A NOTE ON THE PROBABILITY OF GENERATING ALTERNATING OR SYMMETRIC GROUPS

LUKE MORGAN AND COLVA M. RONEY-DOUGAL

Abstract. We improve on recent estimates for the probability of generating the alternating and symmetric groups A_n and S_n. In particular we find the sharp lower bound, if the probability is given by a quadratic in n^{-1}. This leads to improved bounds on the largest number $h(\mathrm{A}_n)$ such that a direct product of $h(\mathrm{A}_n)$ copies of A_n can be generated by two elements.

1. Introduction

For a group $X = \mathrm{S}_n$ or A_n, we write $p(X)$ for the probability that two elements of X generate a group that contains A_n. In [1], Dixon proved that $p(\mathrm{S}_n) \to 1$ as $n \to \infty$. In [2] he sharpened this statement to

$$p(\mathrm{S}_n) = 1 - \frac{1}{n} - \frac{1}{n^2} - \frac{4}{n^3} - \frac{23}{n^4} - \frac{171}{n^5} - \frac{1542}{n^6} + O(n^{-7}).$$

For many applications, numerical results are needed, rather than asymptotics. In [5] Maróti and Tamburini proved explicit upper and lower bounds

$$1 - \frac{1}{n} - \frac{13}{n^2} < p(X) \leqslant 1 - \frac{1}{n} + \frac{2}{3n^2}.$$

In this present note, we find the best possible lower bound of this type, and a close-to-optimal upper bound.

Theorem 1.1. *Let $X = \mathrm{A}_n$ or $X = \mathrm{S}_n$ with $n \geqslant 5$. Then*

$$1 - \frac{1}{n} - \frac{8.8}{n^2} \leqslant p(X) < 1 - \frac{1}{n} - \frac{0.93}{n^2}.$$

Equality holds in the lower bound if and only if $n = 6$.

In fact, for $n \geqslant 14$, we prove that $1 - \frac{1}{n} - \frac{7.5}{n^2} < p(X) < 1 - \frac{1}{n} - \frac{0.93}{n^2}$. The result for smaller n comes from the values for $p(X)$ in Table 1 (taken from [7, Table 4.1]).

Hall [3] considered the largest number $h(S)$ such that a direct product of $h(S)$ copies of a non-abelian finite simple group S can be generated by two elements, and proved that $h(S) = p(S)|S|/|\mathrm{Out}(S)|$. The function $h(S)$ has received considerable attention recently; we refer the reader to [5] for more discussion and references and to [6] for lower bounds on $h(S)$ for all non-abelian finite simple groups S. The new bounds above yield:

Corollary 1.2. *Let n be an integer with $n \geqslant 14$. Then*

$$\left(1 - \frac{1}{n} - \frac{7.5}{n^2}\right)\left(\frac{n!}{4}\right) < h(\mathrm{A}_n) < \left(1 - \frac{1}{n} - \frac{0.93}{n^2}\right)\left(\frac{n!}{4}\right).$$

Let $m(S)$ denote the minimal index of a proper subgroup of a group S. In [4], it is proved that there exist absolute constants c_1 and c_2 such that $1 - c_1/m(S) < p(S) < 1 - c_2/m(S)$, for all non-abelian finite simple groups S. Our main theorem immediately yields the following bounds on c_1 and c_2.

Corollary 1.3. *For $n \geqslant 5$,*

$$1 - \frac{2.468}{n} < p(\mathrm{A}_n) < 1 - \frac{1.186}{n},$$

and hence $c_1 \geqslant 2.468$ and $c_2 \leqslant 1.186$.

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2. Proof of Theorem 1.1

Definition 2.1. For $X = \mathrm{A}_n$ or S_n we let $p_{\mathrm{intrans}}(X)$ and $p_{\mathrm{trans}}(X)$ be the probability that two elements chosen randomly from X generate a subgroup of an intransitive maximal subgroup of X, or a subgroup of a transitive maximal subgroup of X other than A_n, respectively.

Lemma 2.2. *Let $X = \mathrm{A}_n$ or S_n with $n \geqslant 14$. Then*

$$p_{intrans}(X) < \frac{1}{n} + \frac{2.7}{n^2}.$$

Proof. We prove the result for S_n, the arguments for A_n are identical. Let $x, y \in \mathrm{S}_n$ and suppose that $Y := \langle x, y\rangle$ is contained in an intransitive maximal subgroup. Then Y is contained in a subgroup conjugate to $\mathrm{S}_k \times \mathrm{S}_{n-k}$ for some $1 \leqslant k \leqslant \lfloor \frac{n-1}{2} \rfloor$.

Let $k \in \{1, \ldots, n-1\}$. Then the probability that $Y \leqslant \mathrm{S}_k \times \mathrm{S}_{n-k}$ is bounded by

$$\binom{n}{k}\left(\frac{k!(n-k)!}{n!}\right)^2 = \binom{n}{k}^{-1}.$$

So the probability that $Y \leqslant \mathrm{S}_1 \times \mathrm{S}_{n-1}$ is at most $\frac{1}{n}$, and the probability that $Y \leqslant \mathrm{S}_2 \times \mathrm{S}_{n-2}$ and Y is transitive on the orbit of size 2 is bounded by

$$\frac{3}{4}\frac{2}{n(n-1)} = \frac{3}{2n(n-1)}.$$

Similarly, the probability that $Y \leqslant \mathrm{S}_3 \times \mathrm{S}_{n-3}$ and Y is transitive on the orbit of length 3 is

$$\frac{13}{18}\binom{n}{3}^{-1} = \frac{13}{3n(n-1)(n-2)}.$$

Now the probability that $Y \leqslant \mathrm{S}_k \times \mathrm{S}_{n-k}$ for some $4 \leqslant k \leqslant \lfloor \frac{n-1}{2} \rfloor$ is

$$\sum_{k=4}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{\binom{n}{k}} \leqslant \sum_{k=4}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{\binom{n}{4}} \leqslant \frac{12(n-7)}{n(n-1)(n-2)(n-3)}.$$

We now observe that, since $n \geqslant 14$,

$$\frac{3}{2n(n-1)} + \frac{13}{3n(n-1)(n-2)} + \frac{12(n-7)}{n(n-1)(n-2)(n-3)} < \frac{2.7}{n^2}$$

which completes the proof. □

Lemma 2.3. *Let $X = \mathrm{A}_n$ or S_n, with $n \geqslant 14$. Then*

$$p_{intrans}(X) > \frac{1}{n} + \frac{0.93}{n^2}.$$

Proof. We observe that $p_{\mathrm{intrans}}(X)$ is bounded below by the probability that a random pair of elements of X generate a subgroup with a fixed point, or with an orbit of size 2. For $X = \mathrm{S}_n$, we bound $p_{\mathrm{intrans}}(X)$ by doing inclusion-exclusion to depth 2 on the union of the sets $(\mathrm{S}_n)_\alpha$, with $1 \leqslant \alpha \leqslant n$, and $(\mathrm{S}_n)_{\{\alpha,\beta\}} \setminus (\mathrm{S}_n)_{(\alpha,\beta)}$, with $1 \leqslant \alpha < \beta \leqslant n$. We find that $p_{\mathrm{intrans}}(X)$ is greater than

$$\tfrac{1}{n} + \tfrac{3}{4}\tfrac{2(n-2)!}{n!} - \tfrac{(n-2)!}{2n!} - \tfrac{3}{4}\tbinom{n}{1}\tbinom{n-1}{2}\left(\tfrac{2(n-3)!}{n!}\right)^2 - \left(\tfrac{3}{4}\right)^2 \tfrac{\binom{n}{2}\binom{n-2}{2}}{2}\left(\tfrac{4(n-4)!}{n!}\right)^2$$

Thus

$$p_{\mathrm{intrans}}(X) \geqslant \frac{1}{n} + \frac{8n^2 - 52n + 75}{8n(n-1)(n-2)(n-3)}$$

which, since $n \geqslant 14$, is greater than $\frac{1}{n} + \frac{0.93}{n^2}$. □

Proof of Theorem 1.1. For the upper bound we use Lemma 2.3. For the lower bound, note that

$$1 - p(X) = p_{\mathrm{intrans}}(X) + p_{\mathrm{trans}}(X).$$

It follows from the proofs of [5, Lemmas 3.1 and 4.3] that $p_{\mathrm{trans}}(X) \leqslant \frac{4.8}{n^2}$. Combining this with Lemma 2.2 gives the theorem. □

In Table 1 we record the value of $p(\mathrm{A}_n)$ and $p(\mathrm{S}_n)$ for $n \leqslant 13$, together with our lower and upper bounds as stated in Theorem 1.1. All values are correct to three decimal places.

Table 1. Precise values and bounds for $p(X)$

n	5	6	7	8	9	10	11	12	13
$p(\mathrm{A}_n) =$	0.633	0.588	0.726	0.739	0.848	0.875	0.893	0.902	0.913
$p(\mathrm{S}_n) =$	0.633	0.588	0.795	0.796	0.859	0.875	0.894	0.903	0.913
$p(X) \geqslant$	0.448	0.588	0.677	0.737	0.780	0.812	0.836	0.855	0.871
$p(X) \leqslant$	0.763	0.808	0.839	0.861	0.878	0.891	0.902	0.911	0.918

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Luke Morgan, School of Mathematics and Statistics,, University of Western Australia,, 35 Stirling Highway,, Crawley, WA 6009, Australia

E-mail address: luke.morgan@uwa.edu.au

Colva M. Roney-Dougal, School of Mathematics and Statistics,, University of St Andrews,, North Haugh,, St Andrews,, Fife KY16 9SS, U. K.

E-mail address: colva.roney-dougal@st-andrews.ac.uk