LINEAR RESPONSE FOR INTERMITTENT MAPS

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Abstract. We consider the one parameter family $\alpha \mapsto T_\alpha$ ($\alpha \in [0,1]$) of Pomeau-Manneville type interval maps $T_\alpha(x) = x(1 + 2^\alpha x^\alpha)$ for $x \in [0,1/2)$ and $T_\alpha(x) = 2x - 1$ for $x \in [1/2,1]$, with the associated absolutely continuous invariant probability measure $\mu_\alpha$. For $\alpha \in (0,1)$, Sarig and Gouëzel proved that the system mixes only polynomially with rate $n^{1-1/\alpha}$ (in particular, there is no spectral gap). We show that for any $\psi \in L^q$, the map $\alpha \to \int_0^1 \psi \, d\mu_\alpha$ is differentiable on $[0,1-1/q)$, and we give a (linear response) formula for the value of the derivative. This is the first time that a linear response formula for the SRB measure is obtained in the setting of slowly mixing dynamics. Our argument shows how cone techniques can be used in this context. For $\alpha \geq 1/2$ we need the $n^{-1/\alpha}$ decorrelation obtained by Gouëzel under additional conditions.

1. Introduction

Given a family of dynamical systems $T_\alpha$ on a Riemann manifold, depending smoothly on a real parameter $\alpha$, and admitting (at least for some large subset of parameters) an ergodic physical (e.g. absolutely continuous, or SRB) invariant measure $\mu_\alpha$, it is natural to ask how smooth is the dependence of $\mu_\alpha$ on the parameter $\alpha$. In particular, one would like to know whether $\alpha \mapsto \mu_\alpha$ is differentiable and, if possible, compute a formula for the derivative, depending on $\mu_\alpha$, $T_\alpha$, and $v_\alpha = \partial_\alpha T_\alpha$.

This theme of linear response was explored in a few pioneering papers [Ru1, KKPW, Ru] in the setting of smooth hyperbolic dynamics (Anosov or Axiom A), and then further developed, following the influence of ideas of David Ruelle. In the smooth hyperbolic case, the SRB measure $\mu_\alpha$ corresponds to the fixed point of a transfer operator $L_\alpha$ enjoying a spectral gap on a suitable Banach space. In particular, this fixed point is a simple isolated eigenvalue in the spectrum of $L_\alpha$, and linear response can be viewed as an instance of perturbation theory for simple eigenvalues. This is evident in the linear response formulas, which all involve some avatar of the resolvent $(\id - L_\alpha)^{-1} = \sum_k L_\alpha^k$ applied to a suitable vector $Y_\alpha$, depending on the derivative of $\mu_\alpha$ and on $v_\alpha$.

It was soon realised that existence of a spectral gap is not sufficient to guarantee linear response when bifurcations are present (see e.g. [Ma, B1, BS]). In the other direction,
neither the spectral gap nor structural stability is necessary for linear response, as was shown by Dolgopyat [Do] who obtained a linear response formula for some rapidly mixing systems (which were not all exponentially mixing or structurally stable).

The intuition that a key sufficient condition is convergence of the sum $\sum_k \mathcal{L}_k(Y_\alpha)$ was confirmed by [HM, Remark 2.4]. This is of course related to a summable decay of correlations. However, decay of correlation usually only holds for observables with a suitable modulus of continuity, which $Y_\alpha$, being a derivative, does not always enjoy. We confirm this intuition by studying a toy-model, of Pomeau-Manneville type: \[\text{For } \alpha \in \{0, 1\}, \text{ we consider the maps (as in [LSV]) } T_\alpha : [0, 1] \to [0, 1]:\]

\[T_\alpha(x) = \begin{cases} 
  x(1 + 2^\alpha x^\alpha), & x \in [0, 1/2) \\
  2x - 1, & x \in [1/2, 1].
\end{cases}\]

(Of course, $T_0$ is just the angle-doubling map $T_0(x) = 2x$ modulo 1.) It is well-known that each such $T_\alpha$ admits a unique absolutely continuous invariant probability measure $\mu_\alpha = \rho_\alpha \, dx$. (Clearly, $\rho_0(x) \equiv 1$.) Statistical stability (continuity) of $\mu_\alpha$ when $\alpha$ changes is proved in [PT]. The absolutely continuous invariant probability measure $\mu_\alpha = \rho_\alpha \, dx$ is mixing for all $\alpha \in [0, 1)$. For $\alpha = 0$ the mixing rate for Lipschitz observables, say, is exponential (decaying like $1/2^k$). For $\alpha \in (0, 1)$ the mixing rate is only polynomial with rate $n^{1-1/\alpha}$ [Go, Sa]. (In fact, Gouëzel obtains a faster rate $n^{-1/\alpha}$ for $\int (\psi \circ T_\alpha^n) \phi \, d\mu_\alpha$, if $\psi$ is bounded, $\phi$ is Lipschitz and vanishes in a neighbourhood of zero, and $\int \phi \, d\mu_\alpha = 0$, and this property is crucial below when $\alpha \geq 1/2$.) In particular, for any $\alpha \in (0, 1)$, the density $\rho_\alpha$ cannot be the fixed point of a transfer operator with a spectral gap on a Banach space containing all $C^\infty$ functions. However, we are able to prove (Theorem 2.1) that for any $q \in [1, \infty]$ and any $\psi \in L^q$, the map $\alpha \mapsto \int \psi \, d\mu_\alpha$ is continuously differentiable on $[0, 1 - 1/q)$, and we give two expressions (2.6, with a resolvent, (2.7), of susceptibility function type) for the linear response formula, with $Y_\alpha = (X_\alpha N_\alpha(\rho_\alpha))'$, where $X_\alpha = \nu_\alpha \circ (T_\alpha^{-1}|_{[0,1/2)})$ and $N_\alpha$ corresponds to the first branch of the transfer operator $\mathcal{L}_\alpha$. This is the first time that a linear response formula is achieved for a slowly mixing dynamics. The fact that linear response holds for any bounded $\psi$ is relevant since nonsmooth observables appear naturally. For example, if $A$ is smooth and $\Theta$ is a Heaviside function, the expectation value of $\Theta(A(x))$ gives the fraction of the total measure where $A$ has positive value, and more generally such discontinuous observables have probabilistic and physical interpretations, with the work of Lucarini et al. [Lu1, Lu2] showing how the theory of extremes for dynamical systems (in particular regarding climate change) can be cast in this framework.

Our proof is based on the cone techniques from [LSV], and hinges on the new observation that the factor $X_\alpha$, respectively $X'_\alpha$, compensates the singularity at zero of $\rho'_\alpha$, respectively $\rho_\alpha$. Indeed, the compensation is drastic enough so that the $n^{-1/\alpha}$ decorrelation results of Gouëzel [Go, Goth] can be used. It would apply e.g. to the more general one-dimensional maps with finitely many neutral points described in [LSV, Section 5]. Since our goal is to

\[\text{See Remark 2.2 for one possible generalisation.}\]
describe a new mechanism (demonstrating in particular how invariant cone techniques\footnote{Bomfim et al. \cite{BCV} use invariant cones to obtain differentiability of some equilibrium measures (enjoying a spectral gap) of Pomeau–Manneville maps. Their results do not apply to the SRB measure, and thus do not include linear response in the sense of the present work.} can be implemented) for linear response in the presence of neutral fixed points in the simplest setting, we leave such generalisations to further works.

We end this introduction with comments about bifurcations and the singularities of $\rho_\alpha$. By \cite[Theorem 1]{Th1}, for any $\alpha \in (0, 1)$, there exists $0 < c_1 < c_2$ so that
\begin{equation}
 c_1 x^{-\alpha} \leq \rho_\alpha(x) \leq c_2 x^{-\alpha}.
\end{equation}

It is easy to see (e.g. via symbolic dynamics) that the maps $\{T_\alpha \mid \alpha \in [0, 1)\}$ belong to the same topological class, so that bifurcations do not occur. However the conjugacy $h_\alpha$ between $T_0$ and $T_\alpha$ is not differentiable. Indeed, if it were, then we would have $T'_\alpha(0) = T'_0(0)$ at the fixed point 0, but this is impossible\footnote{The upper bound also follows from \cite[Lemma 2.3]{LSV}.} since $T'_0(0) = 2$, while $T'_\alpha(0) = 1$ for $\alpha > 0$. More generally, take arbitrary $\alpha \neq \beta$. The conjugacy $h_{\beta,\alpha}$ between $T_\alpha$ and $T_\beta$ maps the invariant density $\rho_\alpha$ to $\rho_\beta$. Therefore $h_{\beta,\alpha}$ cannot be differentiable, since otherwise it would contradict (1.1).

Another lesson of recent research \cite{Kru2, Ru3, B3, B2, CD} on linear response is that understanding the singularities of the SRB measure is essential. In our application, the density $\rho_\alpha$ is smooth on $(0, 1]$. The only “critical point” of $T_\alpha$ is the neutral fixed point at 0, so that the “postcritical orbit” is reduced to a single point. By (1.1), the singularity type of $\rho_\alpha$ at 0 is $x^{-\alpha}$. So, heuristically, for a bounded observable $\psi$, the contribution of the origin to $\partial_\alpha \int \psi d\mu_\alpha$ should be $\int x^{-\alpha} \log x \cdot \psi(x) \, dx$, which is indeed well defined.\footnote{Also, there are many periodic points $x_0$ for $T_0$ such that if $p$ is the period then $(T_0^p)'(x_0) = 2^p$, but $(T_\alpha^p)'(h_\alpha(x_0)) \neq 2^p$.} Indeed, this heuristic remark sheds some light on the otherwise mysterious singularity cancellation $X_\alpha \rho'_\alpha \sim \log x \sim X'_\alpha \rho_\alpha$. Our approach should extend to give higher order derivatives of $\alpha \to \mu_\alpha$ (using invariant cones with more derivatives).

After the first version of this paper (in which our result was restricted to $\alpha \in [0, 1/2)$ and $L^\infty$ observables) was posted on the arXiv, Korepanov \cite{Ko} obtained linear response (without the formula) for all $\alpha \in (0, 1)$ and $L^q$ observables (for $q > (1 - \alpha)^{-1}$). His method of proof (using inducing) is different from ours.

2. Linear response formula for Pomeau–Manneville maps

2.1. Statement of the main result. We consider the transfer operator $L_\alpha$ defined, e.g. on $L^\infty(dx)$, by (note that $\inf T'_\alpha \geq 1$ so absolute values are not needed)
\begin{equation}
 L_\alpha \varphi(x) = \sum_{T_\alpha(y) = x} \frac{\varphi(y)}{T'_\alpha(y)}.
\end{equation}
The result also holds for \((In particular the right-hand side of \(2.6\))

\begin{align*}
\lim_{\epsilon \to 0} \epsilon^{-1} \left( \int_0^1 \psi d\mu_\alpha - \int_0^1 \psi d\mu_{\alpha + \epsilon} \right) &= - \int_0^1 \psi (\text{id} - \mathcal{L}_\alpha)^{-1} [(X_\alpha N_\alpha(\rho_\alpha))] \, dx.
\end{align*}

(For particular the right-hand side of \(2.6\) is well-defined.) In addition, the right-hand side of \(2.6\) can be written as the following absolutely convergent sum

\begin{align*}
- \sum_{k \geq 0} \int_0^1 \psi \mathcal{L}_\alpha^k [(X_\alpha N_\alpha(\rho_\alpha))] \, dx &= - \sum_{k \geq 0} \int_0^1 (\psi \circ T_{\alpha}^k)(X_\alpha N_\alpha(\rho_\alpha)) \, dx.
\end{align*}

The result also holds for \(\alpha = 0\), taking the limit as \(\epsilon \downarrow 0\) in \(2.6\). For \(p \in [1, \infty)\), the map \(\alpha \mapsto \partial_\alpha \rho_\alpha \in L^p(dx)\) is continuous on \([0, 1/p]\).

Integration by parts allows us to rewrite the convergent sum as

\begin{align*}
- \sum_{k \geq 0} \int_0^1 (\psi \circ T_{\alpha}^k)(X_\alpha N_\alpha(\rho_\alpha)) \, dx &= \sum_{k \geq 0} \int_0^1 (\psi \circ T_{\alpha}^k)' X_\alpha N_\alpha(\rho_\alpha) \, dx.
\end{align*}

We conjecture that the above results also hold for \(\alpha < 0\) in some parameter range, but the proof will require modifications.
Remark 2.2. It seems our proof also applies to the one-parameter family

\[ G_t(x) = \begin{cases} 
(1 + t)x + (1 - t)2^t x^{1+t}, & x \in [0, 1/2), \\
2x - 1, & x \in [1/2, 1),
\end{cases} t \in [0, \epsilon]. \]

Remark 2.3. For the sake of comparison with previous works (e.g. [B3]), we can consider a one-parameter family \( F_\beta \) obtained by perturbations in the image, i.e. so that \( v_\beta = \partial_\beta F_\beta = X_\beta \circ F_\beta \) for some \( X_\beta \). This can be achieved by perturbing the first branch \( x < 1/2 \) in order to have \( T_\beta = F_\beta \) and requiring the second branch \( x \geq 1/2 \) to move "sympathetically" with the first one. More precisely, for fixed \( \alpha \), consider the one-parameter family \( F_{\beta,\alpha} \) which satisfies

\[ F_{\alpha,\alpha}(x) = T_\alpha(x), \forall x \in [0, 1], \quad F_{\beta,\alpha}(x) = T_\beta(x), \forall x \in [0, 1/2), \]

and, setting

\[ X_\beta(x) =: v_\beta \circ g_\beta(x) = \partial_\beta T_\beta \circ g_\beta(x), \forall x \in [0, 1], \]

so that

\[ v_{\beta,\alpha} := \partial_\beta F_{\beta,\alpha} = X_\beta \circ F_{\beta,\alpha}. \]

(For \( x \in [0, 1/2) \) this is automatic, and for \( x \in [1/2, 1] \) it can be obtained by solving the ODE \( \partial_\beta F_{\beta,\alpha} = X_\beta \circ F_{\beta,\alpha} \) with initial condition \( F_{\alpha,\alpha}(x) = 2x - 1 \) on \([1/2, 1]\). By the Picard-Lindelöf theorem, this ODE has a unique solution since \((\beta, y) \mapsto X_\beta(y)\) is continuous in \( \beta \).)

If \( \alpha \) is fixed, slightly abusing notation, we sometimes write \( F_\beta, v_\beta, X_\beta, \) and \( \mathcal{L}_\beta \), instead of \( F_{\beta,\alpha}, v_{\beta,\alpha}, X_{\beta,\alpha}, \) and \( \mathcal{L}_{\beta,\alpha} \), when the meaning is clear. It is not difficult to prove that \( F_\beta \) has a unique absolutely continuous invariant measure \( \hat{\mu}_\beta = \hat{\rho}_\beta \, dx \) satisfying the same properties as \( \mu_\beta \), and the proof of Theorem 2.1 shows that for any \( \alpha \in (0, 1) \) and any \( \psi \in L^\infty(dx) \)

\[ \lim_{\epsilon \to 0} \epsilon^{-1} \left( \int_0^1 \psi \, d\hat{\mu}_\alpha - \int_0^1 \psi \, d\hat{\mu}_{\alpha+\epsilon} \right) = -\int_0^1 \psi (\text{id} - \mathcal{L}_\alpha)^{-1}[ (X_\alpha \hat{\rho}_\alpha)' ] \, dx. \]  

The result also holds for \( \alpha = 0 \), taking the limit as \( \epsilon \downarrow 0 \) in (2.8).

Just like (2.6), the expression (2.8) can be written as an absolutely convergent sum. Integration by parts gives \( \int (\psi' \circ T_\alpha^k) \cdot (T_\alpha^k)'(x) X_\alpha(x) \rho_\alpha(x) \, dx \). This is just \( \Psi(1) \) where \( \Psi(z) \) is the susceptibility function (see e.g. [B3]). It would be interesting to analyze the singularity type of the susceptibility function at \( z = 1 \). (See [BMS] for the corresponding analysis for piecewise expanding maps.)

2.2. Invariant cones. Before, proving the theorem, we introduce notations and state useful results regarding cones adapted from [LSV].

As our proof requires higher derivatives we shall use the following fact:

**Proposition 2.4** (Invariant cone in \( C^2 \)). For fixed \( b_1 \geq \alpha + 1, b_2 \geq b_1, \bar{b}_1 > 0, \bar{b}_2 > 0, \)
define the cone \( C_2 \) to be the set of \( \varphi \in C^2(0, 1) \) so that

\[ \varphi(x) \geq 0, \quad \bar{b}_1 x \varphi(x) \leq -\varphi'(x) \leq \frac{b_1}{x} \varphi(x), \quad \text{and} \quad \frac{\bar{b}_2}{x^2} \varphi(x) \leq \varphi''(x) \leq \frac{b_2}{x^2} \varphi(x), \quad \forall x \in (0, 1]. \]
Then there exists $b_{\max} < \infty$ so that for any $0 \leq \alpha < 1$ there exists $\alpha + 1 \leq b'(\alpha) < b_{\max}$ and $b > 1/b_{\max}$ so that if $b_1 \geq \alpha + 1$, $b_2 \geq b'(\alpha)$, max\{$b_1$, $b_2$\} $\leq 1/b'(\alpha)$ we have

\[(2.9) \quad \varphi \in C_2 \implies \mathcal{L}_\alpha(\varphi) \in C_2 \text{ and } N_\alpha(\varphi) \in C_2.\]

The proof of Proposition 2.4 is given in Appendix A.

For $\varphi \in L^1(dx)$ we set $m(\varphi) = \int_0^1 \varphi(x) \, dx$. For $a \geq 1$, we denote by $C_\ast = C_\ast(\alpha, a)$ the cone

\[(2.10) \quad C_\ast := \left\{ \varphi \in C^1(0, 1) \mid 0 \leq \varphi(x) \leq 2a\rho_\alpha(x)m(\varphi), \quad -\frac{\alpha + 1}{x}\varphi(x) \leq \varphi'(x) \leq 0, \quad \forall x \in (0, 1) \right\}.

By LSV Lemma 2.2, we have\[ (2.11) \quad \mathcal{L}_\alpha(C_\ast(\alpha, a)) \subset C_\ast(\alpha, a), \quad \forall a \geq 2^\alpha(\alpha + 2).\]

Note also that by definition (this will be used to show (2.17))

\[(2.12) \quad \int_0^{1/2} \varphi \, dx \geq \frac{1}{2} m(\varphi), \quad \forall \varphi \in C_\ast.

Finally, for $a \geq 2^\alpha(\alpha + 2)$ and $b_1 \geq \alpha + 1$, we denote by $C_{\ast, 1} = C_{\ast, 1}(\alpha, a, b_1)$ the cone

\[(2.13) \quad C_{\ast, 1} := \left\{ \varphi \in C^1(0, 1) \mid 0 \leq \varphi(x) \leq 2a\rho_\alpha(x)m(\varphi), \quad |\varphi'(x)| \leq \frac{b_1}{x}\varphi(x), \quad \forall x \in (0, 1) \right\}.

Note that (1.1) implies

\[(2.14) \quad \varphi(x) \leq \frac{2a c_2}{x^\alpha} m(\varphi), \quad \forall \varphi \in C_{\ast, 1}(\alpha), \quad \forall x \in (0, 1).\]

By definition, $C_{\ast, 1} \subset L^1(dx)$, and

\[(2.15) \quad \beta \geq \alpha \geq 0 \implies C_{\ast, 1}(\alpha, a, b_1) \subset C_{\ast, 1} \left(\beta, \frac{c_2}{c_1} a, b_1\right).

Also, the arguments of LSV give $a'(\alpha)$ so that

\[(2.16) \quad \rho_\alpha \in C_\ast(\alpha) \cap C_{\ast, 1}(\alpha) \cap C_2 \quad \text{if the parameters satisfy } a \geq a'(\alpha), \quad b_1 \geq \alpha + 1, \quad \text{and } b_2 > b'(\alpha).

Remark 2.5 (Cones $C_\ast$ and $C_{\ast, 1}$). The definition of $C_{\ast, 1}$ will make it easy to check that $-(X_\alpha \varphi)' + c_\varphi$, for suitable $\varphi \in C_\ast$ and constant $c_\varphi > 0$, lies in a cone $C_{\ast, 1}$ (possibly for larger $a$ and $b_1$), via the Leibniz formula. In (2.3) this will be used to get bounds $\varphi(x) \leq C x^{-\alpha}$ and $|\varphi'(x)| \leq C x^{-1-\alpha}$, while in Appendix B these cones play a more important role. This is why we use $C_{\ast, 1}$ instead of the cone $C_\ast$ used in LSV. However, the condition that $\varphi(x)$ is decreasing will be used to get (2.17). (In LSV the condition that $\varphi(x)$ be decreasing is only used in LSV Lemma 2.1, which we do not need in view of Proposition 2.4 and (B.1.).)

We will use the following result (see Appendix A for the proof):

\[6\text{Noting that } -\frac{\alpha + 1}{x} \varphi(x) \leq \varphi'(x) \leq 0 \text{ if and only if } \varphi \text{ is decreasing and } x^{\alpha + 1} \varphi \text{ is increasing.}\]
Proposition 2.6 (Invariance of the cone $C_{*,1}$). Fix $\alpha \in (0, 1)$, $a \geq 1$, and $b_1 \geq \alpha + 1$. Then

$$\mathcal{L}_\alpha(C_{*,1}(\alpha, a, b_1)) \subset C_{*,1}(\alpha, a, b_1),$$

(2.17) \hspace{1cm} \mathcal{N}_\alpha \left( C_{*,1}(\alpha, a, b_1) \cap \left\{ \int_0^{1/2} \varphi \, dx \geq \frac{1}{2} m(\varphi) \right\} \right) \subset C_{*,1}(\alpha, 2a, b_1).

In addition, there exists $C > 0$, independent of $\alpha$, $a$, and $b_1$ so that we have for any $\psi \in L^\infty$ and $\varphi \in C_{*,1}(\alpha) + \mathbb{R}$ with $\int \varphi \, dx = 0$,

$$\int_0^1 \psi \mathcal{L}_0^k(\varphi) \, dx \leq \frac{C_{ab} \psi}{(1 - \alpha)(\log k)^{k-1/\alpha} \|\psi\|_L^\infty \|\varphi\|_1}, \forall k \geq 1.$$ 

(2.18)

Note that for fixed $a_{max} < \infty$ and $b_{max} < \infty$, the expression (2.18) is controlled by

$$\sup_{b_1 \leq b_{max}, a \leq a_{max}} \frac{C_{ab} \psi}{(1 - \alpha)(\log k)^{k-1/\alpha}} < \infty.$$ 

(2.19)

2.3. Proof of Theorem 2.1. We may now prove the theorem:

Proof of Theorem 2.1 Step 0: We show that the right-hand side of (2.6) is well-defined for bounded $\psi$. First observe that integration by parts and $X_\alpha(0) = X_\alpha(1) = 0$ (because $v_\alpha(0) = 0$ and $v_\alpha(1/2) = 0$) imply

$$\int_0^1 (X_\alpha \mathcal{N}_\alpha(\rho_\alpha))' \, dx = 0.$$ 

Then note that

$$\| (X_\alpha \mathcal{N}_\alpha(\rho_\alpha))' \|_1 = \| X_\alpha(\mathcal{N}_\alpha(\rho_\alpha))' + X'_\alpha \mathcal{N}_\alpha(\rho_\alpha) \|_1 \leq C \int_0^1 (|\log x| + 1) \, dx < \infty.$$ 

(2.20) \hspace{1cm} \| (X_\alpha \mathcal{N}_\alpha(\rho_\alpha))' \|_1 = \| X_\alpha(\mathcal{N}_\alpha(\rho_\alpha))' + X'_\alpha \mathcal{N}_\alpha(\rho_\alpha) \|_1 \leq C \int_0^1 (|\log x| + 1) \, dx < \infty.

Indeed, this is easy for $\alpha = 0$ since

$$X_0(x) = \frac{x(\log 2 + \log(x/2))}{2}, \quad X'_0(x) = \frac{1 + \log 2 + \log(x/2)}{2}, \quad \forall x \in [0, 1].$$

Next, $\rho_0[0,1] \equiv 1$, with $(\mathcal{N}_0 \rho_0)[0,1] \equiv 1/2$, so that for any $x \in [0, 1]$,

$$X_0 \rho_0(\rho_0) = \frac{X'_0(x)}{2} = \frac{1 + \log 2 + \log(x/2)}{4}.$$ 

(2.22)

For $\alpha > 0$, on the one hand, (1.1) and Proposition 2.4 imply that

$$|\rho_\alpha(x)| \leq ac_2 x^{-1-\alpha}, \mathcal{N}_\alpha(\rho_\alpha)(x) \leq \rho_\alpha(x) \leq c_2 x^{-\alpha}, \| (\mathcal{N}_\alpha(\rho_\alpha))' \|_1 = \frac{c_2 b_1 x^{-1-\alpha}}{2}.$$ 

On the other hand, the dominant term of $X_\alpha(x)$ is a constant multiple of $x^{1+\alpha} \log x$ (see (2.3)) while the dominant term of $X'_\alpha(x)$ is a constant multiple of $x^{\alpha} \log x$ (see (2.4) and recall (2.2)). This establishes (2.20) for $\alpha > 0$.

Next, write the right-hand side of (2.6) as

$$\sum_{j=0}^\infty \int_0^1 \psi \cdot \mathcal{L}_\alpha^j [(X_\alpha \mathcal{N}_\alpha(\rho_\alpha))'] \, dx \leq \sum_{j=0}^\infty \int_0^1 \psi \cdot \mathcal{L}_\alpha^j [(X_\alpha \mathcal{N}_\alpha(\rho_\alpha))'] \, dx.$$ 

(2.23) \hspace{1cm} \sum_{j=0}^\infty \int_0^1 \psi \cdot \mathcal{L}_\alpha^j [(X_\alpha \mathcal{N}_\alpha(\rho_\alpha))'] \, dx \leq \sum_{j=0}^\infty \int_0^1 \psi \cdot \mathcal{L}_\alpha^j [(X_\alpha \mathcal{N}_\alpha(\rho_\alpha))'] \, dx.$$
The function \( f = -(X_\alpha N_\alpha(\rho_\alpha))'/\rho_\alpha \) is Hölder, and it vanishes at zero, with \( \int f \rho_\alpha \, dx = 0 \).

In addition, for any \( \epsilon \in (0, 1) \) we have \( C \) so that \( |f(x)| \leq C x^{\alpha(1-\epsilon/2)} \). Since

\[
\frac{1}{\alpha}(1 + \alpha(1 - \epsilon/2)) - 1 > \frac{1}{\alpha} - \epsilon,
\]

if \( \epsilon > 0 \) is small enough then \([\text{Goth}] \) Thm 2.4.14] applied \(^7\) to \( f = -(X_\alpha N_\alpha(\rho_\alpha))'/\rho_\alpha \) gives \( K_\alpha > 0 \) so that the \( j \)th term in the right-hand side of \((2.23)\) is bounded by

\[
\sum_{j=0}^{\infty} \left| \int_0^1 (\psi \circ T_\alpha^j) \cdot f \, d\mu_\alpha \right| \leq CK_\alpha \| \psi \|_{L^\infty} \frac{1}{\beta^{(1/\alpha) - \epsilon}}.
\]

Since we may take \( \epsilon < (1/\alpha) - 1 \), this is summable.

If \( \alpha = 0 \), fixing \( \beta > 0 \), it is easy to see that there exists a constant \( C_X > 0 \) so that \( -(X_0 N_0(\rho_0))' + C_X \) belongs to \( C_{s,1}(\beta, a, b_1) \), for suitable \( a \) and \( b_1 \) (see \((2.13)\)). We may apply \((2.18)\) from Proposition 2.6 to \( \varphi = -(X_0 N_0(\rho_0))' \in C_{s,1} + \mathbb{R} \) in order to bound the \( j \)th term in the right-hand side of \((2.23)\).

**Step 1:** Let \( \psi \) be a bounded function so that \( \int \psi \rho_\beta \, dx = 0 \). We first show that \( \beta \mapsto \int \psi \rho_\beta \, dx \) is Lipschitz at \( \beta = \alpha \). Applying the bound on \([\text{LSV}] \) p. 680] to \( g = \psi \) and the zero-average function \( f = \rho_\alpha - 1 \in C_{s,1} + \mathbb{R} \), we have, if \( \alpha > 0 \),

\[
(2.24) \quad \left| \int_0^1 \psi \circ T_\alpha^k \, dx \right| = \left| \int_0^1 \psi L_\alpha^k(\rho_\alpha - 1) \, dx \right| \leq C_{\alpha} \| \psi \|_{L^\infty} \frac{(\log k)^{1/\alpha}}{k^{1+1/\alpha}},
\]

and, for any \( \beta > 0 \),

\[
(2.25) \quad \left| \int_0^1 \psi \circ T_\beta^k \, dx - \int_0^1 \psi \, d\mu_\beta \right| \leq C_{\beta} \| \psi \|_{L^\infty} \frac{(\log k)^{1/\beta}}{k^{1+1/\beta}}.
\]

If \( \alpha = 0 \), then the spectral gap of \( L_0 \) on \( C^1 \) e.g. gives a constant \( C \geq 1 \) so that

\[
(2.26) \quad \left| \int_0^1 \psi \circ T_0^k \, dx \right| \leq C \| \psi \|_{L^\infty} 2^{-k}.
\]

Taking \( k \) large enough, depending on \( \beta \) and \( \alpha \), the three expressions \((2.24)-(2.25)-(2.26)\) are thus \( o(\beta - \alpha) \). More precisely, fixing \( \xi > 0 \), there is \( C \) so that, for all

\[
k > C(C_{\max(\alpha, \beta)}(\beta - \alpha))^{(-1+\xi)/\max(\alpha, \beta)}^{(-1+\xi)/\max(\alpha, \beta)}
\]

we have

\[
(2.27) \quad \left| \int_0^1 \psi \circ T_\alpha^k \, dx \right| + \left| \int_0^1 \psi \circ T_\beta^k \, dx - \int_0^1 \psi \, d\mu_\beta \right| \leq C \| \psi \|_{L^\infty}(\beta - \alpha)^{1+\xi}.
\]

\(^7\)This theorem is a strengthening of \([\text{Goth}] \) Prop. 6.11, Cor. 7.1.
Letting $1$ be the constant function $\equiv 1$, it thus suffices to bound
\[
\frac{1}{\beta - \alpha} \left( \int_0^1 \psi \circ T^k \, dx - \int_0^1 \psi \circ T^k_\alpha \, dx \right) = \frac{1}{\beta - \alpha} \int_0^1 \psi (\mathcal{L}_\beta^k 1 - \mathcal{L}_\alpha^k 1) \, dx 
\]
\[
= \sum_{j=0}^{k-1} \int_0^1 \psi \mathcal{L}_\beta^j \left( \frac{\mathcal{L}_\beta - \mathcal{L}_\alpha}{\beta - \alpha} (\mathcal{L}_\alpha^{k-j-1}(1)) \right) \, dx
\]
(2.28)
uniformly in $\beta \to \alpha$. For this, we shall use below that for any $\varphi \in C^2(0,1)$, any $x \neq 0$, and any $\beta \neq \alpha$, 
\[
\frac{\mathcal{L}_\beta \varphi(x) - \mathcal{L}_\alpha \varphi(x)}{\beta - \alpha} = \partial_\alpha \mathcal{L}_\alpha \varphi(x) + \frac{1}{\beta - \alpha} \int_\alpha^\beta \partial_\gamma^2 \mathcal{L}_\gamma \varphi(x)(\gamma - \alpha) \, d\gamma. 
\]
(2.29)
It is easy to check (see the proof of [B3 Thm 2.2]) that we have
\[
\partial_\alpha g_\alpha(x) = -X_\alpha(x)N_\alpha(1/T'_\alpha)(x) = -\frac{X_\alpha(x)}{T'_\alpha(g_\alpha(x))^2},
\]
and, more generally, for any $\varphi \in C^1(0,1]$ and any $x \neq 0$, we have
\[
\partial_\alpha \mathcal{L}_\alpha(\varphi)(x) = \partial_\alpha N_\alpha(\varphi) = \mathcal{M}_\alpha(\varphi)(x),
\]
where we set for $x \neq 0$
\[
\mathcal{M}_\alpha(\varphi)(x) = -X_\alpha N_\alpha(\varphi) - X_\alpha N_\alpha(\varphi/T'_\alpha)(x) + X_\alpha N_\alpha(\varphi T''_\alpha/(T'_\alpha)^2)(x) 
\]
\[
= -\left( X_\alpha N_\alpha(\varphi) \right)'(x).
\]
(2.30)
Using (2.31) and (2.32) (twice), we also get, for $x \neq 0$ and $\varphi \in C^2(0,1]$, 
\[
\partial_\alpha^2 \mathcal{L}_\alpha(\varphi)(x) = -\partial_\alpha \left( (X_\alpha N_\alpha(\varphi))' \right)(x)
\]
\[
= -\left( (\partial_\alpha X'_\alpha(N_\alpha(\varphi))) (x) - (X'_\alpha(\partial_\alpha N_\alpha(\varphi))) (x) 
\]
\[
= 0 \left( X_\alpha(\partial_\alpha N_\alpha(\varphi)) \right)'(x) - (X_\alpha(\partial_\alpha N_\alpha(\varphi))) (x)
\]
(2.33)
Returning to (2.28), we assume that $\beta > \alpha > 0$. For $k \geq 1$, we get, using (2.29)–(2.32),
\[
\sum_{j=0}^{k-1} \int_0^1 \psi \mathcal{L}_\beta^j \left( \frac{\mathcal{L}_\beta - \mathcal{L}_\alpha}{\beta - \alpha} (\mathcal{L}_\alpha^{k-j-1}(1)) \right) \, dx = - \sum_{j=0}^{k-1} \int_0^1 \psi \mathcal{L}_\beta^{j} \left[ \left. \mathcal{M}_\alpha(\mathcal{L}_\alpha^{k-j-1}(1)) \right\} \right] \, dx 
\]
(2.34)
Consider the first term in the right-hand side of (2.34). Observe that $\mathcal{L}_\alpha 1 \in C_{s,1}(\alpha, a, b_1) \cap C_2$, so that, recalling Proposition 2.4, we have that $\mathcal{L}_\alpha^{k-j-1}(1)$ is in $C_{s,1}(\alpha) \cap C_2$ and thus in $C_{s,1}(\gamma) \cap C_2$ for any $\gamma \geq \alpha$ up to increasing $a$ uniformly in $j$ and $k > j - 1$. Note that $|\mathcal{N}_\alpha(\mathcal{L}_\alpha^{k-j-1}(1))| \leq b_1 c_2 x^{1-\alpha}$. Proceeding as in Step 0 (using (2.11) to invoke (2.12)
to get (2.17), we obtain a constant $C > 1$ so that $|[X_0N_\alpha(L_0^{k-j-1}(1))]|'(x)| \leq C(|\log x| + 1)$ for all $1 \leq j \leq k - 1$ (in particular, $\sup_{1 \leq j \leq k - 1} \|X_0N_\alpha(L_0^{k-j-1}(1))\|_1 < \infty$).

Next, if $0 < \alpha < \beta < 1$, by using (2.29) and replacing (2.29) by $\|\| \leq$, we obtain a constant $10$.

Similarly (2.36)

Finally, we consider the second term of the right-hand side of (2.34). (This is where we require the derivatives of order two in $C_2$.) Applying (2.33) to $\varphi = L_\alpha^{k-j-1}(1) \in C_2 \cap C_{s,1}(\alpha)$, and using Proposition 2.4, we see that for any $\alpha \leq \gamma \leq \beta$,

(2.36) $||\partial^2 \varphi(L_\alpha^{k-j-1}(1))||(x) \leq C(|\log x| + 1)^2$.

Indeed, recalling (2.3) and (2.30), first note that $|\partial_\alpha X_\alpha(x)| \leq cx^{1+\alpha}(|\log x| + 1)^2$, $|\partial_\alpha X_\alpha'(\gamma)| \leq cx^\alpha(\log x)^2$, $|\partial_\alpha X_\alpha''(\gamma)| \leq cx^{\alpha-1}(\log x + 1)^2$.

Labelling the three terms from the right-hand side of (2.33) as $I$, $II$, and $III$, we expand them via the Leibniz equality, obtaining seven functions:

(2.37) $\begin{align*}
I &= -\partial_\alpha X_\alpha'(\gamma)N_\alpha(\varphi) + \partial_\alpha X_\alpha(N_\alpha(\varphi))', \\
II &= (X_\alpha')^2 N_\alpha(\varphi) + X_\alpha'(\gamma)N_\alpha(\varphi)', \\
III &= X_\alpha[X_\alpha''(\gamma)N_\alpha(\varphi) + 2X_\alpha'(\gamma)N_\alpha(\varphi)'] + X_\alpha(N_\alpha(\varphi))'.
\end{align*}$

By the above, since $\varphi = L_\alpha^{k-j-1}(1) \in C_2 \cap C_{s,1}(\alpha)$, we can bound $|I|$ by

$$
\begin{align*}
|I| &\leq (cx^{\alpha}(\log x + 1)^2)(cx^\alpha) + \left((cx^{1+\alpha}(\log x + 1)^2)^2 + \frac{b_1}{x^{\alpha}}m(\varphi)\right).
\end{align*}
$$

Similarly $|II| \leq (cx^\alpha(\log x + 1)(c(\log x + 1))$ and $|III| \leq (cx^{1+\alpha}(\log x + 1))c^x$. This proves (2.36).

Applying [Goth] Thm 2.4.14 once more (as in Step 0) we thus get the bound

(2.38) $\|\psi\|_{L^\infty} \int_\alpha^\beta (\gamma - \alpha)C_\beta \sum_{j=0}^{k-1} \frac{1}{j!(1/\beta - \epsilon)} d\gamma \leq C_\beta \|\psi\|_{L^\infty} (\beta - \alpha)$.

If $\alpha \in (0, 1)$, the case that $\beta < \alpha$ can be handled similarly, substituting

$$
\sum_j L_\beta^j(L_\alpha - L_\beta)L_\alpha^{k-j-1} = -\sum_j L_\alpha^j(L_\alpha - L_\beta)L_\beta^{k-j-1}
$$

in (2.28), and replacing (2.29) by $L_\alpha \varphi(\gamma) - L_\beta \varphi(\gamma) = \partial_\beta \varphi(\gamma) + \frac{1}{\alpha - \beta} \int_\alpha^\gamma \partial_\beta \varphi(\gamma) d\gamma$.

Finally, if $\alpha = 0$, we use $\sum_j L_\beta^j(L_\beta - L_0)L_0^{k-j-1} = -\sum_j L_\alpha^j(L_0 - L_\beta)L_\beta^{k-j-1}$, with (2.29), and exploit (2.18) and (2.19).
This proves that for \( \psi \in L^\infty \) the map \( \beta \mapsto \int \psi(x)\rho_\beta(x)\,dx \) is locally Lipschitz on \([0,1]\).

**Step 2:** Still assuming that \( \psi \) is bounded, we next prove that \( \beta \mapsto \int \psi\rho_\beta\,dx \) is differentiable at \( \beta = \alpha \in [0,1] \) and check that the derivative takes the announced value. To prove differentiability, recalling (2.27) and setting \( k(\beta) = k(\alpha, \beta, \xi) = C(C_\beta(\beta - \alpha)^{-(1+\xi)})^{1/(-1+1/\max(\alpha, \beta))} \), for some small \( \xi > 0 \), it suffices to check that
\[
(2.39) 
\sum_{j=0}^{k(\beta)} \int_0^1 \psi L_\beta^j([X_\alpha N_\alpha(L_\alpha^{k(\beta)-j}(1))'])\,dx + \int_\alpha^\beta \frac{\gamma - \alpha}{\beta - \alpha} \sum_{j=0}^{k(\beta)-1} \int_0^1 \psi L_\beta^j[\partial_\gamma^2 L_\gamma(L_\alpha^{k(\beta)-j-1}(1))]\,dx\,d\gamma,
\]
converges, when \( \beta \to \alpha > 0 \) or \( \beta \downarrow \alpha = 0 \), to
\[
(2.40) 
\sum_{j=0}^{\infty} \int_0^1 \psi L_\alpha^j([X_\alpha N_\alpha(\rho_\alpha')]')\,dx = \int_0^1 \psi \sum_{j=0}^{\infty} L_\alpha^j([X_\alpha N_\alpha(\rho_\alpha')]')\,dx.
\]

By (2.38), the second term in (2.39) converges to zero as \( \beta \to \alpha \). Next, for \( \alpha \in [0,1] \), fixing \( \eta > 0 \) small, by Step 0 we may take \( K = K_\eta \) large enough so that the \( K \)-tail of (2.40) is \( < \eta/4 \), while the \( K \)-tail of the first term of (2.39) is \( < \eta/4 \) uniformly in \( \beta \). It thus suffices, for every fixed \( 0 \leq j \leq K \), to show that the following difference tends to zero as \( \beta \to \alpha > 0 \) or \( \beta \downarrow 0 \):
\[
(2.41) 
\int_0^1 \left[ (\psi \circ T^j_\beta)([X_\alpha N_\alpha(L_\alpha^{k(\beta)}(1))'])' - (\psi \circ T_\alpha^j)([X_\alpha N_\alpha(\rho_\alpha')]') \right] \,dx.
\]

So it is sufficient to show that there exists \( N_\eta \geq 1 \) so that
\[
(2.42) \quad ||[X_\alpha N_\alpha(L_\alpha^{k(\beta)}(1))'] - [X_\alpha N_\alpha(\rho_\alpha')]'||_{L^1} < \frac{\eta}{2K}, \quad \forall k \geq N_\eta.
\]
If \( \alpha = 0 \), this is easy, since \( \rho_\alpha = 1 \) so that the expression (2.42) vanishes trivially.

If \( \alpha \in (0,1) \), setting \( \phi_k := L_\alpha^k(1) \), we note that the bound on [LSV], p. 680 applied to \( g = 1 \) and \( f = 1 - \rho_\alpha \) implies \( ||\phi_k - \rho_\alpha||_1 \leq C_\alpha k^{-1/\alpha} \log k^{1/\alpha} \). (This is not summable if \( \alpha \geq 1/2 \), but it tends to zero for all \( \alpha \in (0,1) \).) Therefore,
\[
(2.43) \quad ||N_\alpha(\phi_k - \rho_\alpha)||_1 \leq C_\alpha ||\phi_{k+1} - \rho_\alpha||_1 \leq C_\alpha \frac{(\log k)^{1/\alpha}}{k^{-1+1/\alpha}}.
\]
Since \( X'_\alpha \in L^\infty \), it thus suffices to show that
\[
||X_\alpha[N_\alpha(L_\alpha^k(1))'] - X_\alpha[N_\alpha(\rho_\alpha')]'||_{L^1} < \frac{\eta}{2K}.
\]
For this, first observe that Proposition 2.4 implies that there exists \( b_1 \) so that, for \( \varphi = \rho_\alpha \) and \( \varphi = 1 \),
\[
||N_\alpha(L_\alpha^k(\varphi))' ||_{L^1} < \frac{b_1}{x} \varphi(x) \leq \frac{b_1 a_2}{x^{1+\alpha}}, \quad \forall k \geq 1, \quad \forall x \in (0,1].
\]
Since \( |X_\alpha(z)| \leq c_\alpha |z|^{1+\alpha} (\log z + \log 2) \), it follows that there exist \( C, \tilde{C} > 0 \) so that for any \( \tilde{x} \in (0,1] \), all \( k \geq 0 \), and, for \( \varphi = \rho_\alpha \) and \( \varphi = 1 \),
\[
(2.44) \quad \int_0^{\tilde{x}} |X_\alpha(z)N_\alpha L_\alpha^k(\varphi)'(z)|\,dz \leq C \int_0^{\tilde{x}} (|\log z + 1)\,dz \leq \tilde{C} \tilde{x}(\log \tilde{x} + 1).
\]
We focus on the first term above, the estimate for the second one being easier. We shall set
\[ x = x_\ell, \] for suitable \( \ell \geq 1 \) to be determined below, where
\[
x_\ell := (g_\alpha)^\ell(1) \leq 2^{1/\alpha^2 + 1/\alpha \ell - 1/\alpha}
\]
by [LSV] Lemma 3.2]. Next, for every \( 1 \leq m \leq k \), setting \( y_m(x_\ell) = (g_\alpha)^m(x_\ell) = x_{\ell + m} \), and \( \phi = \phi_{k+1-m} - \rho_\alpha \), we have
\[
\|\chi_{x > x_\ell}|L_m^\alpha(\phi)'|\|_1 \leq \|L_m^\alpha(\chi_{y > y_m} \phi'/(|T_m^\alpha|))\|_1 + \|L_m^\alpha(\chi_{y > y_m} \phi||(|T_m^\alpha|)'/(|T_m^\alpha|)^2)\|_1
\]
\[
\leq \|\chi_{y > y_m} |\phi' - |(T_m^\alpha)'|^{-1}\|_1 + \|\chi_{y > y_m} \phi||(|T_m^\alpha|)'||/|T_m^\alpha|^{1/2}\|
\]
There exist \( \lambda_m = \lambda_m(x_\ell) < 1 \) and \( \Lambda_m(x_\ell) < \infty \) (both depending on \( \alpha \)) so that the first term in the right-hand side is \( < \lambda_m \|\chi_{y > y_m} \phi\|_1 \) and the second term is \( \leq \Lambda_m \|\phi\|_1 \). In fact, we claim that there exists \( C_\alpha > 0 \) so that for all \( \ell \)
\[
\lambda_m(x_\ell) \leq C_\alpha (1 + m/\ell)^{1/\alpha - 1/\alpha}
\]
Indeed, recalling that \( f_\alpha = T_0[0, 1/2] \), we have \( \lambda_m(x_\ell)^{-1} = (f_m^{\alpha})'(y) \) for some \( y \geq y_m(x_\ell) \), and bounded distortion\(^8\) of \( f_m^{\alpha} \) on \((y_m, f_\alpha(y_m)) = (y_m, y_{m-1})\) gives
\[
\lambda_m(x_\ell) \leq C_f \frac{f_\alpha(y_m) - y_m}{f_\alpha(x_\ell) - x_\ell} = C_f \frac{y_m^{1/\alpha}}{x_\ell^{1+\alpha}} \leq C_\alpha \left( \frac{1}{(1 + m/\ell)^{1/\alpha + 1/\alpha}} \right),
\]
where we used the upper bound \( y_m(x_\ell) = x_{\ell + m} \leq 2^{1/\alpha^2 + 1/\alpha \ell + m - 1/\alpha} \) from (2.45) and the lower bound from [BT] p. 606\(^9\) (replacing their \( x + x_\ell + \alpha \) by our \( x + 2^{\alpha} x_\ell + 1/\alpha \))
(2.47)
\[
x_\ell \geq c(2^\alpha \alpha)^{-1} \frac{1}{\alpha} \ell^{-1/\alpha}.
\]
Recalling that \( \phi = \phi_{k+1-m} - \rho_\alpha \), we get, if \( \alpha \in (0, 1) \),
\[
\|\chi_{x > x_\ell}|L_m^\alpha(\phi)'|\|_1 \leq \lambda_m(x_\ell) \|\chi_{y > y_m} (|\phi_{1-m}'| + |\rho_\alpha'|)\|_1 + \Lambda_m C_\alpha \left( \frac{1}{(k + 1 - m)^{1/\alpha}} \right)
\]
Recall that \( |\phi'(x)| \leq (a/x) \phi \) \( \leq C_0 x^{-\alpha-1} \) so that \( \|\chi_{y > y_m} \phi_{1-m}'\|_1 \leq Cy_m^{\alpha} \) and \( |\rho_\alpha'(x)| \leq c_2 b_1 x^{-\alpha-1} \), giving the same asymptotics, and note that (2.47) gives
\[
y_m(x_\ell) \geq c\alpha^{-1/2} (m + \ell)^{-1/\alpha}
\]
\(^8\)See e.g. [LSV] (2) p. 678\) for the bounded distortion property.
\(^9\)Note that \( \alpha + 1 \) should read \( \alpha - 1 \) in line 7 of the proof of [BT] Prop. 2, p 606\), that \( +\alpha(\alpha + 1)/(2u_{\alpha+1}) \) should be replaced by \( -\alpha(\alpha - 1)/(2u_{\alpha+1}) \) in line 8, that \( +\log n \cdot \alpha(\alpha + 1)/2 \) should be replaced by \( -\log(1 + \alpha n)/(1 + \alpha) \) \( \cdot (\alpha - 1)/2 \) on line 10, and that \( \log n \cdot \alpha(\alpha + 1)/(2n) \) should be replaced by \( -\log(1 + \alpha n) \cdot (\alpha - 1)/(2n) \) in line 12.
uniformly in \( k \). Hence, using (2.47) for \( y_m(\ell) = x_{m+\ell} \), we get

\[
\|\chi_{y>y_m(x_\ell)}(\alpha_{k+1-m} + |\alpha'|)\|_1 \leq C \int_{y_m}^{1} y^{-\alpha} \, dy \leq C y_m^{-\alpha} \leq C \alpha(m + \ell).
\]

Clearly, (2.46) implies

\[
C\alpha(m + \ell)\lambda_m(\alpha\ell) \leq C\alpha \frac{m + \ell}{(1 + m/\ell)^{1+1/\alpha}} \leq C\alpha \frac{\ell}{(1 + m/\ell)^{1/\alpha}}.
\]

Choosing first \( \ell \geq 1 \) to make (2.44) small, then \( m \geq \ell \) to make (2.48) small, and finally taking \( k \geq m \) large enough (i.e., \( \beta \) close enough to \( \alpha \)) so that

\[
\Lambda_mC\alpha(k + 1 - m)^{1/\alpha} = 1
\]

is small, proves (2.42) in view of (2.43) and (2.44).

This proves the result for bounded \( \psi \). If \( \|\psi\|_{L^q} = 1 \) for \( \frac{1}{\alpha} \leq q < \infty \), we observe that \( \text{Leb}(\{\psi(x) > M\}) \leq M^{-q} \) and define \( \psi_M(x) = \min(M, \psi(x)) \), noting that \( \|\psi - \psi_M\|_{L^1} \leq M^{1-q} \) and more generally \( \|\psi - \psi_M\|_{L^r} \leq M^{1-q/r} \) for \( r > 1 \) close to 1. Since \( |\log(2x)| \in L^{r/(r-1)}(dx) \) for all \( r > 1 \), we can generalise Steps 0 and 1 to \( \psi \in L^q(dx) \) the result by taking \( \epsilon > 0 \) very small and \( r > 1 \) so that \( q > r(1-\alpha)/(1-\alpha-\epsilon) \) and choosing \( \eta > 0 \) small enough so that \( \frac{1}{(q/r)-1} < \eta < \frac{1}{\alpha} - \epsilon - 1 \), taking \( M(j) = \psi^\eta \), and decomposing

\[
\psi = (\psi - \psi_M(j)) + \psi_M(j)
\]

in the \( j \)th term of (2.23), (2.35), and (2.34). We get two series each time. The first one is convergent because \( \eta((q/r)-1) > 1 \) while the second one converges because \( \eta + 1 < 1/\alpha - \epsilon \). For Step 2, we take \( M(k) = \psi^\eta \) for \( \eta \in (0, 1/\alpha - 1) \) in (2.41).

Finally, the claim about continuity of the derivative follows from the linear response formula and our control on the tails of the absolutely convergent series therein. \( \square \)

**Appendix A. Proof of Propositions 2.4 and 2.6**

*Proof of Proposition 2.4* The proof for the first derivative is similar to that of the proof of [LSV] Lemma 2.3, Lemma 5.1. We concentrate on the statement for \( \mathcal{N}_\alpha \), the proof for \( \mathcal{L}_\alpha \) follows easily since \( (\mathcal{L}_\alpha - \mathcal{N}_\alpha)(\varphi)(x) = \varphi((x + 1)/2) \). Let \( 0 \leq \alpha < 1 \). We have \( T_\alpha''(x) = 1 + 2\alpha(\alpha+1)x^\alpha \geq 1, T_\alpha''(x) = 2\alpha(\alpha+1)\alpha x^{\alpha-1} \geq 0, \) and \( T_\alpha'''(x) = 2\alpha(\alpha+1)\alpha (\alpha-1) x^{\alpha-2} \leq 0. \) Throughout this proof, we set (recall (2.1)) \( y = g_\alpha(x) \).

For \( \varphi \) as in the statement of the proposition, we have (both terms are positive)

\[
-(\mathcal{N}_\alpha \varphi)'(x) = \frac{T_\alpha''(y)}{(T_\alpha''(y))^2} \varphi(y) - \frac{1}{(T_\alpha''(y))^2} \varphi'(y)
\]

\[
\leq \left( \frac{T_\alpha''(y)}{(T_\alpha''(y))^2} + \frac{b_1}{y(T_\alpha''(y))^2} \right) \varphi(y)
\]

\[
\leq b_1 (\mathcal{N}_\alpha \varphi)(x) \sup_{y \in [0, 1/2]} \left[ \frac{T_\alpha''(y)}{b_1} \left( \frac{T_\alpha''(y)}{(T_\alpha''(y))^2} + \frac{b_1}{y(T_\alpha''(y))^2} \right) \right].
\]
Let \( \Omega_1(y) \) be the term in square brackets, then we find if \( b_1 \geq 1 + \alpha \)

\[
\Omega_1(y) = \frac{T_\alpha(y)}{b_1 T'_\alpha(y)} \left( \frac{y T''_\alpha(y)}{T'_\alpha(y)} + b_1 \right) \leq \frac{1 + 2^\alpha y^\alpha}{1 + 2^\alpha (1 + \alpha) y^\alpha} \cdot \left( \frac{1}{b_1} \frac{2^\alpha (\alpha + 1) y^\alpha}{1 + 2^\alpha (1 + \alpha) y^\alpha} + 1 \right)
\]

\[
= \left( 1 - \frac{2^\alpha y^\alpha}{1 + 2^\alpha (1 + \alpha) y^\alpha} \right) \cdot \left( 1 + \frac{1}{b_1} \frac{2^\alpha (\alpha + 1) y^\alpha}{1 + 2^\alpha (1 + \alpha) y^\alpha} \right)
\]

(A.1)

which is \( \leq 1 \) for all \( y \in [0, 1/2] \) (we used \( (\alpha + 1)/b_1 \leq 1 \) in the last line). Note that if \( \alpha = 0 \) then \( \Omega_1(y) = \frac{T_\alpha(y)}{y T'_\alpha(y)} \left( \frac{y T''_\alpha(y)}{T'_\alpha(y)} + 1 \right) \geq 1 \) if \( b_1 > 0 \) is small enough (just revisit (A.1)).

Next, writing \( T \) instead of \( T_\alpha \) for simplicity \( |(N_\alpha \varphi)''(x)| \) is bounded by (all terms below are nonnegative)

\[
-3 \frac{\varphi''(y) T''(y)}{(T'(y))^4} - \frac{\varphi(y) T'''(y)}{(T'(y))^4} + 3 \frac{\varphi''(y) (T''(y))^2}{(T'(y))^5} + \frac{\varphi''(y)}{(T'(y))^3}
\]

\[
\leq N_\alpha \varphi(x) \left( 3 \frac{b_1}{y} \frac{T''(y)}{(T'(y))^3} - \frac{T'''(y)}{(T'(y))^3} + 3 \frac{(T''(y))^2}{T'(y)^4} + \left( \frac{b_2}{y^2} \right) \frac{1}{(T'(y))^2} \right)
\]

\[
\leq \frac{b_2}{x^2} N_\alpha \varphi(x) \left[ \frac{T(y)^2}{b_2 |T'(y)|^2} \left( 3 \frac{b_1}{y} \frac{2^\alpha (\alpha + 1) y^\alpha - 1}{T'(y)} + \frac{2^\alpha (\alpha + 1) y^\alpha - 1}{T'(y)} + \right.ight.
\]

\[
+ \left. 3 \frac{2^\alpha (\alpha + 1) y^\alpha - 1}{(T'(y))^2} \right] \left( \frac{b_2}{y^2} \right) \right)
\]

The term \( \Omega_2(y) \) in square brackets can be written

\[
\frac{T(y)^2}{y^2 |T'(y)|^2} \left( 1 + \frac{2^\alpha y^\alpha}{(1 + 2^\alpha (\alpha + 1) y^\alpha)} \right) \frac{1}{b_2} \left[ 3 b_1 (\alpha + 1) + (1 - \alpha^2) + 3 \frac{2^\alpha (\alpha + 1) y^\alpha - 1}{1 + 2^\alpha (\alpha + 1) y^\alpha} \right]
\]

We can fix \( b_2 > 3 b_1 (1 + \alpha) + 20 \) large enough so that \( \Omega_2(y) \leq 1 \) for all \( y \in [0, 1/2] \) because

\[
\left( \frac{T(y)}{y T'(y)} \right)^2 = \left( \frac{1 + 2^\alpha y^\alpha}{1 + 2^\alpha (\alpha + 1) y^\alpha} \right)^2 = \left( 1 - \frac{2^\alpha y^\alpha}{1 + 2^\alpha (\alpha + 1) y^\alpha} \right)^2.
\]

(Not e that if \( \alpha = 0 \) then \( \Omega_2(y) \equiv 1 \).) For the reverse inequality, observe that

\[
\overline{\Omega}_2(y) := \left( 1 - \frac{2^\alpha y^\alpha}{1 + 2^\alpha (\alpha + 1) y^\alpha} \right)^2 \cdot \left( 1 + \frac{2^\alpha y^\alpha}{(1 + 2^\alpha (\alpha + 1) y^\alpha)} \right) \frac{1}{b_2} \left[ 3 b_1 (\alpha + 1) + (1 - \alpha^2) + 3 \frac{2^\alpha (\alpha + 1) y^\alpha - 1}{1 + 2^\alpha (\alpha + 1) y^\alpha} \right] \geq 1,
\]

if \( b_1/b_2 > 0 \) is large enough.

\[ \square \]

**Proof of Proposition 2.7** Note that \( m(\varphi) = m(\mathcal{L}_\alpha(\varphi)) \) and \( \mathcal{L}_\alpha(\rho_\alpha) = \rho_\alpha \). Also, \( m(\varphi) \leq 2 m(N_\alpha(\varphi)) \) (using our assumption) and \( N_\alpha(\rho_\alpha) \leq \rho_\alpha \). By (A.1) we have \( \mathcal{L}_\alpha(\mathcal{C}_{*,1}(\alpha)) \subset \mathcal{C}_{*,1}(\alpha, a, b_1) \) and \( N_\alpha(\mathcal{C}_{*,1}(\alpha)) \subset \mathcal{C}_{*,1}(\alpha, 2a, b_1) \).
For the decay claim, use that \( \mathcal{C}_*(0) \subseteq \mathcal{C}_*(\beta) \) for any \( \beta \in (0, 1) \), and fix \( \beta = \alpha \). For \( \epsilon > 0 \) we set
\[
\mathbb{H}_\epsilon \varphi(x) = (2\epsilon)^{-1} \int_{y \in S^1 : |x - y| < \epsilon} \varphi(y) \, dy.
\]
Then, revisiting [LSV] Proposition 3.3 for \( \alpha = 0 \), we see that we can take \( n_\epsilon \) there to be \( |\log \epsilon| / \log 2 \). Since [LSV] Lemma 3.1 implies
\[
\|L_0^n (\text{id} - \mathbb{H}_\epsilon) \varphi\|_1 \leq \frac{18ab_1c_2}{\beta(1-\beta)} \|\varphi\|_1 \quad \text{if} \quad \varphi \in \mathcal{C}_{s1}(\beta, b_1) ,
\]
and since \( \mathcal{C}_*(\alpha) \) is invariant under \( L_0 \), the first paragraph of the proof of [LSV] Thm 4.1, taking \( \epsilon = n^{-1/\alpha} \) gives \( (2.18) \). (Note that [LSV] Lemma 2.4 is not needed when invoking [LSV] Prop 3.3 in the proof of [LSV] Thm 4.1 for \( T_0 \), since we may obtain an easier lower bound.) \( \square \)

**Appendix B. A cone-only proof for \( \alpha \in [0, 1/2) \) and \( \psi \in L^\infty \)**

We show how to modify the proof of Theorem 2.1 to bypass the use of Gouëzel’s results \([\text{Go} \text{ Goth}]\) when \( \alpha < 1/2 \) and \( \psi \in L^\infty \) (exploiting only [LSV]).

We first note that in the setting of Proposition 2.6 there exists \( C_\alpha = C_\alpha(a, b_1) > 0 \), with \( \sup_{\beta \in [a, (1+a)/2]} C_\beta(a, b_1) < \infty \), so that for any \( \psi \in L^\infty \) and \( \varphi \in C_{s1}(\alpha) + \mathbb{R} \) with \( \int \varphi \, dx = 0 \),
\[
\int_0^1 \psi L_k^k(\varphi) \, dx \leq C_\alpha \|\psi\|_{L^\infty} \|\varphi\|_{L^1} \frac{(\log k)^{1/\alpha}}{k^{1-1/\alpha}} , \quad \forall k \geq 1 ,
\]
(\ref{B.1})

Indeed, note that [LSV] Lemma 3.1 applies to \( C_{s1} \) instead of \( C_* \), up to replacing 10\( a \) there by \( 18ab_1c_2 \). (In the computation, just use that \( |\varphi(x) - \varphi(y)| \leq \sup_{z \in [x, y]} |\varphi'(z)| \epsilon \leq 2ab_1c_2 \epsilon x^{-1-\alpha} \), if \( |x - y| \leq \epsilon \) with \( x \leq y \). The original 10 in [LSV] is obtained as \( 4 \times 2 + 2 \): the definition of our cone \( C_{s1} \) incorporates an additional factor of 2, to make \( 4 \times 2 \times 2 + 2 = 18 \) as well as introducing extra factor of \( c_2 \), as in (2.14), while the \( b_1 \) appears since we use the derivative of \( \phi \), as just noted. Finally, apply the argument \([\text{11}]\) in the first paragraph of \([\text{12}]\) the proof of [LSV] Thm 4.1. The proof gives \( C_\alpha = \frac{36ab_1c_2}{\alpha(1-\alpha)} 2^{(2+1/\alpha)(1/\alpha - 1)} \left( \frac{1}{\alpha - 1} \right)^{1/\alpha} \), for some small \( \gamma > 0 \). In particular, \( C_\alpha \) becomes very large as \( \alpha \to 0 \). This ends the proof of (B.1).

We need to introduce the following cone:
\[
C_3 = \left\{ \varphi \in C^3(0, 1) \mid \varphi \in C_2 , \, |\varphi'''(x)| \leq \frac{b_3}{x^3} \varphi(x) , \, \forall x \in (0, 1) \right\} .
\]

If \( b_3 \geq b_1 \) is large enough then the invariance statements of Proposition 2.4 also hold for \( C_3 \), indeed, noting that \( T_\alpha^{(iv)}(x) = 2^\alpha (\alpha + 1) \alpha (\alpha - 1) (\alpha - 2)x^{\alpha - 3} \geq 0 \), we have
\[
|\{N_\alpha \varphi''''(x)\} \leq \frac{|\varphi'''(y)|}{|T'(y)|^4} + 6 \frac{|\varphi''(y) T''(y)|}{|T'(y)|^5} + 4 \frac{|\varphi'(y) T'''(y)|}{|T'(y)|^5} + 15 \frac{|\varphi'(y) (T''(y))^2|}{|T'(y)|^6}
\]
\[
+ \frac{|\varphi(y) T''''(y)|}{|T'(y)|^6} + 6 \frac{|\varphi(y) T'''(y) T''(y)|}{|T'(y)|^6} + 15 \frac{|\varphi(y) (T''(y))^3|}{|T'(y)|^7} ,
\]

\(^{10}\)Identifying \([0, 1]\) with the circle \( S^1 \).
\(^{11}\)There is a typo there and one should take in fact \( \epsilon = n^{-1/\alpha} (2^{2+1/\alpha} \gamma^{-1} (\frac{1}{\alpha} - 1) \log n)^{1/\alpha} \).
\(^{12}\)Just like for [LSV] Prop. 5.4.
and this is bounded by

$$N_{\alpha}\varphi(x) \left( \frac{b_3}{y^3} \right) + \frac{6b_2}{y^2} \frac{1}{|T'(y)|^4} + \frac{1}{y} \left( \frac{|T'''(y)|}{|T'(y)|^4} + 15 \frac{|T''(y)|}{|T'(y)|^5} \right)$$

$$+ \frac{1}{y} \left( \frac{|T''(y)|}{|T'(y)|^4} + 4 \frac{|T'''(y)|}{|T'(y)|^5} + 6 \frac{|T''(y)||T'''(y)|}{|T'(y)|^6} + 15 \frac{|T''(y)|}{|T'(y)|^6} \right)$$

$$\leq \frac{b_3}{x^3} N_{\alpha}\varphi(x) \left[ \frac{T(y)^3}{|T'(y)|^3 y^3} \left( 1 + \frac{6b_2 y}{b_3 |T'(y)|} + \frac{b_1 y^2}{b_3} \left( \frac{4 |T''(y)|}{|T'(y)|} + 15 \frac{|T''(y)|}{|T'(y)|^2} \right) \right.$$

$$\left. + \frac{y^3}{b_3} \left( \frac{T''(y)}{|T'(y)|} + 4 \frac{|T'''(y)|}{|T'(y)|^2} + 6 \frac{|T''(y)||T'''(y)|}{|T'(y)|^3} + 15 \frac{|T''(y)|}{|T'(y)|^3} \right) \right]$$

The term $\Omega_3(y)$ in square brackets is $\leq 1$ for all $y \in [0, 1/2]$ if $b_3$ is large enough because

$$\left( \frac{T(y)}{y T'(y)} \right)^3 \left( \frac{1 + 2^\alpha y^\alpha}{1 + 2^\alpha (\alpha + 1) y^\alpha} \right)^3 \left( \frac{1 - 2^\alpha \alpha y^\alpha}{1 + 2^\alpha (\alpha + 1) y^\alpha} \right)^3.$$

It is easy to see that $\rho_0 \in C_3$, that $L_{\alpha} 1 \in C_{*,1}(\alpha, a, b_1) \cap C_3$, etc. In fact, each occurrence of $\Omega_2$ in the proof of Theorem 2.1 can be replaced by $\Omega_3$.

We now go over the changes in the proof of Theorem 2.1. Consider first Step 0: If $\alpha \in (0, 1/2)$ then, using (2.2), (2.3), (2.4), and (2.5), together with (2.12) and (2.17), and the fact that $|(N_{\alpha}(\rho_0))'| \leq a c_2 b_2 x^{-\alpha}$, it is easy to see that there exists $a > 0$ and a uniformly bounded constant $C_X > 0$ so that $-(X_{\alpha}'(N_{\alpha}(\rho_0)))' + C_X$ belongs to $C_{*,1}(\alpha, a, b_1)$, up to increasing $a$ and $b_1$ (see (2.13)). Next (B.1) applied to the zero-average function $\varphi = -(X_{\alpha}'(N_{\alpha}(\rho_0)))' \in C_{*,1} + \mathbb{R}$ gives $C_\alpha > 0$ so that the $j$th term in the right-hand side of (2.23) is bounded by

$$C_\alpha 3^{-1/\alpha} (\log j)^{1/\alpha} \|\psi\|_{L^\infty} \| (X_{\alpha}'(N_{\alpha}(\rho_0)))' \|_1.$$

Since $\alpha < 1/2$, this is summable.

In Step 1, proceeding as in Step 0 in §2.3 (using (2.11) to invoke (2.12) in order to get (2.17)), we find for any $1 \leq j \leq k - 1$ a real constant $C_{j,k} < \infty$ so that

$$\{ \pm X_{\alpha}'(N_{\alpha}(L_{\alpha}^{k-j-1}(1))) + C_{j,k} \cdot X_{\alpha}[N_{\alpha}(L_{\alpha}^{k-j-1}(1))'] \} \subset C_{*,1}(\alpha, 2a, B_1).$$

Indeed, to show (B.2), setting $\varphi = L_{\alpha}^{k-j-1}(1) \in C_\ast$, and noting that $\varphi \geq 0$ and $\varphi' \leq 0$, so that $N_{\alpha}\varphi \geq 0$ and $\{N_{\alpha}\varphi\}' \leq 0$ so that $X_{\alpha}(N_{\alpha}\varphi)' \geq 0$ it is enough to check that

$$|X_{\alpha}''(N_{\alpha}(\varphi))'(x)| + |X_{\alpha}'(N_{\alpha}(\varphi))'(x)| \leq \frac{B_1}{2x} |X_{\alpha}'(x)| N_{\alpha}(\varphi)(x),$$

(which follows from $-N_{\alpha}(\varphi)'(x) \leq b_1 N_{\alpha}(\varphi)(x)/x$ and (2.4), (2.5)), and

$$|X_{\alpha}'(N_{\alpha}(\varphi))' + X_{\alpha}(N_{\alpha}(\varphi))''(x)| \leq \frac{B_1}{2x} X_{\alpha}(x)(N_{\alpha}(\varphi))'(x),$$

(which follows from $N_{\alpha}(\varphi)''(x) \leq b_2 N_{\alpha}(\varphi)(x)/x^2 \leq -b_2 N_{\alpha}(\varphi)'(x)/b_1 x$ and (2.3), (2.4)).
Next, if $0 < \alpha < \beta < 1/2$, by using $C_{1, \ast}(\alpha) \subset C_{1, \ast}(\beta)$ and \eqref{B.1}, we get summability of the first term of the expression in the right-hand side of \eqref{2.34} as $k \to \infty$:

$$
\left| \sum_{j=0}^{k-1} \int_0^1 \psi \mathcal{L}_j^j(\mathcal{L}_\alpha^{-j-1}(1))' dx \right| \leq C_\beta \|\psi\|_{L^\infty} \sum_{j=0}^{k-1} (\log j)^{1/\beta}.
$$

When we consider the second term of the right-hand side of \eqref{2.34}, we require the derivatives of order three in $C_3$. Applying \eqref{2.33} to $\varphi = \mathcal{L}_\alpha^{-j-1}(1) \in C_3 \cap C_{\ast, 1}(\alpha)$, and using Proposition \ref{2.4}, we see that for any $\alpha \leq \gamma \leq \beta$, up to taking larger $a$ and $b_1$ (uniformly in $1 \leq j \leq k - 1$) the decomposition \eqref{2.37} of $\partial_3^2 L_\gamma(\mathcal{L}_\alpha^{-j-1}(1))$ gives seven functions which, up to multiplying by $-1$ and adding a uniformly bounded constant, all lie in $C_{\ast, 1}(\alpha, a, b_1) \subset C_{\ast, 1}(\beta)$. We proceed as in the proof of \eqref{B.2}, developing the Leibniz inequality. We shall focus on the contribution of $X_\alpha^2(\mathcal{N}_\alpha(\varphi))'$, leaving the other terms to the reader. We need to check that

$$
|2X_\alpha X'_\alpha(\mathcal{N}_\alpha(\varphi))'' + X_\alpha^2(\mathcal{N}_\alpha(\varphi))'''| \leq B_1 |X_\alpha^2(\mathcal{N}_\alpha(\varphi))'''|.
$$

The above bound follows from $(\mathcal{N}_\alpha(\varphi))'''(x) \leq b_3 \mathcal{N}_\alpha(\varphi)(x)/x^3 \leq b_3 (\mathcal{N}_\alpha(\varphi))''(x)/(\bar{b}_2 x)$ and \eqref{2.3}, \eqref{2.4}. Since we are in a cone, we may apply \eqref{B.1} once more, we thus get the bound

$$
\frac{\|\psi\|_{L^\infty}}{\beta - \alpha} \int_0^\beta (\gamma - \alpha) C_\beta \sum_{j=0}^{k-1} (\log j)^{1/\beta} d\gamma \leq C_\beta \|\psi\|_{L^\infty} (\beta - \alpha).
$$

Step 2 does not change, and the proof of Theorem \ref{2.1} bypassing \cite{Go, Goth} is complete.

---

**References**


