

\mathbb{Z}_4 -codes and their Gray map images as orthogonal arrays

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Abstract

A classic result of Delsarte connects the strength (as orthogonal array) of a linear code with the minimum weight of its dual: the former is one less than the latter.

Since the paper of Hammons *et al.*, there is a lot of interest in codes over rings, especially in codes over \mathbb{Z}_4 and their (usually non-linear) binary Gray map images.

We show that Delsarte's observation extends to codes over arbitrary finite commutative rings with identity. Also, we show that the strength of the Gray map image of a \mathbb{Z}_4 code is one less than the minimum Lee weight of its Gray map image.

1 Introduction

Delsarte [2], see also [4, pp. 54–55], showed that the strength (as orthogonal array) of a linear code over a field is one less than the minimum weight of the dual code. Two natural questions are: what happens for codes over rings? and what happens for non-linear codes?

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The first question can be answered completely. We show that a linear code over any finite commutative ring with identity has strength one less than the minimum weight of its dual (Theorem 3.1).

The two questions are related. In a groundbreaking paper, Hammons *et al.* [3] showed that certain non-linear codes which have more codewords (and thus better error-correcting capabilities) than any linear code are images of linear codes over \mathbb{Z}_4 under the non-linear *Gray map*. (These include the Nordstrom–Robinson, Preparata and Kerdock codes.) So we are led to investigate the relationship between the strength of a linear \mathbb{Z}_4 -code and the strength of its Gray map image. We show that the strength of the Gray map image of a linear \mathbb{Z}_4 -code is one less than the minimum Lee weight of the dual \mathbb{Z}_4 -code. As a consequence, if a linear \mathbb{Z}_4 -code C has strength t , then the strength of its Gray map image lies between t and $2t - 1$.

2 Definitions and preliminaries

We will begin with some background material on rings and modules, codes over \mathbb{Z}_4 , the Gray map and orthogonal arrays.

2.1 Rings and modules

All our rings will be finite commutative rings with identity, and our modules are unital. A (linear) code over a ring R is a submodule of the free module R^n ; the dual of C is, as usual,

$$C^\perp = \{(x_1, \dots, x_n) \in R^n : (\forall c_1, \dots, c_n \in C) x_1c_1 + \dots + x_nc_n = 0\}.$$

The definitions of zero-divisor, unit, and ideal in a ring, and annihilator $\text{Ann}(M)$ of a module, are standard: see [5, 6].

A ring is *local* if it has a unique maximal ideal, and *semilocal* if it has only finitely many maximal ideals. If I is an ideal of a ring R , the *I -adic completion* of R is the inverse limit of the rings R/I^n over natural numbers n , relative to the natural projections from R/I^n to R/I^m for $m \leq n$. We use the following result ([6, p. 62]):

Theorem 2.1. *Let R be a semilocal ring with maximal ideals M_1, \dots, M_r , and set $I = M_1 \cdots M_r$ (the radical of R). Then the I -adic completion of R decomposes as*

$$\hat{R} = \hat{R}_1 \times \cdots \times \hat{R}_r,$$

where R_i is the localisation of R at M_i , and \hat{R}_i is its M_i -adic completion.

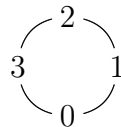
Now let R be any finite commutative ring with identity. Clearly R is semilocal, and satisfies the descending chain condition on ideals; so its radical I is nilpotent, say $I^k = \{0\}$. (See [5, p. 430].) Thus the I -adic completion of R is equal to $R/I^k = R$. Thus, every such ring R is a direct sum of local rings.

2.2 Codes

A code C of length n over a ring R (commutative with identity) is a submodule of the free module R^n . Just as in the classical theory, codewords are n -tuples from R , and addition and scalar multiplication are coordinatewise. We define the *Hamming distance* $d_H(x, y)$, where x and y are the codewords in C , as the number of coordinate places in which x and y differ. The *Hamming weight* w_H of a codeword x is defined to be the number of non-zero entries of x , $w_H(x) = d_H(x, 0)$, where 0 is the all zero word in C . The minimum Hamming distance of C is equal to the minimum Hamming weight among all non-zero codewords in C , which we denote by $w_H(C)$.

2.3 Codes over \mathbb{Z}_4

The four elements of \mathbb{Z}_4 are arranged around a circle. We define the *Lee distance* d_L as the number of steps apart they are between each other:

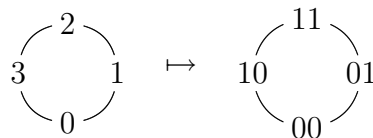


This is extended to codewords over \mathbb{Z}_4 by the rule $d_L(a, b) = \sum_i d_L(a_i, b_i)$. Similarly, the *Lee weight* of a codeword x in C is $w_L(x) = d_L(x, 0)$. Moreover, the minimum Lee distance of C is equal to the minimum Lee weight among all non-zero codewords in C ; this is denoted $w_L(C)$.

The *Gray map*, takes \mathbb{Z}_4 to \mathbb{Z}_2^2 by

$$0 \mapsto 00, \quad 1 \mapsto 01, \quad 2 \mapsto 11, \quad 3 \mapsto 10,$$

and maps \mathbb{Z}_4^n to \mathbb{Z}_2^{2n} coordinatewise. We denote the Gray map image of a linear code C as C' . Note that C' is usually non-linear.



The Lee weight of $v \in C$ is the Hamming weight of $v' \in C'$, also $d_L(v, w) = d_H(v', w')$, where $v, w \in C$ and $v', w' \in C'$. In other words, the Gray map, though non-linear, is an isometry from \mathbb{Z}_4^n to \mathbb{Z}_2^{2n} .

2.4 Orthogonal arrays

An $N \times k$ array A with entries from an alphabet F is said to be an *orthogonal array* with f levels, where $|F| = f$, strength t and index λ (for some t in the range $0 \leq t \leq k$) if every $N \times t$ subarray of A contains each t -tuple based on F exactly λ times as a row. We say that a code D is an *orthogonal array* of strength t and index λ if the codewords of D make up the rows of an orthogonal array A as described above. D may or may not be linear.

Clearly, if D is an orthogonal array of strength t then it is also an orthogonal array of strength s for any $s \leq t$; its index is given by $\lambda_s = |F|^{t-s}\lambda$.

3 Codes over rings as orthogonal arrays

In this section, we prove that Delsarte's theorem extends to codes over rings.

Theorem 3.1. *Let C be a linear code over a finite commutative ring R with identity. Then $\text{Str}(C) = w_H(C^\perp) - 1$.*

Proof . The proof proceeds in several steps. First, we define two properties which a ring may have:

- R has property (+) if, whenever I is an ideal of R with $I \neq R$, we have $\text{Ann}(I) \neq 0$.
- R has property $(*)_t$ if given any $N \times t$ matrix A over ring R with linearly independent columns, the row space of A , $\rho(A)$, is equal to R^t .

Now the conclusion of the main theorem holds if R has property $(*)_t$. For suppose that the minimum weight of C^\perp is d . Let M_C be the $n \times N$ matrix whose rows are all the codewords in C , where $N = |C|$. By assumption, any $d - 1$ columns of M_C are linearly independent, since a linear dependence relation would give a word of weight at most $d - 1$ in C^\perp . Property $(*)_{d-1}$ shows that the row space of the submatrix formed by these columns is R^{d-1} , so that any $(d - 1)$ -tuple occurs in these positions; and the number of times it occurs is equal to the number of codewords with zeros in these positions, since these words form a coset of the submodule consisting of such codewords. Thus M_C is an orthogonal array of strength $d - 1$.

Next we show, by induction on t , that property (+) implies property $(*)_t$ for all t .

To start the induction, note that property $(*)_1$ asserts that if a is an element of R with $\text{Ann}(a) = \{0\}$, then the ideal generated by a is the whole of R ; this is the contrapositive of Property $(+)$ for this ideal.

Now, suppose that $(+)$ implies $(*)_{t-1}$ and let R be a ring satisfying $(+)$ and A an $N \times t$ matrix over R with linearly independent columns. By the induction hypothesis, if \check{A} is the matrix obtained from A by deleting the last column, then $\rho(\check{A}) = R^{t-1}$. This means that, for $1 \leq i \leq t-1$, $\rho(A)$ contains $(0, \dots, 1, 0, \dots, 0, a_i)$ where 1 is in position i and $a_i \in R$. Let $I = \{r \in R : (0, \dots, 0, r) \in \rho(A)\}$. Then, it is easy to check that I is an ideal of R . If it is equal to R , then clearly $\rho(A) = R^t$. So suppose not. Then there is a non-zero element b annihilating I . We claim that

$$\rho(A) \subseteq \{(x_1, \dots, x_t) \in R^t : b(x_t - \sum_{i=1}^{t-1} a_i x_i) = 0\}.$$

For take $(x_1, \dots, x_t) \in \rho(A)$. I know that $(0, \dots, 0, x_i, 0, \dots, 0, a_i x_i) \in \rho(A)$, for $i = 1, \dots, t-1$. I can then subtract the sum of $(0, \dots, 0, x_i, 0, \dots, 0, a_i x_i)$, for all $i = 1, \dots, t-1$, from (x_1, \dots, x_t) , and I get $(0, \dots, 0, x_t - \sum_{i=1}^{t-1} a_i x_i) \in \rho(A)$; so $(x_t - \sum_{i=1}^{t-1} a_i x_i) \in I$, whence $b(x_t - \sum_{i=1}^{t-1} a_i x_i) = 0$. Thus, the vectors in $\rho(A)$ satisfy a non-trivial relation. In particular, this relation holds for the entries in each row of A ; that is, it holds for the columns of A . So the columns are not linearly independent, contrary to assumption.

Our task now is to show that every finite ring R (commutative with identity) has property $(+)$. We observed after Theorem 2.1 that R is a direct product of finite local rings.

Suppose first that R is a local ring, with unique maximal ideal J . Again by finiteness, every element of J is nilpotent, and so there exists an integer k such that $J^k = \{0\}$ (and we can assume that k is minimal with this property). Take $a \in J^{k-1}$ with $a \neq 0$. Then $a \in \text{Ann}(J)$. Since every proper ideal I is contained in J , we have $a \in \text{Ann}(I)$, as required.

Now suppose that $R = R_1 \times \dots \times R_r$, where R_i has property $(+)$ for $i = 1, \dots, r$. Let I be a proper ideal of R . Then $I = I_1 \times \dots \times I_r$, where I_i is an ideal of R_i , and is a proper ideal for at least one value of i , say $i = j$. By $(+)$, R_j contains a non-zero element a_j which annihilates I_j . Then $(0, \dots, 0, a_j, 0, \dots, 0)$ annihilates I .

The theorem is proved. □

We conclude this section with a similar result.

Proposition 3.2. *Let R be a finite commutative ring with identity, in which $|\text{Ann}(I)| = |R|/|I|$ holds for any ideal I . Then, for any code C of length n over R , we have $|C^\perp| = |R|^n/|C|$.*

We omit the proof, which is a straightforward induction. Note that many finite rings (for example, finite fields, rings \mathbb{Z}_n) have this property. An example of a ring not satisfying the property is the ring generated by \mathbb{Z}_4 and an element a satisfying $2a = a^2 = 0$. Then $|R| = 8$; the set $\{0, 2, a, a + 2\}$ is an ideal which is equal to its annihilator.

4 Orthogonal arrays and the Gray map

If D is an orthogonal array, then we denote the strength of D as $\text{Str}(D)$.

4.1 The main theorem

The main result is that the strength of the Gray map image of a linear \mathbb{Z}_4 -code C is one less than the minimum weight of C^\perp .

Theorem 4.1. *Let C be a linear \mathbb{Z}_4 -code and C' its Gray map image. Then $\text{Str}(C') = w_L(C^\perp) - 1$, where w_L denotes the minimum Lee weight.*

Proof . The code C' is the Gray map image of a \mathbb{Z}_4 -linear code, and hence is distance invariant [2]. Thus, its weight enumerator is equal to its distance enumerator, and the *dual distance* is defined; this is the index of the first non-vanishing term in the MacWilliams transform of the weight enumerator of C' . Delsarte's results show that the strength of C' is one less than its dual distance.

On the other hand, the Hamming weight enumerator of C' is equal to the Lee weight enumerator of C . Since the MacWilliams relations hold for the Lee weight enumerators of a linear \mathbb{Z}_4 code and its dual [3], the dual distance of C' is equal to the minimum weight of C^\perp , and the theorem is proved. \square

4.2 A consequence

As a corollary, we have an inequality connecting the strength of a linear \mathbb{Z}_4 -code and that of its Gray map image.

Corollary 4.2. *Let C be a \mathbb{Z}_4 -code and C' its Gray map image. If $\text{Str}(C) = t$, then $t \leq \text{Str}(C') \leq 2t + 1$.*

Proof . If $\text{Str}(C) = t$, then $w_H(C^\perp) = t + 1$. Hence $t + 1 \leq w_L(C^\perp) \leq 2t + 2$, since any non-zero coordinate of a word in C^\perp contributes either 1 or 2 to its Lee weight. The result follows since $\text{Str}(C') = w_L(C^\perp) - 1$ by the main theorem. \square

This corollary leads us to the following question: Given t, t' such that $t \leq t' \leq 2t + 1$, can we find a code C such that $\text{Str}(C) = t$ and $\text{Str}(C') = t'$?

5 Further problems

Connections such as we have explored here probably do not exist between codes over \mathbb{Z}_n or more general rings. Although the Lee distance is defined over \mathbb{Z}_n , for any n , there is no analogue of the Gray map.

Another possible topic would be to find connections between these codes as orthogonal arrays and the t -designs formed by the supports of their code-words.

We also have the problem of deciding which pairs (t, t') are representable as the strengths of a \mathbb{Z}_4 -code and its Gray map image, as discussed after Corollary 4.2.

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