# DIMENSION THEORY OF RANDOM SELF-SIMILAR AND SELF-AFFINE CONSTRUCTIONS 

## Sascha Troscheit

## A Thesis Submitted for the Degree of PhD at the University of St Andrews



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# Dimension Theory of Random Self-similar and Self-affine Constructions 

Sascha Troscheit



## University of <br> St Andrews

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## Declaration

I, Sascha Troscheit, hereby certify that this thesis, which contains approximately 35,000 words, has been written by me, and that it is the record of work carried out by me, or principally by myself in collaboration with others as acknowledged, and that it has not been submitted in any previous application for a higher degree. I was admitted as a research student in September 2013 and as a candidate for the degree of Doctor of Philosophy in September 2013; the higher study for which this is a record was carried out in the University of St Andrews between 2013 and 2017.

## Date

Signature of Candidate
I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate for the degree of Doctor of Philosophy in the University of St Andrews and that the candidate is qualified to submit this thesis in application for that degree.

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## Abstract

This thesis is structured as follows. Chapter 1 introduces fractal sets before recalling basic mathematical concepts from dynamical systems, measure theory, dimension theory and probability theory.

In Chapter 2 we give an overview of both deterministic and stochastic sets obtained from iterated function systems. We summarise classical results and set most of the basic notation.

This is followed by the introduction of random graph directed systems in Chapter 3, based on the single authored paper [T1] to be published in Journal of Fractal Geometry. We prove that these attractors have equal Hausdorff and upper boxcounting dimension irrespective of overlaps. It follows that the same holds for the classical models introduced in Chapter 2. This chapter also contains results about the Assouad dimensions for these random sets.

Chapter 4 is based on the single authored paper [T2] and establishes the boxcounting dimension for random box-like self-affine sets using some of the results and the notation developed in Chapter 3. We give some examples to illustrate the results.

In Chapter 5 we consider the Hausdorff and packing measure of random attractors and show that for reasonable random systems the Hausdorff measure is zero almost surely. We further establish bounds on the gauge functions necessary to obtain positive or finite Hausdorff measure for random homogeneous systems.

Chapter 6 is based on a joint article with J. M. Fraser and J.-J. Miao [FMT] to appear in Ergodic Theory and Dynamical Systems. It is chronologically the first and contains results that were extended in the paper on which Chapter 3 is based. However, we will give some simpler, alternative proofs in this section and crucially also find the Assouad dimension of some random self-affine carpets and show that the Assouad dimension is always 'maximal' in both measure theoretic and topological meanings.

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## List of Symbols

| Symbol | Description | Definition (page) |
| :---: | :---: | :---: |
| $\varnothing$ | empty set, 0 element in semiring $\beth$ |  |
| $\sqcup, \odot, \times$ | binary operations in the definition of the semiring ב | 32, 32 |
| 0 | $n$ by $n$ zero matrix | 31 |
| $0_{\varnothing}$ | $n$ by $n$ zero matrix of arrangements of words | 36 |
| 1 | $n$ by $n$ identity matrix | 31 |
| $\mathbf{1}_{\varepsilon_{0}}$ | $n$ by $n$ identity matrix of arrangements of words | 36 |
| 1 | infinite vector with entries $\mathbf{0}$ and $\mathbf{1}$ | 31 |
| $\mathbb{1}_{l}$ | finite vector of dimension $l$ with entries $\mathbf{0}$ and $\mathbf{1}$ | 31 |
| $\mathbb{1}_{\varepsilon_{0}}$ | infinite vector with arrangement of word matrix entries | 36 |
| $\alpha_{M}(e), \alpha_{m}(e)$ | length of longest and shortest edge of rectangle $e$ | 59 |
| $\mathscr{A}$ | $\sigma$-algebra of events | 3 |
| $\widetilde{\mathscr{A}}$ | $\sigma$-algebra of tail events | 7 |
| $\mathscr{B}$ | Borel $\sigma$-algebra of $\mathbb{R}$ | 4 |
| $\mathscr{B}$ (.) | Borel $\sigma$-algebra of a topological space |  |
| コ | semiring of arrangements of words | 32 |
| $\beth{ }^{\odot}$ | free monoid with generators $\mathcal{G}^{E}$, identity $\varepsilon_{0}$, and operation | 32 |
| $\beth{ }^{\square}$ | free monoid with generators $\beth^{\odot}$, identity $\varnothing$, and operation $\sqcup$ | 32 |
| コ* | set of finite arrangements of words | 32 |
| $\mathcal{C}_{b}(\mathbb{R})$ | class of continuous bounded real functions | 6 |
| $\mathbf{C}_{\varepsilon}(\omega)$ | finite approximation matrix | 47 |
| $\mathrm{C}_{\omega}^{k}$ | 1 -variable $k$-level codings | 59 |
| $\rightarrow_{\text {a.s. }}$ | convergence, almost sure | 5 |
| $\rightarrow_{p}$ | convergence, in probability | 6 |
| $\rightarrow_{d}$ | convergence, in distribution | 6 |
| [w] | cylinder of finite word $w$ | 30 |
| $d$ | dimension of ambient space, generic metric |  |
| $d_{\mathscr{H}}^{l}$ | left-sided Hausdorff distance | 15 |
| $d_{H}$ | Hausdorff metric | 15 |
| $\Delta$ | seed set (usually $[0,1]^{d}$ ) | 30 |
| dev | standard deviation | 5 |
| $\operatorname{dim}_{A}$ | Assouad dimension | 15 |
| $\operatorname{dim}_{B}$ | box-counting dimension | 12 |
| $\operatorname{dim}_{H}$ | Hausdorff dimension | 13 |
| $\operatorname{dim}_{P}$ | packing dimension | 14 |
| $\operatorname{dim}_{T}$ | topological dimension | 12 |


| $e$ | generic edge |  |
| :--- | :--- | :--- |
| $\varepsilon_{0}$ | empty word, unity element in semiring $\beth$ | 11,32 |
| ess $_{\mu}$, ess | the almost sure value (if it exists) with respect to |  |
| $\mathbb{E}$ | a measure $\mu$ |  |
| $\mathbb{E}^{g e o}$ | (arithmetic) expectation | 4 |
| $E, E(i)$, | geometric expectation | edge set of graph $i$ |


| $\omega$ | a random realisation | 3 |
| :---: | :---: | :---: |
| $\Omega$ | set of realisations, code space | 3, 11, 30 |
| (.) ${ }^{+}$ | positive part | 4 |
| $\boldsymbol{\Phi}_{\varepsilon}(s)$ | Hurchinson-Moran matrix pressure | 48 |
| $\vec{\pi}$ | probability vector | 30 |
| $\Pi_{x}, \Pi_{y}$ | projection on the horizontal and vertical, respectively | 58 |
| $P(s)$ | pressure related to the modified singular value function | 60 |
| $\mathbf{P}_{\varepsilon}^{s}(\omega)$ | Hutchinson-Moran sum like random matrix | 37 |
| $\mathbb{P}$ | probability measure on $\Omega$ | 3 |
| $\mathfrak{P}(\varepsilon)$ | $\varepsilon$-joint spectral radius | 40 |
| $\mathcal{P}(A)$ | power set of $A$ |  |
| $\begin{aligned} & \mathscr{P}^{s}(F), \\ & \mathscr{P}^{h}(F) \end{aligned}$ | $s$-packing measure, $h$-packing measure | 14 |
| $\bar{\psi}_{\omega}^{s}(e), \underline{\psi}_{\omega}^{s}(e)$ | upper/lower (random) modified singular value function | 60 |
| $\bar{\Psi}_{\omega}^{k}(s), \underline{\Psi}_{\omega}^{k}(s)$ | modified singular value sum over cylinders of length $k$ | 60 |
| $\boldsymbol{\Psi}_{\omega}(s, \varepsilon)$ | ( $s, \varepsilon$ )-pressure of realisation $\omega$ | 38 |
| $\boldsymbol{\Psi}(s, \varepsilon)$ | $(s, \varepsilon)$-pressure | 38 |
| $\mathbf{Q}_{i}^{j}$ | orthogonal matrix associated with IFS $i$ and index $j$ | 57 |
| $\mathcal{Q}$ | space of random recursive realisations | 41 |
| $\rho_{s}$ | spectral radius of a matrix | 54 |
| $\mathbb{R}$ | real numbers |  |
| $\mathfrak{R}^{s}$ | functor mapping arrangements of words to functions preserving structure | 37 |
| $S$ | (similarity) map |  |
| $s(e, F), s^{x}(F)$ | box-counting dimension of projections | 59 |
| $s_{H, \varepsilon}$ | unique unity of the ( $s, \varepsilon$ )-pressure | 38 |
| $\langle A\rangle_{\sigma}$ | $\sigma$-algebra generated by the elements of $A$ | 7 |
| $T$ | measurable transformation on $\Omega$ | 9 |
| $\mathbf{T}, \mathbf{T}(\omega)$ | infinite upper diagonal matrix matrix | 35 |
| $\tau(e)$ | terminal vertex of edge $e$ | 30 |
| $\tau$ | realisation of random recursive trees | 62 |
| $\mathcal{T}$ | set of realisations of randomly labelled trees | 62 |
| $V, V(i)$ | vertex set of graph $i$, variability of $V$-variable attractor | 30 |
| Var | variance | 5 |
| $W_{i}$ | arrangement of words associated with IFS II | 58 |
| $\mathbf{W}_{\varepsilon}^{s}$ | finite Hutchinson-Moran matrix | 47 |
| $X, X_{i}, Y$ | generic random variable | 4 |
| $\Xi_{\varepsilon}^{i}(\omega)$ | arrangement of words of concatenating $i$ matrices of stopping words | 38 |
| $\Xi_{\omega}(r)$ | 1 -variable coding stopping set consisting of words with images of diameter comparable to $r$ | 86 |

## Introduction

It goes without saying that a mathematical thesis studying a particular class of objects should start by giving a basic definition of the class under investigation. In our case: defining a fractal. This, however, is easier said than done and even though 'fractals' have been studied for several decades, there is no generally accepted definition of what constitutes one. Some authors use non-rectifiability, some define them in terms of dimension theoretic properties, whereas others describe them with the help of complex dimensions.

Since this is a rather unsatisfactory start to a thesis, we will begin with some heuristics about the properties of fractals and why their study is warranted.

Our first example is the Cantor set, named after Georg Cantor who popularised it in 1883, although he was not the first to consider it. This set is constructed in an iterative fashion where one starts with the unit line $[0,1]$ and at step $n$ removes the $2^{n-1}$ open middle thirds of the intervals from level $n-1$, see Figure 1.1. The Cantor set is the limit of this construction. More rigorously, it is the (countable) intersection of the unions of intervals in its construction.


Figure 1.1: The first levels in the construction of the Cantor set

The Cantor set was designed as a topological example of a subset of Euclidean space which is perfect (a closed set that contains no isolated point) but nowhere dense (closure has empty interior). The set is further interesting as an example of an uncountable bounded set that has zero length (one-dimensional Lebesgue measure). Similar shapes were investigated around the turn of the century and here we only mention the von Koch curve and the Sierpiński triangle (or gasket), see Figure 1.2 and 1.3, respectively.

The von Koch curve challenges our notion of a curve by having no tangent at any point, infinite length; yet is connected and contained in a bounded region of the plane. Similarly, the Sierpiński triangle is a bounded, compact set that has infinite length, yet zero area in the sense of $k$-dimensional Lebesgue measure.

These sets were investigated at first solely because they made interesting pathological examples with 'unusual' topological properties and it was not until the 1970s when the term 'fractal' was coined by Mandelbrot and fractal geometry, as a field, was born. Mandelbrot can be considered to be the first to connect the abstract works of Cantor, Hausdorff, von Koch, Weierstraß, and especially Julia and Fatou to natural phenomena such as coastlines and fractal geometry has since grown to a wide ranging discipline spanning both applied and pure mathematics. At the heart of fractal


Figure 1.2: The von Koch curve


Figure 1.3: The Sierpiński triangle
geometry are shapes or structures that share similarities on different levels. These similarities can be as simple as 'looking exactly the same on different scales' as the Cantor set, von Koch curve, and Sierpiński triangle. But many examples from nature exhibit a similar, more 'stochastic similarity' on different scales, see e.g. of lightning ${ }^{1}$ and a riverbed ${ }^{2}$ in Figure 1.4. One can easily imagine how one can alter the iterative description of the Cantor set to a more stochastic version and this lies at the heart of this thesis.


Figure 1.4: Lightning and the shores of Lake Nasser

The usual model employed for deterministic fractals are iterated function systems

[^0](IFSs), which go back to Barnsley and Demko [BD]. Each IFS has a unique associated set known as the attractor with the advantage that the IFS effectively describes the similarities on different scales. We will review the basics of iterated function systems in Chapter 2 but our main interest are random analogues. Unlike deterministic IFSs it is not immediately obvious how to define random iterated function systems (RIFS) and, historically, two randomisations were analysed: the random homogeneous, and the random recursive model. More recently, Barnsley et al. [BHS1, BHS2, BHS3] introduced the notion of $V$-variable attractors with the purpose of interpolating between the two models. A main component of this thesis is Chapter 3 where we introduce the notion of random graph directed systems (RGDS), a general model encompassing random homogeneous, $V$-variable, and random recursive RIFSs. Briefly, random graph directed systems allow us to describe random systems by changing the sub divisions in every step according to auxiliary graphs that are picked randomly. We prove some dimension theoretic results for self-similar RGDS and apply them in Chapter 4 to the projections of random self-affine sets.

Before we investigate these sets we recall the basics of probability theory, ergodic theory, and dimension theory.

### 1.1 Basic Probability Theory

We will now set basic notation, recall terminology from probability theory and state basic results, mostly without proofs. We assume some familiarity with probability theory and will follow the structure in Bauer [Bau] closely. Further material can be found in Billingsley [Bi] and Williams [W].

The general aim is to develop a model of a probabilistic experiment and a formal notion of 'independence'. We start by letting $\Omega$ denote the set of outcomes (set of realisations) of our experiment, where a realisation is generically denoted by $\omega$. We want to assign probabilities to possible events, and for this we introduce the set of events $\mathscr{A}$. We require that $\mathscr{A}$ is a $\sigma$-algebra, i.e. $\varnothing, \Omega \in \mathscr{A}$ and for any countable collection $A_{i} \in \mathscr{A}$, we must have $\bigcup_{i} A, \bigcap_{i} A_{i} \in \mathscr{A}$, and $A_{i}^{c}=\Omega \backslash A_{i} \in \mathscr{A}$. Finally, we need a way to associate probabilities to events. We do this with the set function $\mathbb{P}: \mathscr{A} \rightarrow[0,1]$ satisfying $\mathbb{P}(\varnothing)=0, \mathbb{P}(\Omega)=1$, and for any countable collection of pairwise disjoint $A_{i} \in \mathscr{A}$, that $\mathbb{P}\left(\bigcup_{i} A_{i}\right)=\sum_{i} \mathbb{P}\left(A_{i}\right)$. We call $\mathbb{P}$ a probability measure and call the triple $(\Omega, \mathscr{A}, \mathbb{P})$ a probability space.

Example 1.1.1. To model the throw of a single die we set $\Omega=\{1,2,3,4,5,6\}$ for each of the possible outcomes. We let $\mathscr{A}=\mathcal{P}(\Omega)$ be the set of all subsets of $\Omega$ and define $\mathbb{P}(A)=\# A / 6$, where $\# A$ is the cardinality of the set $A \in \mathscr{A}$. It can be easily checked that this is well defined and $(\Omega, \mathscr{A}, \mathbb{P})$ is indeed a probability space.

Having defined a simple basic probability space, like coin tosses or dice throws, we might want to expand the model to $n$ tosses. This can be done by using the $n$-fold product of the probability triple. Given $n$ probability spaces $\left\{\left(\Omega_{i}, \mathscr{A}_{i}, \mathbb{P}_{i}\right)\right\}_{i=1}^{n}$ we set $\Omega=\Omega_{1} \times \Omega_{2} \times \ldots \times \Omega_{n}, \mathscr{A}=\mathscr{A}_{1} \times \mathscr{A}_{2} \times \ldots \times \mathscr{A}_{n}$, and $\mathbb{P}=\mathbb{P}_{1} \times \ldots \times \mathbb{P}_{n}$, where $\left(\mathbb{P}_{1} \times \ldots \times \mathbb{P}_{n}\right)\left(A_{1} \times \ldots \times A_{n}\right)=\mathbb{P}_{1}\left(A_{1}\right) \cdot \ldots \cdot \mathbb{P}_{n}\left(A_{n}\right)$ for $A_{i} \in \mathscr{A}_{i}$. We write

$$
(\Omega, \mathscr{A}, \mathbb{P})=\bigotimes_{i}\left(\Omega_{i}, \mathscr{A}_{i}, \mathbb{P}_{i}\right)
$$

and this triple is indeed a probability space. In fact, given a probability space $(\Omega, \mathscr{A}, \mathbb{P})$, this can be extended to the infinite product space $\left(\Omega^{\mathbb{N}}, \mathscr{A}^{\mathbb{N}}, \widehat{\mathbb{P}}\right)$ with unique probability measure $\widehat{\mathbb{P}}$ by Kolmogorov's extension theorem.

We note that, when defining a measure, we do not need to specify the value for every possible set in the $\sigma$-algebra. It is sufficient to know the behaviour of a measure on a set that generates the $\sigma$-algebra. In particular, for the Borel $\sigma$-algebra $\mathscr{B}$ it is sufficient to know $\mathbb{P}(\mathcal{O})$, for all $\mathcal{O}$ that form a basis of a topology of $\Omega$.

Proposition 1.1.2 (Carathéodory's extension theorem). Let $R$ be a ring of subsets of $\Omega$ and $\mathbb{P}^{\prime}: R \rightarrow[0,1]$. Then there exists a unique probability measure $\mathbb{P}: \mathscr{A} \rightarrow[0,1]$ for the $\sigma$-algebra $\mathscr{A}$ generated by $R$ such that

$$
\mathbb{P}^{\prime}(A)=\mathbb{P}(A)
$$

for all $A \in R$.
When talking about events we call $\varnothing$ the impossible event and $\Omega$ the sure event. Given an event $A$ we say that $A$ is an almost sure event if $\mathbb{P}(A)=1$, and $A$ is almost impossible or trivial (with respect to $\mathbb{P}$ ) if $\mathbb{P}(A)=0$.

### 1.2 Random Variables

Having defined a probability space with events and outcomes as a model for 'experiments', we might not be interested in the outcome itself but in some form of measurement of the outcome. For example, having full knowledge of an infinite roll of dice might not be what we are after, but maybe the average of the first $n$ throws. This leads us to define the following.

Definition 1.2.1 (Borel measurable). Let $(\Omega, \mathscr{A})$ be a measurable space, i.e. a set $\Omega$ with a $\sigma$-algebra $\mathscr{A}$. We call a map $X: \Omega \rightarrow \mathbb{R}$ (Borel) measurable if $X^{-1}(B) \in \mathscr{A}$ for every $B \in \mathscr{B}$, the Borel $\sigma$-algebra of $\mathbb{R}$.

Definition 1.2.2 (Random Variable). Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space. Then every Borel measurable map $X: \Omega \rightarrow \mathbb{R}$ is called a (real) random variable or r.v. for short.

If the integral of $X$ with respect to a measure $\mathbb{P}$ exists, i.e. $\int|X| d \mathbb{P}<\infty$, we say that $X$ is $\mathbb{P}$-integrable, or just integrable if the measure is clear from context. We are interested in the values taken by this real random variable and the first notion to introduce is the 'mean' or 'expected' value.

Definition 1.2.3 ((Arithmetic) Expectation). Let $X$ be a real random variable on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$. If either $X \geq 0$ or $X$ is $\mathbb{P}$-integrable, we let

$$
\mathbb{E}(X)=\int X d \mathbb{P}=\int_{\Omega} X(\omega) d \mathbb{P}(\omega)
$$

be the expected value (or expectation or mean) of $X$.
When $\mathbb{E}(X)=0$ we say that $X$ is centred. Analogously to the arithmetic mean, we also define the geometric mean.

Definition 1.2.4 (Geometric Expectation). Let $X$ be a real random variable on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ with $X \geq 0$ and $\log X$ is $\mathbb{P}$-integrable. We use the convention that $\log 0=-\infty$ and define

$$
\mathbb{E}^{\text {geo }}(X)=\exp \left(\int \log X d \mathbb{P}\right)=\exp \left(\int_{\Omega} \log X(\omega) d \mathbb{P}(\omega)\right)
$$

as the geometric expectation (or geometric mean) of $X$.
Sometimes, especially when talking about martingales in Section 1.5, we need to talk about moments or the positive part of a random variable.

Definition 1.2.5 (Moments). Let $X$ be a real random variable and $p \geq 1$. We call $\mathbb{E}\left(|X|^{p}\right)$ the $p$-th moment of $X$, if it exists.

Definition 1.2.6 (Positive part). Let $X$ be a real random variable. The positive part is $X^{+}(\omega)=\max \{X(\omega), 0\}$.

For martingales we will also need the notion of conditional expectation.
Definition 1.2.7 (Conditional expectation). Let $X$ be a random variable and $\mathscr{A}^{\prime} \subseteq$ $\mathscr{A}$ be a $\sigma$-algebra. The conditional expectation of $X$ given $\mathscr{A}^{\prime}$, denoted by $\mathbb{E}\left(X \mid \mathscr{A}^{\prime}\right)$, is any $\mathscr{A}^{\prime}$-measurable function $\Omega \rightarrow \mathbb{R}$ satisfying

$$
\int_{A^{\prime}} \mathbb{E}\left(X \mid \mathscr{A}^{\prime}\right) d \mathbb{P}=\int_{A^{\prime}} X d \mathbb{P}
$$

for all $A^{\prime} \in \mathscr{A}^{\prime}$.
The conditional expectation can be interpreted as the expectation with the 'knowledge' of events in the $\sigma$-algebra $\mathscr{A}^{\prime}$. The 'finer' the $\sigma$-algebra, the better our prediction of the outcome. As an example, if $\mathscr{A}^{\prime}=\{\varnothing, \Omega\}$ we have no knowledge and the conditional expectation is a constant function $\mathbb{E}\left(X \mid \mathscr{A}^{\prime}\right)=\mathbb{E}(X)$. However, if $\mathscr{A}^{\prime}=\mathscr{A}$, we have 'total knowledge' and $\mathbb{E}\left(X \mid \mathscr{A}^{\prime}\right)=X$.

It is important to note that $\mathbb{P}$ is a measure on the event space $\Omega$. Given a fixed r.v. we might instead want to talk about the distribution of measurements. The distribution $\mathbb{P}_{X}$ is simply the image of the measure $\mathbb{P}$ under $X$, i.e. $\mathbb{P}_{X}(A)=\mathbb{P}(X \in A)$ for all $A \subset \mathbb{R}$, which is itself a measure.

Definition 1.2.8 (Variance \& Standard Deviation). For every integrable real random variable $X$ with mean $m=\mathbb{E}(X)$, we call

$$
\operatorname{Var}(X)=\mathbb{E}\left([X-m]^{2}\right)
$$

the variance of $X$. If $X$ is centred then the variance is equal to the second moment of $X$. The quantity $\operatorname{dev}(X)=\sqrt{\operatorname{Var}(X)}$ is known as the standard deviation of $X$.

Note that we do not refer to the standard deviation by the more common symbol $\sigma$ which we reserve for the shift operator on symbolic spaces.

Recall that a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is called convex on the interval $I \subset \mathbb{R}$ if $f(\alpha x+(1-\alpha) y) \leq \alpha f(x)+(1-\alpha) f(y)$ for all $x, y \in I$ and $\alpha \in[0,1]$. An example of such a function is $|x|^{p}$ for all $p \geq 1$. Convex functions are particularly useful because of the following well-known theorem.

Proposition 1.2.9 (Jensen's inequality). Let $X$ be an integrable random variable with values in an open interval $I \subset \mathbb{R}$. Then $\mathbb{E}(X) \in I$ and for every convex function $f, f(X)$ is a random variable and

$$
f(\mathbb{E}(X)) \leq \mathbb{E}(f(X))
$$

if $f \circ X$ is integrable.
Corollary 1.2.10. Let $X$ be a random variable. Assume the p-th moment of $X$ exists, then

$$
|\mathbb{E}(X)|^{p} \leq \mathbb{E}\left(|X|^{p}\right)
$$

The final item in this section is the question of convergence. If for every realisation $\omega \in \Omega$ we have $\lim _{i \rightarrow \infty} X_{i}(\omega)=Y(\omega) \in \mathbb{R}$ then we talk of sure convergence to the random variable $Y$, noting that it might be constant. However, this is a very strong statement to make and we require a more nuanced notion of convergence in the probabilistic setting. Often there is a 'big' set of realisations where we cannot say anything. However, we might still know that convergence happens to all but a trivial set of realisations. This is almost sure convergence.

Definition 1.2.11. We say that a sequence of random variables $\left(X_{i}\right)_{i=1}^{\infty}$ converges almost surely to the random variable $Y$ if

$$
\begin{equation*}
\mathbb{P}\left\{\limsup _{i \rightarrow \infty}\left|X_{i}-Y\right|>\varepsilon\right\}=0 \quad \text { for all } \varepsilon>0 \tag{1.2.1}
\end{equation*}
$$

The following are equivalent to (1.2.1):

$$
\begin{aligned}
\lim _{k \rightarrow \infty} \mathbb{P}\left\{\left|X_{i}-Y\right| \leq \varepsilon \text { for all } i>k\right\}=1 & \text { for all } \varepsilon>0, \\
\mathbb{P}\left\{\left|X_{i}-Y\right|>\varepsilon \text { infinitely often }\right\}=0 & \text { for all } \varepsilon>0, \\
\mathbb{P}\left\{\left|X_{i}-Y\right| \leq \varepsilon \text { for all but finitely many } i \in \mathbb{N}\right\}=0 & \text { for all } \varepsilon>0 .
\end{aligned}
$$

We may write $X_{i} \rightarrow_{a . s .} Y$ or $X_{i} \rightarrow Y$ (a.s.) for short.
However if we are interested in the mean, rather than the behaviour of individual realisations, we talk of convergence in mean.

Definition 1.2.12. Let $p \geq 1$. We say that a sequence of random variables $\left(X_{i}\right)_{i=1}^{\infty}$ converges in $p$-th mean to the random variable $Y$ if

$$
\lim _{i \rightarrow \infty} \mathbb{E}\left(\left|X_{i}-Y\right|^{p}\right)=0
$$

The special case $p=1$ is called convergence in mean. By Jensen's inequality (Proposition 1.2.9), convergence in $p$-th mean for some $p>1$ implies convergence in mean.

The next, slightly weaker, notion is convergence in probability.
Definition 1.2.13. We say that a sequence of random variables $\left(X_{i}\right)_{i=1}^{\infty}$ converges in probability to the random variable $Y$ if

$$
\lim _{i \rightarrow \infty} \mathbb{P}\left\{\left|X_{i}-Y\right| \geq \varepsilon\right\}=0 \quad \text { for all } \varepsilon>0
$$

We may write $X_{i} \rightarrow_{p} Y$ for short.
Note the difference to almost sure convergence. Here we only require that the probability tends to zero, rather than the probability of the set where $X_{i}$ does not tend to $Y$ is zero. So, almost sure convergence implies convergence in probability.

The last type of convergence we will mention requires a short diversion to the question of convergence of measures. Let $\left(\mu_{i}\right)_{i=1}^{\infty}$ be a sequence of finite (Borel) measures on $\Omega$ such that $\left(\Omega, \mu_{i} / \mu_{i}(\Omega)\right)$ are probability spaces. We say that this sequence of measures converges weakly to $\mu$, converges in weak* topology to $\mu$, or $\mu_{i} \rightarrow_{w^{*}} \mu$ if

$$
\lim _{i \rightarrow \infty} \int f d \mu_{i}(\omega)=\int f d \mu(\omega)
$$

for every $f \in \mathcal{C}_{b}(\Omega, \mathbb{R})$, where $\mathcal{C}_{b}(\Omega, \mathbb{R})$ denotes the class of bounded, continuous real functions.

Definition 1.2.14. We say that a sequence of random variables $\left(X_{i}\right)_{i=1}^{\infty}$ converges in distribution to the random variable $Y$ if the sequence of distributions $\mathbb{P}_{X_{i}}$ converges weakly to $\mathbb{P}_{Y}$. We may write this as $X_{i} \rightarrow_{d} Y$.

As mentioned above these are progressively weaker notions of convergence and we get the following implications.

$$
\begin{array}{llc}
X_{i} \rightarrow_{\text {a.s. }} Y & \Longrightarrow & X_{i} \rightarrow_{p} Y \\
\text { gence in } p \text {-th mean } & \Longrightarrow & \Longrightarrow \text { convergence in mean }
\end{array}
$$

### 1.3 Probabilistic Laws

Before we delve into probabilistic laws, we discuss independence. Two events $A, B$ are said to be independent if the knowledge of the event $B$ does not affect the probability of the event $A$. That is, $\mathbb{P}(A \mid B)=\mathbb{P}(A)$, where

$$
\mathbb{P}(A \mid B)=\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}
$$

for $\mathbb{P}(B)>0$ and $\mathbb{P}(A \mid B)=0$ otherwise. Extending this notion slightly, we say that a sequence of events $A_{i}$ indexed by $i \in \mathcal{I}$ (not necessarily finite) is independent if

$$
\begin{equation*}
\mathbb{P}\left(\bigcap_{i \in \mathcal{I}^{\prime}} A_{i}\right)=\prod_{i \in \mathcal{I}^{\prime}} \mathbb{P}\left(A_{i}\right) \tag{1.3.1}
\end{equation*}
$$

for all finite non-empty subsets $\mathcal{I}^{\prime} \subseteq \mathcal{I}$.
We extend this notion to include sets of events, so that we can talk about $\sigma$ algebras being independent.

Definition 1.3.1 (independent families). Let $E_{i} \subset \mathscr{A}$ for $i \in \mathcal{I}$ be a family of sets of events. The family is independent if (1.3.1) holds for all $A_{i} \in E_{i}$ for all $i \in \mathcal{I}^{\prime}$ and all finite non-empty subsets $\mathcal{I}^{\prime} \subseteq \mathcal{I}$.

Definition 1.3.2 (tail events). Let $\mathscr{A}_{n} \subseteq \mathscr{A}$ for $n \in \mathbb{N}$ be a sequence of $\sigma$-algebras. Let $\widetilde{\mathscr{A}_{n}}$ be the $\sigma$-algebra generated by $\bigcup_{m \geq n} \mathscr{A}_{m}$, which we denote by

$$
\widetilde{\mathscr{A}_{n}}=\left\langle\bigcup_{m \geq n} \mathscr{A}_{m}\right\rangle
$$

Then $\widetilde{\mathscr{A}_{\infty}}=\bigcap_{n \in \mathbb{N}} \widetilde{\mathscr{A}_{n}}$ is called the $\sigma$-algebra of tail events.
This leads us to a law of great importance.
Theorem 1.3.3 (Kolmogorov zero-one law). Let $\left(\mathscr{A}_{n}\right)_{n \in \mathbb{N}}$ be an independent sequence of $\sigma$-algebras. Then for all $A \in \widetilde{\mathscr{A}_{\infty}}$ either $\mathbb{P}(A)=0$ or $\mathbb{P}(A)=1$.

Colloquially speaking, if the event does not depend on a finite number steps, it is almost sure or almost impossible.

We extend our definition to random variables.
Definition 1.3.4 (independent random variables). A family of random variables $\left(X_{i}\right)_{i \in \mathcal{I}}$ on a common probability space $(\Omega, \mathscr{A}, \mathbb{P})$ is said to be independent if the family of $\sigma$-algebras $\left(\left\langle X_{i}\right\rangle\right)_{i \in \mathcal{I}}$ is independent, where $\left\langle X_{i}\right\rangle$ is the smallest $\sigma$-algebra such that $X_{i}$ is measurable.

This leads to the intuitive and well-known multiplication theorem.
Theorem 1.3.5. Let $X_{1}, \ldots, X_{n}$ be independent, non-negative and $\mathbb{E}\left(X_{i}\right)<\infty$, then

$$
\mathbb{E}\left(\prod_{i=1}^{n} X_{i}\right)=\prod_{i=1}^{n} \mathbb{E}\left(X_{i}\right)
$$

Observe that $\mathbb{E}\left(\sum X_{i}\right)=\sum \mathbb{E} X_{i}$ by linearity of $\mathbb{E}$, irrespective of independence. Similarly, $\mathbb{E}^{\text {geo }}\left(\prod X_{i}\right)=\prod \mathbb{E}^{\text {geo }} X_{i}$, irrespective of independence.

Let $\left(X_{i}\right)_{i \in \mathbb{N}}$ be a sequence of random variables with finite expectation. One of the first questions one might ask is of the long term behaviour of the outcome of the trials. Thinking back to dice we want to be able to say that 'in the limit' the average die roll will be $7 / 2$. We are interested in whether the long term average behaviour coincides with the mean, i.e.

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n}\left(X_{i}-\mathbb{E}\left(X_{i}\right)\right)=0 \quad \text { (a.s.). }
$$

A sequence $\left(X_{i}\right)$ for which this holds almost surely is said to satisfy the strong law of large numbers.

Theorem 1.3.6 (Kolmogorov's strong law of large numbers). Let $\left(X_{i}\right)$ be a sequence of independent and identically distributed random variables with finite expectation $m_{i}=\mathbb{E}\left(X_{i}\right)$. Then

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n}\left(X_{i}-m_{i}\right)=0 \quad \text { (a.s.). }
$$

Standard proofs usually use the Borel-Cantelli lemma which is very useful in its own right. We state it as a theorem below.

Theorem 1.3.7 (Borel-Cantelli). Let $A_{i}$ be a sequence of events. Then,

$$
\sum_{i=1}^{\infty} \mathbb{P}\left(A_{i}\right)<\infty \quad \Longrightarrow \quad \mathbb{P}\left(\left\{A_{n} \text { i.o. }\right\}\right)=0
$$

Additionally, if the events $A_{i}$ are pairwise independent,

$$
\sum_{i=1}^{\infty} \mathbb{P}\left(A_{i}\right)=\infty \quad \Longrightarrow \quad \mathbb{P}\left(\left\{A_{n} \text { i.o. }\right\}\right)=1
$$

Knowing what happens to the average outcome might not be enough. Under some mild conditions on the r.v. $X$, we can say something about the distribution of $\sum X_{i}$. Let $\mathscr{N}(a, v)$ be the normal distribution with mean $a$ and variance $v$. Equivalently it is the measure on $\mathbb{R}$ given by

$$
\mathscr{N}(a, v)(A)=\frac{1}{\sqrt{2 v \pi}} \int_{A} \exp \left(-\frac{(x-a)^{2}}{2 v}\right) d x
$$

Theorem 1.3.8 (Central limit theorem (CLT)). Let $\left(X_{i}\right)_{i \in \mathbb{N}}$ be an i.i.d. sequence of independent random variables with finite and positive variance. Let $m=\mathbb{E}\left(X_{1}\right)$ and $v=\operatorname{Var}\left(X_{1}\right)$. Then

$$
\frac{\sum_{i=1}^{n} X_{i}-n m}{\sqrt{n v}} \rightarrow_{d} \mathscr{N}(0,1)
$$

and we say that the central limit theorem (CLT) holds for $\left(X_{i}\right)_{i \in \mathbb{N}}$.
Thus, scaling $Y_{n}=\sum_{i=1}^{n} X_{i}$ by $n$, the scaled sum converges almost surely to the mean. Scaling by $\sqrt{n}$ instead, the scaled sum converges in distribution to a normal distribution. We can use the latter result to say, for example that for an i.i.d. sequence of random variables and any $K \in \mathbb{R}$,

$$
\left\{\omega \in \Omega \mid \sum_{i=1}^{n} X_{i} \leq K \sqrt{n} \text { for infinitely many } n \in \mathbb{N}\right\}
$$

has full measure. This follows from the CLT and the Borel-Cantelli Lemma as there exists $\varepsilon>0$ and integer $m(\varepsilon, x, n)$ such that the probability of an excursion to less than $K \sqrt{m(\varepsilon, x, n)+n}$ in $m(\varepsilon, x, n)$ steps starting at $x$ has probability greater than $\varepsilon$. We are interested in obtaining a sharper bound on these exceptional excursions and ask: Given a sequence of independent variables with positive and finite variance, does there exist a sequence of real numbers $\left(a_{i}\right)_{i \in \mathbb{N}}$ such that,

$$
\begin{equation*}
\mathbb{P}\left\{\sum_{i=1}^{n} X_{i} \geq(1-\varepsilon) a_{n} \text { infinitely often }\right\}=1 \tag{1.3.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbb{P}\left\{\sum_{i=1}^{n} X_{i} \geq(1+\varepsilon) a_{n} \text { infinitely often }\right\}=0 \tag{1.3.3}
\end{equation*}
$$

for all $\varepsilon>0$ ? This is answered by the law of the iterated logarithm.

Theorem 1.3.9 (Law of the iterated logarithm (LIL)). Let $\left(X_{i}\right)_{i \in \mathbb{N}}$ be a sequence of i.i.d. random variables with mean $m$, and positive and finite variance $v$. Then

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \frac{\sum_{i=1}^{n} X_{i}-n m}{\sqrt{2 n \log \log n}}=\sqrt{v} \tag{1.3.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\liminf _{n \rightarrow \infty} \frac{\sum_{i=1}^{n} X_{i}-n m}{\sqrt{2 n \log \log n}}=-\sqrt{v} \tag{1.3.5}
\end{equation*}
$$

almost surely.
Thus, for $a_{n}=\sqrt{2 n \operatorname{Var}\left(X_{1}\right) \log \log n}$, equations (1.3.2) and (1.3.3) hold. We note that this is not the only sequence for which the conclusion above holds. Any sequence $a_{n}^{\prime}$ such that $\lim _{n}\left(a_{n} / a_{n}^{\prime}\right)=1$ will give the same result. In particular we could have set $a_{n}=\sqrt{2 n \operatorname{Var}(X) \log \log (\operatorname{Var}(X) n)}$. Since $\log \log n$ is not defined for $n \leq 1$ and only greater than 1 for $n \geq 16$, we use the convention that $\log \log x=1$ for $x<e^{e}$ when dealing with the law of the iterated logarithm as we are only concerned with the asymptotic behaviour and want to avoid giving conditions on $x$ for these expressions to be well-defined.

### 1.4 Ergodic Theorems and Subadditivity

We will now look at our probability space from a more dynamical point of view. Let $T: \Omega \rightarrow \Omega$ be a measurable map. We call $T$ measure preserving if $\mathbb{P}\left(T^{-1}(A)\right)=\mathbb{P}(A)$ for every measurable set $A$. If, further, the only sets invariant under $T$ are measure theoretically trivial or their complement is trivial, i.e.

$$
T^{-1}(A)=A \Longrightarrow \mathbb{P}(A)=0 \text { or } 1
$$

we call $T$ ergodic. Intuitively, a map is ergodic if the dynamics it describes are the same on the entire set the measure can see. A major result about ergodic maps is the (pointwise) Birkhoff's ergodic theorem (BET)

Theorem 1.4.1 (Birkhoff's ergodic theorem (BET)). Let $T: \Omega \rightarrow \Omega$ be a measurable, measure preserving, ergodic map. Let $X: \Omega \rightarrow \mathbb{R}$ be a measurable, integrable, real valued function, then

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} X\left(T^{i}(\omega)\right)=\mathbb{E}(X)
$$

for almost every $\omega \in \Omega$.
Thus if $T$ is an ergodic map the average measurement under iteration of $T$ tends to the global spatial average. Compare this with the law of large numbers discussed earlier.

Let $\mathbf{a}=\left(a_{i}\right)_{i \in \mathbb{N}}$ be a real valued sequence. We say that $\mathbf{a}$ is additive if $a_{i+j}=a_{i}+a_{j}$ for all $i, j \in \mathbb{N}$ and subadditive if $a_{i+j} \leq a_{i}+a_{j}$. A basic result, known as Fekete's Lemma, states that the running average of subadditive sequences converges.

Theorem 1.4.2 (Fekete's Lemma). Let $\mathbf{a}=\left(a_{i}\right)_{i \in \mathbb{N}}$ be a subadditive sequence. Then

$$
\frac{a_{n}}{n} \rightarrow \inf _{i}\left\{\frac{a_{i}}{i}\right\} \in[-\infty, \infty) \quad \text { as } \quad n \rightarrow \infty
$$

Note that the limit may be $-\infty$, so under the additional condition that $\inf _{i} a_{i} / i>-\infty$ the limit of $a_{n} / n$ exists.

We recall a standard proof, see e.g. [F4, Proposition 1.1] or [PS, §1 Problem 98], that is used in the derivation of a stronger statement later.

Proof. Fix an integer $p$. Then $n$ can be rewritten as $n=k p+q$ for $k \in \mathbb{N}_{0}$ and $q \in\{0,1, \ldots, p-1\}$. Thus

$$
\frac{a_{n}}{n}=\frac{a_{k p+q}}{k p+q} \leq \frac{a_{k p}+a_{q}}{k p+q} \leq \frac{k a_{p}+a_{q}}{k p} \leq \frac{a_{p}}{p}+\frac{a_{q}}{k} .
$$

As $k$ (and thus $n$ ) increases, the second term becomes negligible. Combining this with the arbitrariness of $p$ we conclude that $\lim a_{n} / n \leq \inf _{p} a_{p} / p$. However, $\inf _{p} a_{p} / p$ is an obvious lower bound to the limit and the result follows.

There is a similar result in the probabilistic setting for random variables known as Kingman's subadditive ergodic theorem $[\mathrm{K}]$.

Theorem 1.4.3 (Kingman's subadditive ergodic theorem). Let $\left\{X_{m, n}\right\}$ be a collection of random variables indexed by $m, n \in \mathbb{N}_{0}$. Assume that

- $X_{k, p} \leq X_{k, l}+X_{l, p}$ whenever $0 \leq k<l<p$,
- the joint distributions of $\left\{X_{m+1, n+1}, 0 \leq m<n\right\}$ are the same as those of $\left\{X_{m, n}, 0 \leq m<n\right\}$ for all $n$,
- for each $n, \mathbb{E}\left|X_{0, n}\right|<\infty$ and $\mathbb{E} X_{0, n} \geq$ cn for some uniform $c>0$,
- for each $k \geq 1$, the process $\left\{X_{n k,(n+1) k}, n \geq 1\right\}$ is stationary and ergodic.

Then $\lim _{n \rightarrow \infty} X_{0, n} / n=\lim _{n \rightarrow \infty}(1 / n) \mathbb{E}\left(X_{0, n}\right)=\inf _{i}(1 / i) \mathbb{E}\left(X_{0, i}\right) \in \mathbb{R}$ almost surely.
However, instead of discussing the result above we state an improvement due to Derriennic.

Proposition 1.4.4 (Derriennic, [D]). Let $X_{m}(\omega)$ be a (measurable) random variable on a probability space $(\Omega, \mathscr{A}, \mu)$ and let $T$ be a measurable, measure preserving map. If the expectation of the subadditive defects is bounded by a sequence of real numbers $\left(c_{m}\right)$, i.e. for all $n, m$,

$$
\mathbb{E}\left(X_{n+m}(\omega)-X_{n}(\omega)-X_{m}\left(T^{n} \omega\right)\right)^{+} \leq c_{m}
$$

where $c_{m}$ satisfies $\lim _{k} c_{k} / k \rightarrow 0$, and $\mathbb{E}_{\inf } X_{k} / k>-\infty$, then $X_{n} / n$ converges in $\mathscr{L}^{1}$ to some random variable $\eta$ taking values in $\mathbb{R}$. If further,

$$
X_{n+m}(\omega)-X_{n}(\omega)-X_{m}\left(T^{n} \omega\right) \leq Y_{m}\left(T^{n} \omega\right) \quad \text { (almost surely) }
$$

for some stochastic process $\left(Y_{m}\right)_{m}$ satisfying $\sup \mathbb{E}\left(Y_{m}\right)<\infty$, then $X_{n} / n \rightarrow \eta$ almost surely.

If $T$ is ergodic with respect to $\mathbb{P}$, then $\eta$ takes a constant value for almost every $\omega$ as

$$
\left\{\omega \in \Omega \mid \liminf _{n \rightarrow \infty} X_{n}(\omega) / n>z\right\}=\left\{\omega \in \Omega \mid \liminf _{n \rightarrow \infty} X_{n}(T \omega) / n>z\right\}
$$

If it exists, the almost sure limit necessarily coincides with

$$
\lim _{k} \frac{\mathbb{E}\left(X_{k}\right)}{k}=\inf _{k} \frac{\mathbb{E}\left(X_{k}\right)}{k}
$$

### 1.5 Martingales

In this section we define the discrete-time martingale and state two convergence results. Assume we are given a stochastic process, i.e. a sequence of random variables, $\left(X_{i}\right)_{i}$. Informally, a martingale is a process where, given information about all previous outcomes, the expectation of the next outcome is the current value.

Definition 1.5.1. A discrete stochastic process $\left(X_{i}\right)_{i}$ is called a (discrete-time) martingale if all $X_{i}$ are integrable and

$$
\mathbb{E}\left(X_{i+1} \mid\left\langle X_{1}, X_{2}, \ldots, X_{i}\right\rangle\right)=X_{i} \quad \text { for all } i .
$$

Equivalently,

$$
\mathbb{E}\left(X_{i+1}-X_{i} \mid\left\langle X_{1}, X_{2}, \ldots, X_{i}\right\rangle\right)=0 \quad \text { for all } i
$$

Similarly, a stochastic process is called a submartingale if it satisfies

$$
\mathbb{E}\left(X_{i+1} \mid\left\langle X_{1}, X_{2}, \ldots, X_{i}\right\rangle\right) \geq X_{i} \quad \text { for all } i
$$

and $a$ supermartingale if

$$
\mathbb{E}\left(X_{i+1} \mid\left\langle X_{1}, X_{2}, \ldots, X_{i}\right\rangle\right) \leq X_{i} \quad \text { for all } i
$$

We note that every martingale is also a supermartingale and submartingale. The first convergence result we mention is due to Doob about pointwise convergence and can be found in [W].

Theorem 1.5.2. Let $\left(X_{i}\right)$ be a supermartingale such that $\inf _{i} X_{i}>-\infty$. Then $X=\lim _{i \rightarrow \infty} X_{i}$ exists and is finite almost surely.

Corollary 1.5.3. Let $\left(X_{i}\right)$ be a supermartingale and $X_{i} \geq 0$ for all $i$. Then $X_{i}$ converges pointwise almost surely.

For uniform convergence we need stronger assumptions and state another result by Doob, see also [W].

Theorem 1.5.4. Let $\left(X_{i}\right)$ be a supermartingale such that $\sup _{i} \mathbb{E}\left|X_{i}\right|^{p}<\infty$ for some $p>1$. Then there exists a random variable $X \in \mathscr{L}^{p}(\Omega, \mathbb{R})$ such that

$$
X_{i} \rightarrow_{a . s .} X \quad \text { and } \quad \int_{\Omega}\left|X_{i}-X\right|^{p} d \mathbb{P} \rightarrow 0
$$

In particular, $\mathbb{E}\left(X_{i}\right) \rightarrow \mathbb{E}(X)$.

### 1.6 Coding Spaces

We now introduce and discuss a very important example of a probability space, the code space $\Omega=\Lambda^{\mathbb{N}}$. Let $\Lambda$ be a finite index set, e.g. $\Lambda=\{1,2, \ldots, N\}$ for some $N \in \mathbb{N}$. We call $\lambda \in \Lambda$ letters and associate a non-trivial probability measure $\mu$ with $\Lambda$, meaning $\mu(\{\lambda\})>0$ for all $\lambda \in \Lambda$. We set $\mathscr{A}_{\Lambda}=\mathcal{P}(\Lambda)$ and write

$$
\Lambda^{k}=\underbrace{\Lambda \times \Lambda \times \ldots \times \Lambda}_{k \text { times }}
$$

for sequences of letters of length $k$. These sequences are called words or codings of length $k$. We call the word of length zero the empty word and denote it by $\varepsilon_{0}$. The set of all finite words is $\Lambda^{*}=\left\{\varepsilon_{0}\right\} \cup \bigcup_{k>0} \Lambda^{k}$ and the set of all infinite words is $\Lambda^{\mathbb{N}}$. This set is the code space we were looking for. It models a sequence of weighted die rolls with faces labelled by a finite collection of letters. While words are formally finite or infinite sequences, we usually concatenate the letters into a single word for readability, so $\omega=\left(\omega_{1}, \omega_{2}, \ldots\right) \in \Omega$ becomes $\omega_{1} \omega_{2} \ldots$ and $\lambda=\left(\lambda_{1}, \ldots, \lambda_{k}\right) \in \Lambda^{*}$ becomes $\lambda=\lambda_{1} \lambda_{2} \ldots \lambda_{k}$. We write $|\lambda|=k$ for the (possibly infinite) length of the word $\lambda \in \Lambda^{k}$. For $x \in \Omega \cup \Lambda^{*}$ write $\left.x\right|_{k}=x_{1} \ldots x_{i}$ with $i=\min \{|x|, k\}$ and we write $\underline{\lambda}=\lambda \lambda \lambda \cdots \in \Lambda^{\mathbb{N}}$. For the remainder of this section let $\Omega=\Lambda^{\mathbb{N}}$ and set $\mathscr{A}=\mathscr{A}_{\Lambda}^{\mathbb{N}}$ and $\mathbb{P}=\mu^{\mathbb{N}}$ to be the natural product $\sigma$-algebra and measure. As discussed earlier, $\mathbb{P}$ is unique, and called the Bernoulli measure induced by $\mu$.

We can define this measure in an alternative way by first considering subsets of $\Omega$ called cylinders.

Definition 1.6.1 (Cylinders). Let $\Omega=\Lambda^{\mathbb{N}}$ and $\lambda \in \Lambda^{*}$. The cylinder of $\lambda$ is defined as

$$
[\lambda]=\left[\lambda_{1} \lambda_{2} \ldots \lambda_{|\lambda|}\right]=\left\{\omega \in \Omega:\left.\omega\right|_{|\lambda|}=\lambda\right\} .
$$

We endow the code space with the metric $d(x, y)=2^{-k}$, where $k=\min \left\{i: x_{i} \neq\right.$ $\left.y_{i}\right\}-1$ for $x \neq y$ and $d(x, y)=0$ if $x=y$. It can be checked that the topology of $\Omega$ can also be generated by the cylinder sets as the basic clopen elements.

We now define a real valued function on the collection $\mathscr{C}$ of all cylinders by $\widehat{\mu}$ : $\mathscr{C} \rightarrow[0,1]$ where $\widehat{\mu}\left(\left[\lambda_{1} \ldots \lambda_{k}\right]\right)=\mu\left(\lambda_{1}\right) \cdot \ldots \cdot \mu\left(\lambda_{k}\right)$. This function satisfies all the axioms of a measure on the cylinders and so, using Carathéodory's extension theorem (Proposition 1.1.2) $\widehat{\mu}$ extends to a unique measure $\mathbb{P}$ on Borel subsets of $\Omega$.

### 1.7 Dimension Theory

The main tool used in this thesis to study random sets is dimension theory to study the geometric scaling properties of sets, measures and other structures.

A classical notion of dimension is the topological dimension, sometimes referred to as the Lebesgue covering dimension. It is one of several proposed ways of defining a dimension that is invariant under homeomorphisms, a modern definition of which can be found in Munkres $[\mathrm{Mu}, \S 50]$. Given a topological space $\mathscr{T}$ and an open cover $\mathcal{U}=\left\{U_{i}\right\}$, a refinement is a new cover $\mathscr{O}=\left\{\mathcal{O}_{i}\right\}$ such that for every $i$ there exists a $j$ such that $\mathcal{O}_{i} \subseteq U_{j}$. The order of a cover $\mathscr{O}$ is the number $n \in \mathbb{N}_{0}$ such that there exists $x \in \mathscr{T}$ that is contained in $n$ elements of $\mathscr{O}$ and no $x \in \mathscr{T}$ is contained in more than $n$ elements of $\mathscr{O}$.

Definition 1.7.1. A topological space $\mathscr{T}$ has topological dimension

$$
\operatorname{dim}_{T} \mathscr{T}=n \in \mathbb{N}_{0} \cup\{-1, \infty\}
$$

if for every open cover of $\mathscr{T}$ there is a refinement which has order $n+1$ and $n$ is the least integer for which this holds. If there is no such $n$, then $\mathscr{T}$ is infinite dimensional.

The topological dimension of $\mathbb{R}^{d}$ is $d$, as one would expect, and for classical geometrical shapes like circles, spheres, cubes, tori, and their higher dimensional analogues, the topological dimension coincides with our intuitive notion of dimension.

Fractals, however, typically have topological dimension strictly less than their ambient space. For example, the topological dimension of the Menger sponge is famously 1, even though the standard construction is embedded in three dimensional Euclidean space. In fact, Menger introduced the Menger sponge in 1926 while studying topological dimension and proved that every (topological) curve, i.e. every compact metric space of topological dimension one, is homeomorphic to a subset of the Menger sponge. The Menger sponge is therefore sometimes also called a universal curve.

### 1.7.1 Box-counting dimension

Our first dimension that combines analysis and geometry is the box-counting dimension. Given a subset $E$ of a metric space, we define $N_{\delta}$ to be the least number of balls with radii less than $\delta$ necessary to cover $E$. Note that the existence of $N_{\delta}$ is guaranteed if the ambient metric space is totally bounded.

Definition 1.7.2 (box-counting dimension). Let $E \subset M$ be a totally bounded subset of a complete metric space $M$. The upper box counting dimension and lower box counting dimension are defined, respectively, by

$$
\overline{\operatorname{dim}}_{B}(E)=\limsup _{\delta \rightarrow 0} \frac{\log N_{\delta}(E)}{-\log \delta}
$$

and

$$
\underline{\operatorname{dim}}_{B}(E)=\liminf _{\delta \rightarrow 0} \frac{\log N_{\delta}(E)}{-\log \delta}
$$

If $\operatorname{dim}_{B} E=\underline{\operatorname{dim}}_{B} E$ we will refer to the box-counting dimension $\operatorname{dim}_{B} E$.
If $E \subset \mathbb{R}^{d}$,the definition does not change if $N_{\delta}(E)$ is substituted by $M_{\delta}(E)$, the number of grid squares in a $\delta$-mesh that intersect $E$. We make use of this fact at various stages. More equivalent definitions of box-counting dimension can be found in Falconer [F6]. Note also that the Minkowski dimension is equivalent to the boxcounting dimension and we will omit its definition.

Example 1.7.3. Consider the countable set $M=\{1 / n: n \in \mathbb{N}\} \cup\{0\}$. Then $\operatorname{dim}_{B}(M)=1 / 2$, see [F6, Example 2.7].

### 1.7.2 Hausdorff measure and dimension

An arguably more interesting notion is the Hausdorff dimension which is defined via the Hausdorff measure and arises from studying the geometry of open covers of sets.

A $\delta$-cover of a set $F \subseteq \mathbb{R}^{d}$ is a countable collection of sets $\left\{U_{i}\right\}$ such that their diameter satisfies $\left|U_{i}\right|<\bar{\delta}$ and

$$
F \subseteq \bigcup_{i} U_{i} .
$$

Definition 1.7.4. Let $F \subseteq \mathbb{R}^{d}$ and $s \in \mathbb{R}_{0}^{+}$, we define the s-dimensional Hausdorff $\delta$-premeasure of $F$ by

$$
\mathscr{H}_{\delta}^{s}(F)=\inf \left\{\sum_{k=1}^{\infty}\left|U_{k}\right|^{s} \mid\left\{U_{i}\right\} \text { is a } \delta \text {-cover of } F\right\}
$$

where the infimum is taken over all countable $\delta$-covers. The s-dimensional Hausdorff measure ${ }^{3}$ of $F$ is then

$$
\mathscr{H}^{s}(F)=\lim _{\delta \rightarrow 0} \mathscr{H}_{\delta}^{s}(F)
$$

The Hausdorff dimension is defined to be

$$
\operatorname{dim}_{H} F=\inf \left\{s \mid \mathscr{H}^{s}(F)=0\right\}
$$

As can readily be seen, this definition is somewhat more involved than that of the box-counting dimension. In fact, our previous example $M=\{1 / n: n \in \mathbb{N}\} \cup\{0\}$ has $\operatorname{dim}_{H} M=0$.

We note that the $t$-Hausdorff measure is 0 for $t$ strictly greater than the Hausdorff dimension and is infinite for $t$ strictly less. The situation at the critical exponent is less clear. In fact, $\mathscr{H}^{\operatorname{dim}_{H}(F)}(F)$ can take any value in $[0, \infty]$. Intuitively this can be interpreted as the geometry not scaling with an exact exponent, but with some additional slow effects. To capture these, the $s$-Hausdorff measure can be extended to use more general gauge function (also called dimension functions). A gauge function is a left continuous, non-decreasing function $h: \mathbb{R}_{0}^{+} \rightarrow \mathbb{R}_{0}^{+}$such that $h(r) \rightarrow 0$ as $r \rightarrow 0$. If there exists a constant $\lambda>1$ such that for $x>0$ we have $f(2 x) \leq \lambda f(x)$ then we say that $f$ is doubling.
Definition 1.7.5. Let $F \subseteq \mathbb{R}^{d}$ and let $h$ be a gauge function. Then the $h$-Hausdorff $\delta$-premeasure of $F$ is

$$
\mathscr{H}_{\delta}^{h}(F)=\inf \left\{\sum_{k=1}^{\infty} h\left(\left|U_{k}\right|\right) \mid\left\{U_{i}\right\} \text { is a } \delta \text {-cover of } F\right\}
$$

[^1]where the infimum is taken over all countable $\delta$-covers. The $h$-Hausdorff measure of $F$ is then
$$
\mathscr{H}^{h}(F)=\lim _{\delta \rightarrow 0} \mathscr{H}_{\delta}^{h}(F)
$$

Obtaining upper bounds for Hausdorff measure and dimensions usually relies on constructing an efficient cover, while lower bounds often make use of the following simple result called the mass distribution principle.

Theorem 1.7.6. Let $\mu$ be a finite measure supported on $F$ and suppose that for some gauge function $h$ there are constants $c>0$ and $r_{0}$ such that $\mu(U) \leq \operatorname{ch}(|U|)$ for all sets $U$ with $|U|<r_{0}$. Then $\mathscr{H}^{h}(F) \geq \mu(F) / c$.

While the proof can be found in a number of places, we recall it for completeness.
Proof. Consider any countable open cover $\left\{O_{i}\right\}$ of $F$. Then

$$
\mu(F) \leq \mu\left(\bigcup_{i} O_{i}\right) \leq \sum_{i} \mu\left(O_{i}\right) \leq c \sum_{i} h\left(\left|O_{i}\right|\right)
$$

But then, taking the infimum for each $\delta>0$, we have $\mathscr{H}^{h}(F) \geq \mathscr{H}_{\delta}^{h}(F) \geq \mu(F) / c$.

This can then be used to obtain lower bounds for the Hausdorff dimension of $F$ by considering $h(r)=r^{s}$ and constructing some finite measure $\mu$ on $F$.

### 1.7.3 Packing measure and dimension

The packing measure was introduced in the late 1970s as the natural dual to the Hausdorff dimension, see Saint Raymond and Tricot [RT] for an early study. While we will not directly work with packing measure and dimension in most cases, we briefly recall their definition.

Definition 1.7.7. Let $F \subseteq \mathbb{R}^{d}$ and $s \geq 0$. Define

$$
\left.\begin{array}{rl}
\mathscr{P}_{\delta}^{s}(F)=\sup \left\{\sum_{i}\left|B_{i}\right|^{s} \mid\left\{B_{i}\right\}\right. \text { is a countable } \\
& \text { collection of disjoint balls centered in } F \text { with radii } r_{i} \leq \delta
\end{array}\right\}
$$

and set $\mathscr{P}_{0}^{s}(F)=\lim _{\delta \rightarrow 0} \mathscr{P}_{\delta}^{s}(F)$. The packing measure is

$$
\mathscr{P}^{s}(F)=\inf \left\{\sum_{i=1}^{\infty} \mathscr{P}_{0}^{s}\left(F_{i}\right) \mid \text { where } F \subseteq \bigcup_{i=1}^{\infty} F_{i}\right\}
$$

Similarly to the Hausdorff dimension, the packing dimension is defined by the abscissa $\operatorname{dim}_{P} F=\inf \left\{s \mid \mathscr{P}^{s}(F)=0\right\}$ and the definition can be suitably extended to $h$-packing measure for each gauge function $h$.

The reason we do not use the packing measure explicitly is that in most of the settings we consider, the packing dimension coincides with the upper box-counting dimension. This will either follow directly from the inequality $\operatorname{dim}_{H} E \leq \operatorname{dim}_{P} E \leq$ $\overline{\operatorname{dim}}_{B} E$ for bounded $E \subset \mathbb{R}^{k}$ or from the following theorem.

Theorem 1.7.8. Let $F \subset \mathbb{R}^{k}$ be compact such that $\overline{\operatorname{dim}}_{B} F \cap \mathcal{O}=\overline{\operatorname{dim}}_{B} F$ for all open $\mathcal{O} \subseteq \mathbb{R}^{d}$ with $F \cap \mathcal{O} \neq \varnothing$. Then $\operatorname{dim}_{P} F=\overline{\operatorname{dim}}_{B} F$.

For a proof see [F6, Corollary 3.10].

### 1.7.4 Assouad dimension

The last dimension we define is the Assouad dimension, introduced by the French mathematician Patrice Assouad in the 1970s [As1, As2]. Assouad's original motivation was to study embedding problems, a subject where the Assouad dimension is still playing a fundamental rôle, see [Ol, OR, Ro]. The concept has also found a place in other areas of mathematics, including the theory of quasi-conformal mappings [ $\mathrm{He}, \mathrm{Lu}, \mathrm{MT}]$, and more recently it is gaining substantial attention in the literature on fractal geometry [AT, Fr2, FHOR, FY, GHM, Ho, KLV, LLMX, M, O2, ORN]. It is also worth noting that, due to its intimate relationship with tangents, it has always been present in the pioneering work of Furstenberg on micro-sets and the related ergodic theory which goes back to the 1960s, see [Fu]. The Assouad dimension also plays a rôle in the fractional Hardy inequality. If the boundary of a domain in $\mathbb{R}^{d}$ has Assouad dimension less than or equal to $d-p$, then the domain admits the fractional $p$-Hardy inequality [A, KZ, LT]. The Assouad dimension gives a coarse and heavily localised description of how 'thick' a given metric space is on small scales; hence its importance for embedding problems. Most of the other popular notions of dimension, like the Hausdorff, packing, or box-counting dimension, give much more global information, taking an 'average thickness' over the whole set. As such, exploring and understanding the relationships, similarities, and differences, between the Assouad dimension and the other global dimensions is of high and increasing interest, and is one of the themes of this thesis.

Definition 1.7.9 (Assouad dimension). Let $(\mathcal{X}, d)$ be a metric space and let $N_{r}(F)$ be the smallest number of sets with diameter less than or equal to $r$ required to cover $F$. The Assouad dimension of a non-empty subset $F$ of $\mathcal{X}$ is given by

$$
\begin{aligned}
& \operatorname{dim}_{A} F=\inf \{\alpha \mid \text { there exists } C>0 \text { such that, } \\
&\text { for all } \left.0<r<R<\infty, \sup _{x \in F} N_{r}(B(x, R) \cap F) \leq C\left(\frac{R}{r}\right)^{\alpha}\right\}
\end{aligned}
$$

Some authors include the existence of a global bound to $R$ in the above definition, i.e. they allow $0<r<R<\rho$ for some uniform $\rho$. The reason for our definition is to guarantee invariance of the Assouad dimension under specific types of maps, for example the involution $x \mapsto x /|x|^{2}(x \in(0,1))$, see [Lu, Theorem A. 10 (1)]. This clearly gives rise to a larger quantity, but for bounded sets $F$ the two notions are equivalent and in this thesis, as with most papers on fractal geometry, we only consider bounded sets. We also note that the Assouad dimension can be defined in a number of slightly different ways, but all leading to the same value. For example, the function $N_{r}(F)$ can be replaced by the maximum size of an $r$-packing of the set $F$, or the minimum number of closed cubes of side length $r$ required to cover $F$.

We formalise a notion of 'zooming in' that gives rise to a structure called tangents, which are useful to determine lower bounds to the Assouad dimension. We will use them to prove that $\operatorname{dim}_{A} M=1$, where $M=\{1 / n \mid n \in \mathbb{N}\} \cup\{0\}$, as before.

### 1.7.5 Weak tangents

Weak tangents and their variants give a powerful technique for finding the Assouad dimension of sets. We start by recalling the one-sided Hausdorff distance.

Definition 1.7.10. Let $K_{1}, K_{2} \subseteq \mathbb{R}^{d}$ be non-empty and compact, and let $[K]_{\varepsilon}$ denote the closure of the $\varepsilon$ neighbourhood of $K$, we write $d_{\mathscr{H}}^{l}\left(K_{1}, K_{2}\right)$ for the left sided Hausdorff distance between two sets $K_{1}$ and $K_{2}$, given by

$$
d_{\mathscr{H}}^{l}\left(K_{1}, K_{2}\right)=\inf \left\{\varepsilon \geq 0 \mid K_{1} \subseteq\left[K_{2}\right]_{\varepsilon}\right\}
$$

Let $d_{\mathcal{H}}$ denote the Hausdorff metric on the space of non-empty compact subsets of $\mathbb{R}^{d}$, defined by

$$
d_{\mathcal{H}}(A, B)=\inf \left\{\varepsilon \geq 0: A \subseteq[B]_{\varepsilon} \text { and } B \subseteq[A]_{\varepsilon}\right\}
$$

We now define very weak pseudo tangents, introduced by Angelevska and Troscheit in [AT] a variant of weak tangents used in [Fr2, FHOR, MT].

Definition 1.7.11. Let $E, \widehat{E} \subseteq \mathbb{R}^{d}$ be compact. If there exists a sequence $\left(T_{i}\right)_{i}$ of bi-Lipschitz maps on $\mathbb{R}^{d}$ such that

$$
\begin{equation*}
\alpha_{i}^{-1}|x-y| \leq \beta_{i}\left|T_{i}(x)-T_{i}(y)\right| \leq \alpha_{i}|x-y| \tag{1.7.1}
\end{equation*}
$$

for all $x, y \in \mathbb{R}^{d}$, where $1 \leq \alpha_{i}<\alpha<\infty$, for some $\alpha \in \mathbb{R}$ and $\beta_{i}>0$, and if $d_{\mathscr{H}}^{l}\left(\widehat{E}, T_{i}(E)\right) \rightarrow 0$ as $i \rightarrow \infty$, we call $\widehat{E} a$ very weak pseudo tangent of $E$.

Example 1.7.12. Let $M=\{1 / n \mid n \in \mathbb{N}\} \cup\{0\}$ and let $I=[0,1]$. Consider the maps $T_{i}=i \cdot x$. Clearly $T_{i}$ are bi-Lipschitz and satisfy (1.7.1) for $\alpha_{i}=1$ and $\beta_{i}=1 / i$. Now $T_{i} M=\{i / n \mid n \in \mathbb{N}\} \cup\{0\}$ and so $d_{\mathscr{H}}^{l}\left(I, T_{i}(M)\right)=1-i /(i+1) \rightarrow 0$ as $i \rightarrow \infty$. Therefore $I$ is a very weak pseudo tangent to $M$.

We note that very weak tangents (convergence to 0 in Hausdorff metric) and weak pseudo tangents ( $\alpha_{i}=1$ and left sided Hausdorff distance converges to 0 ) have been introduced in Fraser [Fr2] and Fraser et al. [FHOR], respectively. Very weak pseudo tangents are a generalisation of both of these types of tangents and we obtain the same useful bound.

Lemma 1.7.13. Let $\widehat{E}$ be a very weak pseudo tangent to some compact set $E \subset \mathbb{R}^{d}$, then $\operatorname{dim}_{A} \widehat{E} \leq \operatorname{dim}_{A} E$.

We reproduce the proof in [AT] which closely follows the argument in [FHOR] and [MT].

Proof. Set $s=\operatorname{dim}_{A} E$, then for all $\varepsilon>0$, there exists constant $C_{\varepsilon}$ such that for all $0<r<R$,

$$
\sup _{x \in E} N_{r}(B(x, r) \cap E) \leq C_{\varepsilon}(R / r)^{s+\varepsilon}
$$

Now $T_{i}$ is a bi-Lipschitz map such that $\underline{\alpha} \leq \alpha_{i} \leq \bar{\alpha}$ for some $0<\underline{\alpha} \leq \bar{\alpha}<\infty$. Therefore

$$
\sup _{x \in E} N_{r}\left(B(x, r) \cap T_{i} E\right) \leq C_{\varepsilon}\left(\frac{\bar{\alpha} \beta_{i} R}{\underline{\alpha} \beta_{i} r}\right)^{s+\varepsilon}=C_{\varepsilon}^{\prime}(R / r)^{s+\varepsilon}
$$

for some $C_{\varepsilon}^{\prime}>0$, independent of $i$. Choose $i$ large enough such that

$$
d_{\mathscr{H}}^{l}\left(\widehat{E}, T_{i}(E)\right)<r
$$

Thus a minimal cover for $T_{i}(E)$ can be extended to a cover of $\widehat{E}$ by covering the $r$-neighbourhood of every $r$-ball with $c^{d}$ balls of radius $r$. We have

$$
\sup _{x \in E} N_{r}(B(x, r) \cap \widehat{E}) \leq c^{d} \sup _{x \in E} N_{r}\left(B(x, r) \cap T_{i}(E)\right) \leq c^{d} C_{\varepsilon}^{\prime}(R / r)^{s+\varepsilon}
$$

So $\operatorname{dim}_{A} \widehat{E} \leq s+\varepsilon$, and as $\varepsilon$ was arbitrary the required conclusion follows.

We can therefore also conclude that $\operatorname{dim}_{A} M \geq \operatorname{dim}_{A} I=1$ and so $\operatorname{dim}_{A} M=1$.

### 1.8 Fractals revisited

We have now developed several notions of dimension and for totally bounded metric spaces we have

$$
\operatorname{dim}_{H} F \leq \underline{\operatorname{dim}}_{B} F \leq \overline{\operatorname{dim}}_{B} F \leq \operatorname{dim}_{A} F
$$

These inequalities hold in general and for 'nice' sets like squares, cubes, $\mathbb{R}^{d}$, etc. these dimensions coincide with our intuitive idea of dimension. However, we cannot hope for this equality to hold in general as our example showed. This motivates the question:

Question 1.8.1. Under what conditions do the Hausdorff, packing, box-counting, and Assouad dimensions of a set coincide?

Is there a general theorem for random sets that gives us details about the coincidence of dimensions and information about their Hausdorff and packing measure?

We will discuss these two question during the course of this thesis. But first we restrict our focus and look at specific examples of deterministic and random fractals in the next chapter.

# Attractors of Iterated Function Systems 

Sets generated by iterated function systems (IFS) and Moran sets are the archetypal fractal sets. The former class of sets was first introduced by Barnsley et al. [BD] and received a lot of attention over the past decades and many generalisations have been proposed with their random versions lying at the heart of this thesis. We refer the reader to Käenmäki and Rossi [KR], and Holland and Zhang [HZ], and references therein for a description of Moran sets and some of their dimension theoretic properties.

In this chapter we define iterated function systems and their associated invariant sets. We provide a brief survey of recent and classical results and proceed by stating the three most common ways of generating random sets or attractors. This will set the scene for the introduction of random graph directed systems in Chapter 3.

### 2.1 Deterministic Attractors

### 2.1.1 Iterated Function Systems

An iterated function system (IFS) is a set of mappings $\mathbb{I}=\left\{f_{i}\right\}_{i \in \mathcal{I}}$, with associated attractor $F$ that satisfies

$$
\begin{equation*}
F=\bigcup_{i \in \mathcal{I}} f_{i}(F) \tag{2.1.1}
\end{equation*}
$$

If $\mathcal{I}$ is a finite index set and each $f_{i}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ is a contraction, then there exists a unique compact and non-empty set $F$ in the family of compact subsets $\mathcal{K}\left(\mathbb{R}^{d}\right)$ that satisfies this invariance, see Hutchinson $[\mathrm{Hu}]$. Let $d_{\mathscr{H}}$ be the Hausdorff metric. Note that $\left(\mathcal{K}\left(\mathbb{R}^{d}\right), d_{\mathscr{H}}\right)$ is itself a complete metric space and the IFS can be considered as a map from compact sets to compact sets. Using the assumptions above, the IFS $\mathbb{I}: \mathcal{K}\left(\mathbb{R}^{d}\right) \rightarrow \mathcal{K}\left(\mathbb{R}^{d}\right)$ is a contraction and, using Banach's Fixed Point Theorem, the map has a unique fixed point in the sets of non-empty compact sets, the attractor $F$.

These assumptions are however still insufficient to give concrete and meaningful dimensional results for IFS attractors and further assumptions on these maps are imposed. If one considers only similitudes, i.e. $|f(y)-f(x)|=c_{i}|y-x|$, where $c_{i} \in(0,1)$ is the Lipschitz constant (contraction rate) of $f_{i}$, the attractors are called self-similar sets. If they are of the form $f_{i}(x)=\mathbf{A}_{i} x+\mathbf{v}_{i}$, for $x, \mathbf{v}_{i} \in \mathbb{R}^{d}$ and nonsingular matrices $\mathbf{A}_{i} \in \mathbb{R}^{d \times d}$ with $\left\|\mathbf{A}_{i}\right\|<1$ we call the attractor and the associated IFS self-affine. Finally, let $\mathcal{O} \subseteq \mathbb{R}^{d}$ be open and let $f_{i}: \mathcal{O} \rightarrow \mathbb{R}^{d}$ be a strictly contracting diffeomorphism with Hölder continuous derivative that preserves angles, equivalently $\left|f^{\prime}(x) y\right|=\left|f^{\prime}(x)\right||y|$ for all $x, y \in \mathcal{O}$. Then we call the attractor and associated IFS self-conformal.

Of particular interest are the dimensional properties of these attractors, with the Hausdorff, packing, and upper and lower box-counting dimension being the main candidates for investigation. We also consider the Assouad dimension in this thesis.

### 2.1.2 Separation conditions

Definition 2.1.1 (strong separation condition (SSC)). Let $\mathbb{I}=\left\{f_{i}\right\}_{i \in \Lambda}$ be a finite IFS consisting of contractive maps with associated attractor $F$. Then $\mathbb{I}$ satisfies the strong separation condition (SSC) if

$$
f_{i}(F) \cap f_{j}(F) \neq \varnothing \quad \Longrightarrow \quad i=j
$$

The finiteness of the IFS and the compactness of $F$ implies that there exists $\varepsilon>0$ such that $d_{H}\left(f_{i}(F), f_{j}(F)\right)>\varepsilon$ for $i \neq j$. So, if a set satisfies the strong separation condition, the first level images are separated by an uniform 'gap'.

Many results that hold for the SSC also hold for a weaker condition, called the open set condition (OSC).

Definition 2.1.2 (open set condition - OSC). Let $\mathbb{I}=\left\{f_{i}\right\}_{i \in \Lambda}$ be a finite IFS consisting of contractive maps with associated attractor $F$. The IFS $\mathbb{I}$ satisfies the open set condition (OSC) if there exists an open set $\mathcal{O}$ such that $f_{i}(\mathcal{O}) \subseteq \mathcal{O}$ for all $i \in \Lambda$ and $f_{i}(\mathcal{O}) \cap f_{j}(\mathcal{O})=\varnothing$ whenever $i \neq j$.

Note that first level images are no longer separated by a gap, but any overlaps must be points in $\partial \mathcal{O}$, the boundary of $\mathcal{O}$.

The last deterministic separation condition we mention is the weak separation property (WSP). It was introduced by Lau and Ngai [LN] with one important alternative characterisation due to Zerner [Z]. Let

$$
\mathcal{E}=\left\{f_{v}^{-1} \circ f_{w}: v, w \in \Lambda^{*}, v \neq w\right\}
$$

where $f_{\varepsilon_{0}}$ is the identity and we only consider the functions' restriction to an open neighbourhood $[F]_{\varepsilon}$. We equip $\mathcal{E}$, a subset of all bounded functions from $[F]_{\varepsilon}$ to $\mathbb{R}^{d}$ with the supremum norm $\|\cdot\|_{\infty}$.

Definition 2.1.3 (weak separation property (WSP)). We say that $\mathbb{I}$ satisfies the weak separation property (WSP) if the identity is not a cluster point of $\mathcal{E}$,

$$
\operatorname{Id} \notin \overline{\mathcal{E} \backslash \mathrm{Id}}
$$

Interestingly, for iterated function systems with similarities, the open set condition is equivalent to Id $\notin \overline{\mathcal{E}}$.

Remark. We note that this definition has only been shown to be equivalent to the WSP in the sense of Lau and Ngai for self-similar attractors of $\mathbb{R}^{d}$, see Zerner $[\mathrm{Z}]$, and self-conformal sets in $\mathbb{R}$, see Angelevska and Troscheit [AT]. The formulation by Zerner did not restrict the functions to a bounded interval and instead considered $\mathcal{E}$ as a subset of the space of all similarities $S\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right)$, endowed with the topology $\mathscr{T}$ of pointwise convergence. Convergence in the topological space $\left(S\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right), \mathscr{T}\right)$ and convergence in the Banach space $\left(C\left([0,1]^{d}, \mathbb{R}^{d}\right),\|\cdot\|_{\infty}\right)$ are equivalent for similarities and our definition of the WSP coincides with the original definition by Lau and Ngai [LN] for self-similar IFSs but also extends to self-conformal IFSs in $\mathbb{R}$.

We end this section by stating that these conditions are progressively weaker, i.e.

$$
\mathrm{SSC} \Longrightarrow \mathrm{OSC} \Longrightarrow \mathrm{WSP}
$$

### 2.1.3 Self-similar sets

Recall the definition of self-similarity.

Definition 2.1.4. A set $F \subset \mathbb{R}^{d}$ is self-similar if there exists a finite IFS $\mathbb{I}=\left\{f_{i}\right\}_{i \in \Lambda}$ such that, for all $i \in \Lambda$, there exist contraction ratios $c_{i} \in(0,1)$, orthogonal $d \times d$ matrices $\mathbf{A}_{i} \in \mathcal{M}_{d \times d}(\mathbb{R})$, and translation vectors $\mathbf{v}_{i} \in \mathbb{R}^{d}$ so that

$$
f_{i}(x)=c_{i} \mathbf{A}_{i}(x)+\mathbf{v}_{i}
$$

Famous examples include the middle third Cantor set, the Sierpiński triangle, the von Koch curve, and the Menger sponge mentioned earlier.

It is a straightforward calculation, assuming the OSC, that the Hausdorff, boxcounting and Assouad dimensions coincide with the similarity dimension. The similarity dimension is the unique $s \in \mathbb{R}_{0}^{+}$satisfying the Hutchinson-Moran formula

$$
\begin{equation*}
\sum_{i \in \mathcal{I}} c_{i}^{s}=1 \tag{2.1.2}
\end{equation*}
$$

see $[\mathrm{Hu}],[\mathrm{Mo}]$. In fact the OSC is not the weakest condition that implies coincidence of Hausdorff and Assouad dimension. The appropriate separation condition here is the weak separation property (WSP), see Fraser et al. [FHOR], Käenmäki and Rossi [KR], and Chapter 6.

One interesting result is that for self-similar (and, more generally, self-conformal) sets the Hausdorff dimension equals the upper box-counting dimension irrespective of separation conditions, and therefore the Hausdorff, packing and box-counting dimensions coincide. This follows from the implicit theorems we will discuss in Section 2.4, see also Falconer [F3]. The Assouad dimension however can jump up if the WSP is not satisfied. For self-similar and self-conformal attractors $F \subset \mathbb{R}$ the Assouad dimension is then always equal to 1 , see Fraser et al. [FHOR] and Angelevska and Troscheit [AT].

### 2.1.4 Self-affine sets

So far, all sets generated by iterated function systems we have seen seem to have the 'nice' property that many notions of dimension coincide. However, it does not take much effort to break the equality of Hausdorff and box-counting dimension. Let $2 \leq n<m<\infty$ and partition the unit square into $n m$ rectangles of width $1 / n$ and height $1 / m$. Let $D$ be the collection of all such rectangles and let $D^{\prime} \subseteq D$ be nonempty. Consider the maps $f_{d}$ which map the unit square onto the rectangle $d \in D$, preserving orientation. Apart from some exceptions (e.g. when $D^{\prime}=D$ ) the attractor associated with $\left\{f_{d}\right\}_{d \in D^{\prime}}$ is not self-similar. These attractors are called BedfordMcMullen sets (or carpets) and have been studied in great detail, see Bedford [Be], McMullen [Mc], and [F5] for a recent survey. In general their Hausdorff and boxcounting dimension differ.

Definition 2.1.5. A set $F$ is called a self-affine set if there exists a finite $I F S \mathbb{I}=$ $\left\{f_{i}\right\}_{i \in \Lambda}$ such that there exist $\mathbf{v}_{i} \in \mathbb{R}^{d}$ and non-singular matrices $\mathbf{A}_{i} \in \mathbb{R}^{d \times d}$ with $\left\|\mathbf{A}_{i}\right\|<1$ so that

$$
f_{i}(x)=\mathbf{A}_{i} x+\mathbf{v}_{i}
$$

If $d=2$ and the $\mathbf{A}_{i}$ preserve the coordinate axes, we call the attractor a self-affine carpet.

Falconer [F2] showed that for almost all (with respect to the translation vector) self-affine sets the Hausdorff and box-counting dimension coincide with the affinity dimension. However, for some Bedford-McMullen carpets the Hausdorff dimension is strictly less than its affinity dimension. This phenomenon is known as a dimension drop.

Further generalisations of Bedford-McMullen carpets have been considered and we refer the reader to a comprehensive survey by Falconer [F5]. We will just briefly mention some of the deterministic examples that succeeded the self-affine carpets by Bedford $[\mathrm{Be}]$ and McMullen [Mc], see Figure 2.1 (left-most). These extensions were


Figure 2.1: From left to right: Bedford-McMullen carpet (left), Feng-Wang carpet (middle) and Fraser carpet (right) showing the difference in the introduction of just one map with rotations.
by Lalley and Gatzouras [LG] who required a grid pattern along the axis of least contraction of rectangles, and Barański $[\mathrm{B}]$ and a grid pattern in both axes (but no restriction on which contracts more). Feng and Wang [FW] extended the analysis to random carpets that map the square onto non-overlapping rectangles, such that the matrix determining the contraction is diagonal. Fraser [Fr1], [Fr4] extended the class to self-affine carpets with IFSs mapping the unit square onto rectangles such that any horizontal or vertical lines get mapped to horizontal or vertical lines, see Figure 2.1. Random analogues of these latter carpets will be treated in Chapter 4.

### 2.1.5 Graph directed iterated function systems

Graph directed systems are a natural extension of the IFS construction that simultaneously describes a finite collection of sets. A directed multi-graph is a finite set of vertices with a finite set of directed edges with no restrictions but that the edges start and end at a vertex in the graph. Given a directed multi-graph $\Gamma=(V, E)$ with finitely many vertices $V$ and edges $E$, we consider a collection of sets $\left\{K_{i}\right\}_{i \in V}$. We say that $\Gamma$ is strongly connected if there exists a path from every vertex $v \in V$ to any other $w \in V$. Let ${ }_{v} E_{w}$ be the set of edges from $v$ to $w$, we associate a mapping $f_{e}$ with every edge and the sets $K_{i}$ are described by an invariance similar to (2.1.1):

$$
\begin{equation*}
K_{i}=\bigcup_{j \in V} \bigcup_{e \in i E_{j}} f_{e}\left(K_{j}\right) \quad \text { for all } i \in V . \tag{2.1.3}
\end{equation*}
$$

Assuming that the maps $f_{e}$ are contractions, then the sets $K_{v}$ are compact and uniquely determined by the graph directed iterated function system.

Note that IFS constructions are also graph directed constructions as these can be modelled by a graph with a single vertex and an edge for every map in the IFS. It can also be shown that there exist graph directed attractors that cannot be the attractors of standard IFSs, see Boore and Falconer [BF]. If one further assumes that $\Gamma$ is strongly connected, then $\operatorname{dim} K_{i}=\operatorname{dim} K_{j}$ for all $i, j \in V$, where $\operatorname{dim}$ refers to any of the Hausdorff, packing, box-counting, and Assouad dimensions. Again, restricting the maps to similarities or conformal contractions, the Hausdorff and upper box-counting dimension coincide by the implicit theorems, see Chapter 5. For more details on graph directed sets we refer the reader to [F4, Chapter 3] and [Bo] for a detailed treatment.

### 2.2 Random Models

All of these models have random analogues, which for standard IFS are either the $V$-variable or the $\infty$-variable construction where $V \in \mathbb{N}$ is a parameter indicating the inhomogeneity at every construction step.

### 2.2.1 Random homogeneous, random recursive, and $V$-variable models

To explain the construction of sets with a Random Iterated Function System (RIFS) one first has to note that the invariant set in (2.1.1) can also be obtained by iteration of the maps of the IFS. Consider the IFS $\mathbb{I}$ as a self-map on compact subsets of $\mathbb{R}^{d}$, $\mathbb{I}: \mathcal{K}\left(\mathbb{R}^{d}\right) \rightarrow \mathcal{K}\left(\mathbb{R}^{d}\right)$, with $X \mapsto \bigcup_{f \in \mathbb{I}} f(X)$. Take a sufficiently large set $\Delta \in \mathcal{K}\left(\mathbb{R}^{d}\right)$, such that $F \subseteq \Delta$, then $F$ can be written as

$$
F=\lim _{N \rightarrow \infty} \bigcap_{k=1}^{N} \mathbb{I}^{(k)}(\Delta)
$$

For the random analogues of this construction we consider a (usually finite) collection of Iterated Functions Systems $\mathbb{L}=\left\{\mathbb{I}_{i}\right\}_{i \in \Lambda}$ with index set that is usually a compact subset $\Lambda \subset \mathbb{R}^{k}$ for some large $k$. Let $\mu$ be a Borel probability measure supported on $\Lambda$. If $\Lambda$ is finite we simply associate a probability $\pi_{i} \in(0,1]$ to each $i \in \Lambda$, such that $\sum \pi_{i}=1$.

Definition 2.2.1 (Random iterated function system (RIFS)). A random iterated function system (RIFS) ( $\mathbb{L}, \mu)$ is a collection of IFSs $\mathbb{L}=\left\{\mathbb{I}_{i}\right\}_{i \in \Lambda}$ each consisting of a finite number of contraction maps together with an associated Borel probability measure $\mu$ supported on $\Lambda$. If $\Lambda$ is finite we write $(\mathbb{L}, \vec{\pi})$, where $\vec{\pi}=\left(\pi_{1}, \pi_{2}, \ldots, \pi_{\# \mathbb{L}}\right)$ is a probability vector.

The 1-variable attractor (also known as random homogeneous attractor) associated with a RIFS $(\mathbb{L}, \mu)$ is the limit set one obtains by choosing an IFS at the $k$-th level independently from previous steps and according to $\mu$ (or the probability vector $\vec{\pi})$. This choice of IFS depends only on the level and the attractor can be written as

$$
F(\omega)=\lim _{N \rightarrow \infty} \bigcap_{k=1}^{N} \mathbb{I}_{\omega_{1}} \circ \mathbb{I}_{\omega_{2}} \circ \cdots \circ \mathbb{I}_{\omega_{N}}(\Delta)
$$

with $\omega=\left(\omega_{1}, \omega_{2}, \ldots\right), \omega_{i} \in \Lambda$, being the infinite sequence chosen according to $\mu$ or $\vec{\pi}$.

The $\infty$-variable attractor of a RIFS (also referred to as the random recursive constructions) differs in the non-uniform application of the same IFS at every level. Choose $\omega(v) \in \Lambda$ independently, according to $\mu$ or $\vec{\pi}$, for every $v \in\{1,2, \ldots\}^{*}=\mathbb{N}^{*}$. We consider a tree rooted at (1) branching into a subbranch for every map in the IFS chosen for node (1). Writing $(1,1),(1,2), \ldots,\left(1, \# \mathbb{I}_{\omega((1))}\right)$ for these nodes. For each of these nodes we have an associated independently chosen IFS $\omega(v)$ and split the branch up into as many subbranches as maps chosen in the parent node, see Figure 2.3. We denote the resulting outcome of labelling the nodes by $\bar{\omega} \in \Lambda^{\left(\mathbb{N}^{*}\right)}$ and write $\mathbb{I}_{i}=\left\{f_{i}^{1}, \ldots, f_{i}^{N_{i}}\right\}$, where $N_{i}=\# \mathbb{I}_{i}$. We set a recursion depth $k$ and associate a set with every node in the tree up to level $k$ recursively,

$$
F_{v}^{k}(\bar{\omega})=f_{\omega(v)}^{1}\left(F_{(v, 1)}^{k}\right) \cup f_{\omega(v)}^{2}\left(F_{(v, 2)}^{k}\right) \cup \cdots \cup f_{\omega(v)}^{N_{\omega(v)}}\left(F_{\left(v, N_{\omega(v)}\right)}^{k}\right)
$$

for all $v$ such that $|v| \leq k$. We set $F_{v}(\bar{\omega})=\Delta$ for all other nodes. For each $k$ we start the recursion at $F_{(1)}^{k}(\bar{\omega})$ and end at $F_{v}^{k}(\bar{\omega})=\Delta$, where $|v|=\left|\left(1, v_{2}, \ldots, v_{k+1}\right)\right|=k+1$. The $\infty$-variable, random recursive, attractor is then

$$
F(\bar{\omega})=\bigcap_{k=1}^{\infty} F_{1}^{k}(\bar{\omega}) .
$$

Thus the attractor is the limit set obtained at node (1) by increasing the recursion depth $k$.

Intuitively, one can consider the union of composite maps as a tree. In the random homogeneous construction a single IFS is chosen for each level and applied to all
branches in the level, whereas for the random recursive an IFS is chosen independently according to the same distribution for every branch, see Figures 2.2 and 2.3.

In general these two approaches give different geometric properties, with $V$ variable attractors introduced by Barnsley, Hutchinson and Stenflo [BHS1, BHS2, BHS3] attempting to interpolate between them. Let $\Lambda$ be finite, then $N=\max _{i \in \Lambda} \# \mathbb{I}_{i}$ exists and is finite. Further assume that $\# \mathbb{I}_{i} \geq 1$ for all $i$. Informally, a $V$-variable set will have at most $V$ different 'patterns' at every step in the construction. This is achieved in the following way: Consider the vector $\vec{F}_{k}=\left\{F_{k}^{1}, F_{k}^{2}, \ldots, F_{k}^{V}\right\}$ of sets for $k \in\{0,-1,-2, \ldots\}$. Again, we define recursively: Fix $k$ and set $F_{l}^{i}=\Delta$ for all $l \leq k$ and $i \in\{1, \ldots, V\}$. For $j>k$ we choose $\lambda_{i} \in \Lambda$ independently according to $\vec{\pi}$ for each $i \in\{1, \ldots, V\}$. Then, for uniformly independently chosen $\omega_{n} \in\{1, \ldots, V\}$,

$$
F_{j}^{i}=f_{\lambda_{i}}^{1}\left(F_{j-1}^{\omega_{1}}\right) \cup f_{\lambda_{i}}^{2}\left(F_{j-1}^{\omega_{2}}\right) \cup \cdots \cup f_{\lambda_{i}}^{\# \mathbb{I}_{\lambda_{i}}}\left(F_{j-1}^{\omega \# \mathbb{I}_{\lambda_{i}}}\right) .
$$

Taking $\bigcap_{k} F_{0}^{i}$ we get a family of $V$ attractors, called the $V$-variable attractor. The description of these last two models might seem complex, but in Chapter 3 we will introduce a unifying notation with the intention to clarify their structure in terms of a conveniently chosen coding space we call arrangements of words.

Originally $V$-variable attractors were introduced to 'interpolate' between random homogeneous and random recursive attractors but we reserve a discussion of whether $V$-variable attractors adequately do this for Chapter 5.


Figure 2.2: Generation of a 1 -variable Cantor set by the iterated function systems $\mathbb{I}_{1}$ and $\mathbb{I}_{2}$. The IFS $\mathbb{I}_{1}$ consists of two maps contracting by $A_{1}$ and $A_{2}$, respectively, whereas the IFS $\mathbb{I}_{2}$ consists of three maps contracting by $a_{1}, a_{2}$, and $a_{3}$. For each level the IFS is independently chosen and applied uniformly to all codings at that level.


Figure 2.3: Generation of an $\infty$-variable Cantor set by applying the iterated function systems $\mathbb{I}_{1}$ and $\mathbb{I}_{2}$ independently for every finite coding in the preceding level. The IFS $\mathbb{I}_{1}$ consists of two maps contracting by $A_{1}$ and $A_{2}$, respectively, whereas the IFS $\mathbb{I}_{2}$ consists of three maps contracting by $a_{1}, a_{2}$, and $a_{3}$.

Example 2.2.2. Figures 2.2 and 2.3 show the difference in construction of 1-variable and $\infty$-variable sets. Both attractors are created by the same RIFS consisting of the two IFSs $\mathbb{I}_{1}=\left\{A_{1} x, A_{2} x+\left(1-A_{2}\right)\right\}$ and $\mathbb{I}_{2}=\left\{a_{1} x, a_{2} x+1 / 2\left(1-a_{2}\right), a_{3} x+\left(1-a_{3}\right)\right\}$, with $\vec{\pi}=\{1 / 2,1 / 2\}$ but in the 1-variable construction the IFS chosen is uniform on every level of the construction, whereas the $\infty$-variable attractor is not subject to this restriction. The Hausdorff dimension of both of these attractors can be calculated to be almost surely $\operatorname{dim}_{H} F_{1-v a r}=0.721057$ and $\operatorname{dim}_{H} F_{\infty-v a r}=0.724952$ (both to 6 s.f.), see below.

### 2.2.2 Random separation conditions \& self-similar random sets

Perhaps contrary to first impression, the independence in $\infty$-variable attractors makes them easier to analyse. We first define the random analogues of the strong separation condition and the open set condition before stating basic results.

Definition 2.2.3. Let $\mathbb{L}=\left\{\mathbb{I}_{i}\right\}_{i \in \Lambda}$ be a (not necessarily finite) collection of IFSs. The RIFS $(\mathbb{L}, \mu)$ satisfies the uniform strong separation condition (USSC) if there exists $\varepsilon>0$ such that $\mathbb{I}_{i}$ satisfies the SSC with individual images separated by at least $\varepsilon$ for all $i \in \Lambda$.

Definition 2.2.4 (uniform open set condition (UOSC)). Let $\mathbb{L}=\left\{\mathbb{I}_{i}\right\}_{i \in \Lambda}$ be a (not necessarily finite) collection of IFSs. Then $(\mathbb{L}, \mu)$ satisfies the uniform open set condition (UOSC) if there exists an open set $\mathcal{O}$ such that $\mathbb{I}_{i}$ satisfies the OSC with $\mathcal{O}$ for all $i \in \Lambda$.

Assuming the UOSC, some natural assumptions on the IFSs and the measure $\mu$ according to which $\mathbb{I}_{i} \in \mathbb{L}$ is picked, we find that in the $\infty$-variable case the Hausdorff dimension is a.s. given by the unique $s$ satisfying

$$
\mathbb{E}\left(\sum_{j \in \mathcal{I}_{\omega_{1}}} c_{j}^{s}\right)=1
$$

see Falconer [F1], Mauldin and Williams [MW1], and Graf [G] whilst in the 1-variable case it is a.s. the unique $s$ satisfying

$$
\begin{equation*}
\mathbb{E}^{\mathrm{geo}}\left(\sum_{j \in \mathcal{I}_{\omega_{1}}} c_{j}^{s}\right)=1 \tag{2.2.1}
\end{equation*}
$$

(see Hambly $[\mathrm{H}]$ ) where the expectation is taken w.r.t. the measure $\mu$. Further, it has been observed that for the $\infty$-variable construction the Hausdorff and upper box dimension coincide almost surely, see Liu and Wu [LW]. The latter result, and the equality of Hausdorff and upper box-counting dimension for deterministic self-similar attractors of Falconer [F3], are the main motivation for Chapter 3, where we prove that the Hausdorff and upper box dimension coincide, independent of overlap, almost surely. The almost sure existence of the box-counting dimension is then used in Chapter 4 where we state dimension theoretic results for random box-like self-affine sets.

Finally, we remark that in the $V$-variable and $\infty$-variable setting there must exist an almost sure Hausdorff, packing, lower box-counting, and upper box-counting dimension. This arises from the fact that $\{\operatorname{dim} F(\omega)>\alpha \mid \omega \in \Omega\}$ is a tail-event and thus the Kolmogorov 0-1 law implies the almost sure existence. However, this does not imply the coincidence of any of the dimensions. Since these almost sure values exist, we often refer to them by writing ess $\operatorname{dim} F(\omega)$.

### 2.2.3 Random self-affine sets

Talking specifically about self-affine sets, several random variants have been considered. In his seminal work, Falconer [F2] considered deterministic self-affine sets
generated from a single IFS with randomly chosen translation vectors and showed that the Hausdorff dimension coincides almost surely (with respect to the chosen translation vectors) with the 'affinity dimension'. The affinity dimension can be considered the 'best guess' for the Hausdorff, packing and box-counting dimension of self-affine sets and it is of major current interest to establish exactly when these notion do, or do not, coincide. Jordan, Pollicott and Simon [JPS], and Jordan and Jurga [JJ] studied limit sets with random perturbations of the translation ('noise') at every level of the construction, recovering the same coincidence with the affinity dimension and Bárány, Käenmäki and Koivusalo [BKK] recently showed that the same coincidence holds if the contracting matrices were randomly chosen for fixed translation vectors.

However, the coincidence of Hausdorff and affinity dimension is not always guaranteed. Fraser and Shmerkin [FS] considered a Bedford-McMullen construction with random translation vectors that keep the column structure intact. Under these conditions they showed that the Hausdorff dimension is strictly less than the affinity dimension, an observation known as a 'dimension drop'.

A 1-variable (homogeneously random) version of Bedford-McMullen carpets was considered by Gui and Li [GL1]. Here an $n \times m$ subdivision of the unit square is fixed and the random iterated function system is created by assigning a probability to all possible collections of rectangles (possibly 0) such that the probabilities add up to 1 . The authors found that in this setting the almost sure Hausdorff and boxcounting dimension equals the mean of the dimensions of the individual deterministic attractors. We will show in Corollary 4.3 .8 that this holds in a more general setting for the box-counting dimension. However, when choosing more general set ups, e.g. by simply choosing different subdivisions $n_{i}$ and $m_{i}$ for the Bedford-McMullen type IFSs, the almost sure box-counting dimension is no longer the mean of the box-counting dimensions of the individual deterministic attractors.

In a later article, Gui and Li [GL2] were looking at a similar 1-variable set up that allowed the subdivisions to vary at different steps in the construction determining the Hausdorff and box-counting dimension as well as sufficient conditions for positive Hausdorff measure. Luzia [Luz] considered a 1-variable construction of self-affine sets of Lalley-Gatzouras type and determined the Hausdorff dimension of these. At this point we refer the reader also to Rams $[R]$ which gave a more general approach for determining the Hausdorff dimension for these 1-variable schemes. On the other hand, Gatzouras and Lalley [GL] were interested in percolation of Bedford-McMullen carpets, which are $\infty$-variable random IFS, also covered in Chapter 4.

Järvenpää et al. [JJKKSS, JJWW, JJLS] used a general model (code-tree fractals) that overlaps somewhat with the $\infty$-variable random model we consider in Chapter 4. However they treat random translations in their construction and recover almost sure coincidence with the affinity dimension, whereas we will fix a translation vector for every realisation. In Chapter 4 we compute the box-counting and packing dimension of random box-like sets without necessarily randomising the translation vectors and thus our results can be used to determine conditions for which there is a 'dimension drop' where Hausdorff and affinity dimension do not coincide almost surely.

### 2.3 Percolation

The last random method of generating sets that we mention is percolation. It can roughly be divided into two families: Mandelbrot percolation and fractal percolation. Note that some authors consider the terms to be synonymous, but we use 'fractal percolation' in the sense of Falconer and Xiong [FJ1, FJ2] as we will describe below.

### 2.3.1 Mandelbrot percolation

Mandelbrot percolation, first appearing in the works of Mandelbrot in the 1970s as a model for intermittent turbulence [Ma], is one of the most well studied and famous examples of a random fractal and is defined as follows. Begin with the unit cube
$Q=[0,1]^{d}$ in $\mathbb{R}^{d}$ and fix an integer $n \geq 2$ and a probability $p \in(0,1)$. Divide the unit cube into a mesh of $n^{d}$ smaller compact cubes each having side lengths $1 / n$. Now choose to keep each smaller cube independently with probability $p$. The result is a compact collection of cubes, which we call $Q_{1}$. Now repeat this process independently with each surviving cube from the first iteration to form another collection of cubes this time of side lengths $1 / n^{2}$, which we denote by $Q_{2}$. Repeating this process gives a decreasing sequence of compact unions of increasingly smaller cubes, $Q_{k}$. The resulting random set, or Mandelbrot percolation, is defined as

$$
F=\bigcap_{k \in \mathbb{N}} Q_{k}
$$

This construction has been studied intensively over the last 40 years, with many interesting phenomena being observed. Initially, most work concerned the classical question of 'percolation', namely, is there a positive probability that one face of $Q$ is connected by $F$ to the opposite face? More recently, a lot of work has been done on generic dimensional properties of $F$, orthogonal (and other) projections of $F$, and slices of $F$. Rather than cite many papers we simply refer the reader to the recent survey by Rams and Simon [RS]. Concerning the dimension of $F$, if $p>1 / n^{d}$ then there is a positive probability that $F$ is non-empty and conditioned on this occurring, the Hausdorff and packing dimension of $F$ are almost surely given by $\log \left(n^{d} p\right) / \log n$.


Figure 2.4: Mandelbrot percolation for $p=0.7$ and $p=0.9(n=2, d=2)$.

### 2.3.2 Fractal percolation

Fractal percolation, in the sense of Falconer and Xiong [FJ1, FJ2] is a generalisation of Mandelbrot percolation. Instead of subdividing a $d$-dimensional cube we start by considering a deterministic set obtained from an iterated function system $\mathbb{I}=$ $\left\{f_{1}, \ldots, f_{N}\right\}$. An obvious way of addressing each set in its construction is by coding each $n$-fold composition of maps by a word $w \in\{1, \ldots, N\}^{*}$. Note that this can be represented as a tree with subbranches $(w, 1),(w, 2) \ldots,(w, N)$. We can now percolate this tree and decide for every node with probability $p$ whether we intend to keep the subbranches or not. With this process we obtain a subset of the original deterministic attractor and one can ask properties about this set, e.g. its dimension theoretic properties, connectedness, etc.. We note that fractal percolation is contained in the class of random recursive constructions if one allows for an IFS to be empty, which indicates that the subbranch will be deleted. See Figure 2.5 for an example of fractal percolation on a deterministic self-affine set.

### 2.4 The implicit theorems

The implicit theorems find their origins in a 1989 article by Falconer [F3] (see also [F4, Theorems 3.1 and 3.2]). They give us information about several of the dimensions introduced earlier without explicitly stating an expression for the dimension.


Figure 2.5: Random fractal percolation on a Bedford-McMullen carpet with extinction probability $p=0.8$.

Theorem 2.4.1 (Falconer [F3]). Let $F$ be a non-empty compact subset of $\mathbb{R}^{d}$ and let $a>0$ and $r_{0}>0$. Write $s=\operatorname{dim}_{H} F$ and suppose that for every set $U$ that intersects $F$ such that $|U|<r_{0}$ there is a mapping $g: U \cap F \rightarrow F$ with

$$
a|x-y| \leq|U| \cdot|g(x)-g(y)|
$$

for every $x, y \in U \cap F$. Then, $\mathscr{H}^{s}(F) \geq a^{s}>0$ and $\overline{\operatorname{dim}}_{B} F=\operatorname{dim}_{H} F$.
Heuristically, this means that if every small enough piece of a set $F$ can be embedded into the entire set $F$ without 'too much distortion', the Hausdorff measure is positive (for the right exponent) and Hausdorff, packing, and box-counting dimensions coincide.

Theorem 2.4.2 (Falconer [F3]). Let $F$ be a non-empty compact subset of $\mathbb{R}^{d}$ and let $a>0$ and $r_{0}>0$. Write $s=\operatorname{dim}_{H} F$ and suppose that for every closed ball $B$ with centre in $F$ and radius $r<r_{0}$ there exists a map $g: F \rightarrow B \cap F$ satisfying

$$
a r|x-y| \leq|g(x)-g(y)|
$$

for all $x, y \in F$. Then $\mathscr{H}^{s}(F) \leq 4^{s} a^{-s}<\infty$ and $\overline{\operatorname{dim}}_{B} F=\operatorname{dim}_{H} F$.
Similarly, the intuitive picture here is that every ball centred in $F$ contains a not too small and not too distorted copy of the entire set $F$. As mentioned above, this implies that self-similar and self-conformal attractors have finite Hausdorff measure and Hausdorff, packing and box-counting dimensions coincide. This is, at least in part, the motivation for proving the equality of Hausdorff and box-counting dimension for 1 -variable and $\infty$-variable self-similar random graph directed constructions we will introduce in Chapter 3. Clearly the distortion condition is important as affine contractions can allow Hausdorff and box-counting dimension to differ, something that we will see again in Chapter 4 for the random setting. In fact, it is not even clear whether the box-counting dimension of self-affine attractors exists. However, these notions of dimension do coincide in many random cases. In particular, the box-counting dimension often coincides with the Hausdorff and affinity dimension.

The results in Chapter 3 as well as many other papers on random self-similar sets, like percolation, random cut-out sets and geometric martingales, see [SS], suggest that there might be a random version of the implicit theorems above. In particular, we would hope that these implicit theorems tell us something about the Hausdorff measure of the random set in question. However, it turns out that there cannot be a statement as strong as Theorems 2.4.1 and 2.4.2. The first observation to make is that the Hausdorff measure for reasonable random systems is 0 and we have to investigate the sets in closer detail using gauge functions in Chapter 5.

## Random Graph Directed Iterated Function Systems

### 3.1 Introduction

Having introduced graph directed attractors, there is, of course, the natural question of a random analogue. The usual model for this considers a fixed directed multi-graph, where for each edge we associate a family of maps with a probability measure and choose a map in a recursive fashion according to this probability measure. This model and its multifractal formalism was extensively studied in Olsen [O1] and we refer to this book and the references contained therein. Here we develop a different natural model that arises in the study of sets with orthogonal projections more complicated than simple self-similar IFSs, in particular the model studied by Troscheit in [T2], the basis of Chapter 4.

Instead of one fixed graph, we consider a finite collection of graphs with an associated probability vector. We consider a 1 -variable random graph directed system (RGDS) and then a $\infty$-variable RGDS, where instead of the maps, the graphs and hence the relations between vertex sets changes in a random fashion, see also Roy and Urbański $[R U]$ for a similar approach ${ }^{1}$ in the 1 -variable setting.

One example of sets whose projections fail to be self-similar RIFS but are random graph directed attractors in our sense, are the $V$-variable extensions of self-affine carpets in the sense of Fraser [Fr1]. Failure here is caused by the non-trivial rotations and the projections cannot be described by the standard RIFS model but can be by the RGDS proposed here, see [T2] or Chapter 4.

Notice that many standard random models can be recovered by setting up the RGDS in the right way. Choosing graphs with a single vertex allows the RGDS setup to be used to analyse 1-variable and random recursive attractors. The class of $V$-variable attractors are specific 1 -variable RGDS in our sense, where one chooses a vertex set with $V$ vertices and the randomly chosen graphs $\Gamma_{i}$ with edges and probabilities appropriately. Results about several other standard models can be deduced from our main theorems, see Corollary 3.2.4. It is a quick calculation to show that $V$-variable constructions satisfy all conditions in Definition 3.2.9 and one can reduce the $V$-variable randomness to the simpler 1 -variable RGDA construction treated here. The model developed here can be further generalised to $V$-variable RGDS and higher order random graph directed systems but we will not deal with the additional complexity of these constructions. We also remark that in the $\infty$-variable case we are allowed to have paths that can become extinct, so choosing the graphs and maps appropriately our model specialises to fractal and Mandelbrot percolation.

[^2]We give basic notation, define the model and give our main results for 1-variable RGDS in Section 3.2. Section 3.3 contains our $\infty$-variable results and proofs are contained in Section 3.4.

### 3.2 Notation and preliminaries for 1-variable RGDS

Let $\boldsymbol{\Gamma}=\left\{\Gamma_{i}\right\}_{i \in \Lambda}$ be a finite collection of graphs $\Gamma_{i}=\Gamma(i)=(V(i), E(i))$ indexed by $\Lambda=\{1, \ldots, n\}$, each with the same number of vertices. For simplicity we will assume that they share the same set of vertices $V(i)=V$. The set $E(i)$ is the set of all directed edges and we write ${ }_{v} E_{w}(i)$ to denote the edges from $v \in V$ to $w \in V$. We write $E_{w}(i)=\bigcup_{v \in V ~}{ }^{2} E_{w}(i)$ and ${ }_{v} E(i)=\bigcup_{w \in V v} E_{w}(i)$ for $i \in \Lambda$. For all edges $e$ we write $\iota(e)$ and $\tau(e)$ to refer to initial and terminal vertex, respectively. The set of all infinite strings with entries in $\Lambda$ we denote by $\Omega=\Lambda^{\mathbb{N}}$, whereas all finite strings of length $k$ are given by $\Lambda^{k}$, and the set of all finite strings is $\Lambda^{*}=\bigcup_{k \in \mathbb{N}} \Lambda^{k}$. For $w \in \Lambda^{*}$ we define as before the $w$-cylinder $[w]=\left\{\omega \in \Omega \mid \omega_{i}=w_{i}\right.$ for $\left.1 \leq i \leq|w|\right\}$. We use the standard metric $d(x, y)=2^{-|x \wedge y|}$ on $\Omega$ and denote by $\mathbb{P}$ the Bernoulli measure on $\Omega$ associated to probability vector $\vec{\pi}=\left\{\pi_{1}, \pi_{2}, \ldots, \pi_{n}\right\}$.

Given a collection of graphs $\boldsymbol{\Gamma}$ we are now interested in the attractor of two associated random processes. We first describe the 1 -variable case. For $v \in V$, we define the random attractor $K_{v}$ for $v \in V$ in terms of paths on the randomly chosen graphs. Let ${ }_{v} E_{u}^{k}(\omega)$ be the set of all paths of length $k$ consisting of edges starting at $v$ and ending at $u$ and traversing through the graph $\Gamma_{\omega_{q}}$ at step $q$, that is

$$
\begin{array}{r}
{ }_{v} E_{u}^{k}(\omega)=\left\{\mathbf{e}=\left(e_{1}, e_{2}, \ldots, e_{k}\right) \mid \iota\left(e_{1}\right)=v, \tau\left(e_{k}\right)=u, \iota\left(e_{l+1}\right)=\tau\left(e_{l}\right)\right. \\
\text { for } \left.1 \leq l \leq k-1 \text { and } e_{i} \in E\left(\omega_{i}\right)\right\} .
\end{array}
$$

To each edge $e \in\{e \in E(i) \mid i \in \Lambda\}$ we associate a strictly contracting self-map $S_{e}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ and choose a compact seed set $\Delta \in \mathcal{K}\left(\mathbb{R}^{d}\right)$ such that $\Delta=\overline{\operatorname{int} \Delta}$ and $S_{e}(\Delta) \subset \Delta$ for all $e \in E(i)$ and $i \in \Lambda$. In this notation we have

$$
K_{v}(\omega)=\bigcap_{l=1}^{\infty} \bigcup_{u \in V} \bigcup_{\mathbf{e} \epsilon_{v} E_{u}^{l}(\omega)} S_{\mathbf{e}}(\Delta)
$$

where $S_{\mathbf{e}}=S_{e_{1}} \circ S_{e_{2}} \circ \ldots \circ S_{e_{|\mathbf{e}|}}$. The set $K_{v}(\omega)$ is well-defined for every $\omega$ and $v$ and it is a simple application of Banach's fixed point theorem to show that $K_{v}(\omega)$ is compact and non-empty. Even though this holds for all collections of contracting maps, we restrict our attention to similarities, i.e. maps such that $\left|S_{e}(x)-S_{e}(y)\right|=c_{e}|x-y|$ for some $0<c_{e}<1$ and all $x, y \in \mathbb{R}^{d}$.

In many places we describe our results in terms of a structure that is an infinite matrix over finite matrices with (semi-)ring element entries. Let $\mathcal{M}_{n \times n}(\mathbb{R})$ be the vector space of all $n \times n$ matrices with real entries and $\mathcal{M}_{n \times n}\left(\mathbb{R}_{0}^{+}\right)$the set of all $n \times n$ matrices with non-negative entries. We also consider the set of square matrices with entries that are finite non-negative matrices

$$
\mathfrak{M}_{k, n}=\mathcal{M}_{k \times k}\left(\mathcal{M}_{n \times n}\left(\mathbb{R}_{0}^{+}\right)\right),
$$

and the (vector) space of countably infinite, upper triangular matrices with entries that are finite real-valued matrices

$$
\mathfrak{M}_{\mathbb{N}, n}^{*}=\mathcal{M}_{\mathbb{N} \times \mathbb{N}}\left(\mathcal{M}_{n \times n}(\mathbb{R})\right),
$$

such that for every $M \in \mathfrak{M}_{\mathbb{N}, n}^{*}$ the number of row entries that are not the zero matrix is uniformly bounded and

$$
\begin{equation*}
\sup _{j \in \mathbb{N}} \sum_{i=0}^{\infty}\left\|M_{i, j}\right\|_{\text {row }}<\infty \tag{3.2.1}
\end{equation*}
$$

where $\|\cdot\|_{\text {row }}$ is the matrix norm, see below. It can be checked that $\mathfrak{M}_{\mathbb{N}, n}^{*}$ is a vector space and we consider the subset consisting of non-negative entries

$$
\mathfrak{M}_{\mathbb{N}, n}=\mathcal{M}_{\mathbb{N} \times \mathbb{N}}\left(\mathcal{M}_{n \times n}\left(\mathbb{R}_{0}^{+}\right)\right) \subset \mathfrak{M}_{\mathbb{N}, n}^{*}
$$

We note that the only infinite matrices we are considering are upper triangular. Further, while the sets $\mathfrak{M}_{k, n}$ and $\mathfrak{M}_{\mathbb{N}, n}$ are not vector spaces per se, they are subsets of vector spaces that are closed under multiplication and addition. We define the following norms and seminorms.
Definition 3.2.1. Let $M \in \mathcal{M}_{n \times n}(\mathbb{R})$, we define

$$
\begin{aligned}
\|M\|_{\text {row }} & =\max _{i} \sum_{j}\left|M_{i, j}\right| \\
\|M\|_{1} & =\sum_{i} \sum_{j}\left|M_{i, j}\right|
\end{aligned}
$$

which can easily seen to be (equivalent) norms. For $M^{*} \in \mathfrak{M}_{\mathbb{N}, n}^{*}$, the space of infinite matrices consisting of matrix entries with real entries, such that only finitely many matrices in each row are not $\mathbf{0}$ (the $n \times n$ zero matrix) and the norm of each row sum is uniformly bounded, we define the norm

$$
\left\|M^{*}\right\|_{\text {sup }}=\sup _{i^{*} \in \mathbb{N}} \sum_{j^{*}=1}^{\infty}\left\|\left(M^{*}\right)_{i^{*}, j^{*}}\right\|_{\text {row }}
$$

Furthermore we define two seminorms. The first $||\mathbb{1} . \||$ is given by (3.2.2) and defined on the same space $\mathfrak{M}_{\mathbb{N}, n}^{*}$ of infinite matrices with real-valued matrix entries such that the number of non-zero matrix entries is uniformly bounded above and (3.2.1) is satisfied. The second seminorm $\left\|\left\|\mathbb{1}_{l} \cdot \mid\right\|_{(1,1)}\right.$, given by (3.2.3), is defined on the space of $l$ by $l$ matrices with $n$ by $n$ real matrix entries. We slightly abuse notation here and concisely write $\|\mathbf{v}\|_{s}$, where $\mathbf{v}$ is a vector with matrix entries, to mean the matrix sum of all, possibly infinite, vector entries. Here $\mathbb{1}=\{\mathbf{1}, \mathbf{0}, \mathbf{0}, \ldots\}$ is an infinite vector and $\mathbb{1}_{l}$ is the vector of dimension $l$ satisfying $\mathbb{1}_{l}=\{\mathbf{1}, \mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}\}$, where $\mathbf{1}$ is the $n \times n$ identity matrix.

$$
\begin{gather*}
\|\mathbb{1} M\|\|=\|\|\mathbb{1} M\|_{s}\left\|_{\text {row }}=\right\| \sum_{k=1}^{\infty}(\mathbb{1} M)_{k} \|_{\text {row }}  \tag{3.2.2}\\
\left\|\mathbb{1}_{l} M\right\|_{(1,1)}=\| \| \mathbb{1}_{l} M\left\|_{s}\right\|_{1}=\sum_{i, j \in\{1, \ldots, n\}} \sum_{k=1}^{l}\left(\left(\mathbb{1}_{l} M\right)_{k}\right)_{i, j} \tag{3.2.3}
\end{gather*}
$$

Before we introduce further necessary notation we refer the reader to two important corollaries of our more general results. First, in Corollary 3.2.24 we state the almost sure Hausdorff dimension of our 1-variable random graph directed systems, assuming the uniform strong separation condition. The quantity $p_{1}^{t}(\omega, 1)$ referred to in (3.2.8) is simply the Hutchinson-Moran matrix for the graph-directed iterated function system associated with $\Gamma\left(\omega_{1}\right)$. Furthermore Corollary 3.2 .4 states that for selfsimilar 1-variable sets, and even $V$-variable sets in the sense of Barnsley et al. [BHS2], we must have $\operatorname{dim}_{H} F_{\omega}=\operatorname{dim}_{B} F_{\omega}$ for almost every $\omega \in \Omega$.

### 3.2.1 Arrangements of words

To describe the cylinders and points in the attractor of iterated function systems and graph directed systems, one uses a natural coding. In this section we give a more abstract way of manipulating words that will become useful in describing the construction in random systems. We introduce two binary operations $\sqcup$ and $\odot$ that take over the rôles of set union and concatenation, respectively, to manipulate strings in a meaningful way.

Definition 3.2.2. Let $\mathcal{G}^{E}$ be a finite alphabet, which in this chapter is the set of letters identifying the edges of the graphs $\Gamma_{i}$, i.e. $\mathcal{G}^{E}=\{e \mid e \in E(i)$ and $i \in \Lambda\}$. We define the prime arrangements $\mathcal{G}$ to be the set of symbols $\mathcal{G}=\left\{\varnothing, \varepsilon_{0}\right\} \cup \mathcal{G}^{E}$.

Define $\beth$ コ to be the free monoid with generators $\mathcal{G}^{E}$ and identity (empty word) $\varepsilon_{0}$, and define $\beth^{\sqcup}$ to be the free commutative monoid with generators $\beth{ }^{\odot}$ and identity $\varnothing$. We define $\odot$ to be left and right multiplicative over $\sqcup$, and $\varnothing$ to annihilate with respect to $\odot$. That is, given an element e of $\beth \odot$, we get $e \odot \varnothing=\varnothing \odot e=\varnothing$. We define $\beth *$ be the set of all finite combinations of elements of $\mathcal{G}$ and operations $\sqcup$ and $\odot$. Using distributivity $\beth=\left(\beth^{*}, \sqcup, \odot\right)$ is the non-commutative free semi-ring with 'addition' $\sqcup$ and 'multiplication' $\odot$ and generator $\mathcal{G}^{E}$ and we will call $\beth$ the semiring of arrangements of words and refer to elements of $\beth^{*}$ as (finite) arrangements of words.

We adopt the convention to 'multiply out' arrangements of words and write them as elements of $\beth \odot$. Furthermore we omit brackets, where appropriate, replace $\odot$ by concatenation to simplify notation, and for arrangements of words $\phi$ write $\varphi \in \phi$ to refer to the maximal subarrangements $\varphi$ that do not contain $\sqcup$ and are thus elements of $\varphi \in \beth$.

Example 3.2.3. Let $\mathcal{G}^{E}=\{0,1\}$. The set of prime arrangements is then $\left\{\varnothing, \varepsilon_{0}, 0,1\right\}$ and the elements of the semiring $\beth^{*}$ are all possible concatenations $\odot$ and unions $\sqcup$, e.g.
$1 \odot 0 \sqcup 1=10 \sqcup 1, \quad\left(110 \sqcup 101 \sqcup \varepsilon_{0}\right) \odot 1=1101 \sqcup 1011 \sqcup 1, \quad \varnothing \odot(10 \sqcup 101)=\varnothing, \ldots$
The usefulness of the description above is that $\beth^{*}$ is ring isomorphic to the set of all cylinders with set union and concatenation as the binary operations and we can use $\odot$ and $\sqcup$ to describe collections of cylinders. For example the set containing all cylinders of length $k$ can be identified with the arrangement of words $(0 \sqcup 1)^{k}$.

We can now use the algebraic structure above to give descriptions of 1 -variable RIFS.

Example 3.2.4. Consider the simple setting of just two Iterated Functions Systems $\mathbb{L}=\left\{\mathbb{I}_{1}, \mathbb{I}_{2}\right\}$ that are picked at random according to probability vector $\vec{\pi}=\left\{\pi_{1}, \pi_{2}\right\}$, $\pi_{i}>0$. Let $\phi_{i}=a_{1}^{i} \sqcup \cdots \sqcup a_{n}^{i}$, where $a_{j}^{i}$ are the letters in the alphabet associated with $\operatorname{IFS} \mathbb{I}_{i}$. The arrangement of words describing the cylinders of length $k$ with realisation $\omega$ is then simply

$$
\phi_{\omega_{1}} \odot \phi_{\omega_{2}} \odot \cdots \odot \phi_{\omega_{k}} .
$$

An arrangement of words is nothing more than a formalisation of the standard alphabet one uses to describe words, where $\odot$ is concatenation of letters and $\sqcup$ is the union of several letters. Before we can apply this construction to our RGDS we need to extend this concept to the natural analogue of matrix multiplication $\times$ and addition, which we also refer to as $\sqcup$.

Definition 3.2.5. Let $\mathbf{M}$ and $\mathbf{N}$ be square $n \times n$ matrices and $\mathbf{v}=\left\{v_{1}, \ldots, v_{n}\right\}$ be a $n$-vector with entries being arrangements of words. We define matrix multiplication in the natural way,

$$
\begin{gathered}
(\mathbf{M} \times \mathbf{N})_{i, j}=\bigsqcup_{k=1}^{n}\left(\mathbf{M}_{i, k} \odot \mathbf{N}_{k, j}\right), \quad(\mathbf{M} \sqcup \mathbf{N})_{i, j}=\mathbf{M}_{i, j} \sqcup \mathbf{N}_{i, j}, \\
(\mathbf{v} \times \mathbf{M})_{i}=\bigsqcup_{k=1}^{n}\left(v_{k} \odot \mathbf{M}_{k, i}\right) .
\end{gathered}
$$

We extend this to multiplication of countable (finite or infinite) square matrices with matrix entries.


Figure 3.1: Graph used in Example 3.2.7

Definition 3.2.6. Let $\mathbf{M}^{*}$ and $\mathbf{N}^{*}$ be elements of $\mathcal{M}_{k, k}\left(\mathcal{M}_{n, n}\left(\left(\beth^{*}\right)\right)\right.$ and $\mathbf{v}^{*} \in$ $\left(\mathcal{M}_{n, n}\left(\left(\beth^{*}\right)\right)^{k}\right.$, where $k \in \mathbb{N} \cup\{\mathbb{N}\}$. We define multiplication and addition by

$$
\left(\mathbf{M}^{*} \times \mathbf{N}^{*}\right)_{i, j}=\bigsqcup_{l=1}^{k}\left(\mathbf{M}_{i, l}^{*} \times \mathbf{N}^{*}{ }_{l, j}\right), \quad\left(\mathbf{M}^{*} \sqcup \mathbf{N}^{*}\right)_{i, j}=\mathbf{M}_{i, j}^{*} \sqcup \mathbf{N}^{*}{ }_{i, j},
$$

and

$$
\left(\mathbf{v}^{*} \times \mathbf{M}^{*}\right)_{i}=\bigsqcup_{l=1}^{k}\left(\mathbf{v}_{l}^{*} \times \mathbf{M}_{l, i}^{*}\right)
$$

For graph directed attractors we can now describe codes as arrangements of words in matrix form. Recall that a graph directed attractor is a collection of sets which is invariant under maps between them, see (2.1.3). The aim of codings in this setting is to describe all paths in the graph and every point in the attractor corresponds to an infinite such path. We apply arrangements of words to succinctly write and modify such paths.

Example 3.2.7. Let $\Gamma_{0}$ be the graph in Figure 3.1. We define

$$
M_{0}=\left(\begin{array}{cc}
\varnothing & e_{1} \sqcup e_{3} \\
e_{2} & e_{4}
\end{array}\right) .
$$

All paths of length, say $k=2$, are the arrangements contained in $\left(M_{0}\right)^{2}=M_{0} \times M_{0}$, that is

$$
\left(M_{0}\right)^{2}=\left(\begin{array}{cc}
e_{1} e_{2} \sqcup e_{3} e_{2} & e_{1} e_{4} \sqcup e_{3} e_{4} \\
e_{4} e_{2} & e_{2} e_{1} \sqcup e_{2} e_{3} \sqcup e_{4} e_{4}
\end{array}\right)
$$

where e.g. $e_{2} e_{1} \sqcup e_{2} e_{3} \sqcup e_{4} e_{4}$ represents nothing but the set of paths starting at vertex $v_{2}$ and ending at $v_{2}$ of length 2.

Slightly more abstractly, we can now take multiple graphs and consider paths that traverse edges of graph $i$ at step $i$ as the following example shows.

Example 3.2.8. Let $\boldsymbol{\Gamma}=\left\{\Gamma_{i}\right\}_{i \in \Lambda}$ be a finite collection of graphs sharing vertex set $V$. Define the matrix $\mathbf{M}(i)$ over arrangements of words by

$$
(\mathbf{M}(i))_{u, v}=\bigsqcup_{e \in_{u} E_{v}(i)} e, \quad u, v \in V
$$

Let $\omega \in \Lambda^{\mathbb{N}}$. The arrangement of words describing all paths of length $k$ starting at $v \in V$, traversing through graph $\Gamma_{\omega_{i}}$ at step $i$, is then simply

$$
\mathbf{v}(v) \mathbf{M}\left(\omega_{1}\right) \mathbf{M}\left(\omega_{2}\right) \ldots \mathbf{M}\left(\omega_{k}\right), \quad \text { where }(\mathbf{v}(v))_{i}= \begin{cases}\varepsilon_{0} & \text { if } i=v \\ \varnothing & \text { otherwise }\end{cases}
$$

Thus the arrangement of words encode paths that in turn will be associated with sets. The limits of these sets as we multiply more and more matrices are the object under investigation and we will expand on them after increasing the level of abstractivication one more level.

### 3.2.2 Stopping graphs

We continue this section by introducing the notion of the $\varepsilon$-stopping graph. Before we can do so we need some conditions on our graphs $\boldsymbol{\Gamma}$.
Definition 3.2.9. Let $\boldsymbol{\Gamma}=\left\{\Gamma_{i}\right\}_{i \in \Lambda}$ be a finite collection of graphs, sharing the same vertex set $V$.
3.2.9.a We say that the collection $\boldsymbol{\Gamma}$ is a non-trivial collection of graphs if for every $i \in \Lambda$ and $v \in V$ we have ${ }_{v} E(i) \neq \varnothing$. Furthermore we require that there exist $i, j \in \Lambda$ and $e_{1} \in \Gamma(i)$ and $e_{2} \in \Gamma(j)$ such that $S_{e_{1}} \neq S_{e_{2}}$.
3.2.9.b If for every $v, w \in V$ there exists $\omega^{v, w} \in \Lambda^{*}$ such that ${ }_{v} E_{w}\left(\omega^{v, w}\right) \neq \varnothing$ and $\mathbb{P}\left(\left[\omega^{v, w}\right]\right)>0$, we call $\boldsymbol{\Gamma}$ stochastically strongly connected.
3.2.9.c We call the Random Graph Directed System (RGDS) associated with $\boldsymbol{\Gamma}$ a contracting self-similar RGDS if for every $e \in E(i), S_{e}$ is a contracting similitude.

Condition 3.2.9.b implies that at each stage of the construction there is a positive probability that one can travel from every vertex to every other in a finite number of steps. As every map for every edge in $\boldsymbol{\Gamma}$ is a strict contraction the maximal similarity coefficient $c_{\max }=\max \left\{c_{\mathbf{e}} \mid \mathbf{e} \in E(i)\right.$ and $\left.i \in \Lambda\right\}$ satisfies $c_{\max }<1$. This gives us that for every $\varepsilon>0$ there exists a least $k_{\max }(\varepsilon) \in \mathbb{N}$ such that $c_{\max }^{k_{\max }(\varepsilon)}<\varepsilon$ and hence every path $\mathbf{e} \in E^{k_{\max }(\varepsilon)}(\omega)$ has an associated contraction $c_{\mathbf{e}}<\varepsilon$. Therefore all paths of length comparable with $\varepsilon$ only depend, at most, on the first $k_{\max }(\varepsilon)$ letters of the random word $\omega \in \Omega$ and thus the set of $\varepsilon$-stopping graphs below is well defined.

Definition 3.2.10. Let $\boldsymbol{\Gamma}$ be a non-trivial, finite collection of graphs sharing vertex set $V$, satisfying Condition 3.2.9.c. Let $E^{*}(\omega, \varepsilon)$ be the set of paths $\mathbf{e}$, corresponding to the realisation $\omega$, such that $S_{\mathrm{e}}$ is a contraction with similarity coefficient comparable to $\varepsilon$ :

$$
\begin{aligned}
& E^{*}(\omega, \varepsilon)=\left\{\mathbf{e} \in \bigcup_{k=1}^{k_{\max }(\varepsilon)} E^{k}(\omega) \mid c_{\mathbf{e}} \leq \varepsilon \text { for } \mathbf{e}=\left(e_{1}, \ldots, e_{|\mathbf{e}|}\right)\right. \\
& \text { but } \left.c_{\mathbf{e}^{\ddagger}}>\varepsilon \text { for } \mathbf{e}^{\ddagger}=\left(e_{1}, \ldots, e_{|\mathbf{e}|-1}\right)\right\} .
\end{aligned}
$$

Now consider all possible subsets of these sets of edges $\mathcal{E}(\omega, \varepsilon)$, such that the images of $\Delta$ are pairwise disjoint in each of the subsets

$$
\mathcal{E}(\omega, \varepsilon)=\left\{U \subseteq E^{*}(\omega, \varepsilon) \mid \text { for all } \mathbf{e}, \mathbf{f} \in U \text { we have } S_{\mathbf{e}}(\Delta) \cap S_{\mathbf{f}}(\Delta)=\varnothing\right\}
$$

As $\mathcal{E}(\omega, \varepsilon)$, and every $U_{i} \in \mathcal{E}(\omega, \varepsilon)$, has finite cardinality we can order $\left\{U_{i}\right\}$ in descending order, i.e. $\left|U_{m}\right| \geq\left|U_{m+1}\right|$. Finally we define $E(\omega, \varepsilon)$ to be the first, and thus maximal, element $E(\omega, \varepsilon)=U_{0}$.
The $\varepsilon$-stopping graph is then defined to be

$$
\boldsymbol{\Gamma}^{\varepsilon}=\left\{\Gamma^{\varepsilon}(\omega) \mid z_{i} \in \Lambda^{k_{\max }(\varepsilon)} \text { and } \omega \in\left[z_{i}\right]\right\}, \text { with } \Gamma^{\varepsilon}(\omega)=(V, E(\omega, \varepsilon))
$$

In fact it does not matter which $\omega \in\left[z_{i}\right]$ is chosen as $\Gamma^{\varepsilon}(\omega)$ only depends on, at most, the first $k_{\max }(\varepsilon)$ letters.

By the arguments above it can easily be seen that the collection $\Gamma^{\varepsilon}$ is finite for every $\varepsilon>0$ and every edge of $\boldsymbol{\Gamma}^{\varepsilon}$ is a finite path in $\boldsymbol{\Gamma}$ for the same $\omega$. However there may be some paths in $\boldsymbol{\Gamma}$ that are not edges of $\boldsymbol{\Gamma}^{\varepsilon}$ for any $\varepsilon$, but for $\varepsilon$ small enough, eventually that path will be a prefix of an edge coding. Note that we consider arrangements to be equivalent if their images under $S$ coincide exactly. In this latter case of exact overlaps we will keep only one of the two paths as they describe an identical subset.

Lemma 3.2.11. Let $\omega \in \Lambda^{\mathbb{N}}$ and let $e=e_{1} e_{2} \ldots e_{k}$ be a path such that $\tau\left(e_{i}\right)=\iota\left(e_{i+1}\right)$ for $1 \leq i \leq k-1$ and $e_{i} \in E\left(\omega_{i}\right)$ and $S_{e}$ is unique. Then there exists $\varepsilon>0$ such that the path $e$ is the prefix of the label of an edge of $\Gamma_{i}^{\varepsilon}$ for some $i$, i.e. ew $\in E(\omega, \varepsilon)$ for some path $w=w_{1} w_{2} \ldots w_{l}$, where $\tau\left(w_{i}\right)=\iota\left(w_{i+1}\right)$ for $1 \leq i \leq l-1$ and $w_{i} \in E\left(\omega_{i+k}\right)$.

Further, for every $\omega \in \Lambda^{\mathbb{N}}$ and $\varepsilon$ the sets $\boldsymbol{\Gamma}^{\varepsilon}$ and $E(\omega, \varepsilon)$ are finite.
We will be considering $\varepsilon$-stopping graphs derived from the original graph and show that if $\Gamma$ has 'nice' properties (it satisfies most assumptions in Definition 3.2.9), then $\Gamma^{\varepsilon}$ also has these properties.

Lemma 3.2.12. Let $\boldsymbol{\Gamma}$ be a non-trivial collection of graphs that is stochastically strongly connected. Then there exists $\varepsilon^{\prime}>0$ such that $\boldsymbol{\Gamma}^{\varepsilon}$ is a non-trivial collection of stochastically strongly connected graphs for all $0<\varepsilon \leq \varepsilon^{\prime}$ and almost every $\omega \in \Omega$.

Proof. Assuming $\boldsymbol{\Gamma}$ is non-trivial implies that for every $v \in V$ and $i \in \Lambda$ there exists at least one edge in ${ }_{v} E(i)$. However as the set $E(\omega, \varepsilon)$ is chosen by non-overlapping images, for a path to be deleted there must be a second path, leaving at least one path. Hence $\left|{ }_{v} E(\omega, \varepsilon)\right| \geq 1$ for all $v$ and $\omega$, i.e. $\Gamma^{\varepsilon}$ is non-trivial.

To show that $\Gamma^{\varepsilon}$ is stochastically strongly connected we note that the only possibility for a path that existed in $\boldsymbol{\Gamma}$ but not in $\boldsymbol{\Gamma}^{\varepsilon}$ is that it had been deleted due to overlapping images. However if $\varepsilon$ is chosen small enough then there will be a different path that is being kept, unless all maps $S_{e}$ are identical. We however exclude this trivial case (Condition 3.2.9.a) as the attractor of such a system would be a singleton.

We can partition the paths in $E(\omega, \varepsilon)$ by initial and terminal vertex and path length and write ${ }_{v} E_{w}^{k}(\omega, \varepsilon)$ to refer to paths $\mathbf{e}$ of length $k$ with $1 \leq k \leq k_{\max }(\varepsilon)$, $\iota(\mathbf{e})=v$ and $\tau(\mathbf{e})=w$. The set $E(\omega, \varepsilon)$ then consists of collections of paths whose images are disjoint under $S_{\mathrm{e}}$.

### 3.2.3 Infinite random matrices

Recall that we succinctly wrote the codings of RGDS by matrices over arrangements of words in Example 3.2.8. We wish to use the stopping graphs we have just introduced to describe a subset of the attractor associated to random graph directed systems. However, we can no longer simply multiply simple matrices, as the paths depend on where along the random process $\omega$ we are. We thus split up the paths in the graph by length and starting realisation. Arranging them as infinite matrices with matrix entries, we can follow the same approach of multiplying (infinite) matrices to get the appropriate codings. Here we will describe these matrices.

Let $\omega \in \Omega$ be a word chosen randomly according to the Bernoulli measure $\mathbb{P}$ associated with the probability vector $\vec{\pi}$, where $\pi_{i}>0$ for all $i \in \Lambda$ and recall that $\sigma$ is the shift map on $\Omega$. For all $i \in\{1,2, \ldots, l\}$ let $t_{i}(\omega) \in \mathcal{M}_{n \times n}\left(\mathbb{R}_{0}^{+}\right)$. Letting $\mathbf{t}(\omega)=\left\{t_{1}(\omega), t_{2}(\omega), \ldots, t_{l}(\omega)\right\}$ we have a random vector with matrix valued entries. Now define $\mathbf{T}(\omega) \in \mathfrak{M}_{\mathbb{N}, n}$ by

$$
\mathbf{T}(\omega)=\left(\begin{array}{ccccc}
t_{1}(\omega) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
t_{2}(\omega) & t_{1}(\sigma \omega) & \mathbf{0} & \mathbf{0} & \ldots \\
\vdots & t_{2}(\sigma \omega) & t_{1}\left(\sigma^{2} \omega\right) & \mathbf{0} & \cdots \\
t_{l}(\omega) & \vdots & t_{2}\left(\sigma^{2} \omega\right) & t_{1}\left(\sigma^{3} \omega\right) & \ddots \\
\mathbf{0} & t_{l}(\sigma \omega) & \vdots & t_{2}\left(\sigma^{3} \omega\right) & \\
\mathbf{0} & \mathbf{0} & t_{l}\left(\sigma^{2} \omega\right) & \vdots & \ddots \\
\vdots & \mathbf{0} & \mathbf{0} & t_{l}\left(\sigma^{3} \omega\right) & \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)^{\top}
$$

The transpose in the definition above is solely to represent $\mathbf{T}$ in a more readable fashion. We also, as indicated in Definition 3.2.1, construct matrices consisting of collections of words. For the 1-variable construction we need two different constructions: a finite and an infinite version corresponding to the $\varepsilon$-stopping graph defined in Definition 3.2.10. We only give the infinite construction here as it is needed to state our main results. Since the finite version is only used in the proof of Theorem 3.2.21 we postpone its definition until then. Let $\Gamma^{\varepsilon}$ be given and consider the partition of edges of $\Gamma^{\varepsilon}(i)$ into the sets ${ }_{v} E_{w}^{k}(\omega, \varepsilon)$. We assign unique letters to each of the paths of $\boldsymbol{\Gamma}$ that are now the edges of the graphs $\boldsymbol{\Gamma}^{\varepsilon}$. For $V=\{1, \ldots, n\}$, let $\eta$ be a $n \times n$ matrix over arrangements of words that are collections of these letters representing the edges. We let, for $1 \leq q \leq k_{\max }(\varepsilon)$,

$$
\eta_{q}(\omega, \varepsilon)=\left(\begin{array}{cccc}
\bigsqcup_{e \in\left({ }_{1} E_{1}^{q}(\omega, \varepsilon)\right)} e & \bigsqcup_{e \in\left({ }_{1} E_{2}^{q}(\omega, \varepsilon)\right)} e & \ldots & \bigsqcup_{e \in\left({ }_{1} E_{n}^{q}(\omega, \varepsilon)\right)} e \\
\bigsqcup_{e \in\left({ }_{2} E_{1}^{q}(\omega, \varepsilon)\right)} e & \ddots & & \vdots \\
\vdots & & \ddots & \vdots \\
\bigsqcup_{e \in\left(_{n} E_{1}^{q}(\omega, \varepsilon)\right)} e & \bigsqcup_{e \in\left({ }_{n} E_{2}^{q}(\omega, \varepsilon)\right)} e & \ldots & \bigsqcup_{e \in\left({ }_{n} E_{n}^{q}(\omega, \varepsilon)\right)} e
\end{array}\right)
$$

We also need to refer to the two elements corresponding to the identity and zero matrix in this setting. Let $\mathbf{0}_{\varnothing}$ and $\mathbf{1}_{\varepsilon_{0}}$ be $n \times n$ matrices such that

$$
\left(\mathbf{0}_{\varnothing}\right)_{i, j}=\varnothing \quad \text { and } \quad\left(\mathbf{1}_{\varepsilon_{0}}\right)_{i, j}= \begin{cases}\varepsilon_{0} & \text { if } i=j \\ \varnothing & \text { otherwise }\end{cases}
$$

Furthermore let $\widehat{\eta}^{i}(\omega, \varepsilon)=\left\{\mathbf{0}_{\varnothing}, \ldots, \mathbf{0}_{\varnothing}, \eta_{1}(\omega, \varepsilon), \ldots \eta_{k_{\max }(\varepsilon)}(\omega, \varepsilon), \mathbf{0}_{\varnothing}, \mathbf{0}_{\varnothing}, \ldots\right\}$ that is the edges in the partitions arranged by length of original paths and prefixed by $i-1$ occurrences of $\mathbf{0}_{\varnothing}$. The matrix $\mathbf{H}^{\varepsilon}(\omega)$ has row entries given by the vectors $\widehat{\eta}^{i}(\omega, \varepsilon)$, in particular the $k$ th row of $\mathbf{H}^{\varepsilon}(\omega)$ is $\widehat{\eta}^{k}\left(\sigma^{k-1} \omega, \varepsilon\right)$ for $k \geq 0$ :

$$
\left(\mathbf{H}^{\varepsilon}(\omega)\right)_{i, j}=\left(\widehat{\eta}^{i}\left(\sigma^{i-1} \omega, \varepsilon\right)\right)_{j} .
$$

We need the structure as described above to construct the words with the stopping graph. The original attractors to $\boldsymbol{\Gamma}$ do not require this structure as words are constructed by multiplying

$$
\eta_{1}\left(\omega_{1}, 1\right) \eta_{1}\left(\omega_{2}, 1\right) \ldots \eta_{k-1}\left(\omega_{k-1}, 1\right)
$$

and then taking the union over each row, cf. Example 3.2.8. However, when taking the $\varepsilon$-stopping graph for non-trivial $\varepsilon$ we have the added complication that edges in $\boldsymbol{\Gamma}^{\varepsilon}$ arise from paths of potentially different lengths in $\boldsymbol{\Gamma}$. This needs to be considered when applying another edge as it does not only need to start with the correct vertex (the terminal vertex of the previous edge), but also on the length of the equivalent path in $\boldsymbol{\Gamma}$ such that the edges of the correct graph are applied, namely for an edge of length $k$ at iteration step $i$, the graph with realisation $\sigma^{k+i+1} \omega$ has to be used. Writing this in terms of matrix notation makes sense as the row a word sits in relates to how long the path was that created it, so that when multiplying with the next random matrix, the correct graph $\Gamma_{i}$ is applied. It can help to visualise this construction of words in a layered iterative fashion, see Figure 3.2.3. Given $\omega \in \Omega$ one starts with the identity empty word matrix $\mathbf{1}_{\varepsilon_{0}}$ and applies the first set of matrices $\left\{\eta_{i}(\omega)\right\}$ to it to get a collection of $k_{\max }(\varepsilon)$ entries (the second row in the figure). The next row is obtained by applying $\left\{\eta_{i}(\sigma \omega)\right\}$ to the collection of words in the first entry, $\left\{\eta_{i}\left(\sigma^{2} \omega\right)\right\}$ to the second, etc., taking $\sqcup$ unions when necessary. The $k$ th entry of the $i$ th row corresponds to the collection of words $\left(\mathbb{1}_{\varepsilon_{0}} \mathbf{H}^{\varepsilon}(\omega) \ldots \mathbf{H}^{\varepsilon}\left(\sigma^{i-1} \omega\right)\right)_{k}$, where the vector $\mathbb{1}_{\varepsilon_{0}} \mathbf{H}^{\varepsilon}(\omega) \ldots \mathbf{H}^{\varepsilon}\left(\sigma^{i-1} \omega\right)$ is the $i$ th row for $\mathbb{1}_{\varepsilon_{0}}=\left\{\mathbf{1}_{\varepsilon_{0}}, \mathbf{0}_{\varnothing}, \mathbf{0}_{\varnothing}, \ldots\right\}$. These words encode a collection of disjoint cylinders that approximate a random attractor 'from the inside out'.


Figure 3.2: Layered construction of words for $k_{\max }(\varepsilon)=3$.

The last construction we require is a generalisation of the Hutchinson-Moran sum (see (2.1.2)) to this infinite setting. Let $\mathfrak{R}^{s}$, defined recursively, map matrices (or vectors) with entries being matrices over arrangements of words into matrices (or vectors) with entries being matrices over real valued, non-negative functions, preserving the matrix (vector) structure.

$$
\begin{gathered}
\varepsilon_{0} \mapsto 1, \quad \varnothing \mapsto 0, \quad \phi_{1} \mapsto c_{\phi_{1}}^{s}, \quad \phi_{1} \sqcup \phi_{2} \mapsto c_{\phi_{1}}^{s}+c_{\phi_{2}}^{s}, \\
\phi_{1} \odot \phi_{2} \mapsto c_{\phi_{1}}^{s} c_{\phi_{2}}^{s}=c_{\phi_{1} \phi_{2}}^{s},
\end{gathered}
$$

where $c_{\phi}$ is the contraction ratio of the similitude $S_{\phi}$. We define $\mathbf{P}_{\varepsilon}^{s}(\omega)=\mathfrak{R}^{s}\left(\mathbf{H}^{\varepsilon}(\omega)\right)$, that is the matrix consisting of rows

$$
\mathbf{p}_{k}^{s}(\omega, \varepsilon)=\left\{\mathbf{0}, \ldots, \mathbf{0}, p_{1}^{s}(\omega, \varepsilon), \ldots, p_{l}^{s}(\omega, \varepsilon), \mathbf{0}, \ldots\right\}
$$

(c.f. $\left.\widehat{\eta}^{i}(\omega, \varepsilon)\right)$ with

$$
p_{q}^{s}(\omega, \varepsilon)=\left(\begin{array}{cccc}
\sum_{e \in\left({ }_{1} E_{1}^{q}(\omega, \varepsilon)\right)} c_{e}^{s} & \sum_{e \in\left({ }_{1} E_{2}^{q}\right)(\omega, \varepsilon)} c_{e}^{s} & \cdots & \sum_{e \in\left({ }_{1} E_{n}^{q}(\omega, \varepsilon)\right)} c_{e}^{s}  \tag{3.2.4}\\
\sum_{e \in\left(_{2} E_{1}^{q}(\omega, \varepsilon)\right)} c_{e}^{s} & \ddots & & \vdots \\
\vdots & & \ddots & \vdots \\
\sum_{e \in\left({ }_{n} E_{1}^{q}(\omega, \varepsilon)\right)} c_{e}^{s} & \sum_{e \in\left({ }_{n} E_{2}^{q}(\omega, \varepsilon)\right)} c_{e}^{s} & \cdots & \sum_{\left.e \in{ }_{n} E_{n}^{q}(\omega, \varepsilon)\right)} c_{e}^{s}
\end{array}\right)
$$

### 3.2.4 Results for 1-variable RGDS

Having established the basic notation, in this section we collate all the important constructive lemmas and theorems. The proofs will be given in Section 3.4. We begin by stating that the norm $\left\|\|\cdot\|_{\text {sup }}\right.$ and seminorm $\|\|\mathbb{1}$.$\| expand almost surely at$ an exponential rate when multiplying the random matrices defined above; in other words the Lyapunov exponent exists.

Lemma 3.2.13. For $\mathbf{T}$ as above we have that

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left\|\mathbf{T}(\omega) \mathbf{T}(\sigma \omega) \ldots \mathbf{T}\left(\sigma^{k-2} \omega\right) \mathbf{T}\left(\sigma^{k-1} \omega\right)\right\|_{\text {sup }}^{1 / k}=\alpha \tag{3.2.5}
\end{equation*}
$$

where $\alpha=\inf _{k} \mathbb{E}^{\mathrm{geo}}\left(\left\|\mathbf{T}(\omega) \ldots \mathbf{T}\left(\sigma^{k-1} \omega\right)\right\| \|_{\text {sup }}^{1 / k}\right)$, for almost every $\omega \in \Omega$. If we use the seminorm defined in (3.2.2), almost surely,

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left\|\mathbb{1} \mathbf{T}(\omega) \mathbf{T}(\sigma \omega) \ldots \mathbf{T}\left(\sigma^{k-2} \omega\right) \mathbf{T}\left(\sigma^{k-1} \omega\right)\right\|^{1 / k}=\beta \tag{3.2.6}
\end{equation*}
$$

where $\beta \in[0, \infty)$ and $\mathbb{1}=\{\mathbf{1}, \mathbf{0}, \mathbf{0}, \ldots\}$. In particular,

$$
\beta=\inf _{k}\left\|\mathbb{1} \mathbf{T}(\omega) \ldots \mathbf{T}\left(\sigma^{k-1} \omega\right)\right\|^{1 / k} \text { for a.e. } \omega
$$

We apply this result to the our RGDS setting and prove that the Lyapunov exponent is independent of the row of the resulting matrix, assuming $\boldsymbol{\Gamma}^{\varepsilon}$ satisfies Condition 3.2.9.b. We define the norm of matrix products in our setting.

Definition 3.2.14. Let $\varepsilon>0$ and define

$$
\Psi_{\omega}^{k}(s, \varepsilon)=\left\|\mathbb{1} \mathbf{P}_{\varepsilon}^{s}(\omega) \mathbf{P}_{\varepsilon}^{s}(\sigma \omega) \ldots \mathbf{P}_{\varepsilon}^{s}\left(\sigma^{k-1} \omega\right)\right\|^{1 / k} \text { and } \mathbf{\Psi}_{\omega}(s, \varepsilon)=\lim _{k \rightarrow \infty} \Psi_{\omega}^{k}(s, \varepsilon)
$$

We call $\boldsymbol{\Psi}_{\omega}(s, \varepsilon)$ the $(s, \varepsilon)$-pressure of realisation $\omega$, if the limit exists, and we write $\boldsymbol{\Psi}(s, \varepsilon)=\mathbb{E}^{\text {geo }} \boldsymbol{\Psi}_{\omega}(s, \varepsilon)$ for the $(s, \varepsilon)$-pressure.

We note at this point that the notion of pressure is usually applied to $\log \mathbf{\Psi}$. However, in the 1 -variable setting it is more natural to talk about Lyapunov exponents and multiplicativity, rather than additivity, and we take the liberty to call these quantities pressures, rather than the more appropriate 'exponential of pressures'.

Lemma 3.2.15. Assume $\boldsymbol{\Gamma}^{\varepsilon}$, together with a non-trivial probability vector $\vec{\pi}$, is a nontrivial collection of graphs that satisfies Condition 3.2.9.b. The exponential expansion rate of the norm of the matrix is identical to the expansion rate of each individual row sum. We have, almost surely, for every $v \in V$ and $\varepsilon>0$

$$
\lim _{k \rightarrow \infty}\left[\sum_{w \in V}\left(\left\|\mathbb{1} \mathbf{P}_{\varepsilon}^{s}(\omega) \mathbf{P}_{\varepsilon}^{s}(\sigma \omega) \ldots \mathbf{P}_{\varepsilon}^{s}\left(\sigma^{k-1} \omega\right)\right\|_{s}\right)_{v, w}\right]^{1 / k}=\mathbf{\Psi}(s, \varepsilon)
$$

Lemma 3.2.16. For almost all $\omega$ we obtain $\boldsymbol{\Psi}(s, \varepsilon)=\boldsymbol{\Psi}_{\omega}(s, \varepsilon)$. Furthermore $\boldsymbol{\Psi}(s, \varepsilon)$ is monotonically decreasing in $s$ and there exists a unique $s_{H, \varepsilon}$ such that

$$
\boldsymbol{\Psi}\left(s_{H, \varepsilon}, \varepsilon\right)=1
$$

For $s=0$ the pressure function is counting the number of cylinders in the construction. However, as we are considering a lower approximation consisting solely of cylinders with diameter comparable to $\varepsilon$ we can find the box counting dimension of $K_{v}(\omega)$ by a supermultiplicative argument.

Theorem 3.2.17. Almost surely the box counting dimension of $K_{v}(\omega)$ exists, is almost surely independent of $v \in V$, and given by

$$
\begin{equation*}
\operatorname{dim}_{B} K_{v}(\omega)=\lim _{\delta \rightarrow 0} \frac{\log \boldsymbol{\Psi}(0, \delta)}{-\log \delta}=\sup _{\varepsilon>0} \frac{\log \boldsymbol{\Psi}(0, \varepsilon)}{-\log \varepsilon} \tag{3.2.7}
\end{equation*}
$$

Using the construction given in Section 3.2 .1 we define the $\varepsilon$-approximation to our attractor. Note that this is not an $\varepsilon$-close set in the sense of Hausdorff distance, but rather an attractor which satisfies the Uniform Strong Separation Condition (USSC) and approximates the attractor from the 'inside out'. Compare this to the approximation of GDA by suitably chosen IFSs, see Farkas [Fa].

Definition 3.2.18. We say that a graph directed attractor satisfies the uniform strong separation condition (USSC) if for every $v \in V, \Gamma_{k} \in \boldsymbol{\Gamma}, \omega \in \Omega$ and $e_{i}, e_{j} \in{ }_{v} E(k)$,

$$
\text { if } S_{e_{i}}\left(K_{v}(\omega)\right) \cap S_{e_{j}}\left(K_{v}(\omega)\right) \neq \varnothing \text {, then } e_{i}=e_{j}
$$

Definition 3.2.19. The $\varepsilon$-approximation attractor $K_{v, \varepsilon}(\omega)$ of $K_{v}(\omega)$ is defined to be the unique compact set that is the limit of words in the $\varepsilon$-stopping graph $\boldsymbol{\Gamma}^{\varepsilon}$ :

$$
K_{v, \varepsilon}(\omega)=\bigcap_{\substack{i=1 \\ \mathbf{e} \in \Xi_{\varepsilon}^{i}(\omega) \\ \iota(\mathbf{e})=v}}^{\infty} S_{\mathbf{e}}(\Delta), \text { where } \Xi_{\varepsilon}^{i}(\omega)=\bigsqcup \mathbb{1}_{\varepsilon_{0}} \mathbf{H}^{\varepsilon}(\omega) \mathbf{H}^{\varepsilon}(\sigma \omega) \ldots \mathbf{H}^{\varepsilon}\left(\sigma^{i-1} \omega\right)
$$

These sets are easily seen to be subsets of $K_{v}(\omega)$.
Lemma 3.2.20. For every $\varepsilon>0$ and $\omega \in \Omega$ we have $K_{v, \varepsilon}(\omega) \subseteq K_{v}(\omega)$. If $K_{v}(\omega)$ satisfies the USSC, then $K_{v, \varepsilon}(\omega)=K_{v}(\omega)$.

Proof. Note that points in the attractor of $K_{v, \varepsilon}(\omega)$ have (unique) coding given by edges of graphs $\Gamma_{i}^{\varepsilon}$ in $E(\omega, \varepsilon)$. To prove the first claim we observe that for every symbol $e_{i}$ in the coding of $x=\left(e_{1}, e_{2}, \ldots\right) \in K_{v, \varepsilon}(\omega)$ we have an equivalent path travelling through $\boldsymbol{\Gamma}$. Starting at the first edge we have $e_{1} \in E^{q_{1}}(\omega, \varepsilon)$ for some $q_{1}$. This means that $e_{1}=\hat{e}_{1}^{1} \hat{e}_{2}^{1} \ldots \hat{e}_{q_{1}}^{1}$ for $\hat{e}_{j}^{1} \in E\left(\omega_{j}\right)$ such that $\tau\left(\hat{e}_{j}\right)=\iota\left(\hat{e}_{j+1}\right)$. Furthermore $e_{2} \in E^{q_{2}}\left(\sigma^{q_{1}} \omega, \varepsilon\right)$ and so $e_{2}=\hat{e}_{1}^{2} \hat{e}_{2}^{2} \ldots \hat{e}_{q_{2}}^{2}$ for $\hat{e}_{j}^{2} \in E\left(\left(\sigma^{q_{1}} \omega\right)_{j}\right)$ for a similarly linked sequence of edges. Inductively we can replace every edge in $x$ by a finite path in the appropriate manner, giving a coding of a point in $K_{v}(\omega)$ and thus $K_{v, \varepsilon}(\omega) \subseteq K_{v}(\omega)$.

Now assume that the maps of $\boldsymbol{\Gamma}$ satisfy the USSC; for all $v \in V$ and $i \in \Lambda$, every $e_{1}, e_{2} \in{ }_{v} E(i)$ satisfy $S_{e_{1}}\left(K_{\tau\left(e_{1}\right)}(\omega)\right) \cap S_{e_{2}}\left(K_{\tau\left(e_{2}\right)}(\omega)\right)=\varnothing$. But then for all $j \in \Lambda, e_{11} \in{ }_{w_{1}} E(j)$ and $e_{21} \in{ }_{w_{2}} E(j)$, where $w_{1}=\tau\left(e_{1}\right)$ and $w_{2}=\tau\left(e_{2}\right)$, we have $S_{e_{1} e_{11}}\left(K_{\tau\left(e_{11}\right)}(\omega)\right) \cap S_{e_{2} e_{21}}\left(K_{\tau\left(e_{21}\right)}(\omega)\right)=\varnothing$. Inductively none of the compositions overlap. But this means that every path traversing through $\boldsymbol{\Gamma}$ must also have an equivalent path traversing through $\Gamma^{\varepsilon}$ as no paths get deleted due to the non-existent overlaps. Hence, assuming the USSC, $K_{v}(\omega) \subseteq K_{v, \varepsilon}(\omega)$.

Having established the almost sure box counting dimension we now consider the Hausdorff dimensions of our approximation sets. These are given by the unique $s$ such that the pressure defined in (3.2.14) equals 1 and form a lower bound of the Hausdorff dimension of $K_{v}(\omega)$.

Theorem 3.2.21. For all $\varepsilon>0$ the almost sure Hausdorff dimension of $K_{v, \varepsilon}(\omega)$ is independent of $v \in V$ and

$$
\operatorname{dim}_{H} K_{v, \varepsilon}(\omega)=s_{H, \varepsilon} \text { where } \boldsymbol{\Psi}\left(s_{H, \varepsilon}, \varepsilon\right)=1
$$

where $s_{H, \varepsilon}$ is given by Lemma 3.2.16.
We get the following important corollary to Lemma 3.2.20 and Theorem 3.2.21.
Corollary 3.2.22. The Hausdorff dimension of the attractor of the 1-variable selfsimilar $R G D S$ is, almost surely, bounded below by $s_{H, \varepsilon}$ for all $\varepsilon>0$

$$
\operatorname{dim}_{H} K_{v}(\omega) \geq \operatorname{dim}_{H} K_{v, \varepsilon}(\omega)=s_{H, \varepsilon}
$$

Our main result is the almost sure equality of Hausdorff, box-counting and therefore also packing dimension, of $K_{v}(\omega)$ for all $v \in V$.

Theorem 3.2.23 (Main Theorem). Let $\boldsymbol{\Gamma}$ be a non-trivial, stochastically strongly connected collection of graphs with associated self-similar attractors $\left\{K_{v}\right\}_{v \in V}$. Then $s_{H, \varepsilon} \rightarrow s_{B}$ as $\varepsilon \rightarrow 0$, where

$$
s_{B}=\lim _{\varepsilon \rightarrow 0} \frac{\log \boldsymbol{\Psi}(0, \varepsilon)}{-\log \varepsilon}
$$

and hence, almost surely,

$$
\operatorname{dim}_{H} K_{v}(\omega)=\operatorname{dim}_{P} K_{v}(\omega)=\operatorname{dim}_{B} K_{v}(\omega)=s_{B}
$$

where $s_{B}$ is independent of $v$.
If the attractor of $\boldsymbol{\Gamma}$ satisfies the USSC we can in addition give an easy description of the almost sure dimension of the attractor.

Corollary 3.2.24. Assume the $U S S C$ is satisfied, then $s_{H, \varepsilon}=s_{B}$ for all $\varepsilon>0$, and, almost surely,

$$
\begin{equation*}
\operatorname{dim}_{H} K_{v}(\omega)=\operatorname{dim}_{B} K_{v}(\omega)=s_{O}, \text { where } \lim _{k \rightarrow \infty}\left\|p_{1}^{s O}(\omega, 1) \ldots p_{1}^{s O}\left(\sigma^{k-1} \omega, 1\right)\right\|_{1}^{1 / k}=1 \tag{3.2.8}
\end{equation*}
$$

Equivalently, so is the unique non-negative real satisfying

$$
\inf _{k}\left(\mathbb{E}^{\mathrm{geo}}\left\|p_{1}^{s O}(\omega, 1) \ldots p_{1}^{s O}\left(\sigma^{k-1} \omega, 1\right)\right\|_{1}\right)^{1 / k}=1
$$

Because $V$-variable self-similar sets are 1-variable self-similar RGDS and under the assumption that $\boldsymbol{\Gamma}$ satisfies the USSC, Corollary 3.2.24 reduces to the results in Barnsley et al. [BHS3]. Additionally we get the following new result:

Corollary 3.2.25. Let $F(\omega)$ be the attractor of a $V$-variable random iterated function system. Irrespective of overlaps, almost surely,

$$
\operatorname{dim}_{H} F(\omega)=\overline{\operatorname{dim}}_{B} F(\omega)=\operatorname{dim}_{B} F(\omega)
$$

This follows since the construction of a $V$-variable set relies on a vector of sets of dimension $V$. Associating a vertex to each of these sets we can chose graphs appropriately.

However, in contrast to all other dimensions, the Assouad dimension 'maximises' the dimension. This phenomenon has been observed in many different settings, which is not surprising as the Assouad dimension 'searches' for the relatively most complex part in the attractor and the random construction allows a very complex pattern to arise on many levels with probability one, even though these events get 'ignored' by the averaging behaviour of Hausdorff and box-counting dimension.

Definition 3.2.26. Let $\boldsymbol{\Gamma}$ be as above. We define the $\varepsilon$-joint spectral radius by

$$
\mathfrak{P}(\varepsilon)=\lim _{k \rightarrow \infty}\left(\sup _{\omega \in \Omega}\left\{\| \| \mathbb{1} \mathbf{P}_{\varepsilon}^{0}(\omega) \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{1} \omega\right) \ldots \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{k-1} \omega\right)\| \|\right\}\right)^{1 / k}
$$

We note that the spectral radius coincides for almost every $\zeta \in \Omega$ with the limit in (3.2.5):

$$
\begin{equation*}
\mathfrak{P}(\varepsilon)=\alpha=\lim _{k \rightarrow \infty}\| \| \mathbf{P}_{\varepsilon}^{0}(\zeta) \ldots \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{k-1} \zeta\right)\| \|_{\sup }^{1 / k} \tag{3.2.9}
\end{equation*}
$$

We demonstrate this in the proof of Theorem 3.2.27.
Theorem 3.2.27. Assume $K_{v}(\omega) \subset \mathbb{R}^{d}$ is not contained in any d-1-dimensional hyperplane for all $v \in V$ and almost all $\omega \in \Omega$. Irrespective of separation conditions, almost surely,

$$
\begin{equation*}
\operatorname{dim}_{A} K_{v}(\omega) \geq \min \left\{d, \sup _{\varepsilon>0} \frac{\log \mathfrak{P}(\varepsilon)}{-\log \varepsilon}\right\} \tag{3.2.10}
\end{equation*}
$$

Further, the USSC implies equality in (3.2.10).

## $3.3 \infty$-variable Random Graph Directed Systems

In this section we introduce and provide results for the $\infty$-variable construction. In a similar fashion to Section 3.2 we start by giving a description of the model and then state the results. For the $\infty$-variable construction many proofs turn out to be simpler and to save space we give less detail in some of the proofs as they follow from standard arguments.

### 3.3.1 Notation and Model

The $\infty$-variable model, sometimes called random recursive or $V$-variable for $V \rightarrow \infty$, is a very intuitive model that is usually defined in a recursive manner (see [F1] and [G]). These are usually described in terms of random code trees. For an overview of that notation we refer the reader to Järvenpää, et al. [JJKKSS, JJWW, JJLS] who studied a slightly different random model with a 'neck structure'. However, to keep notation consistent we will describe the random recursive construction within our framework of arrangements of words. Note that the $\infty$-variable construction overlaps with the notion of random graph directed attractors, considered in Olsen [O1], and some of the results here follow directly from the ones in aforementioned book.

As in Section 3.2 we are given a collection of graphs $\boldsymbol{\Gamma}$ with associated nontrivial probability vector $\vec{\pi}$. We further assume that all the maps given by the edges of the $\Gamma_{i}$ are contracting similitudes and that all conditions in Definition 3.2.9 are satisfied. However, we can generalise the results to include percolation by adapting Condition 3.2.9.a.

Definition 3.3.1. Let $\boldsymbol{\Gamma}=\left\{\Gamma_{i}\right\}_{i \in \Lambda}$ be a finite collection of graphs, sharing the same vertex set $V$. We say that the collection $\boldsymbol{\Gamma}$ is a non-trivial surviving collection of graphs if for every $v \in V$ we have $\mathbb{E}\left(\#_{v} E\left(\omega_{1}\right)\right)>1$ : there exists positive probability that the resulting $\infty$-variable $R G D S$ coding does not consist of only $\varnothing$, and there exist $i, j \in \Lambda$ and $e_{1} \in \Gamma(i)$ and $e_{2} \in \Gamma(j)$ such that $S_{e_{1}} \neq S_{e_{2}}$.

Definition 3.3.2. For $v \in V$ let $\mathbf{F}_{v}^{0}$ be a vector of length $n=|V|$ defined by

$$
\left(\mathbf{F}_{v}^{0}\right)_{i}= \begin{cases}\varepsilon_{0} & \text { if } i=v \\ \varnothing & \text { otherwise }\end{cases}
$$

We then define inductively,

$$
\left(\mathbf{F}_{v}^{k+1}\right)_{i}=\bigsqcup_{j=1}^{n} \bigsqcup_{\mathbf{w} \in\left(\mathbf{F}_{v}^{k}\right)_{j}} \bigsqcup_{e \in_{j} E_{i}\left(\xi_{\mathbf{w}}\right)} \mathbf{w} \odot e
$$

where $\xi_{\mathbf{w}}$ is the random variable given by $\mathbb{P}\left(\xi_{\mathbf{w}}=i\right)=\pi_{i}$ for $i \in \Lambda$ and independent of $\mathbf{w}$.

The $\infty$-variable RGDS coding is then given by $\mathbf{F}_{v}=\lim _{k \rightarrow \infty} \mathbf{F}_{v}^{k}$ and we define the attractor $F_{v}$ of the $\infty$-variable Random Graph Directed System to be the projection of our coding set:

$$
F_{v}=\bigcap_{k=1}^{\infty} \bigcup_{\mathbf{w} \in \mathbf{F}_{v}^{k}} S_{w_{1}} \circ S_{w_{2}} \circ \cdots \circ S_{w_{k}}(\Delta)
$$

Given a collection of graphs satisfying Conditions 3.2.9.b, 3.2.9.c and 3.3.1 that do not necessarily satisfy the USSC we obtain an analogous definition of the $\varepsilon$ approximation.

### 3.3.2 Results for $\infty$-variable RGDS

Let $\mathcal{Q}$ be the space of all possible realisations of the random recursive process, $\mathcal{Q}$ is a labeled tree encoding which graph $\Gamma(i)$ was chosen at each node in the construction of the tree. By the same argument as in Section 3.2.2, for every fixed $\varepsilon>0$ there exists a finite constant $k_{\max }(\varepsilon)$ such that for all $\mathbf{w} \in \mathbf{F}_{v}^{k_{\max }(\varepsilon)}$ we have $\left|S_{\mathbf{w}}(\Delta)\right| \leq \varepsilon$ for all realisations $q \in \mathcal{Q}$. Now $F_{v}$ is a function mapping realisations to compact sets, depending solely on the random variable $q \in \mathcal{Q}$ (picked according to the Borel probability measure induced by $\vec{\pi}$ ) but, in general, we ignore the $q$ in the notation of $F_{v}(q)$.

Definition 3.3.3. Let $\boldsymbol{\Gamma}$ satisfy the conditions in Definition 3.2.9. Let $\mathcal{Q}$ be the space of all possible realisations of the random recursive process, we define the set of edges
(words) of length $j$ for realisation $q$ to be $\mathbf{F}_{v}^{k}(q)$ and the $\varepsilon$-stopping set of edge sets to be

$$
E_{v}^{*}(q, \varepsilon)=\left\{\mathbf{e} \in \bigcup_{i=1}^{k_{\max }(\varepsilon)} \mathbf{F}_{v}^{i}(q) \mid c_{\mathbf{e}} \leq \varepsilon \text { but } c_{\mathbf{e}^{\ddagger}}>\varepsilon\right\} .
$$

Again let the set of all possible subsets such that images under $S$ are pairwise disjoint be

$$
\mathcal{E}(q, \varepsilon)=\left\{U \subseteq E_{v}^{*}(q, \varepsilon) \mid \forall \mathbf{e}, \mathbf{f} \in U \text { we have } S_{\mathbf{e}} \cap S_{\mathbf{f}}=\varnothing\right\} .
$$

Consider the element of maximal cardinality (choosing arbitrarily if there is more than one) $E_{v}(q, \varepsilon) \in \mathcal{E}(q, \varepsilon)$. As $E_{v}(q, \varepsilon)$ only depends, at most, on the first $k_{\max }(\varepsilon)$ entries, the set $\left\{E_{v}(q, \varepsilon)\right\}_{q \in \mathcal{Q}}$ is finite and we write

$$
\boldsymbol{\Gamma}^{\varepsilon}=\left\{\Gamma^{\varepsilon}(q)\right\}_{q \in \mathcal{Q}}=\left\{\left(V, E_{v}(q, \varepsilon)\right)\right\}_{q \in \mathcal{Q}}
$$

for the $\varepsilon$-stopping graph.

As $\Gamma^{\varepsilon}$ is finite we will set up a new code space for each of the graphs $\Gamma^{\varepsilon}(q)$ that we will index by $\Lambda_{\varepsilon}$. Similarly there exists positive probability of picking graph $\Gamma^{\varepsilon}(\lambda)$ for $\lambda \in \Lambda_{\varepsilon}$. Unlike the 1 -variable case, the choice of graph $\boldsymbol{\Gamma}$ is independent for each node, a property which transfers to the setting of the $\varepsilon$-stopping graph.

Lemma 3.3.4. The random recursive algorithm that generates the attractor of the $\varepsilon$-stopping graphs $\boldsymbol{\Gamma}^{\varepsilon}$ is identical to the process that generates the attractor of the $R G D S \boldsymbol{\Gamma}$. Note that for $t \geq 1$ the identity $\boldsymbol{\Gamma}=\boldsymbol{\Gamma}^{t}$ holds and we trivially have that the attractor of the $R G D S \boldsymbol{\Gamma}^{\varepsilon}$ is a subset of the attractor of $\boldsymbol{\Gamma}$, with equality holding if the attractor of $\boldsymbol{\Gamma}$ satisfies the USSC.

We omit a detailed proof as both processes can easily seen to be $\infty$-variable RGDS. Now let $\mathbf{K}^{\varepsilon}(q)$ be the matrix consisting of arrangements of words related to $\Gamma^{\varepsilon}(q)$. Let ${ }_{v} E_{w}(\Gamma(q))$ be the collection of edges $e$ of $\Gamma(q)$ so that $\iota(e)=v$ and $\tau(e)=w$, and define

$$
\mathbf{K}^{\varepsilon}(q)=\left(\begin{array}{ccc}
\bigsqcup_{e \epsilon_{1} E_{1}\left(\Gamma^{\varepsilon}(q)\right)} e & \cdots & \bigsqcup_{e \epsilon_{1} E_{n}\left(\Gamma^{\varepsilon}(q)\right)} e \\
\vdots & \ddots & \vdots \\
\bigsqcup_{e \in_{n} E_{1}\left(\Gamma^{\varepsilon}(q)\right)} e & \cdots & \bigsqcup_{e \in_{n} E_{n}\left(\Gamma^{\varepsilon}(q)\right)} e
\end{array}\right) .
$$

Theorem 3.3.5. Let $\boldsymbol{\Gamma}$ be a finite collection of graphs satisfying Conditions 3.2.9.b, 3.2.9.c and 3.3.1 with associated non-trivial probability vector $\vec{\pi}$. Let $F_{v}$ be the attractor of the random recursive construction, then almost surely the Hausdorff and the upper box counting dimension agree and thus,

$$
\operatorname{dim}_{H} F_{v}=\operatorname{dim}_{P} F_{v}=\operatorname{dim}_{B} F_{v} .
$$

We end this section by stating the Assouad dimension of this construction.

Theorem 3.3.6. Irrespective of overlaps and conditioned on $F_{v} \neq \varnothing$, the Assouad dimension of $F_{v}$ is a.s. bounded below by

$$
\begin{equation*}
\operatorname{dim}_{A} F_{v} \geq \min \left\{d, \sup _{\varepsilon>0} \max _{q \in \mathcal{Q}} \frac{\log \rho_{s}\left(\mathfrak{R}^{0} \mathbf{K}^{\varepsilon}(q)\right)}{-\log \varepsilon}\right\} \tag{3.3.1}
\end{equation*}
$$

where $\rho_{s}$ is the spectral radius of a matrix. If the USSC is satisfied, then equality holds in (3.3.1) almost surely.

### 3.4 Proofs

### 3.4.1 Proof of Lemma 3.2.13

First we prove the convergence in equation (3.2.5). Let $n, m \in \mathbb{N}_{0}, n<m$ and define the random variable $Y_{n, m}$ as

$$
Y_{n, m}(\omega)=\log \left\|\mid \mathbf{T}\left(\sigma^{n} \omega\right) \mathbf{T}\left(\sigma^{n+1} \omega\right) \ldots \mathbf{T}\left(\sigma^{m-1} \omega\right)\right\| \|_{\text {sup }}
$$

Note that, as the row norm is submultiplicative,

$$
\begin{aligned}
Y_{0, n+m}(\omega) & =\log \left\|\mid \mathbf{T}(\omega) \ldots \mathbf{T}\left(\sigma^{n-1} \omega\right) \mathbf{T}\left(\sigma^{n} \omega\right) \ldots \mathbf{T}\left(\sigma^{n+m-1} \omega\right)\right\| \|_{\text {sup }} \\
& \leq \log \left(\left\|\left|\left\|\mathbf{T}(\omega) \ldots \mathbf{T}\left(\sigma^{n-1} \omega\right)\right\|\left\|_{\text {sup }}\right\|\right| \mathbf{T}\left(\sigma^{n} \omega\right) \ldots \mathbf{T}\left(\sigma^{n+m-1} \omega\right)\right\| \|_{\text {sup }}\right) \\
& =\log \left\|\mathbf{T}(\omega) \ldots \mathbf{T}\left(\sigma^{n-1} \omega\right)\right\|\left\|_{\text {sup }}+\log \right\| \mathbf{T}\left(\sigma^{n} \omega\right) \ldots \mathbf{T}\left(\sigma^{n+m-1} \omega\right)\| \|_{\text {sup }} \\
& =Y_{0, n}(\omega)+Y_{n, m}(\omega)
\end{aligned}
$$

As $\mathbb{P}$ is an ergodic probability measure it follows from Kingman's subadditive ergodic theorem that almost surely

$$
\lim _{k \rightarrow \infty} \frac{Y_{0, k}}{k}=\inf _{k} \mathbb{E} \frac{Y_{0, k}}{k}=\inf _{k} \mathbb{E} \log \left\|\mathbf{T}\left(\sigma^{k-1} \omega\right) \ldots \mathbf{T}(\omega)\right\|_{\text {sup }}^{1 / k}=\log \alpha
$$

giving the required result.

The second part is made slightly more difficult because of the interdependence between the steps. We will show stochastic quasi-subadditivity, bounding the subadditive defects, and make use of Proposition 1.4.4.

Writing $\mathbf{u}_{k}(\omega)=\mathbf{T}(\omega) \ldots \mathbf{T}\left(\sigma^{k-1} \omega\right)$ the term $\mathbb{1} \mathbf{u}_{k}(\omega)$ is a matrix-valued vector with at most $l k$ positive entries, all appearing in the first $l k$ rows, where $l \geq 1$ as in Section 3.2.3. We have

$$
\begin{aligned}
\left\|\mathbb{1} \mathbf{u}_{n+m}(\omega)\right\| \| & =\left\|\mathbb{1} \mathbf{u}_{n}(\omega) \mathbf{u}_{m}\left(\sigma^{n} \omega\right)\right\| \\
& =\| \| \mathbb{1} \mathbf{u}_{n}(\omega) \mathbf{u}_{m}\left(\sigma^{n} \omega\right)\left\|_{s}\right\|_{\text {row }} \\
& =\left\|\sum_{j=0}^{n l-1}\left(\mathbb{1} \mathbf{u}_{n}(\omega)\right)_{j}\right\| \mathbb{1} \mathbf{u}_{m}\left(\sigma^{n+j} \omega\right)\left\|_{s}\right\|_{\text {row }} \\
& \leq \sum_{j=0}^{n l-1}\left\|\left(\mathbb{1} \mathbf{u}_{n}(\omega)\right)_{j}\right\| \mathbb{1} \mathbf{u}_{m}\left(\sigma^{n+j} \omega\right)\left\|_{s}\right\|_{\text {row }} \text { by subadditivity of norms, } \\
& \leq \sum_{j=0}^{n l-1}\left\|\left(\mathbb{1} \mathbf{u}_{n}(\omega)\right)_{j}\right\|_{\text {row }}\left\|\mathbb{1} \mathbf{u}_{m}\left(\sigma^{n+j} \omega\right)\right\|
\end{aligned}
$$

by submultiplicativity of the row norm,

$$
\begin{equation*}
\leq n l\left\|\left(\mathbb{1} \mathbf{u}_{n}(\omega)\right)_{j_{\max }(n, m, \omega)}\right\|_{\text {row }}\left\|\mathbb{1} \mathbf{u}_{m}\left(\sigma^{n+j_{\max }(n, m, \omega)} \omega\right)\right\| \tag{3.4.1}
\end{equation*}
$$

$$
\text { for } j_{\max } \text { maximising the sum, }
$$

$$
\begin{equation*}
\leq c n l\| \| \mathbb{1} \mathbf{u}_{n}(\omega)\| \|\left\|\mathbb{1} \mathbf{u}_{m}\left(\sigma^{n} \omega\right)\right\| \| \tag{3.4.2}
\end{equation*}
$$

The last inequality holds for some sufficiently large $c>0$ upon noting that for large $n, m$ the additional shift $j_{\max }$ becomes insignificant as the difference in growth is captured by the 'overestimate' of the first term. Therefore we have quasi-subadditivity and by symmetry

$$
\left\|\mathbb{1} \mathbf{u}_{n+m}(\omega)\right\|\|\leq c m\| \mathbb{1} \mathbf{u}_{n}(\omega)\| \|\left\|\mathbb{1} \mathbf{u}_{m}\left(\sigma^{n} \omega\right)\right\| \|,
$$

for some $c>0$. Considering $\log \left\|\mathbb{1} \mathbf{u}_{n}(\omega)\right\|$ as a random variable, the subadditive defect becomes

$$
c_{m}=\log \left\|\mathbb{1} \mathbf{u}_{n+m}(\omega)\right\|-\log \left\|\mathbb{1} \mathbf{u}_{n}(\omega)\right\|\|-\log \| \mathbb{1} \mathbf{u}_{m}\left(\sigma^{n} \omega\right)\| \| \leq \log c m
$$

Clearly $\mathbb{E}(\log c m)^{+}=\log c m$ and $c_{m} / m \rightarrow 0$. Since $\sigma$ is an (invariant) ergodic transformation with respect to $\mathbb{P}$, applying Proposition 1.4.4 finishes the proof.

### 3.4.2 Proof of Lemma 3.2.15

The boundedness of the entries in the matrix entries of $\mathbb{1} \mathbf{u}_{k}(\omega)$, combined with the linear growth of the number of positive entries of the vector, implies that for some constant $c>0$,

$$
\max _{j}\left\|\left(\mathbb{1} \mathbf{u}_{k}(\omega)\right)_{j}\right\|_{\text {row }} \leq\left\|\mathbb{1} \mathbf{u}_{k}(\omega)\right\| \leq c k \max _{j}\left\|\left(\mathbb{1} \mathbf{u}_{k}(\omega)\right)_{j}\right\|_{\text {row }}
$$

Therefore the value of both terms increase at the same exponential rate. In addition, the $j_{\max }^{k}$ maximising the norm cannot move arbitrarily with increasing $k$. First it must be increasing monotonically, although not necessarily strictly so. But the value can also not jump unboundedly, as the matrices that the matrix with maximal absolute norm is multiplied with have bounded entries as well. Even though we will not prove it here, it can be shown that almost surely $j_{\max }^{k} /(l k) \rightarrow \rho$ as $k \rightarrow \infty$ for some $\rho \in[0,1]$ dependent only on $\Gamma^{\varepsilon}$ and $\vec{\pi}$. Let $R_{v}(k)$ be the row sum for row $v$ in the maximal matrix at multiplication step $k$ and $R_{v}^{T}(k)$ be the total of that row over all matrices. That is

$$
R_{v}(k)=\sum_{i=1}^{n}\left[\left(\mathbb{1} \mathbf{u}_{k}(\omega)\right)_{j_{\max }(n, m, \omega)}\right]_{v, i} \quad \text { and } \quad R_{v}^{T}(k)=\sum_{j=1}^{\infty} \sum_{i=1}^{n}\left[\left(\mathbb{1} \mathbf{u}_{k}(\omega)\right)_{j}\right]_{v, i} .
$$

Furthermore let $R_{\max }(k)=\max _{v \in V} R_{v}(k)$. One immediately has on a full measure set

$$
\left|R_{\max }(k)^{1 / k}-\left\|\mathbb{1} \mathbf{u}_{k}(\omega)\right\|^{1 / k}\right| \rightarrow 0 \text { as } k \rightarrow \infty
$$

so proving Lemma 3.2.15 can be achieved by showing $R_{v}(k) \asymp R_{\max }(k)$ holds almost surely for all $v \in V$. The upper bound $R_{v}(k) \leq R_{\max }(k)$ is trivial.

For the lower bound, since $\boldsymbol{\Gamma}$ is stochastically strongly connected, i.e. satisfies Condition 3.2.9.b, we can construct a finite word $\omega^{r} \in \Lambda^{*}$ that links all vertices, starting at $v=v_{1}$. That is $\omega^{r}=\omega^{v_{1}, v_{2}} \omega^{v_{2}, v_{3}} \ldots \omega^{v_{n}, v_{1}} \omega^{v_{1}, v_{2}} \ldots \omega^{v_{n-1}, v_{n}}$. Clearly $\mathbb{P}\left(\left[\omega^{r}\right]\right)>0$. Consider now the maximal element in the multiplication of $\mathbf{u}_{q k}(\omega)=$ $\mathbf{u}_{k}(\omega) \ldots \mathbf{u}_{k}\left(\sigma^{(q-1) k} \omega\right)$, that is $j_{\max }(q k, k, \omega)$. There exists a random variable, the holding time $H(i)$, that gives the number of multiplication steps $q$ between the $i-1$ and $i$ th time such that $\omega^{r}$ is applied to that element. We have $\sigma^{q k+j_{\max }(q k, k, \omega)}(\omega)=$ $\omega^{r}$. We can without loss of generality assume that $H(i)$ are i.i.d. random variables with finite expectation $\mathbb{E} H(i)<\infty$. Let $W(k)$ be the waiting time for the $k$ th jump, $W(k)=\sum_{i=0}^{k-1} H(i)$ and define $N_{k}$ to be the unique random integer such that

$$
W\left(N_{k}\right) \leq k<W\left(N_{k}+1\right)
$$

There exists a uniform constant $\underline{\lambda}>0$ such that, for all $v \in V$,

$$
R_{v}\left(W\left(N_{k}\right)+\left|\omega^{r}\right|\right) \geq \underline{\lambda} R_{\max }\left(W\left(N_{k}\right)\right) .
$$

Since this holds for all $k$ we can furthermore find a lower bound to the value of $R_{v}$ between occurrences of $\omega^{r}$ by considering the time it takes between occurrences. Condition 3.2.9.a implies non-extinction and there exists contraction rate $\gamma>0$, such that for $k$ and $N_{k}$ as above we have

$$
\liminf _{k \rightarrow \infty} R_{v}^{T}(k)^{1 / k} \geq \liminf _{k \rightarrow \infty}\left(\underline{\lambda} R_{\max }\left(W\left(N_{k}\right)\right) \underline{\gamma}^{H(k)}\right)^{1 / k} \geq \liminf _{k \rightarrow \infty}(\beta-\varepsilon)^{W\left(N_{k}\right) / k} \underline{\gamma}^{H(k) / k}
$$

where the last inequality holds on a set of measure 1 for every $\varepsilon>0$. But we also have that

$$
W\left(N_{k}\right) / k \leq 1<W\left(N_{k}+1\right) / k
$$

and as $W\left(N_{k}\right) / k<1$ and $W\left(N_{k}+1\right) / k=W\left(N_{k}\right) / k+H\left(N_{k}+1\right) / k$ we have by the law of large numbers that almost surely $W\left(N_{k}\right) / k \rightarrow 1$ and $H(k) / k \rightarrow 0$, and hence on a set of measure 1 ,

$$
\liminf _{k \rightarrow \infty} R_{v}^{T}(k)^{1 / k} \geq(\beta-\varepsilon)
$$

for every $\varepsilon$ and $v$. Noting that $R_{v}^{T}(k) \asymp R_{v}(k)$ completes the proof.

### 3.4.3 Proof of Lemma 3.2.16

The almost sure convergence of $\boldsymbol{\Psi}(s, \varepsilon)$ follows directly from Lemma 3.2.13 and we now show that $\boldsymbol{\Psi}_{\omega}(s, \varepsilon)$ is monotonically decreasing in $s$ and continuous for almost all $\omega \in \Omega$. Consider an arbitrary Hutchinson-Moran sum that arises in the Hutchinson-Moran-like matrix in (3.2.4),

$$
\sum_{e \in\left({ }_{i} E_{j}^{q}(\omega, \varepsilon)\right)} c_{e}^{s}
$$

We immediately get

$$
\begin{equation*}
\sum_{e \in\left({ }_{i} E_{j}^{q}(\omega, \varepsilon)\right)} c_{e}^{s+\delta} \leq \bar{\gamma}_{q}^{\delta} \sum_{e \in\left({ }_{i} E_{j}^{q}(\omega, \varepsilon)\right)} c_{e}^{s}, \text { where } \bar{\gamma}_{q}(\omega)=\max _{\substack{i, j \in\{1, \ldots, n\} \\ \mathbf{e} \in\left(\mathcal{C}_{i} E_{j}^{q}(\omega, \varepsilon)\right)}} c_{\mathbf{e}} . \tag{3.4.3}
\end{equation*}
$$

For $\varepsilon>0$ there are only finitely many different $p_{q}^{s}(\omega, \varepsilon)$ and $\mathbf{p}^{s}(\omega)$, see the discussion of Lemma 3.2.11. Thus we can find

$$
\begin{equation*}
\bar{\gamma}=\max _{\substack{q \in\{1, \ldots, l\} \\ \omega \in \Omega}} \bar{\gamma}_{q}(\omega), \tag{3.4.4}
\end{equation*}
$$

where $0<\bar{\gamma}<1$. Similarly we can find the minimal such contraction $0<\underline{\gamma} \leq \bar{\gamma}<1$. Combining this with (3.4.3) we surely deduce, in turn,

$$
\begin{align*}
\underline{\gamma}^{\delta} p_{q}^{s}(\omega, \varepsilon) & \leq p_{q}^{s+\delta}(\omega, \varepsilon) \leq \bar{\gamma}^{\delta} p_{q}^{s}(\omega, \varepsilon) \\
\underline{\gamma}^{\delta} \mathbf{p}^{s}(\omega, \varepsilon) & \leq \mathbf{p}^{s+\delta}(\omega, \varepsilon) \leq \bar{\gamma}^{\delta} \mathbf{p}^{s}(\omega, \varepsilon), \\
\underline{\gamma}^{\delta} \mathbf{P}_{\varepsilon}^{s}(\omega) & \leq \mathbf{P}_{\varepsilon}^{s+\delta}(\omega) \leq \bar{\gamma}^{\delta} \mathbf{P}_{\varepsilon}^{s}(\omega), \tag{3.4.5}
\end{align*}
$$

where $\leq$ is taken to be entry-wise, i.e. for matrices $M \leq N$ if and only if $M_{i, j} \leq N_{i, j}$ for all $i, j$. Using (3.4.5) we can bound the $s+\delta$ pressure

$$
\begin{aligned}
& \Psi_{\omega}^{k}(s+\delta, \varepsilon)=\left\|\mathbb{1} \mathbf{P}_{\varepsilon}^{s+\delta}(\omega) \ldots \mathbf{P}_{\varepsilon}^{s+\delta}\left(\sigma^{k-1} \omega\right)\right\| \|^{1 / k} \\
& \geq \underline{\gamma}^{\delta}\left\|\mathbb{1} \mathbf{P}_{\varepsilon}^{s}(\omega) \ldots \mathbf{P}_{\varepsilon}^{s}\left(\sigma^{k-1} \omega\right)\right\|^{1 / k} \geq \underline{\gamma}^{\delta} \Psi_{\omega}^{k}(s, \varepsilon)
\end{aligned}
$$

and similarly for the upper bound we have $\Psi_{\omega}^{k}(s+\delta, \varepsilon) \leq \bar{\gamma} \Psi_{\omega}^{k}(s, \varepsilon)$. Therefore, if the limit exists, $\gamma^{\delta} \boldsymbol{\Psi}_{\omega}(s, \varepsilon) \leq \boldsymbol{\Psi}_{\omega}(s+\delta, \varepsilon) \leq \bar{\gamma}^{\delta} \boldsymbol{\Psi}_{\omega}(s, \varepsilon)$. Thus as $0<\gamma \leq \bar{\gamma}<1$, $\Psi_{\omega}(s, \varepsilon)$ is strictly decreasing in $s$ and, taking $\delta \rightarrow 0$, is easily seen to be continuous for almost every $\omega$ and thus $\boldsymbol{\Psi}(s, \varepsilon)$ has the same property. Letting $\delta \rightarrow \infty$ we see $\boldsymbol{\Psi}(s+\delta, \varepsilon) \rightarrow 0$ and $\boldsymbol{\Psi}(0, \varepsilon) \geq 1$ by the non-extinction given by Condition 3.2.9.a. The existence and uniqueness of $s_{H, \varepsilon}$ then follows.

### 3.4.4 Proof of Theorem 3.2.17

Note that the proof below directly implies that the box dimension exists almost surely.
Our argument relies on a supermultiplicative property of approximations of $\varepsilon$ stopping graphs given by (3.4.7). Before we derive that expression we establish a connection between the least number of sets of diameter $\varepsilon$ or less needed to cover our attractor $N_{\varepsilon}\left(K_{v}(\omega)\right)$ and the number of edges of our $\varepsilon$-stopping graph $\left.\right|_{v} E(\omega, \varepsilon) \mid$. By the definition of the $\varepsilon$-stopping graph we have that for all $\mathbf{e} \in{ }_{v} E(\omega, \varepsilon)$ the diameter of $S_{\mathbf{e}}(\Delta)$ is of order $\varepsilon$, see Definition 3.2.10. Since we also have that the images of the stopping $\left\{S_{\mathbf{e}}(\Delta)\right\}_{\mathbf{e} \in{ }_{v} E(\omega, \varepsilon)}$ are pairwise disjoint, $\left\{S_{\mathbf{e}}(\Delta)\right\}_{\mathbf{e} \in_{v} E(\omega, \varepsilon)}$ may not form a
cover of $K_{v}(\omega)$. But since the construction is maximal, the image of any word (edge) that was deleted must intersect another image of a word that was kept, which means that to form a cover of $K_{v}(\omega)$ one needs at most $\left.3^{d}\left\lceil\underline{\gamma}^{-1}\right\rceil\right|_{v} E(\omega, \varepsilon) \mid d$-dimensional hypercubes of sidelength $\varepsilon$ to form a cover and hence $N_{\varepsilon}\left(K_{v}(\omega)\right) \leq\left. 3^{d}\left\lceil\underline{\gamma}^{-1}\right\rceil\right|_{v} E(\omega, \varepsilon) \mid$. On the other hand, any element in the minimal cover for $N_{\varepsilon}\left(K_{v}(\omega)\right)$ can intersect at most a uniformly bounded number of elements in $\left\{S_{\mathbf{e}}(\Delta)\right\}_{\mathbf{e}}$, as otherwise the elements in $\left\{S_{\mathbf{e}}(\Delta)\right\}_{\mathrm{e}}$ would intersect. Hence there exists $k_{\text {min }}>0$ such that $N_{\varepsilon}\left(K_{v}(\omega)\right) \geq$ $\left.k_{\min }\right|_{v} E(\omega, \varepsilon) \mid$ and we get the required

$$
\begin{equation*}
N_{\varepsilon}\left(K_{v}(\omega)\right) \asymp\left|{ }_{v} E(\omega, \varepsilon)\right| . \tag{3.4.6}
\end{equation*}
$$

Using the notation of the Hutchinson-Moran matrices introduced in (3.2.4), we can see that for $s=0$, we have $c_{\mathbf{e}}^{s}=1$ and thus the Hutchinson matrix $\mathbf{P}_{\varepsilon}^{0}(\omega)$ 'counts' the number of images in $E(\omega, \varepsilon)$. We have

$$
\left|{ }_{v} E(\omega, \varepsilon)\right|=\sum_{w \in V}\left(\sum_{j}\left(\mathbb{1} \mathbf{P}_{\varepsilon}^{0}(\omega)\right)_{j}\right)_{(v, w)}
$$

The sum above behaves in a supermultiplicative fashion: for some constant $k_{s}>0$ and all $\varepsilon, \delta>0$,

$$
\begin{equation*}
\sum_{w \in V}\left(\sum_{j}\left(\mathbb{1} \mathbf{P}_{\delta \varepsilon}^{0}(\omega)\right)_{j}\right)_{(v, w)} \geq k_{s} \sum_{w \in V}\left(\sum_{j}\left(\mathbb{1} \mathbf{P}_{\varepsilon}^{0}(\omega) \mathbf{P}_{\delta}^{0}(\sigma \omega)\right)_{j}\right)_{(v, w)} \tag{3.4.7}
\end{equation*}
$$

By definition $\bigsqcup \mathbb{1}_{\varepsilon_{0}} \mathbf{H}^{\varepsilon}(\omega)$ is the arrangement of words that describe the cylinders of $\left\{K_{v}(\omega)\right\}_{v \in V}$. Consider an arbitrary word $\mathbf{e}_{1} \mathbf{e}_{2} \in \bigsqcup \mathbb{1}_{\varepsilon_{0}} \mathbf{H}^{\varepsilon}(\omega) \mathbf{H}^{\delta}(\sigma \omega)$, where $\mathbf{e}_{1} \in$ $\bigsqcup \mathbb{1}_{\varepsilon_{0}} \mathbf{H}^{\varepsilon}(\omega)$ and $\mathbf{e}_{2} \in \bigsqcup \mathbf{H}^{\delta}(\sigma \omega)$. Assume $\mathbf{e}_{1}$ is the $(i, j)$ th entry of the matrix at position $k$ of the vector $\mathbb{1}_{\varepsilon_{0}} \mathbf{H}^{\varepsilon}(\omega)$. Since $\mathbf{e}_{1} \mathbf{e}_{2}$ is obtained by regular matrix multiplication, we have that $\mathbf{e}_{2}$ is an entry in one of the matrices in the $k$ th row of $\mathbf{H}^{\delta}(\sigma \omega), \mathbf{e}_{2} \in \hat{\eta}^{k}\left(\sigma^{k} \omega, \delta\right)$. Therefore, for some $v_{1}, v_{2}, v_{3} \in V$, we have $\mathbf{e}_{1} \in{ }_{v_{1}} E_{v_{2}}^{k}(\omega, \varepsilon)$ and $\mathbf{e}_{2} \in{ }_{v_{2}} E_{v_{3}}\left(\sigma^{k} \omega, \delta\right)$. Hence $\mathbf{e}_{1} \mathbf{e}_{2}$ describes a path of $\boldsymbol{\Gamma}$ for realisation $\omega$ and therefore codes a cylinder of $K_{v_{1}}(\omega)$, and as $c_{\min } \varepsilon<c_{\mathbf{e}_{1}} \leq \varepsilon$ and $c_{\min } \delta<c_{\mathbf{e}_{2}} \leq \delta$ we additionally have $c_{\text {min }}^{2} \delta \varepsilon<c_{\mathbf{e}_{1} \mathbf{e}_{2}} \leq \delta \varepsilon$. Recall that $\mathfrak{R}^{s}$ was the operator mapping arrangements of words to the length of the associated image under $S$ to the power $s$. Therefore, applying $\mathfrak{R}^{0}$ to $\left(\mathbb{1}_{\varepsilon_{0}} \mathbf{H}^{\varepsilon}(\omega) \mathbf{H}^{\delta}(\sigma \omega)\right)$, we can express the number of cylinders starting at a given vertex $v$ by

$$
\sum_{w \in V}\left(\sum_{j}\left(\mathbb{1} \mathbf{P}_{\varepsilon}^{0}(\omega) \mathbf{P}_{\delta}^{0}(\sigma \omega)\right)_{j}\right)_{(v, w)}
$$

Obviously these cylinders do not intersect but they do not quite form an $\varepsilon \delta$-stopping graph as some of the edges might have contraction rate $c_{\min }^{2} \delta \varepsilon<c_{\mathbf{e}_{1} \mathbf{e}_{2}} \leq c_{\min } \delta \varepsilon$. However this does not present a problem as one needs to only avoid at most the last branching to recover an $\varepsilon \delta$-stopping graph. Note that the number of subbranches is surely bounded and therefore there exists a constant $k_{s}$, which is the inverse of this maximal splitting bound, such that we have an $\varepsilon \delta$-stopping graph that may not be maximal, hence giving rise to the inequality (3.4.7).

Now given any $\varepsilon>\delta>0$ there exists unique $q \in \mathbb{N}$ and $1 \geq \xi>\varepsilon$ such that $\delta=\varepsilon^{q} \xi$. One can easily generalise equation (3.4.7), using above argument, to show that

$$
\begin{equation*}
\sum_{w \in V}\left(\sum_{j}\left(\mathbb{1} \mathbf{P}_{\delta}^{0}(\omega)\right)_{j}\right)_{(v, w)} \geq k_{s}^{q} \sum_{w \in V}\left(\sum_{j}\left(\mathbb{1} \mathbf{P}_{\xi}^{0}(\omega) \mathbf{P}_{\varepsilon}^{0}(\sigma \omega) \ldots \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{q-1} \omega\right)\right)_{j}\right)_{(v, w)} \tag{3.4.8}
\end{equation*}
$$

The relationship between the expression above and the exponent $\varepsilon$ can be found by an argument akin to that in the proof of Fekete's Lemma, see [PS, §1 Problem 98] and Theorem 1.4.2. Consider

$$
\begin{aligned}
& \frac{\log \sum_{w \in V}\left(\sum_{j}\left(\mathbb{1} \mathbf{P}_{\delta}^{0}(\omega)\right)_{j}\right)_{(v, w)}}{-\log \delta}=\frac{\log \sum_{w \in V}\left(\sum_{j}\left(\mathbb{1} \mathbf{P}_{\varepsilon^{q} \xi}^{0}(\omega)\right)_{j}\right)_{(v, w)}}{-q \log \varepsilon-\log \xi} \\
& \geq \frac{\log k_{s}+\log \left(\sum_{w \in V}\left(\sum_{j}\left(\mathbb{1} \mathbf{P}_{\xi}^{0}(\omega) \mathbf{P}_{\varepsilon}^{0}(\sigma \omega) \ldots \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{q} \omega\right)\right)_{j}\right)_{(v, w)}\right)^{1 / q}}{-\log \varepsilon-(1 / q) \log \xi}
\end{aligned}
$$

Thus for every $\varepsilon>0$, assuming almost sure convergence and stochastically strongly connected graphs,

$$
\liminf _{\delta \rightarrow 0} \frac{\log \sum_{w \in V}\left(\sum \mathbb{1} \mathbf{P}_{\delta}^{0}(\omega)\right)_{(v, w)}}{-\log \delta} \geq \lim _{\varepsilon \rightarrow 0} \frac{\log k_{s}+\log \boldsymbol{\Psi}(0, \varepsilon)}{-\log \varepsilon} \geq \sup _{\varepsilon>0} \frac{\log \boldsymbol{\Psi}(0, \varepsilon)}{-\log \varepsilon}
$$

holding almost surely. For the upper bound simply note that, almost surely,

$$
\limsup _{\delta \rightarrow 0} \frac{\log \sum_{w \in V}\left(\sum \mathbb{1} \mathbf{P}_{\delta}^{0}(\omega)\right)_{(v, w)}}{-\log \delta} \leq \sup _{\delta>0} \frac{\log \boldsymbol{\Psi}(0, \delta)}{-\log \delta}
$$

Therefore, almost surely,

$$
\frac{\log \sum_{w \in V}\left(\sum \mathbb{1} \mathbf{P}_{\delta}^{0}(\omega)\right)_{(v, w)}}{-\log \delta} \rightarrow \sup _{\varepsilon>0} \frac{\log \boldsymbol{\Psi}(0, \varepsilon)}{-\log \varepsilon} \quad \text { as } \delta \rightarrow \infty
$$

Due to (3.4.6) we get the required almost sure result:

$$
\operatorname{dim}_{B} K_{v}(\omega)=\sup _{\varepsilon>0} \frac{\log \boldsymbol{\Psi}(0, \varepsilon)}{-\log \varepsilon}
$$

### 3.4.5 Proof of Theorem 3.2.21

While the construction introduced in Section 3.2 .3 with norm $\|\|\cdot\| \mid$ makes sense in establishing the box counting dimension of RGDS attractors where we wanted all cylinders of diameter comparable to some $\varepsilon>0$, we can also rewrite the system as a finite graph directed system. We employ this idea here to find the lower bound to the Hausdorff dimension of $K_{v, \varepsilon}(\omega)$ by constructing a measure on cylinders obtained in this finite fashion. Since $K_{v, \varepsilon}(\omega) \subseteq K_{v}(\omega)$, the Hausdorff dimension for the approximation will give a lower bound for the Hausdorff dimension of $K_{v}(\omega)$. We use the $\left\|\|\cdot\|_{(1,1)}\right.$ seminorm defined in (3.2.3) on finite matrices with matrix entries.

Consider the system given by the states $A_{1}, H_{2}, H_{3}, \ldots, H_{k_{\max }(\varepsilon)}$, where $k_{\max }(\varepsilon)$ is the maximal length of column specified by $\varepsilon$, see Section 3.2.3. The corresponding graph is shown in Figure 3.3. We record words in either the active $\left(A_{1}\right)$ or a holding state $\left(H_{i}\right)$ as a $k_{\max }(\varepsilon)$-vector with matrix entries and the action given from the active state by right multiplication of $\mathbf{C}_{\varepsilon}(\omega)$ and $\mathbf{W}_{\varepsilon}^{s}(\omega)=\mathfrak{R}^{s} \mathbf{C}_{\varepsilon}(\omega)$, where

$$
\mathbf{C}_{\varepsilon}(\omega)=\left(\begin{array}{ccccc}
\eta_{1}(\omega, \varepsilon) & \eta_{2}(\omega, \varepsilon) & \ldots & \eta_{k_{\max }(\omega)-1}(\omega, \varepsilon) & \eta_{k_{\max }(\omega)}(\omega, \varepsilon) \\
\mathbf{1}_{\varepsilon_{0}} & \mathbf{0}_{\varnothing} & \ldots & \mathbf{0}_{\varnothing} & \mathbf{0}_{\varnothing} \\
\mathbf{0}_{\varnothing} & \mathbf{1}_{\varepsilon_{0}} & \ldots & \mathbf{0}_{\varnothing} & \mathbf{0}_{\varnothing} \\
\vdots & & \ddots & \vdots & \vdots \\
\mathbf{0}_{\varnothing} & \mathbf{0}_{\varnothing} & \ldots & \mathbf{1}_{\varepsilon_{0}} & \mathbf{0}_{\varnothing}
\end{array}\right)
$$



Figure 3.3: Graph for the finite model used in establishing the lower bound.
and

$$
\mathbf{W}_{\varepsilon}^{s}(\omega)=\left(\begin{array}{cccc}
\mathbf{p}_{1}^{s}(\omega, \varepsilon) & \ldots & \mathbf{p}_{k_{\max }(\omega)-1}^{s}(\omega, \varepsilon) & \mathbf{p}_{k_{\max }(\omega)}^{s}(\omega, \varepsilon) \\
\mathbf{1} & \ldots & \mathbf{0} & \mathbf{0} \\
\vdots & \ddots & \vdots & \vdots \\
\mathbf{0} & \ldots & \mathbf{1} & \mathbf{0}
\end{array}\right)
$$

We are now interested in analysing the cylinders given by the (finite) arrangement of words $\mathbf{D}_{\varepsilon}^{k}(\omega)$ and the norm of its Hutchinson-Moran matrix $\mathfrak{R}^{s} \mathbf{D}_{\varepsilon}^{k}(\omega)$,

$$
\mathbf{D}_{\varepsilon}^{k}(\omega)=\mathbb{1}_{\varepsilon_{0}} \mathbf{C}_{\varepsilon}(\omega) \mathbf{C}_{\varepsilon}(\sigma \omega) \ldots \mathbf{C}_{\varepsilon}\left(\sigma^{k-1} \omega\right)
$$

and

$$
\Phi_{\varepsilon}^{k}(s)=\left\|\mathbb{1} \mathbf{W}_{\varepsilon}^{s}(\omega) \ldots \mathbf{W}_{\varepsilon}^{s}\left(\sigma^{k-1} \omega\right)\right\|_{(1,1)}
$$

We first show
Lemma 3.4.1. On a subset of $\Omega$ with full measure we have, for all $\varepsilon>0$,

$$
\mathbf{\Phi}_{\varepsilon}(s):=\lim _{k \rightarrow \infty}\left(\Phi_{\varepsilon}^{k}(s)\right)^{1 / k}=1 \quad \text { if and only if } \quad \boldsymbol{\Psi}(s, \varepsilon)=1 .
$$

Note that these two notions of pressure do not, in general, coincide for $s$ when $\boldsymbol{\Phi}_{\varepsilon}(s) \neq 1$.

Proof. The procedure of picking the multiplications that are applied to the active state $A_{1}$ is determined by the first $k_{\max }(\varepsilon)$ letters of $\omega$, where the individual entries of $\omega$ were chosen independently from $\Lambda$ according to $\vec{\pi}$. However, one can without loss of generality assume that the matrices picked are given by a stochastic process that is Markov. To see this let $\Lambda^{\ddagger}$ be a new alphabet consisting of $|\Lambda|^{k_{\max }(\varepsilon)}$ elements. These elements represent all the different strings one can have that determine the matrices chosen. The full shift on $\Omega$ now induces a subshift of finite type on $\left(\Lambda^{\ddagger}\right)^{\mathbb{N}}$ and $\mathbb{P}$ gives a new Markov measure $\mathbb{P}^{\ddagger}$ with appropriate transition probabilities. It is a simple exercise to show that this subshift is also topologically mixing and we omit it here.

The cylinders given by $\mathbf{D}_{\varepsilon}^{k}(\omega)$ still exhaust all paths (compare with Lemma 3.2.11), however they may no longer have comparable diameter. Given that it is a stopping set we can find certain inclusions if we compare the arrangement of words of this
finite model with the arrangement of words coming from the infinite construction. Let $\mathbf{U}_{\varepsilon}^{k}(\varepsilon)=\mathbf{H}^{\varepsilon}(\omega) \mathbf{H}^{\varepsilon}(\sigma \omega) \ldots \mathbf{H}^{\varepsilon}\left(\sigma^{k-1} \omega\right)$. Then

$$
\begin{equation*}
\bigsqcup_{i=1}^{\lfloor(k+1) / l\rfloor+1}\left(\mathbb{1}_{\varepsilon_{0}} \mathbf{U}_{\varepsilon}^{k-i-1}(\omega)\right)_{i} \subseteq \bigsqcup \mathbf{D}_{\varepsilon}^{k}(\omega) \tag{3.4.9}
\end{equation*}
$$

To see this inclusion we refer the reader back to Figure 3.2.3. The arrangement $\mathbf{D}_{\varepsilon}^{k}(\omega)$ corresponds to taking the off-diagonal of entries that have been decided up to the $k$ th shift. The left hand side of (3.4.9) are exactly those words that were in state $A_{1}$ at the $(k-1)$ th shift and are part of the same off-diagonals in Figure 3.2.3.

The diagonal must also intersect with an element that is within some uniform constant $c>0$ of the maximal element on some level $d_{k}$ from $\lfloor(k+1) / l\rfloor+1$ to $k$, giving the following inclusion:

$$
\bigsqcup \mathbf{D}_{\varepsilon}^{k}(\omega) \subseteq \bigsqcup_{i=\lfloor(k+1) / l\rfloor+1}^{k} \bigsqcup_{j \in \mathbb{N}}\left(\mathbb{1}_{\varepsilon_{0}} \mathbf{U}_{\varepsilon}^{i}(\omega)\right)_{j}
$$

Applying the operator $\mathfrak{R}^{s}$ we get the inequalities

$$
\begin{align*}
\sum_{i=1}^{\lfloor(k+1) / l\rfloor+1}\left\|\left(\mathbb{1} \mathbf{u}_{k-i-1}(\omega)\right)_{i}\right\|_{1} \leq \Phi_{\varepsilon}^{k}(s) & \leq \sum_{i=\lfloor(k+1) / l\rfloor+1}^{k} \sum_{j \in \mathbb{N}}\left\|\left(\mathbb{1} \mathbf{u}_{i}(\omega)\right)_{j}\right\|_{1}  \tag{3.4.10}\\
& \leq n \sum_{i=\lfloor(k+1) / l\rfloor+1}^{k}\left\|\mathbb{1} \mathbf{u}_{i}(\omega)\right\| .
\end{align*}
$$

Let $m_{k}$ refer to the level for which $\boldsymbol{\Psi}_{m_{k}}(s)=\max _{i \in\{L(k+1) / l\rfloor+1, \ldots, k\}} \boldsymbol{\Psi}_{i}(s)$ and $d_{k}$ be as above, then (3.4.10) becomes

$$
\begin{aligned}
\left\|\left(\mathbb{1} \mathbf{u}_{d_{k}}(\omega)\right)_{d_{k}}\right\|_{1} & \leq \Phi_{\varepsilon}^{k}(s) \leq n k\| \| \mathbf{u}_{m_{k}}(\omega) \| \\
\frac{1}{k} \Psi_{\omega}^{d_{k}}(s, \varepsilon) & \leq \Phi_{\varepsilon}^{k}(s) \leq n k \Psi_{\omega}^{m_{k}}(s, \varepsilon) \\
k^{-1 / k} \Psi_{\omega}^{d_{k}}(s, \varepsilon)^{1 / k} & \leq \Phi_{\varepsilon}^{k}(s)^{1 / k} \leq(n k)^{1 / k} \Psi_{\omega}^{m_{k}}(s, \varepsilon)^{1 / k}
\end{aligned}
$$

Now assume $s$ is such that $\boldsymbol{\Psi}_{\omega}(s, \varepsilon)=1$ for all $\omega \in \mathcal{U}$, where $\mathcal{U}$ is a set of measure one. Now,

$$
\underset{k}{\lim \sup } \Phi_{\varepsilon}^{k}(s)^{1 / k} \leq \limsup _{k}(n k)^{1 / k} \Psi_{\omega}^{m_{k}}(s, \varepsilon)^{1 / k} \leq \limsup _{k} k^{1 / k} \Psi_{\omega}^{m_{k}}(s, \varepsilon)^{1 / m_{k}}=1
$$

And similarly

$$
\underset{k}{\liminf } \Phi_{\varepsilon}^{k}(s)^{1 / k} \geq \liminf _{k} k^{-1 / k} \Psi_{\omega}^{d_{k}}(s, \varepsilon)^{1 / k} \geq \liminf _{k} k^{-1 / k} \Psi_{\omega}^{d_{k}}(s, \varepsilon)^{1 / d_{k}}=1
$$

Thus $\boldsymbol{\Psi}(s, \varepsilon)=1 \Rightarrow \boldsymbol{\Phi}_{\varepsilon}(s)=1$. To establish the other direction just note that if $s$ is such that $\boldsymbol{\Psi}(s, \varepsilon)<1$, then eventually $\Psi_{\omega}^{k}(s, \varepsilon)^{1 / k} \leq 1-\delta$ for all $\omega \in \mathcal{U}$ and $\delta>0$ and $k$ large enough and so $\Psi_{\omega}^{k^{\prime}}(s, \varepsilon) \leq 1-\delta$ for large enough $k^{\prime} \geq k$. This gives

$$
\begin{aligned}
\limsup _{k} \Phi_{k}(s)^{1 / k} & \leq \underset{k}{\limsup _{k}} k^{1 / k} \Psi_{\omega}^{m_{k}}(s, \varepsilon)^{1 / k} \\
& \leq \limsup _{k} k^{1 / k} \Psi_{\omega}^{m_{k}}(s, \varepsilon)^{1 /\left(l m_{k}+l+1\right)} \\
& \leq\left(\limsup _{k} k^{1 / k} \Psi_{\omega}^{m_{k}}(s, \varepsilon)^{1 / m_{k}}\right)^{1 /(l+1)}<1 .
\end{aligned}
$$

A similar argument holds for $\boldsymbol{\Psi}(s, \varepsilon)>1$, finishing the proof.
For $t<s_{H, \varepsilon}$ we can define a random mass distribution on $K_{v, \varepsilon}(\omega)$ by constructing a Borel probability measure $\nu$ on the cylinders described by $\mathbf{D}_{\varepsilon}^{k}(\omega)$ that satisfies
$\nu(U) \leq C|U|^{t}$ for some random, almost surely non-zero, constant $C$. We start by defining the (diagonal) $k$-prefractal codings of $K_{v, \varepsilon}(\omega)$ for the vertex $v$ by

$$
\mathcal{F}_{k}^{v}(\omega)=\bigsqcup_{w \in V}\left(\bigsqcup_{j=1}^{l}\left(\mathbb{1}_{\varepsilon_{0}} \mathbf{D}_{\varepsilon}^{k}(\omega)\right)_{i}\right)_{v, w}
$$

Since the words of $\mathcal{F}_{k}^{v}(\omega)$ are in one to one correspondence with the cylinders generating the topology on $K_{v}(\omega)$ it suffices to define our required measure on those (disjoint) cylinders only, as they generate the topology of $K_{v}(\omega)$ and this construction extends to a unique Borel probability measure $\nu_{v}^{s}$. For every word $w \in \mathcal{F}_{k}^{v}(\omega)$ we can describe its 'location' relative to $\mathbf{D}_{\varepsilon}^{k}(\omega)$ by a unique triple $(x, y, z)$, where $x, y \in V$ and $z \in\{1, \ldots, l\}$, such that $w \in\left[\left(\mathbb{1}_{\varepsilon_{0}} \mathbf{D}_{\varepsilon}^{k}(\omega)\right)_{z}\right]_{x, y}$. Let $I$ be an arbitrary word in $\mathcal{F}_{k}^{v}(\omega)$, with coordinates $(x, y, z)$. For any word we define the location matrix as

$$
(\mathbb{V}(I))_{i}=\left\{\begin{array}{ll}
V(I) & \text { for } i=z, \\
\mathbf{0}_{\varnothing} & \text { otherwise; }
\end{array} \quad \text { for } \quad(V(I))_{j, k}= \begin{cases}I & \text { for }(j, k)=(x, y) \\
\varnothing & \text { otherwise }\end{cases}\right.
$$

We set for $I \in \mathcal{F}_{k}^{v}(\omega)$,

$$
\begin{equation*}
\nu_{v}^{s}(I)=\lim _{q \rightarrow \infty} \frac{\left\|\mathfrak{R}^{s}\left(\mathbb{V}(I) \mathbf{C}_{\varepsilon}\left(\sigma^{k} \omega\right) \mathbf{C}_{\varepsilon}\left(\sigma^{k+1} \omega\right) \ldots \mathbf{C}_{\varepsilon}\left(\sigma^{k+q-1} \omega\right)\right) \mid\right\|_{(1,1)}}{\sum_{q_{2}=1}^{n}\left[\sum_{q_{1}=1}^{l}\left(\mathbb{1}_{l} \mathbf{W}_{\varepsilon}^{s}(\omega) \mathbf{W}_{\varepsilon}^{s}(\sigma \omega) \ldots \mathbf{W}_{\varepsilon}^{s}\left(\sigma^{q-1} \omega\right)\right)_{q_{1}}\right]_{v, q_{2}}} \tag{3.4.11}
\end{equation*}
$$

One can check that, almost surely, this limit exists. However as one can derive the properties of $\nu_{v}^{s}$ by defining the measure in terms of lim inf or limsup, we omit details. It is easy to see that $\nu_{v}^{s}$ is in fact a measure. Note that for $I=\varnothing$ we get $\mathfrak{R}^{s} \mathrm{~V}(I)=\mathbf{0}$ and so $\nu_{v}^{s}(\varnothing)=0$. Obviously $\nu_{v}^{s}(I) \geq 0$ and countable stability arises from the construction being an additive set function, where

$$
\nu_{v}^{s}(I)=\lim _{k \rightarrow \infty}\left\{\sum|J|^{s} \mid J \in \mathcal{F}_{k}^{v}(\omega) \text { and } J \subseteq I\right\}
$$

Formally, for any countable collection of disjoint words (no word is a subword of any other) $\bigsqcup w_{i}$ we get, assuming that $w_{i} \in \mathcal{F}_{k_{i}}^{v}(\omega)$ for some length $k_{i}$,

$$
\begin{aligned}
\sum_{i} \nu_{v}^{s}\left(\left[w_{i}\right]\right) & =\sum_{i} \lim _{q \rightarrow \infty} \frac{\left\|\Re^{s}\left(\mathbb{V}\left(w_{i}\right) \mathbf{C}_{\varepsilon}\left(\sigma^{k_{i}} \omega\right) \ldots \mathbf{C}_{\varepsilon}\left(\sigma^{k_{i}+q-1} \omega\right)\right)\right\| \|_{(1,1)}}{\sum_{q_{2}=1}^{n}\left[\sum_{q_{1}=1}^{l}\left(\mathbb{1}_{l} \mathbf{W}_{\varepsilon}^{s}(\omega) \ldots \mathbf{W}_{\varepsilon}^{s}\left(\sigma^{q-1} \omega\right)\right)_{q_{1}}\right]_{v, q_{2}}} \\
& =\lim _{q \rightarrow \infty} \frac{\left\|\Re^{s}\left(\bigsqcup_{i} \mathbb{V}\left(w_{i}\right) \mathbf{C}_{\varepsilon}\left(\sigma^{k_{i}} \omega\right) \ldots \mathbf{C}_{\varepsilon}\left(\sigma^{k_{i}+q-1} \omega\right)\right)\right\| \|_{(1,1)}}{\sum_{q_{2}=1}^{n}\left[\sum_{q_{1}=1}^{l}\left(\mathbb{1}_{l} \mathbf{W}_{\varepsilon}^{s}(\omega) \ldots \mathbf{W}_{\varepsilon}^{s}\left(\sigma^{q-1} \omega\right)\right)_{q_{1}}\right]_{v, q_{2}}} \\
& =\nu_{v}^{s}\left(\left[\bigsqcup_{i} w_{i}\right]\right) .
\end{aligned}
$$

Notice that there exists a uniform constant $C>0$ such that

$$
\nu_{v}^{s}\left(K_{v, \varepsilon}(\omega)\right)=\lim _{q \rightarrow \infty} \frac{\| \| \mathfrak{R}^{s}\left(\mathbb{1}_{l} \mathbf{C}_{\varepsilon}(\omega) \mathbf{C}_{\varepsilon}(\sigma \omega) \ldots \mathbf{C}_{\varepsilon}\left(\sigma^{q-1} \omega\right)\right)\| \|_{(1,1)}}{\sum_{q_{2}=1}^{n}\left[\sum_{q_{1}=1}^{l}\left(\mathbb{1}_{l} \mathbf{W}_{\varepsilon}^{s}(\omega) \mathbf{W}_{\varepsilon}^{s}(\sigma \omega) \ldots \mathbf{W}_{\varepsilon}^{s}\left(\sigma^{q-1} \omega\right)\right)_{q_{1}}\right]_{v, q_{2}}} \leq C
$$

and we conclude that $\nu_{v}^{s}$ is a finite measure, and without loss of generality we rescale such that $\nu_{v}^{s}=1$.

We observe that by virtue of the definition of the measure that there exists a random variable $C^{\dagger}(\omega)$ with $\mathbb{E}_{\omega} C^{\dagger}(\omega)<\infty$ such that

$$
\begin{equation*}
\nu_{v}^{s}(I) \leq C C^{\dagger}(\omega)|I|^{s} \tag{3.4.12}
\end{equation*}
$$

as long as $s<s_{H, \varepsilon}$, such that the denominator in (3.4.11) is almost surely increasing exponentially in $q$. Note that the existence of a Borel measure satisfying (3.4.12) immediately implies that $s_{H, \varepsilon}$ is an almost sure lower bound by the mass distribution principle, Theorem 1.7.6.

### 3.4.6 Proof of Theorem 3.2.23 and Corollary 3.2.24

### 3.4.6.1 Proof of Theorem 3.2.23

Let $\lambda>0$, Theorem 3.2.21 gives us a lower bound on the Hausdorff dimension of the $\lambda$-approximation sets $K_{v, \lambda}(\omega)$. In particular we have that $\operatorname{dim}_{H} K_{v, \lambda}=s_{H, \lambda}$, where

$$
\lim _{k \rightarrow \infty}\| \| \mathbb{1} \mathbf{P}_{\lambda}^{s_{H, \lambda}}(\omega) \ldots \mathbf{P}_{\lambda}^{s_{H, \lambda}}\left(\sigma^{k-1} \omega\right)\| \|^{1 / k}=1
$$

Consider one of the Hutchinson-Moran sums in the matrix $\mathbf{P}_{\lambda}^{s_{H, \lambda}}(\omega)$. They are given by

$$
\sum_{e \in\left({ }_{i} E_{j}^{q}(\omega, \lambda)\right)} c_{\mathbf{e}}^{s_{H, \lambda}} .
$$

But since we have bounds on the size of $c_{\mathbf{e}}$, i.e. $\underline{\gamma}\left\langle c_{\mathbf{e}} \leq \lambda\right.$ we have

$$
\sum_{e \in\left({ }_{i} E_{j}^{q}(\omega, \lambda)\right)} c_{e}^{s_{H, \lambda}} \leq\left.\right|_{i} E_{j}^{q}(\omega, \lambda) \mid \lambda^{s_{H, \lambda}}
$$

and so

$$
\mathbf{P}_{\lambda}^{s_{H, \lambda}}(\omega) \leq \lambda^{s_{H, \lambda}} \mathbf{P}_{\lambda}^{0}(\omega)
$$

Considering the matrices $\lambda^{s} \mathbf{P}_{\lambda}^{0}(\omega)$, dependent on $s$, one can apply the same strategy as in Lemma 3.2.16 to prove that there exists a unique $0 \leq t_{\lambda} \leq s_{H, \lambda}$ such that

$$
\lim _{k \rightarrow \infty}\| \| \mathbb{1} \lambda^{t_{\lambda}} \mathbf{P}_{\lambda}^{0}(\omega) \lambda^{t_{\lambda}} \mathbf{P}_{\lambda}^{0}(\sigma \omega) \ldots \lambda^{t_{\lambda}} \mathbf{P}_{\lambda}^{0}\left(\sigma^{k-1} \omega\right) \|^{1 / k}=1
$$

We leave adapting the proof of Lemma 3.2.16 to the reader. Note that the $t_{\lambda}$ defined above gives an a.s. lower bound to $\operatorname{dim}_{H} K_{v, \lambda}(\omega)$. By linearity,

$$
\begin{aligned}
\left\|\mathbb{1} \lambda^{t_{\lambda}} \mathbf{P}_{\lambda}^{0}(\omega) \lambda^{t_{\lambda}} \mathbf{P}_{\lambda}^{0}(\sigma \omega) \ldots \lambda^{t_{\lambda}} \mathbf{P}_{\lambda}^{0}\left(\sigma^{k-1} \omega\right)\right\|^{1 / k} & =\left\|\mathbb{1} \lambda^{k t_{\lambda}} \mathbf{P}_{\lambda}^{0}(\omega) \ldots \mathbf{P}_{\lambda}^{0}\left(\sigma^{k-1} \omega\right)\right\|^{1 / k} \\
& =\left.\lambda^{t_{\lambda}}\left\|\mathbb{1} \mathbf{P}_{\lambda}^{0}(\omega) \ldots \mathbf{P}_{\lambda}^{0}\left(\sigma^{k-1} \omega\right)\right\|\right|^{1 / k}
\end{aligned}
$$

and so

$$
t_{\lambda}=\lim _{k \rightarrow \infty} \frac{\log \mid\left\|\mathbb{1} \mathbf{P}_{\lambda}^{0}(\omega) \ldots \mathbf{P}_{\lambda}^{0}\left(\sigma^{k-1} \omega\right)\right\| \|^{1 / k}}{-\log \lambda}
$$

But since $\lim _{k \rightarrow \infty}\| \| \mathbb{1} \mathbf{P}_{\lambda}^{0}(\omega) \ldots \mathbf{P}_{\lambda}^{0}\left(\sigma^{k-1} \omega\right)\| \|^{1 / k}=\boldsymbol{\Psi}(0, \lambda)$ we have, comparing with equation (3.2.7), that

$$
t_{\lambda}=\frac{\log \boldsymbol{\Psi}(0, \lambda)}{-\log \lambda}
$$

But $t_{\lambda} \rightarrow \operatorname{dim}_{B} K_{v}(\omega)$ as $\lambda \rightarrow 0$ and so we can, for every $\delta>0$, find a $\lambda$ approximation such that, almost surely,

$$
\overline{\operatorname{dim}}_{B} K_{v}-\delta \leq \operatorname{dim}_{H} K_{v, \varepsilon} \leq \operatorname{dim}_{H} K_{v} \leq \overline{\operatorname{dim}}_{B} K_{v}
$$

Therefore $\operatorname{dim}_{H} K_{v}=\overline{\operatorname{dim}}_{B} K_{v}$ follows for almost all $\omega \in \Omega$.

### 3.4.6.2 Proof of Corollary 3.2.24

If our original graph satisfies the USSC, we can apply Lemma 3.2.20 and have that $K_{v, \varepsilon}(\omega)=K_{v}(\omega)$ for all $\varepsilon>0$ and $\omega \in \Omega$. Therefore $s_{H, 1}=s_{H}$ and the almost sure Hausdorff, packing and box counting dimensions are given by the unique $s_{O}$ such that

$$
\lim _{k \rightarrow \infty}\| \| \mathbb{1} \mathbf{P}_{1}^{s O}(\omega) \mathbf{P}_{1}^{s O}(\sigma \omega) \ldots \mathbf{P}_{1}^{s O}\left(\sigma^{k-1} \omega\right) \|^{1 / k}=1
$$

But as $\varepsilon$ was chosen to be 1 we must necessarily have $k_{\max }(1)=1$ and $\mathbf{P}_{1}^{s o}(\omega)$ reduces to

$$
\mathbf{P}_{1}^{s O}(\omega)=\left(\begin{array}{cccc}
p_{1}^{s O}(\omega, 1) & \mathbf{0} & \mathbf{0} & \cdots \\
\mathbf{0} & p_{1}^{s O}(\sigma \omega, 1) & \mathbf{0} & \cdots \\
\mathbf{0} & \mathbf{0} & p_{1}^{s O}\left(\sigma^{2} \omega, 1\right) & \\
\vdots & \vdots & & \ddots
\end{array}\right)
$$

But then

$$
\left\|\mathbb{1} \mathbf{P}_{1}^{s_{O}}(\omega) \mathbf{P}_{1}^{s_{O}}(\sigma \omega) \ldots \mathbf{P}_{1}^{s_{O}}\left(\sigma^{k-1} \omega\right)\right\|\|=\| p_{1}^{s O}(\omega, 1) p_{1}^{s O}(\sigma \omega, 1) \ldots p_{1}^{s O}\left(\sigma^{k-1} \omega, 1\right) \|_{\text {row }}
$$

giving the required result upon noting that $\|\cdot\|_{\text {row }}$ and $\|\cdot\|_{1}$ are equivalent norms.

### 3.4.7 Proof of Theorem 3.2.27

The proof of the lower bound is a relatively simple adaptation of the almost sure lower bound proof due to Fraser, Miao and Troscheit [FMT], see also Chapter 6.

First note that $\mathfrak{P}(\varepsilon)$ (see Definition 3.2.26) is well-defined by Lemma 3.2.13 since the Lyapunov exponent with respect to the $\|\|\cdot\|\|_{\text {sup }}$ norm exists almost surely. To see that the joint spectral radius takes the same value recall that

$$
\mathbb{1} \mathbf{P}_{\varepsilon}^{0}(\omega) \ldots \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{k-1} \omega\right)=\left(\mathbf{P}_{\varepsilon}^{0}(\omega) \ldots \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{k-1} \omega\right)\right)_{1}
$$

and in general

$$
\mathbb{1} \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{l} \omega\right) \ldots \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{k+l-1} \omega\right)=\left(\mathbf{P}_{\varepsilon}^{0}(\omega) \ldots \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{k-1} \omega\right)\right)_{l}
$$

However this implies that for almost every $\zeta \in \Omega$

$$
\sup _{\omega \in \Omega}\left\{\| \| \mathbb{1} \mathbf{P}_{\varepsilon}^{0}(\omega) \ldots \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{k-1} \omega\right)\| \|\right\}=\sup _{l \in \mathbb{N}}\left(\mathbf{P}_{\varepsilon}^{0}(\zeta) \ldots \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{k-1} \zeta\right)\right)_{l}
$$

The equality in (3.2.9) thus follows. Fix $\varepsilon>0$ and let $\xi_{i} \in \Omega$ be such that

$$
\left\|\left\|\mathbb{1} \mathbf{P}_{\varepsilon}^{0}\left(\xi_{i}\right) \ldots \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{i-1} \xi_{i}\right)\right\|\right\|=\sup _{\omega \in \Omega}\left\|\mathbb{1} \mathbf{P}_{\varepsilon}^{0}(\omega) \ldots \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{i-1} \omega\right)\right\| .
$$

It is easy to check with a standard Borel-Cantelli argument that the set

$$
G=\left\{\omega \in \Omega \mid \exists\left\{j_{i}\right\}_{i=1}^{\infty} \text { such that } j_{i+1} \geq j_{i}+i, \omega_{j_{i}+k_{i}}=\xi_{i}\left(k_{i}\right), \text { for } 1 \leq k_{i} \leq i\right\}
$$

has full measure: all finite words $\xi_{i}$ (in increasing order) are subwords of the infinite word $\omega$ with probability 1 . However this is not the actual set that we have to consider. This is because for every $\xi_{i}$ we also associate a row $v_{i}$ as having the maximal sum that is relevant for the norm. Since we however need a result for every row sum to be maximal we have to consider the family of words $\left\{\xi_{i}^{v}\right\}$, where $\xi_{i}^{v}=\omega^{v, v_{i}} \xi_{i}$. However this modification does not change the fact that the modified good set

$$
\begin{aligned}
G^{*}=\bigcap_{v \in V}\left\{\omega \in \Omega \mid \exists\left\{j_{i}\right\}_{i=1}^{\infty} \text { such that } j_{i+1} \geq j_{i}+i+\left|\omega^{v, v_{i}}\right|\right. \\
\left.\qquad \omega_{j_{i}+k_{i}}=\xi_{i}\left(k_{i}\right), \text { for } 1 \leq k_{i} \leq i+\left|\omega^{v, v_{i}}\right|\right\}
\end{aligned}
$$

still has full measure.
Now assume for a contradiction that

$$
s:=\operatorname{dim}_{A}\left(K_{v}(\omega)\right)<t:=\frac{-\log (\mathfrak{P}(\varepsilon))}{\log (\varepsilon)} .
$$

Let $\left\{w_{i}\right\}$ be any sequence of finite words such that the collection of subcylinders $\mathcal{C}\left(w_{i}\right)$ of $\left[w_{i}\right]$ is given by

$$
\mathcal{C}\left(w_{i}\right)=w_{i} \odot w^{\tau\left(w_{i}\right), v_{i}} \odot\left(\bigsqcup_{j}\left[\bigsqcup_{l}\left(\mathbb{1}_{\varepsilon_{0}} \mathbf{H}^{\varepsilon}\left(\xi_{i}\right) \ldots \mathbf{H}^{\varepsilon}\left(\sigma^{i-1} \xi_{i}\right)\right)_{l}\right]_{v_{i}, j}\right)
$$

where $w^{a, b}$ is a connecting word from vertex $a$ to $b$, that exists because $\boldsymbol{\Gamma}$ is stochastically strongly connected. This sequence of words exists for all $\omega \in G^{*}$, so almost surely, and we consider the sequence of similitudes given by the (unique) mapping $S_{w_{i}}^{-1}$ that takes the cylinder $\left[w_{i}\right]$ and maps it onto $\Delta$. Consider furthermore the sequence of sets $Z_{i}=S_{w_{i}}^{-1}\left(K_{v}(\omega)\right) \cap \Delta$. Since $S_{w_{i}}^{-1}$ is a bi-Lipschitz map we have $\operatorname{dim}_{A} Z_{i} \leq s$ and so by definition there exists a constant $C_{i}\left(s^{*}\right)>0$ such that $\sup _{x \in Z_{i}} N_{r}\left(B(x, R) \cap Z_{i}\right) \leq C_{i}\left(s^{*}\right)(R / r)^{s^{*}}$ for all $0<r<R<\infty$ and $s<s^{*}$. Specifically for $s^{*}$ satisfying $s<s^{*}<t$ there exists uniform constant $C$ such that $\sup _{x \in Z_{i}} N_{r}\left(B(x, R) \cap Z_{i}\right) \leq C(R / r)^{s^{*}}$ and in particular that $N_{r}\left(Z_{i}\right) \leq C^{*} r^{-s^{*}}$ for some $0<C^{*}<\infty$ not depending on $i$, by choosing $R>|\Delta|$. Additionally, it is easy to see that, for some $k_{s}>0$ independent of $i$ and $\varepsilon$ (cf. (3.4.8) and preceding paragraphs) and some $k>0$ related to the difference in length due to the connecting word,

$$
N_{\varepsilon^{i}}\left(Z_{i}\right) \geq k k_{s}^{i}\| \| \mathbf{P}_{\varepsilon}^{0}\left(\xi_{i}\right) \ldots \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{i-1} \xi_{i}\right)\| \| .
$$

Thus there exists $C^{* *}$ such that

$$
k_{s}^{i}\| \| \mathbb{1} \mathbf{P}_{\varepsilon}^{0}\left(\xi_{i}\right) \ldots \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{i-1} \xi_{i}\right) \| \leq C^{* *} \varepsilon^{-i s^{*}}
$$

so

$$
\begin{aligned}
s^{*} & \geq \frac{\log \left[\left(1 / C^{* *}\right) k_{s}^{i}\| \| \mathbb{1} \mathbf{P}_{\varepsilon}^{0}\left(\xi_{i}\right) \ldots \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{i-1} \xi_{i}\right)\| \|\right]}{-i \log \varepsilon} \\
& \geq \frac{\log \left(\left(1 / C^{* *}\right)^{1 / i} k_{s}\| \| \mathbb{1} \mathbf{P}_{\varepsilon}^{0}\left(\xi_{i}\right) \ldots \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{i-1} \xi_{i}\right) \|^{1 / i}\right)}{-\log \varepsilon}
\end{aligned}
$$

for all $i$. However the term on the right converges to $t-\log \left(k_{s}\right) / \log (\varepsilon)$ as $i \rightarrow \infty$. Since $\varepsilon$ was arbitrary, letting $\varepsilon \rightarrow 0$ we have the required contradiction that $t \leq s^{*}<t$.

To prove the upper bound note that since we are assuming the USSC, the $\varepsilon$ approximation sets $K_{v, \varepsilon}(\omega)$ are all equal to the attractor $K_{v}(\omega)$ by Lemma 3.2.20. We first show that for any $z \in \mathbb{R}^{d}$ the number of sets of diameter comparable to $\varepsilon>0$ intersecting the ball $B(z, \varepsilon)$ is uniformly bounded. Let $\Xi^{*}=\left\{x_{i}\right\}$ be the set of words in $\mathbb{1}_{\varepsilon_{0}} \mathbf{H}^{\varepsilon}(\omega)$ whose image $S_{x_{i}}(\Delta)$ intersects $B(z, \varepsilon)$. Let $c_{\min }>0$ be the least contraction rate. We have

$$
\left|\Xi^{*}\right|\left(\varepsilon c_{\min }\right)^{d}=\sum_{x \in \Xi^{*}}\left(\varepsilon c_{\min }\right)^{d} \leq \sum_{x \in \Xi^{*}}\left|S_{x}(\Delta)\right|^{d} \leq|B(z, 2 \varepsilon)|^{d} \leq(4 \varepsilon)^{d}
$$

thus $\left|\Xi^{*}\right| \leq\left(4 / c_{\text {min }}\right)^{d}$ is bounded.
Now let $r$ be such that $0<r<\varepsilon$ and define $k_{r}$ to be the unique integer such that $\varepsilon^{k_{r}+1}<r \leq \varepsilon^{k_{r}}$. For each $x \in \Xi^{*}$ the number of $r$-balls needed to cover $S_{x}(\Delta) \cap$ $K_{v}(\omega)$ is however bounded by $\sum_{i=1}^{n}\left(\left\|\mathbb{1} \mathbf{P}_{\varepsilon}^{0}\left(\xi_{k_{r}}\right) \ldots \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{k_{r}} \xi_{k_{r}}\right)\right\|_{s}\right)_{v, i}$, the maximal way of covering the cylinder with cylinders of diameter $\varepsilon^{k_{r}+1}$ or less. Hence

$$
\begin{aligned}
\sup _{x \in K_{v}(\omega)} N_{r}\left(B(x, \varepsilon) \cap K_{v}(\omega)\right) & \leq\left|\Xi^{*}\right| \sum_{i=1}^{n}\left(\left\|\mathbb{1} \mathbf{P}_{\varepsilon}^{0}\left(\xi_{k_{r}}\right) \ldots \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{k_{r}} \xi_{k_{r}}\right)\right\|_{s}\right)_{v, i} \\
& \leq\left|\Xi^{*}\right|\left\|\mathbb{1} \mathbf{P}_{\varepsilon}^{0}\left(\xi_{k_{r}}\right) \ldots \mathbf{P}_{\varepsilon}^{0}\left(\sigma^{k_{r}} \xi_{k_{r}}\right) \mid\right\| \\
& \leq C\left(\varepsilon^{k_{r}+1}\right)^{-(s+\delta)} \leq C\left(\frac{\varepsilon}{r}\right)^{s+\delta}
\end{aligned}
$$

for some constant $C>0$ for each $\delta>0$ giving the required upper bound to the Assouad dimension.

### 3.4.8 Proof of Theorem 3.3.5

Although we will not prove it here, there exists a nice expression for the Hausdorff dimension of the random attractor.

Lemma 3.4.2. Assume $\boldsymbol{\Gamma}$ satisfies the USSC, then almost surely, conditioned on $F_{v}$ being non-empty, $\operatorname{dim}_{H} F_{v}=s_{h}$, where $s_{h}$ is the unique non-negative real satisfying

$$
\rho_{s}\left[\mathbb{E}\left(\mathfrak{R}^{s_{h}}\left(\begin{array}{ccc}
\bigsqcup_{e \epsilon_{1} E_{1}\left(\omega_{1}\right)} e & \cdots & \bigsqcup_{e \in_{1} E_{n}\left(\omega_{1}\right)} e  \tag{3.4.13}\\
\vdots & \ddots & \vdots \\
\bigsqcup_{e \in_{n} E_{1}\left(\omega_{1}\right)} e & \cdots & \bigsqcup_{e \in_{n} E_{n}\left(\omega_{1}\right)} e
\end{array}\right)\right)\right]=1 .
$$

Here $\rho_{s}$ refers to the spectral radius of a matrix.
Briefly, this can be shown by rewriting the Hutchinson-Moran sum of the $k$ th level as a martingale and a proof strategy almost identical to that of Theorem 15.1 in Falconer [F6]. Compare also with the results in the introduction of Olsen [O1].

Let $\mathbf{K}^{\varepsilon}(q)$ be the matrix of words that corresponds to the graph $\Gamma^{\varepsilon}(q) \in \Gamma^{\varepsilon}$. Since by Lemma 3.3.4 the attractor $F_{v}^{\varepsilon}$ of the approximation is again an $\infty$-variable RGDS which furthermore satisfies the USSC, we can apply Theorem 3.4.2 and get that $\operatorname{dim}_{H} F_{v}^{\varepsilon}=s_{h, \varepsilon}$, where

$$
\rho_{s}\left[\mathbb{E}_{q \in \mathcal{Q}}\left(\mathfrak{R}^{s_{h, \varepsilon}} \mathbf{K}^{\varepsilon}(q)\right)\right]=\lim _{k \rightarrow \infty}\left\|\left[\mathbb{E} \mathfrak{R}^{s_{h, \varepsilon}}\left(\mathbf{K}^{\varepsilon}(q)\right)\right]^{k}\right\|^{1 / k}=1
$$

The second equality holds by Gelfand's Theorem for any suitable matrix norm, such as $\left\|\|\cdot\|_{\text {sup }}\right.$, see for example Arveson [Ar, Theorem 1.7.3]. It can be shown that this expectation is a decreasing, continuous function in $s_{h, \varepsilon}$ and there is a unique value such that the expectation is equal to 1 . The proof is almost identical to that of Lemma 3.2.16 and we will omit it here. Now as $F_{v}^{\varepsilon} \subseteq F_{v}$ we have that $s_{h, \varepsilon} \leq s_{h}$, where $s_{h}=\operatorname{dim}_{H} F_{v}$. We therefore conclude that

$$
\lim _{k \rightarrow \infty}\left\|\left[\mathbb{E} \mathfrak{R}^{s_{h}}\left(\mathbf{K}^{\varepsilon}(q)\right)\right]^{k}\right\|^{1 / k} \leq 1
$$

By an argument similar to that of Theorem 3.2.23, noting that the diameters of the images are comparable to $\varepsilon$, we get

$$
\lim _{k \rightarrow \infty} \varepsilon^{s_{h}}\left\|\left[\mathbb{E} \mathfrak{R}^{0}\left(\mathbf{K}^{\varepsilon}(q)\right)\right]^{k}\right\|^{1 / k}=\varepsilon^{s_{h}} \rho_{s} \mathbb{E}\left(\mathfrak{R}^{0}\left(\mathbf{K}^{\varepsilon}(q)\right)\right) \leq 1
$$

and as $N_{\varepsilon}\left(F_{v}\right) \asymp \sum_{u \in V}\left(\mathfrak{R}^{0}\left(\mathbf{K}^{\varepsilon}\right)\right)_{v, u}$ we have $\mathbb{E} N_{\varepsilon}\left(F_{v}\right) \leq C \varepsilon^{-s_{h}}$. Let $\zeta, \theta>0$ and consider

$$
\sum_{\substack{\delta=\zeta^{k} \\ k \in \mathbb{N}}} \mathbb{P}\left\{N_{\delta}\left(F_{v}\right) \geq \delta^{-\left(s_{h}+\theta\right)}\right\} \leq \sum_{\substack{\delta=\zeta^{k} \\ k \in \mathbb{N}}} \frac{\mathbb{E} N_{\delta}\left(F_{v}\right)}{\delta^{-\left(s_{h}+\theta\right)}} \leq C \sum_{\substack{\delta=\zeta^{k} \\ k \in \mathbb{N}}} \frac{\delta^{-s_{h}}}{\delta^{-s_{h}} \delta^{-\theta}} \leq C \sum_{k \in \mathbb{N}} \zeta^{k \theta}<\infty
$$

Now noting that for all $k$ we have $N_{\zeta^{k}}\left(F_{v}\right) \asymp N_{\zeta^{k+1}}\left(F_{v}\right)$ so by the Borel-Cantelli Lemma with probability 0 the event $N_{\delta}\left(F_{v}\right) \geq \delta^{-\left(s_{h}+\theta\right)}$ happens infinitely often and therefore, almost surely,

$$
\limsup _{\delta \rightarrow 0} \frac{\log N_{\delta}\left(F_{v}\right)}{-\log \delta} \leq \limsup _{\delta \rightarrow 0} \frac{\log \delta^{-\left(s_{h}+\theta\right)}}{-\log \delta}=s_{h}+\theta
$$

But $\theta>0$ was arbitrary, so almost surely $\operatorname{dim}_{B} F_{v}=\operatorname{dim}_{H} F_{v}$, as required.

### 3.4.9 Proof of Theorem 3.3.6

The proof of Theorem 3.3.6 is very similar to that of Theorem 3.2.27 and we only highlight the differences and sketch the rest of the proof. Let $\overline{\mathbf{K}}^{\varepsilon}=\mathbf{K}^{\varepsilon}\left(q_{\max }\right)$, where $q_{\text {max }}$ is such that,

$$
\left\|\mid \mathfrak{R}^{0} \mathbf{K}^{\varepsilon}\left(q_{\max }\right)\right\|\left\|_{\text {sup }}=\max _{q \in \mathcal{Q}}\right\|\left\|\mathfrak{R}^{0} \mathbf{K}^{\varepsilon}(q)\right\| \|_{\text {sup }}
$$

Furthermore let $\mathcal{R}^{\varepsilon}$ be the arrangements of words in the row of $\overline{\mathbf{K}}^{\varepsilon}$ that is maximal with respect to the row norm. Given any finite word $w$ we can therefore construct a
maximal $k$-subtree by appending the letters from $\mathcal{R}^{\varepsilon}$ to $w$, if necessary by connecting them with a connecting word which is bounded in length $l$. Therefore we can construct a subtree of level $k+l$ such that, for some uniform constant $C>0$,

$$
N_{\varepsilon^{k}}\left(S_{w}^{-1}(\Delta)\right) \geq C\| \|_{\Re^{0}}^{\underbrace{\overline{\mathbf{K}}^{\varepsilon} \ldots \overline{\mathbf{K}}^{\varepsilon}}_{k \text { times }}\| \|_{\text {sup }} . . . . ~ . ~}
$$

Noticing that by Gelfand's theorem,

$$
\|\|_{\Re^{0}}^{\underbrace{\overline{\mathbf{K}}^{\varepsilon} \ldots \overline{\mathbf{K}}^{\varepsilon}}_{k \text { times }}\| \|_{\text {sup }}^{1 / k} \rightarrow \rho_{s}\left(\mathfrak{R}^{0} \overline{\mathbf{K}}^{\varepsilon}\right) \quad \text { as } \quad k \rightarrow \infty, ~}
$$

and that for every $k$ we can find a sequence of words $\left\{w_{i}\right\}$ that has this maximal $i+l$ subtree splitting for almost every realisation $q \in \mathcal{Q}$, we can apply the same argument as in Theorem 3.2.27 to conclude that almost surely

$$
\operatorname{dim}_{A} F_{v} \geq \sup _{\varepsilon>0} \frac{\log \rho_{s}\left(\mathfrak{R}^{0} \overline{\mathbf{K}}^{\varepsilon}\right)}{-\log \varepsilon}
$$

Assuming the USSC the upper bound follows immediately as $\rho_{s}\left(\Re^{0} \overline{\mathbf{K}}^{\varepsilon}\right)$ is by definition the largest eigenvalue and hence greatest rate of expansion. The argument is identical to Theorem 3.2.27 and is left to the reader.

## Chapter

## The box-counting dimension of random box-like carpets

### 4.1 Introduction

In this chapter we will determine the almost sure box-counting and packing dimension of self-affine box-like carpets in the sense of Fraser [Fr1]. One of the key elements of our proofs are the results of the previous chapter, stating that self-similar 1-variable and $\infty$-variable random graph directed (RGDS) systems have equal Hausdorff and box-counting dimension almost surely. This will become relevant as the projections of random self-affine box-like sets onto the horizontal and vertical axes are attractors of self-similar RGDSs.

This chapter is structured as follows; In Section 4.2 we introduce additional notation used in this chapter for 1 -variable and $\infty$-variable carpets. Section 4.3 contains our results for random homogeneous (1-variable) attractors and Section 4.4 contains our results for random recursive, or $\infty$-variable, carpets. This is followed by some examples in Section 4.5, while all proofs are contained in Section 4.6.

### 4.2 Notation and basic definitions

The self-affine sets we are considering were introduced by Fraser [Fr1] and are known as box-like self-affine carpets.

Definition 4.2.1. For a given $i$, let $f_{i}^{j}: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ be of the form

$$
f_{i}^{j}(\mathbf{x})=\left(\begin{array}{cc}
a_{i}^{j} & 0 \\
0 & b_{i}^{j}
\end{array}\right) \mathbf{Q}_{\mathbf{i}}^{\mathbf{j}}\binom{x_{1}}{x_{2}}+\binom{u_{i}^{j}}{v_{i}^{j}},
$$

where $0<a_{i}^{j}<1$ and $0<b_{i}^{j}<1, \mathbf{x}=\left(x_{1}, x_{2}\right)$, and $u_{i}^{j}, v_{i}^{j} \in \mathbb{R}$ are such that the unit square $\Delta=[0,1]^{2}$ is mapped into itself, that is $f_{i}^{j}(\Delta) \subset \Delta$ and

$$
\mathbf{Q}_{i}^{j} \in\left\{\left(\begin{array}{cc} 
\pm 1 & 0 \\
0 & \pm 1
\end{array}\right),\left(\begin{array}{cc}
0 & \pm 1 \\
\pm 1 & 0
\end{array}\right)\right\} .
$$

If all maps $f_{i}^{j} \in \mathbb{I}_{i}$ satisfy the criteria above we call the $\operatorname{IFS} \mathbb{I}_{i}=\left\{f_{i}^{j}\right\}_{j=1}^{\# \mathbb{I}_{i}}$ box-like. If all IFSs $\mathbb{I}_{i} \in \mathbb{L}$ are box-like we call the RIFS $(\mathbb{L}, \vec{\pi})$ box-like.

We remark that the matrices $\mathbf{Q}_{i}^{j}$ represent elements of the symmetry group of isometries $D_{8}$ such that $f_{i}^{j}$ maps the square onto rectangles that are still aligned with the $x$ and $y$ axis.

[^3]Definition 4.2.2. Let $(\mathbb{L}, \vec{\pi})$ be box-like as above. If there exist at least two pairs $(i, j)$ and $(k, l)$ with

$$
\mathbf{Q}_{i}^{j} \in\left\{\left(\begin{array}{cc}
0 & \pm 1 \\
\pm 1 & 0
\end{array}\right)\right\}, \text { and } \mathbf{Q}_{k}^{l} \in\left\{\left(\begin{array}{cc} 
\pm 1 & 0 \\
0 & \pm 1
\end{array}\right)\right\}
$$

we call $(\mathbb{L}, \vec{\pi})$ non-separated. Otherwise, we call $(\mathbb{L}, \vec{\pi})$ separated.
We note that this differs from [Fr1] by only requiring one of the $\operatorname{IFSs}$ in $\mathbb{I}_{i} \in \mathbb{L}$ to have one map with a differing diagonal structure. This is the appropriate analogue to consider in the random setting as every IFS is chosen infinitely often with full probability, providing the necessary 'mixing' of projections.

Results in dimension theory usually require some assumptions on the level of overlap. We introduce the random analogue of the condition introduced by Feng and Wang [FW].

Definition 4.2.3. Let $(\mathbb{L}, \vec{\pi})$ be a box-like RIFS. We say that $(\mathbb{L}, \vec{\pi})$ satisfies the uniform open rectangle condition (UORC), if we have, for every $i \in \Lambda$,

$$
f_{i}^{j}(\grave{\Delta}) \cap f_{i}^{k}(\grave{\Delta}) \neq \varnothing \quad \Rightarrow \quad k=j .
$$

Here $\stackrel{\circ}{\Delta}=(0,1)^{2}$ is the open unit square.
To each map $f_{i}^{j}$ we associate a unique symbol $e_{i}^{j}$ to enable us to code points in the random attractor. We adopt the notation of arrangement of words to write sets of codings succinctly. We set $\mathcal{G}^{E}=\left\{e_{i}^{j} \mid i \in \Lambda, 1 \leq j \leq \# \mathbb{I}_{i}\right\}$ and call $\mathcal{G}=\left\{\varnothing, \varepsilon_{0}\right\} \cup \mathcal{G}^{E}$ the prime arrangements. The set of all finite combinations of elements of $\mathcal{G}$ and operations $\sqcup$ and $\odot$ is called $\beth^{*}$. Using distributivity $\beth=\left(\beth^{*}, \sqcup, \odot\right)$ is the noncommutative free semi-ring with 'addition' $\sqcup$ and 'multiplication' $\odot$ and generator $\mathcal{G}^{E}$ and $\beth$ is called the semiring of arrangements of words.

Again, we use the convention to 'multiply out' arrangements of words and write them as elements of $\beth \odot$. We omit brackets, where appropriate, replace $\odot$ by concatenation to simplify notation, and for each arrangement of words $\phi$ write $\varphi_{i} \in \phi$ to refer to the subarrangements $\varphi$ that do not contain $\sqcup$ and are thus elements of $\varphi \in \beth$.
Definition 4.2.4. Given an arrangement of words $\phi$ and a compact set $K \in \mathcal{K}\left(\mathbb{R}^{2}\right)$, we define $f(\phi, K)$ recursively to be the compact set satisfying:

$$
f(\phi, K)= \begin{cases}f\left(\phi_{1}, K\right) \cup f\left(\phi_{2}, K\right), & \text { if } \phi=\phi_{1} \sqcup \phi_{2} ; \\ f\left(\phi_{1}, f\left(\phi_{2}, K\right)\right), & \text { if } \phi=\phi_{1} \odot \phi_{2} ; \\ f_{i}^{j}(K), & \text { if } \phi=e_{i}^{j} ; \\ K, & \text { if } \phi=\varepsilon_{0} ; \\ \varnothing, & \text { if } \phi=\varnothing\end{cases}
$$

To each IFS we associate an arrangement of words.
Definition 4.2.5. Let $W_{i}$ be the arrangement of words that are the letters coding the maps of the IFS $\mathbb{I}_{i}$,

$$
W_{i}=e_{i}^{1} \sqcup e_{i}^{2} \sqcup \cdots \sqcup e_{i}^{\# \mathbb{I}_{i}} .
$$

This representation now allows us to define sets involving the IFSs recursively by right multiplication of $W_{i}$ to existing codings.

### 4.2.1 Projections

Our results depend on the box-counting dimensions of the orthogonal projections onto the $x$ and $y$ axes. We write $\Pi_{x}$ and $\Pi_{y}$ to denote these projections, respectively:

$$
\Pi_{x}: \mathbb{R}^{2} \rightarrow \mathbb{R}, \quad \text { where } \quad \Pi_{x}\left(\left(z_{1}, z_{2}\right)^{\top}\right)=z_{1}, \quad \text { and }
$$

$$
\Pi_{y}: \mathbb{R}^{2} \rightarrow \mathbb{R}, \quad \text { where } \quad \Pi_{y}\left(\left(z_{1}, z_{2}\right)^{\top}\right)=z_{2}
$$

For $e \in \beth \odot$, let

$$
\alpha_{M}(e)=\max _{z \in\{x, y\}}\left|\Pi_{z}(f(e, \Delta))\right| \quad \text { and } \quad \alpha_{m}(e)=\min _{z \in\{x, y\}}\left|\Pi_{z}(f(e, \Delta))\right|
$$

be the length of the longest and shortest edge, respectively, of the rectangle $f(e, \Delta)$. We define $\bar{s}(e, F)$ to be the upper box-counting dimension of the projection of the random set $F$ onto the line parallel to the longest side of $f(e, \Delta)$, that is

$$
\bar{s}(e, F)= \begin{cases}\overline{\operatorname{dim}}_{B}\left(\Pi_{x} F\right), & \text { if }\left|\Pi_{x}(f(e, \Delta))\right| \geq\left|\Pi_{y}(f(e, \Delta))\right|, \\ \operatorname{dim}_{B}\left(\Pi_{y} F\right), & \text { otherwise }\end{cases}
$$

Analogously, let $\underline{s}(e, F)$ be the lower box-counting dimension of the projection of $F$. If the box-counting dimension exists we write $s(e, F)$ for the common value. Let $\bar{s}^{x}(F)=\overline{\operatorname{dim}}_{B}\left(\Pi_{x} F\right)$ and $\bar{s}^{y}(F)=\overline{\operatorname{dim}}_{B}\left(\Pi_{y} F\right)$, with $\underline{s}^{x}(F), \underline{s}^{y}(F), s^{x}(F)$, and $s^{y}(F)$ defined analogously. We will write $\Pi_{e}$ to denote the projection, $\Pi_{x}$ or $\Pi_{y}$, parallel to the long side of the rectangle $f(e, \Delta)$, choosing arbitrarily if they are equal.

### 4.3 Results for 1-variable self-affine carpets

Let $\Omega=\Lambda^{\mathbb{N}}$ be the set of all (infinite) sequences with entries in $\Lambda$ and let $\mathbb{P}$ be the Bernoulli probability measure on $\Omega$ induced by $\vec{\pi}$.

We now define the random set we are investigating in this section. In fact we associate a set $F_{\omega}$ to every $\omega \in \Lambda^{\mathbb{N}}$. Choosing $\omega$ randomly according to $\mathbb{P}$ gives us the random attractor $F_{\omega}$.
Definition 4.3.1. The $k$-level coding with respect to realisation $\omega \in \Lambda^{\mathbb{N}}$ is

$$
\mathbf{C}_{\omega}^{k}=W_{\omega_{1}} \odot W_{\omega_{2}} \odot \cdots \odot W_{\omega_{k}} \quad(k \in \mathbb{N}) \quad \text { and } \quad \mathbf{C}_{\omega}^{0}=\varepsilon_{0}
$$

The arrangement of all finite codings $\mathbf{C}_{\omega}^{*}$ is defined by

$$
\mathbf{C}_{\omega}^{*}=\bigsqcup_{i=0}^{\infty} \mathbf{C}_{\omega}^{i}
$$

Recall that $\sqcup$ represents addition in the semiring $\beth$
Definition 4.3.2. The $k$-level prefractal $F_{\omega}^{k}$ and the 1 -variable random box-like selfaffine carpet $F_{\omega}$ are

$$
F_{\omega}^{k}=f\left(\mathbf{C}_{\omega}^{k}, \Delta\right)=\bigcup_{e \in \mathbf{C}_{\omega}^{k}} f(e, \Delta) \subset \mathbb{R}^{2}
$$

and

$$
F_{\omega}=\bigcap_{k=1}^{\infty} f\left(\mathbf{C}_{\omega}^{k}, \Delta\right)=\bigcap_{k=1}^{\infty} \bigcup_{e \in \mathbf{C}_{\omega}^{k}} f(e, \Delta) \subset \mathbb{R}^{2}
$$

where $\Delta=[0,1]^{2}$.
For reasons of non-triviality we assume that each IFS in $\mathbb{L}$ has at least one map, with at least one IFS containing two maps. This guarantees that $F_{\omega}$ is almost surely not a singleton.


Figure 4.1: Three random realisations using the maps in Figure 2.1.

We define a singular value function for each realisation $\omega \in \Lambda^{\mathbb{N}}$.
Definition 4.3.3. Let $e \in \beth$, we define the upper (random) modified singular value function by

$$
\bar{\psi}_{\omega}^{s}(e)=\alpha_{M}(e)^{\bar{s}\left(e, F_{\omega}\right)} \alpha_{m}(e)^{s-\bar{s}\left(e, F_{\omega}\right)} .
$$

Let $\bar{\Psi}_{\omega}^{k}(s)$ be the sum of the modified singular values over all $k$-level words,

$$
\bar{\Psi}_{\omega}^{k}(s)=\sum_{e \in \mathbf{C}_{\omega}^{k}} \bar{\psi}_{\omega}^{s}(e)
$$

We let $\underline{\psi}_{\omega}^{s}(e)$ and $\underline{\Psi}_{\omega}^{k}(s)$ be the lower modified singular value function and its sum, defined analogously.

We will now introduce the last component, the pressure, which relates to the topological pressure of the associated dynamical system.

Definition 4.3.4. Let $s \in \mathbb{R}_{0}^{+}$, the upper s-pressure for realisation $\omega \in \Lambda^{\mathbb{N}}$ is given by

$$
\bar{P}_{\omega}(s)=\varlimsup_{k \rightarrow \infty}\left(\bar{\Psi}_{\omega}^{k}(s)\right)^{1 / k}
$$

The lower pressure $\underline{P}_{\omega}$ is defined analogously.
Lemma 4.3.5. There exists a function $P(s): \mathbb{R}_{0}^{+} \rightarrow \mathbb{R}_{0}^{+}$, the s-pressure, such that $\bar{P}_{\omega}(s)=\underline{P}_{\omega}(s)=P(s)$ for $\mathbb{P}$-almost every $\omega \in \Omega$. Further, $P(s)$ is continuous and strictly decreasing and there exists a unique $s_{B} \in \mathbb{R}_{0}^{+}$satisfying,

$$
\begin{equation*}
P\left(s_{B}\right)=1 . \tag{4.3.1}
\end{equation*}
$$

Again we note that we are taking the liberty of calling $P$ pressure even though it is more appropriately the exponential of pressure. Section 4.6 contains the proof of above lemma and our main result for the box-counting dimension of 1 -variable random box-like self-affine carpets.

Theorem 4.3.6. Let $(\mathbb{L}, \vec{\pi})$ be a box-like self-affine RIFS that satisfies the UORC. Let $F_{\omega}$ be the associated 1-variable random box-like self-affine carpet. Then

$$
\begin{equation*}
\operatorname{dim}_{B} F_{\omega}=s_{B} \tag{4.3.2}
\end{equation*}
$$

for almost every $\omega \in \Lambda^{\mathbb{N}}$, where $s_{B}$ is the unique solution to $P\left(s_{B}\right)=1$.
Applying Lemma 4.6.7, we get the following corollary.
Corollary 4.3.7. Let $(\mathbb{L}, \vec{\pi})$ be a box-like self-affine RIFS that satisfies the UORC and is of the separated type, with $\alpha_{M}(e)=\left|\Pi_{x} f(e, \Delta)\right|$ for all $e \in \mathbb{I}_{i} \in \mathbb{L}$. Let $F_{\omega}$ be the associated 1-variable random box-like carpet. Then $\bar{\psi}_{\omega}^{t}$ is (stochastically) additive
and the box-counting dimension of $F_{\omega}$ is almost surely given by the unique $s_{B}$ such that,

$$
\begin{equation*}
\exp \mathbb{E}\left(\log \sum_{e \in W_{\omega_{1}}} \bar{\psi}_{\omega}^{s_{B}}(e)\right)=1 \tag{4.3.3}
\end{equation*}
$$

Introducing further conditions, we can express the box-counting dimension in terms of the individual attractors. The following corollary to Corollary 4.3.7 extends the box-counting dimension result from Gui and Li [GL1] which states that for 1-variable Bedford-McMullen carpets with subdivisions $n, m$ the almost sure boxcounting dimension is the mean of the box-counting dimensions of the corresponding deterministic attractors.

Corollary 4.3.8. Let $(\mathbb{L}, \vec{\pi})$ be a box-like self-affine RIFS that satisfies the UORC and is of the separated type, with $\alpha_{M}(e)=\left|\Pi_{x} f(e, \Delta)\right|$ for all $e \in \mathbb{I}_{i} \in \mathbb{L}$. Let $F_{\omega}$ be the associated 1-variable random box-like carpet and write $\underline{i}=(i, i, i, \ldots) \in \Lambda^{\mathbb{N}}$. Assume further that

1. there exists $\eta \in(0,1)$ s.t. $\alpha_{m}(e)=\eta$ for all $e \in \mathbb{I}_{i}$ and $i \in \Lambda$,
2. $s^{x}\left(F_{\omega}\right)=\sum_{i \in \Lambda} \pi_{i} s^{x}\left(F_{\underline{i}}\right)$ almost surely,
3. and the following equality holds:

$$
\mathbb{E}\left(\log \sum_{e \in W_{\omega_{1}}} \alpha_{M}(e)^{s^{x}\left(F_{\omega}\right)}\right)=\mathbb{E}\left(\log \sum_{e \in W_{\omega_{1}}} \alpha_{M}(e)^{s^{x}\left(F_{\omega_{1}}\right)}\right)
$$

Then, almost surely,

$$
\begin{equation*}
\operatorname{dim}_{B} F_{\omega}=\sum_{i \in \Lambda} \pi_{i} \operatorname{dim}_{B} F_{\underline{i}}=\mathbb{E}\left(\operatorname{dim}_{B} F_{\underline{\omega_{i}}}\right) \tag{4.3.4}
\end{equation*}
$$

Proof. First note that $s^{x}\left(F_{\omega}\right)$ is constant almost surely. We denote this value by $s^{x}$ and note from (4.3.3),

$$
\begin{aligned}
1 & =\exp \mathbb{E}\left(\log \sum_{e \in W_{\omega_{1}}} \alpha_{M}(e)^{s^{x}} \alpha_{m}(e)^{s_{B}-s^{x}}\right) \\
& =\exp \mathbb{E}\left(\log \left(\eta^{s_{B}-s^{x}} \sum_{e \in W_{\omega_{1}}} \alpha_{M}(e)^{s^{x}}\right)\right)
\end{aligned}
$$

So

$$
\eta^{-s_{B}}=\eta^{-s^{x}}\left(\sum_{e \in W_{1}} \alpha_{M}(e)^{s^{x}}\right)^{\pi_{1}} \cdots\left(\sum_{e \in W_{N}} \alpha_{M}(e)^{s^{x}}\right)^{\pi_{N}}
$$

but $\eta^{-s^{x}}=\eta^{-\sum_{i \in \Lambda} \pi_{i} s^{x}\left(F_{\underline{i}}\right)}$ almost surely and hence, almost surely,

$$
\eta^{-s_{B}}=\left(\sum_{e \in W_{1}}\left(\alpha_{M}(e) / \eta\right)^{s^{x}\left(F_{\underline{1}}\right)}\right)^{\pi_{1}} \cdots\left(\sum_{e \in W_{N}}\left(\alpha_{M}(e) / \eta\right)^{s^{x}\left(F_{\underline{N}}\right)}\right)^{\pi_{N}}
$$

and

$$
s_{B}=\frac{\sum_{i \in \Lambda} \pi_{i} \log \left(\sum_{e \in W_{i}}\left(\alpha_{M}(e) / \eta\right)^{s^{x}\left(F_{i}\right)}\right)}{-\log \eta}
$$

Thus $s_{B}$ is the weighted average of $\operatorname{dim}_{B} F_{\underline{i}}$.

On first glance these conditions seem very restrictive. However, note that 1variable Bedford-McMullen carpets sharing the same $n, m$ grid subdivision satisfy these conditions (the Gui-Li case). Briefly, this is because

$$
s^{x}\left(F_{\underline{i}}\right)=\frac{-\log (\text { number of non-empty columns })}{\log (\text { column width })}(i \in \Lambda),
$$

and

$$
\left.s^{x}\left(F_{\omega}\right)=\frac{-\log (\text { geometric mean of the number of non-empty columns })}{\log (\text { column width })} \text { (a.s. }\right)
$$

Further, these conditions are satisfied for much more general separated box-like selfaffine RIFS (such as the Lalley-Gatzouras type) if all the individual attractors' projection onto the horizontal have the same box-counting dimension and they contract equally in the direction parallel to the vertical.

Note however, that letting $\alpha_{m}(e)=\eta_{i}$ for every $e \in W_{i}$ is no longer sufficient for the dimension to be the mean of the individual dimensions as Example 4.5.1 in Section 4.5 shows. Another interesting consequence of Theorem 4.3.6 is the following corollary for RIFSs such that the modified singular value function is not stochastically additive.

Corollary 4.3.9. Let $(\mathbb{L}, \vec{\pi})$ be a box-like self-affine RIFS that satisfies the UORC such that $\bar{\psi}_{\omega}^{s}(e)$ is not stochastically additive. Let $F_{\omega}$ be the associated 1-variable random box-like carpet. Then the almost sure box-counting dimension of the attractor can drop below the least box-counting dimension of the individual attractors, that is there exists $(\mathbb{L}, \vec{\pi})$ such that, almost surely,

$$
\operatorname{dim}_{B} F_{\omega}<\min _{i \in \Lambda} \operatorname{dim}_{B} F_{\underline{i}}
$$

Proof. See Example 4.5.2 in Section 4.5.
Of course, in light of Theorem 1.7.8, the box-counting dimension can be replaced by the packing dimension in all the preceding results.

We end this section by commenting that if $s^{x}=s^{y}=1$ a.s. the modified singular value function coincides with the singular value function and $\operatorname{dim}_{B} F_{\omega}$ coincides with the natural affinity dimension. For the separated case with greatest contraction in the vertical direction it is sufficient to have $s^{x}=1$. Conversely, if $s^{x}, s^{y}<1$ the almost sure box-counting dimension (and therefore the almost sure Hausdorff dimension) of $F_{\omega}$ will be strictly less than the associated affinity dimension.

### 4.4 Results for $\infty$-variable box-like carpets

In this section we define an infinite code tree and define the $\infty$-variable attractor of a finite random iterated function system $(\mathbb{L}, \vec{\pi})$. We set $k_{s}=\max _{i \in \Lambda} \# \mathbb{I}_{i}$ and consider the rooted $k_{s}$-ary tree. Each node in this tree we label with a single $i \in \Lambda$, chosen independently, according to probability vector $\vec{\pi}$. We denote the space of all possible labellings of the tree by $\mathcal{T}$ and refer to individual realisations picked according to the induced probability measure, described below, by $\tau \in \mathcal{T}$. In this full tree we address vertices by which branch was taken; if $v$ is a node at level $k$ we write $v=\left(v_{1}, v_{2}, \ldots, v_{k}\right)$, with $v_{i} \in\left\{1, \ldots, k_{s}\right\}$ and root node $v=($.$) . The levels of the$ tree are then:

$$
\{(.)\},\left\{(1),(2), \ldots,\left(k_{s}\right)\right\},\left\{(1,1),(1,2), \ldots,\left(1, k_{s}\right),(2,1), \ldots,\left(k_{s}, k_{s}\right)\right\}, \ldots
$$

We write $\tau(v) \in \Lambda$ to denote the random letter for node $v$ and realisation $\tau$. Given a node $v$ we define $\sigma^{v} \tau$ to be the full subtree starting at vertex $v$, with $\sigma^{(.)} \tau=\tau$. There exists a natural measure $\mathbb{P}$ on the collection of trees, induced by $\vec{\pi}$ which we now describe. Let $[\tau]_{k}$ be the collection of trees $\kappa$ such that $\tau(v)=\kappa(v)$ for all nodes
$v$ in levels up to $k$. Similarly to the 1 -variable setting we refer to this as a cylinder and note that it generates the topology of $\mathcal{T}$. The measure $\mathbb{P}$ is then the unique measure on $\mathcal{T}$ such that $\mathbb{P}\left([\tau]_{k}\right)=\pi_{1}^{\rho(1, k, \tau)} \pi_{2}^{\rho(2, k, \tau)} \ldots \pi_{n}^{\rho(n, k, \tau)}$, where $\rho(i, k, \tau)$ is the number of choices of letter $i \in \Lambda$ for all nodes up to level $k$ in realisation $\tau$.

We note that in this section we relax the requirement that every IFS $\mathbb{I}_{i}$ must contain at least one map, with a single IFS consisting of two maps. We now allow an IFS to have no maps, i.e. $W_{i}=\varnothing$ with positive probability, but we require a non-extinction condition.

Definition 4.4.1. We call the RIFS $(\mathbb{L}, \vec{\pi})$ non-extinguishing if

$$
\sum_{i \in \Lambda} p_{i} \# \mathbb{I}_{i}>1
$$

This implies that there exists positive probability that the associated attractor (defined below) is non-empty. We will later state results 'conditioned on non-extinction' by which we mean 'with respect to the (normalised) measure $\mathbb{P}$ restricted on the set of non-extinction'.

Allowing for extinction we have to extend the definition of the modified singular value function.

Definition 4.4.2. Let $e \in \beth$, we define the upper (random) modified singular value function as

$$
\bar{\psi}_{\tau}^{s}(e)= \begin{cases}\alpha_{M}(e)^{\bar{s}\left(e, F_{\tau}\right)} \alpha_{m}(e)^{s-\bar{s}\left(e, F_{\tau}\right)}, & \text { if } e \neq \varnothing \\ 0, & \text { otherwise }\end{cases}
$$

Again, $\underline{\psi}_{\omega}^{s}(e)$ and $\psi_{\omega}^{s}(e)$ are defined analogously.
Recall that $e_{i}^{j}$ is the letter representing the map $f_{i}^{j} \in \mathbb{I}_{i}$. For each full tree $\tau$ that is randomly labelled by entries in $\Lambda$, we associate another rooted labelled $k_{s}$-ary tree $\mathbf{T}_{\tau}$, where each node is labelled by an arrangement of words that describes the 'coding' of the associated cylinder.

Definition 4.4.3. Let $\mathbf{T}_{\tau}$ be a labelled tree, we write $\mathbf{T}_{\tau}(v)$ for the label of node $v$ of the tree $\mathbf{T}_{\tau}$. The coding tree $\mathbf{T}_{\tau}$ is then defined inductively:

$$
\mathbf{T}_{\tau}((.))=\varepsilon_{0} \text { and } \mathbf{T}_{\tau}(v)=\mathbf{T}_{\tau}\left(\left(v_{1}, \ldots, v_{k}\right)\right)=\mathbf{T}_{\tau}\left(\left(v_{1}, \ldots, v_{k-1}\right)\right) \odot e_{\tau\left(v_{k-1}\right)}^{v_{k}}
$$

for $1 \leq v_{k} \leq \# \mathbb{I}_{\tau\left(v_{k-1}\right)}$ and $e_{\tau\left(v_{k-1}\right)}^{v_{k}}=\varnothing$ otherwise. This 'deletes' this subbranch as $\varnothing$ annihilates under multiplication.

We refer to the arrangement of all labels at the $k$-th level by

$$
\mathbf{T}_{\tau}^{k}=\bigsqcup_{v_{1}, \ldots, v_{k}} \mathbf{T}_{\tau}\left(\left(v_{1}, \ldots, v_{k}\right)\right)
$$

We remark that the resulting tree will almost surely, when conditioned on nonextinction, have an exponentially increasing number of vertices at level $k$ as $k$ increases. We can now define the random recursive, or $\infty$-variable, box-like self-affine carpet.

Definition 4.4.4. Let $(\mathbb{L}, \vec{\pi})$ be a box-like self-affine RIFS and $\tau \in \mathcal{T}$. The $\infty$ variable box-like self-affine carpet $F_{\tau}$ is the compact set satisfying

$$
F_{\tau}=\bigcap_{k=1}^{\infty} f\left(\mathbf{T}_{\tau}^{k}, \Delta\right) .
$$

We note that setting up the RIFS appropriately this models reduces to self-affine fractal percolation.

We write $s^{x}$ and $s^{y}$ for the almost sure box-counting dimension of the projections of $F_{\tau}$ onto the horizontal and vertical axes. In this case the projections are $\infty$ variable RIFSs or random graph directed systems (RGDSs) in the sense of [T1] (see Definition 4.6 .1 below) and in the non-separated case $s^{x}=s^{y}$ almost surely.

Theorem 4.4.5. Let $(\mathbb{L}, \vec{\pi})$ be a box-like self-affine RIFS that satisfies the UORC and is non-extinguishing. Let $F_{\tau}$ be the associated $\infty$-variable random self-affine box-like carpet. The box-counting dimension of $F_{\tau}$, conditioned on non-extinction, is almost surely given by the unique $s_{B}$ satisfying

$$
\begin{equation*}
\mathbb{E}\left(\left(\sum_{e \in \mathbf{T}_{\tau}^{k}} \bar{\psi}_{\tau}^{s_{B}}(e)\right)^{1 / k}\right) \rightarrow 1 \text { as } k \rightarrow \infty \tag{4.4.1}
\end{equation*}
$$

Corollary 4.4.6. Let $(\mathbb{L}, \vec{\pi})$ be a box-like self-affine RIFS that satisfies the UORC and is non-extinguishing. Let $F_{\tau}$ be the associated $\infty$-variable random self-affine box-like carpet. If the modified singular value function is additive, i.e. $\bar{\psi}_{\tau}^{t}(e \odot g)=\bar{\psi}_{\tau}^{t}(e) \bar{\psi}_{\tau}^{t}(g)$, we have, conditioned on non-extinction, $\operatorname{dim}_{B} F_{\omega}=s_{B}$ almost surely, where

$$
\begin{equation*}
\mathbb{E}\left(\sum_{e \in \mathbb{I}_{i}} \bar{\psi}_{\tau}^{s_{B}}(e)\right)=1 \tag{4.4.2}
\end{equation*}
$$

Similarly to the 1 -variable RIFSs we can get a dimension drop for $\infty$-variable carpets.

Corollary 4.4.7. Let $(\mathbb{L}, \vec{\pi})$ be a box-like self-affine RIFS that satisfies the UORC but does not have an additive modified singular value function. Let $F_{\tau}$ be the $\infty$-variable attractor associated with $(\mathbb{L}, \vec{\pi})$. Then the almost sure box-counting dimension of the attractor can drop below the least box-counting dimension of the individual attractors, that is there exists $(\mathbb{L}, \vec{\pi})$ such that, almost surely,

$$
\operatorname{dim}_{B} F_{\tau}<\min _{i \in \Lambda} \operatorname{dim}_{B} F_{\underline{i}}
$$

Proof. See Example 4.5.2 in Section 4.5.
Interestingly, if the modified singular value function is additive, the sequence of sums over the singular value function forms an $\mathscr{L}^{2}$ bounded martingale, allowing us to give an alternative proof of Corollary 4.4.6.

Theorem 4.4.8. Let $(\mathbb{L}, \vec{\pi})$ be a box-like self-affine RIFS that satisfies the UORC and is non-extinguishing. Assume that the modified singular value function is additive, e.g. if $\alpha_{M}(e \odot g)=\alpha_{M}(e) \alpha_{M}(g)$. Then

$$
\left(\sum_{\xi \in \mathbf{T}_{\tau}^{k}} \bar{\psi}_{\tau}^{s_{B}}(\xi)\right)_{k=1}^{\infty}
$$

is an $\mathscr{L}^{2}$ bounded martingale.
Corollary 4.4.6 then follows by an application of the martingale convergence theorem.

Again, we appeal to Theorem 1.7.8, and note that the box-counting dimension can be replaced by the packing dimension in all the results in this section.

Finally, we remark that the the box-counting dimension of $\infty$-variable attractors is always an upper bound to the box-counting dimension of 1 -variable attractors. When $\bar{\psi}$ is additive this can be easily seen by Jensen's Inequality (Proposition 1.2.9), noting that (4.3.3) is a geometric and (4.4.2) an arithmetic average.

### 4.5 Examples

We now use our results to compute the box-counting dimension of some simple random self-affine sets.


Figure 4.2: The two iterated functions systems used in Example 4.5.1. The IFS $\mathbb{I}_{1}$ is to the left and $\mathbb{I}_{2}$ is to the right.


Figure 4.3: The two iterated functions systems used in Example 4.5.2 (left and middle) and the first two iterations, choosing first $\mathbb{I}_{1}$ then $\mathbb{I}_{2}$, which is self-similar.

### 4.5.1 Example

Let $\mathbb{I}_{1}$ be the IFS for a Bedford-McMullen carpet $F_{\underline{1}}$ with subdivision $n_{1}=2$ and $m_{1}=3$, consisting of two maps; one maps the unit square to a rectangle in the left column and one maps the unit square into a rectangle of the right column. Take $\mathbb{I}_{2}$ to be the IFS for a Bedford-McMullen carpet $F_{\underline{2}}$ with subdivision $n_{2}=2$ and $m_{2}=4$, consisting of three maps, two mapping into the left column and one mapping into the right column, see Figure 4.2. Note that for both IFSs the box-counting dimension of the projection onto the horizontal is 1 and consider the 1 -variable box-like self-affine carpet associated with $\mathbb{L}=\left\{\mathbb{I}_{1}, \mathbb{I}_{2}\right\}$ and $\vec{\pi}=\{1 / 2,1 / 2\}$.

The modified singular value function takes the value $\psi_{1}^{t}=(1 / 2)(1 / 3)^{t-1}$ for all elements $e \in \mathbb{I}_{1}$ and $\psi_{2}^{t}=(1 / 2)(1 / 4)^{t-1}$ for all $e \in \mathbb{I}_{2}$. Solving $\sum_{e \in \mathbb{I}_{1}} \psi_{1}^{t}=1$ and $\sum_{e \in \mathbb{I}_{2}} \psi_{2}^{t}=1$ for $t$ we get $\operatorname{dim}_{B} F_{\underline{1}}=1$ and $\operatorname{dim}_{B} F_{\underline{2}}=\log 6 / \log 4$. However, as $\mathbb{P}$ is the ( $1 / 2,1 / 2$ )-Bernoulli measure, substituting into (4.3.3) and solving for $s_{B}$ we get that, almost surely, $\operatorname{dim}_{B} F_{\omega}=\log 18 / \log 12$ and since $\log 18 / \log 12>(1+$ $\log 6 / \log 4) / 2$ equation (4.3.4) fails even in the simple setting of Bedford-McMullen carpets with mixed subdivisions.

### 4.5.2 Example

Let $\mathbb{I}_{1}$ be the IFS for a Bedford-McMullen carpet as in the previous example and let $\mathbb{I}_{2}$ be another Bedford-McMullen carpet but with major contraction in the horizontal, see Figure 4.3. Note that the periodic word $\widetilde{\omega}=(1,2,1,2, \ldots)$ describes a selfsimilar set with Hausdorff and box-counting dimension $\log 4 / \log 6$. It is easy to check that the individual Bedford-McMullen carpets have box-counting dimension 1. Again, $\mathbb{I}=\left\{\mathbb{I}_{1}, \mathbb{I}_{2}\right\}$ is of separated type, but has a non-additive modified singular value function. We can calculate the box-counting dimension for the 1 -variable model explicitly, assuming the ( $1 / 2,1 / 2$ )-Bernoulli measure. First note that both projections
of the attractor have, by symmetry, the same dimension $s_{p}$, which is the unique real satisfying

$$
\exp \mathbb{E} \log \sum_{e \in W_{\omega_{1}}} \operatorname{Lip}\left(\Pi_{x} f(e, K)\right)^{s_{p}}=\left(2(1 / 2)^{s_{p}} 2(1 / 3)^{s_{p}}\right)^{1 / 2}=1
$$

and so, almost surely, $s_{p}=\log 4 / \log 6$. The box-counting dimension of the carpet then becomes the unique $t$ satisfying

$$
\left(2^{n}\left(2^{-k(n)} 3^{k(n)-n}\right)^{\log 4 / \log 6}\left(2^{k(n)-n} 3^{-k(n)}\right)^{t-\log 4 / \log 6}\right)^{1 / n} \rightarrow 1 \quad \text { a.s. as } n \rightarrow \infty
$$

where $k(n)$ is maximum number of 1 s or 2 s in a randomly picked string $\{1,2\}^{n}$. Since $k / n \rightarrow 1 / 2$ a.s. we deduce $t=\log 4 / \log 6$ and, almost surely, the box-counting dimension of the 1 -variable carpet agrees with that of the periodic word $\widetilde{\omega}$ and is strictly less than the box-dimensions of the individual attractors.

Taking the same $\mathbb{I}_{1}, \mathbb{I}_{2}$ and $\vec{\pi}=\{1 / 2,1 / 2\}$ but the $\infty$-variable construction, we can calculate the almost sure box-counting dimension of the projection of the carpet to be the unique $s_{p}$ satisfying

$$
\mathbb{E} \sum_{e \in W_{\omega_{1}}} \operatorname{Lip}\left(\Pi_{x} f(e, K)\right)^{s_{p}}=2^{-s_{p}}+3^{-s_{p}}=1
$$

For tree levels that are odd, the maximal singular value $\alpha_{M}$ cannot equal the lower singular value $\alpha_{m}$. We can explicitly state the expectation of the $2 k+1$ level sum of the modified singular value function for $k \in \mathbb{N}_{0}$ by noting that for a binary tree of level $2 k+1$ with two choices of labels per node there are $2^{1} 2^{2} 2^{4} \ldots 2^{2^{(2 k+1)-1}}=$ $2^{1+2+\ldots+2^{2 k}}=2^{2^{2 k+1}-1}$ choices of trees and thus $2^{2^{2 k}} 2^{2^{2 k+1}-1}=2^{2^{2 k+1}+2^{2 k}-1}=$ $2^{3 \cdot 2^{2 k}-1}$ equally likely paths. These paths correspond to all values of $2 \alpha_{M}^{s_{B}} \alpha_{m}^{t-s_{B}}$ we want to sum up. Notice that $\alpha_{M}=2^{-i} 3^{-j}$ and $\alpha_{m}=2^{-j} 3^{-i}$ for some $i \geq j$ at level $k=i+j$. Thus for subtrees the new singular values can only be $2^{-i-1} 3^{-j}$ if $i=j$, and $2^{-i-1} 3^{-j}$ or $2^{-i} 3^{-j-1}$ otherwise. Therefore the number of choices for $i$ at level $2 k+1$ must be

$$
\binom{2 k+1}{i} 2^{2^{2 k+1}-1}
$$

for $k+1 \leq i \leq 2 k+1$. We can now state the expectation at level $2 k+1$,

$$
\begin{aligned}
\mathbb{E} \sum_{e \in \mathbf{T}_{\tau}^{2 k+1}} \bar{\psi}_{\tau}^{t}(e) & =\frac{1}{2^{2^{2 k+1}-1}} \sum_{i=k+1}^{2 k+1} 2^{2^{2 k+1}-1}\binom{2 k+1}{i} \\
& \cdot 2\left(2^{-i} 3^{-(2 k+1-i)}\right)^{s_{p}}\left(2^{-(2 k+1-i)} 3^{-i}\right)^{t-s_{p}} \\
& =2 \sum_{i=k+1}^{2 k+1}\binom{2 k+1}{i}\left(\frac{3}{2}\right)^{s_{p} 2 i}\left(\frac{3}{2}\right)^{-s_{p}(2 k+1)} 2^{-t(2 k+1)}\left(\frac{2}{3}\right)^{i t} \\
& =2^{1-t(2 k+1)}\left(\frac{2}{3}\right)^{s_{p}(2 k+1)} \sum_{i=k+1}^{2 k+1}\binom{2 k+1}{i}\left(\frac{3}{2}\right)^{i\left(2 s_{p}-t\right)} \\
& \leq 2^{1-t(2 k+1)}\left(\frac{2}{3}\right)^{s_{p}(2 k+1)}\left(1+\left(\frac{3}{2}\right)^{\left(2 s_{p}-t\right)}\right)^{2 k+1}
\end{aligned}
$$

Let $\varepsilon>0$ and set $t=s_{p}+\varepsilon$, then

$$
\leq 2^{1-\varepsilon(2 k+1)} 3^{-s_{p}(2 k+1)}\left(1+\left(\frac{3}{2}\right)^{s_{p}}\left(\frac{3}{2}\right)^{-\varepsilon}\right)^{2 k+1}
$$

$$
\begin{aligned}
& =2^{1-\varepsilon(2 k+1)} 3^{-s_{p}(2 k+1)}\left(1+\left(3^{-s_{p}}-1\right)\left(\frac{3}{2}\right)^{-\varepsilon}\right)^{2 k+1} \\
& \rightarrow 0 \quad \text { as } k \rightarrow \infty
\end{aligned}
$$

Therefore the box-counting dimension $s_{B}$ of the attractor satisfies $s_{p} \leq s_{B} \leq s_{p}+\varepsilon$ and so $s_{B}=s_{p}$. Again this is a dimension drop when compared to the original attractors.

### 4.6 Proofs

This section is divided into two parts; we prove the results of Section 4.3 in the first, followed by the proofs for Section 4.4. First, however, we recall the definition of random graph directed systems introduced in the preceding chapter and alter it slightly to fit in with the notation used in this chapter. We then state and prove Lemmas 4.6.5 and 4.6.6, which apply to both 1 -variable and $\infty$-variable constructions.

Definition 4.6.1. Let $\boldsymbol{\Gamma}=\{\Gamma(1), \ldots, \Gamma(N)\}$ be a finite collection of directed graphs with edge sets $E(i), 1 \leq i \leq N$ and common vertex set $V=\{1, \ldots, M\}$. Associate to each graph a probability $\pi_{i}>0$ such that $\sum \pi_{i}=1$. We represent each edge $\mathbf{e} \in E(i)$ by a unique prime arrangement of words $e \in \mathcal{G}^{E}$ that codes an associated contraction $f_{\mathbf{e}}$. For $v, w \in V$ write ${ }_{v} E_{w}(i)=e_{1} \sqcup \cdots \sqcup e_{n}$, where the edges associated with the prime arrangements $e_{k} \in E(i)$ have initial vertex $v$ and terminal vertex $w$. We set ${ }_{v} E_{w}(i)=\varnothing$ if no such edge exists. Let

$$
\mathbf{E}(i)=\left(\begin{array}{cccc}
{ }_{1} E_{1}(i) & { }_{1} E_{2}(i) & \cdots & { }_{1} E_{M}(i) \\
{ }_{2} E_{1}(i) & \ddots & & { }_{1} E_{M}(i) \\
\vdots & & \ddots & \vdots \\
{ }_{M} E_{1}(i) & { }_{M} E_{2}(i) & \cdots & { }_{M} E_{M}(i)
\end{array}\right)
$$

and $\mathbb{1}_{v}$ be a vector of length $M$ such that $\left(\mathbb{1}_{v}\right)_{k}=\varepsilon_{0}$ if $k=v$ and $\left(\mathbb{1}_{v}\right)_{k}=\varnothing$ otherwise.
Matrix multiplication $\times$ and addition $\sqcup$ for such $n$ by $n$ matrices $\mathbf{M}$ and $\mathbf{N}$, and vectors $\mathbf{v}$ of dimension $n$ is defined in the natural way:

$$
\begin{gather*}
(\mathbf{M} \times \mathbf{N})_{i, j}=\bigsqcup_{k=1}^{n}\left(\mathbf{M}_{i, k} \odot \mathbf{N}_{k, j}\right), \quad(\mathbf{M} \sqcup \mathbf{N})_{i, j}=\mathbf{M}_{i, j} \sqcup \mathbf{N}_{i, j},  \tag{4.6.1}\\
(\mathbf{v} \times \mathbf{M})_{i}=\bigsqcup_{k=1}^{n}\left(v_{k} \odot \mathbf{M}_{k, i}\right) .
\end{gather*}
$$

Definition 4.6.2. Let

$$
\mathbf{E}_{v}^{k}(\omega)=\bigsqcup_{1 \leq j \leq M}\left(\mathbb{1}_{v} \mathbf{E}\left(\omega_{1}\right) \mathbf{E}\left(\omega_{2}\right) \ldots \mathbf{E}\left(\omega_{k}\right)\right)_{j}
$$

be the $k$-level sets for vertex $i \in\{1, \ldots, N\}$. The 1 -variable random graph directed self similar set $K_{v}(\omega)$ associated to $v \in V$ and realisation $\omega \in \Lambda^{\mathbb{N}}$ is given by

$$
K_{v}(\omega)=\bigcap_{k>0} f\left(\mathbf{E}_{v}^{k}(\omega), \Delta\right)
$$

The $\infty$-variable random graph directed system is defined analogously to $\infty$-variable RIFS (see Definitions 4.4.3 and 4.4.4), replacing $e_{\tau\left(v_{k-1}\right)}^{v_{k}}$ by an appropriately chosen matrix $\mathbf{E}(i)$ and taking the sum of arrangements over the $v$-th column.

Definition 4.6.3. Let $\boldsymbol{\Gamma}=\left\{\Gamma_{i}\right\}_{i \in \Lambda}$ be a finite collection of graphs, sharing the same vertex set $V$.
4.6.3.a We say that the collection $\boldsymbol{\Gamma}$ is a non-trivial collection of graphs if for every $i \in \Lambda$ and $v \in V$ we have

$$
\bigsqcup_{w \in V}{ }_{v} E_{w}(i) \neq \varnothing
$$

Furthermore we require that there exist $e_{1} \in E(i)$ and $e_{2} \in E(j)$ such that $f_{e_{1}} \neq f_{e_{2}}$.
4.6.3.b If for every $v, w \in V$ there exists $\omega^{v, w} \in \Lambda^{*}$ such that ${ }_{v} E_{w}\left(\omega^{v, w}\right) \neq \varnothing$ and $\mathbb{P}\left(\left[\omega^{v, w}\right]\right)>0$, we call $\boldsymbol{\Gamma}$ stochastically strongly connected.
4.6.3.c We call the Random Graph Directed System (RGDS) associated with $\boldsymbol{\Gamma}$ $a$ contracting self-similar RGDS if for every $e \in E(i), f_{e}$ is a contracting similitude.

Condition 4.6.4. Let $\boldsymbol{\Gamma}=\left\{\Gamma_{i}\right\}_{i \in \Lambda}$ be a finite collection of graphs, sharing the same vertex set $V$. We say that the collection $\boldsymbol{\Gamma}$ is a non-trivial surviving collection of graphs if for every $v \in V$ we have $\mathbb{E}\left(\#\left\{e \in{ }_{v} E\left(\omega_{1}\right)_{w} \mid w \in V\right\}\right)>1$. Furthermore we require that there exist $e_{1} \in E(i)$ and $e_{2} \in E(j)$ such that $f_{e_{1}} \neq f_{e_{2}}$.

Condition 4.6.4 is similar to a RIFS being non-extinguishing (Definition 4.4.1), guaranteeing the existence of a positive probability that $K_{v}(\omega) \neq \varnothing$.

Lemma 4.6.5. Let $(\mathbb{L}, \vec{\pi})$ be a box-like self-affine RIFS with associated 1-variable ( $\infty$-variable) carpet $F_{\omega}\left(F_{\tau}\right)$. The projections $\Pi_{x}\left(F_{\omega}\right)$ and $\Pi_{y}\left(F_{\omega}\right)$, and $\Pi_{x}\left(F_{\tau}\right)$ and $\Pi_{y}\left(F_{\tau}\right)$ are, in the separated case, random self-similar and, in the non-separated case, random graph-directed self-similar sets ( $R G D S$ ) as above. The 1-variable $R G D S$ satisfies all conditions in Definition 4.6.3 (assuming every IFS has at least one map, with at least one IFS having two maps) and the $\infty$-variable RGDS (assuming it is non-extinguishing) satisfies all conditions in Definition 4.6.3 with Condition 4.6.4 replacing 4.6.3.a.

Proof. We prove the 1 -variable case where $(\mathbb{L}, \vec{\pi})$ is a 1 -variable box-like self-affine RIFS. Assume that $(\mathbb{L}, \vec{\pi})$ is separated; without loss of generality (considering all possible concatenations of two $f_{i}^{j}$ if necessary) for every $i \in \Lambda$ and $j \in \mathbb{I}_{i}$ the matrix $\mathbf{Q}_{i}^{j}$ does not have off-diagonal entries and for some $\widehat{a}_{i}^{j}, \widehat{b}_{i}^{j} \in \mathbb{R} \backslash\{0\}$ each map $f_{i}^{j}$ can be rewritten as

$$
f_{i}^{j}(\mathbf{x})=\left(\begin{array}{cc}
\widehat{a}_{i}^{j} & 0  \tag{4.6.2}\\
0 & \widehat{b}_{i}^{j}
\end{array}\right) \mathbf{x}+\binom{u_{i}^{j}}{v_{i}^{j}}
$$

We define the two induced maps $\widehat{\Pi}_{x} f_{i}^{j}: \mathbb{R} \rightarrow \mathbb{R}$ and $\widehat{\Pi}_{y} f_{i}^{j}: \mathbb{R} \rightarrow \mathbb{R}$ by
$\widehat{\Pi}_{x} f_{i}^{j}(z)=\Pi_{x} \circ f_{i}^{j} \circ \Pi_{x}^{-1}(z)=\widehat{a}_{i}^{j} z+u_{i}^{j} \quad$ and $\quad \widehat{\Pi}_{y} f_{i}^{j}(z)=\Pi_{y} \circ f_{i}^{j} \circ \Pi_{y}^{-1}(z)=\widehat{b}_{i}^{j} z+v_{i}^{j}$.
For every $e \in \mathbf{C}_{\omega}^{k}$

$$
\begin{aligned}
\widehat{\Pi}_{x} f_{e} & =\Pi_{x} \circ f_{e_{1}} \circ f_{e_{2}} \circ \cdots \circ f_{e_{|e|}} \circ \Pi_{x}^{-1} \\
& =\Pi_{x} \circ f_{e_{1}} \circ \Pi_{x}^{-1} \circ \Pi_{x} \circ f_{e_{2}} \circ \ldots \Pi_{x} \circ f_{e_{|e|}} \circ \Pi_{x}^{-1} \\
& =\left(\widehat{\Pi}_{x} f_{e_{1}}\right) \circ\left(\widehat{\Pi}_{x} f_{e_{2}}\right) \circ \cdots \circ\left(\widehat{\Pi}_{x} f_{e_{|e|}}\right),
\end{aligned}
$$

where $f_{e}(x)=f(e, x)$. So the attractor $K_{x, \omega}$ of the iterated function system $\mathbb{L}^{x}=$ $\left\{\mathbb{I}_{i}^{x}\right\}_{i \in \Lambda}$, with $\mathbb{I}_{i}^{x}=\left\{\widehat{\Pi}_{x} f_{i}^{j}\right\}_{j \in \mathcal{I}_{i}}$ satisfies $K_{x, \omega}=\Pi_{x} F_{\omega}$. A similar argument holds for the projection onto the vertical axis and thus the projections of $F_{\omega}$ onto the $x$ and $y$ axes are the attractors of the RIFSs $\mathbb{L}^{x}$ and $\mathbb{L}^{y}$. Finally note that the projections are similitudes and hence the projections of $F_{\omega}$ onto the horizontal and vertical axes are self-similar RIFS.

The argument for the graph directed construction in the non-separated case is similar. For each $i \in \Lambda$ we define a graph $\Gamma_{i}$, where all graphs $\boldsymbol{\Gamma}=\left\{\Gamma_{1}, \ldots, \Gamma_{N}\right\}$ share the same vertex set $V=\left\{V_{h}, V_{v}\right\}$. We define a new set of prime arrangements $\mathcal{G}^{* E}=\left\{e_{i, j}^{z} \mid z \in\{x, y\}\right\}$ for the RGDS, where for every $e_{i}^{j}$ we obtain two new unique words: $e_{i, j}^{x}$ and $e_{i, j}^{y}$. Let $f_{i}^{j}$ be of the same form as (4.6.2), we define
$f\left(e_{i, j}^{x}, z\right)=\Pi_{x} \circ f_{i}^{j} \circ \Pi_{x}^{-1}(z)=\widehat{a}_{i}^{j} z+u_{i}^{j} \quad$ and $\quad f\left(e_{i, j}^{y}, z\right)=\Pi_{y} \circ f_{i}^{j} \circ \Pi_{y}^{-1}(z)=\widehat{b}_{i}^{j} z+v_{i}^{j}$.
For $f_{i}^{j}$ that map vertical lines to horizontal and vice versa, i.e. are of form

$$
f_{i}^{j}(\mathbf{x})=\left(\begin{array}{cc}
0 & \widehat{a}_{i}^{j}  \tag{4.6.3}\\
\widehat{b}_{i}^{j} & 0
\end{array}\right) \mathbf{x}+\binom{u_{i}^{j}}{v_{i}^{j}}
$$

we define
$f\left(e_{i, j}^{y}, z\right)=\Pi_{y} \circ f_{i}^{j} \circ \Pi_{x}^{-1}(z)=\widehat{b}_{i}^{j} z+v_{i}^{j} \quad$ and $\quad f\left(e_{i, j}^{x}, z\right)=\Pi_{x} \circ f_{i}^{j} \circ \Pi_{y}^{-1}(z)=\widehat{a}_{i}^{j} z+u_{i}^{j}$.
Fix $i \in \Lambda$. We define, for all $v, w \in V$ the edge set ${ }_{v} E_{w}(i)$ by

$$
\begin{aligned}
& V_{h} E_{V_{h}}(i)=\bigsqcup\left\{e_{i, j}^{x} \mid f_{i}^{j} \text { is of form (4.6.2) }\right\} \\
& V_{v} E_{V_{v}}(i)=\bigsqcup\left\{e_{i, j}^{y} \mid f_{i}^{j} \text { is of form (4.6.2) }\right\} \\
& V_{h} E_{V_{v}}(i)=\bigsqcup\left\{e_{i, j}^{x} \mid f_{i}^{j} \text { is of form (4.6.3) }\right\} \\
& V_{v} E_{V_{h}}(i)=\bigsqcup\left\{e_{i, j}^{y} \mid f_{i}^{j} \text { is of form (4.6.3) }\right\}
\end{aligned}
$$

It remains to check that $\Pi_{x} F_{\omega}=K_{V_{h}}(\omega)$ and $\Pi_{y} F_{\omega}=K_{V_{v}}(\omega)$, for which it is sufficient to show that, given any finite arrangement of words in $e \in \mathbf{C}_{\omega}^{*}$,

$$
\begin{equation*}
\Pi_{x} f(e, \Delta)=f\left(e^{x},[0,1]\right) \text { and } \Pi_{y} f(e, \Delta)=f\left(e^{y},[0,1]\right) \tag{4.6.4}
\end{equation*}
$$

is satisfied, where $e^{x}$ and $e^{y}$ are the two induced paths, starting at $V_{h}$ and $V_{v}$, respectively.

If $e=\varepsilon_{0}$, then $e^{x}=e^{y}=\varepsilon_{0}$ and if $e=\varnothing$, then $e^{x}=e^{y}=\varnothing$ and in both cases (4.6.4) holds trivially. Let $w=w_{1} \odot \cdots \odot w_{k} \in \mathbf{C}_{\omega}^{k}$ with $w_{1}=e_{i}^{j}$ and write $w^{x}$ and $w^{y}$ for the induced paths. Assume inductively that $\Pi_{x} f\left(w_{2} \odot \cdots \odot w_{k}, \Delta\right)=$ $f\left(\left(w_{2} \odot \cdots \odot w_{k}\right)^{x},[0,1]\right)$ and $\Pi_{y} f\left(w_{2} \odot \cdots \odot w_{k}, \Delta\right)=f\left(\left(w_{2} \odot \cdots \odot w_{k}\right)^{y},[0,1]\right)$. Consider the $\operatorname{map} f_{i}^{j}$ and assume first it is of the form (4.6.2). The map $f(w,$.$) can$ be written as

$$
\begin{aligned}
\binom{z_{1}}{z_{2}} & \mapsto\binom{\widehat{a}_{i}^{j} \cdot \Pi_{x} \circ f\left(w_{2} \odot \cdots \odot w_{k},\left(z_{1}, z_{2}\right)^{\top}\right)+u_{i}^{j}}{\widehat{b}_{i}^{j} \cdot \Pi_{y} \circ f\left(w_{2} \odot \cdots \odot w_{k},\left(z_{1}, z_{2}\right)^{\top}\right)+v_{i}^{j}} \\
& =\binom{\widehat{a}_{i}^{j} \cdot f\left(\left(w_{2} \odot \cdots \odot w_{k} x^{x}, z_{1}\right)+u_{i}^{j}\right.}{\widehat{b}_{i}^{j} \cdot f\left(\left(w_{2} \odot \cdots \odot w_{k}\right)^{y}, z_{2}\right)+v_{i}^{j}} \\
& =\binom{f\left(w^{x}, z_{1}\right)}{f\left(w^{y}, z_{2}\right)} .
\end{aligned}
$$

Analogously, if $f_{i}^{j}$ is of form (4.6.3),

$$
\begin{align*}
\binom{z_{1}}{z_{2}} & \mapsto\binom{\widehat{b}_{i}^{j} \cdot \Pi_{y} \circ f\left(w_{2} \odot \cdots \odot w_{k},\left(z_{1}, z_{2}\right)^{\top}\right)+v_{i}^{j}}{\widehat{a}_{i}^{j} \cdot \Pi_{x} \circ f\left(w_{2} \odot \cdots \odot w_{k},\left(z_{1}, z_{2}\right)^{\top}\right)+u_{i}^{j}} \\
& =\binom{\widehat{b}_{i}^{j} \cdot f\left(\left(w_{2} \odot \cdots \odot w_{k}\right)^{y}, z_{1}\right)+v_{i}^{j}}{\widehat{a}_{i}^{j} \cdot f\left(\left(w_{2} \odot \cdots \odot w_{k}\right)^{x}, z_{2}\right)+u_{i}^{j}} \\
& =\binom{f\left(w^{x}, z_{1}\right)}{f\left(w^{y}, z_{2}\right)}, \tag{4.6.5}
\end{align*}
$$

where we have used that $w_{1}^{x} \in{ }_{V_{h}} E_{V_{v}}\left(\omega_{1}\right)$ and $w_{1}^{y} \in_{V_{v}} E_{V_{h}}\left(\omega_{1}\right)$ in (4.6.5). Therefore, by induction on word length, $\Pi_{x} F_{\omega}^{k}=K_{V_{h}}^{k}(\omega)$ and $\Pi_{y} F_{\omega}^{k}=K_{V_{v}}^{k}(\omega)$. We conclude that $\Pi_{x} F_{\omega}=K_{V_{h}}(\omega)$ and $\Pi_{y} F_{\omega}=K_{V_{v}}(\omega)$, where $K_{i}(\omega)$ is the 1-variable random graph directed system of Definition 4.6.2, as all maps are similitudes.

Finally we check the conditions of Definition 4.6.3. Non-triviality arises from the fact that $p_{i}>0$ for all $i \in \Lambda$, each map $f_{i}^{j}$ induces exactly one map starting at each of the two vertices and that we assume at least two maps to be distinct. Lastly, the stochastically strongly connected condition is satisfied since at least one of the maps is separated and there exists at least one pair $(i, j)$ such that horizontal get mapped to vertical ones.

The result for the $\infty$-variable case is almost identical and left to the reader.
The following lemma follows directly from Theorems 3.2.23 (1-variable) and 3.3.5 ( $\infty$-variable).

Lemma 4.6.6. Let $(\mathbb{L}, \vec{\pi})$ be a box-like self-affine RIFS s.t. every IFS has at least one map, with at least one IFS having two maps (it is non-extinguishing) and let $F_{\omega}$ $\left(F_{\tau}\right)$ be the associated 1-variable ( $\infty$-variable carpet. Let $e \in \mathbf{C}_{\omega}^{*}\left(e \in \mathbf{T}_{\tau}^{*}\right)$ be the level sets of Definition 4.3.1 (Definition 4.4.3). Then $\bar{s}\left(e, F_{\omega}\right)=\underline{s}\left(e, F_{\omega}\right)\left(\bar{s}\left(e, F_{\tau}\right)=\right.$ $\left.\underline{s}\left(e, F_{\tau}\right)\right)$ is constant almost surely and coincides with $s^{x}$ or $s^{y}$, the almost sure boxcounting dimension of the projection of $F_{\omega}\left(F_{\tau}\right)$ onto the horizontal or vertical axis, respectively. If $\mathbb{I}$ is non-separated, then additionally $s^{x}=s^{y}$ almost surely.

Basic dimensional properties give that the box-counting dimension of a set $X$ is bounded above by

$$
\overline{\operatorname{dim}}_{B} X \leq \overline{\operatorname{dim}}_{B}\left(\Pi_{x} X \times \Pi_{y} X\right) \leq \overline{\operatorname{dim}}_{B}\left(\Pi_{x} X\right)+\overline{\operatorname{dim}}_{B}\left(\Pi_{y} X\right)
$$

So, almost surely the box-counting dimension for 1 -variable and $\infty$-variable box-like carpets cannot exceed $s^{x}+s^{y}$. In the proofs below we will however also consider the parameter $s$ for $s>s^{x}+s^{y}$ to show that our results are exactly the unique values $s_{B}$ such that (4.3.2) and (4.4.2) hold, rather than $\min \left\{s_{B}, s^{x}+s^{y}\right\}$. Note that this means that the modified singular value function must be subadditive, although not strictly so.

### 4.6.1 Proofs for Section 4.3

The modified singular value function is in certain cases either stochastically subadditive, additive or superadditive. The proof shares many similarities with [Fr1, Lemma 2.1], although differs in some points because $s^{x}\left(F_{\omega}\right)$ and $s^{y}\left(F_{\omega}\right)$ do not surely coincide.

Lemma 4.6.7. Let $e \in \mathbf{C}_{\omega}^{*}$ and $g \in \beth^{*}$ be such that $e \odot g \in \mathbf{C}_{\omega}^{*}$. Writing $l=|e|$ and $\gamma_{\omega}(e, g), \gamma_{\min }, \gamma_{\max }$ for some quantities that will arise in the proof but are almost surely equal to one, the following statements hold for all $k \geq 0$

1. If $t \in\left[0, \bar{s}^{x}\left(F_{\omega}\right)+\bar{s}^{y}\left(F_{\omega}\right)\right]$ then

$$
\begin{align*}
\bar{\psi}_{\omega}^{t}(e \odot g) & \leq \gamma_{\omega}(e, g) \bar{\psi}_{\omega}^{t}(e) \bar{\psi}_{\sigma^{l} \omega}^{t}(g), \\
\bar{\Psi}_{\omega}^{k+l}(s) & \leq \gamma_{\max }^{(k+l)\left|\bar{s}^{x}\left(F_{\omega}\right)-\bar{s}^{y}\left(F_{\omega}\right)\right|} \bar{\Psi}_{\omega}^{k}(s) \bar{\Psi}_{\sigma^{k} \omega}^{l}(s) \tag{4.6.6}
\end{align*}
$$

2. If $t \geq \bar{s}^{x}\left(F_{\omega}\right)+\bar{s}^{y}\left(F_{\omega}\right)$ then

$$
\begin{aligned}
\bar{\psi}_{\omega}^{t}(e \odot g) & \geq \gamma_{\omega}(e, g) \bar{\psi}_{\omega}^{t}(e) \bar{\psi}_{\sigma^{l} \omega}^{t}(g) \\
\bar{\Psi}_{\omega}^{k+l}(s) & \geq \gamma_{\min }^{(k+l)\left|\bar{s}^{x}\left(F_{\omega}\right)-\bar{s}^{y}\left(F_{\omega}\right)\right|} \bar{\Psi}_{\omega}^{k}(s) \bar{\Psi}_{\sigma^{k} \omega}^{l}(s)
\end{aligned}
$$

An analogous result holds for $\underline{\psi}_{\omega}^{t}$ and $\underline{\Psi}_{\omega}^{k}$.

Proof. We first prove the results concerning the modified singular value function and deal with the separated case, which implies that $\alpha_{M}(e)$ can only take the values of $h(e) h(g)$ or $w(e) w(g)$, where $h(z)=\left|\Pi_{y} f(z, \Delta)\right|$ and $w(z)=\left|\Pi_{x} f(z, \Delta)\right|$ are the height and width of the rectangle $f(z, \Delta)$. Without loss of generality we can assume that $h(e) \geq w(e)$ i.e. $\alpha_{M}(e)=h(e)$.

We therefore have the following cases to check:
(i) $\alpha_{M}(g)=h(g)$ and thus $\alpha_{M}(e \odot g)=h(e) h(g)$,
(ii) $\alpha_{M}(g)=w(g)$ and $\alpha_{M}(e \odot g)=h(e) h(g)$,
(iii) $\alpha_{M}(g)=w(g)$ and $\alpha_{M}(e \odot g)=w(e) w(g)$.

In the separated case we define $\gamma_{\min }=\gamma_{\max }=\gamma_{\omega}(e, g)=1$ and we shall now treat each of the cases above separately.
(i) We have

$$
\begin{align*}
\frac{\bar{\psi}_{\omega}^{t}(e \odot g)}{\bar{\psi}_{\omega}^{t}(e) \bar{\psi}_{\sigma^{l} \omega}^{t}(g)} & =\frac{\left.(h(e) h(g))^{\bar{s}\left(e \odot g, F_{\omega}\right)}(w(e) w(g))\right)^{t-\bar{s}\left(e \odot g, F_{\omega}\right)}}{\left.h(e)^{\bar{s}\left(e, F_{\omega}\right)} w(e)^{t-\bar{s}\left(e, F_{\omega}\right)} h(g)^{\bar{s}\left(g, F_{\sigma}{ }^{l} \omega\right.}\right) b(g)^{t-\bar{s}\left(g, F_{\sigma} l_{\omega}\right)}} \\
& =\frac{(h(e) h(g))^{\bar{s}^{x}\left(F_{\omega}\right)}(w(e) w(g))^{t-\bar{s}^{x}\left(F_{\omega}\right)}}{h(e)^{\bar{s}^{x}\left(F_{\omega}\right)} w(e)^{t-\bar{s}^{x}\left(F_{\omega}\right)} h(e)^{\bar{s}^{x}\left(F_{\omega}\right)} w(g)^{t-\bar{s}^{x}\left(F_{\omega}\right)}}=1 \tag{4.6.7}
\end{align*}
$$

(ii) We have

$$
\begin{aligned}
\frac{\bar{\psi}_{\omega}^{t}(e \odot g)}{\bar{\psi}_{\omega}^{t}(e) \bar{\psi}_{\sigma^{l} \omega}^{t}(g)} & =\frac{(h(e) h(g))^{\bar{s}^{y}\left(F_{\omega}\right)}(w(e) w(g))^{t-\bar{s}^{y}\left(F_{\omega}\right)}}{h(e)^{\bar{s}^{y}\left(F_{\omega}\right)} w(e)^{t-\bar{s}^{y}\left(F_{\omega}\right)} w(g)^{\bar{s}^{x}\left(F_{\omega}\right)} h(g)^{t-\bar{s}^{x}\left(F_{\omega}\right)}} \\
& =\left(\frac{w(g)}{h(g)}\right)^{t-\bar{s}^{x}\left(F_{\omega}\right)-\bar{s}^{y}\left(F_{\omega}\right)}=r^{t-\bar{s}^{x}\left(F_{\omega}\right)-\bar{s}^{y}\left(F_{\omega}\right)}
\end{aligned}
$$

where $r>1$.
(iii) We have

$$
\begin{aligned}
\frac{\bar{\psi}_{\omega}^{t}(e \odot g)}{\bar{\psi}_{\omega}^{t}(e) \bar{\psi}_{\sigma^{l} \omega}^{t}(g)} & =\frac{(w(e) w(g))^{\bar{s}^{x}\left(F_{\omega}\right)}(h(e) h(g))^{t-\bar{s}^{x}\left(F_{\omega}\right)}}{h(e)^{\bar{s}^{y}\left(F_{\omega}\right)} w(e)^{t-\bar{s}^{y}\left(F_{\omega}\right)} w(g)^{\bar{s}^{y}\left(F_{\omega}\right)} h(g)^{t-\bar{s}^{x}\left(F_{\omega}\right)}} \\
& =\left(\frac{h(e)}{w(e)}\right)^{t-\bar{s}^{x}\left(F_{\omega}\right)-\bar{s}^{y}\left(F_{\omega}\right)}=r^{t-\bar{s}^{x}\left(F_{\omega}\right)-\bar{s}^{y}\left(F_{\omega}\right)}
\end{aligned}
$$

where $r>1$.
The required cases then follow by letting $t$ take the appropriate values.
For the non-separated case we have $s^{x}=s^{x}\left(F_{\omega}\right)=s^{y}\left(F_{\omega}\right)$ almost surely. Let

$$
\gamma_{\min }=\min _{\substack{i \in \Lambda \\ h \in \mathbb{I}_{i}}} \frac{\alpha_{m}(h)}{\alpha_{M}(h)} \quad \text { and } \quad \gamma_{\max }=\gamma_{\min }^{-1}
$$

Note that $\alpha_{m}(e \odot g) \geq \alpha_{m}(e) \alpha_{m}(g)$. Equivalently $\alpha_{m}(e \odot g)=c(e, g) \alpha_{m}(e) \alpha_{m}(g)$ for some $c(e, g) \geq 1$ and so we have for all $\omega \in \Lambda^{\mathbb{N}}$,

$$
\begin{align*}
\bar{\psi}_{\omega}^{t}(e \odot g)= & \alpha_{M}(e \odot g)^{\bar{s}\left(e \odot g, F_{\omega}\right)} \alpha_{m}(e \odot g)^{t-\bar{s}\left(e \odot g, F_{\omega}\right)} \\
= & \left(\alpha_{M}(e \odot g) \alpha_{m}(e \odot g)\right)^{\bar{s}\left(e \odot g, F_{\omega}\right)} \alpha_{m}(e \odot g)^{t-2 \bar{s}\left(e \odot g, F_{\omega}\right)} \\
= & \left(\alpha_{M}(e) \alpha_{M}(g) \alpha_{m}(e) \alpha_{m}(g)\right)^{\bar{s}\left(e \odot g, F_{\omega}\right)} \alpha_{m}(e \odot g)^{t-2 \bar{s}\left(e \odot g, F_{\omega}\right)} \\
= & \left(\alpha_{M}(e) \alpha_{M}(g) \alpha_{m}(e) \alpha_{m}(g)\right)^{\bar{s}\left(e \odot g, F_{\omega}\right)} \\
& \quad \cdot\left(\alpha_{m}(e) \alpha_{m}(g)\right)^{t-2 \bar{s}\left(e \odot g, F_{\omega}\right)} c(e, g)^{t-2 \bar{s}\left(e \odot g, F_{\omega}\right)} \\
= & c(e, g)^{t-2 \bar{s}\left(e \odot g, F_{\omega}\right)} \gamma_{\omega}(e, g) \bar{\psi}_{\omega}^{t}(e) \bar{\psi}_{\sigma^{l} \omega}^{t}(g), \tag{4.6.8}
\end{align*}
$$

for some $\gamma_{\omega}(e, g) \in\left[\gamma_{\text {min }}^{|e \odot g| \cdot\left|\bar{s}^{x}\left(F_{\omega}\right)-\bar{s}^{y}\left(F_{\omega}\right)\right|}, \gamma_{\max }^{|e \oslash| \cdot\left|\bar{s}^{x}\left(F_{\omega}\right)-\bar{s}^{y}\left(F_{\omega}\right)\right|}\right]$. The required inequalities follow again by letting $t$ take appropriate values. Note that, while $\gamma_{\omega}(e, g)=$ 1 holds for all $\omega \in \Lambda^{\mathbb{N}}$ in the separated case, in the non-separated case the equality holds only for $\omega \in \mathfrak{G}$, where $\mathfrak{G}$ is the full measure set on which $s^{x}$ and $s^{y}$ exist and coincide (guaranteed to exist by Lemma 4.6.6 for RIFS of the non-separated type). As it turns out this does not make a difference to the convergence of $P$.

We now move on to proving the inequalities involving $\bar{\Psi}_{\omega}^{k+l}(s)$. We have from (4.6.8) for $e \in \mathbf{C}_{\omega}^{k}$ and $g \in \mathbf{C}_{\sigma^{k} \omega}^{l}$ such that $e \odot g \in \mathbf{C}_{\omega}^{k+l}$ and $t \in\left[0, s_{x}+s_{y}\right]$,

$$
\begin{aligned}
\bar{\Psi}_{\omega}^{k+l}(t) & =\sum_{e \odot g \in \mathbf{C}_{\omega}^{k+l}} \bar{\psi}_{\omega}^{t}(e \odot g) \\
& =\sum_{e \in \mathbf{C}_{\omega}^{k}} \sum_{g \in \mathbf{C}_{\sigma^{k} \omega}^{l}} \bar{\psi}_{\omega}^{t}(e \odot g) \leq \sum_{e \in \mathbf{C}_{\omega}^{k}} \sum_{g \in \mathbf{C}_{\sigma^{k} \omega}^{l}} \gamma_{\omega}(e, g) \bar{\psi}_{\omega}^{t}(e) \bar{\psi}_{\sigma^{k} \omega}^{t}(g) \\
& \leq \gamma_{\max }^{(k+l)| |^{x}\left(F_{\omega}\right)-\bar{s}^{y}\left(F_{\omega}\right) \mid}\left(\sum_{e \in \mathbf{C}_{\omega}^{k}} \bar{\psi}_{\omega}^{t}(e)\right)\left(\sum_{g \in \mathbf{C}_{\sigma^{k} \omega}^{l}} \bar{\psi}_{\sigma^{k} \omega}^{t}(g)\right) \\
& =\gamma_{\max }^{(k+l)\left|\bar{s}^{x}\left(F_{\omega}\right)-\bar{s}^{y}\left(F_{\omega}\right)\right| \bar{\Psi}_{\omega}^{k}(t) \bar{\Psi}_{\sigma^{k} \omega}^{l}(t)} .
\end{aligned}
$$

The other inequality follows by a similar argument, again noting that

$$
\gamma_{\max }^{(k+l)\left|\bar{s}^{x}\left(F_{\omega}\right)-\bar{s}^{y}\left(F_{\omega}\right)\right|=1}
$$

for all $\omega \in \Lambda^{\mathbb{N}}$ in the separated case and for all $\omega \in \mathfrak{G}$ in the non-separated case.
Proof of Lemma 4.3.5. Let $t \in\left[0, \bar{s}^{x}\left(F_{\omega}\right)+\bar{s}^{y}\left(F_{\omega}\right)\right]$, and consider $\log \bar{\Psi}_{\omega}^{k}(t)$. We first note that $\log \bar{\Psi}_{\omega}^{k}(t)$ is a (measurable) random variable. Since there exists $c>0$ such that $\bar{\psi}_{\omega}^{t}(x) \geq c$ for all $i$ and $x \in \mathbb{I}_{i}$, we can write

$$
\inf _{k} \frac{\mathbb{E} \log \bar{\Psi}_{\omega}^{k}(t)}{k} \geq \inf _{k} \frac{\log c^{k}}{k}=\log c>-\infty
$$

Hence the condition on bounded infimum is satisfied. Obviously the shift map $\sigma$ is an invariant and ergodic transformation on $\Omega=\Lambda^{\mathbb{N}}$, and the subadditive defect is

$$
\begin{aligned}
\log \bar{\Psi}_{\omega}^{k+l}(t)-\log \bar{\Psi}_{\omega}^{k}(t)-\log \bar{\Psi}_{\sigma^{k} \omega}^{l}(t) & \leq \log \gamma_{\max }^{(k+l)\left|\bar{s}^{x}\left(F_{\omega}\right)-\bar{s}^{y}\left(F_{\omega}\right)\right|} \\
& =0 \quad(\text { a.s. })
\end{aligned}
$$

and therefore $\bar{P}_{\omega}(t)$ converges to some random variable $\widehat{P}_{\omega}$ almost surely by Proposition 1.4.4. Using ergodicity of $\sigma$ we can conclude that $\widehat{P}_{\omega}$ is almost surely constant and so, almost surely,

$$
\begin{equation*}
\bar{P}_{\omega}(t)=\lim _{k} \exp \left(\frac{\mathbb{E} \log \bar{\Psi}_{\omega}^{k}}{k}\right) \tag{4.6.9}
\end{equation*}
$$

The case for $t>\bar{s}^{x}\left(F_{\omega}\right)+\bar{s}^{y}\left(F_{\omega}\right)$ follows by considering the stochastically subadditive sequence of $-\log \bar{\Psi}_{\omega}^{k}$ and we will omit details here. We simply comment that $\bar{\Psi}_{\omega}^{k}=\underline{\Psi}_{\omega}^{k}$ (a.s.) and so $\underline{P}_{\omega}(t)=\bar{P}_{\omega}(t)$ almost surely, and we denote this common, almost sure, constant value by $P(t)$.

Lemma 4.6.8. The $s$-pressure $P(s)$ is strictly decreasing and continuous in $s$, and there exists unique $s_{B}$ such that $P\left(s_{B}\right)=1$.

Proof. Let $\underline{\alpha}=\min _{e \in \mathbb{I}_{i}, i \in \Lambda} \alpha_{m}(e)$ and $\bar{\alpha}=\max _{e \in \mathbb{I}_{i}, i \in \Lambda} \alpha_{M}(e)$. We have for $\varepsilon>0$,

$$
P(s+\varepsilon)=\mathrm{ess} \lim _{k \rightarrow \infty}\left(\sum_{e \in \mathbf{C}_{\omega}^{k}} \alpha_{M}(e)^{\bar{s}\left(e, F_{\omega}\right)} \alpha_{m}(e)^{s+\varepsilon-\bar{s}\left(e, F_{\omega}\right)}\right)^{1 / k}
$$

$$
\leq \operatorname{ess} \lim _{k \rightarrow \infty}\left(\sum_{e \in \mathbf{C}_{\omega}^{k}} \bar{\alpha}^{k \varepsilon} \alpha_{M}(e)^{\bar{s}\left(e, F_{\omega}\right)} \alpha_{m}(e)^{s-\bar{s}\left(e, F_{\omega}\right)}\right)^{1 / k}=\bar{\alpha}^{\varepsilon} P(s)
$$

Similarly we can establish the lower bound to get

$$
\underline{\alpha}^{\varepsilon} P(s) \leq P(s+\varepsilon) \leq \bar{\alpha}^{\varepsilon} P(s)
$$

It immediately follows that $P$ is continuous and strictly decreasing. Furthermore, letting $\varepsilon \rightarrow \infty$ we can see that $P(s) \rightarrow 0$ as $s \rightarrow \infty$. Finally consider

$$
\begin{aligned}
P(0) & =\text { ess } \lim _{k \rightarrow \infty}\left(\sum_{e \in \mathbf{C}_{\omega}^{k}} \alpha_{M}(e)^{\bar{s}\left(e, F_{\omega}\right)} \alpha_{m}(e)^{-\bar{s}\left(e, F_{\omega}\right)}\right)^{1 / k} \\
& \geq \text { ess } \lim _{k \rightarrow \infty}\left(\sum_{e \in \mathbf{C}_{\omega}^{k}} 1\right)^{1 / k}=\exp \mathbb{E}\left(\log \left(\# \mathbb{I}_{i}\right)\right)>1
\end{aligned}
$$

The last inequality follows by our assumption that at least one of our IFSs contains two maps. We can therefore conclude that $s_{B}$ in (4.3.1) is unique, which concludes the proof.

To prove that $s_{B}$ is the almost sure box-counting dimension of $F_{\omega}$ we first define a useful stopping.

Definition 4.6.9. For $0<\delta \leq 1$ we define the $\delta$-stopping:

$$
\Xi_{\omega}^{\delta}=\left\{e=e_{1} \ldots e_{|e|} \in \mathbf{C}_{\omega}^{*} \mid \alpha_{m}(e) \leq \delta \text { and } \alpha_{m}\left(e_{1} e_{2} \ldots e_{|e|-1}\right)>\delta\right\}
$$

This is the collection of arrangements of words such that their associated rectangle has shorter side comparable to $\delta$, i.e. for $e \in \Xi_{\omega}^{\delta}$,

$$
\begin{equation*}
\underline{\alpha} \delta<\alpha_{m}(e) \leq \delta \tag{4.6.10}
\end{equation*}
$$

We can now prove the upper bound of (4.3.2).
Lemma 4.6.10. Let $F_{\omega}$ be the attractor of a box-like self-affine random iterated function system. Irrespective of overlaps, almost surely,

$$
\operatorname{dim}_{B} F_{\omega} \leq s_{B}
$$

Proof. For $e \in \mathbf{C}_{\omega}^{*}$ define

$$
F_{\omega}(e)=f\left(e, F_{\sigma|e| \omega}\right)
$$

Let $\varepsilon>0$ be arbitrary and let $\left\{U_{e, i}\right\}_{i=1}^{N_{\varepsilon}\left(F_{\omega}(e)\right)}$ be a minimal $\varepsilon$-cover of $F_{\omega}(e)$; then

$$
\begin{equation*}
F_{\omega} \subseteq \bigcup_{e \in \Xi_{\omega}^{\varepsilon}} \bigcup_{i=1}^{N_{\varepsilon}\left(F_{\omega}(e)\right)} U_{e, i} \tag{4.6.11}
\end{equation*}
$$

Recall that $\Pi_{e}$ is the projection onto the axes parallel to the longest side of $f(e, \Delta)$. Using 4.6.11 we get

$$
N_{\varepsilon}\left(F_{\omega}\right) \leq \sum_{e \in \Xi_{\omega}^{\varepsilon}} N_{\varepsilon}\left(F_{\omega}(e)\right)=\sum_{e \in \Xi_{\omega}^{\varepsilon}} N_{\varepsilon}\left(f\left(e, F_{\sigma|e| \omega}\right)\right)=\sum_{e \in \Xi_{\omega}^{\varepsilon}} N_{\varepsilon}\left(f\left(e, \Pi_{e} F_{\sigma|e| \omega}\right)\right),
$$

since the rectangles have shortest length equal to $\varepsilon$ or less. Now notice that for $\varepsilon>0$ there exists $C_{\omega}>0$ such that

$$
C_{\omega}^{-1} r^{-\underline{s}_{\omega}(e)+\varepsilon / 2} \leq N_{r}\left(\Pi_{e} F_{\omega}\right) \leq C_{\omega} r^{-\bar{s}_{\omega}(e)-\varepsilon / 2}
$$

with $0<C_{\omega}<\infty$ holding almost surely. We thus get

$$
N_{\varepsilon}\left(F_{\omega}\right) \leq \sum_{e \in \Xi_{\omega}^{\varepsilon}} N_{\varepsilon / \alpha_{M}(e)}\left(\left(\Pi_{e} F_{\sigma|e| \omega}\right)\right) \leq C_{\omega} \sum_{e \in \Xi_{\omega}^{\varepsilon}}\left(\frac{\varepsilon}{\alpha_{M}(e)}\right)^{-\bar{s}\left(e, F_{\sigma|e| \omega}\right)-\varepsilon / 2}
$$

and as $\underline{\alpha} \varepsilon<\alpha_{m}(e)$ for all $e \in \Xi_{\omega}^{\varepsilon}$ we deduce, for $C^{*}=\underline{\alpha}^{-\left(s_{B}+\varepsilon\right)}$,

$$
\begin{aligned}
N_{\varepsilon}\left(F_{\omega}\right) & \leq C_{\omega}\left(\frac{\alpha_{m}(e)}{\underline{\alpha} \varepsilon}\right)^{s_{B}+\varepsilon} \sum_{e \in \Xi_{\omega}^{\varepsilon}}\left(\frac{\varepsilon}{\alpha_{M}(e)}\right)^{-\bar{s}\left(e, F_{\sigma}|e|_{\omega}\right)-\varepsilon / 2} \\
& \leq C_{\omega} \underline{\alpha}^{-\left(s_{B}+\varepsilon\right)} \varepsilon^{-\left(s_{B}+\varepsilon\right)} \sum_{e \in \Xi_{\omega}^{\varepsilon}} \alpha_{M}(e)^{\bar{s}\left(e, F_{\omega}\right)+\varepsilon / 2} \alpha_{m}(e)^{s_{B}+\varepsilon} \varepsilon^{-\bar{s}\left(e, F_{\omega}\right)-\varepsilon / 2} \\
& \leq C_{\omega} C^{*} \varepsilon^{-\left(s_{B}+\varepsilon\right)} \sum_{e \in \Xi_{\omega}^{\varepsilon}} \alpha_{M}(e)^{\bar{s}\left(e, F_{\omega}\right)+\varepsilon / 2} \alpha_{m}(e)^{s_{B}+\varepsilon-\bar{s}\left(e, F_{\omega}\right)-\varepsilon / 2} \\
& \leq C_{\omega} C^{*} \varepsilon^{-\left(s_{B}+\varepsilon\right)} \sum_{e \in \Xi_{\omega}^{\varepsilon}} \bar{\psi}_{\omega}^{s_{B}+\varepsilon / 2}(e) \\
& \leq C_{\omega} C^{*} \varepsilon^{-\left(s_{B}+\varepsilon\right)} \sum_{k=1}^{\infty} \bar{\Psi}_{\omega}^{k}\left(s_{B}+\varepsilon / 2\right)
\end{aligned}
$$

But since $P(s+\varepsilon)<1$ there exists $C_{\omega}^{\prime}$ for almost every $\omega$ such that

$$
\bar{\Psi}_{\omega}^{k}\left(s_{B}+\varepsilon\right) \leq C_{\omega}^{\prime} P(s+\varepsilon / 2)^{k}
$$

Therefore, almost surely,

$$
N_{\varepsilon}\left(F_{\omega}\right) \leq C^{*} C_{\omega} C_{\omega}^{\prime} \varepsilon^{-\left(s_{B}+\varepsilon\right)} \sum_{k} P(s+\varepsilon / 2)^{k}<\infty
$$

and hence $\operatorname{dim}_{B}\left(N_{\varepsilon}\left(F_{\omega}\right)\right) \leq s_{B}+\varepsilon$ as required.
For $t<s_{B}$ the sum over the random modified singular function of the elements in the stopping is bounded from below.

Lemma 4.6.11. Let

$$
L_{\omega}^{\delta}(t)=\sum_{e \in \Xi_{\omega}^{\delta}} \bar{\psi}_{\omega}^{t}(e) .
$$

Then $P(t)<1$ implies $L_{\omega}^{\delta}(t)<1$ for small enough $\delta$, and $P(t)>1$ implies $L_{\omega}^{\delta}(t)>1$ for small enough $\delta$ almost surely.

Proof. We start by introducing the same notation as in [T1] and write the stopping $\Xi_{\omega}^{\delta}$ in an infinite matrix fashion. Define the matrix $\widehat{\Xi}_{\omega}^{\delta}$ entrywise for $i, j \in \mathbb{N}$ by

$$
\left(\widehat{\Xi}_{\omega}^{\delta}\right)_{i, j}=\bigsqcup\left\{e \in \Xi_{\sigma^{i-1} \omega}^{\delta}:|e|=\max \{0, j-i\}\right\}
$$

if there exists $e \in \Xi_{\sigma^{i-1} \omega}^{\delta}$ with $|e|=\max \{0, j-i\}$. Otherwise set $\left(\widehat{\Xi}_{\omega}^{\delta}\right)_{i, j}=\varnothing$. We define the vector $\mathbb{1}_{\varepsilon_{0}}^{k}$ by

$$
\left(\mathbb{1}_{\varepsilon_{0}}^{k}\right)_{i}= \begin{cases}\varepsilon_{0}, & \text { if } i=k \\ \varnothing, & \text { otherwise }\end{cases}
$$

Given $0<\xi<1$, for every $0<\delta<1$ there exists a unique $k \in \mathbb{N}_{0}$ such that $\delta=\xi^{k} \theta$, for $\xi<\theta \leq 1$. Fix such a $\xi$, we start by showing that

$$
\begin{equation*}
L_{\omega}^{\delta}(t)=\sum_{e \in\left(\mathbb{1}_{\varepsilon_{0}}^{1} \hat{\Xi}_{\omega}^{\delta}\right)} \bar{\psi}_{\omega}^{t}(e) \asymp \sum_{e \in\left(\mathbb{1}_{\varepsilon_{0}}^{1} \widehat{\Xi}_{\omega}^{\left(\xi^{k}\right)}\right)} \bar{\psi}_{\omega}^{t}(e), \tag{4.6.12}
\end{equation*}
$$

where we write $g \asymp h$ to indicate $g / h$ and $h / g$ are bounded uniformly away from 0 in $\delta$ (and thus $k$ ). The first equality in (4.6.12) is immediate as

$$
\bigsqcup_{i}\left(\mathbb{1}_{\varepsilon_{0}}^{1} \widehat{\Xi}_{\omega}^{\delta}\right)_{i}=\Xi_{\omega}^{\delta} .
$$

For the asymptotic equality note that there exists $c_{1}>0$ so that for all $\delta>0$ and all $e \in \Xi_{\omega}^{\delta}$ and $g \in \beth^{*}$ such that $e \odot g \in \Xi_{\omega}^{\delta}$ with $|g|=1$ we have

$$
\begin{equation*}
c_{1}^{-1} \bar{\psi}_{\omega}^{t}(e) \leq \bar{\psi}_{\omega}^{t}(e \odot g) \leq c_{1} \bar{\psi}_{\omega}^{t}(e) \tag{4.6.13}
\end{equation*}
$$

Now also note that for every $e \in \Xi_{\omega}^{\delta}$ there exists unique $e^{\dagger} \in \Xi_{\omega}^{\xi^{k}}$ and $e^{\ddagger} \in \Xi_{\omega}^{\xi^{k+1}}$ such that the cylinders satisfy

$$
\begin{equation*}
\left[e^{\ddagger}\right] \subseteq[e] \subseteq\left[e^{\dagger}\right] \tag{4.6.14}
\end{equation*}
$$

There also exists integer $n_{\max }(\xi)$, independent of $\omega \in \Lambda^{\mathbb{N}}$, such that for all $g \in \Xi_{\omega}^{\xi}$ we have $|g|<n_{\max }(\xi)$ as all maps are contractions. We find the following bounds, where, for $e \in \Xi_{\omega}^{\xi^{k}}$, the $g$ is such that $e \odot g \in \Xi_{\omega}^{\delta}$,

$$
\begin{aligned}
\sum_{e \in\left(\mathbb{1}_{\varepsilon_{0}}^{1} \widehat{\Xi}_{\omega}^{\delta}\right)} \bar{\psi}_{\omega}^{t}(e) & =\sum_{e \in\left(\mathbb{1}_{\varepsilon_{0}}^{1} \widehat{\Xi}_{\omega}^{\xi^{k}}\right)} \sum_{\left\{g \mid e \odot g \in \Xi_{\omega}^{\delta}\right\}} \bar{\psi}_{\omega}^{t}(e \odot g) \\
& \leq c_{1}^{n_{\max }(\xi)} \sum_{e \in\left(\mathbb{1}_{\varepsilon_{0}}^{1} \widehat{\Xi}_{\omega}^{\xi^{k}}\right)} \sum_{\left\{g \mid e \odot g \in \Xi_{\omega}^{\delta}\right\}} \bar{\psi}_{\omega}^{t}(e) \\
& \leq\left(n_{\max }(\xi) c_{1}\right)^{n_{\max }(\xi)} \sum_{e \in\left(\mathbb{1}_{\varepsilon_{0}}^{1} \widehat{\Xi}_{\omega}^{\xi^{k}}\right)} \bar{\psi}_{\omega}^{t}(e),
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{e \in\left(\mathbb{1}_{\varepsilon_{0}}^{1} \hat{\Xi}_{\omega}^{\delta}\right)} \bar{\psi}_{\omega}^{t}(e) & =\sum_{e \in\left(\mathbb{1}_{\varepsilon_{0}}^{1} \widehat{\Xi}_{\omega}^{\xi}\right)} \sum_{\left\{g \mid e \odot g \in \Xi_{\omega}^{\delta}\right\}} \bar{\psi}_{\omega}^{t}(e \odot g) \\
& \geq c_{1}^{-n_{\max }(\xi)} \sum_{e \in\left(\mathbb{1}_{\varepsilon_{0}}^{1} \hat{\Xi}_{\omega}^{\left(\xi_{( }^{k}\right)}\right)} \bar{\psi}_{\omega}^{t}(e)
\end{aligned}
$$

Hence the asymptotic estimate in (4.6.12) holds.
Now, by (4.6.13) and (4.6.14) we have for some $c_{2}>0$ that $L_{\omega}^{\delta}(t)$ is related to the sum of the modified singular value function over the $\xi$ approximation codings by

$$
\begin{equation*}
c_{2}^{-k} \mathfrak{L}_{\omega}^{\delta, k}(t) \leq L_{\omega}^{\delta}(t) \leq c_{2}^{k} \mathfrak{L}_{\omega}^{\delta, k}(t) \tag{4.6.15}
\end{equation*}
$$

where

$$
\mathfrak{L}_{\omega}^{\delta, k}(t)=\sum_{e \in_{k} \widehat{\Xi}_{\omega}^{\xi}} \bar{\psi}_{\omega}^{t}(e) \text { and }{ }_{k} \widehat{\Xi}_{\omega}^{\xi}=\mathbb{1}_{\varepsilon_{0}}^{k} \underbrace{\widehat{\Xi}_{\omega}^{\xi} \ldots \widehat{\Xi}_{\omega}^{\xi}}_{k \text { times }} .
$$

Now consider the infinite matrix $\mathcal{M}_{\omega}^{\xi}(t)$ that is defined for all $i, j \in \mathbb{N}$ by

$$
\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{i, j}=\sum_{e \in\left(\hat{\Xi}_{i} \widehat{\Xi}_{\omega}^{\xi}\right)_{j}} \bar{\psi}_{\omega}^{t}(e) .
$$

By definition the sum over all entries in the $k$-th row is $\mathfrak{L}_{\omega}^{\delta, k}$ and we now show that the sums of the $k$-th column is related to $\bar{\Psi}_{\omega}^{k}(t)$, in fact it is easy to see that every entry of the $k$-th column of $\mathcal{M}_{\omega}(t)^{\xi}$ is a lower bound to $\bar{\Psi}_{\omega}^{i}(t)$ as every such entry is given by a word of length $k$. The number of non-empty column entries is at most $n_{\max }(\xi) k$, where $k$ is the column index. Combining this with (4.6.13) and (4.6.14) we get for some $c_{3}>0$

$$
\begin{equation*}
\bar{\Psi}_{\omega}^{k}(t)=\sum_{e \in \mathbf{C}_{\omega}^{k}} \bar{\psi}_{\omega}^{t}(e) \geq \frac{c_{3}^{-1} c_{1}^{-n_{\max }(\xi)}}{n_{\max }(\xi)} \sum_{j=0}^{n_{\max }(\xi)-1} \sum_{i=1}^{\infty}\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{i, j+k} . \tag{4.6.16}
\end{equation*}
$$

Similarly for every $e \in \mathbf{C}^{k}$ there exists $g \in\left({ }_{k+j} \widehat{\Xi}_{\omega}^{\xi}\right)$ for some $j \in\left\{0, \ldots, n_{\max }(\xi)-1\right\}$ such that $[e] \subseteq[g]$ and using (4.6.13) and (4.6.14) again we get for some $c_{4}>0$ that

$$
\bar{\Psi}_{\omega}^{k}(t)=\sum_{e \in \mathbf{C}_{\omega}^{k}} \bar{\psi}_{\omega}^{t}(e) \leq c_{4} c_{1}^{n_{\max }(\xi)} \sum_{j=0}^{n_{\max }(\xi)-1} \sum_{i=1}^{\infty}\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{i, j+k}
$$

We call $\sum_{j=0}^{n_{\max }(\xi)-1} \sum_{i=1}^{\infty}\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{i, j+k}$ the $n_{\max }(\xi)$-corridor at $k$ and denote by $\mathfrak{C}_{\xi}(l)$ all pairs $(i, j) \in \mathbb{N}^{2}$ such that $l \leq i<l+n_{\max }(\xi)$, that is the pairs in $\mathfrak{C}_{\xi}(l)$ are the coordinates of the $n_{\max }(\xi)$-corridor at $l$.

The final ingredient is to compare the rate of growth of the sum of the singular value function of non-empty column entries with the rate of growth of the sum over non-empty row entries. First notice that the rate of growth of the rows is related to the maximal element in the row by

$$
\max _{j}\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{i, j} \leq \sum_{j}\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{i, j} \leq n_{\max }(\xi) i \max _{j}\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{i, j}
$$

Note that elements in the matrix cannot increase arbitrarily from row to row, that is we have for some $c_{5}>0$ and all integers $i, j>1$,

$$
\begin{equation*}
\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{i, j} \leq c_{5} \max _{k \in\left\{1, \ldots, n_{\max }(\xi)\right\}}\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{j-1, i-k} \tag{4.6.17}
\end{equation*}
$$

Combining this with the fact that at least one of $\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{j-1, i-k}$ is positive, the maximal element cannot move arbitrarily and for every column $k$ there exists a row $r$ such that

$$
\max _{i \in\left\{0, \ldots, n_{\max }(\xi)\right\}}\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{r, k+i}=\max _{j \in\{0,1, \ldots\}}\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{r, j}
$$

But using the existence of $n_{\max }(\xi)$ and that the $\Xi_{\omega}^{\xi}$ do not contain the empty word, we also have $\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{i, j}=0$ for $j<i$ and $j>n_{\max }(\xi) i$. We deduce that

$$
\begin{array}{rlrl}
L_{\omega}^{\delta}(t) & \leq c_{2}^{k} \mathfrak{L}_{\omega}^{\delta, k}(t)=c_{2}^{k} \sum_{e \in_{k} \hat{\Xi}_{\omega}^{\xi}} \bar{\psi}_{\omega}^{t}(e) & & \text { by (4.6.15) } \\
& =c_{2}^{k} \sum_{j}\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{k, j} & & \\
& \leq c_{2}^{k} n_{\max }(\xi) k\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{k, j_{\max }} & \text { for some } j_{\max } \in\left\{k, \ldots, n_{\max }(\xi) k\right\} \\
& \leq c_{2}^{k} n_{\max }(\xi) k \sum_{l=0}^{n_{\max }(\xi)-1} \sum_{i=1}^{\infty}\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{i, l+j_{\max }} & \\
& \leq c_{2}^{k} c_{3} c_{1}^{n_{\max }(\xi)} n_{\max }(\xi)^{2} k \bar{\Psi}_{\omega}^{j_{\max }}(t) & & \text { by }(4.6 .16) .
\end{array}
$$

Similarly, we can derive the lower bound,

$$
\begin{align*}
L_{\omega}^{\delta}(t) & \geq c_{2}^{-k} \mathfrak{L}_{\omega}^{\delta, k}(t)=c_{2}^{-k} \sum_{e \in_{k} \widehat{\Xi}_{\omega}^{\xi}} \bar{\psi}_{\omega}^{t}(e)  \tag{4.6.15}\\
& =c_{2}^{-k} \sum_{j}\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{k, j} \\
& \geq c_{2}^{-k}\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{k, j_{\max }} \quad \text { by (4.6.15) } \\
& \geq c_{2}^{-k} c_{5}^{-k-j_{\max }-n_{\max }(\xi)} \max _{(i, j) \in \mathfrak{c}_{\xi}\left(j_{\max }\right)}\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{i, j}  \tag{4.6.18}\\
& \geq \frac{c_{2}^{-k} c_{5}^{-k-n_{\max }(\xi) k-n_{\max }(\xi)}}{n_{\max }(\xi)\left(j_{\max }+n_{\max }(\xi)\right)} \sum_{j=0}^{n_{\max }(\xi)-1} \sum_{i=1}^{\infty}\left(\mathcal{M}_{\omega}^{\xi}(t)\right)_{i, j+j_{\max }} \\
& \geq \frac{c_{2}^{-k} c_{5}^{-k-n_{\max }(\xi) k-n_{\max }(\xi)}}{c_{4} c_{1}^{n_{\max }(\xi)} n_{\max }(\xi)\left(j_{\max }+n_{\max }(\xi)\right)} \bar{\Psi}_{\omega}^{j_{\max }}(t) .
\end{align*}
$$

The inequality in (4.6.18) arises as the maximal element in the $n_{\max }(\xi)$-corridor at $j_{\max }$ must be in one of $j_{\max }+n_{\max }(\xi)$ rows and the maximal element can be no larger than $c_{5}^{j_{\max }+n_{\max }(\xi)}$ times the maximal element in any of the preceding rows by (4.6.17).

Thus we get upper and lower bounds to $L_{\omega}^{\delta}(t)$ and we find for some $c_{6}, c_{7}>0$ such that

$$
\frac{\log L_{\omega}^{\delta}(t)}{-\log \delta} \leq \frac{\log c_{6} c_{7}^{k} \bar{\Psi}_{\omega}^{j_{\max }}(t)}{-\log \xi^{k} \theta} \leq \frac{(1 / k) \log c_{6}+\log c_{7}+\left(j_{\max } / k\right) \log \left(\bar{\Psi}_{\omega}^{j_{\max }}(t)^{1 / j_{\max }}\right)}{-\log \xi}
$$

Thus, for arbitrary $\varepsilon>0$, we can pick $\xi$ small enough such that

$$
\frac{\log L_{\omega}^{\delta}(t)}{-\log \delta} \leq \frac{\left(j_{\max } / k\right) \log \left(\bar{\Psi}_{\omega}^{j_{\max }}(t)^{1 / j_{\max }}\right)}{-\log \xi}+\frac{\varepsilon}{2} .
$$

Similarly, for some $c_{8}, c_{9}>0$ and small enough $\xi$,

$$
\frac{\log L_{\omega}^{\delta}(t)}{-\log \delta} \geq \frac{\log c_{8} c_{9}^{k} \bar{\Psi}_{\omega}^{j_{\max }}(t)}{-k \log \xi} \geq \frac{\left(j_{\max } / k\right) \log \left(\bar{\Psi}_{\omega}^{j_{\max }}(t)^{1 / j_{\max }}\right)}{-\log \xi}-\frac{\varepsilon}{2}
$$

Observe now that almost surely

$$
\limsup _{k}\left(j_{\max } / k\right) \log \left(\bar{\Psi}_{\omega}^{j_{\max }}(t)^{1 / j_{\max }}\right) \leq \max _{p \in\{+1,-1\}} n_{\max }(\xi)^{p} \log P(t) .
$$

Similarly, almost surely,

$$
\liminf _{k}\left(j_{\max } / k\right) \log \left(\bar{\Psi}_{\omega}^{j_{\max }}(t)^{1 / j_{\max }}\right) \geq \min _{p \in\{+1,-1\}} n_{\max }(\xi)^{p} \log P(t)
$$

Now as $\varepsilon$ was arbitrary we conclude that $\left(\log L_{\omega}^{\delta}(t)\right) /(-\log \delta)>0$ if $P(t)>1$, and $\left(\log L_{\omega}^{\delta}(t)\right) /(-\log \delta)<0$ if $P(t)<1$, for small enough $\delta>0$. Therefore the implications in the statement hold.

Lemma 4.6.12. Let $(\mathbb{L}, \vec{\pi})$ be a box-like self-affine RIFS that satisfies the uniform open rectangle condition. Let $F_{\omega}$ be the associated 1-variable random set. Then

$$
\operatorname{dim}_{B} F_{\omega} \geq s_{B} \quad \text { (a.s.). }
$$

Proof. Let $\delta>0$ and consider the $\delta$-mesh on $\Delta$, denoted by $\mathcal{N}_{\delta}$. Since we assume the uniform open rectangle condition, the open rectangles $\mathcal{C}=\left\{f_{e}(\grave{\Delta})\right\}_{e \in \Xi_{\omega}^{\delta}}$ are pairwise disjoint. Furthermore the side lengths of the rectangles $R \in \mathcal{C}$ are bounded below by $\underline{\alpha} \delta$ by definition and thus the number of rectangles of $\mathcal{C}$ each square in the grid of $\mathcal{N}_{\delta}$ can intersect is at most $C^{-1}=\left(\underline{\alpha}^{-1}+2\right)^{2}$. Thus

$$
M_{\delta}\left(F_{\omega}\right) \geq C \sum_{e \in \Xi_{\omega}^{\delta}} N_{\delta}\left(F_{\omega}(e)\right) .
$$

In a similar fashion to the upper bound proof, Lemma 4.6.10, we find

$$
M_{\delta}\left(F_{\omega}\right) \geq C \sum_{e \in \Xi_{\omega}^{\delta}} N_{\delta / \alpha_{M}(e)}\left(F_{\sigma|e| \omega}\right) \geq C C_{\omega}^{-1} \sum_{e \in \Xi_{\omega}^{\varepsilon}}\left(\frac{\delta}{\alpha_{M}(e)}\right)^{-\underline{s}\left(e, F_{\omega}\right)+\varepsilon / 2}
$$

Using (4.6.10),

$$
\begin{aligned}
M_{\delta}\left(F_{\omega}\right) & \geq C C_{\omega}^{-1} \sum_{e \in \Xi_{\omega}^{\varepsilon}}\left(\frac{\underline{\alpha}^{-1} \alpha_{m}(e)}{\alpha_{M}(e)}\right)^{-\underline{s}\left(e, F_{\omega}\right)+\varepsilon / 2} \\
& \geq C C_{\omega}^{-1}\left(\frac{\alpha_{m}(e)}{\delta}\right)^{s_{B}-\varepsilon} \sum_{e \in \Xi_{\omega}^{\varepsilon}}\left(\frac{\underline{\alpha}^{-1} \alpha_{m}(e)}{\alpha_{M}(e)}\right)^{-\underline{s}\left(e, F_{\omega}\right)+\varepsilon / 2}
\end{aligned}
$$

Thus, for some $C_{\omega}^{*}>0$,

$$
\begin{aligned}
M_{\delta}\left(F_{\omega}\right) & \geq C C_{\omega}^{-1} \delta^{-\left(s_{B}-\varepsilon\right)} \underline{\alpha}^{\max _{z \in\{x, y\}} \underline{s}^{z}\left(F_{\omega}\right)-\varepsilon / 2} \sum_{e \in \Xi_{\omega}^{\delta}} \alpha_{M}(e)^{\underline{s}\left(e, F_{\omega}\right)} \alpha_{m}(e)^{s_{B}-\varepsilon / 2-\underline{s}\left(e, F_{\omega}\right)} \\
& \geq C_{\omega}^{*} C_{\omega}^{-1} \delta^{-\left(s_{B}-\varepsilon / 2\right)} \sum_{e \in \Xi_{\omega}^{\delta}} \underline{\psi}_{\omega}^{s_{B}-\varepsilon / 2}=C^{*} C_{\omega}^{-1} \delta^{-\left(s_{B}-\varepsilon / 2\right)} L_{\omega}^{\delta}\left(s_{B}-\varepsilon / 2\right) .
\end{aligned}
$$

This in turn gives

$$
\frac{\log M_{\delta}\left(F_{\omega}\right)}{-\log \delta} \geq \frac{\log C^{*} C_{\omega}}{-\log \delta}+\left(s_{B}-\varepsilon / 2\right)+\frac{\log L_{\omega}^{\delta}\left(s_{B}-\varepsilon / 2\right)}{-\log \delta}
$$

The lower bound follows almost surely because the first term becomes arbitrarily small, and by Lemma 4.6.11, the last term is positive for small enough $\delta$.

### 4.6.2 Proofs for Section 4.4

In this Section we prove the remaining results concerning $\infty$-variable box-like selfaffine carpets. We define a random variable $Y_{k}^{t}$ and show that it behaves similarly to a martingale. For $t>s_{B}$ we have $Y_{k}^{t} \rightarrow 0$ a.s. and $t$ is an almost sure lower bound for the box-counting dimension of $F_{\tau}$. We define a new random variable $Z_{k}^{t}$ that, for $t<s_{B}$, increases exponentially a.s.. This will then allow us to establish the lower bound. We end by showing that additivity also implies that $Y_{k}^{s_{B}}$ is an $\mathscr{L}^{2}$-bounded martingale.

Lemma 4.6.13. Let $Y_{k}^{t}$ be the random variable given by

$$
Y_{k}^{t}(\tau)=\sum_{\xi \in \mathbf{T}_{\tau}^{k}} \bar{\psi}_{\tau}^{t}(\xi) . \quad(\tau \in \mathcal{T})
$$

For all $t \in \mathbb{R}_{0}^{+}$and $l, k \in \mathbb{N}$ and some random variable $c_{k}$ that will be defined in (4.6.22), the sequence of random variables satisfies

$$
\begin{equation*}
\mathbb{E}\left(Y_{k+l}^{t}(\tau) \mid \mathcal{F}^{k}\right)=c_{k}(\tau) Y_{k}^{t}(\tau) \mathbb{E}\left(Y_{l}^{t}\right) \tag{4.6.19}
\end{equation*}
$$

Here $\mathcal{F}^{k}$ refers to the filtration corresponding to the 'knowledge of outcomes up to the $k$-th level'. Furthermore $0<c_{k} \leq 1$ for $t \in\left[0, s^{x}+s^{y}\right]$ and so the sequence $\left\{Y_{q l}^{s}\right\}_{q=1}^{\infty}$ forms a supermartingale if $\mathbb{E}\left(Y_{l}^{t}\right) \leq 1$.

Proof. Let $e \in \mathbf{T}_{\tau}^{k}$ and $g \in \mathbf{T}_{\sigma^{e} \tau}^{l}$, then there exists $C(e, g)$ such that

$$
\bar{\psi}_{\tau}^{t}(e \odot g)=C(e, g) \bar{\psi}_{\tau}^{t}(e) \bar{\psi}_{\sigma^{e} \tau}^{t}(g)
$$

We define the dual of the modified singular value function

$$
\bar{\psi}_{\tau}^{* t}(e)=\alpha_{m}(e)^{\bar{s}\left(\neg e, F_{\tau}\right)} \alpha_{M}(e)^{t-\bar{s}\left(\neg e, F_{\tau}\right)}
$$

where $s\left(\neg e, F_{\tau}\right)$ is the box-counting dimension of the projection of $F_{\tau}$ onto the shorter side of $f(e, \Delta)$. We find that $C(e, g)$ takes only a few possible values, depending on $e$ and $g$.

1. $\alpha_{M}(e \odot g)=\alpha_{M}(e) \alpha_{M}(g)$ and
a) the RIFS is of non-separated type.
b) the RIFS is of separated type and $\Pi_{e}=\Pi_{g}$.
2. $\alpha_{M}(e \odot g)=\alpha_{M}(e) \alpha_{m}(g)$.
3. $\alpha_{M}(e \odot g)=\alpha_{m}(e) \alpha_{M}(g)$.

Case (1a). Since the RIFS is of non-separated type, for almost all $\tau \in \mathcal{T}$, we have $s^{x}=s^{y}$. Furthermore, almost surely,

$$
\begin{aligned}
\bar{\psi}_{\tau}^{t}(e \odot g) & =\alpha_{M}(e \odot g)^{s^{x}} \alpha_{m}(e \odot g)^{t-s^{x}} \\
& =\left(\alpha_{M}(e) \alpha_{M}(g)\right)^{s^{x}}\left(\alpha_{m}(e) \alpha_{m}(g)\right)^{t-s^{x}} \\
& =\bar{\psi}_{\tau}^{t}(e) \bar{\psi}_{\tau}^{t}(g)
\end{aligned}
$$

and so $C(e, g)=1$ a.s..
Case (1b). As the RIFS is of non-separated type and the directions of the maximal modified singular value coincide we can apply (4.6.7) and get $C(e, g)=1$, for all $\tau \in \mathcal{T}$.

Case (2). We can write, for almost every $\tau \in \mathcal{T}$,

$$
\begin{aligned}
\bar{\psi}_{\tau}^{t}(e \odot g) & =\alpha_{M}(e \odot g)^{s(e \odot g)} \alpha_{m}(e \odot g)^{t-s(e \odot g)} \\
& =\left(\alpha_{M}(e) \alpha_{m}(g)\right)^{s(e)}\left(\alpha_{m}(e) \alpha_{M}(g)\right)^{t-s(e)} \\
& =\bar{\psi}_{\tau}^{t}(e) \alpha_{m}(g)^{s(\neg g)} \alpha_{M}(g)^{t-s(\neg g)} \\
& =\bar{\psi}_{\tau}^{t}(e) \bar{\psi}_{\sigma^{e} \tau}^{* t}(g),
\end{aligned}
$$

and so $C(e, g)=\bar{\psi}_{\sigma^{e} \tau}^{t}(g) / \bar{\psi}_{\sigma^{e} \tau}^{* t}(g)=\left(\alpha_{M}(g) / \alpha_{m}(g)\right)^{t-s^{x}-s^{y}}$.
Case (3). Similarly we can write, for almost every $\tau \in \mathcal{T}$,

$$
\begin{aligned}
\bar{\psi}_{\tau}^{t}(e \odot g) & =\alpha_{M}(e \odot g)^{s(e \odot g)} \alpha_{m}(e \odot g)^{t-s(e \odot g)} \\
& =\left(\alpha_{m}(e) \alpha_{M}(g)\right)^{s(g)}\left(\alpha_{M}(e) \alpha_{m}(g)\right)^{t-s(g)} \\
& =\bar{\psi}_{\tau}^{t}(g) \alpha_{m}(e)^{s(\neg e)} \alpha_{M}(e)^{t-s(\neg e)} \\
& =\bar{\psi}_{\tau}^{* t}(e) \bar{\psi}_{\sigma^{e} \tau}^{t}(g),
\end{aligned}
$$

and so $C(e, g)=\bar{\psi}_{\tau}^{t}(e) / \bar{\psi}_{\tau}^{* t}(e)=\left(\alpha_{M}(e) / \alpha_{m}(e)\right)^{t-s^{x}-s^{y}}$.
Therefore

$$
\begin{align*}
\mathbb{E}\left(Y_{k+l}^{t}(\tau) \mid \mathcal{F}^{k}\right) & =\mathbb{E}\left(\sum_{e \in \mathbf{T}_{\tau}^{k}} \sum_{g \in \mathbf{T}_{\sigma e_{\tau}}^{l}} \bar{\psi}_{\tau}^{t}(e \odot g) \mid \mathcal{F}^{k}\right) \\
& =\mathbb{E}\left(\sum_{e \in \mathbf{T}_{\tau}^{k}} \sum_{g \in \mathbf{T}_{\tau}^{l}} C(e, g) \bar{\psi}_{\tau}^{t}(e) \bar{\psi}_{\tau}^{t}(g) \mid \mathcal{F}^{k}\right)  \tag{4.6.20}\\
& =c_{k}(\tau) Y_{k}^{t}(\tau) \mathbb{E}\left(Y_{l}^{t}\right), \tag{4.6.21}
\end{align*}
$$

where

$$
\begin{equation*}
c_{k}(\tau)=\frac{\mathbb{E}\left(\sum_{e \in \mathbf{T}_{\tau}^{k}} \sum_{g \in \mathbf{T}_{\tau}^{t}} C(e, g) \bar{\psi}_{\tau}^{t}(e) \bar{\psi}_{\tau}^{t}(g) \mid \mathcal{F}^{k}\right)}{Y_{k}^{t}(\tau) \mathbb{E}\left(Y_{l}^{t}\right)} . \tag{4.6.22}
\end{equation*}
$$

We will analyse $c_{k}(\tau)$ in Lemma 4.6.15 and here only comment that by inspection we deduce $C(e, g) \leq 1$ for $s \in\left[0, s^{x}+s^{y}\right]$ and so (4.6.21) becomes $\mathbb{E}\left(Y_{k+l}^{s}(\tau) \mid \mathcal{F}^{k}\right) \leq$ $Y_{k}^{s}(\tau) \mathbb{E}\left(Y_{l}^{s}\right)$ and the sequence of random variables $\left\{Y_{q l}^{s}\right\}_{q=1}^{\infty}$ forms a supermartingale if $\mathbb{E}\left(Y_{l}^{s}\right) \leq 1$.

Lemma 4.6.14. Let $s \in\left[0, s^{x}+s^{y}\right]$ and $s>s_{B}$. Then the sequence $\left\{Y_{k}^{s}\right\}$ converges to 0 exponentially fast a.s. and $s_{B}$ is an almost sure upper bound for the box-counting dimension of $F_{\tau}$, so

$$
\operatorname{dim}_{B} F_{\tau} \leq s_{B}
$$

Proof. If $s>s_{B}$ there exists $l$ such that $\mathbb{E}\left(Y_{l}^{s}\right)<1$ by definition, see (4.4.1). Therefore, by Lemma 4.6.13, $\left\{Y_{q l}^{s}\right\}_{q=1}^{\infty}$ is a strict supermartingale. Hence $\mathbb{E}\left(Y_{q l}^{s}\right) \rightarrow 0$ as $q \rightarrow \infty$ and since $Y_{k}^{s} \asymp Y_{\lfloor k / q\rfloor q}^{s}$, we get $Y_{k}^{s} \rightarrow 0$ as $k \rightarrow \infty$, almost surely. This happens at an exponential rate that is there exists $\gamma<1$ and $D_{\tau}>0$ such that $Y_{k} \leq D_{\tau} \gamma^{k}$.

We now define a stopping set $\Xi_{\tau}^{\delta}$ analogously to before

$$
\Xi_{\tau}^{\delta}=\left\{e \in \mathbf{T}_{\tau}^{i} \mid i \in \mathbb{N}, \alpha_{m}(e) \leq \delta \text { and } \alpha_{m}\left(e_{1} e_{2} \ldots e_{|e|-1}\right)>\delta\right\}
$$

and we can modify the argument in Lemma 4.6.10 accordingly to get, for $s=s_{B}+\delta$,

$$
N_{\varepsilon}\left(F_{\tau}\right) \leq C_{\tau} C^{*} \varepsilon^{-\left(s_{B}+\delta\right)} \sum_{k=1}^{\infty} \sum_{e \in \mathbf{T}_{\tau}^{k}} \bar{\alpha}^{k \delta / 2} \bar{\psi}_{\tau}^{s_{B}}(e) \leq C_{\tau} C^{*} \varepsilon^{-\left(s_{B}+\delta\right)} D_{\tau} \sum_{k=1}^{\infty} \bar{\alpha}^{k \delta / 2} \gamma^{k}<\infty
$$

almost surely. We conclude that $s_{B}$ is an almost sure upper bound to the box-counting dimension of $F_{\tau}$.

Lemma 4.6.15. For $c_{k}(\tau)$ as in (4.6.19) we find $s \in\left[0, s^{x}+s^{y}\right]$ implies $c_{k} \nearrow 1$ as $k \rightarrow \infty$ for all $\tau$. If however $s>s^{x}+s^{y}$ we get $c_{k} \searrow 1$ as $k \rightarrow \infty$.

Proof. We first decompose (4.6.20) into

$$
\begin{aligned}
& \mathbb{E}\left(Y_{k+l}^{t}(\tau) \mid \mathcal{F}^{k}\right)=\mathbb{E}\left(\sum_{\substack{e \in \mathbf{T}_{\tau}^{k}, \underline{k} \in \mathbf{T}_{\sigma}{ }_{\sigma}^{l} \tau_{\tau} \\
\alpha_{M}(e \odot g)=\alpha_{M}(e) \alpha_{M}(g)}} \psi_{\tau}^{t}(e) \bar{\psi}_{\sigma^{e} \tau}^{t}(g)\right. \\
& \left.+\sum_{\substack{e \in \mathbf{T}_{\tau}^{k}, g \in \mathbf{T}^{l}{ }^{l} e_{\tau} \\
\alpha_{M}(e \odot g)=\alpha_{M}(e) \alpha_{m}(g)}} \bar{\psi}_{\substack{t}}(e) \bar{\psi}_{\sigma^{e} \tau}^{* t}(g)+\sum_{\substack{e \in \mathbf{T}_{\tau}^{k}, g \in \mathbf{T}_{\sigma}^{l} \\
\alpha_{M}(e \odot g)=\alpha_{m}(e) \alpha_{M} \\
\alpha_{M}(g)}} \bar{\psi}_{\tau}^{* t}(e) \bar{\psi}_{\sigma^{e} \tau}^{t}(g)\right) .
\end{aligned}
$$

Without loss of generality we can assume that the RIFS is strictly self-affine, that is there exists at least one $f_{i}^{j}$ such that $\alpha_{M}\left(e_{i}^{j}\right)>\alpha_{m}\left(e_{i}^{j}\right)$, since otherwise we trivially have $C(e, g)=1$ and so $c_{k}(\tau)=1$. We recall that $l$ is fixed and thus there exists a maximal ratio $\max _{\kappa \in \mathcal{T}} \max _{g \in \mathbf{T}_{\kappa}^{l}} \alpha_{M}(g) / \alpha_{m}(g)$. Now consider a word $e \in \mathbf{T}_{\tau}^{k}$ for large $k$ and consider the case of $\alpha_{M}(e \odot g)=\alpha_{m}(e) \alpha_{M}(g)$. We must, by the bounded length of $g$, have $\alpha_{M}(e) / \alpha_{m}(e) \rightarrow 1$ as $k=|e| \rightarrow \infty$, but then $C(e, g) \rightarrow 1$ as $k \rightarrow \infty$.

Finally consider $\alpha_{M}(e \odot g)=\alpha_{M}(e) \alpha_{m}(g)$. Since $k$ is large, $\alpha_{M}(e)$ is substantially smaller than $\alpha_{M}(g)$ and $\alpha_{m}(g)$ and as $\bar{\psi}$ behaves exponentially to changes in $\alpha_{M}$ the boundedness of $|g|$ gives that $\bar{\psi}_{\tau}^{t}(e) \bar{\psi}_{\tau}^{t}(g)$ behaves as $\bar{\psi}_{\tau}^{t}(e)$ for large $k$ and hence $\bar{\psi}_{\tau}^{t}(e) \bar{\psi}_{\tau}^{* t}(g) \sim \bar{\psi}_{\tau}^{t}(e) \sim \bar{\psi}_{\tau}^{t}(e) \bar{\psi}_{\tau}^{t}(g)$ and thus $c_{k}(\tau) \rightarrow 1$, irrespective of $\tau$.

We remark that the result in Lemma 4.6 .15 shows that the sequence $Y_{k} / \mathbb{E}\left(Y_{1}\right)^{k}$ looks like a martingale 'in the limit'. In the additive case, $c_{k}=1$ (surely) we show in the proof of Theorem 4.4.8 that $Y_{k} / \mathbb{E}\left(Y_{1}\right)^{k}$ is a $\mathcal{L}^{2}$-bounded martingale, but to work in greater generality we will not employ this fact here and prove the general lower bound by a branching argument.

Lemma 4.6.16. For $s<s_{B}$, the sequence of random variables $\left\{Y_{k}^{s}\right\}$ diverges to $+\infty$ almost surely and hence the box-counting dimension of $F_{\tau}$ is bounded below by s. We conclude

$$
\operatorname{dim}_{B} F_{\tau} \geq s_{B}
$$

almost surely.

Proof. Using the definition for the $\infty$-variable stopping set $\Xi_{\tau}^{\boldsymbol{\delta}}$ and using an argument identical to Lemma 4.6.12 we can write for $s=s_{B}-\varepsilon$ and some $C^{*}>0$ and some almost surely positive $C_{\tau}$,

$$
\begin{equation*}
M_{\delta}\left(F_{\tau}\right) \geq C^{*} C_{\tau} \delta^{-\left(s_{B}-\varepsilon / 2\right)} \sum_{e \in \Xi_{\tau}^{\delta}} \bar{\psi}_{\tau}^{s_{B}-\varepsilon / 2}(e) \tag{4.6.23}
\end{equation*}
$$

But one can easily see that for any $\xi>0$, the random variable

$$
Z_{n}^{s}=\sum_{e \in \Xi_{\tau}^{\xi^{n}}} \bar{\psi}_{\tau}^{s_{B}-\varepsilon / 2}(e)
$$

is also an approximate supermartingale, c.f. (4.6.19), for some analogous constant $c_{k}$ with properties as in Lemma 4.6.15,

$$
\begin{equation*}
\mathbb{E}\left(Z_{k+l}^{s} \mid \mathcal{F}^{k}\right)=c_{k} Z_{k}^{s} \mathbb{E}\left(Z_{l}^{s}\right) \tag{4.6.24}
\end{equation*}
$$

Now, for $s<s_{B}$ let $k$ be large enough such that

$$
\mathbb{E}\left(\sum_{e \in \mathbf{T}_{\tau}^{k}} \bar{\psi}_{\tau}^{s}(e)\right)>1
$$

choose $\xi>0$ such that $\xi<\bar{\alpha}^{k}$. Then

$$
\mathbb{E}\left(Z_{l}^{s}\right) \geq \mathbb{E}\left(\sum_{e \in \mathbf{T}_{\tau}^{k}} \bar{\psi}_{\tau}^{s}(e)\right)>1
$$

for all $l$. Now choose $l_{\text {sup }}$ large enough such that $\mathbb{E}\left(c_{l_{\text {sup }}}\right) \mathbb{E}\left(Z_{l_{\text {sup }}}^{s}\right)>1$. Using (4.6.24), we conclude that $\mathbb{E}\left(Z_{q l_{\text {sup }}}^{s}\right)$ increases exponentially as $q$ grows. Similarly, since

$$
Z_{l_{\text {sup }}\left\lfloor k / l_{\text {sup }}\right\rfloor}^{s} \asymp Z_{k}^{s}
$$

there exists some $\beta_{1}, \beta_{2}>1$ and a constant $D$ such that $D^{-1} \beta_{1}^{k} \leq \mathbb{E}\left(Z_{k}^{s}\right) \leq D \beta_{2}^{k}$.
Consider the stopping trees $\Xi_{\tau}^{\xi^{n}}$. Since we are conditioning on non-extinction, a simple Borel-Cantelli argument shows that, almost surely, in every surviving branch there are infinitely many nodes where the branch splits into two or more subbranches. For definiteness let $N(\tau)$ be the least integer such that $\#\left\{\lambda_{i} \in \Xi_{\tau}^{\xi^{N}}\right\}>1$. We know that $\widehat{\tau}_{i}=\sigma^{\lambda_{i}} \tau$ are independent and identical in distribution and hence, for $\zeta_{1}, \zeta_{2}>1$,

$$
\begin{align*}
& \mathbb{P}\left\{\tau \in \mathcal{T} \mid \exists C>0 \text { s.t. for all } n, C^{-1} \zeta_{1}^{n} \leq Z_{n}^{s}(\tau) \leq C \zeta_{2}^{n}\right\} \\
& =1-\prod_{j=1}^{\#\left\{\lambda_{i} \in \Xi_{\tau}^{\xi^{N}}\right\}}\left(1-\mathbb{P}\left\{\tau \in \mathcal{T} \mid \exists C>0 \text { s.t. for all } n, C^{-1} \zeta_{1}^{n} \leq Z_{n}^{s}\left(\widehat{\tau}_{j}\right) \leq C \zeta_{2}^{n}\right\}\right) \tag{4.6.25}
\end{align*}
$$

$=1-\left(1-\mathbb{P}\left\{\tau \in \mathcal{T} \mid \exists C>0 \text { s.t. for all } n, C^{-1} \zeta_{1}^{n} \leq Z_{n}^{s}(\tau) \leq C \zeta_{2}^{n}\right\}\right)^{\#\left\{\lambda_{i} \in \Xi_{\tau}^{\xi^{N}}\right\}}$.

$$
\begin{equation*}
\geq 1-\left(1-\mathbb{P}\left\{\tau \in \mathcal{T} \mid \exists C>0 \text { s.t. for all } n, C^{-1} \zeta_{1}^{n} \leq Z_{n}^{s}(\tau) \leq C \zeta_{2}^{n}\right\}\right)^{2} \tag{4.6.26}
\end{equation*}
$$

Note that the measure $\mathbb{P}$ in (4.6.26) and (4.6.26) is conditioned on $\# \lambda_{i}$. However, since $\#\left\{\lambda_{i} \in \Xi_{\tau}^{\xi^{N}}\right\} \geq 2$ for all such nodes, (4.6.27) holds unconditionally. For $x \in[0,1]$, the only solutions to $x \geq 1-(1-x)^{2}$ are 0 and 1 , and we conclude that the probability that $Z_{n}^{s}$ (eventually) increases at least at exponential rate $\zeta_{1}>1$ and at most at rate $\zeta_{2}>1$ is 0 or 1 . Letting $\zeta_{1}<\beta_{1}$ and $\zeta_{2}>\beta_{2}$, it is easy to see that the probability must be 1 by noticing that $\underline{\alpha} Z_{n}^{s} \leq Z_{n+1}^{s} \leq \bar{\alpha} Z_{n}^{s}$ and

$$
\mathbb{P}\left\{\tau \mid Z_{n}^{s}(\tau) \geq\left(\zeta_{2}+\varepsilon\right)^{n} \text { or } Z_{n}^{s}(\tau) \leq\left(\zeta_{1}-\varepsilon\right)^{n}\right\} \rightarrow 0 \text { as } n \rightarrow \infty
$$

for sufficiently small $\varepsilon>0$. This, and the arbitrariness of $\zeta_{1}$ imply that there exists $\gamma>1$ and a random constant $D_{\tau}$ such that $Z_{n}^{s} \geq D_{\tau} \gamma^{n}$ almost surely.

We can now bound the expression in (4.6.23) for $\delta<\xi$, where $k$ is such that $\xi^{k+1}<\delta \leq \xi^{k}$, and redefining $C^{*}$ if necessary, by

$$
\begin{aligned}
M_{\delta}\left(F_{\tau}\right) & \geq C^{*} C_{\tau} \delta^{-\left(s_{B}-\varepsilon / 2\right)} \sum_{e \in \Xi_{\tau}^{\xi^{k}}} \bar{\psi}_{\tau}^{s_{B}-\varepsilon / 2}(e) \\
& \asymp C^{*} C_{\tau} \delta^{-\left(s_{B}-\varepsilon / 2\right)} Z_{k}^{s_{B}-\varepsilon / 2} \\
& \geq C^{*} C_{\tau} D_{\tau} \delta^{-\left(s_{B}-\varepsilon / 2\right)} \gamma^{k} .
\end{aligned}
$$

And so

$$
\begin{aligned}
\liminf _{\delta \rightarrow 0} \frac{\log M_{\delta}\left(F_{\tau}\right)}{-\log \delta} & \geq \liminf _{\delta \rightarrow 0} \frac{\log \left(\delta^{-\left(s_{B}-\varepsilon / 2\right)} \gamma^{k}\right)}{-\log \delta} \\
& =s_{B}-\varepsilon / 2+\liminf _{\delta \rightarrow 0} k \frac{\log \gamma}{-\log \delta} \\
& =s_{B}-\varepsilon / 2-\frac{\log \gamma}{\log \xi} \geq s_{B}-\varepsilon / 2
\end{aligned}
$$

Using the same idea one can extend Lemma 4.6.14 and drop the condition $s \leq$ $s^{x}+s^{y}$ by picking $l_{\text {sup }}$ large enough such that $c_{l_{\text {sup }}} \mathbb{E}\left(Y_{l_{\text {sup }}}^{s}\right)<1$, we omit details. Combining Lemmas 4.6 .14 and 4.6.16 we conclude that Theorem 4.4.5 holds.

Finally we prove Theorem 4.4 .8 by proving this general result.
Lemma 4.6.17. Let $s_{B}$ be the unique value such that (4.4.2) is satisfied. The sequence $\left\{Y_{k}^{s_{B}}\right\}$ forms an $\mathscr{L}^{2}$ bounded martingale and hence converges to an $L^{1}$ random variable $Y$ almost surely. We have $0 \leq Y<\infty$ for almost every $\tau \in \mathcal{T}$ and $Y>0$ with positive probability.

Proof. As remarked earlier we have already established that $\left\{Y_{k}^{s_{B}}\right\}$ is a martingale. It remains to prove $\mathscr{L}^{2}$ boundedness. Let $s \in\left[0, s^{x}+s^{y}\right]$, then

$$
\begin{align*}
& \mathbb{E}\left(\left(Y_{k+1}^{s}(\tau)\right)^{2} \mid \mathcal{F}^{k}\right)=\mathbb{E}\left(\left(\sum_{e \in \mathbf{T}_{\tau}^{k}} \sum_{g \in \mathbf{T}_{\sigma^{e} \tau}} C(e, g) \psi_{\tau}^{s}(e) \psi_{\sigma^{e} \tau}^{s}(g)\right)^{2} \mid \mathcal{F}^{k}\right) \\
& =\mathbb{E}\left(\left(\sum_{e \in \mathbf{T}_{\tau}^{k}}\left(\psi_{\tau}^{s}(e) \sum_{g \in \mathbf{T}_{\sigma^{e} \tau}^{1}} \psi_{\sigma^{e} \tau}^{s}(g)\right)\right)^{2} \mid \mathcal{F}^{k}\right) \\
& =\mathbb{E}\left(\sum _ { e _ { 1 } \neq e _ { 2 } \in \mathbf { T } _ { \tau } ^ { k } } \left(\psi_{\tau}^{s}\left(e_{1}\right) \psi_{\tau}^{s}\left(e_{2}\right)\left(\sum_{g \in \mathbf{T}_{\sigma^{e_{1}}(\tau)}^{1}} \psi_{\sigma^{e_{1}}(\tau)}^{s}(g)\right)\right.\right. \\
& \left.\cdot\left(\sum_{g \in \mathbf{T}_{\sigma^{e_{2}}(\tau)}^{1}} \psi_{\sigma^{e_{2}}(\tau)}^{s}(g)\right)\right) \\
& \left.+\sum_{e \in \mathbf{T}_{\tau}^{k}}\left(\psi_{\tau}^{s}(e)^{2}\left(\sum_{g \in \mathbf{T}_{\sigma^{e} \tau}^{1}} \psi_{\sigma^{e} \tau}^{s}(g)\right)^{2}\right) \mid \mathcal{F}^{k}\right) \\
& =\left(\mathbb{E}\left(\sum_{g \in \mathbf{T}_{\tau}^{1}} \psi_{\tau}^{s}(g)\right)\right)^{2} \sum_{e_{1} \neq e_{2} \in \mathbf{T}_{\tau}^{k}} \psi_{\tau}^{s}\left(e_{1}\right) \psi_{\tau}^{s}\left(e_{2}\right)+C \sum_{e \in \mathbf{T}_{\tau}^{k}} \psi_{\tau}^{s}(e)^{2}, \tag{4.6.28}
\end{align*}
$$

for $C=\mathbb{E}\left[\left(\sum_{e \in \mathbf{T}_{\tau}^{l}} \psi_{\tau}^{s}(e)\right)^{2}\right]$. Note that equality (4.6.28) holds for any $s \in\left[0, s^{x}+\right.$ $\left.s^{y}\right]$. Then for $s_{B}$ the first term is equal to 1 and

$$
\begin{align*}
\mathbb{E}\left(\left(Y_{k+1}^{s_{B}}(\tau)\right)^{2} \mid \mathcal{F}^{k}\right) & =\sum_{e_{1} \neq e_{2} \in \mathbf{T}_{\tau}^{k}} \psi_{\tau}^{s_{B}}\left(e_{1}\right) \psi_{\tau}^{s_{B}}\left(e_{2}\right)+C \sum_{e \in \mathbf{T}_{\tau}^{k}} \psi_{\tau}^{s_{B}}(e)^{2} \\
& \leq\left(\sum_{e \in \mathbf{T}_{\tau}^{k}} \psi_{\tau}^{s_{B}}(e)\right)^{2}+C \sum_{e \in \mathbf{T}_{\tau}^{k}} \psi_{\tau}^{s_{B}}(e)^{2} \\
& =Y_{k}(\tau)^{2}+C \sum_{e \in \mathbf{T}_{\tau}^{k}} \psi_{\tau}^{s_{B}}(e)^{2} \tag{4.6.29}
\end{align*}
$$

Furthermore the unconditional expectation becomes

$$
\begin{equation*}
\mathbb{E}\left(Y_{k+1}^{s_{B}}(\tau)^{2}\right) \leq \mathbb{E}\left[Y_{k}(\tau)^{2}\right]+C \mathbb{E}\left(\sum_{e \in \mathbf{T}_{\tau}^{k}} \psi_{\tau}^{s_{B}}(e)^{2}\right) \tag{4.6.30}
\end{equation*}
$$

Note that $\psi_{\tau}^{t}(e \odot g)^{2}=\left(\alpha_{M}(e \odot g) / \alpha_{m}(e \odot g)\right)^{s(e)} \psi_{\tau}^{2 t}(e \odot g)$ and so, writing $\beta=$ $\mathbb{E}\left[\sum_{e \in \mathbf{T}_{\tau}^{1}} \psi_{\tau}^{2 s_{B}}(e)\right]$, we obtain

$$
\mathbb{E}\left(Y_{k+1}^{s_{B}}(\tau)^{2}\right) \leq \mathbb{E}\left(Y_{k}^{s_{B}}(\tau)^{2}\right)+C \beta
$$

and so by induction

$$
\mathbb{E}\left(Y_{k+1}^{s_{B}}(\tau)^{2}\right) \leq \mathbb{E}\left(Y_{1}^{s_{B}}(\tau)^{2}\right)+C\left(\beta^{k}+\beta^{k-1}+\cdots+1\right)<\infty,
$$

and therefore $\left\{Y_{k}^{s_{B}}\right\}$ is an $\mathscr{L}^{2}$ bounded martingale.

## Hausdorff and packing measure for random attractors

In this chapter we summarise recent and classical work regarding the Hausdorff and packing measure of $\infty$-variable, random recursive sets and analyse the 1 -variable, random homogeneous, case. We prove some bounds on the gauge functions that give positive and finite Hausdorff and packing measure in this setting. We relate this to the implicit theorems by showing that any potential random implicit theorem cannot be 'as strong' as their deterministic counterpart.

### 5.1 Almost deterministic attractors

Let $\mathbb{L}=\left\{\mathbb{I}_{i}\right\}_{i \in \Lambda}$ be a (not necessarily finite) collection of IFSs with at most $N$ similarities, i.e. $\left|f_{i}^{j}(x)-f_{i}^{j}(y)\right|=c_{i}^{j}|x-y|$. Let $\Lambda$ be a compact metric space such that $\mu$ is a compactly supported Borel probability measure on $\Lambda$. We now construct 1 -variable and $\infty$-variable sets as before by picking realisations $\omega \in \Omega$ and $\tau \in \mathcal{T}$ according to the natural measure induced by $\mu$. Thus $\mathbb{P}=\mu^{\mathbb{N}}$ in the 1 -variable case and it is the natural measure on the random recursive labelled tree for $\mathcal{T}$ in the random recursive setting, see Section 4.4. We write ( $\mathbb{L}, \mu$ ) for a random iterated function system. Since random graph directed systems are inherently discrete we did not extend our proofs in Chapter 3 to measures supported on more than finitely many points, but using the UOSC and natural assumptions on contractions it is easy to show that the Hausdorff dimension of 1 -variable attractors is almost surely given by the unique $s$ satisfying

$$
\mathbb{E}^{\mathrm{geo}}\left(\sum_{j=1}^{\left.\# \mathbb{\mathbb { I } _ { \omega _ { 1 } }}\left(c_{\omega_{1}}^{j}\right)^{s}\right)=1 . . . . . . .}\right.
$$

For $\infty$-variable sets the almost sure Hausdorff dimension is the unique $s$ satisfying

$$
\mathbb{E}\left(\sum_{j=1}^{\# \mathbb{I}_{\omega_{1}}}\left(c_{\omega_{1}}^{j}\right)^{s}\right)=1
$$

To ease notation we write $\mathfrak{S}_{\lambda}^{s}=\sum_{j=1}^{\# \mathbb{I}}\left(c_{\lambda}^{j}\right)^{s}$ for $\lambda \in \Lambda$. We state a condition that will be used later.

Condition 5.1.1. Let $(\mathbb{L}, \mu)$ be a random iterated function system. We assume that there exists $N$ such that $\# \mathbb{I}_{\lambda} \leq N$ for all $\lambda \in \Lambda$ and $c_{\min }>0$ such that $c_{i}^{j} \geq c_{\min }$ for all $i \in \Lambda$ and $j \in\left\{1, \ldots, \mathbb{I}_{i}\right\}$. For the associated $\infty$-variable set $F_{\tau}$ and $\tau \in \mathcal{T}$ we further assume $\mathbb{E}\left(\mathfrak{S}_{\tau_{1}}^{0}\right)>1$. For the associated 1-variable set $F_{\omega}$ with $\omega \in \Omega$ we similarly stipulate $\mathbb{E}^{\text {geo }}\left(\mathfrak{S}_{\omega_{1}}^{0}\right)>1$.

We note that Condition 5.1.1 implies that $c_{\min }^{s} \leq \mathfrak{S}_{\lambda}^{s}<N$ and $s \log c_{\min } \leq$ $\log \mathfrak{S}_{\lambda}^{s}<\log N$ for all $s \geq 0$. We immediately obtain that $\mathbb{E}\left(\mathfrak{S}_{\tau_{1}}^{s}\right)<\infty$ and
$\mathbb{E}^{\text {geo }}\left(\mathfrak{S}_{\omega_{1}}^{s}\right)<\infty$ for all $s \geq 0$. Under these conditions we have $\operatorname{Var}\left(\log \mathfrak{S}_{\omega_{1}}^{s}\right)<\infty$ for all $s \geq 0$.

Definition 5.1.2. A random iterated function system $(\mathbb{L}, \mu)$ is called almost deterministic if there exists $s$ such that $\mathfrak{S}_{\lambda}^{s}=1$ for $\mu$-almost every $\lambda \in \Lambda$.

If such $s$ exists it must necessarily be the almost sure Hausdorff dimension $s=$ ess $\operatorname{dim}_{H}\left(F_{\tau}\right)$.

Theorem 5.1.3 (Graf $[\mathrm{G}])$. Let $(\mathbb{L}, \mu)$ be a random iterated function system satisfying the UOSC (Definition 2.2.2) and Condition 5.1 .1 with associated $\infty$-variable set $F_{\tau}$ and write $s_{0}=\operatorname{ess} \operatorname{dim}_{H} F_{\tau}$. If $(\mathbb{L}, \mu)$ is almost deterministic then

$$
0<\mathscr{H}^{s_{0}}\left(F_{\tau}\right)<\infty \quad \text { (a.s.) }
$$

and $\mathscr{H}^{s_{0}}\left(F_{\tau}\right)=0$ (a.s.) otherwise.
For 1 -variable attractors we similarly obtain the following result ${ }^{1}$.
Theorem 5.1.4. Let $(\mathbb{L}, \mu)$ be a random iterated function system satisfying the UOSC and Condition 5.1 .1 with associated 1-variable set $F_{\omega}$ with almost sure Hausdorff dimension $s_{0}=\operatorname{ess} \operatorname{dim}_{H} F_{\omega}$. If $(\mathbb{L}, \mu)$ is almost deterministic then

$$
0<\mathscr{H}^{s_{0}}\left(F_{\omega}\right)<\infty \quad \text { (a.s.) }
$$

and $\mathscr{H}^{s_{0}}\left(F_{\omega}\right)=0$ (a.s.) otherwise.
Before proving this we prove the following lemma which shows that the number of sets $\overline{f(\phi, \mathcal{O})}$ with diameter approximately $r$ that intersect a closed ball $B(z, r)$ centred at $z \in F_{\omega}$ of radius $r$ is bounded by a constant not depending on $r$ and $z$. This is a simple generalisation of results in [Hu, O2] to the random setting and is included here for completeness.

Lemma 5.1.5. Assume that $(\mathbb{L}, \mu)$ satisfies the UOSC and set

$$
\Xi_{\omega}(r)=\bigsqcup\left\{\phi \in \bigsqcup_{k \in \mathbb{N}_{0}} \mathbf{C}_{\omega}^{k} \mid c_{\omega}^{\phi}<r \leq c_{\omega}^{\phi^{\dagger}}\right\}
$$

where $\phi^{\dagger}=\phi_{1} \phi_{2} \ldots \phi_{|\phi|-1}$. Then

$$
\#\left\{\phi \in \Xi_{\omega}(r) \mid \overline{f(\phi, \mathcal{O})} \cap B(z, r) \neq \varnothing\right\} \leq\left(4 / c_{\min }\right)^{d}
$$

for all $z \in F_{\omega}$ and $r \in(0,1]$, where $\mathcal{O}$ is the open set guaranteed by the UOSC.
Proof. Fix $z \in F_{\omega}$ and $r>0$. Let $\Xi=\left\{\phi \in \Xi_{\omega}(r) \mid \overline{f(\phi, \mathcal{O})} \cap B(z, r) \neq \varnothing\right\}$ and suppose the ambient space is $\mathbb{R}^{d}$. We have

$$
\# \Xi\left(r c_{\min }\right)^{d}=\sum_{\phi \in \Xi}\left(r c_{\min }\right)^{d} \leq \sum_{\phi \in \Xi}|\overline{f(\phi, \mathcal{O})}|^{d}
$$

But since $\overline{f(\phi, \mathcal{O})} \cap B(z, r) \neq \varnothing$ and $|\overline{f(\phi, \mathcal{O})}|<r$ we find $f(\phi, \mathcal{O}) \subseteq B(z, 2 r)$ for all $\phi \in \Xi$ and since the sets $f(\phi, \mathcal{O})$ are pairwise disjoint we have

$$
\# \Xi\left(r c_{\min }\right)^{d} \leq \sum_{\phi \in \Xi}|f(\phi, \mathcal{O})|^{d} \leq \mathcal{L}^{d}(B(z, 2 r)) \leq(4 r)^{d}
$$

where $\mathcal{L}^{d}$ is the $d$-dimensional Lebesgue measure. It follows that $\# \Xi \leq\left(4 / c_{\min }\right)^{d}$ as required.

[^4]Proof of Theorem 5.1.4. First assume that $(\mathbb{L}, \mu)$ is almost deterministic. Using the natural covering by images $\overline{f(\phi, \mathcal{O})}$, where $\phi \in \mathbf{C}_{\omega}^{k}$ and $\mathcal{O}$ is the open set given by the UOSC we have, almost surely,

$$
\begin{align*}
\mathscr{H}_{\delta}^{s_{0}}\left(F_{\omega}\right) & \leq \sum_{\phi \in \mathbf{C}_{\omega}^{k(\delta)}}|\overline{f(\phi, \mathcal{O})}|^{s}=|\mathcal{O}|^{s_{0}} \sum_{\phi \in \mathbf{C}_{\omega}^{k(\delta)}}\left(c_{\omega_{1}}^{\phi_{1}} c_{\omega_{2}}^{\phi_{2}} \ldots c_{\omega_{k(\delta)}}^{\phi_{k(\delta)}}\right)^{s_{0}}  \tag{5.1.1}\\
& =|\mathcal{O}|^{s_{0}} \mathfrak{S}_{\omega_{1}}^{s_{0}} \mathfrak{S}_{\omega_{2}}^{s_{0}} \ldots \mathfrak{S}_{\omega_{k(\delta)}}^{s_{0}}=|\mathcal{O}|^{s_{0}}<\infty \tag{5.1.2}
\end{align*}
$$

where $k(\delta)$ is such that $|\overline{f(\phi, \mathcal{O})}|<\delta$ for all $\phi \in \mathbf{C}_{\omega}^{k(\delta)}$. Note that $k(\delta)<\infty$ for all $\delta>0$ for almost every $\omega$. Our estimate immediately implies $\mathscr{H}^{s_{0}}\left(F_{\omega}\right) \leq 1$ almost surely.

Let $\nu_{\omega}$ be the natural (random) probability measure induced by assigning weight $\left|f\left(\phi,[0,1]^{d}\right)\right|^{s_{0}}$ to each cylinder $\phi \in \beth^{*}$. This is a measure if $\mathfrak{S}_{\sigma^{k} \omega}^{s_{0}}=1$ for all cylinders. Since there are countably many cylinders and the intersection of countably many full measure sets has full measure, $\nu_{\omega}$ is well defined for almost every $\omega$. Let $U \subset \mathbb{R}^{d}$ be a bounded $\nu_{\omega}$-measurable open set such that $|U| \leq 1$. We want to show that $\nu_{\omega}(U) \leq c_{\omega}|U|^{s_{0}}$ for some constant $c_{\omega} \geq 1$ that is almost surely finite. Assume $U \cap F_{\omega} \neq \varnothing$ as there is nothing to prove otherwise. Clearly, there exists $z \in F_{\omega}$ such that $U \cap F_{\omega} \subseteq B(z, u) \cap F_{\omega}$, where $u=2|U|$. Therefore

$$
\begin{aligned}
\nu_{\omega}(U) \leq & \nu_{\omega}\left(B(z, u) \cap F_{\omega}\right) \leq \nu_{\omega}\left(\bigcup_{\frac{\phi \in \Xi_{\omega}(u)}{f(\phi, \mathcal{O}) \cap B(z, u) \neq \varnothing}} f(\phi, \mathcal{O})\right) \\
& =\sum_{\substack{\phi \in \Xi_{\omega}(u) \\
f(\phi, \mathcal{O}) \cap B(z, u) \neq \varnothing}} \nu_{\omega}(f(\phi, \mathcal{O}))=\sum_{\substack{\phi \in \Xi_{\omega}(u) \\
f(\phi, \mathcal{O}) \cap B(z, u) \neq \varnothing}}\left(c_{\omega}^{\phi}\right)^{s_{0}} \\
\leq & \sum_{\substack{\phi \in \Xi_{\omega}(u) \\
f(\phi, \mathcal{O}) \cap B(z, u) \neq \varnothing}}(u)^{s_{0}} \leq\left(4 / c_{\min }\right)^{d}(u)^{s_{0}}=\left(4 / c_{\min }\right)^{d} 2^{s_{0}}|U|^{s_{0}} .
\end{aligned}
$$

Therefore, using the mass distribution principle (Theorem 1.7.6) we obtain

$$
\mathscr{H}^{s_{0}}\left(F_{\omega}\right) \geq\left(4 / c_{\min }\right)^{-d} 2^{-s_{0}}>0
$$

almost surely.
Finally, we show that if $(\mathbb{L}, \mu)$ is not almost deterministic then $\mathscr{H}^{s_{0}}\left(F_{\omega}\right)=0$ almost surely. Consider the random variable $Y_{\lambda}=\log \mathfrak{S}_{\lambda}^{s_{0}}$. Since $c_{\min }>0, \mathbb{E}\left(Y_{\omega_{1}}\right)=$ 0 , and $\mathbb{P}\left\{Y_{\omega_{1}}=0\right\}<1$ we must have $0<\operatorname{Var}\left(Y_{\omega_{1}}\right)<\infty$ and we can apply the central limit theorem, Theorem 1.3.8, to obtain, for any $C>0$,

$$
\begin{equation*}
\mathbb{P}\left(\left\{\omega \in \Omega \mid \sum_{k=1}^{n} Y_{\omega_{k}}<-C \text { for infinitely many } n \in \mathbb{N}\right\}\right)=1 \tag{5.1.3}
\end{equation*}
$$

But then, using the same covering idea as in (5.1.1), we obtain

$$
\mathscr{H}_{\delta}^{s_{0}}\left(F_{\omega}\right) \leq \sum_{\phi \in \mathbf{C}_{\omega}^{k(\delta)}}|\overline{f(\phi, \mathcal{O})}|^{s_{0}}=|\mathcal{O}|^{s_{0}} \mathfrak{S}_{\omega_{1}}^{s_{0}} \mathfrak{S}_{\omega_{2}}^{s} \ldots \mathfrak{S}_{\omega_{k(\delta)}}^{s_{0}}=|\mathcal{O}|^{s_{0}} \exp \left(\sum_{l=1}^{k(\delta)} Y_{l}\right)
$$

for large enough $k(\delta)$. Applying (5.1.3), we note that $\mathscr{H}^{s_{0}}\left(F_{\omega}\right) \leq|\mathcal{O}|^{s_{0}} e^{-C}$ is finite almost surely and the desired conclusion follows as $C$ was arbitrary.

### 5.2 Hausdorff and packing measure for $\infty$-variable constructions

Recall the general $h$-Hausdorff measure stated in Definition 1.7 .5 where we replace the exponential function $|U|^{s}$ with a gauge function $h(U)$. Note that this function need only be defined on $\left[0, r_{0}\right]$ for some $r_{0}>0$. Graf, Mauldin, and Williams determined the natural gauge function for which we obtain a positive and finite Hausdorff measure in the $\infty$-variable setting.
Theorem 5.2.1 (Graf - Mauldin - Williams [GMW, MGW]). Let $(\mathbb{L}, \mu)$ be a random iterated function system that is not almost deterministic. Let $F_{\tau}$ be the associated $\infty$ variable attractor. Assume that

$$
\mathbb{E}\left(\sum_{j}\left(c_{\omega_{1}}^{j}\right)^{0}\right)>1
$$

Let

$$
\begin{equation*}
h_{\beta}^{s}(t)=t^{s}(\log \log (1 / t))^{1 / \beta} \text { and } \beta_{0}=\sup \left\{\beta \mid \sum_{j}\left(c_{\omega_{1}}^{j}\right)^{s /(1-1 / \beta)} \leq 1(\text { a.s. })\right\} . \tag{5.2.1}
\end{equation*}
$$

Then, $\mathscr{H}^{h_{\beta}^{s_{0}}}\left(F_{\tau}\right)<\infty$ for all $\beta>\beta_{0}$, where $s_{0}=\operatorname{ess} \operatorname{dim}_{H} F_{\tau}$.
The authors then proceed to give technical conditions under which $\beta_{0}=1-s / d$, where $d$ is the dimension of the ambient space. Under these conditions the $h_{\beta_{0}}^{s}$ Hausdorff measure of $F_{\tau}$ is positive and finite almost surely. Checking the conditions one obtains that Mandelbrot percolation of $[0,1]^{d}$ has positive and finite measure at this critical value $\beta_{0}$.

Liu [Liu] investigated the Gromov boundary of Galton-Watson trees with i.i.d. randomised descendants. Let $m=\mathbb{E}(N)$, where $N$ is the number of descendants, $\alpha=\log m$, and assume that $\mathbb{E}(N \log N)<\infty$. If $\bar{m}=\operatorname{ess} \sup N<\infty$, then the appropriate gauge function for which one obtains positive and finite measure of the boundary (with respect to a natural metric) is

$$
h(t)=t^{\alpha}(\log \log (1 / t))^{\beta}, \text { where } \beta=1-\frac{\log m}{\log \bar{m}} .
$$

For the packing measure to be positive and finite the appropriate gauge function is

$$
h^{*}(t)=t^{\alpha}(\log \log (1 / t))^{\beta^{*}}, \text { where } \beta^{*}=1-\frac{\log m}{\log \underline{m}}
$$

with $\underline{m}=\operatorname{ess} \inf N>1$
Berlinkov and Mauldin [BM] point out an error in the proof of positivity and provide the following, more general result. Under the same almost deterministic condition they show that the $s$-dimensional packing measure is positive and finite almost surely. When this fails the packing measure is $\infty$ almost surely, assuming the UOSC in both cases. Let $\alpha$ denote the almost sure packing dimension. The authors prove that for the gauge function

$$
h_{\beta}^{\alpha}(t)=t^{\alpha}(\log \log (1 / t))^{\beta}, \text { where } \beta \text { satisfies } 0<\liminf _{a \rightarrow 0}-a^{-1 / \beta} \log \mathbb{P}\left(\mathbb{S}_{\lambda}^{\alpha}<a\right)<\infty,
$$

the packing measure is almost surely finite. We remark that the constant $\beta$ may not exist and only coincides with the $\beta_{0}$ in the Hausdorff measure statement in trivial cases.

Additionally, Berlinkov and Mauldin give an integral test [BM, Theorem 6] to determine whether the packing measure is 0 almost surely. They further conjecture a lower bound that Berlinkov proved in [Ber]: If the random variable $\mathfrak{S}_{\lambda}^{\alpha}$ is of exponential type, i.e. if

$$
C^{-1} a^{1 / \beta} \leq-\log \mathbb{P}\left(0<\mathfrak{S}_{\lambda}^{\alpha} \leq a\right) \leq C a^{1 / \beta}
$$

for some $C, \beta>0$ and all $a \in(0,1)$, then the packing measure is positive and finite almost surely with gauge function $h_{\beta}^{\alpha}(t)$.

### 5.3 Hausdorff measure for 1-variable constructions

Recall that the implicit theorems (Theorems 2.4.1 and 2.4.2) give us information about the Hausdorff measure of attractors satisfying conditions like e.g. the open set condition. In particular if the attractor is self-similar or self-conformal the Hausdorff measure will be positive and finite. One might expect that an implicit theorem for random systems might be of a similar nature and that a gauge function of the form $t^{\alpha}(\log \log (1 / t))^{\beta}$ for some exponent $\beta$ should work for all reasonable natural random constructions, including the 1 -variable model.

However, we will show that this turns out not to be the case. Indeed, for $\alpha=$ ess $\operatorname{dim}_{H} F_{\omega}$, we argue that the correct gauge function for positive and finite Hausdorff dimension should be of the form

$$
h_{1}(t)=t^{\alpha} \exp (\sqrt{(\log (1 / t))(\log \log \log (1 / t))}) .
$$

Let $\beta, \gamma \in \mathbb{R}$, we similarly define

$$
h_{1}(t, \beta, \gamma)=t^{\alpha} \exp \left(\sqrt{2 \beta(\log (1 / t))\left(\log \log \log \left(1 / t^{\beta}\right)\right)}\right)^{1-\gamma}
$$

The first thing to note is that $h_{1}(t, \beta, \gamma)$ is doubling in $t$.
Lemma 5.3.1. Fix $\beta, \gamma>0$. There exists $t_{0}, \rho>0$ such that

$$
h_{1}(t, \beta, \gamma) \leq \rho h_{1}(2 t, \beta, \gamma) \leq \rho^{2} h_{1}(t, \beta, \gamma)
$$

for all $0<t<t_{0}$.
Proof. Let $\kappa \in \mathbb{R}$ and

$$
h_{*}(x+\kappa)=\sqrt{\beta(x+\kappa) \log \log (\beta(x+\kappa))} .
$$

This is well defined for $\log \log \beta(x+\kappa)>1 \Longrightarrow x>e^{e} / \beta-\kappa$. It can easily seen that this function is strictly increasing in $x$, and differentiating we obtain,

$$
h_{*}^{\prime}(x+\kappa)=\sqrt{\beta} \cdot \frac{1 /(\log (\beta(x+\kappa))+\log \log (\beta(x+\kappa))}{2 \sqrt{(x+\kappa) \log \log (\beta(x+\kappa))}} .
$$

Then, for $\kappa>0$,

$$
\begin{aligned}
\frac{h_{*}^{\prime}(x)}{\sqrt{\beta}} & =\frac{1 /(\log (\beta x)+\log \log (\beta x)}{2 \sqrt{x \log \log (\beta x)}} \\
& >\frac{1 /(\log (\beta(x+\kappa))+\log \log (\beta(x+\kappa))}{2 \sqrt{x \log \log (\beta(x+\kappa))}}=\frac{h_{*}^{\prime}(x+\kappa)}{\sqrt{\beta}}
\end{aligned}
$$

and so $h_{*}^{\prime}(x+\kappa)-h_{*}^{\prime}(x)<0$ and $h_{*}(x+\kappa)-h_{*}(x)$ is decreasing, i.e. there exists some $\rho_{0}$ such that

$$
0 \leq h_{*}(x+\kappa)-h_{*}(x) \leq \rho_{0} .
$$

Now substituting $\kappa=-\log 2$ and $x=-\log t$, i.e. $x+\kappa=\log (1 / 2 t)$, we obtain, for $0<t<t_{0}$ and $t_{0}>0$ small enough,

$$
0 \leq \sqrt{\beta \log (1 /(2 t) \log \log (\beta \log 1 /(2 t))}-\sqrt{\beta \log (1 / t) \log \log (\beta \log 1 / t)} \leq \rho_{0}
$$

and

$$
\begin{array}{r}
2^{\alpha} \leq \frac{(2 t)^{\alpha}}{t^{\alpha}} \cdot e^{(1-\gamma) \sqrt{2} \cdot(\sqrt{\beta \log (1 /(2 t) \log \log (\beta \log 1 /(2 t))}-\sqrt{\beta \log (1 / t) \log \log (\beta \log 1 / t)})} \\
\leq 2^{\alpha} e^{(1-\gamma) \sqrt{2} \rho_{0}}
\end{array}
$$

But then

$$
2^{\alpha} \leq \frac{h_{1}(2 t, \beta, \gamma)}{h_{1}(t, \beta, \gamma)} \leq 2^{\alpha} e^{(1-\gamma) \sqrt{2} \rho_{0}}
$$

as required.

For ease of exposition we deal with the basic case where all maps in a fixed IFS contract equally. Note that we assume $\mathbb{E}\left(\mathfrak{S}_{\omega_{1}}^{0}\right)>1$ throughout.

Theorem 5.3.2. Let $F_{\omega}$ be the random homogeneous attractor associated to the RIFS $(\mathbb{L}, \mu)$ satisfying the UOSC and suppose that $c_{\lambda}^{i}=c_{\lambda} \in\left[c_{\min }, c_{\max }\right]$ for every $i \in\left\{1, \ldots, \# \mathbb{I}_{\lambda}\right\}$ and $\lambda \in \Lambda$, where $0<c_{\min } \leq c_{\max }<1$. Let $\varepsilon>0, \alpha=\operatorname{ess} \operatorname{dim}_{H} F_{\omega}$ and $\beta=\operatorname{Var}\left(\log \mathfrak{S}_{\omega_{1}}^{\alpha}\right) / \eta$ for some $\eta \in \mathbb{R}$ (arising in the proof), then

$$
\mathscr{H}^{h_{1}(t, \beta, \varepsilon)}\left(F_{\omega}\right)=0,
$$

almost surely.
Proof. Let $\mathcal{O}$ be the open set guaranteed by the UOSC, we assume without loss of generality that $|\mathcal{O}|=1$. From the definition of Hausdorff measure

$$
\mathscr{H}^{h_{1}(t, \beta, \varepsilon)}\left(F_{\omega}\right) \leq \sum_{\phi \in \mathbf{C}_{\omega}^{k}} h_{1}(|\overline{f(\phi, \mathcal{O})}|, \beta, \varepsilon)
$$

for all $k \in \mathbb{N}$. So, writing $v=\operatorname{Var}\left(\log \mathfrak{S}_{\omega_{1}}^{\alpha}\right)$,

$$
\begin{aligned}
& \mathscr{H}^{h_{1}(t, \beta, \varepsilon)}\left(F_{\omega}\right) \leq \liminf _{k \rightarrow \infty} \sum_{\phi \in \mathbf{C}_{\omega}^{k}} h_{1}(|\overline{f(\phi, \mathcal{O})}|, \beta, \varepsilon) \\
&= \liminf _{k \rightarrow \infty}\left(\prod_{i=1}^{k} \# \mathbb{I}_{\omega_{i}}\right)\left(c_{\omega_{1}} c_{\omega_{2}} \ldots c_{\omega_{k}}\right)^{\alpha} \\
& \cdot \exp \left(\sqrt{2 \beta \log \left(1 /\left(c_{\omega_{1}} \ldots c_{\omega_{k}}\right)\right) \log \log \beta \log \left(1 /\left(c_{\omega_{1}} \ldots c_{\omega_{k}}\right)\right)}\right)^{1-\varepsilon} \\
&=\liminf _{k \rightarrow \infty} \exp \left[\left(\sum_{i=1}^{k} \log \mathfrak{S}_{\omega_{i}}^{\alpha}\right)\right. \\
&\left.\quad+(1-\varepsilon) \sqrt{2 k \beta \log \left(C_{\omega}^{k}\right) \log \log \left(\beta k \log \left(C_{\omega}^{k}\right)\right)}\right]
\end{aligned}
$$

for $C_{\omega}^{k}=\left(c_{\omega_{1}} c_{\omega_{2}} \ldots c_{\omega_{k}}\right)^{-1 / k}$, and so

$$
\begin{aligned}
& =\liminf _{k \rightarrow \infty} \exp \left[\left(\sum_{i=1}^{k} \log \mathfrak{S}_{\omega_{i}}^{\alpha}\right)\right. \\
& \\
& \left.\quad+(1-\varepsilon) \sqrt{2 \frac{\log C_{\omega}^{k}}{\eta} k v \log \log \left(\frac{\log C_{\omega}^{k}}{\eta} k v\right)}\right]
\end{aligned}
$$

Note that we can apply the law of the iterated logarithm, Theorem 1.3.9, to sums over the random variables $Y_{i}=\log \mathfrak{S}_{\omega_{i}}^{\alpha}$ where $Y_{i}$ are i.i.d. with $\mathbb{E}\left(Y_{1}\right)=0$ and $0<\operatorname{Var}\left(Y_{1}\right)<\infty$. Thus

$$
\mathbb{P}\left\{\sum_{i=1}^{k} Y_{i} \leq-(1-\varepsilon / 2) \sqrt{2 v k \log \log (v k)} \text { for infinitely many } k \in \mathbb{N}\right\}=1
$$

Let $\left(i_{1}, i_{2}, \ldots\right)$ be a sequence of indices where the above inequality holds. Note that $c_{\min } \leq C_{\omega}^{k}<c_{\max }$ for all $\omega$ and $k$, and so $\log c_{\min } \leq \log C_{\omega}^{k}<\log c_{\max }$. Therefore, for some uniform $\widetilde{\eta} \in\left[\log c_{\min }, \log c_{\max }\right]$, we have $\log C_{\omega}^{i_{k}} / \widetilde{\eta} \geq 1$ for infinitely many $k$, for almost all $\omega$. We can thus choose $\eta$ the greatest value for which this is satisfied.

We get, almost surely,

$$
\mathscr{H}^{h_{1}(t, \beta, \varepsilon)}\left(F_{\omega}\right) \leq \lim _{k \rightarrow \infty} \exp \left(-\frac{\varepsilon}{3} \sqrt{2 \operatorname{Var}\left(\log \mathfrak{S}_{\omega_{1}}^{\alpha}\right) k \log \log \left(\operatorname{Var}\left(\log \mathfrak{S}_{\omega_{1}}^{\alpha}\right) k\right)}\right)=0
$$

completing the proof.
We note that if $c_{\lambda}=\widetilde{c}$ for every $\lambda$, then $\eta=\log \widetilde{c}$. Note also the following corollary which implies that the 'fine dimension', i.e. the dimension according to the gauge function, is distinct from the $\infty$-variable case.

Corollary 5.3.3. Let $F_{\omega}$ be the random homogeneous attractor associated to the $\operatorname{RIFS}(\mathbb{L}, \mu)$ satisfying the UOSC and suppose that $c_{\lambda}^{i}=c_{\lambda} \in\left[c_{\min }, c_{\max }\right]$ for every $i \in$ $\left\{1, \ldots, \# \mathbb{I}_{\lambda}\right\}$ and $\lambda \in \Lambda$, where $0<c_{\min } \leq c_{\max }<1$. Let $h_{\beta}^{\alpha}(t)=t^{\alpha}(\log \log (1 / t))^{\beta}$, where $\alpha=\operatorname{ess} \operatorname{dim}_{H} F_{\omega}$ and $\beta>0$, then

$$
\mathscr{H}^{h_{\beta}^{\alpha}(t)}\left(F_{\omega}\right)=0 . \quad \text { (a.s.) }
$$

Proof. We check

$$
\begin{aligned}
\lim _{t \rightarrow 0} \frac{h_{\beta}^{\alpha}(t)}{h_{1}\left(t, \beta^{\prime}, \varepsilon\right)} & =\lim _{t \rightarrow 0} \frac{t^{\alpha}(\log \log (1 / t))^{\beta}}{t^{\alpha} \exp \left(\sqrt{2 \beta^{\prime}(\log (1 / t))\left(\log \log \log \left(1 / t^{\beta^{\prime}}\right)\right)}\right)^{1-\varepsilon}} \\
& =\lim _{t \rightarrow 0} \frac{(\log \log (1 / t))^{\beta}}{\exp \left((1-\varepsilon) \sqrt{2 \beta^{\prime}(\log (1 / t))\left(\log \log \log \left(1 / t^{\beta^{\prime}}\right)\right)}\right)} \\
& \leq \lim _{t \rightarrow 0} \frac{(\log (1 / t))^{\beta}}{\exp \left((1-\varepsilon) \sqrt{2 \beta^{\prime}(1 / t)}\right)}=0
\end{aligned}
$$

This holds for all $\beta, \beta^{\prime}, \varepsilon>0$ and the behaviour of the limits is sufficient for the desired result

Considering $h_{1}(t, \beta,-\varepsilon)$ the law of the iterated logarithm guarantees a similar lower bound where the sum diverges and one can define a mass distribution on the random set, implying infinite Hausdorff measure for $\varepsilon>0$.

Theorem 5.3.4. Let $F_{\omega}$ be the random homogeneous attractor associated to the RIFS $(\mathbb{L}, \mu)$ satisfying the UOSC and suppose that $c_{\lambda}^{i}=c_{\lambda} \in\left[c_{\min }, c_{\max }\right]$ for every $i \in\left\{1, \ldots, \# \mathbb{I}_{\lambda}\right\}$ and $\lambda \in \Lambda$, where $0<c_{\min } \leq c_{\max }<1$. Let $\varepsilon>0, \alpha=$ ess $\operatorname{dim}_{H} F_{\omega}$ and $\beta_{0}=\eta_{0} \operatorname{Var}\left(\log \mathfrak{S}_{\omega_{1}}^{\alpha}\right)$ for some $\eta_{0} \in \mathbb{R}$ (arising in the proof), then

$$
\mathscr{H}^{h_{1}\left(t, \beta_{0},-\varepsilon\right)}\left(F_{\omega}\right)=\infty
$$

holds almost surely.
Proof. We use the same notation of the proof of Theorem 5.3.2. Let $\varepsilon>0$ be given and write $v=\operatorname{Var}\left(Y_{1}\right)$. Then the law of the iterated logarithm, Theorem 1.3.9, implies

$$
\mathbb{P}\left\{\sum_{i=1}^{k} Y_{i} \leq-(1+\varepsilon) \sqrt{2 v k \log \log (v k)} \text { for infinitely many } k \in \mathbb{N}\right\}=0
$$

and so, writing $D_{k}(\omega)=\left(c_{\omega_{1}} c_{\omega_{2}} \ldots c_{\omega_{k}}\right)^{-1}$, and $\eta_{k}(\omega)=k / \log D_{k}(\omega)$,

$$
\begin{align*}
& \mathbb{P}\left\{\mathfrak{S}_{\omega_{1}}^{\alpha} \mathfrak{S}_{\omega_{2}}^{\alpha} \ldots \mathfrak{S}_{\omega_{k}}^{\alpha} \geq C \exp \left(-(1+\varepsilon) \sqrt{2 v \eta_{k}(\omega) \log \left(D_{k}(\omega)\right)}\right.\right. \\
& \left.\left.\quad \cdot \sqrt{\log \log \left(v \eta_{k}(\omega) \log \left(D_{k}(\omega)\right)\right.}\right) \text { for all } k \geq l_{0}(\omega) \text { where } l_{0}(\omega) \in \mathbb{N}\right\}=1 \tag{5.3.1}
\end{align*}
$$

for any $C \in \mathbb{R}$. Since $c_{\lambda}$ is bounded away from 0 and 1 , the sequence $\eta_{k}(\omega)$ is uniformly bounded in $k$ and $\omega$. Therefore there exists uniform $\eta_{0}$ such that (5.3.1) holds with $\eta_{k}(\omega)$ replaced by $\eta_{0}$. Let $\mathscr{N}(i)=\# \mathbb{I}_{i}$ then, on a full measure set,

$$
\begin{align*}
\left(\prod_{i=1}^{k} \mathscr{N}\left(\omega_{i}\right)\right) & \geq C D_{k}(\omega)^{\alpha} \exp \left(-(1+\varepsilon) \sqrt{2 v \eta_{0} \log \left(D_{k}(\omega)\right) \log \log \left(v \eta_{0} \log \left(D_{k}(\omega)\right)\right.}\right) \\
& =\frac{C}{h_{1}\left(D_{k}(\omega)^{-1}, \beta_{0},-\varepsilon\right)} \tag{5.3.2}
\end{align*}
$$

holds for all $k \geq l_{0}(\omega)$.
We define a random measure $\nu_{\omega}$ on $F_{\omega}$. Assume $\phi \in \mathbf{C}_{\omega}^{k}$ for some $k \in \mathbb{N}$, for every basic cylinder we set

$$
\widetilde{\nu}_{\omega}(f(\phi, \mathcal{O}))=\left(\prod_{i=1}^{k} \mathscr{N}\left(\omega_{i}\right)\right)^{-1}
$$

This extends to a unique random measure $\nu_{\omega}$ on $F_{\omega}$ for every $\omega \in \Omega$ by Carathéodory's extension theorem. We now show that, almost surely, there exists $C_{\omega}>0$ such that $\nu_{\omega}(U) \leq\left(C_{\omega} / C\right) h_{1}\left(|U|, \beta_{0},-\varepsilon\right)$ for all small enough open $U$ that intersect $F_{\omega}$. Let $U$ be such that $u=2|U|<\left(c_{\text {min }}\right)^{l_{0}(\omega)}$ and choose $z \in\left(U \cap F_{\omega}\right)$, then

$$
\begin{aligned}
\nu_{\omega}(U) \leq & \nu_{\omega}(B(z, u)) \leq \nu_{\omega}\left(\bigcup_{\substack{\phi \in \Xi_{\omega}(u) \\
f(\phi, \mathcal{O}) \cap B(z, u) \neq \varnothing}} f(\phi, \mathcal{O})\right) \\
& =\sum_{\substack{\phi \in \Xi_{\omega}(u) \\
f(\phi, \mathcal{O}) \cap B(z, u) \neq \varnothing}}\left(\prod_{i=1}^{k(u)} \mathscr{N}\left(\phi_{i}\right)\right)^{-1} \leq\left(\frac{4}{c_{\text {min }}}\right)^{d}\left(\prod_{i=1}^{k(u)} \mathscr{N}\left(\phi_{i}\right)\right)^{-1},
\end{aligned}
$$

by Lemma 5.1.5, where $k(u)$ is the common length of all $\phi \in \Xi_{\omega}(u)$. Note that by assumption $k(u) \geq l_{0}(\omega)$. Therefore, using (5.3.2),

$$
\nu_{\omega}(U) \leq\left(\frac{4}{c_{\min }}\right)^{d} C^{-1} h_{1}\left(D_{k(u)}(\omega)^{-1}, \beta_{0},-\varepsilon\right)
$$

Recall that $h_{1}$ is doubling, $c_{\omega_{1}} c_{\omega_{2}} \ldots c_{\omega_{k}}=D_{k}(\omega)<u$, and so there exists $\kappa>0$ such that

$$
\nu_{\omega}(U) \leq(\kappa / C) h_{1}\left(|U|, \beta_{0},-\varepsilon\right)
$$

Now, using the mass distribution principle, Theorem 1.7.6, we conclude

$$
\mathscr{H}^{h_{1}\left(t, \beta_{0},-\varepsilon\right)}\left(F_{\omega}\right) \geq \frac{C}{\kappa} .
$$

The desired conclusion follows from the fact that $C$ was arbitrary.
Note that the constants $\eta$ and $\eta_{0}$ might not coincide, and thus $\beta=\beta_{0}$ might not hold.

Both Theorems 5.3.2 and 5.3.4 seem to suggest that $h_{1}(t, \beta, 0)$ with $\beta=\beta_{0}$ is the correct function that gives positive and finite Hausdorff measure. However, it seems unlikely that there exists $\beta$ such that

$$
0<\mathscr{H}^{h_{1}(t, \beta, 0)}\left(F_{\omega}\right)<\infty .
$$

We conjecture that the situation in the random graph directed setting should be similar.

Conjecture 5.3.5. Let $K_{v}(\omega)$ be an attractor of a 1-variable random graph directed system $(\boldsymbol{\Gamma}, \vec{\pi})$ satisfying the UOSC, all conditions in Definition 3.2.9 and Condition 5.1.1. Then there exists $\beta, \beta^{\prime} \in \mathbb{R}$ such that for all $\varepsilon>0$, almost surely,

$$
\mathscr{H}^{h_{1}(t, \beta, \varepsilon)}\left(K_{v}(\omega)\right)=0 \quad \text { and } \quad \mathscr{H}^{h_{1}\left(t, \beta^{\prime},-\varepsilon\right)}\left(K_{v}(\omega)\right)=\infty .
$$

Since $V$-variable attractors, see Section 2.2.1, are nothing but a special case of random graph directed systems, this would show that $V$-variable sets behave intrinsically like 1-variable sets. This means that these systems cannot truly interpolate between 1 -variable and $\infty$-variable attractors.

### 5.4 Packing measure for 1-variable constructions

Recall that $\operatorname{dim}_{P}(F)=\overline{\operatorname{dim}}_{B}(F)$ if $F$ is compact and $\overline{\operatorname{dim}}_{B} F \cap O=\overline{\operatorname{dim}}_{B} F$ for every open $O$ that intersects $F$, see Theorem 1.7.8. Similarly, we can prove the following Lemma.

Lemma 5.4.1. Let $h$ be a doubling gauge function and let $F_{\omega}$ be the 1-variable attractor of a RIFS $(\mathbb{L}, \mu)$. Assume that all maps in the IFSs are strict contractions such that there exist $0<c_{\min } \leq c_{\max }<1$ such that $c_{\min }|x-y| \leq\left|f_{i}^{j}(x)-f_{i}^{j}(y)\right| \leq$ $c_{\max }|x-y|$ for all $i, j$ and all $x, y \in \mathbb{R}^{d}$. Let $\omega \in \Omega$, then

$$
\mathscr{P}_{0}^{h}\left(F_{\omega}\right)=\infty \quad \Longrightarrow \quad \mathscr{P}^{h}\left(F_{\omega}\right)=\infty
$$

and

$$
\mathscr{P}_{0}^{h}\left(F_{\omega}\right)=0 \quad \Longrightarrow \quad \mathscr{P}^{h}\left(F_{\omega}\right)=0 .
$$

Note that we did not make any assumption on the contractions and separation conditions in this Lemma.

Proof. The second claim follows by definition of $\mathscr{P}^{h}$ and it remains to prove the first, i.e. we need to show that

$$
\inf \left\{\sum_{i=1}^{\infty} \mathscr{P}_{0}^{h}\left(E_{i}\right) \mid F_{\omega} \subseteq \bigcup_{i=1}^{\infty} E_{i}\right\}=\infty
$$

if $\mathscr{P}_{0}^{h}\left(F_{\omega}\right)=\infty$. Now $F_{\omega}$ is compact, and so we can assume the subcover of $\left\{E_{i}\right\}$ is finite. Thus there exists $j$ and $\phi \in \mathbf{C}_{\omega}^{k}$ for some $k$ such that $f\left(\phi, F_{\sigma^{k} \omega}\right) \subset E_{j}$ and so, for some $n$ dependent on the cover,

$$
\begin{aligned}
\mathscr{P}^{h}\left(F_{\omega}\right) & =\inf \left\{\sum_{i=1}^{n} \mathscr{P}_{0}^{h}\left(E_{i}\right) \mid F_{\omega} \subseteq \bigcup_{i=1}^{n} E_{i}\right\} \\
& \geq \inf \left\{\mathscr{P}_{0}^{h}\left(E_{j}\right) \mid F_{\omega} \subseteq \bigcup_{i=1}^{n} E_{i}\right\} \\
& \geq \inf \left\{\mathscr{P}_{0}^{h}\left(f\left(\phi, F_{\sigma^{k} \omega}\right)\right) \mid F_{\omega} \subseteq \bigcup_{i=1}^{n} E_{i}\right\} \\
& \geq \inf \left\{\lim _{\delta \rightarrow 0} \kappa \mathscr{P}_{\delta}^{h}\left(F_{\sigma^{k} \omega}\right)=\infty \mid F_{\omega} \subseteq \bigcup_{i=1}^{n} E_{i}\right\}
\end{aligned}
$$

where the infimum is taken over all finite covers and $\kappa$ is a finite constant arising from the maximal distortion of the map $f(\phi,$.$) (bounded by c_{\min }^{k}$ and $c_{\max }^{k}$ ) and the doubling of $h$, see Lemma 5.3.1.

Inspired by the recent progress on the packing measure of random recursive attractors mentioned above, we would hope that using the gauge $h_{1}(t, \beta, \gamma)$ should give
similar similar convergence and divergence, depending on the sign of $\gamma$. This can be achieved by considering the natural dual to $h_{1}$. Let $\alpha \geq 0, \gamma \in \mathbb{R}$ and $\beta>0$, we set

$$
h_{1}^{*}(t, \beta, \gamma)=t^{\alpha} \exp (-\sqrt{2 \beta \log (1 / t) \log \log (\beta \log (1 / t))})^{1-\gamma}
$$

We remark that, in light of Lemma 5.4.1, we only sketch proofs.
Theorem 5.4.2. Let $F_{\omega}$ be the random homogeneous attractor associated to the selfsimilar RIFS $(\mathbb{L}, \mu)$ satisfying the UOSC and suppose that $c_{\lambda}^{i}=c_{\lambda} \in\left[c_{\min }, c_{\max }\right]$ for every $i \in\left\{1, \ldots, \# \mathbb{I}_{\lambda}\right\}$ and $\lambda \in \Lambda$, where $0<c_{\min } \leq c_{\max }<1$. Let $\varepsilon>0$, $\alpha=\operatorname{ess} \operatorname{dim}_{H} F_{\omega}=\operatorname{ess} \operatorname{dim}_{P} F_{\omega}$ and $\beta_{0}^{*}=\eta_{0} \operatorname{Var}\left(\log \mathfrak{S}_{\omega_{1}}^{\alpha}\right)$ for some $\eta_{0}^{*} \in \mathbb{R}$ (arising in the proof). Then $\mathscr{P}^{h_{1}^{*}\left(t, \beta_{0}^{*}, \varepsilon\right)}\left(F_{\omega}\right)=\infty$ almost surely.

Proof. By Lemma 5.4.1 we only have to analyse $\lim _{\delta \rightarrow 0} \mathscr{P}_{\delta}^{h_{1}^{*}\left(t, \beta_{0}^{*}, \varepsilon\right)}\left(F_{\omega}\right)$. Let $\langle X\rangle$ denote the compact convex hull of $X$. Since $c_{\lambda}$ is uniformly bounded away from 0 and 1 and $\# \mathbb{I}_{\lambda}$ is uniformly bounded above there exist $l, M<\infty$ such that there exists at least one $\phi_{c h}(\omega) \in \mathbf{C}_{\omega}^{l}$ for which $f\left(\phi_{c h}(\omega),\left\langle F_{\omega}\right\rangle\right) \subset\left\langle F_{\omega}\right\rangle$. Thus we get, in a similar fashion to the Hausdorff measure argument,

$$
\begin{aligned}
\lim _{\delta \rightarrow 0} \mathscr{P}_{\delta}^{h_{1}^{*}\left(t, \beta_{0}^{*}, \varepsilon\right)}\left(F_{\omega}\right)= & \lim _{\delta \rightarrow 0} \sup \left\{\sum_{i=1}^{\infty} h_{1}^{*}\left(2 r_{i}, \beta_{0}^{*}, \varepsilon\right) \mid\left\{B\left(x_{i}, r_{i}\right)\right\}\right. \text { is a disjoint } \\
& \text { collection of balls with } \left.2 r_{i}<\delta \text { and } x_{i} \in F_{\omega}\right\} \\
\geq & \limsup _{k \rightarrow \infty} \sum_{\phi \in \mathbf{C}_{\omega}^{k}} h_{1}^{*}\left(\left|f\left(\phi \odot \phi_{c h}\left(\sigma^{k} \omega\right), F_{\sigma^{k+l} \omega}\right)\right|, \beta_{0}^{*}, \varepsilon\right) \\
\geq & \limsup _{k \rightarrow \infty}\left(\prod_{i=1}^{k} \mathscr{N}_{\omega_{i}}\right) h_{1}^{*}\left(c_{\omega_{1}} c_{\omega_{2}} \ldots c_{\omega_{k}} c_{\min }^{l}, \beta_{0}^{*}, \varepsilon\right) \\
\geq & \limsup _{k \rightarrow \infty}\left(\prod_{i=1}^{k} \mathscr{N}_{\omega_{i}}\right) \kappa\left(c_{\omega_{1}} c_{\omega_{2}} \ldots c_{\omega_{k}}\right)^{\alpha} \exp (-(1-\varepsilon) \\
& \left.\cdot \sqrt{\beta_{0}^{*} \log \left(1 /\left(c_{\omega_{1}} \ldots c_{\omega_{k}}\right) \log \log \left(\beta_{0}^{*} \log \left(1 /\left(c_{\omega_{1}} \ldots c_{\omega_{k}}\right)\right)\right.\right.}\right) \\
& \geq \limsup _{k \rightarrow \infty} \kappa \exp \left(\sum_{i=1}^{k} \log \mathfrak{S}_{\omega_{i}}^{\alpha}-(1-\varepsilon) \sqrt{v k \log \log v k}\right) \\
& =\infty,
\end{aligned}
$$

writing $v=\operatorname{Var}\left(\mathfrak{S}_{\omega_{1}}^{\alpha}\right)$ and having used he law of the iterated logarithm in the last step.

Finally, we also obtain an upper bound.
Theorem 5.4.3. Let $F_{\omega}$ be the random homogeneous attractor associated to the selfsimilar RIFS ( $\mathbb{L}, \mu$ ) satisfying the UOSC and suppose that $c_{\lambda}^{i}=c_{\lambda} \in\left[c_{\min }, c_{\max }\right]$ for every $i \in\left\{1, \ldots, \# \mathbb{I}_{\lambda}\right\}$ and $\lambda \in \Lambda$, where $0<c_{\min } \leq c_{\max }<1$. Let $\varepsilon>0$, $\alpha=\operatorname{ess} \operatorname{dim}_{H} F_{\omega}=\operatorname{ess} \operatorname{dim}_{P} F_{\omega}$ and $\beta^{*}=\eta \operatorname{Var}\left(\log \mathfrak{S}_{\omega_{1}}^{\alpha}\right)$ for some $\eta^{*} \in \mathbb{R}$ (arising in the proof), then $\mathscr{P}^{h_{1}^{*}\left(t, \beta^{*}, \varepsilon\right)}\left(F_{\omega}\right)=0$ holds almost surely.

Proof. By the homogeneity of the construction

$$
\sup \left\{\sum_{i=1}^{\infty} h_{1}^{*}\left(2 r_{i}, \beta_{0}^{*}, \varepsilon\right) \mid\left\{B\left(x_{i}, r_{i}\right)\right\} \text { are disjoint balls with } 2 r_{i}<\delta \text { and } x_{i} \in F_{\omega}\right\}
$$

$$
\leq \kappa \sup _{n \geq k(\delta)}\left\{\left(\prod_{i=1}^{n} \mathscr{N}_{\omega_{i}}\right) h_{1}^{*}\left(c_{\omega_{1}} \ldots c_{\omega_{n}}, \beta^{*}, \varepsilon\right)\right\}
$$

for some $\kappa>0$ depending on the diameter of $F_{\omega}$ and the doubling properties of $h_{1}$ only. So, for an appropriately chosen $\eta$, we obtain the desired conclusion from the law of the iterated logarithm in a similar fashion to results above.

Clearly $\lim _{\delta \rightarrow 0}\left(h_{1}^{*}(t, \beta, \gamma)\right) /\left(h_{1}(t, \beta, \gamma)\right)=0$ and so $h_{1}^{*}$ and $h_{1}$ are not equivalent gauge functions. This, however, means that while Hausdorff and packing dimensions coincide, in this simple setting they also require different but related gauge functions for finite and infinite measure.

This of course means that while both the 1 -variable as well as the $\infty$-variable constructions are very natural and the Hausdorff and packing dimensions coincide, their precise asymptotic behaviour measured by the gauge functions differ immensely. This means that any implicit theorem that was to capture these fine details must take into account the random process defining them.

## The Assouad dimension of randomly generated sets

### 6.1 Introduction

In this last chapter, we study the generic Assouad dimension for a variety of different models for generating random fractal sets. We start by considering the 1 -variable random iterated function systems model. Recall that we already established some results regarding the Assouad dimension in the self-similar setting in Chapter 3. In Section 6.2 we revisit these results and give more precise results in the 1-variable RIFS case, in particular we establish a sharp bound using the uniform open set condition as opposed to the uniform strong separation condition, $c f$. Theorem 3.2.27. We compute the Hausdorff dimension of the exceptional set where this value is not attained.

We then consider the setting of self-affine carpets in Section 6.3 and establish the almost sure Assouad dimension of 1-variable Bedford-McMullen carpets. In particular, these sections seek the generic dimension from a measure theoretic point of view. In Section 6.4 we consider 1 -variable attractors from a topological point of view and compute the Assouad dimension for a residual subset of the sample space. This approach was initiated by Fraser [Fr3] and we compute the generic dimension for a finite collection of IFSs with bi-Lipschitz contractions.

In Section 6.5 we will return to Mandelbrot percolation and compute, conditioned on non-extinction, the almost sure Assouad dimension of fractal percolation as well as the almost sure Assouad dimension of all orthogonal projections of the percolation simultaneously. While the first conclusion follows directly from Theorem 3.3.6 we prove this specific example here on its own. A somewhat surprising corollary of our results is that, conditioned on non-extinction, almost surely the fractal percolation cannot be embedded in any lower dimensional Euclidean space, no matter how small the almost sure Hausdorff dimension is.

The key common theme throughout this chapter is that the Assouad dimension is always generically as large as possible. In the measure theoretic setting this behaviour is completely different from that observed by other important notions of dimension, such as Hausdorff, packing or box-counting, where these dimensions are generically an intermediate value, which take the form of an appropriately weighted average of deterministic values, cf. Theorem 3.2.23, Corollary 3.2.24, Theorem 3.4.2, Corollary 4.3.7, and Corollary 4.4.6.

In the topological setting, the generic dimensions of random fractals were shown to be 'extremal' in [Fr3]: some are generically as small as possible and others are generically as large as possible. Interestingly, the Assouad dimension of random attractors agree in both the measure theoretic and topological framework. This is also in stark contrast with what is 'usually' the case. A classical example being that

The content of this chapter is based on The Assouad dimension of randomly generated fractals in collaboration with Jonathan M. Fraser and Jun-Jie Miao, and to appear in Ergodic Theory and Dynamical Systems, see [FMT].

Lebesgue almost all real numbers are normal, but a residual set of real numbers are as far away from being normal as possible [HLOPS, S].

### 6.2 The self-similar setting

As before we let $\mathbb{L}=\left\{\mathbb{I}_{i}\right\}_{i \in \Lambda}$ be a collection of IFSs indexed by $\Lambda=\{1, \ldots, N\}$. Let $\vec{\pi}=\left\{\pi_{1}, \ldots, \pi_{\# \Lambda}\right\}$ be a probability vector such that $\pi_{i}>0$ for all $i \in \Lambda$. We write $\mathbb{P}$ for the product probability measure on $\Omega=\Lambda^{\mathbb{N}}$ induced by $\vec{\pi}$, see Section 1.6 and 5.1. For $\mathbb{I}_{i}=\left\{f_{i}^{j}\right\}_{j=1}^{\# \mathbb{I}_{i}}$ we code the map $f_{i}^{j}$ by the letter $e_{i}^{j}$ and refer to the arrangement of words encoding the IFS $\mathbb{I}_{i}$ by $W_{i}=e_{i}^{1} \sqcup e_{i}^{2} \sqcup \cdots \sqcup e_{i}^{\# \mathbb{I}_{i}}$. The 1-variable attractor associated with $\omega \in \Omega$ can then be expressed as

$$
F_{\omega}=\bigcap_{k=1}^{\infty} f\left(\mathbf{C}_{\omega}^{k}, \Delta\right),
$$

where $\mathbf{C}_{\omega}^{k}=W_{\omega_{1}} \odot \cdots \odot W_{\omega_{k}}$ and $f$ is defined recursively as in Definition 4.3.2.
First we obtain a sure upper bound, i.e. an upper bound which holds for all realisations.

Theorem 6.2.1. Let $(\mathbb{L}, \vec{\pi})$ be a RIFS consisting of IFSs of similarities such that $0<\# \mathbb{I}_{i}<\infty$ for all $i \in \Lambda$. Assume that $(\mathbb{L}, \vec{\pi})$ satisfies the UOSC and let $F_{\omega}$ denote the associated 1 -variable attractor. Then, for all $\omega \in \Omega$,

$$
\operatorname{dim}_{\mathrm{A}} F_{\omega} \leq \sup _{i \in \Lambda} \operatorname{dim}_{\mathrm{A}} F_{\underline{i}} .
$$

The proof of Theorem 6.2.1 will be given in Section 6.6.1.1. Note that for each $i \in \Lambda, \operatorname{dim}_{\mathrm{A}} F_{\underline{i}}$ is the Assouad dimension of the deterministic self-similar set $F_{\underline{i}}$, which may be computed via the Hutchinson-Moran formula since the OSC is satisfied. We will provide an example in Section 6.2 .1 showing that this upper bound can fail if we do not assume the UOSC. We are also able to obtain an almost sure lower bound.

Theorem 6.2.2. Let $(\mathbb{L}, \vec{\pi})$ be a RIFS consisting of IFSs of similarities such that $0<\# \mathbb{I}_{i}<\infty$ for all $i \in \Lambda$. Let $F_{\omega}$ denote the associated 1-variable attractor. Then, for almost all $\omega \in \Omega$, we have

$$
\begin{equation*}
\operatorname{dim}_{\mathrm{A}} F_{\omega} \geq \sup _{i \in \Lambda} \operatorname{dim}_{\mathrm{A}} F_{\underline{i}} . \tag{6.2.1}
\end{equation*}
$$

The proof of Theorem 6.2.2 will be given in Section 6.6.1.2. Note that the Theorem 6.2.2 requires no separation conditions, whereas Theorem 6.2.1 requires the UOSC. Combining the upper and lower estimates immediately yields our main result on random self-similar sets.

Theorem 6.2.3. Let $(\mathbb{L}, \vec{\pi})$ be a RIFS consisting of IFSs of similarities such that $0<\# \mathbb{I}_{i}<\infty$ for all $i \in \Lambda$. Assume that $(\mathbb{L}, \vec{\pi})$ satisfies the UOSC and let $F_{\omega}$ denote the associated 1-variable attractor. Then

$$
\operatorname{dim}_{\mathrm{A}} F_{\omega}=\sup _{i \in \Lambda} \operatorname{dim}_{\mathrm{A}} F_{\underline{i}}=\max _{i \in \Lambda} \operatorname{dim}_{\mathrm{A}} F_{\underline{i}},
$$

for almost all $\omega \in \Omega$.
The results above are in stark contrast to the analogous almost sure formulae for the Hausdorff, packing and box-counting dimension which are some form of weighted average of the deterministic values. As can be deduced from Corollary 3.2.24 the Hausdorff dimension of a random 1 -variable self-similar set satisfying the UOSC is almost surely given by the unique zero of the weighted average of the logarithm of the Hutchinson-Moran formulae for the individual IFSs. A neat consequence of this is that the Assouad dimension and the Hausdorff dimension can be almost surely
distinct, no matter which separation condition you assume. Recall that in the deterministic setting the WSP is sufficient to guarantee equality, and in the random setting the Hausdorff and box-counting dimensions almost surely coincide, even if there are overlaps. In fact the only way the Assouad and Hausdorff dimensions can almost surely coincide in the UOSC case is if all of the deterministic IFSs had the same similarity dimension. Also, apart from this special situation, our result shows that random self-similar sets are almost surely not Ahlfors regular, as for Ahlfors regular sets the Hausdorff and Assouad dimensions coincide. Finally we obtain precise information on the size of the exceptional set of Theorem 6.2.3.

Theorem 6.2.4. Let $(\mathbb{L}, \vec{\pi})$ be a RIFS consisting of IFSs of similarities such that $0<\# \mathbb{I}_{i}<\infty$ for all $i \in \Lambda$. Assume that $(\mathbb{L}, \vec{\pi})$ satisfies the UOSC and let $F_{\omega}$ denote the associated 1-variable attractor. Assume further that $\operatorname{dim}_{\mathrm{A}} F_{\underline{i}}$ is not the same for all $i \in \Lambda$, i.e. the similarity dimensions of the deterministic attractors are not all the same. Then the exceptional set

$$
E=\left\{\omega \in \Omega \mid \operatorname{dim}_{\mathrm{A}} F_{\omega}<\max _{i \in \Lambda} \operatorname{dim}_{\mathrm{A}} F_{\underline{i}}\right\}
$$

is a set of full Hausdorff dimension, despite being a $\mathbb{P}$-null set, i.e. $\operatorname{dim}_{\mathrm{H}} E=\operatorname{dim}_{\mathrm{H}} \Omega$.
The proof of Theorem 6.2.4 can be found in Section 6.6.1.3. The following two figures depict some examples of random self-similar sets. The RIFS is made up of three deterministic IFSs, which are shown in Figure 6.1. Dotted squares indicate the (homothetic) similarities used. In Figure 6.2, three different random realisations are shown, which will (almost surely) all have the same Assouad dimension as the maximum of the three deterministic values.


Figure 6.1: Deterministic self-similar attractors $F_{\underline{1}}, F_{\underline{2}}$ and $F_{\underline{3}}$.


Figure 6.2: Random self-similar attractors $F_{\alpha}, F_{\beta}$ and $F_{\gamma}$ for different realisations $\alpha=(1,2,3,1,2,1,3,3, \ldots), \beta=(2,1,2,1,1,1,1,3, \ldots), \gamma=(2,3,3,2,1,1,1,3, \ldots) \in$ $\Omega$.

We finish this section by mentioning that Li et al. [LLMX] studied the Assouad dimension of Moran sets $E$ generated by two sequences

$$
\left\{n_{k} \in \mathbb{N}\right\}_{k=1}^{\infty} \quad \text { and } \quad\left\{\phi_{k} \in \mathbb{R}^{n_{k}}\right\}_{k=1}^{\infty}
$$

where $n_{k}$ indicates the number of contractions, and $\xi_{k}=\left(c_{k, 1}, \cdots, c_{k, n_{k}}\right)$ gives the contraction ratios at the $k$ th level. They show that

$$
\operatorname{dim}_{\mathrm{A}} E=\lim _{m \rightarrow \infty} \sup _{k} s_{k, k+m}
$$

where $s_{k, k+m}$ is the unique solution to the equation

$$
\prod_{i=k+1}^{k+m} \sum_{j=1}^{n_{i}}\left(c_{i, j}\right)^{s}=1
$$

By choosing $\xi_{k}=\left(c_{k, 1}, \cdots, c_{k, n_{k}}\right)$ from a fixed number of patterns, such a Moran set may be regarded as a particular realisation of our random self-similar sets. Therefore this result gives information about specific realisations, whereas our results study the generic situation.

### 6.2.1 An example with overlaps

Here we provide an example showing that the assumption of some separation condition in Theorem 6.2.1 is necessary. Let the RIFS $\mathbb{L}$ be the system consisting of two IFSs of similarities, $\mathbb{I}_{1}$ and $\mathbb{I}_{2}$. Let $\mathbb{I}_{1}$ be the IFS consisting of the three maps $S_{1,1}, S_{1,2}$ and $S_{1,3}$ and $\mathbb{I}_{2}$ consist of the three maps $S_{2,1}, S_{2,2}$ and $S_{2,3}$, where $S_{i, j}: \mathbb{R} \rightarrow \mathbb{R}$ and

$$
\begin{array}{lll}
S_{1,1}=\frac{1}{2} x, & S_{1,2}=\frac{1}{4} x, & S_{1,3}=\frac{1}{16} x+\frac{15}{16} \\
S_{2,1}=\frac{1}{3} x, & S_{2,2}=\frac{1}{9} x, & S_{2,3}=\frac{1}{81} x+\frac{80}{81}
\end{array}
$$

As $S_{i, 1}$ and $S_{i, 2}$ have the same fixed point for $i=1,2$, both $\mathbb{I}_{1}$ and $\mathbb{I}_{2}$ fail the OSC. Note that

$$
\operatorname{Id}=\left(S_{i, 1} \circ S_{i, 1}\right)^{-1} \circ S_{i, 2} \in \mathcal{E}
$$

where $\mathcal{E}$ is the set of composition maps, see Definition 2.1.3. We conclude that the two IFSs fail the OSC and so $\mathbb{L}$ fails to satisfy the UOSC, see the discussion following Definition 2.1.3. Let $c_{i}^{j}$ be the contraction rate of $S_{i, j}$. If one considers the individual IFSs, since $\left(\log c_{i}^{1}\right) /\left(\log c_{i}^{2}\right) \in \mathbb{Q}$ for $i=1,2$, one can show directly from the definition that the WSP is satisfied. Therefore, for both systems the Assouad and Hausdorff dimensions coincide and are therefore no greater than their similarity dimensions, see [FHOR]. That is $\operatorname{dim}_{\mathrm{A}} F_{\underline{i}} \leq s_{i}$ for $i=1,2$, where $s_{i}$ is given implicitly by the Hutchinson-Moran formula $\sum_{j=1}^{3}\left(c_{i}^{j}\right)^{s_{i}}=1$. Solving numerically we find that $s_{1} \approx 0.81137$ and $s_{2} \approx 0.511918$ and so $\max _{i \in \Lambda} \operatorname{dim}_{\mathrm{A}} F_{\underline{i}}<1$. Consider however $\omega=(1,2,1,2,1, \ldots)$. This is equivalent to the deterministic IFS consisting of the 9 possible compositions of a map from $\mathbb{I}_{1}$ with a map from $\mathbb{I}_{2}$. Consider just the two maps

$$
\begin{aligned}
& T_{1}=S_{1,1} \circ S_{2,2}=\frac{1}{18} x \\
& T_{2}=S_{1,2} \circ S_{2,1}=\frac{1}{12} x
\end{aligned}
$$

One can check that $\log 18 / \log 12 \notin \mathbb{Q}$ and therefore using an argument similar to the one in [Fr2, Section 3.1] one can show that $\operatorname{dim}_{\mathrm{A}} F_{(1,2,1, \ldots)}=1$, which is strictly greater than the maximum given by the deterministic IFS, showing that if the UOSC is not satisfied, then the Assouad dimension of particular realisations can exceed the maximum of the deterministic values.

### 6.3. ALMOST SURE ASSOUAD DIMENSION FOR RANDOM SELF-AFFINE CARPETS



Figure 6.3: Top and middle: the two deterministic attractors which both have Assouad dimension strictly smaller than 1 . Bottom: the random self-similar set for the realisation $\omega=(1,2,1,2, \ldots)$ which has Assouad dimension 1. Stretching the imagination slightly, one can see the unit interval emerging as a tangent at the origin for the third set, but the rational dependence between the contraction ratios prevents this happening for the first two examples.

### 6.3 Almost sure Assouad dimension for random self-affine carpets

We have already answered some dimension theoretic properties of random self-affine sets in Chapter 4. Here we investigate the Assouad dimension of special class of self-affine carpets, a random version of Bedford-McMullen sets.

Fraser and Shmerkin [FS] considered the dimensions of random self-affine carpets where for them the randomness was obtained by randomly translating the column structure. They computed the almost sure Hausdorff and box-counting dimensions and remarked that the situation for the Assouad dimension was not clear because the Assouad dimension could 'jump up' above the initial value. It turns out that in our model, the Assouad dimension is similarly sensitive to 'jumping up' and we show that, in a different context, the Assouad dimension of random self-affine carpets can again 'jump up' above the initial and expected values, see the example in Section 6.3.2.

### 6.3.1 Notation

For each $i \in \Lambda$, let $m_{i}, n_{i}$ be fixed integers with $n_{i}>m_{i} \geq 2$. Then, for each $i \in \Lambda$, divide the unit square $[0,1]^{2}$ into a uniform $m_{i} \times n_{i}$ grid and select a subset of the sub-rectangles formed. Let the IFS $\mathbb{I}_{i}$ be made up of the affine maps which take the unit square onto each chosen sub-rectangle without any rotation or reflection. Thus the constituent maps $f_{i}^{j}:[0,1]^{2} \rightarrow[0,1]^{2}$ are of the form

$$
f_{i}^{j}=\left(\frac{x}{m_{i}}+\frac{a_{i, j}}{m_{i}}, \frac{y}{n_{i}}+\frac{b_{i, j}}{n_{i}}\right),
$$

for integers $a_{i, j}, b_{i, j}$, where $0 \leq a_{i, j}<m_{i}$, and $0 \leq b_{i, j}<n_{i}$. For each $i \in \Lambda$, let $A_{i}$ be the number of distinct integers $a_{i, j}$ used for maps in $\mathbb{I}_{i}$, i.e. the number of non-empty columns in the defining pattern for the $i$ th IFS. Also, for each $i \in \Lambda$, let $B_{i}=\max _{k \in\{0, \ldots, m-1\}} \#\left\{S_{i, j} \in \mathbb{I}_{i}: a_{i, j}=k\right\}$, i.e. the maximum number of rectangles chosen in a particular column of the defining pattern for the $i$ th IFS. For the deterministic IFS $\mathbb{I}_{i}$ with attractor $F_{\underline{i}}$, it was shown by Mackay $[\mathrm{M}]$ that

$$
\begin{equation*}
\operatorname{dim}_{\mathrm{A}} F_{\underline{i}}=\frac{\log A_{i}}{\log m_{i}}+\frac{\log B_{i}}{\log n_{i}} \tag{6.3.1}
\end{equation*}
$$

One interpretation of this is that the Assouad dimension is the dimension of the projection of $F_{\underline{i}}$ onto the first coordinate plus the maximal dimension of a vertical slice through $\overline{F_{\underline{i}}}$. A reasonable first guess for the almost sure Assouad dimension of the random attractors of $\mathbb{L}$ would be to take the maximum of equation (6.3.1) over all $i \in \Lambda$. Surprisingly this is not the correct answer, as we shall see in this section. First we prove a sure upper bound, which at first sight does not look particularly sharp.

Theorem 6.3.1. Let $(\mathbb{L}, \vec{\pi})$ be a RIFS with individual IFSs satisfying the conditions above. Then for all $\omega \in \Omega$

$$
\operatorname{dim}_{\mathrm{A}} F_{\omega} \leq \max _{i \in \Lambda} \frac{\log A_{i}}{\log m_{i}}+\max _{i \in \Lambda} \frac{\log B_{i}}{\log n_{i}}
$$

Theorem 6.3.1 will be proved in Section 6.6.2.2. It turns out that this upper bound is almost surely sharp and this is the content of our main self-affine result in this section.

Theorem 6.3.2. Let $\mathbb{I}$ be as above. Then for almost all $\omega \in \Omega$, we have

$$
\operatorname{dim}_{\mathrm{A}} F_{\omega}=\max _{i \in \Lambda} \frac{\log A_{i}}{\log m_{i}}+\max _{i \in \Lambda} \frac{\log B_{i}}{\log n_{i}}
$$

Theorem 6.3.2 will be proved in Section 6.6.2.3. As remarked above this is in stark contrast to results concerning the classical dimensions of attractors of RIFS, but still in keeping with the 'almost surely maximal' philosophy. An example of the 'averaging' that happens with Hausdorff, packing and box-counting dimension, which was discussed and expanded upon in Chapter 4, is the result by Gui and Li [GL1], where if $m_{i}=m<n=n_{i}$ for all $i \in \Lambda$, the almost sure dimension is given by the weighted average of the dimensions of the individual attractors:

$$
\operatorname{dim} F_{\omega}=\sum_{i \in \Lambda} p_{i} \operatorname{dim} F_{\underline{i}},
$$

where dim can refer to any of the Hausdorff, packing or box-counting dimension, see also Corollary 4.3.8. The key difference between the case considered here and the self-similar case is that, despite whatever separation conditions one wishes to impose, the 'maximal value' is not generally the maximum of the deterministic values. We construct a very simple example to illustrate this difference in Section 6.3.2.

The following two figures depict some examples of random self-affine BedfordMcMullen carpets. The RIFS is made up of three deterministic IFSs, which are shown in Figure 6.4. We chose $m_{1}=2, n_{1}=3, m_{2}=3, n_{2}=5, m_{3}=2$ and $n_{3}=4$ and indicate the chosen affine maps with rectangles. In Figure 6.5, three different random realisations are shown.


Figure 6.4: Deterministic Bedford-McMullen Carpets $F_{\underline{1}}, F_{\underline{2}}$ and $F_{\underline{3}}$.



Figure 6.5: Random Bedford-McMullen Carpets $F_{\alpha}, F_{\beta}$ and $F_{\gamma}$ for realisations $\alpha=$ $(1,1,3,3,1,3,1,3, \ldots), \beta=(1,2,1,2,2,3,2,1, \ldots), \gamma=(2,2,3,2,1,2,2,2, \ldots) \in \Omega$.

### 6.3.2 An example with larger Assouad dimension than expected

In this section we briefly elaborate on our belief that the formula for the almost sure Assouad dimension of random self-affine carpets returns a surprisingly large value. Consider the following very simple example. Let $\mathbb{L}=\left\{\mathbb{I}_{1}, \mathbb{I}_{2}\right\}$, where $m=2$ and $n=3$ for both deterministic IFSs. Let $\mathbb{I}_{1}$ consist of the maps corresponding to the two rectangles in the top row of the defining grid and let $\mathbb{I}_{2}$ consist of the maps corresponding to the three rectangles in the right hand column of the defining grid. Both the deterministic attractors are not very interesting; they are both line segments. In particular, they both have Assouad dimension equal to 1. Moreover, it is a short calculation to show that the Assouad dimension of $F_{\omega}$ is no larger than 1 for any eventually periodic word $\omega$. This means that, unlike the self-similar example in Section 6.2.1, the Assouad dimension cannot increase by taking a finite combination of the initial IFSs. However, Theorem 6.3.2 shows that the Assouad dimension of $F_{\omega}$ is almost surely 2 .


Figure 6.6: The left-most image is the random self-affine carpet associated to the above RIFS for the realisation $(3,3,3,1,1,1,1,3, \ldots)$. The other images show small parts of the set blown back up to the unit square. One can see the zoomed in images filling up more and more space, leading to the unit square being a very weak tangent to the random self-affine set. This is what causes the Assouad dimension to be maximal, see Section 6.6.2.3.

### 6.4 Typical Assouad dimension for random attractors

In this section we consider an alternative approach to deciding the 'generic properties' of random fractals. This approach is topological rather than measure theoretic and was first considered by Fraser [Fr3]. Let $\left(Y, d_{Y}\right)$ be a complete metric space. A set $N \subseteq Y$ is nowhere dense if for all $y \in N$ and for all $r>0$ there exists a point $x \in Y \backslash N$ and $t>0$ such that $B(x, t) \subseteq B(y, r) \backslash N$. A set $M$ is said to be of the first category, or, meagre, if it can be written as a countable union of nowhere dense
sets. We think of a meagre set as being small and the complement of a meagre set as being big. A set $T \subseteq Y$ is residual or co-meagre, if $Y \backslash T$ is meagre. A property is called typical if the set of points which have the property is residual. In many ways a residual set behaves a lot like a set of full measure. For example, the intersection of a countable number of residual sets is residual and the space cannot be broken up into the disjoint union of two sets which are both residual. As such it is a reasonable replacement for the notion of almost all in describing generic properties in a complete metric space. In Section 6.6 we will use the following theorem to test for typicality without mentioning it explicitly.

Theorem 6.4.1. In a complete metric space, a set $T$ is residual if and only if $T$ contains a countable intersection of open dense sets or, equivalently, $T$ contains a dense $\mathcal{G}_{\delta}$ subset of $Y$.

For a proof of this result and for a more detailed account of Baire Category the reader is referred to $[\mathrm{Ox}]$. By applying these notions to the complete metric space $(\Omega, d)$ we can replace "full measure" with "residual" to gain our new notion of genericity. In [Fr3] it was shown that these two approaches differ immensely in the context of Hausdorff and packing dimension. Indeed, it was shown that there exists a residual set $R \subseteq \Omega$ such that for all $\omega \in R$

$$
\operatorname{dim}_{H} F_{\omega}=\inf _{u \in \Omega} \operatorname{dim}_{H} F_{u}
$$

and

$$
\operatorname{dim}_{P} F_{\omega}=\sup _{u \in \Omega} \operatorname{dim}_{P} F_{u}
$$

for any 1 -variable RIFS consisting of bi-Lipschitz contractions without assuming any separation conditions. This is very different from the measure theoretic approach, which tends to favour convergence rather than divergence, with the almost sure packing and Hausdorff dimensions often equal to some sort of average over the parameter space, rather than opposite extremes. Our main result in this section proves an analogous result for Assouad dimension. In the wider context of the paper, the main interest of this result is that in the setting of Assouad dimension, the topological and measure theoretic approaches seem to agree. Observe that we are able to compute the typical Assouad dimension in a much more general context than the almost sure Assouad dimension, but this is not surprising in view of [Fr3].

Theorem 6.4.2. Let $\mathbb{L}$ be an RIFS consisting of deterministic IFSs of bi-Lipschitz contractions. Then there exists a residual set $R \subseteq \Omega$ such that for all $\omega \in R$

$$
\operatorname{dim}_{\mathrm{A}} F_{\omega}=\sup _{u \in \Omega} \operatorname{dim}_{\mathrm{A}} F_{u}
$$

We will prove Theorem 6.4.2 in Section 6.6.3. Notice that the above result assumes no separation properties and the mappings can be much more general than similarities or even affine maps.

An immediate and perhaps surprising corollary of this is that Theorems 6.2.3 and 6.3.2 remain true even if the measure theoretic approach is replaced by the topological approach adopted in this section.

### 6.5 Random recursive constructions

We answered several questions about random recursive constructions in Chapter 3. Here we will give some complementary results for Mandelbrot percolation defined in Section 2.3. Recall that Mandelbrot percolation is simply a random recursive RIFS and Theorem 3.3.6 implies that, if $p>1 / n^{d}$, then, conditioned on non-extinction,

$$
\operatorname{dim}_{A} F=d
$$

We remark that this can also be obtained by studying the notion of porosity, see Berlinkov and Järvenpää [BJ].

Recently, there has also been a lot of work on almost sure properties of the orthogonal projections of fractal percolation. In particular, one wants to obtain a 'Marstrand type result' for all projections $\pi \in \Pi_{d, k}$ rather than just almost all. Here $\Pi_{d, k}$ is the Grassmannian manifold consisting of all orthogonal projections from $\mathbb{R}^{d}$ to $\mathbb{R}^{k}(k \leq d)$ identified with $k$ dimensional subspaces of $\mathbb{R}^{d}$ in the natural way and equipped with the usual Grassmann measure. Our main result is the following, which gives, conditioned on non-extinction, the almost sure Assouad dimension of $F$ as well as an optimal projection result.

Theorem 6.5.1. Let $p>1 / n^{d}$. Then, conditioned on $F$ being nonempty, we have that almost surely

$$
\operatorname{dim}_{\mathrm{A}} F=d
$$

and for all $k \leq d$ and $\pi \in \Pi_{d, k}$ simultaneously,

$$
\operatorname{dim}_{\mathrm{A}} \pi F=k
$$

We will prove Theorem 6.5.1 in Section 6.6.4. Observe that, provided $p>1 / n^{d}$, the almost sure Assouad dimension does not depend on $p$. This is in stark contrast to the Hausdorff and packing dimension case, but by now not surprising. An immediate corollary of Theorem 6.5.1 is the following embedding theorem for Mandelbrot percolation.

Corollary 6.5.2. Let $p>1 / n^{d}$. Then, conditioned on $F$ being nonempty, almost surely $F$ cannot be embedded in any lower dimensional Euclidean space via a biLipschitz map, i.e. there does not exists a bi-Lipschitz $\operatorname{map} \phi: F \rightarrow \mathbb{R}^{d-1}$.

This follows from Theorem 6.5.1 and the fact that bi-Lipschitz maps cannot decrease Assouad dimension [ Lu , Theorem A.5.1]. This result is somewhat surprising in that given any $\varepsilon>0$, one can choose $p$ sufficiently close to (but greater than) $n^{-d}$, such that almost surely (conditioned on non-extinction) the Hausdorff dimension of $F$ is smaller than $\varepsilon$, but yet $F$ still cannot be embedded in any Euclidean space with dimension less than that of the initial ambient space.

### 6.6 Proofs

### 6.6.1 Proofs concerning random self-similar sets

### 6.6.1.1 Proof of Theorem 6.2.1

The proof of Theorem 6.2.1 will closely follow the strategy of Olsen [O2], who gave a simple argument demonstrating the sharp upper bound for the Assouad dimension of a deterministic self-similar set satisfying the OSC. Before beginning the proof we recall some notation. Let $\mathcal{O}$ be the uniform open set given by the UOSC and write $|X|$ for the diameter of a set $X$ and let $u=|\mathcal{O}|$. Fix a realisation $\omega=\left(\omega_{1}, \omega_{2}, \ldots\right) \in \Omega$ and define the arrangements of words in the usual way, see the discussion just after Definition 4.2.3. Observe that

$$
F_{\omega}=\bigcap_{k} \bigcup_{\phi \in \mathbf{C}_{\omega}^{k}} f(\phi, \overline{\mathcal{O}})=\bigcap_{k} \bigcup_{\phi \in \mathbf{C}_{\omega}^{k}} \overline{f(\phi, \mathcal{O})}
$$

Write $c_{i}^{j}=\operatorname{Lip}\left(f_{i}^{j}\right)$, then $|\overline{f(\phi, \mathcal{O})}|=c_{\omega_{1}}^{\phi_{1}} c_{\omega_{2}}^{\phi_{2}} \ldots c_{\omega_{l}}^{\phi_{l}} u$, where $l=|\phi|$. We write $c_{\omega}^{\phi}=$ $|\overline{f(\phi, \mathcal{O})}|$ for brevity. For $r \in(0,1]$, let $\Xi_{\omega}(r)$ be the arrangement of words $\phi$ for which the associated $\overline{f(\phi, \mathcal{O})}$ has diameter approximately $r$, that is

$$
\Xi_{\omega}(r)=\bigsqcup\left\{\phi \in \bigsqcup_{k \in \mathbb{N}_{0}} \mathbf{C}_{\omega}^{k} \mid c_{\omega}^{\phi}<r \leq c_{\omega}^{\phi^{\dagger}}\right\}
$$

where $\phi^{\dagger}=\phi_{1} \phi_{2} \ldots \phi_{|\phi|-1}$. First, recall from Lemma 5.1.5 that the number of sets $\overline{f(\phi, \mathcal{O})}$ with diameter approximately $r$ that intersect a closed ball $B(z, r)$ centred at $z \in F_{\omega}$ of radius $r$ is bounded by a constant not depending on $r$ and $z$.

Write $s=\max _{i \in \Lambda}\left(\operatorname{dim}_{\mathrm{A}} F_{\underline{i}}\right)$.
Lemma 6.6.1. Under the same assumptions as Theorem 6.2.1,

$$
\# \Xi_{\omega}(r) \leq u^{s} c_{\min }^{-s} r^{-s}
$$

for all $r \in(0,1]$.
Proof. Fix $r \in(0,1]$ and observe that since the Assouad dimension of each deterministic attractor is given by the appropriate version of the Hutchinson-Moran formula, we have, for every $i \in \Lambda$,

$$
\sum_{j=1}^{\# \mathbb{I}_{i}}\left(c_{i}^{j}\right)^{s} \leq 1
$$

for all $i \in \Lambda$. By repeated application of this, it follows that

$$
u^{s} \geq \sum_{\phi \in \Xi_{\omega}(r)}\left(c_{\omega}^{\phi}\right)^{s} \geq \sum_{\phi \in \Xi_{\omega}(r)}\left(c_{\min } r\right)^{s}=\# \Xi_{\omega}(r) c_{\min }^{s} r^{s},
$$

which proves the lemma.
We can now prove Theorem 6.2.1. Let $\mathcal{O}$ be the open set given by the UOSC. Fix $z \in F_{\omega}, R \in(0,1]$ and $r \in(0, R]$. Clearly

$$
B(z, R) \cap F_{\omega} \subseteq \bigcup_{\substack{\phi \in \Xi_{\omega}(R) \\ f(\phi, \mathcal{O}) \cap B(z, R) \neq \varnothing}} \overline{f(\phi, \mathcal{O})}
$$

and for each such set $\overline{f(\phi, \mathcal{O})}$ in the above decomposition we have

$$
\overline{f(\phi, \mathcal{O})} \subseteq \bigcup_{\varphi \in \Xi_{\sigma(|\phi|, \omega)}(r / R)} \overline{f(\phi \odot \varphi, \mathcal{O})}
$$

where for clarity we have written $\sigma(|\phi|, \omega)=\sigma^{|\phi|}(\omega)$. These observations combine to give

$$
B(z, R) \cap F_{\omega} \subseteq \bigcup_{\substack{\phi \in \Xi_{\omega}(R) \\ f(\phi, \mathcal{O}) \cap B(z, R) \neq \varnothing}} \bigcup_{\substack{ \\\varphi \in \Xi_{\sigma(|\phi|, \omega)}(r / R)}} \overline{f(\phi \odot \varphi, \mathcal{O})}
$$

which is an $r$-cover of $B(z, R) \cap F_{\omega}$, yielding

$$
\begin{aligned}
& N_{r}\left(B(z, R) \cap F_{\omega}\right) \leq \sum_{\substack{\phi \in \Xi_{\omega}(R) \\
f(\phi, \mathcal{O}) \cap B(z, R) \neq \varnothing}} \# \Xi_{\sigma(\phi, \omega)}(r / R) \\
& \leq \sum_{\substack{\phi \in \Xi_{\omega}(R) \\
f(\phi, \mathcal{O}) \cap B(z, R) \neq \varnothing}} u^{s} c_{\min }^{-s}\left(\frac{R}{r}\right)^{s} \quad \text { by Lemma 6.6.1 } \\
& \leq\left(4 / c_{\min }\right)^{d} u^{s} c_{\min }^{-s}\left(\frac{R}{r}\right)^{s} \quad \text { by Lemma 5.1.5, }
\end{aligned}
$$

which proves the theorem.

### 6.6.1.2 Proof of Theorem 6.2.2

In order to prove the almost sure lower bound we identify a 'good set' of full measure within which we can prove the lower bound surely. Fix $i \in \Lambda$ which maximises $\operatorname{dim}_{A} F_{\underline{i}}$. The good set $G_{i}$ is the set of all realisations $\omega \in \Omega$ such that there are arbitrarily long subwords consisting only of the letter $i$. Equivalently, let

$$
\begin{aligned}
G_{i}=\left\{\omega=\left(\omega_{1}, \omega_{2}, \ldots\right) \in \Omega \mid \forall n \in \mathbb{N}, \exists k\right. & \in \mathbb{N}, \exists m \geq n \\
& \text { such that } \left.\omega_{j}=i \text { for all } k \leq j<k+m\right\} .
\end{aligned}
$$

The following Lemma follows from a standard Borel-Cantelli argument.
Lemma 6.6.2. Let $\mathbb{P}$ be the Bernoulli probability measure on $\Omega$ induced by $\vec{\pi}$. Then for all $i \in \Lambda$, the set $G_{i} \subseteq \Omega$ has full measure, i.e. $\mathbb{P}\left(G_{i}\right)=1$.
Proof. Fix $i \in \Lambda$ and let $C_{j}$ be an increasing sequence of integers such that $C_{j+1}>$ $C_{j}+j$ and set

$$
\begin{aligned}
& A_{i}(j)=\left\{\omega \in \Omega \mid \text { there exists } l_{0} \text { such that } \omega_{k}=i \text { for all } k \text { s.t. } l_{0} \leq k \leq l_{0}+j\right. \\
& \left.\quad \text { and } l_{0} \text { satisfies } \sum_{l=1}^{j} C_{j} \leq l_{0}<\left(\sum_{l=1}^{j+1} C_{j}\right)-j\right\},
\end{aligned}
$$

that is, given a $j \in \mathbb{N}$, realisations in the event $A_{i}(j)$ have a subword consisting only of the letter $i$ of length $k$ starting between positions $\sum_{l=1}^{j} C_{j}$ and $\sum_{l=1}^{j+1} C_{j}-j$. Given that the fixing of letters are on disjoint intervals the events $\left\{A_{i}(j)\right\}_{j \in \mathbb{N}}$ are pairwise independent. Given a string of length $C_{j+1}$, we divide it into blocks of length $\lfloor j / 2\rfloor$ (with a possibly shorter final block). The probability of choosing $j$ consecutive letters equal to $i$ in a string of length $C_{j+1}$ is at least the probability that at least one of these blocks consists just of the letter $i$, that is

$$
\mathbb{P}\left(A_{i}(j)\right) \geq 1-\left(1-\pi_{i}^{\lfloor j / 2\rfloor}\right)^{\left\lfloor C_{j+1} / j\right\rfloor}
$$

We can now choose the constants $C_{j}$ large enough such that $\left(1-\pi_{i}^{\lfloor j / 2\rfloor}\right)^{\left\lfloor C_{j+1} / j\right\rfloor}<1 / 2$. Then $\sum_{j \in \mathbb{N}} \mathbb{P}\left(A_{i}(j)\right) \geq \sum_{j \in \mathbb{N}} 1 / 2=\infty$ and, using the Borel-Cantelli Lemma,

$$
\mathbb{P}\left(\left\{\omega \in \Omega \mid \omega \in A_{i}(j) \text { for infinitely many } j\right\}\right)=1
$$

But,

$$
\left\{\omega \in \Omega \mid \omega \in A_{i}(j) \text { for infinitely many } j\right\}=\bigcap_{k \in \mathbb{N} j \geq k} \bigcup_{j} A_{i}(j) \subseteq G_{i}
$$

and so the desired result follows.
Let $i \in \Lambda$ and $\omega=\left(\omega_{1}, \omega_{2}, \ldots\right) \in G_{i}$. We will show that the deterministic attractor $F_{\underline{i}}$ is a very weak pseudo tangent to $F_{\omega}$ and this is sufficient to prove Theorem 6.2.2 in light of Lemma 1.7.13. Note that, since we do not assume any separation conditions, the existence of complicated overlaps mean that $F_{\underline{i}}$ may not be a weak (or very weak) tangent to $F_{\omega}$.

Since $\omega \in G_{i}$, for every $n$ we can find $k_{n}$ such that $\omega_{j}=i$ for all $k_{n}+1 \leq j \leq k_{n}+n$. Choose any arrangement $\phi=\phi_{1} \odot \phi_{2} \odot \cdots \odot \phi_{k_{n}}$ with $\phi \in \mathbf{C}_{\omega}^{k_{n}}$ and let $T_{k_{n}}$ be the similarity given by $T_{k_{n}}(x)=f(\phi, x)^{-1}$. Write $c_{\max }^{i}=\max _{j \in\left\{1, \ldots, \mathbb{I}_{i}\right\}} c_{i}^{j} \in(0,1)$. It follows that

$$
F_{\sigma^{k_{n}}(\omega)} \subseteq T_{k_{n}}\left(F_{\omega}\right)
$$

and therefore, since the first $n$ symbols in $\sigma^{k_{n}}(\omega)$ are all $i$,

$$
F_{\underline{i}} \subseteq\left[T_{k_{n}} F_{\omega}\right]_{\left(c_{\max }^{i}\right)^{n}}
$$

This proves that

$$
d_{\mathscr{H}}^{l}\left(F_{\underline{i}}, T_{k_{n}} F_{\omega}\right) \leq\left(c_{\max }^{i}\right)^{n} \rightarrow 0
$$

as $n \rightarrow \infty$. Thus $F_{\underline{i}}$ is a weak pseudo tangent to $F_{\omega}$, choosing the sequence of maps $\left\{T_{k_{n}}\right\}_{n \in \mathbb{N}}$.

### 6.6.1.3 Proof of Theorem 6.2.4

Using the mass distribution principle, Theorem 1.7.6, it is easy to see that $\operatorname{dim}_{H} \Omega=$ $\log N / \log 2$ where $N$ is the cardinality of $\Lambda$ and the ' 2 ' comes from our choice of metric $d(\omega, \nu)=2^{-\omega \wedge \nu}$. Let $\beta=\max _{i \in \Lambda} \operatorname{dim}_{\mathrm{A}} F_{\underline{i}}$ and

$$
\Lambda_{\mathscr{E}}=\left\{i \in \Lambda \mid \operatorname{dim}_{\mathrm{A}} F_{\underline{i}}<\beta\right\}
$$

which by assumption is non-empty and by definition a proper subset of $\Lambda$. Let

$$
\begin{aligned}
& \mathscr{E}_{n}=\left\{\omega \in \Omega \mid \text { if, for some } k \in \mathbb{N}, \text { whenever } \omega_{j} \notin \Lambda_{\mathscr{E}}\right. \\
& \text { for all } \left.j=k, k+1, \ldots, k+n-1, \text { then } \omega_{k+n} \in \Lambda_{\mathscr{E}}\right\},
\end{aligned}
$$

i.e. the set of all sequences such that the length of subwords consisting only of letters which maximise the Assouad dimension is bounded above by $n$. First we will show that for $\omega \in \mathscr{E}_{n}$, we have $\operatorname{dim}_{\mathrm{A}} F_{\omega}<\beta$. Let $\Lambda_{n}^{\dagger}$ be a new alphabet consisting of all combinations of words of length at most $n$ (including length zero) over $\Lambda \backslash \Lambda_{\mathscr{E}}$ concatenated with an element of $\Lambda_{\mathscr{E}}$, that is

$$
\Lambda_{n}^{\dagger}=\left\{v w \mid v \in \cup_{k=0}^{n}\left(\Lambda \backslash \Lambda_{\mathscr{E}}\right)^{k} \text { and } w \in \Lambda_{\mathscr{E}}\right\}
$$

To each word (now identified as a letter) in $\Lambda_{n}^{\dagger}$ we associate the IFS formed by composing the IFSs corresponding to $\Lambda$ in the natural way. Since the UOSC was satisfied, it is easy to see that the similarity dimension of each such deterministic IFS is strictly less than $\beta$ since they are all influenced by an IFS associated to an element of $\Lambda_{\mathscr{E}}$. Moreover, every word in $\mathscr{E}_{n}$ can be obtained as a word over $\Lambda_{n}^{\dagger}$ and so we obtain a lower bound from Theorem 6.2.2. It follows from this that the exceptional set from Theorem 6.2.4 contains

$$
\mathscr{E}:=\bigcup_{n=1}^{\infty} \mathscr{E}_{n}
$$

and so it suffices to prove that the Hausdorff dimension of $\mathscr{E}$ is $\log N / \log 2$. Now consider the finite set $\Lambda^{\prime}$ consisting of all possible words of length $\lceil n / 2\rceil$. We could have equivalently defined $\Omega$ in terms of those words rather than the individual symbols $\Lambda$ where, abusing notation slightly, $\Omega=\Lambda^{\mathbb{N}}=\Lambda^{\prime \mathbb{N}}$. Consider $\Lambda^{\prime}$ and remove the words consisting only of letters from $\Lambda \backslash \Lambda_{\mathscr{E}}$ forming a new set $\Lambda^{\prime \prime}$. If one considers $\mathscr{E}_{n}^{\prime}=\Lambda^{\prime \prime \mathbb{N}}$ one notes that several combinations are now no longer possible. Crucially it restricts the length of subwords over $\Lambda \backslash \Lambda_{\mathscr{E}}$ to $2(\lceil n / 2\rceil-1)$, which corresponds to two concatenated elements of $\Lambda^{\prime \prime}$, one starting with symbol $j \in \Lambda_{\mathscr{E}}$ followed by letters from $\Lambda \backslash \Lambda_{\mathscr{E}}$ and the second word starting with letters from $\Lambda \backslash \Lambda_{\mathscr{E}}$ but ending with $j \in \Lambda_{\mathscr{E}}$. Since $2(\lceil n / 2\rceil-1) \leq n$ we have that elements of $\mathscr{E}_{n}^{\prime}$ have more restrictive conditions than $\mathscr{E}_{n}$ and so $\mathscr{E}_{n}^{\prime} \subseteq \mathscr{E}_{n}$. Let $\nu$ be the uniform Bernoulli measure on $\mathscr{E}_{n}^{\prime}$, given by a uniform probability vector associated with $\Lambda^{\prime \prime}$, and let

$$
\alpha_{n}=\frac{\log \left(N^{\lceil n / 2\rceil}-\left|\Lambda \backslash \Lambda_{\mathscr{E}}\right|^{\lceil n / 2\rceil}\right)}{\lceil n / 2\rceil \log 2} .
$$

Let $U_{k} \subseteq \mathscr{E}_{n}^{\prime \prime}$ be a cylinder of length $k$ (over $\Lambda^{\prime \prime}$ ) and observe that $\nu\left(U_{k}\right)=\left(N^{\lceil n / 2\rceil}-\right.$ $\left.\left|\Lambda \backslash \Lambda_{\mathscr{E}}\right|^{\lceil n / 2\rceil}\right)^{-k}$ and $\left|U_{k}\right|=2^{-k\lceil n / 2\rceil}$ and so

$$
\begin{aligned}
\left|U_{k}\right|^{\alpha_{n}} & =2^{-k\lceil n / 2\rceil \log \left(N^{\lceil n / 2\rceil}-\left|\Lambda \backslash \Lambda_{\mathscr{E}}\right|^{\lceil n / 2\rceil}\right) /(\lceil n / 2\rceil \log 2)} \\
& =2^{-k \log \left(N^{\lceil n / 2\rceil}-\left|\Lambda \backslash \Lambda_{\mathscr{E}}\right|^{\lceil n / 2\rceil}\right) / \log 2} \\
& =\left(N^{\lceil n / 2\rceil}-\left|\Lambda \backslash \Lambda_{\mathscr{E}}\right|^{\lceil n / 2\rceil}\right)^{-k} \\
& =\nu\left(U_{k}\right)
\end{aligned}
$$

and thus by the mass distribution principle $\operatorname{dim}_{\mathrm{H}} \mathscr{E}_{n}^{\prime} \geq \alpha_{n}$. Finally $\operatorname{dim}_{\mathrm{H}} \mathscr{E}=$ $\sup _{n} \operatorname{dim}_{H} \mathscr{E}_{n} \geq \sup _{n} \alpha_{n}=\log N / \log 2$, as required.

### 6.6.2 Proofs concerning random self-affine carpets

### 6.6.2.1 Preliminary results and random approximate $R$-squares

In this section we introduce random approximate $R$-squares, which will be heavily relied on in both the upper bound and the lower bound. Fix $\omega=\left(\omega_{1}, \omega_{2}, \ldots\right) \in \Omega$ and $R \in(0,1)$ and let $k_{1}^{\omega}(R)$ and $k_{2}^{\omega}(R)$ be the unique natural numbers satisfying

$$
\begin{equation*}
\prod_{i=1}^{k_{1}^{\omega}(R)} n_{\omega_{i}}^{-1} \leq R<\prod_{i=1}^{k_{1}^{\omega}(R)-1} n_{\omega_{i}}^{-1} \tag{6.6.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\prod_{i=1}^{k_{2}^{\omega}(R)} m_{\omega_{i}}^{-1} \leq R<\prod_{i=1}^{k_{2}^{\omega}(R)-1} m_{\omega_{i}}^{-1} \tag{6.6.2}
\end{equation*}
$$

respectively with notation as in Section 6.3.1. Also let

$$
m_{\max }=\max _{i \in \Lambda} m_{i} \quad \text { and } \quad n_{\max }=\max _{i \in \Lambda} n_{i}
$$

A rectangle $[a, b] \times[c, d] \subseteq[0,1]^{2}$ is called a random approximate $R$-square if it is of the form

$$
S\left([0,1]^{2}\right) \cap\left(\Pi_{x}\left(T\left([0,1]^{2}\right)\right) \times[0,1]\right)
$$

where $\Pi_{x}:(x, y) \mapsto x$ is projection onto the first coordinate and

$$
S(x)=f(\phi, x) \quad \text { for some } \phi \in \mathbf{C}_{k_{1}^{\omega}(R)}
$$

and

$$
T(x)=f(\varphi, x) \quad \text { for some } \varphi \in \mathbf{C}_{k_{2}^{\omega}(R)}
$$

The use of the term 'random' indicates that the family of approximate $R$-squares depends on the random sequence $\omega$ and observe that such rectangles are indeed approximately squares of side length $R$ because the base

$$
b-a=\prod_{i=1}^{k_{2}^{\omega}(R)} m_{\omega_{i}}^{-1} \in\left(m_{\max }^{-1} R, R\right] \quad \text { by }(6.6 .2)
$$

and the height

$$
d-c=\prod_{i=1}^{k_{1}^{\omega}(R)} n_{\omega_{i}}^{-1} \in\left(n_{\max }^{-1} R, R\right] \quad \text { by }(6.6 .1)
$$

These approximate squares are a standard tool in the study of self-affine carpets, see e.g. [M, Fr2].

### 6.6.2.2 Proof of Theorem 6.3.1

Fix $\omega=\left(\omega_{1}, \omega_{2}, \ldots\right) \in \Omega, R \in(0,1)$ and $r \in(0, R)$. For $k, l \in \mathbb{N}$ and $i \in \Lambda$ let

$$
\mathcal{N}_{i}(k, l)=\#\left\{j=k, k+1, \ldots, l: \omega_{j}=i\right\}
$$

We wish to bound $N_{r}\left(B(x, R) \cap F_{\omega}\right)$ up to a constant uniformly over $x \in F_{\omega}$, but since there exists a constant $K \geq 1$ depending on $m_{\max }$ and $n_{\max }$ such that for any $x \in F_{\omega}$, $B(x, R)$ is contained in fewer than $K$ random approximate $R$-squares, it suffices to bound $N_{r}\left(Q \cap F_{\omega}\right)$ up to a constant uniformly over all random approximate $R$-squares, $Q$. We will adopt the version of $N_{r}(\cdot)$ which uses covers by squares of sidelength $r$.

Fix such a $Q$ and observe that $Q \cap F_{\omega}$ can be decomposed as the union of the parts of $F_{\omega}$ contained inside rectangles of the form

$$
\mathcal{X}=f\left(e_{\omega_{1}}^{i_{1}} \odot e_{\omega_{2}}^{i_{2}} \odot \cdots \odot e_{\omega_{k_{2}^{\prime}}^{\omega}(R)}^{i_{k_{2}^{\omega}(R)}},[0,1]^{2}\right)
$$

for some $i_{1}, i_{2}, \ldots$ with $i_{j} \in\left\{1, \ldots, \# \mathbb{I}_{\omega_{j}}\right\}$ for all $j$. Moreover, the number of such rectangles in this decomposition can be bounded above by

$$
\prod_{i \in \Lambda} B_{i}^{\mathcal{N}_{i}\left(k_{1}^{\omega}(R)+1, k_{2}^{\omega}(R)\right)}
$$

Now, let us continue to iterate the construction of $F_{\omega}$ inside such a rectangle $\mathcal{X}$, i.e. by breaking it up into smaller basic rectangles. Assuming $k_{1}^{\omega}(r)>k_{2}^{\omega}(R)$ continue iterating until level $k_{1}^{\omega}(r)$ where each $\mathcal{X} \cap F_{\omega}$ can be written as the union of parts of $F_{\omega}$ inside rectangles of the form

$$
\mathcal{Y}=f\left(e_{\omega_{1}}^{i_{1}} \odot e_{\omega_{2}}^{i_{2}} \odot \cdots \odot e_{\omega_{k_{1}}(r)}^{i_{k}^{\omega}(r)},[0,1]^{2}\right)
$$

for some $i_{1}, i_{2}, \ldots$ with $i_{j} \in\left\{1, \ldots, \# \mathbb{I}_{\omega_{j}}\right\}$ for all $j$. Note that this time we use words of length $k_{1}^{\omega}(r)$. Writing $N_{i}=\# \mathcal{I}_{i}(i \in \Lambda)$, we can bound the number of rectangles of the form $\mathcal{Y}$ used to decompose a rectangle of the form $\mathcal{X}$ by

$$
\prod_{i \in \Lambda} N_{i}^{\mathcal{N}_{i}\left(k_{2}^{\omega}(R)+1, k_{1}^{\omega}(r)\right)}
$$

If $k_{1}^{\omega}(r) \leq k_{2}^{\omega}(R)$, then we leave $\mathcal{X}$ alone and set $\mathcal{Y}=\mathcal{X}$, corresponding to $\mathcal{N}_{i}\left(k_{2}^{\omega}(R)+\right.$ $\left.1, k_{1}^{\omega}(r)\right)=0$ for each $i$. Note that each rectangle $\mathcal{Y}$ in the new decomposition is a rectangle with height

$$
\prod_{i=1}^{k_{1}^{\omega}(r)} n_{\omega_{i}}^{-1} \leq r
$$

and we are trying to cover it by squares of side length $r$. Thus to give an efficient estimate on $N_{r}(\mathcal{Y})$ we need only worry about covering $\Pi_{x}(\mathcal{Y})$ and we can certainly do this using no more than

$$
\prod_{i \in \Lambda} A_{i}^{\mathcal{N}_{i}\left(k_{1}^{\omega}(r)+1, k_{2}^{\omega}(r)\right)}
$$

such squares. Combining the above estimates and using the fact that for all $i \in \Lambda$, $N_{i} \leq A_{i} B_{i}$ yields

$$
\begin{aligned}
& N_{r}\left(Q \cap F_{\omega}\right) \leq\left(\prod_{i \in \Lambda} B_{i}^{\mathcal{N}_{i}\left(k_{1}^{\omega}(R)+1, k_{2}^{\omega}(R)\right)}\right) \cdot\left(\prod_{i \in \Lambda} N_{i}^{\mathcal{N}_{i}\left(k_{2}^{\omega}(R)+1, k_{1}^{\omega}(r)\right)}\right) \\
& \leq \prod_{i \in \Lambda} A_{i \in \Lambda}^{\mathcal{N}_{i}\left(k_{2}^{\omega}(R)+1, k_{2}^{\omega}(r)\right)} B_{i}^{\mathcal{N}_{i}\left(k_{1}^{\omega}(R)+1, k_{1}^{\omega}(r)\right)}
\end{aligned}
$$

Now that this estimate has been established, the desired upper bound follows by careful algebraic manipulation. In particular,

$$
\begin{aligned}
N_{r}\left(Q \cap F_{\omega}\right) & \leq\left(\prod_{i \in \Lambda} A_{i}^{\mathcal{N}_{i}\left(k_{2}^{\omega}(R)+1, k_{2}^{\omega}(r)\right)}\right)\left(\prod_{i \in \Lambda} B_{i}^{\mathcal{N}_{i}\left(k_{1}^{\omega}(R)+1, k_{1}^{\omega}(r)\right)}\right) \\
& =\prod_{i \in \Lambda}\left(m_{i}^{\mathcal{N}_{i}\left(k_{2}^{\omega}(R)+1, k_{2}^{\omega}(r)\right)}\right)^{\log A_{i} / \log m_{i}}
\end{aligned}
$$

$$
\begin{aligned}
& \prod_{i \in \Lambda}\left(n_{i}^{\mathcal{N}_{i}\left(k_{1}^{\omega}(R)+1, k_{1}^{\omega}(r)\right)}\right)^{\log B_{i} / \log n_{i}} \\
& \leq\left(\prod_{i \in \Lambda} m_{i}^{\mathcal{N}_{i}\left(k_{2}^{\omega}(R)+1, k_{2}^{\omega}(r)\right)}\right)^{\max _{i \in \Lambda} \log A_{i} / \log m_{i}} \\
& \cdot\left(\prod_{i \in \Lambda} n_{i}^{\mathcal{N}_{i}\left(k_{1}^{\omega}(R)+1, k_{1}^{\omega}(r)\right)}\right)^{\max _{i \in \Lambda} \log B_{i} / \log n_{i}} \\
& =\left(\frac{\prod_{i \in \Lambda} m_{i}^{-\mathcal{N}_{i}\left(1, k_{2}^{\omega}(R)\right)}}{\prod_{i \in \Lambda} m_{i}^{-\mathcal{N}_{i}\left(1, k_{2}^{\omega}(r)\right)}}\right)^{\max _{i \in \Lambda} \log A_{i} / \log m_{i}} \\
& \cdot\left(\frac{\prod_{i \in \Lambda} n_{i}^{-\mathcal{N}_{i}\left(1, k_{1}^{\omega}(R)\right)}}{\prod_{i \in \Lambda} n_{i}^{-\mathcal{N}_{i}\left(1, k_{1}^{\omega}(r)\right)}}\right)^{\max _{i \in \Lambda} \log B_{i} / \log n_{i}} \\
& =\left(\frac{\prod_{i=1}^{k_{2}^{\omega}(R)} m_{\omega_{i}}^{-1}}{\prod_{i=1}^{k_{2}^{\omega}(r)} m_{\omega_{i}}^{-1}}\right)^{\max _{i \in \Lambda} \log A_{i} / \log m_{i}}\left(\frac{\prod_{i=1}^{k_{1}^{\omega}(R)} n_{\omega_{i}}^{-1}}{\prod_{i=1}^{k_{1}^{\omega}(r)} n_{\omega_{i}}^{-1}}\right)^{\max _{i \in \Lambda} \log B_{i} / \log n_{i}} \\
& \leq\left(\frac{R}{m_{\max }^{-1} r}\right)^{\max _{i \in \Lambda} \log A_{i} / \log m_{i}}\left(\frac{R}{n_{\max }^{-1} r}\right)^{\max _{i \in \Lambda} \log B_{i} / \log n_{i}}
\end{aligned}
$$

by (6.6.1) and (6.6.2) and so

$$
\leq m_{\max } n_{\max }\left(\frac{R}{r}\right)^{\max _{i \in \Lambda} \log A_{i} / \log m_{i}+\max _{i \in \Lambda} \log B_{i} / \log n_{i}}
$$

which proves that

$$
\operatorname{dim}_{\mathrm{A}} F_{\omega} \leq \max _{i \in \Lambda} \frac{\log A_{i}}{\log m_{i}}+\max _{i \in \Lambda} \frac{\log B_{i}}{\log n_{i}}
$$

and since $\omega \in \Omega$ was arbitrary this proves the desired result.

### 6.6.2.3 Proof of Theorem 6.3.2

In light of Theorem 6.3.1, all that remains is to prove the almost sure lower bound. In order to do this we identify a 'good set' of full measure within which we can prove the lower bound surely, similar to Theorem 6.2.2. First fix $i \in \Lambda$ which maximises $\log A_{i} / \log m_{i}$ and $j \in \Lambda$ which maximises $\log B_{j} / \log n_{j}$. Of course $i$ and $j$ may be different, and this is the more interesting case which leads to examples such as those in Section 6.3.2.

A first guess for the good set might be the set of strings containing arbitrarily long runs of $j$ followed by the same number of $i$. This is philosophically the correct approach, but does not work because the point where the string is required to change from $j$ to $i$ depends crucially on the stage one is at in the sequence. Since one may have to wait much longer than $O(n)$ steps to get a string of $n j$ s followed by $n$ is, by the time it occurs the eccentricity of the rectangles in the construction will be so large that switching from $j$ to $i$ after $n$ steps in the approximate square is not enough to obtain the desired tangent. A second approach might be to look for strings of $j$ s followed by is where the number of $j$ s depends on the starting point of the string (in fact the dependence would be linear), however, this approach also fails because one cannot guarantee that such strings exist infinitely often almost surely. Our solution
is to recognise that one needs a long string of $i$ s and a long string of $j$ s to get the necessary tangent, but these strings do not have to be next to each other.

The good set $G_{i, j} \subseteq \Omega$ is defined to be
$G_{i, j}=\left\{\omega=\left(\omega_{1}, \omega_{2}, \ldots\right) \in \Omega \mid\right.$ there exists a sequence of pairs $\left(R_{l}, n_{l}\right) \in(0,1) \times \mathbb{N}$ with $R_{l} \rightarrow 0$ and $n_{l} \rightarrow \infty$ with $n_{l} \leq k_{2}^{\omega}\left(R_{l}\right)-k_{1}^{\omega}\left(R_{l}\right)$ such that

$$
\begin{gathered}
\omega_{i^{\prime}}=j \text { for all } i^{\prime}=k_{1}^{\omega}\left(R_{l}\right)+1, \ldots, k_{1}^{\omega}\left(R_{l}\right)+n_{l} \text { and } \\
\left.\omega_{i^{\prime}}=i \text { for all } i^{\prime}=k_{2}^{\omega}\left(R_{l}\right)+1, \ldots, k_{2}^{\omega}\left(R_{l}\right)+n_{l}\right\} .
\end{gathered}
$$

Lemma 6.6.3. Let $i, j \in \Lambda$ be the maximising indices, as above. The good set has full measure in $\Omega$, i.e. $\mathbb{P}\left(G_{i, j}\right)=1$.

Proof. For $n \in \mathbb{N}$ let

$$
l(n)=\left\lceil\frac{-\log 2}{\log \left(1-p_{j}^{n} p_{i}^{n}\right)}\right\rceil
$$

and let

$$
\theta=\frac{\max _{i \in \Lambda} \log n_{i}}{\min _{i \in \Lambda} \log m_{i}}>1
$$

Also, for $n \in \mathbb{N}$ and $m=1, \ldots, l(n)+1$, we define numbers $K(n), K_{n}(m) \in \mathbb{N}$ inductively by

$$
\begin{aligned}
K(1) & =1, \\
K_{n}(1) & =K(n), \\
K_{n}(m+1) & =\theta K_{n}(m)+n \\
K(n+1) & =K_{n}(l(n)+1) .
\end{aligned}
$$

These numbers are arranged as follows and will form partitions of the natural numbers:

$$
\cdots<K(n)=K_{n}(1)<K_{n}(2)<\cdots<K_{n}(l(n)+1)=K(n+1)<\cdots .
$$

For $\omega \in \Omega, n \in \mathbb{N}$ and $m \in\{1, \ldots, l(n)\}$, let $K_{n}^{\omega}(m)=k_{2}^{\omega}(R)$ for

$$
R=\prod_{i=1}^{K_{n}(m)} n_{\omega_{i}}^{-1}
$$

and let

$$
\begin{array}{r}
\mathscr{E}_{n}(m)=\left\{\omega=\left(\omega_{1}, \omega_{2}, \ldots\right) \in \Omega \mid \omega_{i^{\prime}}=j \text { for all } i^{\prime}=K_{n}(m)+1, \ldots, K_{n}(m)+n\right. \\
\text { and } \left.\omega_{i^{\prime}}=i \text { for all } i^{\prime}=K_{n}^{\omega}(m)+1, \ldots, K_{n}^{\omega}(m)+n\right\}
\end{array}
$$

observing that $n \ll K_{n}^{\omega}(m)-K_{n}(m)$ for large $n$. Finally, let

$$
\mathscr{E}_{n}=\bigcup_{m=1}^{l(n)} \mathscr{E}_{n}(m)
$$

It follows from these definitions that

$$
\bigcap_{k \in \mathbb{N}} \bigcup_{n>k} \mathscr{E}_{n} \subseteq G_{i, j}
$$

and, moreover, the events $\left\{\mathscr{E}_{n}\right\}_{n \in \mathbb{N}}$ are independent because they concern properties of $\omega$ at disjoint parts of the sequence. This can be seen since $K_{n}^{\omega}(m)+n \leq \theta K_{n}(m)+n=$ $K_{n}(m+1)$. Also, for a fixed $n$, the events $\left\{\mathscr{E}_{n}(m)\right\}_{m=1}^{l(n)}$ are independent. We have

$$
\mathbb{P}\left(\mathscr{E}_{n}\right)=1-\prod_{m=1}^{l(n)} \mathbb{P}\left(\Omega \backslash \mathscr{E}_{n}(m)\right)
$$

$$
\begin{aligned}
& =1-\left(1-p_{j}^{n} p_{i}^{n}\right)^{l(n)} \\
& \geq 1-\left(1-p_{j}^{n} p_{i}^{n}\right)^{-\log 2 / \log \left(1-p_{j}^{n} p_{i}^{n}\right)}=1 / 2
\end{aligned}
$$

Therefore

$$
\sum_{n \in \mathbb{N}} \mathbb{P}\left(\mathscr{E}_{n}\right) \geq \sum_{n \in \mathbb{N}} 1 / 2=\infty
$$

and since the events $\mathscr{E}_{n}$ are independent the Borel-Cantelli Lemma implies that

$$
\mathbb{P}\left(G_{i, j}\right) \geq \mathbb{P}\left(\bigcap_{k \in \mathbb{N}} \bigcup_{n>k} \mathscr{E}_{n}\right)=1
$$

as required.
We can now prove Theorem 6.3.2. Fix $\omega \in G_{i, j}$ and consider a column of the defining pattern for $\mathbb{I}_{j}$ containing a maximal number of chosen rectangles $B_{j}$. If there is more than one such column, then choose one arbitrarily. This column induces a natural IFS of similarities on the unit interval, consisting of $B_{j}$ maps with contraction ratios $n_{j}^{-1}$ and satisfying the OSC. Let $K_{j}$ denote the self-similar attractor of this IFS and for $l \in \mathbb{N}$, let $K_{j}^{l}$ denote the $l$ th level of the construction, i.e. the union of $\left(B_{j}\right)^{l}$ intervals of length $n_{j}^{-l}$ corresponding to images of $[0,1]$ under compositions of $l$ maps from the induced column IFS. Also, consider the IFS $\mathbb{I}_{i}$ and let $\Pi_{x}\left(F_{\underline{i}}\right)$ denote the projection onto the first coordinate of the attractor of $\mathbb{I}_{i}$, which is also a self-similar set satisfying the OSC. We will now show that $\Pi_{x}\left(F_{\underline{i}}\right) \times K_{j}$ is a very weak tangent to $F_{\omega}$.

For a random approximate square $Q$, let $T^{Q}$ be the uniquely defined affine map given by the composition of a non-negative diagonal matrix and a translation which maps $Q$ to $[0,1] \times[0,1]$. Let $\left(R_{l}, n_{l}\right) \in(0,1) \times \mathbb{N}$ be a pair which together with $\omega$ satisfy the definition of $G_{i, j}$ and consider the family of random approximate $R_{l}$ squares. Since $\omega_{i^{\prime}}=j$ for all $i^{\prime}=k_{1}^{\omega}\left(R_{l}\right)+1, \ldots, k_{1}^{\omega}\left(R_{l}\right)+n_{l}$, by keeping track of the maximising column mentioned above we can choose $Q$ satisfying

$$
T^{Q}(Q) \subseteq \Pi_{x}\left(F_{\sigma_{2}^{k_{2}^{\omega}(R)}(\omega)}\right) \times K_{j}^{n_{l}}
$$

Moreover, by decomposing $K_{j}^{n_{l}}$ into its basic intervals of length $n_{j}^{-l}$, we see that within each corresponding rectangle in $T^{Q}(Q)$ (which has height $n_{j}^{-l}$ ), one finds affinely scaled copies of $F_{\sigma^{k_{2}^{\omega}(R)}(\omega)}$. Since $\omega_{i^{\prime}}=i$ for all $i^{\prime}=k_{2}^{\omega}\left(R_{l}\right)+1, \ldots, k_{2}^{\omega}\left(R_{l}\right)+n_{l}$, this implies that $T^{Q}(Q)$ occupies every basic rectangle at the $n^{l}$ th stage of the construction of $\Pi_{x}\left(F_{\underline{i}}\right) \times K_{j}$. Since such rectangles have base $m_{i}^{-n_{l}}$ and height $n_{j}^{-n_{l}}$ this yields

$$
d_{\mathscr{H}}^{l}\left(T^{Q}(Q), \Pi_{x}\left(F_{\underline{i}}\right) \times K_{j}\right) \leq\left(m_{i}^{-2 n_{l}}+n_{j}^{-2 n_{l}}\right)^{1 / 2}
$$

This is sufficient to show that $\Pi_{x}\left(F_{\underline{i}}\right) \times K_{j}$ is a very weak tangent to $F_{\omega}$ because we can choose our sequence of maps to be $T^{Q}$ for a sequence of random approximate squares $Q$ satisfying the above inequality, but with $n_{l} \rightarrow \infty$, giving the desired convergence. Moreover, for any random approximate $R$-square $Q$ we have

$$
R^{-1}|x-y| \leq\left|T^{Q}(x)-T^{Q}(y)\right| \leq n_{\max } R^{-1}|x-y| \quad\left(x, y \in \mathbb{R}^{2}\right)
$$

and so the maps satisfy the conditions required in Definition 1.7.11. It follows that

$$
\begin{align*}
\operatorname{dim}_{\mathrm{A}} F_{\omega} & \geq \operatorname{dim}_{\mathrm{A}}\left(\Pi_{x}\left(F_{\underline{i}}\right) \times K_{j}\right) & & \text { by Lemma } 1.7 .13 \\
& \geq \operatorname{dim}_{\mathrm{H}}\left(\Pi_{x}\left(F_{\underline{i}}\right) \times K_{j}\right) & & \\
& \geq \operatorname{dim}_{\mathrm{H}} \Pi_{x}\left(F_{\underline{i}}\right)+\operatorname{dim}_{\mathrm{H}} K_{j} & & \text { by [F6, Corollary } 7.4] \\
& =\frac{\log A_{i}}{\log m_{i}}+\frac{\log B_{j}}{\log n_{j}} & &
\end{align*}
$$

as required.

### 6.6.3 Proof of Theorem 6.4.2

Let $s=\sup _{u \in \Omega} \operatorname{dim}_{\mathrm{A}} F_{u}$. We will show that the set

$$
A=\left\{\omega \in \Omega \mid \operatorname{dim}_{\mathrm{A}} F_{\omega} \geq s\right\}
$$

is residual, from which Theorem 6.4.2 follows.
First we recall some useful functions. Consider $F_{\omega}=F(\omega)$ as a function between two metric spaces i.e. $F:(\Omega, d) \rightarrow\left(\mathcal{K}\left(\mathbb{R}^{k}\right), d_{\mathscr{H}}\right)$ and observe that it is continuous. For $x \in \mathbb{R}^{k}$ and $R \in(0,1]$ let $\beta_{x, R}^{o}: \mathcal{K}\left(\mathbb{R}^{k}\right) \rightarrow \mathcal{P}\left(\mathbb{R}^{k}\right)$ be given by

$$
\beta_{x, R}^{o}(F)=B^{o}(x, R) \cap F,
$$

where $B^{o}(x, R)$ is the open ball centered at $x$ with radius $R$, and $\mathcal{P}\left(\mathbb{R}^{k}\right)$ is the power set of $\mathbb{R}^{k}$ (the images need not be compact). Also, for $r \in(0,1]$, let $\mathscr{M}_{r}(F)$ denote the maximum number of closed sets in an $r$-packing of $F \subseteq \mathbb{R}^{k}$, where an $r$-packing of $F$ is a pairwise disjoint collection of closed balls centered in $F$ of radius $r$. It was shown in [Fr2, Lemma 5.2] that the map $\mathscr{M}_{r} \circ \beta_{x, R}^{o}: \mathcal{K}\left(\mathbb{R}^{k}\right) \rightarrow \mathbb{R}$ is lower semicontinuous. It thus follows from the continuity of $F$, that the function $\mathscr{C}:=\mathscr{M}_{r} \circ \beta_{x, R}^{o} \circ F: \Omega \rightarrow \mathbb{R}$ is lower semicontinuous. We have

$$
\begin{aligned}
& A=\left\{\omega \in \Omega \mid \text { for all } n \in \mathbb{N}, C, \rho>0, \text { there exists } x \in \mathbb{R}^{k} \text { and } 0<r<R<\rho,\right. \\
& \text { such that } \left.M_{r}\left(B^{o}(x, R) \cap F_{\omega}\right)>C\left(\frac{R}{r}\right)^{s-1 / n}\right\} \\
& =\cap \cap \cap \cup
\end{aligned}
$$

The set $\mathscr{C}^{-1}\left(\left(C(R / r)^{s-1 / n}, \infty\right)\right)$ is open by the lower semicontinuity of $\mathscr{C}^{-1}$ and therefore $A$ is a $\mathcal{G}_{\delta}$ subset of $\Omega$.

To complete the proof that $A$ is residual, it remains to show that $A$ is dense in $\Omega$. For $n \in \mathbb{N}$ let

$$
A_{n}=\left\{\omega \in \Omega: \operatorname{dim}_{\mathrm{A}} F_{\omega} \geq s-1 / n\right\}
$$

It follows that $A_{n}$ is $\mathcal{G}_{\delta}$ by the same argument as above, and since

$$
A=\bigcap_{n \in \mathbb{N}} A_{n}
$$

it follows from the Baire Category Theorem that it suffices to show that $A_{n}$ is dense in $\Omega$ for all $n$. Let $n \in \mathbb{N}, \omega=\left(\omega_{1}, \omega_{2}, \ldots\right) \in \Omega$, and $\varepsilon>0$. Let $u=\left(u_{1}, u_{2}, \ldots\right) \in \Omega$ be such that $\operatorname{dim}_{\mathrm{A}} F_{u}>s-1 / n$, choose $l \in \mathbb{N}$ such that $2^{-l}<\varepsilon$ and let $v=$ $\left(\omega_{1}, \ldots, \omega_{l}, u_{1}, u_{2}, \ldots\right)$. It follows that $d(v, \omega)<\varepsilon$ and, furthermore,

$$
F_{v}=\bigcup_{\phi \in \mathbf{C}_{\omega}^{l}} f\left(\phi, F_{u}\right) .
$$

Since, for all $\phi_{0} \in W_{i}$ and $i \in \Lambda$ the map $f\left(\phi_{0}, \cdot\right)$ is a bi-Lipschitz contraction, it follows from basic properties of the Assouad dimension that $\operatorname{dim}_{\mathrm{A}} F_{v} \geq \operatorname{dim}_{\mathrm{A}} F_{u}>s-1 / n$ and so $v \in A_{n}$, proving that $A_{n}$ is dense.

### 6.6.4 Proof of Theorem 6.5.1

The upper bound is trivial, and we prove the lower bound here. As we condition on non-extinction, we may assume there exists $x \in F$ and hence also a sequence of nested compact cubes $Q_{k}^{x}$ that each contain $x$, have sidelengths equal to $n^{-k}$ and are such that $x=\cap_{k \in \mathbb{N}} Q_{k}^{x}$. We start by introducing some additional notation. At the $(k+1)$ th stage in the construction of $F$ the cube $Q_{k}^{x}$ was split into $N=n^{d}$ compact cubes. We will index these cubes by $\mathcal{I}=\{1,2, \ldots, N\}$ (ordered lexicographically by their midpoints) and keep track of the tree structure of subcubes by words that give their position in the iteration. That is for words of length $m$ we write $Q_{k}^{x}(w)$, where $w \in \mathcal{I}^{m}$, to mean the uniquely determined cube at the $(k+m)$ th stage of the construction lying inside $Q_{k}^{x}$ at position $w$ starting from $Q_{k}^{x}$. We also write $Q_{k}^{x}=Q_{k}^{x}(\varnothing)$. Let $p_{\neg e}>0$ be the probability that any cube which has survived up to some point in the construction does not go on to become extinct. Due to the independence and homogeneity of the construction, this is the same for any surviving cube at any level. Moreover, it is strictly positive due to our assumption on $p$. The following lemma is similar in spirit to Lemma 6.6.2.

Lemma 6.6.4. Let $x$ be as above. Almost surely there exists an increasing sequence of natural numbers $\left(M_{i}\right)_{i=1}^{\infty}$ such that, for all $i \in \mathbb{N}$, all cubes

$$
Q_{M_{i}}^{x}(w) \text { where } w \in\{\varnothing\} \cup \bigcup_{a=1}^{i} \mathcal{I}^{a}
$$

survive and each of the last cubes $\left\{Q_{M_{i}}^{x}(w)\right\}_{w \in \mathcal{I}^{i}}$ in this iteration do not become extinct.

Proof. Let $m, r \in \mathbb{N}$ be given. First we establish the probability of all cubes $Q_{r}^{x}(w)$ for $w \in\{\varnothing\} \cup \bigcup_{a=1}^{m} \mathcal{I}^{a}$ surviving and not becoming extinct. By the homogeneity of the construction the probability of those cubes surviving is independent of $r$ and is the number of '(weighted) coin tosses' needed for all cubes to survive. As we are given that at least one path (the one for $x$ ) survives, the number of 'tosses' is

$$
L_{N}^{m}=\sum_{a=1}^{m}\left(N^{a}-1\right)=\frac{N^{m+1}-N}{N-1}-m
$$

and so the probability of all of the cubes surviving is $p^{L_{N}^{m}}$. We also have to take into account the non-extinction criteria. Given that they have survived to the $(r+m)$ th level, the probability that all of the cubes $\left\{Q_{r}^{x}(w)\right\}_{w \in \mathcal{I}^{m}}$ will not become extinct is $p_{\neg e}^{N^{m}-1}$. Thus the probability of all cubes $Q_{r}^{x}(w)$ for $w \in\{\varnothing\} \cup \bigcup_{a=1}^{m} \mathcal{I}^{a}$ surviving and not becoming extinct is $\widehat{p}_{m}=p^{L_{N}^{m}} p_{\neg e}^{N^{m}-1}$. Now define $l(m+1)=l(m)+k(m)$, where $l(1)=1$ and

$$
k(m)=m\left\lceil\frac{-\log 2}{\log \left(1-\widehat{p}_{m}\right)}\right\rceil .
$$

Let $\mathcal{E}_{m}$ be the event

$$
\begin{aligned}
\mathcal{E}_{m}= & \{\text { for at least one of } j \in\{0, m, 2 m, \ldots, k(m)-m\} \text { we have that all } \\
& \left.Q_{l(m)+j}^{x}(v) \text { survive and are non-extinct in the limit for } v \in\{\varnothing\} \cup \bigcup_{a=1}^{m} \mathcal{I}^{a}\right\} .
\end{aligned}
$$

Given that the cubes $Q_{k}^{x}$ all survive, it is evident that the behaviour of one $k(m) / m$ block is independent of the next and so

$$
\mathbb{P}\left(\mathcal{E}_{m}\right)=1-\left(1-\widehat{p}_{m}\right)^{k(m) / m} \geq 1 / 2
$$

Lemma 6.6.4 now follows immediately by the Borel Cantelli Lemma and the fact that $\mathcal{E}_{m}$ are easily seen to be independent.

Using Lemma 6.6 .4 we now show that, almost surely and conditioned on nonextinction, that $[0,1]^{d}$ is a weak tangent to $F$. The required lower bound on the dimension of $F$ then follows from Lemma 1.7.13. Let $T_{i}$ be the homothetic similarity that maps the cube $Q_{M_{i}}^{x}$ to $X$. By Lemma 6.6 .4 we have that, almost surely conditioned on non-extinction, each of the subcubes $Q_{M_{i}}^{x}(v)$ for $v \in \mathcal{I}^{i}$ survive and are non-empty in the limit. Now $T_{i}(F) \cap X$ is the union of all blow ups of these subcubes under $T_{i}$ and, since each blown up subcube contains at least one point and has diameter $\sqrt{d} n^{-i}$, it follows that

$$
d_{\mathscr{H}}\left(T_{i}(F) \cap X, X\right) \leq \sqrt{d} n^{-i}
$$

and so $d_{\mathscr{H}}\left(T_{i}(F) \cap X, X\right) \rightarrow 0$ as $i \rightarrow \infty$ as required.
The optimal projection result now follows as a simple consequence of $F$ being almost surely of full dimension. In particular, for all $k \leq d$ and $\Pi \in \Pi_{d, k}$ we have

$$
F \subset \Pi F \times \Pi^{\perp}
$$

where $\Pi^{\perp}$ is the $(d-k)$-dimensional orthogonal complement of (the $k$ dimensional subspace identified with) $\Pi$, and so by basic properties of how Assouad dimension behaves concerning products [Ro, Lemma 9.7] it follows that, for all realisations where $\operatorname{dim}_{\mathrm{A}} F=d$,

$$
d=\operatorname{dim}_{\mathrm{A}} F \leq \operatorname{dim}_{\mathrm{A}} \Pi F+\operatorname{dim}_{\mathrm{A}} \Pi^{\perp}=\operatorname{dim}_{\mathrm{A}} \Pi F+d-k
$$

which gives $\operatorname{dim}_{\mathrm{A}} \Pi F \geq k$. The opposite inequality is trivial and since $\operatorname{dim}_{\mathrm{A}} F=d$ occurs almost surely conditioned on non-extinction, the result follows.

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[^0]:    ${ }^{1}$ ©User:Griffinstorm / Wikimedia Commons / CC-BY-SA-4.0
    ${ }^{2}$ by NASA International Space Station Imagery

[^1]:    ${ }^{3}$ Note that the Hausdorff measure is an outer measure as opposed to a bona fide measure and we treat the Hausdorff measure mostly as a set function. Countable additivity may fail and in the rare cases that we use this property it is understood to mean for all measurable sets w.r.t. the Hausdorff measure. See also the discussion in [Rog].

[^2]:    The content of this chapter is based on On the dimensions of attractors of random self-similar graph directed iterated function systems and will appear in Journal of Fractal Geometry, see [T1].
    ${ }^{1}$ We note that we were originally unaware of the work in [RU], but we are dealing with the removal of separation conditions that were not considered by the aforementioned authors.

[^3]:    The content of this chapter is based on The box dimension of random box-like self-affine sets by the author, see [T2].

[^4]:    ${ }^{1}$ We note that this result is also proven in greater generality in $[\mathrm{RU}]$ but we will prove it here to set the scene for gauge functions in the following sections.

