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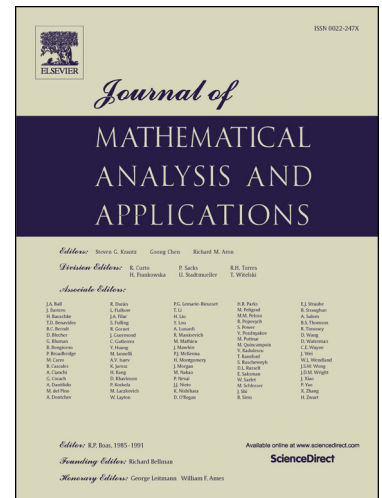
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MIXED MOMENTS AND LOCAL DIMENSIONS OF MEASURES

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ABSTRACT. We analyse the asymptotic behaviour of the mixed moments of Borel probability measures on $[0, 1]^d$. In particular, we prove that the asymptotic behaviour of the mixed moments of a measure is intimately related to the local dimensions of the measure.

1. STATEMENT OF RESULTS.

The purpose of this paper is to analyse the limiting behaviour of the mixed moments of measures. In particular, we show that there is a surprising relationship between the asymptotic behaviour of the mixed moments of a measure and the so-called local dimensions of the measure.

1.1. Local dimensions of measures. If X is a metric space, then we write $\mathcal{P}(X)$ for the family of Borel probability measures on X , i.e. metric space X , write

$$\mathcal{P}(X) = \left\{ \mu \mid \mu \text{ is a Borel probability measure on } X \right\}.$$

Next, if $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we define the lower and upper local dimension of μ at x by

$$\underline{\dim}_{\text{loc}}(\mu; x) = \liminf_{r \searrow 0} \frac{\log \mu(B(x, r))}{\log r} \quad (1.1)$$

and

$$\overline{\dim}_{\text{loc}}(\mu; x) = \limsup_{r \searrow 0} \frac{\log \mu(B(x, r))}{\log r}, \quad (1.2)$$

respectively. If the lower and upper local dimension of μ at x coincide, then we write $\dim_{\text{loc}}(\mu; x)$ for the common value, i.e. we write

$$\dim_{\text{loc}}(\mu; x) = \lim_{r \searrow 0} \frac{\log \mu(B(x, r))}{\log r} \quad (1.3)$$

provided the limit exists. The detailed study of the local dimensions of measures is known as multifractal analysis and has received enormous interest the past 20 years; the reader is referred to the texts [Fa, Pe] for a more thorough discussion of this. It is now generally believed by experts that local dimensions provide important information about the geometric properties of measures.

1.2. Absolute moments of measures. For $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $q > 0$, we define the q 'th absolute moment of μ by

$$M_q(\mu) = \int |\mathbf{x}|^q d\mu(\mathbf{x}); \quad (1.4)$$

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here and below we use the following notation, namely, if $\mathbf{x} \in \mathbb{R}^d$, then $|\mathbf{x}|$ denotes the Euclidean norm of \mathbf{x} . Absolute moments play an important role in probability theory, see, for example, [Sh, pp. 182]. It is clear that if $\mu \in \mathcal{P}([0, 1])$ satisfies $\mu(\{1\}) = 0$, then $M_q(\mu) \rightarrow 0$ as $q \rightarrow \infty$. It is therefore natural and of interest to ask for estimates of the rate at which $M_q(\mu)$ converges to 0 as $q \rightarrow \infty$, i.e. we ask for estimates of $\liminf_{q \rightarrow \infty} \frac{\log M_q(\mu)}{-\log q}$ and $\limsup_{q \rightarrow \infty} \frac{\log M_q(\mu)}{-\log q}$. This problem was investigated in [Ols]. For example, an application of the main results from [Ols] gives the following result providing estimates of $\liminf_{q \rightarrow \infty} \frac{\log M_q(\mu)}{-\log q}$ and $\limsup_{q \rightarrow \infty} \frac{\log M_q(\mu)}{-\log q}$ in terms of the lower and upper local dimensions of μ at 1.

Theorem A [Ols]. *Let $\mu \in \mathcal{P}([0, 1])$ with $1 \in \text{supp } \mu$. Then*

$$\underline{\dim}_{\text{loc}}(\mu; 1) \leq \liminf_{q \rightarrow \infty} \frac{\log M_q(\mu)}{-\log q} \leq \limsup_{q \rightarrow \infty} \frac{\log M_q(\mu)}{-\log q} \leq \overline{\dim}_{\text{loc}}(\mu; 1).$$

In particular, if the local dimension $\dim_{\text{loc}}(\mu; 1)$ exists then the limit $\lim_{q \rightarrow \infty} \frac{\log M_q(\mu)}{-\log q}$ exists and

$$\lim_{q \rightarrow \infty} \frac{\log M_q(\mu)}{-\log q} = \dim_{\text{loc}}(\mu; 1).$$

Results for measures in higher dimensional Euclidean spaces are also presented in [Ols]. However, these results are more involved and not relevant for the present discussion.

1.3. Mixed moments of measures. The absolute moments $M_q(\mu) = \int |\mathbf{x}|^q d\mu(\mathbf{x})$ are in some (admittedly) vague sense “1 dimensional” constructions: they are obtained by integrating the q 'th power of the “1 dimensional” distance $|\mathbf{x}|$ from the origin to \mathbf{x} . The “1 dimensional” nature of the absolute moments $M_q(\mu)$ was utilised implicitly in the proofs in [Ols] reducing the arguments in [Ols] to a careful analysis (of the fractal geometric properties) of the set of points \mathbf{x} with $|\mathbf{x}| = \sup_{\mathbf{y} \in \text{supp } \mu} |\mathbf{y}|$.

However, there is an equally common type of moments, namely, the mixed moments that are genuinely “higher dimensional”. Mixed moments play an important part in many different areas of mathematics including, for example, probability theory (see [Sh, pp. 289]) and harmonic analysis (see [BeChRe]), and are defined as follows. For $\mathbf{x} = (x_1, \dots, x_d) \in [0, \infty)^d$ and $\mathbf{q} = (q_1, \dots, q_d) \in (0, \infty)^d$, write

$$\mathbf{x}^{\mathbf{q}} = \prod_i x_i^{q_i}.$$

Next, for $\mu \in \mathcal{P}([0, 1]^d)$ and $\mathbf{q} = (q_1, \dots, q_d) \in (0, \infty)^d$, we define the \mathbf{q} 'th mixed moment of μ by

$$N_{\mathbf{q}}(\mu) = \int_{[0, 1]^d} \mathbf{x}^{\mathbf{q}} d\mu(\mathbf{x}).$$

The purpose of this paper is to analyse the asymptotic behaviour of the mixed moments $N_{\mathbf{q}}(\mu)$ as $|\mathbf{q}| \rightarrow \infty$, and, in particular, to obtain results analogous to Theorem A for the mixed moments $N_{\mathbf{q}}(\mu)$. The mixed moments $N_{\mathbf{q}}(\mu)$ are obtained by integrating $\mathbf{x}^{\mathbf{q}}$, and this expression clearly intertwines (or mixes) the contributions from the coordinates x_i of \mathbf{x} in a subtle “higher dimensional” multiplicatively way making the analysis of the asymptotic behaviour of $N_{\mathbf{q}}(\mu)$ as $|\mathbf{q}| \rightarrow \infty$ much more delicate than the corresponding analysis of the absolute moments in [Ols]. In particular, Lemma 3.1 and Lemma 3.2 providing efficient coverings of the set

$$E_{u, \mathbf{w}} = \left\{ \mathbf{x} \in [0, 1]^d \mid \mathbf{x}^{\mathbf{w}} \geq u \right\},$$

for $\mathbf{w} \in (0, \infty)^d$, are needed for the analysis of the mixed moments $N_{\mathbf{q}}(\mu)$. Indeed, the crux of the analysis of the absolute moments $M_q(\mu)$ in [Ols] is to find efficient coverings of the set

$$E_{u,q} = \left\{ \mathbf{x} \in [0, 1]^d \mid |\mathbf{x}|^q \geq u \right\}.$$

Similarly, the main issue when analysing the mixed moments $N_{\mathbf{q}}(\mu)$, is to find efficient coverings of the analogous set, namely, the set

$$E_{u,\mathbf{w}} = \left\{ \mathbf{x} \in [0, 1]^d \mid \mathbf{x}^{\mathbf{w}} \geq u \right\}$$

for $\mathbf{w} \in (0, \infty)^d$. It is not difficult to see that the set $E_{q,u}$ is “comparable” to the ball with centre at $(1, \dots, 1) \in \mathbb{R}^d$ and radius equal to $1 - u$ (when u is sufficiently close to 1); see [Ols] for more details. However, since the expression $\mathbf{x}^{\mathbf{w}}$ intertwines the contributions from the coordinates x_i of \mathbf{x} in a subtle “higher dimensional” multiplicatively way, this is not necessarily the case for the set $E_{u,\mathbf{w}}$, and the arguments from [Ols] can therefore not be applied. In stead, the more delicate “covering” result in Lemma 3.2 is needed and the subsequent arguments need to be adapted appropriately.

We will now state our main results. For brevity we first introduce the following notation, namely, we let $\mathbf{1}$ denote the element in \mathbb{R}^d whose coordinates are all equal to 1, i.e. we write

$$\mathbf{1} = (1, \dots, 1).$$

Also, for $\mathbf{q} = (q_1, \dots, q_d) \in (0, \infty)^d$, write $\min(\mathbf{q}) = \min_i q_i$. It is now clear that if $\mu \in \mathcal{P}([0, 1]^d)$ satisfies $\mu(\{\mathbf{1}\}) = 0$ and $\mathbf{q} = (q_1, \dots, q_d) \in (0, \infty)^d$, then we have

$$\begin{aligned} N_{\mathbf{q}}(\mu) &= \int_{[0,1]^d} \mathbf{x}^{\mathbf{q}} d\mu(\mathbf{x}) \\ &= \int_{[0,1]^d} \prod_i x_i^{q_i} d\mu(x_1, \dots, x_d) \\ &\leq \int_{[0,1]^d} \prod_i x_i^{\min(\mathbf{q})} d\mu(x_1, \dots, x_d) \\ &= \int_{[0,1]^d} \left(\prod_i x_i \right)^{\min(\mathbf{q})} d\mu(x_1, \dots, x_d) \\ &\rightarrow 0 \text{ as } \min(\mathbf{q}) \rightarrow \infty. \end{aligned}$$

It is therefore natural and of interest to ask for estimates of the rate at which $N_{\mathbf{q}}(\mu)$ converges to 0 as $\min(\mathbf{q}) \rightarrow \infty$, i.e. we ask for estimates of $\liminf_{q \rightarrow \infty} \frac{\log N_{\boldsymbol{\gamma}(q)}(\mu)}{-\log q}$ and $\limsup_{q \rightarrow \infty} \frac{\log N_{\boldsymbol{\gamma}(q)}(\mu)}{-\log q}$ for curves $\boldsymbol{\gamma} : (0, \infty) \rightarrow (0, \infty)^d$ with $|\boldsymbol{\gamma}(q)| \rightarrow \infty$ as $q \rightarrow \infty$. The main result in this paper, namely Theorem 1.1 below, provides estimates of $\liminf_{q \rightarrow \infty} \frac{\log N_{\boldsymbol{\gamma}(q)}(\mu)}{-\log q}$ and $\limsup_{q \rightarrow \infty} \frac{\log N_{\boldsymbol{\gamma}(q)}(\mu)}{-\log q}$ in terms of the lower and upper local dimensions of μ at $\mathbf{1}$ for a large class of (not necessarily continuous) curves $\boldsymbol{\gamma} : (0, \infty) \rightarrow (0, \infty)^d$ with $|\boldsymbol{\gamma}(q)| \rightarrow \infty$ as $q \rightarrow \infty$.

Theorem 1.1. *Let $\mu \in \mathcal{P}([0, 1]^d)$ with $\mathbf{1} \in \text{supp } \mu$. Let $\mathbf{w} : (0, \infty) \rightarrow (0, \infty)^d$ be a (not necessarily continuous) function satisfying the following:*

- (i) $\overline{\{\mathbf{w}(q) \mid q \in (0, \infty)\}} \subseteq (0, \infty)^d$;
- (ii) $\overline{\{\mathbf{w}(q) \mid q \in (0, \infty)\}}$ is compact.

Define $\gamma : (0, \infty) \rightarrow (0, \infty)^d$ by

$$\gamma(q) = q \mathbf{w}(q).$$

Then

$$\underline{\dim}_{\text{loc}}(\mu; \mathbf{1}) \leq \liminf_{q \rightarrow \infty} \frac{\log N_{\gamma(q)}(\mu)}{-\log q} \leq \limsup_{q \rightarrow \infty} \frac{\log N_{\gamma(q)}(\mu)}{-\log q} \leq \overline{\dim}_{\text{loc}}(\mu; \mathbf{1}).$$

In particular, if the local dimension $\dim_{\text{loc}}(\mu; \mathbf{1})$ exists, then the limit $\lim_{q \rightarrow \infty} \frac{\log N_{\gamma(q)}(\mu)}{-\log q}$ exists and

$$\lim_{q \rightarrow \infty} \frac{\log N_{\gamma(q)}(\mu)}{-\log q} = \dim_{\text{loc}}(\mu; \mathbf{1}).$$

The proof of Theorem 1.1 is given in Sections 2–3.

Remark. Theorem 1.1 shows, somewhat surprisingly, that the limiting behaviour of $N_{\gamma(q)}(\mu)$ as $q \rightarrow \infty$ is independent of the “exponent” $\gamma(q)$. We now present two examples illustrating the diverse nature of the “exponents” $\gamma(q)$ satisfying Conditions (i)–(ii) in Theorem 1.1.

Example: The “exponent” $\gamma(q)$ tends to infinity along a straight line. As an application of Theorem 1.1 we obtain Corollary 1.2 providing estimates of $\liminf_{q \rightarrow \infty} \frac{\log N_{q\mathbf{v}}(\mu)}{-\log q}$ and $\limsup_{q \rightarrow \infty} \frac{\log N_{q\mathbf{v}}(\mu)}{-\log q}$, i.e. Corollary 1.2 provides estimates of the limiting behaviour of $N_{\mathbf{q}}(\mu) = N_{q\mathbf{v}}(\mu)$ when the “exponent” $\mathbf{q} = q\mathbf{v}$ tends to infinity along the straight line passing through the origin and parallel to \mathbf{v} .

Corollary 1.2. Let $\mu \in \mathcal{P}([0, 1]^d)$ with $\mathbf{1} \in \text{supp } \mu$. Let $\mathbf{v} \in (0, \infty)^d$. Then

$$\underline{\dim}_{\text{loc}}(\mu; \mathbf{1}) \leq \liminf_{q \rightarrow \infty} \frac{\log N_{q\mathbf{v}}(\mu)}{-\log q} \leq \limsup_{q \rightarrow \infty} \frac{\log N_{q\mathbf{v}}(\mu)}{-\log q} \leq \overline{\dim}_{\text{loc}}(\mu; \mathbf{1}).$$

In particular, if the local dimension $\dim_{\text{loc}}(\mu; \mathbf{1})$ exists, then the limit $\lim_{q \rightarrow \infty} \frac{\log N_{q\mathbf{v}}(\mu)}{-\log q}$ exists and

$$\lim_{q \rightarrow \infty} \frac{\log N_{q\mathbf{v}}(\mu)}{-\log q} = \dim_{\text{loc}}(\mu; \mathbf{1}).$$

Proof.

This corollary follows from applying Theorem 1.1 to the function $\mathbf{w} : (0, \infty) \rightarrow (0, \infty)^d$ defined by $\mathbf{w}(q) = \mathbf{v}$ for $q > 0$ since the set $\{\mathbf{w}(q) \mid q \in (1, \infty)\} = \{\mathbf{v}\}$ satisfies Conditions (i)–(ii) in Theorem 1.1. \square

Theorem A is clearly a special case of Corollary 1.2. Indeed, if we put $d = 1$ and $\mathbf{v} = 1$, then Corollary 1.2 simplifies to Theorem A.

Example: The “exponent” $\gamma(q)$ tends to infinity while oscillating “wildly”. As a further application of Theorem 1.1 we obtain Corollary 1.3 providing estimates of $\liminf_{q \rightarrow \infty} \frac{\log N_{\gamma(q)}(\mu)}{-\log q}$ and $\limsup_{q \rightarrow \infty} \frac{\log N_{\gamma(q)}(\mu)}{-\log q}$ when $\gamma(q)$ tends to infinity (as $q \rightarrow \infty$) while oscillating “wildly”. For $0 < \varepsilon < \frac{1}{2}$, define $\ell : (0, \infty) \rightarrow \mathbb{R}$ by $\ell(q) = \frac{\pi}{2}((1 - 2\varepsilon)|\sin q| + \varepsilon)$, and put

$$\mathbf{w}(q) = e^{i\ell(q)}.$$

It clear that the function $\gamma : (0, \infty) \rightarrow (0, \infty)^2$ defined by $\gamma(q) = q \mathbf{w}(q)$ tends to infinity (as $q \rightarrow \infty$) while oscillating “wildly” inside the cone

$$\begin{aligned} C &= \left\{ \mathbf{x} \in (0, \infty)^2 \mid \text{dist}(\mathbf{x}, \Delta) \leq \frac{\cos(\frac{\pi}{2}\varepsilon) - \sin(\frac{\pi}{2}\varepsilon)}{\sqrt{2}} |\mathbf{x}| \right\} \\ &= \left\{ \mathbf{x} \in (0, \infty)^2 \mid \text{dist}(\mathbf{x}, \Delta) \leq \sin\left(\left(1 - \varepsilon\right)\frac{\pi}{2}\right) |\mathbf{x}| \right\}, \end{aligned}$$

where $\Delta = \{(x, x) \mid x \in \mathbb{R}\}$ denotes the diagonal in \mathbb{R}^2 . The asymptotic behaviour of $N_{q e^{i\ell(q)}}(\mu)$ is given by the next corollary.

Corollary 1.3. *Let $\mu \in \mathcal{P}([0, 1]^2)$ with $\mathbf{1} \in \text{supp } \mu$. Let $\varepsilon > 0$ and define $\ell : (0, \infty) \rightarrow \mathbb{R}$ by $\ell(q) = \frac{\pi}{2}((1 - 2\varepsilon)|\sin q| + \varepsilon)$. Then*

$$\underline{\dim}_{\text{loc}}(\mu; \mathbf{1}) \leq \liminf_{q \rightarrow \infty} \frac{\log N_q e^{i\ell(q)}(\mu)}{-\log q} \leq \limsup_{q \rightarrow \infty} \frac{\log N_q e^{i\ell(q)}(\mu)}{-\log q} \leq \overline{\dim}_{\text{loc}}(\mu; \mathbf{1}).$$

In particular, if the local dimension $\dim_{\text{loc}}(\mu; \mathbf{1})$ exists, then the limit $\lim_{q \rightarrow \infty} \frac{\log N_q e^{i\ell(q)}(\mu)}{-\log q}$ exists and

$$\lim_{q \rightarrow \infty} \frac{\log N_q e^{i\ell(q)}(\mu)}{-\log q} = \dim_{\text{loc}}(\mu; \mathbf{1}).$$

Proof.

This corollary follows from applying Theorem 1.1 to the function $\mathbf{w} : (0, \infty) \rightarrow (0, \infty)^2$ defined by $\mathbf{w}(q) = e^{i\ell(q)}$ for $q > 0$ since the set $\{\mathbf{w}(q) \mid q \in (1, \infty)\} = \{e^{i\theta} \mid \theta \in [\varepsilon\frac{\pi}{2}, (1 - \varepsilon)\frac{\pi}{2}]\}$ satisfies Conditions (i)–(ii) in Theorem 1.1. \square

2. PROOF OF THEOREM 1.1. PRELIMINARY RESULTS

The purpose of this section is to provide various auxiliary results that will be used in the proofs of Theorem 1.1. The two main results are Lemma 2.2 and Lemma 2.4. Lemma 2.2 provides an alternative expression for the mixed moments $N_{\mathbf{q}}(\mu)$. This expression will (see Section 3) allow us to bound $N_{\gamma(q)}(\mu)$ by an integral of the form $\int_{1-\delta}^1 qu^q(1-u)^a du$ for suitable choices of δ and a , and in Lemma 2.4 we establish the asymptotic behaviour of the integral $\int_{1-\delta}^1 qu^q(1-u)^a du$ as $q \rightarrow \infty$.

Before stating and proving the first main auxiliary result, namely Lemma 2.2, we first recall the following well-known result from analysis.

Lemma 2.1. *Let X be a separable metric space and let m be a Borel measure on X . If $f : X \rightarrow [0, \infty)$ is a positive Borel function, then*

$$\int f dm = \int_0^\infty m(\{f \geq t\}) dt.$$

Proof.

This result is proven in [Ma, Theorem 1.15]. \square

Lemma 2.2. *Let $\mu \in \mathcal{P}([0, 1]^d)$ with $\mathbf{1} \in \text{supp } \mu$ and let $\mathbf{w} \in (0, \infty)^d$. Fix $0 < \delta < 1$. Then there is a function $h : (0, \infty) \rightarrow \mathbb{R}$ such that*

$$N_{q\mathbf{w}}(\mu) = \int_{1-\delta}^1 qu^{q-1} \mu\{\mathbf{x} \in [0, 1]^d \mid \mathbf{x}^{\mathbf{w}} \geq u\} du + h(q)$$

and $|h(q)| \leq (1 - \delta)^q$ for all $q > 0$.

Proof.

For $q > 0$, define $f : [0, 1]^d \rightarrow [0, \infty)$ by $f(\mathbf{x}) = \mathbf{x}^{q\mathbf{w}}$. It now follows from Lemma 2.1 that

$$\begin{aligned} N_{q\mathbf{w}}(\mu) &= \int f d\mu \\ &= \int_0^\infty \mu(\{f \geq t\}) dt \\ &= \int_0^\infty \mu\{\mathbf{x} \in [0, 1]^d \mid \mathbf{x}^{q\mathbf{w}} \geq t\} dt \\ &= \int_0^\infty \mu\{\mathbf{x} \in [0, 1]^d \mid \mathbf{x}^{\mathbf{w}} \geq t^{\frac{1}{q}}\} dt. \end{aligned} \tag{2.1}$$

Introducing the substitution $u = t^{\frac{1}{q}}$ into the integral in (2.1), it now follows that

$$N_{q\mathbf{w}}(\mu) = \int_0^\infty qu^{q-1} \mu\{\mathbf{x} \in [0, 1]^d \mid \mathbf{x}^{\mathbf{w}} \geq u\} du,$$

and the assumption $\text{supp } \mu \subseteq [0, 1]^d$, therefore implies that

$$N_{q\mathbf{w}}(\mu) = \int_0^1 qu^{q-1} \mu\{\mathbf{x} \in [0, 1]^d \mid \mathbf{x}^{\mathbf{w}} \geq u\} du. \quad (2.2)$$

It follows immediately from (2.2) that

$$N_{q\mathbf{w}}(\mu) = \int_{1-\delta}^1 qu^{q-1} \mu\{\mathbf{x} \in [0, 1]^d \mid \mathbf{x}^{\mathbf{w}} \geq u\} + h(q),$$

where $h(q) = \int_0^{1-\delta} qu^{q-1} \mu\{\mathbf{x} \in [0, 1]^d \mid \mathbf{x}^{\mathbf{w}} \geq u\} du$. In particular, we conclude that $|h(q)| \leq \int_0^{1-\delta} qu^{q-1} du = (1-\delta)^q$ for all $q > 0$. \square

Next, we state and prove the second main auxiliary result in this section, namely, Lemma 2.4. In order to prove Lemma 2.4 we first prove Lemma 2.3 below. We note that both Lemma 2.3 and Lemma 2.4 are proved in [Ols]; however, we have decided to include the short proofs for completeness.

Lemma 2.3 [Ols, Lemma 3.2]. *Fix $0 < \delta < 1$ and $a > 0$. Then there are functions $f, g : (0, \infty) \rightarrow \mathbb{R}$ and a real number c such that*

$$\int_{1-\delta}^1 qu^{q-1}(1-u)^a du = cf(q)q^{-a} + g(q)$$

and $f(q) \rightarrow 1$ as $q \rightarrow \infty$ and $|g(q)| \leq (1-\delta)^q$ for all $q > 0$.

Proof.

Define the function $f : (0, \infty) \rightarrow \mathbb{R}$ and the real number c by $f(q) = q^a \frac{\Gamma(q+1)}{\Gamma(q+a+1)}$ and $c = \Gamma(a+1)$, and note that it follows from [Olv, p. 119] that $f(q) \rightarrow 1$ as $q \rightarrow \infty$.

Also, define the function $g : (0, \infty) \rightarrow \mathbb{R}$ by $g(q) = -\int_0^{1-\delta} qu^{q-1}(1-u)^a du$, and note that $|g(q)| \leq \int_0^{1-\delta} qu^{q-1}(1-u)^a du \leq \int_0^{1-\delta} qu^{q-1} du = (1-\delta)^q$ for all $q > 0$.

Finally, we observe that it follows from [Olv, p. 36, (1.10)] that $\int_0^1 u^{q-1}(1-u)^a du = \frac{\Gamma(q)\Gamma(a+1)}{\Gamma(q+a+1)}$, whence

$$\begin{aligned} \int_{1-\delta}^1 qu^{q-1}(1-u)^a du &= \int_0^1 qu^{q-1}(1-u)^a du - \int_0^{1-\delta} qu^{q-1}(1-u)^a du \\ &= q \frac{\Gamma(q)\Gamma(a+1)}{\Gamma(q+a+1)} + g(q) \\ &= \frac{\Gamma(q+1)\Gamma(a+1)}{\Gamma(q+a+1)} + g(q) \\ &= cf(q)q^{-a} + g(q) \end{aligned}$$

for all $q > 0$. \square

Lemma 2.4 [Ols, Lemma 3.4]. *Fix $0 < \delta < 1$, $a > 0$ and $m > 0$. Let $h : (0, \infty) \rightarrow \mathbb{R}$ be a function and assume that $|h(q)| \leq (1-\delta)^q$ for all $q > 0$. Then*

$$\lim_{q \rightarrow \infty} \frac{\log \left(m \int_{1-\delta}^1 qu^{q-1}(1-u)^a du + h(q) \right)}{-\log q} = a.$$

Proof.

It follows from Lemma 3.3 there are functions $f, g : (0, \infty) \rightarrow \mathbb{R}$ and a real number c such that

$$\int_{1-\delta}^1 qu^{q-1}(1-u)^a du = cf(q)q^{-a} + g(q)$$

and $f(q) \rightarrow 1$ as $q \rightarrow \infty$ and $|g(q)| \leq (1-\delta)^q$ for all $q > 0$. In particular, this shows that

$$m \int_{1-\delta}^1 qu^{q-1}(1-u)^a du + g(q) = mc f(q)q^{-a} + mg(q) + h(q) = q^{-a} \varphi(q)$$

where the function $\varphi : (0, \infty) \rightarrow \mathbb{R}$ is defined by $\varphi(q) = mc f(q) + mq^a g(q) + q^a h(q)$, and so

$$\frac{\log \left(m \int_{1-\delta}^1 qu^{q-1}(1-u)^a du + h(q) \right)}{-\log q} = a - \frac{\log \varphi(q)}{\log q}. \quad (2.3)$$

However, we clearly have $|q^a g(q)| \leq q^a(1-\delta)^q \rightarrow 0$ as $q \rightarrow \infty$ and $|q^a h(q)| \leq q^a(1-\delta)^q \rightarrow 0$ as $q \rightarrow \infty$, and so $\varphi(q) = mc f(q) + mq^a g(q) + q^a h(q) \rightarrow mc$ as $q \rightarrow \infty$. The desired result follows from this and (2.3). \square

3. PROOF OF THEOREM 1.1

The purpose of this section is to prove Theorem 1.1. However, we first prove the following two auxiliary lemmas.

Lemma 3.1. *Let $c > 0$. Then there are constants $a, b, \delta_0 > 0$ such that if $0 < r < \delta_0$, then $1 - ar > 0$, $1 - br > 0$ and*

$$(1 - ar)^c > 1 - r > (1 - br)^c.$$

Proof.

Since $\frac{1-(1-r)^{\frac{1}{c}}}{r} \rightarrow \frac{1}{c}$ as $r \searrow 0$, we can find $\rho_0 > 0$ such that $\frac{1}{2c} < \frac{1-(1-r)^{\frac{1}{c}}}{r} < \frac{2}{c}$ for all $0 < r < \rho_0$. Now put $a = \frac{1}{2c}$, $b = \frac{2}{c}$ and $\delta_0 = \min(\rho_0, \frac{1}{a}, \frac{1}{b})$, and observe that if $0 < r < \delta_0$, then $1 - ar > 0$, $1 - br > 0$ and $a < \frac{1-(1-r)^{\frac{1}{c}}}{r} < b$. Rearranging the previous inequality gives the desired result. \square

Lemma 3.2. *Let $\mu \in \mathcal{P}([0, 1]^d)$. Let $W \subseteq (0, \infty)^d$ satisfy the following:*

- (i) $\overline{W} \subseteq (0, \infty)^d$;
- (ii) \overline{W} is compact.

For $u \in (0, 1)$ and $\mathbf{w} \in (0, \infty)^d$, write

$$E_{u, \mathbf{w}} = \left\{ \mathbf{x} \in [0, 1]^d \mid \mathbf{x}^{\mathbf{w}} \geq u \right\}.$$

Then there are constants $a, b, \delta_0 > 0$ such that the following is satisfied: for all $1 - \delta_0 < u < 1$ and all $\mathbf{w} \in W$, we have

$$B(\mathbf{1}, a(1-u)) \cap [0, 1]^d \subseteq E_{u, \mathbf{w}} \subseteq B(\mathbf{1}, b(1-u)) \cap [0, 1]^d.$$

Proof.

We first introduce the following notation. Namely, for $\mathbf{w} = (w_1, \dots, w_d) \in (0, \infty)^d$, we write $\min(\mathbf{w}) =$

$\min_i w_i$ and $\max(\mathbf{w}) = \max_i w_i$. Next, note that since $\overline{W} \subseteq (0, \infty)^d$ and \overline{W} is compact, we conclude that there are constants w_{\min} and w_{\max} such that

$$0 < w_{\min} \leq \min(\mathbf{w}) \leq w_i \leq \max(\mathbf{w}) \leq w_{\max} < \infty$$

for all $\mathbf{w} = (w_1, \dots, w_d) \in W$ and all i .

It follows from Lemma 3.1 that there are constants $a_0, b_0, \delta_0 > 0$ such that if $0 < r < \delta_0$, then $1 - a_0 r > 0$, $1 - b_0 r > 0$ and

$$\begin{aligned} (1 - a_0 r)^{dw_{\max}} &> 1 - r, \\ 1 - r &> (1 - b_0 r)^{w_{\min}}. \end{aligned} \quad (3.1)$$

Now, put $a = a_0$ and $b = \sqrt{d} b_0$. Below we prove that if $1 - \delta_0 < u < 1$ and $\mathbf{w} \in W$, then

$$B(\mathbf{1}, a(1 - u)) \cap [0, 1]^d \subseteq E_{u, \mathbf{w}} \subseteq B(\mathbf{1}, b(1 - u)) \cap [0, 1]^d.$$

Claim 1. If $1 - \delta_0 < u < 1$ and $\mathbf{w} \in W$, then $B(\mathbf{1}, a(1 - u)) \cap [0, 1]^d \subseteq E_{u, \mathbf{w}}$.

Proof of Claim 1. Fix $1 - \delta_0 < u < 1$ and $\mathbf{w} = (w_1, \dots, w_d) \in W$. Next, let $\mathbf{x} = (x_1, \dots, x_d) \in B(\mathbf{1}, a(1 - u)) \cap [0, 1]^d$. Since $\mathbf{x} = (x_1, \dots, x_d) \in B(\mathbf{1}, a(1 - u)) \cap [0, 1]^d$, we conclude that $|\mathbf{x} - \mathbf{1}| \leq a(1 - u)$, whence $|x_i - 1| \leq a(1 - u)$ for all i , and so $1 - a(1 - u) \leq x_i \leq 1$ for all i . We deduce from this inequality that

$$\begin{aligned} \mathbf{x}^{\mathbf{w}} &= \prod_i x_i^{w_i} \\ &\geq \prod_i (1 - a(1 - u))^{w_i} \\ &\geq \prod_i (1 - a(1 - u))^{w_{\max}} \\ &= (1 - a(1 - u))^{dw_{\max}}. \end{aligned} \quad (3.2)$$

However, since $1 - u \leq \delta_0$, we see from (3.1) that $(1 - a(1 - u))^{dw_{\max}} \geq 1 - (1 - u) = u$, and it therefore follows from (3.2) that

$$\mathbf{x}^{\mathbf{w}} \geq u.$$

This shows that $\mathbf{x} \in E_{u, \mathbf{w}}$. This completes the proof of Claim 1.

Claim 2. If $1 - \delta_0 < u < 1$ and $\mathbf{w} \in W$, then $E_{u, \mathbf{w}} \subseteq B(\mathbf{1}, b(1 - u)) \cap [0, 1]^d$.

Proof of Claim 2. Fix $1 - \delta_0 < u < 1$ and $\mathbf{w} = (w_1, \dots, w_d) \in W$. Let $\mathbf{x} = (x_1, \dots, x_d) \in E_{u, \mathbf{w}}$. We now claim that

$$x_i^{w_i} \geq u \quad \text{for all } i. \quad (3.3)$$

We will now prove (3.3). Indeed, if (3.3) is not satisfied, then there is an index i_0 such that $x_{i_0}^{w_{i_0}} < u$, and since $x_i \leq 1$ for all i , this implies that $\mathbf{x}^{\mathbf{w}} = \prod_i x_i^{w_i} \leq x_{i_0}^{w_{i_0}} < u$. However, the inequality $\mathbf{x}^{\mathbf{w}} < u$ clearly contradicts the fact that $\mathbf{x} \in E_{u, \mathbf{w}}$. This proves (3.3).

Combining (3.1) and (3.3) now gives

$$\begin{aligned} x_i^{w_i} &\geq u \\ &= 1 - (1 - u) \\ &> (1 - b_0(1 - u))^{w_{\min}} \\ &\geq (1 - b_0(1 - u))^{w_i}. \end{aligned} \quad (3.4)$$

It follows immediately from (3.4) that $|x_i - 1| < b_0(1 - u)$ for all i , whence $\mathbf{x} \in B(\mathbf{1}, b_0(1 - u)) \times \cdots \times B(\mathbf{1}, b_0(1 - u)) \subseteq B(\mathbf{1}, \sqrt{d} b_0(1 - u))$. This completes the proof of Claim 2.

The desired statement follows immediately from claim 1 and Claim 2. \square

We can now prove Theorem 1.1.

Proof of Theorem 1.1.

We must prove that $\underline{\dim}_{\text{loc}}(\mu; \mathbf{1}) \leq \liminf_{q \rightarrow \infty} \frac{\log N_{\gamma(q)}(\mu)}{-\log q} \leq \limsup_{q \rightarrow \infty} \frac{\log N_{\gamma(q)}(\mu)}{-\log q} \leq \overline{\dim}_{\text{loc}}(\mu; \mathbf{1})$.

However, since $\liminf_{q \rightarrow \infty} \frac{\log N_{\gamma(q)}(\mu)}{-\log q} \geq 0$ (because $N_{\gamma(q)}(\mu) \leq 1$), we may clearly assume that $\underline{\dim}_{\text{loc}}(\mu; \mathbf{1}) > 0$.

Fix $\varepsilon > 0$ with $0 < \varepsilon < \underline{\dim}_{\text{loc}}(\mu; \mathbf{1})$.

First, note that we can choose $r_\varepsilon > 0$ such that

$$\underline{\dim}_{\text{loc}}(\mu; \mathbf{1}) - \varepsilon \leq \frac{\log \mu(B(\mathbf{1}, r))}{\log r} \leq \overline{\dim}_{\text{loc}}(\mu; \mathbf{1}) + \varepsilon \quad (3.5)$$

for all $0 < r < r_\varepsilon$.

Next, write

$$W = \left\{ \mathbf{w}(q) \mid q \in (0, \infty) \right\},$$

and observe that $\overline{W} \subseteq (0, \infty)^d$ and that \overline{W} is compact.

Also, for $u \in (0, 1)$ and $q \in (0, \infty)$, let $E_{u, \mathbf{w}(q)}$ be defined as in Lemma 3.2, i.e. $E_{u, \mathbf{w}(q)} = \{\mathbf{x} \in [0, 1]^d \mid \mathbf{x}^{\mathbf{w}(q)} \geq u\}$. Since $\overline{W} \subseteq (0, \infty)^d$ and \overline{W} is compact, it follows from Lemma 3.2 that there are constants $a, b, \delta_0 > 0$ such that the following is satisfied: for all $1 - \delta_0 < u < 1$ and all $q \in (0, \infty)$, we have

$$B(\mathbf{1}, a(1 - u)) \cap [0, 1]^d \subseteq E_{u, \mathbf{w}(q)} \subseteq B(\mathbf{1}, b(1 - u)) \cap [0, 1]^d. \quad (3.6)$$

Now let $\delta_\varepsilon = \min(\frac{r_\varepsilon}{a}, \frac{r_\varepsilon}{b}, \delta_0)$.

It now follows from Lemma 2.2 that there is a function $h_\varepsilon : (0, \infty) \rightarrow \mathbb{R}$ such that

$$N_{\gamma(q)}(\mu) = N_{q \mathbf{w}(q)}(\mu) = I_\varepsilon(q) + h_\varepsilon(q) \quad (3.7)$$

where

$$I_\varepsilon(q) = \int_{1-\delta_\varepsilon}^1 qu^{q-1} \mu \left\{ \mathbf{x} \in [0, 1]^d \mid \mathbf{x}^{\mathbf{w}(q)} \geq u \right\} du$$

and $|h_\varepsilon(q)| \leq (1 - \delta_\varepsilon)^q$ for all $q > 0$. We will now estimate $I_\varepsilon(q)$. This is done in Claim 1 and Claim 2 below. For brevity we write

$$\begin{aligned} \underline{\alpha}_\varepsilon &= \underline{\dim}_{\text{loc}}(\mu; \mathbf{1}) - \varepsilon, \\ \overline{\alpha}_\varepsilon &= \overline{\dim}_{\text{loc}}(\mu; \mathbf{1}) + \varepsilon. \end{aligned}$$

Claim 1. We have $I_\varepsilon(q) \leq M \int_{1-\delta_\varepsilon}^1 qu^{q-1} (1 - u)^{\underline{\alpha}_\varepsilon} du$ where $M = b^{\underline{\alpha}_\varepsilon}$.

Proof of Claim 1. We first note that it follows from (3.6) that

$$\begin{aligned} I_\varepsilon(q) &= \int_{1-\delta_\varepsilon}^1 qu^{q-1} \mu \left\{ \mathbf{x} \in [0, 1]^d \mid \mathbf{x}^{\mathbf{w}(q)} \geq u \right\} du \\ &= \int_{1-\delta_\varepsilon}^1 qu^{q-1} \mu(E_{u, \mathbf{w}(q)}) du \\ &\leq \int_{1-\delta_\varepsilon}^1 qu^{q-1} \mu(B(\mathbf{1}, b(1 - u))) du. \end{aligned} \quad (3.8)$$

Next, we observe that if $u \in (1 - \delta_\varepsilon, 1)$, then $b(1 - u) \leq b\delta_\varepsilon \leq r_\varepsilon$, whence (using (3.5)) $\underline{\dim}_{\text{loc}}(\mu; \mathbf{1}) - \varepsilon \leq \frac{\log \mu(B(\mathbf{1}, b(1-u)))}{\log b(1-u)}$ and so $\mu(B(\mathbf{1}, b(1-u))) \leq (b(1-u))^{\underline{\dim}_{\text{loc}}(\mu; \mathbf{1}) - \varepsilon}$. This and (3.8) clearly imply that

$$\begin{aligned} I_\varepsilon(q) &\leq \int_{1-\delta_\varepsilon}^1 qu^{q-1} (b(1-u))^{\underline{\dim}_{\text{loc}}(\mu; \mathbf{1}) - \varepsilon} du \\ &= M \int_{1-\delta_\varepsilon}^1 qu^{q-1} (1-u)^{\alpha_\varepsilon} du. \end{aligned}$$

This completes the proof of Claim 1.

Claim 2. We have $I_\varepsilon(q) \geq m \int_{1-\delta_\varepsilon}^1 qu^{q-1} (1-u)^{\bar{\alpha}_\varepsilon} du$ where $m = a^{\bar{\alpha}_\varepsilon}$.

Proof of Claim 2. The proof of Claim 2 is similar to the proof of Claim 1 and is therefore omitted.

Combining Claim 1, Claim 2 and (3.7) yields

$$\begin{aligned} N_{\gamma(q)}(\mu) &\leq M \int_{1-\delta_\varepsilon}^1 qu^{q-1} (1-u)^{\alpha_\varepsilon} du + h_\varepsilon(q), \\ N_{\gamma(q)}(\mu) &\geq m \int_{1-\delta_\varepsilon}^1 qu^{q-1} (1-u)^{\bar{\alpha}_\varepsilon} du + h_\varepsilon(q), \end{aligned} \tag{3.9}$$

for all $\varepsilon > 0$, where $|h_\varepsilon(q)| \leq (1 - \delta_\varepsilon)^q$ for all $q > 0$, and $\alpha_\varepsilon > 0$ (because $0 < \varepsilon < \underline{\dim}_{\text{loc}}(\mu; \mathbf{1})$) and $\bar{\alpha}_\varepsilon > 0$. It therefore follows from Lemma 2.4 and (3.9) that

$$\begin{aligned} \liminf_{q \rightarrow \infty} \frac{\log N_{\gamma(q)}(\mu)}{-\log q} &\geq \lim_{q \rightarrow \infty} \frac{\log \left(M \int_{1-\delta_\varepsilon}^1 qu^{q-1} (1-u)^{\alpha_\varepsilon} du + h_\varepsilon(q) \right)}{-\log q} \\ &= \alpha_\varepsilon \\ &= \underline{\dim}_{\text{loc}}(\mu; \mathbf{1}) - \varepsilon \end{aligned}$$

and

$$\begin{aligned} \limsup_{q \rightarrow \infty} \frac{\log N_{\gamma(q)}(\mu)}{-\log q} &\leq \lim_{q \rightarrow \infty} \frac{\log \left(m \int_{1-\delta_\varepsilon}^1 qu^{q-1} (1-u)^{\bar{\alpha}_\varepsilon} du + h_\varepsilon(q) \right)}{-\log q} \\ &= \bar{\alpha}_\varepsilon \\ &= \overline{\dim}_{\text{loc}}(\mu; \mathbf{1}) + \varepsilon \end{aligned}$$

for all $\varepsilon > 0$. Letting $\varepsilon \searrow 0$ now gives the desired result. \square

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