NUMERICAL SIMULATIONS OF SUNSPOT ROTATION
DRIVEN BY MAGNETIC FLUX EMERGENCE

Zoe Sturrock

A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

2016

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Zoe Sturrock

University of St Andrews

This thesis is submitted in partial fulfilment for the degree of PhD at the
University of St Andrews

December 8, 2016
Abstract

Magnetic flux continually emerges from the Sun, rising through the solar interior, emerging at the photosphere in the form of sunspots and expanding into the atmosphere. Observations of sunspot rotations have been reported for over a century and are often accompanied by solar eruptions and flaring activity. In this thesis, we present 3D numerical simulations of the emergence of twisted flux tubes from the uppermost layers of the solar interior, examining the rotational movements of sunspots in the photospheric plane. The basic experiment introduces the mechanism and characteristics of sunspot rotation by a clear calculation of rotation angle, vorticity, magnetic helicity and energy, whereby we find an untwisting of the interior portion of the field, accompanied by an injection of twist into the atmospheric field. We extend this model by altering the initial field strength and twist of the sub-photospheric tube. This comparison reveals the rotation angle, helicity and current show a direct dependence on field strength. An increase in field strength increases the rotation angle, the length of fieldlines extending into the atmosphere, and the magnetic energy transported to the atmosphere. The fieldline length is crucial as we predict the twist per unit length equilibrates to a lower value on longer fieldlines, and hence possesses a larger rotation angle. No such direct dependence is found when varying the twist but there is a clear ordering in rotation angle, helicity, and energy, with more highly twisted tubes undergoing larger rotation angles. We believe the final angle of rotation is reached when the system achieves a constant degree of twist along the length of fieldlines. By extrapolating the size of the modelled active region, we find rotation angles and rates comparable with those observed. In addition, we explore sunspot rotation caused by sub-photospheric velocities twisting the footpoints of flux tubes.
Declarations

I, Zoe Sturrock, hereby certify that this thesis, which is approximately 60,000 words in length, has been written by me, and that it is the record of work carried out by me, or principally by myself in collaboration with others as acknowledged, and that it has not been submitted in any previous application for a higher degree.

I was admitted as a research student in September 2013 and as a candidate for the degree of Ph.D in September 2014; the higher study for which this is a record was carried out in the University of St Andrews between 2013 and 2016.

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I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate for the degree of Ph.D in the University of St Andrews and that the candidate is qualified to submit this thesis in application for that degree.

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Acknowledgements

My time in St Andrews would not have been possible without many people. I am grateful for the support and friendship of all those with whom I have shared my postgraduate studies.

Firstly, I would like to gratefully acknowledge the financial support of the Carnegie Trust for the Universities of Scotland, which made this work possible.

I would like to thank my supervisor Alan for his help and guidance throughout the course of my PhD. I would also like to extend my gratitude and thanks to all the staff and students in the Maths department who made the last three years so enjoyable. Particular thanks go to Crisi, Jenny and Rachael, whose friendship has brightened both my work days and weekends. I do not know how I would have survived without our long coffee and lunch breaks.

I wish to thank my family for all their support throughout my studies in St Andrews. Especially to my parents for always believing in me and helping me to strive for my goals. Special thanks also go to my brothers, grandparents and aunties who have offered a welcome distraction from all things work related.

Most of all, thank you to Mark for all his love and support during my studies. Thank you for always being there to listen to my problems, and making me smile through the highs and lows. I could not have done this without you.
# Contents

1 Introduction ............................................................. 1

1.1 The Sun ......................................................................... 1

1.1.1 Structure of the Sun ................................................... 2

1.1.2 Sun’s magnetic field .................................................... 5

1.2 The MHD equations ....................................................... 8

1.2.1 Maxwell’s equations ................................................... 10

1.2.2 Fluid equations ........................................................ 11

1.2.3 Ohm’s Law ............................................................... 12

1.2.4 Summary of MHD equations ....................................... 12

1.2.5 Derived equations and quantities ................................. 13

1.3 Flux tubes .................................................................. 18

1.3.1 Definition of a flux tube ............................................. 18

1.3.2 Properties of a flux tube .............................................. 19

1.3.3 Gold-Hoyle flux tubes ................................................. 20

1.4 Magnetic buoyancy ....................................................... 23

1.4.1 Buoyancy instability ................................................... 24

1.4.2 Magnetic buoyancy instability ..................................... 28

1.5 Modelling active regions ................................................. 32

1.5.1 2D and 2.5D simulations ............................................ 34
1.5.2 3D simulations .......................................................... 35
1.6 Rotation of sunspots ..................................................... 39
1.7 Outline ................................................................... 42

2 Numerical code ................................................................. 45
  2.1 Lare1d ................................................................. 45
     2.1.1 Grid .......................................................... 46
     2.1.2 Lagrangian step ............................................ 48
     2.1.3 Remap step .................................................. 50
     2.1.4 Gradient limiters .......................................... 52
     2.1.5 Shock viscosity ............................................ 54
  2.2 Test cases ................................................................. 54
     2.2.1 1D Euler experiment .................................... 54
     2.2.2 Riemann problem ........................................ 56
  2.3 Lare3d ................................................................. 59
     2.3.1 Equations and normalisation ......................... 60
     2.3.2 Lagrangian step and remap step .................... 62
  2.4 Summary ............................................................... 64

3 Initial set-up ................................................................. 65
  3.1 Analytic stratification ............................................... 65
     3.1.1 Solar interior ............................................... 66
     3.1.2 Atmosphere ............................................... 68
  3.2 Numerical stratification ............................................. 71
  3.3 Choice of sub-photospheric magnetic flux tube ............ 74
     3.3.1 Cylindrical magnetic field ............................ 75
     3.3.2 Toroidal magnetic field ............................... 76
3.4 Summary ................................................................. 82

4 Sunspot rotation due to flux emergence 83

4.1 Parameter choice .................................................. 83
4.2 General analysis ..................................................... 84
  4.2.1 Rise through solar interior ................................. 84
  4.2.2 Arrival at the photosphere ................................. 85
4.3 Rotation analysis .................................................. 88
  4.3.1 Evolution of magnetic field ................................. 88
  4.3.2 Rotation angle ............................................... 90
  4.3.3 Driver of rotational motion ............................... 94
  4.3.4 Plasma vorticity ........................................... 99
  4.3.5 Current density ............................................ 101
  4.3.6 Fieldline twist ............................................. 103
  4.3.7 Force-free parameter ..................................... 107
  4.3.8 Magnetic helicity .......................................... 111
  4.3.9 Magnetic energy ........................................... 114
  4.3.10 Propagation of torsional Alfvén wave ................. 115
4.4 Summary ............................................................ 115

5 Effects of varying the field strength and twist of an emerging flux tube 117

5.1 Parameter choice .................................................. 119
5.2 Group 1 - Varying $B_0$ with fixed $\alpha$ ...................... 121
  5.2.1 General evolution ........................................... 121
  5.2.2 Torque ....................................................... 123
  5.2.3 Rotation angle ............................................. 124
  5.2.4 Twist ......................................................... 126
5.2.5  Vorticity ................................................................. 130
5.2.6  Current density ..................................................... 131
5.2.7  Magnetic helicity .................................................. 133
5.2.8  Magnetic energy ................................................... 135
5.3  Group 2 - Varying $\alpha$ with fixed $B_0$ ......................... 136
  5.3.1  General evolution ................................................ 136
  5.3.2  Rotation angle .................................................... 138
  5.3.3  Twist ............................................................... 139
  5.3.4  Vorticity .......................................................... 140
  5.3.5  Current density .................................................. 141
  5.3.6  Magnetic helicity ................................................ 142
  5.3.7  Magnetic energy ................................................ 143
5.4  Comparison with observations ..................................... 143
5.5  Summary .............................................................. 148

6  Evolution of idealised twisted magnetic flux tube ............... 151
  6.1  Initial set-up ....................................................... 152
  6.2  Length of box comparison ....................................... 154
  6.3  Magnetic field strength comparison ............................ 161
  6.4  Summary ............................................................ 164

7  Sunspot rotation due to sub-photospheric velocities .......... 167
  7.1  Standard case ...................................................... 168
  7.2  Case 1 - vary size of driver .................................... 174
  7.3  Case 2 - vary velocity of driver ............................... 176
  7.4  Case 3 - vary field strength of tube ......................... 177
  7.5  Case 4 - vary number of drivers .............................. 179
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.6</td>
<td>Case 5 - switch off drivers</td>
<td>181</td>
</tr>
<tr>
<td>7.7</td>
<td>Summary</td>
<td>182</td>
</tr>
<tr>
<td>8</td>
<td>Conclusions and future work</td>
<td>185</td>
</tr>
<tr>
<td>A</td>
<td>Calculation of magnetic helicity</td>
<td>191</td>
</tr>
<tr>
<td>A.1</td>
<td>DeVore’s method</td>
<td>191</td>
</tr>
<tr>
<td>A.2</td>
<td>Moraitis’ method</td>
<td>192</td>
</tr>
<tr>
<td>A.3</td>
<td>Comparison of two methods</td>
<td>193</td>
</tr>
<tr>
<td>A.4</td>
<td>Resistivity comparison</td>
<td>194</td>
</tr>
<tr>
<td>B</td>
<td>Asymmetry in sunspot formation</td>
<td>195</td>
</tr>
<tr>
<td>B.1</td>
<td>Two loop model</td>
<td>196</td>
</tr>
<tr>
<td>B.1.1</td>
<td>Initial set-up</td>
<td>196</td>
</tr>
<tr>
<td>B.1.2</td>
<td>Preliminary results</td>
<td>197</td>
</tr>
<tr>
<td>B.2</td>
<td>Varying buoyancy</td>
<td>199</td>
</tr>
<tr>
<td>B.3</td>
<td>Summary and future work</td>
<td>200</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

The scientific research comprising this thesis considers three-dimensional (3D) numerical simulations to model and investigate emerging magnetic flux on the Sun. In this chapter, we provide any necessary background and a motivation for the study of flux emergence, with a particular emphasis on the influence of the magnetic field on photospheric velocities. To begin, we describe basic details of the Sun in Section 1.1, with a focus on solar features relevant to flux emergence. In order to model the Sun on a macro-scale, we solve the magnetohydrodynamics (MHD) equations. The MHD approximation, together with its associated equations and the assumptions under which it is valid is presented in Section 1.2. In addition, we discuss derived quantities from the MHD equations, for example magnetic helicity and energy as these have important consequences later in this thesis. Flux tubes are the building blocks of magnetic field on the Sun, and as such we consider features and properties of flux tubes in Section 1.3. The key process underpinning the journey of magnetic flux tubes through layers of the Sun, namely magnetic buoyancy, is introduced in Section 1.4. In Section 1.5, a short summary of past efforts of numerical simulations of flux emergence is given. We introduce the topic of sunspot rotation, detailing previous observations and simulations investigating this phenomenon in Section 1.6. Finally in Section 1.7, we draw together all introductory information to present the aims of this thesis as well as a brief outline of the chapters that follow.

1.1 The Sun

As the star at the centre of our Solar System, the Sun is by far the most important source of energy for life on Earth. Although it is one of billions of stars in our galaxy, its close proximity to Earth makes it the most studied star in the universe. The Sun has been studied to varying degrees since 2000BC and provides a natural laboratory for studying plasmas and large scale energy events.

The Sun (shown in Fig. 1.1) is a large, almost perfect, sphere of plasma with a radius of \(6.96 \times 10^8\) m, roughly 109 times that of the Earth’s radius. Furthermore, the Sun has a mass about 330,000 times that of Earth, containing more than 99% of the total mass of the Solar System. The Sun is estimated to be
1.1 The Sun

Figure 1.1: An image of the Sun taken using the Atmospheric Imaging Assembly (AIA) 304 Å filter on Solar Dynamics Observatory (SDO) taken on the 12th of September 2014 with the Earth drawn for a scale comparison [Image courtesy NASA].

approximately $4.6 \times 10^9$ years old and is mainly made up of hydrogen. As stated before, the Sun is made up of the fourth state of matter, known as plasma. A plasma is formed when a gas is heated to high temperatures, stripping atoms of their electrons. This results in a sea of positive ions and negative electrons. The movement of these particles within the plasma can create electric and magnetic fields. In reality, a local charge imbalance produces an electric field with a spatial range defined as the Debye length, $\lambda_D$. In fact, a plasma may be defined as an ionised gas for which the number of particles in a sphere of radius $\lambda_D$ is very large (Priest, 2014).

Solar observations date back to 2000 BC, when the Chinese first recorded solar eclipses (Priest, 1982). However, it was not until 1530 that Copernicus understood the importance of the Sun in our solar system, when he suggested that the six known planets revolved around the Sun at the centre of our Solar System. More recently, the first telescope observations were performed in 1610. Since this time, our understanding of the Sun has vastly improved due to huge advances in solar telescopes that have provided both quantity and quality of solar observations. Over the past fifty years, there have been numerous exciting satellite missions such as Skylab, Solar and Heliospheric Observatory (SOHO), Transition Region and Coronal Explorer (TRACE), and Solar Dynamics Observatory (SDO), amongst others. The combination of developments in solar observations with huge advancements in theory and computational modelling of solar processes has lead to leaps in our understanding. This makes the Sun a particularly fascinating object to study.

1.1.1 Structure of the Sun

The Sun’s structure is commonly divided into the solar interior and solar atmosphere, as separated by the visible surface of the Sun named the photosphere. Within each of these subdomains, the structure is further divided into different zones depending on the dominant physical processes and properties of each region,
as shown clearly in Fig. 1.2.

Solar interior

The interior of the Sun is split into three domains: the core, the radiative zone, and the convection zone as shown in Fig. 1.2. The dense core of the Sun extends to 0.25 solar radii ($R_\odot$) and is the region in which nuclear processes occur. The high temperature (ranging from $7 \times 10^6$ K to $1.5 \times 10^7$ K) of the core allows fast moving hydrogen to fuse together to produce helium. The high energy gamma rays produced by this process travel outward to the rigidly rotating radiative zone, the zone just above the core that extends to $0.7R_\odot$. The high energy gamma-rays are continually absorbed and emitted as they bounce from particle to particle, shifting to longer wavelengths as they do so, over a timescale of hundreds of thousands of years before they leave the radiative zone. This leads to a temperature gradient where the temperature drops from $7 \times 10^6$ K to $2 \times 10^6$ K at the base of the next region, the convection zone. On the lower boundary of the convection zone, there is a strong shear layer named the tachocline.

Figure 1.2: Artistic interpretation of the layers comprising the solar interior and atmosphere [Image courtesy NASA].

The convection zone extends from $0.7R_\odot$ (200,000 km below the surface) to the photosphere as the temperature drops to approximately 6000 K. This drop in temperature allows the hot plasma at the bottom of the convection zone to rise up through the convection zone then sink as it cools. As the cool plasma sinks to the bottom of the convection zone it is heated and once again rises through the zone, thus transporting energy through convection. This region is convectively unstable and hence is subject to vigorous convective motions that efficiently transport energy from the interior to the solar surface. This occurs over a much shorter time scale (period of days) and produces a bubbling effect at the photosphere at the top of the convection zone.
1.1 The Sun

The Sun

Figure 1.3: (a) Example of photospheric granulation. High-resolution G-band image from the Swedish 1 m Solar Telescope and Institute of Theoretical Physics, Oslo. (b) An image of the Sun taken using the AIA 171 Å filter on SDO taken on the 12th of September 2014 [Image courtesy NASA].

Solar atmosphere

The atmosphere of the Sun lies above the solar surface and extends to $10R_\odot$. The solar atmosphere is split into three main regions with different physical properties: the photosphere; chromosphere; and corona. The photosphere, as named after the Greek word for light “photos”, is seen from the Earth as it emits light on the visible spectrum (see Fig. 1.4a). It is a relatively dense, thin layer of plasma (only a few 100 km thick with a density of approximately $10^{-3}$ kg/m$^3$) at which photons can finally escape from the Sun and hence is where the majority of light emission comes from. At a temperature of approximately 6000 K, the solar surface appears to be continually changing as hot plasma from below rises up and bubbles at the Sun’s surface. This effect is known as granulation. The photosphere is tiled with millions of irregularly shaped granules (see Fig. 1.3a) at any one time on a variety of scales (Priest, 2014). Found at the top of convection cells, the centre of granules are bright due to hot rising material that subsequently flows horizontally to the boundaries (intergranular lanes) which are dark due to cool falling plasma (as intensity and temperature can be related by Stefan-Boltzmann law). Granules continually form and disappear in a turbulent manner with a mean life-time of only 5 − 10 minutes. A typical granule is of size 1 Mm but super-granules can be as large as 30 Mm, with a lifetime of about a day.

The layer above the photosphere, known as the chromosphere, is approximately 2500 km thick and and is marked by a slight increase in temperature with height, reaching about 20000 K, and a rapid drop off in density. Above this lies a very narrow region, known as the transition region which is characterised by a sharp increase in temperature to around 1 MK. The outer-most layer of the solar atmosphere is known as the solar corona which extends from the transition region into interplanetary space. Accompanied by
the increase in temperature, the density also drops off in the corona to about $10^{-11}$ kg/m$^3$. Although the solar corona was first seen in visible light during an eclipse, the corona can always be observed in extreme ultra-violet (EUV) and soft X-rays, as shown in Fig. 1.3b. A particularly interesting feature of the corona is the increase in temperature moving away from the solar surface. The solar corona’s surprisingly high temperature has become one of the exciting, open questions in the solar physics community. We direct the reader to De Moortel and Browning (2015) for a full review of the coronal heating problem as we do not consider this in this thesis.

### 1.1.2 Sun’s magnetic field

Magnetic fields thread in and out of the photosphere, permeating through the entire Sun. The Sun exhibits magnetic activity over a wide variety of spatial scales. Almost all of the interesting visible features on the Sun are influenced by the magnetic field. Hence we discuss key features and phenomena relevant to the scientific research contained in this thesis.

![Figure 1.4: Comparisons of the sunspot and magnetic field configurations at the photosphere. (a) Continuum image of the Sun taken using the Helioseismic and Magnetic Imager (HMI) on SDO on the 12th of September 2014 and (b) shows the HMI magnetogram taken at the same time. [NASA]](image)

**Magnetic flux emergence**

The Sun’s magnetic field is generated within the interior by the solar dynamo due to a rotation of plasma. Dynamo action is said to occur if a magnetic field is continuously regenerated by inductive motions within an electrically-conducting fluid. A key component of solar dynamo theory focuses on the way in which toroidal (azimuthal) and poloidal (meridional) field components are generated. See Priest (2014) for details of several dynamo theories. The tachocline lies between the rigidly rotating radiative zone and differentially
rotating convection zone. The change in rotational regimes across the tachocline causes shear motions to amplify the diffuse magnetic field into strong toroidal flux ropes. Once their field strength exceeds about $10^5$ G, undular instabilities form along the tubes allowing for the formation of loops which can rise to the surface over a timescale of months (Murray, 2007). Some magnetic flux loops rise to the surface, uninhibited by turbulent flows associated with convection, and flux emergence results from their intersection with the photosphere. The intersection of magnetic flux tubes with the photosphere results in the formation of sunspots, and active regions (Zwaan, 1985 and references therein).

### Sunspots and active regions

As mentioned previously, the photosphere emits the majority of the visible light on the Sun and, to most of us, may appear as a perfect globe without a blemish. However, detailed scrutiny of observations indicate that the white light photosphere is marked with dark spots known as sunspots, shown in Fig. 1.4a.

The observations and study of sunspots has had a wide history. As a feature that can be detected by the naked eye, the earliest records of sunspot observations date back to 325 BC by Theophrastus, a pupil of Aristotle in Athens. From 23 BC, Chinese astronomers continued to chart observations of sunspots but had little understanding of the phenomena. It was not until the invention of the telescope in 1610 that sunspots could be studied in greater detail. The first telescopic observations of sunspots were undertaken by Thomas Harriot and Johannes and David Fabricius independently. David Fabricius, alongside his eldest son, Johannes discovered that sunspots were moving, thereby securing the first evidence that the Sun rotated on its axis. Later, Heinrich Schwabe’s detailed sketches of sunspot activity lead to the discovery of the periodic variation of the number of sunspots in 1843. Later in 1908, the breakthrough that sunspots possess a strong magnetic field was made by American solar astronomer, George Ellery Hale, with the invention of the spectroheliograph.

Sunspots are the most readily visible manifestations of solar magnetic field concentrations and appear darker and cooler than the surrounding plasma (Solanki, 2003). They are cooler than the rest of the photosphere due to the inhibition of convection by strong magnetic fields in the interior (Bray and Loughhead, 1964). Sunspots have the strongest magnetic field of any feature in the solar atmosphere. An example of such sunspots is shown in Figs. 1.4a and 1.4b. These images show the same observation as a continuum image and a magnetogram, respectively. The magnetogram shows the line of sight magnetic field, where the white regions represent magnetic field pointing out of the Sun and the black regions represent magnetic field pointing into the Sun. From the images, it is clear that the regions of strong magnetic field align with the dark features on the visible surface, shown in the continuum image.

Observations show that magnetic fields emerge onto the solar surface on a broad range of scales (Schrijver et al., 1998). Groups of sunspots form active regions that arise from coherent flux bundles. Table 1.1 classifies active regions by the amount of photospheric flux and the lifetime of the region. Ephemeral regions are at the smallest end of the size-spectrum of active regions with typical magnetic fluxes less than $1 \times 10^{20}$ Mx (Harvey and Martin, 1973). Small active regions are larger than ephemeral regions and contain more magnetic flux to the upper bound of $5 \times 10^{21}$ Mx. They do not contain sunspots but instead contain pores (smaller magnetic features described below). Large active regions contain much larger amounts of
1.1 The Sun

magnetic flux and have a much longer lifetime than smaller regions.

Table 1.1: Typical magnetic fluxes and lifetimes of active regions (based on van Driel-Gesztelyi and Green, 2015).

<table>
<thead>
<tr>
<th>Region</th>
<th>Magnetic Flux (Mx)</th>
<th>Lifetime</th>
<th>Rise/Lifetime</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large (with sunspots)</td>
<td>$5 \times 10^{21} - 3 \times 10^{22}$</td>
<td>weeks - months</td>
<td>15 – 30%</td>
</tr>
<tr>
<td>Small (with pores, no sunspots)</td>
<td>$1 \times 10^{20} - 5 \times 10^{21}$</td>
<td>days - weeks</td>
<td>15 – 27%</td>
</tr>
<tr>
<td>Ephemeral</td>
<td>$3 \times 10^{18} - 1 \times 10^{20}$</td>
<td>hours - days</td>
<td>$\approx$ 30%</td>
</tr>
</tbody>
</table>

Figure 1.5: (a) Close up of a sunspot taken by the Big Bear Solar Observatory in July 2010 and (b) a schematic illustration of the magnetic field of a sunspot.

Sunspots come in a range of shapes and sizes with diameters from 3 Mm to 60 Mm (Priest, 2014). Sunspots are comprised of a central dark umbra with a strong, almost vertical magnetic field surrounded by a lighter penumbra characterised by a weaker more horizontal magnetic field as shown schematically in Fig. 1.5b. Fig. 1.5a shows the fine detail of a sunspot in a recent observation, clearly depicting the dark umbral centre and brighter filamentary penumbra that surrounds it. In general, sunspot umbrae have diameters ranging from 10 Mm to 20 Mm with the strongest vertical magnetic field found here (typically 2000–3000 G). The penumbra is comprised of dark radial filaments, typically 5000 to 7000 km long and 300 to 400 km wide (Priest, 1982). The magnetic field geometry of the penumbral region is surprisingly complex and is a consequence of overturning convective motions (Rempel, 2012). Pores are penumbra-less strong magnetic field features on the photosphere less than 5Mm in diameter (van Driel-Gesztelyi and Green, 2015).

Based on a seven year study, fifty-three percent of sunspots observed appeared in bipolar sunspot pairs (Benestad, 2006). An example of a bipolar sunspot pair is shown in Fig. 1.4b as a pair of white and black patches representing the positive and negative polarities, respectively. Bipolar sunspot pairs mark the intersection of rising magnetic flux tubes breaking through the photosphere, and join arched magnetic structures extending high into the atmosphere, termed coronal loops. Coronal loops can be seen by considering the emission of hot plasma due to the high temperature of the corona. The high temperature means
1.2 The MHD equations

the plasma becomes highly ionised and therefore plasma cannot easily flow across the magnetic field, and hence follows the structure of the magnetic field.

Dynamic solar activity and events occur and vary over different time periods. However, there is a dominant periodic cycle governing the overall temperament of the Sun. This twenty-two year cycle is split into two consecutive eleven year cycles as distinguished by a reversal of the poles polarities. Throughout each cycle, the number of sunspots or regions of magnetic activity rise and fall before the pole reversal signifies the start of the next eleven year cycle. The start of the cycle is characterised by very few visible features on the Sun and is termed solar minimum. Half way through the cycle, the number of visible features increase until the Sun is at its most active at solar maximum. There is a significant variation in the number of sunspots during the solar cycle. The number of sunspots is a good indication of the number of magnetic loops extending into the corona and hence a good indicator of magnetic activity. At solar minimum, the majority of sunspots lie at mid-latitudes, whereas at solar maximum sunspot groups migrate towards the equator accompanied by a reversal of the polar polarities.

Observation of flux emergence event

Fig. 1.6 demonstrates the evolution of a typical large-scale flux emergence event. These images, taken from an event from May 13th-15th 2011, show both the evolution of magnetic flux in the HMI magnetogram on the left, and the evolution of the coronal loops seen in 171 Å on the right. The first stages of emergence are seen in the $t = 0$ images, where a bipolar sunspot pair, consisting of positive and negative polarities, begins to emerge at the left hand side of the image. At this stage, there is some brightening in the coronal image, but the coronal loops lie close to the photosphere so are hard to see. However, 12 hours later, the sunspot pair has grown and there is a clear formation of an arcade of coronal loops. These loops have expanded into the atmosphere, both horizontally and vertically. 24 hours later, the loops have fully expanded into the atmosphere, and the bipolar sunspot pair has grown in size.

Flux emergence is of great importance to events on the Sun as the emerged magnetic field couples various atmospheric layers of the Sun. For example, the magnetic field can cause photospheric motions that propagate into the corona. In addition, flux emergence can destabilise existing magnetic structures and lead to the initiation of coronal mass ejections (CMEs) and flaring activity.

1.2 The MHD equations

Plasmas are comprised of electrons, neutrals and protons. Therefore, models exist that consider each species individually and their interaction with one another. Microscopic descriptions are very useful when considering small-scale problems and understanding the in-depth physics driving events. However, the microscopic description is quite limiting and often problems require a macroscopic description. As we are modelling magnetic flux emergence, we must model multiple layers of the Sun. This requires a macroscopic approach, namely the magnetohydrodynamics (MHD) approximation developed by Alfvén in the 1940s, as this allows us to consider large-scale situations.
1.2 The MHD equations

The MHD equations are a combination of the Navier-Stokes equations of fluid dynamics, and Maxwell’s equations of electromagnetism. Before we explicitly discuss the MHD equations, we must first consider the various assumptions made in formulating the equations, as follows:

1. The plasma is assumed to be **quasi-neutral**. Explicitly, we assume there are an equal number of ions and electrons to allow us to use a one-fluid model.

2. The **speeds** involved are much **smaller than the speed of light**, $c$ (non-relativistic).

3. The **length scales** involved are much **larger than kinetic length scales**.

4. The **time scales** involved are much **longer than kinetic time scales** (collision times).

5. The **gas pressure** is assumed to be a **scalar** and hence isotropic.
1.2 The MHD equations

Here we have only briefly outlined the assumptions required for MHD. However, we refer the reader to Goedbloed and Poedts (2004) or Priest (2014) for a detailed description of the assumptions and applicability of MHD. In order to formulate the MHD equations used in this thesis, we first consider Maxwell’s equations.

1.2.1 Maxwell’s equations

The four principles in electromagnetism are given as follows:

- **Ampere’s Law** (with Maxwell’s correction)
  \[ \nabla \times \mathbf{B} = \mu \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}, \quad (1.1) \]

- **Faraday’s Law**
  \[ \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \quad (1.2) \]

- **Solenoidal Constraint**
  \[ \nabla \cdot \mathbf{B} = 0, \quad (1.3) \]

- **Gauss’ Law**
  \[ \nabla \cdot \mathbf{E} = \rho^*/\epsilon, \quad (1.4) \]

where \( \mathbf{B} \) is the magnetic field, \( \mu \) is the magnetic permeability, \( \mathbf{j} \) is the current density, \( c \) is the speed of light \( (3 \times 10^8 \text{ m/s}) \), \( \mathbf{E} \) is the electric field, \( t \) is time, \( \rho^* \) \( (= e(z^+ n^+ - n^-)) \) is the charge density where \( e \) is the electron charge, \( z^+ \) is the ion number and \( n^+ \) and \( n^- \) are the positive and negative ion numbers densities per unit volume respectively and finally, \( \epsilon \) is the permittivity of free space. It is assumed that the solar atmosphere is close to a vacuum and hence the magnetic permeability and permittivity of free space are taken as their values in a vacuum:

\[ \mu \equiv \mu_0 = 4\pi \times 10^{-7} \text{ H/m}, \]
\[ \epsilon \equiv \epsilon_0 = 8.9 \times 10^{-12} \text{ F/m}. \]

Ampere’s Law, given by Eq. 1.1, states that gradients in the magnetic field can create electric currents. However, Eq. 1.1 is altered under the MHD approximation. The second assumption we state is that characteristic speeds are much smaller than the speed of light, \( c \). To simplify Ampere’s Law, we first express Faraday’s Law (Eq. 1.2) in terms of its dimensions:

\[ E_0 = \frac{L_0}{l_0} B_0, \]
where $E_0$ has units of the electric field, $L_0$ and $t_0$ are typical length and time scales, respectively, and $B_0$ has units of the magnetic field. This allows us to rewrite Ampere’s Law as

$$\frac{B_0}{L_0} = \mu_0 j_0 + \frac{1}{c^2} \frac{E_0}{t_0} = \mu_0 j_0 + \frac{1}{c^2} \frac{L_0}{t_0} B_0 = \mu_0 j_0 + \frac{v_0^2}{c^2} B_0,$$

where we have defined $v_0$ as a typical plasma velocity. Hence, if we assume the typical velocity is much smaller than the speed of light, i.e. $v_0^2 \ll c^2$, this tells us that

$$|\nabla \times B| \gg \frac{1}{c^2} \frac{\partial |E|}{\partial t},$$

and hence simplifies Ampere’s Law to $\nabla \times B = \mu j$ and so the current is divergence free.

Faraday’s Law (Eq. 1.2) tells us that spatially varying electric fields can induce magnetic fields. The solenoidal constraint (Eq. 1.3) states that there can exist no magnetic monopoles, $\nabla \cdot B = 0$ is actually an initial condition as it can be shown that if the solenoidal constraint is satisfied initially it is satisfied for all time. This can be shown by taking the divergence of Faraday’s Law (Eq. 1.2),

$$\frac{\partial}{\partial t} (\nabla \cdot B) = -\nabla \cdot (\nabla \times E) = 0.$$

Finally, Gauss’ Law (Eq. 1.4) can also be simplified as the plasma is assumed to be quasi-neutral (equal numbers of ions and electrons) and so, $\rho^* \approx 0$. Gauss’ Law therefore becomes $\nabla \cdot E = 0$, which tells us that the electric field can only be induced by a changing magnetic field.

### 1.2.2 Fluid equations

There are four fluid equations, known as the Navier-Stokes equations:

- **Continuity equation**
  $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$  
  (1.5)

- **Equation of motion**
  $$\rho \frac{D \mathbf{v}}{Dt} = -\nabla p + j \times B + \mathbf{F},$$  
  (1.6)

- **Energy equation**
  $$\frac{\rho^\gamma}{\gamma - 1} \frac{D}{Dt} \left( \frac{p}{\rho^\gamma} \right) = -\mathbf{L},$$  
  (1.7)

- **Equation of state**
  $$p = \frac{k_B}{\mu_m} \rho T.$$  
  (1.8)
1.2 The MHD equations

Here $\rho$ is the plasma density, $v$ the plasma velocity, $p$ the plasma pressure, $F$ represents the sum of any additional forces such as gravitational and viscous forces, $\gamma$ is the ratio of specific heats ($5/3$ for an ideal gas), $\mathcal{L}$ is the energy loss function, $k_B$ is the Boltzmann constant ($1.38 \times 10^{-23}$ m$^2$ kg s$^{-2}$ K$^{-1}$), $\mu_m$ is the reduced mass of all particles in the plasma and $T$ is the plasma temperature. The reduced mass, $\mu_m$, is the average mass of all particles in the plasma. Hence $\mu_m = m_p$ for neutral hydrogen atoms and $\mu_m = 0.5m_p$ for fully ionised hydrogen ($m_p = 1.67 \times 10^{-27}$ kg is the proton mass). In our simulations, we assume $\mu_m = m_p$ for neutral hydrogen.

The continuity equation (Eq. 1.5) states that mass can neither be created nor destroyed. Equivalently, the rate of mass leaving a system is equal to the rate of mass entering a system. Eq. 1.6, the equation of motion, is essentially Newton’s Second Law, i.e. that the mass of an object multiplied by its acceleration is equal to the net force acting on that object. The third fluid equation is the energy equation as given by Eq. 1.7. This tells us that the rate of change of entropy, $p/\rho \gamma$, is due to the net effect of energy sources and sinks. If the energy loss function, $\mathcal{L}$, is equal to zero, there are no thermal exchanges between the plasma and its surroundings and hence, the plasma is adiabatic. In this case, the entropy is conserved. If the plasma is non-adiabatic, the energy loss function can be written as $L = -j^2/\sigma + Q_{\text{visc}}$. $-j^2/\sigma$ is the energy loss due to Ohmic heating and $Q_{\text{visc}}$ is energy loss due to viscous heating. Finally, the equation of state (Eq. 1.8) is the ideal gas law.

1.2.3 Ohm’s Law

Ohm’s Law couples the electromagnetic equations to the plasma fluid equations through $v$, the plasma velocity. It states that the current density is proportional to the total electric field, which consists of the electric field that would act on a stationary plasma plus the electric field produced by a moving magnetic field. The classical form of Ohm’s Law is

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j},$$

where $\eta$ is the resistivity defined as $\eta = 1/\sigma$ where $\sigma$ is the conductivity.

1.2.4 Summary of MHD equations

The resistive, non-adiabatic form of the equations of MHD are as follows:

- **Mass Continuity (conservation of mass)**

  $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

  (1.9)

- **Equation of motion**

  $$\rho \frac{\text{Dv}}{\text{Dt}} = -\nabla p + \mathbf{j} \times \mathbf{B} + \mathbf{F},$$

  (1.10)
1.2 The MHD equations

- **Ampere’s law**
  \[ j = \frac{1}{\mu} \nabla \times B, \]  
  (1.11)

- **Faraday’s law**
  \[ \frac{\partial B}{\partial t} = -\nabla \times E, \]  
  (1.12)

- **Ideal gas law**
  \[ p = \frac{k_B}{\mu m} \rho T, \]  
  (1.13)

- **Ohm’s law**
  \[ E + v \times B = \eta j, \]  
  (1.14)

- **Energy equation**
  \[ \frac{\rho^\gamma}{\gamma - 1} \frac{D}{Dt} \left( \frac{p}{\rho^\gamma} \right) = -\mathcal{L}, \]  
  (1.15)

  with the initial condition for the magnetic field,

- **Solenoidal constraint**
  \[ \nabla \cdot B = 0, \]  
  (1.16)

where all quantities are defined as described in the previous sections. To reiterate, these equations are valid under the assumptions that the plasma is quasi-neutral, the pressure is a scalar, MHD time scales are much longer than kinetic time scales, and MHD velocities are much less than the speed of light, \( c \).

1.2.5 Derived equations and quantities

The MHD equations lead to several characteristic quantities and equations, which we readily encounter in our work. We now list some specific examples of equations and quantities that we use in later investigations.

**Equation of motion**

\[ \rho \frac{Dv}{Dt} = -\nabla p + j \times B + F. \]

In the equation of motion, many forces act on the plasma, including:
1.2 The MHD equations

- the gas pressure gradient \((-\nabla p)\) which acts to smooth out gradients in pressure, i.e. from high to low pressure
- the Lorentz force exerted by the magnetic field, \(j \times B\)
- External forces, \(F\), can be made up of:
  - gravitational force, \(\rho g\) (where \(g\) is the gravitational acceleration)
  - the viscous force, \(\rho \nu \left[ \nabla^2 v + \frac{1}{3} \nabla (\nabla \cdot v) \right]\) where \(\nu\) is the coefficient of kinematic viscosity

In this experiments performed in this thesis, gravity and viscosity are included. We may rewrite the Lorentz force as

\[
j \times B = \frac{1}{\mu} (\nabla \times B) \times B = \frac{1}{\mu} (B \cdot \nabla) B - \nabla \left( \frac{B^2}{2 \mu} \right),
\]

so that we have a magnetic tension force and magnetic pressure force in the equation of motion. The magnetic tension force appears wherever fieldlines are curved and acts to straighten out irregularities in the field. The magnetic pressure force occurs when there are variations in the magnitude of \(B\) and acts from high \(B\) to low \(B\).

**Induction equation**

Taking the curl of Ohm’s law (Eq. 1.14), and combining Eqs. 1.11 and 1.12 yields the induction equation,

\[
\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \eta' \nabla^2 B, \tag{1.17}
\]

where \(\eta' = \eta/\mu\), the magnetic diffusivity, is assumed to be constant. Eq. 1.17 describes how the magnetic field changes in time due to two effects expressed on the right hand side of the equation, that is, the advection of the magnetic field with the plasma and the diffusion of the magnetic field through the plasma, respectively.

**Magnetic Reynolds number**

The magnetic Reynolds number expresses the order of magnitude of the two terms on the right hand side of the induction equation, namely the advection and diffusion terms:

\[
R_m = \frac{|\nabla \times (v \times B)|}{|\eta' \nabla^2 B|},
\]

or equivalently, in terms of dimensions,

\[
R_m = \frac{v_0 L_0}{\eta'},
\]
where \( v_0 \) is a typical velocity and \( L_0 \) is a typical length scale. This important parameter gives an indication of the interaction of the magnetic field with the plasma. In most of the universe, \( R_m \gg 1 \) so the advection term dominates and the magnetic field is frozen into the plasma (Alfvén’s frozen-in-flux theorem). Alternatively, if \( R_m \ll 1 \), then diffusion is most important. On the Sun, typically \( R_m \gg 1 \) as the length scales (\( L_0 \)) and velocities (\( v_0 \)) are generally large, and the change in magnetic field is dominated by plasma motions. However, there are exceptions to this, at current sheets for instance, where diffusion becomes important. For details of this process termed magnetic reconnection, we refer the reader to Priest et al. (2003) and Pontin (2011) and references therein.

**Plasma \( \beta \)**

Another important dimensionless parameter is the ratio of the plasma pressure (\( p \)) to the magnetic pressure (\( B^2/2\mu \)), named the plasma beta, defined as

\[
\beta = \frac{p}{B^2/2\mu} = \frac{2\mu p}{B^2}.
\]

This term can be recovered by considering the equation of motion without gravity or flows (\( v = 0 \)) and comparing the magnitudes of the pressure gradient and Lorentz force. Therefore, if \( \beta \ll 1 \) we can neglect the pressure term and, if \( \beta \gg 1 \) we can neglect the Lorentz force. In the solar interior, \( \beta > 1 \) and the plasma is considered to dominate the magnetic field. Conversely, in the solar atmosphere \( \beta < 1 \) and the magnetic field dominates the plasma.

**Characteristic wave speeds**

Waves are very important on the Sun and have been observed in a variety of solar phenomena. For the work contained within this thesis, it is therefore important that we define two characteristic wave speeds. Sound waves are longitudinal waves that exist because of a restoring pressure force. In a uniform plasma, they propagate isotropically (in all directions) at the sound speed \( c_s = \sqrt{\gamma p_0/\rho_0} \). Alfvén waves are a type of MHD wave that exist in response to the magnetic tension force. They are transverse waves that travel in the direction of the magnetic field at the Alfvén wave speed, \( v_A = B_0/\sqrt{\mu \rho_0} \). The Alfvén speed and equivalent Alfvén time are important characteristic plasma quantities. These quantities arise in the work contained within this thesis.

**Potential and force-free magnetic fields**

A comparison of the magnitude of the terms in the equation of motion (Eq. 1.10) is interesting to consider, with typical plasma and magnetic field quantities denoted with a subscript 0. The velocity variations on the left hand side can be neglected if the flow speed is much less than the characteristic magnetic speed, termed the Alfvén speed \( (v_0 \ll v_A = B_0/\sqrt{\mu_0 \rho_0}) \), and the sound speed \( v_A \ll c_s = \sqrt{\gamma p_0/\rho_0} \). Let us also consider the case with the only external force being gravity. Therefore the equation of motion reduces
1.2 The MHD equations

\[ 0 = -\nabla p + \frac{1}{\mu} \mathbf{j} \times \mathbf{B} + \rho g. \]

To make the next comparison, we must define the pressure scale height, \( H \), as the distance over which the pressure decreases by a factor of \( e \). The gravity term can be neglected when the length scales are much less than the pressure scale height \( (L_0 \ll H = p_0/\rho_0 g) \), to reduce the equation to

\[ 0 = -\nabla p + \frac{1}{\mu} \mathbf{j} \times \mathbf{B}. \]

Similarly, when \( \beta \ll 1 \) the pressure force can be neglected in favour of the Lorentz force, and the equation reduces to the force-free approximation:

\[ \mathbf{j} \times \mathbf{B} = 0. \]

Magnetic fields satisfying the above equation are known as force-free. This is a good approximation for the magnetic field in the corona since \( \beta \ll 1 \) in this region. These are several solutions to this equation, including the simple solution \( \mathbf{j} = 0 \), where the current density is zero everywhere and the magnetic field is termed potential. A method for calculating the potential field is included in Appendix A. The potential magnetic field constructed from the normal component of the magnetic field given on a closed surface is the minimum energy state of the magnetic field (Priest, 1982). This result is used in later chapters.

**Total energy equation**

In order to check conservation of energy across the domain, we must introduce the total energy density equation, that can be derived by putting the MHD equations (Eq. 1.9 - Eq. 1.16) into conservative form,

\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \rho v^2 + \frac{p}{\gamma - 1} + \frac{B^2}{2\mu} + \rho \Phi \right) + \nabla \cdot \left( \frac{1}{2} \rho v^2 \mathbf{v} + \frac{\gamma p}{\gamma - 1} \mathbf{v} + \frac{\mathbf{E} \times \mathbf{B}}{\mu} + \rho \Phi \mathbf{v} \right) = 0. \]

Integrating this equation over the domain, and using the divergence theorem, this becomes

\[ \frac{dE}{dt} + \int_S \mathbf{F} \cdot d\mathbf{S} = 0, \]

where \( E(t) = \int_V \frac{1}{2} \rho v^2 + \frac{p}{\gamma - 1} + \frac{B^2}{2\mu} + \rho \Phi \, dV \) is the total energy of the plasma within a fixed volume, \( V \), made up of the kinetic energy, internal energy, magnetic energy, and gravitational potential energy respectively. Note \( \Phi \) is gravitational potential such that \( \mathbf{g} = -\nabla \Phi \). The flux of energy is given by \( \mathbf{F} = \frac{1}{2} \rho v^2 \mathbf{v} + \frac{\gamma p}{\gamma - 1} \mathbf{v} + \frac{\mathbf{E} \times \mathbf{B}}{\mu} + \rho \Phi \mathbf{v} \), made up of the kinetic energy flux, enthalpy flux, Poynting flux of energy and gravitational potential energy flux, respectively. Hence, the total energy of a plasma in a fixed volume can only change if there is a flow of energy into or out of the volume. The Poynting flux is an important quantity which we return to later in this thesis.
Magnetic helicity

As we are concerned with the emergence of twisted magnetic flux tubes (defined in the next section) in this thesis, we are interested in the twist of the magnetic field. The twist, shear or braidedness of a magnetic flux tube can be quantified using a special mathematical property of the magnetic field known as the magnetic helicity. Magnetic helicity has the special property that it is conserved under ideal MHD and approximately conserved under reconnection (Berger and Field, 1984). Magnetic helicity is defined as

\[ H_m = \int A \cdot B \, dV, \]

where \( B \) is the magnetic field and \( A \) is the vector potential such that \( B = \nabla \times A \). This form of helicity is, however, gauge-dependent. Explicitly, this means that the value of the helicity \( H_m \) depends on the choice of gauge of \( A \). Therefore, to find a form of helicity with physical meaning, we must find a gauge-independent helicity quantity. To solve this, Finn and Antonsen (1985) defined the relative magnetic helicity as

\[ H_r = \int (A + A_p) \cdot (B - B_p) \, dV, \]  \hspace{1cm} (1.18)

where \( B \) is the magnetic field, \( A \) is the vector potential of \( B \), \( B_p \) is the reference potential field with the same normal flux distribution as \( B \) on the \( z = 0 \) plane and \( A_p \) is the vector potential of \( B_p \). The relative magnetic helicity is advantageous over the standard magnetic helicity as it is gauge-invariant with respect to the gauges for \( A \) and \( A_p \).

The rate of change of magnetic helicity can also be calculated analytically to understand the sources relating to the change in helicity. The production and depletion of helicity in the atmosphere is dominated by contributions of vertical flows that advect twisted magnetic fields up into the corona and by horizontal flows that shear and twist magnetic fields (Berger and Field, 1984). The rate of change of relative magnetic helicity, \( H_r \), can be evaluated analytically by differentiating the expression in Eq. 1.18 (Berger and Field, 1984).

\[
\frac{dH_r}{dt} = -2\eta \int j \cdot B \, dV + 2\eta \int (A_p \times j) \cdot n \, dS \\
+ 2 \int [(A_p \cdot v)(B \cdot n) - (A_p \cdot B)(v \cdot n)] \, dS, \]  \hspace{1cm} (1.19)

where \( n \) is the outward pointing normal to the surface \( S \). The first term on the right-hand side of Eq. 1.19 relates to the depletion of helicity by internal dissipation (dissipation term), the second corresponds to a surface correction to the resistive dissipation (surface correction term), the third relates to the generation of helicity by horizontal motions of the boundary (shear term) and the last corresponds to the injection of helicity by direct emergence (emergence term). Pariat et al. (2015) adds corrections to this derivative but this is discussed in more detail in Section 4.3.8.
1.3 Flux tubes

Convective motions are believed to arrange the sub-surface magnetic field into unorganised bundles of twisted magnetic fieldlines. In order to model this, we represent these bundles as organised twisted magnetic flux tubes and as such we introduce and discuss flux tubes in this section. Most of the magnetic flux that emerges in the Sun is confined to isolated flux tubes. Flux tubes are a fundamental element of flux emergence experiments, and as such we discuss their definition and properties. As discussed previously, it is now widely believed that sunspot pairs are formed as a result of flux tubes rising through the convection zone and emerging into the atmosphere. Observations and theory of sunspots suggest that the emerging tubes are actually a collection of many smaller flux tubes (Parker, 1979). However, in this thesis, we capture the large-scale structure of sunspots by emerging a single flux tube to produce a bipolar active region.

1.3.1 Definition of a flux tube

A magnetic flux tube is comprised of a bundle of magnetic fieldlines. Hence, before we define a magnetic flux tube, we must first define a magnetic fieldline. Given a magnetic field, \( \mathbf{B} \), a magnetic fieldline is a curve whose tangent is in the direction of \( \mathbf{B} \) at any given point. The magnetic lines of force, called magnetic fieldlines, for a known magnetic field \( \mathbf{B} = (B_x, B_y, B_z) \), are defined by

\[
\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z}.
\]

A magnetic flux tube is defined as a cylindrical volume enclosed by a set of fieldlines, in which every fieldline intersects the same simple closed curve. For illustration purposes, we have included a schematic in Fig. 1.7. The total flux of the flux tube, \( F \), is defined as the magnetic flux crossing a surface, \( S \),

\[
F = \int_S \mathbf{B} \cdot d\mathbf{S},
\]  

(1.20)

where \( S \) is the surface shown in Fig. 1.7 and \( d\mathbf{S} \) is directed along the normal, \( \mathbf{n} \), to the surface \( S \).

Figure 1.7: Illustration of the flux threading a flux tube, where \( \mathbf{B} \) is the magnetic field, \( S \) is the surface whose perimeter encompasses all of the fieldlines passing through the tube and \( \mathbf{n} \) is the normal to the surface \( S \).
1.3.2 Properties of a flux tube

1. The total flux of a tube remains constant along its length.

This is a consequence of the solenoidal constraint, \( \nabla \cdot \mathbf{B} = 0 \). To illustrate this result, consider measuring the flux across a surface \( S \) enclosing a volume \( V \), as shown in Fig. 1.8. In order to calculate the flux, we must define the normal to the surface. For ease, we split the surface \( S \) into three surfaces; \( S_1 \) and \( S_2 \) are the two ends of the cylinder and \( S_3 \) is the curved surface. The total flux can then be rewritten as

\[
F = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_{S_1} \mathbf{B} \cdot d\mathbf{S} + \int_{S_2} \mathbf{B} \cdot d\mathbf{S} + \int_{S_3} \mathbf{B} \cdot d\mathbf{S}. \tag{1.21}
\]

By applying the divergence theorem to the left-hand side, we find

\[
\int_S \mathbf{B} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{B} \ dV = 0,
\]

where we have made use of the solenoidal constraint, \( \nabla \cdot \mathbf{B} = 0 \). We also note that the normal, \( n_3 \), of the curved surface is perpendicular to the direction of the magnetic field, \( \mathbf{B} \) and hence \( \mathbf{B} \cdot d\mathbf{S} = \mathbf{B} \cdot n_3 \ dS = 0 \) and the third integral does not contribute. Therefore, Eq. 1.21 reduces to

\[
0 = \int_{S_1} \mathbf{B} \cdot d\mathbf{S} + \int_{S_2} \mathbf{B} \cdot d\mathbf{S},
\]

and consequently

\[
\int_{S_1} \mathbf{B} \cdot d\mathbf{S} = -\int_{S_2} \mathbf{B} \cdot d\mathbf{S}.
\]

In other words, the flux through \( S_1 \) equals the flux through \( S_2 \). The negative sign accompanies the \( S_2 \) flux integral as the outward pointing normal of \( S_2 \) points downwards and is hence negative. This can be shown for any two surfaces along the flux tube and hence proves that the total flux of the tube is constant along its length.

![Figure 1.8: Illustrating of flux tube split into three surfaces; \( S_1 \) and \( S_2 \) are the two ends of the cylinder while \( S_3 \) is the curved surface.](image)
2. **The mean field strength of a flux tube increases when it narrows and decreases when it widens.**
   This can be shown to be a direct consequence of the definition of flux given in Eq. 1.20. This can equivalently be written as \( F = BA \) where \( B \) is the mean field strength of the flux tube and \( A \) is the cross sectional area. Hence, if the flux tube narrows, \( A \) decreases. However, to ensure \( F \) remains constant, the mean field strength, \( B \), must increase. Similarly, if the cross-sectional area increases as the tube widens, \( A \) increases and \( B \) must decrease. Stronger fields have fieldlines closer together whereas weaker fields have fieldlines further apart (Priest, 2014).

3. **An isolated magnetic flux tube is surrounded by a region of reverse current.**
   This can be shown by integrating the magnetic field along a closed contour encircling the flux tube in the surrounding plasma (Priest, 2014). The magnetic field vanishes outside the flux tube, by definition of an isolated flux tube. Hence
   \[
   \int \mathbf{B} \cdot d\mathbf{l} = 0,
   \]
   and by Stokes’ theorem,
   \[
   \int \mathbf{B} \cdot d\mathbf{l} = \int \nabla \times \mathbf{B} \cdot d\mathbf{S} = \int \mathbf{\mu j} \cdot d\mathbf{S} = 0.
   \]
   The integral involving \( j \) describes the total current through a cross section of the flux tube. Specifically, if the current is directed in one particular direction in the interior of the flux tube, there must be an equal but opposite current on the surface of the flux tube.

The terms magnetic flux tubes and magnetic flux ropes are often used interchangeably in literature. They both define twisted magnetic structures. In the literature, authors often use the notation such that tubes refer to those in the interior and ropes refer to those in the atmosphere. However, we will use the term flux tube in the rest of this thesis.

### 1.3.3 Gold-Hoyle flux tubes

This is one of the most common types of flux tube used in numerical MHD simulations. Although the original definition was constrained as a particular set of equations with varying constants, the definition of a Gold-Hoyle flux tube has since been generalised and now describes a large set of magnetic flux tubes. Although we do not use this type of flux tube in our simulations, the ideas and concepts used to derive this field are useful for further chapters, and the literature review in Section 1.5.

The original intent of the Gold and Hoyle (1960) paper was to investigate the origin of solar flares. In trying to understand how the large amounts of energy released in flares were stored in the chromosphere, they tried to formulate the structure of a force-free, twisted flux tube. The authors made the following assumptions:

1. The field is force free, \( \mathbf{j} \times \mathbf{B} = (\nabla \times \mathbf{B}) \times \mathbf{B} = 0 \).
2. The curvature of the axis is sufficiently small over a short distance such that we can assume it to be straight.

3. All fieldlines have the same number of turns per unit length of the axis (central fieldline).

4. All points along a fieldline are the same distance from the axis.

5. The field is independent of the azimuthal coordinate of the axis and the distance coordinate in the direction of the axis. Thus, the field is only a function of radius.

The authors defined a cylindrical coordinate system \((r, \theta, z)\) with the axis of the tube aligned along the \(z\)-direction. This results in a form \(\mathbf{B} = (0, B_\theta(r), B_z(r))\) that describes a magnetic field where all the fieldlines rotate about the axis of the tube. The twist of the tube, \(\alpha\), is defined as the rate of rotation of the fieldlines, and provides a measure of the angle (in radians) through which the fieldlines rotate over one unit of length along the axis. The twist per unit length is given by

\[
\alpha = \frac{d\theta}{rdz} = \frac{B_\theta}{rB_z}, \quad (1.22)
\]

and this is related to the number of turns of a fieldline per axial unit length, \(n\), by

\[
\alpha = 2\pi n. \quad (1.23)
\]

In order to proceed, we define \(\phi\) as the angle a fieldline makes with the plane normal to the axis, as shown in Fig. 1.9. As \(B_z\) is in the direction of the axis and \(B_\theta\) is normal to the axis, we can relate \(\phi\) to the magnetic field by

\[
\cot(\phi) = \frac{\cos \phi}{\sin \phi} = \frac{B_\theta}{B_z}.
\]

Using this result in Eq. 1.22, and also noting Eq. 1.23, then gives us

\[
\cot \phi = \alpha r = 2\pi nr. \quad (1.24)
\]

This can be rewritten in terms of \(\sin \phi\) and \(\cos \phi\), by the use of basic trig identities, as

\[
\sin \phi = \frac{1}{\sqrt{1 + 4\pi^2n^2r^2}}, \quad (1.25)
\]
\[ \cos \phi = \frac{2\pi nr}{\sqrt{1 + 4\pi^2 n^2 r^2}}. \quad (1.26) \]

This result is used later. Continuing with Gold and Hoyle (1960)’s derivation, we next consider the first assumption, i.e. that the field is force-free \((\mathbf{j} \times \mathbf{B} = 0)\). In cylindrical coordinates, this can be written as

\[ \left( B_\theta \frac{1}{r} \frac{d}{dr} (r B_\theta) + B_z \frac{d}{dr} (B_z) \right) \hat{r} = 0 \hat{r}. \]

Noting \(B_\theta = B \cos(\phi)\) and \(B_z = B \sin(\phi)\), and using \(\cos^2 \phi + \sin^2 \phi = 1\), this simplifies to

\[ \frac{1}{B} \frac{dB}{dr} = -\frac{\cos^2 \phi}{r}. \]

Replacing \(\cos \phi\) with the expression given in Eq. 1.26, yields

\[ B = \exp \left( -\int \frac{4\pi^2 n^2 r}{1 + 4\pi^2 n^2 r^2} dr \right) = \exp \left( -\frac{1}{2} \log \left( 1 + 4\pi^2 n^2 a^2 \right) + C \right) = D / \sqrt{1 + 4\pi^2 n^2 r^2}, \quad (1.27) \]

where \(C\) and \(D\) are constants of integration. Note, we have assumed \(n\), the number of turns of the fieldline per axial unit length, is independent of \(r\). This is valid as this is one of the initial assumptions the authors used to formulate the magnetic field.

The authors also assume the flux tube has radius \(a\), and hence \(B = 0\) for \(r > a\). The flux of the tube, \(F\), is given by

\[ F = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} B_z r dr d\theta = 2\pi \int_{r=0}^{a} B_z r dr, \]

and by rewriting \(B_z = B \sin(\phi)\) using \(B\) from Eq. 1.27 and \(\sin \phi\) from Eq. 1.25, the flux becomes

\[ F = 2\pi D \int_{r=0}^{a} \frac{r}{1 + 4\pi^2 n^2 r^2} dr = \frac{D}{4\pi n^2} \log \left( 1 + 4\pi^2 n^2 a^2 \right). \]

By noting \(\alpha = 2\pi n\), we can rearrange for the constant \(D\) as

\[ D = \frac{\alpha^2 F}{\pi \log \left( 1 + \alpha^2 a^2 \right)}. \]

This constant of integration can now be removed from the formula and by noting \(B_\theta = B \cos \phi\) and \(B_z = B \sin \phi\), the magnetic field components are as follows:

\[ B_\theta = B \cos \phi = \frac{\alpha^2 F}{\pi \log \left( 1 + \alpha^2 a^2 \right)} \frac{\alpha r}{1 + \alpha^2 r^2}, \]

\[ B_z = B \sin \phi = \frac{\alpha^2 F}{\pi \log \left( 1 + \alpha^2 a^2 \right)} \frac{1}{1 + \alpha^2 r^2}. \]
Or more simply, if we absorb all constants into one unknown $B_0$, the resulting magnetic field components are

$$B_r = 0,$$

$$B_\theta = \alpha r B_z,$$

$$B_z = \frac{B_0}{1 + \alpha^2 r^2}.$$  

This defines the original Gold-Hoyle flux tube. Note, that $B_\theta = 0$ at $r = 0$ so the magnetic field is solely in the axial direction at the axis of the flux tube as we would expect. Although, we stated that the flux tube would be confined to a radius of $a$, this field does not satisfy this as there will still be some field at larger radii. The field is in fact primarily azimuthal at larger radii. We also stated that the authors assumed that the number of turns about the axis per axial unit distance would be constant. If $n$ is constant, $\cot \phi = 2\pi nr$ (Eq. 1.24) demands that $\phi$ varies with $r$. At small $r$, the fieldlines are directed towards the axial direction. However, as $r$ increases, the fieldlines turn away from the axial direction and turn towards the direction normal to the axis. Hence, $\phi$ decreases from $\pi/2$ to approximately 0 with increasing $r$.

Now that we have derived the Gold-Hoyle flux tube formulated in Gold and Hoyle (1960) it is important to note that a much more general set of Gold-Hoyle flux tubes exist today. A Gold-Hoyle flux tube now describes any force-free, uniformly twisted flux tube. In our buoyant magnetic field simulations, we do not consider a force-free flux tube and only constrain it to be uniformly twisted. Force-free flux tubes are more useful when considering flux tubes in the solar corona. However, many flux tubes in the literature use this type of tube and hence it deserves to be studied.

### 1.4 Magnetic buoyancy

Bipolar active regions are the largest concentrations of magnetic flux on the solar surface and have captivated the interest of observers and theorists alike. Parker (1955) first suggested the universally recognised model of active region formation; a flux tube stored in the solar interior rises, under the influence of magnetic buoyancy, and creates a pair of bipolar sunspots when it breaks through the photosphere. If a magnetic flux tube is in pressure balance and thermal equilibrium with its surroundings, the tube will be less dense than its surroundings, and will therefore be buoyant. This mechanism allows a flux tube to rise to the photosphere where it can rise no further until the tube is able to enter the atmosphere by initiation of a second instability, namely the magnetic buoyancy instability. Bipolar sunspots mark the intersection of the magnetic field at the surface.

Currently, it is believed that the Sun’s magnetic field is created by a dynamo action in the tachocline (the layer between the convection zone and the radiative zone). It is widely accepted that magnetic flux, in the form of flux tubes, rises up through the solar interior and emerges at the solar surface to form active regions. In this thesis, we model this process using numerical simulations. However, in this section, we describe the process of magnetic buoyancy both qualitatively and mathematically to enhance our understanding allowing us to model this key process.
1.4 Magnetic buoyancy

The first stage of the process involves the rise of an isolated magnetic flux tube in the convection zone. A simple model illustrates this concept. Consider an isolated horizontal flux tube in pressure equilibrium with its surroundings. The pressure balance equation takes the form

\[ p_t + \frac{B_t^2}{2\mu} = p_e, \]

with \( p_t \), the gas pressure of the tube, \( p_e \), the gas pressure of the environment and \( B_t \), the magnitude of the flux tube’s magnetic field. We assume the surrounding environment is unmagnetised so we do not need to take into account the magnetic pressure here. This assumption seems reasonable in the interior due to the high plasma \( \beta \). If, for the sake of argument, we also assume that the flux tube is in thermal equilibrium with its surroundings, it follows from the dimensions of the ideal gas law,

\[ \frac{k_B T}{\mu_m} p_e = \frac{k_B T}{\mu_m} p_t + \frac{B_t^2}{2\mu}, \]

that \( p_t < p_e \) as \( B_t^2 / 2\mu > 0 \) for a non-zero magnetic field. Hence, the tube is lighter than its surroundings and will rise with the buoyancy force \((p_e - p_t)g\) under the influence of gravity. As the tube rises, it becomes arched and feels a restoring magnetic tension force. The restoring force will not inhibit the rising of the flux tube if, for the radius of curvature \( L \),

\[ (p_e - p_t)g > \frac{B_t^2}{\mu L}. \]

1.4.1 Buoyancy instability

Now that we have expressed a simple qualitative argument for the rise of a flux tube due to buoyancy, we look at this instability more rigorously. Pressure forces dominate magnetic forces in the convection zone resulting in \( \beta \gg 1 \). We can therefore neglect magnetic pressure. Consider an adiabatic, ideal plasma \((\eta = 0)\) in equilibrium where the pressure is given by \( p_0(z) \), the density by \( \rho_0(z) \) and the temperature by \( T_0(z) \). The hydrostatic balance equation (found by balancing \( \nabla p \) with \( \rho g \)) and ideal gas law are then expressed as

\[ \frac{dp_0}{dz} = -\rho_0 g, \]

\[ p_0 = \frac{\rho_0 k_B T_0}{\mu_m}. \]

Our aim is to determine the criteria for the onset of the buoyancy instability. We now wish to perturb the MHD equations (Eq. 1.9 - Eq. 1.16) about this equilibrium by assuming perturbations of the form \( f_1(z) \exp(i(kx - \omega t)) \), where \( f_1 \) represents a perturbation of any of the variables. For example, \( p = p_0(z) + p_1(z) \exp(i(kx - \omega t)) \) and \( v_x = v_{x1}(z) \exp(i(kx - \omega t)) \). Note, \( v_{x0}(z) = 0 \) as the tube is in equilibrium. Hence, we replace time and spatial derivatives by

\[ \frac{\partial}{\partial x} \rightarrow ik, \]
We perform a linearisation on the following subset of the MHD equations where we have excluded the influence of the magnetic field $B$ (due to the high plasma $\beta$):

\[
\begin{align*}
\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} &= -\nabla p - \rho g \mathbf{\hat{z}}, \\
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0, \\
\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p &= -\gamma p \nabla \cdot \mathbf{v},
\end{align*}
\]

to find the following set of linear equations,

\[
\begin{align*}
-i \omega \rho_0 v_{z1} &= -ik p_1, \\
-i \omega \rho_0 v_{z1} &= -\frac{dp_1}{dz} - \rho_1 g, \\
-i \omega \rho_1 &= -v_{z1} \frac{dp_0}{dz} - \rho_0 \left( ik v_{x1} + \frac{dv_{z1}}{dz} \right), \\
-i \omega \rho_1 &= -v_{z1} \frac{dp_0}{dz} - \gamma p_0 \left( ik v_{x1} + \frac{dv_{z1}}{dz} \right),
\end{align*}
\]

where the subscript 1 denotes the perturbation values and the subscript 0 denotes the equilibrium values. Note, we have used the adiabatic energy equation with $\mathcal{L} = 0$ to give $\frac{D}{Dt} \left( \frac{p}{\rho^\gamma} \right) = 0$, which can be rewritten as $\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p = -\gamma p \nabla \cdot \mathbf{v}$. In addition, we have used $\frac{dp_0}{dz} = -\rho_0 g$ to cancel the equilibrium terms in Eq. 1.29. In order to proceed, we shall consider a particular case in which perturbations are very narrow in the $x$ direction compared with the scale in the $z$ direction. In other words, we let $k \to \infty$ in such a way that $kv_{x1}$ remains of the order of unity to ensure $\nabla \cdot \mathbf{v}$ remains finite. This implies that $v_{x1} \sim 1/k$ and, in turn, implies that $p_1 \sim 1/k^2$ from Eq. 1.28. Hence, we can set the pressure perturbations equal to zero and the plasma is, therefore, in pressure balance. Thus, if we neglect the left hand side of Eq. 1.31 we find

\[
\begin{align*}
\left( ik v_{x1} + \frac{dv_{z1}}{dz} \right) &= -v_{z1} \frac{dp_0}{dz}.
\end{align*}
\]

Substituting this into Eq. 1.30 yields

\[
\begin{align*}
iv_1 &= v_{z1} \frac{dp_0}{dz} - \frac{p_0 v_{z1}}{\gamma p_0} \frac{dp_0}{dz},
\end{align*}
\]

and by use of Eq. 1.29 and noting $p_1 = 0$, we can find an equation in terms of $v_{z1}$ only,

\[
\begin{align*}
\frac{-\omega^2}{g} v_{z1} - \frac{dp_0}{dz} + \frac{p_0}{\gamma p_0} \frac{dp_0}{dz} = 0,
\end{align*}
\]

and ignoring the trivial solution $v_{z1} = 0$, the equation becomes

\[
\begin{align*}
\omega^2 = g \left( \frac{1}{\gamma p_0} \frac{dp_0}{dz} - \frac{1}{\rho_0} \frac{dp_0}{dz} \right).
\end{align*}
\]
Let us consider the implication of this on the form of \( \omega \) on the perturbations \( f(z)\exp(i(kx - \omega t)) \). Consider the case \( \omega^2 < 0 \), this implies that \( \omega = \pm i \sigma \). Inserting this form of \( \omega \) into the perturbations, the solution either grows or decays exponentially in time. Therefore, the plasma is unstable if

\[
\frac{1}{\gamma p_0} \frac{dp_0}{dz} < \frac{1}{\rho_0} \frac{d\rho_0}{dz}.
\]

If we notice \( \frac{d}{dz} \log p_0 = \frac{1}{p_0} \frac{dp_0}{dz} \) and \( \frac{d}{dz} \log \rho_0 = \frac{\gamma}{\rho_0} \frac{d\rho_0}{dz} \), we can rewrite the dispersion relation as

\[
\omega^2 = \frac{g}{\gamma} \frac{d}{dz} \left\{ \log \left( \frac{p_0}{\rho_0} \right) \right\}.
\]

This is usually expressed as

\[
\omega^2 = N^2, \quad \text{where} \quad N^2 = \frac{g}{\gamma} \frac{d}{dz} \left\{ \log \left( \frac{p_0}{\rho_0} \right) \right\}, \tag{1.33}
\]

is the *Brunt-Väisälä* frequency, also known as the *buoyancy frequency*. If \( N \) is real (i.e. \( N^2 > 0 \)), the solutions are oscillations and the system is stable. If \( N \) is imaginary (i.e. \( N^2 < 0 \)), the solutions correspond to an exponentially growing and exponentially decaying mode, and the plasma is unstable. This type of instability is referred to as the *buoyancy instability*. This is the type of instability that is responsible for bringing the magnetic field from its source in the tachocline, through the convection zone and up to the base of the photosphere. \( N^2 < 0 \) in the interior, allowing the magnetic field to rise due to the instability. At the photosphere, \( N^2 > 0 \) (the plasma is convectively stable) and the magnetic field cannot rise any further.

We have now determined a condition for the stability of the plasma in terms of the gradients of \( p \) and \( \rho \). However, a more intuitive criteria can be derived in terms of temperature gradients. In order to do this we define an *adiabatic* temperature gradient. *Adiabatic* describes a process that occurs without the transfer of heat or matter between a system and its surroundings. If we assume the gas is ideal, this implies that \( p = \rho k_B T/\mu_n \). We have dropped the subscripts for ease and simplicity. If we move a gas parcel upward a small distance \( dz \), then the change in pressure is given by

\[
dp = \left( \frac{p}{T} \frac{dT}{dz} + \frac{p}{\rho} \frac{d\rho}{dz} \right) dz.
\]

If the gas is adiabatic, this tells us that no entropy is generated, i.e. \( p/\rho^\gamma = A \) where \( A \) is constant. This gives us a change in pressure of

\[
dp = A \gamma \rho^{\gamma - 1} \frac{d\rho}{dz} dz = \gamma \frac{dp}{\rho} \frac{d\rho}{dz} dz.
\]

Equating the two expressions for \( dp \) and rearranging for \( dT/dz \), yields

\[
\left( \frac{dT}{dz} \right)_{ad} = \left( \frac{\gamma - 1}{\gamma} \right) \frac{T}{p} \frac{dp}{dz}.
\]
known as the adiabatic temperature gradient. This can also be expressed as

\[
\left( \frac{dT}{dz} \right)_{ad} = -\frac{(\gamma - 1) \mu m}{\gamma k_B g}.
\]  

(1.35)

An alternative form of the adiabatic temperature gradient can be found in terms of a logarithmic derivative of \(T\) with respect to \(p\). We can gain this expression by rearranging the adiabatic temperature gradient (Eq. 1.34),

\[
\frac{dT}{dp} \frac{p}{T} = \frac{(\gamma - 1)}{\gamma},
\]

to give

\[
\left( \frac{d \log T}{d \log p} \right)_{ad} = \frac{\gamma - 1}{\gamma}.
\]

Now that we have constructed the adiabatic value for the logarithmic temperature gradient, we look at how we can rearrange the stability condition in terms of the logarithmic temperature gradient. Using the ideal gas law,

\[
\frac{d \rho}{dz} = \rho \frac{dp}{dz} - \rho \frac{dT}{T} \frac{dT}{dz},
\]

allows us to rewrite the buoyancy frequency from Eq. 1.32 as

\[
N^2 = g \left( \frac{1}{T} \frac{dT}{dz} - \frac{(\gamma - 1)}{\gamma} \frac{1}{p} \frac{dp}{dz} \right).
\]

We know the condition for a buoyancy instability to occur is \(N^2 < 0\). This can now be expressed as

\[
\frac{1}{T} \frac{dT}{dz} < \frac{1}{p} \frac{(\gamma - 1)}{\gamma} \frac{dp}{dz},
\]

\[
\Rightarrow \frac{dT}{dz} < \left( \frac{dT}{dz} \right)_{ad}.
\]

Remember the adiabatic temperature gradient is negative. Hence, it is more intuitive to rewrite this as

\[
-\frac{dT}{dz} > -\left( \frac{dT}{dz} \right)_{ad} = \frac{(\gamma - 1) \mu m g}{\gamma k_B}.
\]

Therefore the buoyant magnetic field will continue to rise towards the photosphere provided the gradient in the background temperature stratification is greater than the adiabatic temperature gradient. Or, equivalently, in terms of the logarithmic temperature gradient with respect to \(p\),

\[
\frac{d \log T}{d \log p} > \frac{\gamma - 1}{\gamma}.
\]

Plasmas with a temperature gradient steeper than the adiabatic temperature gradient are known to be superadiabatic. We have shown that super-adiabatic temperature gradients are convectively unstable. Furthermore, if the temperature gradient is shallower than its adiabatic value we call the gradient subadiabatic.
1.4 Magnetic buoyancy

and the plasma is convectively stable. As introduced earlier, the photosphere is an example of a convectively stable atmosphere and hence is stable against the buoyancy instability (sometimes described as the convective instability). This concludes our discussion of the first stage of emergence to the photosphere.

1.4.2 Magnetic buoyancy instability

Next, we consider the instability that allows magnetic fields to rise into the solar atmosphere. As introduced earlier, the photosphere is convectively stable so flux tubes can no longer rise by means of the buoyancy instability. This is because the temperature gradient is no longer sufficiently decreasing and the plasma is no longer buoyant. Hence, the magnetic field must find another pathway to rise into the atmosphere. This is achieved by means of a magnetic buoyancy instability. We now derive the general conditions under which this instability ensues.

Consider an ideal plasma in equilibrium, where all variables are functions of \( z \) solely. The initial equilibrium consists of a horizontal magnetic field, \( \mathbf{B} = (B_0(z), 0, 0) \) in a gravitationally stratified plasma. The field satisfies magnetohydrostatic balance:

\[
\frac{dp_0}{dz} + \frac{d}{dz} \left( \frac{B_0^2}{2\mu} \right) = -\rho_0 g. 
\]

Following the same linearisation process as before, using perturbations of the form \( f(z) \exp(i(kx + ly - \omega t)) \), the linearised ideal MHD equations become

\[
\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0), \\
\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \frac{1}{\mu} (\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 + \frac{1}{\mu} (\nabla \times \mathbf{B}_0) \times \mathbf{B}_1 - \rho_1 g \hat{z}, \\
\nabla \cdot \mathbf{B}_1 = 0, \\
\frac{\partial p_1}{\partial t} = -\rho_0 \nabla \cdot \mathbf{v}_1 - \mathbf{v}_1 \cdot \nabla \rho_0, \\
\frac{\partial \rho_1}{\partial t} = -\mathbf{v}_1 \cdot \nabla \rho_0 - \gamma \rho_0 \nabla \cdot \mathbf{v}_1. 
\]

This set of equations can be rearranged algebraically to give

\[
\rho_0 \frac{\partial^2 \mathbf{v}_1}{\partial t^2} = \nabla (\mathbf{v}_1 \cdot \nabla \rho_0 + \gamma \rho_0 \nabla \cdot \mathbf{v}_1) + \frac{1}{\mu} (\nabla \times \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0)) \times \mathbf{B}_0 \\
+ \frac{\nabla \times \mathbf{B}_1}{\mu} \times \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0) + g(\rho_0 \nabla \cdot \mathbf{v}_1 + \mathbf{v}_1 \cdot \nabla \rho_0) \hat{z}. 
\]

Note, in this case, \( x \) derivatives are replaced by \( ik \) and \( y \) derivatives are replaced by \( il \). This can be expressed in terms of its \( x \), \( y \) and \( z \) components:

\[
-\rho_0 \omega^2 v_{x1} = ik \left( \frac{dp_0}{dz} + \frac{B_0 \, dB_0}{\mu \, dz} \right) v_{z1} + ik \gamma \rho_0 \nabla \cdot \mathbf{v}_1, \\
-\rho_0 \omega^2 v_{y1} = il \left[ \left( \frac{dp_0}{dz} + \frac{B_0 \, dB_0}{\mu \, dz} \right) v_{z1} + \left( \gamma \rho_0 + \frac{B_0^2}{\mu} \right) \nabla \cdot \mathbf{v}_1 - i \frac{k B_0^2}{\mu} v_{x1} \right] - \frac{k^2 B_0^2}{\mu} v_{y1},
\]

(1.36)

(1.37)
Then, the dispersion relation given in Eq. 1.41 can be expressed as

\[ \omega^4 (c_s^2 + v_A^4) - \omega^2 \left[ k^2 (v_A^4 + 2c_s^2 v_A^2) + v_A^2 g \frac{d}{dz} \log \left( \frac{B_0}{\rho_0} \right) + c_s^2 N^2 \right] + k^2 v_A^2 \left[ k^2 c_s^2 + g \frac{d}{dz} \log B_0 + \frac{c_s^2}{v_A^2} N^2 \right] = 0, \]  

(1.41)

where \( c_s = \sqrt{\gamma p_0 / \rho_0} \) is the sound speed and \( v_A = B_0 / \sqrt{\mu_0 \rho_0} \) is the Alfvén speed.

This dispersion relation is fairly complex and very difficult to solve in the form given, and hence we consider specific cases which correspond to different forms of the instability. We first consider the case \( k = 0 \), where we assume there is no perturbation along the field lines, or more generally \( k \perp \mathbf{B} \). We are therefore considering the case where perturbations are perpendicular to the field. This simplifies the dispersion relation to

\[ \omega^4 (c_s^2 + v_A^4) - \omega^2 \left[ v_A^2 g \frac{d}{dz} \log \left( \frac{B_0}{\rho_0} \right) + c_s^2 N^2 \right] = 0. \]  

An instability will ensue if \( \omega^2 < 0 \), or, equivalently, if the following instability condition is satisfied,

\[ -g \frac{d}{dz} \log \left( \frac{B_0}{\rho_0} \right) > \frac{N^2}{v_A^2}. \]

The \( k = 0 \) case is defined as the interchange mode and differs from the buoyancy instability in that this type of instability can arise in convectively stable atmospheres where \( N^2 > 0 \), i.e. the photosphere. If the field strength falls off faster than the density, an instability is triggered as the field needs to support more mass than is possible.

Next, we assume \( k \neq 0 \) to investigate what is known as the undular magnetic buoyancy instability. This does not simplify the dispersion relation given in Eq. 1.41. However, the instability condition can be
determined by the final constant term in the expression (independent of $\omega$). As the dispersion relation is a quadratic in $\omega^2$, the product of the roots of this equation must equal the final term. If the final term is positive this suggests one of two possibilities: both roots are negative or both roots are positive. If both roots are positive, the system is stable. However, if the final term is negative, one root is positive and the other is negative. This can be thought of as the easiest condition to satisfy to ensure the instability will occur. Hence, the instability condition for the undular magnetic buoyancy instability is satisfied if the final term is less than zero, i.e.

$$k^2 + \frac{g}{c_s^2} \frac{d}{dz} \log B_0 + \frac{N^2}{v_A^2} < 0,$$

or perhaps more intuitively if

$$-\frac{g}{c_s^2} \frac{d}{dz} \log B_0 > k^2 + \frac{N^2}{v_A^2}.$$ (1.42)

This instability can also occur in convectively stable atmospheres, if the derivative of $B_0$ with height is negative and sufficiently large. A large $k^2$ can prevent this instability from occurring as magnetic tension becomes important when the field lines are curved. Note, the undular magnetic buoyancy instability is often referred to as the Parker instability in the literature (Kim et al., 2004).

Note, in the instability relations, there are derivatives of $\log B_0$ in the undular relation, and derivatives of $\log \left( \frac{B_0}{\rho_0} \right)$ in the interchange relation. This implies that the undular modes require a decrease in $B_0$ with height whereas the interchange modes require a decrease in $B_0/\rho_0$ with height. The condition for undular modes is more easily satisfied than that of interchange modes, and hence, undular modes are a more common type of instability.

A more detailed investigation into the criterion for the magnetic buoyancy instability is undertaken by Acheson (1979) where the effects of diffusion and rotation are taken into account. In addition, see Newcomb (1961) and Thomas and Nye (1975) for further details on the magnetic buoyancy instability. For example, for the undular mode of the magnetic buoyancy instability, Acheson (1979) uses perturbations to the equilibrium of the form $\sin mz \exp i(kx + ly - \omega t)$ using our notation for a wave-vector $k = (k,l,m)$. Using these perturbations, the following instability condition is derived:

$$-\frac{g}{c_s^2} \frac{d}{dz} \log B_0 > k^2 \left( 1 + \frac{m^2}{l^2} \right) + \frac{N^2}{v_A^2},$$ (1.43)

where $N^2$ is the Brunt-Väisälä frequency as given in Eq. 1.33. In our analysis, we consider the case as $l \to \infty$, this forces $m^2/l^2$ to be negligible and Acheson’s condition above reduces to the one we derived earlier for the undular mode (Eq. 1.42). This instability condition is used in Archontis et al. (2004) in a slightly different form to describe the instability condition that allows tubes trapped at the photosphere to rise into the higher atmosphere. The equivalent expression introduced in Archontis et al. (2004) is

$$-H_p \frac{d}{dz} \log B_0 > k_||^2 \left( 1 + \frac{k_\perp^2}{k_\perp^2} \right) \frac{\gamma}{2} \beta \delta,$$ (1.44)
1.4 Magnetic buoyancy

where \( H_p = k_B T/\mu g \) is the pressure scale height at the photosphere, \( \tilde{z} \) is the dimensionless height \( z/H_p \), \( \beta \) is the ratio of the plasma pressure over the magnetic pressure and \( \tilde{k}_||, \tilde{k}_\perp \) and \( \tilde{k}_z \) are the three components of the perturbation wave vector (normalised by \( 1/H_p \)) with \( \tilde{k}_|| \) and \( \tilde{k}_\perp \) being the two horizontal components parallel and perpendicular to the local magnetic field direction and \( \tilde{k}_z \) being the \( z \)-component.

The superadiabaticity of the atmosphere, denoted by \( \delta \) is given by \( \nabla - \nabla_{ad} \) where \( \nabla \) is the logarithmic temperature gradient specified as \( d \log T/d\tilde{z} \) and \( \nabla_{ad} \) is its adiabatic value.

In order to verify our derivation with the expression quoted in Archontis et al. (2004), we rearrange Eq. 1.43 to obtain the form of Eq. 1.44. First, we derive the adiabatic value for the logarithmic temperature gradient using the adiabatic temperature gradient specified in Eq. 1.35,

\[
\left( \frac{dT}{d\tilde{z}} \right)_{ad} = -\frac{(\gamma - 1) \mu g}{\gamma k_B g H_p} = -\frac{(\gamma - 1) \mu g H_p}{\gamma} \Rightarrow \left( \frac{dT}{d\tilde{z}} \right)_{ad} = -\frac{(\gamma - 1) \mu g H_p}{\gamma} \Rightarrow -\frac{T}{(\gamma - 1) \gamma}.
\]

Now, rewriting Eq. 1.43 in terms of the dimensionless quantities \( \tilde{z}, \tilde{k}_||, \tilde{k}_\perp \) and \( \tilde{k}_z \) defined above, where we have redefined \( k|| = k, k\perp = l \) and \( k_z = m \),

\[-\frac{1}{\gamma} \frac{d}{d\tilde{z}} \log B_0 > \tilde{k}_||^2 \left( 1 + \frac{\tilde{k}_z^2}{\tilde{k}_\perp^2} \right) + H_p^2 N^2 v_A^2 \.
\]

In order to proceed, we re-express the final term as

\[
H_p^2 N^2 v_A^2 = \frac{c_s^4}{\gamma^2 g^2} \frac{N^2}{v_A^2} = H_p \beta/2 \left( \frac{1}{T} \frac{d T}{d \tilde{z}} - \frac{1}{p} \frac{d p}{d \tilde{z}} \frac{(\gamma - 1)}{\gamma} \right) = \beta/2 \left( \frac{d \log T}{d \tilde{z}} - \frac{1}{\rho g} \frac{d \rho}{d \tilde{z}} \frac{(\gamma - 1)}{\gamma} \right) = \beta/2 \left( \frac{d \log T}{d \tilde{z}} + \frac{(\gamma - 1)}{\gamma} \right) = \beta/2 (\nabla - \nabla_{ad}) = \beta \delta^2.
\]

This allows us to rewrite our instability criteria as

\[-\frac{d}{d\tilde{z}} \log B_0 > \gamma k||^2 \left( 1 + \frac{\tilde{k}_z^2}{\tilde{k}_\perp^2} \right) + \gamma \frac{\beta \delta}{2} \quad \text{(1.45)}
\]

We note a couple of minor differences between this expression and that derived in Archontis et al. (2004),
namely the first term on the right hand side is multiplied by a $\gamma$ and the final term is added on rather than subtracted. Also, note that the superadiabatic excess, $\delta = \nabla - \nabla_{ad}$ is equal to $+0.4$ for an isothermal stratification given $\gamma = 5/3$. This contradicts that quoted in Archontis et al. (2004) with $\delta = -0.4$ defined for an isothermal stratification. Nonetheless, an isothermal stratification can cause strong stabilisation. However, as it is multiplied by the plasma $\beta$ the term becomes small when the magnetic field reaches the photosphere and the magnetic pressure begins to dominate the plasma pressure. Murray et al. (2006) noticed that this criteria was found to be satisfied when the plasma $\beta$ drops to approximately one. This is investigated later in Chapter 4.

Table 1.2: Summary of instability conditions for both the interior and atmospheric regions.

<table>
<thead>
<tr>
<th></th>
<th>Solar Interior</th>
<th>Photosphere</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brunt-Väisälä frequency</td>
<td>$N^2 &lt; 0$</td>
<td>$N^2 &gt; 0$</td>
</tr>
<tr>
<td>Temperature gradient</td>
<td>$-\frac{dT}{dz} &gt; -\left(\frac{dT}{dz}\right)_{ad}$</td>
<td>$-\frac{dT}{dz} = 0 &lt; -\left(\frac{dT}{dz}\right)_{ad}$</td>
</tr>
<tr>
<td>Convective stability</td>
<td>Convectively unstable</td>
<td>Convectively stable</td>
</tr>
<tr>
<td>Buoyancy Instability</td>
<td>✓</td>
<td>x</td>
</tr>
<tr>
<td>Magnetic buoyancy Instability</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

A summary of the instability conditions is given in Table 1.2. The table is split into two columns, namely the solar interior and the solar photosphere. As explained earlier in the chapter, the buoyancy instability is responsible for the rise of flux tubes to the photosphere and the magnetic buoyancy instability for the emergence past the photosphere. We do, however, note it is possible for the magnetic buoyancy instability to be initiated in the solar interior. Having said that, the plasma $\beta$ is large in the solar interior which makes this criteria harder to satisfy than the buoyancy instability criteria. Hence, we expect the rise of the tubes in the solar interior to be governed by the buoyancy instability.

### 1.5 Modelling active regions

Many simulations have been performed with the aim of aiding our understanding of the flux emergence process. In this thesis, we are primarily concerned with the large-scale emergence of single active regions, rather than global multiple emergence events. Simulations of flux emergence have been carried out for almost 30 years now and with the recent rapid increase in computing power over the last decade, the detail and size of experiments has improved vastly. In this section, we shall briefly review the results of previous flux emergence simulations. However, this is by no means an extensive review and as such we direct the reader to appropriate literature throughout the section. We shall only give a brief synopsis of each paper.

Simulations of flux emergence are usually set up in a similar manner. They consist of a stratified domain, including a convection zone and two or more atmospheric layers. Magnetic flux is then initialised in the form of a flux sheet or a flux tube within the solar interior. Through the use of reducing the density of the tube compared with the environment or imposing velocity perturbations, the flux sheet/tube rises through the solar interior. In some experiments, flux emerges into the atmosphere while in others it remains...
stuck at the photosphere. There are specific conditions that need to be met for full flux emergence into the corona (see Section 1.4.2 for further details).

Flux tube simulations can be categorised into two main types given the way in which they are initialised in the solar interior. Typically, authors start with an equilibrium and modify the state in order to facilitate emergence.

1. **Thermal Equilibrium**
   In this type of equilibrium, as the name suggests, the temperature of the tube is chosen to match the external temperature. We do not choose the flux tube to be in force balance, i.e. the magnetic forces do not balance each other. However, we define the gas pressure such that its gradient together with the magnetic forces and gravitational forces, results in force balance:
   \[ -\nabla p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g} = 0. \]
   This results in a pressure deficit added on at the location of the tube. In order to modify the equilibrium to allow the flux tube to emerge, a density deficit is imposed by maintaining the pressure deficit and thermal equilibrium in the ideal gas law. Depending on the direction of the gas pressure gradient, the tube will either be made lighter or heavier than its surroundings. Depending on whether it is buoyant or more dense, the tube will either rise or sink when the simulation begins. In all flux emergence experiments, the tube is made buoyant in order to begin the emergence of flux.

2. **Mechanical Equilibrium**
   In this case, the flux tube is chosen to be force-free, e.g. a Gold-Hoyle flux tube. This means that the magnetic forces balance, i.e. \( \mathbf{j} \times \mathbf{B} = 0 \). We also choose the density and pressure to match the surroundings, and hence by definition, the temperature of the tube matches the surroundings. The tube is therefore in equilibrium so any initial perturbation that causes the tube to rise must come outwith the tube. When tubes are in mechanical equilibrium, the tube is usually disturbed by a vertical velocity perturbation that causes it to rise to the surface.

Although both of these types of equilibrium are used in the papers we will discuss, the thermal equilibrium set-up is used more often. This is due to the fact that the plasma \( \beta \) is greater than 1 in the convection zone on the Sun as plasma forces dominate over magnetic forces. In the derivation of the force-free approximation in Section 1.2.5, we assume \( \beta \ll 1 \) along with other assumptions. Hence, force-free fields are used more often in the corona due to the low \( \beta \) value found here. Therefore, mechanical equilibrium is used less often and thermal equilibrium is often seen as the best choice for initialising a flux tube in the solar interior (Murray, 2007).

We shall proceed discussing papers in approximately chronological order but will split this review into two sections: 2D/2.5D simulations and 3D simulations.
1.5 Modelling active regions

1.5.1 2D and 2.5D simulations

In the late 1980s and early 1990s, a series of papers were published regarding the simulations of magnetic flux sheets in two dimensions. Shibata et al. (1989a) considered a simple two-layer atmosphere comprising of a cool photosphere/chromosphere and a hot corona, and solved the ideal MHD equations. This set-up is typical of simulations in this era. Shibata et al. (1989a) chose the tube to be placed in the photosphere such that the sheet satisfies magnetostatic balance (mechanical equilibrium) and imposed a velocity perturbation in order to drive the rise of the flux sheet. The velocity perturbation used is of the form

$$v_x = A \sin \left( \frac{2\pi (x - X_{\text{max}})}{\lambda} \right)$$

where $\lambda$ is the horizontal wavelength and $X_{\text{max}}$ is the length of the domain in the horizontal direction. Shibata et al. (1989a) chose to study flux emergence in order to understand the mechanism which allows flux to emerge, which they referred to as the Parker instability. In our discussion of magnetic buoyancy, we referred to this instability as the undular mode ($k \neq 0$) of the magnetic buoyancy instability. As discussed previously, this is the mode of instability that is easiest to achieve and can occur even when the system is stable to interchange modes ($k = 0$). The loop accelerates during the rise phase because it becomes unstable to the undular perturbation. The background pressure falls off sharply with height in the photosphere, then less steeply in the corona. As the loop expands vertically through the photosphere, the magnetic pressure continually exceeds the gas pressure, and in turn the magnetic buoyancy force is greater than the restoring tension force caused by the curvature of the loop. The loop then decelerates when it reaches the corona because the pressure difference is lower in the corona, and so the magnetic buoyancy force is decreased.

Shibata et al. (1989b) continued on from the previous work including the extra effects of resistivity and a magnetised corona. The rest of the set-up is the same as that of Shibata et al. (1989a). The rise of the loop follows the same pattern as before. They divided the results into two separate cases. When the coronal field is parallel to the loop field, the loop undergoes sufficient deceleration as it tries to push through the coronal field. Alternatively, when the coronal and loop fields are anti-parallel, there is reconnection at the interface of the two fields.

The work described above consists of initial conditions in which horizontal sheets are inserted into photospheric layers, stable to convective motions. These pioneering simulations addressed the initiation of the magnetic buoyancy instability but did not address how the magnetic flux reached the photospheric layer. Shibata et al. (1990) chose to initialise the flux sheet in the convection zone, a convectively unstable layer. This is chosen to lie beneath a simplified two layer atmosphere consisting of a photosphere/chromosphere and corona. The combined effects of magnetic buoyancy and convection carry magnetic flux from the convection zone into the photosphere, after which it expands into the chromosphere. The authors showed that an initially weak magnetic field ($\approx 600$ G) can be amplified to $> 1000$ G (typical strength of umbra in observations Solanki, 2003) by the convective collapse of the flux tube by the downflows of the rising loop. They attribute this to the addition of a radiative cooling term at the photosphere.

Over ten years later, Magara (2001) performed a 2.5D experiment of the emergence of a twisted flux tube in a four layer model. In 2.5D simulations, the coordinates and associated vectors have three dimen-
sions but all quantities are assumed to be invariant or uniform in one direction. In this case, the axis is aligned along the uniform $y$ coordinate. The domain is stratified into four layers: a convection zone; photosphere; transition region; and corona. The authors insert a Gold-Hoyle flux tube at a depth of 1 Mm in the convection zone and impose a density deficit by reducing the pressure within the tube and maintaining temperature equilibrium with its environment. Initially, the flux tube rises through the convection zone by buoyancy but decelerates when it reaches the convectively stable photosphere. The top of the tube flattens as it cannot proceed in the photosphere but the following tube plasma is still rising below it. The rising plasma becomes deflected sideways by horizontal surface flows. The authors then noticed the surface became subject to a Rayleigh-Taylor instability as a heavier photospheric plasma layer is located on top of a magnetic layer with lighter plasma. By this instability, the magnetic field emerges through the photosphere, whereupon it begins to expand as the background gas pressure is weaker than that of the magnetic pressure of the layer. The underlying magnetic field is then free to emerge under the influence of the magnetic buoyancy instability (introduced in Section 1.4.2). However, the axis of the flux tube (central fieldline) remains trapped at photospheric heights.

Most simulations assume the corona is fully ionised throughout the computational domain. However, in reality certain regions of the atmosphere are at sufficiently low temperatures to be only partially ionised, in particular the photosphere and low chromosphere. Here, the plasma has a substantial neutral hydrogen fraction. The 2.5D emergence simulations of Leake and Arber (2006) take into account the partial ionisation of the Sun’s chromosphere by including Cowling resistivity ($1/\sigma_c$) in addition to Coulomb resistivity ($\eta' = 1/\sigma$). Cowling resistivity gives a measure of the diffusion of the magnetic field due to collisions of ions and electrons with neutrals and acts perpendicular to the field. Coulomb resistivity, on the other hand, only considers collisions between ions and electrons and acts on the current parallel to the magnetic field. In addition to removing the perpendicular current, partial ionisation also modifies the gas law and, for given pressure and density, increases the temperature, resulting in a larger pressure scale-height. Thus, the pressure and density do not drop off with height as quickly as the equivalent for a fully ionised plasma. The authors find that the rates of emergence in partially ionised regions are greatly increased, and the resulting magnetic field is more diffuse. This is due to the dissipation of energy of cross-field currents by ion-neutral interactions resulting in a more force-free field.

In these 2D and 2.5D simulations, the flux tube must rise as a whole due to the reduced dimensionality of the experiment. This prevents draining of material from any portion of the tubes axis and Magara (2001) suggested this as the reason for the failure of the axis to emerge into the atmosphere. Furthermore, the magnetic buoyancy instability occurs along the whole length of the tube and, therefore, there is no possibility of forming distinct sunspots. Solutions to these limitations are addressed in the next section, in which we consider 3D simulations. Although the work reviewed above was carried out in 2 or 2.5 dimensions, it laid the foundations for the subsequent 3D simulations.

### 1.5.2 3D simulations

Matsumoto et al. (1998) were the first to carry out a 3D simulation of the emergence of a magnetic flux tube from the convection zone into the atmosphere. They used a different approach, by considering the rise
of a kink-unstable (Hood and Priest, 1979) flux tube. This study (along with others including Fan et al., 1998) suggests that kinked structures resulting from this instability may result in sigmoidal loops observed in X-rays (Canfield et al., 1999).

The first 3D experiment to simulate the emergence of a magnetic flux tube due to buoyancy was carried out by Fan (2001). This experiment was groundbreaking and shaped the next ten years of flux emergence experiments. The set-up of the magnetic field is similar to that of the Gold-Hoyle flux tube in that the tube has a component of the magnetic field along its axis ($B_{\text{axis}}$) and an azimuthal component proportional to $B_{\text{axis}}$ (i.e. $B_{\theta} \propto \alpha B_{\text{axis}}$, where $\alpha$ is the twist of the magnetic field). However, the field chosen is not force-free and is of the form $B_{\text{axis}} = B_0 e^{-r^2/a^2}$ and $B_{\theta} = \alpha r B_{\text{axis}}$. The tube has a uniform twist profile and takes the shape of a cylindrical magnetic field (discussed in more detail in Section 3.3.1). Fan (2001) chose a twist value that makes the tube marginally stable to convection with a left-hand twist profile. All subsequent simulations contain a solar interior marginally stable to convection, unless otherwise stated. By balancing radial pressure gradients with magnetic forces, the authors find a pressure deficit within the tube, and use this to impose a density deficit. This is the thermal equilibrium approach. Rather than making the entire tube buoyant, Fan (2001) multiplied the density deficit by $\exp\left(-x^2/\lambda^2\right)$, where $x$ is the direction of the axis. This shapes the emerging flux tube into an $\Omega$ shape. Hence $\lambda$ controls the size of the active region, and for $|x| \gg \lambda$, the tube is in mechanical equilibrium.

The buoyant part of the flux tube rises, and the front and axis of the tube decelerate as the tube reaches the photosphere, due to its stabilising isothermal stratification. The front of the tube then undergoes a magnetic buoyancy instability due to the build up of magnetic flux, and expands into the unmagnetised atmosphere. However, the axis of the flux tube remains stuck at low photospheric heights. Fan (2001)’s experiment also successfully reproduces many observed features of newly emerged active regions reported by Strous (1994) and Strous et al. (1996). Specifically, they correctly identify the orientation of the arch-filament system, the distribution of the photospheric vertical magnetic field, the locations of sunspots and the shear flows that appear in the photospheric horizontal velocity field. This correctly orientated sigmoid structure was also found in the flux emergence experiments performed by Magara and Longcope (2001). Both experiments found an inverse $S$ shape in the coronal field, due to the left-hand twisted flux tube inserted. This corroborates observational studies that have shown active regions with negative helicity (or left-hand twisted fields) have soft X-rays with inverse $S$ shaped structures (Canfield et al., 1999).

A study by Archontis et al. (2004) used the same initial condition as Fan (2001). In a similar experiment, they found the axial magnetic field strength of the flux tube decreases during its rise due to the law of magnetic flux conservation, as follows:

$$B_{y}(z) = B_{y}(z_0)\rho(z)/\rho(z_0),$$

where $z_0$ is the initial depth of the axis, $B_{y}$ is the axial field strength and $\rho$ is the density. Although all components of the magnetic field decrease when the flux tube rises due to a decrease in density, the azimuthal component decreases more slowly due to the compression of the front of the tube. Hence, by the time the tube reaches the photosphere, the fieldlines’ pitch is pointing away from the axial direction. Archontis et al. (2004) also provide detail about the criteria for the onset of magnetic buoyancy described in Section 1.4.2.
Archontis et al. (2004) found that once emergence was underway, the field expanded in a runaway fashion in both the horizontal and vertical directions. In fact, horizontal growth occurs at a faster rate than vertical growth as gravity acts to slow growth in the vertical direction. The expansion of the field into the atmosphere causes the gas pressure and temperature to fall below that of the original stratification. However, the density at the front of the tube is $100$ times greater than the original atmospheric stratification due to the dense plasma lifted from the interior. Furthermore, the observed photospheric signatures produced by Fan (2001) and Magara and Longcope (2001) were reproduced by Archontis et al. (2004). By using a right-hand twisted field (opposite twist to that of Fan, 2001 and Magara and Longcope, 2001), they reveal a forward S shape current concentration lying in the atmosphere.

Manchester et al. (2004) used the same set-up as Fan (2001), except they reduced the length of the buoyant section of the flux tube by a factor of $2$ (by decreasing $\lambda$). The shorter section of the flux tube bends more sharply and has fewer turns, which allows plasma to drain more efficiently. The signatures of flux emergence are again similar to that of Fan (2001) with a distribution of $B_z$ that resembles sunspots and a horizontal velocity shear pattern. However, there are significant differences between their results and that of Fan (2001). The sunspots are closer together, which is not surprising given the smaller buoyant region. The efficient draining of plasma means that the magnetic flux tube reaches higher into the photosphere before the onset of the magnetic buoyancy instability. In addition, the photospheric magnetic field forms a quadrupolar pattern.

Murray et al. (2006) performed a series of experiments using the initial condition founded by Fan (2001), with the aim of understanding the effects of varying the magnetic field strength and twist of an emerging tube as it rises through the solar interior and emerges into the atmosphere. The authors noted a self-similar evolution in the rise and emergence of the tube as its magnetic field strength is modified. A quantity is said to be self-similar when it is similar to itself at a different time, or a copy of itself on a different scale. Specifically, they found that properties such as the height and velocity of the crest and axis and amount of unsigned flux can be scaled against the initial magnetic field strength. However, no such self-similarity is seen when varying the twist of the tube due to the non-linear dependence of the tension force on the twist parameter. For low magnetic field strengths and twists, the authors found that flux tubes could not emerge beyond the photosphere since the magnetic buoyancy instability could not be fulfilled. In all cases, the axis of the flux tube does not emerge beyond photospheric heights.

Murray and Hood (2008) extended this study by considering the emergence of tubes with non-constant twist. They found that, irrespective of the twist profile, sufficiently weakly twisted fields experience a greater expansion due to the lower tension force, and hence a weakening of the magnetic field during the rise through the interior. Therefore, when the field reaches the photosphere, the field strength is too weak to initiate the magnetic buoyancy instability, and hence the flux tubes do not emerge into the atmosphere. For those that do emerge into the atmosphere, little difference is found in the atmospheric field for the different twist profiles. However, a couple of general trends were found. Tubes with strong tension forces tend to have a faster growth rate due to the easier initiation of the magnetic buoyancy instability. However, tubes with weaker tension forces tend to expand horizontally to a greater degree. In general, the particular details of the twist profile are lost post emergence. Only in cases with a sufficiently low tension force is it possible to make any distinction in post emergence magnetograms.
1.5 Modelling active regions

Carrying on with the same initial condition, Archontis and Török (2008) investigated the formation of flux ropes and eruptions from magnetic flux emergence. They considered the emergence of magnetic flux into both unmagnetised and magnetised atmospheres. Of course, a magnetised atmosphere is a more realistic model of the Sun’s corona but an unmagnetised atmosphere is often used for simplicity, in order to pinpoint individual modifications. The authors find the formation of a flux rope within the expanding magnetic volume, due to the shearing and reconnection of fieldlines at low atmospheric heights. Without a pre-existing magnetised atmosphere, the flux tube rises, confined within the expanding volume. However, with a magnetised atmosphere, the flux tube reconnects with the pre-existing field and experiences an eruption, reminiscent of filament eruptions or coronal mass ejections.

Hood et al. (2009) departed from the common cylindrical field introduced by Fan (2001) and derived a flux tube in a toroidal geometry. That is, they consider a flux tube in the shape of a half-torus with its feet anchored on the base of the domain. However, the same field components are still employed, with $B_{\text{axis}} = B_0 e^{-r^2/a^2}$ and $B_\theta = \alpha r B_{\text{axis}}$, but the axis is now defined along the centre of the semi-torus. This is the model we use for the experiments performed in this thesis and as such this model is described in greater detail in Chapter 3. In this model, the authors found a rotation of nearly 360$^\circ$ of sunspots during the emergence of the toroidal flux tube. We plan to extend this work, and as such introduce and discuss this topic in Chapter 4.

MacTaggart and Hood (2009) compared the toroidal model used by Hood et al. (2009) with the cylindrical model used before then. Firstly, a half-torus shaped flux tube imitates a field anchored deeper within the solar interior, rather than at the very top of the convection zone. A common shortcoming of the cylindrical model in simulations without convective flows is that the axis of the tube does not fully emerge. Altering the geometry of the flux tube to a curved shape allows for the axis of the tube to rise into the corona. In addition, the authors find that the pair of sunspots do not continually drift apart but instead drift to a fixed distance determined by the major radius of the flux tube (defined in Chapter 3). In contrast, the sunspot pair continually drift in the cylindrical model as the flux tube is weakly buoyant along its length.

Archontis and Hood (2010) revisit the model of Archontis et al. (2004) with an ambient magnetic field to investigate the effect of varying magnetic field strength, and buoyant region parameters on the resulting photospheric distribution. In their results they observe the photospheric flux distribution to be made up of two opposite polarities, and elongated magnetic tails on the two sides of the polarity inversion line (PIL) that separates the two polarity regions. This configuration is in qualitative agreement with observations (Canou et al., 2009). For cases with small twists, the tails are less pronounced, and easily disturbed by photospheric flows. However, for cases with larger twists, the tails are defined, robust features of the sunspots. Hence, the presence of tails may tell us something about the twist profile of the emerging magnetic flux.

Toriumi et al. (2011) build on the parameter study of Murray et al. (2006) using the same initial condition but considering ten different cases, covering the three that were investigated by Murray et al. (2006). They term the emergence process a two-step process due to the pause often found before the initiation of the magnetic buoyancy instability (Toriumi and Yokoyama, 2011). The authors find that, in the strongly twisted case, the deceleration of the emerging flux in the photosphere is not significant. In contrast, they find that magnetic flux does not emerge into the atmosphere for the weakly twisted case. In the intermediate twist cases, the flux remains temporarily trapped at the photosphere, and after a small delay it rapidly emerges
into the corona due to the initiation of the magnetic buoyancy instability.

All the simulations described thus far model the emergence of magnetic flux into an unmagnetised corona or an ambient coronal field. However, in reality there will be cases where magnetic flux emerges into pre-existing active regions. MacTaggart (2011) addressed this by building on previous models of Fan (2001) and Murray et al. (2006) and considering the emergence of twisted magnetic flux into a mature active region. They choose a potential magnetic field to model the existing mature active region, and place the emerging $\Omega$-loop closer to the existing negative polarity. MacTaggart (2011) found that the expansion of the new negative polarity is restricted due to the curvature of the overlying magnetic field. They also found a build up of pressure between the two systems, along with the reconnection, to inhibit and resist the expansion of the new flux system. Of course, these results will vary depending on the initial set-up and positioning and strength of the flux systems.

All simulations have been performed under the MHD approximation and, with the exception of Leake and Arber (2006), are known as idealised studies. The idealised experiments performed may seem naive and simple in comparison to complex observations. However, these experiments allow us to study the effects of discrete physical processes. Hence, by considering an idealised simple state, an equilibrium for example, and modifying the state in a controlled fashion, we can understand the key physical processes this affects. If too many modifications are made simultaneously, the important physical processes underpinning the results are obscured. We direct the reader to a recent review of idealised emergence experiments in Hood et al. (2012). On the contrary, a new class of realistic simulations have been performed in recent years that include a more detailed energy equation, with radiative transfer, thermal conduction and partial ionisation. Realistic simulations may be perceived the most readily useful as they may produce comparable results with observations. However, idealised simulations are an incredibly important tool to build on our knowledge. We focus on idealised simulations in this thesis. See Cheung and Isebe (2014) for a review of both idealised and realistic emergence simulations performed in recent years.

1.6 Rotation of sunspots

The rotation of sunspots has intrigued many authors over the years from both an observational and modelling approach. We define the rotation of a sunspot as the circular movement of a sunspot (region of strong magnetic field) around its own vertical axis (or umbral centre).

Whilst working at the Kodaikanal Solar Observatory early in the twentieth century, Evershed (1909) first discovered evidence of the rotational movements of a sunspot around its own vertical axis based on spectral observations. Several studies have since been conducted in order to observe this phenomena in view of the initial findings and have found that it is not a rare occurrence in solar observations (see Brown et al., 2003, Zhao and Kosovichev (2003), Tian and Alexander (2006), Yan and Qu, 2007, Min and Chae, 2009, Yan et al., 2009 amongst others). All studies have shown, by the tracking of distinguishing features of active regions, that sunspots can experience rotations of the order of hundreds of degrees over a period of days.
Brown et al. (2003) studied seven sunspots and their rotation rate using white light data from TRACE. This detailed investigation considered active regions from 1999 to 2002 and involved finding sunspot centres and following pronounced penumbral features to calculate rotation rates. To do this, they calculated sunspot centres and unwrapped images to create $r - \theta$ plots of white-light data. Time slices on constant radii show rotations of sunspots as diagonal streaks of light and dark distinguishable features. Over the seven active regions, they calculated rotation angles varying from $40^\circ$ to $200^\circ$ over time periods of three to five days, corresponding to an average rotation rate of a few degrees per hour. Six of the seven sunspots led to recorded flaring activity and there was evidence that the rotating spots contributed to the energisation of the corona.

Yan et al. (2008), on the other hand, conducted a larger statistical study using TRACE, Hinode, and MDI (Michelson Doppler Imager) magnetograms from a twelve year period spanning from 1996 to 2007. By separately analysing 2959 active regions, the authors noted 153 active regions contained 182 sunspots that they deemed to exhibit significant rotations. Based on this statistical study, the authors found 5% of active regions contain significantly rotating sunspots. Although this may seem like a small percentage, this corresponds to a considerable number of observations that deserve to be studied and understood. In addition, we believe a sizeable number of rotating sunspots may have been missed for two reasons; sunspots may fragment before the minimum recorded time and rotational velocities smaller than that of differential rotation are excluded from the study.

All studies show the same general trend with rotations of the order of hundreds of degrees over days. However, there is clearly work to be done in accurately resolving and measuring the photospheric velocities produced due to sunspot rotation, as observations appear to give differing results based on the method and rotation threshold employed. For instance, Min and Chae (2009) and Yan et al. (2009) both analysed the same active region, NOAA 10930, and found significantly different results. Both sets of authors found a counter-clockwise rotation but Min and Chae (2009) found a rotation of $540^\circ$ over five days whereas Yan et al. (2009) measured a rotation of $259^\circ$ over four days. In addition, although two separate studies by Brown et al. (2003) and Yan and Qu (2007) found that different regions of sunspots often rotate at different speeds, Brown et al. (2003) measured that the highest rotation was found in the penumbra whereas Yan and Qu (2007) found the highest in the umbra. Yan and Qu (2007) concluded that this variation in rotation rate with radius can create twist that may be injected into the corona triggering flare activity.

Understanding the driver of photospheric velocities can help in our understanding of coronal heating and large-scale energy release events (see De Moortel and Galsgaard, 2006 and references therein). Periodic motions at the photosphere, like the rotations observed, can generate waves and lead to reconnection events that supply magnetic energy to the atmosphere. There are a wealth of numerical simulations exploring both of these mechanisms and the heating in the atmosphere (see the recent review paper De Moortel and Browning, 2015 and references therein). In particular, CMEs and high flaring activity often accompany the rapid rotation of sunspots. The study of sunspot rotation is therefore vital to our understanding of such explosive events on the Sun. For example, Török et al. (2014) considered the initiation of CMEs by sunspot rotation in detail from both an observational and modelling perspective, by considering a simulation with an ambient magnetic field. They concluded that eruptions are caused by the weakening of the magnetic tension of the overlying field by the rotation of sunspots. This demonstrates that the rotation of sunspots
can contribute to eruptive activity by several mechanisms.

The experiment described in Chapter 4 aims to discuss and simulate a possible mechanism for the rotation of sunspots. Although an observable feature, we do not include the rotation of sunspots around each other in a sunspot group in our experiments and instead solely consider the rotation of parts of the sunspot about its centre. Photospheric flows and magnetic flux emergence were put forward as possible mechanisms for the rotation of sunspots by Brown et al. (2003). The authors suggested that photospheric flows are mainly due to the large scale effect of differential rotation and localised motions resulting from magneto-convective dynamics. They also stress that the effects of differential rotation are kept to a minimum in their study as the images were de-rotated prior to measuring the velocities. Instead, we focus on the second mechanism suggested, namely the emergence of magnetic flux. The authors explained that the photospheric footprints of a flux tube are observed to rotate as a twisted flux tube emerges. A case study by Min and Chae (2009) corroborates that flux emergence is a suitable mechanism for this observed rotation. The authors noticed that the rotation speed of the spot increased in close relation to the growth in area of the sunspot of interest. If we assume the growth of the sunspot is caused by the emergence of flux, we can relate the rotation speed of the sunspot to the rate of flux emergence.

Now that we have determined that the emergence of flux may be a viable mechanism for spot rotation, the next logical step is to determine how flux emergence drives sunspot rotation. Min and Chae (2009) proposed this may be explained by two possible mechanisms: an apparent rotation or a real rotation driven by torque. An apparent rotation is a virtual motion caused by the vertical advection of a twisted magnetic flux tube passing through the photospheric boundary. The fieldlines pass through the photospheric boundary at slightly different locations at each point in time and the twisted structure of the fieldlines causes this to appear as a circular rotation at the photosphere. If the vertical velocity moving the field upwards is large enough, this can correspond to a significant rotation. The direction of the rotation is dependent on the sign of the twist in the initial flux tube. If we adopt this justification, there is no real horizontal motion of the plasma; instead this rotation is a virtual effect caused by continued displacement of fieldlines. Alternatively, Min and Chae (2009) proposed that the observed rotation may represent a real horizontal motion of the plasma caused by a net torque. The authors suggested that this torque may originate in the interior and drive the plasma to rotate on the photospheric boundary. Both of these mechanisms are investigated later in Chapter 4 determine the driver of the photospheric velocity in our experiment.

Many authors have investigated this phenomenon in recent years, with a focus on the photospheric velocities produced as the flux tube emerges. Longcope and Welsch (2000) presented an idealised analytic model that connects a twisted sub-photospheric flux tube to a force-free coronal field. They then separated the photospheric footprints, connected by the coronal field, to imitate flux emergence and found the sunspots to rotate. They demonstrate that a torsional Alfvén wave propagates along the tube from the twisted interior to the stretched coronal field at the instance of emergence. Furthermore, Longcope and Welsch (2000) predict that this rotation will continue until the interior twist per unit length matches that of the corona. Later, Gibson et al. (2004) conducted a numerical MHD experiment and explained the rotation as an observational consequence of the emergence of flux through the photosphere.

There have, in fact, been many numerical experiments performed in recent years to study this phenomena. An interesting investigation was conducted by Magara and Longcope (2003) in which they focused
on the movement of magnetic energy and helicity through the domain in their emerging flux model. The injection of energy and helicity in the atmosphere is contributed to by both horizontal and vertical flows created through flux emergence. Although the vertical flows are the dominant contributor initially, the horizontal flows become the primary source of helicity and energy transport later. To examine this in greater detail, Magara (2006) conducted a study of horizontal velocities at the photosphere, and found distinct rotational flows to develop within the flux concentrations after the intersecting magnetic field aligns vertically. Furthermore, Fan (2009) also found significant rotational motions developed within their model and attributed the flows to the propagation of torsional Alfvén waves along the tube, transporting twist from the highly twisted interior to the stretched coronal field. In this simulation, a cylindrical flux tube is inserted into the solar interior as described in Section 3.3.1.

1.7 Outline

The aim of this thesis is to study the emergence of twisted magnetic flux tubes through the use of 3D numerical simulations, with a particular emphasis on the rotational motions that develop within the photospheric footpoints and the distribution of twist, helicity, and energy across the system. We extend this idea by independently altering the magnetic field strength and twist of the initial flux tube and investigate the impact of each of these modifications.

Before we go on to discuss the numerical experiments performed in this thesis, we begin in Chapter 2 by introducing the numerical code used to solve the MHD equations. This code has been used in a number of previous studies including flux emergence experiments. In Chapter 3, we present the initial numerical set-up used in the simulations in this thesis. We derive the basis for this background stratification and show how it relates to the realistic solar profile, clearly highlighting the simplifications required to model this numerically. In addition, we define the twisted magnetic flux tube used to initialise these experiments.

In Chapter 4, we present a general 3D flux emergence experiment with the initial set-up outlined in the previous chapter. Using this experiment, we introduce the general features of sunspot rotation by calculating the rotation angle, magnetic helicity, and energy amongst various other quantities.

We proceed to examine how the magnetic field strength and twist of the emerging flux tube affects the timing and magnitude of sunspot rotation, as well as the energetics of the system. Specifically, in Chapter 5, we split the experiments into two groups, one where we vary the magnetic field strength and one where we vary the twist of the magnetic flux tube. Later, in Chapter 5, we compare our results to observations by extrapolating the small ephemeral region we model to a larger solar active region, and discuss how this impacts the amount of rotation. In addition, we consider the effect of increasing the depth of the flux tube on the amount of rotation at the photosphere.

Given the complex evolution of the three-dimensional magnetic field as well as the computational expense of performing the emergence experiments in Chapters 4 and 5, some of the simple underlying ideas are obscured. We address this in Chapter 6 by presenting a simplified set-up consisting of a straight cylindrical flux tube with a twisted interior region and straight coronal section, with a simplified density strati-
fication. The insights gained from this experiment can be applied to help in our understanding of the more complex processes influencing the emergence experiments.

In Chapter 7, we adopt a slightly different approach by modelling the emergence of an untwisted magnetic field that is subsequently twisted by spinning vortical motions lower down in the convection zone. This helps us to understand how twisted magnetic structures may be formed and emphasises the impact of twist on the coherence and roundness of sunspots at the photosphere. We also use these results, together with modelling attempts described in Appendix B, to try and model an asymmetry often found in observations of sunspots.

Finally, in Chapter 8, we summarise the findings of Chapters 4, 5, 6 and 7 and present possible directions for future work.
Chapter 2

Numerical code

The main aspect of this thesis is to investigate the rotational velocities produced at the photosphere during numerical experiments of flux emergence. Therefore, we are required to use numerical techniques to determine the MHD behaviour of our emergence model. To perform our numerical experiments, we use the 3D MHD code, *Lare3d* (Arber et al., 2001). This chapter introduces and summarises the main steps included in this 3D Lagrangian-eulerian REmap scheme used to simulate the emergence process. *Lare3d* solves the MHD equations using finite differencing methods and allows the user to control the initial set-up as well as various global parameters, such as resistivity and viscosity. The code uses a staggered grid and is second order accurate in space and time. The *Lare3d* code can be divided into two main steps: the Lagrangian step and the remap step. In the Lagrangian step the equations are solved in a frame that moves with the fluid. This causes the grid to be distorted, so in order to put the variables back on to the original grid, a remap step is used. At the remap step, gradient limiters can be used to preserve monotonicity and shock viscosity can also be implemented to help resolve shocks. The remainder of this chapter is laid out as follows. First, we introduce the basics of the Lare code in a one dimensional problem to demonstrate the main features then we apply the 1D scheme to the Riemann problem in order to test its effectiveness in dealing with interesting features such as shocks. Finally, a summary is given of the three-dimensional scheme used in our experiments.

2.1 Lare1d

First we apply the basic 1D scheme to solve the Euler equations; highlighting the key features of this scheme. Consider the 1D Euler equations in Lagrangian form, with spatial coordinate $x$ and temporal coordinate $t$,

$$\frac{D\rho}{Dt} = -\rho \frac{\partial v}{\partial x},$$  \hspace{1cm} \text{(continuity equation)}  \hspace{1cm} (2.1)

$$\frac{Dv}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x},$$  \hspace{1cm} \text{(equation of motion)}  \hspace{1cm} (2.2)
\[
\frac{D\epsilon}{Dt} = -\frac{p}{\rho} \frac{\partial v}{\partial x},
\]
(energy equation) (2.3)

where \(\rho\) denotes density, \(p\) pressure, \(v\) velocity, \(\epsilon = p/(\rho(\gamma - 1))\) specific energy density and \(\gamma\) ratio of specific heats. The material derivative (or Lagrangian derivative) is given by \(D/Dt = \partial/\partial t + v\partial/\partial x\). The Lagrangian specification of a flow field is a way of examining fluid motion where the observer tracks an individual fluid particle as it moves, whereas the Eulerian specification of a flow field is a way of examining fluid motion that focuses on specific locations in the space through which the fluid flows as time passes. This is perhaps best illustrated by an example. The Eulerian specification of a flow can be thought of as sitting on the river bank and watching the water pass whereas the Lagrangian specification can be thought of as sitting in a boat and drifting down the river. We now consider how these two types of flows can affect and distort a numerical grid.

2.1.1 Grid

Computational grids can generally be divided into two main categories; Eulerian grids and Lagrangian grids. The types of grid are related to the specification of the flow field, as discussed above.

Eulerian and Lagrangian grids

When using an Eulerian grid (i.e. \(\partial/\partial t\)), variables are updated and placed at fixed locations. For example, in Fig. 2.1a, the updated density at time \(n + 1\) \((\rho_{i}^{n+1})\) is located at the same point as the density at time \(n\) \((\rho_{i}^{n})\). The grid is therefore uniform and does not change in time. However, in the case of a Lagrangian grid \((D/Dt)\), the updated variables move with the fluid and distort the grid (Fig. 2.1b). The updated variables are now placed at a different location on the grid. In the Lare1d code, we utilise a Lagrangian grid and hence need to consider how a Lagrangian flow can move and distort the grid.

Cell centres and boundaries

Rather than using a typical grid comprised of a select number of grid points, we build the grid using cells. The grid points now become cell centres, denoted by \(x_{c}\), and the divisions between cells are defined as cell boundaries, denoted by \(x_{b}\). Note that if we have a grid made up of \(N\) cells, there will be \(N\) cell centres and \(N + 1\) cell boundaries. Ghost cells are also utilised when performing calculations close to the outer boundaries.

Staggered grid

We should also stress that the grid used in this numerical scheme is staggered. In other words, the variables are not necessarily defined at the same positions on the grid. Vector quantities are defined at one particular location and scalar quantities are defined at another. In this case, we define \(v\) as the velocity of cell bound-
Figure 2.1: Schematics of (a) Eulerian and (b) Lagrangian grids detailing the positioning of density at time \( t = n \) and updated density at time \( t = n + 1 \) (based on Arber, 2007).

aries and \( \rho \), \( p \), and \( \epsilon \) as the cell averaged density, pressure, and energy density defined at cell centres. The layout of the grid is displayed in Fig. 2.2. The dashes denote the cell boundaries and the dots represent the cell centres.

Figure 2.2: The staggered grid with \( x_{ci} \) denoting the cell centre labelled \( i \) and \( x_{bi} \) denoting the cell boundary (based on Arber, 2007).

As shown in Fig. 2.2, \( dx_{bi} \) is the distance between the boundaries of cell \( i \) and \( dx_{ci} \) is the distance between the centre of cells \( i \) and \( i + 1 \). The scheme makes use of mass conservation so we note the mass in cell \( i \) is given by \( \rho_i dx_{bi} \). Conservation of mass is key in determining the updated density so we must calculate the change in the volume of a cell in one time step. As we are currently working in one spatial dimension, this is simply the width of a cell. After one time step, \( dt \), the width of cell \( i \) becomes

\[
dx_{bi}^{n+1} = dx_{bi}^n + \left(v_{i}^{n+1/2} - v_{i-1}^{n+1/2}\right) dt,
\]

where velocity \( v^{n+1/2} \) is the temporally centred velocity given by \( \frac{v^n + v^{n+1}}{2} \). This form of velocity has
been used to make the width of the cell second order accurate. This quantity can now be used to calculate the fractional change in the cell’s width, denoted by $\Delta$.

$$\Delta = \frac{dx b_i^{n+1}}{dx b_i^n} = \frac{dx b_i^n + (v_i^{n+1/2} - v_i^{n+1/2}) dt}{dx b_i^n}.$$  

This in turn can be used to determine the density at the next time step

$$\rho_i^{n+1} = \rho_i^n \frac{dx b_i^n}{dx b_i^{n+1}} = \rho_i^n \left( \frac{dx b_i^n}{dx b_i^n + (v_i^{n+1/2} - v_i^{n+1/2}) dt} \right),$$

as $\rho_i dx b_i$ remains constant through time by use of mass conservation. Thus, we do not in fact need to solve the density differential equation directly. Instead, the use of mass conservation and careful treatment of the grid is enough to find the updated density.

### 2.1.2 Lagrangian step

Derivatives in equations can be approximated by finite differences on a numerical grid. Using this idea, we express the Euler equations as a set of finite difference equations (FDEs). Second order spatial differencing is simple when using a staggered grid, as all spatial derivatives are centred due to the use of cell centres and boundaries as discussed in Section 2.1.1. For example, if we consider the uniform staggered grid in Fig. 2.2, spatial derivatives of pressure and velocity are given by

$$\left( \frac{\partial p}{\partial x} \right)_{xb_i} = \frac{p_{i+1} - p_i}{dx c_i},$$

$$\left( \frac{\partial v}{\partial x} \right)_{xc_i} = \frac{v_i - v_{i-1}}{dx b_i},$$

respectively. We note that derivatives of cell centred quantities are positioned on cell boundaries, and equivalently derivatives of quantities located on cell boundaries are placed on cell centres.

The Lagrangian step is a straightforward predictor-corrector scheme, where predicted values are calculated from an Eulerian step with time step $dt/2$ and then corrected at the full time step $dt$.

#### Predictor step

In order for the scheme to be second order accurate in time as well as space, we need to define variables at the half time step $n + 1/2$. For example, in order for the velocity differential equation (Eq. 2.2) to be second order accurate in time, we require density and pressure at the half time step:

$$\frac{v_i^{n+1} - v_i^n}{dt} = -\frac{1}{\rho_i^{n+1/2}} \frac{p_i^{n+1/2} - p_i^{n+1/2}}{dx c_i^{n+1/2}}.$$
A half time step is used in the predictor step, in order to calculate $p_i^{n+1/2}$. To find $v_i^{n+1}$, $\epsilon_i^{n+1}$ and $\rho_i^{n+1}$ to second order, we only need $p_i^{n+1/2}$ to be first order accurate. To do this we must first find expressions for $\rho_i^{n+1/2}$ and $\epsilon_i^{n+1/2}$. The discretisation of the energy differential equation (Eq. 2.3) at the half time step is given by

$$
\frac{\epsilon_i^{n+1/2} - \epsilon_i^n}{dt/2} = \frac{p_i^n v_i^n - v_i^{n-1}}{\rho_i^n} \frac{dxb_i^n}{dx}.
$$

Note that $p$ and $\rho$ are defined at the cell centre as this is where the velocity derivative is centred. In order to find $\rho_i^{n+1/2}$, we must determine the grid’s width at the half time step, $n + 1/2$,

$$
dxb_i^{n+1/2} = dxb_i^n + dt/2 (v_i^n - v_i^{n-1}).
$$

This can, in turn, be used to calculate the density at the half time step

$$
\rho_i^{n+1/2} = \frac{\rho_i^n dxb_i^n}{dxb_i^{n+1/2}},
$$

using mass conservation. Hence, the pressure at the half time step is given by

$$
p_i^{n+1/2} = (\gamma - 1) \epsilon_i^{n+1/2} \rho_i^n \frac{dxb_i^n}{dxb_i^{n+1/2}}.
$$

We now have the specific energy density, pressure, density, and grid spacing at the half time step concluding our discussion of the predictor step.

**Corrector step**

The velocity equation (Eq. 2.2) can then be discretised to find the update for velocity at time $n + 1$

$$
\frac{v_i^{n+1} - v_i^n}{dt} = -\frac{1}{\rho_i^{n+1/2}} \frac{p_i^{n+1/2} - p_i^{n+1}}{dxe_i^{n+1/2}}.
$$

Note, we must use density averaged at boundary $i$ as other terms in this equation are located on this boundary. The boundary-averaged density can be expressed by weighting density by the cell width

$$
\rho_{i+1/2} = \frac{dxb_i \rho_i + dxb_{i+1} \rho_{i+1}}{dxb_i + dxb_{i+1}}.
$$

Since the mass inside any cell is constant throughout time, we can rewrite $\rho_{i+1/2} dxe_i^{n+1/2} = \rho_i^n dxe_i^n$. This enables all derivatives to be performed on the original Eulerian grid at time $n$. In order to find energy at the $n + 1$th time step, we require velocity at the half time step

$$
v_i^{n+1/2} = \frac{v_i^n + v_i^{n+1}}{2}.
$$
2.1 Lare1d

The update for energy using Eq. 2.3 can now be found from the FDE

$$\frac{\epsilon_i^{n+1} - \epsilon_i^n}{dt} = -\frac{p_i^n}{\rho_i^n} \left( \frac{v_i^{n+1/2} - v_{i-1}^{n+1/2}}{dx_i^n} \right).$$

The spatial derivatives are, once again, performed on the original Eulerian grid. The final quantities left to calculate in the corrector step are the updated grid and density. Explicitly,

$$dx_i^{n+1} = dx_i^n + dt \left( v_i^{n+1/2} - v_{i-1}^{n+1/2} \right),$$

and

$$\rho_i^{n+1} = \rho_i^n \frac{dx_i^n}{dx_i^{n+1}}.$$

This completes the corrector step of the Lagrangian step of the scheme. From these variables, the distance between cell centres, $dx_c^n$, can be easily calculated as

$$dx_c^n = \frac{dx_i^n + dx_i^{n+1}}{2},$$

and the grid points $x_b$ (cell boundaries) and $x_c$ (cell centres) can be updated.

2.1.3 Remap step

At the end of the Lagrangian step described above, all updated variables are located on a grid that has moved with the fluid. In order to place updated variables back on the original Eulerian grid, a remap step is required. This step is solely geometrical and is required as the grid can become highly distorted throughout the process and grid points may gather in particular locations while missing other locations (Arber, 2007). We briefly outline the density remap in order to illustrate the remap process.

Density remap

The law of mass conservation is utilised throughout the scheme in order to update density. Explicitly, we calculate mass that has moved into or out of cell $i$. That is, mass that has left cell $i$ and moved into cell $i + 1$ and mass that has moved from cell $i - 1$ to cell $i$. For practical reasons the size of the time step is restricted such that mass cannot cross more than one boundary during one time step. This is ensured by the Courant Friedrichs Lewy (CFL) condition. This states a requirement for the system’s numerical stability is that $dt \leq \min(dx) / \max(\sqrt{v^2 + c_s^2})$, where $c_s = \sqrt{\gamma p/\rho}$ is the sound speed. In other words, mass cannot cross more than one cell during a time step $dt$. Note, we have added $c_s$ to the plasma velocity as the combined speed is the speed at which information travels by waves. We consider density at time $n$ and $n + 1$ in order to calculate the change in mass of cell $i$ over one time step. This is illustrated in Fig. 2.3, where we have assumed $v_i^{n+1/2} > 0$. 

In Fig. 2.3, \( M_A \) represents mass that has moved from cell \( i-1 \) to cell \( i \) and \( M_B \) represents mass that has moved from cell \( i \) to cell \( i+1 \). In order to calculate \( M_A \) and \( M_B \) effectively, we assume \( \rho \) is a linear function of position within each cell. This assumption is often used in subgrid models (Dullemond, 2008). Another possibility would be to assume that density is constant within each cell, known as a donar-cell scheme. The assumption of approximating quantities by piecewise linear functions is revisited in a later section on gradient limiters. If \( \rho^n_i \Delta x_{hi} \) is the mass in Eulerian cell \( i \) at time \( n \), the mass in the Lagrangian cell at time \( n+1 \) is

\[
\rho^{n+1}_i = \rho^n_i - (M_A - M_B).
\]

To perform the remap we need to make use of mass conservation, i.e. that mass of the Eulerian cell at time \( n \) is equal to the mass of the Lagrangian cell at time \( n+1 \). Note that \( M_A \) is equivalent to \( M_B \) with the spatial index \( i \) replaced by \( i-1 \), so we can label the masses \( dM_{i-1} \) and \( dM_i \) respectively. As mass can neither be created nor lost, the remapped \( \rho^{n+1}_E \) on the original (Eulerian) grid follows the relation

\[
\rho^{n+1}_E \Delta x_i = \rho^{n+1}_i \Delta x_{hi} - dM_i + dM_{i-1},
\]

where \( \rho^{n+1}_E \) denotes density of cell \( i \) at time \( n+1 \) on the Eulerian grid and \( \rho^{n+1}_i \) denotes density on the Lagrangian grid. Using mass conservation, \( \rho^n_i \Delta x_{hi} = \rho^{n+1}_i \Delta x_{hi} + dM_i \), allows us to rewrite the remapped density as

\[
\rho^{n+1}_E = \rho^n_i + \frac{1}{\Delta x_i} (dM_{i-1} - dM_i).
\]

We can now adopt the original notation \( \rho^{n+1}_i \) to denote density at time \( n+1 \) remapped on to the original grid.

This is just a very brief overview of the density remap. Of course, the masses \( dM_{i-1} \) and \( dM_i \) need to be calculated by using gradients in density as discussed in the next section. The reader is directed to Arber (2007) and Mactaggart (2010) for further details of the specifics of the remap process.
2.1.4 Gradient limiters

The quantity $dM_i$ is left undefined in the previous section. However, to calculate $dM_i$ we need to find gradients in density. Gradients in energy and velocity also arise when remapping these quantities so we must have an effective way of dealing with gradients. As introduced earlier, we are using a piecewise linear approximation of quantities in cells. A drawback of this assumption is that piecewise linear elements can have overshoots, as seen in Fig. 2.4. An overshoot is found when the value of a function at the end of a piecewise linear element $i$ is greater than the value of the function at the beginning of cell $i + 1$. A successful method for dealing with these overshoots is to make use of gradient limiters (also known as slope limiters or flux limiters). These numerical tools are required to maintain monotonicity in the scheme and help resolve sharp gradients. They can also prevent artificial oscillations in the code. An example of the use of a gradient limiter is shown by the red line in Fig. 2.4. It is clear that in this case, the overshoot (the middle black line) has been prevented by use of the gradient limiter.

Figure 2.4: Schematic of example of gradient limiter, shown in red.

There are several different types of slope limiters. The main choices for a piecewise linear function $\frac{df_i}{dx}$ are displayed in Table 2.1, assuming that $v_i^{n+1/2} > 0$. The upwind and downwind slopes are simply interchanged for $v_i^{n+1/2} < 0$. Notice, in Table 2.1, we have assumed $f$ is defined on a cell centre. If we wanted to find the gradient of a variable defined on a boundary, i.e. the velocity, the spacing $dxc$ would be replaced by $dxb$. Also, the superscript $n$ has been removed here as it is important to note that these gradients are applicable at any time step.

Table 2.1: Choices of gradients.

<table>
<thead>
<tr>
<th>Gradient</th>
<th>$m_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upwind slope</td>
<td>$m_i = \frac{f_i - f_{i-1}}{dxc_{i-1}}$</td>
</tr>
<tr>
<td>Downwind slope</td>
<td>$m_i = \frac{f_{i+1} - f_i}{dxc_i}$</td>
</tr>
<tr>
<td>Centred slope</td>
<td>$m_i = \frac{f_{i+1} - f_{i-1}}{dxc_{i-1} + dxc_i}$</td>
</tr>
</tbody>
</table>

Next, we discuss possible strategies to limit the value of these gradients in order to deal with overshoots.
One of the simplest choices is to consider both upwind and downwind gradients and choose the gradient with the smallest absolute value if they are of the same sign. If they are of different signs, this suggests there is either a maximum or minimum and so we set the gradient equal to zero. This can be expressed mathematically as

\[ m_i = \minmod \left( \frac{f_i - f_{i-1}}{dx_{ci-1}}, \frac{f_{i+1} - f_i}{dx_{ci}} \right), \]

where the \( \minmod \) function is defined as

\[
\minmod(a, b) = \begin{cases} 
  a, & |a| < |b| \text{ and } ab > 0, \\
  b, & |b| < |a| \text{ and } ab > 0, \\
  0, & ab < 0.
\end{cases}
\]

This limiter is very proficient in dealing with shocks; shocks need to be kept tight and not smeared over too many cells.

In \textit{Lare1d}, a van Leer limiter is used (Arber et al., 2001). This type of limiter also preserves the monotonicity of a function and helps to resolve shocks. In the application of this limiter, a higher order gradient is required. We again use a piecewise linear variable \( f \) in order to represent a general gradient we are interested in determining, i.e., \( D_i = df/dx \). In order to use a van Leer limiter, we need to introduce a new third-order upwind gradient, following the work of Arber et al. (2001),

\[
|D_i| = \frac{(2 - \phi_i)}{3} \left| \frac{f_{i+1} - f_i}{dx_{ci}} \right| + \frac{(1 + \phi_i)}{3} \left| \frac{f_i - f_{i-1}}{dx_{ci-1}} \right|, \quad v_i^{n+1/2} > 0,
\]

where \( \phi_i = |v_i^{n+1/2}| dt / dx_{bi+1} \). For the velocity gradient, \( dx_{ci} \) is replaced by \( dx_{bi} \) as the control volume. This gradient can then be limited, if necessary, using a condition described by van Leer (1977). In order for the gradient to be monotonic, the value of \( f_i \) on the boundaries of cell \( i \) must be less than \( f_{i+1} \) (value in the next cell) and greater than \( f_{i-1} \) (value in the previous cell). This can be expressed mathematically as

\[
\begin{align*}
& f_i + \frac{dx_{bi}}{2} D_i \quad < \quad f_{i+1}, \\
& f_i + \frac{dx_{bi}}{2} D_i \quad > \quad f_{i-1}.
\end{align*}
\]

This can be accomplished by using the van Leer limiter

\[
D_i = s\text{MIN} \left( |D_i|, 2 \left| \frac{f_{i+1} - f_i}{dx_{ci}} \right|, 2 \left| \frac{f_i - f_{i-1}}{dx_{ci-1}} \right| \right),
\]

where \( s \) is given by

\[
s = \begin{cases} 
  \text{sign}(f_{i+1} - f_i), & \text{if sign}(f_{i+1} - f_i) = \text{sign}(f_i - f_{i-1}), \\
  0, & \text{otherwise}.
\end{cases}
\]

The intention of the variable \( s \) is to assign a sign to the gradient, i.e. a positive sign if \( df/dx > 0 \) and a negative sign if \( df/dx < 0 \). If the upwind and downwind gradients are of opposite signs, the gradient is set...
2.2 Test cases

2.2.1 1D Euler experiment

In a first attempt to test the accuracy of Lare1d, we solve the 1D Euler equations presented in Eq. 2.1 - Eq. 2.3 by two methods. We consider a small amplitude wave solution and a numerical solution obtained through the implementation of Lare1d.

Consider the following problem with initial conditions:

\[ v(x_0, 0) = 0, \]

This limiter can be easily applied to gradients in density, energy density, and velocity needed for the remaps. The velocity limiter is slightly different in that \( dx_b \) is the control volume in replacement of \( dx_c \). We direct the reader to van Leer (1997) for further details of this limiter.

2.1.5 Shock viscosity

Shocks are discontinuous solutions to equations so we need to be able to deal with them effectively. Shocks are jump discontinuities that cause an irreversible transition between supersonic and subsonic flow. Mass, energy, and momentum are conserved across shocks. At shocks, gradients become singular so can no longer be represented by finite difference equations. This can cause artificial post-shock oscillations to form and ultimately lead to the breakdown of the numerical scheme. As an alternative, we can use the integrated form of the equations to get jump conditions across shocks. These are known as the Rankine-Hugoniot relations in 1D and describe the relationship between the states on the two sides of a shock wave. Shock viscosity can be implemented using these jump conditions, which in turn will introduce some diffusion where required. Note, we only require the effects of shock viscosity when the fluid is compressed, i.e. at shocks. This indicates that this viscosity should only be imposed provided that \( \nabla \cdot v < 0 \). The addition of artificial viscosity makes the shock broader, so that it can be resolved across several grid cells. To introduce some dissipation, only at steep gradients, we add a scalar to the pressure:

\[
p_{\text{shock}} = \nu_1 \rho c_s \Delta x |\nabla \cdot v| + \nu_2 \rho \Delta x^2 (\nabla \cdot v)^2,
\]

provided that \( \nabla \cdot v < 0 \), i.e. the cell is being compressed. As we are adding this to the pressure term, we must also add an associated heating term \(-p_{\text{shock}} \nabla \cdot v / \rho\) to the energy equation. The coefficients \( \nu_1 \) and \( \nu_2 \) are fixed through experimentation, although \( \nu_1 = 0.1 \) and \( \nu_2 = 0.5 \) are usually acceptable (Arber, 2014). These are the values used throughout this thesis.

2.2 Test cases

2.2.1 1D Euler experiment

In a first attempt to test the accuracy of Lare1d, we solve the 1D Euler equations presented in Eq. 2.1 - Eq. 2.3 by two methods. We consider a small amplitude wave solution and a numerical solution obtained through the implementation of Lare1d.

Consider the following problem with initial conditions:

\[ v(x_0, 0) = 0, \]
2.2 Test cases

\[ p(x_c, 0) = \begin{cases} 
 1 + a + a \cos(2\pi(x_c - 2.5)), & 2 < |x_c| < 3, \\
 1, & \text{elsewhere},
\end{cases} \]

and

\[ \rho(x_c, 0) = (p(x_c, 0) - 1)/\gamma + 1. \]

We have therefore assumed there is no initial flow and pressure and density are set up as shown in Fig. 2.5 where we have chosen \( a = 0.01 \) and \( \gamma = 5/3 \).

![Figure 2.5: Initial set-up with distribution of pressure (blue solid line) and density (red asterisks) against \( x \).](image)

Small amplitude wave solution

Before we look at solving this problem numerically, we look at the linear solution found by considering small amplitude waves.

**Equilibrium**

We choose the equilibrium to be \( v_0 = 0, \rho_0 = 1, \) and \( p_0 = 1 \).

**Perturbations**

Consider small disturbances from the equilibrium

\[ p = p_0 + p_1; \rho = \rho_0 + \rho_1; v = v_1, \]

where all perturbations are small, i.e. \( p_1 \ll 1, \rho_1 \ll 1, \) and \( v_1 \ll 1 \). We can then linearise the Euler equations by noting that equilibrium values are constant and neglecting products of perturbation terms. The linearised set of equations take the form

\[ \frac{\partial p_1}{\partial t} = -\rho_0 \frac{\partial v_1}{\partial x}, \]

\[ \frac{\partial \rho_1}{\partial t} = \frac{\partial}{\partial x} \left( \rho_0 \frac{\partial v_1}{\partial x} \right), \]

\[ \frac{\partial v_1}{\partial t} = \frac{1}{\gamma - 1} \frac{\partial}{\partial x} \left( \rho_0 \frac{\partial p_1}{\partial x} \right). \]
2.2 Test cases

\[
\frac{\partial v_1}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p_1}{\partial x},
\]
\[
\frac{\partial \epsilon_1}{\partial t} = -\frac{p_0}{\rho_0} \frac{\partial v_1}{\partial x}.
\]

Through manipulation of this set of linear equations, we find that the perturbed pressure solves the wave equation:

\[
\frac{\partial^2 p_1}{\partial t^2} = c_s^2 \frac{\partial^2 p_1}{\partial x^2},
\]

where \( c_s = \sqrt{\gamma p_0 / \rho_0} \) is the sound speed. This wave equation can be solved simply given the initial condition presented at the beginning of the problem to give the solution

\[
p_1 = \frac{F(x_c - c_s t)}{2} + \frac{F(x_c + c_s t)}{2},
\]

where

\[
F(x_c) = \begin{cases} 
  a + a \cos(2\pi(x_c - 2.5)), & 2 < |x_c| < 3, \\
  0, & \text{elsewhere}.
\end{cases}
\]

Using this solution, similar expressions can be found for the perturbed density and velocity:

\[
v_1 = \frac{1}{2c_s} \left( F(x_c - c_s t) - F(x_c + c_s t) \right),
\]
\[
\rho_1 = \frac{1}{2c_s^2} \left( F(x_c - c_s t) + F(x_c + c_s t) \right).
\]

This suggests that the wave splits with one wave propagating to the left and another to the right with speed \( c_s \).

**Numerical solution**

This same problem can be solved numerically using the *Lare1d* code by entering the initial conditions described at the beginning of the section. With both the numerical and analytic solutions, we compare pressure, density, and velocity profiles as shown in Fig. 2.6 with the analytic solution depicted in solid blue and the numerical solution shown by red asterisks. The code has clearly accurately described the solution as the two solutions agree very well. In both cases, the initial disturbance splits and move outwards.

2.2.2 Riemann problem

In order to test the effectiveness of *Lare1d* in dealing with shocks, we attempt to solve the Riemann problem. This is a very useful problem to consider as it gives an exact solution to the Euler equations and interesting properties such as shocks and contact discontinuities appear as characteristics in the solution. To do this we consider the domain \([0, 1]\) and set \( \gamma = 2 \) by considering a simple initial state that consists of two uniform
Figure 2.6: Pressure, density, and velocity profiles solved using *Lare1d* (red asterisks) and an analytic solution (solid blue line) for time \( t = 0.8 \).

regions separated by an interface at \( x = 1/2 \),

\[
\begin{align*}
    x < \frac{1}{2} & \quad \left\{ \begin{array}{l}
        p_L = 128, \\
        \rho_L = \frac{256}{49}, \\
        v_L = 0, \\
    \end{array} \right. \\
    x > \frac{1}{2} & \quad \left\{ \begin{array}{l}
        p_R = 1, \\
        \rho_R = \frac{2}{25}, \\
        v_R = 0, \\
    \end{array} \right.
\end{align*}
\]

where subscripts \( L \) and \( R \) denote the left and right hand side of the interface at \( x = 1/2 \) respectively. The density and pressure interfaces are illustrated in Fig. 2.7.

The interface will, in general, split into three waves; a shock wave, a contact discontinuity and an
expansion wave. *Shocks* are jump discontinuities that cause an irreversible transition between supersonic and subsonic flow. As stated before, mass, energy, and momentum are conserved across shocks. The simplest wave to understand is a *contact discontinuity*. Contact discontinuities form when pressure and flow speed are constant across a boundary but there is a jump in density. They are similar to shocks in that they are also governed by the Rankine-Hugoniot relations. However, there are several distinctions between *contact discontinuities* and *shocks*. In order for a discontinuity to be a shock there must be a flow of plasma through the shock surface and there must be some compression across the shock. *Expansion waves* are waves in which velocity increases and pressure and density decrease.

**Analytic solution**

The analytic solution to the Riemann problem, given the initial conditions stated above, is separated into five domains,

\[
\begin{align*}
&x < \frac{1}{2} - 7t \\
&\quad \left\{ \begin{array}{l}
p = 128, \\
\rho = \frac{256}{49}, \\
v = 0,
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
&\frac{1}{2} - 7t < x < \frac{1}{2} + \frac{7}{2}t \\
&\quad \left\{ \begin{array}{l}
p = 128 \left( \frac{1}{4} \left( \frac{(x-1/2)}{t} + 7 \right) - \frac{(x-1/2)}{t} \right) \right), \\
\rho = \frac{256}{49} \left( \frac{1}{4} \left( \frac{(x-1/2)}{t} + 7 \right) - \frac{(x-1/2)}{t} \right) \right)^2, \\
v = \frac{2}{3} \left( \frac{(x-1/2)}{t} + 7 \right),
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
&\frac{1}{2} + \frac{7}{2}t < x < \frac{1}{2} + 7t \\
&\quad \left\{ \begin{array}{l}
p = 8, \\
\rho = \frac{64}{49}, \\
v = 7,
\end{array} \right.
\end{align*}
\]
The four uniform regions represent the flow behind the expansion wave, region between the expansion wave and the contact discontinuity, between the contact discontinuity and the shock and the region ahead of the shock, respectively. Finally, the non-uniform region signifies the expansion wave. The size of these regions vary temporally as governed by time, $t$. 

### Comparison of analytic and numerical solutions

Using 1000 grid points, the analytic and numerical solutions for $v$, $\rho$, and $p$ are plotted for $t = 0.0364$ in Fig. 2.8. This also displays the effect of the addition of shock viscosity and van Leer slope limiters as we show the solution without on the left and the solution with these additions on the right. The analytic and numerical solutions appear to agree very well for the most part in both cases. To examine this more accurately, we have zoomed in on the shock in the velocity profile in Fig. 2.9 to look at the effect of shock viscosity.

In the scheme without shock viscosity, the numerical solution inaccurately determines the position of the shock and there is a substantial overshoot at the shock. In the scheme with shock viscosity, on the other hand, the scheme accurately locates the positions and propagation speeds of the shock, contact discontinuity, and expansion fan. The expansion fan is represented well in all three plots. In the velocity profile, the shock is captured well with the implementation of shock viscosity and van Leer limiters. The shock is also captured well in the density profile, although there is a slight smearing across the contact discontinuity. The accuracy of the numerical solution could be improved by increasing the number of grid points.

In summary, we show that through the use of shock viscosity and van Leer limits, the Lare1d code successfully resolves shocks and is an effective scheme to use to solve the MHD equations when scaled up to three dimensions.

### 2.3 Lare3d

Now that the basic steps of the Lare1d scheme have been discussed, we look at how these steps are implemented in the full three-dimensional Lare3d code when we apply it to the MHD equations. Although the details are far more complex when we add two extra spatial dimensions, the basic elements are essentially the same. We direct the reader to Arber et al. (2001) and the LareXd manual (Arber, 2007) for further
Figure 2.8: The profiles for $v$, $\rho$, and $p$ at $t = 0.0368$ where the blue line represents the exact analytic solution and the red asterisks denote the numerical solution. The left column shows the comparison with the numerical solution without shock viscosity or gradient limiters and the right with both included.

details.

2.3.1 Equations and normalisation

The standard resistive MHD equations, which we wish to solve, are described in Section 1.2 in Eq. 1.9 – Eq. 1.15. Before solving any set of equations numerically it is very important to first non-dimensionalise
Figure 2.9: Zoomed in profiles for $v$ at $t = 0.0368$ where the blue line represents the exact analytic solution and the red asterisks denote the numerical solution. The left column shows the solution without shock viscosity and the right column with shock viscosity.

the equations to remove any physical units. This ensures that all the computed quantities are of relatively similar magnitudes and makes the computation much more manageable. The normalisation is performed through the choice of three characteristic values: a typical magnetic field $B_0$; a typical density $\rho_0$; and a typical length scale $L_0$ given by the pressure scale height. By substituting these values into the equations above, we find the following relations:

$$
\begin{align*}
  v_0 &= \frac{B_0}{\sqrt{\mu_0 \rho_0}}, \\
  j_0 &= \frac{B_0}{\rho_0 L_0}, \\
  p_0 &= \frac{B_0^2}{\mu_0}, \\
  E_0 &= v_0 B_0, \\
  \epsilon_0 &= v_0^2, \\
  t_0 &= \frac{L_0}{v_0}, \\
  \bar{m} &= \frac{\epsilon_0 \bar{m}}{k_B}, \\
  \eta_0 &= \rho_0 L_0 v_0, \\
  g_0 &= \frac{v_0^2}{L_0}, \\
\end{align*}
$$

(2.4)

where $\bar{m}$ is the average mass of ions in the plasma. In the corona, $\bar{m} = 1.2m_p$ is often quoted as a typical value as this includes both hydrogen and helium in the plasma composition. This parameter can be varied in the Lare3d code and as such we choose to ignore helium and set $\bar{m} = m_p$ to represent pure hydrogen. The quantities are redefined in such a way that

$$
\begin{align*}
  B &= \hat{B}_0, \\
  v &= \hat{v}_0, \\
  j &= \hat{j}_0, \\
  p &= \hat{p}_0, \\
\end{align*}
$$

where the subscript 0 describes the dimensional conversion quantity and variables with a hat refer to the dimensionless quantities. In Lare3d the variable $\eta$ refers to the resistivity given by $1/\sigma$ and not the magnetic diffusivity ($\eta' = 1/(\sigma \mu)$) as often quoted in the literature. This notation is adopted throughout this thesis.
From the equations above, the plasma beta, $\beta$, can be written as
\[ \beta = \frac{2\hat{p}}{B^2}. \]

To demonstrate the process of the non-dimensionalisation of equations, we use the ideal gas law as an example, as presented below:
\[
\begin{align*}
p &= \frac{k_B}{\mu_m} \rho T, \\
\frac{B_0^2}{\mu_0} \hat{\rho} &= \frac{k_B}{\mu_m} \rho_0 \hat{\rho}_0 T_0, \\
\frac{B_0^2}{\mu_0} \hat{\rho} &= v_0^2 \frac{\hat{m}}{\mu_m} \hat{\rho} T, \\
\hat{p} &= \frac{\hat{m}}{\mu_m} \hat{\rho} \hat{T}.
\end{align*}
\]

We remind the reader that $\hat{m}$ is the average mass of ions in the plasma, and hence $\hat{m} = m_p$ for pure hydrogen. On the other hand, $\mu_m$ is the average mass of all particles in the plasma, and hence for neutral hydrogen $\mu_m = m_p$ as well. Hence, as we assume neutrality, the ideal gas law reduces to
\[ \hat{p} = \hat{\rho} \hat{T}. \]

A similar non-dimensionalisation is performed for the rest of the equations, though this is left out for brevity. In summary, the final set of normalised resistive MHD equations we solve in Lare3d (dropping the hats on normalised variables) are:
\[
\begin{align*}
\frac{D\rho}{Dt} &= -\rho \nabla \cdot v, \\
\frac{Dv}{Dt} &= \frac{1}{\rho} (\nabla \times B) \times B - \frac{1}{\rho} \nabla p - g \hat{z} + \frac{1}{\rho} F_\nu, \\
\frac{DB}{Dt} &= (B \cdot \nabla)v - B(\nabla \cdot v) - \nabla \times (\eta \nabla \times B), \\
\frac{De}{Dt} &= -\frac{p}{\rho} \nabla \cdot v + \frac{\eta}{\rho} j^2 + \frac{1}{\rho} H_\nu.
\end{align*}
\]

with $p = \rho T$, where we have assumed that gravity is pointing downwards. $\epsilon = p/(\rho(\gamma - 1))$ is the specific energy density and all other variables are as defined in Section 1.2. Here $F_\nu = \nu p(\nabla^2 v + \frac{1}{3} \nabla(\nabla \cdot v))$ is the viscous force, where $\nu$ is the coefficient of kinematic viscosity, and $H_\nu = \nu p(1/2 \epsilon v_{ij} e_{ij} - \frac{2}{3}(\nabla \cdot v)^2)$ is the associated viscous heating term, such that $e_{ij} = \partial v_i/\partial x_j + \partial v_j/\partial x_i$ is the rate of strain tensor.

### 2.3.2 Lagrangian step and remap step

As in the 1D case, the normalised equations are solved in the Lagrangian step using a staggered grid. The staggered grid in three dimensions is shown on Fig. 2.10. As in the 1D Lagrangian remap scheme, different variables are defined at different positions on the grid. Scalars, such as $\rho$ and $\epsilon$, are defined on cell centres,
velocities on cell vertices and magnetic field components on face centres. Gravity is defined on cell vertices in the same position as velocity.

Mirroring the 1D case, variables sometimes need to be defined at locations other than those prescribed in Fig. 2.10. For example, we often make use of the density averaged on the boundary. The type of averaging we use in three dimensions is dependent on where the variable is defined, i.e. if it is a volume average (e.g. \( \rho \)) or a surface average (e.g. \( B_z \)). In order to tackle this averaging problem, we shall define the volume of each cell to be \( cvol_{i,j,k} \) following the notation used in Arber et al. (2001). The use of subscripts to denote cells is important as volumes can change from cell to cell as the grid can become stretched. For a cell corresponding to indices \((i,j,k)\), \( \rho_{i,j,k} \), and \( \epsilon_{i,j,k} \) are the averages over the volume of density and specific energy defined at the cell centre. \( B_{x_{i,j,k}} \) is the \( x \)-component of the magnetic field and is defined on the face centred at \( xc_{i,j,k} + dx_{i,j,k}/2 \) where \( xc_{i,j,k} \) is the \( x \) coordinate of the centre of the cell and \( dx_{i,j,k}/2 \) is the length of the cell in the \( x \) direction. Similarly, \( B_{y_{i,j,k}} \) and \( B_{z_{i,j,k}} \) are defined using \( dy_{i,j,k} \) and \( dz_{i,j,k} \) respectively. All three components of the velocity vector are defined at the same place, on the vertex \((xc_{i,j,k} + dx_{i,j,k}/2, yc_{i,j,k} + dy_{i,j,k}/2, zc_{i,j,k} + dz_{i,j,k}/2)\).

If, for example, we needed to calculate density at a cell vertex, we would use control volume averaging

\[
\rho_{i,j,k}^v = \frac{1}{8 cvol_{i,j,k}^v} \sum_{l=i}^{i+1} \sum_{m=j}^{j+1} \sum_{n=k}^{k+1} \rho_{l,m,n} cvol_{l,m,n}.
\]

where \( cvol_{i,j,k}^v \) is the velocity cell control volume calculated by taking the average of the 8 neighbouring cells to the vertex, i.e.

\[
cvol_{i,j,k}^v = \frac{1}{8} \sum_{l=i}^{i+1} \sum_{m=j}^{j+1} \sum_{n=k}^{k+1} cvol_{l,m,n}.
\]
If we wanted the magnetic field components at the cell centre, we would simply average the values on opposing faces. Similarly, the velocity components on the cell faces, are found by averaging over the four surrounding vertex values. Using the averaging described above, the normalised MHD equations (Eq. 2.5 – Eq. 2.8) can be written as finite difference equations in a similar manner to the 1D Euler example.

The remap step in the LARE scheme is very similar to that of the 1D case. In three dimensions, the remap is performed in 1D sweeps, i.e. $x$, $y$, and $z$ remaps. However, the order of the 1D sweeps is interchanged in order to avoid any bias. Therefore the $x$, $y$ and $z$ remaps will be comparable with the 1D remap described in Section 2.1.3.

2.4 Summary

This chapter has introduced a Lagrangian remap scheme (Lare) for solving the MHD equations. Lare3d is ideally suited to problems in flux emergence and in the solar atmosphere, in general. A brief overview of the main elements of the scheme have been given for both the Euler and MHD equations. As a test of the effectiveness of the scheme, the Riemann problem is solved and compared to an analytical solution. With the inclusion of shock viscosity and gradient limiters, an accurate solution is produced. Without these, however, the scheme fails to resolve shocks accurately.
Chapter 3

Initial set-up

In this chapter, we present the initial state employed in the flux emergence simulations performed in this thesis. The simulation domain is divided into four horizontal regions representing the convection zone, photosphere, transition region, and corona. The analytic atmosphere is introduced and non-dimensionalised in order to be implemented in the numerical code \textit{Lare3d} described in Chapter 2. The analytic environment is set up in hydrostatic equilibrium and therefore remains static. However, when the equilibrium is imposed numerically, the resulting environment is not perfectly static.

The choice of magnetic flux tube used in the solar interior is important when initialising magnetic flux emergence experiments. However, we cannot observe the interior magnetic field, and hence we must choose a simple model for the magnetic structure of the field. As stated previously, we mimic the large-scale structure of sunspots by emerging a single flux tube. When modelling active regions, it is typically assumed that the magnetic field is generated in the tachocline at the base of the convection zone. Many authors have suggested that in order for buoyant magnetic flux tubes to remain coherent during their rise through the convection zone, they must be twisted by a sufficient amount (Moreno-Insertis and Emonet, 1996). Hence, for decades, the typical choice for the initial magnetic flux tube is a uniformly twisted cylindrical flux tube. Note, flux tubes used to initialise emergence experiments are not necessarily force-free (as in the case of the Gold-Hoyle flux tube). For convenience, flux tubes are often placed just below the base of the photosphere. However, in recent years an alternative model for the initial magnetic field was introduced, namely a toroidal flux tube. This is a half-torus shaped twisted flux tube that is anchored on the base of the simulation domain in the solar interior. This model adds a further reality to our simulations as the toroidal tube models an initial magnetic field rooted much deeper in the convection zone.

3.1 Analytic stratification

We use a Cartesian \((x, y, z)\) system in our model, which is valid over small regions on the Sun. The simulation domain is separated into four horizontal layers: the convection zone; photosphere; transition
region; and corona. We define \( z \) to be the height from the solar surface, and \( x \) and \( y \) to be coordinates within planes of constant height. We assume the solar surface is represented by the \( z = 0 \) plane (the base of the photosphere). This defines \( z < 0 \) as the solar interior and \( z \geq 0 \) as the solar atmosphere. We define quantities to be continuous across boundaries at each region, but not necessarily continuously differentiable. In addition, all quantities are assumed to be uniform in the \( x \) and \( y \) directions and functions of \( z \) only.

We prescribe a temperature profile which approximates the actual solar temperature values found by using the average quiet Sun stratification from the solar surface, up through the chromosphere, to the start of the transition region based on the Vernazza, Avrett and Loeser (VAL) model (Vernazza et al., 1981). To model this computationally, a simplified representation is utilised. Using the temperature profile, the corresponding density and pressure profiles can be derived analytically using the hydrostatic pressure balance equation \( (\nabla p = \rho g) \), and ideal gas law, i.e.

\[
\frac{dp}{dz} = -\rho g, \\
p = \frac{k_B}{\mu_m} \rho T,
\]

(3.1)

where \( k_B \) and \( \mu_m \) are constants defined in Section 1.2.2. Note, we have used \( g = -g k \).

### 3.1.1 Solar interior

The only layer of the solar interior we are considering is the solar convection zone (see Section 1.1 for more details). The stability of plasma to convection is determined by the Schwarzschild criterion,

\[
-\frac{dT}{dz} \geq - \left( \frac{dT}{dz} \right)_{ad} = \frac{g \mu_m \gamma - 1}{k_B}.
\]

(3.2)

This states that plasma is unstable to convection if the absolute value of the temperature gradient is greater than its adiabatic value defined on the right hand side of Eq. 3.2. This is also defined in our discussion of the buoyancy instability as discussed in Section 1.4. An adiabatic process is one in which no heat is gained or lost from a system. If the temperature gradient is steeper than the adiabatic value, the plasma is convectively unstable and if the gradient is shallower, the plasma is stable to convection. In the simulations performed in this thesis, the temperature gradient in the convection zone is set equal to its adiabatic value, so the criterion is only just satisfied. The plasma is therefore marginally stable to convection, as used in many flux emergence experiments (Fan, 2001, Manchester et al., 2004 etc.). In practice, the mean stratification is close to adiabatic (Priest, 2014). The effects of convection are an interesting feature to explore but we must first understand the evolution without convection. Hence, we leave the inclusion of convection for further work.

In order to derive the temperature profile for the convection zone (given as the adiabatic temperature gradient in Eq. 3.2), we assume that the entropy of the system is constant or equivalently that the convection zone is adiabatic as discussed above. Note, this assumption, together with the hydrostatic balance equation,
forces the temperature profile to be linear. This can be expressed mathematically as

\[ T = T_0 - mz, \quad \frac{p}{\rho^\gamma} = A, \] (3.3)

where \( T_0 \), \( m \) and \( A \) are constants to be determined. At \( z = 0 \), the temperature is equal to \( T_0 \) which we set as \( T_s \) (the temperature of the solar surface). Using the linear temperature profile specified in Eq. 3.3 and the ideal gas law (Eq. 3.1), an analytic expression for the pressure can be derived as

\[ \frac{dp}{dz} = -\frac{\rho g}{\gamma} = -\xi p = \frac{-\xi p}{(T_s - mz)}, \]

\[ \Rightarrow p = B(T_s - mz)^{\xi/m}, \]

where we have defined \( \xi = \frac{g \mu m}{k_B} \) for convenience and introduced a constant, \( B \). Hence, from Eq. 3.3, the density is given by

\[ \rho = \left( \frac{p}{A} \right)^{1/\gamma} = C(T_s - mz)^{\xi/\gamma m}, \]

where we have introduced a new constant \( C \). Setting \( z = 0 \) in both the pressure and density equations, we can evaluate \( B = \frac{p_s}{T_s^{\xi/m}} \) and \( C = \frac{\rho_s}{T_s^{\xi/m}} \), defining \( p_s \) and \( \rho_s \) as the values of pressure and density at the solar surface, respectively. Substituting the above pressure and density into Eq. 3.1 yields the following expression for temperature,

\[ T = \frac{\mu m}{k_B} \rho = \frac{\mu m}{k_B} \frac{p_s}{\rho_s} \frac{T_s^{\xi/m}(1/\gamma - 1)}{(T_s - mz)^{\xi/m(1-1/\gamma)}}. \]

This can be set equal to the original expression for temperature, \( T \), from Eq. 3.3, which forces the powers of \((T_s - mz)\) to be equal

\[ \frac{\xi}{m} (1 - 1/\gamma) = 1, \]

\[ m = \frac{\xi (\gamma - 1)}{\gamma}. \]

This yields our final expression for the temperature in the convection zone,

\[ T = T_s - \frac{g \mu m (\gamma - 1)}{k_B \gamma} z. \]

Similarly, analytic expressions for pressure and density in the convection zone can be easily derived as

\[ p = \frac{p_s}{T_s^{\xi/(\gamma - 1)}} \left( T_s - \frac{g \mu m (\gamma - 1)}{k_B \gamma} z \right)^{\gamma/(\gamma - 1)}, \]

\[ \rho = \frac{\rho_s}{T_s^{1/(\gamma - 1)}} \left( T_s - \frac{g \mu m (\gamma - 1)}{k_B \gamma} z \right)^{1/(\gamma - 1)}. \]

With the analytic expressions for temperature, pressure and density prescribed in the convection zone, we
3.1 Analytic stratification

move on to the atmosphere.

3.1.2 Atmosphere

Next, we prescribe the temperature stratification in the atmosphere (photosphere, chromosphere, transition region, and out to the lower corona). The basic structure of the temperature profile is based on the VAL model, which displays the average quiet Sun temperature profile from the solar surface, up through the photosphere and chromosphere to the start of the transition region (Vernazza et al., 1981). Fig. 3.1 shows us the temperature decreases from the solar surface until reaching a minimum at the top of photosphere/bottom of chromosphere. The temperature then rises swiftly from the temperature minimum through the low chromosphere, where temperature increase falls off and the temperature is approximately constant. The high chromosphere is characterised by a large jump in temperature from 7000K to 24000K, proceeded by a comparatively small region of approximately constant temperature. Lastly, the transition region is characterised by a sizeable jump in temperature over a very short distance to 1MK at the base of the corona.

Figure 3.1: The average quiet Sun temperature profile with height from the solar surface at $h = 0$ to the start of the transition region is shown in black (taken from Vernazza et al., 1981) with our simplified temperature profile, $T(z)$, overplotted in red.

Simplifications must be made in order to effectively model this temperature profile computationally. Firstly, we choose to ignore the variation in temperature in the photosphere and low chromosphere, and assume a constant temperature profile in these regions. We, henceforth, combine the photosphere, low chromosphere, and middle chromosphere in our analytic model and refer to them collectively as the “pho-
tosphere”. Secondly, we model all variations in temperature in the high chromosphere and transition region as one single rise in temperature with height. We again combine these two regions and refer to them collectively as the “transition region”. We model the lower corona with a uniform temperature profile. This seems to be a valid simplification as the temperature is approximately uniform in the low corona, except at explosive events (Murray, 2007). The simplified temperature profile is overplotted in Fig. 3.1 in red to highlight the approximations made when modelling this region. To summarise, the analytic temperature profile for all regions is given below,

\[
T(z) = \begin{cases} 
  T_{ph} - \frac{g \mu m}{k_B} \frac{(\gamma - 1)}{\gamma} z & z < 0, \\
  T_{ph} & 0 \leq z < z_{tr}, \\
  T_{ph} \left( \frac{T_{cor}}{T_{ph}} \right) \frac{z - z_{tr}}{z_{cor} - z_{tr}} & z_{tr} \leq z < z_{cor}, \\
  T_{cor} & z \geq z_{cor},
\end{cases}
\]

where \( T_{ph} \) and \( T_{cor} \) are the constant photospheric and coronal temperatures, respectively. Here, we use a power law profile to describe the steep rise in temperature of the transition region, where \( z_{tr} \) and \( z_{cor} \) denote the heights of the transition region and corona, respectively.

We now derive analytic expressions for pressure and density in the photosphere, transition region, and corona. Details of these derivations are only summarised due to their similarities with the pressure and density derivation in the convection zone. At the photosphere, the hydrostatic pressure equation becomes

\[
\frac{dp_{ph}}{dz} = -\frac{p_{ph}}{H_{ph}},
\]

where \( H_{ph} = k_B T_{ph}/g \mu m \) is the pressure scale height at the solar surface. Integrating the equation above gives the photospheric gas pressure,

\[
p_{ph} = p_s\exp(-z/H_{ph}),
\]

and similarly the photospheric density,

\[
\rho_{ph} = \rho_s\exp(-z/H_{ph}). \tag{3.4}
\]

Hence, pressure and density decrease exponentially at the photosphere on a scale of \( H_{ph} \). Deriving the pressure and density profiles in the transition region is less straightforward due to the power law description of the temperature in this region. Nevertheless, the analytic expressions for pressure and density are given by

\[
p_a = p_{ph}(z = z_a)\exp \left( \frac{(z_{cor} - z_{tr})}{H_{ph} \ln \left( \frac{T_{cor}}{T_{tr}} \right)} \left( \frac{T_{ph}}{T_{tr}} - 1 \right) \right),
\]
and 

\[ \rho_{tr} = \rho_{ph}(z = z_{tr}) \frac{T_{ph}}{T_{tr}} \exp \left( \frac{(z_{cor} - z_{tr})}{H_{cor}} \ln \left( \frac{T_{cor}}{T_{ph}} \right) \left( \frac{T_{ph}}{T_{tr}} - 1 \right) \right). \]

The density and pressure in the transition region have a more complex \( z \) dependence, given the form of 

\[ T_{tr} = T_{ph} \left( \frac{T_{cor}}{T_{ph}} \right)^{ \frac{z_{cor} - z_{tr}}{z_{cor} - z_{tr}} } \].

However, it is important to note that density falls off more quickly than pressure due to the \( T_{ph}/T_{tr} \) factor multiplied by density.

Lastly, we derive the corresponding density and pressure profiles in the low corona. The coronal hydrostatic pressure equation is given by 

\[ \frac{dp_{cor}}{dz} = -\frac{p_{cor}}{H_{cor}}, \]

where \( H_{cor} = k_B T_{cor}/g\mu_m \) which yields the following expressions for pressure and density:

\[ p_{cor} = p_{tr}(z = z_{cor}) \exp \left( \frac{z_{cor} - z}{H_{cor}} \right), \]

\[ \rho_{cor} = \rho_{tr}(z = z_{cor}) \exp \left( \frac{z_{cor} - z}{H_{cor}} \right). \]

The coronal density and pressure follow a similar profile to that of the photospheric density and pressure, in that they both fall off exponentially. However, in this region of the domain, they fall off on a scale \( H_{cor} \).

In general, \( H_{cor} \gg H_{ph} \) and hence, the pressure and density fall off much more slowly in the atmosphere.

The values of \( T_{ph}, T_{cor}, z_{tr}, \) and \( z_{cor} \) are taken to be approximately in line with real solar values. The basis for our choice of heights is displayed in Fig. 3.2. We have chosen \( T_{ph} = 5.6 \times 10^3 \) K, \( T_{cor} = 8.4 \times 10^5 \) K, \( z_{tr} = 1.7 \times 10^6 \) m, and \( z_{cor} = 3.4 \times 10^6 \) m. Similarly, we choose the density at the solar surface to be \( 3 \times 10^{-4} \) kg/m\(^3\) and gravity to be \( 2.7 \times 10^2 \) m/s\(^2\). Of course, the heights, temperature and density are not perfectly in line with observations. For instance, we model the transition region to be much larger than...
3.2 Numerical stratification

it is in reality in an attempt to resolve the steep temperature gradient observed here. This concludes our
discussion of the analytic stratification of the solar interior and atmosphere we use in our experiments.

3.2 Numerical stratification

As discussed in the previous section, we implement this analytic solar interior and atmosphere into a
numerical code *Lare3d* which solves the dimensionless MHD equations. We must first, therefore, non-
dimensionalise the function \( T(z) \) and all constants. In order to do this, we define normalising values for
length, magnetic field and density. We perform the normalisation by using the temperature and density
prescribed at the end of Section 3.1, precisely \( T_{ph} = 5.6 \times 10^3 \text{ K} \) and \( \rho_{ph} = 3 \times 10^{-4} \text{ kg/m}^3 \), and choosing
the quantities to possess dimensionless values of unity at the solar surface. We also choose gravity to have
a dimensionless value of unity at the solar surface. To convert quantities from dimensional to dimension-
less quantities, we divide by their values at the solar surface, for example, \( \hat{T} = T/T_{ph} \), where we have
newly defined \( \hat{T} \) as the dimensionless temperature and \( T_{ph} \) as its photospheric value. Similarly, \( \hat{z} = z/L_{ph} \),
\( \hat{\rho} = \rho/\rho_{ph} \), etc.

We note, by imposing the dimensionless value of gravity to be unity, this forces the length scale we
use to be the pressure scale height at the solar surface, i.e., \( L_{ph} = H_{ph} = \frac{k_B T_{ph}}{\mu_m g} = 170 \text{ km} \). Using the
dimensional conversions \( T_{ph} = 5.6 \times 10^3 \text{ K} \), \( \rho_{ph} = 3 \times 10^{-4} \text{ kg/m}^3 \), and \( L_{ph} = 170 \text{ km} \) together with
the relations described in Section 2.3.1, it is possible to derive conversions for all the basic quantities as sum-
marised in Table 3.1. Applying these conversions, the dimensional values and corresponding dimensionless
values for all the basic variables are easily derived as shown in Table 3.2. Note, these are the normalising
values used for all simulations within this thesis, unless otherwise stated. We perform experiments on a
uniform Cartesian grid \((x, y, z)\) spanning from \(-50\) to \(50\) in \(x\) and \(y\), and \(-25\) to \(75\) in \(z\), where \(z = 0\) is
the solar surface dividing the solar interior from the atmosphere.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Physical Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
<td>( \rho_{ph} )</td>
<td>( 3 \times 10^{-4} \text{ kg/m}^3 )</td>
</tr>
<tr>
<td>Temperature</td>
<td>( T_{ph} )</td>
<td>( 5.6 \times 10^3 \text{ K} )</td>
</tr>
<tr>
<td>Length</td>
<td>( L_{ph} )</td>
<td>( 170 \times 10^3 \text{ m} )</td>
</tr>
<tr>
<td>Velocity</td>
<td>( v_{ph} )</td>
<td>( 6.8 \times 10^3 \text{ m/s} )</td>
</tr>
<tr>
<td>Time</td>
<td>( t_{ph} )</td>
<td>( 25 \text{ s} )</td>
</tr>
<tr>
<td>Pressure</td>
<td>( p_{ph} )</td>
<td>( 1.4 \times 10^4 \text{ Pa} )</td>
</tr>
<tr>
<td>Magnetic field</td>
<td>( B_{ph} )</td>
<td>( 0.13 \text{ T} )</td>
</tr>
<tr>
<td>Gravity</td>
<td>( g_{ph} )</td>
<td>( 274 \text{ m/s}^2 )</td>
</tr>
</tbody>
</table>

Let us now consider the dimensionless form of the analytic temperature profile developed in the previous
3.2 Numerical stratification

Table 3.2: Dimensional and dimensionless values for constants in the background atmosphere.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Dimensional value</th>
<th>Dimensionless value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{ph}$</td>
<td>$5.6 \times 10^4$ K</td>
<td>1</td>
</tr>
<tr>
<td>$T_{cor}$</td>
<td>$8.4 \times 10^5$ K</td>
<td>150</td>
</tr>
<tr>
<td>$z_{tr}$</td>
<td>$1.7 \times 10^6$ m</td>
<td>10</td>
</tr>
<tr>
<td>$z_{cor}$</td>
<td>$3.4 \times 10^6$ m</td>
<td>20</td>
</tr>
<tr>
<td>$H_{ph}$</td>
<td>$1.7 \times 10^5$ m</td>
<td>1</td>
</tr>
<tr>
<td>$H_{cor}$</td>
<td>$2.6 \times 10^7$ m</td>
<td>150</td>
</tr>
<tr>
<td>$g$</td>
<td>274 m/s$^2$</td>
<td>1</td>
</tr>
</tbody>
</table>

section. The temperature profile reduces to

$$
\hat{T}(\hat{z}) = \begin{cases} 
1 - \frac{(\gamma - 1)}{\gamma} \hat{z} & \hat{z} < 0, \\
1 & 0 \leq \hat{z} < 10, \\
\frac{\hat{z} - 10}{10} & 10 \leq \hat{z} < 20, \\
150 & \hat{z} \geq 20.
\end{cases}
$$

NOTE: From now on, we shall remove the hats and all quantities will be dimensionless unless otherwise stated. Dimensional quantities are identifiable by their accompanying physical units.

Rather than using analytic expressions for pressure and density, we calculate the quantities using finite differencing. This is preferable as it means the atmosphere will be closer to “numerical equilibrium” as opposed to “analytic equilibrium”. We note that we have included the analytic profiles for pressure and density in the previous section solely to make a comparison of the effectiveness of finite differencing. As an example, a comparison of the density stratification calculated analytically and numerically using finite differencing is shown in Fig. 3.3. The maximum difference between the two curves is 0.01. This is for the photospheric region specifically, and as such is calculated using the analytic expression in Eq. 3.4 and the finite differencing method described below. Clearly, the finite differencing method has successfully reproduced the expected density stratification in the photosphere.

We calculate the density and pressure profiles by solving

$$
\frac{dp}{dz} = -\rho g,
$$

by finite differencing and making use of the dimensionless ideal gas law, $p = \rho T$,

$$
\rho_i T_i - \rho_{i+1} T_{i-1} \frac{dz}{dz_{i+1}} = -\left(\frac{dz b_i + dz b_{i-1} \rho_{i+1}}{dz b_i + dz b_{i-1}}\right) g_{i-1/2}.
$$

(3.5)

Notice, we have averaged the cell-centred density here as the finite difference equation is centred on the cell edge. Details of finite differencing methods are discussed in Chapter 2. Eq. 3.5 can be solved for the
3.2 Numerical stratification

density by either going up into the atmosphere from the surface $z = 0$ or down into the interior from $z = 0$. The resulting profiles for temperature (red), density (black), and pressure (blue) are plotted on a log-scale against height as shown in Fig. 3.4. The dimensional and dimensionless profiles are presented side by side in Fig. 3.4a and Fig. 3.4b respectively. By non-dimensionalising the quantities, the range of orders of magnitude are reduced and hence the experiment becomes more computationally manageable. The density varies over 8 orders of magnitude from the highly dense interior to the rarefied corona. The initial profiles for density and pressure closely follow the analytic expressions found in Section 3.1 where we find steep gradients in density and pressure within the photosphere as they fall off on a scale of $H_{ph} = 1$ and a very shallow decrease in pressure and density in the corona as the scale height $H_{cor}$ is much longer.

The initial configuration is set up such that its acting forces are balanced and the background environment is in equilibrium. In order to check how robust this equilibrium is, we have plotted the vertical velocity divided by the local sound speed $c_s$ along the central line given by $x = 0$ and $y = 0$ as displayed in Fig. 3.5.
We note that this is a low resolution experiment, with 128 grid points in each direction. The experiments performed in this thesis typically contain $512^3$ grid points. The $z$-component of velocity has been plotted over 200 normalised time units, which is equivalent to approximately 83 minutes. Although the vertical velocity increases in magnitude over time, it remains below $10^{-10}$, or equivalently $2 \times 10^{-12} \, \text{c}$. We shall therefore conclude that spurious oscillations in velocity remain much smaller than the sound speed and use this as sufficient evidence to state that the background atmosphere is in hydrostatic equilibrium.

To summarise, the equations are solved on a uniform Cartesian grid of physical size $17 \, \text{Mm} \times 17 \, \text{Mm} \times 17 \, \text{Mm}$. The background stratification consists of an upper layer of the solar interior of thickness 4.25 Mm governed by an adiabatic temperature gradient, an isothermal photosphere/chromosphere (5600 K) of thickness 1.7 Mm, a transition region of thickness 1.7 Mm with a steep temperature gradient and finally an isothermal corona ($10^6$ K) which is 150 times hotter than the photosphere with thickness 9.35 Mm. We note this is a common background stratification and as such has been used in many flux emergence simulations, including Fan (2001), Archontis et al. (2004), and MacTaggart and Hood (2009). The boundaries of the box are periodic in the horizontal directions and closed on the top and bottom of the box. Specifically, $\nabla \cdot \mathbf{v} = 0$ on the top and bottom boundaries, and the normal derivatives of all quantities are set to zero. In addition, we set resistivity as $\eta = 0.005$ and the kinematic viscosity as $\nu = 0.05$ throughout the volume.

### 3.3 Choice of sub-photospheric magnetic flux tube

Next, we add a sub-photospheric magnetic flux tube to our simulation domain. We choose to leave the solar atmosphere unmagnetised and concentrate on the interior flux tube. However, the inclusion of a magnetised atmosphere can lead to interesting phenomena such as jets and eruptions. For details of the effect of including an ambient coronal magnetic field in 3D numerical simulations, see Archontis et al. (2004) and Lee et al. (2015), and references therein. To set up an equilibrium, we require force balance and set $\nabla \cdot \mathbf{v} = 0$ in the equation of motion

$$-\nabla p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g} = 0.$$
3.3 Choice of sub-photospheric magnetic flux tube

Given that the external environment is in hydrostatic pressure balance as prescribed above, we split the gas pressure into a component that balances gravity ($\nabla p_b = -\rho g$) and a component that balances the Lorentz force such that this reduces to

$$\nabla p_{\text{exc}} = j \times B,$$

(3.6)

where $p_{\text{exc}}$ is the pressure excess such that the gas pressure in the tube, $p_t$, is defined to be $p_b + p_{\text{exc}}$ where $p_b$ is the background gas pressure. The next hurdle is which form of magnetic field to prescribe in the solar interior. The interior magnetic field cannot be observed so simple models of magnetic fields are chosen to initiate emergence. Having discussed the Gold-Hoyle flux tube in Section 1.3.3, we now discuss two other popular choices of magnetic flux tubes used to initialise emergence experiments.

3.3.1 Cylindrical magnetic field

We first discuss the placement of a twisted cylindrical flux tube lying horizontally in the middle of the model convection zone. This form of magnetic field was first discussed in Fan (2001) and has been utilised in countless emergence simulations since (including Archontis et al., 2004, Manchester et al., 2004, Murray et al., 2006, Moreno-Insertis et al., 2008, and Archontis and Hood, 2012). In a cylindrical coordinate system $(r, \theta, y)$, the magnetic field is prescribed as $B = (B_r, B_\theta, B_y)$ where the individual components are defined as

$$B_r = 0,$$

$$B_\theta = \alpha r B_y,$$

$$B_y = B_0 e^{-r^2/a^2},$$

where $a$ is the radius of the flux tube, $B_0$ is the axial field strength, and $\alpha = B_\theta / r B_y$ is the twist per unit length. We define the axis of the flux tube to be the fieldline threading through the centre at $r = 0$. The twist per unit length, $\alpha$, is the radian angle through which fieldlines rotate over one axial unit in length. Note, this quantity is equivalent to $\alpha$ defined in Eq. 1.22. The radial coordinate, $r$, is defined as $r^2 = x^2 + (z - z_{\text{base}})^2$, the azimuthal coordinate, $\theta$, is given by $\theta = \tan^{-1}(x/(z - z_{\text{base}}))$, and $y$ is the coordinate describing the direction of the tube axis. The base of the simulation domain is defined as $z = z_{\text{base}}$. The equivalent Cartesian magnetic field is defined as

$$B_x = \alpha (z - z_{\text{base}}) B_y,$$

$$B_y = B_0 e^{-r^2/a^2},$$

$$B_z = -\alpha x B_y.$$

This prescribed magnetic field ensures that the field strength is set to zero at large distances from the tube as the surrounding plasma is unmagnetised. The parameter $\alpha$ is constant for all simulations in this thesis and so the flux tubes are uniformly twisted. An example set of fieldlines in a typical cylindrical flux tube is shown in Fig. 3.6 where the fieldlines are coloured by radius. In the same manner as the Gold-Hoyle flux tube, the fieldlines turn from axial to azimuthal with increasing radius.
3.3 Choice of sub-photospheric magnetic flux tube

Figure 3.6: Example set of fieldlines within a cylindrical flux tube, where fieldlines are coloured by radius.

Given the magnetic field prescribed in cylindrical coordinates, the force balance equation given in Eq. 3.6 becomes

\[
\frac{1}{2} \frac{d}{dr} B_r^2 + \frac{B_\theta}{r} \frac{d}{dr} (r B_\theta) + \frac{dp_{exc}}{dr} = 0,
\]

which can be solved to give an excess pressure, relative to the background gas pressure, of

\[
p_{exc}(r) = \frac{B_0^2}{4} e^{-2r^2/a^2} (\alpha^2 a^2 - 2\alpha^2 r^2 - 2).
\]

The pressure excess is negative for all \(r\) if \(\alpha^2 a^2 - 2 < 0\) and hence if \(|\alpha| < \sqrt{2}/a\). Therefore, if this is satisfied there is always a pressure deficit at the flux tube. This completes the equilibrium of the cylinder model. Hence with this prescribed magnetic field and pressure excess, the flux tubes sits in equilibrium.

In order to form an \(\Omega\)-shaped loop that can rise into the atmosphere and form two sunspots, a density excess with a Gaussian profile is implemented as

\[
\rho_{exc} = \frac{p_{exc}}{T} \exp\left(-\frac{y^2}{\lambda^2}\right),
\]

following the work of Fan (2001) and many others. Note \(y\) is the distance along the axial direction of the flux tube and \(\lambda\) is the desired length of the buoyant region. Similarly, the density excess is negative, and hence a deficit, if \(|\alpha| < \sqrt{2}/a\) and then the tube is lighter than its surroundings. However, there is one issue with this initial set-up; the whole flux tube is weakly buoyant due to the exponential profile. This issue will be addressed in the next section as we consider a different structure of magnetic tube in the solar interior.

Although we do not directly use the cylindrical model in our experiments, the insights gained from this model have been vast (Fan, 2001, Magara, 2001, Archontis et al., 2004, and references therein). The parallels between this simple cylindrical model and the more complex initial tube we choose are very useful in the next section.

3.3.2 Toroidal magnetic field

An alternative choice of initial flux tube is a half-torus shaped tube comprised of twisted fieldlines, henceforth referred to as a toroidal tube. This models the emergence of the upper portion of an \(\Omega\)-shaped loop that is rooted much deeper in the solar interior. This choice of initial flux tube has been used in simulations by Hood et al. (2009) and MacTaggart and Hood (2009). For a clear comparison between the two models, a simple schematic of a selection of twisted fieldlines is shown in Fig. 3.7, again coloured by radius. The magnetic field configuration was first derived in Hood et al. (2009) and we follow this derivation in order to construct a toroidal flux tube. First, we express Cartesian coordinates \((x, y, z)\), such that the tube lies
3.3 Choice of sub-photospheric magnetic flux tube

along the $y$ direction and $z$ denotes height from the solar surface as previously, in terms of cylindrical coordinates $(R, \phi, x)$. Note, this corrects an error in Hood et al. (2009) where they transformed to a left-handed coordinate system $(R, \phi, -x)$.

![Figure 3.7: Example set of fieldlines within a toroidal flux tube, as coloured by radius.](image)

Explicitly,

$$R^2 = y^2 + (z - z_{\text{base}})^2, \quad \text{with} \quad y = R \cos \phi \quad \text{and} \quad z - z_{\text{base}} = R \sin \phi,$$

where $z_{\text{base}}$ again denotes the value of $z$ at the base of the computational domain. Schematics to demonstrate the Cartesian to cylindrical transformation are shown in Fig. 3.8a and Fig. 3.8b respectively. The magnetic field is expressed in terms of a flux function, $A = A(R, x)$, as

$$B = \nabla A \times \nabla \phi + B_\phi \mathbf{e}_\phi$$

$$= \left( \frac{\partial A}{\partial R} \frac{1}{R} \frac{\partial A}{\partial \phi} \frac{1}{R} \frac{\partial A}{\partial x} \right) \times \left( 0, \frac{1}{R}, 0 \right) + B_\phi \mathbf{e}_\phi$$

$$= -\frac{1}{R} \frac{\partial A}{\partial x} \mathbf{e}_R + B_\phi \mathbf{e}_\phi + \frac{1}{R} \frac{\partial A}{\partial R} \mathbf{e}_x.$$

In order to make progress, we assume the magnetic field is rotationally invariant to ensure $B_\phi$ is independent of $\phi$ and in turn that the magnetic field satisfies the solenoidal constraint, $\nabla \cdot \mathbf{B} = 0$. Taking the dot product
3.3 Choice of sub-photospheric magnetic flux tube

of Eq. 3.6 with \( \mathbf{B} \) yields

\[
\mathbf{B} \cdot \nabla p_{\text{exc}} = (\nabla A \times \nabla \phi + B_\phi e_\phi) \cdot \nabla p_{\text{exc}} = \mathbf{B} \cdot (\mathbf{j} \times \mathbf{B}) = 0,
\]

or more fully,

\[
\left( \frac{1}{R} \frac{\partial A}{\partial x} e_R + B_\phi e_\phi + \frac{1}{R} \frac{\partial A}{\partial R} e_x \right) \cdot \left( \frac{\partial p_{\text{exc}}}{\partial R} e_R + \frac{\partial p_{\text{exc}}}{\partial x} e_x \right) = 0,
\]

\[
- \frac{1}{R} \frac{\partial A}{\partial x} \frac{\partial p_{\text{exc}}}{\partial R} + \frac{1}{R} \frac{\partial A}{\partial R} \frac{\partial p_{\text{exc}}}{\partial x} = 0,
\]

where we have noted that \( \frac{\partial p_{\text{exc}}}{\partial \phi} = 0 \) due to the assumption of rotational invariance. It is worth noting that this is equivalent to

\[
\nabla p_{\text{exc}} \times \nabla A = 0,
\]

and hence, \( \nabla p_{\text{exc}} \) is parallel to \( \nabla A \) and so the contours of \( p_{\text{exc}} \) and \( A \) overlap. There is therefore some mapping between contours of \( p_{\text{exc}} \) and \( A \) which allows us to rewrite the pressure excess as a function of \( A, p_{\text{exc}} = F(A(R,x)) \) for an arbitrary function \( F \). This allows us to rewrite the force balance equation (Eq. 3.6) for the flux tube as

\[
\left\{ - \frac{1}{R^2} \frac{\partial^2 A}{\partial x^2} - \frac{1}{R} \frac{\partial}{\partial x} \left( \frac{1}{R} \frac{\partial A}{\partial R} \right) \right\} \nabla A + \left( \frac{1}{R} \frac{\partial A}{\partial R} \frac{\partial B_\phi}{\partial x} - \frac{1}{R^2} \frac{\partial A}{\partial x} \frac{\partial}{\partial R} (RB_\phi) \right) \hat{e}_\phi
\]

\[
- \frac{B_\phi}{R} \nabla (RB_\phi) = \frac{d p_{\text{exc}}}{dA} \nabla A.
\]

Let us consider the \( \phi \) component

\[
\frac{\partial A}{\partial R} \frac{\partial B_\phi}{\partial x} - \frac{1}{R} \frac{\partial A}{\partial x} \frac{\partial}{\partial R} (RB_\phi) = \frac{1}{R} \left( \frac{\partial A}{\partial R} \frac{\partial (RB_\phi)}{\partial x} - \frac{\partial A}{\partial x} \frac{\partial}{\partial R} (RB_\phi) \right) = 0,
\]

or, equivalently \( \nabla (RB_\phi) \times \nabla A = 0 \). This tells us that \( \nabla (RB_\phi) \) is parallel to \( \nabla A \) and in turn

\[
RB_\phi = G(A(R,x)),
\]

given an arbitrary function \( G \). This allows us to rewrite Eq. 3.8 as

\[
- \frac{1}{R} \left\{ 1 \frac{\partial^2 A}{\partial x^2} + \frac{\partial}{\partial x} \left( \frac{1}{R} \frac{\partial A}{\partial R} \right) \right\} \nabla A - \frac{1}{R^2} (RB_\phi) \frac{d (RB_\phi)}{dA} \nabla A = \frac{d p_{\text{exc}}}{dA} \nabla A.
\]

Notice, all the terms in Eq. 3.9 are in the direction of \( \nabla A \), so by ignoring the trivial solution \( \nabla A = 0 \), the Grad-Shafranov equation is of the form

\[
R \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial A}{\partial R} \right) + \frac{\partial^2 A}{\partial x^2} + RB_\phi \frac{d RB_\phi}{dA} + R^2 \frac{d p_{\text{exc}}}{dA} = 0.
\]

Following the strategy used by Hood et al. (2009), we convert to a local toroidal coordinate system
3.3 Choice of sub-photospheric magnetic flux tube

\[(r, \theta, \phi)\) (shown schematically in Fig. 3.8c), such that

\[r^2 = x^2 + (R - R_0)^2\]

with \(R - R_0 = r \cos \theta\) and \(x = -r \sin \theta\),

where \(R_0\) is the major axis of the toroidal loop, as shown schematically in Fig. 3.8b. To change into this coordinate system, we replace the \(R\) and \(x\) derivatives in Eq. 3.10, as follows:

\[
\frac{\partial}{\partial R} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta},
\]

\[
\frac{\partial}{\partial x} = -\sin \theta \frac{\partial}{\partial r} - \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}.
\]

This allows us to re-express Eq. 3.10 in terms of the new toroidal system as

\[
\frac{\partial^2 A}{\partial r^2} + \frac{1}{r} \frac{\partial A}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A}{\partial \theta^2} = -\frac{1}{R_0 + r \cos \theta} \left( \cos \theta \frac{\partial A}{\partial r} - \frac{\sin \theta}{r} \frac{\partial A}{\partial \theta} \right) + (RB_{\phi}) \frac{d(RB_{\phi})}{dA} + (R_0 + r \cos \theta) \frac{dp_{exc}}{dA} = 0. \tag{3.11}
\]

In order to proceed analytically, we shall assume \(a << R_0\) and in turn \(r << R_0\). In other words, we assume that the minor radius of the torus is much smaller than the major radius. Note, the minor radius of the flux tube, \(a\), is shown in Fig. 3.8b. We can then take a regular expansion, in powers of \(a/R_0\), such that

\[A \sim A_0(r) + \frac{a}{R_0} A_1(r, \theta) + \frac{a^2}{R_0^2} A_2(r, \theta) + \ldots\]

To leading order, \(A\) can be approximated as \(A_0(r)\) and Eq. 3.11 becomes

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial A_0}{\partial r} \right) + \frac{1}{2R_0^2} \frac{d(RB_{\phi})}{dA_0} \frac{d(RB_{\phi})}{dA_0} + R_0 \frac{dp_{exc}}{dA_0} = 0.
\]

To proceed, we multiply the above equation by \(B_{\theta} = -\frac{1}{R_0} \frac{\partial A_0}{\partial r}\) yielding

\[
R_0 \frac{B_{\theta}}{r} \frac{d}{dr} (rB_{\theta}) + \frac{1}{2R_0^2} \frac{d(RB_{\phi})}{dA_0} \frac{d(RB_{\phi})}{dA_0} + R_0 \frac{dp_{exc}}{dA_0} = 0,
\]

or equivalently

\[
\frac{B_{\theta}}{r} \frac{d}{dr} (rB_{\theta}) + \frac{1}{2} \frac{dB_{\phi}^2}{dr} + \frac{dp_{exc}}{dr} = 0,
\]

where we have cancelled \(R/R_0\) to leading order given the definition \(R = R_0 + r \cos \theta \sim R_0 + O(a/R_0)\). This has exactly the same form as the standard cylindrical equation found in Eq. 3.7 with the transformation \(B_{\theta} \rightarrow B_{\phi}\). Note, in both cases this represents the field in the axial direction. We can therefore choose the solutions to be the same as the cylindrical flux tube used in Archontis et al. (2004). Specifically,

\[B_{\phi} = B_{0\phi} e^{-r^2/a^2} \quad \text{and} \quad B_{\theta} = \alpha r B_{\phi} = \alpha B_{0\phi} e^{-r^2/a^2}. \tag{3.12}\]

With these approximations to the local toroidal field, the pressure excess can be calculated by exact com-
3.3 Choice of sub-photospheric magnetic flux tube

Comparison with the cylindrical model

\[ p_{\text{exc}}(r) = \frac{B_0^2}{4} e^{-2r^2/a^2} (\alpha^2 a^2 - 2\alpha^2 r^2 - 2). \]

This balances Eq. 3.6 to leading order. The expression is negative for all \( r \) if \( |\alpha| < \sqrt{2}/a \). In all the parameter choices used for the numerical experiments in this thesis, this is satisfied and so the pressure excess is negative and is instead a pressure deficit. Equivalently, this means that the outwards directed magnetic pressure force is larger than the inwards directed magnetic tension force, and hence the gas pressure gradient acts inwards to balance the forces.

The toroidal tube is made buoyant by a similar technique to that of the cylindrical model. We set the temperature of the tube equal to the external temperature and calculate the excess density based on the excess pressure,

\[ \rho_{\text{exc}} = \frac{p_{\text{exc}}}{T(z)} = \frac{B_0^2}{4 T(z)} e^{-2r^2/a^2} (\alpha^2 a^2 - 2\alpha^2 r^2 - 2). \]  

(3.13)

Note, we do not need to multiply the density excess by an exponential as we did in the cylindrical case. In the cylindrical model, this controlled the density deficit in such a way that the maximum buoyancy effect is at the centre of the tube. However, due to the curvature of the toroidal tube, the field near the base of the box has further to travel than the field at the top (Hood et al., 2009). At the same time, the background density and pressure decreases towards the top of the tube, so that the plasma beta decreases and the effects of buoyancy becomes more important higher up the tube. This is shown in Fig. 3.10, discussed in more detail at the end of the section. The combination of these two effects ensure that the upper part of the toroidal loop has a larger buoyancy effect and rises more quickly. Again, \( |\alpha| < \sqrt{2}/a \) ensures the density excess is negative and the tube is buoyant rather than over-dense with its surroundings.

Lastly, we re-express the magnetic field in Cartesian coordinates as this is the coordinate system we are working in for our numerical experiments. Before we can do this directly, we must first express the magnetic field in cylindrical coordinates \((R, \phi, x)\) as

\[ B_R = -B_\theta(r) \sin \theta = B_\theta(r) \frac{x}{r}, \]
\[ B_x = -B_\theta(r) \cos \theta = -B_\theta(r) \frac{R - R_0}{r}. \]

This can then be converted to Cartesian coordinates:

\[ B_x = -B_\theta \frac{R - R_0}{r}, \]
\[ B_y = -B_\phi \frac{z - z_{\text{base}}}{R} + B_R \frac{y}{R} = -B_\phi \frac{z - z_{\text{base}}}{R} + B_\theta \frac{x}{r} \frac{y}{R}, \]
\[ B_z = B_\phi \frac{y}{R} + B_R \frac{z - z_{\text{base}}}{R} = B_\phi \frac{y}{R} + B_\theta \frac{x}{r} \frac{z - z_{\text{base}}}{R}, \]  

(3.14)

where \( B_\phi = B_0 e^{-r^2/a^2} \) and \( B_\theta = \alpha B_0 e^{-r^2/a^2} \). The error in the coordinate transformation stated in Hood et al. (2009) follows through to change the sign of \( B_R \) and \( B_x \) in the final expression. Fortunately this error is not significant as it is equivalent to using an initial twist, \( \alpha \), of the opposite sign. If we do not impose a
3.3 Choice of sub-photospheric magnetic flux tube

The density deficit, the flux tube is sitting in approximate equilibrium. However, we note that the equilibrium is only first-order accurate. We are not concerned with this as we impose a density deficit that makes the tube buoyant. If we wished to impose a more accurate equilibrium, we would need to derive extra terms in the expansion.

Figure 3.9: Initial set-up of example experiment. (a) The initial profiles of the temperature, density, gas, and magnetic pressures through \( x = 0 \) and \( y = 0 \) as a function of height. (b) A 3D visualisation of the experiment with log profiles of the temperature on the back wall and density on the right wall as coloured by key on the left. A magnetogram of the vertical magnetic field is shown on the base of the domain as well as a select set of fieldlines shown in red and an isosurface of magnetic field overplotted. The solar surface is also highlighted at \( z = 0 \) in grey.

An example of a typical initial set-up with the insertion of the sub-photospheric magnetic field is given in Fig. 3.9a for the parameters: initial axial field strength \( B_0 = 9 \); twist \( \alpha = 0.4 \); minor radius \( a = 2.5 \); major radius \( R_0 = 15 \); and base value \( z_{\text{base}} = -25 \). Note, the criteria \( |\alpha| < \sqrt{2}/a \) is satisfied given the parameter choice and so the tube is made buoyant. We notice, in Fig. 3.9a, the insertion of a sub-photospheric magnetic field in the interior adds magnetic pressure to the convection zone as well as lowering the gas pressure and density at the apex of the tube. This can be compared with Fig. 3.4b where we considered the hydrostatic atmosphere without a magnetic field. A 3D schematic of the initial set-up is also shown in Fig. 3.9b, in order to visualise the 3D structure of the magnetic field with respect to the surrounding environment. The background stratification of temperature and density is shown on the back and right walls respectively and a magnetogram of the vertical magnetic field is shown on the base of the domain where white represents a positive \( B_z \) and black a negative \( B_z \).

Finally, we have plotted the distribution of density, temperature, and pressure along the axis of the flux tube (at \( r = 0 \)) in Fig. 3.10. The density and pressure are already adjusted by the deficits described earlier. However, since the deficits are uniform along the axis, this only shifts the curve by a constant value. Clearly, the density, pressure, and temperature of the flux tube vary depending on the height within the domain. The
legs of the flux tube are embedded in hot, dense plasma at a higher pressure. However, at the apex of the flux tube, the magnetic field is surrounded by cooler, less dense plasma at a lower pressure.

### 3.4 Summary

This chapter has introduced the general initial set-up used for the majority of experiments performed in this thesis. We introduced the background stratification of the plasma, separating the domain into an interior and atmospheric region as separated by the solar surface at \( z = 0 \). The atmospheric region is further split into three regions: the photosphere; transition region; and corona. First, we presented the analytic stratification and, given the normalising values, we showed how to reach the initial numerical stratification. By considering background velocities that arise within the domain, we surmise that the velocities remain small over the length of a typical experiment. With the background in hydrostatic equilibrium, we discussed two types of magnetic flux tube that can initialise emergence. For the experiments performed in this thesis, we use a toroidal magnetic flux tube as it models a magnetic flux tube rooted lower in the interior. In the next chapter, we use this set-up to initialise an emergence experiment and present a thorough analysis of the evolution of the magnetic flux tube and plasma.
Chapter 4

Sunspot rotation due to flux emergence

The results of this chapter have been published in


In this chapter, we perform a resistive 3D MHD simulation of an arched twisted flux tube (the toroidal model introduced in Chapter 3) placed in the solar interior and track its emergence through the photosphere and lower atmosphere. Our primary aim is to study the rotation at the photosphere (see Section 1.6) by explicitly calculating the angle of rotation, identifying the cause of this rotation and studying the distribution of twist, helicity, and energy across the simulation domain. Moreover, we aim to investigate (i) what governs the final angle of rotation and (ii) what causes the rotation to cease. Also, we seek to compare the rotation rates with those found in observations. We achieve these aims by investigating a variety of quantities relating to the magnetic field and plasma.

The remainder of the chapter is structured as follows. First of all, we specify the particular parameters used for this general case in Section 4.1. In Sections 4.2 and 4.3, the simulation results are presented. Section 4.2 focuses on the general evolution of the flux tube as it emerges whereas Section 4.3 focuses on the rotational motions that develop within the two polarities on the photospheric plane. The rotation angle of the sunspots and twist of the fieldlines are among the quantities we calculate. The rotational analysis also includes an investigation of the flow vorticity at the photosphere, current, twist, and relative magnetic helicity. Finally, in Section 4.4 we conclude the chapter with a summary of our findings.

4.1 Parameter choice

The magnetic field and background stratification of this experiment are summarised in Chapter 3 and the code used to solve the equations is described in Chapter 2. In this general experiment, we set the magnetic field strength at the apex as $B_0 = 9 \times 11700$ G and the twist as $\alpha = 0.4$ (1 turn in 2.67 Mm). A positive $\alpha$
corresponds to a right-hand twisted flux tube. The base of the computational domain is set at \( z = -25 \). The major radius of the torus is \( R_0 = 15 \) (2550 km) and the minor radius is \( a = 2.5 \) (425 km). The initial set-up of the experiment is summarised in Fig. 3.9b. A summary of the parameter choice is given in Table 4.1. The total flux through a cross-section of the flux tube is \( 6.6 \times 10^{11} \) Wb (\( 6.6 \times 10^{10} \) Mx), typical of a large ephemeral region or small active region.

<table>
<thead>
<tr>
<th>Magnetic field parameters</th>
<th>Global parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_0 = 9 ), ( \alpha = 0.4 )</td>
<td>( 512^3 ) grid points</td>
</tr>
<tr>
<td>( R_0 = 15 ), ( a = 2.5 )</td>
<td>( \eta = 0.005 ) everywhere</td>
</tr>
</tbody>
</table>

### 4.2 General analysis

Before we analyse the rotational motion of the sunspots at the photosphere in this experiment, let us consider a general overview of the evolution of the flux tube as it rises through the interior and emerges into the atmosphere.

#### 4.2.1 Rise through solar interior

The density deficit introduced (see Eq. 3.13) disrupts the equilibrium and allows the flux tube to start to rise buoyantly to the photosphere. This deficit is implemented by setting the temperature of the tube equal to the temperature of the surroundings and maintaining the negative pressure excess (or pressure deficit) found by balancing the pressure gradient with the Lorentz force. Further details of this can be found in Chapter 3. The flux tube continues to rise through the solar interior due to the buoyancy instability until it reaches the convectively stable photosphere. The isothermal stratification in this layer results in an \( N^2 > 0 \). See Section 1.4 for a definition of the buoyancy frequency \( N^2 \) and further details on both the buoyancy and magnetic buoyancy instability.

The height-time profiles for the axis and leading edge of the system are shown in Fig. 4.1. The rise of the flux tube to photospheric heights is governed by the buoyancy instability as shown in the height-time plot up until \( t = 25 \). Following the method of Hood et al. (2009), the axis can be identified by plotting the zero contour of \( B_x \) and \( B_z \) in the mid plane, and identifying the intersection of the two contours. In this particular case \( B_z \) is zero along the line at \( x = 0 \) and \( y = 0 \). Thus, we track the axis by any zero of \( B_x \) along the line prescribed by \( x = 0 \) and \( y = 0 \). This has also been checked against tracing the field from the centre of both footpoints and is found to agree for most of the experiment. We believe the formation of a new flux rope and, in turn, new axis is responsible for the divergence of the two methods. New flux ropes form due to shearing flows and reconnection (see Mactaggart, 2010 for an explanation of the mechanism). We warn the reader that tracing from the left and right footpoints produce almost identical results and as such the blue and pink colourings are difficult to identify. The leading edge of the expanding volume is
calculated as the height where the field strength first increases above $10^{-7}$. The leading edge of the system is determined by the pressure balance boundary, where the total pressure within the tube equals the gas pressure. Initially, the flux tube rises relatively slowly until the leading edge reaches the photosphere. The tube then expands more quickly due to the density drop off at the photosphere. Later, the magnetic bubble expands very quickly due to the initiation of the magnetic buoyancy instability. The divergence of the two methods at later times is likely to be due to the kinking of the axis from the $y = 0$ plane.

Figure 4.1: The height-time profiles of the axis of the flux tube traced in the $x = 0, y = 0$ plane using $B_x = 0$ (x symbol), the leading edge of the flux system (dashed), the intersection of the magnetic field with the $y = 0$ plane as traced from the centre of the left footpoint (blue) and from the centre of the right footpoint (pink). The three horizontal lines at $z = 0$, $z = 10$ and $z = 20$ signify the solar surface, the start of the transition region and the start of the corona, respectively.

### 4.2.2 Arrival at the photosphere

The rise of the axis of the flux tube appears to slow when the tube reaches photospheric heights due to the change in stratification. At the photosphere, the plasma is stably stratified with a constant temperature and the flux tube is no longer able to rise by means of the buoyancy instability. The temperature gradient is no longer sufficiently decreasing and is therefore strongly sub-adiabatic. Therefore, the magnetic field must find another way to rise and expand into the corona, and it does, specifically, through the magnetic buoyancy instability. In order to initiate this instability, a criterion must be satisfied as derived in Chapter 1.

Typically, the onset of this instability occurs when the plasma $\beta$, defined as the gas pressure divided by the magnetic pressure, drops to one (Murray et al., 2006). Therefore, this can only occur if the initial field strength is large enough. This suggests that the properties of the emerging flux are highly dependent on the strength of the original interior field, a concept which we will investigate in Chapter 5. At this stage, the plasma $\beta$ dropping below one means the magnetic pressure exceeds the gas pressure and the field expands into the atmosphere. As a reminder, the criterion for the magnetic buoyancy instability is shown below in terms of the plasma $\beta$,

$$-\frac{1}{\beta} \frac{d}{dz} \log B_0 > \frac{\gamma k^2}{\beta} \left(1 + \frac{k^2 - 2}{k_\perp^2}\right) + \frac{\gamma \delta}{2},$$  

(4.1)
where we have divided the criterion given in Eq. 1.45 by $\beta$ and all variables used here are outlined in Section 1.4. Before the flux tube reaches the photosphere the criterion $N^2 < 0$ determines whether the flux tube rises by means of the buoyancy instability. However, at $z = 0$, $N^2 > 0$ and the buoyancy instability can no longer be triggered. Hence, the magnetic buoyancy instability is the only instability that could allow the flux tube to rise. Note from Eq. 1.43 that even if $N^2 > 0$, as at the photosphere, the criterion can still be satisfied if the magnetic field gradient is significantly steep.

Figure 4.2: The left (red) and right (blue) hand terms of Eq. 4.1 plotted against height for (a) $t = 11$, (b) $t = 17$ and, (c) $t = 20$. Also overplotted is the plasma $\beta$ as shown in green. The grey term in Eq. 4.1 is not plotted as it is not comparable with other terms. A movie of this figure is attached in the electronic version.

Figure 4.3: The change in the density excess at the axis of the flux tube, as a fraction of the unsigned initial density excess at the axis, plotted against height.

In Fig. 4.2, the different terms in Eq. 4.1 are shown against height for selected times. The terms of the criterion are coloured as they are in the equation. The red line displays the magnetic field gradient divided by $\beta$ and the blue line shows the superadiabatic excess $\delta$ multiplied by $\gamma/2$. We note the term involving wave-numbers is shown in grey as this term is negligible in comparison to the rest of the equation, and in turn has been excluded from the plot. Note, the red term is positive as the magnetic field strength decreases with height. As the magnetic field reaches the photosphere, the magnetic field gradient increases, and at the same time the plasma beta ($\beta = p/((B|^2)/2)$ shown in green) decreases. This combined effect causes the red term to increase and allows the criterion to be satisfied. There is often a delay in the initiation of this instability as the magnetic field builds up at $z = 0$ and spreads horizontally allowing the plasma $\beta$ to lower. Here we can see that the criterion for the magnetic buoyancy instability is satisfied when the plasma
\( \beta \) drops to unity as predicted by Murray et al. (2006). Only at this point, when the instability is satisfied, does the field rise above the photosphere and emerge into the corona. As the plasma \( \beta \) has now dropped below one, the magnetic pressure exceeds the background pressure, allowing the field to expand rapidly into the corona. This is made easier by the exponential decrease in pressure with height in the atmosphere.

Due to the expansion of the magnetic field into the corona, plasma drains from the top of the emerging bubbles and flows down fieldlines to the photospheric plane. To explore this, a plot of the weighted density excess, which is calculated as \( \rho(x = 0, y = 0, z_{\text{axis}}) - \rho(x = -50, y = -50, z_{\text{axis}}) \) divided by its initial magnitude, is shown in Fig. 4.3. Moving up through the solar interior, the surrounding plasma density decreases, and hence the excess decreases in magnitude until the tube is over-dense with its surroundings when it reaches the photosphere. This is an improvement on the cylindrical case, where in the experiments performed by Murray et al. (2006) the tube became over-dense much lower in the solar interior. This is a consequence of the geometry of the toroidal loops allowing for efficient draining of plasma.

One important distinction between the cylindrical and toroidal model is that the cylindrical tube was initiated by a density deficit that made the tube maximally buoyant at the centre and reduced towards the edges. However due to the exponential profile of the density deficit in the cylindrical case, the edges of the tube are made weakly buoyant which allows the sunspots to drift continually until they reach the edge of the box. This is not the case in the toroidal simulations, whereby the sunspots drift to a fixed distance, namely the major diameter of the torus. The separation of the sunspots is shown in Fig. 4.4. To estimate the separation of the sunspots, we have plotted the separation in the \( y \) direction between the maximum and minimum of \( B_z \), and find that it levels off after \( t = 60 \). This result is corroborated by observational studies, including Kosovichev and Stenflo (2008) and Wallace Hartshorn (2012). Through the analysis of a sample of active regions from a study of 715 active regions, Kosovichev and Stenflo (2008) found a general trend where polarity separation increases to a maximum and then starts to very gradually decrease. Wallace Hartshorn (2012) analysed the polarity separation of 57 active regions in her PhD thesis and found a similar trend. The polarity separation reached a maximum and started to level off with a slight decrease. This highlights one of the advantages of the toroidal model over the corresponding cylindrical model.

Figure 4.4: The \( y \)–separation in time of the maximum and minimum of \( B_z \) at the base of the photosphere \( (z = 0) \).
4.3 Rotation analysis

In order to examine and quantify the rotational movements around the vertical axis of the sunspots in this general experiment, we investigate numerous quantities involving the plasma and magnetic field. Furthermore, as well as investigating the horizontal velocities at the photosphere, we aim to explore the ramifications this has for both the interior and atmospheric field, with a particular emphasis on the twist of the magnetic field.

4.3.1 Evolution of magnetic field

In an attempt to visualise the fieldlines in the experiment, we have included three figures illustrating the evolution of the interior portion of the field as the flux tube emerges in Fig. 4.5. Three fieldlines are traced from the locations \((0, -14, -25)\), \((0, -15, -25)\), and \((0, -16, -25)\) coloured in blue, black, and red respectively. The black fieldline represents the axis of the flux tube defined at \(r = 0\). At the beginning of the experiment (see Fig. 4.5a) there are three full turns of twist in the flux tube. In Fig. 4.5b at \(t = 40\), the flux tube reaches the photosphere and the flux tube’s legs start to straighten. At this time, approximately half a turn of twist is contained within the emerged section, which subsequently expands into the corona. However, there is still a considerable amount of twist submerged; approximately a full twist in each leg. Later, the submerged twist unwinds resulting in a final state with virtually straight fieldlines in the interior as evidenced by Fig. 4.5c. Of course this is merely a visual estimate and we calculate the fieldline twist more accurately later in this chapter.

![Figure 4.5: Visualisation of the field in the interior at times (a) \(t = 0\), (b) \(t = 40\), and (c) \(t = 100\) respectively as traced from the lower negative footpoint (left). A movie of this figure is included in the electronic version.](image)

In addition to analysing the interior field, we present a general overview of the magnetic field at the photosphere, analysing the direction in which the field is twisted and how the field evolves with time. Synthetic magnetograms are shown at the base of the photosphere as shown in Fig. 4.6, with the horizontal fieldline vectors overplotted in red. The sources first appear at a slight angle to the North-South (left to right) direction. Very quickly, the sources separate, sunspot tails form (see Archontis and Hood, 2010), and the sources drift towards the East-West (top to bottom) direction. Due to the toroidal configuration, the sources separate to a fixed distance as the footpoints are anchored at the base of the domain (see Fig. 4.4). Analysing
Figure 4.6: Coloured contours of the $z$-component of the magnetic field with red horizontal fieldlines superimposed on top. These plots act as synthetic magnetograms where white represents the positive vertical field and black represents the negative vertical field.

the projection of the fieldlines on the photosphere, an interesting feature to note is that the sunspots are not completely circular. The fieldlines actually exhibit an $S$ shaped configuration, spiralling up from the upper sunspot with a counter-clockwise motion and spiralling down to the lower sunspot exhibiting a clockwise motion. This corresponds to a right-hand twisted field as we set up the field with a positive twist parameter $\alpha$. If we started with a straight, untwisted magnetic field in the shape of a semi-torus and wanted to create our initial field configuration, a clockwise rotation of both footpoints would be required. This is a helpful concept to keep in mind when we discuss the rotation at the photosphere. For a positive $B_z$, a right-hand twisted field appears to rotate clockwise when viewed from above (see Fig 4.17 for an example of a right-hand twisted field).

In an effort to understand the interaction of the magnetic field and plasma flows at the photosphere, we compare the horizontal magnetic fieldline arrows with the horizontal velocity arrows at $z = 0$, as shown in Fig. 4.7a. In Fig. 4.7b, some individual fieldlines have been drawn for comparison and to highlight the S shaped structure of the photospheric field. In the lower negative polarity source, the velocity and magnetic field arrows appear to be in the same direction threading around the sunspot in a clockwise direction. In contrast, for the upper positive polarity, the magnetic field and velocity field arrows are in opposite
4.3 Rotation analysis

Figure 4.7: (a) Comparison of the projection of the magnetic field on the horizontal plane (red) and the projection of the velocity field (blue) at $t = 50$, and (b) fieldlines of the vector $(B_x, B_y, 0)$ at $z = 0$ traced from the minimum and maximum of $B_z$ at $t = 50$.

directions. The velocity vectors again follow a clockwise pattern whereas the magnetic fieldlines spiral out counter-clockwise. The velocity arrows correspond to a clockwise rotation of the plasma at the photosphere. This is investigated later as we analyse the movement of plasma in more detail.

4.3.2 Rotation angle

To build on our analysis of the horizontal photospheric magnetic field, a calculation of the angle of rotation at the photosphere is necessary. As stated in Section 1.6, the rotation angle is an observable quantity and is therefore of interest to both modellers and observers. In order to calculate this, we have again traced three fieldlines from the base of the simulation domain to the photosphere in an attempt to track the general behaviour of fieldlines threading the sunspots. The axis of the flux tube has been traced from the lower negative footpoint as well as two fieldlines either side of the axis in the $y$–direction, i.e. we have traced fieldlines from $(0, -14.5, -25)$, $(0, -15, -25)$, and $(0, -15.5, -25)$ coloured in blue, black, and red respectively. Note, the axis fieldline threads through the centre of the sunspot. A schematic of the traced fieldlines is shown in Fig. 4.8a and Fig. 4.8b for times $t = 40$ and times $t = 80$ respectively. As evident from Fig. 4.8, the outer fieldlines coloured in red and blue appear to move around the central black fieldline over the course of the experiment. Both the red and blue fieldlines (henceforth referred to as the outer fieldlines) appear to have rotated through an angle of at least $\pi$ radians over 40 normalised time units. This visual estimation is not sufficient and hence we calculate the rotation angle more rigorously.

In order to track selected fieldlines undergoing this rotation, we have traced the $x$ and $y$ coordinates of the locations of the red, black, and blue fieldlines as they pass through the photospheric plane with time. The fieldlines are traced using a fourth-order Runge-Kutta scheme (a numerical technique used to solve ordinary differential equations). The $(x, y)$ trajectories of these fieldlines are shown in Fig. 4.9a. Initially,
4.3 Rotation analysis

Figure 4.8: Visualisation of the axis of the flux tube (black fieldline) as well as two fieldlines (red and blue) spaced either side of the axis for comparison at (a) $t = 40$ and (b) $t = 80$. A movie of this figure is included in the electronic version.

the three fieldlines drift outwards in a line as the sunspots separate. Subsequently, the fieldlines start to move across the photosphere more slowly then start to move around one another. This figure is not very helpful in quantifying the rotation as it is hard to envisage how the outer fieldlines move with respect to the central fieldline as they all translate across the photosphere as the sunspots separate. To rectify this issue, we consider the relative positions of the outer fieldlines with respect to the central axis by redefining $\bar{x} = x - x_{\text{axis}}$ and $\bar{y} = y - y_{\text{axis}}$, as presented in Fig. 4.9b. This plot is much more helpful in visualising the rotation and indicates that the outer fieldlines have in fact rotated around the central fieldline axis by almost $360^\circ$. This is a significant rotation similar in magnitude to those seen in observations (see Section 1.6).

With the $x$ and $y$ coordinates of the photospheric intersections of select fieldlines stored, we can easily calculate the angle of rotation as

\[ \tan \phi = \frac{y_0 - y_{\text{axis}}}{x_0 - x_{\text{axis}}}, \tag{4.2} \]

where $x_{\text{axis}}$ and $y_{\text{axis}}$ are the $x$ and $y$ coordinates of the axis of the tube (black fieldline) and $x_0$ and $y_0$ are the coordinates of the outer fieldline we are investigating, e.g. the red or blue fieldline. In Fig. 4.10 a schematic has been included to help us visualise the rotation angle $\phi$. This shows that as the blue fieldline moves clockwise the angle $\phi$ will decrease passing through zero when it is in line with the axis. The rotation angle for the red fieldline is calculated in exactly the same way. Through the use of Eq. 4.2, the angle $\phi$ is calculated for the outer red and blue fieldlines as displayed in Fig. 4.11a. As the red and blue fieldlines are initially equally spaced on either side of the axis, the rotation angles are $\pi$ out of phase at the beginning as expected. Both fieldlines undergo a rotation of between $7\pi/4$ and $2\pi$ over 90 normalised time units. Specifically, the red fieldline undergoes a rotation of $340^\circ$ and the blue fieldline undergoes a rotation of $353^\circ$ before the experiment is terminated. These rotations can certainly be seen as significant given that the motion is not prescribed and is a direct result of the twist contained within the tube. The specific driver of this rotation is investigated in Section 4.3.3.

Another interesting aspect to explore is the rate of change of the angle of rotation, $d\phi/dt$, as shown in
4.3 Rotation analysis

Figure 4.9: (a) The trajectories of the fieldlines as they pass through the photospheric plane coloured with increasing intensity as time progresses and (b) the relative trajectories with the location of the black axis subtracted. The colour scale on the right shows the times during the evolution.

Figure 4.10: Representation of angle $\phi$ with the black, red, and blue dots representing the photospheric intersections of the respective fieldlines.

Fig. 4.11b. Although this illustrates that different regions of the sunspot are rotating at slightly different rates, both fieldlines are found to rotate most quickly from approximately $t = 40$ to $t = 70$ where we find a peak in the rotation rate. This peak in rotation rate occurs at about $t = 44$ for the red fieldline and at about $t = 62$ for the blue fieldline. The rate of rotation diminishes as the experiment proceeds until it reaches zero indicating that the fieldlines have essentially stopped rotating. The reason behind the final rotation angle value is an interesting concept to explore. Is the interior field completely untwisting or is the twist per unit length tending to a constant along the field? This is investigated in later sections.

In order to investigate if the sunspot is rotating as a whole, i.e. that the rotation angle does not depend on the radius of the fieldline, we check the assumption of solid body rotation. To proceed, we note that the velocity in the $\phi$ direction, at radius $r$, is given by

$$v_\phi = r \frac{d\phi}{dt},$$
4.3 Rotation analysis

and the $z-$component of the vorticity is given by

$$
\omega_z = (\nabla \times \mathbf{v})_z = \frac{1}{r} \frac{\partial}{\partial r} (rv_\phi) - \frac{1}{r} \frac{\partial v_r}{\partial \phi} \approx \frac{1}{r} \frac{\partial}{\partial r} (rv_\phi) = \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \frac{d\phi}{dt} \right).$$

It is worth noting that we have ignored the term $\partial v_r/\partial \phi$ as the sunspots are essentially axisymmetric. If we assume that the rotation is solid body, and hence that $\phi$ does not depend on $r$, we can relate the vertical vorticity and the rate of change of the angle $\phi$ by

$$
\omega_z = \frac{1}{r} 2r \frac{d\phi}{dt} \Rightarrow \frac{d\phi}{dt} = \frac{\omega_z}{2}.
$$

(4.3)

To check if the assumption of solid-body rotation is valid, we investigate Eq. 4.3 by plotting $d\phi/dt$ and $\omega_z/2$ for both the red and blue fieldlines. In both panels of Fig. 4.12, the two terms are approximately in phase with each other suggesting that Eq. 4.3 is approximately valid and the rotation angle may not have a large dependence on the radius from the axis of the tube. This analysis is very simple and brief and hence...
further investigation is necessary in order to state whether the rotation is solid body given that different fieldlines appear to rotate at slightly different rates.

Now that we have calculated the angle of rotation and rate of rotation for two specific fieldlines, we can generalise this to considering a larger selection of fieldlines traced from a footpoint within a given radius. This can then be averaged to gain a more accurate representation of the rotation rate and in turn we can use this to investigate further how the rotation rate varies with radius. To achieve this, we have traced 100 fieldlines from within a circle on the base with centre \((x_c = 0, y_c = -15)\) and radius one. We have, in fact, traced fieldlines from four different radii on the base within the left footpoint. A schematic of the starting locations of the traced fieldlines is shown in Fig. 4.13. The fieldlines are coloured by their starting radius as described in Fig. 4.13. The time evolution of the rotation angle for the 100 traced fieldlines is shown in Fig. 4.14a and displays that all fieldlines show the same general trend, though they all appear at different locations on the surface and hence have varying initial angles. The average of this set of fieldlines is shown in Fig. 4.14b where we have subtracted off the initial angle and the final average angle through which the fieldlines rotate is 394°.

For completeness, we have also shown a plot of how the rotation angle and rate differ with radius in Fig. 4.15. As evident from this plot, the radius within the sunspot has little effect on the rotation angle observed. This supports our earlier argument that the rotation is indeed a solid body motion. However, there is perhaps a slight trend with the fieldlines traced from the outer edge of the footpoint rotating slightly slower during the peak rotation phase. This is in support of observations from Yan and Qu (2007) where the authors found the greatest rotation rate in the umbra.

### 4.3.3 Driver of rotational motion

Now that we have established a clear rotation at the photosphere, we discuss the underlying cause for a photospheric rotation when a twisted magnetic structure emerges. As mentioned in Section 1.6, two possible mechanisms introduced by Min and Chae (2009) need to be tested in order to understand the
4.3 Rotation analysis

Figure 4.14: Evolution of (a) the angle of rotation for 100 fieldlines traced from footpoint of radius one on the base of the simulation domain and (b) the average angle of rotation.

Figure 4.15: Evolution of (a) the average rotation angle for the four radii, namely $r = 0.24$ as shown in black, $r = 0.48$ as shown in blue, $r = 0.71$ shown in green and $r = 0.95$ shown in red and (b) the corresponding rotation rate.

source of this rotation, as follows:

Torque-driven rotation

Min and Chae (2009) suggested observed rotational motions may be real horizontal motions caused by net torque. The fundamental source of this torque and in turn rotational motion is the behaviour of the Lorentz force. We can think of torque as a measure of the tendency of a force to rotate an object about an axis. Torque is defined as $\mathbf{r} \times \mathbf{F}$, where $\mathbf{r}$ is the displacement vector from the axis of the sunspot and $\mathbf{F}$ is any given force. To investigate the effect of the Lorentz force, we follow the argument introduced in Cheung and Isobe (2014) and consider a closed curve lying on the photospheric plane enclosing some point $P$ denoting the location of the maximum of $B_z$. This has been checked and is representative of the location of the axis of the sunspot.
Let us consider the torque due to various forces acting on the plasma and magnetic field through the surface confined by this closed contour, and, in the process, correct a statement made in Cheung and Isobe (2014), in which they considered a contour integral instead of a surface integral. Explicitly, the surface integral of the torque due to gas pressure ($\tau_P$), magnetic pressure ($\tau_{MP}$), and magnetic tension ($\tau_{MT}$) are given by

\[\begin{align*}
\tau_P & = \iiint \mathbf{r} \times \nabla (-p_{gas}) \cdot d\mathbf{S}, \\
\tau_{MP} & = \iiint \mathbf{r} \times \nabla \left(-\frac{B^2}{2}\right) \cdot d\mathbf{S}, \\
\tau_{MT} & = \iiint \mathbf{r} \times \left[(\mathbf{B} \cdot \nabla)\mathbf{B}\right] \cdot d\mathbf{S},
\end{align*}\] (4.4)

where $\mathbf{r}$ is the displacement vector of a point on the curve from $P$. Let us focus on the surface integral of the torque due to the magnetic and gas pressure. For generality, we consider the surface integral of torque, $\tau_F$, caused by a force of the form $\mathbf{F} = \nabla f$ as this describes the form of both the magnetic and gas pressures. Using the vector identity

\[\mathbf{r} \times \nabla f = f \nabla \times \mathbf{r} - \nabla \times (f \mathbf{r}),\]

and noting that $\nabla \times \mathbf{r} = 0$, we can rewrite the surface integral as

\[\begin{align*}
\tau_F & = \iint \mathbf{r} \times \nabla (f) \cdot d\mathbf{S} \\
& = -\iint \nabla \times (f \mathbf{r}) \cdot d\mathbf{S} \\
& = -\oint_C f \mathbf{r} \cdot d\mathbf{l}.
\end{align*}\]

We note the use of Stokes’ theorem to convert the surface integral to a contour integral in the last line of the equation. Now, if $\mathbf{r} \cdot d\mathbf{l} = 0$ we can state that contributions to the surface integral of torque from gas and magnetic pressures vanish. However, this is only true for specific contours, for example circular contours. Also, if the magnetic pressure gradient is symmetric, other contours may give zero values. Square contours, on the other hand, may give non-zero contributions due to the nature of $\mathbf{r} \cdot d\mathbf{l}$ for this contour. This clarifies the argument made in Cheung and Isobe (2014) where they did not highlight the assumption that this result only holds for specific contours.

To proceed, we consider a closed circular contour and integrate the torque due to the magnetic forces introduced above. Hence, we find the torque contributions from the magnetic pressure and gas pressure forces through this surface vanish and any non-zero surface integral of torque is due to the magnetic tension force. Explicitly, $\tau_P = 0$ and $\tau_{MP} = 0$ and any non-zero contribution is from $\tau_{MT}$, as described in Eq. 4.4. To verify this result numerically, we have calculated the surface integral of torque due to magnetic tension and magnetic pressure within a circular contour of radius 2.5 surrounding the location of the maximum of $B_z$, as displayed in Fig. 4.16. In this case, it is clear that there is no contribution from the magnetic pressure force. Consequently, we speculate that the driving motion of the rotation at the photosphere may be governed by the unbalanced torque produced by the magnetic tension force. This is characteristic of a torsional Alfvén wave which we will discuss in more detail later in this chapter. Overall, the surface integral of torque is predominantly negative indicative of the force generating a clockwise motion. However, later, at the end
of the experiment the surface integral of torque due to magnetic tension changes sign. We speculate that this is not due to a change in sign of the rotation direction but rather a damping of the clockwise rotation, similar to what we observed in the rotation angle. The experiment should be performed for longer in order to investigate whether we find a rotation in the opposite sense.

Caution must be taken when interpreting this result as it is important to note that we have chosen a particular shape of contour. This result is not robust to using different contours as alluded to earlier. We have also calculated the surface integral of torque within a square contour and find that although there is a non-zero contribution by magnetic pressure, it is significantly smaller than that of the tension and is in the opposite direction.

**Apparent rotation**

As noted earlier, Min and Chae (2009) also speculated that the observed rotation of sunspots due to flux emergence may be an apparent effect when a twisted field rises and each fieldline appears at a slightly different position at the photosphere. To demonstrate this effect, we have included a schematic in Fig. 4.17 to illustrate how the vertical rise of a twisted flux tube might manifest itself as a rotation of the fieldlines. In the panel on the left, there is a screenshot of a vertical twisted flux tube with red and blue fieldlines twisted around a straight black axis fieldline intersected by a green plane. The lower panel displays the same figure as viewed from above with the intersections of the field through the plane coloured according to the intersecting fieldline. In the middle panel, the green plane has been lowered to imitate the vertical advection of the flux tube. In the lower middle panel, the intersections according to the figure above are shown in red and blue. This allows us to see the horizontal movement of the fieldlines as the green plane is lowered or equivalently as the flux tube rises. Similarly, in the right panel, the plane has been lowered again and we can again see the apparent movement of the fieldline intersections with the photosphere. It is quite clear in this case that the fieldlines appear to be rotating in a clockwise direction.
4.3 Rotation analysis

Figure 4.17: Schematic to illustrate the phenomena known as “apparent rotation”. The top panel displays three screenshots of a twisted flux tube with three fieldlines highlighted in red, black and blue and the bottom panel shows their intersection at the green plane. The black fieldline represents the axis of the flux tube.

To estimate the contribution to the rotation by apparent effects, we quantify the vertical advection of the flux tube by averaging the vertical velocity over the area where $B_z > 3/4\max(B_z)$ to obtain an average denoted by $\langle v_z \rangle$. To find an upper bound for our estimate, we take the vertical speed of the tube to be the maximum of $\langle v_z \rangle$ through time and assume that the vertical leg has a full turn of twist at $t = 40$ when the field intersects the photospheric plane. This is equivalent to the field being advected vertically by 2.4 units by the end of the experiment, resulting in an “apparent” 34.6° rotation as the fieldlines intersect the photosphere as governed by their helical structure. We note that this is an over-estimate for the apparent rotation angle as we have taken the maximum velocity for all time, and yet this is still significantly smaller than the calculated rotation angle. In addition, this estimate assumes the field remains twisted throughout the experiment, which is not the case. From our preliminary analysis of the interior field, it appears to be untwisting. This low estimate for the apparent rotation helps us dismiss this theory and explain the rotation in our simple experiment as a dynamical, rather than geometrical, consequence of the emergence of flux, with the torque driving the fieldlines to rotate on the photospheric boundary.
Now that we have analysed the evolution of the magnetic field at the photosphere and studied the rotation angle, it is important to analyse the movement of the plasma in this plane. Thus the vorticity, calculated as the curl of the plasma velocity, is examined given that this quantifies the rotation of the plasma. As we are concerned with horizontal velocities in the photospheric plane, the vertical component of the vorticity is of interest as it measures rotation in the $x-y$ plane. This is expressed as

$$\omega_z = (\nabla \times \mathbf{v})_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}. \quad (4.5)$$

The sign of $\omega_z$ is vital to our interpretation of the rotation of the plasma. A positive $\omega_z$ represents a counter-clockwise motion while a negative $\omega_z$ represents a clockwise motion. As discussed earlier, if this twisted field was created from a straight field, the footpoints would have been rotated in a clockwise motion.

In order to visualise the field, we refer the reader to an earlier schematic of selected fieldlines after 40 normalised time units shown in Fig. 4.5b. At this point, the legs of the flux tube have started to straighten out as the tube emerges and almost resemble two cylindrical tubes originating at $z = -25$ and intersecting the photosphere at $z = 0$. As described earlier in the chapter, although our magnetic field is initially twisted, a clockwise rotation of both footpoints of an untwisted toroidal magnetic field would create a similar twist profile, i.e. an injection of negative vorticity creates a magnetic field with positive helicity. Hence, one may expect that a counter-clockwise rotation would be needed to untwist this magnetic field. However, this is not the case at the photospheric plane at least. As the magnetic field is fixed at the base of the simulation domain, a clockwise rotation at the photosphere unwinds the twisted cylinder-like magnetic fieldlines within the interior. However, this same rotational motion at the photosphere adds twist to the atmospheric magnetic field.

Three coloured contours of the $z$-component of vorticity are displayed in Fig. 4.18 at times $t = 40$, $t = 60$, and $t = 80$ respectively. For reference, several line contours of $B_z$ have been overplotted in black to display the location of the sunspots. At the centre of each of the polarities, a red concentration of negative $\omega_z$ appears, corresponding to a clockwise rotation. This accumulation of strong $\omega_z$ appears when the field first emerges and builds with time until it peaks at about $t = 60$ before decaying as the experiment continues. This suggests there is some bulk rotation of the sunspots, similar to the sunspot
4.3 Rotation analysis

rotations observed by Brown et al. (2003) and Yan et al. (2009). We suggest that these sunspot rotations are
due to the untwisting of the interior field injecting twist into the atmosphere. We also note that there is a
red streak of negative vorticity between the sunspots. Streaks of vorticity are more likely to correspond to
shearing motions than rotational motions. Therefore we explain these longer streaks by shear flows between
the sunspots. In addition, the blue tails of positive vorticity located on the inner side of each sunspot are
again typical of a shearing motion. This agrees with the type of shearing we expect when there are two
clockwise rotating bodies in close proximity. Furthermore, we must highlight that the same sign of vorticity
is found within both of the flux concentrations, even though they are of opposite magnetic polarities. As
the field within the spots is inclined oppositely, this results in an uncoiling of the interior field in both legs
of the flux tube.

Now that we have established concentrations of negative vorticity on both sunspots, we try to express
the evolution of vertical vorticity at the photosphere in a more quantifiable manner. We achieve this by
considering the variation of mean vertical vorticity averaged over the area of the upper sunspot where \( B_z \)
is greater than \( 3/4 \) of its peak value, \( \langle \omega_z \rangle \). Explicitly

\[
\langle \omega_z \rangle = \frac{1}{N} \left( \sum_{k=1}^{N} \omega_z(x_k, y_k, z = 0) \right),
\]

where \( x_k \) and \( y_k \) are the \( x \) and \( y \) coordinates of the region where \( B_z > \frac{3}{4} \max(B_z) \) and \( N \) is the number
of points that satisfy this criterion. This is displayed in Fig. 4.19.

![Figure 4.19: Evolution of the mean vertical vorticity \( \langle \omega_z \rangle \) averaged over the area of each polarity concentration where \( B_z \) is above 75% of the maximum of \( B_z \) on \( z = 0 \) as described by Eq. 4.6.](image)

From the average vertical vorticity at the photosphere in Fig. 4.19, it is clear that the vorticity remains
negative throughout the experiment. This is in agreement with our analysis suggesting that the dominant
motion is a clockwise rotation. We find that a clockwise vertical motion appears in each polarity source
when the field reaches the photosphere. The average vorticity quickly rises to a peak in magnitude at \( t = 50 \)
soon after the emerged field becomes vertical and the photospheric footprints have reached their maximum
separation. The horizontal velocity at this time is approximately 0.5 in magnitude which corresponds to
a physical horizontal velocity of 3.4 km/s. Soon after the peak, the vorticity steadily begins to drop in
4.3 Rotation analysis

magnitude towards zero. Again, it would be interesting to extend this general experiment to see if this is in fact the end of the rotation. This significant vortical motion twists up the emerged fieldlines in the atmosphere transporting twist from the tube’s interior portion to its stretched coronal portion.

Although not included here, a similar analysis can be performed on the azimuthal velocity field. We find that there are clearly very strong azimuthal flows rotating the plasma in a clockwise direction in the sunspot, similar to the sunspot rotations seen in observations. This helps us to dismiss the apparent rotation argument brought forward earlier. Though our analysis of the plasma flows and vorticity do give us a useful insight into the rotational properties of plasma at the photosphere, further examination is necessary in order to study the distribution of twist, helicity, and energy across the flux system.

4.3.5 Current density

As a measure of twist of the magnetic field, we study the $z$ component of the current density, given by

$$j_z = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y},$$

as this describes the twist of the magnetic field in the $x - y$ plane. We consider coloured contours of $j_z$ at a height half way down the solar interior at $z = -12.5$ and at the base of the photosphere, as shown in Fig. 4.20. The coloured contours of $j_z$, in Fig. 4.20, shows that although most of each sunspot is dominated by one sign of $j_z$, the outer edge is dominated by the opposite sign of $j_z$. As the initial sub-photospheric flux tube is isolated, and hence surrounded by unmagnetised plasma, Faraday’s law requires that the flux tube must carry no net current (see Section 1.3.2). As the flux tube carries current inside the tube due to its coiled structure, a region of reverse current surrounds the sunspot to ensure a zero net current in the overall area. In the top panel of Fig. 4.20, there is some evidence that the two concentrations of strong $j_z$ centred on the interior legs of the tube are depleting with time. Similarly, considering the bottom panel of Fig. 4.20, although the concentrations of $j_z$ at the photosphere intensify when the field first emerges the concentrations diminish as the experiment proceeds. As $j_z$ is equivalent to the twist of the field in the $x - y$ direction and is linked to the azimuthal magnetic field, a decrease in $j_z$ may indicate a decline in the amount of twist. The reverse current is now spread in the thin rims on the outside of the flux tube.

The time evolution of the maximum value of $j_z$ for specific heights below the photosphere is displayed in Fig. 4.21. There is an initial increase in the maximum of $j_z$ for all heights due to the emergence of the field before a steady decline as the experiment proceeds. This steady decrease can be explained by one of two mechanisms. The decrease could be caused by the expansion and stretching of the field as it emerges or by a decline in the amount of twist stored in the field. The stretching of the field results in a decrease in the gradients of $B_x$ and $B_y$, and lowers $j_z$. To evaluate the extent of the expansion of the field, we estimate the diameter of the contour $B_z = 1$ as shown in Fig. 4.22. At first, the separation increases for heights near the photosphere as the flux tube buoyantly rises and the legs of the tube straighten. Later, there is very little change in the separation of the legs for heights deep in the solar interior. There is however some expansion of the magnetic field for the photospheric height $z = 0$ and 5 units below the boundary as expected as the magnetic field expands into the low density atmosphere. This helps us disregard the expansion of field as a
4.3 Rotation analysis

![Figure 4.20: Coloured contours of $j_z$ at the plane in the middle of the interior ($z = -12.5$) in the top panel and at the solar surface ($z = 0$) in the bottom panel for the specified times, as well as line contours of $B_z$ for comparison of the size of sunspots.](image)

cause for the drop in the maximum of $j_z$ deep in the interior and instead explain it solely by an untwisting of the field. To reiterate, a decrease in $j_z$ is equivalent to a decline in the azimuthal component of the field. However, a decrease in $j_z$ at the photosphere is not due a decline in twist, but rather an expansion of the field above the photosphere.

To conclude our discussion of current, we examine the total $j_z$ within one of the sunspots. As we noted from the coloured contours, although the centre of each sunspot is predominantly one sign of current, the outer boundary consists of reverse current. Therefore, the total positive or negative current does not give us an accurate representation of the current within the spot. Instead, we estimate the current in the centre of the upper sunspot by averaging the vertical current over the area where the vertical magnetic field is greater than $3/4$ of its maximum in a similar fashion to our calculation of the average vorticity. The evolution of $\langle j_z \rangle$ is shown in Fig. 4.23 for the interior plane ($z = -12.5$) and the photospheric plane respectively. At a plane located at the centre of the interior in Fig. 4.23a, $\langle j_z \rangle$ generally decreases. However, as the legs straighten there is a slight increase in $\langle j_z \rangle$ as the negative outer boundary plays a less significant role. In contrast, in Fig. 4.23b at the photospheric plane, $\langle j_z \rangle$ increases as field emerges then drops as the magnetic bubble expands.
4.3 Rotation analysis

In order to estimate the twist of the magnetic field within different subvolumes of the domain, we calculate the twist of individual fieldlines as introduced in Section 1.3.3. The fieldlines are given by

\[ r \frac{d\psi}{dn} = \frac{B_\psi}{B_n}, \]

and the amount by which a given fieldline is twisted is

\[ \Psi_{total} = \int d\psi = \int_0^L \frac{B_\psi}{rB_n} \, dn, \]

along the length of a fieldline, L, in a local cylindrical coordinate system \((r, \psi, n)\). Note, this is not the fieldline twist per unit of axial length as introduced in Eq. 1.22 but rather the total fieldline twist along the length of the field. This can be related to the number of turns around the fieldline axis, \(N_{total}\), by \(\Psi_{total} = 2\pi N_{total}\). Note, this is not a straightforward calculation due to the curvature of the toroidal field.
4.3 Rotation analysis

Figure 4.23: Time evolution of the vertical current \( \langle j_z \rangle \) averaged over the area of the positive polarity flux source where \( B_z \) is greater than 75\% of its maximum on (a) the \( z = -12.5 \) plane and (b) the \( z = 0 \) plane.

and hence the curvature of the axis.

Figure 4.24: Schematic of twisted toroidal loop with representative planes drawn perpendicular to the axis fieldline with (a) the full view and (b) a close up of a particular plane.

To illustrate this method, consider a set of fieldlines that have been traced using a Runge-Kutta fourth-order scheme. For simplicity, let us suppose the axis fieldline is traced using \( m \) steps. In order to calculate \( \alpha = B_\psi / r B_n \), we need to convert to a new coordinate system where the direction of the axis replaces \( \hat{z} \). In order to do this, we need to define a set of \( m \) planes in which the plane’s normal component is tangential to the fieldline axis at each point along the axis. A set of representative planes are drawn for reference in Fig. 4.24a. Note, we actually define a plane at each small step we take along the axis in the fieldline tracer. We define the normal as

\[
n = \left( \frac{dx}{dn}, \frac{dy}{dn}, \frac{dz}{dn} \right) = \frac{B}{|B|} \text{ on the axis,}
\]
where \( n \) is a parameter defined along the axis fieldline. Using this normal and the location of the axis, we can define a plane at each point along the axis with the normal orientated in the direction of the axis. The equation of the plane is given by

\[
(r - r_{axis}) \cdot n = (x - x_{axis}) \cdot \frac{dr}{dl}_{axis} + (y - y_{axis}) \cdot \frac{dy}{dl}_{axis} + (z - z_{axis}) \cdot \frac{dz}{dl}_{axis} = 0.
\]

We then check where the other fieldlines of interest (for example, the red or blue fieldline in Fig. 4.24) intersect each of the \( m \) planes. From this point, we are working in a plane orientated with the normal and we have the intersections of the axis and the other fieldlines of interest as shown in Fig. 4.24b.

Before we transform into the local cylindrical coordinate system, we first convert to a new Cartesian system \((x, s, n)\). \( \hat{x} \) is simply the transverse direction across the field as before and \( \hat{n} \) is the direction of the normal to the plane, along the axis. The new coordinate \( \hat{s} \) is defined as \( \hat{n} \times \hat{x} \) to complete the orthogonal system. Both \( \hat{x} \) and \( \hat{n} \times \hat{x} \) lie in the plane and \( \hat{n} \) is perpendicular to the plane. We convert to this system using

\[
s = \pm \sqrt{(y - y_{axis})^2 + (z - z_{axis})^2},
\]

where the sign of \( s \) is determined by which half of the plane it lies in as separated by the \( x \) axis. This allows us to define the magnetic field in this coordinate system as

\[
B = B_x \hat{x} + B_s \hat{s} + B_n \hat{n},
\]

where

\[
B_s = B \cdot (\hat{n} \times \hat{x}) \quad \text{and} \quad B_n = B \cdot \hat{n}.
\]

Finally to get into the local cylindrical system \((r, \psi, n)\) expressed at the beginning of this section, we use the following coordinate transformation:

\[
r = \sqrt{(x - x_{axis})^2 + s^2} = \sqrt{(x - x_{axis})^2 + (y - y_{axis})^2 + (z - z_{axis})^2},
\]

\[
\psi = \arctan \left( \frac{s}{x - x_{axis}} \right) = \arctan \left( \pm \sqrt{(y - y_{axis})^2 + (z - z_{axis})^2} \right),
\]

and furthermore, we calculate the magnetic field in the \( \psi \) direction, corresponding to the azimuthal field within the plane, using

\[
B_\psi = B_s \cos(\psi) - B_x \sin(\psi).
\]

With \( B_n, B_\psi, \) and \( r \) now prescribed, we can integrate along the axis fieldline to gain the expression

\[
\Psi_{total} = \int \Psi = \int_0^L \frac{B_\psi}{r B_n} \, dn.
\]

This is the total twist within the toroidal loop, and by dividing this expression by \( 2\pi \) we obtain the number
4.3 Rotation analysis

of turns around the axis.

This method has been checked for the simple twisted cylinder used to illustrate the apparent rotation caused by the vertical advection of a twisted tube shown in Fig. 4.17. Note, in the case of a straight cylinder, the method returns to a \((r, \psi, z)\) coordinate system as the axis lies in the \(z\)-direction. The method correctly identified that the tube contained \(5/\pi\) turns of twist. To test the effectiveness of this method for calculating the fieldline twist in the toroidal model, we can estimate the initial twist of the field in this experiment. The initial twist is defined as \(\Psi_{\text{initial}} = \alpha L\), where \(\alpha\) is the twist per unit length and \(L\) is the length of the axis. In this case \(\alpha = 0.4\) and \(L\) is half of the circumference of the torus \((L = R_0 \pi = 15\pi)\). Therefore \(\Psi_{\text{initial}} = 0.4 \cdot 15\pi = 6\pi\). This results in three full turns of twist in the initial field. To verify this, we have plotted the \(x\) and \(y\) locations of the fieldlines against one another as shown in Fig. 4.25. We have stretched out the arched fieldlines vertically ignoring the \(z\) variation as this allows us to readily estimate the twist. From this schematic, it is quite clear that the red and blue fieldlines wrap around the black axis three times. The numerical method correctly calculates \(\alpha = B_\psi/rB_n = 0.4\) along the toroidal loop, within each of the planes. Furthermore, when we integrate this along the field, we correctly identify the loop contains 3 turns of twist.

![Figure 4.25: Schematic of initial fieldlines with the black axis traced from \((0, -15, -25)\), the red fieldline traced from \((0, -15.5, -25)\) and the blue fieldline traced from \((0, -14.5, -25)\).](image)

Note, this method assumes that the coordinate \(\hat{x}\) lies within the plane and hence that the axis does not vary spatially in the \(\hat{x}\) direction. This is valid for early times, but later the axis does start to kink in the atmosphere. However, this is not a huge problem for us as it is not the total twist of the flux tube that we are interested in at later times, rather the twist within the interior of the domain. This allows us to understand how the rotation at the photosphere affects the interior magnetic field. Rather than just considering a couple of representative fieldlines, we have instead traced 100 fieldlines from the left footpoint within a radius of unity (see Fig. 4.13). We then average the fieldline twist, given by

\[
\Psi_1 = \langle \Psi_i \rangle = \frac{1}{100} \sum_{i=1}^{100} \Psi_i,
\]

where \(\Psi_i = \int_{y<0,z<0} B_\psi/rB_n \, dn\) is evaluated for each fieldline \(i\). Therefore, the average number of turns
Figure 4.26: Time evolution of \( N_I \) calculated in the left leg below \( z = 0 \), averaged over 100 fieldlines traced from a footpoint of radius one.

the fieldlines makes around the axis in the interior is given by

\[
N_I = \frac{\Psi_I}{2\pi}.
\]

The time evolution of \( N_I \) is shown in Fig. 4.26. As expected, the number of turns of twist in the interior, \( N_I \), decreases over the course of the experiment after it emerges. This proves the theory that rotational motions at the photosphere extract twist from the interior transporting it to the atmosphere. Although this method relies on the fact that the axis does not vary in \( x \) and we know that there is a slight variation in \( x \) at later times, the method seems to calculate the interior fieldline twist interior accurately. Initially, there appears to be just over one full turn of twist in the interior whereas towards the end of the experiment there is very little twist present, only about a fifth of a turn of twist to be precise. There is, however, a levelling off of \( N_I \) from \( t = 40 \) to \( t = 50 \). This is most likely caused by the straightening of the field.

Unfortunately, we cannot accurately calculate the fieldline twist within the atmospheric section of the tube as the axis kinks from the transverse \( x \) direction and the assumptions we require to change into our local planar coordinate system are no longer valid. Hence, this has been excluded from our discussion of fieldline twist. Now that we have an accurate representation of the evolution of interior twist, we seek to understand what controls the final state of twist across the field.

4.3.7 Force-free parameter

To study the composition of twist along specific fieldlines, an estimate of the local rate of twist is presented. Understanding how the local twist rate varies along the length of fieldlines helps us to understand the mechanism controlling the final level of rotation. Consider the quantity, \( \alpha_L \), normally referred to as the force-free parameter or sometimes the fieldline torsion parameter,

\[
\alpha_L = (\nabla \times \mathbf{B}) \cdot \mathbf{B} / B^2.
\]
Note, we have added a subscript $L$ to differentiate the force-free parameter, $\alpha_L$, and the twist parameter in our model, $\alpha$. The force-free parameter (usually referred to as $\alpha$) is often used in the literature as a proxy for the twist (see numerical experiments such as Fan, 2009) and as a measure of the twist associated with active regions in observational studies (see Hahn et al., 2005 and Liu et al., 2014). For a force-free magnetic field, $j \times B = 0$ is satisfied and hence $j = \nabla \times B$ is parallel to $B$ so we can write

$$\nabla \times B = \alpha_L B.$$  \hfill (4.7)

The geometrical meaning of $\alpha_L$ for a force-free field can be derived as follows. Integrating both sides of Eq. 4.7 and assuming a constant $\alpha_L$ yields

$$\int \nabla \times B \cdot dS = \alpha_L \int B \cdot dS,$$

or equivalently, using Stokes’ theorem,

$$\oint B \cdot dl = \alpha_L \int B \cdot dS.$$

Hence, $\alpha_L$ is given by

$$\alpha_L = \frac{\oint B \cdot dl}{F},$$

where $F$ is the flux of the tube. In cylindrical coordinates $(r, \psi, z)$, $F = \int B \cdot dS = \pi r^2 B_z$ for a uniform $B_z$ and $\oint B \cdot dl = 2\pi r B_\psi$, assuming $B_\psi$ is constant, simplifying the expression for $\alpha_L$ to

$$\alpha_L = \frac{2B_\psi}{r B_z},$$

or equivalently

$$\alpha_L = 2 \frac{d\psi}{dz}.$$

Hence, for linear force-free fields, the force-free parameter is equal to twice the degree of twist per unit length. In addition, Longcope and Klapper (1997) show that, for a thin flux tube with a flat field profile, $\alpha_L$ can again be shown to be equal twice the twist per unit length. As far as we are aware, these are the only links between $\alpha_L$ and the rate of twist. In our case, the field is not force-free and $\alpha_L$ is not constant for much of our experiment so our interpretation of $\alpha_L$ is not exactly known. However, it is important to note that although the parameter $\alpha_L$ appears in the force-free field equation, the determination of $\alpha_L$ does not assume that the magnetic field is explicitly force-free. Many authors see a direct link between $\alpha_L$ and the twist. Further investigation is required to understand the exact meaning of $\alpha_L$ in an experiment where the field is not force-free. For the remainder of this chapter, we assume that $\alpha_L$ is related to the rate of change of twist.

For visualisation purposes, Fig. 4.27 shows a selection of fieldlines coloured according to their value of $\alpha_L$. Low magnitude $\alpha_L$ values between 0 and 0.2 are shown in blue and high magnitude $\alpha_L$ values between 0.2 and 0.4 are shown in red. Although not shown here, the initial field is coloured red indicating...
4.3 Rotation analysis

Figure 4.27: Visualisation of magnetic fieldlines traced from both footpoints coloured by the parameter $\alpha L$ for (a) $t = 40$ and (b) $t = 100$, such that red represents a strong twist ($0.2 < \alpha L < 0.4$) and blue denotes a weaker twist ($0 < \alpha L < 0.2$). (c) shows the $t = 100$ field from above.

that the field is highly twisted in a uniform fashion. Later, at $t = 40$, the buoyant magnetic field rises and reaches the photosphere. As the magnetic field enters the low density corona it rapidly expands resulting in an initially untwisted state. As $\alpha \sim 1/L$, an increase in the length scale results in a decrease in $\alpha L$. This is a common feature of emergence; see Longcope and Welsch (2000) for further details. Hence, a gradient is now established between the highly twisted interior and untwisted atmosphere. It is this twist imbalance that produces a torque that drives the rotational motions at the photosphere (Fan, 2009). This result has been proven by Longcope and Klapper (1997) and Longcope and Welsch (2000) suggest that this motion will continue until the gradient in $\alpha L$ is removed. It is not particularly clear from this figure if $\alpha L$ tends to a constant. However, it is clear the interior $\alpha L$ drops significantly and in turn the coronal $\alpha L$ increases owing to emergence and the rotational motions. The transport of twist to the atmosphere changes the orientation
of the field to an S-type configuration as shown in the view of the magnetic field from above in Fig. 4.27c.

In order to study this more closely, we consider the variation in $\alpha_L$ with height at four different times as traced along the axis fieldline from the lower left footpoint to the apex of the fieldlines. Note, the number of symbols does not reflect the number of steps taken along the fieldline. The symbols have been plotted less frequently for visualisation purposes. At $t = 0$, in Fig. 4.28a, the magnetic field extends to $z = -10$ at its apex and $\alpha_L$ is constant along the length of the field, equal to twice the initial twist per unit length. This was the relationship quoted for a force-free field above but this still appears to hold for our non force-free initial state. This can be proven for the initial magnetic field used in our experiments. Later, at $t = 40$ in Fig. 4.28b, magnetic flux has reached photospheric levels and as such the apex of the field is now at $z = 5$. As the axis reaches the low density atmosphere, the fieldlines stretch and the twist per unit length drops significantly. There is now a gradient in $\alpha_L$ from highly twisted interior field to stretched coronal field, similar to the trend we saw when colouring the fieldlines by their $\alpha_L$ value. Rotational motions at the photosphere then transport twist from the interior to the atmosphere, smoothing out this gradient. By $t = 110$ in Fig. 4.28d, $\alpha_L$ is heading to a constant, especially in the corona as the field extends high into the atmosphere. However, the interior $\alpha_L$ appears to have dropped below that of the coronal $\alpha_L$. This suggests that $\alpha_L$ may have passed through a constant $\alpha_L$ state and would need to rotate in the opposite sense to equilibrate the interior and coronal $\alpha_L$. This point is crucial as this helps us to understand the factors that

\begin{figure}[h]
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\includegraphics[width=0.4\textwidth]{fig4_28a}
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\includegraphics[width=0.4\textwidth]{fig4_28c}
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\hfil
\subfloat[Time=110]{
\includegraphics[width=0.4\textwidth]{fig4_28d}
\label{fig4_28d}}
\caption{The quantity $\alpha_L$ traced along the axis fieldline against the height from the solar surface, $z$ at times (a) $t = 0$, (b) $t = 40$, (c) $t = 70$, and (d) $t = 110$.} 
\end{figure}
govern the final rotation angle. We return to this concept in Chapter 6.

### 4.3.8 Magnetic helicity

In order to study the distribution of twist across the magnetic flux system, we now calculate the magnetic helicity, as introduced in Section 1.2.5. The magnetic helicity essentially describes the geometric twist or shear of a flux tube. We are concerned with the relative magnetic helicity and hence calculate

\[
H_r = \int (\mathbf{A} + \mathbf{A}_p) \cdot (\mathbf{B} - \mathbf{B}_p) \, dV.
\]

See Section 1.2.5 for a full explanation of this quantity and Appendix A for a description of the two methods used for the numerical calculation of helicity. In this section, we discuss the results using the latter method employed by Moraitis et al. (2014).

![Figure 4.29: Comparison of a selection of fieldlines from (a) the initial twisted magnetic field and (b) the corresponding potential magnetic field sharing the same normal distribution on the boundary.](image)

Using Moraitis’ method (Moraitis et al., 2014), we can calculate the corresponding potential and vector potentials for the magnetic field, and hence the relative magnetic helicity within different subvolumes of the total simulation domain. To emphasise the difference between the initial magnetic field and the corresponding potential magnetic field, we have plotted them side by side in Fig. 4.29. Note, the potential field is the minimum magnetic energy state. The potential field fills a larger portion of the domain as opposed to the confined twisted state of the initial magnetic field.

To investigate the movement of helicity across the domain, we first calculate the atmospheric magnetic helicity above the photosphere, as shown in Fig. 4.30. As expected, the atmospheric helicity is zero until \( t = 20 \) as the field has not yet reached the photospheric plane. When the field reaches the photosphere, there is a linear increase in helicity as twist is steadily injected into the atmosphere by two mechanisms.
4.3 Rotation analysis

We note the transport of helicity to the atmosphere is contributed to by the emergence of flux and the rotational motions at the photosphere. This is discussed in more detail when considering the rate of change of helicity. By the end of the experiment, the magnetic helicity injected into the atmosphere has reached $3.6 \times 10^{23} \text{ Wb}^2 (3.6 \times 10^{39} \text{ Mx}^2)$, typical of a small event. As a point of reference, we compare this value with observations. Min and Chae (2009) quote a helicity transport of $4 \times 10^{42} \text{ Mx}^2$ but it is important to note that they are considering a much larger active region than the one we model in our experiment.

To investigate the normalised magnetic helicity transported to the atmosphere, we divide the atmospheric helicity by $F_{\text{tube}}^2$ and find that the evolution follows the same trend reaching a value of $0.83$ by $t = 120$. This corresponds to almost one full twist of flux transported to the atmosphere as we expect from the angle of rotation calculated earlier.

Figure 4.30: Evolution of the relative helicity $H_r$ when calculated above $z = 0$ in the solar atmosphere.

Figure 4.31: Evolution of the time derivative of the relative helicity $H_r$ when calculated above $z = 0$ in the solar atmosphere using Eq. 4.8. The total time derivative (black solid line) is split into the dissipation term (purple solid line), the surface correction term (yellow solid line), the shear term (red solid line), and emergence term (blue solid line). The dashed black line is the derivative of the curve from Fig. 4.30 calculated using finite differencing.
4.3 Rotation analysis

In addition, we calculate the rate of change of magnetic helicity above the photosphere, both numerically using finite difference and analytically. The analytic expression for the rate of change of magnetic helicity is given in Eq. 1.19 in Section 1.2.5. In the case of the rate of change of helicity in the atmosphere, the outward pointing normal is $\mathbf{n} = -\mathbf{k}$. Therefore the time derivative of helicity above $z = 0$ is given by

$$
\left( \frac{dH_r}{dt} \right)_{z>0} = -2\eta \int_{z>0} \mathbf{j} \cdot \mathbf{B} \, dV + 2 \int_{z=0} \left[ (\mathbf{A}_p \cdot \mathbf{B}) \mathbf{v}_z - (\mathbf{A}_p \cdot \mathbf{v}) B_z \right] \, dx dy
- 2\eta \int_{z=0} (\mathbf{A}_p \times \mathbf{j}) \cdot \mathbf{k} \, dx dy.
$$

We must proceed with caution when considering the rate of change of helicity. See Pariat et al. (2015) for the full derivative including additional terms. Clearly, from Fig. 4.31, the additional terms are not important in this particular case as the flux through the surfaces closely follows the time derivative of the helicity. Pariat et al. (2015) also notes that care must be taken when dividing the helicity flux into individual terms as although their sum is gauge-independent the individual terms are not, hence limiting their physical meaning.

Consider the rate of change of atmospheric helicity, divided into the four main source terms, in Fig. 4.31. During the initial stages of emergence, the helicity flux is dominated by the emergence term (blue). However, later, the horizontal shearing and rotational motions at the photospheric footpoints (red) are the primary sources of helicity change, in agreement with previous studies including Fan (2009) and Liu and Schuck (2012). The contributions by internal helicity dissipation (purple) and the surface correction term (yellow) are much less significant and do not affect the overall change in helicity in the atmosphere. To compare the two approaches for calculating $dH_r/dt$, we have numerically differentiated $H_r$ and plotted this as a dashed line in Fig. 4.31 for comparison. We speculate that the fluctuations in the rate of change of helicity are due to Alfvén waves reflecting off the bottom of the domain.

![Figure 4.32: Evolution of (a) the relative helicity, $H_r$, and (b) the rate of change of helicity, $dH_r/dt$, when calculated below $z = 0$ in the solar interior.](image)

Now that we have confirmed the atmospheric helicity increases monotonically, we apply the same method to the calculation of the magnetic helicity in the solar interior. This is possible with the method described in Moraitis et al. (2014) due to the fact that the determination of the potential field takes into account all boundaries. Our analysis of the fieldline twist shows a depletion of twist in the interior and as
such we expect the magnetic helicity in the interior to decrease due to both the emergence of magnetic flux and the rotational motions on the photospheric boundary. Both the relative magnetic helicity and the change in helicity in the interior region are shown in Fig. 4.32. Throughout the experiment, the interior magnetic helicity decreases in a linear manner. Before the flux reaches the photosphere, the change in helicity is governed by internal helicity dissipation. However, later as the flux emerges, the interior helicity decreases more sharply due to both the emergence of flux and rotational velocities at photosphere.

4.3.9 Magnetic energy

To complement our study of the magnetic helicity, we consider the magnetic energy and its distribution across the domain. Magnetic energy and its role in the total energy equation is discussed in Section 1.2.5. To gain an understanding of the amount of magnetic energy accessible for solar eruptive events, we calculate the free magnetic energy relative to the potential field. Explicitly, we calculate the excess magnetic energy contained within the field by subtracting the energy associated with the potential field with the same normal flux distribution on all boundaries, i.e.

$$E_{\text{free}} = \int \frac{B^2}{2} dV - \int \frac{B^p}{2} dV.$$  

We can again make use of the potential field determined using Moraitis’ code. The evolution of free magnetic energy above $z = 0$ is shown in Fig. 4.33a. Free atmospheric energy builds from the time the field first emerges. At the end of the experiment, the free magnetic energy transported to the atmosphere has reached $8.2 \times 10^{22}$ J ($8.2 \times 10^{29}$ ergs). To investigate the contributions to magnetic energy by flux through the photospheric boundary we consider the Poynting flux through $z = 0$ as given by

$$F_P = \int_{z=0} B^2 v_z \, dx \, dy - \int_{z=0} (\mathbf{v} \cdot \mathbf{B}) B_z \, dx \, dy + \eta \int_{z=0} (\mathbf{j} \times \mathbf{B})_k \, dx \, dy.$$  

The first term in Eq. 4.9 corresponds to the contribution to Poynting flux by vertical flows owing to emergence, the second denotes the generation of magnetic energy by rotational flows and the third term is a result of resistive effects. The vertical Poynting flux (total in black) is split into emergence (blue) and rotational

![Figure 4.33](image-url)
4.4 Summary

The resistive term is shown in purple, but is negligible compared to the other terms. The rate of increase of energy is largest during the initial stages of the experiment due primarily to the emergence term. However, later the rotational term becomes the dominant contributor to magnetic energy increase in the atmosphere. This pattern corroborates the trend that appeared in the helicity flux whereby vertical flows dominate the flux initially and horizontal flows become important later. Keeping in mind the precaution in Pariat et al. (2015) about splitting the helicity flux into individual terms as they are gauge-dependent, the behaviour of the Poynting flux helps us to trust that the helicity flux trend may have physical meaning.

4.3.10 Propagation of torsional Alfvén wave

Our results indicate that the rotational motions we observe may be governed by some form of torsional Alfvén wave. We believe that an upward and downward propagating wave are launched at the instance of emergence due to a gradient in the twist per unit length. The upward propagating wave transports twist from the highly twistier interior to the stretched coronal field. The travel time for an Alfvén wave to propagate from the photosphere to the base of the domain is approximately \(20\) normalised time units. This suggests that an Alfvén wave would take approximately \(40\) normalised time units to travel down to the base, reflect and return to the photospheric plane. We propose that sunspot rotation will be largest during the time this propagating Alfvén wave returns to the photosphere. The rotation will only slow down once the reflected wave has returned to the photosphere. This appears to be in fairly good agreement with Fig. 4.19 where the rapid rotation and large \(|\omega_z|\) occurs from about \(t = 50\) to \(t = 90\).

4.4 Summary

In this chapter, we present a 3D MHD numerical experiment of the emergence of an arched twisted flux tube from the solar interior, through the photosphere where sunspots are formed and into the solar corona. The primary objective of this chapter is to consider the general evolution of the experiment and to investigate the various manifestations of sunspot rotation. Through our analysis of this particular experiment, there is strong evidence that the toroidal flux tube’s interior field untwists while at the same time the atmospheric field is twisted by means of a rotation of both polarities at the photosphere.

Our detailed investigation involved examining various quantities relating to both the magnetic field and plasma. Our analysis of the plasma flow vorticity at the photospheric plane revealed that significant vortical motions develop in the centre of the sunspots. A definitive rotation of the outer field around the sunspot centre is also demonstrated by tracking the fieldlines and calculating the rotation rate of the fieldlines within the sunspot. Rotations of the order of one full rotation \(360^\circ\) are observed in our experiment. This is similar in magnitude to the angles of rotation reported in studies that concluded a direct relationship between quick sunspot rotation and enhanced eruptive activity (Brown et al., 2003, Yan and Qu, 2007, Yan et al., 2009 etc.). In observations the rotation angles seen were in general observed over a period of days. However, in our experiment the rotation occurs over the course of about forty minutes so our timescales, and hence the rate of rotation, are clearly not in line with what we observe. We believe this is linked to the size of the
4.4 Summary

... emerging active region we are modelling. The sunspots in the active region we model are only about 2 Mm wide, much smaller than a typical active region. If we scale up our experiment to a more typical active region, we predict the timescales to be in line with observations. This is investigated later in Section 5.4.

The combination of flux emergence with the continual rotation of the photospheric footpoints transports magnetic helicity and energy from the solar interior to the atmosphere. We note that $8.2 \times 10^{22}$ J of magnetic energy is transported to the atmosphere over the course of the experiment. To understand the main contributors to magnetic energy transport to the atmosphere, we also investigate the Poynting flux of energy across the photospheric boundary. This is split into horizontal shearing and vertical emergence terms. Initially, the flux of energy across the photospheric boundary is dominated by emergence but latterly the dominant contributor is the horizontal shearing. The rate of change of relative magnetic helicity atmosphere also has two primary sources: namely helicity flux due to emergence and helicity flux due to rotational motions. This follows a similar trend to the magnetic energy with the initial predominant source being the emergence of the magnetic flux tube but later the flux due to rotational motions at the photospheric level is dominant. The magnetic helicity transported to the atmosphere reaches a value of $3.6 \times 10^{19}$ Mx$^2$. As well as the production of helicity in the atmosphere, we find a clear decrease in the magnetic helicity in the interior, supporting our understanding that this portion of the field undergoes an untwisting motion as also evidenced by a clear decrease in the vertical current and, in turn, the azimuthal magnetic field in the interior.

To quantify this untwisting motion, we also calculated the twist of individual fieldlines within an interior section of the flux tube. We found that the interior twist drops from $1\frac{1}{5}$ turns around the axis at the point of emergence to a fifth of a turn by the end of the experiment. To try and understand the mechanism controlling this final state of twist, we also investigated the twist per unit length, $\alpha_L$, along the axis. We found that this tended to a constant in the atmosphere but found that the interior $\alpha_L$ dropped below that of the corona. This suggests that the sunspots may have over-rotated and may need to rotate in the opposite sense to allow the twist per unit length to reach a constant as predicted by Longcope and Welsch (2000). We cannot make concrete conclusions about this as the experiment has not been executed for enough time units. However, we return to this idea in Chapters 5 and 6.

Furthermore, we wish to try and understand why the rotation rate we calculate in our simulation is much larger than those calculated in observations. There are many possibilities for this discrepancy, including the size of active region which was discussed earlier. In addition, varying the strength or twist of the tube may change the time it takes for the flux tube to rise to the photosphere and hence govern the rotation rate of the tube. The model presented by Longcope and Welsch (2000) predicts that the level of rotation will depend on the rapidity of flux emergence so we plan to investigate how this affects the rotation. The length of time for the rotation may also be related to the depth at which the flux tube is anchored; this is another approach that requires investigation. These ideas are explored in Chapter 5.
Chapter 5

Effects of varying the field strength and twist of an emerging flux tube

The results of this chapter have been published in

*Sunspot rotation. II. Effects of varying the field strength and twist of an emerging flux tube, Z. Sturrock and A. W. Hood, Astronomy and Astrophysics, 593 (2016)*

In this chapter, we aim to investigate the impact of varying the initial field strength and twist of the sub-photospheric magnetic flux tube, introduced in Chapter 3, on the rotation of the sunspots at the photosphere. Similar parameter studies have been conducted in the past, with a different focus. However, most studies use the cylinder field definition described in Section 3.3.1. Many studies have focused on understanding the effects of varying the degree of twist on the rise of flux tubes through the convection zone (Moreno-Insertis and Emonet, 1996; Emonet and Moreno-Insertis, 1998; Abbett et al., 2000; Cheung et al., 2006). Specifically, they aimed to understand the effect of twist on the fragmentation of flux tubes during their rise. In addition, Dorch (2003) considered instabilities that arise as tubes rise through the convection zone due to different amounts of initial twist.

Fan et al. (2003) performed a more comprehensive parameter study in which the authors considered the effects of variations in the magnetic field strength and twist of a flux tube rising through a convective flow. The authors consider the effect of downdrafts on the emergence of twisted flux tubes, considering how the distortion of the tube can be affected by a change in field strength or twist of the tube. Cheung et al. (2007), on the other hand, describe a set of realistic simulations in which they consider the effects of these parameters as the flux tube approaches and passes through the solar surface.

Murray et al. (2006) provides the most extensive parameter study with the cylindrical field definition in which the authors investigate the effects of varying the magnetic field strength and twist of a sub-photospheric tube with the aim of understanding how these parameters affect the evolution of the flux tube on its rise to the atmosphere. They found a self-similar evolution in the rise and emergence of the tube.
when the magnetic field strength is varied. However, they did not find such a simple self-similar evolution
when varying the twist due to the non-linear interaction between the twist and the tension force acting on
the tube. Nevertheless, if the field strength or twist is low enough, they found the flux tube cannot fully
emerge into the atmosphere.

Another interesting parameter study was carried out by MacTaggart and Hood (2009) where they used a
different initial condition, namely the toroidal flux tube introduced in Chapter 3, which we intend to use in
our parametric study. Although the parameters used in this study do have some overlap with the parameters
used in ours, there are five different parameter runs and the aim of the authors is very different. MacTaggart
and Hood (2009) aimed to study the evolution of the axis of the flux tube, as well as considering the
sunspot separation, making clear comparisons with the cylindrical model. However, we aim to understand
the effects of these parameters on the rotation of the photospheric polarities.

Although sunspot rotation has been a very attractive topic to both observers and theorists in recent years,
the rotation’s dependence on field strength and twist of the initial sub-photospheric field has, to the best of
our knowledge, been left unexplored. Given that there are no current observations of sunspot rotation for
varying initial strengths and twist, the results we find based on this simple model should be checked against
future observations. We can, however, make predictions on the effect of these parameters from previous
studies (for example Murray et al., 2006 and MacTaggart and Hood, 2009). For instance, we predict that the
twist of the sub-photospheric tube should have a substantial effect on the rate at which the sunspots rotate
as we expect a larger rotation for more highly twisted fields, as there is more twist stored in the interior
field to unwind. However, as we discuss later, the density deficit’s non-linear dependence on the twist $\alpha$
will complicate this effect. In addition, we predict the role of the magnetic field strength may not be so
important to the magnetic flux tube’s rotation at the photosphere. Nonetheless, the density deficit is directly
proportional to the field strength of the tube squared and, hence, the stronger fields emerge more fully in
our experiments. This allows the axis to align vertically which may impact on the rotation.

In this chapter, we present results from 3D MHD simulations of buoyant magnetic flux tubes rising
through the solar interior and emerging at the photosphere. We are particularly interested in the rotational
motions of the photospheric footpoints of the tube, i.e. the sunspots. We have varied two of the parameters
defining the magnetic structure of the sub-photospheric flux tube, namely the magnetic field strength at the
axis, $B_0$, and the twist of the tube, $\alpha$. Our aim is to identify the effect of these parameters on a number
of quantities relating to the rotation of the sunspots. Furthermore, we seek to understand the process
controlling the amount of sunspot rotation. We hope to discover that the amount of rotation at the surface
may tell us something about the interior magnetic field when analysing observations. To determine the
individual effect of these parameters, we vary the field strength and twist independently of each other.

The remainder of this chapter is laid out as follows. In Section 5.1, we introduce the two parameter
groups we choose for the parametric study. Section 5.2 and Section 5.3 present the results of varying the
magnetic field strength and twist of the flux tube respectively. In Section 5.4, we compare the rotation
angle magnitude and timing with specific observations and discuss reasons for discrepancies between the
two. Finally, we conclude the chapter with a summary of the results and main conclusions in Section 5.5.
5.1 Parameter choice

In this study, we fix the base of the computational domain at \( z_{\text{base}} = -25 \). The major radius of the torus is \( R_0 = 15 \) and the minor radius is \( a = 2.5 \) for all experiments (same structure as in Chapter 4). We choose to vary the magnetic field strength at the axis of the interior tube, \( B_0 \), and the degree of twist, \( \alpha \). The twist is assumed positive in all experiments, ensuring that all tubes are right-hand twisted in the initial set-up. Each fieldline rotates about the axis through an angle of \( \alpha \) radians over one unit of distance along the axis. The initial set-up of a representative experiment is summarised in Fig. 3.9b, with \( B_0 = 9 \) (axial field strength of 11700 G) and \( \alpha = 0.4 \) (three full turns of twist in interior tube).

The experiments are split into two groups: Group 1 where \( \alpha \) is kept fixed and \( B_0 \) is varied and Group 2 where \( B_0 \) is fixed and \( \alpha \) is varied. A summary of the \( B_0 \) and \( \alpha \) values under consideration is given in Table 5.1. We note we only consider a relatively small range of \( B_0 \) values as a consequence of our model choice. As found in previous parameter studies (see MacTaggart and Hood, 2009), if we pick a lower \( B_0 \) value, the flux tube will fail to fully emerge, and if we choose a much higher \( B_0 \) value, we may encounter unphysical negative pressures. In fact, if we increase the initial axial field strength to \( B_0 = 12 \), we encounter negative pressure in this particular initial set-up. It is, however, important to note that the range of allowable field strengths will vary depending on the background stratification chosen for the experiment. The twist values range from \( \alpha = 0.2 \) corresponding to a turn and a half of twist to \( \alpha = 0.4 \) that corresponds to three full turns of twist in the initial field. The total flux threading a cross section of the tubes we study range from \( 3.7 \times 10^{11} \text{ Wb} \) (\( 3.7 \times 10^{19} \text{ Mx} \)) to \( 7.4 \times 10^{11} \text{ Wb} \) (\( 7.4 \times 10^{19} \text{ Mx} \)). This is typical of a small active region or large ephemeral region.

Table 5.1: Parameter space under investigation.

<table>
<thead>
<tr>
<th>Group 1</th>
<th>Group 2</th>
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<tbody>
<tr>
<td>( B_0 = [5, 6, 7, 8, 9, 10] )</td>
<td>( B_0 = 7 )</td>
</tr>
<tr>
<td>( \alpha = 0.3 )</td>
<td>( \alpha = [0.2, 0.3, 0.4] )</td>
</tr>
</tbody>
</table>

All experiments are performed for different numbers of normalised time units and hence different final times. The Group 1 experiments are carried out as follows: \( B_0 = 5 \) for 216 normalised time units (90 minutes); \( B_0 = 6 \) for 180 normalised time units (75 minutes); \( B_0 = 7 \) for 154 normalised time units (64 minutes); \( B_0 = 8 \) for 135 normalised time units (56 minutes); \( B_0 = 9 \) for 120 normalised time units (50 minutes); and \( B_0 = 10 \) for 108 normalised time units (45 minutes). This ensures all experiments are performed for the same rescaled time, i.e. \( \bar{t} = B_0 t \). More details of rescaling the time follow in Section 5.2. The Group 2 experiments, on the other hand, are all executed for the same number of time units, 120 normalised time units or equivalently 50 minutes.

Before we proceed to discuss the results of our experiments, we analyse how the parameter choices affect the structure of the magnetic field and resulting density. In order to initiate emergence, we need to introduce a density deficit as prescribed in Chapter 3. From the density excess in Eq. 3.13, it is clear that both the twist and magnetic field strength play an important role in controlling the buoyancy of the flux tube. For Group 1, the variation in \( B_0 \) significantly affects the magnetic field strength for all radii until a
5.1 Parameter choice

Figure 5.1: Radial distribution of the initial magnetic field strength, $|B|$, at $z = -25$ for varying (a) $B_0$ and (b) $\alpha$.

Figure 5.2: Radial distribution of the initial density excess, $\rho_{\text{exc}}$, at $z = -25$ for varying (a) $B_0$ and (b) $\alpha$.

significant distance from the axis is reached, i.e. the edge of the tube, as shown in Fig. 5.1a. The initial field strength of the tube is, of course, directly proportional to $B_0$. The buoyancy profile, too, is strongly dependent on $B_0$ because the density deficit is proportional to $B_0^2$. The $B_0 = 10$ tube will therefore be 4 times more buoyant than the $B_0 = 5$ tube (see Fig. 5.2a).

In Group 2, the variation in $\alpha$ has a minimal effect on field strength for all radii, leaving the field strength at the axis of the tube unchanged, as displayed in Fig. 5.1b. However, reducing $\alpha$ increases the rate at which the field strength falls off with radius. The buoyancy profile, on the other hand, is strongly influenced by the value of $\alpha$ as increasing the amount of twist increases the inward acting magnetic tension force more than the outward acting magnetic pressure force, therefore altering the Lorentz force and in turn the density deficit. This results in the higher $\alpha$ cases having a smaller inwardly acting gas pressure gradient and hence being less buoyant at the centre of the flux tube, as demonstrated in Fig. 5.2b. However, the plasma is more buoyant away from the axis at outer radii due to the larger field strength here.
5.2 Group 1 - Varying $B_0$ with fixed $\alpha$

In this group, we fix $\alpha$ at 0.3 (2/4 turns of twist or 1 turn in 3.56 Mm) and vary the field strength at the axis of the tube, $B_0$, from $B_0 = 5$ (6500 G) to $B_0 = 10$ (13000 G) in steps of 1 (1300 G). This allows us to understand the isolated effect that the magnetic field strength of the sub-photospheric tube has on the rotation of the sunspots. Altering the initial field strength, $B_0$, changes the evolution of the field in two ways. Firstly, changing $B_0$ alters the initial density deficit thereby controlling the speed at which the flux tube rises through the solar interior (see Fig. 5.2a). Secondly, the tube’s evolution is altered on its journey from the photosphere. In order for the flux tube to emerge, the magnetic buoyancy instability must be triggered which occurs when the plasma $\beta$ is lowered to one. This occurs more quickly for stronger fields due to their higher magnetic pressure. For a weaker field, on the other hand, the magnetic pressure is built up more slowly as the flux tube is squashed and the field is spread at the photosphere. Rotational motions are manifested in several different ways, and hence we investigate a variety of different quantities. Before delving into the rotational properties, we first analyse the general evolution of the magnetic field.

5.2.1 General evolution

![Figure 5.3: The evolution of (a) the scaled mean of $B_z$ on the $z = 0$ plane and (b) the scaled mean of $B_h$ on the $z = 0$ plane over rescaled time for the parameters outlined in the legend for Group 1.](image)

To try and understand the influence of the interior magnetic field strength on the evolution of the tube as it rises, we consider how this affects the magnetic field strength at the photosphere. We consider two proxies for the magnetic field strength at the photosphere, namely the mean vertical field strength, $\langle B_z \rangle_{z=0}$, and the mean horizontal field strength, $\langle B_h \rangle_{z=0}$. Both expressions have been plotted in Fig. 5.3a and Fig. 5.3b over rescaled time for all six Group 1 simulations as coloured by the key. We calculate the mean by averaging over the photospheric region where $B_z > 3/4 \times \text{max} (B_z)$, in a similar manner to quantities calculated in the previous chapter. Hence, we assume that contributions from the positive sunspot is included and that weaker undular field regions outside the spots are excluded. This has been compared with other proxies for the magnetic field, such as the maximum field strength and we found the same general behaviour in this case. In order to take into account the density deficit’s dependence on $B_0$, we rescale the horizontal axis by redefining time as $\bar{t} = B_0 t$. This is equivalent to measuring time on an Alfvén timescale rather than...
5.2 Group 1 - Varying $B_0$ with fixed $\alpha$

a sound timescale. Before we proceed, we note that the symbols on this plot and all subsequent plots do not reflect the spatial or temporal resolution of the experiments. Unless stated otherwise, symbols are only plotted every five grid points (or time units) for visualisation purposes.

All three components of the initial magnetic field are proportional to $B_0$ (see the expression for the magnetic field in Eq. 3.14) and as such we might expect that the field will still be proportional to $B_0$ when the tube reaches the photosphere. Interestingly, from Fig. 5.3a, we find that the vertical field strength at the photosphere can instead be scaled by $B_0^2$, suggesting that stronger initial fields tend to concentrate and strengthen in the vertical direction at the photosphere. We find that stronger fields emerge more fully with a vertical axis and hence possess a larger vertical field, $B_z$, at the photosphere. Flux tubes with weaker fields, on the other hand, tend to spread at the photosphere before the magnetic buoyancy instability is initiated. This could be responsible for a smaller than expected $B_z$ at the photosphere for lower $B_0$ values.

For completeness, we have also included the horizontal field strength, $B_h = \sqrt{B_x^2 + B_y^2}$ and find that it is proportional to $B_0$. As we are averaging over the region where $B_z > \frac{3}{4}\max(B_z)$, we do not see the effects of the horizontal expansion of the field for weaker $B_0$ values. It is important that we bear these scalings in mind when analysing later results as the magnetic field is altered on its journey to the photosphere, and hence we may not find the scalings we expect. Observations often only consider the line of sight magnetic field, the vertical magnetic field in our case, and so caution must be used when making deductions about the interior magnetic field.

![Figure 5.4](image.png)

Figure 5.4: The evolution of the height of the leading edge of the flux system for Group 1 over (a) unscaled time and (b) rescaled time.

In order to investigate the magnetic field’s journey to the atmosphere, we also consider the leading edge of the flux system over time for the Group 1 cases in Fig. 5.4a. There is clearly a time-lag associated with the weaker fields and so we plot the leading edge of the flux system over rescaled time, $\tilde{t} = B_0 t$ in Fig. 5.4b. We find the evolution to be self-similar until approximately $\tilde{t} = 500$. The $B_0 = 5$ and $B_0 = 6$ case appear to plateau at a fixed height, not reaching the top of the box. The height that the flux tubes reach is determined by pressure balance on the boundary, i.e. where the total pressure equals the background gas pressure. We note that the total pressure is calculated as the sum of the gas and magnetic pressure, i.e. $p_{\text{total}} = p + B^2/2$. Emergence slows for the weaker $B_0$ experiments and there is not enough magnetic pressure to push the boundary upward. Hence, the maximum height reached is lower for weaker field experiments. It is hard
to determine where stronger fields reach their pressure balance boundary as they reach the top of the box during the experiment.

![Graphs showing the evolution of the length of the axis fieldline in the atmosphere above z = 0 for Group 1 over (a) unscaled time and (b) rescaled time.](image)

Figure 5.5: The evolution of the length of the axis fieldline in the atmosphere above z = 0 for Group 1 over (a) unscaled time and (b) rescaled time.

Consequently, the fieldlines in the stronger $B_0$ experiments extend further into the atmosphere and so their axes are longer as shown in Fig. 5.5a. This plot shows the length of the axis fieldline as measured from the point the fieldline enters the $z = 0$ plane to the point where it leaves through $z = 0$ after passing through the atmosphere. Again, we can rescale the x-axis to show rescaled time as shown in Fig. 5.5b. The axis of the $B_0 = 5$ tube appears to plateau at a fixed length, much shorter than that of the stronger $B_0$ tubes given that they extend higher into the corona. This has important consequences later when calculating twist.

### 5.2.2 Torque

In the previous experiment conducted in Chapter 4, a full analysis is performed of the unbalanced torque caused by magnetic forces. This was concluded to be the driver of the rotational motion at the photosphere. If we consider a closed circular curve surrounding the maximum of $B_z$ on the photospheric plane, and calculate a surface integral of the torque within this integral, we find the magnetic tension force to be the only contributor. Explicitly, we find the surface integral of torque due to magnetic forces, $\tau_F$, to be equal to the surface integral of torque due to magnetic tension, $\tau_{MT}$, as follows:

$$\tau_F = \tau_{MT} = \int \mathbf{r} \times ((\mathbf{B} \cdot \nabla)\mathbf{B}) \cdot d\mathbf{S},$$

where $S$ is the surface contained within a circular contour of radius $a = 2.5$ surrounding the maximum of $B_z$. Hence, we speculate that the unbalanced torque produced by the magnetic tension force drives the rotation. This has been plotted in Fig. 5.6a for all of the Group 1 cases. It is not clear if the evolution of the surface integral of torque is showing self-similar behaviour. In order to investigate this, we rescale both quantities as follows in Fig. 5.6b. We again rescale time as $\bar{t} = B_0 t$ and we also redefine the torque integral by scaling it with respect to $B_0^2$ as $\bar{\tau}_{MT} = \tau_{MT}/B_0^2$, given the magnetic tension force is proportional to $B_0^2$. 
5.2 Group 1 - Varying $B_0$ with fixed $\alpha$

As stated earlier, all experiments are executed with different final times, in order to ensure they have the same final $\bar{t}$. This should ensure that we capture the same period of magnetic evolution for all of the varying $B_0$ experiments and that we are not missing some phases of the evolution for lower $B_0$ values. Generally, it appears that all six simulations demonstrate a self-similar torque imbalance which drives the rotation. The largest magnitude of torque is found between $\bar{t} = 200$ and $\bar{t} = 700$, after which the torque decreases suggesting a slowing of the rotation after this point.

5.2.3 Rotation angle

As discussed in Chapter 4, in the general experiment performed, both sunspots experienced significant rotations. This is true for all $B_0$ cases to varying degrees. In order to calculate the rotation angle, we trace the photospheric location of a series of fieldlines using a fourth-order Runge-Kutta method from the base of the computational domain. In particular, we trace the axis fieldline from the centre of the negative flux source at $(0, -15, -25)$ and, as we expect, it follows the centre of the sunspot. In order to calculate the rotation angle, we also trace a selection of fieldlines from the base within a radius of one around the axis. Given the $x$ and $y$ coordinates of the intersections of selected fieldlines through the photosphere, we can calculate the angle of rotation using

$$\phi = \tan^{-1} \left( \frac{y - y_{axis}}{x - x_{axis}} \right),$$

where $(x_{axis}, y_{axis})$ is the location of the axis and $(x, y)$ is the location of another fieldline we have traced. To calculate the angle, we trace 100 fieldlines from a circular footpoint of radius one on the base. This is explained in more detail in Section 4.3.2 and shown clearly in Fig. 4.13. We can then calculate the mean rotation angle by averaging the rotation angle over the traced fieldlines within the footpoint of interest. However, as all traced fieldlines intersect at different locations on the photosphere, and hence have different initial rotation angles, we first subtract off the initial rotation angle and the rotation angle average begins...
at $\phi = 0$ for all cases. The resulting rotation angles are shown in Fig. 5.7a for the six cases we are investigating.

![Graph showing rotation angles](image)

(a)

![Graph showing rescaled rotation angles](image)

(b)

![Graph showing rotation rate](image)

(c)

Figure 5.7: Rotation angles for various $B_0$ cases with (a) the unscaled rotation angles measured over time, (b) the rescaled rotation angles, $\bar{\phi} = \phi/B_0$, measured over rescaled time, $\bar{t} = B_0 t$, and (c) the unscaled rotation rate, $d\phi/dt$ measured over rescaled time.

After the emergence of the fields at the photosphere, there is a short period with little change in rotation angle while the sunspots drift apart. From Eq. 3.13, the buoyancy force is proportional to $B_0^2$ and so flux tubes with larger $B_0$ values appear at the photosphere first. Consequently, the time taken for the flux tubes to reach the photosphere is inversely proportional to $B_0$. To incorporate this, we again redefine time as $\bar{t} = B_0 t$ and rescale the horizontal axis as shown in Fig. 5.7b. We also notice a direct relationship between $\phi$ and $B_0$ so we redefine $\bar{\phi} = \phi/B_0$ and find the scaled rotation angles are approximately similar to each other. The scaled time evolution of the scaled rotation angles is shown in Fig. 5.7b.

In Table 5.2, we select the magnitude of the final angles of rotation for the various $B_0$ cases. The second column of the table contains the unscaled time, $t$, the third column the rescaled time, $\bar{t}$, the fourth the unscaled rotation angle, $\phi$ and the fifth the rescaled rotation angle, $\phi/B_0$, to take into account the rotation angle’s dependence on the magnetic field strength. We have chosen to consider the rotation angles at the final rescaled time of $\bar{t} = 1080$ as we expect the flux tubes to be in a similar stage of their evolution here. This is presented in the fifth column of the table and shows that the final scaled rotation angles are approximately constant.
5.2 Group 1 - Varying $B_0$ with fixed $\alpha$

Table 5.2: Rotation angles for varying $B_0$ experiments with $\alpha = 0.3$

<table>
<thead>
<tr>
<th>$B_0$</th>
<th>$t$</th>
<th>$\bar{t}$</th>
<th>$\phi$</th>
<th>$\phi/B_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>216</td>
<td>1080</td>
<td>124°</td>
<td>25°</td>
</tr>
<tr>
<td>6</td>
<td>180</td>
<td>1080</td>
<td>163°</td>
<td>27°</td>
</tr>
<tr>
<td>7</td>
<td>154</td>
<td>1080</td>
<td>183°</td>
<td>26°</td>
</tr>
<tr>
<td>8</td>
<td>135</td>
<td>1080</td>
<td>211°</td>
<td>26°</td>
</tr>
<tr>
<td>9</td>
<td>120</td>
<td>1080</td>
<td>254°</td>
<td>28°</td>
</tr>
<tr>
<td>10</td>
<td>108</td>
<td>1080</td>
<td>265°</td>
<td>26°</td>
</tr>
</tbody>
</table>

By rescaling $\phi$ with respect to $B_0$, the only varied quantity in this model, we are able to remove any dependency on $B_0$ and reveal a self-similar behaviour as the fieldlines threading the sunspot rotate around the centre. This result may be surprising on first inspection as all flux tubes have the same initial helical structure since the degree of twist, $\alpha$, is constant in this group. Hence, from the structure of the field, we may have expected all experiments to have the same final rotation angle. This suggests that varying the field strength not only affects the timing at which key processes occur but also the amount by which the fieldlines rotate.

Furthermore, the rotation rate, $d\phi/dt$, as displayed in Fig. 5.7c, drops off towards the end of all experiments, suggesting that the rotation might not significantly persist if the experiments were continued. By demonstrating the rotation angle’s dependence on the field strength, this corroborates the theory introduced by Min and Chae (2009) that the rotation is a consequence of the torque on the photospheric boundary rather than by apparent effects. If the rotation was due to apparent effects, altering the field strength of the flux tube would not vary the amount of rotation. A similar analysis has been conducted for the other sunspot with comparable results. The relationship between the rotation angle and $B_0$ and the reasons behind this dependence are investigated in further detail in later sections.

5.2.4 Twist

In order to estimate the twist of the magnetic field, we investigate a number of twist-related quantities. To begin, we calculate the twist of individual fieldlines. To be precise, we consider the fieldline twist of 100 fieldlines originating in a footpoint of radius one surrounding the axis of the sunspot. To determine the twist of a fieldline we calculate

$$\Psi_{\text{total}} = \int \frac{d\psi}{\bar{t}} = \int_{0}^{L} \frac{B_{\psi}}{rB_{n}} \, dn,$$

along the length of a fieldline in a local cylindrical coordinate system $(r, \psi, n)$. Consider a distance of one unit along the axis, henceforth referred to as an axial unit length. Then, $\alpha = B_{\psi}/(rB_{n})$ is the angle through which the fieldlines rotate over one axial unit length. Hence, by summing this quantity over the length of the fieldline, we calculate the angle in radians through which fieldlines rotate over the axial length. We can then deduce the number of turns, $N_{\text{total}}$, the fieldlines pass through over the axial length by noting...
5.2 Group 1 - Varying $B_0$ with fixed $\alpha$

$\Psi_{\text{total}} = 2\pi N_{\text{total}}$.

Using the method described in Section 4.3.6 in the previous chapter, the fieldline twist, and hence the number of turns the fieldlines pass through, is calculated for 100 fieldlines originating from the left footpoint within a radius of one. To obtain a mean value, we average over the number of turns the fieldlines pass through for all fieldlines traced. This is plotted against scaled time in Fig. 5.8a. More precisely, we have plotted average the number of turns, $N_1$, fieldlines pass through within an interior section of a leg of the flux tube, i.e. up until $z = 0$,

$$N_1 = \frac{\Psi_1}{2\pi} = \frac{1}{2\pi} \frac{1}{100} \sum_{i=1}^{100} \Psi_i,$$

(5.1)

where $\Psi_i = \int_{y<0, z<0} B_\psi r B_\n d\n$ is evaluated for each fieldline $i$.

![Figure 5.8](image.png)

Figure 5.8: Evolution of (a) average number of turns, $N_1$, and b) scaled average number of turns $(N_1 - N_1|_{t=0})/B_0$ fieldlines undergo within the interior portion of one leg of the tube measured over rescaled time, $\bar{t} = B_0 t$ for Group 1 cases.

Notice, we are again measuring this quantity over rescaled time to take into account the slower evolution associated with weaker initial field strengths. Initially all fields have the same number of turns around the axis within the interior as expected as they contain the same initial twist. The initial evolution is similar for all cases but the twist drops off more sharply for higher $B_0$ tube cases. This result is again surprising as we may expect the final interior twist to be the same for all cases as they all contain the same amount of initial twist. However, this discrepancy is likely to be related to the expansion of the field in the atmosphere. The stronger $B_0$ cases expand higher into the atmosphere distributing the atmospheric twist along a greater length (see Fig. 5.5a) resulting in a smaller twist per unit length in the atmospheric portion of the field. This produces a larger gradient in the twist per unit length, driving the rotation and untwisting the interior field. Notice, there are large amounts of residual twist in the submerged legs of the tube for the weaker field cases. This is related to the distribution of the twist per unit length across the domain. To scale this quantity against $B_0$, we have subtracted the initial twist $N_1|_{t=0}$ so that $N_1 - N_1|_{t=0}$ begins at zero, and hence can be scaled with $B_0$. This is plotted in Fig. 5.8b, and shows a clear self-similarity with the fieldline twist scaling with $B_0$. 

As mentioned in Chapter 4, we cannot accurately calculate the fieldline twist within the atmospheric section of the tube as the axis kinks towards the transverse $x$ direction and the assumptions we require to change coordinate systems are no longer valid. Hence this is excluded from this study.

In an attempt to explain the large amount of twist left in the interior for lower $B_0$ experiments, we present an analysis of a proxy for the local twist referred to as the force-free parameter, defined as

$$\alpha_L = (\nabla \times \mathbf{B}) \cdot \mathbf{B}/B^2.$$  

This quantity has been shown to be closely linked to the twist per unit length. In fact, in particular circumstances, $\alpha_L$ is equal to twice the twist per unit length (see Section 4.3.7 for full details of $\alpha_L$). We reiterate this is not the $\alpha$ we vary in Group 2 and hence we have denoted this as $\alpha_L$ to differentiate between the two.

In order to visualise the distribution of $\alpha_L$ along different sections of the tube, we have traced $\alpha_L$ along the axis passing through the centre of the sunspot from the left footpoint to the apex of the fieldline. To try and compare the different $B_0$ cases, we have considered a snapshot of $\alpha_L$ along the axial fieldline at three scaled times, $\bar{t} = 0$, $\bar{t} = 450$ and $\bar{t} = 1080$ as shown in Fig. 5.9. Notice, we have plotted $\alpha_L$ against the height of the tube. Hence, the stronger $B_0$ tubes have reached further into the atmosphere at $\bar{t} = 1080$. In addition, one should note that we have plotted the symbols much less frequently in these particular plots due to the large number of steps we have taken along the axis fieldline.

In Fig. 5.9a, at $\bar{t} = 0$, $\alpha_L$ is constant along the axis fieldline at a value of 0.6 (twice the initial twist per unit length). However, this value drops as the field begins to expand into the atmosphere as shown in Fig. 5.9b at $\bar{t} = 450$ soon after the field has emerged. We see that the value of $\alpha_L$ drops off with increasing height for all experiments, and hence a gradient develops in $\alpha_L$. Longcope and Welsch (2000) predicted that this gradient in $\alpha_L$ produces a torque (as we found earlier) that drives the torsional motion of the flux tube intersecting the photosphere. They suggested that the torsional motion will continue until this gradient in $\alpha_L$ is removed.

At the final scaled time, $\bar{t} = 1080$, (shown in Fig. 5.9c) we find the axis of the flux tubes reach higher into the atmosphere. The magnitude of $\alpha_L$ appears approximately constant in the coronal portion of the field, indicating that this section of the field is almost force-free. The low-magnitude in $\alpha_L$ likely occurs because of the stretching and expansion of the field (see Fan, 2009). Interestingly, the magnitude of $\alpha_L$ in the interior is ordered by $B_0$ such that stronger magnetic flux tubes process a lower magnitude of $\alpha_L$ and weaker magnetic flux tubes retain a higher magnitude of $\alpha_L$ in this region. As mentioned earlier, it is often conjectured that the cause of the rotation at the photosphere is related to $\alpha_L$ trying to equilibrate between the twisted interior and the stretched coronal field. However, this is not yet the case in the weaker field experiments as a higher magnitude $\alpha_L$ persists in the interior at the final time. Furthermore, the strongest experiments ($B_0 = 9$ and $B_0 = 10$) appear to have distributed their twist to the stage where $\alpha_L$ is larger in the atmosphere. This suggests that the experiments have passed through an equilibrium state as the tubes have “over-rotated”. The experiments would need to be performed for a longer time to definitively determine whether the twist is equilibrating on a longer timescale or whether the lower strength field cases are unable to unwind their interior twist to match their coronal twist.

It is worth noting that although Fig. 5.9c is plotted at the same scaled time for all experiments, all
experiments appear to be at slightly different stages in their evolution. The reasons behind this are two-fold. Firstly, the weaker fields are equilibrating on a shorter distance and secondly, the Alfvén speed is also proportional to $1/\sqrt{\rho}$. Given the density deficit’s dependence on $B_0$, the density deficit of the tube is non-linearly related to $B_0$ with weaker tubes possessing a larger density. This in turn means weaker tubes have a slightly smaller $v_A$, and as such may be evolving on a slower timescale than predicted by $\bar{t} = B_0 t$.

Additionally, it is worth noting that as the weaker fields do not extend as far into the atmosphere, the axis fieldline is shorter. If the twist per unit length, $\alpha_L$, equilibrates on a shorter fieldline, the average value of $\alpha_L$ will be significantly larger in both the interior and atmospheric regions. Explicitly, if we assume that $\alpha_L$ tends to a constant value and the total twist is conserved, then $\alpha_i l_i = \alpha_f l_f$ where $\alpha_i$ is the initial twist per unit length, $\alpha_f$ the final, $l_i$ the initial axial length, and $l_f$ the final. In this case the predicted final twist per unit length is $\alpha_f = \alpha_i l_i/l_f$. Since all experiments possess the same $\alpha_i$ and $l_i$ values, an increase in $l_f$ decreases the final twist per unit length, $\alpha_f$. In the weaker cases, the final rotation angle will be smaller as the tube does not need to unwind as much interior twist to achieve its larger final twist per unit length. This effect is not yet apparent but again may be seen if the experiments had been performed for a longer time.

In an attempt to predict the long term behaviour of the experiments, we assume that the total twist along
the fieldlines is conserved and take the final axis length to be the axis length at $\tilde{t} = 1080$ shown in Fig. 5.5. Given these assumptions, we calculate the final $\alpha_L$ along the field if the quantity were to tend to a constant (this hypothesis is tested in detail in Chapter 6). $\alpha_L$ is plotted as a function of height in Fig. 5.10 and shows that experiments with weaker magnetic fields have a shorter axial field length, and hence reach a state with a higher twist per unit length. Conversely, experiments with stronger magnetic fields extend higher into the atmosphere and possess a greater axial length and lower twist per unit length. It is worth noting that the total twist is the same for all cases (by assumption), but it is the twist per unit length that varies. In addition, the $B_0 = 9$ case has a lower $\alpha_L$ value than the $B_0 = 10$ case as the final axial length is longest for the $B_0 = 9$ case as shown in Fig. 5.5. This is likely to arise because of boundary issues as the $B_0 = 10$ case reaches the top of the box first.

![Figure 5.10: Final predicted constant values of $\alpha_L$ along axis given final axis length calculated in Fig. 5.5.](image)

### 5.2.5 Vorticity

As discussed in Section 5.2.3, all six simulations in Group 1 exhibit rotation in their sunspots, quantifiable in terms of an angle. To examine this further, we have also calculated the mean vertical vorticity within each sunspot, as given by

$$\langle \omega_z \rangle = \langle (\nabla \times \mathbf{v})_z \rangle = \langle \frac{dv_y}{dx} - \frac{dv_x}{dy} \rangle,$$

where we have averaged over the photospheric region where $B_z > \frac{3}{4} \max(B_z)$. This quantifies the rotation of the plasma within the upper polarity source. The temporal evolution has been plotted for all six Group 1 experiments in Fig. 5.11a.

The average vertical vorticity is consistently negative for all $B_0$ cases indicating that the dominant motion within the sunspots is a clockwise rotation, consistent with the theory suggested in Chapter 4. Precisely, this rotation acts to untwist the interior magnetic field and inject twist into the atmospheric field. A very clear trend develops in that tubes with a stronger initial field strength emerge more quickly and significant vortical motions develop within their sunspot centres. To try and explore the relationship
5.2 Group 1 - Varying $B_0$ with fixed $\alpha$

Figure 5.11: Average vorticity for Group 1 experiments with (a) the unscaled average vertical vorticity measure over time and (b) the rescaled average vorticity, $\langle \omega_z \rangle = \langle \omega_z \rangle / B_0^3$, measured over rescaled time, $\bar{t} = B_0 t$.

between $\langle \omega_z \rangle$ and $B_0$, we have rescaled $\langle \omega_z \rangle$ with respect to $B_0^3$ as well as redefining the time as $\bar{t} = B_0 t$. The rescaled plot is shown in Fig. 5.11b where again we find a self-similar evolution, apart from during the latter stages of the $B_0 = 5$ case. The difference in this case may be because we are not capturing the correct area within which the vorticity lies. In the weak $B_0 = 5$ case, the field spreads over the photosphere until it builds up sufficient field strength to initiate the magnetic buoyancy instability. This is not captured when considering the area where $B_z > 3/4 \max(B_z)$. The $B_0^3$ scaling is surprising and may be related to how we calculate $\langle \omega_z \rangle$. Future studies should repeat this to see if this is in fact a real trend. In this particular case, the scaling itself is not of great significance. More importantly, we should conclude that stronger $B_0$ tubes tend to have stronger vortical motions developing in their polarity sources.

5.2.6 Current density

We consider another estimate for the twist of the magnetic field by calculating the electric current density, specifically the $z$-component,

$$j_z = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y},$$

as this is linked to how twisted the magnetic field is in the photospheric $x-y$ plane. We analyse the vertical current density at two different planes: the photospheric plane, $z = 0$, and a plane at the centre of the interior domain, $z = -\frac{12}{5}$. In this case, we plot the temporal evolution of the maximum of $j_z$ for the $z = 0$ plane in Fig. 5.12a. All cases show an initial peak in the maximum of $j_z$ at the photosphere due to the emergence of the field. The timing of this maximum is clearly dependent on the value of $B_0$ as the emergence timescale is inversely proportional to $B_0$. The magnitude of the peak is also dependent on $B_0$ as we find the peak in the curve to be higher for larger $B_0$ values. This is as we expected given that $j_z$ is proportional to $B_0$. Later, all plots show a steady decline due to the expansion of the field into the higher atmosphere. To investigate the self-similarity in this plot, we rescale the maximum of $j_z$ with respect to $B_0$ and also rescale the time to become an Alfvén time as discussed before. The result of the
5.2 Group 1 - Varying $B_0$ with fixed $\alpha$

Figure 5.12: Maximum of $j_z$ over photospheric $z = 0$ plane for various $B_0$ cases with (a) the unscaled maximum of $j_z$ measured over time and (b) the scaled maximum of $j_z$, $\text{Max of } j_z / B_0$ measured over rescaled time, $\bar{t} = B_0 t$. Additionally, (c) shows the scaled average of $j_z$, $\langle j_z \rangle / B_0$, measured over rescaled time.

rescaling is shown in Fig. 5.12b where we find a clear self-similarity in the curves as they lie on top of one another. From Fig. 5.12c, however, we note that the average of $j_z$ does not directly scale with $B_0$. Stronger field experiments have larger averaged currents than predicted by the $B_0$ scaling, perhaps due to the larger rotation angle and vortical motions seen for stronger experiments.

Similarly, we have plotted the spatial maximum of $j_z$ half way down the solar interior at $z = -12.5$, as displayed in Fig. 5.13a. In all cases, after an initial decrease in the maximum of $j_z$, there is a slight increase due to the straightening of the legs of the tube. However, later there is steady decrease in the maximum of $j_z$. Again, we find the timing of the maximum to be dependent on the initial field strength $B_0$. To take this into account, we again rescale the current and time to produce Fig. 5.13b. A clear self-similarity is seen here.
5.2 Group 1 - Varying $B_0$ with fixed $\alpha$

5.2.7 Magnetic helicity

To investigate the distribution of twist across the domain, we analyse the relative magnetic helicity within different subvolumes of the domain. This quantitatively describes the degree of twist and shear of magnetic fieldlines. For full details on the quantity for magnetic helicity and details of its numerical calculation, see Section 1.2.5 and Appendix A.

Figure 5.14: Relative magnetic helicity calculated within the atmospheric region $z > 0$ for various $B_0$ cases. (a) shows the unscaled helicity measured over time and (b) details the rescaled magnetic helicity, $H_r = H_r / B_0^2$ measured over rescaled time, $t = B_0 t$.

From the general experiment in Chapter 4, we expect a linear increase in magnetic helicity in the atmosphere accompanied by a depletion of magnetic helicity in the interior region. This is a result of the direct emergence of flux and rotations of sunspots at the photosphere that twist and stress the atmospheric field while untwisting the interior portion of the field. Fig. 5.14a considers the temporal evolution of atmospheric helicity for the different cases, and as expected, it increases for all cases. The injection of helicity is clearly ordered by the value of initial field strength. Both $B$ and $B_p$ are directly proportional to
Table 5.3: Final atmospheric magnetic helicity for varying $B_0$ with $\alpha = 0.3$.

<table>
<thead>
<tr>
<th>$B_0$</th>
<th>$H_r$</th>
<th>$H_r$ (Wb$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>922</td>
<td>$1.3 \times 10^{22}$ Wb$^2$</td>
</tr>
<tr>
<td>6</td>
<td>2453</td>
<td>$3.4 \times 10^{22}$ Wb$^2$</td>
</tr>
<tr>
<td>7</td>
<td>4533</td>
<td>$6.4 \times 10^{22}$ Wb$^2$</td>
</tr>
<tr>
<td>8</td>
<td>7683</td>
<td>$1.1 \times 10^{23}$ Wb$^2$</td>
</tr>
<tr>
<td>9</td>
<td>14367</td>
<td>$2.0 \times 10^{23}$ Wb$^2$</td>
</tr>
<tr>
<td>10</td>
<td>21291</td>
<td>$3.0 \times 10^{23}$ Wb$^2$</td>
</tr>
</tbody>
</table>

the axial field strength, $B_0$, in the initial set-up and hence, the vector potentials are also proportional to $B_0$ as evident by the expressions quoted in Eq. A.1 and Eq. A.2. Therefore, the helicity is initially proportional to $B_0^3$. The rescaled atmospheric helicity is plotted against rescaled time in Fig. 5.14b. There still seems to be a larger amount of $B_0$-scaled helicity for larger $B_0$ cases. By doubling the initial magnetic field strength of the sub-photospheric tube, $B_0$, the helicity transported to the atmosphere is increased by over 23 times. As this is a volume-integrated quantity, the larger helicity injection may be explained by stronger fields occupying a larger portion of the volume due to their higher magnetic pressure. In addition, the period of strong rotation injects more helicity than expected by the $B_0^2$ scaling.

Similarly, the interior helicity is plotted in Fig. 5.15a, depicting a reduction in helicity in a similar way. The initial interior helicities are clearly ordered by the value of $B_0$ and we actually find this quantity can be directly scaled by $B_0^2$. We must bear this in mind when interpreting the helicity as magnetic fields with the same amount of twist may have different amounts of helicity as scaled by their initial magnetic field strength. Furthermore, the stronger experiments clearly undergo a sharper drop-off in helicity. Despite this drop-off, the stronger $B_0$ experiments have considerably larger amounts of interior helicity compared with the weaker $B_0$ experiments during the latter stages of the experiment. For instance, in the $B_0 = 10$ experiment, there is $6.23 \times 10^{23}$ Wb$^2$ ($6.23 \times 10^{19}$ Mx$^2$) of helicity left in the interior. This may seem
surprising as helicity is a measure of the twist of fieldlines and the fieldline twist is considerably lower for the \( B_0 = 10 \) case by the end of the experiment (see Fig. 5.8). However, it is important to note that although a non-zero fieldline twist implies a non-zero relative helicity, the opposite is not necessarily true (MacTaggart, 2015). A magnetic field may have a non-zero helicity without containing a large amount of twist. For example, an untwisted magnetic field may be stretched and not exactly potential and hence may possess a non-zero relative helicity.

We again rescale the helicity by \( B_0^2 \) as we have plotted in Fig. 5.15b. In this case, the helicity demonstrates self-similar behaviour, unlike the atmospheric helicity. The change in helicity by resistive dissipation is much larger in the solar interior, so the scaling with \( B_0^2 \) shows a better fit. For more details on the dominant contributors to the change in magnetic helicity, including the resistive dissipation term, see Section 1.2.5 or we refer the reader to Pariat et al. (2015) for full details of additional terms included in the rate of change of helicity.

### 5.2.8 Magnetic energy

Finally, an analysis of the magnetic energy is presented to assess how much free energy is produced by the rotational motions at the photosphere. To quantify the distribution of energy across the domain, we calculate the free magnetic energy above \( z = 0 \), as

\[
E_{\text{free}} = \int \frac{B^2}{2} \, dV - \int \frac{B_p^2}{2} \, dV,
\]

where \( V \) is defined as the volume above \( z = 0 \). This is essentially the excess energy stored in the field as we have subtracted off the energy stored in the potential field with the same normal distribution on the photospheric boundary. Both the unscaled and scaled free magnetic energy are plotted against unscaled and scaled time in Fig. 5.16a and Fig. 5.16b respectively. The self-similar evolution is followed strictly for the first 275 scaled time units. However, the different \( B_0 \) cases deviate from the original trend suggesting a
different $B_0$ dependence for later times. This agrees with the trend we see in the helicity with larger than expected amounts of energy transported to the atmosphere for stronger initial fields. As the diffusion time is independent of $B_0$, we see greater diffusion for weaker experiments as they have run for a longer unscaled time. This may explain the drop in free magnetic energy in the latter stages of the experiment. The free magnetic energy transported to the atmosphere ranges from $7.2 \times 10^{21}$ J to $6.9 \times 10^{22}$ J over the range of $B_0$ values.

5.3 Group 2 - Varying $\alpha$ with fixed $B_0$

In Group 2, we fix $B_0$ at 7 (an axial field strength of 9100 G) and vary the initial twist of the tube, $\alpha$, as shown in Table 5.1. We vary $\alpha$ from $\alpha = 0.2$ (1 turn in 5.34 Mm) to $\alpha = 0.4$ (1 turn in 2.67 Mm). Using a similar approach to the last group, we pinpoint the effect that the initial twist, $\alpha$, has on the rotation of sunspots. We again investigate a variety of different features related to the rotational movements at the photosphere.

The twist of the magnetic field of the tube alters the magnetic tension force acting on the tube. When we modify the degree of twist, $\alpha$, the change in magnetic tension force alters the magnetic buoyancy profile of the tube. This was discussed earlier in Section 5.1 where Fig. 5.2b displays a comparison of the density deficit for different values of $\alpha$. This reveals that as the value of $\alpha$ is reduced the axial region becomes more buoyant but the surrounding plasma is less buoyant due to the smaller field strength here (see Fig. 5.1b). Therefore, we need to consider that the buoyancy profile is non-linearly altered in this group suggesting that $\alpha$ may have a more complex effect on the dynamics of the experiment. The density excess at the axis is given by

$$\rho_{exc} = \frac{B_0^2 (\alpha^2 a^2 - 2)}{4T(z)}.$$  

As discussed before, this density excess varies with $r$ and with height due to the monotonic increase in temperature moving down to the base. The non-linear dependence of the density deficit on $\alpha$ makes it difficult to rescale the time to remove this effect as we did in the Group 1 case. For brevity, we have excluded the methods behind calculating the quantities in this section. Unless stated otherwise, all quantities are calculated as introduced in Group 1 (see Section 5.2).

5.3.1 General evolution

Previous studies have investigated the degree of twist required for flux tubes to rise through the solar interior without suffering distortion. This is because a degree of twist provides a tension force that protects the magnetic field from breaking up during its rise. Murray et al. (2006) found that tubes with a low twist ($\alpha = 0.1$) can be distorted due to its weaker tension force enabling material to be dragged from the front of the tube towards its rear, as described by Moreno-Insertis and Emonet (1996). In all cases we analyse, the tubes are sufficiently twisted to rise through the solar interior. Before we analyse the rotational properties
of the magnetic flux tubes within the Group 2 experiments, we first investigate how the axis of the tubes behave throughout the experiments. In Fig. 5.17, we have plotted both the time evolution of the height of the axis and the length of the axis fieldline above $z = 0$ in Fig. 5.17a and Fig. 5.17b respectively. We find that the axis of the tube with the smallest $\alpha$ value reaches the photosphere in the shortest amount of time. This is because the tube has the largest buoyancy force at the axis and a lower tension force throughout.

However, the axes of the $\alpha = 0.3$ and $\alpha = 0.4$ tubes overtake the $\alpha = 0.2$ tube reaching higher into the atmosphere. The differences here lie with the timing and spatial distribution of the initiation of the magnetic buoyancy instability. The $\alpha = 0.2$ tube has a smaller azimuthal field and the fastest decreasing axial field strength due to the horizontal expansion of the tube. This is similar to that found with the cylinder model in Murray et al. (2006). This means that the magnetic buoyancy instability is satisfied later, and its emergence is less rapid. On the other hand, the larger field strength of the $\alpha = 0.4$ tube combined with the fact that it reaches higher in the photosphere means that the external gas pressure is lower and the plasma beta is reduced to allow the initiation of the magnetic buoyancy instability. This allows the axis to emerge to very large heights into the atmosphere by the end of the experiment, thus with a greater axial length.

![Figure 5.17](image)

Figure 5.17: The evolution of (a) the height of the axis and (b) the length of the axis fieldline above $z = 0$ for Group 2 experiments.

Given that the axis of the $\alpha = 0.2$ tube rises the most quickly through the interior, it could be assumed that the front of this tube will also rise the most quickly since the velocity behind it gives it momentum. However, from the height of the leading edge (or front) of the system shown in Fig. 5.18, there is little difference in the rate of the rise through the interior. This may be related to the less buoyant plasma on the outside of the $\alpha = 0.2$ tube. The horizontal expansion at the photosphere is less constrained for weakly twisted tubes due to lower tension force throughout. In addition, the $\alpha = 0.4$ tube has a stronger field initially and throughout, and hence has a higher magnetic pressure which allows it to expand more fully into the atmosphere. This allows the leading edge of the $\alpha = 0.4$ case to reach higher than the other two tubes.
5.3 Group 2 - Varying $\alpha$ with fixed $B_0$

5.3.2 Rotation angle

From Fig. 5.19, it is clear that the mean rotation angle, calculated in the same way as in Section 5.2.3, has some dependence on the initial twist, $\alpha$. Changing the initial degree of twist changes the helical structure of the fieldlines and means that fieldlines traced from the same location on the base appear at different locations at the photosphere in all three cases. To try and account for this effect, we have artificially moved all the starting angles to 0 and the subsequent evolution has been shifted. The time at which the field reaches the photosphere is also affected by $\alpha$ with the $\alpha = 0.2$ tube reaching the photosphere first due to its larger axial buoyancy. In Fig. 5.19, there is clearly some trend in that the sunspots in the higher $\alpha$ experiment undergo a larger rotation. If we surmise that the rotation of sunspots is due to a propagation of twist, and we expect the rotation to attempt to equilibrate the twist imbalance, we predict the rotation angle to be largest for the highest twist case.

Figure 5.19: Rotation angles measured over time for various $\alpha$ cases, as depicted in the key.
Summarised in Table 5.4, at the end of the section, are the final rotational angles at $t = 120$. Notice, in this case all experiments have been run for the same amount of time and we do not rescale $t$. This clearly shows some ordering of the rotation angle with the parameter $\alpha$. By doubling the initial twist, the final rotation angle is more than doubled. However, the relationship is not as clear as we found in Group 1 due to the more complicated effect of $\alpha$ on the tube’s evolution.

5.3.3 Twist

In order to investigate the distribution of twist along fieldlines in the three Group 2 cases, we have plotted the twist per unit length, $\alpha_L$, along the axial fieldline at four different times, similar to the analysis performed for Group 1. As we are unable to rescale time to take into account the density deficit’s dependence on $\alpha$, we are forced to plot $\alpha_L$ against height on the same plot at the same time $t$ for different $\alpha$ cases. For clarity, we must reiterate the difference between $\alpha$ and $\alpha_L$. The twist parameter we vary in the model is $\alpha$ and $\alpha_L = (\nabla \times \mathbf{B}) \cdot \mathbf{B}$ is the force-free parameter that gives a measure of the twist per unit length. See Section 4.3.7 for further details.

Figure 5.20: The quantity $\alpha_L$ traced along the axis fieldline against height, $z$, of the axis for four unscaled times, (a) $t = 0$, (b) $t = 40$, (c) $t = 70$, and (d) $t = 120$. The dashed black line denotes the height at the solar surface, $z = 0$. Symbols are plotted every 250 points for visualisation purposes.

Initially, in Fig. 5.20a at $t=0$, $\alpha_L$ is plotted as a function of distance along the axis against the height
of the axis for the three Group 2 cases. For all cases, $\alpha_L$ is constant along the fieldline at a value twice the initial twist per unit length, $\alpha$, as found in the Group 1 experiments. As the flux tube emerges at the photosphere, a gradient begins to develop in $\alpha_L$ from the highly dense interior to photosphere (see Fig. 5.20b). By the end of the experiments, at $t = 120$ in Fig. 5.20d, $\alpha_L$ is tending to a constant in all three cases. However, there is still a gradient in $\alpha_L$ in the interior, with the value of $\alpha_L$ ordered by the value of $\alpha$. Interestingly, the value of $\alpha_L$ in the interior for the $\alpha = 0.4$ tube is close to that of the $\alpha = 0.3$ tube. This is due to the $\alpha = 0.4$ tube distributing its twist over a longer axial length. The changes in twist per unit length are governed by two parameters, the initial twist $\alpha_i$ and the final length of field, $l_f$. If we assume that $l_i$ is constant, which it is for the initial fields, we find that $\alpha_i - \alpha_f = \alpha_i (1 - l_i/l_f)$, so we find a larger change in $\alpha_L$ for a final state with a longer axial field. Again, we expect the twist per unit length to approach a constant in all three cases if we performed the experiments for longer. The constant value will be determined by the final length of the field and initial twist per unit length.

As discussed in the previous section, the rotational motions at the photosphere extract twist from the interior. The number of turns that the field takes around the axis within an interior section of the tube, $N_I$, is presented in Fig. 5.21 as calculated in Eq. 5.1. Given that all experiments start with a different initial twist, $\alpha$, they all contain differing amounts of twist within the interior when they first intersect the photosphere. In addition, weakly twisted tubes reach the photosphere first as they are more buoyant. Due to the larger rotation seen for more highly twisted fields, the interior twist is extracted more efficiently. Interestingly, the twist contained within the interior for the $\alpha = 0.3$ and $\alpha = 0.4$ tube are approximately the same by the end of the experiment. We believe this is related to the axis of the $\alpha = 0.4$ tube extending higher into the atmosphere, distributing its twist more readily.

![Figure 5.21: Average number of turns, $N_I$, fieldlines undergo within the interior portion of one leg of the tube, measured over time for Group 2 cases.](image)

5.3.4 Vorticity

As the flux tubes reach the photosphere, vortical motions develop on the sunspot centres, as shown in Fig. 5.22. As $B_0$ is constant throughout this group of simulations, the region where $B_z > 3/4 \text{max}(B_z)$ is
approximately constant due to $B_z$’s weak dependence on $\alpha$. There is, however, a clear trend indicating that tubes with a higher initial degree of twist have larger vortical motions developing on their sunspot centres. This is expected as we predict vortical motions at the photosphere untwist interior field in an attempt to equilibrate the twisted interior with the stretched atmospheric field. If the initial flux tube is highly twisted, the fieldlines threading through the sunspot must rotate through a larger angle (see Fig. 5.19 and Table 5.4), producing a higher magnitude of vorticity.

![Figure 5.22: Average vorticity over time for various $\alpha$ cases.](image)

### 5.3.5 Current density

The mean of the electric current density on the photospheric $z = 0$ and interior $z = -12.5$ planes are presented in Fig. 5.23a and Fig. 5.23b respectively, as they evolve through time. We have again averaged over the area where $B_z > \frac{3}{4} \max(B_z)$. In Fig. 5.23a, we find that the lower twist $\alpha = 0.2$ case appears

![Figure 5.23: Evolution of maximum of $j_z$ over (a) the photospheric $z = 0$ plane and (b) $z = -12.5$ plane measured over time for varying $\alpha$ cases.](image)
first as it is the most buoyant due to the large density deficit. All three tube’s average current then increase sharply due to their emergence at the photosphere followed by a gradual decrease as the flux tube expands into the atmosphere. As expected, we discover the magnitude of \( \langle j_z \rangle \) is ordered by the value of twist, \( \alpha \).

The temporal evolution of the average of \( j_z \) at \( z = -12.5 \), half-way into the solar interior, is also interesting. The three \( \alpha \) cases follow the same general trend with the most twisted tube carrying the most current as expected. The small increase in \( \langle j_z \rangle \) is likely to be related to the straightening of the legs of the tube causing a more vertical alignment of the field. This agrees with the results of the general experiment described in Chapter 4 and the Group 1 experiments in Section 5.2.

### 5.3.6 Magnetic helicity

Adding to our analysis of the Group 2 simulations, we present an analysis of magnetic helicity within two distinct sections of the domain, namely the solar interior and atmosphere, as separated by the \( z = 0 \) photospheric boundary. The temporal evolution of this quantity, as calculated using Eq. 1.18, is shown in Figs. 5.24a and 5.24b for the atmosphere and interior respectively. As expected, there is a linear increase in atmospheric helicity in all cases due to the emergence of vertical flux into the atmosphere and the twisting of the atmospheric field caused by horizontal photospheric flows. The degree of twist clearly alters the amount of magnetic helicity transported to the atmosphere. The amount of helicity in the atmosphere tends to saturate more quickly for the lower twist cases (red and green). The highly twisted (blue) case, on the other hand, continues to increase over the whole experiment. The final helicity values in physical units for the three cases are summarised in Table 5.4. There is some evidence that the helicity may be proportional to \( \alpha^2 \). However, there is not a direct relationship given the non-linear dependence of the magnetic field on \( \alpha \), made clear by the initial field, \( B_0 \), outlined in Eq. 3.12.

The initial magnetic helicity stored in the interior is clearly altered by the degree of initial twist of the field as evidenced in Fig. 5.24b. Initially, the total magnetic helicity within the volume is given solely by the magnetic helicity stored within the interior as the magnetic flux tube is yet to emerge and enter
the atmosphere. The starting interior helicities range from $3.25 \times 10^{23}$ Wb$^2$ for the $\alpha = 0.2$ case to $6.5 \times 10^{23}$ Wb$^2$ for the $\alpha = 0.4$ case. The rate at which the helicity in the interior decreases also appears to be dependent on the initial twist with a steeper decline for the highest $\alpha$ experiment. This directly links to the more rapid rotation observed in the highly twisted case.

### 5.3.7 Magnetic energy

Finally, we present the atmospheric free energy for the Group 2 experiments in Fig. 5.25. In a similar trend, we see the largest transport of magnetic energy to the atmosphere for the most highly twisted case. Given that the largest rotation angle is passed through for the most highly twisted case, and a larger amount of flux is transported to the atmosphere, the free magnetic energy in the atmosphere is highest for $\alpha = 0.4$. The magnetic energies in physical units are summarised at $t = 120$ in the fourth column of Table 5.4.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\phi$</th>
<th>$H_r$ (Wb$^2$)</th>
<th>$E_{\text{free}}$ (J)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>112°</td>
<td>$3.3 \times 10^{22}$ Wb$^2$</td>
<td>$1.1 \times 10^{22}$ J</td>
</tr>
<tr>
<td>0.3</td>
<td>177°</td>
<td>$6.0 \times 10^{22}$ Wb$^2$</td>
<td>$1.6 \times 10^{22}$ J</td>
</tr>
<tr>
<td>0.4</td>
<td>268°</td>
<td>$1.4 \times 10^{23}$ Wb$^2$</td>
<td>$3.4 \times 10^{22}$ J</td>
</tr>
</tbody>
</table>

Figure 5.25: Free magnetic energy calculated above $z = 0$ for varying $\alpha$ cases.

### Table 5.4: Summary of key results at $t = 120$ for Group 2 experiments.

### 5.4 Comparison with observations

The model constructed in the previous two chapters successfully reproduces a rotation of the same order as those found in observations (i.e. between 40° and 540°). In particular, we find an average rotation of 394° of the sunspot around its centre in Chapter 4, with a strong, highly twisted initial field. In this chapter, we found that varying the initial magnetic field strength and twist of the emerging flux tube affects the final
5.4 Comparison with observations

angle of rotation. Specifically, we found rotation angles ranging from $124^\circ$ to $268^\circ$ over the parameter space we studied. However, at this stage it is important to compare the timescale of the rotational evolution with that observed on the Sun. In the introduction to Chapter 4, we discussed various observations, which are now summarised in Table 5.5.

<table>
<thead>
<tr>
<th>Source</th>
<th>Active region /parameters</th>
<th>Umbral radius</th>
<th>Penumbral radius</th>
<th>Angle</th>
<th>Timescale</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brown et al. (2003)</td>
<td>AR 8688</td>
<td>7.25 Mm</td>
<td>16.31 Mm</td>
<td>80 – 120$^\circ$</td>
<td>4 days</td>
</tr>
<tr>
<td>Brown et al. (2003)</td>
<td>AR 9004</td>
<td>4.35 Mm</td>
<td>12.33 Mm</td>
<td>90 – 120$^\circ$</td>
<td>3 days</td>
</tr>
<tr>
<td>Brown et al. (2003)</td>
<td>AR 9077</td>
<td>3.63 Mm</td>
<td>7.25 Mm</td>
<td>70 – 120$^\circ$</td>
<td>3 days</td>
</tr>
<tr>
<td>Brown et al. (2003)</td>
<td>AR 9114</td>
<td>5.44 Mm</td>
<td>14.5 Mm</td>
<td>120 – 150$^\circ$</td>
<td>3 days</td>
</tr>
<tr>
<td>Brown et al. (2003)</td>
<td>AR 9280</td>
<td>5.44 Mm</td>
<td>12.69 Mm</td>
<td>60 – 160$^\circ$</td>
<td>6 days</td>
</tr>
<tr>
<td>Brown et al. (2003)</td>
<td>AR 9354</td>
<td>3.63 Mm</td>
<td>12.69 Mm</td>
<td>40 – 60$^\circ$</td>
<td>3 days</td>
</tr>
<tr>
<td>Brown et al. (2003)</td>
<td>AR 0030</td>
<td>5.44 Mm</td>
<td>14.5 Mm</td>
<td>160 – 200$^\circ$</td>
<td>5 days</td>
</tr>
<tr>
<td>Yan and Qu (2007)</td>
<td>AR 10930</td>
<td>1.6 Mm</td>
<td>5 Mm</td>
<td>250$^\circ$</td>
<td>4 days</td>
</tr>
<tr>
<td>Min and Chae (2009)</td>
<td>AR 10930</td>
<td>1.6 Mm</td>
<td>5 Mm</td>
<td>540$^\circ$</td>
<td>5 days</td>
</tr>
<tr>
<td>Chapter 4</td>
<td>$B_0 = 9$, $\alpha = 0.4$</td>
<td>1.275 Mm</td>
<td>394$^\circ$</td>
<td>40 minutes</td>
<td></td>
</tr>
<tr>
<td>Chapter 5</td>
<td>$B_0 = 5$, $\alpha = 0.3$</td>
<td>1.02 Mm</td>
<td>124$^\circ$</td>
<td>71 minutes</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.5: Summary of observations of sunspot rotation as well as some key simulation results.

Seven of the observations shown in Table 5.5 are taken from Brown et al. (2003), and the other two from Yan and Qu (2007) and Min and Chae (2009). The umbral and penumbral radii are shown in the 3rd and 4th columns of the table, as given in the respective papers. However, as AR 10930 is growing during the rotation period, the umbral and penumbral radii are an estimate. It is worth noting that the rotating sunspot in AR 10930 is a particularly small sunspot in a larger active region. Although we have compared the magnitude of rotation angles with observations, we have not compared the active region size or timescale. In the final two lines of the table, we have presented the size, angle and timescale of simulations from Chapters 4 and 5. As the sunspots in our simulation do not exhibit umbral and penumbral features, we have only included one radius. This is the radius of the line of sight magnetic field at a saturation of 390 G. As the Chapter 5 experiment is weaker at $B_0 = 5$, it appears smaller at a fixed magnetic field saturation. In fact, by comparing a white light image with its equivalent magnetogram for AR 10930 in Gopasyuk (2015) we notice that the radius of the line of sight magnetic field is larger than the umbral radius but smaller the penumbral radius. However, this radius clearly depends on the field saturation chosen so the radii quoted are just rough approximations. Two differences are clear between our experiments and observations - the radius of the tube in our experiment is much smaller than those quoted in observations, and the timescale is much shorter. In fact, the active region we model is at least 10 times smaller than most observations quoted in Table 5.5, particularly those from Brown et al. (2003).

There are many possible reasons why the rotation rates we calculate in our simulations are much larger than those calculated in observations. Firstly, as evident in this chapter, varying the twist and field strength of the emerging tube changes the amount of time for the rotation, and secondly the size of active region may affect the time of rotation. The final row of the table displays the experiment with the smallest rotation
over the longest period of time, specifically $B_0 = 5$ and $\alpha = 0.3$ from the Group 1 experiments, where we calculate a rotation of $124^\circ$ over 71 minutes. However, even with a smaller field strength, the time period for the rotation is much shorter than those observed. Hence, the next factor to investigate is the size of the active region.

In order to model a larger region effectively we must change the normalisation we use to non-dimensionalise the equations. Earlier, in Section 2.3.1, we outlined the normalisation procedure and in Section 3.2 we specified three key normalising values as $B_0 = 1300$ G, $L_0 = 170$ km and $\rho_0 = 3 \times 10^{-4}$ kg/m$^3$. This ensured gravity had a normalising value of $g_0 = 274$ m/s$^2$ forcing $L_0 = 170$ km to be the pressure scale height. To test the effect of modelling a larger region, we double the normalising length scale $L_0$ to give $L_0 = 340$ km. If we assume $v_0$ does not change, this forces $g_0 = v_0^2/L_0 = 137$ m/s$^2$ and $t_0 = L_0/v_0 = 50$ s. However, as we have changed the normalising value of gravity, the pressure scale height is no longer equal to $L_0$, and hence the stratification at the photosphere and transition region must change. To ensure the same physical stratification, the width of the photosphere and transition region must be halved as shown in Fig. 5.26b. In addition, to ensure $g = 274$ m/s$^2$ the dimensionless gravity is increased from $\hat{g} = 1$ to $\hat{g} = 2$. Fig. 5.26 shows the initial stratification in both dimensionless and dimensional units, for both the original experiment and the larger experiment.

The main difference between Figs. 5.26a and 5.26b is that the photosphere and transition region are reduced from a width of 10 in Fig. 5.26a to a width of 5 in Fig. 5.26b. However, when we convert these plots to their dimensional equivalents, shown in Fig. 5.26c and Fig. 5.26d, the width of the photosphere and transition region are the same in Mm. The increase in $L_0$ increases the length of region we are examining, and means the convection zone is much deeper and we model a larger section of the corona. Most importantly, the flux tube is twice as wide in the larger experiment so we can compare the effects of doubling the size of the active region on the evolution of the rotation angle.

The rotation angles for both experiments are shown against normalised time in Fig. 5.27. Although the “doubled” experiment was only executed for 140 normalised time units, it is clear that the evolution of the rotation angle for the two experiments are very similar. This suggests that doubling the length scale has no real effect on the evolution of the rotation angle, except for the timescale over which it occurs. Although the normalised time over which the rotation angle evolves is similar, $t_0$ is double that of the standard experiment. If we assume the “doubled” experiment continues to evolve in the same manner, we can surmise a rotation angle of $124^\circ$ over 220 normalised time units, or equivalently 142 minutes. Hence, by doubling the size of the active region to a radius of 2.5 Mm, the timescale for the rotation has doubled. Furthermore, this concept can be extrapolated to predict the behaviour of our model in even larger regions. If we increased $L_0$ by a factor of 10 to give us an active region of radius 12.75 Mm, we predict $t_0$ would also increase by a factor of 10, to give a rotation of $124^\circ$ over 11 hours and 48 minutes. This gives us a timescale of about half a day, more comparable to those found in observations.

In addition, the model presented by Longcope and Welsch (2000) predicts that the level of rotation will depend on the rapidity of flux emergence so we plan to investigate how this affects the rotation. The length of time for the rotation may also be related to the depth at which the flux tube is anchored; we have briefly investigated this, as presented below. However, to fully understand the effects of a deeper convection zone a full parametric study would be necessary.
5.4 Comparison with observations

Figure 5.26: Top panel shows the dimensionless initial stratification of (a) the original simulation as introduced in Chapter 3 and (b) the new experiment with $L_0 = 340$ km. Bottom panel shows the dimensional initial stratification of (c) the original stratification introduced in Chapter 3, and (d) the new experiment with $L_0 = 340$ km.

Figure 5.27: Comparison of rotation angle against normalised time for the parameters $B_0 = 5$ and $\alpha = 0.3$ for (a) the standard experiment with stratification described in Chapter 3 and (b) the larger experiment with $L_0 = 340$ km.
We used the same parameters as given in Chapter 4 ($B_0 = 9$ and $\alpha = 0.4$) but increased the depth of the convection zone to $z = -40$, i.e., increasing the size of the convection zone by 2.55 Mm. The height-time profile of this axis is shown in Fig. 5.28a, and the corresponding photospheric rotation angle in Fig. 5.28b. The axis of the tube sits at $z = -25$ initially as the feet of the flux tube sits on the base at $z = -40$. The tube then slowly rises through the convection zone until it reaches the photosphere at about $t = 50$. The magnetic field then builds up at photospheric heights for some time until the magnetic buoyancy instability is initiated at about $t = 90$. At this point, the axis of the flux tube rapidly rises, as well as the leading edge of the system. The rotation angle passed through by the fieldlines steadily decreases (due to the clockwise rotation) until $t = 90$ where there is a rapid increase in the rate of change of the rotation angle, due to the rapid emergence of the axis.

![Height-time profile of axis (a) and photospheric rotation angle (b)](image)

Figure 5.28: Plots of experiment with deeper convection zone with (a) height-time profile of axis (symbols) and leading edge of system (dashed line) and (b) the photospheric rotation angle as function of time.

For clarity, we have compared the deeper convection zone simulation with the equivalent experiment from Chapter 4 (with $B_0 = 9$ and $\alpha = 0.4$), as shown in Fig. 5.29a. The model presented by Longcope and Welsch (2000) predicts that the level of rotation depends on the rapidity of flux emergence. By changing the thickness of the convection zone, we have lengthened the amount of time the flux tube spends in the convection zone leading to a later emergence at $t = 50$. The flux tube then pauses at the photosphere until $t = 90$ while the magnetic field strength builds up to initiate the magnetic buoyancy instability. This takes considerably longer than the equivalent experiment with the shallow convection zone shown in Fig. 4.1, as $B_z$ becomes weaker by the time the magnetic field passes through the deeper connection zone. Hence it takes longer for $\beta$ to lower to one.

During this delay at the photosphere, before the full expansion of the field into the atmosphere, the rotation rate is steady but slow. However, at $t = 90$ the rotation rate increases and the rotation becomes more rapid during the sudden rise of the axis of the flux tube. To investigate the rate of change of rotation angle, we have plotted the rotation rate in Fig. 5.29b. To try and remove the effect of the later emergence, we have shifted the deep interior rotation rate 23 units to the left. There is a clear difference in the rotation rates when the rotation slows for the deep interior case while the tube sits at the photosphere. However, when the field fully emerges, the rotation rates for both cases are very similar. Hence, the depth of the flux tube in the convection zone only appears to affect the initial phase of rotation while the magnetic field spreads at the photosphere.
5.5 Summary

In this chapter, we presented results from a parametric study by performing a series of 3D MHD simulations of buoyant twisted toroidal flux tubes as they rise through the solar interior and emerge into the atmosphere. Our primary aim was to investigate the rotation of the photospheric footpoints. We have varied the magnitudes of two parameters governing the magnetic structure of the tube, namely the axial magnetic field strength, $B_0$, and the twist, $\alpha$. Our focus was to identify the distinct effect of each of these parameters on the rotational motion at the photosphere and the many ramifications of this photospheric velocity.

To investigate this effect, we analyse various quantities relating to the plasma and magnetic field. To directly measure the rotation, we calculate the rotation angle based on the axis of the flux tube as the centre of the sunspot. This allows us to make a direct comparison as to how magnetic field strength and twist affect rotation rates. Similarly, we look at how the field strength and twist affect the plasma vorticity within the sunspots. In addition, we analyse the twist of individual fieldlines, the magnetic energy and helicity to study the twist contained within different subvolumes of the domain. This allows us to understand the distribution of twist across the system, and the transport of twist from the interior to atmosphere of the model.

Many interesting relationships were found for Group 1 in which we kept the twist, $\alpha$, constant and varied the axial magnetic field strength, $B_0$. This parameter investigation provides us with an insight into how the initial magnetic field strength affects the amount by which the flux tube rotates. We reveal $t = B_0 \ell$ to be the natural time unit for the evolution of the magnetic field. Surprisingly, we find the vertical photospheric magnetic field strength to scale with $B_0^2$ when we vary the initial axial field strength, $B_0$, of the sub-photospheric field. All components of the magnetic field are initially proportional to $B_0$ but by the time the tube reaches the photosphere the magnetic field’s magnitude and direction are adapted as governed by the initial $B_0$. Stronger fields tend to emerge more fully with a vertically directed axis, whereas weaker fields tend to spread horizontally at the photosphere to allow the magnetic buoyancy instability to initiate. In brief, the magnetic field is altered on its journey to the photosphere, and hence the scalings we find with

Figure 5.29: (a) Comparison of rotation angle with and without deep convection zone shown in red and black respectively, and (b) the rotation rate with and without the deep convection zone where the deep interior rotation rate is shifted 23 units to the left.
Another particularly interesting result we find in Group 1 is that the rotation angle varies with $B_0$. The timescale over which the rotation occurs is dependent on $B_0$ due to the density deficit’s dependence on $B_0$. Hence, in a fixed time, a larger rotation angle is passed through by sunspots in higher $B_0$ experiments. To remove this time dependence, we scale the time as $\tilde{t} = B_0t$, but this does not reveal a self-similarity. Instead, we discover that the rescaled rotation angle, $\tilde{\phi} = \phi/B_0$, is self-similar. This result may seem surprising on first inspection as we might expect the final rotation angle to be the same for varying magnetic field strength as the initial fields share the same twist and hence helical structure. The basis for this relationship was difficult to ascertain at first but became clearer when considering the twist results. It is conceivable that if we performed the experiments for a longer time, the rotation would cease for stronger experiments and continue for weaker experiments until a plateau was reached for all experiments. Due to the diffusion timescale and computational expense, we are unable to check this at the current time. However, this seems unlikely as the rotation rate drops off significantly to almost zero by the end of the weaker $B_0$ experiments. If the rotation rate does, in fact, cease for the weaker field experiments and there is not any variation in rotation angle for later times, it is possible that the magnetic fields in the weaker experiments are unable to extract as much interior twist as the larger experiments.

The investigations of twist were also in agreement with the rotation angle results. By considering the fieldline twist and the force-free parameter $\alpha_L$, we find a considerable amount of twist left in the interior for the weaker twist experiments. If we adopt Longcope and Welsch (2000)’s conjecture, we expect $\alpha_L$ to equilibrate along the fieldline. This theory is tested in detail in Chapter 6. In order for the twist per unit length to equilibrate, the sunspot rotation would need to continue for the weaker fields. However, as the axis fieldline is shorter for weaker fields, the twist per unit length, $\alpha_L$, is larger so it seems plausible that even once $\alpha_L$ has equilibrated, there will still be a considerable amount of twist left in the interior. This could help us explain why the rotation angles are smaller for the weaker experiments. In addition, we found the helicity and magnetic energy to be ordered by $B_0$. In stronger field experiments, we see a larger transport of magnetic energy and helicity to the atmosphere.

Varying the initial degree of twist also has an effect on the amount of rotation we see within the sunspots. However, it is difficult to find direct relationships with $\alpha$, given the non-linear dependence of the initial field on $\alpha$ and the fact that the magnetic buoyancy profile is also altered by the degree of twist. We find the helicity, current and vorticity to be ordered by the degree of twist, $\alpha$. Larger vortical motions develop in the highly twisted experiment with larger rotation angles transporting more helicity and energy to the atmosphere. Work must be done to understand the non-linear effect of the twist on the evolution of the tube. However as the twist and magnetic tension force are inherently linked non-linearly it is difficult to scale quantities in a simple linear manner.

This chapter has shown a clear correlation between rotation angle, and the twist and magnetic field strength of the sub-photospheric field. Hence, by understanding this relation, we can make predictions about the sub-photospheric magnetic field based on observations of sunspot rotations at the photosphere. However, given that an increase in both twist and field strength increase the angle of rotation, it may be hard to distinguish which combination of these parameters is responsible for the observed rotation.
5.5 Summary

Finally, by comparison with observations, we found that the rotation angles we calculate occur over much shorter timescales than those found in observations. However, we note the active regions we model are much smaller than observed regions that contain rotating sunspots. Hence, by performing an experiment modelling a flux tube twice the size, we find that the fieldlines threading the sunspot rotate through the same angle in twice the length of time. Consequently, by scaling our model up to the active region size found in observations, we find a rotation of a few hundred degrees (depending on field strength) over a period of half a day. This is more in line with observations. We also explored the impact of lengthening the convection zone on the timescale of rotation, and concluded that a deeper convection zone lead to less rapid emergence and in turn a slower rotation. However, this did not strongly impact the timescale of rotation.
Chapter 6

Evolution of idealised twisted magnetic flux tube

To try and understand the complex processes controlling the magnetic flux tube in our experiments, it is important that we simplify the numerical set-up. Although this set-up does not mimic realistic conditions on the Sun, it is a great tool to help us understand the behaviour of the magnetic field and plasma in a simplified model. An idealised experiment allows us to investigate various theories presented in Chapters 4 and 5. Primarily, we aim to demonstrate the theory introduced by Longcope and Welsch (2000). Explicitly, the authors used an idealised analytic model to predict the evolution of the force-free parameter, $\alpha_L$, and how this relates to the propagation of a non-linear torsional Alfvén wave. The magnitude of $\alpha_L = j \cdot B / B^2$, a measure of the rate of twist per unit length, decreases dramatically when the magnetic field enters the atmosphere and stretches. Hence, as a result, a gradient in $\alpha_L$ is established between the highly twisted interior portion and stretched coronal portion. The authors predict this gradient produces a torque responsible for the rotation observed. Ultimately, the authors predict the system will reach a state in which the interior $\alpha_L$ equilibrates with its coronal value. This is yet to be proven in a numerical simulation, to the best of our knowledge. Hence, by considering a simple configuration, we aim to investigate this hypothesis by studying the distribution of $\alpha_L$ during the latter stages of an experiment, at a time we cannot follow in our complex emergence experiments due to computational restrictions. In addition, we aim to determine whether the length of box, and hence length of fieldlines is related to the distribution of twist across the domain.

The remainder of this chapter is laid out as follows. In Section 6.1, we outline the model set-up detailing the initial magnetic flux tube and simplified background stratification. We then describe the two cases we consider in the length of box comparison. The results of this comparison are presented in Section 6.2. Later, in Section 6.3, we consider a comparison of the initial axial magnetic field strength of the flux tube. In each comparison, we independently vary the length of domain and magnetic field strength in order to understand the individual effects of these modifications on the evolution of the experiment. This is following the idealised modelling philosophy, as if we choose to make too many modifications at once, we
lose the important physics underlying the experiment. Finally, in Section 6.4, we conclude the chapter with a summary of our findings.

6.1 Initial set-up

In our general experiments (with the initial set-up described in Chapter 3), we model a curved twisted flux tube in the solar interior and allow it to emerge and rise into the atmosphere. Upon emergence, the twisted fieldlines enter the atmosphere and thereafter stretch and straighten into the lower density coronal atmosphere. In this chapter, we simplify the model by ignoring the emergence stage and only considering the evolution from the end of emergence onwards. This is achieved by modelling the set-up with a straight vertical flux tube with twisted fieldlines in the interior and untwisted fieldlines in the atmosphere. This essentially models one leg of the flux tube, ignoring the evolution of the other as well as the curvature of the system. Although this model makes substantial simplifications, the illustrative benefits help us to pinpoint the physical processes underpinning the behaviour of the magnetic flux tube.

Table 6.1: Initial set-up of idealised twisted experiment.

<table>
<thead>
<tr>
<th>Interior ((z &lt; 0))</th>
<th>Atmosphere ((z \geq 0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(B_z = B_0 e^{-r^2/a^2})</td>
<td>(B_z = B_0 e^{-r^2/a^2})</td>
</tr>
<tr>
<td>(B_\phi = \alpha r B_z)</td>
<td>(B_\phi = 0)</td>
</tr>
<tr>
<td>(B_r = 0)</td>
<td>(B_r = 0)</td>
</tr>
<tr>
<td>(\rho = 10)</td>
<td>(\rho = 0.1)</td>
</tr>
<tr>
<td>(p_{exc} = \frac{B_0^2}{4} e^{-2r^2/a^2} \left(\alpha^2 a^2 - 2 - 2\alpha^2 r^2\right))</td>
<td>(p_{exc} = -\frac{B_0^2}{2} e^{-2r^2/a^2})</td>
</tr>
</tbody>
</table>

The initial set-up of the experiment in cylindrical coordinates, \((r, \phi, z)\), is summarised in Table 6.1, as separated into two regions: the interior below \(z = 0\) and the atmosphere above \(z = 0\), with \(z = 0\) modelling the solar surface or photosphere. This can be related to a Cartesian \((x, y, z)\) coordinate system by

\[
\begin{align*}
    r^2 &= x^2 + y^2, \\
    \phi &= \tan^{-1}\left(\frac{y}{x}\right).
\end{align*}
\]

The vertical component of the magnetic field is independent of height, \(z\), and only varies with radius, \(r\), following a Gaussian profile. However, the \(B_\phi\) component is set to zero in the atmosphere and only switched on in the interior. Hence, the experiment is set-up with a twisted interior field with \(5/\pi\) turns around the axis and a straight coronal field. In the following discussions, we refer to two experiments as Experiment 1 and Experiment 2 as introduced in Table 6.2. A selection of fieldlines is shown in Fig. 6.1b for Experiment 1. In order to study the effects of the length of the fieldlines, we have also considered the effects of doubling the length of the entire box in Experiment 2 (see Fig. 6.1c).

To keep the set-up as simple as possible, we stratify the background plasma using a \(\tanh\) profile to switch from the very dense interior \((\rho = 10)\) to the rarefied atmosphere \((\rho = 0.1)\) over two orders of
6.1 Initial set-up

Figure 6.1: Initial set-up of idealised experiments with (a) log scale of plasma density, pressure and temperature along axis of flux tube. 3D visualisation of (b) Experiment 1 with a selection of fieldlines traced from the lower boundary in purple. The left wall shows the temperature on a log scale and the right wall shows the density. A filled contour of the vertical magnetic field is shown on the base of the domain. (c) shows a similar schematic of Experiment 2 where we have doubled the height of the domain.

magnitude. The background pressure is kept constant at a value of $p_b = 20$, and hence the resulting temperature profile varies from 2 to 200 from the interior to atmosphere. We continue to use the same normalising values as outlined in Chapter 3. We are therefore not attempting to model the observed solar profile here, simply modelling a change in density and temperature. However the gas pressure is changed
Table 6.2: Length of box comparison set-up.

<table>
<thead>
<tr>
<th>Experiment 1 (Standard)</th>
<th>Experiment 2 (Double)</th>
</tr>
</thead>
<tbody>
<tr>
<td>128 × 128 × 256 gridpoints</td>
<td>128 × 128 × 512 gridpoints</td>
</tr>
<tr>
<td>−25 &lt; x &lt; 25, −25 &lt; y &lt; 25, −25 &lt; z &lt; 75</td>
<td>−25 &lt; x &lt; 25, −25 &lt; y &lt; 25, −25 &lt; z &lt; 175</td>
</tr>
<tr>
<td>( a = 2.5 ), ( R_0 = 15 ), ( B_0 = 5 ), ( \alpha = 0.4 )</td>
<td></td>
</tr>
</tbody>
</table>

at the flux tube location by the pressure excess described in the last row of Table 6.1. We note the excess pressure is negative for all the parameters we choose. Hence there exists a pressure deficit in both the interior and atmospheric portions of the domain in order to balance the magnetic forces in the equation of motion (Eq. 1.10). This inward acting gas pressure gradient then balances the two other forces, namely the outward acting magnetic pressure gradient and inwardly acting magnetic tension force.

The initial stratification of the gas pressure, density, and temperature along the axis of the flux tube \((r = 0)\) is shown in Fig. 6.1a for Experiment 1. Due to the discontinuity in \( B_0 \) at \( z = 0 \), there is a discontinuity in the excess pressure, \( p_{\text{exc}} \), and hence the resulting gas pressure at the photosphere. Lare3d accurately resolves shocks using a combination of shock viscosity and Van Leer flux limiters (see Chapter 2 for further details). An example to demonstrate the properties of a shock wave is given in Section 2.2.2. Although not shown, the plasma stratification in Experiment 2 is the same with a discontinuity at \( z = 0 \). However, the atmospheric region is extended by 100 units in the \( z \) direction.

Both experiments are set up such that the flux tube is sitting in horizontal equilibrium and allowed to evolve over 300 normalised time units (125 minutes). However, the flux tube is not in vertical equilibrium with a pressure gradient that acts along the axis from the high pressure interior to lower pressure atmosphere. Note that although the length of the box has been doubled in the \( z \) direction and the number of grid points are doubled from 256 to 512, the length of the interior remains unchanged and hence the length of the atmospheric region is increased by \( 2^{1/3} \). However, the grid resolution is consistent through the domain in both experiments. All boundary conditions are the same as those described in Chapter 3.

6.2 Length of box comparison

The two experiments are allowed to evolve over 300 normalised time units. This is considerably longer than the run time of emergence experiments performed in earlier chapters (ranging from 108 to 216 normalised time units). We assume that the field has entered the atmosphere and straightened to produce the simplified set-up we have constructed. In earlier chapters we were unable to perform the experiments any longer due to computational limitations. The timestep became too small to continue executing experiments using available computational resources. The experiment performed in this chapter allows us to make predictions about the flux tube and plasma behaviour later in other experiments.

Initially both flux tubes are sitting in horizontal equilibrium. However, as soon as the experiment begins, twist within the tightly twisted interior field propagates along the flux tube, by means of a torsional Alfvén wave. This wave is launched due to a mismatch in current between the highly twisted interior and straight
6.2 Length of box comparison

coronal field. This is also seen in the emergence case as only a fraction of the interior current is carried to the corona. The propagation of this wave transports twist to the atmosphere, and results in a rotation of the plasma at the $z = 0$ plane, similar to that of the “sunspot rotation” seen in earlier emergence experiments. The propagation of the torsional Alfvén wave (tangential discontinuity) is accompanied by the launch of a shock wave (due to a jump in gas pressure).

Consider the quantity $\alpha_L = (\nabla \times \mathbf{B}) \cdot \mathbf{B} / B^2$, which gives a measure of the local rate of twist of the
magnetic field. See Section 4.3.7 for further details of the relationship of this quantity and the fieldline twist. In Fig. 6.2, we have plotted $\alpha_L$ as a function of depth along the axis fieldline at $r = 0$ for both Experiments 1 and 2. Initially at $t = 0$, there is a discontinuity in $\alpha_L$ in both experiments. In the interior $\alpha_L = 0.8$, twice the initial twist per unit length, $\alpha = 0.4$. For a force-free field, $\alpha_L$ is equal to twice the twist per unit length. However, the interior field is not force-free and hence, further investigation is required to interpret the relation here. On the contrary, in the atmosphere, $\alpha_L$ is equal to zero as the magnetic fieldlines are straight here with no twist. A gradient in $\alpha_L$ (or equivalently the current) is established. The disturbance splits into two waves: one moving upwards and one downwards. The downward propagating torsional Alfvén wave reaches the base at about $t = 10$, consistent with the Alfvén wave travel time. Based on the initial set-up, we would expect the wave to take approximately 16 time units to propagate to the base for both experiments. However, the density in the solar interior is reduced, thereby increasing the Alfvén speed and lowering the travel time to about 10 time units. An Alfvén wave is also launched upwards, and based on the initial field and stratification it should reach the top of the box after five and eleven normalised time units for Experiments 1 and 2 respectively. However the pressure gradient between the interior and atmosphere causes an upward force increasing the density and pressure in the atmosphere. The tubes try to remain in total pressure balance and hence the magnetic field strength of the flux tube decreases to ensure this. Consequently, the Alfvén speed is slowly reduced and so the wave takes longer to reach the top of the box.

The gradient in $\alpha_L$ produces a torque that drives a rotational motion of the flux tube in the $z = 0$ plane (shown later). Once the downward propagating wave has reached the bottom of box, $\alpha_L$ decreases due to a reduction in twist as the fieldlines unwind in the interior. Simultaneously, $\alpha_L$ increases in the atmospheric portion. At this point, it is important to distinguish between the two experiments. $\alpha_L$ is proportional to $1/L$ where $L$ is a typical length scale. Hence, $\alpha_L$ is reduced by a greater amount in Experiment 2 due to the longer length scales as it is distributed over the axial length. In Fig. 6.2d, at $t = 30$, $\alpha_L$ is approximately constant along the axial length for Experiment 1. However, the torsional motion continues until $\alpha_L$ is lower in the interior and approximately constant in the atmosphere as shown at $t = 80$ in Fig. 6.2e. This indicates the magnetic field is approximately force-free in the atmosphere. However, there is now a positive gradient in $\alpha_L$ that produces a torque of the opposite sign to equilibrate $\alpha_L$. Ultimately, at the end of the experiment at $t = 300$, $\alpha_L$ appears to be approximately constant along the axis fieldline, indicative of a force-free field. The length of the axis clearly has an effect on the final value of $\alpha_L$. The final mean values are 0.094 and 0.047 for Experiments 1 and 2 respectively. Hence, the local rate of twist per unit length for Experiment 1 is exactly double that of Experiment 2 as we have halved the length of the domain.

These experiments have explicitly demonstrated that the interior and coronal values of $\alpha_L$ equilibrate during the latter stages of the experiment, thereby confirming the hypothesis suggested by the analytical model put forward by Longcope and Welsch (2000) in a numerical experiment. This result helps us to predict the behaviour of flux tubes in earlier experiments performed in Chapters 4 and 5 at a later normalised time.

Now that we have established the evolution of $\alpha_L$ over the course of the experiments, we investigate the rotation at the interior-atmospheric boundary, i.e. $z = 0$. The mean rotation angle, calculated by averaging over 100 traced fieldlines, is shown in Fig. 6.3a for both experiments. In both experiments, the fieldlines originating from the left footpoint within a radius of 2.5 experience rotations of the order of $3\pi$
Figure 6.3: Comparison of (a) the average rotation angle at z=0 as calculated using the method described in Section 4.3.2 and (b) the rate of change of rotation angle, $d\phi/dt$, for Experiments 1 and 2, as coloured by the key.

(540°), or one and a half turns. As seen earlier, $\alpha_L$ equilibrates to a lower value in Experiment 2 and hence the flux tube must rotate through a larger angle to sufficiently reduce the interior $\alpha_L$. By the end of the experiments, the final rotation angle differs by 38.6° for the two cases. However, the rotation angles differ by larger amounts earlier in the experiment. In particular, at $t = 118$, the rotation angles differ by 62°. Hence, we can conclude that the length of fieldlines may have a significant effect on the rotation observed. This is important in the parameter study performed in Chapter 5 as the axes of the tubes with varying field strengths extend into the atmosphere by different amounts.

Another important point to highlight is the period of opposite rotation seen after about $t = 70$, made clear by the rotation rate plotted in Fig. 6.3b, where $d\phi/dt$ is positive during the latter stages. As discussed earlier, this is due to an over-rotation and hence the fieldlines need to rotate in the opposite direction to equilibrate $\alpha_L$. We speculate that the over-rotation is due to the inertia of plasma but this is an area that needs more investigation.

Although these experiments have the same initial twist per unit length as the general experiment described in Chapter 4, the total twist within one leg of the flux tube in Chapter 4 was approximately 7.5 radians. However, in the experiments discussed in this chapter, the total twist within the interior section of the domain is 10 radians. Hence, we expect a larger rotation for this case. The final rotation angle for the general emergence experiment was approximately $2\pi$ compared to $11\pi/4$ in Experiment 1 in this chapter. Note, we have chosen to discuss Experiment 1 as this has the same length of domain as the earlier emergence experiments. In addition to the larger initial twist, the fieldlines in this experiment extend to the top of the domain, slightly further than the other experiments due to the boundary conditions on the top boundary. As we have just discussed, longer fieldlines experience a larger rotation angle. Furthermore, we may have missed some of the evolution in the emergence experiments as we have not reached the stage where $\alpha_L$ is constant along the domain.

For completeness, we also include the full analysis of the rotation angle for Experiment 1 in Fig. 6.4b using the starting points shown in Fig. 6.4a. In Fig. 6.5, we have ignored the variation in $\phi$ before $t = 50$ so
we can clearly see differences in rotation angles for varying starting radius. Although the difference is very small, there is a clear trend showing that fieldlines closer to the sunspot centre tend to rotate slightly more rapidly, in agreement with the trend seen in Chapter 4. We assume that the difference is due to the stronger field at the centre of the flux tube. However, the difference is very small and the rotation is generally solid body.

In order to study the effect of this rotation on the movement of plasma within the flux source centred at $x = 0$ and $y = 0$ on the $z = 0$ plane, we present an analysis of vertical vorticity (see Eq. 4.5) for both Experiments 1 and 2. Using the method described in Section 4.3.4, we plot the average vertical vorticity in Fig. 6.6. Note, $v = 0$ at the beginning of all experiments run using the Lare3d code so we should ignore the first time unit. During the first 50 normalised time units, there is a large magnitude of vorticity consistent with the larger rotation angle passed though by the magnetic field. The negative sign of vorticity is consistent with a clockwise motion as observed. If we were to choose an initially left-hand twisted magnetic field, we would find an anti-clockwise rotation at the photosphere and a positive vorticity within the flux source. Later, as made clear by Fig. 6.6b, the magnitude of vorticity reduces and the sign becomes positive to reflect the counter-rotation seen during this stage of the experiment. This is consistent with our analysis of both the rotation angle and the twist per unit length.

To understand the distribution of twist across the domain, we have calculated the average number of turns of fieldline twist using the method outlined in Section 4.3.6. In this case, the axis remains vertical and hence the plane in which we calculate the fieldline angle, and hence twist is simply the $x - y$ plane. The fieldline twist has been calculated for both the interior and atmospheric regions in Figs. 6.7a and 6.7b respectively. Specifically, we find an initial interior twist of a turn and a half which rapidly decreases during the first 50 normalised time units before reaching a constant during the latter stages. As the flux tube untwists, the inward magnetic tension force reduces and the flux tube spreads out due to the outwardly directed magnetic pressure force. We again find the final fieldline twist of Experiment 1 to be approximately
6.2 Length of box comparison

Figure 6.5: Evolution of (a) the average rotation angle for four different radii introduced in Fig. 6.4a and (b) the rate of change of rotation angle, averaged at each of the radii.

Figure 6.6: Comparison of average vorticity, \( \langle \omega_z \rangle \), on \( z = 0 \) for Experiment 1 and Experiment 2 with (a) full evolution, and (b) second half of evolution.

Twice that of Experiment 2 as expected from our previous analyses. The atmospheric twist, on the other hand, builds from zero as twist propagates upwards into the atmosphere. In both experiments, the atmospheric twist reaches a maximum after 60 normalised time units then decreases due to the decay of the total twist of the system. We return to this concept later in the section. A larger amount of twist is transported to the atmosphere in Experiment 2 as the atmospheric region is 100 normalised units longer. Even though the twist per unit length, \( \alpha_L \), is lower in this experiment, the total twist across the atmospheric region is consistently larger.

In a similar manner, we integrate the local rate of twist, \( \alpha_L \), along different sections of the axis fieldline, as another proxy for twist. Precisely, we have integrated this quantity along the interior section of the fieldline and plotted the time evolution in Fig. 6.8a. Similarly we have plotted the time evolution of the integral over the atmospheric section of the fieldline in Fig. 6.8b. As argued earlier, this is hypothesised to be twice the total twist. Comparing the two, we find the two methods follow each other very closely with a maximum 15% difference over the duration of the experiment. This suggests that \( \alpha_L \) is in fact a good proxy for fieldline twist. Again, we find the interior-integrated \( \alpha_L \) equilibrates to a higher value in Experiment
6.2 Length of box comparison

Figure 6.7: Evolution of averaged number of turns the field takes around the axis in (a) the interior region below $z = 0$, and (b) the atmospheric region above $z = 0$ for both Experiment 1 and Experiment 2.

than 2. Due to the differing lengths of the atmospheric domain in Experiments 1 and 2, there is a larger proportion of $\alpha_L$ distributed in the atmospheric portion for Experiment 2, and hence a consistently larger amount of twist is transported to the atmosphere.

In addition the time evolution of the total twist is presented for both cases in Fig. 6.8c. The total twist halves over the run-time of the experiment. This is due to diffusive effects (see Eq. 1.17 in Chapter 1) given the resistivity, $\eta$, present. Fieldlines, and hence flux, are brought into the O-point in the middle of the flux tube where the flux dissipates. There is a similar drop off in both the positive and negative azimuthal flux, typical of flux cancellation. To check this, we have compared the experiment with an ideal ($\eta = 0$) experiment and find that the twist remains constant in the ideal case. As the rate of decay of twist is constant across simulations, it does not impact our results.

To complete our discussion of this series of idealised experiments, we calculate the magnetic helicity contained within different subvolumes of the system using Moraitis’ method described in Appendix A. We remind the reader that the potential magnetic field is calculated such that it has the same normal magnetic field component as the original field on all boundaries surrounding the volume but with a zero electric current. An example of the calculation of a potential magnetic field is shown in Fig. 6.9, with the original field shown in Fig. 6.9a and the potential magnetic field shown in Fig. 6.9b. Clearly, the electric current associated with the twisted structure of the fieldlines is removed in the potential magnetic field. As discussed in Section 1.2.5, the potential magnetic field has zero current and the minimum magnetic energy, and spreads to fill the volume.

The relative magnetic helicity, a measure of the stress or twist of a magnetic field, is shown in Fig. 6.10 for both the interior and atmospheric regions. Initially, there is a large amount of helicity contained in the interior as a result of the twisted field and no helicity in the atmosphere in both experiments. As the experiments progress, and plasma begins to rotate at the photosphere, the helicity steadily declines in the interior as it is injected into the atmosphere. After approximately 50 normalised time units, the helicity contained within the interior of Experiment 2 begins to drop off more rapidly as it is more readily injected into the atmosphere. This is a consequence of the final state of equilibrium of the twist per unit length,
6.3 Magnetic field strength comparison

Now that the effect of altering the length of domain is established, we seek to understand how a change in magnetic field strength affects this idealised model. By considering the model in this manner we separate the results of changing initial field strength from that of a change in length of fieldlines. In the experiments performed in Chapters 4 and 5, a change in magnetic field strength allows the magnetic field to extend higher into the atmosphere, resulting in longer fieldlines. However, by considering this simple model we can understand which factor (field strength or fieldline length) is responsible for the difference in rotational velocities at the photosphere.

We use the same background stratification as in the length of box comparison and split this study into a comparison of two experiments as shown in Table 6.3 with the only difference being the initial field strength.
of the flux tube, $B_0$. Note, the $B_0 = 5$ experiment is the same as Experiment 1 discussed in the previous section with one small change. We had wished to keep these experiments the same but when we double $B_0$ we find that the pressure deficit required for equilibrium exceeds the background gas pressure, resulting in negative pressures. As this is unphysical, we need to triple the background gas pressure to $p_b = 60$. Although this is only a problem in the $B_0 = 10$ case, we used the same background gas pressure in the $B_0 = 5$ experiment to make a closer comparison.

A comparison of the time evolution of rotation angles for the $B_0 = 5$ and $B_0 = 10$ experiments is shown in Fig. 6.11a. We note that in this case there is no emergence phase so the rotation angle begins to evolve at $t = 0$. Clearly, a difference in the initial field strength alone has a dramatic impact on the evolution of the rotation angle but there is very little difference in the final angle of rotation. The photospheric rotation angle of the $B_0 = 10$ tube drops more rapidly, experiencing an over-rotation before increasing to reach an approximately constant rotation angle. This indicates the fieldlines threading through the sunspot rotate clockwise initially, causing an over-rotation, after which they begin to rotate anti-clockwise until a constant angle is reached. The $B_0 = 5$ tube, on the other hand, behaves slightly differently. The initial change
Table 6.3: Magnetic field strength comparison set-up.

<table>
<thead>
<tr>
<th>$B_0 = 5$</th>
<th>$B_0 = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>128 $\times$ 128 $\times$ 256 gridpoints</td>
<td></td>
</tr>
<tr>
<td>$-25 &lt; x &lt; 25$, $-25 &lt; y &lt; 25$, $-25 &lt; z &lt; 75$</td>
<td></td>
</tr>
<tr>
<td>$a = 2.5$, $R_0 = 15$, $\alpha = 0.4$</td>
<td></td>
</tr>
</tbody>
</table>

in rotation angle is both slower and smaller in magnitude, and the tube does not experience a definite over-rotation and subsequent rotation reversal.

To compare the two cases on a more suitable timescale, we again redefine a scaled time $\bar{t} = tB_0$ to taken into account the effect of $B_0$ on the Alfvén speed, shown in Fig. 6.11b. This method for redefining time is discussed in more detail in Section 5.2. As we have doubled $B_0$ from one case to the other, we have also doubled the final scaled time. Even with this rescaling, there is still a discrepancy between the evolution of rotation angles. We note that the pressure within the tubes are different in the two cases, owing to the difference in pressure deficits as scaled by $B_0^2$.

To analyse this comparison in more detail, we also present the evolution of the twist per unit length parameter, $\alpha_L$, as a function of depth along the axis of the tube in Fig. 6.12. It should be noted that each of the twist per unit lengths, $\alpha_L$, are plotted at the same scaled time $\bar{t}$ to try and compare them at similar stages in their evolution. The interior twist per unit length decreases by a larger amount and more rapidly in the $B_0 = 10$ case before increasing to match that of the corona. By the end of the experiment, the interior and coronal twist per unit length match in the $B_0 = 10$ case, indicating that a state of equilibrium has been reached causing the flux tube to cease rotating. The $B_0 = 5$ case, on the other hand, is evolving on a slower timescale and has not yet reached a constant $\alpha_L$ along the axis. We predict that $\alpha_L$ will tend to a constant value in the $B_0 = 5$ case and this will lead to the final agreement of rotation angles for the $B_0 = 5$ and $B_0 = 10$ cases.
6.4 Summary

In this chapter we have conducted a series of simplified experiments to investigate the propagation of twist across an idealised interior-atmospheric region. In all cases, the set-up is split into a dense interior region and a rarefied atmosphere, and a vertical magnetic flux tube is defined such that all of the magnetic field’s...
twist is contained within the interior connected to a straight untwisted atmospheric field. The flux tubes are set up in horizontal force balance and allowed to evolve. This results in an initial discontinuity in the twist per unit length, $\alpha_L$, and pressure, $p$, across the interior-atmosphere boundary. In all cases, torsional Alfvén waves are launched and a final state is reached in which the twist per unit length tends to a constant value along the axis fieldline. By considering domains of different lengths, we demonstrate the length of fieldlines are vital in determining the constant $\alpha_L$ to which the system tends. Specifically, we find that the twist per unit length is inversely dependent on the length of fieldlines. For example, if the twist per unit length tends to a value of $\alpha$ along a fieldline of length $L$, the twist per unit length would tend to a value of $\alpha/2$ over a fieldline of length $2L$. As the rotation angle at the photosphere originates due to a twist imbalance, the rotation angle is dependent on the final twist per unit length. Hence, we also find larger rotation angles for longer fieldlines. This is consistent with our conclusions from Chapter 5 where we predicted that the fieldline length was crucial in determining the angle of rotation, and in turn the magnetic energy transported to the atmosphere.

We also investigate the impact of a change in the initial axial magnetic field strength on the evolution of twist and rotation angle, without altering the length of domain. By considering a change in magnetic field strength in this simplified set-up, we can separate the effects of changing the magnetic field strength from changing the length of fieldlines. These two effects are inherently linked in the earlier emergence experiments as stronger magnetic fields emerge more fully and their larger magnetic pressure allows the fields to expand higher into the atmosphere. In this case, we find that an increase in magnetic field strength changes the evolution of rotation angle, but not necessarily the final angle of rotation. In the small sample we have considered, we find that the tube in the strong field ($B_0 = 10$) case experiences an over-rotation while the weak field ($B_0 = 5$) case rotates more slowly tends towards the final angle of rotation without over-rotating. However, both cases appear to tend towards the same final angle of rotation. The final angle of rotation, does not necessarily give clues to the strength of field that lies under the photosphere, but rather the length of fieldlines threading through the region. However, as the two are inherently linked, longer fieldlines are most likely to originate from a strong sub-photospheric flux tube.
Chapter 7

Sunspot rotation due to sub-photospheric velocities

In all previous chapters, we have considered single flux tubes that have been twisted prior to emergence. As we do not yet know the structure of the interior magnetic field, the addition of twist is merely an assumption based on simulations that predict distortion of untwisted magnetic fields on their rise through the interior. We have not yet investigated a case where the sub-photospheric flux tube is initially untwisted and how rotational convective velocities may influence the twist of the magnetic field, and the resulting rotation rate at the photosphere. A recent study by Syntelis et al. (2015) investigated the emergence of untwisted magnetic flux tubes and their subsequent expansion into the atmosphere accompanied by the onset of jets and heating of the plasma. However, this study did not consider the influence of sub-photospheric velocities, specifically the impact of these on the magnetic flux tube as it rises from below the photosphere.

The granular pattern of convection, described in Section 1.1.1, can result in interesting convective velocities in the interior. The overturning flow at the edges of granules can produce horizontal vortices at the interface between the granule and intergranular lane (Nordlund et al., 2009). By taking the curl of the equation of momentum (Eq. 1.10), it is clear that vorticity is generated by the cross product of gradients in density and pressure. Hence, vorticity is produced in locations where density and pressure gradients are not parallel, for instance at the mushroom heads of downdrafts (Nordlund et al., 2009). Downdrafts are sinks where cool plasma returns to the solar interior (Bonet et al., 2008). As the matter has angular momentum with respect to the draining point, it must spin up when nearing the sink, giving rise to a “bathtub” like whirl flow. Hence, we can compare these sub-photospheric vortices predicted by numerical simulations of convection to bath tub vortices (Bonet et al., 2008).

In order to incorporate these vortices into our model, we use rotational velocity drivers on the base of the domain. Velocity drivers on the boundary have been applied to magnetic fields in a variety of scenarios, mostly in coronal experiments where the driver is inserted on the photospheric boundary (e.g. Priest et al., 2002, Wilmot-Smith, 2015 etc.). In the experiments in this chapter, we begin with an untwisted magnetic flux tube and allow it to emerge. However, at the same time we impose rotational velocity drivers on the
footpoints of the flux tube on the lower boundary to inject twist into the flux tube. In all experiments, we impose a sub-photospheric velocity at a boundary 4.25 Mm below the solar surface. In order to study the effect of these rotations, we vary the size, magnitude, and number of velocity drivers, as well as the duration of driving.

The set-ups we choose are very highly idealised and it may seem overly simple to assume that the rotational drivers are at the footpoints of the flux tubes. However, as Meyer et al. (1979) and Schmidt et al. (1985) noted, the tendency of emerging flux tubes to migrate towards the boundaries of super-granules and the congregation in the network is well established. Hence, it is reasonable to assume the footpoints of flux tubes could be caught in vortical motions at downdrafts.

The chapter is laid out as follows. In Section 7.1, we describe the initial set-up of the standard case, with a strong magnetic flux tube and two fast, confined rotational velocity drivers that continue throughout the duration of the experiment. We outline the key results of this experiment before going on to discuss several cases in which we make one modification from the standard case. In Section 7.2, we describe Case 1, in which we use a very similar set-up varying only the size of the rotational drivers, opting for a more spread driver surrounding the flux tube in its entirety. In Section 7.3, Case 2 uses the same set-up as the standard case with two slower velocity drivers on the base. Similarly, in Case 3 we take the standard case and vary only the magnetic field strength of the flux tube, as described in Section 7.4. Next, in Section 7.5, we alter the number of drivers by considering only one driver on one of the footpoints. This experiment is, henceforth, referred to as Case 4. Finally, in Section 7.6, Case 5 investigates an experiment in which we vary the driving time, only driving the experiment for the first 75 normalised time units. To conclude, in Section 7.7, we briefly summarise the main findings of this chapter.

7.1 Standard case

The experiment begins with an untwisted magnetic flux tube (with $\alpha = 0$ in the initial magnetic field given in Eq. 3.14). This results in a density excess of

$$\rho_{\text{exc}} = -\frac{B_0^2}{2} e^{-2\pi^2/a^2}.$$  

This is a larger density deficit than any non-zero twist values as the zero twist field has no inward magnetic tension force counteracting the outward magnetic pressure force. In this standard case, we set the axial field strength as $B_0 = 10$ to model a strong magnetic flux tube. The rest of the parameters are set up as before with $\alpha = 2.5$ and $R_0 = 15$. The axial field strength of the magnetic flux tube will be varied in Case 3 described later. A schematic of the initial set-up is shown in Fig. 7.1 with the same density and temperature stratification as that outlined in Chapter 3. The only difference between the initial set-up of this experiment and those described in Chapters 4 and 5 is that $\alpha$ is set to zero. This causes the fieldlines threading through the flux tube to appear straight, following the axis of the flux tube.

In order to twist the magnetic fieldlines and model a vortex at the centre of each footpoint, we subject the initial set-up to a driving velocity on the lower boundary of the domain. Specifically, we use spinning
solid body footpoint motions described in De Moortel and Galsgaard (2006) and Wilmot-Smith and De Moortel (2007). In their experiments, they were modelling a photospheric rotation (like the sunspot rotations produced by emergence in earlier chapters) but the same general profile can be used lower down with a smaller driving speed. This is prescribed as the $\phi$ component of velocity (in a local $(r, \phi, z)$ cylindrical system) on the base of the domain:

$$v_\phi(r) = v_0 r_1 [1 + \tanh(A(1 - Br_1))] + v_0 r_2 [1 + \tanh(A(1 - Br_2))] = v_{\phi_1} + v_{\phi_2}$$

(7.1)

where $r_1^2 = (x - x_1)^2 + (y - y_1)^2$ and $r_2^2 = (x - x_2)^2 + (y - y_2)^2$ such that $(x_1, y_1)$ and $(x_2, y_2)$ are the centres of the footpoints. We have split the velocity into $v_{\phi_1}$ and $v_{\phi_2}$, as given by the first and second terms of $v_\phi$ respectively. This results in a Cartesian velocity driver of

$$v_x = -v_{\phi_1} \sin(\phi_1) - v_{\phi_2} \sin(\phi_2),$$

$$v_y = v_{\phi_1} \cos(\phi_1) + v_{\phi_2} \cos(\phi_2),$$

where $\phi_1 = \arctan((y - y_1)/(x - x_1))$ and $\phi_2 = \arctan((y - y_2)/(x - x_2))$. The coefficient $v_0$ affects the magnitude and direction of the driving speed, and $A$ and $B$ describe the steepness and location of the drop off in $v$ outside the sources, respectively. Note the velocity profile is designed to increase linearly with radial distance from the centre of each of the sources, in order to maintain the shape of the flux concentrations as they are rotated. We choose this type of spinning rotation as it is localised only affecting the footpoint it is concentrated on. Other studies, such as De Moortel and Galsgaard (2006), consider rotational velocity drivers that rotate multiple flux tubes with a single larger driver. In all experiments we set $A$ as 16.8 and vary $v_0$ and $B$ throughout the experiments. We set $(x_1, y_1) = (0, -15)$ and $(x_2, y_2) = (0, 15)$ as these are the centres of the footpoints of the flux tube (as outlined in Chapter 3). In the standard case, we set $v_0 = 0.05$ for a counter-clockwise rotation and $B = 0.4$ which we refer to as a fast, confined rotation.
7.1 Standard case

To demonstrate the velocity drivers, a coloured contour of vorticity \( \omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \) is shown at \( z = -25 \) in Fig. 7.2a. The positive sign of vorticity signifies a counter-clockwise rotation, and the black contours overplotted show the magnetic field concentrations. It is important to note in this case that the magnetic field contours surround the region of rotation and both footpoints are rotated. In Fig. 7.2b, we have plotted \( v_\phi \) against \( r \) calculated on the base of the simulation domain. This shows that the velocity increases linearly with radius from zero at \( r = 0 \) until \( r = a = 2.5 \) where it drops off with a \( \tanh \) profile. The velocity reaches a maximum of 0.2 or 1.2 km/s in physical units.

![Figure 7.2: Initial set-up of standard case with (a) the \( z \)-component of vorticity on base of simulation domain and (b) the \( \phi \) component of velocity against radius \( r \) on the base of simulation domain.](image)

To start the rotation smoothly, the driver is built up with a \( \tanh \) profile:

\[
v_\phi(r, t) = v_\phi(r) \left[ \frac{1}{2} \left( 1 + \tanh \left( \frac{t - \frac{2}{2}}{2} \right) \right) \right],
\]

using \( v_\phi(r) \) prescribed in Eq. 7.1. The purpose of this is to increase the velocity gradually, in an attempt to reduce shocks due to a sudden onset of velocity. The experiment is performed for 150 normalised time units, or 62.5 minutes. In order to work out the amount of twist injected, we use the relation \( v_\phi = r \frac{d\phi}{dt} \) to work out the expected angle of rotation on the base. Let us focus on the footpoint subscripted 1. For \( r_1 < 2.5, v_\phi \approx 2v_0r_1 \). Hence, \( \frac{d\phi}{dt} = \frac{v_\phi}{r_1} \approx 2v_0 \) and \( \phi \approx 2v_0t \) where we have set the constant of integration equal to zero as \( \phi = 0 \) at \( t = 0 \). Hence, after 150 normalised units, we expect each footpoint to have rotated \( \phi \approx 15 \) radians or approximately \( \phi = 4.77\pi \) radians. Due to the nature of the driver, the rotation is solid body and hence \( \phi \) does not vary with \( r \).

The temporal evolution of the rotation angle, \( \phi \), and the corresponding temporal derivative on the base are shown in Fig. 7.3. We note this is the numerical calculation of the angle \( \phi \) as introduced in Section 4.3.2 but calculated a grid point up on the base rather than at the photosphere. In addition, in order to trace the same fieldlines throughout the experiment when calculating the rotation angle, we use the \( v_x \) and \( v_y \) components on the base to follow the same fieldlines. We neglect any viscous or resistive slippage of fieldlines on the boundary.

As expected the angle \( \phi \) increases linearly in time with an approximately constant \( \frac{d\phi}{dt} = 0.1 = 2v_0 \).
7.1 Standard case

Figure 7.3: Rotation on base in footpoint one of standard case with the temporal evolution of (a) the rotation angle, $\phi$ and (b) the rate of change of rotation angle, $d\phi/dt$.

This is what we predict given the form of the velocity driver input on the boundary and our estimation of the angle $\phi$. In this standard case, two rotational drivers are inserted on the base and hence twist is injected into both footpoints. This means that double the amount of twist is inserted into the flux tube, specifically 30 radians over 150 normalised time units.

As soon as the experiment begins, the flux tube starts to rise buoyantly to the photosphere due to the density deficit introduced. Simultaneously, twist is injected into the flux tube. By approximately $t = 25$, all traced fieldlines have reached the photospheric plane, as shown in Fig. 7.4. At this stage, 2.5 radians of twist has been injected into each footpoint but the fieldlines in Fig. 7.4a show little evidence of this. However, by $t = 50$, in Fig. 7.4b, there is qualitative evidence of twist due to the kinking of fieldlines as a combined total of 10 radians of twist has been injected into the field.

Figure 7.4: Selected set of traced fieldlines from both footpoints at (a) $t = 25$ and (b) $t = 50$. It is important to note that the fieldlines shown are not the same traced fieldlines in both images due to the velocity driver moving fieldlines on the base.

In order to understand how this rotation on the base translates to the photosphere, we have followed
7.1 Standard case

particular fieldlines on the base and calculated their intersection with the photospheric plane through time. If the photospheric rotation is the same as the base rotation, the angle will follow the same trend as in Fig. 7.3a. The rotation angle and rate of change of rotation angle at the photospheric plane are shown in Fig. 7.5. These quantities are only plotted from \( t = 25 \) onwards when the field first emerges. Therefore the final angle of rotation at the photosphere is less than the base rotation angle. Hence we consider the rate of change of rotation angle shown in Fig. 7.5b. The rotation rate takes about 25 normalised time units to build up to its maximum, of 0.08. The time taken to reach this is related to the travel time of an Alfvén wave (about 12.5 normalised time units to reach base with \( v_A \approx 2 \) over leg of flux tube). Furthermore, the constant value reached by \( d\phi/dt \) is 0.08. The rotation angle has reached an approximately linear phase but the rotation rate is only \( 4/5 \) of that input by the velocity driver on the base. This supports Longcope and Welsch (2000)’s theory that only a fraction of the interior twist reaches the corona.

![Figure 7.5: Rotation at photosphere in footpoint one of standard case with the temporal evolution of (a) the rotation angle, \( \phi \) and (b) the rate of change of rotation angle, \( d\phi/dt \).](image)

We believe the discrepancy in the rotation rate at the base and photosphere may arise owing to two reasons. It may be due to viscous effects damping waves before they reach the photosphere, or more likely, it may be due to the flux tube attempting to equilibrate the twist per unit length along fieldlines. If the rotation rate was to equal that on the base, all twist input into the interior would propagate to the atmosphere. As the axis reaches a length of approximately 275, a proportion of \( 50/275 \) or \( 18\% \) would need to remain below the surface to balance the twist per unit length. Hence, only \( 82\% \) of the twist should be transported to the photosphere, approximately consistent with the rotation rate observed.

To investigate this, we have plotted \( \alpha_L \) along the axis at three separate times in Fig. 7.6. We direct the reader to Section 4.3.7 for further details of the relation between \( \alpha_L \) and the twist per unit length, and to Section 6.2 for a demonstration of this relationship. At \( t = 0 \), in Fig. 7.6a, \( \alpha_L = 0 \), consistent with the zero twist field inserted in the initial conditions. The axis fieldline is drawn from the left footpoint to its apex. Hence, at \( t = 0 \) the fieldline extends to \( z = -10 \). As the experiment proceeds and the footpoints are rotated, twist is injected into the flux tube and \( \alpha_L \) tends to a non-zero negative value due to the counter-clockwise rotation. In Fig. 7.6b, by \( t = 15 \), the axis has almost reached photospheric heights and \( \alpha_L \) has dropped to about \(-0.1\) along the axis. It is worth noting that the counter-clockwise rotation of the footpoints at the photosphere produces a left-hand twisted field with a negative \( \alpha_L \) value. Hence, this field is twisted in the opposite sense to that of the flux tube described in Chapter 3. By the end of the experiment, in Fig. 7.6c,
$\alpha_L$ has tended to a constant value in the atmosphere with a gradient in the interior. This signifies that the interior magnetic field is more highly twisted. Hence $\alpha_L$ has not yet tended to a constant along the field.

Figure 7.6: Variation of $\alpha_L$ as a function of $z$ along the axis fieldline for times (a) $t = 0$, (b) $t = 15$, and (c) $t = 150$. The solar surface is highlighted with a vertical line at $z = 0$.

Another interesting observational signature to explore is a line of sight synthetic magnetogram (i.e. a coloured contour of the vertical magnetic field) in an attempt to understand the role of the injection of twist on the roundness and coherence of the magnetic sources at the photosphere. Two magnetograms are shown at $t = 50$ and $t = 80$ in Fig. 7.7a and Fig. 7.7b respectively. In the first magnetogram, the sunspots are relatively squashed in $y$ and spread in $x$ with a non-circular shape but by $t = 80$, the sunspots are well rounded. The contribution of twist to this effect is investigated in further detail in Case 4 in Section 7.5 when we vary the number of drivers.

Figure 7.7: Magnetograms of standard case at time (a) $t = 50$ and (b) $t = 80$. Blue and red contours are overplotted at $B_z = \{0.2, 0.4, 0.6\}$ and $B_z = \{-0.2, -0.4, -0.6\}$ respectively.

Additionally, as a control case, we have executed this experiment with no velocity drivers on the base and find that the untwisted flux tube rises through the interior, before plateauing at the photosphere creating undefined, non-circular sunspots. There is no rotation of the sunspots in the photospheric plane in the control case. A comparison of magnetograms, with and without the addition of rotational drivers is shown in Fig. 7.8. The case with no velocity drivers (Fig. 7.8b) exhibits oval sunspots that appear to spread in the $x$ direction. The addition of rotation on the base rounds off and concentrates the sunspots (see Fig. 7.8a). This clearly demonstrates the addition of twist, and hence a magnetic tension force acting on the magnetic
Now that the features of the standard case have been determined, we vary a number of properties of the driver and magnetic field configuration, and study the effects of these on the photospheric rotation and structure of the sunspots.

7.2 Case 1 - vary size of driver

In Case 1, we vary the size of the drivers, keeping the driver velocity and number of drivers the same as well as the magnetic field configuration of the flux tube. Hence, this case describes a fast, spread rotation of both footpoints. To illustrate this, we have again plotted $\omega_z$ and $v_\phi$ on the lower boundary of the domain in Fig. 7.9. In this case, the velocity driver extends to approximately $r = 4.5$, surrounding the magnetic flux tube which extends to $r = 2.5$. This is achieved by setting $B = 0.2$ in $v_\phi$ prescribed in Eq. 7.1. The vorticity is again positive signifying a counter-clockwise rotation but there is a distinct region of negative vorticity surrounding the positive vorticity. This corresponds to the region where $v_\phi$ decreases and is expected from the form of $v_\phi$. It is important to note that the $d\phi/dt$ inserted on the base is the same as the standard case (shown in Fig. 7.3b). However, the main difference is that the velocity driver surrounds the magnetic field, moving the plasma around the magnetic field as well as the flux tube itself. This differs from the other case, where there was some weak magnetic field encompassing the vortical motions.

By following traced fieldlines on the base, we find that the fieldlines undergo a smaller rotation angle than that of the standard case (see Fig. 7.10a). By analysing the rate of change of rotation angle presented in Fig. 7.10b, we find that the rotation rate again reaches $4/5$ of that input on the boundary but then slowly drops off during the later stages of the experiment. The reason for this drop off in rotation is not known.

To understand the effect of the size of the driver on the structure of the sunspots, we have shown two magnetograms side by side in Fig. 7.11 for the standard case and Case 1, respectively. Although the magnetograms are fairly similar, there are two distinct differences. The sunspots are larger in Case 1 where

![Magnetograms at t = 80 for (a) standard and (b) B_0 = 10 with no velocity drivers inserted. Blue and red contours are overplotted at B_z = {0.2, 0.4, 0.6} and B_z = {-0.6, -0.4, -0.2} respectively.](image-url)
7.2 Case 1 - vary size of driver

Figure 7.9: Initial set-up of Case 1 with (a) the $z$-component of vorticity on base of simulation domain and (b) the $\phi$ component of velocity against radius $r$ on the base of simulation domain.

Figure 7.10: Rotation at photosphere in footpoint one of standard case with the temporal evolution of (a) the rotation angle, $\phi$, and (b) the rate of change of rotation angle, $d\phi/dt$.

Figure 7.11: Magnetograms at $z = 0$ at $t = 150$ for (a) the standard case and (b) Case 1. Blue and red contours are overplotted at $B_z = \{0.2, 0.4, 0.6\}$ and $B_z = \{-0.6, -0.4, -0.2\}$ respectively.
the rotating plasma surrounds the magnetic field. This suggests that the size of the velocity driver may control the size of the resulting sunspots. In the standard case, the region outside the confined rotation appears to diffuse leaving smaller sunspots. Secondly, the spread rotation appears to have created larger, more defined streaks of weak magnetic field between the spots.

### 7.3 Case 2 - vary velocity of driver

In this next case, we vary the magnitude of the velocity driver using a driving speed $v_0 = 0.025$ (half that prescribed in Section 7.1), and keep the size and number of drivers the same as the standard case. The vorticity contour and $v_\phi$ line plot are shown in Fig. 7.12 on the same scale as the standard case, to reflect the smaller vortical motions inserted on the base, as well as the smaller peak in $v_\phi$. The spread and position of the velocity driver is identical to the standard case, as is the magnetic field configuration used to set up the magnetic flux tube. The level of rotation inserted on the base is half that of the standard case, as shown in Fig. 7.13a. The rotation rate at the photosphere is shown in Fig. 7.13b and shows a similar trend to the standard case. The only notable difference is that it reaches a plateau half that of the standard case. Again, there are small oscillations, indicative of the transport of twist by torsional Alfvén waves.

To show the similarities in the evolution, we have plotted the two cases on the same scale in Fig. 7.14a. The red line shows the evolution of the standard case divided by two to reflect the change in the driving speed, and the black line shows the evolution of Case 2. These lines follow a very self-similar evolution, and suggest that the speed of the velocity driver at the base is proportional to the velocity of the rotation at photospheric heights.

Furthermore, to compare the photospheric magnetic field for this case and the standard case, we compare magnetograms at $t = 80$. This magnetogram is shown for the standard case in Fig. 7.7b and for Case 2 in Fig. 7.14b. By comparing these magnetograms, the effect of doubling the speed of the driver is clear. The sunspots for the standard case are quite well rounded and coherent by $t = 80$ but the sunspots in Case...
7.4 Case 3 - vary field strength of tube

Next, we modify the field strength of the flux tube by halving the value of the axial field strength, i.e. lowering $B_0$ to 5. In this case, the size, distribution, and number of drivers is kept the same as the standard case discussed in Section 7.1. Hence, the $d\phi/dt$ input on the base is identical to that shown in Fig. 7.3. The corresponding angle $\phi$ and rotation rate $d\phi/dt$ at the photosphere are shown in Fig. 7.16. The rotation rate again reaches a similar level but it takes much longer to reach this level due to the slower Alfvén wave associated with the lower $B_0$ value. We have again plotted a comparison with the standard case in Fig. 7.16a.
7.4 Case 3 - vary field strength of tube

Figure 7.15: Rotation at photosphere in footpoint one of Case 3 with the temporal evolution of (a) the rotation angle, $\phi$, and (b) the rate of change of rotation angle, $d\phi/dt$.

plotted in red. The rotation rate appears later in the $B_0 = 5$ case due to the later emergence. The rate builds up more slowly but ultimately reaches the same level, $4/5$ of the rate inserted. Another difference is the quantity of fluctuations in the $B_0 = 5$ case.

In addition, we present a magnetogram at $t = 80$ for Case 3, and find the sunspots are not as well rounded as the equivalent in the $B_0 = 10$ experiment. The lower field strength means that magnetic forces are not as dominant, and the vortex motions can control the position of the flux tube. This may be the reason why the sunspots are not centred along the $x = 0$ line, and the sunspots are not as round or coherent.

Figure 7.16: (a) Evolution of the rate of change of rotation angle, $d\phi/dt$ for Case 3 shown in black with the temporal evolution of $d\phi/dt$ for the standard case shown in red both calculated at the photosphere. (b) Magnetogram at $t = 80$ with blue and red contours overplotted at $B_z = \{0.05, 0.1, 0.2\}$ and $B_z = \{-0.2, -0.1, -0.05\}$ respectively.
7.5 Case 4 - vary number of drivers

In this case, we reduce the number of velocity drivers, using a confined driver on the right footpoint and no driver on the left. This will answer two interesting questions. Firstly, we are able to see a clear comparison in the two sunspots: one that is rotated from below and one that isn’t. Will there be a clear difference in their structure? Secondly, will the twist injected travel to the other leg of the tube?

Figure 7.17: Initial set-up of Case 4 with the $z$-component of vorticity on base of simulation domain.

The $z$-component of vorticity in the $z = -25$ plane is plotted in Fig. 7.17, demonstrating no rotation in the lower negative footpoint and a confined rotation in the upper positive footpoint. The rate of rotation, $d\phi/dt$, is therefore zero in the lower negative footpoint and $d\phi/dt = 0.1$ in the upper right footpoint.

Figure 7.18: Rotation at photosphere of Case 4 with the temporal evolution of the rate of change of rotation angle, $d\phi/dt$, for (a) the left (lower) footpoint, and (b) the right (upper) footpoint.

From Fig. 7.18a, the rotation rate in the left footpoint is very close to zero suggesting that the photospheric field does not rotate. In addition, the minimal rotation rate changes sign suggesting that there is no preferred direction for the rotation. The right footpoint, on the other hand, experiences a similar rotation to that of the standard case (see Fig. 7.18b). This suggests that although the twist injected into the right footpoint of the flux tube by the velocity driver travels along the tube, it does not reach the neighbouring footpoint. The disturbance is reflected by the high density interior and is unable to rotate the neighbouring
spots’ interior field. The effect of this on the photospheric sunspots is shown in Fig. 7.19a, where we have plotted a magnetogram at $t = 80$, equivalent to that plotted for the standard case in Fig. 7.7b (with the colour table reversed). The difference between the two sunspots, and hence the photospheric consequences of the addition of twist, is quite remarkable. The lower negative sunspot has become quite oddly shaped and undefined without the addition of twist, while the upper positive sunspot is very round and defined due to injection of twist from rotation at the base. Later, at $t = 125$ the upper positive sunspot remains round and concentrated while the lower negative sunspot shows signs of starting to break apart.

This can be compared with an observation shown in Fig. 7.19c. Many well-known asymmetries exist with regards to solar active regions. The leading polarity and following polarity sunspots (with respect to the rotation of the Sun) have inherent differences in their stability, size, and evolution. One particularly interesting property of bipolar active regions is the asymmetry in the structure between the leading and following polarities. The leading polarity of active regions tend to be in the form of large, coherent sunspots, whilst the following sunspots appear more fragmented and dispersed. In Fig. 7.19c, a magnetogram of a bipolar active region from June 10th 2014 is displayed. This sunspot pair is a perfect example of the
asymmetry that often arises on the Sun. The leading sunspot (negative polarity) is large and circular in appearance whereas the following sunspot (positive polarity) is very fragmented and scattered. The sunspots created in this model show a similar trend. We have reversed the colour table in the synthetic magnetograms to match the opposite polarities in the observation. Clearly, the model used here does not treat the underlying asymmetry in sunspot formation and relies on a sub-photospheric velocity forming below one of the sunspots. This is, however, not completely unrealistic. We have delved slightly deeper into the subject of asymmetry in sunspot formation with other experiments but this is definitely a topic that requires further investigation. Our preliminary attempts to understand this process are outlined in Appendix B, but the topic is left for future study.

7.6 Case 5 - switch off drivers

In the final case, we use the exact set-up of the standard case and switch off the rotation at \( t = 75 \), in a similar manner to the way it is switched on, i.e. by using the \( \tanh \) profile outlined in Eq. 7.2. The rotation rates at the base and the photosphere are displayed in Fig. 7.20, on the left and right panels respectively.

![Figure 7.20](image)

(a) Figure 7.20: Rotation in footpoint one of Case 5 with the temporal evolution of the rate of change of rotation angle, \( d\phi/dt \) calculated on (a) the base, and (b) the photosphere.

The switch off in the rotation is very clear in Fig. 7.20a. During the first 75 normalised time units, the evolution at the photosphere is identical to that of the standard case. However, the rotation continues at the photosphere until approximately \( t = 90 \), followed by a gradual decay in rotation rate until \( t = 140 \). The delay between the rotation switch off at \( t = 75 \) and the photospheric rotation starting at \( t = 90 \) is the time for an Alfvén wave to propagate from \( z = -25 \) to \( z = 0 \). In an attempt to understand the evolution of the rotation rate, we have also plotted the twist per unit length, \( \alpha_L \), against height at three different times in Fig. 7.21.

Initially, in Fig. 7.21a, \( \alpha_L = 0 \) indicates the magnetic flux tube is untwisted and reaches an apex at \( z = -10 \). It is worth noting that we only trace fieldlines to their apex in these plots. By \( t = 50 \) in Fig. 7.21b, each footpoint has been rotated by \( 5/\pi \) radians and \( \alpha_L \) has reached a constant value of about \(-0.25 \) in the interior. The field has started to reach the atmosphere and hence a non-zero \( \alpha_L \) develops
Figure 7.21: Variation of $\alpha_L$ as a function of $z$ along the axis fieldline for times (a) $t = 0$, (b) $t = 50$, and (c) $t = 150$. The solar surface is highlighted with a vertical line at $z = 0$.

indicating the presence of some twist. However, after the footpoint driving is switched off at $t = 75$, the rotation at the photosphere slows dropping to $d\phi/dt = 0$. By this stage $\alpha_L$ has tended to a constant along the field (see Fig. 7.21c).

### 7.7 Summary

This chapter has studied several different cases considering rotational velocities in the convection zone and their effect on the emergence of a sub-photospheric flux tube. The sub-photospheric rotational driver is inserted on the base in all cases, at a physical depth of 4.25 Mm below the photosphere. The standard case considers a fast, confined velocity driver on both footpoints of an untwisted sub-photospheric flux tube emerging through various layers of the Sun. The velocities found at the photosphere are proportional to that inserted on the base, approximately $4/5$ of that inserted on the base. We do not expect all of the twist to reach the photosphere in order to keep the twist per unit length constant. In addition, the sunspots formed are well rounded in structure, compared to the case with no driver. The addition of twist provides the magnetic field with a tension force that constricts the horizontal expansion of the tube.

In Case 1, we varied the size of the driver, opting for a driver that surrounds the magnetic field on the base. This inserts the same rotation rate on the base but over a larger radius. In this case, the photospheric rotation reached the same level as the standard case described above but the rotation rate dropped off during the later stages. We notice the size of driver helps to control the size of the resulting sunspots, as we see larger sunspots in Case 1. The velocity of the driver is varied in Case 2 by halving the value of the driving speed. The rotation rate at the photosphere follows a similar scaling but the sunspots formed are less defined as a smaller amount of twist is inserted into the tube.

In Case 3, the magnetic field strength of the tube is halved and the velocity driver is kept constant. Halving the field strength halves the Alfvén speed (see Section 1.2.5) and hence slows the transport of twist along the field, meaning that the rotation reaches the photosphere at a later time. The number of drivers is reduced to one in Case 4, where the velocity driver on the left footpoint is removed. In this experiment we find that the sunspot above the rotated footpoint rotates but the field threading the other sunspot remains relatively untwisted. This results in a well defined, concentrated sunspot and a spread, irregular sunspot.
This reflects an asymmetry often found in sunspot formation, with one coherent, strong sunspot and another scattered, spread sunspot.

Finally, we repeated the standard case but switched off the footpoint rotation half way through the experiment. The photospheric rotation slowed from this point on until $\alpha_L$ settled to a constant along the fieldlines. Most importantly, the rotation rate at the photosphere is proportional to that input on the base in most cases. The addition of a velocity driver can control the shape and concentration of the sunspots, as is particularly evident in Case 4 in which we varied the number of velocity drivers. This chapter also tells us that sunspot rotation may not necessarily arise due to the untwisting of pre-twisted magnetic fields, but instead by the transmission of twist by a sub-photospheric velocity.
Chapter 8

Conclusions and future work

In this thesis, we have investigated sunspot rotation through the use of three-dimensional magnetic flux emergence experiments. We achieved this by numerically solving the time-dependent, resistive MHD equations using the *Lare3d* code described in Chapter 2. To begin, in Chapter 3, a background stratification is created modelling four layers of the Sun: the convection zone; photosphere/chromosphere; transition region; and corona. This set-up is in hydrostatic balance, and has been used extensively in previous emergence experiments (Archontis et al., 2004; Murray et al., 2006; Archontis and Hood, 2010 etc.) and as such is used in all experiments (apart from Chapter 6) contained within this thesis. A magnetic flux tube, first derived by Hood et al. (2009), is then inserted into the marginally stable convection zone. To balance the forces acting on the tube, a pressure deficit is required. The flux tube is then made buoyant by initialising the tube in thermal equilibrium and imposing a density deficit by maintaining the pressure deficit. This is the basic premise of all of the emergence experiments performed in this thesis, apart from Chapter 6 where we investigate the evolution of a straight twisted flux tube, and do not perform an emergence.

The basic experiment is described in Chapter 4, in which we perform the emergence of a twisted toroidal flux tube. We use this experiment to illustrate a mechanism for the rotation of sunspots, an observable feature reviewed in numerous observational studies described at the beginning of Chapter 4. We demonstrate sunspot rotation by a clear calculation of rotation angle and the presence of vorticity contours. The appearance of vortical motions centred on both sunspots in MHD simulations has been discussed in past work such as Magara (2006) and Fan (2009). Fan (2009) also explained these rotations as a consequence of torsional Alfvén wave propagation and established an increase in helicity in the atmosphere. Our work, however, explicitly discusses the effect this rotational motion has on the interior portion of the field by establishing a depletion in the magnetic helicity stored in the interior segment of the domain and a drop in the vertical current in this region. We also show the magnetic tension force may govern this rotational motion as it appears to produce an unbalanced torque that drives the rotation. By demonstrating the cause of the rotation, we show that it is not an apparent motion, but rather a physical rotation of plasma. In addition, we trace fieldlines from the base of the domain as they pass through the photosphere and explicitly calculate their angles of rotation which are approximately in line with the angles calculated in observations. By considering the trajectories of these selected fieldlines, we find a very helpful visual representation of two fieldlines
rotating in a circular motion around the axis (the centre of the sunspot), as shown in Fig. 4.9b.

We calculate a rotation angle of the order of one full turn (360°) over a period of the order of an hour. Two interesting questions are raised from this result. The magnitude of rotation angle is comparable with those found in observations, but what controls the final angle of rotation? Would the rotation continue if the experiments continued? Secondly, the time period of rotation is much shorter than found in observations and hence what are the reasons behind this? Both of these questions motivated a parametric study performed in Chapter 5, in which we independently vary both the initial magnetic field strength and twist of the emerging tube.

Varying the magnetic field strength and twist of the interior flux tube has a profound effect on the evolution of the flux tube in our experiments as well as the rotational properties at the photosphere. By modifying the initial magnetic field strength, we reveal the natural timescale of evolution to be the Alfvén time, and as such find time can be scaled by the initial magnetic field strength. The rotation angle, vorticity, and current also show a direct dependence on the initial magnetic field strength. Specifically, we find that an increase in field strength increases the angle through which fieldlines rotate, the length of fieldlines extending into the atmosphere, and the amount of magnetic energy and helicity transported to the atmosphere. The rotation angle’s dependence on field strength was surprising on first inspection as the twist is kept constant in this group, and we may have expected the final angle of rotation to be consistent for cases of the same twist and helical structure. Interestingly, we find the length of fieldlines are crucial in determining the angle of rotation. Stronger magnetic fields extend higher into the atmosphere owing to their larger magnetic pressure which exceeds the surrounding gas pressure. The simple hypothesis proposed by Longcope and Welsch (2000) suggests that only a fraction of twist or current carried by a twisted flux tube will pass into the corona, and a rotation of the spots continues until the twist evenly distributes along the field. Adopting this hypothesis, a longer atmospheric portion of the field results in a smaller degree of twist along the field, and hence a larger rotation angle is required to reach this degree of twist. Furthermore, we find large amounts of residual twist in the interior at the end of the experiments initiated with weaker flux tubes, again indicating the weaker fields are equilibrating to a higher degree of twist. We deduce that it is the height of axis and in turn length of fieldlines that hinder the transport of twist in weaker flux tubes. In addition, we note the final state of the rate of twist for stronger fields has a lower coronal value than interior value. This suggests that stronger tubes undergo an over-rotation, before reaching a constant rate of twist along the field.

Modifying the twist of the interior flux tube also has a significant impact on the evolution of the emergence experiments. However, as the twist of the flux tube is non-linearly related to the magnetic tension force acting on the tube, it is hard to find scaling relations between physical quantities and twist as we did in the magnetic field strength cases. Nonetheless, we do find clear trends in the simulations. An increase in the twist contained within the initial flux tube results in more helicity, energy, and current transported to the atmosphere, accompanied by a larger rotation of sunspots. Hence, we conclude that larger rotation angles are passed through for stronger, more highly twisted flux tubes.

The parametric study displayed a clear correlation between rotation angle, and the magnetic field strength and twist of the sub-photospheric field. This relation could prove very important, as this allows us to make predictions about the unobservable sub-photospheric magnetic field based on the observed sunspot rotation. As an increase in both twist and field strength increase the angle of rotation, it is hard to distin-
guish which combination of these parameters is responsible for any observed rotation. When considering observations, it is important to compare different rotating sunspots regions and use the magnetograms to estimate the strength of the sub-photospheric magnetic field.

Furthermore, the parameter study did not address the disparity between rotation rates calculated in our simulations and those observed. Even with a lower field strength and twist, the rotation rate is reduced from 9.85°/min on average (in Chapter 4) to 1.75°/min (1st experiment in Group 1 in Chapter 5), still not in line with observations that find rotation rates of a few degrees per hour. To address this, we analysed the size of observed active regions containing rotating sunspots and found that, in general, they are much larger than the active region we model in our simulations. The computational expense of running larger experiments meant that it was not feasible to check this. However, in order to test the effect of increasing the size of active region on the evolution of rotation angle, we increased the normalising length scale $L_{ph}$ by a factor of two. This modelled a flux tube twice as large, and hence an active region twice as large. With this modification, we found the evolution of rotation angle did not change. However, an increase in $L_{ph}$ results in an increase in $t_{ph}$, and hence the same rotation angle is passed through in twice the time. This halves the rotation rate experienced by the sunspots. Hence, by extrapolating this method to model an active region ten times the size, the rotation rate will be decreased by a factor of ten, producing a rotation rate more comparable to observations.

In summary, a myriad of factors affect the rotation angle and time period, and we need to get the right combination of factors to correctly reproduce the rotation rates found in observations. These factors include the magnetic field strength and twist of the tube, the size of active region and the depth at which the flux tube is inserted. We have provided an extensive study of varying the first two parameters. However, the last two parameters, namely the size of active region and depth at which the flux tube is anchored have not been fully explored in this study. We only considered individual experiments to highlight the differences here. Ideally, a full parametric study should be performed to explore the effect of these modifications. Again, an observational study would compliment this numerical investigation.

In the experiments performed in Chapters 4 and 5, it seemed reasonable to suggest the final angle of rotation was governed by the system reaching a state in which the flux tube’s twist was evenly distributed along the fieldlines. However, this was not explicitly demonstrated in these chapters as the twist per unit length had not quite reached an equilibrium along the field. To illustrate this, we performed a simple experiment of a cylindrical flux tube split into twisted and untwisted regions in Chapter 6. The twisted section of the field was placed into a densely stratified “interior” region, while the untwisted section is surrounded by a less dense “atmospheric” region. The tube was sitting in horizontal equilibrium and evolved over 300 normalised time units. The twist propagated from the twisted section of the field to the atmospheric section, until reaching a constant rate of twist along fieldlines, consistent with Longcope and Welsch (2000)’s theory. In addition, this experiment corroborated the parameter study, indicating that fieldlines with different lengths equilibrate to different values, and hence undergo different rotation angles. By altering the field strength in this experiment, we were able to isolate the effect of varying field strength from a change in fieldline length. Importantly, we recorded the same final rotation angle for two experiments of different field strengths. However, the intermediate behaviour was very different, as the strong tube reached the final angle much more quickly, producing an over-rotation before settling to the final angle, while the weak tube slowly rotated to reach the angle, without experiencing an over-rotation. This confirms the over-rotation
predicted for stronger fields in earlier chapters.

Finally, in Chapter 7, we considered sunspot rotation by a different means. Instead of emerging a pre-twisted flux tube, we emerge an untwisted flux tube and rotate the footpoints using rotational drivers on the base of the domain. This models the presence of sub-photospheric vortices at downdrafts. In this case, we observed sunspot rotation due to the transmission of twist from the interior. Importantly, we conclude that the addition of a rotational driver and in turn twist can help to round off emerging sunspots, and use this to show how asymmetries can develop in emerging sunspots. This is accomplished by considering several cases in which we vary the size, magnitude and number of velocity drivers on the base.

In Chapters 4, 5, 6 and 7, we demonstrate a rotation of sunspots. However, the source of this rotation varies in the different experiments. In Chapters 4, 5 and 6, a clockwise rotation at the photosphere originates due to the untwisting of a right-hand twisted field upon emergence. Whereas, in Chapter 7, a counter-clockwise photospheric rotation originates from the creation of a left-hand twisted field due to a counter-clockwise rotation of its footpoints. This experiment could easily have been performed with a clockwise footpoint rotation, and would have produced a right-hand twisted field with a clockwise photospheric rotation, akin to those calculated in previous chapters. Therefore, the same direction of rotation is calculated for a right-hand twisted flux tube in both cases. Hence, the cause of rotation (an untwisting of interior field or a transmission of a sub-photospheric velocity driver) cannot be deciphered from the direction of rotation and sign of twist.

The work contained within this thesis has provided us with an insight into flux emergence, and the rotation of sunspots. However, there are a number of questions that have been raised and future studies that should be carried out. Our flux emergence model for sunspot rotation predicts that the sunspots of every twisted flux tube should rotate. However, as discussed in the study by Yan et al. (2008), this is not the proportion found in observations. In fact, only 5% of active regions were found to contain rotating sunspots. What are the reasons behind this? This may be related to the threshold chosen for the observational study, in that slower rotations are missed. It also could be accounted for by the idealised symmetry of our model, which is not likely to be found in reality. However, the disparity may indicate that a large proportion of emerged magnetic fields are not highly twisted, or lose their twist on their rise to the solar photosphere. Nonetheless, the addition of convection to the interior of numerical simulations may address this, due to the distortion of the tube in the interior. A future study should perform additional simulations and conduct observational analysis to address this.

In all experiments contained within this thesis, we assume the atmosphere is unmagnetised but this is certainly not the case within the Sun’s corona. It would be interesting to investigate whether the addition of a magnetised corona affects the rate and amount of rotation within our experiments. The addition of an ambient magnetic field may diminish or enhance the rates of rotation based on the inclination of the existing field. Moreover, the rotating sunspots may lead to eruptive activity when interacting with ambient field. Hence, we suggest this could make for an interesting future study.

Furthermore, all flux tubes have been uniformly twisted in the experiments performed in this thesis, and hence a study of non-uniformly twisted tubes would be of interest. The effect of varying twist with radius may influence the rotation of sunspots at the photosphere, and hence the transport of magnetic energy to
the atmosphere.

In Appendix B, we highlight our basic preliminary efforts to model an observed asymmetry in sunspots. However, we were unsuccessful in reproducing the desired asymmetry. Future studies should address this on two levels. Firstly, the underlying source of the asymmetry should be investigated through use of observations. This will help decipher the underlying cause for the break-up of the following sunspot. Secondly, a flux tube with varying cross-sectional area should be derived to investigate if this is in fact a viable option. This would produce a concentrated strong source, and a weaker spread source.

Another possible avenue for further research is to use the rotational motions produced at the photosphere and insert them as a photospheric driver in coronal heating simulations. Studies such as De Moortel and Galsgaard (2006) analyse the coronal heating produced by rotational footpoint motions on the photospheric plane. However, by using the velocity field output from the emergence simulations, this would add a further realism to their model, and complete a more self-consistent approach.

These are just a few areas that could be investigated further, based on the findings of this thesis. With the huge advances in computing power and the abundance and quality of observations, this is an exciting time for both theorists and observers. Through the use of simulations, we learn more and more about the nature of the sub-photospheric field, and can make predictions about this based on observed phenomena.
Appendix A

Calculation of magnetic helicity

As the magnetic field evolves over the course of the experiment, we cannot calculate the magnetic helicity analytically and instead calculate it numerically at each time step. We compare and contrast two methods for calculating the magnetic helicity given a prescribed magnetic field $B$ numerically, namely DeVore (2000)’s method and Moraitis et al. (2014)’s method. We first discuss the method employed by DeVore in 2000 in which he investigates the magnetic helicity generated by differential rotation on the Sun.

A.1 DeVore’s method

As the relative magnetic helicity is gauge-invariant, we are free to choose the following $A$ and $A_p$ using the method employed by DeVore (2000):

$$A(x, y, z) = A_p(x, y, z = 0) - \hat{z} \times \int_0^z B(x, y, z') \, dz',$$

$$A_p(x, y, z) = \nabla \times \hat{z} \int_z^{\infty} \phi(x, y, z') \, dz',$$

where

$$\phi(x, y, z) = \frac{1}{2\pi} \int \int \frac{B_z(x', y', z = 0)}{[(x - x')^2 + (y - y')^2 + z^2]^{1/2}} \, dx' \, dy'.$$

For this form of $A_p$, it can be shown that $\nabla \cdot A_p = 0$ and $A_p \cdot n = 0$ on $S$. This then simplifies the expression for the relative magnetic helicity to the standard expression for magnetic helicity, given by

$$H_r = \int (A + A_p) \cdot (B - B_p) \, dV = \int A \cdot B \, dV.$$

An advantage of this method is that it does not explicitly calculate the potential magnetic field, $B_p$, only its vector potential, $A_p$. It is important to note that the derivation of relative magnetic helicity by Berger and Field (1984) and DeVore (2000) assumes an unbounded half space above the lower boundary. This scheme
A.2 Moraitis’ method

is therefore only valid in the atmospheric portion of the domain above \( z = 0 \) if we assume that the field outside the domain is zero. This assumption is only valid before the emerging field expands to hit the top and side boundaries. This derivation is therefore not applicable to the interior portion of the domain as the top boundary has flux passing through it.

As this calculation involves a triple integral of a double integral at its most complicated, the computation is very expensive and so we reduce the resolution of the simulation in order to complete the calculations in a manageable time. We have reduced the dimensions by 8 in the \( x, y, \) and \( z \) directions. This has been compared with reducing the dimensions by 4 and the results are comparable, as shown in Fig. A.1.

![Figure A.1: Evolution of the rate of change of relative magnetic helicity when reducing the dimensions of the variables by 4 (black solid line) and 8 (dashed line).](image)

A.2 Moraitis’ method

This method varies slightly from that of Devore’s in that this method can be used on a magnetic field within any type of domain as it takes into account all boundaries. In calculating the potential field within the volume \( V = [x_1, x_2] \times [y_1, y_2] \times [z_1, z_2] \), the numerical procedure utilised in Moraitis et al. (2014) takes into account all boundaries within the finite volume. This has advantages over DeVore’s method which is only valid for a semi-infinite space above a lower boundary. The potential field satisfies \( j_p = \nabla \times B_p = 0 \) within \( V \), thus implying \( B_p = -\nabla \phi \) where \( \phi \) is a scalar potential. The solenoidal constraint \( \nabla \cdot B_p = 0 \) then implies that the scalar potential is a solution of Laplace’s equation \( \nabla^2 \phi = 0 \) in \( V \). The condition that \( B \) and \( B_p \) have the same normal components along the boundaries of the volume translates to Neumann boundary conditions for \( \phi \), i.e. \( \frac{\partial \phi}{\partial n}|_{\partial V} = -\hat{n} \cdot B|_{\partial V} \). Laplace’s equation is then solved numerically using a standard FORTRAN routine included in the FISHPACK library (Swarztrauber and Sweet, 1979).

The original and potential fields are now stored for the given time step and desired volume. The next step is to calculate the corresponding vector potentials given the method proposed by Valori et al. (2012). As the relative magnetic helicity is gauge-independent, we are free to choose the gauge \( A \cdot \hat{z} = 0 \) throughout \( V \) so that the \( x \) and \( y \) components of \( B = \nabla \times A \) are integrated over the interval \((z_1, z)\) to

\[
A = A_0 - \hat{z} \times \int_{z_1}^{z} B(x, y, z') \, dz',
\]  

(A.1)
where \( \mathbf{A}_0 = \mathbf{A}(x, y, z = z_1) = (A_{0x}, A_{0y}, 0) \) is a solution to the \( z \)-component of \( \mathbf{B} = \nabla \times \mathbf{A} \), i.e.

\[
\frac{\partial A_{0y}}{\partial x} - \frac{\partial A_{0x}}{\partial y} = B_z(x, y, z = z_1).
\]

Following Valori’s method we choose the simplest solution to the above equation, given by

\[
A_{0x} = -\frac{1}{2} \int_{y_1}^{y} B_z(x, y', z = z_1) \, dy',
\]

\[
A_{0y} = \frac{1}{2} \int_{x_1}^{x} B_z(x', y, z = z_1) \, dx'.
\]

Similarly, the vector potential of the potential field is calculated using

\[
\mathbf{A}_p = \mathbf{A}_0 - \hat{z} \times \int_{z_1}^{z_2} \mathbf{B}_p(x, y, z) \, dz',
\]

where we have noted that \( \mathbf{A}_{p0} = \mathbf{A}_0 \) as \( \mathbf{B} \) and \( \mathbf{B}_p \) share the same normal component on the boundary at \( z = z_1 \).

An alternative solution for the vector potentials can be obtained if we use the top boundary, i.e. integrating over the interval \( (z, z_2) \) as

\[
\mathbf{A} = \tilde{\mathbf{A}}_0 + \hat{z} \times \int_{z}^{z_2} \mathbf{B}(x, y, z') \, dz'.
\]

This has been checked for comparison and there is no notable difference between the two solutions.

### A.3 Comparison of two methods

Figure A.2: Comparison of atmospheric helicity calculated using DeVore’s method (shown in red) and Moraitis’ method (shown in black).

To compare the two methods, we have calculated the magnetic helicity in the atmospheric portion of the domain for the experiment described in Chapter 4, as shown in Fig. A.2. The general trend in magnetic
helicity is the same for both methods, with very few deviations in the rate of change. Hence, we conclude that either method is suitable for calculation of the magnetic helicity. As Moraitis’ method is less computationally expensive and exploits the full resolution of the experiments, we employ this method to calculate the helicity in all experiments executed in this thesis.

A.4 Resistivity comparison

In an attempt to assess the role of numerical resistivity in our experiment, we have calculated the helicity dissipation in the entire volume for a range of $\eta$ values. In our experiment, we have used a uniform resistivity of $\eta = 0.005$ as depicted by the black solid line in Fig. A.3. Reducing the resistivity to $\eta = 0.0001$ and $\eta = 0.001$ results in a much smaller rate of change due to resistive dissipation as expected. The initial rate of change of helicity for $\eta = 0.001$ is approximately $1/5$ of the initial rate of change for $\eta = 0.005$. This suggests that this dissipation is due to real resistivity rather than numerical effects. Increasing the value of $\eta$ to 0.01 results in a larger decrease in relative magnetic helicity.

![Figure A.3: Evolution of the relative magnetic helicity and rate of change of helicity over the entire volume for four different $\eta$ values. A uniform resistivity of $\eta = 0.005$ is denoted by the black line (our experiment), a resistivity of $\eta = 0.0001$ is shown in blue, $\eta = 0.001$ is shown in green and $\eta = 0.01$ is shown in red.](image-url)
Appendix B

Asymmetry in sunspot formation

Many well-known asymmetries exist with regards to solar active regions. The leading polarity and following polarity sunspots (with respect to the rotation of the Sun) have inherent differences in their stability, size and evolution. One particularly interesting property of bipolar active regions is the asymmetry in structure between the leading and following polarities. The leading polarity of active regions tend to be in the form of large, coherent sunspots, whilst the following sunspots appear more fragmented and dispersed. Furthermore, the leading spots tend to outlive the following spots. An observation that depicts this asymmetry is shown below in Fig. B.1. In this appendix, we outline our initial efforts to try and model this observed feature.

Figure B.1: Close up of observation of an asymmetric sunspot pair from a HMI magnetogram on June 10th 2014.

Leading sunspots in a sunspot pair or group tend to outlast following sunspots. Meyer et al. (1977) suggested this is a result of the leading part of the rising tube becoming almost vertical and the following part becoming inclined to the vertical. This means the leading spot is more stable and coherent while the following spot is more likely to break up. Fan et al. (1993) also proposed an explanation for the asymmetry we have outlined. Through the use of thin flux tube simulations of emerging $\Omega$-loops through a solar convection zone model that includes rotation, they found that a remarkable asymmetry develops between
the field strength of leading and following polarities. Fan et al. (1993) found that, in general, the leading side of the loop has a field strength twice as strong as that of the following side. They also found that the cross section of the leading leg is about half as large as that of the following side. With a greater field strength, the leading side is much more stable against perturbations, and hence tends to outlive the following spot.

In this appendix, we describe various approaches we used to try and model this sunspot asymmetry. Precisely, we attempt to model one strong, coherent sunspot accompanied by a more fragmented, weak sunspot. To do this we first consider a configuration of two loops to try and reproduce this observed effect. We then go on to discuss a case in which we vary the buoyancy in order to make one leg of the tube more buoyant than the other. To conclude, we summarise our preliminary findings and discuss future approaches that are worth implementing.

### B.1 Two loop model

#### B.1.1 Initial set-up

In this section we use a similar set-up to that introduced in Chapter 3. However, in order to insert two flux tubes into the interior, we extend the base of the domain to \( z = -35 \) keeping the solar surface at \( z = 0 \). As shown in Fig. B.2a, Tube A has a major radius of \( R_0 = 27 \) and a minor radius of \( a = 2.5 \) with an axial field strength of \( B_0 = 8 \) and twist of \( \alpha = 0.4 \). The second tube inserted below Tube A, henceforth referred to as Tube B, has a major radius of \( R_0 = 15 \) and a minor radius of \( a = 2.5 \) with an axial field strength of \( B_0 = 15 \) and a twist of \( \alpha = 0.4 \). To ensure the tubes are not initially interfering with one another, Tube A is shifted 6 units to the left in the \( y \) direction. The previous \( B_\phi = B_0 e^{-r^2/a^2} \) prescribed in Section 3.3.2 has an exponential form and hence the tubes’ magnetic fields overlap due to their close proximity. In order to prevent the tubes from interfering, we prescribe a piece-wise function for \( B_\phi \) as

\[
B_\phi = \begin{cases} 
B_0(1 - r^2/a^2)^2 & r \leq a, \\
0 & r > a.
\end{cases}
\]

This ensures \( B = 0 \) outside the flux tube, i.e. for \( r > a = 2.5 \). In addition, this ensures there are no unwanted forces in the initial setup of our experiment as there is no interaction of the two flux tubes. This change in \( B_\phi \) also alters the pressure excess to

\[
p_{exc} = \frac{B_0^2}{4}(1 - r^2/a^2)^4\left(\frac{2}{5}(\alpha a)^2 - \frac{12}{5}(\alpha r)^2 - 2\right).
\]

The two tubes are initially set up so they are in equilibrium to a first order approximation as described in Chapter 3. In order to initiate this experiment, we impose a density deficit to allow both of the tubes to rise, as displayed in Fig. B.2b. Tube B is made more buoyant due to its larger \( B_0 \) value, and hence has a larger density deficit. We have chosen to set the flux tubes up in this way to allow Tube B to rise through the interior more quickly than Tube A in an attempt for the two flux tubes to emerge at a similar time. The density deficit is normally found by setting the internal temperature to the external temperature
and calculating $\rho_{exc} = \rho_{exc}/T$. This is the case for Tube A, however for Tube B we have doubled the density deficit, thereby increasing the temperature inside this loop. This adjustment makes Tube B even more buoyant allowing it to quickly catch up with Tube A.

Figure B.2: (a) Cross section of inserted flux tubes labelled A and B, and (b) shows the density excess ($\rho_{exc}$) used to initiate the emergence against height from the solar surface $z$.

Figure B.3: Illustration of initial setup of our model. The simulation domain is displayed with the density distribution shown on the right wall, the temperature variation on the back wall and a contour outlining the magnetic field. The solar surface is also highlighted at $z = 0$.

A summary of the initial set-up is shown as a 3D visualisation in Fig. B.3, where the placement of the two flux tubes is shown clearly. This also shows the temperature and density stratification on the back and right walls.

B.1.2 Preliminary results

Synthetic magnetograms with horizontal velocity vectors are investigated at the solar surface ($z = 0$) in Fig. B.4 in an attempt to determine if we observe this asymmetric feature. Tube A emerges at the
photosphere first with its sources at a slight angle to the N-S direction. Very quickly, sunspot tails form and the sources start to drift towards the E-W direction. Subsequently, Tube B emerges and its sources begin to stray outwards. As these sunspots move, they appear to disturb the existing sunspots causing them to fragment and migrate outwards. The positive (white) polarity sources appear to merge to an extent to produce a concentrated flux source. However, the sunspot is not well rounded and is clearly made up of two spots of different sizes. On the contrary, the two flux tubes have merged together more clearly further down in the interior as evidenced in a cut at \( z = -5 \) as shown in Fig. B.5a. Although they merge to a degree it is clear that a lot of the existing field has been forced outwards at photospheric heights.

![Figure B.4: Coloured contours of the vertical magnetic field (also known as synthetic magnetograms) at times (a) \( t = 15 \), (b) \( t = 20 \), (c) \( t = 25 \), and (d) \( t = 35 \). Horizontal velocity vectors are overplotted in black.](image)

Now that we have considered the photospheric magnetic field and shape and configuration of the sunspots, we can also examine a 3D visualisation of fieldlines as shown in Fig. B.5b. The red fieldlines are traced from both footpoints of Tube A, and the blue fieldlines are traced from both footpoints of Tube B. From this figure, it is clear that most of the field is trapped at photospheric heights. Tube B is more buoyant but is restricted by the trapped field of Tube A above. The right footpoints of the flux tubes are closer than the left due to the way in which we set up the flux tubes. By \( t = 40 \), as displayed in Fig. B.5b, the right legs of the flux tubes have merged together just below the photosphere, whereas the left legs of the flux
tube remain separate. There is also some reconnection as some fieldlines traced from a red footpoint reach a blue footpoint and vice versa.

![Figure B.5: (a) Coloured contour of vertical magnetic field at $z = -5$ and (b) 3D visualisation of fieldlines as traced from all four footpoints coloured by whether they are traced from a footpoint of Tube A (in red) or Tube B (in blue). Both images are taken at $t = 40$.](image)

Although this experiment did produce an asymmetry in the sunspots, it was not the desired asymmetry seen in observations (see Fig. B.1). That is, we did not manage to reproduce a strong, concentrated sunspot accompanied by a weak, fragmented collection of oppositely signed flux. There was some evidence that the positive sunspots merged, but they did not produce a strong coherent, circular sunspot. The negative sunspot of Tube A breaks apart, but that of Tube B remains round and coherent, therefore not producing the fragmented structure desired. The drawback of this model is that the emergence of the second flux tube caused the first sunspots to break apart and drift.

This model tells us the use of two close sources at the base does not produce one coherent sunspot. Instead a common source is needed to produce the main coherent sunspot.

### B.2 Varying buoyancy

Another method we tested to try and model this asymmetry is to use a different form of buoyancy profile. In particular, we consider a set-up in which the leading sunspot sits above a vertical leg while the following sunspot sits above a slanted leg. In our experiments, neither sunspot is leading or following but the general behaviour can be seen by treating the sunspots differently.

In order to make one leg of the tube more buoyant than the other, we use the density excess displayed...
We note that this is similar to the density excess used in Chapter 3 with a tanh(y) + 3/2 factor multiplying the expression. This ensures that one leg is more buoyant than the other, and changes the shape of the emerging flux tube before it reaches \( z = 0 \). We have lowered the base to \( z = -30 \) to allow the density deficit to affect the tube’s shape before it reaches photospheric heights.

![Figure B.6: Density excess as a function of y at (a) \( z = -30 \) and (b) \( z = -15 \) used to initiate the emergence.](image)

To demonstrate the effect of the \( y \)-dependent density deficit, we have taken a cut of a coloured contour of \( |B| = \sqrt{B_x^2 + B_y^2 + B_z^2} \) in the \( x = 0 \) plane in Fig. B.7. This demonstrates the shape of the emerging flux tube at an early and late stage of the experiment. The density deficit introduced makes the right leg more buoyant than the left, and hence causes the right leg to be straighter, while the left leg becomes curved (see Fig. B.7a). Simultaneously, the right leg’s magnetic field becomes stronger and more concentrated while the left leg remains weaker and disperses. By \( t = 130 \), in Fig. B.7b, the right leg remains straight and concentrated, while the left leg continues to drift until the left sunspot has drifted further than the left footpoint. The left leg also appears wider with a lower field strength.

Corresponding synthetic magnetograms are shown in Fig. B.8, for the same snapshots as in Fig. B.7. At \( t = 30 \), there is little difference in the sunspots formed, except the left (lower) sunspot is slightly larger. However, by \( t = 130 \), the lower sunspot associated with the less buoyant leg remains quite round and coherent, while the upper sunspot associated with the more buoyant leg starts to break apart and fragment. This is the opposite of what we would expect given the theory put forward by Meyer et al. (1977). These results are preliminary and require further investigation to be understood fully.

### B.3 Summary and future work

In this appendix, we discussed two separate approaches for modelling the asymmetries observed in sunspot formation. In the first case, we emerged two tubes of different sizes (one on top of the other). We positioned
the tubes such that two of the footpoints are close together and the others are far apart. In this experiment, we found that the emergence of the second tube caused the first sunspots to break apart. Although this experiment did yield some kind of asymmetry, it is not the asymmetry we are looking to model. The neighbouring sunspots do not merge to form a coherent sunspot.

In the second experiment, we choose a different buoyancy profile in which we make one leg of the tube more buoyant than the other. This causes there to be one strong vertical leg and one inclined weaker leg. Again, an asymmetry develops; the sunspot lying above the vertical, buoyant leg begins to separate while the sunspot lying above the slanted, less buoyant leg remains quite coherent. This is the opposite of what we expect given Meyer et al. (1977)’s theory and requires further investigation to understand.

We also planned to try a third approach. Specifically, to produce a flux tube with varying cross sectional area, as presented schematically in Fig. B.9. As discussed in Section 1.3.2, as the flux of a tube remains constant along its length, the magnetic field strength of the tube increases as it narrows and decreases as
B.3 Summary and future work

Figure B.8: Coloured contour of $B_z$ (synthetic magnetogram) in the $z = 0$ $x - y$ plane (a) $t = 30$ and (b) $t = 130$.

Figure B.9: Schematic demonstrating type of flux tube that should be used to initiate emergence experiment that may have asymmetric sunspots.

it widens. Hence, the flux tube shown in Fig. B.9 would have one strong, narrow leg and another weaker, wider leg. Upon emergence, this may produce the type of asymmetry in sunspots we are looking to model. However, we struggled to come up with an analytic form for the initial magnetic field where the radius of the flux tube $a$ is a function of $y$. This is a possible avenue for future work.
Bibliography


