

# Note on Prodi-Serrin-Ladyzhenskaya type regularity criteria for the Navier-Stokes equations

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In this article we prove new regularity criteria of the Prodi-Serrin-Ladyzhenskaya type for the Cauchy problem of the three-dimensional incompressible Navier-Stokes equations. Our results improve the classical  $L^r(0, T; L^s)$  regularity criteria for both velocity and pressure by factors of certain negative powers of the scaling invariant norms  $\|u\|_{L^3}$  and  $\|u\|_{\dot{H}^{1/2}}$ .

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## I. INTRODUCTION

The Cauchy problem of the three-dimensional incompressible Navier-Stokes equations play an important role in not only mathematical fluid mechanics but also the development of the theory of general evolutionary equations. The system reads

$$u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u, \quad (x, t) \in \mathbb{R}^3 \times (0, \infty), \quad (1)$$

$$\operatorname{div} u = 0, \quad (x, t) \in \mathbb{R}^3 \times (0, \infty), \quad (2)$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^3, \quad (3)$$

where  $u : \mathbb{R}^3 \mapsto \mathbb{R}^3$  is the velocity field,  $p : \mathbb{R}^3 \mapsto \mathbb{R}$  is the pressure, and  $\nu > 0$  is the (dimensionless) viscosity.

Systematic study of this problem began in 1934 with the classical paper [22] by Jean Leray, where it is shown that for arbitrary  $T \in (0, \infty]$  there is at least one function  $u(x, t)$  satisfying the following.

- i.  $u \in L^\infty(0, T; L^2(\mathbb{R}^d)) \cap L^2(0, T; H^1(\mathbb{R}^d))$ ;
- ii.  $u$  satisfies (1) and (2) in the sense of distributions;
- iii.  $u$  takes the initial value in the  $L^2$  sense:  $\lim_{t \searrow 0} \|u(\cdot, t) - u_0(\cdot)\|_{L^2} = 0$ ;
- iv.  $u$  satisfies the energy inequality

$$\|u(\cdot, t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(\cdot, \tau)\|_{L^2}^2 d\tau \leq \|u_0\|_{L^2}^2 \quad (4)$$

for all  $0 \leq t \leq T$ .

Such a function  $u(x, t)$  is called a Leray-Hopf weak solution for (1)–(3) in  $\mathbb{R}^d \times [0, T)$ .

It is easy to show that if a Leray-Hopf weak solution is smooth, then it is a classical solution and is furthermore unique (in the class of Leray-Hopf weak solutions). However the smoothness of Leray-Hopf weak solutions is still a completely open problem. On the other hand, it has been long known that various additional assumptions guarantee such smoothness. One important class of such assumptions is the following so-called Prodi-Serrin-Ladyzhenskaya criteria, developed over three decades in [11], [21], [24], [23], [25], [26]. If a Leray-Hopf solution  $u(x, t)$  further satisfies

$$u \in L^r(0, T; L^s(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{r} + \frac{3}{s} \leq 1, \quad 3 < s \leq \infty, \quad (5)$$

then  $u(x, t)$  is smooth. Here the norm of the mixed Lebesgue space  $L^r(0, T; L^s(\mathbb{R}^3))$  is defined as

$$\|u\|_{L^r(0, T; L^s)} := \begin{cases} \left( \int_0^T \|u(\cdot, t)\|_{L^s}^r dt \right)^{1/r} & 1 \leq r < \infty \\ \operatorname{esssup}_{t \in (0, T)} \|u(\cdot, t)\|_{L^s} & r = \infty \end{cases}. \quad (6)$$

The proof of the criterion (5) is quite straightforward through standard energy estimate, though it should be mentioned that it is the much more non-trivial “localized” version of (5) that was proved in the references above. The borderline case  $u \in L^\infty(0, T; L^3(\mathbb{R}^3))$  that is missing in (5) turned out to be much more complicated due to the criticality of  $\|u\|_{L^3}$  under the rescaling transformation  $u(x, t) \mapsto \lambda u(\lambda x, \lambda^2 t)$  that keeps (1)–(3) invariant. This case could not be dealt with using the method that established (5), and was only recently settled by Escauriaza, Seregin, and Šverák in

[10] via a novel approach based on deep understanding of backward uniqueness of heat equations. Many generalizations and refinements of (5) have been proved, see e.g. [3], [5], [8], [12], [31], [34].

Mathematically the pressure  $p$  serves as the Lagrange multiplier of the incompressibility constraint  $\operatorname{div} u = 0$ . As a consequence there is no explicit equation governing the evolution of  $p$  in (1)–(3). The lack of such an equation is partially compensated through the following relation between  $u$  and  $p$  obtained via taking divergence of (1),

$$-\Delta p = \sum_{i,j=1}^3 \partial_i \partial_j (u_i u_j). \quad (7)$$

For the convenience of analyzing (1)–(3) in the framework of functional analysis, (7) is usually written as

$$p = \sum_{i,j=1}^3 R_i R_j (u_i u_j), \quad (8)$$

where  $R_i, i = 1, 2, 3$  are the Riesz transforms. As Riesz transforms are zeroth order pseudo-differential operators, there holds

$$\|p\|_{L^s} \leq C \|u\|_{L^{2s}}^2 \quad \text{for all } s \in (1, \infty). \quad (9)$$

From (9) it is natural to conjecture that

$$p \in L^r(0, T; L^s) \quad \text{with} \quad \frac{2}{r} + \frac{3}{s} \leq 2, \quad \frac{3}{2} < s \leq \infty \quad (10)$$

may guarantee the smoothness of  $u$ . Note that thanks to (9), (10) is a weaker assumption than (5) as it is implied by the latter. The affirmative answer to this conjecture was established in [2], [7] and later refined in many follow-up papers, including [1], [3], [9], [13], [16], [17], [19], [27], [28], [30].

Roughly speaking, most of the aforementioned improvements of (5) or (10) can be categorized into two types. The first type replaces the Lebesgue norm  $L^s$  and/or  $L^r$  by weaker norms with the same scaling property. For example in [13]  $\|p\|_{L^s}$  is replaced by the homogeneous Besov norm  $\|p\|_{\dot{B}_{s,\sigma}^0}$  for some appropriate  $\sigma$ , and in [3]  $\|u\|_{L^s}$  is replaced by  $\|u\|_{L^{s,\infty}}$  where  $L^{s,\infty}$  are the weak Lebesgue spaces. The second type of improvement weakens the conditions by a logarithmic factor. For example in [31] it is shown that  $u$  is smooth as long as

$$\int_0^T \frac{\|u\|_{L^s}^r}{\log(e + \|u\|_{L^\infty})} dt < \infty, \quad \frac{2}{r} + \frac{3}{s} \leq 1, \quad 3 < s \leq \infty. \quad (11)$$

Possibility of combining these two types of improvement has been well explored, see e.g. [12], [16]. We must also mention that criteria similar to (5) or (10) have also been proved for other physically meaningful quantities such as  $\nabla u, \omega := \operatorname{curl} u$ , and  $\nabla p$ , see e.g. [4], [6], [14], [33], [35].

In this article we present and prove a new type of improvement of (5) and (10), of the form

$$\int_0^T \frac{\|u\|_{L^s}^r}{(1 + \|u\|_X)^\kappa} dt < \infty, \quad \frac{2}{r} + \frac{3}{s} \leq 1, \quad 3 < s < \infty, \quad (12)$$

or

$$\int_0^T \frac{\|p\|_{L^s}^r}{(1 + \|u\|_X)^\kappa} dt < \infty, \quad \frac{2}{r} + \frac{3}{s} \leq 2, \quad \frac{3}{2} < s \leq \infty, \quad (13)$$

where  $\|\cdot\|_X$  is a scaling-invariant norm for (1)–(3) and  $\kappa > 0$ . More specifically, we will prove the following theorems.

**Theorem 1.** *Let  $u_0 \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$  for some  $q > 3$ , and satisfy  $\operatorname{div}u_0 = 0$ . Let  $u(t, x)$  be a Leray-Hopf solution of NSE in  $[0, T)$ . If for some  $s \in (3, \infty)$  and  $\frac{2}{r} + \frac{3}{s} = 1$  there holds*

$$\int_0^T \frac{\|u\|_{L^s}^r}{(1 + \|u\|_{\dot{H}^{1/2}})^\kappa} dt < \infty, \quad (14)$$

where  $\kappa = \begin{cases} 2 & 3 < s \leq 5 \\ \frac{4}{s-3} & 5 < s < \infty \end{cases}$ , then  $u(t, x)$  is smooth up to  $T$  and could be extended beyond  $T$ .

**Theorem 2.** *Let  $u_0 \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$  for some  $q > 3$ , and satisfy  $\operatorname{div}u_0 = 0$ . Let  $u(t, x)$  be a Leray-Hopf solution of NSE in  $[0, T)$ . If for some  $s \in (3, \infty)$  and  $\frac{2}{r} + \frac{3}{s} = 1$  there holds*

$$\int_0^T \frac{\|u\|_{L^s}^r}{(1 + \|u\|_{L^3})^\kappa} dt < \infty, \quad (15)$$

where  $\kappa = \begin{cases} 3 & 3 < s \leq 5 \\ \frac{6}{s-3} & 5 < s < \infty \end{cases}$ , then  $u(t, x)$  is smooth up to  $T$  and could be extended beyond  $T$ .

**Theorem 3.** *Let  $u_0 \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$  for some  $q > 3$ , and satisfy  $\operatorname{div}u_0 = 0$ . Let  $u(t, x)$  be a Leray-Hopf solution of NSE in  $[0, T)$ . If for some  $s \in (\frac{3}{2}, \infty)$  and  $\frac{2}{r} + \frac{3}{s} = 2$  there holds*

$$\int_0^T \frac{\|p\|_{L^s}^r}{(1 + \|u\|_{L^3})^\kappa} dt < \infty, \quad (16)$$

where  $\kappa = \begin{cases} \frac{6}{2s-3} & s \geq 3 \\ \frac{2s}{2s-3} & \frac{9}{4} \leq s \leq 3 \\ 3 & \frac{3}{2} < s \leq \frac{9}{4} \end{cases}$ , then  $u(t, x)$  is smooth up to  $T$  and could be extended beyond  $T$ .

A few remarks are in order.

**Remark 1.** *It is straightforward to cover the sub-critical cases. For example, if  $u$  satisfies (14) for some  $r, s$  satisfying  $s \in (3, \infty)$  and  $\frac{2}{r} + \frac{3}{s} < 1$ , then there is  $r' < r$  such that  $\frac{2}{r'} + \frac{3}{s} = 1$  and (14) holds for  $r', s$  thanks to Hölder's inequality.*

**Remark 2.** *We emphasize that unlike (5) and (10), where one is weaker than the other, Theorems 1, 2, and 3 are independent of each other. For example, although  $\|u\|_{L^3} \leq C\|u\|_{\dot{H}^{1/2}}$  by Sobolev embedding, Theorem 2 does not follow from Theorems 1, as the values of  $\kappa$  are different.*

**Remark 3.** *When  $s = 6$ , criterion (15) can be seen to be comparable but not equivalent to the following criterion*

$$\int_0^T \frac{\|u\|_{L^6}^6}{\|u\|_{L^4}^4 + 2\|u\|_{\dot{H}^1}^2} < \infty, \quad (17)$$

which has been reported recently [29]. The two criteria would be equivalent if we could establish that  $\|u\|_{L^4}^4 + 2\|u\|_{\dot{H}^1}^2 \approx (\|u\|_{L^3}^2 + 2)\|u\|_{L^6}^2$ , which is consistent in scaling yet does not hold for general divergence free vector field  $u$ . For all other  $s > 3$ , (15) may be regarded as an extension of (17). It is also of interest to explore the possibility of weakening (16) through replacing the pressure  $p$  by the "effective pressure"  $p + \mathcal{P}$  in [29].

**Remark 4.** *It is not clear whether the splitting into different cases in Theorems 1, 2, and 3 is purely technical, due to the limitations of the energy method, or reflects deeper properties of the Navier-Stokes dynamics.*

**Remark 5.** *From the proofs we will see that a logarithmic factor could easily be added “for free”. For example, (15) could be replaced by*

$$\int_0^T \frac{\|u\|_{L^s}^r}{(1 + \|u\|_{L^3})^\kappa \log(e + \|u\|_{L^3})} dt < \infty. \quad (18)$$

*However it does not seem likely that the  $\|u\|_{L^3}$  inside the logarithm could be replaced by  $\|u\|_{L^\infty}$ . Thus our results are not stronger than, though still independent of, the previous logarithmic improvement results such as (11).*

**Remark 6.** *Criteria in a sense similar to Theorems 1, 2, and 3 have been proved in [18] and [32]. There it is shown that smoothness of the solution  $u$  is guaranteed if one of the following holds.*

- [32].

$$\frac{p}{1 + |u|^\delta} \in L^r(0, T; L^s), \quad \frac{2}{r} + \frac{3}{s} = \frac{5 - 3\delta}{2}, \quad \frac{6}{5 - 3\delta} < s \leq \infty, \quad 1 \leq \delta \leq \frac{5}{3}; \quad (19)$$

- [18].

$$\frac{p}{1 + |u|^\delta} \in L^r(0, T; L^s), \quad \text{with } \frac{2}{r} + \frac{3}{s} = \frac{4 - 3\delta}{2}, \quad \frac{18}{8 - 9\delta} \leq s \leq \frac{6}{2 - 3\delta}, \quad 0 \leq \delta < \frac{2}{3}. \quad (20)$$

- [18].

$$\frac{p}{1 + |u|^\delta} \in L^r(0, T; L^s), \quad \text{with } \frac{2}{r} + \frac{3}{s} = \frac{4 - 3\delta}{2}, \quad \frac{18}{8 - 9\delta} \leq s \leq \frac{6}{2 - 3\delta}, \quad \frac{2}{3} \leq \delta \leq \frac{8}{9}. \quad (21)$$

*We briefly discuss their relations to Theorems 1, 2, and 3.*

1. (19)–(21) are sub-critical from a scaling point of view and therefore does not improve (10) except for the case  $\delta = 1$  in (19) (and  $\delta = 0$  in (20) which reduces (20) to (10)). To see this we recall (8) which dictates that  $p$  scales as  $|u|^2$ . This makes  $\frac{p}{1 + |u|^\delta} \in L^r(0, T; L^s)$  roughly equivalent to, from the scaling point of view,  $|u|^{2-\delta} \in L^r(0, T; L^s)$  that is  $u \in L^{(2-\delta)r}(0, T; L^{(2-\delta)s})$ . Thus for example (20) corresponds to

$$\frac{2}{(2-\delta)r} + \frac{3}{(2-\delta)s} = \frac{4-3\delta}{4-2\delta} < 1 \quad (22)$$

*for all values of  $\delta$  except  $\delta = 0$ . Similarly, in (19) we have*

$$\frac{2}{(2-\delta)r} + \frac{3}{(2-\delta)s} = 1 + \frac{1-\delta}{4-2\delta} < 1 \quad (23)$$

*unless  $\delta = 1$ .*

2. On the other hand, in (19)–(21) the factor  $(1 + |u|^\delta)^{-1}$  is inside the whole space-time integral, while in our theorems  $(1 + \|u\|_{L^3})^{-\kappa}$  or  $(1 + \|u\|_{\dot{H}^{1/2}})^{-\kappa}$  is only inside the time integral. Thus the conditions (19)–(21) are in a sense more “localized”.

**Remark 7.** It is quite straightforward to generalize Theorems 1, 2, and 3 to  $d$ -dimensional Navier-Stokes equations. For simplicity of presentation we will focus on the physical case  $d = 3$  in this article.

In the next section we prove Theorems 1, 2, and 3.

## II. PROOF OF THEOREMS

### A. Preliminaries

Without loss of generality, we take  $\nu = 1$  in (1) to simplify the presentation. We apply the following result, summarized from [15], [20], to guarantee short-time smoothness of the solution and thus relieving us from worrying about the legitimacy of the various integral and differential manipulations below.

**Theorem 4.** Let  $u_0 \in L^s(\mathbb{R}^3)$ ,  $s \geq 3$ . Then there exists  $T > 0$  and a unique classical solution  $u \in \text{BC}(0, T; L^s(\mathbb{R}^3))$ . Moreover, let  $(0, T_*)$  be the maximal interval such that the solution  $u$  stays in  $C(0, T_*; L^s(\mathbb{R}^3))$ ,  $s > 3$ . Then for any  $t \in (0, T_*)$ ,

$$\|u(\cdot, t)\|_{L^s} \geq \frac{C}{(T_* - t)^{\frac{s-3}{2s}}} \quad (24)$$

where the constant  $C$  is independent of  $T_*$  and  $s$ .

We will also need the following simple lemma.

**Lemma 1.** Let  $X(t) \in C^1(0, T) \cap C([0, T])$  be non-negative and solve  $\dot{X}(t) \leq \frac{A(t)}{X(t)^\kappa} X(t) + \frac{C}{X(t)^\kappa}$  for some  $A(t) \geq 0$ ,  $\kappa > 0$ . Assume

$$\int_0^T \frac{A(t)}{(1 + X(t))^\kappa} dt < \infty. \quad (25)$$

Then  $\limsup_{t \rightarrow T^-} X(t) < \infty$ .

*Proof.* Denote  $B(t) := \frac{A(t)}{\max\{1, X(t)\}^\kappa}$ . It is clear that (25) is equivalent to  $\int_0^T B(t) dt < \infty$ . Let  $Y(t) := \max\{1, X(t)\}$ . Then  $Y(t) = X(t)$  on the union of at most countably many open intervals  $(t_{iL}, t_{iR})$  with  $Y(t_{iL}) = 1$ . Now on  $(t_{iL}, t_{iR})$  we have  $Y(t) > 1$  and therefore

$$\begin{aligned} \dot{Y}(t) &\leq \frac{A(t)}{X(t)^\kappa} X(t) + \frac{C}{X(t)^\kappa} \\ &= \frac{A(t)}{Y(t)^\kappa} Y(t) + \frac{C}{Y(t)^\kappa} \\ &\leq B(t)Y(t) + C. \end{aligned} \quad (26)$$

The conclusion immediately follows.  $\square$

Finally we need the following result which is a special case of Theorem 1.3 in [10].

**Theorem 5.** *Suppose that  $u$  is a weak Leray-Hopf solution of the Cauchy problem (1)–(3). If furthermore  $u \in L^\infty(0, T; L^3(\mathbb{R}^3))$ , then  $u$  is smooth up to  $T$  and can be extended beyond  $T$ .*

**Notation.** *In the following we will use  $A \lesssim B$  to denote  $A \leq cB$  when the constant  $c$  is inconsequential to the validity of the proof.*

## B. Proof of Theorem 1

Assume the contrary. By Theorem 4 there is  $T^* \in (0, T)$  such that  $u(x, t)$  is smooth for  $t \in (0, T^*)$  but cease to be so at  $t = T^*$ . Thanks to Theorem 5 and the Sobolev embedding  $\dot{H}^{1/2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$ , this implies  $\limsup_{t \nearrow T^*} \|u\|_{\dot{H}^{1/2}} = \infty$ . Therefore to obtain contradiction it suffices to prove that  $\|u\|_{\dot{H}^{1/2}} \leq C$ ,  $\forall t \in (0, T^*)$ , for some constant  $C > 0$ .

Let  $\Lambda := (-\Delta)^{1/2}$ . Multiplying the equation by  $\Lambda u$  and then integrate over  $\mathbb{R}^3$ , we reach

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{\dot{H}^{1/2}}^2 &= - \int_{\mathbb{R}^3} u \cdot \nabla u \cdot \Lambda u dx - \int_{\mathbb{R}^3} \Lambda u \cdot \Lambda^2 u dx \\ &\leq \int_{\mathbb{R}^3} |u| |\nabla u| |\Lambda u| dx - \|u\|_{\dot{H}^{3/2}}^2. \end{aligned} \quad (27)$$

In what follows we discuss the two regimes of  $s$  stated in the theorem separately.

- $3 < s \leq 5$ .

Let  $s'$  be the conjugate to  $s$ , that is  $\frac{1}{s} + \frac{1}{s'} = 1$ . We start by estimating using Hölder's inequality

$$I := \int_{\mathbb{R}^3} |u| |\nabla u| |\Lambda u| dx \leq \|u\|_{L^s} \|\nabla u\|_{L^{2s'}} \|\Lambda u\|_{L^{2s'}}. \quad (28)$$

Next we notice that as  $3 < s \leq 5 \implies 2s' \in [\frac{5}{2}, 3) \subset (1, \infty)$ , the boundedness of Riesz transforms on  $L^p(\mathbb{R}^3)$  for  $1 < p < \infty$  now yields  $\|\Lambda u\|_{L^{2s'}} \lesssim \|\nabla u\|_{L^{2s'}}$  since  $\Lambda u = -\Lambda^{-1} \nabla \cdot (\nabla u) = -\sum_{i=1}^3 R_i(\partial_i u)$ . Consequently

$$I \lesssim \|u\|_{L^s} \|\nabla u\|_{L^{2s'}}^2. \quad (29)$$

Now thanks to the the interpolation inequality

$$\|\nabla u\|_{L^{2s'}} \lesssim \|u\|_{L^s}^{\frac{s-3}{6}} \|u\|_{\dot{H}^{3/2}}^{\frac{9-s}{6}}, \quad (30)$$

we further obtain

$$I \lesssim \|u\|_{L^s} \|u\|_{L^s}^{(s-3)/3} \|u\|_{\dot{H}^{3/2}}^{(9-s)/3}. \quad (31)$$

Finally by Young's inequality we conclude that

$$I \leq C \|u\|_{L^s}^r + \frac{1}{2} \|u\|_{\dot{H}^{3/2}}^2 \quad (32)$$

with  $r = \frac{2s}{s-3}$ .

Substituting (32) into (27), we have

$$\|u\|_{\dot{H}^{1/2}} \frac{d}{dt} \|u\|_{\dot{H}^{1/2}} \lesssim \|u\|_{L^s}^r \quad (33)$$

which gives

$$\frac{d}{dt} \|u\|_{\dot{H}^{1/2}} \lesssim \frac{\|u\|_{L^s}^r}{\|u\|_{\dot{H}^{1/2}}^2} \|u\|_{\dot{H}^{1/2}} \quad (34)$$

and the conclusion follows from Lemma 1.

- $5 < s < \infty$ .

We first notice that

$$\int_{\mathbb{R}^3} |u| |\nabla u| |\Lambda u| dx \leq \int_{\mathbb{R}^3} |u| |\nabla u|^2 dx + \int_{\mathbb{R}^3} |u| |\Lambda u|^2 dx =: I + II. \quad (35)$$

We estimate  $I$  first. By Hölder inequality we have

$$\begin{aligned} I &= \int_{\mathbb{R}^3} |u| |\nabla u|^{\frac{2(s-5)}{s-2}} |\nabla u|^{\frac{6}{s-2}} dx \\ &\leq \|u\|_{L^s} \|\nabla u\|_{L^{\frac{s-2}{s-5}}}^{\frac{2(s-5)}{s-2}} \|\nabla u\|_{L^{\frac{s-2}{s-5}}}^{\frac{6}{s-2}} \\ &= \|u\|_{L^s} \|\nabla u\|_{L^2}^{\frac{2(s-5)}{s-2}} \|\nabla u\|_{L^{\frac{3s}{s+1}}}^{\frac{6}{s-2}}. \end{aligned} \quad (36)$$

Note that as  $s > 5$ , there holds  $1 < \frac{s-2}{s-5} < \infty$  and  $1 < \frac{s}{4} < \frac{s(s-2)}{2(s+1)} < \infty$ . Therefore the application of Hölder inequality is justified.

Next we apply the following Gagliardo-Nirenberg inequalities,

$$\|\nabla u\|_{L^2} \lesssim \|u\|_{\dot{H}^{1/2}}^{1/2} \|u\|_{\dot{H}^{3/2}}^{1/2}, \quad \|\nabla u\|_{L^{3s/(s+1)}} \lesssim \|u\|_{L^s}^{1/3} \|u\|_{\dot{H}^{3/2}}^{2/3}, \quad (37)$$

to obtain

$$I \lesssim \|u\|_{L^s}^{s/(s-2)} \|u\|_{\dot{H}^{1/2}}^{(s-5)/(s-2)} \|u\|_{\dot{H}^{3/2}}^{(s-1)/(s-2)}. \quad (38)$$

Young's inequality now yields

$$I \leq C \|u\|_{L^s}^{2s/(s-3)} \|u\|_{\dot{H}^{1/2}}^{2(s-5)/(s-3)} + \frac{1}{2} \|u\|_{\dot{H}^{3/2}}^2. \quad (39)$$

Through almost identical arguments, the same estimate could be obtained for  $II$ .

$$II \leq C \|u\|_{L^s}^{2s/(s-3)} \|u\|_{\dot{H}^{1/2}}^{2(s-5)/(s-3)} + \frac{1}{2} \|u\|_{\dot{H}^{3/2}}^2. \quad (40)$$

Substituting these into (27) and dividing both sides by  $\|u\|_{\dot{H}^{1/2}}$ , we obtain

$$\frac{d}{dt} \|u\|_{\dot{H}^{1/2}} \lesssim \frac{\|u\|_{L^s}^r}{\|u\|_{\dot{H}^{1/2}}^{4/(s-3)}} \|u\|_{\dot{H}^{1/2}} \quad (41)$$

and the conclusion now follows from Lemma 1.

### C. Proof of Theorem 2

Similar to the proof of Theorem 1, we assume that  $u$  blows up at  $T^* \in (0, T)$ , and it suffices to show that  $\|u\|_{L^3} \leq C$  on  $(0, T^*)$  for some  $C > 0$  independent of  $t$ .

We multiply (1) by  $|u|u \cdot$  and integrate in  $\mathbb{R}^3$  to obtain

$$\begin{aligned} \|u\|_{L^3}^2 \frac{d}{dt} \|u\|_{L^3} &= - \int_{\mathbb{R}^3} |u|u \cdot \nabla p dx + \int_{\mathbb{R}^3} |u|u \cdot \Delta u dx \\ &= \int_{\mathbb{R}^3} p u \cdot \nabla(|u|) dx + \int_{\mathbb{R}^3} |u|u \cdot \Delta u dx \\ &= \int_{\mathbb{R}^3} p |u| (\hat{u} \cdot \nabla |u|) dx + \int_{\mathbb{R}^3} |u|u \cdot \Delta u dx. \end{aligned} \quad (42)$$

where  $\hat{u} := \frac{u}{|u|}$  (if  $u = 0$ , just defined  $\hat{u} = 0$  too).

Recalling the identity

$$u \cdot \Delta u = \nabla \cdot (|u| \nabla |u|) - |\nabla u|^2, \quad (43)$$

we easily derive

$$\int_{\mathbb{R}^3} |u|u \cdot \Delta u dx = -\frac{4}{9} \|\nabla |u|^{3/2}\|_{L^2}^2 - \|\nabla u |u|^{1/2}\|_{L^2}^2, \quad (44)$$

and reach the following estimate

$$\frac{d}{dt} \|u\|_{L^3}^3 + \|\nabla u |u|^{1/2}\|_{L^2}^2 + \|\nabla |u|^{3/2}\|_{L^2}^2 \lesssim \left| \int_{\mathbb{R}^3} p |u| (\hat{u} \cdot \nabla |u|) dx \right|. \quad (45)$$

which gives

$$\|u\|_{L^3}^2 \frac{d}{dt} \|u\|_{L^3} \lesssim C \int_{\mathbb{R}^3} |p| |u|^{1/2} |\nabla |u|^{3/2}| dx - \| |u|^{3/2} \|_{\dot{H}^1}^2. \quad (46)$$

Application of Young's inequality and then Hölder and Sobolev inequalities to (46) gives

$$\begin{aligned} \|u\|_{L^3}^2 \frac{d}{dt} \|u\|_{L^3} &\lesssim C \int_{\mathbb{R}^3} |p|^2 |u| dx - \| |u|^{3/2} \|_{\dot{H}^1}^2 \\ &\lesssim C \|p^2\|_{L^{5/4}} \|u\|_{L^5} - \|u\|_{L^9}^3 \\ &= C \|p\|_{L^{5/2}}^2 \|u\|_{L^5} - \|u\|_{L^9}^3 \\ &\lesssim C \|u\|_{L^5}^5 - \|u\|_{L^9}^3. \end{aligned} \quad (47)$$

Note that in the last inequality we have used (9).

In what follows we discuss the two regimes of  $s$  stated in the theorem separately.

- $3 < s \leq 5$ . In this case we apply the interpolation inequality

$$\|u\|_{L^5} \leq \|u\|_{L^s}^\theta \|u\|_{L^9}^{1-\theta} \quad (48)$$

where  $\theta = \frac{4}{5} \frac{s}{9-s}$ . This gives

$$\begin{aligned} \|u\|_{L^3}^2 \frac{d}{dt} \|u\|_{L^3} &\lesssim C \|u\|_{L^s}^{4s/(9-s)} \|u\|_{L^9}^{9(5-s)/(9-s)} - \|u\|_{L^9}^3 \\ &\lesssim C \|u\|_{L^s}^{2s/(s-3)} \end{aligned} \quad (49)$$

after application of Young's inequality. Dividing both sides by  $\|u\|_{L^3}^2$  we have

$$\frac{d}{dt} \|u\|_{L^3} \lesssim \frac{\|u\|_{L^s}^{2s/(s-3)}}{\|u\|_{L^3}^3} \|u\|_{L^3}. \quad (50)$$

The conclusion now follows from Lemma 1.

- $5 < s < \infty$ . In this case we apply the interpolation inequality

$$\|u\|_{L^5} \leq \|u\|_{L^3}^\theta \|u\|_{L^s}^{1-\theta} \quad (51)$$

where  $\theta = \frac{3}{5} \frac{s-5}{s-3}$ . This gives

$$\|u\|_{L^3}^2 \frac{d}{dt} \|u\|_{L^3} \lesssim \|u\|_{L^3}^{3(s-5)/(s-3)} \|u\|_{L^s}^{2s/(s-3)}. \quad (52)$$

Dividing both sides by  $\|u\|_{L^3}^2$  we have

$$\frac{d}{dt} \|u\|_{L^3} \lesssim \frac{\|u\|_{L^s}^{2s/(s-3)}}{\|u\|_{L^3}^{6/(s-3)}} \|u\|_{L^3}, \quad (53)$$

and the conclusion immediately follows from Lemma 1.

#### D. Proof of Theorem 3

Again we assume that  $u$  blows up at  $T^* \in (0, T)$ , and try to show that  $\|u\|_{L^3} \leq C$  on  $(0, T^*)$  for some  $C > 0$  independent of  $t$ .

Following (42)–(45) we have

$$\begin{aligned} \|u\|_{L^3}^2 \frac{d}{dt} \|u\|_{L^3} &\leq \int_{\mathbb{R}^3} |p| |u| |\nabla u| dx - \|\nabla |u|^{3/2}\|_{L^2}^2 \\ &= \int_{\mathbb{R}^3} |p|^\alpha |p|^{1-\alpha} |u|^{1/2} |\nabla |u|^{3/2}| dx - \|\nabla |u|^{3/2}\|_{L^2}^2 \\ &\leq \| |p|^\alpha \|_{L^a} \| |p|^{1-\alpha} \|_{L^b} \| |u|^{1/2} \|_{L^c} \|\nabla |u|^{3/2}\|_{L^2} - \|\nabla |u|^{3/2}\|_{L^2}^2 \\ &= \|p\|_{L^{a\alpha}}^\alpha \|p\|_{L^{b(1-\alpha)}}^{1-\alpha} \|u\|_{L^{c/2}}^{1/2} \|\nabla |u|^{3/2}\|_{L^2} - \|\nabla |u|^{3/2}\|_{L^2}^2 \\ &\leq C \|p\|_{L^s}^\alpha \|u\|_{L^{2b(1-\alpha)}}^{2(1-\alpha)} \|u\|_{L^{c/2}}^{1/2} \|\nabla |u|^{3/2}\|_{L^2} - \|\nabla |u|^{3/2}\|_{L^2}^2 \\ &=: A - \|\nabla |u|^{3/2}\|_{L^2}^2. \end{aligned} \quad (54)$$

for appropriate  $\alpha, a, b, c$  with  $a\alpha = s, \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}$ . We deal with the three cases stated in the theorem one by one.

- $3 < s < \infty$ . We take  $\alpha = \frac{s}{2s-3}, a = 2s-3, b = \frac{3(2s-3)}{2s-6}, c = 6$ . This gives

$$A = C \|p\|_{L^s}^{\frac{s}{2s-3}} \|u\|_{L^3}^{\frac{6s-15}{4s-6}} \|\nabla |u|^{3/2}\|_{L^2} \leq C \|p\|_{L^s}^{\frac{2s}{2s-3}} \|u\|_{L^3}^{\frac{6s-15}{2s-3}} + \|\nabla |u|^{3/2}\|_{L^2}^2. \quad (55)$$

Here we have applied Young's inequality. Substituting (55) into (54) and dividing both sides by  $\|u\|_{L^3}^2$ , we reach

$$\frac{d}{dt} \|u\|_{L^3} \leq \frac{\|p\|_{L^s}^{\frac{2s}{2s-3}}}{\|u\|_{L^3}^{\frac{6}{2s-3}}} \|u\|_{L^3}. \quad (56)$$

The conclusion now follows from Lemma 1.

- $\frac{9}{4} < s < 3$ . We take  $\alpha = 1$ ,  $a = s$ ,  $c = \frac{2s}{s-2}$ . Note that in this case the term involving  $b$  is not present. Application of the following interpolation inequality and Sobolev inequality

$$\|u\|_{L^{\frac{s}{s-2}}} \leq \|u\|_{L^3}^{\frac{4s-9}{s}} \|u\|_{L^9}^{\frac{9-3s}{s}}, \quad \|u\|_{L^9}^3 \lesssim \|\nabla|u|^{3/2}\|_{L^2}^2 \quad (57)$$

gives

$$A = C \|p\|_{L^s} \|u\|_{L^{\frac{s}{s-2}}}^{1/2} \|\nabla|u|^{3/2}\|_{L^2} \lesssim \|p\|_{L^s} \|u\|_{L^3}^{\frac{4s-9}{2s}} \|\nabla|u|^{3/2}\|_{L^2}^{\frac{3}{2}}. \quad (58)$$

Application of Young's inequality now gives

$$A \leq C \|p\|_{L^s}^{\frac{2s}{2s-3}} \|u\|_{L^3}^{\frac{4s-9}{2s-3}} + \|\nabla|u|^{3/2}\|_{L^2}^2. \quad (59)$$

Substituting this into (54) and dividing both sides by  $\|u\|_{L^3}^2$ , we reach

$$\frac{d}{dt} \|u\|_{L^3} \leq \frac{\|p\|_{L^s}^{\frac{2s}{2s-3}}}{\|u\|_{L^3}^{\frac{2s}{2s-3}}} \|u\|_{L^3}. \quad (60)$$

The conclusion now follows from Lemma 1.

- $\frac{3}{2} < s \leq \frac{9}{4}$ . We take  $\alpha = \frac{2s}{9-2s}$ ,  $a = \frac{9-2s}{2}$ ,  $b = \frac{9}{2} \frac{9-2s}{9-4s}$ ,  $c = 18$ . Now application of Sobolev inequality  $\|u\|_{L^9}^3 \lesssim \|\nabla|u|^{3/2}\|_{L^2}^2$  gives

$$A = C \|p\|_{L^s}^{\frac{2s}{9-2s}} \|u\|_{L^9}^{\frac{45-18s}{18-4s}} \|\nabla|u|^{3/2}\|_{L^2} \lesssim \|p\|_{L^s}^{\frac{2s}{9-2s}} \|\nabla|u|^{3/2}\|_{L^2}^{\frac{24-8s}{9-2s}}. \quad (61)$$

Note that since  $\frac{3}{2} < s \leq \frac{9}{4}$ , there holds  $\frac{24-8s}{9-2s} < 2$ . Thus we can apply Young's inequality and obtain

$$A \leq C \|p\|_{L^s}^{\frac{2s}{2s-3}} + \|\nabla|u|^{3/2}\|_{L^2}^2. \quad (62)$$

Substituting this into (54) and dividing both sides by  $\|u\|_{L^3}^2$ , we reach

$$\frac{d}{dt} \|u\|_{L^3} \leq \frac{\|p\|_{L^s}^{\frac{2s}{2s-3}}}{\|u\|_{L^3}^3} \|u\|_{L^3}. \quad (63)$$

The conclusion now follows from Lemma 1.

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