1 y = v_1 y_2,$$

$$\cdots\cdots$$

$$x_n\varphi(s_n)v_n = x', \qquad u_n y_n = v_n y'.$$

and so from Lemma I.4.4 we have $x \otimes y = x' \otimes y'$ in $X \otimes Y$ as required.

We have defined the extension property in the category of monoids. It will be useful to extend the definition to the category of semigroups. To this end, let U be a subsemigroup of a semigroup S. We shall say that U <u>has the extension property in</u> S if ${}^1U$ has the extension property in ${}^1S$ (as monoids). When necessary we can distinguish between the <u>semigroup extension property</u> and the <u>monoid extension property</u>. However it is not normally necessary to make this distinction, by virtue of

THEOREM 1.11 *Let* U *be a submonoid of a monoid* S *with identity* e. *Then* U *has the monoid extension property in* S *if and only if* U *has the semigroup extension property in* S.

Proof Suppose that U has the semigroup extension property in S and suppose that $X \in \text{ENS-}U$ and $Y \in U\text{-ENS}$. Then $X \in \text{ENS-}{}^1U$ and $Y \in {}^1U\text{-ENS}$ if we define

$$x.1 = x, \quad 1.y = y, \quad \text{for all } x \text{ in } X,\ y \text{ in } Y.$$

It is easy to check that $X \otimes_U Y \simeq X \otimes_{{}^1U} Y$. Also the map $X \otimes_U S \otimes_U Y \to X \otimes_{{}^1U} {}^1S \otimes_{{}^1U} Y$ is well-defined and we have a commutative diagram

$$\begin{tikzcd}
X \otimes_U Y \arrow[r] \arrow[d] & X \otimes_U S \otimes_U Y \arrow[dl] \\
X \otimes_{{}^1U} {}^1S \otimes_{{}^1U} Y &
\end{tikzcd}$$

Hence $X \otimes_U Y \to X \otimes_U S \otimes_U Y$ is 1-1 as required.

Conversely, suppose that U has the monoid extension property in S and let $X \in \text{ENS-}{}^1U$ and $Y \in {}^1U\text{-ENS}$. Notice that X and Y need *not* be U-sets since we may have $xe \neq x$ and $ey \neq y$. However, if we let $X' = XU$ and $Y' = UY$, then X' and Y' *are* U-sets. Now suppose that

$$x \otimes 1 \otimes y = x' \otimes 1 \otimes y' \quad \text{in} \quad X \otimes_{{}^1U} {}^1S \otimes_{{}^1U} Y.$$

If $x \notin X'$ and $y \notin Y'$ then it is clear that $x = x'$ and $y = y'$, and hence $x \otimes y = x' \otimes y'$ in $X \otimes_{{}^1U} Y$ as required. A similar conclusion holds if $x' \notin X'$ and $y' \notin Y'$. We are therefore left with four possibilities:

(1) $x,x' \in X'$; (2) $x \in X'$, $y' \in Y'$;

(3) $y \in Y'$, $x' \in X'$; (4) $y,y' \in Y'$.

It is easy to check that the following maps are well-defined

(a) $X \otimes_{^1U} {}^1S \otimes_{^1U} Y \to X' \otimes_U S \otimes_U Y'$, given by $x \otimes s \otimes y \mapsto xe \otimes ese \otimes ey$,

(b) $X' \otimes_U Y' \to X \otimes_{^1U} Y$, given by $x \otimes y \mapsto x \otimes y$.

For example in (a), the map $\varphi : X \times {}^1S \times Y \to X' \otimes_U S \otimes_U Y'$ given by $\varphi(x,s,y) = xe \otimes ese \otimes ey$, is "trilinear" with respect to ${}^1U$, in that

$$\begin{aligned}
\varphi(xu,s,y) &= (xu)e \otimes ese \otimes ey, \\
&= (x(eu))e \otimes ese \otimes ey, \\
&= ((xe)u)e \otimes ese \otimes ey, \\
&= (xe)u \otimes ese \otimes ey, \\
&= xe \otimes u(ese) \otimes ey, \\
&= \ldots \\
&= xe \otimes e(us)e \otimes ey = \varphi(x,us,y).
\end{aligned}$$

Similarly $\varphi(x,s,uy) = \varphi(x,su,y)$.

The four cases above can now be taken separately to deduce that $x \otimes y = x' \otimes y'$ in $X \otimes_{^1U} Y$. For example, in case (1) we have, on applying the map in (a) above,

$$xe \otimes e \otimes ey = x'e \otimes e \otimes ey' \quad \text{in} \quad X' \otimes_U S \otimes_U Y'.$$

Since U has the monoid extension property in S we see that $xe \otimes ey = x'e \otimes ey'$ in $X' \otimes_U Y'$ and hence in $X \otimes_{^1U} Y$ on applying the map given in (b) above. That is, $xe \otimes y = x'e \otimes y$ in $X \otimes_{^1U} Y$

(since $e^2 = e$). But $x, x' \in X'$ implies $xe = x$, $x'e = x'$. Hence the result.

As a consequence we have

COROLLARY 1.12 *Let* U *be a quasi-unitary subsemigroup of a semigroup* S. *Then* U *has the extension property in* S.

Let $f : A \to B$ be a monomorphism. Say that f *splits* if there exists a map $g : B \to A$ such that $g \circ f = 1_A$.

LEMMA 1.13 *Let* U *be a subsemigroup of a semigroup* S. *Suppose that the map* $U \to S$ *splits either* (1) *in the category of semigroups, or* (2) *in the category* $\underline{U}$-ENS-$\underline{U}$. *Then* U *is quasi-unitary in* S.

Proof Notice that if $U \to S$ splits in the category of semigroups then it splits in $\underline{U}$-ENS-$\underline{U}$. Hence we need only consider case (2). Define $\varphi : {}^1S \to {}^1S$ by $\varphi(1) = 1$, $\varphi(s) = f(s)$ where $f : S \to U$ is the splitting map. It is straightforward to check that U is quasi-unitary in S.

Let $X \in$ ENS-$\underline{U}$. Then it is easy to see that if X is injective then every monomorphism $X \to Y$ splits.

Let $\underset{\sim}{K}$ be a class of semigroups and let $U \in \underset{\sim}{K}$. Say that U is $\underset{\sim}{K}$-injective if for all monomorphisms $f : S \to T$ with $S, T \in \underset{\sim}{K}$ and all morphisms $g : S \to U$, there exists a morphism $h : T \to U$ such that $h \circ f = g$. The next theorem is an easy consequence of Theorem 1.10 and Lemma 1.13.

THEOREM 1.14 (1) *Let* U *be a monoid and suppose that* U *is injective in* $\underline{U}$-ENS-$\underline{U}$. *Then* U *is absolutely extendable*.

(2) *Let* $U \leq S \in \underset{\sim}{K}$ *where* U *is* $\underset{\sim}{K}$-*injective*. *Then* U *has the extension property in* S.

Let $f : X \to Y$ be a right U-map and $\lambda : A \to B$ a left U-map, and consider the diagram

$$\begin{tikzcd} X \otimes A \arrow[r, "1_X \otimes \lambda"] \arrow[d, "f \otimes 1_A"'] & X \otimes B \arrow[d, "f \otimes 1_B"] \\ Y \otimes A \arrow[r, "1_Y \otimes \lambda"'] & Y \otimes B \end{tikzcd}$$

We shall say that the pair (f,λ) is *stable* if

$$\operatorname{im}(f \otimes 1_B) \cap \operatorname{im}(1_Y \otimes \lambda) = \operatorname{im}(f \otimes \lambda).$$

It is clear that (f,λ) is stable if and only if whenever $y \otimes \lambda(a) = f(x) \otimes b$ in $Y \otimes B$, then there exists $x_1 \in X$, $a_1 \in A$ such that $y \otimes \lambda(a) = f(x_1) \otimes \lambda(a_1)$. We see from Lemma I.3.12 that (f,λ) is stable if and only if

$$\begin{tikzcd} \operatorname{im}(f \otimes \lambda) \arrow[r] \arrow[d] & \operatorname{im}(f \otimes 1_B) \arrow[d] \\ \operatorname{im}(1_Y \otimes \lambda) \arrow[r] & Y \otimes B \end{tikzcd}$$

is a pullback.

Let $X, Y \in$ ENS-$\underline{U}$ and let $f : X \to Y$ be a U-monomorphism. We say that f is *right pure* if for all $B \in \underline{U}$-ENS the map $f \otimes 1 : X \otimes B \to Y \otimes B$

is 1-1. Left purity is defined dually. Let $X,Y \in \underline{U}\text{-ENS-}\underline{U}$ and let $f : X \to Y$ be a (U,U)-monomorphism. We say that f is pure if for all $A \in \text{ENS-}\underline{U}$ and for all $B \in \underline{U}\text{-ENS}$ the map $1 \otimes f \otimes 1 : A \otimes X \otimes B \to A \otimes Y \otimes B$ is 1-1. The following are clear.

LEMMA 1.15 *If $f : X \to Y$ is a pure monomorphism then f is both right and left pure.*

LEMMA 1.16 *If U is a submonoid of a monoid S then U has the (right,left) extension property in S if and only if the inclusion $U \to S$ is (left,right) pure.*

Let $f : X \to Y$ be a left U-monomorphism. Then we shall say that f is stable if for all right U-monomorphisms $\lambda : A \to B$, the pair (λ, f) is stable.

THEOREM 1.17 *Let $f : X \to Y$ be a left pure monomorphism. Then f is stable.*

Proof Let $\lambda : A \to B$ be a right U-monomorphism and consider the pushout diagram

$$\begin{tikzcd} A \arrow[r, "\lambda"] \arrow[d, "\lambda"'] & B \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"'] & P \end{tikzcd}$$

By Theorem I.4.7,

$$\begin{tikzcd} A \otimes X \arrow[r, "\lambda \otimes 1"] \arrow[d, "\lambda \otimes 1"'] & B \otimes X \arrow[d, "\alpha \otimes 1"] \\ B \otimes X \arrow[r, "\beta \otimes 1"'] & P \otimes X \end{tikzcd}$$

is also a pushout. Suppose then that

$$b \otimes f(x) = \lambda(a) \otimes y \quad \text{in} \quad B \otimes Y.$$

Then
$$\begin{aligned} \alpha(b) \otimes f(x) &= \alpha\lambda(a) \otimes y, \\ &= \beta\lambda(a) \otimes y, \\ &= \beta(b) \otimes f(x) \text{ in } P \otimes Y. \end{aligned}$$

But f is left pure and so the map $P \otimes X \to P \otimes Y$ is 1-1. Hence $\alpha(b) \otimes x = \beta(b) \otimes x$ in $P \otimes X$. It follows from Lemma I.3.8 that there exists $a' \otimes x'$ in $A \otimes X$ such that $b \otimes x = \lambda(a') \otimes x'$ in $B \otimes X$. Hence $b \otimes f(x) = \lambda(a') \otimes f(x')$ in $B \otimes Y$ and (λ, f) is stable.

COROLLARY 1.18 Let U be a submonoid of a monoid S and suppose that U has the right extension property in S. Let $\lambda : X \to Y$ be a right U-map and suppose that $y \otimes 1 = \lambda(x) \otimes s$ in $Y \otimes S$. Then $y \in \operatorname{im} f$.

Proof By Theorem 1.17, $y \otimes 1 = \lambda(x_1) \otimes 1$ in $Y \otimes S$ for some $x_1 \in X$. Hence, since $Y \to Y \otimes S$ is 1-1, we see that $y \in \operatorname{im}\lambda$.

COROLLARY 1.19 Let U be a submonoid of a monoid S and suppose that U has the right extension property in S. Then U is closed in S.

Proof Suppose that $s \otimes 1 = 1 \otimes s$ in ${}^1S \otimes_{1_U} {}^1S$. Then if e is the identity of U it is easy to check that $s \otimes e = e \otimes s$ in $S \otimes_U S$. Hence from Corollary 1.18 we see that $s \in U$ (take $X = U$, $Y = S$ and $\lambda = i : U \to S$). Hence $\mathrm{Dom}_S U = U$ and U is closed in S.

It is of interest at this point to note that if U is a submonoid S, then U is closed in S if and only if $s \in U$ whenever $s \otimes 1 = 1 \otimes s$ in $S \otimes_U S$. An interesting and easily proved consequence of this is:

THEOREM 1.20 Let U be a submonoid of a monoid S and let $i : U \to S$ be the inclusion. Then U is closed in S if and only if the pair (i,i) is stable.

Let $A \subseteq B \in$ U-ENS. Say that A is (left) relatively unitary in B if for all non-empty right ideals I of U, $A \cap IB = IA$. The definition of right relatively unitary is dual.

The following is reasonably clear.

THEOREM 1.21 Let $f : X \to Y$ be a left U-monomorphism. Then $\mathrm{im} f$ is left relatively unitary in Y if and only if for all right ideals I of U, the pair (i_I, f) is stable, where $i_I : I \to U$ is the inclusion.

From Theorem 1.17 we can therefore deduce

COROLLARY 1.22 (i) Let $f : X \to Y$ be a left pure monomorphism. Then $\mathrm{im} f$ is left relatively unitary in Y.

(ii) Let U be a subsemigroup of a semigroup S and suppose that

U has the extension property in S. Then U is relatively unitary in S.

The following connection between purity and direct limits will prove useful later.

THEOREM 1.23 Let U be a monoid and let (X_i, φ_j^i) be a direct system in U-ENS-U with directed index set and direct limit (X, α_i). Then α_i is (right) pure if and only if φ_k^i is (right) pure for all $k \geq i$.

Proof The result follows from Corollary I.3.18 and Corollary I.4.9.

THEOREM 1.24 Let U be a monoid and let $f : A \to B$ and $g : C \to D$ be (U,U)-monomorphisms. Then the induced monomorphism $f \dot{\cup} g : A \dot{\cup} C \to B \dot{\cup} D$ is pure if and only if f and g are pure.

Proof The proof is immediate from Lemma I.4.8.

We end this section with a connection between free extensions and the extension property which will prove extremely useful in the next section. Later we shall provide a similar connection between the perfect property and free extensions.

THEOREM 1.25 Let U be a submonoid of a monoid S. Then U has the extension property in S if and only if for all $X \in$ U-ENS-S, all $Y \in$ U-ENS-U and all pure (U,U)-monomorphisms $f : X \to Y$, there exists $Z \in$ U-ENS-S, a (U,S)-monomorphism $h : X \to Z$ and a pure (U,U)-monomorphism $g : Y \to Z$ such that $g \circ f = h$.

Proof Suppose that U has the extension property in S. Let $Z = F(S;X,Y)$ and let $g : Y \to Z$, $h : X \to Z$ be as in the proof of Theorem II.1.1. We use Theorem I.3.14. Let $A \in \text{ENS-}\underline{U}$ and $B \in \underline{U}\text{-ENS}$ and consider the commutative diagram

$$\begin{array}{ccc} A \otimes X \otimes B & \longrightarrow & A \otimes Y \otimes B \\ \downarrow & & \downarrow \\ A \otimes X \otimes S \otimes B & \longrightarrow & A \otimes Y \otimes S \otimes B \end{array}$$

Notice that since both $f : X \to Y$ and $i : U \to S$ are pure we can deduce that

(1) $X \otimes S \to Y \otimes S$ is pure,

(2) $Y \to Y \otimes S$ is pure,

and (3) $B \to S \otimes B$ is left pure (and hence stable by Theorem 1.17).

Using (3) and Lemma I.3.12 it is an easy matter to deduce that the above diagram is a pullback. But by Theorem II.1.2 and Theorem I.4.7 the diagram

$$\begin{array}{ccc} A \otimes X \otimes S \otimes B & \longrightarrow & A \otimes Y \otimes S \otimes B \\ \downarrow & & \downarrow \\ A \otimes X \otimes B & \longrightarrow & A \otimes Z \otimes B \end{array}$$

is a pushout. Hence by Theorem I.3.14 we see that the map $A \otimes Y \otimes B \to A \otimes Z \otimes B$ is 1-1, for all $A \in \text{ENS-}\underline{U}$ and $B \in \underline{U}\text{-ENS}$, i.e. $Y \to Z$ is a pure monomorphism.

Conversely, consider the pure monomorphism $S \to S \dot{\cup} U$. By assumption, there exists an S-set Z, an S-monomorphism $\beta : S \to Z$ and a pure U-monomorphism $\alpha : S \dot{\cup} U \to Z$ such that

$$\begin{tikzcd} S \arrow[rr] \arrow[dr, "\beta"'] & & S \dot{\cup} U \arrow[dl, "\alpha"] \\ & Z & \end{tikzcd}$$

commutes. From Lemma II.1.5 we see that $F(S;S,S \dot{\cup} U) \simeq S \dot{\cup} S$ and so there exists a unique S-map $\psi : S \dot{\cup} S \to Z$ such that

$$\begin{tikzcd} S \arrow[rr] \arrow[dr, "\beta"'] \arrow[ddr] & & S \dot{\cup} U \arrow[dl, "\alpha"] \arrow[ddl] \\ & Z & \\ & S \dot{\cup} S \arrow[u, "\psi"'] & \end{tikzcd}$$

commutes. Consequently, we deduce that $S \dot{\cup} U \to S \dot{\cup} S$ is pure and so from Theorem 1.24, $U \to S$ is pure as required.

2. The extension property and amalgamations

We proceed in this section to show that the extension property "implies amalgamation". From this we are able to deduce many of the principal results on semigroup amalgams (See Section I.1). We conclude with a rather surprising characterisation of amalgamation bases.

THEOREM 2.1 <u>Let</u> $[U;S_1,S_2]$ <u>be a monoid amalgam</u>. <u>If</u> U <u>has the extension property in</u> S_1 <u>and</u> S_2 <u>then the amalgam is strongly embeddable and</u> U <u>has the extension property in</u> $S_1 *_U S_2$.

<u>Proof</u> Construct the sequence (W_n,f_n) as in Theorem II.2.1. Then $g^{(1)} : S_2 \to S_1 \otimes S_2$ is 1-1 by the left extension property in S_1 and $f_1 : S_1 \to S_1 \otimes S_2$ is 1-1 by the right extension property in S_2. Also $f_1 : W_1 \to W_2$ is pure since, if $X \in$ ENS-$\underline{U}$ and $Y \in \underline{U}$-ENS, then $X \otimes S_1 \otimes Y \to X \otimes S_1 \otimes S_2 \otimes Y$ is 1-1 by the extension property in S_2. Hence by Theorem 1.25, $f_n : W_n \to W_{n+1}$ is a pure monomorphism for all $n \geq 1$ (since $W_n = F(S_i, W_{n-2}, W_{n-1})$, $i \equiv n (\mathrm{mod}\ 2)$). By Theorem II.2.5 the amalgam is weakly embeddable and by Corollary I.3.18 the map $\varphi_2 : W_2 \to S_1 *_U S_2$ is 1-1. Suppose then that $s_1 \otimes 1 = 1 \otimes s_2$ in $S_1 \otimes_U S_2$. From Corollary 1.18 it follows that $s_1 \in U$. (Take $X = U$, $Y = S_1$ and λ as the inclusion from U to S_1.) Hence $1 \otimes s_2 = 1 \otimes s_1$ in $S_1 \otimes S_2$ and so $s_2 = s_1 \in U$. By Lemma II.2.7 the amalgam is strongly embeddable. By Theorem 1.23 we see that in particular the map $\varphi_1 : S_1 \to S_1 *_U S_2$ is pure. But the map $U \to S_1$ is pure and so $U \to S_1 *_U S_2$ is pure and U has the extension property in $S_1 *_U S_2$.

The result extends from monoids to semigroups.

THEOREM 2.2 <u>Let</u> $[U;S_1,S_2]$ <u>be a semigroup amalgam if</u> U <u>has the extension property in</u> S_1 <u>and</u> S_2 <u>then the amalgam is strongly embeddable and</u> U <u>has the extension property in</u> $S_1 *_U S_2$.

<u>Proof</u> This is a consequence of Theorems 2.1, 1.11, I.1.7 and I.1.8.

From Theorem I.1.6 we can extend the result to 'special' amalgams with more than two semigroups.

COROLLARY 2.3 *Let* $[U;\{S_i : i \in I\}]$ *be an amalgam such that* U *has the extension property in each* S_i. *Then the amalgam is strongly embeddable*.

Also by virtue of Theorem 1.7 we have

COROLLARY 2.4 (See [Hall, 13]). *Let* $[U;S_i]$ *be an amalgam such that* U *has the orbit preserving extension property in each* S_i. *Then the amalgam is strongly embeddable*.

Since unitary subsemigroups have the orbit preserving extension property [Hall, 13, Theorem 2.9] we could deduce Howie's result on unitary amalgams. Alternatively this will follow from Lemma 1.8 and

COROLLARY 2.5 *Let* $[U;S_i]$ *be an amalgam such that* U *is quasi-unitary in each* S_i. *Then the amalgam is strongly embeddable*.

Proof A direct consequence of Corollary 1.12.

Using Lemma 1.13 we then obtain

COROLLARY 2.6 *Let* $[U;S_i]$ *be an amalgam such that the maps* $U \to S_i$ *split (either in the category of semigroups or the category of* (U,U)-*bisets*). *Then the amalgam is strongly embeddable*.

From Theorem 1.14 we have

COROLLARY 2.7 (1) Let $\underset{\sim}{K}$ be a class of semigroups and let $U \leq S_i \in \underset{\sim}{K}$, with U an injective element of $\underset{\sim}{K}$. Then the amalgam $[U;S_i]$ is strongly embeddable.

(2) If U is injective in U-ENS-U, then U is an amalgamation base.

From Theorem 1.6 (3) we have

COROLLARY 2.8 [Hall, 13, Howie, 23] Let U be (right,left) absolutely perfect. Then U is an amalgamation base.

From [Bulman-Fleming and McDowell, 4] we have that inverse semigroups are absolutely flat and hence by Theorem 1.3 they are absolutely extendable, and so we have

COROLLARY 2.9 [See Howie 21, 23; Hall 13]. Every inverse semigroup is an amalgamation base in the category of semigroups.

T E Hall [13] gave the following definition. Let U be a subsemigroup of a semigroup S. Say that (U,S) is a (weak) amalgamation pair if every amalgam of the form [U;S,T] is (weakly) embeddable. He proved that if (U,S) is a weak amalgamation pair then U has both the right and left extension properties in S. (See Theorem I.1.19). In fact we have:

THEOREM 2.10 (cf. P M Cohn 7, Theorem 5.1). Let (U,S) be a weak amalgamation pair in the category of monoids [semigroups]. Then U has the extension property in S.

Proof We prove the theorem for the category of monoids. The semigroup case is similar. Let $X \in \text{ENS-}\underline{U}$ and $Y \in \underline{U}\text{-ENS}$. We need to show that the map $X \otimes Y \to X \otimes S \otimes Y$ is 1-1.

Let $W = X \dot{\cup} Y$ and make W a (U,U)-biset by defining $ux = x$, $yu = y$ for all x in X, y in Y and u in U. Let $W^{(0)} = U$, $W^{(1)} = W$ and $W^{(n)} = W^{(n-1)} \otimes_U W$ for $n \geq 2$. Put $T = \dot{\bigcup}_{n \geq 0} W^{(n)}$ (the tensor algebra over W) and define a multiplication on T by

$$(w_1 \otimes \ldots \otimes w_r).(z_1 \otimes \ldots \otimes z_s) = w_1 \otimes \ldots \otimes w_r \otimes z_1 \otimes \ldots \otimes z_s$$

$$u(w_1 \otimes \ldots \otimes w_r) = (uw_1) \otimes \ldots \otimes w_r$$

$$(w_1 \otimes \ldots \otimes w_r).u = w_1 \otimes \ldots \otimes (w_r u).$$

Then T is a monoid with U as submonoid.

The following are obvious

LEMMA 2.11 (1) The map $X \otimes_U Y \to T$ given by $x \otimes y \mapsto xy$ is 1-1.

(2) The map $X \otimes_U S \otimes_U Y \to T \otimes_U S \otimes_U Y$ given by $x \otimes s \otimes y \mapsto x \otimes s \otimes y$ is well-defined.

By assumption we have a commutative diagram of monomorphisms

$$\begin{array}{ccc} U & \longrightarrow & S \\ \big\downarrow & & \big\downarrow{\scriptstyle\alpha} \\ T & \xrightarrow[\beta]{} & S *_U T \end{array}$$

Consider the well-defined map $\varphi : T \otimes_U S \otimes_U T \to S *_U T$ given by $\varphi(t \otimes s \otimes t') = \beta(t)\alpha(s)\beta(t')$. Suppose that $x \otimes 1 \otimes y = x' \otimes 1 \otimes y'$ in $X \otimes S \otimes Y$. Then by Lemma 2.11 (2) we see that $\beta(x)\alpha(1)\beta(y) =$

$\beta(x')\alpha(1)\beta(y')$ in $S *_U T$, i.e. $\beta(xy) = \beta(x'y')$. Since β is 1-1 then we deduce by Lemma 2.11 (1) that $x \otimes y = x' \otimes y'$ in $X \otimes_U Y$.

COROLLARY 2.12 [Hall, 13] *If* (U,S) *is a weak amalgamation pair, then* U *has the right and left extension properties in* S.

From Corollary 1.22 we have

COROLLARY 2.13 [Howie, 15] *If* (U,S) *is a weak amalgamation pair, then* U *is relatively unitary in* S.

From Theorems 2.10 and 2.1, we have the following rather surprising result.

THEOREM 2.14 *Let* U *be a monoid [semigroup]. Then* U *is an amalgamation base in the category of monoids [semigroups] if and only if* U *is absolutely extendable in this class.*

Finally, C J Ash [1] gave the following definition and theorem. Let $\underline{M}$ be a class of semigroups with subclass $\underline{M}_0$. Then $\underline{M}_0$ is *cofinal* in $\underline{M}$ if for all $S \in \underline{M}$, there exists $T \in \underline{M}_0$ with $S \leq T$.

THEOREM 2.15 [See, 1, page 171, Theorem 3.3]. *If* $\underline{M}_0$ *is cofinal in* $\underline{M}$ *then* $U \in \underline{M}_0$ *is an amalgamation base for* $\underline{M}_0$ *if and only if it is an amalgamation base for* $\underline{M}$.

We now have

THEOREM 2.16 *Let* $\underline{M}$ *be a cofinal subclass of the class of all semigroups. Then* $U \in \underline{M}$ *is an amalgamation base for* $\underline{M}$ *if and only if it is absolutely extendable in* $\underline{M}$.

Proof The direct half is an immediate consequence of Theorems 2.15 and 2.14.

Conversely, let S be any semigroup containing U. Then there exists $T \in \underline{M}$ with $U \leq S \leq T$. By assumption, U has the extension property in T and hence in S, i.e. U is absolutely extendable in the class of all semigroups. The result now follows from Theorems 2.15 and 2.14.

Since the class of regular semigroups is cofinal in the class of all semigroups we thus have

COROLLARY 2.17 Let U be a regular semigroup. Then U is an amalgamation base in the class of regular semigroups if and only if it is absolutely extendable in this class.

CHAPTER IV FLATNESS AND AMALGAMATION

1. Flatness, quasi-flatness and free-extensions

In this section we provide a collection of results on flat and quasi-flat U-sets which will enable us in Section IV.2 to deduce results on flatness and amalgamations.

LEMMA 1.1 *Let* $f : X \to Y$ *be a right* U-*monomorphism and suppose that* X *and* Y/X *are quasi-flat. Then* Y *is quasi-flat*.

Proof Let $\lambda : A \to B$ be a left U-monomorphism with B flat and suppose that $y \otimes \lambda(a) = y' \otimes \lambda(a')$ in $Y \otimes_U B$. Then $\bar{y} \otimes \lambda(a) = \bar{y}' \otimes \lambda(a')$ in $(Y/X) \otimes B$ and so, since Y/X is quasi-flat, $\bar{y} \otimes a = \bar{y}' \otimes a'$ in $(Y/X) \otimes A$. From Lemma I.4.10 we see that either $y \otimes a = y' \otimes a'$, as required, or there exists $x_1, x_2 \in X$, $a_1, a_2 \in A$ such that $y \otimes a = f(x_1) \otimes a_1$ and $y' \otimes a' = f(x_2) \otimes a_2$. Hence $f(x_1) \otimes \lambda(a_1) = f(x_2) \otimes \lambda(a_2)$ in $Y \otimes B$. Since B is flat, the map $f \otimes 1 : X \otimes B \to Y \otimes B$ is 1-1, and since X is quasi-flat, the map $1 \otimes \lambda : X \otimes A \to X \otimes B$ is 1-1. Hence we see that $x_1 \otimes a_1 = x_2 \otimes a_2$ in $X \otimes A$ and so $y \otimes a = y' \otimes a'$ in $Y \otimes A$.

THEOREM 1.2 *Let* U *be a right reversible monoid. Let* $f : X \to Y$ *be a right* U-*monomorphism and* $\lambda : A \to B$ *a left* U-*monomorphism. Suppose that the map* $1 \otimes \lambda : Y \otimes A \to Y \otimes B$ *is* 1-1. *Then the map* $1 \otimes \lambda : (Y/X) \otimes A \to (Y/X) \otimes B$ *is* 1-1 *if and only if* (f, λ) *is stable*.

Proof Suppose that $1 \otimes \lambda : (Y/X) \otimes A \to (Y/X) \otimes B$ is 1-1. Suppose also that $y \otimes \lambda(a) = f(x) \otimes b$ in $Y \otimes B$. From Lemma I.4.14 we see that

$\overline{y}\otimes\lambda(a) = \overline{f(x)}\otimes\lambda(a)$ in $(Y/X)\otimes B$. Hence, we have $\overline{y}\otimes a = \overline{f(x)}\otimes a$ in $(Y/X)\otimes A$. From Lemma I.4.10 we deduce that there exists $x_1 \in X$, $a_1 \in A$ such that $y\otimes a = f(x_1)\otimes a_1$ in $Y\otimes A$. Hence $y\otimes\lambda(a) = f(x_1)\otimes\lambda(a_1)$ and (f,λ) is stable.

Conversely, suppose that $\overline{y}\otimes\lambda(a) = \overline{y}'\otimes\lambda(a')$ in $(Y/X)\otimes B$. From Lemma I.4.10 we have two cases to consider: either

(i) $y\otimes\lambda(a) = y'\otimes\lambda(a')$ in $Y\otimes B$, or

(ii) $y\otimes\lambda(a) = f(x_1)\otimes b_1$, $y'\otimes\lambda(a') = f(x_2)\otimes b_2$, for some $x_1,x_2 \in X$ and $b_1,b_2 \in B$.

In case (i) we see that since $Y\otimes A \to Y\otimes B$ is 1-1, then $y\otimes a = y'\otimes a'$ in $Y\otimes A$. Hence $\overline{y}\otimes a = \overline{y}'\otimes a'$ in $(Y/X)\otimes A$, as required.

In case (ii) we deduce, by stability of (f,λ) that there exists x_3,x_4 in X, a_3,a_4 in A such that

(1) $y\otimes\lambda(a) = f(x_3)\otimes\lambda(a_3)$, and

(2) $y'\otimes\lambda(a') = f(x_4)\otimes\lambda(a_4)$.

Since $Y\otimes A \to Y\otimes B$ is 1-1, we have

(3) $y\otimes a = f(x_3)\otimes a_3$, and

(4) $y'\otimes a' = f(x_4)\otimes a_4$.

Now by Lemma I.4.5 $\lambda(a) \sim \lambda(a')$ in B and so we see that $\lambda(a_3) \sim \lambda(a_4)$ in B, from (1) and (2). By the dual of Corollary I.2.7, $a_3 \sim a_4$ in A and so by Lemma I.4.10, $\overline{y}\otimes a = \overline{y}'\otimes a'$ in $(Y/X)\otimes A$.

If $f : X \to Y$ is a right U-monomorphism then we shall say that f is <u>right quasi-stable</u> if for all left U-sets A, all <u>flat</u> left U-sets

B and all U-monomorphisms $\lambda : A \to B$, the pair (f,λ) is stable. We can now deduce

COROLLARY 1.3 *Let* $f : X \to Y$ *be a right* U-*monomorphism*. *Suppose that* Y *is [quasi-] flat*. *Then* Y/X *is [quasi-] flat if and only if* U *is right reversible and* f *is [quasi-] stable*.

Proof Suppose that Y/X is [quasi-] flat. Then we see from Lemma I.5.19 that U is right reversible. Hence from Theorem 1.2 f is [quasi-] stable.

The converse follows immediately from Theorem 1.2.

We also have connections between flatness and purity.

LEMMA 1.4 *Let* $f : X \to Y$ *be a right pure monomorphism and let* Y *be right [quasi-] flat*. *Then* X *is right [quasi-] flat and* f *is [quasi-] stable*.

Proof Let $\lambda : A \to B$ be a left U-map [with B flat], and consider the commutative diagram,

$$\begin{tikzcd} X \otimes A \arrow[r, "1 \otimes \lambda"] \arrow[d, "f \otimes 1"'] & X \otimes B \arrow[d, "f \otimes 1"] \\ Y \otimes A \arrow[r, "1 \otimes \lambda"'] & Y \otimes B \end{tikzcd}$$

Then $f \otimes 1 : X \otimes A \to Y \otimes A$ is 1-1 by right purity of f, and $1 \otimes \lambda : Y \otimes A \to Y \otimes B$ is 1-1 by right flatness of Y. Hence we see that $1 \otimes \lambda : X \otimes A \to X \otimes B$ is 1-1 and X is right [quasi-] flat.

Suppose then that $y \otimes \lambda(a) = f(x) \otimes b$ in $Y \otimes B$, we see that $f(x) \otimes \overline{\lambda(a)} = f(x) \otimes \overline{b}$ in $Y \otimes (B/A)$, by the dual of Lemma I.4.14. Hence, since f is right pure, $x \otimes \overline{\lambda(a)} = x \otimes \overline{b}$ in $X \otimes (B/A)$. From Lemma I.4.10, we deduce that there exists $x' \in X$, $a' \in A$ such that $x \otimes b = x' \otimes \lambda(a')$ in $X \otimes B$. Hence $y \otimes \lambda(a) = f(x) \otimes b = f(x') \otimes \lambda(a')$ and f is [quasi-] stable.

THEOREM 1.5 <u>Let</u> $f : X \to Y$ <u>be a right pure</u> U-<u>monomorphism</u>. <u>Then</u> X <u>and</u> Y/X <u>are [quasi-] flat if and only if</u> Y <u>is [quasi-] flat and</u> U <u>is right reversible</u>.

<u>Proof</u> Suppose that X and Y/X are [quasi-] flat. Then U is right reversible by Lemma I.5.22. If X and Y/X are quasi-flat then Y is quasi-flat by Lemma 1.1. Suppose then that X and Y/X are flat. Let $\lambda : A \to B$ be a left U-monomorphism and suppose that $y \otimes \lambda(a) = y' \otimes \lambda(a')$ in $Y \otimes B$. Then $\overline{y} \otimes \lambda(a) = \overline{y}' \otimes \lambda(a')$ in $(Y/X) \otimes B$ and so, since (Y/X) is flat, $\overline{y} \otimes a = \overline{y}' \otimes a'$ in $(Y/X) \otimes A$. From Lemma I.4.10 we see that either (i) $y \otimes a = y' \otimes a'$ in $Y \otimes A$ as required, or (ii) $y \otimes a = f(x_1) \otimes a_1$, $y' \otimes a' = f(x_2) \otimes a_2$ and $a_1 \sim a_2$. In this case we have $f(x_1) \otimes \lambda(a_1) = f(x_2) \otimes \lambda(a_2)$. Now $f \otimes 1 : X \otimes B \to Y \otimes B$ is 1-1 by purity and $1 \otimes \lambda : X \otimes A \to X \otimes B$ is 1-1 by flatness of X. Hence $x_1 \otimes a_1 = x_2 \otimes a_2$ and so $y \otimes a = y' \otimes a'$ as required.

The converse follows from Lemma 1.4 and Corollary 1.3.

Let $X \in$ ENS-$\underline{U}$. Say that X is <u>weakly flat</u> if for all non-empty left ideals I of U, $X \otimes I \to X \otimes U$ is 1-1. Notice that X is weakly flat if and only if $X \otimes I \simeq XI$ for all left ideals I of U.

THEOREM 1.6 Let X be a sub U-set of a U-set Y, and suppose that Y is weakly flat. Then Y/X is weakly flat if and only if U is right reversible and X is relatively unitary in Y.

Proof Let I be a non-empty left ideal of U and consider the maps $\alpha : (Y/X)\otimes I \to (Y/X)I$, $\beta : (Y/X)\otimes I \to YI/XI$ and $\gamma : (Y/X)I \to (YI)/(X \cap YI)$, given by $\alpha(\overline{y}\otimes i) = \overline{y}.i$, $\beta(\overline{y}\otimes i) = \overline{yi}$ and $\gamma(\overline{y}.i) = \overline{yi}$. It is not too difficult to show that α and β are well-defined U-epimorphisms and that γ is a well-defined U-isomorphism. We therefore have a commutative diagram

$$\begin{tikzcd}
Y/X \otimes I \arrow[r, "\beta"] \arrow[d, "\alpha"'] & YI/XI \arrow[d, "\delta"] \\
(Y/X)I \arrow[r, "\gamma"'] & YI/(X \cap YI)
\end{tikzcd}$$

where $\delta(\overline{yi}) = \overline{yi}$ for $y \in Y$, $i \in I$. It is reasonably straightforward to show

(1) Y/X is weakly flat if and only if α is an isomorphism for every left ideal I of U;

(2) α is an isomorphism if and only if β and δ are isomorphisms;

(3) δ is an isomorphism for all left ideals I of U if and only if X is relatively unitary in Y.

The result will therefore follow if we can show

LEMMA 1.7 The map β is an isomorphism for all left ideals I of U if and only if U is right reversible.

<u>Proof</u> Suppose that β is an isomorphism for all left ideals I of U. Let $u,v \in U$ and let $I = Uu \cup Uv$. Then it is clear that if $x \in X$ then $\overline{x.u} = \overline{x.v}$ in YI/XI and so $\overline{x} \otimes u = \overline{x} \otimes v$ in $Y/X \otimes I$ since β is 1-1. Hence $u \sim v$ in $I = Uu \cup Uv$ and from this we easily deduce that $Uu \cap Uv \neq \varphi$ and so U is right reversible.

Conversely, suppose U is right reversible and let I be any left ideal of U. Suppose that $\overline{yi} = \overline{y'j}$ in YI/XI. Then we have two possibilities: either (i) $yi = y'j$ in YI or (ii) $yi = xk$, $y'j = x'k'$ for some $x,x' \in X$, $k,k' \in I$. In case (i) we see that $y \otimes i = y' \otimes j$ in $Y \otimes I$, since Y is weakly flat. Hence $\overline{y} \otimes i = \overline{y}' \otimes j$ in $Y/X \otimes I$ as required. In case (ii) we have by weak flatness of Y that

$$y \otimes i = x \otimes k \quad \text{and} \quad y' \otimes j = x' \otimes k' \quad \text{in} \quad Y \otimes I.$$

But since U is right reversible we see that $k \sim k'$ in I (in fact $k \sim k'$ in $Uk \cup Uk' \subseteq I$). Hence by Lemma I.4.10, $\overline{y} \otimes i = \overline{y}' \otimes j$ as required.

The following results will be of use in the next section.

THEOREM 1.8 <u>Let</u> U <u>be a submonoid of a monoid</u> S, <u>and suppose that</u> S <u>is flat</u>. <u>Let</u> $f : X \to Y$ <u>be a right</u> U-<u>monomorphism and suppose that</u> Y/X <u>is [quasi-] flat</u>. <u>Then the map</u> $f \otimes 1 : X \otimes S \to Y \otimes S$ <u>is [quasi-] stable</u>.

<u>Proof</u> We deal with the case where Y/X is quasi-flat. The other case is similar. Let $\lambda : A \to B$ be a left U-monomorphism with B flat. Suppose that $y \otimes s \otimes \lambda(a) = f(x) \otimes s' \otimes b$ in $Y \otimes S \otimes B$. By Lemma I.4.14 we see that $\overline{y} \otimes s \otimes \lambda(a) = \overline{f(x)} \otimes s \otimes \lambda(a)$ in $(Y/X) \otimes S \otimes B$.

Since $S\otimes B$ is flat and Y/X is quasi-flat we deduce that $\overline{y}\otimes s\otimes a = \overline{f(x)}\otimes s\otimes a$ in $Y/X\otimes S\otimes A$. From Lemma I.4.10 we see that there exists $x_1 \in X$, $s_1\otimes a_1 \in S\otimes A$ such that $y\otimes s\otimes a = f(x_1)\otimes s_1\otimes a_1$. Hence $y\otimes s\otimes \lambda(a) = f(x_1)\otimes s_1\otimes\lambda(a_1)$ and $f\otimes 1$ is quasi-stable.

THEOREM 1.9 Let U be a submonoid of a monoid S and suppose that S and S/U are flat. Let $Y \in$ ENS-$\underline{U}$ be [quasi-] flat. Then the map $Y \to Y \otimes_U S$ is a [quasi-] stable monomorphism.

Proof We deal with the case Y quasi-flat. The other case is similar. First of all notice that the map $Y \to Y\otimes_U S$ is indeed 1-1, since $Y\otimes U \to Y\otimes S$ is 1-1 by quasi-flatness of Y and $Y \simeq Y\otimes_U U$. Let $\lambda : A \to B$ be a left U-monomorphism with B flat and suppose that $y\otimes s\otimes\lambda(a) = y'\otimes 1\otimes b$ in $Y\otimes S\otimes B$. Then $y\otimes\overline{s}\otimes\lambda(a) = y'\otimes\overline{1}\otimes\lambda(a)$ in $Y\otimes(S/U)\otimes B$ by Lemma I.4.15. But the map $Y\otimes(S/U)\otimes A \to Y\otimes(S/U)\otimes B$ is 1-1 since Y is quasi-flat and $(S/U)\otimes B$ is flat. Hence $y\otimes\overline{s}\otimes a = y'\otimes\overline{1}\otimes a$ in $Y\otimes(S/U)\otimes A$ and so from Corollary I.4.13 there exists $y_1 \in Y$, $a_1 \in A$ such that $y\otimes s\otimes a = y_1\otimes 1\otimes a_1$. Hence the result follows.

LEMMA 1.10 Consider the following commutative diagram in ENS-$\underline{U}$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\alpha"] \arrow[ddr, "\varepsilon"] & \\
C \arrow[r, "\beta"'] \arrow[drr, "\psi"'] & P \arrow[dr, "\delta"] & \\
 & & D
\end{tikzcd}$$

where P is the pushout of

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \\
C &
\end{tikzcd}
$$

Let $\lambda : E \to F$ be a left U-map and suppose that the pairs (ε,λ) and (ψ,λ) are stable. Then (δ,λ) is stable.

Proof Suppose that $d \otimes \lambda(e) = \delta(p) \otimes x$ in $D \otimes F$. Then from example I.3.3 we see that there are two cases:

(1) $p = \alpha(b)$, $b \in B$,

(2) $p = \beta(c)$, $c \in C$.

In case (1) we have $d \otimes \lambda(e) = \delta\alpha(b) \otimes x = \varepsilon(b) \otimes x$. Since (ε,λ) is stable, there exists $b_1 \in B$, $e_1 \in E$ such that $d \otimes \lambda(e) = \varepsilon(b_1) \otimes \lambda(e_1)$. Hence $d \otimes \lambda(e) = \delta(\alpha(b_1)) \otimes \lambda(e_1)$ and (δ,α) is stable. Case (2) is similar to case (1).

From Corollary 1.3 we deduce

THEOREM 1.11 Let U be right reversible. Consider the following commutative diagram of U-sets and U-monomorphisms

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\alpha"] \arrow[ddr, "\varepsilon"] & \\
C \arrow[r, "\beta"'] \arrow[drr, "\psi"'] & P \arrow[dr, "\delta"] & \\
& & D
\end{tikzcd}
$$

where P is the pushout of

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \\ C & & \end{array}$$

If D is [quasi-] flat and if ε and ψ are [quasi-] stable then D/B, D/C and D/P are all [quasi-] flat.

We now provide a connection between free extensions and quasi-flatness similar to that between purity and free extensions (Theorem III.1.25).

THEOREM 1.12 Let U be a submonoid of a monoid S and suppose that S and S/U are flat. Let $f : X \to Y$ be a U-map with $X \in$ ENS-S, $Y \in$ ENS-U and suppose that the following is a free S-extension diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Suppose also that (1) f is 1-1 and (2) X and Y/X are quasi-flat. Then (a) g is 1-1 and (b) Y and Z/Y are quasi-flat.

Proof First Y is quasi-flat by Lemma 1.1. Consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
X \otimes S \arrow[r, "f \otimes 1"] \arrow[d] & Y \otimes S \arrow[d] \\
X \arrow[r] & Z
\end{tikzcd}$$

By Theorem II.1.2, the bottom square is a pushout. The map $f \otimes 1 : X \otimes S \to Y \otimes S$ is 1-1 by flatness of S and the map $Y \to Y \otimes S$ is 1-1 by quasi-flatness of Y, and flatness of S. Also from Lemma I.3.12 and Lemma I.5.17 we easily deduce that the top square is a pullback. Hence by Theorem I.3.14 (4) we see that $g : Y \to Z$ is 1-1.

We now require to show that Z/Y is quasi-flat. Let P be the pushout of the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \\
X \otimes S &
\end{tikzcd}$$

Then by Lemma I.3.13 we see that there exists a unique U-monomorphism $\delta : P \to Y \otimes S$ such that

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\gamma"'] & Y \arrow[d, "\alpha"] \arrow[ddr, "\vartheta"] & \\
X \otimes S \arrow[r, "\beta"'] \arrow[drr, "f \otimes 1"'] & P \arrow[dr, "\delta"] & \\
& & Y \otimes S
\end{tikzcd}$$

commutes, where $\gamma(x) = x \otimes 1$ and $\vartheta(y) = y \otimes 1$. We use Lemma I.3.15 to show that $Z/Y \simeq (Y \otimes S)/P$. First, consider the U-epimorphism $\sigma^\natural : Y \otimes S \to Z$. Then by Lemma I.3.7 we see that

$$\sigma \subseteq \rho_{f \otimes 1} \quad (= \mathrm{im}(f \otimes 1) \times \mathrm{im}(f \otimes 1) \cup 1_{Y \otimes S}).$$

Secondly, $\mathrm{im}(\sigma^\natural \circ (f \otimes 1)) \subseteq \mathrm{im}(\sigma^\natural \circ \vartheta)$, since the diagram

$$\begin{tikzcd}
X \otimes S \arrow[r, "f \otimes 1"] \arrow[d] & Y \otimes S \arrow[d, "\sigma^\natural"] \\
X \arrow[r, "h"'] & Z
\end{tikzcd}$$

commutes and so $\sigma^\natural(f(x) \otimes s) = h(xs) = (g \circ f)(xs) = (\sigma^\natural \circ \vartheta)(f(xs))$. Lastly, we have already demonstrated that $\sigma^\natural \circ \vartheta$ (= g) is 1-1. Hence by Lemma I.3.15, $Z/Y \simeq (Y \otimes S)/P$. But U is reversible (Corollary I.5.20) and so by Theorems 1.8 and 1.9, the maps $f \otimes 1$ and ϑ are quasi-stable. Hence by Theorem 1.11, $(Y \otimes S)/P$ is quasi-flat.

2. Flatness and amalgamation

THEOREM 2.1 *Let $[U;S_1,S_2]$ be a monoid amalgam and suppose that S_i and S_i/U are flat $(i = 1,2)$. Then the amalgam is strongly embeddable.*

Proof Construct the sequence (W_n, f_n) as in Theorem II.2.1. The map $f_1 : W_1 \to W_2$ is 1-1 by flatness of S_1 and the map $g^{(1)} : S_2 \to W_2$

is 1-1 by flatness of S_2. W_1 is quasi-flat, since it is flat and by Theorem 1.9, f_1 is quasi-stable (in fact it is stable). By Corollary 1.3 and Corollary I.5.20 we see that W_2/W_1 is quasi-flat. We now deduce from Theorem 1.12 that for all $n \geq 1$, f_n is 1-1 and W_n and W_{n+1}/W_n are quasi-flat. By Theorem II.2.5 the amalgam is weakly embeddable and by Corollary I.3.18 the map $\varphi_2 : W_2 \to S_1 *_U S_2$ is 1-1. Suppose then that $s_1 \otimes 1 = 1 \otimes s_2$ in $S_1 \otimes_U S_2$. We see from Lemma I.5.17 that $s_1 \in U$ (take $X = U$, $Y = S_1$ and $f = i : U \to S_1$). Hence $1 \otimes s_2 = 1 \otimes s_1$ and $s_2 = s_1 \in U$, since $g^{(1)}$ is 1-1. By Lemma II.2.7 the amalgam is strongly embeddable.

Notice that in Theorem 2.1, S_i and S_i/U <u>must</u> be both right <u>and</u> left flat. This follows from Hall [13, Theorem 25 (ii) and (iii)] where he shows that right absolutely flat semigroups need not be amalgamation bases.

From Lemma 1.1 and Lemma I.1.12 we can deduce

LEMMA 2.2 <u>If</u> $[U;S_1,S_2]$ <u>is as above then</u> W_n <u>and</u> W_n/U <u>are quasi-flat for all</u> $n \geq 1$.

<u>Proof</u> From Lemma I.1.12 we see that $W_n/W_{n-1} \simeq (W_n/W_{n-2})/(W_{n-1}/W_{n-2})$. By Lemma 1.1 we see that W_n/W_{n-2} is quasi-flat. Similarly, $W_n/W_{n-3},\ldots,W_n/W_1$ are quasi-flat. Hence $W_n/W_1 = (W_n/U)/(W_1/U)$ and so W_n/U is quasi-flat.

From Theorem I.5.13 and Theorem I.3.19 we can deduce immediately

COROLLARY 2.3 *Let* $[U;S_1,S_2]$ *be a monoid amalgam with* S_i *and* S_i/U *flat* $(i = 1,2)$. *Then* $S_1 *_U S_2$ *and* $(S_1 *_U S_2)/U$ *are quasi-flat*.

We do not know whether $S_1 *_U S_2$ and $(S_1 *_U S_2)/U$ are flat.

CHAPTER V PERFECT SUBMONOIDS AND AMALGAMATION

In view of the techniques described in Chapters II, III and IV we intend, in this section, to review some of the work concerning the perfect property. We shall provide a new proof that 'perfect implies amalgamation', similar in character to these of Theorems III.2.1 and IV.2.1. We shall also deduce as a corollary that U is perfect in $S_1 *_U S_2$.

Recall ([Theorem III.1.4]), that a submonoid U of a monoid S is said to be right perfect if U has the right extension property in S and S is left flat as a U-set.

We mention at this stage that Hall [13] has given an example of an amalgam [U;S,T] such that U is right perfect in S, U has the orbit preserving extension property in T (and hence the extension property in T) but such that the amalgam [U;S,T] is not weakly embeddable. Hence we see that the extension property and the perfect property are independent, in that neither implies the other.

We shall need the following rather technical lemma.

LEMMA 1.1 Let

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \varphi\downarrow & & \downarrow\sigma \\ C & \xrightarrow[f]{} & P \end{array} \quad \text{and} \quad \begin{array}{ccc} D & \xrightarrow{\delta} & E \\ \vartheta\downarrow & & \downarrow\tau \\ F & \xrightarrow[k]{} & Q \end{array}$$

be pushouts in ENS-U and suppose that there exists monomorphisms $\beta : A \to D$, $\gamma : B \to E$ and $g : C \to F$, such that the diagram

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[dr] \arrow[dd] & \\
& D \arrow[rr] \arrow[dd] & & E \arrow[dd] \\
C \arrow[rr] \arrow[dr] & & P & \\
& F \arrow[rr] & & Q
\end{tikzcd}$$

commutes. Suppose also that the top square

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "\gamma"] \\
D \arrow[r, "\delta"'] & E
\end{tikzcd}$$

is a pullback and that $\vartheta : D \to F$ and $\varphi : A \to C$ are both onto. Then these exists a unique monomorphism $h : P \to Q$ such that the completed 'cube' commutes.

Proof Notice first that by Lemma I.3.7, $P \simeq B/\rho$ and $Q \simeq E/\varepsilon$, where $\rho = \{(\alpha(a),\alpha(a')) : (a,a') \in \mathrm{Ker}\varphi\} \cup 1_B$ and $\varepsilon = \{(\delta(d),\delta(d')) : (d,d') \in \mathrm{Ker}\vartheta\} \cup 1_E$. Define $h : P \to Q$ by $h(b\rho) = (\gamma(b))\varepsilon$. Then it is clear that h is a well-defined U-map which will complete the above 'cube'. Suppose then that $(\gamma(b),\gamma(b')) \in \varepsilon$. Then either (i) $\gamma(b) = \gamma(b')$, in which case $b = b'$ and so $b\rho = b'\rho$ as required, or (ii) $\gamma(b) = \delta(d)$, $\gamma(b') = \delta(d')$ for some $(d,d') \in \mathrm{Ker}\vartheta$. In this case we see that since

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "\gamma"] \\ D \arrow[r, "\delta"'] & E \end{tikzcd}$$

is a pullback, then there exists unique $a,a' \in A$ such that

$$b = \alpha(a), \quad d = \beta(a), \quad b' = \alpha(a'), \quad d' = \beta(a').$$

Now we have
$$\begin{aligned} g\varphi(a) &= \vartheta\beta(a), \\ &= \vartheta(d), \\ &= \vartheta(d'), \\ &= \vartheta\beta(a') = g\varphi(a'). \end{aligned}$$

But g is 1-1 and so $(a,a') \in \mathrm{Ker}\varphi$. Hence $b\rho = b'\rho$ as required.

Let $f : X \to Y$ be a right U-monomorphism. Say that f is *perfect* if f is right pure and Y is right flat. We readily see

LEMMA 1.2 *Let U be a submonoid of a monoid S. Then U has the left perfect property in S if and only if the inclusion $U \to S$ (considered as a right U-map) is perfect.*

THEOREM 1.3 *Let U be a submonoid of a monoid S. Then U is left perfect in S if and only if the following two conditions hold:*

(1) *there exists $A \in$ ENS-S such that A is flat in ENS-U,*

(2) *for all $X \in$ ENS-S, all $Y \in$ ENS-U and all perfect monomorphisms $f : X \to Y$, the natural map $g : Y \to F(S;X,Y)$ is a perfect monomorphism.*

<u>Proof</u> Let U be left perfect in S. Then S is right flat as a U-set and so condition (1) holds.

Suppose that $f : X \to Y$ is a perfect monomorphism, where $X \in \text{ENS-}\underline{S}$, $Y \in \text{ENS-}\underline{U}$. Let $Z = F(S;X,Y)$ and let $g : Y \to Z$ and $h : X \to Z$ be as in the proof of Theorem II.1.1. We use Theorem I.3.14 to show that g is pure. Let $A \in \underline{U}\text{-ENS}$ and consider the commutative diagram

$$\begin{tikzcd} X \otimes A \arrow[r] \arrow[d] & Y \otimes A \arrow[d] \\ X \otimes S \otimes A \arrow[r] & Y \otimes S \otimes A \end{tikzcd}$$

Notice that all the maps in this diagram are 1-1, the only difficult case being the map $X \otimes A \to X \otimes S \otimes A$. But this is 1-1 since $A \to S \otimes A$ is 1-1 (U has the left extension property in S) and X is right flat (Lemma IV.1.4). Since f is right pure and hence stable, by the dual of Theorem III.1.17, it is easy to deduce from Lemma I.3.12 that the above diagram is a pullback. By Theorem II.1.2 and Theorem I.4.7 the diagram

$$\begin{tikzcd} X \otimes S \otimes A \arrow[r] \arrow[d] & Y \otimes S \otimes A \arrow[d] \\ X \otimes A \arrow[r] & Z \otimes A \end{tikzcd}$$

is a pushout. Hence by Theorem I.3.14 we see that the map $g : Y \to Z$ is right pure. We now require to show that Z is flat. Let

$\lambda : A \to B$ be a left U-monomorphism and consider the incomplete 'cube'

$$\begin{tikzcd}
X \otimes S \otimes A \arrow[rr] \arrow[dr] \arrow[dd] & & Y \otimes S \otimes A \arrow[dr] \arrow[dd] & \\
& X \otimes S \otimes B \arrow[rr] \arrow[dd] & & Y \otimes S \otimes B \arrow[dd] \\
X \otimes A \arrow[rr] \arrow[dr] & & Z \otimes A & \\
& X \otimes B \arrow[rr] & & Z \otimes B
\end{tikzcd}$$

It is not at all difficult to check that the conditions of Lemma 1.1 are satisfied. Hence there exists a unique U-monomorphism $Z \otimes A \to Z \otimes B$ which will complete the cube, and so Z is right flat as required.

Conversely, suppose that conditions (1) and (2) are satisfied. Let $A \in$ ENS-$\underline{S}$ be flat in ENS-$\underline{U}$ and consider the perfect monomorphism $A \to A \dot{\cup} U$. We see from Lemma II.1.5 that $F(S;A,A \dot{\cup} U) \simeq A \dot{\cup} S$ and hence we have that the map $A \dot{\cup} U \to A \dot{\cup} S$ is a perfect monomorphism. It is now easy to deduce that $U \to S$ is perfect as required.

We are now in a position to deduce

THEOREM 1.4 [Hall, 13; Howie, 23]. *Let* $[U;S_1,S_2]$ *be a monoid amalgam and suppose that* U *is left perfect in* S_1 *and* S_2. *Then the amalgam is strongly embeddable and* U *is left perfect in* $S_1 *_U S_2$.

<u>Proof</u> Construct the sequence $(W_n, f_n)_{n \geq 1}$ as in the proof of Theorem II.2.1. Then $f_1 : W_1 \to W_2$ is 1-1 since S_1 is right flat and $g^{(1)} : S_2 \to W_2$ is 1-1 since U has the left extension property in S_1. Also, W_2 is right flat since S_1 and S_2 are and f_1 is right pure since if $A \in \underline{U}$-ENS then $f_1 \otimes 1 : S_1 \otimes A \to S_1 \otimes S_2 \otimes A$ is 1-1 by right flatness of S_1 and by the left extension property of U in S_2. Hence by Theorem 1.3, we deduce that f_n is a perfect monomorphism for each $n \geq 1$. From Theorem II.2.5, the amalgam is weakly embeddable and by Corollary I.3.18 the map $\varphi_2 : W_2 \to S_1 *_U S_2$ is 1-1. Suppose then that $s_1 \otimes 1 = 1 \otimes s_2$ in $S_1 \otimes S_2$. By the dual of Corollary III.1.18, $s_2 \in U$ and so $s_1 \otimes 1 = s_2 \otimes 1$ in $S_1 \otimes S_2$. Hence $s_1 = s_2 \in U$ since f_1 is 1-1 and so the amalgam is strongly embeddable by Lemma II.2.7. Also, by Theorem III.1.23 and Theorem I.5.13 we see that $\varphi_1 : W_1 \to S_1 *_U S_2$ is perfect and so the map $U \to S_1 *_U S_2$ is perfect, i.e. U is left perfect in $S_1 *_U S_2$.

From Theorem I.1.6 we can now deduce

COROLLARY 1.5 <u>Let</u> $[U;S_i]$ <u>be an amalgam such that</u> U <u>is left perfect in each</u> S_i. <u>Then the amalgam is strongly embeddable</u>.

We end this chapter with a rather interesting connection with Chapter IV.

THEOREM 1.6 <u>Let</u> U <u>be a right reversible monoid</u>. <u>Let</u> X,Y $\in$ ENS-$\underline{U}$ <u>and suppose that</u> $f : X \to Y$ <u>is a perfect monomorphism</u>. <u>Then</u> X,Y <u>and</u> Y/X <u>are right flat</u>.

Proof The proof follows immediately from Theorem IV.1.5.

COROLLARY 1.7 *Let* U *be a left perfect submonoid of a monoid* S *and suppose that* U *is right reversible. Then* S *and* S/U *are right flat.*

The converse of Corollary 1.7, and hence of Theorem 1.6 is false. This follows from the dual of the example given in the note after Theorem III.1.6, where it is shown that left absolutely flat monoids are not necessarily right absolutely perfect.

CHAPTER VI EXTENSION AND AMALGAMATION IN RINGS

In this final chapter we examine the case of rings. All rings will be associative rings with 1 and all maps will be 1-preserving ring homomorphisms. We shall denote the category of right R-modules by MOD-R and call its maps (right) R-maps, while the category of (R,S)-bimodules will be denoted by R-MOD-S and its maps called (R,S)-maps.

Many of the results in this chapter have already been proved for the semigroup case. Consequently, we shall omit some of the proofs and simply refer the reader to an earlier section for more details.

1. Direct limits, pushouts and pullbacks

THEOREM 1.1 [Rotman, 35, Corollary 2.20] Let (X_i, φ^i_j) be a direct system in R-MOD-S with direct limit (X, α_i) and let $A \in$ MOD-R, $B \in$ S-MOD. Then $(A \otimes_R X \otimes_S B,\ 1 \otimes \alpha_i \otimes 1)$ is the direct limit of $(A \otimes_R X_i \otimes_S B,\ 1 \otimes \varphi^i_j \otimes 1)$.

LEMMA 1.2 [cf. Corollary I.3.5] Let R be a subring of a ring S. Then any pushout in MOD-S is also a pushout in MOD-R.

LEMMA 1.3 [cf. Lemmas I.3.8 and I.3.9] Let

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "\alpha"] \\ C \arrow[r, "\beta"'] & D \end{tikzcd}$$

be a pushout diagram and suppose that $\alpha(b) = \beta(c)$. *Then there exists* $a \in A$ *such that* $b = g(a)$, $c = f(a)$.

LEMMA 1.4 [cf. Lemma I.3.12] *The commutative diagram*

$$\begin{tikzcd} D \arrow[r, "\alpha"] \arrow[d, "\beta"'] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

is a pullback if and only if whenever $g(c) = f(b)$, *there exists a unique* $d \in D$ *such that* $c = \alpha(d)$, $b = \beta(d)$.

THEOREM 1.5 [cf. Theorem I.3.14] *Consider the commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\gamma"'] & B \arrow[d, "\varepsilon"] \\ C \arrow[r, "\psi"'] \arrow[d, "\varphi"'] & D \arrow[d, "\alpha"] \\ E \arrow[r, "\beta"'] & F \end{tikzcd}$$

in MOD-R, where the top square is a pullback and the bottom square is a pushout. Suppose that φ is onto and that ψ is 1-1. Then $\alpha \circ \varepsilon$ is 1-1 if and only if ε and $\varphi \circ \gamma$ are 1-1.

THEOREM 1.6 [cf. Lemma V.1.1] Let

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & B \arrow[d, "\sigma"] \\ C \arrow[r, "f"'] & P \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} D \arrow[r, "\delta"] \arrow[d, "\vartheta"'] & E \arrow[d, "\tau"] \\ F \arrow[r, "k"'] & Q \end{tikzcd}$$

be pushouts in MOD-R and suppose that there exist monomorphisms $\beta : A \to D$, $\gamma : B \to E$ and $g : C \to F$, such that the diagram

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[dr] \arrow[dd] & \\ & D \arrow[rr] \arrow[dd] & & E \arrow[dd] \\ C \arrow[rr] \arrow[dr] & & P & \\ & F \arrow[rr] & & Q \end{tikzcd}$$

commutes. Suppose also that the top square

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "\gamma"] \\ D \arrow[r, "\delta"'] & E \end{tikzcd}$$

is a pullback and that $\vartheta : D \to F$ and $\varphi : A \to C$ are both onto. Then there exists a unique monomorphism $h : P \to Q$ such that the completed 'cube' commutes.

2. Extensions and free extensions

Let R be a subring of a ring S. We say that R has the *right extension property* in S if for all $X \in \text{MOD-}\underline{R}$, the map $X \to X \otimes_R S$, given by $x \mapsto x \otimes 1$ is 1-1. The *left extension property* is defined dually. We say that R has the *extension property* in S if for all $X \in \text{MOD-}\underline{R}$ and all $Y \in \underline{R}\text{-MOD}$, the map $X \otimes_R Y \to X \otimes_R S \otimes_R Y$, given by $\Sigma x \otimes y \mapsto \Sigma x \otimes 1 \otimes y$ is 1-1. We shall say that a ring R is *(right, left) absolutely extendable* if R has the (right,left) extension property in every containing ring.

THEOREM 2.1 *Let* R *be a subring of a ring* S. *If* R *has the extension property in* S *then* R *has both the right and left extension properties in* S.

The next result will prove useful later.

LEMMA 2.2 [cf. Corollary III.1.18] *Let* R *be a subring of a ring* S *and suppose that* R *has the left extension property in* S. *Let* $f : X \to Y$ *be a left* R*-monomorphism and suppose that* $1 \otimes y = \Sigma s \otimes f(x)$ *in* $S \otimes Y$. *Then* $y \in \text{im} f$.

A ring R is *(von-Neumann) regular* if for all $a \in R$, $a \in aRa$, or equivalently, [35, Theorem 4.16] if every (right) R-module is flat.

We can in fact show

LEMMA 2.3 <u>A commutative ring</u> R <u>is regular if and only if every containing ring of</u> R <u>is flat as an</u> R-<u>module</u>.

<u>Proof</u> The 'only if' part is clear. Suppose then that every containing ring of R is flat. Let $X \in$ MOD-$\underline{R}$ and let T be the tensor algebra over X. It is straightforward to show that X is flat if and only if T is flat. Since T is a containing ring of R, then X is in fact flat. Hence the result.

THEOREM 2.4 <u>If</u> R <u>is a regular ring, then</u> R <u>is absolutely extendable</u>.

Let R be a subring of a ring S. Let $X \in$ MOD-$\underline{S}$, $Y \in$ MOD-$\underline{R}$ and let $f : X \to Y$ be an R-map. The <u>free</u> S-<u>extension of</u> X <u>and</u> Y, is a right S-module $F(S;X,Y)$ together with an S-map $h : X \to F(S;X,Y)$ and an R-map $g : Y \to F(S;X,Y)$ such that:

(1) $g \circ f = h$;

(2) whenever there is an S-module Z, an S-map $\beta : X \to Z$ and an R-map $\alpha : Y \to Z$ such that $\alpha \circ f = \beta$, then there exists a unique S-map $\psi : F(S;X,Y) \to Z$ such that $\psi \circ g = \alpha$ and $\psi \circ h = \beta$.

THEOREM 2.5 [cf. Theorem II.1.2] Let $X \in$ MOD-$\underline{S}$, $Y \in$ MOD-$\underline{R}$ and let $f : X \to Y$ be an R-map. Then the pushout in MOD-$\underline{S}$ (and hence in MOD-$\underline{R}$) of the diagram

$$\begin{tikzcd} X \otimes_R S \arrow[r, "f \otimes 1"] \arrow[d, "\varphi"'] & Y \otimes_R S \\ X & \end{tikzcd}$$

where $\varphi(\Sigma x \otimes s) = \Sigma xs$, is isomorphic to $F(S;X,Y)$.

Let R be a subring of a ring S. We say that R is right level in S if for all $X \in$ MOD-$\underline{S}$, all $Y \in$ MOD-$\underline{R}$ and all R-monomorphisms $f : X \to Y$, there exists $Z \in$ MOD-$\underline{S}$, an R-monomorphism $g : Y \to Z$ such that $h = g \circ f : X \to Z$ is an S-monomorphism. Left level subrings are defined dually. A ring R is (right,left) absolutely level if it is (right,left) level in every containing ring.

THEOREM 2.6 [cf. Theorem III.1.4] Let R be a subring of a ring S. Then R is right level in S if and only if R has the right extension property in S and S is left flat in MOD-$\underline{R}$.

Proof Suppose that R is a right level subring of S. Let $X \in$ MOD-$\underline{R}$ and set $Y = X \oplus S$. Then $Y \in$ MOD-$\underline{R}$ and the map $f : S \to Y$

given by $f(s) = (0,s)$ is an R-monomorphism. By assumption, there exists $Z \in \underline{S}$-MOD and an R-monomorphism $g : Y \to Z$. Define $\beta : X \to Z$ and $\delta : X \otimes_R S \to Z$ by $\beta(x) = g(x,0)$ and $\delta(x \otimes s) = \beta(x).s$. Then β is an R-monomorphism, δ is well-defined and the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\beta"'] & X \otimes_R S \arrow[dl, "\delta"] \\ Z & \end{tikzcd}$$

commutes. Hence $X \to X \otimes_R S$ is 1-1 and R has the right extension property in S.

Suppose now that $f : X \to Y$ is a right R-monomorphism and consider the following pushout diagram in MOD-$\underline{R}$,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "\beta"] \\ X \otimes_R S \arrow[r, "\alpha"'] & P \end{tikzcd}$$

Since f is 1-1, then so is α and hence there exists $Z \in$ MOD-$\underline{S}$ and an R-monomorphism $g : P \to Z$ such that $h = g \circ \alpha$ is an S-monomorphism. Define $\varphi : Y \otimes_R S \to Z$ by $\varphi(y \otimes s) = (g \circ \beta)(y).s$. Then φ is a well-defined S-map and

$$\begin{aligned}(\varphi \circ (f \otimes 1))(x \otimes s) &= (g \circ \beta \circ f)(x).s, \\ &= (g \circ \alpha)(x \otimes 1).s, \\ &= h(x \otimes 1).s, \\ &= h(x \otimes s).\end{aligned}$$

Hence $\varphi \circ (f \otimes 1) = h$ and so $f \otimes 1$ is 1-1 as required.

Conversely, let $f : X \to Y$ be a right R-monomorphism with $X \in$ MOD-$\underline{S}$ and $Y \in$ MOD-$\underline{R}$. We see that the maps $X \to X \otimes_R S$, $Y \to Y \otimes_R S$ and $X \otimes_R S \to Y \otimes_R S$ are all 1-1. From Lemma 1.4 and the dual of Lemma 2.2 we can easily deduce that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X \otimes_R S \arrow[r] & Y \otimes_R S \end{tikzcd}$$

is a pullback. Hence from Theorem 1.5 and Theorem 2.5 we see that the map $g : Y \to F(S;X,Y)$ is 1-1. The result now follows.

From Lemma 2.3 and Theorems 2.4 and 2.6 we can deduce

COROLLARY 2.7 *Let* R *be a commutative ring. Then the following are equivalent*:

(i) R *is (von-Neumann) regular*,

(ii) R *is right absolutely level*,

(iii) R *is left absolutely level*.

Let $f : X \to Y$ be a right R-monomorphism. We say that f is *right pure* if for all left R-modules B, the map $f \otimes 1 : X \otimes_R B \to Y \otimes_R B$ is 1-1. Let $f : X \to Y$ be an (R,R)-monomorphism. Then we say that f is *pure* if for all $A \in$ MOD-$\underline{R}$ and all $B \in \underline{R}$-MOD, the map $1 \otimes f \otimes 1 : A \otimes_R X \otimes_R B \to A \otimes_R Y \otimes_R B$ is 1-1. The following are clear.

LEMMA 2.8 *If* $f : X \to Y$ *is a pure monomorphism then* f *is both right and left pure*.

LEMMA 2.9 If R is a subring of a ring S, then R has the (right,left) extension property in S if and only if the inclusion $R \to S$ is (left,right) pure.

LEMMA 2.10 [Rotman, 35, Theorem 2.7] Let $f : A \to B$ be an R-monomorphism. Then f splits if and only if $B \simeq A \oplus C$ for some R-module C.

We can now deduce:

THEOREM 2.11 Let $A,B \in$ MOD-$\underline{R}$ [respectively $\underline{R}$-MOD-$\underline{R}$] and let $f : A \to B$ be a split R-monomorphism. Then f is right pure [respectively pure].

Proof The proof follows from Lemma 2.10 on noticing that $C \otimes_R (D \oplus E) \simeq (C \otimes_R D) \oplus (C \otimes_R E)$ for all $C \in$ MOD-$\underline{R}$, and all $D,E \in \underline{R}$-MOD.

THEOREM 2.12 [cf. Theorem III.1.23] Let (X_i,φ_j^i) be a direct system with directed index set I and suppose that (X,α_i) is the direct limit. If $k \in I$, then α_k is a [right] pure monomorphism if and only if φ_ℓ^k is a [right] pure monomorphism for all $\ell \geq k$.

The definition of *stability* of R-module maps is the same as that for S-set maps. See section III.1.

THEOREM 2.13 [cf. Theorem III.1.17] Let R be a ring and let $f : X \to Y$ be a left pure monomorphism. Then f is stable.

The following characterisation of the extension property is similar to that in the semigroup case.

THEOREM 2.14 [cf. Theorem III.1.25] Let R be a subring of a ring S. Then R has the extension property in S if and only if for all $X \in \underline{R}\text{-MOD-}\underline{S}$, for all $Y \in \underline{R}\text{-MOD-}\underline{R}$ and for all pure R-monomorphisms $f : X \to Y$, there exists $Z \in \underline{R}\text{-MOD-}\underline{S}$, an (R,S)-monomorphism $h : X \to Z$ and a pure (R,R)-monomorphism $g : Y \to Z$ such that $g \circ f = h$.

We now return to the notion of level subring. In view of Theorem 2.6, the following definition seems reasonable. Let $f : X \to Y$ be a right R-monomorphism. Say that f is level if f is right pure and Y is right flat. The following is easy to prove.

LEMMA 2.15 Let $f : X \to Y$ be a level monomorphism. Then X is flat.

THEOREM 2.16 Let $f : X \to Y$ be a right R-monomorphism and suppose that Y is flat. Then the following are equivalent:

(1) Y/X is flat,

(2) f is pure,

(3) f is level,

(4) f is stable,

(5) for all left R-monomorphisms $\lambda : A \to B$, the diagram

$$\begin{tikzcd} X \otimes A \arrow[r] \arrow[d] & Y \otimes A \arrow[d] \\ X \otimes B \arrow[r] & Y \otimes B \end{tikzcd}$$

is a pullback.

Proof (1) implies (2). Let $A \in \underline{R}$-MOD and consider the exact homology sequence

$$\ldots\ \mathrm{TOR}_1(Y/X,A) \to X \otimes A \to Y \otimes A \to (Y/X) \otimes A \to 0$$

Since Y/X is flat, then $\mathrm{TOR}_1(Y/X,A) = 0$ and so exactness implies that f is pure.

(2) and (3) are clearly equivalent.

(2) implies (5). The proof follows from the dual of Theorem 2.13 and Lemma 1.6 on noting that all the maps in the above diagram are 1-1.

(5) implies (4). This follows immediately from Lemma 1.6.

(4) implies (1). Consider the commutative diagram

$$\begin{tikzcd} X \otimes A \arrow[r, "f \otimes 1_A"] \arrow[d, "1_X \otimes \lambda"'] & Y \otimes A \arrow[r] \arrow[d, "1_Y \otimes \lambda"] & (Y/X) \otimes A \arrow[d] \\ X \otimes B \arrow[r, "f \otimes 1_B"'] & Y \otimes B \arrow[r] & (Y/X) \otimes B \end{tikzcd}$$

Consider also the well-defined map $Y/X \otimes B \to (Y \otimes B)/\mathrm{im}(f \otimes 1_B)$ given by $\Sigma(y + \mathrm{im} f \otimes b) \mapsto \Sigma(y \otimes b + \mathrm{im}(f \otimes 1_B))$. Suppose that $\Sigma(y + \mathrm{im} f \otimes \lambda(a)) = 0$ in $Y/X \otimes B$. Then $\Sigma y \otimes \lambda(a) \in \mathrm{im}(f \otimes 1_B)$ on applying the above mentioned map. Hence, by stability of f and flatness of Y we deduce that $\Sigma y \otimes a \in \mathrm{im}(f \otimes 1_A)$, and so $\Sigma(y + \mathrm{im} f \otimes a) = 0$ in $Y/X \otimes A$.

COROLLARY 2.17 Let $f : X \to Y$ be a right R-monomorphism and suppose that Y/X is flat. Then X is flat if and only if Y is flat.

Proof The proof follows from the exact homology sequence.

COROLLARY 2.18 Let $f : X \to Y$ be a right R-monomorphism. Then f is level if and only if X and Y/X are flat.

As a special consequence of the above result, we have

COROLLARY 2.19 Let R be a subring of a ring S. Then R is left level in S if and only if S/R is right flat.

THEOREM 2.20 [cf. Theorem V.1.3] Let R be a subring of a ring S. Then R is left level in S if and only if

(1) there exists $C \in$ MOD-$\underline{S}$ such that C is flat in MOD-$\underline{R}$, and

(2) for all $X \in$ MOD-$\underline{S}$, all $Y \in$ MOD-$\underline{R}$ and all level R-monomorphisms $f : X \to Y$, the natural map $g : Y \to F(S;X,Y)$ is a level monomorphism.

Finally, recall that a ring R is normally said to be (right) perfect if every flat right R-module is projective. We have the following rather interesting result.

THEOREM 2.21 [Fieldhouse, 9, Proposition 10.2] A ring R is right perfect if and only if every level R-monomorphism splits.

3. Free products with amalgamation

We describe a construction, first given in Cohn [7] and derive a necessary and sufficient condition for a ring amalgam to be embeddable.

Let $[R;S_1,S_2]$ be an amalgam of rings. Let $W_1 = S_1$, $W_2 = S_1 \otimes_R S_2$ and define $f_1 : W_1 \to W_2$ by $f_1(s_1) = s_1 \otimes 1$. Suppose, by way of induction, that we have a sequence of R-modules and R-maps

$$W_1 \xrightarrow{f_1} W_2 \xrightarrow{f_2} W_3 \longrightarrow \cdots \xrightarrow{f_{n-2}} W_{n-1},$$

and suppose that

(1) W_k is an (S_1,S_i)-bimodule, $i \equiv k (\mathrm{mod}\ 2)$,

(2) f_k is an (S_1,R)-map, $\qquad k = 1,\ldots,n-2$.

Let $i \equiv n(\mathrm{mod}\ 2)$ and define $W_n = F(S_i;W_{n-2},W_{n-1})$. Then we have an (S_1,R)-map $f_{n-1} : W_{n-1} \to W_n$ and so by induction we have a direct system (W_n,f_n) of R-modules and R-maps.

The following was proved by Cohn [7]. See also [2, pages 324-325] and Theorem II.2.1.

THEOREM 3.1 Let $[R;S_1,S_2]$ be an amalgam of rings and construct the system (W_n,f_n) as above. Then $S_1 *_R S_2$, the free product of the amalgam, is the direct limit in R-MOD-R of the system (W_n,f_n).

Notice that the direct limit, $S_1 *_R S_2$, comes naturally equipped with maps $\varphi_n : W_n \to S_1 *_R S_2$ such that

$$\varphi_n \circ f_{n-1} = \varphi_{n-1}, \qquad n = 2,3,\ldots .$$

Let $f^{(n-1)} = f_{n-1} \circ f_{n-2} \circ \cdots \circ f_1 : W_1 \to W_n$, let $g^{(1)} : S_2 \to W_2$ be given by $g^{(1)}(s_2) = 1 \otimes s_2$ and let $g^{(n-1)} = f_{n-1} \circ \cdots \circ f_2 \circ g^{(1)} : S_2 \to W_n$.

THEOREM 3.2 [cf. Theorem II.2.5] Let $[R;S_1,S_2]$ be an amalgam of rings. Then the amalgam is weakly embeddable if and only if the maps $f^{(n)}$ and $g^{(n)}$ are 1-1 for all $n \geq 1$.

THEOREM 3.3 [cf. Lemma II.2.7] Let $[R;S_1,S_2]$ be an amalgam of rings. Suppose that the amalgam is weakly embeddable and suppose that the map $\varphi_2 : W_2 \to S_1 *_R S_2$ is 1-1. Then the amalgam is strongly embeddable if and only if whenever $s_1 \otimes 1 = 1 \otimes s_2$ in $S_1 \otimes S_2$, then $s_1 = s_2 \in R$.

4. Extensions and amalgamations

THEOREM 4.1 [P M Cohn, 7, Theorem 4.4] Let $[R;S_1,S_2]$ be an amalgam of rings and suppose that S_i/R is flat in MOD-R, $(i = 1,2)$. Then the amalgam is strongly embeddable and $(S_1 *_R S_2)/R$ is again flat.

Proof Construct the system $(W_n,f_n)_{n \geq 1}$ as in Theorem 3.1. Notice that $g^{(1)} : S_2 \to W_2$ is 1-1 since R has the left extension property in S_1 (Corollary 2.19). Also, since S_1 is right flat and since R has the left extension property in S_2, then it is easy to check that $f_1 : S_1 \to S_1 \otimes S_2$ is a right pure monomorphism. But $S_1 \otimes_R S_2$ is flat, since both S_1 and S_2 are, and so by Theorem 2.20 we deduce that $f_n : W_n \to W_{n+1}$ is level, for all $n \geq 1$. By Theorem 3.2 the amalgam is weakly embeddable. Suppose then that $s_1 \otimes 1 = 1 \otimes s_2$ in $S_1 \otimes_R S_2$. Then by Lemma 2.2 we see that $s_2 \in R$, (take $X = R$, $Y = S_2$ and $f = i : R \to S_2$). Hence $s_1 \otimes 1 = s_2 \otimes 1$ and so

$s_2 = s_1 \in R$, since f_1 is 1-1. By Theorem 3.3, the amalgam is strongly embeddable. From Theorem 2.12 we see that $\varphi_1 : W_1 \to S_1 *_R S_2$ is right pure and since direct limits of flats are flat [Rotman, 35, Theorem 3.47] we see that $S_1 *_R S_2$ is flat. That is to say, the map $\varphi_1 : W_1 \to S_1 *_R S_2$ is level. But $R \to S_1$ $(= W_1)$ is level and so the inclusion $R \to S_1 *_R S_2$ is level and $(S_1 *_R S_2)/R$ is flat by Corollary 2.19.

By the associativity of free products with amalgamation (see Theorem I.1.6) we can deduce

THEOREM 4.2 [P M Cohn, 7, Theorem 4.5] Let $[R;S_i]$ be an amalgam of rings and suppose that S_i/R is right flat for all i. Then the amalgam is strongly embeddable.

THEOREM 4.3 [cf. Theorem III.2.1] Let $[R;S_1,S_2]$ be an amalgam of rings and suppose that R has the extension property in S_1 and S_2. Then the amalgam is strongly embeddable and R has the extension property in $S_1 *_R S_2$.

THEOREM 4.4 [cf. Corollary III.2.3] Let $[R;S_i]$ be an amalgam of rings and suppose that R has the extension property in each S_i. Then the amalgam is strongly embeddable.

From Theorem 2.4 we deduce

COROLLARY 4.5 [Cohn, 7, Theorem 4.7] Let R be a regular ring. Then R is an amalgamation base.

Theorem 2.11 gives us

COROLLARY 4.6 *Let* $[R;S_i]$ *be an amalgam of rings such that the inclusions* $R \to S_i$ *split in* $\underline{R}$-MOD-$\underline{R}$. *Then the amalgam is strongly embeddable*.

An immediate corollary of this is

COROLLARY 4.7 *Let* R *be an injective* (R,R)-*bimodule*. *Then* R *is an amalgamation base*.

THEOREM 4.8 [Cohn, 7, Theorem 5.1] *Let* (R,S) *be a weak amalgamation pair*. *Then* R *has the extension property in* S.

We can now deduce the rather surprising result

THEOREM 4.9 [cf. Theorem III.2.14] *A ring* R *is an amalgamation base if and only if it is absolutely extendable*.

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