INTERSECTION PROBLEMS IN COMBINATORICS

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INTERSECTION PROBLEMS IN COMBINATORICS

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ABSTRACT

With the publication of the famous Erdős-Ko-Rado Theorem in 1961, intersection problems became a popular area of combinatorics. A family of combinatorial objects is t-intersecting if any two of its elements mutually t-intersect, where the latter concept needs to be specified separately in each instance. This thesis is split into two parts; the first is concerned with intersecting injections while the second investigates intersecting posets.

We classify maximum 1-intersecting families of injections from $\{1,\ldots,k\}$ to $\{1,\ldots,n\}$, a generalisation of the corresponding result on permutations from the early 2000s. Moreover, we obtain classifications in the general $t>1$ case for different parameter limits: if n is large in terms of k and t, then the so-called fix-families, consisting of all injections which map some fixed set of t points to the same image points, are the only t-intersecting injection families of maximal size. By way of contrast, fixing the differences $k-t$ and $n-k$ while increasing k leads to optimal families which are equivalent to one of the so-called saturation families, consisting of all injections fixing at least $r+t$ of the first $2r+t$ points, where $r=\lfloor (k-t)/2\rfloor$. Furthermore we demonstrate that, among injection families with t-intersecting and left-compressed fixed point sets, for some value of r the saturation family has maximal size .

The concept that two posets intersect if they share a comparison is new. We begin by classifying maximum intersecting families in several isomorphism classes of posets which are linear, or almost linear. Then we study the union of the almost linear classes, and derive a bound for an intersecting family by adapting Katona's elegant cycle method to posets. The thesis ends with an investigation of the intersection structure of poset classes whose elements are close to the antichain.

The overarching theme of this thesis is fixing versus saturation: we compare the sizes and structures of intersecting families obtained from these two distinct principles in the context of various classes of combinatorial objects.

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CONTENTS

Part I

Introduction

Chapter 1

Intersecting Set Families

1.1 Basic Concepts

Let $\mathcal{A}$ be a collection of subsets of $\{1, 2, \ldots, n\}$ such that any two elements of $\mathcal{A}$ have non-empty intersection. How large can such a collection be? Since each set X is disjoint from its *complement*

$$\overline{X} = \{1, 2, \ldots, n\} \setminus X = \{\, x \in \{1, 2, \ldots, n\} : x \notin X \,\},$$

we can have no more than 2^{n-1} sets in $\mathcal{A}$.

Let us now suppose that members of $\mathcal{A}$ have size at most k and do not contain each other. To simplify, we may study families whose members all have size k. The following definitions, notations and conventions will be used throughout the thesis.

- The set of the first n natural numbers is denoted by $[n]$. Zero is not considered a natural number.
- A set of size k is called a *k-set*.
- Two sets A, B are said to *t-intersect* if $|A \cap B| \geq t$.
- A collection $\mathcal{A}$ of sets is an *antichain* if for all $X, Y \in \mathcal{A}$, we have $X \subseteq Y$ only if $X = Y$.
- The finite families, collections and classes in this thesis are just finite sets, so their members are distinct.

The objects mentioned in the following definitions are not necessarily sets. We will discuss in Chapter 2 what it means for other combinatorial objects to intersect.

- A collection of objects is t-*intersecting* if its members mutually t-intersect.
- To *intersect* means to 1-intersect.
- A t-intersecting subset $\mathcal{A}$ of a class $\mathcal{X}$ is *maximal* if, for any $X \in \mathcal{X} \setminus \mathcal{A}$, the set $\mathcal{A} \cup \{X\}$ is not t-intersecting.
- A t-intersecting subset $\mathcal{A}$ of a class $\mathcal{X}$ is *maximum* if there exists no t-intersecting subset of $\mathcal{X}$ which is larger than $\mathcal{A}$.

For each class of objects, there is a natural maximal t-intersecting family, which many authors refer to as the 'trivial family'. In the context of k-subsets of $[n]$, this is

$$F_0 = \{ X \subset [n] : |X| = k, [t] \subseteq X \}$$

with

$$|F_0| = \binom{n-t}{k-t}.$$

Since F_0 is obtained by fixing t points, we refer to F_0 as the *fix-family*.

Paul Erdős, Chao Ko and Richard Rado published two theorems in their pioneering article [EKR61]. The first implies that the fix-family is a maximum intersecting family of k-subsets of $[n]$ when $k \leq n/2$. Note that if $k > n/2$ then the collection of all k-subsets of $[n]$ is intersecting by the pigeonhole principle.

Theorem 1.1.1. *(Erdős, Ko, Rado [EKR61]).*

Let $k \leq n/2$ and let $\mathcal{F}$ be an intersecting antichain of subsets of $[n]$ which have size at most k. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ and equality implies that each member of $\mathcal{F}$ has size k.

The second result states that the fix-family is also maximum for $t > 1$, provided n is large.

Theorem 1.1.2. *(Erdős, Ko, Rado [EKR61]).*

Let t, k, n be natural numbers with $n \geq n_0(k,t)$. If $\mathcal{F}$ is a t-intersecting antichain of subsets of $[n]$ which have size at most k then

$$|\mathcal{F}| \leq \binom{n-t}{k-t}.$$

Erdős explained in [Erd87] that he had proved these now famous theorems with Ko & Rado by 1938, but did not publish them until 1961 due to what they perceived as a lack of interest in combinatorics at the time. It seems that they chose the right moment, because there continues to be an abundance of interest in [EKR61]. This seminal paper ends with an extensive section of concluding remarks which has led to a branch in extremal set theory that could be called *Erdős-Ko-Rado Theory*. We will discuss these questions and conjectures here, as well as presenting the progress which has been made over the last half century. Whilst we try our best to give due credit to all authors involved, we cannot guarantee, for any of the results discussed in this thesis, that a given list of sources is exhaustive.

1.1.1 From Fixing to Saturation

Erdős, Ko & Rado remarked in [EKR61] that the n_0 they give in Theorem 1.1.2 is not best possible. Frankl [Fra78a] made considerable progress on this problem, and Wilson [Wil84] completed the proof that

$$n_0(k,t) = (k-t+1)(t+1) \tag{1.1.3}$$

is the smallest $n_0(k,t)$ for which Theorem 1.1.2 holds. Moreover, Wilson proved that for $n > n_0(k,t)$, no family other than the fix-family attains the bound given in Theorem 1.1.2.

To demonstrate that a different bound holds for small n, Erdős, Ko & Rado quoted in [EKR61] the following example due to S. H. Min: let $\mathcal{F}$ be the set of 4-subsets of $[8]$ which contain at least 3 elements of $[4]$. Then $\mathcal{F}$ is 2-intersecting of size 16, while in this case the fix-family

$$\{\, X \subset [8] : 1, 2 \in X, |X| = 4 \,\}$$

has size $\binom{8-2}{4-2} = 15$. The idea behind Min's example is central to this thesis: we regard the concept of *fixing* as a special case of a *saturation* process. To be more precise, for $0 \le r \le (n-t)/2$, let $F_r(t,k,n)$ be the collection of all k-subsets of $[n]$ which contain at least $t+r$ elements of $[t+2r]$. Then the pigeonhole principle implies that $F_r(t,k,n)$ is t-intersecting. Note also that $F_0(t,k,n)$ is the fix-family.

We will see throughout this thesis that either saturation, or the special case of fixing, usually yield optimal t-intersecting families. However, these ideas were very new in 1961 and Erdős, Ko & Rado simply conjectured that $F_{m-1}(2, 2m, 4m)$ is maximum, which was often referred to as the $4m$-Conjecture in survey papers such as [DF83, Ahl01]. Among the open problems from [EKR61], we have presented this one first for convenience of notation. However, it remained the last open problem from [EKR61], as Erdős pointed out in his article 'Some of my favourite unsolved problems' [Erd90], published 6 years before he died. Indeed, Erdős had offered \$500 for the solution of the $4m$-Conjecture [Fra88b, Ahl01].

1.2 Classifications

By Theorem 1.1.1, the fix-family is a maximum intersecting family of k-subsets of $[n]$. Are there any other optimal families? From the discussion beginning this chapter, we see that when $k = n/2$, $\mathcal{A}$ is a maximum intersecting family of k-subsets of $[n]$ if, and only if, for each k-subset X of $[n]$, precisely one of $X \in \mathcal{A}$ or $\overline{X} \in \mathcal{A}$ holds. But what about $k < n/2$? Let us establish some further definitions which will be used throughout this thesis:

- A t-intersecting family is *equivalent* to the fix-family if it can be obtained from the fix-family by a permutations of the labels. For example, the set

$$\{ X \subset [22] : |X| = k, \{1, 2, 17\} \subseteq X \}$$

  is equivalent to the fix-family $F_0(3, k, 22)$.

- We say that fixing is *optimal* if the fix-family is maximum. In this context, *unique* means unique up to permutations of the labels: if every maximum family is equivalent to the fix-family, we say that fixing is the unique optimal strategy. In the same way, we may talk about some other family as being the *(unique) optimal* subset of some class of objects.

Erdős, Ko & Rado conjectured that every maximum intersecting family of k-subsets of $[n]$ is equivalent to the fix-family, and this was proved three years later by Katona [Kat64]. Since then, mentions of 'the Erdős-Ko-Rado Theorem' are usually referring to the following result:

Theorem 1.2.1. *(Erdős, Ko, Rado [EKR61]; Katona [Kat64]).*

Let $k \leq n/2$ and let $\mathcal{F}$ be an intersecting family of k-subsets of $[n]$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ and equality implies that all members of $\mathcal{F}$ have a fixed element of $[n]$ in common.

1.2.1 Unrestricted Size of Members

In fact, Katona proved another much more general result in [Kat64]: he obtained a bound for t-intersecting subsets of the power set of $[n]$. Erdős, Ko & Rado had conjectured in [EKR61] that if $n + t$ is even then

$$K_0(t, n) = \{ X \subseteq [n] : |X| \geq (n + t)/2 \}$$

is maximum. Katona not only proved this conjecture, but also found an optimal family for the case where $n+t$ is odd: set

$$\begin{aligned} K_1(t,n) &= \{\, X \subseteq [n] : |X| > (n+t)/2 \,\} \\ &\quad \cup \{\, X \subseteq [n-1] : |X| = (n+t-1)/2 \,\}. \end{aligned}$$

Two elements X, Y of the first set satisfy

$$|X \cap Y| \geq |X| + |Y| - n > t$$

and, similarly, the second set is t-intersecting. If X and Y are elements of the first and second sets above respectively, then there are at most $(n-t-1)/2$ points in $[n]$ which are not in X. Thus

$$|X \cap Y| \geq |Y| - |\overline{X}| \geq \frac{n+t-1}{2} - \frac{n-t-1}{2} = t,$$

so $K_1(t,n)$ is t-intersecting.

Indeed, by considering the case $r = \lfloor (n-t)/2 \rfloor$ of the saturation families $F_r(t,k,n)$ from Section 1.1.1, we see that $K_0(t,n)$ and $K_1(t,n)$ are just saturation families interpreted in the context of t-intersecting subsets of $[n]$:

$$K_p(t,n) = \bigcup_k F_{(n-t-p)/2}(t,k,n).$$

Theorem 1.2.2. *(Katona [Kat64]).*

If $\mathcal{F}$ is a t-intersecting family of subsets of $[n]$ then

$$|\mathcal{F}| \leq |K_p(t,n)|$$

where $p = n + t \bmod 2$.

This settles the more general case of set families whose members have arbitrary size. Ahlswede & Khachatrian published another four different proofs of Theorem 1.2.2 over the next few decades [AK99, AK05]. Indeed, these two authors were instrumental in completely characterising t-intersecting families of k-subsets of $[n]$ as originally considered by Erdős, Ko, and Rado.

1.2.2 Fixed Size of Members

Seventeen years after the publication of [EKR61], Frankl generalised the $4m$-Conjecture: he conjectured in [Fra78a] that for all values of $n \geq k \geq t$, a maximum t-intersecting family of k-subsets of $[n]$ is equivalent to $F_r(t,k,n)$ for some $0 \leq r \leq (n-t)/2$. Thus Frankl was the first to realise that any optimal family of k-subsets of n can be expressed as a saturation family. After Erdős drew the attention of the combinatorial community back to this problem in [Erd90], Frankl renewed his efforts and proved, together with different coauthors [FF91, CF92], that his conjecture holds in various special cases.

Ahlswede & Khachatrian finally published a general proof of Frankl's conjecture one year after Erdős' death. It is proved in [AK97], for all parameter values which are not covered by Wilson's generalisation of the Erdős-Ko-Rado result, that saturation, including the case of fixing, is the unique optimal strategy.

Theorem 1.2.3. *(Wilson [Wil84]).*

Let $\mathcal{F}$ be a maximum t-intersecting family of k-subsets of $[n]$ for $n > (k-t+1)(t+1)$. Then $\mathcal{F}$ is equivalent to $F_0(t,k,n)$.

Fixing is also optimal for $n = (k-t+1)(t+1)$, but not uniquely so, as can be seen from the theorem below in which Ahlswede & Khachatrian specify the optimal *saturation constant* r as a function of n, k, t. The statement of their 'Complete Intersection Theorem for Systems of Finite Sets' uses the convention $(t-1)/r = \infty$ if $r = 0$.

Theorem 1.2.4. *(Ahlswede, Khachatrian [AK97]).*

Let $\mathcal{F}$ be a maximum t-intersecting family of k-subsets of $[n]$ and set

$$f(r) = (k-t+1)\left(2 + \frac{t-1}{r+1}\right).$$

- *If $f(r) < n < f(r-1)$ for some non-negative integer r, then $\mathcal{F}$ is equivalent to $F_r(t,k,n)$.*

- *If $n = f(r)$ for some non-negative integer r, then $\mathcal{F}$ is equivalent to $F_r(t,k,n)$ or $F_{r+1}(t,k,n)$.*

Thus the results of Wilson and Ahlswede & Khachatrian together amount to a complete classification of maximum t-intersecting families of k-subsets of $[n]$. This classification has been applied in many other areas of combinatorics, for example it has been used to develop better algorithms for approximating minimum vertex covers in graphs, an NP-hard problem [DS05].

We now turn our attention to other questions from [EKR61]. For instance, now that we know the optimal families for all parameter values, we might ask what the next best families are.

1.2.3 Non-Fix Families

Erdős, Ko & Rado inspired an investigation of so-called nontrivial families. As usual, we consider the case $t = 1$ first: if we are not allowed to fix a point, then what is the largest size of intersecting family we can achieve? It was conjectured in [EKR61] that if $\mathcal{F}$ is an intersecting family of k-subsets of $[n]$ for some $3 \leq k \leq n/2$, then $\bigcap_{X\in\mathcal{F}} = \emptyset$ implies

$$|\mathcal{F}| \leq 3\binom{n-3}{k-2} + \binom{n-3}{k-3},$$

a bound which is attained by $F_1(t, k, n)$, as Erdős, Ko & Rado pointed out. However, this conjecture was disproved by Hilton & Milner in [HM67] where they showed that the second best strategy is still very close to fixing: G is the family consisting of $[k]$ along with all k-subsets of $[n]$ containing $k + 1$ and intersecting $[k]$.

Theorem 1.2.5. *(Hilton, Milner [HM67]).*

Let $k \leq n/2$ and set

$$\mathcal{G} = \{[k]\} \cup \{\, X \subset [n] : |X| = k,\ k+1 \in X,\ X \cap [k] \neq \emptyset \,\}\,.$$

If $\mathcal{F}$ is a maximal intersecting family of k-subsets of $[n]$ which is not equivalent to the fix-family, then $|\mathcal{F}| \leq |\mathcal{G}|$.

Frankl & Füredi later presented a very short proof of the Hilton-Milner Theorem in [FF86].

Having settled the case $t = 1$, the next natural investigation focus are optimal non-fix families for $t > 1$. Again, Frankl was the first to tackle the problem. In [Fra78c], he characterised the maximum t-intersecting families $\mathcal{F}$ of k-subsets of $[n]$ which satisfy the property that no element of $[n]$ is contained in more than $c|\mathcal{F}|$ members of $\mathcal{F}$, for various fixed values of c, provided n exceeded some function $n_1(k, t)$. The complementary result for small n was not established until 1996, when Ahlswede & Khachatrian realised that their recent proof of the Complete Intersection Theorem 1.2.4 could be modified to yield a characterisation of maximum non-fix families. Their main contribution to Theorem 1.2.6 was to determine that $n_1(k, t) = n_0(k, t)$, see (1.1.3).

Frankl generalised Hilton & Milner's optimal family $\mathcal{G}$ as follows: set

$$G(t, k, n) = A(t, k, n) \cup B(t, k, n)$$

where $A(t, k, n)$ is the set of k-subsets of $[n]$ which contain $[t]$ and intersect $\{t + 1, \ldots, k + 1\}$, and

$$B(t, k, n) = \{\, [k+1] \setminus \{i\} : 1 \leq i \leq t \,\}\,.$$

It is easily seen that both $A(t,k,n)$ and $B(t,k,n)$ are t-intersecting for $t \leq k-1$. To demonstrate that $G(t,k,n)$ is also t-intersecting, note that if $X \in A(t,k,n)$ and $Y \in B(t,k,n)$ then X contains at least $t+1$ elements of $[k+1]$, and hence at least t elements of Y.

Theorem 1.2.6. *(Frankl; Ahlswede, Khachatrian [Fra78c, AK96]).*
Let $\mathcal{F}$ be a maximum t-intersecting family of k-subsets of $[n]$ with $\left| \bigcap_{X \in \mathcal{F}} X \right| < t$.

- *If $2k - t < n \leq n_0(k,t)$ then $\mathcal{F}$ is characterised by Theorem 1.2.4.*
- *If $n > n_0(k,t)$ and $k \leq 2t+1$ then $\mathcal{F}$ is equivalent to $F_1(t,k,n)$.*
- *If $n > n_0(k,t)$ and $k > 2t+1$ then $\mathcal{F}$ is equivalent to either $F_1(t,k,n)$ or $G(t,k,n)$.*

Together with Theorem 1.2.4, this result constitutes a complete characterisation of maximum t-intersecting families of k-subsets of $[n]$ which are not equivalent to the fix-family.

Building on the approach of Frankl & Füredi in [FF86], Balogh & Mubayi recently demonstrated in [BM08] that the original methods of Erdős, Ko & Rado in [EKR61] yield a simpler proof of Theorem 1.2.6, at least for the case $k \leq 2t+1$.

1.3 Further Generalisations

1.3.1 s-wise t-intersecting Families

The final concluding remark in [EKR61] presents a short proof of the following result, with Erdős, Ko & Rado noting that various other authors had given alternative proofs. Let $\mathcal{F}$ be a family of subsets of $[n]$ such that

$$X \cap Y \cap Z \neq \emptyset$$

for all $X, Y, Z \in \mathcal{F}$. Then $|\mathcal{F}| \leq 2^{n-1}$, and equality implies that $\mathcal{F}$ is equivalent to the fix-family.

The following definition is a natural generalisation:

- A family $\mathcal{F}$ is s-wise t-intersecting if the intersection of any s members of $\mathcal{F}$ has size at least t.

With the classifications of the previous section in mind, the following examples seem natural. Let $C_r(s,t,n)$ be the collection of subsets of $[n]$ which contain at least $t+(s-1)r$ elements of $[t+sr]$, and set

$$D_r(s,t,k,n) = \{ X \in C_r(s,t,n) : |X| = k \}.$$

It has long been conjectured that there are functions c, d of the parameters such that both $C_c(s,t,n)$ and $D_d(s,t,k,n)$ are maximum families in their respective settings. There has been much research in this area, with recent progress mainly due to Frankl and Tokushige [FT02, FT03, FT05, Tok05, FT06, Tok06, Tok07b], though many authors have settled different specific cases since the publication of [HM67]. For a good survey of 35 important contributions to this area by various authors over the last 41 years, see [Tok07a, Tok07c].

1.3.2 Other Directions

There are over 150 mathematical publications which reference [EKR61], a testimony to the fact that there are many alternative angles on what it means to generalise the Erdős-Ko-Rado Theorems 1.1.1 and 1.1.2. We will simply mention some of these approaches here, to give a flavour of the variety of combinatorial viewpoints.

- Hajnal & Rothschild characterised families of k-subsets of $[n]$ satisfying the property that no more than r of its members have pairwise fewer than s elements in common in [HR73].
- Various papers such as [Fra78b] have combined conditions on the intersection size with restrictions on the size of the unions of members of a set family.
- Together with Frankl & Deza, Erdős investigated in [DEF78] families $\mathcal{F}$ of k-subsets of $[n]$ such that the size of intersection of two members of $\mathcal{F}$ must be in a fixed set of admissible values, rather than simply being larger than t.
- Generalising again from there, Chung, Graham, Frankl & Shearer obtained a bound on the size of a family F of subsets of $[n]$ such that the intersection of any two members of F contains some member of a fixed family B. For instance, they show in [CGFS86] that if B is the family of n k-subsets of $[n]$ formed by choosing k cyclically consecutive elements of $\mathbb{Z}_n$ then the fix-family $F_0(t,k,n)$ is optimal. Interestingly, they also use these results to obtain bounds on the size of families of graphs whose members intersect in triangles.
- Numerous papers have been written on cross t-intersecting set families, i.e. families where $|A \cap B| \geq t$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$. The interested reader can find more details in [MT89].
- In [DR94] Duke & Rödl investigated some Ramsey type EKR questions, such as the following: for $|\mathcal{A}|$ linear in n, how large does k need to be to guarantee that each family $\mathcal{A}$ of k-subsets of $[n]$ contains a t-intersecting family of some guaranteed minimum size? They extended their results to s-wise t-intersecting families together with Erdős in [DER03].

- Intersecting families in direct products of sets can be found in [Fra96, AAK98].
- Bollobás & Leader showed in [BL97] that fixing is the unique optimal strategy for intersecting families of signed r-sets on $[n]$.
- Together with various co-authors, Talbot generalised the Erdős-Ko-Rado Theorem from a graph-theoretic perspective in [Tal04, HST05].

There are also plenty of surveys discussing generalisations of the Erdős-Ko-Rado Theorem, see for instance [DF83, Fra88a, Fra88b, BE00, Ahl01, BK08] or Chapter 5 in [And87].

So far we have described the origins of Erdős-Ko-Rado Theory in the study of set families. The subject of this thesis, however, will be intersecting families of injections or posets. The next chapter gives an overview of the extension of Erdős-Ko-Rado Theory to other combinatorial structures.

Chapter 2

Extending Erdős-Ko-Rado Theory

Ever since Erdős, Ko, Rado, Katona and others investigated t-intersecting families of subsets of a set in the 1960s, classifying the largest t-intersecting subsets of some class of objects has been a standard problem of extremal set theory. Building on the original question, one approach has been to impose additional restrictions on the collection, as we have seen in the previous chapter. An alternative approach is to vary the combinatorial objects which are the elements of the underlying set. Structures which have been considered more recently in this context are graphs, (partial) permutations, words and set partitions. Investigations in this field usually begin by finding the size of the largest intersecting family in the set of all objects of a given type. Having found a bound, the next natural aim is to classify all intersecting families attaining the bound — and then to extend the results to $t > 1$.

This chapter gives an overview of results in this area which are relevant to this thesis, i.e. the study of intersecting posets and intersecting injections. While the former concept seems to be new, our study of t-intersecting injection families builds on previous research into intersecting permutation sets and word families.

2.1 Words

Consider integer sequences $a_1 a_2 \dots a_k$ with $1 \leq a_i \leq n$, and let $[n]^k$ be the set of these words of length k over the alphabet $[n]$.

Definition 2.1.1. For two words $a = a_1a_2 \dots a_k$, $b = b_1b_2 \dots b_k \in [n]^k$, set

$$\mathrm{int}(a, b) = \{ i \in [k] : a_i = b_i \} .$$

Then a and b *t-intersect* if $|\,\mathrm{int}(a, b)| \geq t$.

It follows from the definitions in Section 1.1 that a family $\mathcal{F} \subseteq [n]^k$ is *t-intersecting* if every pair of words in $\mathcal{F}$ is t-intersecting.

2.1.1 A Bound on Intersecting Word Families

Words were the first objects after sets to be considered in the context of intersecting families. In [Kle66b] Kleitman proved a conjecture of Erdős that if the elements of $\mathcal{F}$ are sequences of length k of zeros and ones which differ from one another in at most $2x$ positions then

$$|\mathcal{F}| \leq \sum_{i=0}^{x} \binom{k}{i}.$$

The earliest proof of the fact that an intersecting subset of $[n]^k$ has size at most n^{k-1} is generally credited to [Ber74]. Livingston showed in [Liv79] that the only families attaining this bound are the ones whose elements all have a fixed position in common.

In this section, we present our own proof of the classification of maximum intersecting word families, though the idea of what we call *orbits* below is fairly standard, see [DF77, BK08]. The methods introduced for this purpose will be used again in Section 3.1 to classify maximum intersecting injection families.

Let $\pi = (1\,2\,\dots\,n)$ denote the n-cycle in the symmetric group $\mathcal{S}_n$ of permutations on n points, and let permutations act on words in $[n]^k$ by acting on each position separately:

$$(w_1w_2 \dots w_k)\pi = (w_1\pi)(w_2\pi) \dots (w_k\pi).$$

Note that the image of an element of $[n]^k$ is again an element of $[n]^k$.

Definition 2.1.2. For any $w \in [n]^k$, the *orbit* of w in $[n]^k$ is the set $\{w\pi^i \,:\, i \in \mathbb{N}\}$, denoted by $O(w)$.

For example, if $n = 5$ and $k = 4$, then

$$O(2551) = \{2551, 3112, 4223, 5334, 1445\}.$$

Theorem 2.1.3. *If $\mathcal{F} \subseteq [n]^k$ is intersecting then $|\mathcal{F}| \leq n^{k-1}$.*

Moreover, the set of orbits $\{O(w) \,:\, w \in [n]^k\}$ forms a partition of $[n]^k$ into disjoint sets of size n. If $|\mathcal{F}| = n^{k-1}$ then $|\mathcal{F} \cap O(w)| = 1$ for all $w \in [n]^k$.

Proof. Let $w \in [n]^k$ be arbitrary. We can deduce three statements from the fact that π is a permutation: firstly, $|O(w)|$ is equal to the order of π, namely n. Secondly, if u, v are two distinct elements of $O(w)$ then u and v do not intersect. Thirdly, if v and w are two distinct elements of $[n]^k$ which are not images of one another under π — that is $v \notin O(w)$ — then $O(v)$ and $O(w)$ are disjoint: otherwise, we have $v\pi^i = w\pi^j$ for some $i, j \in \mathbb{N}$, giving $v = w\pi^{j-i}$, contradicting our choice of v and w. Hence the set of orbits partitions $[n]^k$ into disjoint sets of size n.

By the second observation, $\mathcal{F}$ contains at most one word from each orbit, so

$$|\mathcal{F}| \leq |\{O(w) \,:\, w \in [n]^k\}|.$$

Our first observation was that all orbits have the same size, and so it follows from the third observation that

$$|\{O(w) \,:\, w \in [n]^k\}| = \frac{|[n]^k|}{|O(w)|} = \frac{n^k}{n} = n^{k-1}$$

as required. If equality holds, then $\mathcal{F}$ must contain precisely one word from each orbit. $\square$

We will see in Chapter 3 that the above arguments apply to injections as well as words, see Theorem 3.1.1. The above result was recently presented in [BK08], where Brockman & Kay use orbits to derive the bound in Theorem 2.1.3 from Theorem 3.1.1, after having obtained the latter in a different way. In [BK08], they do not investigate the structure of maximum families; we will, however, do precisely that in the following section.

2.1.2 Classification of Maximum Families

Having found a bound on the size of an intersecting subset, we would like to characterise the families attaining it. We begin by explaining how the concepts of Section 1.2.1 transfer to the context of words. The *saturation family* is defined as follows: for $0 \leq r \leq (k-t)/2$, set

$$H_r(t, k, n) = \left\{\, w \in [n]^k : w \text{ has 1 in at least } t + r \text{ of its first } t + 2r \text{ positions} \,\right\}.$$

Note that for any $i \in [n]$, $j \in [k]$, the set

$$\{w_1 w_2 \dots w_k \in [n]^k \,:\, w_j = i\}$$

can be obtained from $H_0(1, k, n)$ simply by permuting letters and positions. Thus the two are structurally equivalent and we refer to either of them as a *fix-family*. More generally, any word family which can by obtained from $H_r(t, k, n)$ by permutations of the letters and positions is said to be *equivalent* to $H_r(t, k, n)$. In this section, we show that for $n \geq 3$, the fix-families are the only intersecting subsets of size n^{k-1}.

If $\mathcal{A} = \{ A_i : 1 \leq i \leq m \}$ is a family of m sets, then a collection of m elements $T = \{ a_i : 1 \leq i \leq m \}$ is called a *transversal* of $\mathcal{A}$ if $a_i \in A_i$ for all $1 \leq i \leq m$. Also, for $a \in [n]$ and $l \in \mathbb{N}$, a^l denotes the word $\underbrace{aa\ldots a}_{l \text{ times}}$. For instance, $a^3b^0c^2 = aaacc$.

Theorem 2.1.4. *If $n \geq 3$ and $\mathcal{F}$ is a maximum intersecting subset of $[n]^k$, then $\mathcal{F}$ is equivalent to the fix-family $H_0(1, k, n)$.*

Proof. Recall that by Theorem 2.1.3, an intersecting subset of $[n]^k$ whose size attains the bound is a transversal of the orbits, so there exists a unique $z \in [n]$ satisfying $\mathcal{F} \cap O(1^k) = \{z^k\}$. We can assume without loss of generality that $z = 1$, so $1^k \in \mathcal{F}$.

Let $\mathcal{L}$ be the list of words in $[n]^k$ given by

$$\begin{array}{rcl} 111\ldots1 & = & 1^k \\ n11\ldots1 & = & n1^{k-1} \\ nn1\ldots1 & = & n1^{k-2} \\ & \vdots & \\ nn\ldots n11 & = & n^{k-2}1^2 \\ nnn\ldots n1 & = & n^{k-1}1 \end{array}$$

then $\mathcal{F} \cap \mathcal{L} \neq \emptyset$ since $1^k \in \mathcal{F}$. So let $n^{j-1}1^{k-j+1}$ be the last element of $\mathcal{L}$ belonging to $\mathcal{F}$, in the sense that $l > j$ implies $n^{l-1}1^{k-l+1} \notin \mathcal{F}$, then $1 \leq j \leq k$. (We do not necessarily have $n^{l-1}1^{k-l+1} \in \mathcal{F}$ for all $l < j$.)

For arbitrary $r \in \{1, \ldots, k-1\}$, $O(n^r1^{k-r})$ contains precisely two words which intersect with 1^k, namely n^r1^{k-r} and 1^r2^{k-r}. Thus the choice of j implies

$$X = \{1^k, 1^{k-1}2, 1^{k-2}2^2, 1^{k-3}2^3, \ldots, 1^j2^{k-j}, n^{j-1}1^{k-j+1}\} \subseteq \mathcal{F}. \tag{2.1.5}$$

Let $a = a_1 \ldots a_k \in [n]^k$ such that $a_1, \ldots, a_{j-1} \in \{1, n\}$, $a_{j+1}, \ldots, a_k \in \{1, 2\}$ and $a_j = 1$. We claim that $a \in \mathcal{F}$.

Note that if $k = 1$, the theorem is trivially true. If $k = 2$, then a is an element of X and hence belongs to $\mathcal{F}$, so suppose $k \geq 3$. We will show that none of the elements of $O(a)$ other than a itself can belong to $\mathcal{F}$.

Firstly, observe that $a\pi$ has its first $j-1$ positions in $\{2, 1\}$, its jth position is 2 and its last $k-j$ positions are in $\{2, 3\}$. Since $n \geq 3$, this means that $a\pi$ does not intersect $n^{j-1}1^{k-j+1}$ which is in $\mathcal{F}$ by (2.1.5).

Next, let $i \in \{2, \ldots, n-1\}$ and consider $a\pi^i$. This element of $O(a)$ does not have 1 in any of its first j positions, and none of its last $k-j$ positions are equal to 2. Hence $a\pi^i$ does not intersect $1^j 2^{k-j}$ which is in $\mathcal{F}$ by (2.1.5).

In conclusion, $a\pi^i \notin \mathcal{F}$ for $1 \leq i \leq n-1$, so we must have $a\pi^n = a \in \mathcal{F}$.

Now let $w = w_1 w_2 \ldots w_k \in [n]^k$ with $w_j = 1$ and consider an element $v = v_1 v_2 \ldots v_k$ of $O(w)$ which is distinct from w. Since $v_j \neq 1$, there exists $b = b_1 b_2 \ldots b_k \in [n]^k$ with $b_1, \ldots, b_{j-1} \in \{1, n\}$, $b_{j+1}, \ldots, b_k \in \{1, 2\}$ and $b_j = 1$ such that b and v do not intersect. By the above arguments, $b \in \mathcal{F}$, so $v \notin \mathcal{F}$. Since v was an arbitrary element of $O(w)$ distinct from w, and $\mathcal{F}$ contains one word from each orbit, this implies $w \in \mathcal{F}$. Since w was an arbitrary element of $[n]^k$ with 1 in position j, this yields

$$\mathcal{F} = \{w_1 w_2 \ldots w_k \in [n]^k \ : \ w_j = 1\}$$

as required. □

Different Maximum Intersecting Subsets

We have shown that every maximum intersecting subset of $[n]^k$ is equivalent to the fix-family when $n \geq 3$. The case $n < 2$ is trivial. However, when $n = 2$, different maximum intersecting subsets exist.

For instance, the family

$$\mathcal{F} = \{111, 112, 122, 212\}$$

is an intersecting subset of $\{1,2\}^3$ of maximal size $|\mathcal{F}| = n^{k-1} = 2^2 = 4$, but $\mathcal{F}$ is not equivalent to the fix-family $H_0(1,3,2)$: suppose otherwise, then since $111 \in \mathcal{F}$ the letter being fixed must be 1. In view of $122 \in \mathcal{F}$ we thus conclude that elements of $\mathcal{F}$ have 1 in the first position, but this contradicts $212 \in \mathcal{F}$.

Indeed, this example illustrates the more general result given in the following theorem, which is that if $n = 2$, then any transversal of the orbits of $[n]^k$ is intersecting. This shows that the condition $n \geq 3$ in Theorem 2.1.4 is best possible.

Proposition 2.1.6. *Let $A = \{1, 2\}$ and $k \in \mathbb{N}$. Then $\mathcal{F} \subseteq A^k$ is intersecting with $|\mathcal{F}| = 2^{k-1}$ if, and only if, $\mathcal{F}$ is a transversal of the orbits $O(w)$, $w \in A^k$.*

Proof. The forward implication is given by Theorem 2.1.3, so let $\mathcal{F}$ be a transversal of the orbits. Consider $v, w \in \mathcal{F}$ with $v = v_1 v_2 \ldots v_k$, $w = w_1 w_2 \ldots w_k$ and suppose that v and w do not intersect.

Then since $|A| = 2$, we have

$$v_i = 1 \iff w_i = 2 \text{ and } v_i = 2 \iff w_i = 1, \ \forall i \in \{1, \dots, k\}.$$

But this means $v = w\pi$, that is $v \in O(w)$ and so $\mathcal{F}$ contains two words from $O(w)$ which is a contradiction. Thus $\mathcal{F}$ is intersecting. The maximality of $\mathcal{F}$ follows from the fact that $\mathcal{F}$ contains one word from each orbit, since there are precisely 2^{k-1} orbits by Theorem 2.1.3. □

There is a natural correspondence between words of length k over a two letter alphabet and subsets of $[k]$. Formally, the bijection ϕ from $\{0,1\}^k$ to the power set $\mathcal{P}([k])$ is given by

$$x \in \phi(w_1 \dots w_k) \iff w_x = 1$$

for $1 \leq x \leq k$ and $w_1 \dots w_k \in \{0,1\}^k$.

However, it is important to note that the intersection structures of $\{0,1\}^k$ and $\mathcal{P}([k])$ are quite different: two words in $\{0,1\}^k$ intersect in a position where they both have 0, but two subsets of $[k]$ cannot intersect in a point which neither of them contain. To obtain a more appropriate correspondence between a family of subsets of $[k]$ and words under the Hamming distance, Ahlswede & Katona endowed set families with the symmetric difference as a distance function in [AK77]: if A and B are sets then their symmetric difference $A \Delta B$ is the set of points contained in one but not both of A and B:

$$A \Delta B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

We will consider symmetric differences again in Section 3.2.2. For now, having completed our classification of intersecting word families, we will describe what is known about the case $t > 1$ in the next section.

2.1.3 Increasing the Intersection Parameter

One year after Livingston's classification of maximum 1-intersecting word families in [Liv79], Frankl & Füredi proved in [FF80] that for $t \geq 15$, the fix-family is optimal if and only if $n \geq t+1$. Moreover, [FF80] concludes with the general conjecture that if $\mathcal{F}$ is a t-intersecting subset of $[n]^k$ then

$$|\mathcal{F}| \leq \max_{0 \leq r \leq (k-t)/2} |H_r(t, k, n)|.$$

In [Moo82], Moon used induction on cross-t-intersecting families to show that for $n \geq t+2$, all maximum t-intersecting families in $[n]^k$ are equivalent to the fix-family. (Two families $\mathcal{F}, \mathcal{F}'$ are cross-t-intersecting if every $x \in \mathcal{F}$ t-intersects every $y \in \mathcal{F}'$.)

Following their complete classification of maximum t-intersecting set families in [AK97], Ahlswede & Khachatrian proved Frankl & Füredi's conjecture by showing that the principle of saturation also applies to words. Once again, the statement of the theorem uses the convention that $(t-1)/(n-2) = \infty$ if $n = 2$.

Theorem 2.1.7. *(Ahlswede, Khachatrian [AK98]).*

For $n \geq 2$ let $\mathcal{F}$ be a t-intersecting family in $[n]^k$.

Set $q := (t-1)/(n-2)$ and let r be the largest non-negative integer such that

$$t + 2r < \mathit{min}\{k+1,\ t+2q\}.$$

If $t > 1$, $t + 2q \leq k$ and q is integer valued, then $\mathcal{F}$ is equivalent to $H_q(t,k,n)$ or $H_{q-1}(t,k,n)$.

Otherwise, $\mathcal{F}$ is equivalent to $H_r(t,k,n)$.

One year later, Frankl and Tokushige published an alternative proof of this result in [FT99]. In Chapter 4, we adapt Ahlswede & Khachatrian's proof from [AK98] to show that, provided injection families can be standardised, one of the saturation families is optimal t-intersecting.

2.2 Mappings

Throughout this thesis, $\mathcal{S}_n$ denotes the symmetric group of permutations on n points, and $\mathcal{I}_n^k$ is the set of injections from $[k]$ to $[n]$, so $\mathcal{I}_n^n = \mathcal{S}_n$. Note also that $\mathcal{I}_n^k$ may be viewed as a subset of $[n]^k$, and the definition of intersection is the same: two injections in $\mathcal{I}_n^k$ *t-intersect* if they agree on the image of at least t domain points.

Assuming that t is clear from the context, we say that a family $\mathcal{F}$ in $\mathcal{I}_n^k$ is *equivalent to the fix-family* if there exists a subset T of $[k] \times [n]$ of size $|T| = t$ such that

$$\mathcal{F} = \left\{ \alpha \in \mathcal{I}_n^k : \alpha(x) = y \ \text{ for all } (x,y) \in T \right\},$$

giving $|\mathcal{F}| = (n-t)!/(n-k)!$.

2.2.1 Permutations

Studying the intersection structure of $\mathcal{I}_n^k$ began with research into intersecting permutation families in the 1970s.

Theorem 2.2.1. *(Deza, Frankl [DF77].)*

If $\mathcal{F}$ is an intersecting subset of $\mathcal{S}_n$ then $|\mathcal{F}| \leq (n-1)!$.

Deza & Frankl also showed in [DF77] that the fix-family is optimal t-intersecting in $\mathcal{S}_n$ whenever there exists a sharply t-transitive set of permutations in $\mathcal{S}_n$, and give examples of such parameter values. Moreover, [DF77] first applied the idea of saturation to permutation families: when $k-t$ is even, let

$$\mathcal{G}(t,k,n) = \left\{ w \in \mathcal{I}_n^k : w \text{ moves at most } (k-t)/2 \text{ points} \right\},$$

and if $k-t$ is odd, set

$$\mathcal{G}(t,k,n) = \left\{ w \in \mathcal{I}_n^k : w \text{ moves at most } (k-t-1)/2 \text{ elements of } [k-1] \right\}.$$

Then $\mathcal{G}(t,k,n)$ is t-intersecting by the pigeonhole principle.

Considering that an injection which moves at most $(k-t)/2$ points fixes at least

$$k - (k-t)/2 = (k+t)/2$$

points, we observe a distinct similarity between $\mathcal{G}(t,k,n)$ and the Katona family $K_p(t,k)$ from Section 1.2.1.

Theorem 2.2.2. *(Deza, Frankl [DF77])*

For each $T \in \mathbb{N}$ with $T \geq 3$, there exists $k_0(T) \in \mathbb{N}$ such that for $k \geq k_0(T)$, the saturation family $\mathcal{G}(k-T,k,k)$ is maximum $(k-T)$-intersecting in $\mathcal{S}_k$.

The proof of Theorem 2.2.2 depends on the Erdős-Ko-Rado Theorem 1.1.1. Recall from Chapter 1 that Katona proved in [Kat64] that a family attaining the bound in Theorem 1.1.1 must be a fix-family, see Theorem 1.2.1. Using this structural version of the Erdős-Ko-Rado Theorem in Deza & Frankl's proof of Theorem 2.2.2 demonstrates that for T and k as in Theorem 2.2.2, the saturation family $\mathcal{G}(k-T,k,k)$ is in fact the unique maximum $(k-T)$-intersecting subset of $\mathcal{S}_k$. This argument will be presented in detail in the concluding paragraphs of the proof of Theorem 3.2.11 which generalises Theorem 2.2.2 to injections.

After Deza & Frankl's paper [DF77], intersecting permutation families were almost forgotten for a quarter century until, in the early 2000s, Cameron & Ku as well as Larose & Malvenuto independently obtained the classification of maximum intersecting permutation families.

Theorem 2.2.3. *(Cameron, Ku [CK03]; Larose, Malvenuto [LM04].)*

If $n \geq 2$ and $\mathcal{F}$ is an intersecting subset of $\mathcal{S}_n$ with $|\mathcal{F}| = (n-1)!$ then $\mathcal{F}$ is equivalent to the fix-family.

This result inspired numerous investigations of intersecting permutation families. It has since been shown that fixing is the unique optimal strategy for obtaining large intersecting subsets of the following global sets:

- the set of k-partial permutations of $[n]$ [KL06, LW07],
- the alternating group $\mathcal{A}_n \subset \mathcal{S}_n$ [KW07],
- a direct product $\mathcal{S}_{n_1} \times \cdots \times \mathcal{S}_{n_q}$ of symmetric groups [KW07],
- Coxeter groups of types B and D [WZ08].

We point out that $\mathcal{I}_n^k$ is strictly contained in the set of k-partial permutations on n points studied in [KL06, LW07], since the domain of a k-partial permutation is not fixed to be $[k]$, but can be any k-subset of $[n]$. Finally, consider a different definition of intersection: two elements of $\mathcal{S}_n$ *t-cycle intersect* if, when written in disjoint cycle form, they share at least t cycles. Ku & Renshaw showed in [KR08] that for sufficiently large n, all maximum t-cycle intersecting subsets of $\mathcal{S}_n$ are equivalent to the family fixing t singleton cycles.

2.2.2 Injections

In this thesis we prove that, with the original definition of intersection for injections, every maximum intersecting subset of $\mathcal{I}_n^k$ is equivalent to the fix-family, a fact which was recently conjectured in [Bor08], an article about labelled sets building on [BL97]. Moreover, we show in Chapter 3 that

- if n is large in terms of k and t, fixing is the unique optimal strategy;
- if k is large in terms of $k-t$ and $n-k$, the saturation family $\mathcal{G}(t,k,n)$ is the unique maximum t-intersecting subset of $\mathcal{I}_n^k$.

In view of Ahlswede & Khachatrian's results concerning the optimality of saturation families for small parameter values in the context of sets and words, we are not surprised to find that computational evidence suggests that the same is true for injections. Unfortunately, the well-known proof methods cannot be applied to injection families as we will see in Section 5.2.2. In Chapter 4, we prove that saturation, including fixing, is optimal among so-called exemplary injection families for all parameter values except $k = n$. Whether there are any injection families which cannot be standardised in this way remains an open question.

2.3 Relational Structures

The following definitions will be used throughout the thesis. A (binary) *relation* R on $[n]$ is a subset of $[n] \times [n]$. R is furthermore:

- *reflexive* if $(x, x) \in R$ for all $x \in [n]$;
- *irreflexive* if $(x, x) \notin R$ for all $x \in [n]$;
- *symmetric* if for all distinct $x, y \in [n]$, $(x, y) \in R$ implies $(y, x) \in R$;
- *antisymmetric* if for all distinct $x, y \in [n]$, $(x, y) \in R$ implies $(y, x) \notin R$;
- *transitive* if for all $x, y, z \in [n]$, $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$.

A reflexive, symmetric and transitive relation E is called an *equivalence relation*. The set of its *equivalence classes*

$$\mathcal{C}(E) := \{ \{ y \in [n] : (x, y) \in E \} : x \in [n] \}$$

forms a *partition* of $[n]$, which is a collection $P = \{X_1, \ldots, X_k\}$ such that the *classes* X_i are disjoint, non-empty and their union is $[n]$.

An (undirected) *graph* is a symmetric relation G on $[n]$. In this context, the elements of $[n]$ are called *vertices* and the elements of G are referred to as *edges*. Unless otherwise stated, we consider graphs to be *simple*, i.e. G is irreflexive. Regrettably, graphs only make sporadic appearances in this thesis. The study of intersecting properties of graphs is an old and well-developed area of combinatorics which we will not survey here; instead, we refer the interested reader to [Szw03].

An antisymmetric and transitive relation P on $[n]$ which is either reflexive or irreflexive is called a *(partial) order*. If all distinct elements of $[n]$ are *comparable* under P, i.e. for all distinct $x, y \in [n]$ we have either $(x, y) \in P$ or $(y, x) \in P$, then P is a *linear* order.

2.3.1 Partial Orders

Part III is concerned with intersecting orders. For the combinatorial structures considered so far in this introduction, the generalisation from fixing to saturation does not become relevant before we move from considering 1-intersecting sets to studying t-intersecting families for $t > 1$. We will see in Chapter 7 that this is not necessarily the case for poset classes. Perhaps it is the fact that there are several conceivable definitions of intersection for partial orders which sets the intersection structure of posets apart from that of other structures. These issues are discussed in detail in Section 6.3.1, so rather than pre-empting our observations on partial orders at this point, we now turn our attention to equivalence relations to see why the definition of intersection is not necessarily straightforward in the context of relational structures.

2.3.2 Equivalence Relations

Since each equivalence relation on $[n]$ leads to a partition of $[n]$ and vice versa, we have at least two alternative approaches in this context: if two partitions of $[n]$ share t classes, it seems natural to say that they t-intersect. On the other hand, we might say that two equivalence relations on $[n]$ intersect if there are two distinct elements of $[n]$ which are equivalent under both relations. How this second notion extends to the case $t > 1$ once again depends on whether one's background is primarily in relational structures, or whether one's main motivations lie in Chapter 1.

These alternative intersection definitions may be summarised as follows: let E_1 and E_2 be equivalence relations on $[n]$ with associated partitions $\mathcal{C}(E_1)$ and $\mathcal{C}(E_2)$, then

1. E_1 and E_2 have property $\mathscr{I}_1(t)$ if $|\mathcal{C}(E_1) \cap \mathcal{C}(E_2)| \geq t$,
2. E_1 and E_2 have property $\mathscr{I}_2(t)$ if there exist $C_1 \in \mathcal{C}(E_1)$, $C_2 \in \mathcal{C}(E_2)$ such that $|C_1 \cap C_2| \geq t$,
3. E_1 and E_2 have property $\mathscr{I}_3(t)$ if $|E_1 \cap E_2| \geq n + t$. (Since equivalence relations are reflexive, any two of them intersect in n pairs of the form (x, x).)

Further alternative definitions are discussed in [ES00]. As usual, a family of equivalence relations has property $\mathscr{I}_j(t)$ if this is the case for any two of its elements.

To describe what is know about these properties in various contexts, let

$$\begin{aligned}
\mathcal{B}_n &= \{\text{partitions of } [n]\}, \\
\mathcal{P}_k^n &= \{\text{partitions of } [n] \text{ with } k \text{ classes}\}, \\
\mathcal{U}_k^n &= \{\text{partitions of } [n] \text{ with } k \text{ classes, each of size } n/k\}.
\end{aligned}$$

(In the definition of $\mathcal{U}_k^n$, k must be a divisor of n.)

Property $\mathscr{I}_1(t)$

Families with this property are almost entirely classified. Péter Erdős & Székely demonstrated in [ES00] that the $\mathscr{I}_1(t)$-*fix-family*, which consists of all partitions in $\mathcal{P}_k^n$ containing t fixed singleton classes, is the largest subset of $\mathcal{P}_k^n$ with property $\mathscr{I}_1(t)$. In [MM05], Meagher & Moura showed that, for n sufficiently large, no other subset of $\mathcal{U}_k^n$ with property $\mathscr{I}_1(t)$ is as large as the $\mathscr{I}_1(t)$-fix-family, and if $t = 1$ then this holds for all n. Ku & Renshaw proved the analogue of the Meagher-Moura result for $\mathcal{B}_n$ in [KR08].

Property $\mathscr{I}_2(t)$

It seems that this case is much more complex: we have a couple of conjectures but are not aware of any results. Czabarka conjectured in [Cza99] that when $k \geq (n+1)/2$, the $\mathscr{I}_2(2)$-*fix-family*, consisting of all elements of $\mathcal{P}_k^n$ which have 1 in the same class as 2, is maximum among $\mathscr{I}_2(2)$-families in $\mathcal{P}_k^n$. Meagher & Moura conjectured in [MM05] that for $t \leq n/k$, the $\mathscr{I}_2(t)$-fix-family is the unique maximum $\mathscr{I}_2(t)$-family in $\mathcal{U}_k^n$ up to permutations of $[n]$. We will refer to this as the [MM05]-Conjecture.

Note that if two partitions P_1, $P_2 \in \mathcal{U}_k^n$ contain classes $C_1 \in P_1$, $C_2 \in P_2$ with $|C_1 \cap C_2| \geq n/k$, then we must have $C_1 = C_2$ since all classes of partitions in $\mathcal{U}_k^n$ have size n/k. In other words, if a subset $\mathcal{F}$ of $\mathcal{U}_k^n$ has property $\mathscr{I}_2(n/k)$, then $\mathcal{F}$ has property $\mathscr{I}_1(1)$. Hence the case $t = n/k$ of the [MM05]-Conjecture is confirmed by Meagher & Moura's result on subsets of $\mathcal{U}_k^n$ with property $\mathscr{I}_1(1)$ described above.

However, the [MM05]-Conjecture is not true in general: consider once more the case $t = 2$, let n be an even number greater than 4 and set $k = 2$. Then two arbitrary partitions $P_1, P_2 \in \mathcal{U}_2^n$ each have two classes of size $n/2$, say

$$P_1 = \{C_{11}, C_{12}\}, \quad P_2 = \{C_{21}, C_{22}\},$$

where $C_{i1} = \overline{C_{i2}}$. Therefore

$$|C_{11} \cap C_{21}| < 2 \implies |C_{11} \cap C_{22}| \geq n/2 - 1 > 1$$

since $n > 4$. In other words, we have either $|C_{11} \cap C_{21}| \geq 2$ or $|C_{11} \cap C_{22}| \geq 2$. This shows that for $n > 2^2$, the whole of $\mathcal{U}_2^n$ has property $\mathscr{I}_2(2)$ and so the $\mathscr{I}_2(2)$-fix-family, being strictly contained in $\mathcal{U}_2^n$, cannot be maximum. Similarly, for all $n > 3^2 = 9$ which are divisible by 3, the class $\mathcal{U}_3^n$ has property $\mathscr{I}_2(2)$ and is therefore a counterexample to the [MM05]-Conjecture .

Property $\mathscr{I}_3(t)$

We are not aware of any results or conjectures concerning property $\mathscr{I}_3(t)$ and our own investigation of the problem did not come to any substantial conclusions. However, we present a few preliminary observations here.

Note that property $\mathscr{I}_2(2)$ coincides with property $\mathscr{I}_3(1)$. Thus we deduce from the above counterexamples to the [MM05]-Conjecture that fixing is not always optimal with respect to $\mathscr{I}_3(t)$ in $\mathcal{U}_k^n$. Let us consider how similar examples can be constructed in $\mathcal{B}_n$.

For positive integers m_i with $m_1 \geq m_2 \geq \cdots \geq m_k$ and $\sum_{i=1}^k m_i = n$, denote by $(m_1, \ldots, m_k)$ the collection of partitions of $[n]$ which have k classes of respective sizes m_i. Conversely, if P is a partition of $[n]$ then $m_i(P)$ denotes the size of its i^{th} largest class, including multiplicities. For the sake of simplicity, let us refer to subsets of $\mathcal{B}_n$ which have the $\mathscr{I}_3(1)$-property as *intersecting families* for the moment. It is easy to see that the fix-family has size $|\mathcal{B}_{n-1}|$, the $(n-1)^{st}$ *Bell number*.

Thus we are interested in finding intersecting families in $\mathcal{B}_n$ which are larger than $|\mathcal{B}_{n-1}|$. Finding these by hand for small n is easy, but it is not clear how a maximal saturation family for larger n would be defined. To see this, consider saturating over m_1, the size of the largest class. If P, Q are partitions of $[n]$ with $m_1(P), m_1(Q) \geq n/2+1$ then P and Q have property $\mathscr{I}_3(t)$; for if $A \in P$, $B \in Q$ are classes of sizes $m_1(P)$, $m_1(Q)$ respectively, then

$$|A \cap B| \geq m_1(P) + m_1(Q) - n \geq 2(n/2+1) - n = 2$$

by the pigeonhole principle. To see that the bound $n/2+1$ is sharp, let $n = 2x+1$ and consider the following two elements of $(x+1, 1, 1, \ldots, 1)$:

$$\begin{aligned} P &= \{\{1, 2, \ldots, x, n\}, \{x+1\}, \{x+2\}, \ldots, \{n-1\}\}, \\ Q &= \{\{1\}, \{2\}, \ldots, \{x\}, \{x+1, x+2, \ldots, n\}\}. \end{aligned}$$

Then P, Q are partitions of $[n]$ with $m_1(P), m_1(Q) = (n+1)/2 < n/2+1$ which do not have property $\mathscr{I}_3(t)$.

Although the bound $n/2+1$ is sharp, the family

$$G(n) = \{\, P \in \mathcal{B}_n : m_1(P) \geq n/2+1 \,\}$$

is usually not maximal. For instance, when $n = 7$ the family

$$\mathcal{F} = G(7) \cup (4,3) \cup (4,2,1) \cup (3,3,1) \cup \{\, P \in (4,1,1,1) : \{7\} \in P \,\}$$

is intersecting and strictly larger than $G(7)$. We have $|\mathcal{F}| = 275$ and $|\mathcal{B}_6| = 203$, but it is not clear if the definition of $\mathcal{F}$ could be extended to yield saturation families which are larger than the fix-family for general n.

Even considering a fixed choice of all m_i, it is difficult to describe exactly when $(m_1, \ldots, m_k)$ is intersecting. The condition $m_1 > \sqrt{n}$ is necessary (proof omitted) but insufficient, since e.g. $(4,3,1,1,1)$

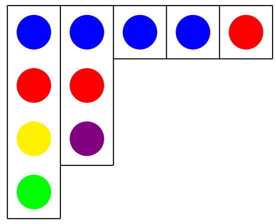


Figure 2.3.1: it is possible to fit 4 blue, 3 red, 1 yellow, 1 green and 1 violet ball into five boxes of respective sizes 4, 3, 1, 1, 1 in such a way that no box contains two balls of the same colour. This distribution of balls into boxes corresponds to the example shown in (2.3.1), demonstrating that $(4,3,1,1,1)$ is not intersecting.

is not intersecting:

$$\{\{1,2,3,4\},\{5,6,7\},\{8\},\{9\},\{10\}\},$$
$$\{\{1,5,8,9\},\{2,6,10\},\{3\},\{4\},\{7\}\} \tag{2.3.1}$$

are two elements of $(4,3,1,1,1)$ which do not have property $\mathscr{I}_3(t)$, c.f. Figure 2.3.1. On the other hand, if $m_1 > k$ then $(m_1,\ldots,m_k)$ is intersecting: if $P,Q \in (m_1,\ldots,m_k)$ and $A \in P$ with $|A| = m_1 > k$, then at least two elements of A must be in the same class of Q by the pigeonhole principle. Thus the condition $m_1 > k$ is sufficient, but unnecessary since e.g. $(5,4,1,1,1,1)$ is intersecting (see Figure 2.3.2), despite the fact that $m_1 = 5 < 6 = k$.

Recall that if P is a partition of $[n]$ then $m_i(P)$ is the size of its i^{th} largest class, and denote by $l_i(P)$ the number of classes in P which have size at least i. In Figures 2.3.1 and 2.3.2, we represent one partition by coloured balls, the other by empty boxes, and two balls of the same colour in the same

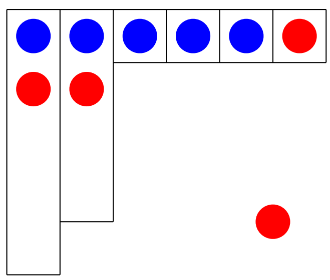


Figure 2.3.2: it is impossible to fit 5 blue and 4 red balls into six boxes of respective sizes 5, 4, 1, 1, 1, 1 in such a way that no box contains two balls of the same colour. (This remains impossible if 4 balls of distinct colours are added to the scenario.) Hence $(5,4,1,1,1,1)$ is intersecting.

box correspond to an intersection of the two partitions. Thinking about intersecting partitions in this way a little longer leads us to the following observation: let P and Q be partitions of $[n]$, not necessarily distinct. If

$$\sum_{i=1}^{j} m_i(X) \leq \sum_{i=1}^{j} l_i(Y) \text{ for all } 1 \leq j \leq b_1(Y) \tag{2.3.2}$$

holds for either $(X, Y) = (P, Q)$ or for $(X, Y) = (Q, P)$ then $P \cup Q$ is not intersecting. Conversely, if (2.3.2) fails for both $(X, Y) = (P, Q)$ and for $(X, Y) = (Q, P)$ then $P \cup Q$ is intersecting, provided P and Q are individually intersecting, of course. (Note P is intersecting if (2.3.2) fails for $X = Y = P$.)

Condition (2.3.2) formalises the observation illustrated in Figures 2.3.1 and 2.3.2 that if not all balls of the same colour can be distributed into different boxes, or if this cannot be done simultaneously for all colours, then the two corresponding equivalence relations intersect. It presents some progress, but it is unclear whether it can be used to find large intersecting families in $\mathcal{B}_n$.

We conclude that there is more work to be done in this area and, recalling the purpose of this excursion into the world of equivalence relations, that even the definition of intersection can be ambiguous in the study of relational structures. Chapter 6 further explores this issue by considering different definitions of intersection for partial orders.

2.4 Conclusion

This chapter has surveyed results on intersection properties of combinatorial structures which are relevant to this thesis; therefore our overview is by no means exhaustive. For instance, [Hsi75, FW86] present analogues of the Erdős-Ko-Rado Theorem for collections of intersecting subspaces of a finite vector space; Stanton's corresponding result in [Sta80] is concerned with Chevalley groups; and Rands' findings regarding designs in [Ran82] are analogous to the Erdős-Ko-Rado Theorem 1.1.2 for the t-intersecting case. However, it is now time to concern ourselves in detail with one of the two main subjects of this thesis: intersecting injections.

Part II

Injections

CHAPTER 3

BOUNDS AND STRUCTURE IN THE LIMIT

Throughout Part II of this thesis, k and n will be positive integers with $1 \leq k \leq n$. Also, $\mathcal{I}_n^k$ will be the set of injections from $[k]$ to $[n]$ or, equivalently, the set of words of length k over $[n]$ with no repeated symbols. So

$$\mathcal{I}_n^k = \{a_1 a_2 \dots a_k \mid a_i \in [n] \text{ and } i \neq j \implies a_i \neq a_j\}$$

with

$$|\mathcal{I}_n^k| = \prod_{i=0}^{n-k+1} (n-i) = \frac{n!}{(n-k)!}.$$

The definition of intersection is the same for injections as it is for permutations in e.g. [DF77, CK03].

Definition 3.0.1. For $a = a_1 a_2 \dots a_k$, $b = b_1 b_2 \dots b_k \in \mathcal{I}_n^k$, set

$$\operatorname{int}(a, b) = \{\, i \in [k] : a_i = b_i \,\}.$$

A subset $\mathcal{F}$ of $\mathcal{I}_n^k$ is *t-intersecting* if, for all $a, b \in \mathcal{F}$, we have $|\operatorname{int}(a, b)| \geq t$. When $t = 1$, we usually say *intersecting* rather than 1-intersecting.

The aim of Part II is to determine the maximum t-intersecting families in $\mathcal{I}_n^k$, so we need to develop some concept of when two subsets of $\mathcal{I}_n^k$ are equivalent. To this end, we let permutations act on injections as they acted on words in Chapter 2, namely by acting on each image point separately: for $w_1 w_2 \dots w_k \in \mathcal{I}_n^k$ and $\sigma \in \mathcal{S}_n$,

$$(w_1 w_2 \dots w_k)\sigma = (w_1\sigma)(w_2\sigma)\dots(w_k\sigma).$$

Then clearly

$$\left\{ w\sigma : w \in \mathcal{I}_n^k, \sigma \in \mathcal{S}_n \right\} = \mathcal{I}_n^k.$$

We may also permute the positions as follows: for $w \in \mathcal{I}_n^k$ and $\rho \in \mathcal{S}_k$, define a new injection w^ρ by

$$w^\rho(x) = w(x\rho), \qquad \forall x \in [k].$$

We consider two subsets of $\mathcal{I}_n^k$ to be equivalent if they may be obtained from each other by permuting the positions and image points:

Definition 3.0.2. A subset X of $\mathcal{I}_n^k$ is *equivalent* to $Y \subseteq \mathcal{I}_n^k$ if there exist $\sigma \in \mathcal{S}_n$ and $\rho \in \mathcal{S}_k$ such that

$$\{ (w\sigma)^\rho : w \in X \} = Y.$$

Let $\mathcal{K}_0(t,k,n)$ be the t-intersecting subset of $\mathcal{I}_n^k$ obtained by including all injections which fix the first t points:

$$\mathcal{K}_0(t,k,n) = \left\{ v \in \mathcal{I}_n^k : v(i) = i,\ 1 \le i \le t \right\}.$$

We will refer to this and any family equivalent to $\mathcal{K}_0(t,k,n)$ as a *fix-family*.

To consider an example, the following subset of $\mathcal{I}_5^4$ is 2-intersecting but not 3-intersecting:

$$F = \{ \quad 1234, \quad 1245, \quad 1435, \quad 4235 \quad \}.$$

Note that F is not a fix-family since for each position $i \in [4]$, there exist injections $v, w \in F$ such that $v(i) \neq w(i)$. For these parameter values, the fix-family is given by

$$\mathcal{K}_0(2,4,5) = \{1234, 1235, 1243, 1245, 1253, 1254\}.$$

Investigating the intersecting structure of $\mathcal{I}_n^k$ requires that we alternate between viewing an element $a \in \mathcal{I}_n^k$ as a word made up of k distinct letters a_i, or as the permutation of $\mathrm{im}(a) = \{a_1, a_2, \ldots, a_k\}$ which maps $i \mapsto a_i = a(i)$ for $1 \le i \le k$, written in image form.

We begin by studying 1-intersecting injection families: after giving a bound, we show that if $\mathcal{F}$ is a maximum intersecting subset of $\mathcal{I}_n^k$ and k is small, then all words in $\mathcal{F}$ have a fixed position in common. The complementary result for large domains will be proved at the end of Part II.

Next we move on to considering larger t in Section 3.2.1. There we adapt the Meagher-Moura version of the kernel method as applied to equivalence relations in [MM05] to show that, for n large enough in terms of k and t, fixing is the unique optimal strategy for building large t-intersecting families in $\mathcal{I}_n^k$. In Section 3.2.2 we build on Theorem 2.2.2, a result of Deza & Frankl from [DF77] concerning permutations, to classify the maximum t-intersecting injection families when k is large in terms of $k-t$ and $n-t$. Here the maximum families are equivalent to the saturation family $\mathcal{G}$ from page 20, which is not equivalent to the fix-family.

3.1 Intersection Size 1

3.1.1 A Bound for Intersecting Injection Families

Replicating the approach of Section 2.1, for $w \in \mathcal{I}_n^k$, let $O(w)$ denote the *orbit* of w in $\mathcal{I}_n^k$:

$$O(w) = \{w(1\,2\,\ldots n)^i \;:\; i \in \mathbb{N}\}.$$

Note that $\mathcal{I}_n^k$ is closed under permutations, so $O(w) \subseteq \mathcal{I}_n^k$.

Theorem 3.1.1. *If $\mathcal{F} \subseteq \mathcal{I}_n^k$ is intersecting then $|\mathcal{F}| \leq \frac{(n-1)!}{(n-k)!}$.*

Moreover, the set of orbits $\{O(w) \mid w \in \mathcal{I}_n^k\}$ forms a partition of $\mathcal{I}_n^k$ into disjoint sets of size n and if $|\mathcal{F}| = \frac{(n-1)!}{(n-k)!}$ then $\mathcal{F}$ is a transversal of the orbits.

Proof. This proof is very similar to the proof of Theorem 2.1.3. Let $w \in \mathcal{I}_n^k$ be arbitrary. If u, v are two distinct elements of $O(w)$ then u and v do not intersect. Thus $\mathcal{F}$ contains at most one word from each orbit, which implies

$$|\mathcal{F}| \leq |\{O(w) \;:\; w \in \mathcal{I}_n^k\}|.$$

Moreover, the orbits are disjoint: Let a and b be two distinct elements of $\mathcal{I}_n^k$ satisfying $O(a) \neq O(b)$. Then at least one of $O(a)$, $O(b)$ is not contained in the other, so we may assume, without loss of generality, that for some $i \in \mathbb{Z}$, we have $a\pi^i \neq b\pi^l$ for any $l \in \mathbb{Z}$. Now suppose $O(a) \cap O(b) \neq \emptyset$. Then there exist $r, s \in \mathbb{Z}$ with $a\pi^r = b\pi^s$. But then

$$a\pi^i = a\pi^r\pi^{i-r} = b\pi^s\pi^{i-r} = b\pi^{s+i-r}$$

which contradicts the fact that $a\pi^i \neq b\pi^l$ for any $l \in \mathbb{Z}$. Thus we conclude that two orbits are either equal or disjoint.

Note $|O(w)|$ is equal to the order of π which is n. Hence all orbits have the same size, and they are pairwise disjoint, so

$$|\{O(w) \;:\; w \in \mathcal{I}_n^k\}| = \frac{|\mathcal{I}_n^k|}{|O(w)|} = \frac{|\mathcal{I}_n^k|}{n}.$$

Since $|\mathcal{I}_n^k| = \frac{n!}{(n-k)!}$, combining the above equations and inequalities gives

$$|\mathcal{F}| \leq |\{O(w) \,:\, w \in \mathcal{I}_n^k\}| = \frac{|\mathcal{I}_n^k|}{n} = \frac{(n-1)!}{(n-k)!}.$$

If equality holds, then $\mathcal{F}$ must contain precisely one word from each orbit. □

Brockman & Kay also considered intersecting subsets of $\mathcal{I}_n^k$ recently in [BK08]. They use a Katona-type argument involving cyclic permutations to prove the bound of Theorem 3.1.1, but make no attempt at the structural result of this thesis: that up to permutations of $[k]$ and $[n]$, the fix-family is the only maximum intersecting subset of $\mathcal{I}_n^k$. We prove this for $k \leq (n+1)/2$ in Section 3.1.2 and for $k \geq (n+1)/2$ in Chapter 5.

Note that if $k = n$ then $\mathcal{I}_n^k = \mathcal{S}_n$ and our structural result is equivalent to the main result of [CK03]. Thus we may assume $k < n$, giving $n \geq 2$ and if $n = 2$ then $k = 1$. In the latter case, the bound of Theorem 3.1.1 gives $|\mathcal{F}| \leq 1$ and the result is trivial. In summary, we assume $1 \leq k < n$ and $n \geq 3$ in all remaining proofs in this chapter.

3.1.2 Classification for Small Domains

By investigating some simple consequences of the orbit approach of Theorem 3.1.1, this section proves that fixing is the only optimal intersection strategy in $\mathcal{I}_n^k$ for small k.

Definition 3.1.2. Two words a, b in $\mathcal{I}_n^k$ are said to *strictly t-intersect* if they t-intersect, but do not $(t+1)$-intersect.

Lemma 3.1.3. *If $\mathcal{F}$ is a maximal intersecting subset of $\mathcal{I}_n^k$ for $1 \leq k \leq n$, then there exist two words $a, b \in \mathcal{F}$ which strictly 1-intersect.*

Proof. By Theorem 3.1.1, $\mathcal{F}$ contains precisely one word from $O(12\dots k)$. Denote this word by c and let c_1 be the first letter in c. Instead of $\mathcal{F}$, we will now investigate

$$\mathcal{F}' = \mathcal{F}\pi^{n-c_1+1}$$

whose intersection with $O(12\dots k)$ is $\{12\dots k\}$. Since π^{n-c_1+1} is a permutation, $\mathcal{F}'$ is a maximal intersecting subset of $\mathcal{I}_n^k$. Moreover, suppose there are two words $u, v \in \mathcal{F}'$ which strictly 1-intersect. Then $u\pi^{c_1-1}, v\pi^{c_1-1}$ are elements of $\mathcal{F}$ which strictly 1-intersect. In other words, it suffices to prove the lemma about $\mathcal{F}'$.

Using Theorem 3.1.1 again, we know that $\mathcal{F}'$ contains precisely one word from $O(n(n-1)\dots(n-k+1))$. Denote this word by u and set $v := 12\dots k$. There are only two fundamentally different forms which u can take.

Suppose firstly that u is strictly decreasing, that is $u = l(l-1)\dots(l-k+1)$ for some $l \in [n]$. Then if u and v intersect in position p, we must have $u_p = v_p = p$ since $v_i = i$ for all i. In v, all entries in positions left of p are strictly less than p. In u on the other hand, all entries in positions left of p are strictly greater than p. Therefore, u and v cannot intersect in any position left of p. Similarly, u and v cannot intersect anywhere to the right of position p, so u and v strictly 1-intersect.

If u is not strictly decreasing then $u_j = 1$ for some $j \in [k-1]$ and

$$u = j(j-1)\dots 1\, n(n-1)\dots(n-k+j+1).$$

In this case, there is only one among the first j positions in which u and v can intersect: for $p \in [j]$ we have $u_p = j-p+1$, so $u_p = v_p$ requires

$$j - p + 1 = p \implies p = (j+1)/2$$

since $v_i = i$ for all i. For the remaining positions q with $j < q \le k$, we have $u_q = n - q + j + 1$, so u and v can intersect only in position $q = (n+j+1)/2$.

Suppose u and v intersect in both positions p and q. Since $n > k$, the word w obtained from v by replacing $(j+1)/2$ by $k+1$ is an element of $\mathcal{I}_n^k$. The element of $O(w) \cap \mathcal{F}'$ is unique by Theorem 3.1.1 and must intersect v. Thus either $w \in \mathcal{F}'$ or $z \in \mathcal{F}'$ where z is the unique element of $O(w)$ which has $(j+1)/2$ in position $(j+1)/2$, that is $z = w\pi^i$ where $i = n - (k+1) + (j+1)/2$.

Since u and v strictly 2-intersect and one of their intersecting positions is $(j+1)/2$, u and w strictly 1-intersect. Also, v and w only differ in one position, so v and $z = w\pi^i$ strictly 1-intersect. Thus in the case $w \in \mathcal{F}'$, the Lemma is satisfied with $(a,b) = (u,w)$ and if $z \in \mathcal{F}'$, then the result holds with $(a,b) = (v,z)$. □

Simply using the fact that these two strictly 1-intersecting words are in $\mathcal{F}$, it can be deduced that $\mathcal{F}$ contains a much larger set of mutually 1-intersecting elements:

Lemma 3.1.4. *Any maximal intersecting subset $\mathcal{F}$ of $\mathcal{I}_n^k$ has a subset*

$$X = \{\alpha\} \cup \{(a_1 a_2 \dots a_{p-1} d a_{p+1} \dots a_k)\pi^{c-d} \;:\; d \in [n] \setminus \mathrm{im}(\alpha)\}$$

for some $\alpha = a_1 \dots a_{p-1} c\, a_{p+1} \dots a_k \in \mathcal{F}$.

Moreover, $|X| = n - k + 1$, and any two elements of X strictly 1-intersect.

Proof. By Lemma 3.1.3, there exist $\alpha, \beta \in \mathcal{F}$ such that, for some $p \in [k]$,

$$\alpha = a_1 a_2 \dots a_{p-1} c\, a_{p+1} \dots a_k, \quad \beta = b_1 b_2 \dots b_{p-1} c\, b_{p+1} \dots b_k$$

with $a_i \neq b_i$ for all i. Let $d \in [n] \setminus \mathrm{im}(\alpha)$ and set

$$\delta = a_1 a_2 \dots a_{p-1} d\, a_{p+1} \dots a_k,$$

then $\delta \in \mathcal{I}_n^k$. Since p is the only position in which α and δ differ, there are only two words in $O(\delta)$ which intersect $\alpha \in \mathcal{F}$. These two words are δ and $\delta\pi^{c-d}$ which has c in position p. Since $d \neq c$ and $a_i \neq b_i$ for all i, it is clear that δ does not intersect $\beta \in \mathcal{F}$. Therefore $\delta\pi^{c-d} \in \mathcal{F}$ which proves $X \subseteq \mathcal{F}$. We have $|[n] \setminus \mathrm{im}(\alpha)| = n-k$, so $|X \setminus \{\alpha\}| = n-k$. Moreover, α is distinct from all elements of $X \setminus \{\alpha\}$, so $|X| = n-k+1$.

It remains to be shown that elements of X are mutually strictly 1-intersecting. Let us label the elements of $[n] \setminus \mathrm{im}(\alpha)$ as $d_1, d_2, \dots, d_{n-k}$ in such a way that this labelling corresponds to their ordering as natural numbers, i.e. for $i, j \in [n-k]$, we have $d_i < d_j$ whenever $i < j$. Using this notation, X consists of the following words:

$$\begin{gathered}
a_1 a_2 \dots a_{p-1}\, c\, a_{p+1} \dots a_k = \alpha \\
(a_1 a_2 \dots a_{p-1} d_1 a_{p+1} \dots a_k)\pi^{c-d_1} \\
(a_1 a_2 \dots a_{p-1} d_2 a_{p+1} \dots a_k)\pi^{c-d_2} \\
\vdots \\
(a_1 a_2 \dots a_{p-1} d_{n-k} a_{p+1} \dots a_k)\pi^{c-d_{n-k}}
\end{gathered}$$

All of the above words have c in position p. Since $d_i \neq c$ and the d_i are distinct for all $i \in [n-k]$, it is apparent from the above list that X is a set of $n-k+1$ elements all of which mutually strictly 1-intersect. $\square$

We are now in a position to classify the maximum intersecting subsets of $\mathcal{I}_n^k$ for $k \leq (n+1)/2$.

Theorem 3.1.5. *For $1 \leq k \leq (n+1)/2$, if $\mathcal{F}$ is a maximal intersecting subset of $\mathcal{I}_n^k$ then all words in $\mathcal{F}$ have a fixed position in common.*

Proof. Let

$$X = \{\alpha\} \cup \{(a_1 a_2 \dots a_{p-1} d a_{p+1} \dots a_k)\pi^{c-d} \;:\; d \in [n] \setminus \mathrm{im}(\alpha)\}$$

be the subset of $\mathcal{F}$ given by Lemma 3.1.4 and suppose there exists $w \in \mathcal{F}$ such that $w(p) \neq c$. Since two distinct elements of X do not intersect in any position other than p, w can intersect at most one element of X in position i, for any $i \in [k]$. Since w does not intersect any element of X in position p, this implies that w intersects with at most $k-1$ elements of X. Since $k < (n+2)/2$, we have

$$k - 1 = (2k-2) - k + 1 < n - k + 1 = |X|.$$

Thus w does not intersect all elements of X, contradicting the intersecting property of $\mathcal{F}$. We conclude that $w(p) = c$ for all $w \in \mathcal{F}$. □

We will prove the complementary result on large domains much later in Theorem 5.3.9, after having introduced the more complicated traditional machinery of Chapter 4. For now, let us turn our attention to t-intersecting injection families.

3.2 Arbitrary Intersection Size: Classifications in the Limit

3.2.1 Injections with Large Images

In this section, we prove that for large n, fixing eventually becomes the unique optimal strategy for t-intersecting subsets of $\mathcal{I}_n^k$. We use a version of the so-called *kernel method* as presented in the context of partition systems in [MM05], where Meagher & Moura attribute the origins of this method to Hajnal & Rothschild [HR73].

Lemma 3.2.1. *Let $\mathcal{F}$ be a t-intersecting subset of $\mathcal{I}_n^k$. If there do not exist $x \in [k]$, $y \in [n]$ such that all elements of $\mathcal{F}$ map x to y, then*

$$|\mathcal{F}| \leq \frac{k!(n-t-1)!}{t!(k-t-1)!(n-k)!}.$$

Proof. Let $\alpha \in \mathcal{F}$ and $1 \leq x \leq k$. By assumption, there exists $\beta \in \mathcal{F}$ such that $\alpha(x) \neq \beta(x)$. Setting

$$\mathcal{F}_{\alpha(x)} = \{ \gamma \in \mathcal{F} : \gamma(x) = \alpha(x) \},$$

it is then clear that $\mathrm{int}(\gamma, \beta) \subseteq [k] \setminus \{x\}$ for all $\gamma \in \mathcal{F}_{\alpha(x)}$. On the other hand, $\mathrm{int}(\gamma, \beta)$ has size at least t and so

$$|\mathcal{F}_{\alpha(x)}| \leq \binom{k-1}{t} \frac{(n-(t+1))!}{(n-k)!}.$$

By the intersecting property of $\mathcal{F}$, we have $\mathcal{F} = \bigcup_{x=1}^{k} \mathcal{F}_{\alpha(x)}$, giving

$$|\mathcal{F}| \leq \sum_{x=1}^{k} |\mathcal{F}_{\alpha(x)}| = k \cdot |\mathcal{F}_{\alpha(x)}| = \frac{k!(n-t-1)!}{t!(k-t-1)!(n-k)!}$$

as required. □

Theorem 3.2.2. *Let $1 \leq t \leq k \leq n$ and suppose that*

$$(k-c)! < (n-t)(t-c)!(k-t-1)!$$

for all $0 \leq c < t$. Then any maximum t-intersecting subset of $\mathcal{I}_n^k$ is equivalent to the fix-family $\mathcal{K}_0(t,k,n)$.

Proof. Let $\mathcal{F}$ be a t-intersecting subset of $\mathcal{I}_n^k$ which is not equivalent to $\mathcal{K}_0(t,k,n)$. It suffices to show that $|\mathcal{F}| < |\mathcal{K}_0(t,k,n)| = (n-t)!/(n-k)!$.

Let $\mathcal{C}$ be the intersection of all elements of $\mathcal{F}$, so

$$\mathcal{C} = \{\, (x,y) \in [k] \times [n] : \alpha(x) = y \text{ for all } \alpha \in \mathcal{F} \,\},$$

and set $c = |\mathcal{C}|$, $X = \{\, x : (x,y) \in \mathcal{C} \,\}$ and $Y = \{\, y : (x,y) \in \mathcal{C} \,\}$. Then $0 \leq c < t$ since $\mathcal{F}$ is not a fix-family.

Let $\mathcal{F}'$ be the family obtained from $\mathcal{F}$ by first deleting all elements of $\mathcal{C}$ from each element of $\mathcal{F}$, and then relabelling $[k] \setminus X$ and $[n] \setminus Y$ to eliminate the resulting gaps. Then $\mathcal{F}'$ is a $(t-c)$-intersecting subset of $\mathcal{I}_{n-c}^{k-c}$ with $|\mathcal{F}'| = |\mathcal{F}|$. Thus we may employ Lemma 3.2.1 to obtain

$$\begin{aligned} |\mathcal{F}| = |\mathcal{F}'| \quad &\leq \quad \frac{(k-c)!(n-t-1)!}{(t-c)!(k-t-1)!(n-k)!} \\ &= \frac{(k-c)!}{(n-t)(t-c)!(k-t-1)!} \cdot \frac{(n-t)!}{(n-k)!} \end{aligned}$$

as required. □

Corollary 3.2.3. *There exists a function $n_0(k,t) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $n > n_0(k,t)$, every maximum t-intersecting subset of $\mathcal{I}_n^k$ is equivalent to the fix-family $\mathcal{K}_0(t,k,n)$.*

Proof. Given $0 \leq c < t \leq k \leq n$, we have $(k-c)! \leq k!$ and $(t-c)! \geq 1$, and these bounds cannot be simultaneously achieved since c is fixed. Thus if

$$k! \leq (n-t)(k-t-1)! \tag{3.2.4}$$

then

$$(k-c)! \leq k! \leq (n-t)(k-t-1)! \leq (n-t)(k-t-1)!(t-c)!$$

and one of these inequalities is strict. By Theorem 3.2.2, inequality (3.2.4) therefore implies that no t-intersecting subset of $\mathcal{I}_n^k$ is larger than $\mathcal{K}_0(t,k,n)$. For fixed k and t, inequality (3.2.4) can clearly be achieved by taking

$$n > n_0(k,t) = t + \frac{k!}{(k-t-1)!},$$

which completes the proof. □

Thus t-intersecting injection families eventually behave like the t-intersecting set families studied in the second half of the previous century in the sense that for large n, fixing is the unique optimal strategy.

Note however, that Corollary 3.2.3 is a result strictly about injections, not including the case of permutations, since n is required to be large in terms of t as well as k. Corollary 3.2.3 tells us what happens to the intersection structure of $\mathcal{I}_n^k$ if we fix two parameter values and increase the third. By way of contrast, Theorem 2.2.2 sets $n = k$, fixes the difference between k and t and increases k: recall from Chapter 2 that in this case, the t-intersecting saturation family $\mathcal{G}$ is maximum in $\mathcal{S}_n$, see page 20. The remainder of this chapter is devoted to generalising Theorem 2.2.2 to injection families: Theorem 3.2.11 classifies the optimal t-intersecting injection families for large k, given that both $k-t$ and $n-k$ are fixed. For these parameter values, Theorem 3.2.11 shows that fixing is not optimal, since $\mathcal{G}$ is not equivalent to $\mathcal{K}_0$.

3.2.2 Injections with Large Domains

Before we can prove Theorem 3.2.11, we need to establish a lower bound for the size of $\mathcal{G}(t, k, n)$ in Lemma 3.2.8. For that we need to find the number of injections from $[k]$ to $[n]$ with no fixed points, which we denote by $d(k, n)$ throughout Part II. The function $d(k, n)$ is given by the following lemma, which requires the convention that there is one injection with no fixed points from the empty set into any other set. Note that $d(n, n)$ is the number of derangements of $[n]$.

Lemma 3.2.5. *The number $d(k, n)$ of injections from $[k]$ to $[n]$ with no fixed points is given by*

$$d(k, n) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{(n-i)!}{(n-k)!}.$$

Proof. Denote the product of the first k factors of $n!$ by $(n)_k$. Then by the inclusion-exclusion principle,

$$d(k, n) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (n-i)_{k-i} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{(n-i)!}{(n-k)!}$$

as required. □

Before continuing our discussion, let us briefly establish the technical properties of $d(k, n)$ required throughout Part II.

Lemma 3.2.6. *For $0 \leq a \leq b$, let $d(a, b)$ be the number of injections from $[a]$ to $[b]$ with no fixed points. Then*

1. $d(a, b+1) = d(a, b) + ad(a-1, b)$;

2. $d(a, b) \le (b-a+1)d(a-1, b)$.

Proof. Let $D(a, b)$ denote the set of all injections from $[a]$ to $[b]$ with no fixed points. If w is a word of length k, we denote by wx the word obtained from w by adjoining the letter x to the end of w.

1. Partition $D(a, b+1)$ into

$$\begin{aligned} X_0 &= \{ w \in D(a, b+1) : (b+1) \notin \operatorname{im}(w) \}, \\ X_i &= \{ w \in D(a, b+1) : w_i = b+1 \}, \quad 1 \le i \le a. \end{aligned}$$

Clearly, $X_0 = D(a, b)$. To determine $|X_i|$ for $1 \le i \le a$, note that any point in $[a]$ may be mapped to $b+1 > a$, so there are $d(a-1, b)$ elements of $D(a, b+1)$ which map i to $b+1$. Thus

$$|D(a, b+1)| = \sum_{i=0}^{a} |X_i| = |D(a, b)| + \sum_{i=1}^{a} d(a-1, b)$$

and the result follows.

2. Let $E = \{ wx : w \in D(a-1, b),\ x \in [b] \setminus \operatorname{im}(w) \}$, then

$$|E| = (b-a+1)d(a-1, b).$$

It is clear that $D(a, b) \subseteq E$, which yields the required inequality. $\square$

Later we will use the fact that Lemma 3.2.6 (1) may be rewritten as

$$d(a+1, b+1) = (a+1)d(a, b) + d(a+1, b),$$

so

$$a \ge 1 \implies d(a+1, b+1) > d(a, b) \tag{3.2.7}$$

for $0 \le a \le b$.

The *moved point set* of an injection $w \in \mathcal{I}_n^k$ is defined as

$$E(w) = \{ x \in [k] : w(x) \ne x \}.$$

Throughout the thesis, if f is a function from A to B and X is a subset of A, we may use the notation

$$f(X) = \{ f(x) : x \in X \}.$$

Thus if S is a subset of $\mathcal{I}_n^k$ then $E(S) = \{E(w) \ : \ w \in S\}$ is a family of subsets of $[k]$.

We are now ready to generalise Theorem 2.2.2.

Lemma 3.2.8. *For fixed natural numbers T, N and $c_{N,T}$ with $T \geq 2$, there exists $k_0(T, N) \in \mathbb{N}$ such that*

$$|\mathcal{G}(k - T, k, k + N)| > c_{N,T} \sum_{j=0}^{\lfloor T/2 \rfloor - 1} \binom{k}{j}$$

for all $k \geq k_0(T, N)$.

Proof. Setting $t = k - T$ and $n = k + N$, we abbreviate $\mathcal{G}(t, k, n)$ by $\mathcal{G}$. The bulk of this proof is concerned with establishing expressions for $|\mathcal{G}|$.

Case 1 *$T = 2h$ is even.*

Setting $\mathcal{A} = \{ X \subseteq [k] : |X| \leq h \}$, we have

$$\mathcal{G} = \bigcup_{X \in \mathcal{A}} \{ w \in \mathcal{I}_n^k : E(w) = X \}$$

and this union is disjoint, so

$$|\mathcal{G}| = \sum_{X \in \mathcal{A}} |\{ w \in \mathcal{I}_n^k : E(w) = X \}|.$$

For given $X \subseteq [k]$, an injection w in $\mathcal{I}_n^k$ with $E(w) = X$ must fix all elements of $[k] \setminus X$, so the image points of X under w are all in $[n] \setminus ([k] \setminus X)$. Hence

$$\begin{aligned} |\mathcal{G}| &= \sum_{X \in \mathcal{A}} |\{ w \in \mathcal{I}_n^k : E(w) = X \}| \\ &= \sum_{X \in \mathcal{A}} d(|X|, n - (k - |X|)) \\ &= \sum_{j=0}^{h} \binom{k}{j} d(j, n - k + j) \\ &= \sum_{j=0}^{T/2} \binom{k}{j} d(j, N + j). \end{aligned}$$

Since both $d(j, N + j)$ and $c_{N,T}$ depend only on the fixed constants N and T, they are constants themselves. Since $T \geq 2$, we may thus choose k sufficiently large to ensure

$$|\mathcal{G}| > c_{N,T} \sum_{j=0}^{T/2-1} \binom{k}{j} = c_{N,T} \sum_{j=0}^{\lfloor T/2 \rfloor - 1} \binom{k}{j},$$

completing the proof for the case that T is even.

Case 2 *$T = 2h + 1$ is odd.*

Then

$$\begin{aligned} \mathcal{G} &= \{ w \in \mathcal{I}_n^k : |E(w) \cap [k-1]| \leq h \} \\ &= \{ w \in \mathcal{I}_n^k : |E(w)| \leq h \} \cup \{ w \in \mathcal{I}_n^k : |E(w)| = h + 1,\ k \in E(w) \}, \end{aligned} \tag{3.2.9}$$

so we set

$$\mathcal{B} = \{\, X \subseteq [k] : |X| \leq h \,\} \cup \{\, X \subseteq [k] : |X| = h+1,\ k \in X \,\}.$$

This gives

$$\begin{aligned} |\mathcal{G}| &= \sum_{X \in \mathcal{B}} |\{\, w \in \mathcal{I}_n^k : E(w) = X \,\}| = \sum_{X \in \mathcal{B}} d(|X|, n-(k-|X|)) \\ &= \sum_{j=0}^{h} \binom{k}{j} d(j, n-k+j) + \binom{k-1}{h} d(h+1, n-k+h+1) \qquad (3.2.10) \\ &= \sum_{j=0}^{(T-1)/2} \binom{k}{j} d(j, N+j) + \binom{k-1}{h} d(h+1, N+h+1) \\ &> \sum_{j=0}^{(T-1)/2} \binom{k}{j} d(j, N+j). \end{aligned}$$

Since $d(j, N+j)$ depends only on the constants N and T, we may again choose k sufficiently large to ensure

$$|\mathcal{G}| > c_{N,T} \sum_{j=0}^{(T-1)/2-1} \binom{k}{j} = c_{N,T} \sum_{j=0}^{\lfloor T/2 \rfloor - 1} \binom{k}{j},$$

since $T \geq 2$, and the proof is complete. $\square$

Note in the statement of Theorem 3.2.11 below that the case $N = 0$ and $T \geq 3$ would be equivalent to Theorem 2.2.2 about permutation families in [DF77], so there is no need for us to prove this fact again: we concentrate on injections here. Moreover, it is easy to see that for $T < 2$, a maximum $(k-T)$-intersecting family in $\mathcal{I}_n^k$ must be equivalent to the fix-family irrespective of k and n.

Recall that the symmetric difference $A \Delta B$ of two sets A and B is the set of points contained in one but not both of A and B:

$$A \Delta B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

Theorem 3.2.11. *For fixed positive integers T and N with $T \geq 2$, there exists $k_0(T, N) \in \mathbb{N}$ such that for $k \geq k_0(T, N)$, every maximum $(k-T)$-intersecting subset of $\mathcal{I}_{k+N}^k$ is equivalent to the saturation family $\mathcal{G}(k-T, k, k+N)$.*

Proof. Set $t = k - T$, $n = k + N$, abbreviate $\mathcal{G}(t, k, n)$ by $\mathcal{G}$ as before and let $\mathcal{F}$ be a maximum t-intersecting subset of $\mathcal{I}_n^k$. Following the proof outline of Theorem 2.2.2 in [DF77], we begin by establishing some useful technicalities (3.2.12 – 3.2.15) before picking a set $\mathcal{W}$ of $3T+1$ elements of $\mathcal{F}$. We then prove that in the case where T is even, all elements of $\mathcal{W}$ act in the same way on the set of points moved by all of them, and applying the inverse of this action to the whole of $\mathcal{F}$ maps $\mathcal{F}$

into $\mathcal{G}$. In the case where T is odd we proceed similarly, though mapping $\mathcal{F}$ into $\mathcal{G}$ in this situation requires an application of the Erdős-Ko-Rado Theorem.

Note that two injections cannot intersect in a point which is moved by one and fixed by the other, i.e. an element of the symmetric difference of their fixed point sets. Since elements of $\mathcal{F}$ t-intersect, this means that the size of their symmetric difference can be at most $k - t = T$. In the case of equality, the two injections not only intersect in each point they both fix, which is always the case, but additionally they intersect in each point they both move. Generalising this argument, we see that two elements of $\mathcal{F}$ whose symmetric difference has size $T - j$ can disagree in at most j of the positions they both move. We have proved that for all $v, w \in \mathcal{F}$,

$$|E(v)\Delta E(w)| \leq T, \tag{3.2.12}$$

$$|(E(v) \cap E(w)) \setminus \mathrm{int}(v,w)| \leq T - |E(v)\Delta E(w)|. \tag{3.2.13}$$

Both facts will be used frequently throughout this proof.

Without loss of generality, we may assume that the identity $12\ldots k$ is an element of $\mathcal{F}$. Then each $w \in \mathcal{F}$ must t-intersect the identity, so

$$|E(w)| \leq k - t = T, \quad \forall w \in \mathcal{F}. \tag{3.2.14}$$

Pick $w_0 \in \mathcal{F}$ with $|E(w_0)|$ maximal. We wish to show that all remaining $w \in \mathcal{F}$ move at most $\lfloor T/2 \rfloor$ of the points which are fixed by w_0. So suppose the opposite holds for some $w \in \mathcal{F}$, then the maximality of $|E(w_0)|$ forces

$$|E(w_0) \setminus E(w)| \geq |E(w) \setminus E(w_0)| > \left\lfloor \frac{T}{2} \right\rfloor .$$

But this implies that the symmetric difference of $E(w)$ and $E(w_0)$ is larger than T, contradicting (3.2.12). Thus we have shown that

$$|E(w) \setminus E(w_0)| \leq \left\lfloor \frac{T}{2} \right\rfloor \tag{3.2.15}$$

for all $w \in \mathcal{F}$.

Picking the Elements of $\mathcal{W}$

We wish to pick $w_1 \in \mathcal{F}$ which achieves equality in (3.2.15), and subsequently continue to pick $w_{i+1} \in \mathcal{F}$ such that w_{i+1} moves exactly $\lfloor T/2 \rfloor$ points which are not moved by any of the injections $w_0, \ldots, w_i$ chosen so far:

$$\left| E(w_{i+1}) \setminus \bigcup_{j=0}^{i} E(w_j) \right| = \left\lfloor \frac{T}{2} \right\rfloor . \tag{3.2.16}$$

We will use the maximality of $\mathcal{F}$ as a t-intersecting subset of $\mathcal{I}_n^k$ to show, by contradiction, that we can pick elements of $\mathcal{F}$ in this way. Suppose that for some

$$i < 3T, \tag{3.2.17}$$

we cannot find such a w_{i+1} in $\mathcal{F} \setminus \{w_0, \ldots, w_i\}$. Then we must have

$$\left| E(w) \setminus \bigcup_{j=0}^{i} E(w_j) \right| \neq \left\lfloor \frac{T}{2} \right\rfloor$$

for all $w \in \mathcal{F} \setminus \{w_0, \ldots, w_i\}$. Also,

$$\left| E(w) \setminus \bigcup_{j=0}^{i} E(w_j) \right| \leq |E(w) \setminus E(w_0)| \leq \left\lfloor \frac{T}{2} \right\rfloor$$

by (3.2.15), and combining the previous two equations gives

$$\left| E(w) \setminus \bigcup_{j=0}^{i} E(w_j) \right| < \left\lfloor \frac{T}{2} \right\rfloor \tag{3.2.18}$$

for all $w \in \mathcal{F} \setminus \{w_0, \ldots, w_i\}$.

Note that due to condition (3.2.16) according to which the elements $w_0, \ldots, w_i$ were picked, we have

$$\left| \bigcup_{j=0}^{i} E(w_j) \right| = |E(w_0)| + i \left\lfloor \frac{T}{2} \right\rfloor < T + 3T \left\lfloor \frac{T}{2} \right\rfloor = T \left(1 + 3 \left\lfloor \frac{T}{2} \right\rfloor \right)$$

by (3.2.14, 3.2.17). Moreover, since $T \geq 2$ we have

$$1 + 3 \left\lfloor \frac{T}{2} \right\rfloor \leq 1 + 1.5T \leq T - 1 + 1.5T = 2.5T - 1 < 3T,$$

and combining the previous two inequalities gives

$$\left| \bigcup_{j=0}^{i} E(w_j) \right| < 3T^2. \tag{3.2.19}$$

We use these arguments to establish an upper bound on the size of

$$E(\mathcal{F}) = \{ E(w) : w \in \mathcal{F} \}$$

as follows: denote $\bigcup_{j=0}^{i} E(w_j)$ by U. Then $w_0, \ldots, w_i$ move no points outside U. Moreover, (3.2.18) tells us that whilst each element of $\mathcal{F}$ may move an arbitrary number of elements of U, it moves less than $\lfloor T/2 \rfloor$ of the points which are not in U. Since U has less than 2^{3T^2} subsets by (3.2.19), this yields

$$|E(\mathcal{F})| < 2^{3T^2} \sum_{j=0}^{\lfloor T/2 \rfloor - 1} \binom{k}{j}. \tag{3.2.20}$$

Clearly we have

$$\mathcal{F} \subseteq \bigcup_{X \in E(\mathcal{F})} \{\, w \in \mathcal{I}_n^k : E(w) = X \,\}$$

and, since this union is disjoint,

$$\begin{aligned} |\mathcal{F}| &\leq \sum_{X \in E(\mathcal{F})} |\{\, w \in \mathcal{I}_n^k : E(w) = X \,\}| \\ &\leq \sum_{X \in E(\mathcal{F})} d(|X|, n-k+|X|) \\ &= \sum_{X \in E(\mathcal{F})} d(|X|, N+|X|). \end{aligned} \tag{3.2.21}$$

Recall from (3.2.7) that $|X| \geq 1$ implies $d(|X|, N+|X|) < d(|X|+1, N+|X|+1)$ and since $N \geq 1$ we also have

$$d(0, N) = 1 \leq d(1, N+1) = 1.$$

Therefore we may use (3.2.14) together with the above bound on $|\mathcal{F}|$ to conclude

$$\begin{aligned} |\mathcal{F}| &\leq \sum_{X \in E(\mathcal{F})} d(T, N+T) \\ &= d(T, N+T) \cdot |E(\mathcal{F})| \\ &< d(T, N+T) 2^{3T^2} \sum_{j=0}^{\lfloor T/2 \rfloor - 1} \binom{k}{j} \end{aligned}$$

by (3.2.20). Since $c_{N,T} = d(T, N+T)2^{3T^2}$ depends only on the fixed constants T and N, Lemma 3.2.8 now implies that $|\mathcal{F}| < |\mathcal{G}|$. This contradicts the fact that $\mathcal{F}$ is maximum $(k-T)$-intersecting in $\mathcal{I}_{N+k}^k$, so we conclude that we can indeed pick $w_0, \ldots, w_{3T}$ as described above.

Note that if $|E(w_0)| < \lfloor T/2 \rfloor$ then the maximality of $|E(w_0)|$ would force all elements of $E(\mathcal{F})$ to have size less than $\lfloor T/2 \rfloor$, making it impossible to pick the w_{i+1} according to (3.2.16). Since we have just shown that we can pick such w_{i+1} for $i < 3T$, we conclude that

$$|E(w_0)| \geq \left\lfloor \frac{T}{2} \right\rfloor \tag{3.2.22}$$

and set

$$\mathcal{W} = \{w_0, \ldots, w_{3T}\}. \tag{3.2.23}$$

It is clear from (3.2.16) that the w_i are distinct, so $|\mathcal{W}| = 3T + 1$. As in the proof of Lemma 3.2.8, we need to consider the possible parities of T separately.

Case 1 $T = 2h$ is even

We will show that w_0 t-intersects all other elements w_i of $\mathcal{W}$ in the same t positions. In the process, we establish the sizes of the moved point sets $E(w_i)$ as well as their respective intersections and symmetric differences with $E(w_0)$.

The Intersection of w_0 with Other Elements of $\mathcal{W}$

By (3.2.22, 3.2.14) the number of points moved by w_0 is between h and $2h$. Thus setting

$$s = |E(w_0)| - h,$$

we have $0 \leq s \leq h$ and the maximality of $|E(w_0)|$ implies $|E(w)| \leq h + s$ for all $w \in \mathcal{F}$. Indeed, our next claim is that all $w \in \mathcal{W}$ satisfy $|E(w)| = h + s$.

For $w_i \in \mathcal{W} \subseteq \mathcal{F}$, it follows from the way the w_i were picked (3.2.16) that

$$|E(w_i) \setminus E(w_0)| \geq \left| E(w_i) \setminus \bigcup_{j=0}^{i-1} E(w_j) \right| = h.$$

Therefore setting $|E(w_i)| = h + s - j$ for some $j \geq 0$, we have

$$\begin{aligned} |E(w_0) \setminus E(w_i)| &= |E(w_0)| - |E(w_0) \cap E(w_i)| \\ &= h + s - (|E(w_i)| - |E(w_i) \setminus E(w_0)|) \\ &= h + s - (h + s - j) + |E(w_i) \setminus E(w_0)| \\ &\geq j + h. \end{aligned}$$

Thus the size of the symmetric difference $E(w_i)\Delta E(w_0)$ is given by

$$|E(w_i) \setminus E(w_0)| + |E(w_0) \setminus E(w_i)| \geq 2h + j = T + j.$$

Using (3.2.12, 3.2.13) this implies $j = 0$ and

$$\operatorname{int}(w_i, w_0) = [k] \setminus (E(w_i)\Delta E(w_0)), \tag{3.2.24}$$

i.e. w_0 and w_i intersect in all points which they both move. Observe that by proving $j = 0$ we have shown

$$\begin{aligned} |E(w_i)| &= h + s, & (3.2.25) \\ |E(w_0)\Delta E(w_i)| &= T = 2h, & (3.2.26) \\ |E(w_0) \setminus E(w_i)| &= h \end{aligned}$$

for all $w_i \in \mathcal{W}$. Therefore

$$\begin{aligned} |E(w_i) \cap E(w_0)| &= |E(w_0)| - |E(w_0) \setminus E(w_i)| \\ &= h + s - h = s, \qquad \forall w_i \in \mathcal{W}. \end{aligned} \tag{3.2.27}$$

Together with the arguments preceding (3.2.12), (3.2.26) implies that w_0 strictly t-intersects (i.e. does not $(t+1)$-intersect) each element of $\mathcal{W} \setminus \{w_0\}$. It remains to be shown that all of these intersections coincide.

A Common Intersection

We concluded in (3.2.24) that w_i and w_0 agree on each point in $E(w_i) \cap E(w_0)$. Indeed, suppose that for some $w_i \in \mathcal{W}$, $i \geq 2$, we had

$$E(w_i) \cap E(w_0) \neq E(w_1) \cap E(w_0).$$

Then since both intersections have the same size by (3.2.27), we must have

$$|(E(w_i) \cap E(w_0)) \setminus (E(w_1) \cap E(w_0))| = |(E(w_i) \cap E(w_0)) \setminus E(w_1)| > 0.$$

But then

$$\begin{aligned} |E(w_i) \cap E(w_1)| &= |E(w_i)| - |E(w_i) \setminus (E(w_0) \cup E(w_1))| - |(E(w_i) \cap E(w_0)) \setminus E(w_1)| \\ &< |E(w_i)| - |E(w_i) \setminus (E(w_0) \cup E(w_1))| \\ &\leq |E(w_i)| - \left| E(w_i) \setminus \bigcup_{j=0}^{i-1} E(w_j) \right| \\ &= h + s - h = s \end{aligned}$$

by (3.2.25, 3.2.16). But both $E(w_i)$ and $E(w_1)$ are sets of size $h + s$ by (3.2.25), so if their intersection has size less than s, then their symmetric difference must have size greater than $2h = T$, contradicting (3.2.12).

In conclusion, there must exists a set $X \subseteq [k]$ such that

$$E(w_i) \cap E(w_0) = X, \qquad \forall i \in [3T],$$

which has size s by (3.2.27). Clearly this implies

$$X \subseteq E(w_i) \cap E(w_j), \qquad \forall i, j \in [3T] \cup \{0\},\ i \neq j. \tag{3.2.28}$$

Indeed, it does not require much further effort to show that we have equality there: we already know this when $i = 0$, so suppose that for some $1 \leq i < j \leq 3T$, the sets $E(w_i)$ and $E(w_j)$ intersect in some point outside X. Then combining this with (3.2.28), we see that at least $|X|+1 = s+1$ of the points moved by w_j are also moved by w_i. But w_j only moves $h+s$ points in total by (3.2.25), so w_j moves at most

$$h + s - (s+1) = h - 1$$

of the points which are not moved by w_i, contradicting the way w_j was picked (3.2.16) since $i < j$. Hence

$$E(w_i) \cap E(w_j) = X, \qquad \forall i < j \in [3T] \cup \{0\}. \tag{3.2.29}$$

Moreover, combining this with (3.2.24) gives

$$X = E(w_i) \cap E(w_0) \subseteq [k] \setminus (E(w_i) \Delta E(w_0)) = \operatorname{int}(w_i, w_0),$$

telling us that all elements of $\mathcal{W}$ act on X in the same way as w_0, i.e. X is invariant under $\mathcal{W}$.

Mapping $\mathcal{F}$ into $\mathcal{G}$

Let $\sigma \in \mathcal{S}_n$ be the permutation which coincides with the elements of $\mathcal{W}$ on X and with the identity elsewhere:

$$\sigma(x) = \begin{cases} w_0(x) & x \in X \\ x & x \in [n] \setminus X \end{cases},$$

and let σ^{-1} be the inverse of σ in $\mathcal{S}_n$. We let permutations act on injections as in Section 3.1, so a permutation acts on each image point of an injection separately, and set

$$\mathcal{F}_\sigma = \left\{ v\sigma^{-1} : v \in \mathcal{F} \right\}.$$

Since all elements w_i of $\mathcal{W}$ as well as σ agree on X, the effect of postmultiplying w_i by σ^{-1} is to fix the elements of X:

$$|E(w_i\sigma^{-1})| = |E(w_i)| - |X| = h + s - s = h, \tag{3.2.30}$$

as each w_i moves $h+s$ points by (3.2.25) and X has size s by (3.2.27). Applying the same argument to (3.2.29) gives

$$E(w_i\sigma^{-1}) \cap E(w_j\sigma^{-1}) = \emptyset, \qquad 0 \le i < j \le 3T. \tag{3.2.31}$$

By definition σ, and therefore also σ^{-1}, move $|X| = s$ points and any $v \in \mathcal{F}$ moves at most $T = 2h$ points by (3.2.14). Moreover, $v\sigma^{-1}$ certainly cannot move more points than the sum of those moved by v and σ^{-1}, i.e.

$$|E(v\sigma^{-1})| \le |E(v)| + |E(\sigma^{-1})| \le 2h + s \le 3h, \qquad \forall v \in \mathcal{F}. \tag{3.2.32}$$

It follows from the definition of $\mathcal{G}$ that $|E(v\sigma^{-1})| \leq h$ for all $v\sigma^{-1} \in \mathcal{F}_\sigma$ would imply $\mathcal{F}_\sigma \subseteq \mathcal{G}$. Showing this is our final objective in Case 1, so suppose that $|E(v\sigma^{-1})| > h$ for some $v\sigma^{-1} \in \mathcal{F}_\sigma$. For any $w_i \in \mathcal{W}$ the symmetric difference of $E(v\sigma^{-1})$ and $E(w_i\sigma^{-1})$ has size at most $2h$ by (3.2.12). But if two sets, one of size larger than h by assumption, the other of size h by (3.2.30), have symmetric difference of size at most $2h$, then their intersection must be non-empty. In other words, $E(v\sigma^{-1})$ intersects each of the $3T + 1$ sets $E(w_i\sigma^{-1})$, which are mutually disjoint by (3.2.31). This gives

$$|E(v\sigma^{-1})| \geq 3T + 1 = 6h + 1,$$

clearly contradicting (3.2.32). We have completed Case 1.

Case 2 $\quad$ $T = 2h + 1$ is odd

By (3.2.22, 3.2.14) the number of points moved by w_0 is between h and $2h + 1$, so setting

$$s = |E(w_0)| - h$$

as in Case 1, we have $0 \leq s \leq h + 1$ here.

Once again, the maximality of $|E(w_0)|$ implies $|E(w)| \leq h + s$ for all $w \in \mathcal{F}$. We wish to show that the moved point set of each $w_i \in \mathcal{W}$ has size either $h + s$ or $h + s - 1$. So suppose that for some $1 \leq i \leq 3T$, the injection $w_i \in \mathcal{W}$ moves at least two points less than w_0. Then

$$|E(w_0) \setminus E(w_i)| \geq |E(w_i) \setminus E(w_0)| + 2 \geq h + 2$$

because Condition 3.2.16, according to which w_i was picked, ensures that w_i moves at least h of the points not moved by w_0. The symmetric difference of the two moved point sets then has size

$$\begin{aligned} |E(w_i)\Delta E(w_0)| &= |E(w_0) \setminus E(w_i)| + |E(w_i) \setminus E(w_0)| \\ &\geq 2h + 2 > T, \end{aligned}$$

contradicting (3.2.12). Thus we may partition $\mathcal{W}$ according to the cardinalities of the moved point sets: setting

$$\begin{aligned} W_0 &= \{\, w_i \in \mathcal{W} : |E(w_i)| = h + s \,\}, \\ W_1 &= \{\, w_i \in \mathcal{W} : |E(w_i)| = h + s - 1 \,\}, \end{aligned}$$

we have $\mathcal{W} = W_0 \cup W_1$. Now we reconsider the arguments employed in Case 1 with the new scenario in mind.

The Intersection of w_0 with Elements of W_0 and W_1

It follows from the way the w_i were picked (3.2.16) that any $w_i \in \mathcal{W}$ moves at least h of the points not moved by w_0. We therefore obtain

$$\begin{aligned}|E(w_0) \setminus E(w_i)| &= |E(w_0)| - |E(w_i)| + |E(w_i) \setminus E(w_0)| \\ &\geq h + s - |E(w_i)| + h \\ &= \begin{cases} h & w_i \in W_0 \\ h+1 & w_i \in W_1 \end{cases}, \end{aligned} \tag{3.2.33}$$

implying

$$|E(w_i) \Delta E(w_0)| \geq \begin{cases} 2h & w_i \in W_0 \\ 2h+1 & w_i \in W_1 \end{cases}.$$

Recall that two elements of $\mathcal{F}$ cannot have symmetric difference larger than $T = 2h + 1$ by (3.2.12). Thus we conclude as in Case 1 that for $w_i \in W_1$,

$$|E(w_0) \setminus E(w_i)| = h + 1, \tag{3.2.34}$$

$$|E(w_i) \setminus E(w_0)| = h,$$

$$|E(w_i) \cap E(w_0)| = s - 1. \tag{3.2.35}$$

For elements of W_0 the situation is slightly different. Reconsidering how we obtained (3.2.33), it soon becomes clear that for $w_i \in W_0$,

$$|E(w_0) \setminus E(w_i)| = h + 1 \iff |E(w_i) \setminus E(w_0)| = h + 1.$$

Hence

$$|E(w_i) \Delta E(w_0)| = |E(w_0) \setminus E(w_i)| + |E(w_i) \setminus E(w_0)|$$

cannot be equal to $2h + 1$, so we apply (3.2.12) to conclude that for all $w_i \in W_0$,

$$|E(w_i) \Delta E(w_0)| = 2h,$$

$$|E(w_0) \setminus E(w_i)| = |E(w_i) \setminus E(w_0)| = h, \tag{3.2.36}$$

$$|E(w_0) \cap E(w_i)| = s. \tag{3.2.37}$$

Next we investigate to what extent the intersections of elements of $E(\mathcal{W})$ overlap. The arguments used to investigate W_0 do not differ from those concerning W_1 in the next section, so we investigate the two sets simultaneously.

A Common Intersection

Let $p \in \{0,1\}$ and let a_p be the smallest positive integer such that $w_{a_p} \in W_p$. Suppose there exists $w_i \in W_p$ with

$$E(w_i) \cap E(w_0) \neq E(w_{a_p}) \cap E(w_0).$$

Neither of these intersections can be contained in the other since they have the same size by (3.2.35, 3.2.37). Also, $E(w_i)$ has size $h+s-p$ and so

$$\begin{aligned}
|E(w_i) \cap E(w_{a_p})| &= |E(w_i)| - |E(w_i) \setminus (E(w_0) \cup E(w_{a_p}))| - |(E(w_i) \cap E(w_0)) \setminus E(w_{a_p})| \\
&\leq |E(w_i)| - |E(w_i) \setminus (E(w_0) \cup E(w_{a_p}))| - 1 \\
&\leq h + s - p - \left| E(w_i) \setminus \bigcup_{\lambda=0}^{i-1} E(w_\lambda) \right| - 1 \\
&= s - p - 1.
\end{aligned}$$

This yields

$$\begin{aligned}
|E(w_i) \Delta E(w_{a_p})| &= |E(w_i)| + |E(w_{a_p})| - 2|E(w_i) \cap E(w_{a_p})| \\
&> 2(h+s-p) - 2(s-p-1) \\
&= 2h + 2 > T,
\end{aligned}$$

our familiar contradiction to (3.2.12). Hence we have

$$E(w_i) \cap E(w_0) = E(w_{a_p}) \cap E(w_0), \qquad \forall w_i \in W_p,$$

implying that the intersection of any two elements of $E(W_p)$ contains

$$X_p = E(w_{a_p}) \cap E(w_0).$$

If some $w_i, w_j \in W_p$ with $i < j$ both move a point outside X_p, then $E(w_i) \cap E(w_j)$ has size at least

$$|X_p| + 1 = s - p + 1$$

by (3.2.35, 3.2.37). Therefore the maximum number of points moved by w_j and not moved by w_i is

$$|E(w_j)| - (|X_p| + 1) = (h + s - p) - (s - p + 1) = h - 1.$$

This contradicts the way w_j was picked (3.2.16) and so we conclude that any two elements of $E(W_p) \cup \{E(w_0)\}$ have intersection precisely X_p.

This section may now be summarised as follows: let $p \in \{0,1\}$. For distinct $w_i, w_j \in W_p \cup \{w_0\}$,

$$(E(w_i) \cap E(w_j)) = X_p \subset E(w_0) \tag{3.2.38}$$

where $|X_p| = s - p$.

Mapping $\mathcal{F}$ into $\mathcal{G}$

We define $\sigma_p \in \mathcal{S}_n$ and $\mathcal{F}_p$ analogously to σ and $\mathcal{F}_\sigma$ in Case 1: let

$$\sigma_p(x) = \begin{cases} w_0(x) & x \in X_p \\ x & x \in [n] \setminus X_p \end{cases},$$

let σ_p^{-1} be the inverse of σ_p in $\mathcal{S}_n$ and set

$$\mathcal{F}_p = \left\{ v\sigma_p^{-1} : v \in \mathcal{F} \right\}.$$

Let $w_i \in W_p$ with $i > 0$. Clearly w_i intersects w_0 in at most $|X_p| = s - p$ elements of X_p, implying that postmultiplying w_i by ${\sigma_p}^{-1}$ can fix at most $s - p$ of the points moved by w_i. That is,

$$|E(w_i{\sigma_p}^{-1})| \geq |E(w_i)| - (s-p) = (h+s-p) - (s-p) = h, \qquad \forall w_i \in W_p. \qquad (3.2.39)$$

Moreover, since elements of $E(W_p)$ do not intersect in points outside X_p by (3.2.38), we have

$$E(w_i{\sigma_p}^{-1}) \cap E(w_j{\sigma_p}^{-1}) = \emptyset, \qquad w_i, w_j \in W_p,\ i \neq j. \qquad (3.2.40)$$

And for $v \in \mathcal{F}$ we have

$$\begin{aligned} |E(v{\sigma_p}^{-1})| &\leq |E(v)| + |E({\sigma_p}^{-1})| \\ &\leq h + s + |X_p| = h + 2s - p \\ &\leq 3h + 2 - p \end{aligned} \qquad (3.2.41)$$

since $|X_p| = s - p$ by the summary of the previous section and $s \leq h + 1$ by definition.

Since T is odd, in order to prove $\mathcal{F}_p \subseteq \mathcal{G}$ we must demonstrate that for some $p \in \{0, 1\}$, all $v \in \mathcal{F}_p$ satisfy

$$|E(v) \cap [k-1]| \leq h.$$

We begin by proving that for at least one value of $p \in \{0, 1\}$, all $v \in \mathcal{F}_p$ satisfy $|E(v)| \leq h + 1$.

So suppose, for a contradiction, that for both $p = 0$ and $p = 1$ there exists $v_p \in \mathcal{F}_p$ with $|E(v_p)| > h + 1$. By (3.2.12) we have

$$|E(v_p) \Delta E(w_i{\sigma_p}^{-1})| \leq T = 2h + 1, \qquad \forall w_i \in W_p,$$

since the size of the symmetric difference is constant under the action of a permutation. Using (3.2.39), all $w_i \in W_p$ therefore satisfy

$$\begin{aligned} |E(v_p) \cap E(w_i{\sigma_p}^{-1})| &= \frac{1}{2}\left(|E(v_p)| + |E(w_i{\sigma_p}^{-1})| - |E(v_p)\Delta E(w_i{\sigma_p}^{-1})|\right) \\ &> \frac{1}{2}\left(h + 1 + h - (2h+1)\right) = 0. \end{aligned}$$

Combining this with (3.2.40) we see that $E(v_p)$ intersects each of the mutually disjoint sets $E(w_i{\sigma_p}^{-1})$ for $w_i \in W_p$, implying

$$|E(v_p)| \geq |W_p|. \tag{3.2.42}$$

If $|W_1| > 3h + 1$ then considering the case $p = 1$ in (3.2.42) gives

$$|E(v_1)| > 3h + 1 = 3h + 2 - p,$$

contradicting (3.2.41). Therefore we must have $|W_1| \leq 3h + 1$ which, together with (3.2.23, 3.2.42), yields

$$\begin{aligned} |E(v_0)| &\geq |W_0| = |\mathcal{W}| - |W_1| \\ &\geq 3T + 1 - (3h + 1) = 3(2h + 1) - 3h \\ &= 3h + 3 > 3h + 2, \end{aligned}$$

this time contradicting (3.2.41) for $p = 0$. Hence we conclude that there exists $p^* \in \{0, 1\}$ such that all $v \in \mathcal{F}_{p^*}$ satisfy $|E(v)| \leq h + 1$.

The family $\mathcal{F}_{p^*}$ is t-intersecting, so if two elements $u, v \in \mathcal{F}_{p^*}$ do not intersect in any points they move, they must jointly fix at least t positions. Suppose, for a contradiction, that two elements $u, v \in \mathcal{F}_{p^*}$ have moved point sets of size $h + 1$ which do not intersect. Then the number of points fixed by both u and v is

$$\begin{aligned} k - |E(u)| - |E(v)| &= k - 2h - 2 \\ &= k - (k - t - 1) - 2 = t - 1, \end{aligned}$$

a contradiction. We conclude that for $u, v \in \mathcal{F}_{p^*}$,

$$|E(u)| = |E(v)| = h + 1 \implies E(u) \cap E(v) \neq \emptyset,$$

so

$$\mathcal{A} = \{\, A \in E(\mathcal{F}_{p^*}) : |A| = h + 1 \,\}$$

is intersecting. Furthermore,

$$h + 1 = \frac{k - t - 1}{2} + 1 \leq \frac{k - 2}{2} + 1 = \frac{k}{2},$$

so we may apply the Erdős-Ko-Rado Theorem 1.2.1 to deduce

$$|\mathcal{A}| \leq \binom{k - 1}{h}. \tag{3.2.43}$$

If this inequality is strict, we combine (3.2.21) with the fact that all elements of $E(\mathcal{F}_{p^*})$ have size at most $h+1$ to obtain

$$\begin{aligned} |\mathcal{F}_{p^*}| &\leq \sum_{X \in E(\mathcal{F}_{p^*})} d(|X|, N+|X|) \\ &< \sum_{j=0}^{h} \binom{k}{j} d(j, N+j) + \binom{k-1}{h} d(h+1, N+h+1) \quad = \quad |\mathcal{G}| \end{aligned}$$

by (3.2.10), contradicting the fact that $\mathcal{F}$, and therefore also $\mathcal{F}_{p^*}$, is maximum. ($\mathcal{F}_{p^*}$ has the same size as $\mathcal{F}$ since σ is a permutation.)

Hence we must have equality in (3.2.43), so Theorem 1.2.1 implies that all elements of $\mathcal{A}$ have a fixed point z in common: we have

$$E(\mathcal{F}_{p^*}) \quad \subseteq \quad \{\, A \subseteq [k] : |A| \leq h \,\} \cup \{\, A \subseteq [k] : |A| = h+1, \ z \in A \,\}$$

and comparing this with (3.2.9), we conclude that $(z\ k)\mathcal{F}_{p^*} \subseteq \mathcal{G}$, where $(z\ k) \in \mathcal{S}_k$ is the transposition swapping z and k. We have demonstrated that $\mathcal{F}$ is equivalent to $\mathcal{G}$. Finally, this completes the proof of Theorem 3.2.11. □

The main results of this chapter were Corollary 3.2.3 and Theorem 3.2.11, two limit results concerning the t-intersection structure of $\mathcal{I}_n^k$ for large parameter values. The remainder of Part II investigates the t-intersection structure of $\mathcal{I}_n^k$ for small parameter values. We begin the next chapter by generalising the concept of saturation for injections.

CHAPTER 4

A COMPLETE BOUND ON EXEMPLARY FAMILIES

4.1 Introduction

The previous chapter demonstrated that if n is large in terms of k and t, fixing is the unique optimal strategy, whereas if k is large in terms of $k-t$ and $n-k$, then the saturation family $\mathcal{G}$ is the unique maximum t-intersecting subset of $\mathcal{I}_n^k$. To complete the picture, we now investigate what happens for small parameter values. This investigation could be regarded as analogous to Ahlswede & Khachatrian's study of set families following the work of Erdős, Ko, Rado, Katona and Wilson, which we described in Chapter 1. This chapter establishes sufficient conditions for a bound on the size of t-intersecting sets of injections with $k < n$. Unfortunately, we do not have an analogous result for permutations.

4.1.1 Saturation for Injections

Recall the definitions of the fix-family $\mathcal{K}_0$ and the saturation family $\mathcal{G}$ in $\mathcal{I}_n^k$ from pages 32 and 20. The smallest value of n for which fixing is not the unique optimal strategy for t-intersection in $\mathcal{I}_n^k$ is $n = 6$. Here

$$\begin{aligned}
K_0(3,6,6) &= \{\, \alpha \in \mathcal{S}_6 : \alpha(i) = i, 1 \le i \le 3 \,\}, \\
\mathcal{G}(3,6,6) &= \{\, \alpha \in \mathcal{S}_6 : \alpha \text{ moves at most one of the first five points} \,\}.
\end{aligned}$$

Both are 3-intersecting subsets of $\mathcal{S}_6$, and we clearly have $|K_0(3,6,6)| = 3!$. To determine the size of the saturation family, note that the identity permutation is the only element of $\mathcal{S}_6$ which moves none of the first five points. The remaining injections in $\mathcal{G}(3,6,6)$ move precisely one element of $[5]$. There are five choices for this moved point, and once it has been picked, the permutation is completely determined by the fact that the remaining elements of $[5]$ are fixed. In other words, we have

$$|\mathcal{G}(3,6,6)| = 5 + 1 = |K_0(3,6,6)|,$$

demonstrating that, while these two families may or may not be maximum, neither of them is uniquely optimal.

We generalise the concept of saturation for injections as follows: setting

$$\mathcal{K}_r(t,k,n) = \left\{\, w \in \mathcal{I}_n^k : w \text{ fixes at least } t+r \text{ elements of } [t+2r] \,\right\}$$

is consistent with our notation $\mathcal{K}_0$ for the fix-family, and we generally abbreviate $\mathcal{K}_r(t,k,n)$ by $\mathcal{K}_r$. Recalling that $d(k,n)$ denotes the number of injections from $[k]$ to $[n]$ with no fixed points, it is not difficult to employ some of the arguments from the proof of Lemma 3.2.8 to find that

$$|\mathcal{K}_r| = \frac{(n-t-2r)!}{(n-k)!} \cdot \sum_{j=0}^{r} \binom{t+2r}{t+r+j} \cdot d(r-j, n-t-r-j).$$

Since we wish to discover the largest possible t-intersecting subsets of $\mathcal{I}_n^k$, we restrict ourselves to considering $\mathcal{K}_r$ for which $[t+2r]$ is a subset of $[k]$, so

$$r \leq \frac{k-t}{2}. \tag{4.1.1}$$

Indeed, it can easily be demonstrated that

$$\mathcal{G}(t,k,n) = \mathcal{K}_{\lfloor (k-t)/2 \rfloor}(t,k,n)$$

and we saw in Theorem 3.2.11 that these are the unique maximum families if k is large in terms of $k-t$ and $n-k$. Indeed, we conjecture that one of the $\mathcal{K}_r$ is always maximum in $\mathcal{I}_n^k$, but we have not succeeded in proving this. However, there are many instances of the parameters where it is easily demonstrated that $\mathcal{K}_r$ is larger than the fix-family $\mathcal{K}_0$ for some $r > 0$, see for instance the proposition below.

Proposition 4.1.2. *If $n/2 \leq t \leq (2n-4)/3$ then $|\mathcal{K}_1| > |\mathcal{K}_0|$ for all k.*

Proof. We have $|\mathcal{K}_0| = (n-t)!/(n-k)!$ and

$$\begin{aligned} |\mathcal{K}_1| &= \frac{(n-t-2)!}{(n-k)!} \cdot \left(\binom{t+2}{t+1} \cdot d(1, n-t-1) + \binom{t+2}{t+2} \cdot d(0, n-t-2) \right) \\ &= \frac{(n-t-2)!}{(n-k)!} \cdot \left((t+2)(n-t-2) + 1 \right). \end{aligned}$$

Thus it suffices to show that

$$(t+2)(n-t-2) \geq (n-t)(n-t-1).$$

Now

$$(t+2)(n-t-2) - (n-t)(n-t-1) = (n-t-1)(2t-n+2) - (t+2)$$

and since $t \geq n/2$, this cannot be less than

$$2(n-t-1) - (t+2) = 2n - 3t - 4,$$

which is non-negative since $t \leq (2n-4)/3$. □

Proposition 4.1.2 gives us specific values of t, k, and n for which fixing is not optimal, and Theorem 3.2.11 guarantees the existence of many more such parameter values. However, neither of these two results give us any indication which saturation parameter r yields the largest $\mathcal{K}_r$ in general. Figure 4.1.1 compares the sizes of the families $\mathcal{K}_r$ for $n = k = 30$. In this case, $\mathcal{I}_n^k$ is the set of permutations on 30 points and $1 \leq t \leq 30$, giving $[(k-t)/2] \subset [15]$ for all t. So let $r^* : [30] \to [15]$ such that for each $t \in [30]$, we have

$$\max \{ |\mathcal{K}_r| : 0 \leq r \leq (k-t)/2 \} = |\mathcal{K}_{r^*(t)}|.$$

Then $r^*(t)$ is given by the blue points in Figure 4.1.1. From our computational evidence, it seems that the plot in Figure 4.1.1 is typical for small parameter values in the following sense: fixing is optimal for $t < n/2$; then from $t = \lceil n/2 \rceil$, the optimal r starts increasing quadratically, until it hits the linear condition (4.1.1) which the remaining r^* are determined by, as we would expect from Theorem 3.2.11. (There seems to be only one exception to this: whenever $n = k$ we have $r^*(k-2) = 0$ instead of 1. Again, we would expect this from Theorems 2.2.2 and 3.2.11 since the former requires $k - t \geq 3$, while the latter requires $k - t \geq 2$.)

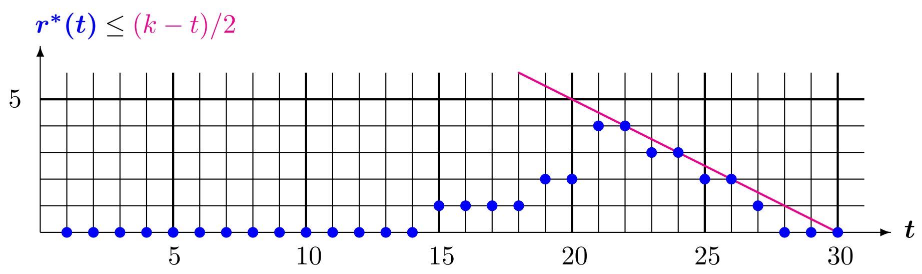


Figure 4.1.1: For fixed $n = k = 30$, the optimal r depends on t.

Figure 4.1.1 shows that the upper bound for t in Proposition 4.1.2 is not sharp. For a better bound, see Conjecture 5.1.3.

4.1.2 Exemplary Injection Families

In this chapter we prove that, under certain conditions, one of the $\mathcal{K}_r$ is optimal. To do this, we will build on the extensive knowledge of t-intersecting set families presented in Chapter 1, similarly to the way that Theorem 3.2.11 used the Erdős-Ko-Rado Theorem 1.2.1. Once again, we therefore express injections in terms of sets: for an injection $w \in \mathcal{I}_n^k$, its *fixed point set* is the set of points in $[k]$ which are fixed under w. That is,

$$\mathrm{Fix}(w) = \{x \in [k] \ : \ w(x) = x\}$$

and if S is a subset of $\mathcal{I}_n^k$ then $\mathrm{Fix}(S) = \{\mathrm{Fix}(w) \ : \ w \in S\}$. Clearly $\mathrm{Fix}(w) = [k] \setminus E(w)$, so sets have been used to represent injections or permutations in this way since [DF77].

With the following definition we will be able to describe the objective of this chapter in more detail. The first part of the definition is well-known.

Definition 4.1.3. A family $\mathcal{A}$ of subsets of $[k]$ is *left-compressed* if for each $A \in \mathcal{A}$ and all $1 \leq i < j \leq k$ with $i \notin A$, $j \in A$, the set obtained by removing j and adding i is a member of $\mathcal{A}$, that is $((A \setminus \{j\}) \cup \{i\}) \in \mathcal{A}$.

A t-intersecting family $\mathcal{F} \subseteq \mathcal{I}_n^k$ is *left-compressed* if its fixed point set $\mathrm{Fix}(\mathcal{F})$ is left-compressed.

Left-compression maps have traditionally been popular as a method of deriving bounds for maximum t-intersecting families of combinatorial objects. Where the intersecting objects are themselves maps, such as injections or permutations, a left-compression map $\mathcal{L}$ would be applied in conjunction with a map $\mathcal{T}$ ensuring that a t-intersecting family has an intersecting fixed point set. In short, one would transform an arbitrary t-intersecting family into an *exemplary* one, and derive a bound for the reduced case of exemplary families.

Definition 4.1.4. A t-intersecting subset $\mathcal{F}$ of $\mathcal{I}_n^k$ is *exemplary* if

1. $\mathcal{F}$ is maximal under set inclusion,
2. $\mathrm{Fix}(\mathcal{F})$ is t-intersecting and
3. $\mathrm{Fix}(\mathcal{F})$ is left-compressed.

Unfortunately, it is difficult to find such maps $\mathcal{T}$ or $\mathcal{L}$ for injections, given that they must preserve both the cardinality and the intersecting property of a set in order to be useful. We discuss some of these difficulties in Section 5.2.2. The main result of this chapter is Theorem 4.4.4 which states that, provided we are not considering the permutation case $k = n$, if $\mathcal{F}$ is an exemplary t-intersecting subset of $\mathcal{I}_n^k$, then it cannot be larger than all the $\mathcal{K}_r$, which are themselves exemplary.

Before we prove this last claim, note that by (4.1.1), the restriction $r \neq k-t-1$ of Lemma 4.1.5 only applies if

$$k - t - 1 \leq (k-t)/2 \iff k - t \leq 2$$

and $k = n$, that is $t \geq n-2$. We already noted previously that in these marginal cases, it is easily proved that all maximum t-intersecting subsets of $\mathcal{I}_n^k$, not just the exemplary ones, are equivalent to the fix-family.

Lemma 4.1.5. *For $1 \leq t \leq k \leq n \in \mathbb{N}$ and $0 \leq r \leq (k-t)/2$, the family $\mathcal{K}_r(t,k,n)$ is exemplary, unless $r = k-t-1$ and $k = n$.*

Proof. Since $\mathcal{K}_r$ consists of injections which fix at least $t+r$ of the first $t+2r$ points, any two injections in $\mathcal{K}_r$ share at least

$$2(t+r) - (t+2r) = t$$

fixed points. Thus $\mathrm{Fix}(\mathcal{K}_r)$ is t-intersecting, which also demonstrates that $\mathcal{K}_r$ is a t-intersecting subset of $\mathcal{I}_n^k$. We have

$$\mathrm{Fix}(\mathcal{K}_r) = \{\, X \subseteq [k] : |X \cap [t+2r]| \geq t+r \,\}. \tag{4.1.6}$$

For $X \in \mathrm{Fix}(\mathcal{K}_r)$ and $1 \leq i < j \leq k$ with $i \notin X$, $j \in X$, set

$$Y = (X \setminus \{j\}) \cup \{i\}.$$

Then since $i < j$, Y contains at least as many elements of $[t+2r]$ as X does, so $Y \in \mathrm{Fix}(\mathcal{K}_r)$ by (4.1.6). Hence $\mathrm{Fix}(\mathcal{K}_r)$ is left-compressed.

To demonstrate that $\mathcal{K}_r$ is maximal, we will show that for any $\alpha \in \mathcal{I}_n^k \setminus \mathcal{K}_r$, there exists $\beta \in \mathcal{K}_r$ which does not t-intersect α. So let

$$A = \mathrm{Fix}(\alpha) \cap [t+2r],$$

then $|A| = t+r-x$ where $1 \leq x \leq t+r$ if α is not an element of $\mathcal{K}_r$. Let B be any $(t-x)$-subset of A and let $\beta \in \mathcal{I}_n^k$ be an injection which satisfies the following two properties: firstly, β fixes all elements of

$$C := ([t+2r] \setminus A) \;\cup\; B,$$

ensuring that $\beta \in \mathcal{K}_r$ since C is a subset of $[t+2r]$ of size

$$|C| = t + 2r - (t+r-x) + t - x = t + r.$$

Secondly, β must not fix any elements of

$$[t + 2r] \setminus C = A \setminus B,$$

and β must not intersect α in any positions beyond $t + 2r$ either. Note that this second condition may be summarised as $\mathrm{int}(\alpha, \beta) = B$ and can therefore be satisfied unless the set $[k] \setminus C$ has size 1 and $k = n$ (in which case we have no choice where to map the single point). But

$$|[k] \setminus C| = k - t - r \neq 1$$

if $k = n$ since $r \neq k - t - 1$ by assumption. We now have

$$|\,\mathrm{int}(\alpha, \beta)| = |B| < t,$$

so $\mathcal{K}_r$ is indeed maximal and the proof is complete. $\square$

4.1.3 Methodology

The approach of this chapter is based on the paper [AK98], where Ahlswede & Khachatrian characterise maximum t-intersecting families of words. However, many of the arguments in [AK98] need to be adapted to injections, and it turns out that their left-compression and fixing maps cannot be used successfully in this context. In Sections 4.2 – 4.4 we present our proof of the main result in this chapter, Theorem 4.4.4, which states that the saturation families $\mathcal{K}_r$ are maximum among exemplary t-intersecting injection families for $k < n$. Possible future directions from there are discussed in Chapter 5.

Throughout much of this chapter, we represent an exemplary t-intersecting set $\mathcal{F}$ of injections by $\mathrm{Fix}(\mathcal{F})$ and investigate $\mathcal{M}(\mathcal{F})$, the set of minimal elements of $\mathrm{Fix}(\mathcal{F})$ under set inclusion. The lemmas in Section 4.2 clarify why $\mathcal{M}(\mathcal{F})$ can be considered to 'generate' $\mathcal{F}$ in some sense. Section 4.3 presents Proposition 4.3.2, the main structural result concerning $\mathcal{M}(\mathcal{F})$. In Section 4.4, we use a method of [FF80] together with a result from [AK98] to express $|\mathcal{K}_r|$ in terms of a t-intersecting family of sets, enabling us to use Proposition 4.3.2. Finally, we combine this expression of $|\mathcal{K}_r|$ with some simple properties of $d(k, n)$ to conclude that Theorem 4.4.4 holds.

4.2 From Injections to Sets

4.2.1 An Antichain at the Heart of the Fixed Point Set

This section begins to shift the focus of this chapter from the injection family $\mathcal{F}$ to the set family $\mathcal{M}(\mathcal{F})$. To pursue this approach successfully, we need maps going both ways. The notion of $\mathrm{Fix}(\mathcal{F})$ provides a map from injections to sets. Conversely, we introduce a map $\mathcal{V}$ which can be regarded as a map from sets back to injections. For a subset A of $[k]$, we denote by $\mathcal{V}(A)$ the set of injections in $\mathcal{I}_n^k$ which fix all elements of A:

$$\mathcal{V}(A) = \left\{ w \in \mathcal{I}_n^k : A \subseteq \mathrm{Fix}(w) \right\}.$$

Note that individual injections in $\mathcal{V}(A)$ may fix more points than just the elements of A. For a family $\mathcal{A}$ of subsets of $[k]$,

$$\mathcal{V}(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} \mathcal{V}(A).$$

If $\mathcal{F}$ is a t-intersecting subset of $\mathcal{I}_n^k$, we refer to the set of minimal elements of $\mathrm{Fix}(\mathcal{F})$ under set inclusion by

$$\mathcal{M}(\mathcal{F}) = \{ X \in \mathrm{Fix}(\mathcal{F}) : \text{no element of } \mathrm{Fix}(\mathcal{F}) \text{ is strictly contained in } X \}.$$

The following lemma clarifies why we refer to $\mathcal{M}(\mathcal{F})$ as the *basis* of $\mathcal{F}$.

Lemma 4.2.1. *If $\mathcal{F}$ is a maximal t-intersecting subset of $\mathcal{I}_n^k$ such that* $\mathrm{Fix}(\mathcal{F})$ *is t-intersecting then $\mathcal{F} = \mathcal{V}(\mathcal{M}(\mathcal{F}))$ and*

$$|\mathcal{F}| \leq \sum_{X \in \mathcal{M}(\mathcal{F})} \frac{(n - |X|)!}{(n-k)!}.$$

Proof. The fixed point set of any element w of $\mathcal{F}$ contains some element X of $\mathcal{M}(\mathcal{F})$. Thus $w \in \mathcal{V}(X) \subseteq \mathcal{V}(\mathcal{M}(\mathcal{F}))$. For the reverse containment, let $w \in \mathcal{V}(\mathcal{M}(\mathcal{F}))$ and X be an element of $\mathcal{M}(\mathcal{F})$ such that $w \in \mathcal{V}(X)$. Since X is an element of the t-intersecting set $\mathrm{Fix}(\mathcal{F})$, we have $|X \cap Y| \geq t$ for all $Y \in \mathrm{Fix}(\mathcal{F})$, so w t-intersects all elements of $\mathcal{F}$. Since $\mathcal{F}$ is maximal, this implies $w \in \mathcal{F}$.

Hence $\mathcal{F} = \mathcal{V}(\mathcal{M}(\mathcal{F}))$, giving

$$|\mathcal{F}| \leq \sum_{X \in \mathcal{M}(\mathcal{F})} |\mathcal{V}(X)| = \sum_{X \in \mathcal{M}(\mathcal{F})} \frac{(n - |X|)!}{(n-k)!},$$

as required. □

4.2.2 Intersecting Families are Generated by their Bases

We use the following notation: the maximum element of a subset A of the natural numbers $\mathbb{N}$ is denoted by $\max(A)$. More generally, if $\mathcal{A}$ is a collection of subsets of $\mathbb{N}$, we denote the maximum element of all the sets by $\max(\mathcal{A})$:

$$\max(\mathcal{A}) = \max_{A\in\mathcal{A}} (\max(A)) = \max\left(\bigcup_{A\in\mathcal{A}} A\right).$$

Also recall that $[\max(A)] = \{1, 2, \ldots, \max(A)\}$.

The definition of $D(X)$ below is slightly counterintuitive at first, but the concept turns out to be very useful in what follows, since the map D refines the map $\mathcal{V}$: $D(X)$ is the set of injections whose fixed point sets, in the smallest initial segment of $[k]$ containing X, agree precisely with X.

Lemma 4.2.2. *An exemplary t-intersecting subset $\mathcal{F}$ of $\mathcal{I}_n^k$ is a disjoint union*

$$\mathcal{F} = \dot{\bigcup_{X\in\mathcal{M}(\mathcal{F})}} D(X)$$

where

$$D(X) = \left\{ w \in \mathcal{I}_n^k : \mathrm{Fix}(w) \cap [\max(X)] = X \right\}.$$

Proof. We will use Lemma 4.2.1 to show that $\mathcal{F} \subseteq \bigcup_{X\in\mathcal{M}(\mathcal{F})} D(X)$, so let $w \in \mathcal{F}$ and let $\mathcal{A}$ be the set of elements of $\mathcal{M}(\mathcal{F})$ contained in $\mathrm{Fix}(w)$. Moreover, let A be the element of $\mathcal{A}$ with the smallest maximum element, i.e. $\max(B) \geq \max(A), \forall B \in \mathcal{A}$. Note that since $A \subseteq \mathrm{Fix}(w)$,

$$\mathrm{Fix}(w) \cap [\max(A)] \supseteq A.$$

Now suppose, for a contradiction, that there exists $x \in \mathrm{Fix}(w) \cap [\max(A)]$ with $x \notin A$. Note $x < \max(A)$.

Since $\mathcal{F}$ is exemplary, $\mathrm{Fix}(\mathcal{F})$ is left-compressed. So $A \in \mathrm{Fix}(\mathcal{F})$ together with $x \notin A$, $\max(A) \in A$ implies that

$$B = A \setminus \{\max(A)\} \cup \{x\} \quad \in \mathrm{Fix}(\mathcal{F}).$$

Note $B \subseteq \mathrm{Fix}(w)$. Now $B \in \mathrm{Fix}(\mathcal{F})$ means there exists $C \in \mathcal{M}(\mathcal{F})$ with $C \subseteq B$, so $C \in \mathcal{A}$. By construction of B, we have $\max(B) < \max(A)$ which implies $\max(C) < \max(A)$. But $C \in \mathcal{A}$ with $\max(C) < \max(A)$ contradicts the definition of A, so we conclude

$$\mathrm{Fix}(w) \cap [\max(A)] = A.$$

This shows $w \in D(A)$ and since $A \in \mathcal{M}(\mathcal{F})$ this concludes the proof of

$$\mathcal{F} \subseteq \bigcup_{X \in \mathcal{M}(\mathcal{F})} D(X).$$

Conversely, let $w \in \bigcup_{X \in \mathcal{M}(\mathcal{F})} D(X)$ and let X be an element of $\mathcal{M}(\mathcal{F})$ such that $w \in D(X)$, that is $\mathrm{Fix}(w) \cap [\max(X)] = X$. Then $\mathrm{Fix}(w) \supseteq X$ which implies $w \in \mathcal{F}$ by Lemma 4.2.1. Thus

$$\mathcal{F} = \bigcup_{X \in \mathcal{M}(\mathcal{F})} D(X).$$

Finally, we show that this union is disjoint. Let $w \in \mathcal{F}$ and $A, B \in \mathcal{M}(\mathcal{F})$ such that $w \in D(A) \cap D(B)$. Then

$$\mathrm{Fix}(w) \cap [\max(A)] = A \text{ and } \mathrm{Fix}(w) \cap [\max(B)] = B.$$

Without loss of generality, suppose $\max(A) \leq \max(B)$. Then

$$\mathrm{Fix}(w) \cap [\max(A)] \subseteq \mathrm{Fix}(w) \cap [\max(B)],$$

i.e. $A \subseteq B$. Since A and B are both minimal elements of $\mathrm{Fix}(\mathcal{F})$ under inclusion, we cannot have $A \subset B$, therefore we conclude $A = B$. □

The next lemma connects $D(X)$ to the map $\mathcal{V}$ which was defined at the beginning of this section.

Lemma 4.2.3. *Let $\mathcal{F}$ be an exemplary t-intersecting subset of $\mathcal{I}_n^k$, let $X \in \mathcal{M}(\mathcal{F})$ be such that $\max(X) = \max(\mathcal{M}(\mathcal{F}))$ and consider the set $\mathcal{F}_X$ of elements of $\mathcal{F}$ which are generated by X only:*

$$\mathcal{F}_X = \mathcal{V}(X) \setminus \mathcal{V}(\mathcal{M}(\mathcal{F}) \setminus \{X\}).$$

Then $\mathcal{F}_X = D(X)$ (see Lemma 4.2.2) and

$$|\mathcal{F}_X| = d(\max(X) - |X|, n - |X|) \cdot \frac{(n - \max(X))!}{(n-k)!}$$

where d is given in Lemma 3.2.5.

Proof. Showing that $\mathcal{F}_X \subseteq D(X)$ is very similar to the proof of the previous lemma: let $w \in \mathcal{F}_X$, then $\mathrm{Fix}(w) \supseteq X$ and

$$\text{there does not exist } Y \in \mathcal{M}(\mathcal{F}) \text{ with } Y \neq X \text{ such that } \mathrm{Fix}(w) \supseteq Y. \tag{4.2.4}$$

Since $\mathrm{Fix}(w) \supseteq X$ we have $\mathrm{Fix}(w) \cap [\max(X)] \supseteq X$. Suppose there exists $p \in \mathrm{Fix}(w) \cap [\max(X)]$ such that $p \notin X$. Then since $\mathrm{Fix}(\mathcal{F})$ is left-compressed and $X \in \mathrm{Fix}(\mathcal{F})$,

$$Z = X \setminus \{\max(X)\} \cup \{p\} \quad \in \mathrm{Fix}(\mathcal{F}).$$

Note that by construction, $Z \subseteq \mathrm{Fix}(w)$. So if Y is an element of $\mathcal{M}(\mathcal{F})$ contained in Z then $Y \subseteq \mathrm{Fix}(w)$. By (4.2.4) this implies $Y = X$. But $\max(X) \notin Z$ implies $\max(X) \notin Y$ which means $Y \neq X$. By this contradiction we must have no such Z and hence so such p. Therefore

$$\mathrm{Fix}(w) \cap [\max(X)] = X$$

which means $w \in D(X)$. Thus we have shown $\mathcal{F}_X \subseteq D(X)$.

For the reverse containment, let $w \in D(X)$, then

$$\mathrm{Fix}(w) \cap [\max(X)] = X, \tag{4.2.5}$$

so $\mathrm{Fix}(w) \supseteq X$ and $w \in \mathcal{V}(X)$. We need to show that $w \notin \mathcal{V}(Y)$ for any $Y \in \mathcal{M}(\mathcal{F})$, $Y \neq X$, so suppose $w \in \mathcal{V}(Y)$ with $Y \in \mathcal{M}(\mathcal{F})$, then $\mathrm{Fix}(w) \supseteq Y$. Now $\max(X) = \max(\mathcal{M}(\mathcal{F}))$ forces $\max(X) \geq \max(Y)$. Thus $\mathrm{Fix}(w) \supseteq Y$ together with (4.2.5) implies

$$\mathrm{Fix}(w) \cap [\max(X)] \supseteq Y.$$

This gives $X \supseteq Y$ which contradicts $X, Y \in \mathcal{M}(\mathcal{F})$ unless $X = Y$.

Hence $\mathcal{F}_X = D(X)$ and so

$$\begin{aligned} |\mathcal{F}_X| &= |\left\{ w \in \mathcal{I}_n^k : \mathrm{Fix}(w) \cap [\max(X)] = X \right\}| \\ &= d(\max(X) - |X|, n - |X|) \cdot \frac{(n - \max(X))!}{(n-k)!} \end{aligned}$$

as asserted. □

4.3 The Structure of the Basis

We begin this section by establishing a lemma which we use, in conjunction with the lemmas of the previous section, to prove a key fact about the basis of an intersecting family $\mathcal{F}$ in Proposition 4.3.2.

Lemma 4.3.1. *Let $\mathcal{F}$ be an exemplary t-intersecting subset of $\mathcal{I}_n^k$, let $A_1, A_2 \in \mathrm{Fix}(\mathcal{F})$ have the property that there exist $i, j \in [k]$ with $i < j$, such that neither A_1 nor A_2 contain i, but both contain j. Then*

$$|A_1 \cap A_2| \geq t + 1.$$

Proof. Denoting $(A_1 \setminus \{j\}) \cup \{i\}$ and $(A_2 \setminus \{j\}) \cup \{i\}$ by A_1', A_2' respectively, we have $A_1', A_2' \in \mathrm{Fix}(\mathcal{F})$ since $\mathrm{Fix}(\mathcal{F})$ is left-compressed. Moreover, $\mathrm{Fix}(\mathcal{F})$ is t-intersecting, and so

$$|A_1' \cap A_2| \geq t.$$

Since neither i nor j are elements of $A_1' \cap A_2$, but $j \in A_1 \cap A_2$, this gives

$$|A_1 \cap A_2| \geq |A_1' \cap A_2| + 1 \geq t + 1$$

which completes the proof. $\square$

4.3.1 Basis Elements

We are now in a position to prove our main structural result concerning $\mathcal{M}(\mathcal{F})$. The following proposition constitutes most of the proof of Theorem 4.4.4, the main result in this chapter.

Proposition 4.3.2. *Let $\mathcal{F}$ be a maximum exemplary t-intersecting subset of $\mathcal{I}_n^k$ and denote $\max(\mathcal{M}(\mathcal{F}))$ by l. Then there exists $r \in \{0, 1, \ldots, (k-t)/2\}$ such that $l = 2r + t$ and all elements of $\mathcal{M}(\mathcal{F})$ containing l have size $r + t$.*

Proof. The result will follow fairly easily from the fact that all elements of $\mathcal{M}(\mathcal{F})$ containing l have size $(l+t)/2$, which we will prove here. Partition $\mathcal{M}(\mathcal{F})$ into

$$\mathcal{M}(\mathcal{F}) = \mathcal{M}_0 \cup \mathcal{M}_1$$

according to whether the elements of $\mathcal{M}(\mathcal{F})$ contain the maximum element l:

$$\mathcal{M}_0 = \{ A \in \mathcal{M}(\mathcal{F}) : \max(A) = l \},$$
$$\mathcal{M}_1 = \{ A \in \mathcal{M}(\mathcal{F}) : \max(A) < l \}.$$

Due to the choice of l, $\mathcal{M}_0$ is non-empty. Moreover, elements of $\mathcal{M}_0$ have the following property:

(P) *If $A_1, A_2 \in \mathcal{M}_0$ with $|A_1 \cap A_2| = t$ then $|A_1| + |A_2| = l + t$.*

To see this, note that if $|A_1 \cap A_2| = t$ then by Lemma 4.3.1, there exist no $i, j \in [k]$ with $i < j$ such that neither A_1 nor A_2 contain i, but both contain j. In other words, since both A_1 and A_2 contain l, it is true for all $i \in [k], i < l$ that i is contained in at least one of A_1, A_2. Since l is the maximum element of both A_1 and A_2, this is simply saying

$$|A_1| + |A_2| = l + |A_1 \cap A_2| = l + t.$$

Now we further partition $\mathcal{M}_0$ according to the cardinalities of its members:

$$\mathcal{M}_0^{(i)} = \{ A \in \mathcal{M}_0 : |A| = i \}.$$

Finally, consider the collection of sets obtained by removing the maximum element l from all members of $\mathcal{M}_0^{(i)}$:

$$\widehat{\mathcal{M}_0^{(i)}} = \left\{ X \subseteq [l-1] : X \cup \{l\} \in \mathcal{M}_0^{(i)} \right\}.$$

To prove that all elements of $\mathcal{M}(\mathcal{F})$ containing l have size $(l+t)/2$, we need to demonstrate that $\mathcal{M}_0^{(i)}$ is empty unless $i = (l+t)/2$. So let $i \in [k]$ with $\mathcal{M}_0^{(i)} \neq \emptyset$ and suppose $i \neq (l+t)/2$. We will derive a contradiction by constructing two t-intersecting families larger than $\mathcal{F}$.

Consider the set $\mathcal{S}_1$ which is obtained from $\mathcal{M}(\mathcal{F})$ by removing l from all members of size i, and removing all members of size $l+t-i$ completely if they contain l:

$$\mathcal{S}_1 = \mathcal{M}(\mathcal{F}) \setminus \left(\mathcal{M}_0^{(i)} \cup \mathcal{M}_0^{(l+t-i)}\right) \cup \widehat{\mathcal{M}_0^{(i)}}. \tag{4.3.3}$$

Similarly, $\mathcal{S}_2$ is obtained from $\mathcal{M}(\mathcal{F})$ by removing l from all sets in $\mathcal{M}(\mathcal{F})$ of size $l+t-i$, and deleting all members of size i which contain l:

$$\mathcal{S}_2 = \mathcal{M}(\mathcal{F}) \setminus \left(\mathcal{M}_0^{(i)} \cup \mathcal{M}_0^{(l+t-i)}\right) \cup \widehat{\mathcal{M}_0^{(l+t-i)}}. \tag{4.3.4}$$

Each of $\mathcal{S}_1, \mathcal{S}_2$ is a t-intersecting set of sets; we will show this for $\mathcal{S}_1$. The only elements of $\mathcal{S}_1$ which are not members of the t-intersecting set $\mathcal{M}(\mathcal{F})$ are the elements of $\widehat{\mathcal{M}_0^{(i)}}$, so let $E_1 \in \widehat{\mathcal{M}_0^{(i)}}$ and $E_2 \in \mathcal{S}_1$. We need to show $|E_1 \cap E_2| \geq t$, so recall $E_1 \cup \{l\} \in \mathcal{M}_0$.

- If $E_2 \in \mathcal{M}_1$ then E_2 does not contain l, so the fact that $E_1 \cup \{l\} \in \mathcal{M}(\mathcal{F})$ t-intersects $E_2 \in \mathcal{M}(\mathcal{F})$ implies $|E_1 \cap E_2| \geq t$.

- If $E_2 \in \mathcal{M}_0$ and $|(E_1 \cup \{l\}) \cap E_2| \geq t+1$ then $|E_1 \cap E_2| \geq t$.

- If $E_2 \in \mathcal{M}_0$ and $|(E_1 \cup \{l\}) \cap E_2| = t$ then $|E_1 \cup \{l\}| + |E_2| = l+t$ by Property (P), so $|E_2| = l+t-i$ since $E_1 \in \widehat{\mathcal{M}_0^{(i)}}$. But $E_2 \in \mathcal{M}_0$ with $|E_2| = l+t-i$ contradicts $E_2 \in \mathcal{S}_1$, since all elements of size $l+t-i$ containing l have been removed from $\mathcal{S}_1$.

- If $E_2 \in \widehat{\mathcal{M}_0^{(i)}}$ and $|(E_1 \cup \{l\}) \cap (E_2 \cup \{l\})| \geq t+1$ then $|E_1 \cap E_2| \geq t$.

- If $E_2 \in \widehat{\mathcal{M}_0^{(i)}}$ and $|(E_1 \cup \{l\}) \cap (E_2 \cup \{l\})| = t$ then Property (P) implies

  $$|E_1 \cup \{l\}| + |E_2 \cup \{l\}| = l+t.$$

  Since $E_1, E_2 \in \widehat{\mathcal{M}_0^{(i)}}$ they both have size $i-1$, so the above equation reduces to $2i = l+t$ which contradicts the assumption $i \neq (l+t)/2$.

Hence $\mathcal{S}_1$ is t-intersecting, and it follows from similar arguments that $\mathcal{S}_2$ is also t-intersecting. This implies that the sets

$$\mathcal{F}_j = \mathcal{V}(\mathcal{S}_j), \quad j = 1, 2$$

are also t-intersecting.

We now have three t-intersecting families: $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$, with the latter two arising from (4.3.3) and (4.3.4). It is intuitively clear from these two equations that neither $\mathcal{F}_1$ nor $\mathcal{F}_2$ differ greatly from $\mathcal{F}$; however, none is contained in another. To make this more formal, we examine the set differences, beginning with $\mathcal{F} \setminus \mathcal{F}_1$. It follows from (4.3.3) that the elements of $\mathcal{F} \setminus \mathcal{F}_1$ are those words which are generated by the sets in $\mathcal{M}_0^{(l+t-i)}$ only:

$$\mathcal{F} \setminus \mathcal{F}_1 = \mathcal{V}(\mathcal{M}(\mathcal{F})) \setminus \mathcal{V}(\mathcal{S}_1) = \bigcup_{X \in \mathcal{M}_0^{(l+t-i)}} \mathcal{V}(X) \setminus \mathcal{V}(\mathcal{M}(\mathcal{F}) \setminus X).$$

Since $\max(X) = \max(\mathcal{M}(\mathcal{F}))$ for all $X \in \mathcal{M}_0^{(l+t-i)}$, we may apply Lemma 4.2.3 to obtain

$$\mathcal{F} \setminus \mathcal{F}_1 = \bigcup_{X \in \mathcal{M}_0^{(l+t-i)}} D(X) \tag{4.3.5}$$

where

$$\begin{aligned} D(X) &= \left\{ w \in \mathcal{I}_n^k : \mathrm{Fix}(w) \cap [\max(X)] = X \right\} \\ &= \left\{ w \in \mathcal{I}_n^k : \mathrm{Fix}(w) \cap [l] = X \right\}, \end{aligned}$$

implying that (4.3.5) is a disjoint union. By Lemma 4.2.3 this gives

$$\begin{aligned} |\mathcal{F} \setminus \mathcal{F}_1| &= \sum_{X \in \mathcal{M}_0^{(l+t-i)}} d(\max(X) - |X|, n - |X|) \cdot \frac{(n - \max(X))!}{(n-k)!} \\ &= \sum_{X \in \mathcal{M}_0^{(l+t-i)}} d(l - (l+t-i), n - (l+t-i)) \cdot \frac{(n-l)!}{(n-k)!} \\ &= \left| \mathcal{M}_0^{(l+t-i)} \right| \cdot d(i-t, n-l-t+i) \cdot \frac{(n-l)!}{(n-k)!}. \end{aligned} \tag{4.3.6}$$

Similarly,

$$\begin{aligned} |\mathcal{F} \setminus \mathcal{F}_2| &= \sum_{X \in \mathcal{M}_0^{(i)}} d(\max(X) - |X|, n - |X|) \cdot \frac{(n - \max(X))!}{(n-k)!} \\ &= \left| \mathcal{M}_0^{(i)} \right| \cdot d(l-i, n-i) \cdot \frac{(n-l)!}{(n-k)!}. \end{aligned} \tag{4.3.7}$$

Next we establish $|\mathcal{F}_1 \setminus \mathcal{F}|$. By the construction of $\mathcal{S}_1$ in (4.3.3), the elements of $\mathcal{F}_1 \setminus \mathcal{F}$ are those which are generated by the sets in $\widehat{\mathcal{M}_0^{(i)}}$ only, i.e. injections whose fixed point set contains some element of $\widehat{\mathcal{M}_0^{(i)}}$ but which do not fix l. So let $Y \in \widehat{\mathcal{M}_0^{(i)}}$; such a Y exists since $\mathcal{M}_0^{(i)} \neq \emptyset$ implies $\widehat{\mathcal{M}_0^{(i)}} \neq \emptyset$, though Y may be empty. Defining

$$D'(Y) = \left\{ w \in \mathcal{I}_n^k : \mathrm{Fix}(w) \cap [l] = Y \right\},$$

we have

$$D'(Y) \subseteq \mathcal{F}_1 \setminus \mathcal{F}.$$

Now $|Y| = i - 1$ so

$$|D'(Y)| = d(l - i + 1, n - i + 1) \cdot \frac{(n-l)!}{(n-k)!}.$$

For distinct $Y, Z \in \widehat{\mathcal{M}_0^{(i)}}$ we have $D'(Y) \cap D'(Z) = \emptyset$ and so we conclude

$$\begin{aligned} |\mathcal{F}_1 \setminus \mathcal{F}| &\geq \sum_{Y \in \widehat{\mathcal{M}_0^{(i)}}} |D'(Y)| \\ &= \left|\mathcal{M}_0^{(i)}\right| \cdot d(l - i + 1, n - i + 1) \cdot \frac{(n-l)!}{(n-k)!} \end{aligned} \tag{4.3.8}$$

since $\left|\widehat{\mathcal{M}_0^{(i)}}\right| = \left|\mathcal{M}_0^{(i)}\right|$.

Analogously,

$$\begin{aligned} |\mathcal{F}_2 \setminus \mathcal{F}| &\geq \left|\mathcal{M}_0^{(l+t-i)}\right| \cdot d(l - (l+t-i-1), n - (l+t-i-1)) \cdot \frac{(n-l)!}{(n-k)!} \\ &= \left|\mathcal{M}_0^{(l+t-i)}\right| \cdot d(i - t + 1, n - l - t + i + 1) \cdot \frac{(n-l)!}{(n-k)!}. \end{aligned} \tag{4.3.9}$$

Since $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}$ are t-intersecting subsets of $\mathcal{I}_n^k$ and $\mathcal{F}$ has largest possible size, we must have $|\mathcal{F}_i \setminus \mathcal{F}| \leq |\mathcal{F} \setminus \mathcal{F}_i|$ for $i = 1, 2$. Thus equations (4.3.6) - (4.3.9) yield

$$\left|\mathcal{M}_0^{(i)}\right| \cdot d(l - i + 1, n - i + 1) \leq \left|\mathcal{M}_0^{(l+t-i)}\right| \cdot d(i - t, n - l - t + i) \tag{4.3.10}$$

and

$$\left|\mathcal{M}_0^{(l+t-i)}\right| \cdot d(i - t + 1, n - l - t + i + 1) \leq \left|\mathcal{M}_0^{(i)}\right| \cdot d(l - i, n - i). \tag{4.3.11}$$

Recall that i was chosen to ensure that $\mathcal{M}(\mathcal{F})$ contains a set X of size i. Since $\mathcal{M}(\mathcal{F}) \subseteq \mathrm{Fix}(\mathcal{F})$ is t-intersecting, this gives $i \geq t$. If $i = t$, then in order to t-intersect X, all elements of $\mathrm{Fix}(\mathcal{F})$ must contain X, giving $\mathcal{M}(\mathcal{F}) = \{X\}$, so $l = \max(X)$. Since $\mathrm{Fix}(\mathcal{F})$ is left-compressed, the fact that X is the only minimal element of $\mathrm{Fix}(\mathcal{F})$ means that it cannot have 'gaps', i.e. $X = [\max(X)] = [l]$, giving $l = |X| = i$. But now we have $i = l = t$, contradicting $i \neq (l + t)/2$.

Thus $i > t$, so we may use (3.2.7) to obtain

$$d(i - t, n - l - t + i) \leq d(i - t + 1, n - l - t + i + 1).$$

We then deduce from (4.3.10) and (4.3.11) that

$$\left|\mathcal{M}_0^{(i)}\right| \cdot d(l-i+1, n-i+1) \leq \left|\mathcal{M}_0^{(i)}\right| \cdot d(l-i, n-i).$$

Since $\left|\mathcal{M}_0^{(i)}\right| \neq 0$ this gives

$$d(l-i+1, n-i+1) \leq d(l-i, n-i),$$

implying $l = i$ by (3.2.7). But then $\mathcal{M}(\mathcal{F})$ contains an element X of size $l = \max(\mathcal{M}(\mathcal{F}))$ which, since all elements of $\mathcal{M}(\mathcal{F})$ must be subsets of $[l]$, is only possible if $X = [l]$. Since $\mathcal{M}(\mathcal{F})$ is an antichain this forces $\mathcal{M}(\mathcal{F}) = \{[l]\}$. Once again, since $\mathrm{Fix}(\mathcal{F})$ is left-compressed this forces $l = t$, contradicting $i \neq (l+t)/2$.

We have shown that all elements of $\mathcal{M}(\mathcal{F})$ containing l have size $(l+t)/2$. Since $\mathcal{M}(\mathcal{F})$ must have an element containing $\max(\mathcal{M}(\mathcal{F})) = l$, this implies that $l+t$ as well as

$$l - t = l + t - 2t$$

are divisible by 2. Moreover, $\mathcal{M}(\mathcal{F})$ is t-intersecting, giving $l \geq t$. Hence there exists $r \in \mathbb{N} \cup \{0\}$ such that

$$l = 2r + t.$$

Note also that $l \in \mathrm{Fix}(w)$ for some $w \in \mathcal{F}$ which implies $l \leq k$ and therefore

$$r \in \{0, 1, \ldots, (k-t)/2\}$$

which completes the proof. □

4.4 From Sets back to Injections

To make the link back from sets to injections, the following proposition expresses the maximal size of a t-intersecting subset of $\mathcal{I}_n^k$ in terms of t-intersecting sets of sub$sets$ of $[k]$. This method was inspired by [FF80] which contains a similar proposition linking t-intersecting subsets of words in $[n]^k$ to t-intersecting elements of the power set of $[n]$.

If G is a collection of sets, let

$$G^{(i)} = \{\, X \in G : |X| = i \,\}.$$

Recall that $d(k, n)$ denotes the number of injections from $[k]$ to $[n]$ with no fixed points.

Proposition 4.4.1. *Let $\mathcal{G}(k,t)$ be the set of all t-intersecting families of subsets of $[k]$. If $\mathcal{F} \subseteq \mathcal{I}_n^k$ is exemplary and t-intersecting then*

$$|\mathcal{F}| \le \max_{G \in \mathcal{G}(t+2r,t)} \frac{(n-t-2r)!}{(n-k)!} \cdot \sum_{i=t}^{t+2r} \left|G^{(i)}\right| \cdot d(t+2r-i, n-i)$$

where $r \in \{0,1,\ldots,(k-t)/2\}$.

Proof. Let $\mathcal{F}$ be a maximum exemplary t-intersecting subset of $\mathcal{I}_n^k$. For each $w \in \mathcal{F}$, $\mathrm{Fix}(w)$ contains some element Y of $\mathcal{M}(\mathcal{F})$, and $Y \subseteq [t+2r]$ for some $r \in \{0,1,\ldots,(k-t)/2\}$ by Proposition 4.3.2. Thus setting

$$\mathrm{Fix}(w) \cap [t+2r] = X \tag{4.4.2}$$

we have $X \subseteq [t+2r]$ and $X \supseteq Y$ for some $Y \in \mathcal{M}(\mathcal{F})$. Hence upon defining

$$H(\mathcal{F}) = \{\, X \subseteq [t+2r] : X \supseteq Y, Y \in \mathcal{M}(\mathcal{F}) \,\},$$

it follows that

$$\mathcal{F} \subseteq \bigcup_{X \in H(\mathcal{F})} \left\{\, w \in \mathcal{I}_n^k : \mathrm{Fix}(w) \cap [t+2r] = X \,\right\}.$$

In fact, we have equality here: if $w \in \mathcal{I}_n^k$ satisfies (4.4.2) for some $X \in H(\mathcal{F})$, then its fixed point set $\mathrm{Fix}(w)$ contains some $Y \in \mathcal{M}(\mathcal{F})$. This implies that w t-intersects all elements of $\mathcal{F}$, and therefore $w \in \mathcal{F}$ since $\mathcal{F}$ is maximal. We conclude that

$$\mathcal{F} = \bigcup_{X \in H(\mathcal{F})} \left\{\, w \in \mathcal{I}_n^k : \mathrm{Fix}(w) \cap [t+2r] = X \,\right\}$$

and note that this union is disjoint.

For elements X of $H(\mathcal{F})$, we have

$$\left|\left\{\, w \in \mathcal{I}_n^k : \mathrm{Fix}(w) \cap [t+2r] = X \,\right\}\right| = d(t+2r-|X|, n-|X|) \cdot \frac{(n-t-2r)!}{(n-k)!}.$$

Moreover, $H(\mathcal{F})$ is t-intersecting, so elements of $H(\mathcal{F})$ have size $t \le |X| \le t+2r$, giving

$$|\mathcal{F}| = \frac{(n-t-2r)!}{(n-k)!} \cdot \sum_{i=t}^{t+2r} \left|H(\mathcal{F})^{(i)}\right| \cdot d(t+2r-i, n-i)$$

Since each element of $H(\mathcal{F})$ contains an element of the t-intersecting set family $\mathcal{M}(\mathcal{F})$, the maximum value of this sum over all possible sets of t-intersecting subsets of $[t+2r]$ is an upper bound. $\square$

Let

$$G^* = \{\, X \subseteq [t+2r] : |X| \ge t+r \,\}.$$

We would like to show that the expression in Proposition 4.4.1 is maximised by the saturation family G^*. For this we need the following proposition, a consequence of results proved in [AK98].

Proposition 4.4.3. *(Ahlswede, Khachatrian [AK98]).*

Let $\gamma_t, \gamma_{t+1}, \ldots, \gamma_{t+2r} \in \mathbb{R}^+$ *be such that*

$$\frac{\gamma_i}{\gamma_{i+1}} \leq n-1 \;\; \textit{for} \;\; i = t, \ldots, t+2r-1.$$

Then

$$\max_{G \in \mathcal{G}(t+2r,t)} \sum_{i=t}^{t+2r} \left| G^{(i)} \right| \cdot \gamma_i$$

is attained at $G = G^*$.

Proof. This follows from Lemma 7 on page 443 and the corollary on page 446 together with the concluding work on pages 447-8 of [AK98]. □

4.4.1 The Bound on Exemplary Families

At last, we are now in a position to prove the main result of this chapter.

Theorem 4.4.4. *If* $\mathcal{F}$ *is an exemplary* t*-intersecting subset of* $\mathcal{I}_n^k$ *with* $1 \leq k < n$ *then*

$$|\mathcal{F}| \leq \max_{0 \leq r \leq (k-t)/2} |\mathcal{K}_r| \,.$$

Proof. Recall that $\mathcal{K}_r$ consists of the injections in $\mathcal{I}_n^k$ which fix at least $t+r$ of the first $t+2r$ points. So $\mathcal{K}_r$ is t-intersecting with

$$\mathcal{M}(\mathcal{K}_r) = \{\, X \subseteq [t+2r] : |X| = t+r \,\} \,.$$

Then, using the language and arguments of the proof of Proposition 4.4.1, we have $H(\mathcal{K}_r) = G^*$ and

$$\mathcal{K}_r \subseteq \bigcup_{X \in G^*} \left\{\, w \in \mathcal{I}_n^k : \mathrm{Fix}(w) \cap [t+2r] = X \,\right\} \subseteq \mathcal{K}_r,$$

giving

$$|\mathcal{K}_r| = \frac{(n-t-2r)!}{(n-k)!} \cdot \sum_{i=t}^{t+2r} \left| G^{*(i)} \right| \cdot d(t+2r-i, n-i).$$

Thus by Propositions 4.4.1 and 4.4.3, to prove Theorem 4.4.4 all that remains to be shown is

$$\frac{d(t+2r-i, n-i)}{d(t+2r-i-1, n-i-1)} \leq n-1 \;\; \text{for} \;\; i = t, \ldots, t+2r-1. \tag{4.4.5}$$

Recall that $d(a,b)$ is the number of injections from $[a]$ to $[b]$ with no fixed points. Therefore the denominator in (4.4.5) is non-zero if, and only if, not both of its arguments are equal to 1. When $t+2r-i-1=1$ we have $i=t+2r-2$, and substituting this value of i into $n-i-1=1$ gives

$$n = t + 2r \leq t + (k-t) = k.$$

Since $k < n$ we conclude that $d(t+2r-i-1, n-i-1) \neq 0$ for $i = t, \ldots, t+2r-1$.

Using Lemma 3.2.6(1), the left hand side of (4.4.5) becomes

$$\begin{aligned}
& \frac{d(t+2r-i, n-i-1) + (t+2r-i)d(t+2r-i-1, n-i-1)}{d(t+2r-i-1, n-i-1)} \\
= \; & \frac{d(t+2r-i, n-i-1)}{d(t+2r-i-1, n-i-1)} + (t+2r-i) \\
\leq \; & \frac{(n-t-2r)d(t+2r-i-1, n-i-1)}{d(t+2r-i-1, n-i-1)} + (t+2r-i) \\
= \; & n-i
\end{aligned}$$

by Lemma 3.2.6(2). Hence

$$\frac{d(t+2r-i, n-i)}{d(t+2r-i-1, n-i-1)} \leq n-i \leq n-t \leq n-1$$

as required. $\square$

4.4.2 The Saturation Constant

In view of Theorem 4.4.4 we would like to find a function which, given t, k and n, returns the value of r which yields the largest saturation family $\mathcal{K}_r$. The following result makes considerable progress in this direction: our computational evidence confirms that for $1 \leq t \leq k \leq 30$, the optimal r is given by the largest $r \in \{0, 1, \ldots, (k-t)/2\}$ satisfying (4.4.7).

Proposition 4.4.6. *If $\mathcal{F}$ is a maximum exemplary t-intersecting subset of $\mathcal{I}_n^k$ then* $\max(\mathcal{M}(\mathcal{F})) = 2r + t$ *where $r \in \{0, 1, \ldots, (k-t)/2\}$ satisfies*

$$\begin{aligned}
(2r+t-1) \cdot \sum_{i=0}^{r} (-1)^i \binom{r}{i} (n-r-t-i)! \geq \\
r \cdot \sum_{i=0}^{r} (-1)^i \binom{r}{i} (n-r-t+1-i)!. \qquad (4.4.7)
\end{aligned}$$

Proof. By Proposition 4.3.2 it suffices to show that r satisfies (4.4.7). Proceeding similarly to the proof of Proposition 4.3.2, we will therefore construct another t-intersecting subset of $\mathcal{I}_n^k$ from $\mathcal{F}$, and Inequality 4.4.7 will follow from the fact that this new family cannot be larger than $\mathcal{F}$. So let $\max(\mathcal{M}(\mathcal{F})) = l$ as before and recall the definitions of $\mathcal{M}_0$, $\mathcal{M}_1$, $\mathcal{M}_0^{(i)}$ and $\widehat{\mathcal{M}_0^{(i)}}$ from page 65. Proposition 4.3.2 implies that $\mathcal{M}_0 = \mathcal{M}_0^{(t+l)/2}$, or

$$\mathcal{M}(\mathcal{F}) = \mathcal{M}_0^{(t+l)/2} \cup \mathcal{M}_1.$$

All elements X of $\widehat{\mathcal{M}_0^{(t+l)/2}}$ are subsets of $[l-1]$ of size $|X| = \frac{t+l}{2} - 1$. Thus using the pigeonhole principle, there exists $p \in [l-1]$ which is contained in at most

$$\frac{\left|\widehat{\mathcal{M}_0^{(t+l)/2}}\right| \cdot |X|}{l-1}$$

members of $\widehat{\mathcal{M}_0^{(t+l)/2}}$. Fix such a p and let $\mathcal{Q}$ be the set of elements of $\widehat{\mathcal{M}_0^{(t+l)/2}}$ which do *not* contain p. Then

$$\begin{aligned} |\mathcal{Q}| &\geq \left|\widehat{\mathcal{M}_0^{(t+l)/2}}\right| - \frac{\left|\widehat{\mathcal{M}_0^{(t+l)/2}}\right| \cdot |X|}{l-1} \\ &= \left|\widehat{\mathcal{M}_0^{(t+l)/2}}\right| \cdot \left(1 - \frac{1}{l-1} \cdot \left(\frac{t+l}{2} - 1\right)\right) \\ &= \left|\widehat{\mathcal{M}_0^{(t+l)/2}}\right| \cdot \frac{2(l-1) - t - l + 2}{2(l-1)} = \left|\widehat{\mathcal{M}_0^{(t+l)/2}}\right| \cdot \frac{l-t}{2(l-1)}. \end{aligned}$$

In order to show that $\mathcal{Q}$ is t-intersecting, let $X_1, X_2 \in \mathcal{Q}$ and consider the sets $X_1^* = X_1 \cup \{l\}$ and $X_2^* = X_2 \cup \{l\}$ which are elements of $\mathcal{M}_0$. Now $p < l$ and neither X_1^* nor X_2^* contain p, but both contain l, so by Lemma 4.3.1,

$$|X_1^* \cap X_2^*| \geq t + 1.$$

It follows that $|X_1 \cap X_2| \geq t$ and so $\mathcal{Q}$ is t-intersecting. Note also that $\mathcal{M}(\mathcal{F})$ is t-intersecting and elements of $\mathcal{M}_1$ do not contain l. Hence

$$\mathcal{N} = \mathcal{M}_1 \cup \mathcal{Q}$$

is t-intersecting which implies that $\mathcal{V}(\mathcal{N}) \subseteq \mathcal{I}_n^k$ is t-intersecting.

By Lemma 4.2.1, we have $\mathcal{F} = \mathcal{V}(\mathcal{M}(\mathcal{F}))$, so

$$\mathcal{F} = G_1 \cup G_2$$

where G_2 consists of the injections generated by nothing other than $\mathcal{M}_0^{(t+l)/2}$ and G_1 contains the words generated by the rest of $\mathcal{M}(\mathcal{F})$:

$$G_1 = \mathcal{V}(\mathcal{M}_1) = \mathcal{V}\left(\mathcal{M}(\mathcal{F}) \setminus \mathcal{M}_0^{(t+l)/2}\right),$$

$$G_2 = \bigcup_{X \in \mathcal{M}_0^{(t+l)/2}} \left(\mathcal{V}(X) \setminus \mathcal{V}\left(\mathcal{M}(\mathcal{F}) \setminus X\right)\right).$$

Similarly,

$$\mathcal{V}(\mathcal{N}) = G_1 \cup G_3$$

where

$$G_3 = \mathcal{V}(\mathcal{Q}) \setminus \mathcal{V}(\mathcal{M}_1).$$

Clearly G_2 is a disjoint union. Therefore

$$\begin{aligned} |G_2| &= \sum_{X \in \mathcal{M}_0^{(t+l)/2}} \left|\mathcal{V}(X) \setminus \mathcal{V}\left(\mathcal{M}(\mathcal{F}) \setminus X\right)\right| \\ &= \sum_{X \in \mathcal{M}_0^{(t+l)/2}} d(\max(X) - |X|, n - |X|) \cdot \frac{(n - \max(X))!}{(n-k)!} \end{aligned}$$

by Lemma 4.2.3 since $\max(X) = \max(\mathcal{M}(\mathcal{F}))$ for all $X \in \mathcal{M}_0^{(t+l)/2}$. That is,

$$\begin{aligned} |G_2| &= \sum_{X \in \mathcal{M}_0^{(t+l)/2}} d\left(l - \frac{t+l}{2}, n - \frac{t+l}{2}\right) \cdot \frac{(n-l)!}{(n-k)!} \\ &= \left|\mathcal{M}_0^{(t+l)/2}\right| \cdot d\left(\frac{l-t}{2}, n - \frac{t+l}{2}\right) \cdot \frac{(n-l)!}{(n-k)!}. \end{aligned}$$

We establish a lower bound on $|G_3|$ by examining a collection of its subsets. For $X \in \mathcal{Q}$, let

$$C(X) = \left\{ w \in \mathcal{I}_n^k : \mathrm{Fix}(w) \cap [l-1] = X \right\}.$$

Two facts follow immediately from this definition:

$$\begin{aligned} |C(X)| &= d(l - 1 - |X|, n - |X|) \cdot \frac{(n-l+1)!}{(n-k)!} \\ &= d\left(\frac{l-t}{2}, n - \frac{t+l}{2} + 1\right) \cdot \frac{(n-l+1)!}{(n-k)!} \end{aligned} \tag{4.4.8}$$

and

$$X_1 \neq X_2 \quad \Rightarrow \quad C(X_1) \cap C(X_2) = \emptyset. \tag{4.4.9}$$

To show that $C(X) \subseteq G_3$ for all $X \in \mathcal{Q}$, let $w \in C(X)$. Certainly $w \in \mathcal{V}(\mathcal{Q})$, so we need to show that $w \notin \mathcal{V}(\mathcal{M}_1)$. Suppose then, for a contradiction, that there exists $Y \in \mathcal{M}_1$ such that $\mathrm{Fix}(w) \supseteq Y$. Since $Y \in \mathcal{M}_1$, Y is a subset of $[l-1]$. But $\mathrm{Fix}(w) \cap [l-1] = X$ which forces $Y \subseteq X$. This contradicts the fact that $X \cup \{l\}$ is a minimal element of $\mathrm{Fix}(\mathcal{F})$ and so we conclude $C(X) \subseteq G_3$.

Combining this with (4.4.8) and (4.4.9) gives

$$|G_3| \geq \sum_{X \in \mathcal{Q}} |E(X)| \; = \; |\mathcal{Q}| \cdot d\left(\frac{l-t}{2}, n - \frac{t+l}{2} + 1\right) \cdot \frac{(n-l+1)!}{(n-k)!}.$$

Since $\mathcal{F}$ is maximum we have $|\mathcal{F}| \geq |\mathcal{V}(\mathcal{N})|$. This requires $|G_2| \geq |G_3|$, giving

$$\begin{aligned}
\left|\mathcal{M}_0^{(t+l)/2}\right| \cdot d\left(\frac{l-t}{2}, n-\frac{t+l}{2}\right) \cdot \frac{(n-l)!}{(n-k)!} &\geq \\
|\mathcal{Q}| \cdot d\left(\frac{l-t}{2}, n-\frac{t+l}{2}+1\right) \cdot \frac{(n-l+1)!}{(n-k)!} &\geq \\
\left|\widehat{\mathcal{M}_0^{(t+l)/2}}\right| \cdot \frac{l-t}{2(l-1)} \cdot d\left(\frac{l-t}{2}, n-\frac{t+l}{2}+1\right) \cdot \frac{(n-l+1)!}{(n-k)!}&.
\end{aligned}$$

Since $\left|\mathcal{M}_0^{(t+l)/2}\right| = \left|\widehat{\mathcal{M}_0^{(t+l)/2}}\right| \neq 0$ this simplifies to

$$d\left(\frac{l-t}{2}, n-\frac{t+l}{2}\right) \quad \geq \quad \frac{l-t}{2(l-1)} \cdot d\left(\frac{l-t}{2}, n-\frac{t+l}{2}+1\right) \cdot (n-l+1).$$

Writing this in terms of r we obtain

$$d(r, n-r-t) \quad \geq \quad \frac{r}{2r+t-1} \cdot d(r, n-r-t+1) \cdot (n-2r-t+1).$$

By Lemma 3.2.5 this is equivalent to

$$\begin{aligned}
(2r+t-1) \cdot \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{(n-r-t-i)!}{(n-2r-t)!} &\geq \\
r \cdot \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{(n-r-t+1-i)!}{(n-2r-t+1)!} \cdot (n-2r-t+1)& \\
= r \cdot \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{(n-r-t+1-i)!}{(n-2r-t)!}&
\end{aligned}$$

and the result follows. □

In this chapter, we have shown that for $k < n$, $\mathcal{K}_r$ has largest possible size among the exemplary t-intersecting subsets of $\mathcal{I}_n^k$, where r satisfies (4.1.1) and (4.4.7). To determine the optimal r completely, we would need to demonstrate that it is given by the *largest* non-negative integer satisfying these two inequalities. We begin the following chapter by summarising our conjectures.

CHAPTER 5

TOWARDS A COMPLETE CLASSIFICATION

5.1 Structural Conjectures

In the previous chapter it was proven that the size of an exemplary t-intersecting set of injections from $[k]$ to $[n]$ is bounded above by

$$\max_{0 \leq r \leq (k-t)/2} |\mathcal{K}_r|,$$

where $k < n$ and r satisfies (4.4.7). In fact, we conjecture that $\mathcal{K}_r$ is the only maximum exemplary t-intersecting subset of $\mathcal{I}_n^k$, up to permutations of the saturation set and its image points, and that this holds for $k = n$ also:

Conjecture 5.1.1. *Let $t \leq k \leq n$ be natural numbers with $n \geq 8$.*

If $\mathcal{F}$ is a maximum exemplary t-intersecting subset of $\mathcal{I}_n^k$ then $\mathcal{F}$ is equivalent to $\mathcal{K}_r$ where r is the largest integer in $\{0, 1, \ldots, (k-t)/2\}$ satisfying (4.4.7).

We require $n \geq 8$ since for $n \in \{6, 7\}$, some exceptions occur: when $n = 6$, $k \in \{5, 6\}$ and $t = 3$, both $\mathcal{K}_0$ and $\mathcal{K}_1$ are optimal, see page 55.

For $n = k = 7$ and $t = 3$, the largest t-intersecting subset of $\mathcal{I}_n^k$ is $\mathcal{K}_0$. Here $r = 0$ is the largest element of $\{0, 1, 2 = (k-t)/2\}$ satisfying *strict* inequality in (4.4.7). Now $r = 2$ gives equality in (4.4.7) but $|\mathcal{K}_2| < |\mathcal{K}_0|$ and so this case does not fit the conjecture, since we need (4.4.7) to be non-strict otherwise.

5.1.1 Sufficient Conditions

Chapter 4 made considerable progress towards Conjecture 5.1.1. By Propositions 4.3.2 and 4.4.6, the optimal r is an element of $\{0, 1, \ldots, (k-t)/2\}$ satisfying (4.4.7), though we have not proved that the largest r satisfying these restrictions is the optimal one. Moreover, Proposition 4.3.2 makes further progress towards Conjecture 5.1.1 by showing that all elements of

$$\mathcal{M}_0 = \{\, X \in \mathcal{M}(\mathcal{F}) : t + 2r \in X \,\}$$

have size $t+r$ and are subsets of $[2r+t]$. This means that all elements of $\mathcal{F}$ which are generated by elements of $\mathcal{M}_0$ fix at least $t+r$ of the first $t+2r$ positions and are thus elements of $\mathcal{K}_r$. Extending the result of Proposition 4.3.2 about the size of elements of $\mathcal{M}_0$ to hold for *all* elements of $\mathcal{M}(\mathcal{F})$ would thus prove Conjecture 5.1.1.

In other words, it would suffice to show that $\mathcal{M}(\mathcal{F})$ is the set of minimal elements of the Katona family from Theorem 1.2.2 to prove the conjecture: since $l+t$ is even, it follows from [Kat64] that the set

$$\left\{ X \subseteq [l] \ : \ |X| \geq \frac{l+t}{2} \right\}$$

is a maximum t-intersecting family of subsets of $[l]$.

5.1.2 Conjectures on the Optimality of Fixing

At the end of [CK03], Cameron & Ku asked the following question:

> Given $t \geq 1$, is there a number $n_0(t)$ such that, if $n > n_0(t)$, then a t-intersecting subset of $\mathcal{S}_n$ has cardinality at most $(n-t)!$, and that a set meeting the bound is a coset of the stabiliser of t points?

This is a long-standing conjecture of Deza & Frankl from [DF77]. Since Corollary 3.2.3 does not apply to the case $k = n$, we have not answered this question. However, we make the following conjecture. It is based on computational comparisons of $|\mathcal{K}_r|$ in [GAP07] for

$$0 \leq r \leq \frac{k-t}{2}, \ \ 1 \leq t \leq k \leq n \leq 120.$$

Conjecture 5.1.2. *For integers t, n with $1 \leq t \leq n$ and $n > 6$, the following are equivalent:*

- *A t-intersecting subset of $\mathcal{S}_n$ has cardinality at most $(n-t)!$, and a set meeting the bound is a coset of the stabiliser of t points.*

- *t does not lie in the interval $[n/2, n-3]$.*

Our corresponding conjecture for injections is very similar:

Conjecture 5.1.3. *For integers t, k, n with $1 \leq t \leq k < n$ and $n > 6$, the following are equivalent:*

- *A t-intersecting subset of $\mathcal{I}_n^k$ has cardinality at most $(n-t)!/(n-k)!$, and a set meeting the bound is equivalent to*
$$\mathcal{K}_0 = \{\textit{injections from } [k] \textit{ to } [n] \textit{ which fix all elements of } [t]\}.$$
- *t does not lie in the interval $[n/2, k-2]$.*

As was noted in our discussion of Figure 4.1.1, the difference between Conjectures 5.1.2 and 5.1.3 corresponds to the difference between Theorems 2.2.2 and 3.2.11.

5.2 Removing the Exemplary Restriction

Chapter 4 focussed on exemplary families, but we expect Conjecture 5.1.1 to hold for arbitrary families: we should be able to transform any maximum t-intersecting injection family into an exemplary one, without changing the size of the original family. Indeed, this assumption lies behind Conjectures 5.1.2 and 5.1.3.

5.2.1 Standardising Injection Families

Recall that a t-intersecting subset $\mathcal{F}$ of $\mathcal{I}_n^k$ is exemplary if it is maximal under set inclusion and $\mathrm{Fix}(\mathcal{F})$ is t-intersecting and left-compressed. Thus we would like to standardise an arbitrary maximal t-intersecting subset $\mathcal{F}$ of $\mathcal{I}_n^k$ as follows: first, we require a map $\mathcal{T}$ to transform $\mathcal{F}$ into the family $\mathcal{T}(\mathcal{F})$ whose elements fix 'as many points in $[k]$ as possible', to ensure that $\mathrm{Fix}(\mathcal{F})$ is t-intersecting. We would then apply a left-compression map $\mathcal{L}$ to $\mathcal{T}(\mathcal{F})$ to ensure that in the resulting subset of $\mathcal{I}_n^k$, the positions relevant to the t-intersection property occur at the beginning of each word. That is, elements of $\mathcal{L}(\mathcal{T}(\mathcal{F}))$ should t-intersect not just anywhere, but in their first l positions, where l is specified in Proposition 4.3.2. Transformations similar to $\mathcal{T}$ or $\mathcal{L}$ have been applied to various combinatorial structures by Kleitman [Kle66b], Frankl & Füredi [FF80], Ahlswede & Khachatrian [AK98], Cameron & Ku [CK03] and many others since the 1960s. For a survey of 'The shifting technique in extremal set theory' see [Fra87].

A Fixing Operation

To remove the restriction of Theorem 4.4.4 to exemplary families, let us concentrate first on finding a map $\mathcal{T}$ which turns an arbitrary t-intersecting family into one with t-intersecting fixed point set. In other words, to ensure that we can use a set of sets to accurately represent a set of injections, we need to permute the image point labels of the injections in such a way as to ensure that the injections fix 'as many points as possible'. For this purpose, we introduce a fixing operation, based on traditional shifting maps, which is formally specified in Definition 5.2.1. Intuitively, for $x \in [n]$ and $w \in \mathcal{I}_n^k$, we obtain the injection $f(w, x)$ from w as follows: no changes are made if x is already fixed under w, or cannot be fixed because it is not an element of the domain. If no point maps to x under w, then we may fix x without having to make any further changes. Finally, if some point $y \in [k]$, distinct from x, maps to x, then we swap the images of y and x.

To formalise this fixing operation, we use the image notation for injections: in (5.2.2) and (5.2.3), the image point is given underneath the corresponding domain point.

Definition 5.2.1. Let $x \in [n]$ and $w \in \mathcal{I}_n^k$.

- If either $x \leq k$ and $w(x) = x$, or if $x > k$, then $f(w, x) = w$.
- If $x \leq k$ and $x \notin \operatorname{im}(w)$, then

$$f(w, x) = \begin{pmatrix} x & \lambda \\ x & w(\lambda) \end{pmatrix}, \quad \lambda \in [k] \setminus \{x\}. \tag{5.2.2}$$

- If $x \leq k$ and $w(y) = x$ for some $y \in [k]$ with $y \neq x$, then

$$f(w, x) = \begin{pmatrix} x & y & \lambda \\ x & w(x) & w(\lambda) \end{pmatrix}, \quad \lambda \in [k] \setminus \{x, y\}. \tag{5.2.3}$$

It is fairly easy to see that $f(w, x)$ is an injection in $\mathcal{I}_n^k$ which fixes x.

The powerful technique of fixing operations was introduced by Kleitman in [Kle66a] to prove that the size of the union of m intersecting families of subsets of $[n]$ is at most $2^n - 2^{n-m}$. Our map f on injections combines previous fixing maps for words and permutations: the naive 'insertion' of the second case is based on fixing maps for words in e.g. [Kle66b, AK98], while the swapping map for the permutation case is taken straight from Cameron & Ku's paper [CK03].

We may apply a sequence of fixing operations by using the inductive definition

$$f(w; x_1, \ldots, x_q) = f(f(w; x_1, \ldots, x_{q-1}), x_q).$$

If S is a subset of $\mathcal{I}_n^k$ such that $f(w, x) \in S$ for all $x \in [n]$ and $w \in S$, then we say that S is *closed under the fixing operation*.

The following result is standard in the study of t-intersecting families of combinatorial structures other than sets, see for instance [FF80, AK98, CK03]. It shows how fixed point sets together with the fixing operation may enable us to build on the theory of t-intersecting set families.

Theorem 5.2.4. *If $\mathcal{F}$ is a t-intersecting subset of $\mathcal{I}_n^k$ which is closed under the fixing operation then* $\mathrm{Fix}(\mathcal{F})$ *is t-intersecting.*

Proof. Suppose $\mathrm{Fix}(\mathcal{F})$ is not t-intersecting. Then there exist $v, w \in \mathcal{F}$ with $|\mathrm{Fix}(v) \cap \mathrm{Fix}(w)| < t$. Note that

$$\mathrm{int}(v, w) = \{x_1, x_2, \dots, x_s\}$$

has size $t > 0$ and that $u = f(v; x_1, \dots, x_s) \in \mathcal{F}$ since $\mathcal{F}$ is closed under the fixing operation. We will show that u cannot t-intersect w.

First we consider positions $y \in [k] \setminus \mathrm{int}(v, w)$. It follows from Definition 5.2.1 that for an injection $a \in \mathcal{I}_n^k$ and points $x, z \in [k]$, if the images of z under a and $f(a; x)$ are different, then we must have either $z = x$ or $a(z) = x$. Thus unless v maps y to one of the points x_i which we are trying to fix, the image of y remains unchanged: if $v(y) \notin \mathrm{int}(v, w)$ then

$$u(y) = v(y) \neq w(y)$$

since $y \notin \mathrm{int}(v, w)$, as claimed.

If on the other hand $v(y) \in \mathrm{int}(v, w)$, say $v(y) = x_l$, then

$$\begin{aligned} f(v; x_1, \dots, x_{l-1})(y) &= v(y) = x_l, \\ f(v; x_1, \dots, x_l)(y) &= f(v; x_1, \dots, x_{l-1})(x_l). \end{aligned}$$

Now whether or not the image of y is changed again under the fixing operation depends on whether or not $f(v; x_1, \dots, x_l)(y)$ is one of the elements of $\mathrm{int}(v, w)$ which have not yet been fixed. In any case, we end up with $u(y) = v(z)$ for some $z \in \mathrm{int}(v, w)$. Therefore

$$\begin{aligned} u(y) &= v(z) \ \text{ for some } z \in \mathrm{int}(v, w) \\ &= w(z) \ \text{ by definition of } \mathrm{int}(v, w) \\ &\neq w(y) \end{aligned}$$

since $y \notin \mathrm{int}(v, w)$ implies $y \neq z$. We have shown that u and w do not intersect in positions $y \in [k] \setminus \mathrm{int}(v, w)$.

Finally, suppose $u(x_i) = w(x_i)$ for some $i \in [s]$. Then since u fixes all elements of $\mathrm{int}(v, w)$, we have

$$x_i = u(x_i) = w(x_i) = v(x_i)$$

because $x_i \in \mathrm{int}(v, w)$. Since $|\operatorname{Fix}(v) \cap \operatorname{Fix}(w)| < t$, this can occur for at most $t - 1$ values of i.

Hence u and w do not t-intersect, so the result follows from this contradiction to the t-intersection property of $\mathcal{F}$. □

A Left-Compression Map

Similarly to our work in the previous section, one can define left-compression maps $l_{i,j}$ on elements of $\mathcal{F}$ which enable us to prove a result analogous to Theorem 5.2.4: namely that t-intersecting injection families which are closed under these left-compression maps have left-compressed fixed point sets.

Intuitively, for $w \in \mathcal{I}_n^k$ and $i, j \in [k]$ with $i < j$, the injection $l_{i,j}(w)$ is obtained from w as follows: if w either fixes i or does not fix j, then we simply set $l_{i,j}(w) = w$. On the other hand, if j is fixed under w and i is not, then the left compression of w differs from w as follows: $l_{i,j}(w)$ fixes i and maps j to the image of i under w. The fixing of i under $l_{i,j}(w)$ is achieved in the same way as the fixing of x under $f(w, x)$: if no domain point maps to i then we can simply insert it; if some $p \in [k]$ maps to i under w, then $l_{i,j}(w)$ maps p to j instead. This last situation can be illustrated in image notation as follows:

$$l_{i,j} : \begin{pmatrix} p & i & j & \lambda \\ i & w(i) & j & w(\lambda) \end{pmatrix} \mapsto \begin{pmatrix} p & i & j & \lambda \\ j & i & w(i) & w(\lambda) \end{pmatrix}, \quad \lambda \in [k] \setminus \{p, i, j\}.$$

Unfortunately, this traditional approach leads to considerable difficulties later in the process, as we will see in the next section.

5.2.2 Traditional Shifting Maps in the Injection Setting

Many of the results on injections in this thesis, such as Theorems 4.4.4 and 5.3.9, are obtained by examining the basis $\mathcal{M}(\mathcal{F})$ associated with an exemplary injection family $\mathcal{F}$. Thus we would like to use Theorem 5.2.4 to show that any t-intersecting injection family $\mathcal{F}$ has a t-intersecting fixed point set, and proceed similarly with regards to left-compression. However, it is very difficult to show that any t-intersecting injection family can be sensibly mapped into one which is closed under either the fixing or the left-compression maps. In this section, we illustrate these challenges by discussing potential fixing maps.

Potential Fixing Maps

We would like to define a map $\mathcal{T}$ on t-intersecting injection families $\mathcal{F}$ such that $\mathcal{T}(\mathcal{F})$ is t-intersecting, closed under the fixing operation, and has size $|\mathcal{F}|$. Considering previous work in the area, such a map $\mathcal{T}$ is likely to be the result of sequential applications of maps T_i focussed on specific positions in $[k]$. As a first attempt to define the T_i, it seems most natural to stay as close as possible to the shifting maps which work for set families: for $w \in \mathcal{F}$, define

$$T_i(w) = \begin{cases} f(w,i) & f(w,i) \notin \mathcal{F} \\ w & \text{otherwise} \end{cases}$$

and set $T_i(\mathcal{F}) = \{\, T_i(w) : w \in \mathcal{F} \,\}$.

It is not difficult to show that if $\mathcal{F}$ is t-intersecting, then $T_i(\mathcal{F})$ is t-intersecting also. However, T_i does not preserve the cardinality of $\mathcal{F}$ in general since we may have $f(v,i) = f(w,i) \notin \mathcal{F}$ for distinct elements v, w of $\mathcal{F}$. For example,

$$F_0 = \{213, 214, 243, 413\}$$

is a maximal 1-intersecting subset of $\mathcal{I}_4^3$ with

$$T_1(F_0) = \{123, 124, 143\}$$

since $f(243,1) = 143 = f(413,1)$. Thus T_1 does not preserve the size of F_0.

In an attempt to fix this problem, we define a map which sequentially replaces elements w of $\mathcal{F}$ by $f(w,i)$ unless doing so would reduce the size of $\mathcal{F}$: for $w \in \mathcal{F}$ let

$$seq_{\mathcal{F}} T_i(w) = \begin{cases} f(w,i) & f(w,i) \notin \mathcal{F} \\ w & \text{otherwise} \end{cases}.$$

Recall that elements of $\mathcal{I}_n^k$ can be regarded as words of length k over $[n]$ with no repeats. Thus we may label the elements of $\mathcal{F}$ in lexicographic order by $v_1, \ldots, v_{|\mathcal{F}|}$ and set

$$\mathcal{F}_1 = \mathcal{F} \setminus \{v_1\} \cup \{seq_{\mathcal{F}} T_i(v_1)\}, \quad \mathcal{F}_{j+1} = \mathcal{F}_j \setminus \{v_{j+1}\} \cup \{seq_{\mathcal{F}_j} T_i(v_{j+1})\}$$

for $j = 0, \ldots, |\mathcal{F}| - 1$. Finally, define $seqT_i(\mathcal{F}) = \mathcal{F}_{|\mathcal{F}|}$.

Clearly, $seqT_i$ preserves the cardinality of $\mathcal{F}$. However, it does not necessarily preserve the t-intersecting structure of a family of injections: considering F_0 once more, observe that

$$seqT_1(F_0) = \{123, 124, 143, 413\}$$

but 413 does not intersect 124.

Permutations

We have seen two maps which shift $\mathcal{F}$ into an injection family closed under the fixing operation. The first, more traditional map shifts one t-intersecting family into another, but fails to preserve the size of $\mathcal{F}$. The second map preserves the cardinality, but not the t-intersecting property of $\mathcal{F}$. Note that this problem cannot easily be fixed by restricting our attention to permutations, i.e. to the case $n = k$, as the following example shows. The set

$$E = \{24135, 41235, 42135, 43125, 45132\}$$

is a maximal 2-intersecting subset of $\mathcal{S}_5 = \mathcal{I}_5^5$. But $T_1(E)$ has size $|E| - 1$ since $f(24135, 1) = 14235 = f(41235, 1) \notin E$. Similarly, $seqT_1(E)$ is 1-intersecting but not 2-intersecting, since

$$seqT_1(E) = \{14235, 41235, 12435, 13425, 15432\}$$

and $|\operatorname{int}(41235, 15432)| = 1$. (Note that although the proof of our bound on exemplary families in Theorem 4.4.4 requires $k < n$, the remaining results in Chapter 4 apply to permutations as well as more general injections.)

Learning from Maps on Sets

So why do these maps work for sets? Let $\mathcal{A}$ be a t-intersecting set of subsets of $[n]$. Since $\mathcal{A}$ is already a set family, we do not need a fixing operation, but combinatorialists have used the following left-compression map to study intersecting set families since the publication of the Erdős-Ko-Rado paper [EKR61].

Let $1 \leq i < j \leq n$. To obtain $H_{ij}(\mathcal{A})$ from $\mathcal{A}$:

- for each $X \in \mathcal{A}$ containing j but not i,
- replace it by $X_{ij} = X \setminus \{j\} \cup \{i\}$,
- unless $X_{ij} \in \mathcal{A}$.

Then $H_{ij}(\mathcal{A})$ is t-intersecting. Furthermore, if $X, Y \in \mathcal{A}$ are both replaced by $X_{ij} = Y_{ij}$, then we must have $X = Y$. Thus $H_{ij}(\mathcal{A})$ has size $|\mathcal{A}|$.

So perhaps we need to consider more positions at once? Let $g \in \mathcal{F}$ with $g(j) = i$. Define a permutation $h_{ij}(g)$ by swapping the images of i and j in g, then $h_{ij}(g)$ fixes i. To obtain $H_{ij}(\mathcal{F})$ from $\mathcal{F}$, replace g by $h_{ij}(g)$ unless $h_{ij}(g) \in \mathcal{F}$; then $H_{ij}(\mathcal{F})$ has size $|\mathcal{F}|$. However, $H_{ij}(\mathcal{F})$ is not necessarily

t-intersecting: if a family F_1 contains 2134 and 2413 then F_1 does not contain $h_{12}(2134) = 1234$ since this permutation does not intersect 2413. However, 2413 is fixed under h_{12} and so $H_{12}(F_1)$ is not intersecting.

Conclusions

There are many other variations of these maps, but we were unable to find one with the desired properties. We conclude that standardising injection or permutation families remains a challenging problem. However, all is not lost: in [CK03], Cameron & Ku present a way of overcoming these challenges in the case $t = 1$ for permutations by using Latin Squares and graph theory. Their methods transfer to injections with large domains, enabling us to complete the classification of maximum 1-intersecting injection families from Chapter 3.

5.3 Classification of Maximum 1-Intersecting Families for Large Domains

Recall from Section 3.1.2 that if $\mathcal{F}$ is a maximal intersecting subset of $\mathcal{I}_n^k$ with $k \leq (n+1)/2$, then all words in $\mathcal{F}$ have a fixed position, or image point, in common. In order to establish the same result for arbitrary k, this section employs the approach of Cameron & Ku in [CK03], where it is proved that all elements of a maximum intersecting set of permutations from $\mathcal{S}_n$ have a fixed image point in common.

5.3.1 Cliques, Cocliques and Latin Squares

As this section shows, the arguments in [CK03] only require slight modifications to apply to injections with large domain sizes. We will examine the intersection structure of $\mathcal{I}_n^k$ from a graph theoretic point of view: an intersection of two injections will correspond to an edge in a graph with vertex set $\mathcal{I}_n^k$. We start with the relevant definitions:

- A *graph automorphism* is a bijection, between the vertex sets of two graphs, which preserves edges and non-edges. That is, two vertices are adjacent in the domain if, and only if, they are adjacent in the image.
- A graph is called *vertex-transitive* if any vertex can be mapped into any other by some graph automorphism.

- A *clique* in a graph is a set of vertices all of which are mutually adjacent.
- Dually, a *coclique* is a set of vertices in a graph none of which are mutually adjacent.

The idea of a clique-coclique bound is not new — for instance, it is the first lemma in [DF77]. Here we use the version of Proposition 5.3.1 stated and proved in [CK03].

Proposition 5.3.1. *(Deza, Frankl [DF77]; Cameron, Ku [CK03])*

Let C be a clique and A a coclique in a vertex-transitive graph on m vertices. Then $|C|\cdot|A| \leq m$ and equality implies that $|C \cap A| = 1$.

We now set out to find the sets C and A appropriate to our context.

Definition 5.3.2. A *Latin Square* of order n is an $n \times n$ array in which each row and each column contain each symbol $1, 2, \ldots, n$ precisely once.

Let $r_1, r_2, \ldots, r_n$ be the rows of some Latin Square $\mathcal{L}$ of order n. The *ith k-row of $\mathcal{L}$* is the word of length k obtained by taking the first k symbols of r_i.

Theorem 5.3.3. *If $\mathcal{F}$ is a maximum intersecting subset of $\mathcal{I}_n^k$ then $\mathcal{F}$ contains exactly one k-row of each Latin Square of order n.*

Proof. Form a graph Γ with vertex set $\mathcal{I}_n^k$ where vertices v and w are joined by an edge if, and only if, the words v and w intersect. Let permutations act on words in $\mathcal{I}_n^k$ by permuting the letters, as before. Then any permutation in the symmetric group $\mathcal{S}_n$ is a graph automorphism of Γ. Since $\mathcal{S}_n$ acts transitively on itself, it clearly acts transitively on $\mathcal{I}_n^k$, and so Γ is vertex-transitive.

Let $\mathcal{R}_k$ be the set of k-rows of some Latin Square of order n. Then $\mathcal{R}_k$ is a coclique of size n. On the other hand, $\mathcal{F}$ is a clique in Γ, and $|\mathcal{F}| = \frac{(n-1)!}{(n-k)!}$ by Theorem 3.1.1 since $\mathcal{F}$ is maximum. Thus

$$|\mathcal{R}_k| \cdot |\mathcal{F}| = n \cdot \frac{(n-1)!}{(n-k)!} = \frac{n!}{(n-k)!} = |\mathcal{I}_n^k|,$$

and we apply Proposition 5.3.1 to conclude $|\mathcal{R}_k \cap \mathcal{F}| = 1$. □

We need another two results before we can prove closure under the fixing operation (Theorem 5.3.7), and we simply quote these here.

For any word $w \in \mathcal{I}_n^k$, denote by $N(w)$ the set of all words in $\mathcal{I}_n^k$ which do not intersect with w.

Proposition 5.3.4. *(Cameron, Ku [CK03])*

Let s be an integer satisfying $2s \leq n$. Then for $g_1, g_2, \ldots, g_s \in \mathcal{S}_n$, we have $N(g_1) \cap N(g_2) \cap \cdots \cap N(g_s) \neq \emptyset$.

Theorem 5.3.5. ***(Hall 1945)*** *Every $k \times n$ Latin rectangle can be extended to an $n \times n$ Latin square.*

We need to formalise the correspondence between $\mathcal{I}_n^k$ and $\mathcal{S}_n$ more precisely, so let us introduce some notation. Recall that a word $w \in \mathcal{I}_n^k$ may be viewed as a bijection

$$w : [k] \to \{w(1), w(2), \ldots, w(k)\}.$$

It is then clear that w can be extended to a permutation in $\mathcal{S}_n$.

Definition 5.3.6. For $w \in \mathcal{I}_n^k$, a permutation $\sigma \in \mathcal{S}_n$ is called an *extension of w in $\mathcal{S}_n$* if $\sigma(i) = w(i)$ for all $i \in [k]$. Conversely, we may refer to w as the *restriction of σ to $\mathcal{I}_n^k$*.

5.3.2 Closure under the Fixing Operation

Note that applying permutations of $\mathcal{S}_n$ to a subset of $\mathcal{I}_n^k$ does not alter the cardinality or intersecting structure of that subset. Thus when considering an intersecting subset of $\mathcal{I}_n^k$, we can assume without loss of generality that it contains the identity $12\ldots k$.

Theorem 5.3.7. *If $n \geq 6$ and $\mathcal{F}$ is a maximum intersecting subset of $\mathcal{I}_n^k$ containing $12\ldots k$ then $\mathcal{F}$ is closed under the fixing operation.*

Proof. Suppose $\mathcal{F}$ is not closed under the fixing operation. Then there exist $x \in [n]$ and $w \in \mathcal{F}$ such that $f(w,x) \notin \mathcal{F}$, requiring $f(w,x) \neq w$. Thus $x \leq k$ and $w(x) \neq x$.

Let $g = w_1 \ldots w_x \ldots w_y \ldots w_n$ be an extension of w in $\mathcal{S}_n$ with $w_y = x$. (Note we are assuming $x < y$ without loss of generality, for simplicity of notation.) Then

$$f(g,x) = w_1 \ldots w_{x-1} w_y w_{x+1} \ldots w_{y-1} w_x w_{y+1} \ldots w_n$$

is an extension of $f(w,x)$ in $\mathcal{S}_n$.

Following the proof of Theorem 8 in [CK03], we consider two cases in turn. Note that if ϕ is a map from a set X to a set Y, we denote the restriction of ϕ to a subset $X' \subset X$ by $\phi|_{X'}$.

Case 1 $\quad w_x = y$.

Set $M = [n] \setminus \{x, y\}$, $\bar{id} = 12\ldots n|_M$ and $\bar{g} = g|_M = f(g,x)|_M$. Then $\bar{id}$ and $\bar{g}$ are elements of the symmetric group on M, $\mathrm{Sym}(M) \cong \mathcal{S}_{n-2}$. By Proposition 5.3.4, since $2 \cdot 2 \leq n-2$, there exists

$\bar{h} \in N(\bar{id}) \cap N(\bar{g})$. Consider the permutation $h \in \mathcal{S}_n$ given by

$$h(i) = \begin{cases} \bar{h}(i) & i \in M \\ y & i = x \\ x & i = y \end{cases}.$$

Note $f(g, x)$ and h form a $2 \times n$ Latin rectangle which, by Theorem 5.3.5, can be extended to some $n \times n$ Latin Square $\mathcal{L}$. We will show that $\mathcal{F}$ cannot contain any k-row of $\mathcal{L}$.

Recall that $f(g, x)|_{[k]} = f(w, x) \notin \mathcal{F}$ by assumption. Moreover, h does not intersect $12 \ldots n$ by construction, so $h|_{[k]}$ does not intersect $12 \ldots k$, which we assumed is in $\mathcal{F}$, giving $h|_{[k]} \notin \mathcal{F}$. Finally, for any row r of $\mathcal{L}$ other than $f(g, x)$ or h, we have $r \in N(f(g, x)) \cap N(h)$ which implies $r \in N(g)$. Thus $r|_{[k]} \in N(w)$, giving $r|_{[k]} \notin \mathcal{F}$ since $w \in \mathcal{F}$. Hence $\mathcal{F}$ does not contain any k-row of the Latin Square $\mathcal{L}$ of order n, which contradicts Theorem 5.3.3.

Case 2 $\quad w_x = z \neq y$.

We present an abbreviated version of the arguments on p. 884 of [CK03], giving details where we make the transfer from permutations to general injections.

Let $M = [n] \setminus \{x, z\}$ and let $\bar{id} = 12 \ldots n|_M$ denote the identity of $\mathrm{Sym}(M)$. Define another permutation $\bar{g}$ on M by

$$\bar{g}(i) = \begin{cases} g(i) & i \neq y \\ g(z) & i = y \end{cases}.$$

Again, $|M| = n - 2 \geq 4$, so by Proposition 5.3.4, there exists a permutation $\bar{h} \in \mathrm{Sym}(M)$ satisfying $\bar{h} \in N(\bar{id}) \cap N(\bar{g})$. From $\bar{h}$, we construct the permutation h_* on $[n]$ as

$$h_*(i) = \begin{cases} \bar{h}(i) & i \in M \\ z & i = x \\ x & i = z \end{cases},$$

and from $h_* \in \mathcal{S}_n$ we construct the permutation $h \in \mathcal{S}_n$ given by

$$h(i) = \begin{cases} h_*(i) & i \notin \{y, z\} \\ h_*(z) = x & i = y \\ h_*(y) = \bar{h}(y) & i = z \end{cases}.$$

By [CK03], $f(g, x)$ and h form a $2 \times n$ Latin rectangle, so by Theorem 5.3.5 there exists a Latin Square

$\mathcal{L}$ of order n containing $f(g,x)$ and h. Again, we wish to show that no k-row of $\mathcal{L}$ can be contained in $\mathcal{F}$.

Let r be any row in $\mathcal{L}$ other than $f(g,x)$ and h and recall that w is the restriction of g to $[k]$. Since $\mathrm{int}(g,h) = \{x,y\}$ and those are precisely the positions where g differs from $f(g,x)$, the fact that r intersects neither $f(g,x)$ nor h implies that r cannot intersect g. Thus $r|_{[k]}$ does not intersect w and so $w \in \mathcal{F}$ implies $r|_{[k]} \notin \mathcal{F}$.

Moreover $f(g,x)|_{[k]} = f(w,x)$ is not an element of $\mathcal{F}$ by assumption. Finally, h does not intersect $12\ldots n$: this is true for positions in M since $\bar{h} \in N(\bar{id})$. Also $h(x) = z \neq x$ and $h(z) = h_*(y)$ which is not equal to z since $h_*(x) = z$ and $x \neq y$. Thus $h|_{[k]}$ does not intersect $12\ldots k \in \mathcal{F}$, implying that $h|_{[k]}$ is not in $\mathcal{F}$. This constitutes the required contradiction to Theorem 5.3.3. □

We expect Theorem 5.3.7 to hold for $n < 6$ as well, but this is not required for the proof of our classification: small cases are checked separately in Section 5.3.4.

5.3.3 Injections with Large Images

The following lemma follows from the so-called LYM inequality; see [CK03] for details.

Lemma 5.3.8. *(Cameron, Ku [CK03])*

If $\mathcal{Z}$ is an antichain of subsets of a k-set such that $|A| \geq j$ for all $A \in \mathcal{Z}$ then

$$\sum_{A \in \mathcal{Z}} (k - |A|)! \leq k!/j!.$$

We are now in a position to classify the maximum intersecting subsets of $\mathcal{I}_n^k$ for $k > n/2$.

Theorem 5.3.9. *For $n/2 < k \leq n$, let $\mathcal{F}$ be a maximum intersecting subset of $\mathcal{I}_n^k$. Then all words in $\mathcal{F}$ have a fixed position in common.*

Proof. We noted previously that we may assume $12\ldots k \in \mathcal{F}$ without loss of generality. Moreover, if $k = n$ then Theorem 5.3.9 is equivalent to the main result of [CK03], so we assume $k < n$. Lastly, small values of k and n can be checked by an elementary case analysis (see Section 5.3.4), so we will assume within the proof that $n \geq 6$ and $k \geq 4$.

By Theorems 5.3.7 and 5.2.4, $\mathrm{Fix}(\mathcal{F})$ is intersecting. Moreover, $12\ldots k \in \mathcal{F}$ and so $\mathcal{M}(\mathcal{F})$ is a non-empty, intersecting antichain of subsets of $[k]$. We will establish bounds on the size of the elements of $\mathcal{M}(\mathcal{F})$. Since $\mathrm{Fix}(\mathcal{F})$ is intersecting, $\emptyset \notin \mathrm{Fix}(\mathcal{F})$. Moreover, if $\mathrm{Fix}(\mathcal{F})$ contains an element of size

1, then Theorem 5.3.9 follows by the intersection property of $\mathrm{Fix}(\mathcal{F})$. Thus we may assume that all elements of $\mathcal{M}(\mathcal{F})$ have size at least 2.

Pursuing a similar argument, if $Y = \bigcap_{X \in \mathcal{M}(\mathcal{F})} X$ is non-empty, then all elements of $\mathcal{F}$ fix all elements of Y, and Theorem 5.3.9 is immediate. We therefore assume $\bigcap_{X \in \mathcal{M}(\mathcal{F})} X = \emptyset$, implying $|\mathcal{M}(\mathcal{F})| \geq 2$. Since $\mathcal{M}(\mathcal{F})$ is an antichain of subsets of $[k]$, this gives $[k] \notin \mathcal{M}(\mathcal{F})$, and we have shown that all $X \in \mathcal{M}(\mathcal{F})$ satisfy $2 \leq |X| \leq k-1$.

For the remainder of this proof, the aim is to derive a contradiction to the assumption that $\mathcal{F}$ attains the bound given in Theorem 3.1.1, but there exists no $i \in [k]$ such that $w(i) = i$ for all $w \in \mathcal{F}$. As in [CK03], we consider two cases.

Case 1 *$\mathcal{M}(\mathcal{F})$ contains no element of size 2.*

By Lemma 4.2.1 we have

$$\begin{aligned} |\mathcal{F}| \cdot (n-k)! &\leq \sum_{X \in \mathcal{M}(\mathcal{F})} (n-|X|)! \\ &= \sum_{\substack{X \in \mathcal{M}(\mathcal{F}) \\ 3 \leq |X| \leq \lfloor k/2 \rfloor}} (n-|X|)! + \sum_{\substack{X \in \mathcal{M}(\mathcal{F}) \\ |X| > \lfloor k/2 \rfloor}} (n-|X|)! \\ &\leq \sum_{i=3}^{\lfloor k/2 \rfloor} |\mathcal{M}^{(i)}(\mathcal{F})|(n-i)! + \frac{n!}{(\lfloor k/2 \rfloor + 1)!} \end{aligned}$$

where $|\mathcal{M}^{(i)}(\mathcal{F})|$ is the number of elements in $\mathcal{M}(\mathcal{F})$ of size i. The inequality follows from Lemma 5.3.8 upon noting that $X \subseteq [k] \subset [n]$ for all $X \in \mathcal{M}(\mathcal{F})$. Using the Erdős-Ko-Rado Theorem 1.1.1, this inequality becomes

$$|\mathcal{F}| \cdot (n-k)! \quad \leq \quad \sum_{i=3}^{\lfloor k/2 \rfloor} \binom{k-1}{i-1} (n-i)! + \frac{n!}{(\lfloor k/2 \rfloor + 1)!}.$$

We are assuming that $|\mathcal{F}| = (n-1)!/(n-k)!$, so this gives

$$(n-1)! \quad \leq \quad \sum_{i=3}^{\lfloor k/2 \rfloor} \frac{(k-1)!(n-i)!}{(i-1)!(k-i)!} + \frac{n!}{(\lfloor k/2 \rfloor + 1)!}. \tag{5.3.10}$$

Let us denote the right hand side of (5.3.10) by $f(n,k)$.

To provide the required contradiction to (5.3.10), straightforward numerical calculation demonstrates that $f(n,k) < (n-1)!$ for $n < 16$, unless

$$(n,k) \in \{(6,4), (6,5), (7,4), (7,5), (8,5), (9,5)\}.$$

These special cases have been checked by a more involved recursive algorithm using a computer package [GAP07], see Section 5.3.4. For the remainder of Case 1, we therefore assume $n \geq 16$.

Since $k < n$, we have

$$\begin{aligned} f(n,k) &= \sum_{i=3}^{\lfloor k/2 \rfloor} \frac{(k-1)(k-2)\ldots(k-i+1)(n-i)!}{(i-1)!} + \frac{n!}{(\lfloor k/2 \rfloor + 1)!} \\ &< \sum_{i=3}^{\lfloor k/2 \rfloor} \frac{(n-1)(n-2)\ldots(n-i+1)(n-i)!}{(i-1)!} + \frac{n!}{(\lfloor k/2 \rfloor + 1)!} \\ &= (n-1)! \sum_{i=3}^{\lfloor k/2 \rfloor} \frac{1}{(i-1)!} + \frac{n!}{(\lfloor k/2 \rfloor + 1)!} \\ &\leq (n-1)! \sum_{i=3}^{\lfloor n/2 \rfloor} \frac{1}{(i-1)!} + \frac{n!}{(\lfloor k/2 \rfloor + 1)!}. \end{aligned}$$

Now if e is the natural exponent then

$$e = \sum_{i=0}^{\infty} \frac{1}{i!} = 2 + \sum_{i=3}^{\infty} \frac{1}{(i-1)!}$$

and so

$$\sum_{i=3}^{\lfloor n/2 \rfloor} \frac{1}{(i-1)!} < e - 2 < \frac{4}{5}.$$

Since $k > n/2$, this gives

$$\begin{aligned} f(n,k) &< (n-1)! \cdot \frac{4}{5} + \frac{n!}{(\lfloor k/2 \rfloor + 1)!} \\ &< (n-1)! \cdot \frac{4}{5} + \frac{n!}{(\lfloor n/4 \rfloor + 1)!} = (n-1)! \left(\frac{4}{5} + \frac{n}{(\lfloor n/4 \rfloor + 1)!} \right). \end{aligned}$$

It is easily verified that

$$\frac{n}{(\lfloor n/4 \rfloor + 1)!} < \frac{1}{5}$$

for $n \geq 16$, and so $f(n,k) < (n-1)!$, giving the required contradiction to (5.3.10).

Case 2 $\quad$ *$\mathcal{R}_2 = \{X \in \mathcal{M}(\mathcal{F}) \ : \ |X| = 2\}$ is non-empty.*

If $\bigcap_{X \in \mathcal{R}_2} X = \emptyset$ then, by the intersection property of $\mathcal{M}(\mathcal{F})$, there exist distinct $a, b, c \in [k]$ such that

$$\{\{a,b\}, \{a,c\}, \{b,c\}\} \subseteq \mathcal{R}_2.$$

Suppose there exists $X \in \mathcal{M}(\mathcal{F}) \setminus \{\{a,b\}, \{a,c\}, \{b,c\}\}$. Since $X \cap \{b,c\} \neq \emptyset$, we have either $b \in X$ or $c \in X$. This implies $a \notin X$ because otherwise either $\{a,b\} \subseteq X$ or $\{a,c\} \subseteq X$ which would contradict the fact that $\mathcal{M}(\mathcal{F})$ is an antichain. However, we must also have $X \cap \{a,b\} \neq \emptyset$ and

$X \cap \{a, c\} \neq \emptyset$, so $a \notin X$ implies $\{b, c\} \subseteq X$ which again contradicts the antichain property of $\mathcal{M}(\mathcal{F})$. We conclude

$$\mathcal{M}(\mathcal{F}) = \mathcal{R}_2 = \{\{a, b\}, \{a, c\}, \{b, c\}\}$$

and applying Lemma 4.2.1 gives

$$|\mathcal{F}| \leq \sum_{X \in \mathcal{M}(\mathcal{F})} \frac{(n - |X|)!}{(n - k)!} = 3\frac{(n - 2)!}{(n - k)!} < \frac{(n - 1)!}{(n - k)!}$$

for $n \geq 5$, giving the contradiction $|\mathcal{F}| < |\mathcal{F}|$.

Hence we must have $\bigcap_{X \in \mathcal{R}_2} X \neq \emptyset$, so we may assume without loss of generality that

$$\mathcal{R}_2 = \{\{1, i\} : 2 \leq i \leq c\}$$

for some $c \in \{2, 3, \ldots, k\}$. Set

$$\mathcal{Y} = \{X \in \mathcal{M}(\mathcal{F}) \setminus \mathcal{R}_2 \ : \ 1 \in X\}, \quad \mathcal{N} = \{X \in \mathcal{M}(\mathcal{F}) \setminus \mathcal{R}_2 \ : \ 1 \notin X\}.$$

Then it follows from the definition of $\mathcal{Y} \subset \mathcal{M}(\mathcal{F})$ that each $Y \in \mathcal{Y}$ satisfies $\{1, x, y\} \subseteq Y$ for some distinct $x, y \in [k] \setminus [c]$ since $\mathcal{M}(\mathcal{F})$ is an antichain. If $w \in \mathcal{I}_n^k$ is a word whose fixed point set $\mathrm{Fix}(w)$ contains Y, then $w \in \mathcal{V}(\{1, x, y\})$.

By the intersection property of $\mathcal{M}(\mathcal{F}) \supseteq \mathcal{R}_2$, we have $\{2, 3, \ldots, c\} \subseteq N$ for all $N \in \mathcal{N}$. Thus if $w \in \mathcal{I}_n^k$ is a word whose fixed point set $\mathrm{Fix}(w)$ contains $N \in \mathcal{N}$, then $w \in \mathcal{V}(\{2, 3, \ldots, c\})$. By an argument analogous to the proof of Lemma 4.2.1, we therefore have

$$\begin{aligned} |\mathcal{F}| \quad &\leq \quad \sum_{X \in \mathcal{R}_2} \frac{(n - |X|)!}{(n - k)!} + \sum_{\substack{x \neq y \\ x, y \in \{c+1, \ldots, k\}}} |\mathcal{V}(\{1, x, y\})| \ + \ |\mathcal{V}(\{2, 3, \ldots, c\})| \\ &= \quad (c - 1)\frac{(n - 2)!}{(n - k)!} + \binom{k - c}{2}\frac{(n - 3)!}{(n - k)!} + \frac{(n - c + 1)!}{(n - k)!}. \end{aligned}$$

Since $|\mathcal{F}| = \frac{(n-1)!}{(n-k)!}$, this may be simplified to

$$(n - 1)! \ \leq \ (c - 1)(n - 2)! + \binom{k - c}{2}(n - 3)! + (n - c + 1)!. \tag{5.3.11}$$

We will now investigate the range of values which c can take. Firstly, suppose $3 \leq c \leq k - 2$. Then $(n - c + 1)! \leq (n - 2)!$ and so

$$(n - 1)! \leq c(n - 2)! + \binom{k - c}{2}(n - 3)! := f(c),$$

i.e. denote the right hand side of this inequality by $f(c)$. Now $c > 2$ implies

$$\frac{n - c}{2} < \frac{n - 2}{2} < n - 2,$$

giving

$$\frac{(n-c)(n-c-1)}{2} < (n-2)(n-c-1)$$

since $n-c-1>0$. Using $k<n$, this gives

$$\binom{k-c}{2}(n-3)! < \binom{n-c}{2}(n-3)! < (n-2)!(n-c-1)$$

which yields $f(c) < (n-1)!$. We now have the contradiction $(n-1)! \leq f(c) < (n-1)!$, so we conclude that we cannot have $3 \leq c \leq k-2$.

Next suppose $c \geq k-1$. Recall that each $Y \in \mathcal{Y}$ satisfies $\{1,x,y\} \subseteq Y$ for some distinct $x,y \in [k]\setminus[c]$. Thus $c \geq k-1$ implies $\mathcal{Y} = \emptyset$ and $\mathcal{M}(\mathcal{F}) = \mathcal{R}_2 \cup \mathcal{N}$. If $c = k-1$ then

$$\begin{aligned}
|\mathcal{F}| &\leq \sum_{X\in\mathcal{R}_2} \frac{(n-|X|)!}{(n-k)!} \ + \ |\mathcal{V}(\{2,3,\ldots,k-1\})| \\
&= (k-2)\frac{(n-2)!}{(n-k)!} \ + \ (n-k+2)(n-k+1)
\end{aligned}$$

and multiplying through by $(n-k)!$ gives

$$\begin{aligned}
(n-1)! &\leq (k-2)(n-2)! \ + \ (n-k+2)! \\
&< (n-2)(n-2)! \ + \ (n-2)! \ = \ (n-1)!
\end{aligned}$$

since $4 \leq k < n$. Similarly, if $c = k$ then

$$\begin{aligned}
|\mathcal{F}| &\leq |\mathcal{R}_2| \cdot \frac{(n-2)!}{(n-k)!} \ + \ |\mathcal{V}(\{2,3,\ldots,k\})| \\
&= (k-1)\frac{(n-2)!}{(n-k)!} \ + \ (n-k+1),
\end{aligned}$$

so

$$\begin{aligned}
(n-1)! &\leq (k-1)(n-2)! \ + \ (n-k+1)! \\
&< (k-1)(n-2)! \ + \ (n-k)(n-2)! \ = \ (n-1)!.
\end{aligned}$$

It follows from these contradictions that $c = 2$.

Hence we have $\mathcal{R}_2 = \{\{1,2\}\}$ which implies $\mathcal{M}(\mathcal{F}) = \mathcal{R}_2 \cup \mathcal{B}_1 \cup \mathcal{B}_2$ where

$$\mathcal{B}_1 = \{X \in \mathcal{M}(\mathcal{F}) \setminus \mathcal{R}_2 \ : \ 1 \in X\}, \quad \mathcal{B}_2 = \{X \in \mathcal{M}(\mathcal{F}) \setminus \mathcal{R}_2 \ : \ 2 \in X\}.$$

Since $\mathcal{M}(\mathcal{F})$ is an antichain, $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$. Moreover, for $i \in \{1,2\}$, each $X \in \mathcal{B}_i$ satisfies $\{i,a,b\} \subseteq X$ for some $a,b \in [k]\setminus\{1,2\}$. Therefore we deduce

$$\begin{aligned}
|\mathcal{F}| &\leq \sum_{X\in\mathcal{R}_2} \frac{(n-|X|)!}{(n-k)!} + \sum_{\substack{a\neq b\\ a,b\in[k]\setminus\{1,2\}}} |\mathcal{V}(\{1,a,b\})| + \sum_{\substack{a\neq b\\ a,b\in[k]\setminus\{1,2\}}} |\mathcal{V}(\{2,a,b\})| \\
&= \frac{(n-2)!}{(n-k)!} + 2\binom{k-2}{2}\frac{(n-3)!}{(n-k)!},
\end{aligned}$$

and simplifying yields the usual contradiction:

$$\begin{aligned}(n-1)! &\leq (n-2)! + 2\binom{k-2}{2}(n-3)! \\ &< (n-2)! + 2\binom{n-2}{2}(n-3)! \\ &= (n-2)! + \frac{(n-2)!}{(n-4)!}(n-3)! \\ &= (n-2)(n-2)! \; < \; (n-1)!.\end{aligned}$$

We started the proof by assuming that not all elements of $\mathcal{F}$ have a fixed position in common. We have shown that this assumption leads to a contradiction in all possible cases, so the result now follows. □

5.3.4 Injections with Small Images

Within the proof of Theorem 5.3.9, we noted that the cases $n \leq 5$, $k \leq 3$ and

$$(n,k) \in \{(6,4),(6,5),(7,4),(7,5),(8,5),(9,5)\}$$

can be proved by hand or using a computer package. For the sake of completeness, we will do this here. Using [CK03] as before, we may assume $k < n$ throughout, so $n/2 < k < n$. Moreover, $12\ldots k \in \mathcal{F}$ without loss of generality.

Short Words

We need to prove the result for $k \leq 3$ and arbitrary n. When $k = 1$, Theorem 5.3.9 is immediate, so consider the case $k = 2$.

Since $k < n$, we have $n \geq 3$. If $n = 3$ then

$$|\mathcal{F}| = \frac{(n-1)!}{(n-k)!} = \frac{2!}{1!} = 2$$

so Theorem 5.3.9 is again trivial. If $n \geq 4$ then by Theorem 3.1.1, each maximal intersecting subset of $\mathcal{I}_n^k$ must be a transversal of the orbits. Now $12 \in \mathcal{F}$, and $\mathcal{F}$ must also contain an element of $O(13)$. This word, call it v, intersects 12, so we have $v \in \{13, n2\}$.

Suppose firstly that $v = 13$, so $\mathcal{F} \supseteq \{12, 13\}$. Then in order to intersect both 12 and 13, any other element of $\mathcal{F}$ must have its first position equal to 1. When $v = n2$ the same argument holds: Any other element of $\mathcal{F}$ must intersect both 12 and $n2$, so it must have 2 in the second position since $n \neq 1$. Thus Theorem 5.3.9 holds for $k = 2$ and n arbitrary.

Next we consider the domain size $k = 3$. We add the case $n = 4$ to the list of (n, k)-combinations to be checked with a computer package and assume $n \geq 5$. From Theorem 3.1.1 we know that $v \in \mathcal{F}$ for some $v \in O(124)$, and $123 \in \mathcal{F}$ implies $v \in \{124, n13\}$. Now simply by using the two facts that $\mathcal{F}$ is a transversal of the orbits and any two elements of $\mathcal{F}$ intersect, we will see that Theorem 5.3.9 must hold for $k = 3$ and $n \geq 5$.

Case 1 $\quad v = n13$

Suppose there exists $abc \in \mathcal{F}$ with $c \neq 3$. Then $123 \in \mathcal{F}$ implies either $a = 1$ or $b = 2$. If $a = 1$ then the fact that $abc = 1bc$ intersects $n13 \in \mathcal{F}$ implies $b = 1$, contradicting the fact that abc is an injection. Thus $b = 2$ and $abc = a2c$ must intersect $n13 \in \mathcal{F}$, which implies $a = n$. In summary, any element of $\mathcal{F}$ whose last position is not 3 must be of the form $n2c$ for some $c \in [n] \setminus \{n, 2, 3\}$.

Now $\mathcal{F}$ must contain an element of $O(125)$, call it w, since $n \geq 5$. Now $123 \in \mathcal{F}$ implies $w \in \{125, (n-1)n3\}$ and since w intersects $n13$ also, we must have $w = (n-1)n3 \in \mathcal{F}$. But $(n-1)n3$ does not intersect $n2c$ for $c \neq 3$, so $\mathcal{F}$ contains no word whose last position is different from 3.

Case 2 $\quad v = 124$

Let u be the unique word in the intersection of $\mathcal{F}$ with $O(134)$, then since $124 \in \mathcal{F}$, we have $u \in \{134, n23\}$.

Case 2.a $\quad u = 134$

Then $123, 124, 134 \in \mathcal{F}$. Suppose there is a word $abc \in \mathcal{F}$ with $a \neq 1$. Then $123 \in \mathcal{F}$ implies $b = 2$ or $c = 3$. If $c = 3$ then $abc = ab3$ cannot intersects 134, so $b = 2$. Since $abc = a2c$ intersects $134 \in \mathcal{F}$, we must have $c = 4$. Hence if $\mathcal{F}$ contains a word whose first position is not 1, then it must be of the form $a24$.

Denote the element of $\mathcal{F} \cap O(534)$ by w. Since $134 \in \mathcal{F}$ we have $w \in \{534, 1(n-1)n\}$. Now 123 does not intersect 534 and so $w = 1(n-1)n \in \mathcal{F}$. But $n \geq 5$, so $n - 1 \neq 2$ and $n \neq 4$, which means that $1(n-1)n$ does not intersect $a24$ for $a \neq 1$. Hence $\mathcal{F}$ cannot contain a word whose first position is not equal to 1.

Case 2.b $\quad u = n23$

Then $123, 124, n23 \in \mathcal{F}$. Suppose there exists $abc \in \mathcal{F}$ with $b \neq 2$, then $a = 1$ or $c = 3$ since abc must intersect $123 \in \mathcal{F}$. Now abc must also intersect $n23 \in \mathcal{F}$, so if $a = 1$ we must have $c = 3$. Conversely, if $c = 3$, then $124 \in \mathcal{F}$ forces $a = 1$. Thus any element of $\mathcal{F}$ whose second position is not 2 must be of the form $1b3$.

Now $\mathcal{F}$ must contain an element w of $O(154)$, and $124 \in \mathcal{F}$ implies $w \in \{154, (n-2)21\}$. Since 154 does not intersect $n23$, it follows that $w = (n-2)21$. Clearly $(n-2)21$ does not intersect $1b3$ for $b \neq 2$ and so $\mathcal{F}$ cannot contain any injection which does not fix 2. This concludes the case $k = 3$.

An Algorithm for the Remaining Cases

We have proved Theorem 5.3.9 for $k \leq 3$ apart from the single case $k = 3$, $n = 4$. This settles all cases of $n \leq 4$ since $k < n$, and when $n = 5$, we have $2.5 < k < 5$. In summary, the remaining cases are

$$(n, k) \in \{(4,3), (5,4), (6,4), (6,5), (7,4), (7,5), (8,5), (9,5)\} \tag{5.3.12}$$

and these have been checked using GAP [GAP07]: for given k and n, the set `Ink` corresponds to $\mathcal{I}_n^k$ and the function `intersect` returns the number of positions in which its two arguments intersect.

```
inc:=[1..k];
Sn:=SymmetricGroup(n);
Ink:=[];
  for s in Sn do
    Add(Ink,OnTuples(inc,s));
  od;
Ink:=Set(Ink);;
intersect := function(a,b)
  return Number([1..Length(a)],i->a[i]=b[i]);
end;
```

Given an intersecting subset of `Ink`, we need a test function which returns 1 if its argument is a fix-family, and 0 otherwise. This was implemented in GAP [GAP07] using the function `isfix` below.

```
isfix := function(path)
  local k, j, ch, res;
  k := Length(path[1]);
  j := 1;
  while j <= k do
    ch := Collected(List(path,x->x[j]));
    if Length(ch) = 1 then break; fi;
    j := j+1;
```

```
  od;
  if j > k then
    res := 0;
  else
    res := 1;
  fi;
  return res;
end;
```

By Theorem 3.1.1, each maximal intersecting subset $\mathcal{F}$ of $\mathcal{I}_n^k$ must be a transversal of the orbits.

```
p := [2..n]; Add(p,1); p := PermList(p);
G:=Group(p);
o := Orbits(G,Ink,OnTuples);;
```

We need to show that all words in $\mathcal{F}$ have a fixed position in common. The function `checkallfams` implements a backtrack search to find all transversals of the orbits which are intersecting. Using `isfix`, the function then checks whether each of these maximum intersecting subsets is a fix-family.

```
checkallfams := function(depth,path)
  local i,j,nr,x;
  if depth = Length(o) then
    if isfix(path)=1 then
      return;
    else
      Error("MAYDAY, MAYDAY, Counterexample!\n");
    fi;
  fi;
  for i in [1..Length(o[depth+1])] do
      x := o[depth+1][i];
      j := 1;
      while j <= depth do
          nr := intersect(x,path[j]);
          if nr = 0 then break; fi;
          j := j + 1;
```

```
      od;
      if j > depth then
          path[depth+1] := o[depth+1][i];
          checkallfams(depth+1,path);
          Unbind(path[depth+1]);
      fi;
  od;
  return;
end;
```

Running `checkallfams` in GAP [GAP07] confirms that for the remaining values of n and k given in (5.3.12), each maximum intersecting subset of $\mathcal{I}_n^k$ is indeed a fix-family.

The author would like to express her sincere gratitude to Dr Max Neunhöffer for his significant help in writing `checkallfams`.

5.3.5 Conclusion

The previous two sections have completed the proof of the following result:

> **Theorem 5.3.9.** For $n/2 < k \leq n$, let $\mathcal{F}$ be a maximum intersecting subset of $\mathcal{I}_n^k$. Then all words in $\mathcal{F}$ have a fixed position in common.

Recall also our complementary result from Chapter 3:

> **Theorem 3.1.5.** For $1 \leq k \leq (n+1)/2$, if $\mathcal{F}$ is a maximal intersecting subset of $\mathcal{I}_n^k$ then all words in $\mathcal{F}$ have a fixed position in common.

This completes the classification of maximum intersecting injection families:

Corollary 5.3.13. *If $\mathcal{F}$ is a maximum intersecting subset of $\mathcal{I}_n^k$ then all words in $\mathcal{F}$ have a fixed position in common.*

5.4 Increasing the Intersection Parameter

In Section 5.2.2 we concluded that standardising t-intersecting injection families is generally difficult. Thus we would like to generalise the Cameron-Ku approach to larger intersection parameters

$t > 1$. This section presents ideas due to Prof. Peter Cameron. We use properties of mutually orthogonal Latin Squares to prove that as n increases, fixing eventually becomes optimal when $t = 2$ and k is fixed. Even though we have already proved a more general result in Section 3.2.1, the generalisation of the Cameron-Ku method from Latin Squares and permutations to mutually orthogonal Latin Squares (*MOLS*) and injections in this section is interesting as a technique in its own right. We begin with some definitions.

Definition 5.4.1. Denote the (i, j) entry of a Latin Square L by $(L)_{ij}$. Two Latin Squares L_1, L_2 are mutually *orthogonal* if the set

$$\{ ((L_1)_{ij}, (L_2)_{ij}) : 1 \le i, j \le n \}$$

contains each pair in $[n] \times [n]$ precisely once.

We will use the following beautiful result from the 1960s.

Theorem 5.4.2. *(Bose, Shrikhande, Parker [BSP60]).*

For every $k \in \mathbb{N}$ there exists $n_0(k) \in \mathbb{N}$ such that for all $n \ge n_0(k)$, there exist k mutually orthogonal Latin Squares of order n, all with diagonal $1, 2, \ldots, n$.

To illustrate how Theorem 5.4.2 may be used to construct injections, we construct words of length 4 from the two mutually orthogonal Latin Squares L_1 and L_2 in Figure 5.4.1 as follows: the respective $(1, 2)$ entries of L_1 and L_2 are 3 and 4, so we read 1234. Similarly, the last two cells in the first row lead to the words 1342 and 1423. Since both L_1 and L_2 are Latin Squares with diagonal 1, 2, 3, 4, the words resulting from this process will be injections if we do not use diagonal cells for our construction. Thus the second row gives 2143, 2314 and 2431, and rows 3 and 4 can be used to construct injections in the same way. The following proposition generalises this construction and establishes a special property of the resulting injections.

Proposition 5.4.3. *For $n \ge n_0(k-2)$ there exists a subset A of $\mathcal{I}_n^k$ with $|A| = n(n-1)$ such that any two elements $v, w \in A$ satisfy $|\operatorname{int}(v, w)| \le 1$.*

1, 1	3, 4	4, 2	2, 3
4, 3	2, 2	1, 4	3, 1
2, 4	4, 1	3, 3	1, 2
3, 2	1, 3	2, 1	4, 4

Figure 5.4.1: L_1 and L_2 are two mutually orthogonal Latin Squares of order 4 with diagonal 1, 2, 3, 4.

Proof. By Theorem 5.4.2 there exist $k-2$ mutually orthogonal Latin Squares $L_1, \ldots, L_{k-2}$ all having diagonal $1, 2, \ldots, n$. Define

$$A = \{\, i\, j\, (L_1)_{ij} \ldots (L_{k-2})_{ij} : i \neq j,\ 1 \leq i, j \leq n \,\},$$

then $|A| = n(n-1)$.

Since $L_1, \ldots, L_{k-2}$ all have $1, 2, \ldots, n$ on the diagonal, for $i \neq j$ none of these Latin Squares have their (i, j) entry equal to i or j. Moreover, $L_1, \ldots, L_{k-2}$ are mutually orthogonal, and so for distinct l, m, the fact that $(L_l)_{ii} = (L_m)_{ii}$ for all $i \in [n]$ means that this can occur in no other position. In other words, $(L_l)_{ij} = (L_m)_{ij}$ implies $i = j$. Thus for $i \neq j,\ 1 \leq i, j \leq n$, the elements

$$i, j, (L_1)_{ij}, \ldots, (L_{k-2})_{ij}$$

of n are all distinct and we conclude $A \subseteq \mathcal{I}_n^k$.

It remains to be shown that no two elements of A 2-intersect, so let v, w be distinct elements of A. Suppose v and w intersect in position 1. Then

$$\begin{aligned} v &= i\, j\, (L_1)_{ij} \ldots (L_{k-2})_{ij}, \\ w &= i\, l\, (L_1)_{il} \ldots (L_{k-2})_{il} \end{aligned}$$

cannot intersect in any other position since row i of each Latin Square contains each symbol $1, 2, \ldots, n$ precisely once, and $j \neq l$ since $v \neq w$.

Similarly, if $2 \in \operatorname{int}(v, w)$, then v and w cannot intersect in any other position since any column of a Latin Square contains each symbol precisely once.

So let $i \neq l$, $j \neq m$ and suppose

$$\begin{aligned} v &= i\, j\, (L_1)_{ij} \ldots (L_{k-2})_{ij}, \\ w &= l\, m\, (L_1)_{lm} \ldots (L_{k-2})_{lm} \end{aligned}$$

intersect in some position a where $3 \leq a \leq k$. Then $(L_{a'})_{ij} = (L_{a'})_{lm}$ where $a' = a-2$, so we cannot have $(L_b)_{ij} = (L_b)_{lm}$ for any $b \neq a'$, $1 \leq b \leq k-2$ since $L_{a'}$ and L_b are mutually orthogonal. $\square$

We are now in a position to derive a bound on 2-intersecting families in $\mathcal{I}_n^k$ for large n, using Proposition 5.4.3 together with Proposition 5.3.1 from [CK03].

Theorem 5.4.4. *Let $n \geq n_0(k-2)$ and let $G \subset \mathcal{I}_n^k$ be 2-intersecting. Then*

$$|G| \leq \frac{(n-2)!}{(n-k)!}.$$

Proof. Let Γ be a graph on the vertex set $\mathcal{I}_n^k$ in which two vertices are joined by an edge if, and only if, the corresponding injections 2-intersect. Then Γ is vertex-transitive for reasons we have already discussed when we formed a graph on 1-intersecting injections in the proof of Theorem 5.3.3: for two injections $v, w \in \mathcal{I}_n^k$, there exists a permutation σ in the symmetric group $\mathcal{S}_n$ such that $v\sigma = w$. Moreover,

$$\mathcal{I}_n^k \sigma = \left\{ v\sigma : v \in \mathcal{I}_n^k \right\} = \mathcal{I}_n^k$$

and two elements v, w of $\mathcal{I}_n^k$ 2-intersect if, and only if, $v\sigma$ 2-intersects $w\sigma$. Hence each permutation in $\mathcal{S}_n$ is a graph automorphism of Γ and $\mathcal{S}_n$ acts transitively on $\mathcal{I}_n^k$.

By Proposition 5.4.3, there exists a subset A of $\mathcal{I}_n^k$ with $|A| = n(n-1)$ such that no two elements of A mutually 2-intersect; therefore A is a coclique in Γ. Note also that G is a clique in Γ, and so we may use Proposition 5.3.1 to obtain

$$|G| \leq \frac{|\mathcal{I}_n^k|}{|A|} = \frac{n!}{(n-k)!} \cdot \frac{1}{n(n-1)} = \frac{(n-2)!}{(n-k)!}$$

as required. □

Since the size of $\mathcal{K}_0$ is equal to the bound of Theorem 5.4.4, this bound is sharp and we have the following corollary:

Corollary 5.4.5. *For every $k \in \mathbb{N}$ there exists $n_1(k) = n_0(k-2) \in \mathbb{N}$ such that the fix-family $\mathcal{K}_0$ is a maximum 2-intersecting subset of $\mathcal{I}_n^k$ for all $n \geq n_1(k)$.*

Unfortunately, the above theorem and corollary do not apply to permutations, since they require that n is large in terms of k, just like the more general result of Corollary 3.2.3.

In this section, we have shown how to generalise the Cameron-Ku approach of [CK03] to derive a bound for the case $t = 2$. It is unclear whether further investigations in this direction could yield structural results, or whether the approach could be extended to larger t. In [DF77] Deza & Frankl used the clique-coclique bound of Proposition 5.3.1 to show that if there exists a sharply t-transitive set of permutations, then no t-intersecting set of permutations is larger than the fix-family. However, as Cameron & Ku point out at the end of [CK03], their method is unsuitable for classification results. Thus the problem of obtaining a complete classification of maximum t-intersecting injection families, or even permutation families, is still open.

Part III

Partial Orders

CHAPTER 6

INTERSECTING FAMILIES OF ORDERS

6.1 Definitions

This chapter investigates the intersecting structure of various classes of orders. Before we start, let us recall some definitions from Section 2.3, and agree on some conventions.

A *(partial) order* R on $[n]$ is a set of ordered pairs, $R \subseteq [n] \times [n]$, which is irreflexive, antisymmetric and transitive: for all $x, y, z \in [n]$ we have $(y, x) \notin R$ whenever $(x, y) \in R$, and also $(x, y), (y, z) \in R$ implies $(x, z) \in R$. A *partially ordered set*, or *poset* for short, is a pair $p = ([n], R)$ where R is a partial order on $[n]$. Since the ground set $[n]$ is usually clear from the context, however, we simply refer to R as a poset for brevity. If (x, y) is an element of the poset p, this is interpreted as $x < y$ under p, so we often write $x <_p y$ instead of $(x, y) \in p$.

Two elements x, y of $[n]$ are *comparable* under the poset p if either $x <_p y$ or $y <_p x$. If $x \not<_p y$ and $y \not<_p x$ then we say that x and y are *incomparable* under p and denote this by $x \,||_p\, y$. We call $x < y$ a *comparison*, and $x \,||\, y$ a *non-comparison*. If all pairs of elements of $[n]$ are comparable under p, we say that p is a *linear* order. For some labelling $\{x_1, x_2, \ldots, x_n\} = [n]$ we may use the notation $x_1 x_2 \ldots x_n$ for the linear order under which $x_1 < x_2 < \cdots < x_n$. A *chain* in P is a subset C of $[n]$ such that P restricted to C is linear. Dually, a subset of $[n]$ whose elements are mutually *incomparable* under P is called an *antichain*. The set of all linear orders on $[n]$ is denoted by $\mathcal{L}_n$, while the set of all partial orders on $[n]$ is denoted by $\mathcal{P}_n$, so $\mathcal{L}_n \subset \mathcal{P}_n$. Considering our notation $x_1 x_2 \ldots x_n$ for linear orders, it is hardly surprising that $|\mathcal{L}_n| = n!$. On the other hand, $\mathcal{P}_n$ is difficult to enumerate and its size is only known for fairly small values of n.

6.2 Intersecting Families of Linear Orders

We say that two linear orders *intersect* if they share a comparison. The following definition will play an important role in our study of the intersecting structure of order classes.

Definition 6.2.1. For $p \in \mathcal{P}_n$ define the *reverse* of p to be

$$\mathrm{rev}(p) = \{ (x, y) \in [n] \times [n] : (y, x) \in p \} .$$

Note that $\mathrm{rev}(p)$ is irreflexive, antisymmetric and transitive because p has these properties, and so $\mathrm{rev}(p) \in \mathcal{P}_n$.

Given Definition 6.2.1, the classification of maximum intersecting subsets of $\mathcal{L}_n$ is straightforward:

Theorem 6.2.2. *If $\mathcal{F} \subseteq \mathcal{L}_n$ is intersecting then $|\mathcal{F}| \leq n!/2$. In particular, $\mathcal{F}$ has maximal size if and only if $\mathcal{F}$ is a transversal of*

$$\{ \{\sigma, \mathrm{rev}(\sigma)\} : \sigma \in \mathcal{L}_n \} .$$

Proof. It is not hard to see that two linear orders do not intersect if, and only if, one is the reverse of the other. The result follows. □

Clearly the fix-family $\{ \sigma \in \mathcal{L}_n : x <_\sigma y \}$ has maximal size $n!/2$. So to obtain a maximal intersecting subset of $\mathcal{L}_n$, we can fix a comparison — but we do not have to.

Example 6.2.3. Consider the following subset of $\mathcal{L}_4$:

$$\mathcal{F} = \{1234, 3421, 4231, 1342, 1423, 2341, 4312, 2143, 3124, 2413, 3214, 4132\}.$$

It is easily checked that $\mathcal{F}$ is a transversal of $\{ \{\sigma, \mathrm{rev}(\sigma)\} : \sigma \in \mathcal{L}_4 \}$, so this is a maximum intersecting subset of $\mathcal{L}_4$.

What makes $\mathcal{F}$ interesting is that for all distinct $x, y \in \{1, 2, 3, 4\}$, precisely $|\mathcal{F}|/2$ elements of $\mathcal{F}$ satisfy $x < y$, and the other $|\mathcal{F}|/2$ elements of $\mathcal{F}$ satisfy $y < x$. Thus maximum intersecting families in $\mathcal{L}_n$ do not necessarily have 'dominant' comparisons (comparisons which occur more often than others) as fixing and saturation families do.

Example 6.2.3 will be of interest again when we investigate partial orders. For now, we conclude that many different intersecting families are optimal in $\mathcal{L}_n$.

6.3 Intersecting Families of Partial Orders

6.3.1 Different Definitions of Intersection

Péter Erdős with various co-authors as well as Czabarka consider in [EFK92, ESS94, Cza99, ESS00] families of k-chains in a fixed labelled poset $P \in \mathcal{P}_n$ which they call t-intersecting if any two of the k-chains share at least t elements of $[n]$. Our approach is different: we are interested in the scenario where two distinct labelled posets t-intersect each other. Nevertheless, two alternative definitions of intersection are conceivable in $\mathcal{P}_n$:

1. We could say that two partial orders on $[n]$ intersect if they share a comparison.
2. Alternatively, one might say that two partial orders on $[n]$ intersect if they share either a comparison or a non-comparison. That is, $a, b \in \mathcal{P}_n$ intersect if there exist $x, y \in [n]$ such that
   - either $x <_a y$ and $x <_b y$
   - or $x \,||_a\, y$ and $x \,||_b\, y$.

As one would expect, these two definitions yield different bounds for intersecting subsets: using Definition 1, the largest intersecting subset of $\mathcal{P}_3$ is

$$F_{1,2}(\mathcal{P}_3) = \{\, \pi \in \mathcal{P}_3 : 1 <_\pi 2 \,\}$$

with $|F_{1,2}(\mathcal{P}_3)| = 6$. Using Definition 2, on the other hand, the maximal size of an intersecting subset of $\mathcal{P}_3$ is $7 > 6$. It can be attained either by fixing a non-comparison

$$N_{1,2} = \{\, \pi \in \mathcal{P}_3 : 1 \,||_\pi\, 2 \,\},$$

or by saturation:

$$\begin{aligned} G &= \{\, \pi \in \mathcal{P}_3 : \pi \text{ contains at least 2 distinct non-comparisons} \,\} \\ &= \{\, \pi \in \mathcal{P}_3 : \pi \text{ contains at most 1 comparison} \,\}. \end{aligned}$$

Note that Definition 1 is equivalent to the following:

Definition 6.3.1. Two partial orders $p, q \in \mathcal{P}_n$ *intersect* if there exist $x, y \in [n]$ such that $(x, y) \in p \cap q$.

According to this definition, two posets intersect if, and only if, they intersect as sets. To develop a theory of intersecting posets which is compatible with its motivations and origins, most notably the Erdős-Ko-Rado Theorem, we therefore choose Definition 1. As the reader would expect, a set of partial orders $\mathcal{F} \subseteq \mathcal{P}_n$ is *intersecting* if every pair of elements of $\mathcal{F}$ is intersecting.

6.3.2 Fixing vs. Saturation: Preliminary Observations

Having determined our definition of intersection for posets, the first natural investigation is to compare fixing with saturation in $\mathcal{P}_n$. So let a, b be fixed elements of $[n]$ and define the fix-family by

$$F_{a,b}(\mathcal{P}_n) = \{\, p \in \mathcal{P}_n : a <_p b \,\}\,.$$

In Lemma 6.3.12 we use linear extensions to show that $F_{a,b}(\mathcal{P}_n)$ is maximal as an intersecting subset of $\mathcal{P}_n$.

On the other hand, the definition of a generic saturation family G is not immediately clear: $G(S, n)$ contains all posets in $\mathcal{P}_n$ which contain at least $r+1$ of the comparisons in some set S of size $2r+1$. Due to transitivity, however, the choice of the saturation base S is not arbitrary for posets as it is for some other combinatorial structures such as sets or permutations. Indeed, different saturation bases of the same size may lead to families of different sizes: if

$$p = \{(1,2), (1,3), (1,4)\}, \quad q = \{(1,2), (1,3), (2,3)\}$$

then both $G(p, 4)$ and $G(q, 4)$ are saturation families over posets of size 3, but it is easily computed that $|G(p, 4)| = 58 > |G(q, 4)| = 54$.

Thus as n gets large, it is unclear whether we obtain larger families by saturating over a set of independent comparisons such as

$$\{(1,2), (3,4), \ldots, (a,b)\},$$

which impose no transitive restrictions on the posets which the resulting family may contain; or whether we should strive to restrict as few of the labels as possible by choosing a saturation set such as

$$\{(1,2), (1,3), \ldots, (1,c)\}.$$

Alternatively, we could saturate over linear orders, as in the example $G(q, n)$ above, or indeed over sets of comparisons which are not partial orders. Additional difficulties arise from the fact that saturation families in $\mathcal{P}_n$ are not necessarily maximal.

For small values of n, the largest saturation families are attained by taking the first few comparisons in the sequence nat_n as the saturation base.

Definition 6.3.2. Given a linear order $\sigma = x_1 x_2 \ldots x_n \in \mathcal{L}_n$, let $\alpha(\sigma, x_j)$ be the sequence of comparisons listing all points which are less than x_j under σ, in the order in which they appear in σ:

$$\alpha(\sigma, x_j) = x_1 < x_j, \;\; x_2 < x_j, \;\; \ldots, \;\; x_{j-1} < x_j.$$

Note $\alpha(\sigma, x_1)$ is the empty sequence. Set $\iota_n = 12\ldots n$, the natural order on the first n integers, and

$$\begin{aligned}\mathrm{nat}_n &= \alpha(\iota_n, 1), \alpha(\iota_n, 2), \ldots, \alpha(\iota_n, n) \\ &= 1 < 2,\ 1 < 3, 2 < 3,\ 1 < 4, \ldots, n-1 < n.\end{aligned}$$

Finally, if

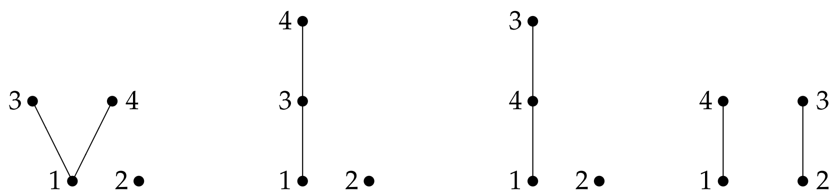
$$K_r(\mathcal{P}_n) = \{\, p \in \mathcal{P}_n : |p \cap S| \geq r+1 \,\}$$

where S is the set of the first $2r+1$ comparisons of nat_n, then $K_r(\mathcal{P}_n)$ is intersecting, though we will see below that $K_r(\mathcal{P}_n)$ is not necessarily maximal.

It is easily shown that $K_1(\mathcal{P}_n)$ is strictly contained in $F_{1,3}(\mathcal{P}_n)$. On the other hand, one can show with a little more effort that

$$K_2(\mathcal{P}_n) = \{\, p \in \mathcal{P}_n : p \text{ satisfies at least three of } 1 < 2, 3, 4; 2 < 3, 4 \,\}$$

is not contained in $F_{a,b}(\mathcal{P}_n)$ for any a, b, n. However, $K_2(\mathcal{P}_n)$ is not maximal. For instance, one of the maximal closures of $K_2(\mathcal{P}_4)$ in $\mathcal{P}_4$ is the set obtained by adding the following four posets to $K_2(\mathcal{P}_4)$:



For $3 \leq n \leq 5$, the largest saturation families in $\mathcal{P}_n$ are maximal closures of some $K_r(\mathcal{P}_n)$. However, the respective fix-families are larger in each of these three cases, and so we devote the next section to a search for an injection from an arbitrary intersecting subset of $\mathcal{P}_n$ into the fix-family, despite the fact that we have not yet found the optimal interpretation of saturation in $\mathcal{P}_n$. Recall also that the most successful method of Part II was to represent injections by sets, and study the resulting set families instead. It may be that the same is true for posets, and we simply have not yet thought of a suitable map from posets to sets. After all, an injection is technically set (namely a subset of $[k] \times [n]$), yet the successful approach in Part II was to represent injections by their fixed point sets, as opposed to considering them as sets themselves.

6.3.3 In Search of an Injection into the Fix-Family

We will soon need to start considering different types of posets separately, so let us partition $\mathcal{P}_n$ into *isomorphism classes*.

Definition 6.3.3. Two labelled posets $p, q \in \mathcal{P}_n$ are *isomorphic* if one can be obtained from the other by a permutation of the labels. Formally, we have $p \cong q$ if there exists $\pi \in \mathcal{S}_n$ such that $p = \pi q$. Informally, p and q are isomorphic if they have the same unlabelled Hasse diagram; see Figure 6.3.1.

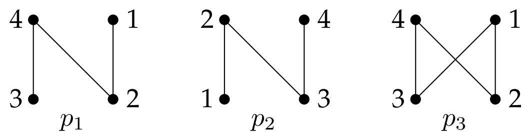


Figure 6.3.1: p_1 and p_2 are isomorphic but p_3 is not isomorphic to either of them.

Now we begin our search for a way of mapping any intersecting family into the fix-family. Proposition 6.3.4 specifies the map we are looking for.

For a subset $X \subseteq \mathcal{P}_n$ and a comparison or non-comparison r, define the *restriction* of X to r by

$$X_r = \{ p \in X : r \in p \} .$$

Proposition 6.3.4. *Let $G \subseteq \mathcal{P}_n$ be intersecting. If, for some $a, b \in [n]$, there exists an injection $f : G_{a||b} \to F_{a,b}(\mathcal{P}_n)$ such that*

$$f(G_{a||b}) \cap G_{a<b} = \emptyset \text{ and } f(G_{a||b}) \cap \operatorname{rev}(G_{b<a}) = \emptyset \tag{6.3.5}$$

then $|G| \le |F_{a,b}(\mathcal{P}_n)|$.

Proof. Define a map ϕ on elements p of G as follows:

$$\phi(p) = \begin{cases} p & a <_p b \\ \operatorname{rev}(p) & b <_p a \\ f(p) & a \,||_p\, b \end{cases}$$

We wish to show that ϕ is an injection from G to $F_{a,b}(\mathcal{P}_n)$. Partition G according to the different cases of ϕ:

$$G = G_{a<b} \cup G_{b<a} \cup G_{a||b}.$$

For $p \in G$, we clearly have $\phi(p) \in F_{a,b}(\mathcal{P}_n)$, thus we need to show that ϕ is an injection. So let $p, q \in G$ and suppose $\phi(p) = \phi(q)$.

- If $p, q \in G_{a<b}$ then $\phi(p) = \phi(q)$ implies $p = q$.

- If $p \in G_{a<b}$, $q \in G_{b<a}$, then $\phi(p) = \phi(q)$ gives $p = \mathrm{rev}(q)$. But q does not intersect $\mathrm{rev}(q) = p$ which contradicts the fact that G is intersecting.
- If $p \in G_{a<b}$, $q \in G_{a||b}$ then $\phi(p) = \phi(q)$ gives $p = f(q)$ which contradicts $f(G_{a||b}) \cap G_{a<b} = \emptyset$.
- If $p, q \in G_{b<a}$ then $\phi(p) = \phi(q)$ becomes $\mathrm{rev}(p) = \mathrm{rev}(q)$ which implies $p = q$.
- If $p \in G_{b<a}$, $q \in G_{a||b}$, then $\phi(p) = \phi(q)$ gives $\mathrm{rev}(p) = f(q)$ which contradicts $f(G_{a||b}) \cap \mathrm{rev}(G_{b<a}) = \emptyset$.
- If $p, q \in G_{a||b}$ then $\phi(p) = \phi(q)$ implies $p = q$ since f is injective.

This completes the case analysis and the result follows. □

Note that if there exist $a, b \in [n]$ such that all elements of G compare a to b, then $G_{a||b} = \emptyset$, so f exists vacuously. In this case, simply reversing all $(b < a)$-posets of G is an injection from G to the fix-family $F_{a,b}$. Thus we need to concentrate our efforts on maximal intersecting families G which, for any two points $a, b \in [n]$, contain a poset under which $a||b$. If fixing is optimal in $\mathcal{P}_n$ then we may be able to resolve these cases using the injection f from Proposition 6.3.4.

Finding such an f is easily done by inspection for small n, but difficult in general: in order to keep the map as simple as possible, we would ideally like $f(p) \cong p$. If that is impossible, the unlabelled Hasse diagrams of $f(p)$ and p should look very similar at least. Now the condition $f(G_{a||b}) \cap G_{a<b}$ forbids f to map into G, so for any particular $p \in G_{a||b}$, f needs to destroy a sufficient number of 'intersections' (comparisons of p which ensure p intersects all other elements of G) to exclude $f(p)$ from G. Since we do not know in general how many non-comparisons other than $a||b$ we have available in the Hasse diagram of p, the easiest way of destroying such 'intersections' is to reverse existing comparisons. But this approach runs the risk of causing $\mathrm{rev}(f(p)) \in G$, which is exactly what the second condition $f(G_{a||b}) \cap \mathrm{rev}(G_{b<a})$ forbids.

Linear Extensions

We conclude that we need to know more about an intersecting family G before we can hope to map it into $F_{a,b}$. One approach is to deduce properties of the partial orders from properties of their linear extensions.

Definition 6.3.6. For a poset $p \in \mathcal{P}_n$ we denote by $\mathcal{L}_n(p)$ the set of its linear extensions:

$$\mathcal{L}_n(p) = \{\, \sigma \in \mathcal{L}_n : p \subseteq \sigma \,\}\,.$$

It is fairly clear that each partial order has a linear extension, but we would like to keep control of the comparisons in such a linear extension. Corollary 6.3.11 summarises that this is indeed possible, but we begin by proving the slightly more general result given in Lemma 6.3.9. To do so, we require the concept of height, which will make an appearance at several points over the coming chapters.

Definition 6.3.7. For $p \in \mathcal{P}_n$ and $x \in [n]$ the *height of* x under p, denoted by $h_p(x)$, is defined to be one less than the greatest number of elements in a chain whose largest member is x.

Lemma 6.3.8. *Elements of equal height are incomparable.*

Proof. Let $p \in \mathcal{P}_n$, $x, y \in [n]$ with $h(x) = h(y)$ and suppose $x <_p y$. By the definition of $h(x)$, there exists a chain

$$z_1 <_p z_2 <_p \cdots <_p z_{h(x)} <_p x$$

all of whose elements satisfy $z_i <_p x <_p y$ by transitivity. But then y would have height at least $h(x) + 1 = h(y) + 1$, a contradiction. □

We are now ready to prove the lemma which underpins the way we think about building linear extensions.

Lemma 6.3.9. *Let $p \in \mathcal{P}_n$ and $a, b \in [n]$ with $a||_p b$.*

Then there exists a partition of $[n]$ into k parts, some of which may be empty, such that

1. *a and b are in the same part;*

2. *each part is an antichain;*

3. *there exists a linear order $\prec$ on the parts of the partition such that if X, Y are parts with $X \prec Y$ then for all $x \in X$ and all $y \in Y$ we have either $x <_p y$ or $x||_p y$.*

Proof. Let $<_{\mathbb{N}}$ denote the natural order on $\mathbb{N}$. Suppose, without loss of generality, that $h(a) \leq_{\mathbb{N}} h(b)$, and let

$$m = \max_{x \in [n]} h(x)$$

be the maximal height in p.

For all $i \in \{0, 1, \ldots, h(a) - 1\} \cup \{h(b) + 1, h(b) + 2, \ldots, m\}$, set

$$L_i = \{ x \in [n] : h(x) = i \} .$$

If $h(a) = h(b)$, set $L_{h(a)} = \{\, x \in [n] : h(x) = h(a) \,\}$. Otherwise, set

$$\begin{aligned} L_{h(a)} &= \{\, x \in [n] : h(x) = h(a) \,\} \setminus \{a\}, \\ L_{h(b),1} &= \{\, x \in [n] : h(x) = h(b) \text{ and } x \not>_p a \,\} \cup \{a\}, \\ L_{h(b),2} &= \{\, x \in [n] : h(x) = h(b) \text{ and } x >_p a \,\}, \end{aligned}$$

and for $j \in \{h(a)+1, h(a)+2, \ldots, h(b)-1\}$, set

$$\begin{aligned} L_{j,1} &= \{\, x \in [n] : h(x) = j \text{ and } x \not>_p a \,\}, \\ L_{j,2} &= \{\, x \in [n] : h(x) = j \text{ and } x >_p a \,\}. \end{aligned}$$

Although some of the sets $L_{j,1}, L_{j,2}$ may be empty, clearly

$$\Lambda = \{\, L_i, L_{j,1}, L_{j,2} : i = 0, 1, \ldots, h(a), h(b)+1, \ldots, m,\ j = h(a)+1, \ldots, h(b) \,\}$$

is a partition of $[n]$ with a and b in the same part, or 'level'.

To show that each element of Λ is an antichain, note that if two points are in the same level then they must have equal height in p, unless one of the points is equal to a, and we are done by Lemma 6.3.8.

It remains to be shown that all $w \in L_{h(b),1}$ satisfy $w||_p a$. We have $w \not>_p a$ by the definition of $L_{h(b),1}$, so suppose $w <_p a$. Then any chain in p with maximal element w can be extended to a chain with maximal element a, so $h(w) = h(b) <_{\mathbb{N}} h(a)$ which contradicts $h(a) \leq_{\mathbb{N}} h(b)$.

Thus, with the convention that the empty set is an antichain, we conclude that all of the levels $L_i, L_{j,1}, L_{j,2}$ are antichains.

To prove part 3 of the lemma, define a total order $\prec$ on elements of Λ as follows: for $z \in \{1, 2\}$, $i, i' \in \{0, 1, \ldots, h(a)\} \cup \{h(b)+1, \ldots, m\}$ and $j \in \{h(a)+1, \ldots, h(b)\}$, we have

$$\begin{aligned} L_i \prec L_{i'} &\iff i <_{\mathbb{N}} i', \\ L_i \prec L_{j,z} &\iff i <_{\mathbb{N}} j, \\ L_{j,1} \prec L_{j',2}, &\qquad \forall j, j', \\ L_{j,z} \prec L_{j',z} &\iff j <_{\mathbb{N}} j'. \end{aligned}$$

Note that for $c \in [n] \setminus \{a\}$, $z \in \{1, 2\}$, we have

$$c \in L_i \implies h(c) = i \ \text{ and } \ c \in L_{j,z} \implies h(c) = j. \tag{6.3.10}$$

We need to show that for $X, Y \in \Lambda$ with $X \prec Y$, we have $x \not>_p y$ for all $x \in X, y \in Y$. So suppose firstly that $a \notin \{x, y\}$ and $x >_p y$. Then by the argument used in part two, $h(x) >_{\mathbb{N}} h(y)$ which, together with (6.3.10) and $X \prec Y$, implies $X = L_{h(x),1}$ and $Y = L_{h(y),2}$. Thus, by definition of the $L_{j,z}$, we have $x \not>_p a$ and $y >_p a$. But $x >_p y >_p a$ gives $x >_p a$, which is impossible.

It remains to be shown that part 3 of the lemma holds for a. So let $X \in \Lambda$ with $X \prec L_{h(b),1}$. Then we must have one of the following two cases:

- $X = L_i$ with $i <_{\mathbb{N}} h(b)$. Since $i \in \{0, 1, \ldots, h(a)\} \cup \{h(b)+1, \ldots, m\}$, this implies $i \leq_{\mathbb{N}} h(a)$. All elements x of $X = L_i$ have height $i \leq_{\mathbb{N}} h(a)$ and so $x \not>_p a$ as required.
- $X = L_{j,1}$ for some j. Then we are done since elements x of $L_{j,1}$ satisfy $x \not>_p a$ by definition of $L_{j,1}$.

Finally, let $Y \in \Lambda$ with $L_{h(b),1} \prec Y$. Since for all $L_{j,z} \in \Lambda$, we have $j \leq_{\mathbb{N}} h(b)$, one of the following must hold:

- $Y = L_{j,2}$ for some j. Then by definition of $L_{j,2}$, we have $a <_p y$ for all $y \in L_{j,2}$.
- $Y = L_i$ with $i >_{\mathbb{N}} h(b)$. Then all $y \in Y$ have $h(y) >_{\mathbb{N}} h(b) \geq_{\mathbb{N}} h(a)$ and so $y \not<_p a$ as required.

This completes the proof. $\square$

Lemma 6.3.9 enables us to build linear extensions of partial orders as follows:

Corollary 6.3.11. *Let $p \in \mathcal{P}_n$ and $a, b \in [n]$ with $a||_p b$. Then p has a linear extension σ with $a <_\sigma b$.*

Proof. We adjoin comparisons to p until we obtain σ. Let $\mathcal{A}$ be the partition of $[n]$ given by Lemma 6.3.9. First, we ensure that points of different levels are comparable under σ in the way which agrees with the linear order $\prec$ on the parts of $\mathcal{A}$. We may then specify $a <_\sigma b$ since the part containing a and b is an antichain, and this was not affected by the comparisons between elements of different parts which we already added. The resulting relation has a transitive closure, and σ is any linear extension of that order. $\square$

Indeed, it is not hard to see from the proof of Corollary 6.3.11 that if points $x, y \in [n]$ are incomparable under $p \in \mathcal{P}_n$, then p has linear extensions σ and σ' which disagree only on the ordering of x and y. Such observations lie at the heart of many simple arguments, e.g. the following:

Lemma 6.3.12. *$F_{a,b}(\mathcal{P}_n)$ is maximal in $\mathcal{P}_n$.*

Proof. Let $\mathcal{F}$ be a maximal closure of $F_{a,b}(\mathcal{P}_n)$ in $\mathcal{P}_n$ and let $p \in \mathcal{P}_n$. If $a <_p b$ then p is an element of $F_{a,b}(\mathcal{P}_n)$. If $b <_p a$ then $\mathrm{rev}(p)$, which does not intersect p, is an element of $F_{a,b}(\mathcal{P}_n)$, so $p \notin \mathcal{F}$. If $a||_p b$ then there exists a linear extension σ of p with $b <_\sigma a$ by Corollary 6.3.11. Since $p \subset \sigma$, we then have

$$(p \cap \mathrm{rev}(\sigma)) \subseteq (\sigma \cap \mathrm{rev}(\sigma)) = \emptyset,$$

so $p \notin \mathcal{F}$ since $\mathrm{rev}(\sigma)$ is an element of $F_{a,b}(\mathcal{P}_n)$. We conclude that $\mathcal{F} = F_{a,b}(\mathcal{P}_n)$ as required. □

Having gained an insight into linear extensions, we would like to use them in our investigation of the intersection structure of $\mathcal{P}_n$. It is clear from Definition 6.3.6 that if two partial orders intersect, then any of their linear extensions intersect. Thus if $\mathcal{F}$ is a maximal intersecting family in $\mathcal{P}_n$, it must contain $\mathcal{L}(\mathcal{F})$. But we can say more than that.

Proposition 6.3.13. *Let $\mathcal{X}$ be a subclass of $\mathcal{P}_n$ closed under taking reverses, and let $\mathcal{F}$ be a maximal intersecting subset of $\mathcal{X}$. Then $\mathcal{F}$ contains a transversal of*

$$\{\, \{\pi, \mathrm{rev}(\pi)\} : \pi \in \mathcal{L}_n \cap \mathcal{X} \,\}.$$

Proof. Since $\mathcal{X}$ is closed under taking reverses, we have $\mathrm{rev}(\sigma) \in \mathcal{X}$ for all $\sigma \in \mathcal{X}$. So suppose there exists $\sigma \in \mathcal{L}_n \cap \mathcal{X}$ with neither σ nor $\mathrm{rev}(\sigma)$ in $\mathcal{F}$. Then the maximality of $\mathcal{F}$ implies that there exist $p, q \in \mathcal{F}$ such that p does not intersect σ and q does not intersect $\mathrm{rev}(\sigma)$. This means that for all $x, y \in [n]$, whenever $x <_p y$ we must have $x \not<_\sigma y$, implying $y <_\sigma x$ since σ is linear. In other words, for all $x, y \in [n]$, $x <_p y$ implies $x <_{\mathrm{rev}(\sigma)} y$, that is, $\mathrm{rev}(\sigma)$ is a linear extension of p. Similarly, $\mathrm{rev}(\mathrm{rev}(\sigma)) = \sigma$ is a linear extension of q.

Since $\mathcal{F}$ is intersecting, p and q must intersect, which implies that their linear extensions intersect. But that contradicts the fact that two linear orders intersect if, and only if, one is not the reverse of the other. □

Taking $\mathcal{X} = \mathcal{P}_n$ in particular, Proposition 6.3.13 says that any maximal intersecting subset of $\mathcal{P}_n$ contains a maximum intersecting family of linear orders. Moreover, these linear orders lie at the core of $\mathcal{F}$, in the sense that knowledge about the linear family gives us much information about $\mathcal{F}$ itself, as we will see in the following proposition. Set $F_{a,b}(\mathcal{L}_n) = \{\, \sigma \in \mathcal{L}_n : (a, b) \in \sigma \,\}$.

Proposition 6.3.14. *Let $\mathcal{F}$ be a maximal intersecting subset of $\mathcal{P}_n$. If $\mathcal{F} \cap \mathcal{L}_n = F_{a,b}(\mathcal{L}_n)$ for some $a, b \in [n]$ then $\mathcal{F} = F_{a,b}(\mathcal{P}_n)$.*

Proof. Recall that $\mathcal{L}(\mathcal{F}) \subseteq \mathcal{F}$ by the maximality of $\mathcal{F}$, so $\mathcal{L}(\mathcal{F})$ must be contained in $\mathcal{F} \cap \mathcal{L}_n = \{\, \sigma \in \mathcal{L}_n : a <_\sigma b \,\}$. That is, every linear extension of every element of $\mathcal{F}$ has $a < b$. Clearly this means that there cannot exist $p \in \mathcal{F}$ with $b <_p a$. Moreover, any partial order p with $a||_p b$ has a linear extension σ with $b <_\sigma a$ by Corollary 6.3.11, so $\mathcal{F}$ cannot contain posets under which $a||b$ either. The result then follows from the fact that $\mathcal{F}$ is maximal. $\square$

The previous two propositions suggest that in order to work towards a classification of maximum intersecting subsets $\mathcal{F}$ of $\mathcal{P}_n$, it may be a good idea to investigate the structure of the linear families $\mathcal{F} \cap \mathcal{L}_n$ contained within them. Since we tend to describe poset families in terms of the comparisons occurring most frequently within its members, this raises the following question:

> Given a maximum intersecting subset $\mathcal{F}'$ of $\mathcal{L}_n$, how can the comparisons occurring most frequently in $\mathcal{F}'$ be identified?

Such a procedure would also help to define more efficient saturation families in $\mathcal{P}_n$. Unfortunately, we already know that that this is not possible in general: see Example 6.2.3.

6.3.4 Restriction to Poset Classes

Having seen that classifying the optimal families in $\mathcal{P}_n$ in general seems rather difficult, we proceed to consider subclasses. Our successful observations on linear orders point in two possible directions here. One viewpoint is that what we did in Section 6.2 was to fix an unlabelled poset, in this case a chain of length n, and consider the class of all permutations of the labels. To further pursue this approach, we will consider the intersection structures of various isomorphism classes of posets in the following chapters.

An alternative direction to pursue is to attempt to classify poset classes whose intersection structure we can determine by applying the proof method of Theorem 6.2.2, or some generalisation thereof. It is hardly surprising that these two approaches overlap sometimes, but this is not always the case. Our work in Section 7.1 is an example where the two approaches coincide: our method of classifying the maximum intersecting subsets of the fixed isomorphism classes considered there is based on the idea of reverse pairings.

Chapter 7

Posets which are Almost Linear

By making the simple observation that a linear order intersects any other except its reverse, we succeeded in classifying all maximum intersecting subsets of $\mathcal{L}_n$. The main purpose of this chapter is to investigate what happens if we remove just one of the comparisons in a linear order. In Section 7.1 we fix such a poset and take our class to be all permutations of the labels. We partition this class according to whether the individual posets intersect. By describing the blocks of this partition, we obtain a complete classification of maximum intersecting families in this class. To get an idea of where these intersecting families lie on the fixing – saturating spectrum, we show that fixing is not optimal in this class whereas, in all but a few marginal cases, a saturation family does attain the bound.

In Section 7.2 we no longer fix the poset, but consider the union of the classes in the previous section. Here we use the classical method of cyclic orderings to obtain a bound on intersecting subsets of this class. The comparisons on $[n]$ are arranged on a circle, and it is shown that posets in the class are equivalent to intervals on the cyclic orderings. Finally, we show that both fixing and saturating give optimal intersecting families in this class.

The chapter concludes with Section 7.3, where we investigate which poset classes the reverse pairings method of Section 7.1 could be extended to.

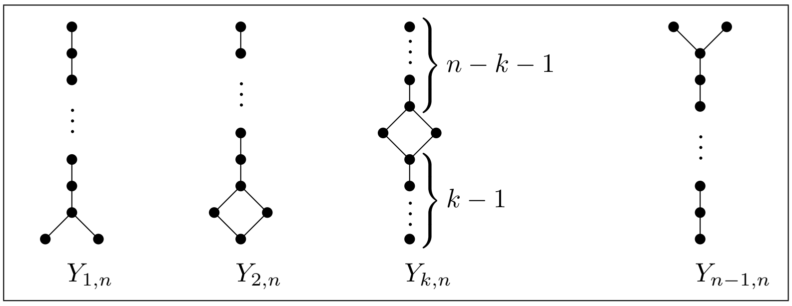


Figure 7.1.1: Hasse diagrams of elements of $Y_{k,n}$, $1 \le k \le n-1$, $n \ge 3$.

7.1 Fixing the Isomorphism Class

Consider those isomorphism classes in $\mathcal{P}_n$ whose elements are chains with one point replaced by an antichain of size 2: for a linear order $\sigma = x_1x_2\dots x_n \in \mathcal{L}_n$, $n \ge 3$, and $1 \le k \le n-1$, define

$$y_k(\sigma) = \sigma \setminus \{(x_k, x_{k+1})\}$$

and set $Y_{k,n} = \{\, y_k(\sigma) : \sigma \in \mathcal{L}_n \,\}$.

7.1.1 Partitioning the Class

To characterise the intersecting subsets of $Y_{k,n}$, we wish to partition the class in such a way that for any $p \in Y_{k,n}$, the elements of the class which p does not intersect are in the same block as p. Therefore we begin by considering the set

$$N(p) = \{\, q \in Y_{k,n} : p \cap q = \emptyset \,\}\,,$$

which is the subject of the lemma below.

Lemma 7.1.1. *If $p \in Y_{k,n}$ then $N(p) = \{\, y_k(\mathrm{rev}(\sigma)) : \sigma \in \mathcal{L}_n(p) \,\}$.*

Proof. Let $p, q \in Y_{k,n}$ be such that p and q do not intersect. Then there exist linear extensions $\hat{p}$ of p and $\hat{q}$ of q such that $\hat{p}$ and $\hat{q}$ do not intersect. But two linear orders do not intersect if, and only if, one is the reverse of the other. Thus $\hat{q} = \mathrm{rev}(\hat{p})$, which gives $q = y_k(\mathrm{rev}(\hat{p}))$. In other words,

$$N(p) \subseteq \{\, y_k(\mathrm{rev}(\sigma)) : \sigma \in \mathcal{L}_n(p) \,\}\,.$$

Conversely, for $p \in Y_{k,n}$ and $\sigma \in \mathcal{L}_n(p)$, we have $p \subset \sigma$ and $y_k(\mathrm{rev}(\sigma)) \subset \mathrm{rev}(\sigma)$ by definition. Therefore $p \cap y_k(\mathrm{rev}(\sigma)) = \emptyset$ since $\sigma \cap \mathrm{rev}(\sigma) = \emptyset$. $\square$

To obtain the blocks of the desired partition of $Y_{k,n}$, we keep adding all such posets to $N(p)$ which do not intersect with some poset that is already in $N(p)$: for a set of posets $X \subseteq Y_{k,n}$, define

$$N(X) = \{\, q \in Y_{k,n} : p \cap q = \emptyset \text{ for some } p \in X \,\}$$

and set

$$B(p) = \bigcup_{i \in \mathbb{N}} N^i(p).$$

Intuitively, $B(p)$ is obtained from linear extensions of p by successively applying the following operations: taking the reverse, de-coupling at the k^{th} level, and swapping at the $(n-k)^{th}$ level. In fact, it is not difficult to convince oneself that when these two levels do not overlap, $B(p)$ contains

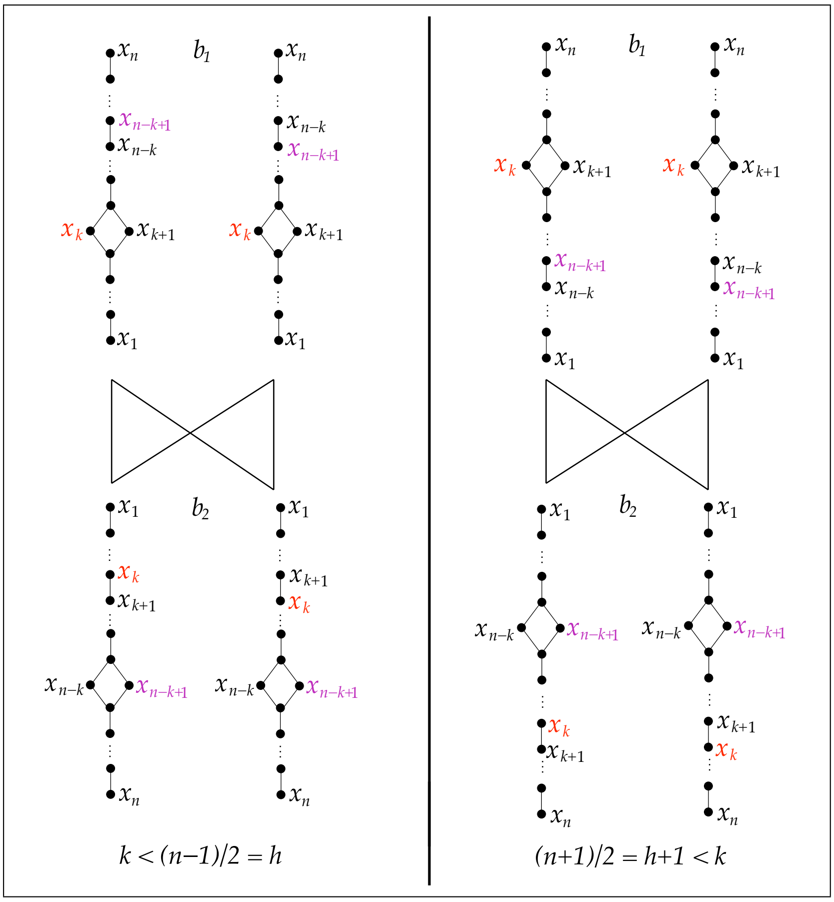


Figure 7.1.2: elements of $B(p)$ where $p = y_k(x_1 \dots x_n)$ and $k < h := (n-1)/2$ on the left, or $k > h+1$ on the right. Posets which do not intersect are joined by a line. On either side, the elements of $b_1(B(p))$ and $b_2(B(p))$ are shown at the top and bottom respectively.

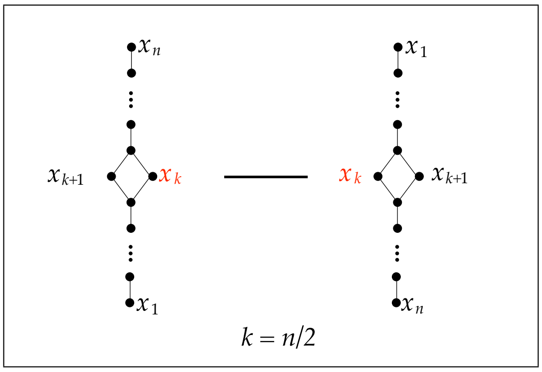


Figure 7.1.3: elements of $B(p)$ when $k = n/2$ and $p = y_k(x_1 \dots x_n)$. Posets which do not intersect are joined by a line. The set $b_1(B(p))$ merely contains the poset on the left. Similarly, $b_2(B(p))$ consists of the poset on the right.

the posets in Figure 7.1.2. If $k = n - k$ then $B(p)$ has only one element other than p, which is obtained by turning p upside down: see Figure 7.1.3. A slightly more complicated situation occurs when the k^{th} and $(n-k)^{th}$ level overlap but do not coincide; see Figure 7.1.4.

To summarise, we have

$$|B(p)| = \begin{cases} 4 & k \notin \{\frac{n}{2}, \frac{n\pm 1}{2}\} \\ 2 & k = \frac{n}{2} \\ 6 & k = \frac{n\pm 1}{2} \end{cases}$$

and the precise elements of $B(p)$ are given by Lemma 7.1.3, which is the main auxiliary result enabling us to obtain a bound on the size of intersecting subsets of $Y_{k,n}$.

Before stating the lemma, note that we already have the de-coupling operator y_k. To introduce a swapping operator, we simply let transpositions $(i\ j) \in \mathcal{S}_n$ act on orders by permuting the labels: for $p \in \mathcal{P}_n$,

$$(i\ j)p = \{\, (i\ j)(x, y) : (x, y) \in p \,\}\,.$$

This action commutes with the operators rev and y_k: the following lemma is easily proved.

Lemma 7.1.2. *Let $\sigma \in \mathcal{L}_n$ and $i, j \in [n]$. Then $\operatorname{rev}((i\ j)\sigma) = (i\ j)\operatorname{rev}(\sigma)$ and $y_k((i\ j)\sigma) = (i\ j)y_k(\sigma)$.*

Proof. It is clear that

$$\begin{aligned} \operatorname{rev}((i\ j)\sigma) &= \operatorname{rev}(\{\, (i\ j)(x, y) : (x, y) \in \sigma \,\}) \\ &= \{\, (i\ j)(y, x) : (x, y) \in \sigma \,\} = (i\ j)\operatorname{rev}(\sigma). \end{aligned}$$

The second fact is equally simple if $k \notin \{i, j\}$. Moreover, for $\sigma = x_1 \ldots x_n$, Figure 7.1.5 shows that $y_i((i\ j)\sigma) = (i\ j)y_i(\sigma)$. The case $k = j$ is very similar. □

The following lemma formalises the partition.

Lemma 7.1.3. *For positive integers k and n with $1 \leq k \leq n-1$, set $h = \frac{n-1}{2}$. Let $p \in Y_{k,n}$ and $\sigma = x_1 \ldots x_n \in \mathcal{L}_n$ such that $p = y_k(\sigma)$.*

1. *If $k \notin \{h, h+1\}$ then*

$$B(p) = \{y_k(\sigma),\ y_k(\mathrm{rev}(\sigma)),\ (x_{n-k}\ x_{n-k+1})y_k(\sigma),\ (x_k\ x_{k+1})y_k(\mathrm{rev}(\sigma))\}.$$

2. *If $k \in \{h, h+1\}$ then $B(p) = \{\, y_k(\omega), y_k(\mathrm{rev}(\omega)) : \omega \in \Omega_\sigma \,\}$, where*

$$\Omega_\sigma = \{\, x_1 \ldots x_{h-1} u\, v\, w\, x_{h+3} \ldots x_n : \{u, v, w\} = \{x_h, x_{h+1}, x_{h+2}\}\,\}\,.$$

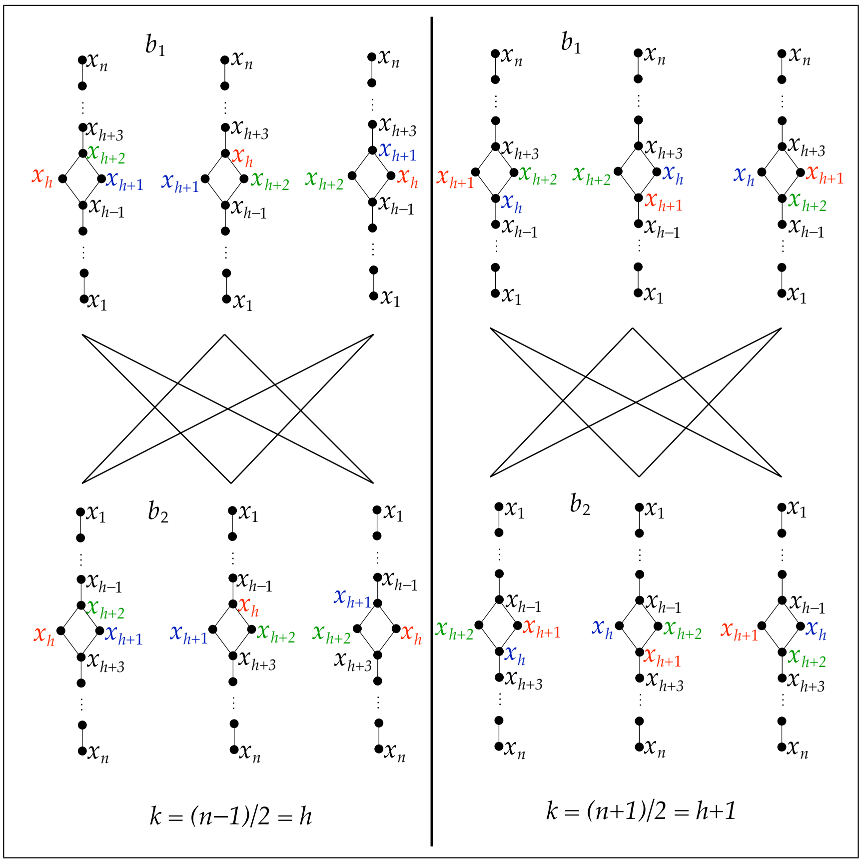


Figure 7.1.4: elements of $B(p)$ where $p = y_k(x_1 \ldots x_n)$ and $k = h$ or $k = h+1$. Posets which do not intersect are joined by a line. On either side, the elements of $b_1(B(p))$ and $b_2(B(p))$ are shown at the top and bottom respectively.

Note that the case $k = n/2$ is subsumed in (1) in the statement of Lemma 7.1.3. In this case, the swapping operator $(x_{n-k}\ x_{n-k+1}) = (x_k\ x_{k+1})$ swaps the two incomparable points in elements of $Y_{k,n}$ and hence does not change the poset: we have $B(p) = \{y_k(\sigma),\ y_k(\mathrm{rev}(\sigma))\}$ when $k = n/2$.

A formal proof of Lemma 7.1.3 is included here for completeness, though referring to Figures 7.1.2 - 7.1.4 may in fact be more enlightening for the reader.

Proof. We will use Lemma 7.1.2 frequently throughout this proof, without necessarily making this explicit. Let $p = y_k(\sigma) \in Y_{k,n}$ with $\sigma = x_1 \ldots x_n$. Then $\mathcal{L}_n(p) = \{\sigma, (x_k\ x_{k+1})\sigma\}$, so it follows from Lemma 7.1.1 that

$$N(p) \quad = \quad \{y_k(\mathrm{rev}(\sigma)),\ y_k(\mathrm{rev}((x_k\ x_{k+1})\sigma))\}. \tag{7.1.4}$$

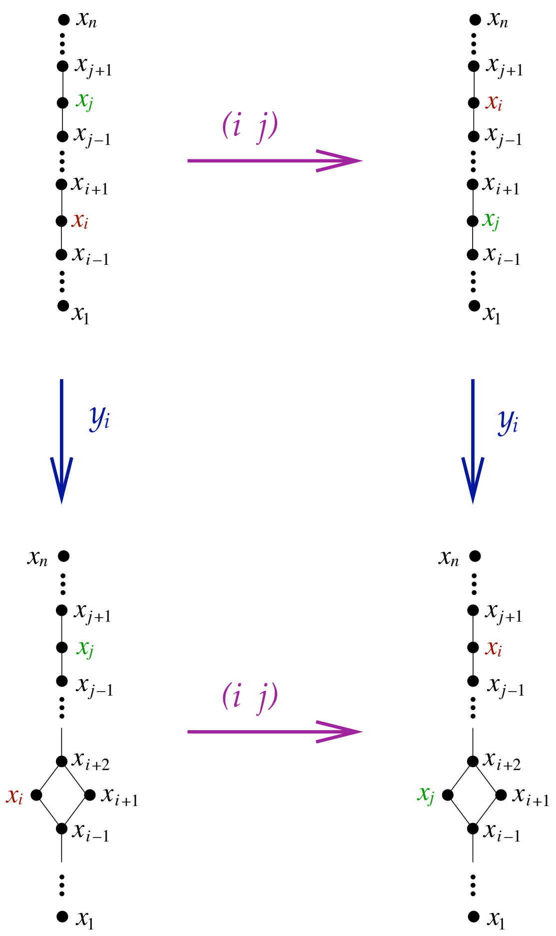


Figure 7.1.5: the proof of Lemma 7.1.2 uses the fact that this diagram commutes.

To determine $B(p)$, we need to investigate

$$N(N(p)) = N(q_1) \cup N(q_2)$$

where $q_1 = y_k(\mathrm{rev}(\sigma))$ and $q_2 = y_k(\mathrm{rev}((x_k\ x_{k+1})\sigma))$. Since the k^{th} and $(k+1)^{st}$ smallest points in $\mathrm{rev}(\sigma) = x_n \dots x_1$ are x_{n-k+1} and x_{n-k} respectively, we have

$$\mathcal{L}_n(q_1) = \{\mathrm{rev}(\sigma),\ (x_{n-k+1}\ x_{n-k})\,\mathrm{rev}(\sigma)\}, \tag{7.1.5}$$

so we use Lemmas 7.1.1 and 7.1.2 to obtain

$$\begin{aligned} N(q_1) &= \{y_k(\mathrm{rev}(\mathrm{rev}(\sigma))),\ y_k(\mathrm{rev}((x_{n-k+1}\ x_{n-k})\,\mathrm{rev}(\sigma)))\} \\ &= \{y_k(\sigma),\ (x_{n-k+1}\ x_{n-k})y_k(\sigma)\} \\ &= \{p,\ (x_{n-k+1}\ x_{n-k})p\}. \end{aligned} \tag{7.1.6}$$

Case 1: $k = \frac{n}{2}$.

Then for all $\pi \in \mathcal{L}_n$,

$$(x_k\ x_{k+1})y_k(\pi) = y_k(\pi) = (x_{n-k}\ x_{n-k+1})y_k(\pi), \tag{7.1.7}$$

so we need to show

$$B(p) = \left\{y_{\frac{n}{2}}(\sigma),\ y_{\frac{n}{2}}(\mathrm{rev}(\sigma))\right\} = \{p,\ y_k(\mathrm{rev}(\sigma))\}.$$

Applying (7.1.7) to (7.1.4) and (7.1.6), we see that $N(p) = \{y_k(\mathrm{rev}(\sigma))\}$, so $q_1 = q_2$ and

$$N(N(p)) = N(q_1) = \{y_k(\sigma)\} = \{p\}.$$

Thus for natural numbers i,

$$N^i(p) = \begin{cases} \{p\} & i \text{ even} \\ N(p) & i \text{ odd} \end{cases},$$

which implies $B(p) = \{p\} \cup N(p) = \{p\} \cup \{y_k(\mathrm{rev}(\sigma))\}$ as required.

Case 2: $k \notin \left\{\frac{n-1}{2}, \frac{n}{2}, \frac{n+1}{2}\right\}$.

Recall that we need to investigate $N(q_1) \cup N(q_2)$, and $N(q_2)$ depends on $\mathcal{L}_n(q_2)$ where

$$q_2 = y_k((x_k\ x_{k+1})\,\mathrm{rev}(\sigma)).$$

Reconsidering the arguments preceding (7.1.5), what are the k^{th} and $(k+1)^{st}$ smallest points in

$$(x_k\ x_{k+1})\,\mathrm{rev}(\sigma) = (x_k\ x_{k+1})x_n \dots x_1?$$

By the definition of case 2, either $k < \frac{n-1}{2}$, in which case $n-k > k+1$, or $k > \frac{n+1}{2}$ which implies $n-k+1 < k$. Therefore in case 2 overall, the elements x_k, x_{k+1}, x_{n-k}, x_{n-k+1} of $[n]$ must be distinct. Thus just as in $\mathrm{rev}(\sigma)$, the k^{th} and $(k+1)^{st}$ smallest points in $(x_k\ x_{k+1})\,\mathrm{rev}(\sigma)$ are x_{n-k+1} and x_{n-k} respectively, and so

$$\mathcal{L}_n(q_2) = \{(x_k\ x_{k+1})\,\mathrm{rev}(\sigma),\ (x_{n-k+1}\ x_{n-k})(x_k\ x_{k+1})\,\mathrm{rev}(\sigma)\}.$$

We use this together with Lemmas 7.1.1 and 7.1.2 to obtain

$$\begin{aligned} N(q_2) &= \{y_k((x_k\ x_{k+1})\sigma),\ y_k((x_{n-k+1}\ x_{n-k})(x_k\ x_{k+1})\sigma)\} \\ &= \{y_k(\sigma),\ y_k((x_{n-k+1}\ x_{n-k})\sigma)\} \\ &= \{p,\ (x_{n-k+1}\ x_{n-k})p\} = N(q_1). \end{aligned}$$

Hence

$$N^2(p) = N(q_1) \cup N(q_2) = \{p,\ (x_{n-k+1}\ x_{n-k})p\}$$

and $N^3(p) = N(p) \cup N((x_{n-k+1}\ x_{n-k})p)$.

Again using the fact that $\{x_k, x_{k+1}\}$ and $\{x_{n-k}, x_{n-k+1}\}$ are disjoint, we see that

$$\begin{aligned} \mathcal{L}_n((x_{n-k+1}\ x_{n-k})p) &= \mathcal{L}_n(y_k((x_{n-k+1}\ x_{n-k})\sigma)) \\ &= \{(x_{n-k+1}\ x_{n-k})\sigma,\ (x_k\ x_{k+1})(x_{n-k+1}\ x_{n-k})\sigma\} \end{aligned}$$

and so $N((x_{n-k+1}\ x_{n-k})p)$ must be equal to

$$\begin{aligned} &\{y_k(\mathrm{rev}((x_{n-k+1}\ x_{n-k})\sigma)),\ y_k(\mathrm{rev}((x_k\ x_{k+1})(x_{n-k+1}\ x_{n-k})\sigma))\} \\ =\ &\{(x_{n-k+1}\ x_{n-k})y_k(\mathrm{rev}(\sigma)),\ (x_{n-k+1}\ x_{n-k})y_k((x_k\ x_{k+1})\,\mathrm{rev}(\sigma))\}. \end{aligned}$$

But recall that in both posets $\mathrm{rev}(\sigma)$ and $(x_k\ x_{k+1})\,\mathrm{rev}(\sigma)$, the k^{th} and $(k+1)^{st}$ smallest points are x_{n-k+1} and x_{n-k} respectively, so

$$\begin{aligned} (x_{n-k+1}\ x_{n-k})y_k(\mathrm{rev}(\sigma)) &= y_k(\mathrm{rev}(\sigma)) = q_1, \\ (x_{n-k+1}\ x_{n-k})y_k((x_k\ x_{k+1})\,\mathrm{rev}(\sigma)) &= y_k((x_k\ x_{k+1})\,\mathrm{rev}(\sigma)) = q_2. \end{aligned}$$

Thus

$$N^3(p) = N(p) \cup N((x_{n-k+1}\ x_{n-k})p) = N(p) \cup \{q_1,\ q_2\} = N(p),$$

in other words, we do not get any new elements in $N^i(p)$ when $i \geq 3$. Hence

$$\begin{aligned} B(p) &= N(p) \cup N^2(p) \qquad (7.1.8) \\ &= \{y_k(\mathrm{rev}(\sigma)),\ y_k((x_k\ x_{k+1})\,\mathrm{rev}(\sigma))\} \cup \{p,\ (x_{n-k+1}\ x_{n-k})p\} \end{aligned}$$

as required.

The remaining two cases, when $k = \frac{n\pm 1}{2}$, are more complicated because the k^{th} and $(k+1)^{st}$ smallest points in $(x_k\ x_{k+1})\,\mathrm{rev}(\sigma)$ are not the same as those in $\mathrm{rev}(\sigma)$.

For the remainder of this proof, let

$$\begin{array}{rcl} \omega_{123} = \sigma & , & \omega_{132} = \left(x_{\frac{n+1}{2}}\ x_{\frac{n+3}{2}}\right)\sigma, \\ \omega_{213} = \left(x_{\frac{n-1}{2}}\ x_{\frac{n+1}{2}}\right)\sigma & , & \omega_{231} = \left(x_{\frac{n-1}{2}}\ x_{\frac{n+1}{2}}\ x_{\frac{n+3}{2}}\right)\sigma, \\ \omega_{312} = \left(x_{\frac{n-1}{2}}\ x_{\frac{n+3}{2}}\ x_{\frac{n+1}{2}}\right)\sigma & , & \omega_{321} = \left(x_{\frac{n-1}{2}}\ x_{\frac{n+3}{2}}\right)\sigma. \end{array}$$

Then $\Omega_\sigma = \{\, \omega_{abc} : \{a,b,c\} = \{1,2,3\} \,\}$ and $q_1 = y_k(\mathrm{rev}(\omega_{123}))$. Set

$$B' = \{\, y_k(\omega), y_k(\mathrm{rev}(\omega)) : \omega \in \Omega_\sigma \,\}\,.$$

We need to demonstrate that $B(p) = B'$.

<u>Case 3: $k = \frac{n-1}{2}$.</u>

Then $q_2 = y_k((x_k\ x_{k+1})\,\mathrm{rev}(\sigma)) = y_k(\mathrm{rev}(\omega_{213}))$, so

$$N(p) = \{q_1,\ q_2\} = \{y_k(\mathrm{rev}(\omega_{123})),\ y_k(\mathrm{rev}(\omega_{213}))\}.$$

In particular,

$$\begin{array}{rcl} q_2 & = & y_{\frac{n-1}{2}}\left(x_n \ldots x_{\frac{n+3}{2}} x_{\frac{n-1}{2}} x_{\frac{n+1}{2}} x_{\frac{n-3}{2}} \ldots x_1\right) \\ & = & x_n \ldots x_{\frac{n+3}{2}} x_{\frac{n-1}{2}} x_{\frac{n+1}{2}} x_{\frac{n-3}{2}} \ldots x_1 \ \setminus \left\{\left(x_{\frac{n+3}{2}}, x_{\frac{n-1}{2}}\right)\right\}. \end{array}$$

Therefore $\mathcal{L}_n(q_2) = \{\mathrm{rev}(\omega_{213}), \mathrm{rev}(\omega_{231})\}$ which, by Lemma 7.1.1, implies

$$N(q_2) = \{y_k(\omega_{213}),\ y_k(\omega_{231})\}.$$

Simply substituting $k = \frac{n-1}{2}$ into (7.1.6) gives

$$N(q_1) \quad = \quad \{y_k(\omega_{123}),\ y_k(\omega_{132})\}.$$

So far, we have

$$\begin{array}{rcl} B(p) & \supseteq & N(p) \cup N^2(p) \\ & = & N(p) \cup (N(q_1) \cup N(q_2)) \\ & = & \{y_k(\mathrm{rev}(\omega_{123})),\ y_k(\mathrm{rev}(\omega_{213})), \\ & & \quad y_k(\omega_{123}),\ y_k(\omega_{132}),\ y_k(\omega_{213}),\ y_k(\omega_{231})\}. \end{array} \tag{7.1.9}$$

Note that $k = \frac{n-1}{2}$ implies

$$\begin{aligned} y_k(\omega_{abc}) &= y_k(\omega_{bac}), && (7.1.10) \\ y_k(\mathrm{rev}(\omega_{abc})) &= y_k(\mathrm{rev}(\omega_{acb})) && (7.1.11) \end{aligned}$$

for all $\{a, b, c\} = \{1, 2, 3\}$. Thus (7.1.9) becomes

$$B(p) \supseteq \{y_k(\omega),\ y_k(\mathrm{rev}(\omega_{1b_1c_1})),\ y_k(\mathrm{rev}(\omega_{2b_2c_2}))\} \tag{7.1.12}$$

where $\omega \in \Omega_\sigma$ and $\{b_1, c_1\} = \{2, 3\}$, $\{b_2, c_2\} = \{1, 3\}$.

From (7.1.12) we see that $y_k(\omega_{321}) \in B(p)$. Clearly, $\omega_{321} \cap \mathrm{rev}(\omega_{321}) = \emptyset$, and so $y_k(\omega_{321})$ and $y_k(\mathrm{rev}(\omega_{321}))$ do not intersect either. Hence $y_k(\omega_{321}) \in B(p)$ implies

$$y_k(\mathrm{rev}(\omega_{321})) = y_k(\mathrm{rev}(\omega_{312})) \in B(p),$$

which completes (7.1.12) to

$$B(p) \supseteq \{\, y_k(\omega),\ y_k(\mathrm{rev}(\omega)) : \omega \in \Omega_\sigma \,\} = B'.$$

To prove $B(p) \subseteq B'$, it suffices to show that B' is closed under N, i.e. that $N(q) \subseteq B'$ for any $q \in B'$. Now for any

$$q \in B' = \left\{ y_{\frac{n-1}{2}}(\omega),\ y_{\frac{n-1}{2}}(\mathrm{rev}(\omega)) : \omega \in \Omega_\sigma \right\},$$

if $a, b \in [n]$ are incomparable under q then $a, b \in \left\{ x_{\frac{n-1}{2}}, x_{\frac{n+1}{2}}, x_{\frac{n+3}{2}} \right\}$. Thus

$$\mathcal{L}_n(q) \subseteq \{\, \omega, \mathrm{rev}(\omega) : \omega \in \Omega_\sigma \,\}$$

which, together with Lemma 7.1.1, gives

$$N(q) = \{\, y_k(\mathrm{rev}(\pi)) : \pi \in \mathcal{L}_n(q) \,\} \subseteq \{\, y_k(\mathrm{rev}(\omega)), y_k(\omega)) : \omega \in \Omega_\sigma \,\} = B'$$

as required.

Case 4: $k = \frac{n+1}{2}$.

This is very similar to case 3 and therefore omitted. □

Observe from Figures 7.1.2 - 7.1.4 that B is generated by any of its elements: for any $q \in B(p)$, we have $B(q) = B(p)$ and so

$$\mathcal{B}_{k,n} = \{\, B(p) : p \in Y_{k,n} \,\}$$

is a partition of $Y_{k,n}$.

7.1.2 A Bound and a Maximum Family

To characterise the maximum intersecting subsets of $Y_{k,n}$, we partition each $B(p)$ in two halves, such that for every $q \in B(p)$, the posets which do not intersect with q are not in the same half as q: see Figures 7.1.2 - 7.1.4. More formally, for $p \in Y_{k,n}$ with $p = y_k(\sigma)$, $\sigma = x_1 \ldots x_n$, let τ_{n-k} be the transposition swapping x_{n-k} with x_{n-k+1} and set

$$
\begin{aligned}
b_1(B(p)) &= \begin{cases} \{y_k(\sigma),\ y_k(\tau_{n-k}\sigma)\} & k \neq \frac{n\pm1}{2} \\ \{\, y_k(\omega) : \omega \in \Omega_\sigma \,\} & k = \frac{n\pm1}{2} \end{cases}, \\
b_2(B(p)) &= \begin{cases} \{y_k(\mathrm{rev}(\sigma)),\ y_k(\tau_{n-k}\,\mathrm{rev}(\sigma))\} & k \neq \frac{n\pm1}{2} \\ \{\, y_k(\mathrm{rev}(\omega)) : \omega \in \Omega_\sigma \,\} & k = \frac{n\pm1}{2} \end{cases},
\end{aligned}
$$

where Ω_σ is given in Lemma 7.1.3. Note that the partition $\{b_1(B), b_2(B)\}$ of $B = B(p)$ does not depend on the choice of p.

Theorem 7.1.13. *Let k, n be natural numbers with $1 \leq k < n$ and $n \geq 4$.*

Then $\mathcal{F}$ is a maximum intersecting subset of $Y_{k,n}$ if, and only if, $\mathcal{F}$ is the union of a transversal of

$$\{\, \{b_1(B), b_2(B)\} : B \in \mathcal{B}_{k,n} \,\}\,.$$

Proof. Let $B \in \mathcal{B}_{k,n}$. It is clear from the definitions of N and B that for any $q \in B$, all posets in $Y_{k,n}$ which do not intersect with q are also in B. Thus if $\{\, \mathcal{F}'(B) : B \in \mathcal{B}_{k,n} \,\}$, is a collection of intersecting families with $\mathcal{F}'(B) \subseteq B$, then their union is also intersecting. Conversely, any intersecting subset of $Y_{k,n}$ can be decomposed in this way.

Indeed, let $\mathcal{F} \subseteq Y_{k,n}$ be maximum intersecting. Then we must have

$$\mathcal{F} = \bigcup_{B \in \mathcal{B}_{k,n}} \mathcal{F}'(B)$$

where for each $B \in \mathcal{B}_{k,n}$, the set $\mathcal{F}'(B) = \mathcal{F} \cap B$ is a maximum intersecting subset of B. So to prove the proposition, we need to demonstrate that for $B \in \mathcal{B}_{k,n}$, if $\mathcal{F}'$ is a maximum intersecting subset of B, then either $\mathcal{F}' = b_1(B)$ or $\mathcal{F}' = b_2(B)$.

We begin by showing that $b_1(B)$ and $b_2(B)$ are intersecting and maximal in terms of set inclusion. Let $p \in Y_{k,n}$, $B = B(p)$ and $\sigma = x_1 \ldots x_n \in \mathcal{L}_n$ such that $p = y_k(\sigma)$. We begin by considering the case $k \neq (n \pm 1)/2$ and let $\pi \in \{\sigma, \mathrm{rev}(\sigma)\}$. The transposition $\tau_i := (x_i\ x_{i+1})$ only replaces a single comparison in π by its reverse, so provided $|y_k(\pi)| \geq 2$, $y_k(\pi)$ must intersect $y_k(\tau_i(\pi))$. Similarly, the operator y_k only removes a single comparison from π, so $|y_k(\pi)| \geq 2$ is satisfied whenever $|\pi| \geq 3$. Since $\pi \in \mathcal{L}_n$, we have $|\pi| \geq 3$ for all $n \geq 3$. We have shown

$$n \geq 3,\ k \neq \frac{n \pm 1}{2},\ q \in b_i(B) \implies N(q) \subseteq b_j(B),\ \{i,j\} = \{1,2\}, \tag{7.1.14}$$

in other words, $b_1(B)$ and $b_2(B)$ are intersecting.

Now suppose $k = (n \pm 1)/2$. This clearly requires n to be odd, so we have $n \geq 5$, which guarantees

$$(x_1, x_n) \in \bigcap_{\omega \in \Omega_\sigma} y_k(\omega),\quad (x_n, x_1) \in \bigcap_{\omega \in \Omega_\sigma} y_k(\mathrm{rev}(\omega)).$$

Thus again, $b_1(B)$ and $b_2(B)$ are intersecting for all $B \in \mathcal{B}_{k,n}$.

Recall that posets are only added to B if they do not intersect with some element of B. Indeed, it is clear from Figures 7.1.2 - 7.1.4 that

$$q \in b_i(B) \implies \exists q' \in b_j(B) \text{ such that } q \cap q' = \emptyset,\ \{i,j\} = \{1,2\}.$$

Therefore both $b_1(B)$ and $b_2(B)$ are maximal under set inclusion as intersecting subsets of B.

It remains to be shown that $b_1(B)$ and $b_2(B)$ are maximum among intersecting subsets of B. If $k = n/2$ then this follows immediately from Figure 7.1.3. If $k \notin \left\{\frac{n-1}{2}, \frac{n}{2}, \frac{n+1}{2}\right\}$ then it follows from Figure 7.1.2 that (7.1.14) becomes

$$q \in b_i(B) \implies b_j(B) = N(q),\ \{i,j\} = \{1,2\}.$$

Thus any intersecting subsets of B must be contained in either $b_1(B)$ or $b_2(B)$, as required.

Finally, let $k = \frac{n \pm 1}{2}$ and let $\mathcal{F}'$ be an intersecting subset of B of size $|\mathcal{F}'| \geq |b_i(B)| = 3$. By the pigeonhole principle, $|\mathcal{F}' \cap b_i(B)| = 2$ for some $i \in \{1,2\} = \{i,j\}$; say $\mathcal{F}' \cap b_i(B) = \{q_1, q_2\}$. But then it is clear from Figure 7.1.4 that any element of $b_j(B)$ does not intersect with at least one of q_1, q_2. Thus $\mathcal{F}' \cap b_j(B) = \emptyset$, which contradicts $|\mathcal{F}'| \geq |b_i(B)| = 3$. □

Corollary 7.1.15. *Let $\mathcal{F}$ be an intersecting subset of $Y_{k,n}$. Then*

$$|\mathcal{F}| \leq \frac{|Y_{k,n}|}{2} = \frac{n!}{4}.$$

Proof. Any maximum intersecting $\mathcal{F} \subseteq Y_{k,n}$ must be the union of a transversal of

$$\{\, \{b_1(B), b_2(B)\} : B \in \mathcal{B}_{k,n} \,\}$$

by Theorem 7.1.13. Let $\gamma : \mathcal{B}_{k,n} \to \{1, 2\}$ be the function assigning the appropriate sub-blocks $b_i(B)$ to $\mathcal{F}$, that is

$$\mathcal{F} = \bigcup_{B \in \mathcal{B}_{k,n}} b_{\gamma(B)}(B).$$

We noted before that any block $B \in \mathcal{B}_{k,n}$ is generated by any of its elements and it is clear from Figures 7.1.2 - 7.1.4 that for fixed k, all $B \in \mathcal{B}_{k,n}$ have the same size. Thus $\mathcal{B}_{k,n}$ is a partition of $Y_{k,n}$ into parts of equal size. Moreover, for any $B \in \mathcal{B}_{k,n}$, both $b_1(B)$ and $b_2(B)$ have half the size of B. In conclusion,

$$|\mathcal{F}| = \sum_{B \in \mathcal{B}_{k,n}} |b_{\gamma(B)}(B)| = \sum_{B \in \mathcal{B}_{k,n}} \frac{|B|}{2} = \frac{|Y_{k,n}|}{2} = \frac{n!}{4}$$

as required. $\square$

As we have seen in previous chapters, fixing and saturating are two common forms in which solutions to extremal problems occur. For the classes $Y_{k,n}$, we will prove that saturation is optimal unless k is around $n/2$, but fixing is never optimal.

Remark 7.1.16. The fix-family

$$F_{i,j}(Y_{k,n}) = \{\, p \in Y_{k,n} : (i,j) \in p \,\}$$

is not optimal in $Y_{k,n}$ for any $i, j \in [n]$: by Theorem 7.1.13, to show that $F_{i,j}$ does not attain the bound of Corollary 7.1.15, it suffices to find $B \in \mathcal{B}_{k,n}$ such that neither $b_1(B)$ nor $b_2(B)$ are entirely contained in $F_{i,j}$.

Let $p \in Y_{k,n}$ such that $i||_p j$. By the definition of $Y_{k,n}$, such a p exists for any $k \in [n-1]$. Then p cannot be an element of $F_{i,j}$, so $b_1(B(p)) \not\subset F_{i,j}$ since $p \in b_1(B(p))$. Set $h = \frac{n-1}{2}$ as before.

- If $h \le k \le h+1$ then it is easily seen from Figures 7.1.3 and 7.1.4 that $b_2(B(p))$ must contain an element q with $i||_q j$, which implies $q \notin F_{i,j}$ by the definition of $F_{i,j}$.
- Otherwise, it follows from Figure 7.1.2 that there exists $q \in b_2(B(p))$ with $i >_q j$ so again, $q \notin F_{i,j}$.

Hence fix-families are not optimal in $Y_{k,n}$. On the other hand, Propositions 7.1.17 - 7.1.18 show that, provided the k^{th} and $(n-k)^{th}$ level do not interact, we can find saturation families which are optimal.

Proposition 7.1.17. *For positive integers k, n with $k < n$ and n even, let $v_n \in \mathcal{P}_n$ be the poset $v_n = \{\, (i,n) : 1 \le i \le n-1 \,\}$ and define*

$$G(Y_{k,n}) = \{\, p \in Y_{k,n} : |p \cap v_n| \ge n/2 \,\}.$$

Then $G(Y_{k,n})$ is an intersecting subset of $Y_{k,n}$ of size

$$|G(Y_{k,n})| = \begin{cases} (n-2)\cdot(n-1)!/4 & \text{if } k = n/2 \\ n!/4 & \text{otherwise} \end{cases}.$$

Proof. Firstly, observe that $G(Y_{k,n})$ is intersecting: any two elements of $G(Y_{k,n})$ contain at least $n/2$ elements of the $(n-1)$-element set v_n, so they must have at least one in common by the pigeonhole principle.

Now let $p \in G(Y_{k,n})$ and let $\sigma = x_1 \dots x_n \in \mathcal{L}_n$ be such that $p = y_k(\sigma)$. Then $|p \cap v_n| \geq n/2$, in other words, p must contain at least $n/2$ distinct comparisons (a, n) for some $a \in [n-1]$.

If $k \neq n/2$ then clearly $|p \cap v_n| \geq n/2$ if, and only if, $n \in \{x_{n/2+1}, \dots, x_n\}$. Precisely half of the elements of $Y_{k,n}$ have n in the top half, and so

$$|G(Y_{k,n})| = \frac{|Y_{k,n}|}{2} = \frac{n!}{4}, \qquad k \neq n/2.$$

If $k = n/2$ and $n \in \{x_k, x_{k+1}\}$, then there are only $n/2 - 1$ elements below n with respect to p, and so $p \notin G(Y_{k,n})$. Thus we must have $n \in \{x_{n/2+2}, \dots, x_n\}$, giving

$$|G(Y_{k,n})| = (n/2-1)\cdot\frac{(n-1)!}{2} = \frac{(n-2)\cdot(n-1)!}{4}, \qquad k = n/2,$$

as required. □

Proposition 7.1.18. *For positive integers k, h, n with $k < n = 2h+1$, let $v_n \in \mathcal{P}_n$ be the poset $v_n = \{\, (i, n) : 1 \leq i \leq n-1 \,\}$ and define*

$$G(Y_{k,n}) = \{\, p \in Y_{k,n} : \textit{either } |p \cap v_n| \geq h+1 \textit{ or } |p \cap v_n| = h,\ 1 <_p n \,\}.$$

Then $G(Y_{k,n})$ is an intersecting subset of $Y_{k,n}$ of size

$$|G(Y_{k,n})| = \begin{cases} (n-1)\cdot(n-1)!/4 & k \in \{h, h+1\} \\ n!/4 & \text{otherwise} \end{cases}.$$

Proof. Firstly, we show that $G(Y_{k,n})$ is intersecting: set

$$\begin{aligned} A_{k,n} &= \{\, p \in Y_{k,n} : |p \cap v_n| \geq h+1 \,\}, \\ B_{k,n} &= \{\, p \in Y_{k,n} : 1 <_p n,\ |p \cap v_n| = h \,\}, \end{aligned}$$

so $G(Y_{k,n}) = A_{k,n} \cup B_{k,n}$, and let $p, q \in G(Y_{k,n})$. If $p, q \in A_{k,n}$ then they both contain at least $h+1$ elements of the $2h$-element set v_n, which guarantees $|p \cap q| \geq 2$. If $p, q \in B_{k,n}$ then $(1, n) \in p \cap q$. If, without loss of generality, $p \in A_{k,n}$ and $q \in B_{k,n}$ then

$$|p \cap v_n| + |q \cap v_n| \geq (h+1) + h = n > |v_n| = n-1.$$

Thus, again by the pigeonhole principle, p, q and v_n must share at least one comparison. In conclusion, $G(Y_{k,n})$ is intersecting.

Now let $p \in A_{k,n}$ and let $\sigma = x_1 \dots x_n \in \mathcal{L}_n$ such that $p = y_k(\sigma)$. Then p must contain at least $h+1$ distinct comparisons (a, n) for some $a \in [n-1]$. If $k \neq h+1$ then clearly $|p \cap v_n| \geq h+1$ if, and only if, $n \in \{x_{h+2}, \dots, x_n\}$. Thus

$$|A_{k,n}| = (n-(h+1)) \cdot (n-1)!/2 = (n-1) \cdot (n-1)!/4, \qquad k \neq h+1.$$

If $k = h+1$ and $n \in \{x_{h+1}, x_{h+2}\}$, then there are only h elements below n with respect to p, so $p \notin A_{k,n}$. Thus we must have $n \in \{x_{h+3}, \dots, x_n\}$, giving

$$|A_{k,n}| = (n-(h+2)) \cdot (n-1)!/2 = (n-3) \cdot (n-1)!/4, \qquad k = h+1.$$

Now let $p \in B_{k,n}$ and $\sigma = x_1 \dots x_n \in \mathcal{L}_n$ such that $p = y_k(\sigma)$. Then p contains precisely h comparisons of the form (a, n) for some $a \in [n-1]$. If $k = h$ then such a p does not exist. For $k \neq h$ we have $|p \cap v_n| = h$ if, and only if, $x_{h+1} = n$. The other condition for $p \in B_{k,n}$ is $1 <_p n$ which is now clearly equivalent to $1 \in \{x_1, \dots, x_h\}$.

If $k = h+1$, then $Y_{k,n}$ has $(n-1)!$ elements p with $x_{h+1} = n$ and precisely half of them have $1 \in \{x_1, \dots, x_h\}$. If $k \notin \{h, h+1\}$, then $Y_{k,n}$ has $(n-1)!/2$ elements p with $x_{h+1} = n$, and precisely half of these have $1 \in \{x_1, \dots, x_h\}$.

In summary, we have

$$|B_{k,n}| = \begin{cases} (n-1)!/2 & k = h+1 \\ 0 & k = h \\ (n-1)!/4 & \text{otherwise} \end{cases}$$

and, since $A_{k,n}$ and $B_{k,n}$ are disjoint,

$$|G(Y_{k,n})| = |A_{k,n}| + |B_{k,n}| = \begin{cases} (n-1) \cdot (n-1)!/4 & k \in \{h, h+1\} \\ n!/4 & \text{otherwise} \end{cases}$$

as required. □

Summary

In this section, we obtained a complete classification of maximum intersecting subsets of $Y_{k,n}$ in Theorem 7.1.13. Moreover, we saw that

- if $h \le k \le h+1$, where $h = (n-1)/2$, then neither the fix-family $F_{i,j}(Y_{k,n})$ nor the saturation family $G(Y_{k,n})$ are optimal in $Y_{k,n}$.
- For the remaining values of k, $G(Y_{k,n})$ is optimal but $F_{i,j}(Y_{k,n})$ is not.

We now turn our attention to a larger class of posets. So far, our approach was to fix the poset and take our class to be all permutations of the labels. The following section investigates what happens if we consider the posets in all classes $Y_{k,n}$ together.

7.2 The Union of the Almost Linear Posets

We now wish to study maximum intersecting subsets of the class $\mathcal{M}_n$ of posets which are one comparison away from being linear. It is easy to see that a linear order on n points contains $r_n = n(n-1)/2$ comparisons, so we define $\mathcal{M}_n$ as

$$\mathcal{M}_n = \{\, p \in \mathcal{P}_n : |p| = r_n - 1 \,\}\,.$$

Our first lemma relates $\mathcal{M}_n$ to the classes $Y_{k,n}$ from the previous section.

Lemma 7.2.1. $\mathcal{M}_n = \bigcup_{i=1}^{n-1} Y_{i,n}$.

Proof. Clearly, each $p \in Y_{k,n}$ contains precisely $r_n - 1$ comparisons. Conversely, any element p of $\mathcal{M}_n$ can be obtained from some linear order σ by removing one comparison, say (a, b). Suppose there exists $c \in [n]$ with $a <_\sigma c <_\sigma b$. Since p contains all comparisons of σ other than (a, b), we have $a <_p c$ and $c <_p b$ which implies $a <_p b$, a contradiction. Thus such a c cannot exist, which implies $p \in Y_{k,n}$ for some $k \in [n-1]$. $\square$

7.2.1 Cyclic Arrangements

In extremal combinatorics, one standard method of obtaining bounds for the size of 1-intersecting sets of certain combinatorial structures is to arrange elements of the ground set on a circle. As is often the case in mathematics, this method was refined into its present simple state by various contributions over the years. It was introduced in [Kat72] where Katona built on a method of Lubell from [Lub66] by arranging the elements of $[n]$ on a circle to give a simple proof of the bound in the Erdős-Ko-Rado Theorem 1.1.1. Subsequent applications of the method improved its presentation, most recently in [KL06, LW07, BK08]. Our proof of Theorem 7.2.15 is modelled on the presentation in [FG89].

The method of cyclic arrangements relies on the fact that two arbitrary cyclic arrangements have a t-interval in common. Thus it works only for $t = 1$, although some attempts have been made to extend this proof method to higher values of t, see e.g. [HKS01]. Katona himself surveyed the use of his cycle method in [Kat00]. Despite its limitations however, it is a very elegant method well worth of exposition here.

In our context, we wish to arrange the comparisons (a, b) for distinct $a, b \in [n]$ on a circle, so let

$$\mathrm{Comp}_n = \{\, (a, b) : a \neq b,\ a, b \in [n] \,\} \,.$$

Definition 7.2.2. Recall the sequence of comparisons $\alpha(\sigma, x_j)$ from Definition 6.3.2 and denote by $c(\sigma)$ the *cyclic arrangement* of the comparisons on $[n]$ obtained as follows: on one half of the circle, we have

$$\alpha(\sigma, x_2),\ \ \alpha(\sigma, x_3),\ \ \ldots,\ \ \alpha(\sigma, x_n)$$

clockwise in that order, and for all comparisons $x < y \in \mathrm{Comp}_n$, we have $y < x$ directly opposite $x < y$ on $c(\sigma)$.

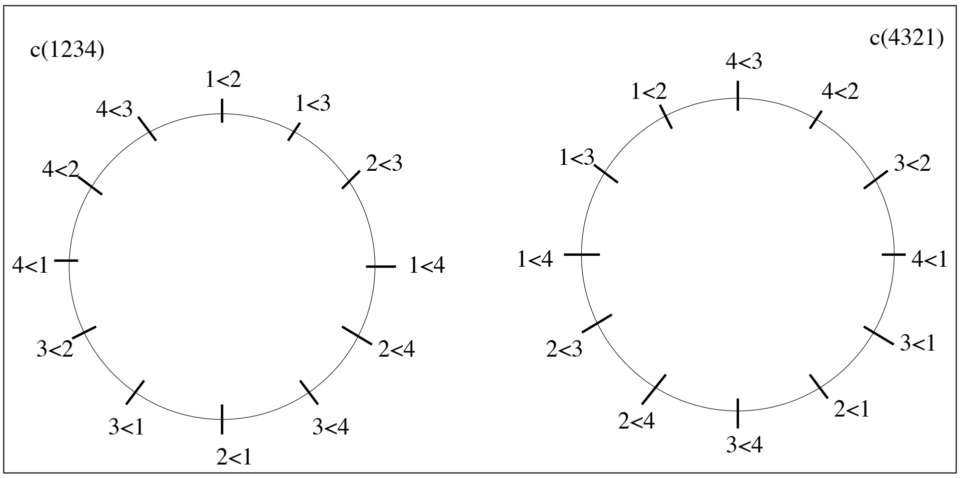


Figure 7.2.1: $c(1234)$ and $c(\mathrm{rev}(1234))$.

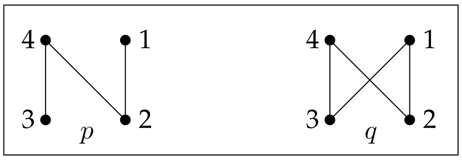


Figure 7.2.2: posets p and q are respectively defined by the 3-interval and 4-interval starting clockwise at $2 < 4$ in $c(1234)$.

We collect these cyclic arrangements or *circles* together in

$$\mathcal{C}_n := \{\, c(\sigma) : \sigma \in \mathcal{L}_n \,\}$$

and make some further, intuitive definitions.

- An *interval* A on a cyclic arrangement $c \in \mathcal{C}_n$ is a sequence of elements of Comp_n which are consecutive on c. Sometimes A will refer to the set containing the elements of the sequence; it will be clear from the context whether we consider A as a set or as a sequence.

  We say that A has *length* $|A|$ and an l-interval is simply an interval of length l. Finally, we make the convention that all intervals are read clockwise.

- Note that for $\sigma \in \mathcal{L}_n$ and $n \geq 4$, the circle $c(\mathrm{rev}(\sigma))$ cannot be obtained from $c(\sigma)$ by combinations of rotations and reflections. For instance, note from Figure 7.2.1 that $[(1,4),(2,4)]$ is an interval in $c(1234)$, whereas any interval in $c(\mathrm{rev}(1234))$ containing $(1,4)$ and $(2,4)$ must contain at least one other comparison.

- Given a set of comparisons $X \subseteq \mathrm{Comp}_n$ and a partial order $p \in \mathcal{P}_n$, we say that X *defines* p if p is the transitive closure of X.

Observe that there are $|\,\mathrm{Comp}_n\,| = n(n-1) = 2r_n$ points on each circle $c \in \mathcal{C}_n$. No interval on c longer than r_n can define an order, since it contains both (a,b) and (b,a) for some $a,b \in [n]$.

Example 7.2.3. Consider the 3-interval starting clockwise at $2 < 4$ in $c(1234)$ (see Figure 7.2.1): $A = [2 < 4, 3 < 4, 2 < 1]$. As a set, A is the poset p whose Hasse diagram is shown in Figure 7.2.2.

Now consider the 4-interval starting clockwise at $2 < 4$ in $c(1234)$. We have p together with the comparison $3 < 1$, which gives the poset q in Figure 7.2.2.

Note that neither of these posets contains any comparisons other than the ones from the interval which originally defined them. The following proposition shows that this is true in general: any interval of length up to r_n not only defines an order, but coincides precisely with some order p.

Proposition 7.2.4. *Let A be an interval of some $c = c(\sigma) \in \mathcal{C}_n$ of length at most $r_n = n(n-1)/2$. Then A defines an order $p \in \mathcal{P}_n$ such that as sets, $A = p$.*

Proof. Let S be the semicircle of $c = c(\sigma)$ defining σ and let R be the semicircle complementing S. We need to show that A is closed under transitivity, so suppose (x, y), $(y, z) \in A$ for some $x, y, z \in [n]$.

Case 1: $x <_\sigma y$, $y <_\sigma z$.

Since σ is transitive, we have $x <_\sigma z$. Now $y <_\sigma z$ implies that reading clockwise, (x, y) comes before both of (y, z) and (x, z) in S. Similarly, since $x <_\sigma y$ we must have (x, z) before (y, z) in S by the definition of $\alpha(\sigma, z)$. In other words, S contains the subinterval

$$(x, y), \ldots, (x, z), \ldots, (y, z). \tag{7.2.5}$$

Since A is an interval and (x, y), $(y, z) \in A$, it follows that A must contain either the interval (7.2.5) or the interval

$$(y, z), \ldots, (x, y). \tag{7.2.6}$$

Now (7.2.5) is contained in S, and so can have length at most $|S| = r_n$. Since $|c| = 2r_n$, this means that (7.2.6) is strictly longer than r_n and thus cannot be contained in A. Hence A contains (7.2.5) and so $(x, z) \in A$.

Case 2: $x \not<_\sigma y$, $y \not<_\sigma z$.

Then $z <_\sigma y$ and $y <_\sigma x$ since σ is linear. By relabelling $x \leftrightarrow z$ in the arguments of Case 1, we see that S must contain the subinterval

$$(z, y), \ldots, (z, x), \ldots, (y, x).$$

Thus R contains the subinterval $(y, z), \ldots, (x, z), \ldots, (x, y)$. This subinterval must then be contained in A, since (x, y), $(y, z) \in A$ and $|A| \leq r_n = |R|$.

Case 3: $x <_\sigma y$, $y \not<_\sigma z$.

Since σ is linear, we must have $z <_\sigma y$. We do not know how x and z compare under σ, but since both x and z are less than y under σ, the comparison relating x and z must come before both of

(x, y), (z, y) in S: one of the intervals

$$(x, z), \ldots, (x, y), \ldots, (z, y), \tag{7.2.7}$$

$$(z, x), \ldots, (z, y), \ldots, (x, y) \tag{7.2.8}$$

must be contained in S.

Suppose $x <_\sigma z$ and consider the semicircles on c clockwise *ending* at (x, y) and (y, x) respectively, call them D_1 and D_2. Since (7.2.7) is a subinterval of the r_n-interval S, we have $(x, z) \in D_1$ and $(z, y) \in D_2$. Since (y, z) occurs directly opposite (z, y) on c, $(z, y) \in D_2$ implies $(y, z) \in D_1$. Thus the r_n-interval D_1 contains the subinterval $(y, z), \ldots, (x, z), \ldots, (x, y)$, which implies $(x, z) \in A$ as required. This situation is illustrated in Figure 7.2.3 (Case $3.a$).

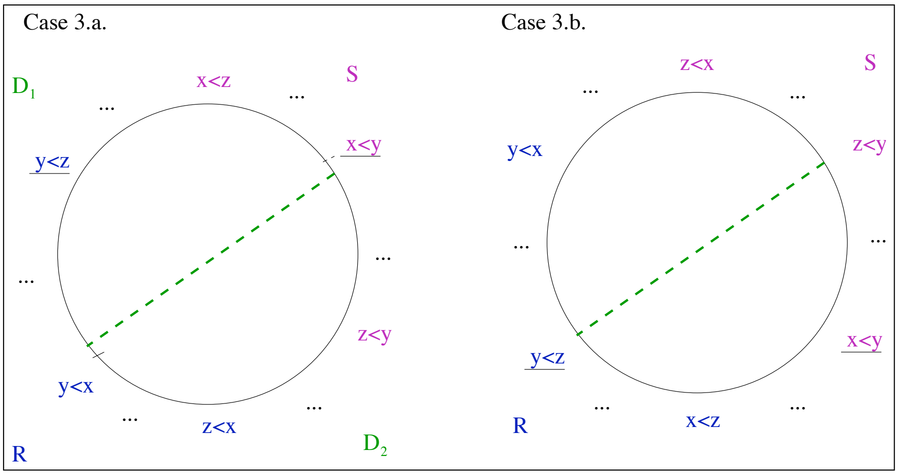


Figure 7.2.3: possibilities for $c(\sigma)$ in Case 3. Elements of A are underlined.

Case $3.b$, when $z <_\sigma x$, is very similar to Case $3.a$: here (7.2.8) is a subinterval of S. Thus if we draw a line on c connecting (y, z) and (z, y), we must have (x, y) and (x, z) on the same semicircle with respect to that line, because reverse comparisons are opposite each other on c. Since A cannot cover more than half of c, this implies $(x, z) \in A$.

Case 4: $x \not<_\sigma y$, $y <_\sigma z$

Since both x and z are greater than y under σ, the comparison comparing x and z must come after both (y, x) and (y, z) in S. The two sub-cases of Case 4 are very similar to the two sub-cases of Case 3.

If $x <_\sigma z$ then the interval $(y, x), \ldots, (y, z), \ldots, (x, z)$ occurs in S, see Case $4.a$ in Figure 7.2.4.

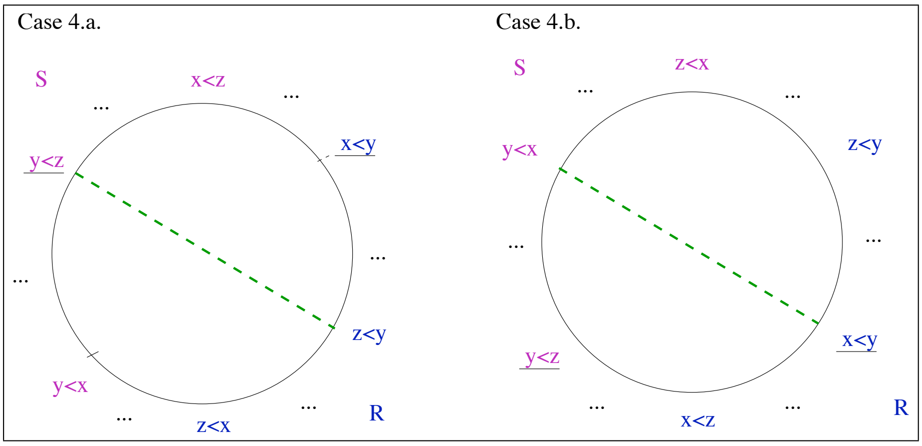


Figure 7.2.4: possibilities for $c(\sigma)$ in Case 4. Elements of A are underlined.

Reflecting (y, x) over the half circles with respect to the (y, z) - (z, y) axis shows that (x, z) and (x, y) are on the same semicircle, which forces $(x, z) \in A$ since (x, y), $(y, z) \in A$.

Finally, in Case $4.b$, the interval $(y, z), \ldots, (y, x), \ldots, (z, x)$ occurs in S. Considering the two half circles with respect to (x, y) and (y, x), we see that (y, z) is less than r_n away from (x, y), and (x, z) lies between them. So again, $|A| \leq r_n$ implies $(x, z) \in A$.

We have shown that A is closed under transitivity. Moreover, $|A| \leq r_n$ means that A cannot cover both of two opposite points on the circle $c(\sigma)$. In other words, $(x, y) \in A \implies (y, x) \notin A$ for all $x, y \in [n]$, so there exists $p \in \mathcal{P}_n$ with $A = p$. □

Definition 7.2.9. We say that $p \in \mathcal{P}_n$ and $c \in \mathcal{C}_n$ are *compatible*, denoted $p \prec c$, if the set of comparisons of p can be found as an interval on c.

For any $c \in \mathcal{C}_n$, $a, b \in [n]$ and $1 \leq l \leq r_n$, denote by $[(a, b)]_c^l$ the interval of length l starting clockwise at (a, b) in c.

To obtain a bound on intersecting subsets of $\mathcal{M}_n$, we need to investigate the posets arising from elements of $\mathcal{C}_n$ in some more detail.

Definition 7.2.10. For $\sigma = x_1 x_2 \ldots x_n \in \mathcal{L}_n$ and $1 \leq i < j \leq n$, define a set of comparisons $\lambda(\sigma, i, j)$ as follows:

(a) the chain $x_{j+1} < x_{j+2} < \cdots < x_{n-1} < x_n$ is preserved from σ;

(b) the chain on $x_1, x_2, \ldots, x_{j-1}$ is reversed: in $\lambda(\sigma, i, j)$ we have $x_{j-1} < x_{j-2} < \cdots < x_2 < x_1$;

(c) all elements of (b) are less than all elements of (a); and

(d) $x_i < x_j < x_{i-1}$ when $i > 1$, and $x_1 < x_j < x_{j+1}$ if $i = 1$.

It is not difficult to see that this description defines the linear order depicted in Figure 7.2.5.

Proposition 7.2.11. *Let $\sigma = x_1x_2 \dots x_n \in \mathcal{L}_n$ and let $1 \le i < j \le n$. Then*

$$[(x_i, x_j)]^{r_n}_{c(\sigma)} = \lambda(\sigma, i, j); \quad \textit{and}$$
$$[(x_j, x_i)]^{r_n}_{c(\sigma)} = \mathrm{rev}(\lambda(\sigma, i, j)).$$

Proof. Let $r = r_n$ and $c = c(\sigma)$. By Proposition 7.2.4, any r-interval on a cyclic ordering is equivalent to some linear order on $[n]$. The second claim, namely $[(x_j, x_i)]^r_c = \mathrm{rev}(\lambda(\sigma, i, j))$, will follow from the first, since for $a, b \in [n]$ it is clear that

$$(a, b) \in [(x_j, x_i)]^r_c \iff (b, a) \in [(x_i, x_j)]^r_c.$$

So to prove the proposition, it suffices to show that $[(x_i, x_j)]^r_c = \lambda(\sigma, i, j)$.

Now it follows from the definition of the cyclic ordering $c(\sigma)$ that $[(x_i, x_j)]^r_c$ consists of the following comparisons:

(i) the intervals $\alpha(\sigma, x_{j+1}), \dots, \alpha(\sigma, x_n)$,

(ii) the reverse of each comparison in the intervals $\alpha(\sigma, x_1), \dots, \alpha(\sigma, x_{j-1})$,

(iii) the comparisons $(x_i, x_j), (x_{i+1}, x_j), \dots, (x_{j-1}, x_j)$,

(iv) and the comparisons $(x_j, x_{i-1}), \dots, (x_j, x_2), (x_j, x_1)$.

Note that (i) implies (a) and (c) in Definition 7.2.10, and (ii) implies (b). We have (x_i, x_j) by (iii) and (x_j, x_{i-1}) by (iv) for $i > 1$. Finally, $x_j < x_{j+1}$ follows from (i), so we conclude that $(i) - (iv)$ also imply (d) in Definition 7.2.10. Thus $[(x_i, x_j)]^r_c$ contains $\lambda(\sigma, i, j)$, which means they must be equal since both are linear orders on $[n]$. □

In the next proposition we will show how elements of $Y_{k,n}$, and hence elements of $\mathcal{M}_n$, arise as intervals of cyclic arrangements. For $\sigma = x_1 \dots x_n \in \mathcal{L}_n$ and $1 \le i < j \le n$, define posets $\mu(\sigma, i, j) \in \mathcal{P}_n$ by their Hasse diagrams in Figure 7.2.5.

Remark 7.2.12. Observe that for any $\sigma \in \mathcal{L}_n$, we have

$$\begin{aligned} \mu(\sigma, 1, 2) &\in Y_{n-1,n}, \\ \mu(\sigma, 1, j) &\in Y_{1,n}, \quad 3 \le j \le n, \\ \mu(\sigma, i, j) &\in Y_{j-i+1,n}, \quad 1 < i < j \le n. \end{aligned}$$

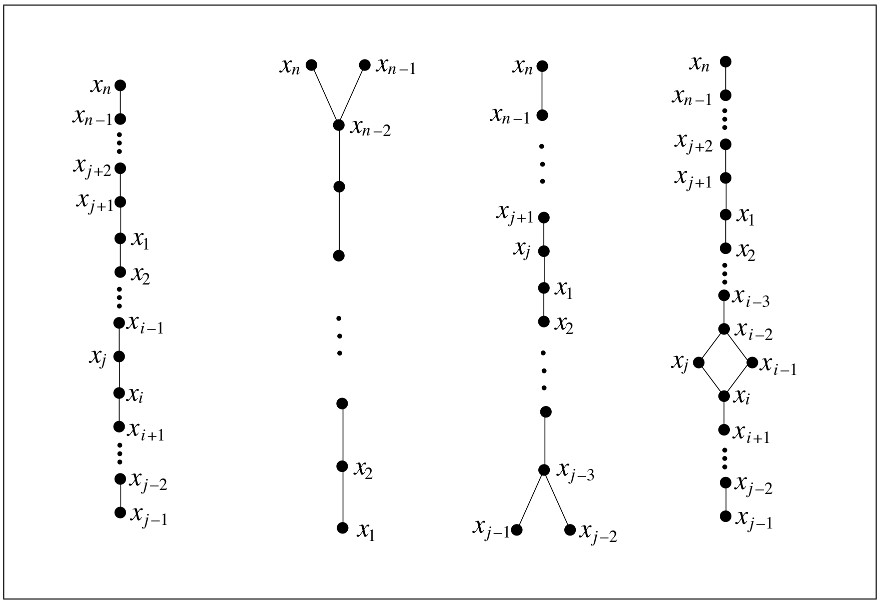


Figure 7.2.5: for $\sigma = x_1 \dots x_n \in \mathcal{L}_n$ we have, from left to right, Hasse diagrams of posets $\lambda(\sigma, i, j)$, $\mu(\sigma, 1, 2)$, $\mu(\sigma, 1, j)$ for $j \geq 3$, and $\mu(\sigma, i, j)$ for $1 < i < j \leq n$.

Note also that the statement of Proposition 7.2.13 is only concerned with intervals of $c(\sigma)$ starting at elements of σ. The remaining ones can be determined by recalling that as a poset, the l-interval starting at (x_j, x_i) is isomorphic to the reverse of the l-interval starting at (x_i, x_j).

Proposition 7.2.13. *Let $\sigma = x_1 x_2 \dots x_n \in \mathcal{L}_n$ and $1 \leq i < j \leq n$. Then*

$$[(x_i, x_j)]_{c(\sigma)}^{r_n - 1} = \mu(\sigma, i, j).$$

Proof. Set $r = r_n$, $c = c(\sigma)$, and let p be the poset equivalent to $[(x_i, x_j)]_c^{r-1}$.

Suppose firstly that $i > 1$ and recall that there are precisely $2r$ points on c. Since (x_{i-1}, x_j) is the comparison clockwise preceding (x_i, x_j) on c, the last point in $[(x_i, x_j)]_c^r$ is (x_j, x_{i-1}). Considering intervals as sets, we therefore have

$$p = [(x_i, x_j)]_c^{r-1} = [(x_i, x_j)]_c^r \setminus \{(x_j, x_{i-1})\}.$$

But $[(x_i, x_j)]_c^r = \lambda(\sigma, i, j)$ by Proposition 7.2.11, and so

$$p = \lambda(\sigma, i, j) \setminus \{(x_j, x_{i-1})\}.$$

Considering Figure 7.2.5, the previous equation implies $p = \mu(\sigma, i, j)$.

The case $i = 1$ is very similar; we need to show that $p = \mu(\sigma, 1, j)$. When $j = 2$, clearly $[(x_1, x_j)]_c^{r-1}$ contains all elements of σ except (x_{n-1}, x_n), so $p = \mu(\sigma, 1, 2)$.

When $j > 2$, the comparison preceding (x_1, x_j) in c is (x_{j-2}, x_{j-1}) and so

$$p = [(x_1, x_j)]_c^r \setminus \{(x_{j-1}, x_{j-2})\} = \lambda(\sigma, 1, j) \setminus \{(x_{j-1}, x_{j-2})\}$$

by Proposition 7.2.11. Reconsidering Figure 7.2.5, we see that removing the comparison (x_{j-1}, x_{j-2}) from $\lambda(\sigma, 1, j)$ gives $\mu(\sigma, 1, j)$. □

Finally, we are now ready to prove the regularity result which enables us to use the method of cyclic orderings for the class $\mathcal{M}_n$.

Proposition 7.2.14. *Let $c \in \mathcal{C}_n$ and $1 \leq k \leq n-1$. Then c is compatible with n elements of $Y_{k,n}$.*

Proof. Let $\sigma = x_1 x_2 \dots x_n \in \mathcal{L}_n$ be such that $c = c(\sigma)$. By Proposition 7.2.13, we obtain elements of $Y_{k,n}$ from c simply by picking appropriate starting points of $(r_n - 1)$-intervals. Denote by $p(a, b)$ the poset equivalent to the $(r_n - 1)$-interval on c starting at (a, b).

Case $1 < k < n-1$.

We begin by considering intervals starting at elements of σ. By Proposition 7.2.13 and Remark 7.2.12, we have $p(x_i, x_j) \in Y_{k,n}$ if, and only if, $k = j - i + 1$ for some $1 < i < j \leq n$. For given i, we therefore have $j = k + i - 1 \leq n$, which implies $i \leq n - k + 1$. Thus there are $|\{2, 3, \dots, n-k+1\}| = n - k$ values of i that determine intervals of c which start at elements of σ and are equivalent to elements of $Y_{k,n}$.

Since $Y_{k,n} \cong \mathrm{rev}(Y_{n-k,n})$, it follows by symmetry that there are $n - (n-k) = k$ such intervals whose starting points are not in σ. Thus c is compatible with a total of $n - k + k = n$ elements of $Y_{k,n}$.

Case $k = 1$ or $k = n-1$.

Consider the case $k = 1$. Since the isomorphism classes $Y_{k,n}$ are only defined for $n \geq 3$, Remark 7.2.12 and Proposition 7.2.13 tell us that for $i < j$, we have $p(x_i, x_j) \in Y_{1,n}$ if, and only if, $i = 1$ and $3 \leq j \leq n$. Hence there are $n - 2$ intervals of c starting at elements of σ which are equivalent to elements of $Y_{1,n}$.

The remaining ones are precisely those intervals opposite the $(r_n - 1)$-intervals on c starting at elements of σ and equivalent to elements of $Y_{n-1,n}$. Using Remark 7.2.12 and Proposition 7.2.13 again, we have $p(x_i, x_j) \in Y_{n-1,n}$ for $i < j$ if, and only if, either $i = 1$ and $j = 2$ or $j - i + 1 = n - 1$

for some $1 < i < j \leq n$. In the second situation, we have $j = n + i - 2 \leq n$ which implies $i \leq 2$, forcing $i = 2$ and $j = n + i - 2 = n$. We have shown that there are two $(r_n - 1)$-intervals on c starting at elements of σ and equivalent to elements of $Y_{n-1,n}$. Reversing these gives elements of $Y_{1,n}$. In summary, then, c is compatible with a total of $n - 2 + 2 = n$ elements of $Y_{1,n}$.

The case $k = n - 1$ follows by symmetry. $\square$

7.2.2 A Bound and Some Maximum Families

Recall that we are investigating the class $\mathcal{M}_n = \{ p \in \mathcal{P}_n : |p| = r_n - 1 \}$ where

$$r_n - 1 = \frac{n(n-1)}{2} - 1 = \frac{(n-2)(n+1)}{2}.$$

Theorem 7.2.15. *If $\mathcal{F} \subseteq \mathcal{M}_n$ is intersecting then*

$$|\mathcal{F}| \leq \frac{(n-2)(n+1)(n-1)!}{4}.$$

Proof. Let $\mathcal{F} \subseteq \mathcal{M}_n$ be intersecting. We count pairs consisting of a poset in $\mathcal{F}$ and a cyclic ordering in $\mathcal{C}_n$ which are compatible with one another:

$$|\{ (p, c) : p \in \mathcal{F}, c \in \mathcal{C}_n, p \prec c \}|.$$

Elements of $\mathcal{C}_n$ are obtained from each other by relabellings of $[n]$, and the same is true for elements of $Y_{k,n}$ for fixed k. Thus by Proposition 7.2.14, each poset in $Y_{k,n}$ is compatible with $n \cdot |\mathcal{C}_n|/|Y_{k,n}|$ elements of $\mathcal{C}_n$. Since $|Y_{k,n}| = |\mathcal{L}_n|/2 = n!/2$ for all $k \in [n-1]$, each poset in $Y_{k,n}$ is therefore compatible with

$$\frac{n \cdot |\mathcal{C}_n|}{n!/2} = \frac{2|\mathcal{C}_n|}{(n-1)!} \tag{7.2.16}$$

elements of $\mathcal{C}_n$. Since $\mathcal{F} \subseteq \mathcal{M}_n = \bigcup_{k=1}^{n-1} Y_{k,n}$ by Lemma 7.2.1, and since (7.2.16) is independent of k, we conclude that each element of $\mathcal{F}$ is compatible with $2|\mathcal{C}_n|/(n-1)!$ elements of $\mathcal{C}_n$. Thus

$$|\{ (p, c) : p \in \mathcal{F}, c \in \mathcal{C}_n, p \prec c \}| = |\mathcal{F}| \cdot \frac{2|\mathcal{C}_n|}{(n-1)!}. \tag{7.2.17}$$

Conversely, fix $c \in \mathcal{C}_n$ and set $\mathcal{F}'(c) = \{ p \in \mathcal{F} : p \prec c \}$. Let $v \in \mathcal{F}'(c)$ and let $A = [a_1, \ldots, a_{r_n - 1}]$ be the interval of c equivalent to v, so each a_i is an element of Comp_n. Since all other posets in $\mathcal{F}'(c)$ intersect v, they must either start or end in A. Each point in A can be the starting point of at most one element of $\mathcal{F}'(c)$ and the end point of at most one element of $\mathcal{F}'(c)$. So to ensure that these posets are distinct from v, we require that they do not start at the starting point of v or end at the end point of v. We obtain

$$|\mathcal{F}'(c) \setminus \{v\}| \leq 2|A| - 2 = 2(r_n - 2).$$

Now for each $i \in [r_n - 1]$, we pair up the $(r_n - 1)$-interval ending at a_i with the one starting at a_{i+1}. These paired intervals do not intersect since they cannot cover all points on c:

$$2(r_n - 1) < 2r_n = |c|.$$

Hence $|\mathcal{F}'(c)| \leq r_n - 1$, implying

$$|\,\{\,(p, c) : p \in \mathcal{F}, c \in \mathcal{C}_n, p \prec c\,\}\,| \leq |\mathcal{C}_n| \cdot (r_n - 1),$$

and combining this with (7.2.17) yields the result. □

Next we show that one way of obtaining a maximum intersecting family is by fixing a comparison: consistently with previous notation, set

$$F_{i,j}(\mathcal{M}_n) = \{\,p \in \mathcal{M}_n : (i, j) \in p\,\}$$

for some fixed $i, j \in [n]$.

Proposition 7.2.18. *The fix-family $F_{i,j}(\mathcal{M}_n)$ is maximum intersecting in $\mathcal{M}_n$.*

Proof. It suffices to show that $F_{i,j}(\mathcal{M}_n)$ attains the bound in Theorem 7.2.15. For fixed k, $Y_{k,n}$ contains $(n-2)!$ posets with $i||j$, and exactly half of the remaining elements of $Y_{k,n}$ have $i < j$. Thus

$$|\,\{\,p \in Y_{k,n} : i <_p j\,\}\,| = \frac{|Y_{k,n}| - (n-2)!}{2} = \frac{(n-2)! \cdot (n-2)(n+1)}{4}$$

since $|Y_{k,n}| = n!/2$. Combining this with Lemma 7.2.1 gives

$$|F_{i,j}(\mathcal{M}_n)| \;=\; \sum_{i=1}^{n-1} |\,\{\,p \in Y_{k,n} : i <_p j\,\}\,| = \frac{(n-1)! \cdot (n-2)(n+1)}{4},$$

as required. □

The next proposition shows that $\mathcal{M}_n$ also contains optimal saturation families.

Proposition 7.2.19. *Let $v_n \in \mathcal{P}_n$ be the poset $v_n = \{\,(i, n) : 1 \leq i \leq n-1\,\}$.*

- *If n is even, set $G(\mathcal{M}_n) = \{\,p \in \mathcal{M}_n : |p \cap v_n| \geq n/2\,\}$.*
- *If $n = 2h + 1$, set $G(\mathcal{M}_n) = \{\,p \in \mathcal{M}_n :$ either $|p \cap v_n| \geq h + 1$ or $|p \cap v_n| = h$, $1 <_p n\,\}$.*

Then $G(\mathcal{M}_n)$ is maximum intersecting in $\mathcal{M}_n$.

Proof. Since $\mathcal{M}_n = \bigcup_{k=1}^{n-1} Y_{k,n}$, it follows that $G(\mathcal{M}_n) = \bigcup_{k=1}^{n-1} G(Y_{k,n})$ where $G(Y_{k,n})$ is defined in Propositions 7.1.17 and 7.1.18. Moreover, this union is disjoint since $Y_{i,n} \cap Y_{j,n} = \emptyset$ for $i \neq j$, and so

$$\begin{aligned} |G(\mathcal{M}_n)| &= \sum_{k=1}^{n-1} |G(Y_{k,n})| \\ &= \sum_{k=1}^{n-1} \begin{cases} (n-2)\cdot(n-1)!/4 & n \text{ even, } k = n/2 \\ (n-1)\cdot(n-1)!/4 & n = 2h+1,\ k \in \{h, h+1\} \\ n!/4 & \text{otherwise} \end{cases} \end{aligned}$$

by Propositions 7.1.17 and 7.1.18. When n is even, this gives

$$|G(\mathcal{M}_n)| = (n-2)\cdot\frac{n!}{4} + \frac{(n-2)\cdot(n-1)!}{4} = \frac{(n-2)(n+1)(n-1)!}{4},$$

and when n is odd,

$$|G(\mathcal{M}_n)| = (n-3)\cdot\frac{n!}{4} + 2\cdot\frac{(n-1)\cdot(n-1)!}{4} = \frac{(n-2)(n+1)(n-1)!}{4}.$$

Hence $|G(\mathcal{M}_n)|$ attains the bound in Theorem 7.2.15.

Finally, the definitions of $G(Y_{k,n})$ for n even and odd do not depend on k, and so the arguments used in Propositions 7.1.17 and 7.1.18 to show that $G(Y_{k,n})$ is intersecting also imply that $G(\mathcal{M}_n)$ is intersecting. □

7.2.3 Conclusion

Since the method of cyclic orderings is aesthetically appealing, it would be nice to extend it to other poset classes. However, the reason why the bound in Theorem 7.2.15 is sharp is that the set of the $r_n - 1$ posets arising as $(r_n - 1)$-intervals around one certain comparison on a cycle is indeed a subset of the class $\mathcal{M}_n$. This is clearly not the case for a general isomorphism class — for example, it is not the case for any of the classes $Y_{k,n}$. Or, to consider a more straightforward example, let $I_3(4)$ be the subclass of $\mathcal{P}_4$ with elements defined by Figure 7.2.6.

Since the comparisons in an element of $I_3(4)$ actually form a linear order on three points, it is easy to see that any cyclic ordering $c(x_1x_2x_3x_4) \in \mathcal{C}_4$ contains precisely two elements of $I_3(4)$, namely the 3-intervals starting at $x_1 < x_2$ and $x_2 < x_1$ respectively. So the size of an element of $I_3(4)$ is three, but given a fixed comparison on $c(x_1x_2x_3x_4)$, at most one of the three 3-intervals containing it defines an element of $I_3(4)$.

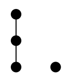


Figure 7.2.6: Hasse diagram of elements of $I_3(4)$.

Thus if the bound on an intersecting subset of a class $\mathcal{X}$ of posets given by our method of cyclic orderings is to be sharp, we need to ensure that $\mathcal{X}$ includes *all* posets of a certain size s (i.e. with s comparisons), because an intersecting subset of $\mathcal{X}$ can then include the set of all s-intervals around a certain comparison on a cyclic ordering. But then we can no longer assume that each poset in $\mathcal{X}$ is contained in the same number of cyclic orderings. Indeed, it seems that the only value of s for which a regularity result such as Proposition 7.2.14 holds, without introducing a weight function, is in fact $s = r_n - 1$. A weight function could control the probability of certain intervals on the cycle being picked over others, but we do not pursue this approach here.

To summarise, the method of cyclic orderings presented in this chapter will not deliver sharp bounds for poset classes other than $\mathcal{M}_n$. Whether it has applications in the investigation of slightly different combinatorial structures, such as pre-orders for example, is a separate question.

On the other hand, whilst perhaps lacking the inherently satisfying element of sophistication displayed by the cyclic arrangements, the method from Section 7.1 of partitioning a class $\mathcal{X}$ of posets into blocks appears a lot more promising in terms of applicability. We will extend its use from single isomorphism classes to a larger class in the following section.

7.3 The Split Ends Class

Gradually increasing the number of antichains in poset classes we consider, let Υ_n be the *'split ends'* subclass of $\mathcal{P}_n$, containing linear orders and partial orders obtained from linear ones by replacing the two highest and/or the two lowest points in their Hasse diagrams by an antichain of length two. So

$$\Upsilon_n = \mathcal{L}_n \cup Y_{1,n} \cup Y_{n-1,n} \cup Z_n$$

where the elements of Z_n are chains with an antichain of size two at the top and bottom. It is easy to see from Figure 7.3.1 that the class Z_n exists for $n \geq 5$, so we consider Υ_n for $n \geq 5$.

Recall that two linear orders $\sigma, \rho \in \mathcal{L}_n$ do not intersect if, and only if, $\sigma = \mathrm{rev}(\rho)$. We obtain the

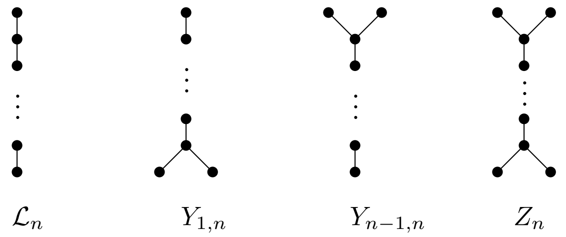


Figure 7.3.1: Hasse diagrams of elements of $\Upsilon_n = \mathcal{L}_n \cup Y_{1,n} \cup Y_{n-1,n} \cup Z_n$.

following analogous condition for partial orders from Lemma 6.3.9.

Lemma 7.3.1. *Two partial orders p and q in $\mathcal{P}_n$ do not intersect if, and only if, there exist linear extensions $\rho \in \mathcal{L}(p)$ and $\sigma \in \mathcal{L}(q)$ with $\sigma = \mathrm{rev}(\rho)$.*

Proof. ($\Leftarrow$) We have $(p \cap q) \subseteq (\rho \cap \sigma) = (\rho \cap \mathrm{rev}(\rho)) = \emptyset$.

($\Rightarrow$) Suppose there do not exist $\rho \in \mathcal{L}(p)$ and $\sigma \in \mathcal{L}(q)$ such that $\sigma = \mathrm{rev}(\rho)$; then $\mathcal{L}(p) \cup \mathcal{L}(q)$ is intersecting. Choose $\rho \in \mathcal{L}(p)$, $\sigma \in \mathcal{L}(q)$ with minimal intersection size and let $(a, b) \in \sigma \cap \rho$, then neither of p, q have $b < a$.

Suppose, for a contradiction, that $a||_p b$. Then it follows from Lemma 6.3.9 that p has a linear extension ρ' which is identical to ρ apart from the comparison $b <_{\rho'} a$. But this implies

$$|\rho' \cap \sigma| = |\rho \cap \sigma| - 1$$

which contradicts our choice of ρ and σ.

Thus $a <_p b$, and it follows by the same arguments that $a <_q b$, which shows that p and q intersect. □

7.3.1 Partitioning the Class

We partition Υ_n into blocks indexed by elements of Z_n: for $p \in Z_n$, B_p contains all elements of Υ_n containing it, i.e.

$$B_p = \{\, q \in \Upsilon_n : p \subseteq q \,\}\,.$$

Note that B_p consists of $\mathcal{L}(p)$ along with any element of Υ_n with a linear extension in $\mathcal{L}(p)$.

Lemma 7.3.2. *For $n \geq 5$, $\{\, B_p : p \in Z_n \,\}$ is a partition of Υ_n. Moreover, if two posets $q_1, q_2 \in \Upsilon_n$ do not intersect, then $q_1 \in B_p$, $q_2 \in B_{\mathrm{rev}(p)}$ for some $p \in Z_n$.*

Proof. It is easy to see from Figure 7.3.1 that every $q \in \Upsilon_n$ is contained in some block B_p, and also that two distinct $p_1, p_2 \in Z_n$ cannot be contained in the same $q \in \Upsilon_n$. Hence every $q \in \Upsilon_n$ is contained in precisely one B_p, proving the first sentence of the lemma.

Now let q_1, q_2 be non-intersecting elements of Υ_n with $q_i \in B_{p_i}$ for $i = 1, 2$. By Lemma 7.3.1, there exist $\sigma_1 \in \mathcal{L}(q_1)$ and $\sigma_2 \in \mathcal{L}(q_2)$ such that $\sigma_1 = \mathrm{rev}(\sigma_2)$. It is clear from the definition that each B_p is closed under taking linear extensions, so $\sigma_i \in B_{p_i}$. Moreover, σ_i must be a linear extension of p_i by transitivity. Thus if $\sigma_1 = x_1 \dots x_n$ then $\sigma_2 = \mathrm{rev}(\sigma_1)$ gives

$$p_1 = \sigma_1 \setminus \{(x_1, x_2), (x_{n-1}, x_n)\} = \mathrm{rev}(\sigma_2 \setminus \{(x_2, x_1), (x_n, x_{n-1})\}) = \mathrm{rev}(p_2)$$

as required. □

The isomorphism class Z_n is closed under taking reverses. It therefore follows from Lemma 7.3.2 that

$$\mathcal{B}(n) = \left\{ B_p \cup B_{\mathrm{rev}(p)} : p \in Z_n \right\}$$

is a partition of Υ_n.

7.3.2 A Bound and a Maximum Family

Before we state the next result, recall that we are not interested in the class Υ_n for $n < 5$. Also note that every block B_p is intersecting, since each of its elements contains p.

Theorem 7.3.3. *For $n \geq 5$, $\mathcal{F}$ is a maximum intersecting subset of Υ_n if, and only if, $\mathcal{F}$ is the union of a transversal of*

$$\left\{ \{B_p, B_{\mathrm{rev}(p)}\} : p \in Z_n \right\}.$$

Proof. It follows from Lemma 7.3.2 that $\mathcal{F}$ is the disjoint union of $|Z_n|/2$ families $\mathcal{F}'(p)$, $p \in Z_n$, with each $\mathcal{F}'(p)$ a maximum intersecting subset of $B_p \cup B_{\mathrm{rev}(p)}$, and $\mathcal{F}'(p) = \mathcal{F}'(q)$ if, and only if, $q = \mathrm{rev}(p)$. To prove the theorem, we therefore need to show that B_p and $B_{\mathrm{rev}(p)}$ are the only maximum intersecting families of $B_p \cup B_{\mathrm{rev}(p)}$ for all $p \in Z_n$.

Let a, b, c, d be the four distinct points in $[n]$ with $a||_p b$ and $c||_p d$. For any assignment

$$\{ x_i : 1 \leq i \leq 4 \} = \{a, b, c, d\},$$

denote by $l(x_1 x_2 x_3 x_4)$ the linear extension of p with $x_1 < x_2 < x_3 < x_4$. Recall that B_p consists of

$\mathcal{L}(p)$ along with any element of Υ_n with a linear extension in $\mathcal{L}(p)$, so

$$\begin{aligned} B_p &= \{p, l(abcd), l(bacd), l(abdc), l(badc), \\ &\qquad y_1(l(abcd)), y_1(l(abdc)), y_{n-1}(l(abcd)), y_{n-1}(l(bacd))\}. \end{aligned}$$

Obtaining $B_{\mathrm{rev}(p)}$ in a similar way, we note that $B_{\mathrm{rev}(p)} = \mathrm{rev}(B_p)$. Thus $B_p \cup B_{\mathrm{rev}(p)}$ is a subset of $\mathcal{P}_n$ closed under taking reverses and we can apply Proposition 6.3.13 to conclude that $\mathcal{F}'(p)$ contains a transversal of

$$\begin{aligned} &\{\, \{\sigma, \mathrm{rev}(\sigma)\} : \sigma \in \mathcal{L}_n \cap (B_p \cup B_{\mathrm{rev}(p)})\,\} \\ =\ &\{\, \{\sigma, \mathrm{rev}(\sigma)\} : \sigma \in \mathcal{L}_n \cap (B_p \cup \mathrm{rev}(B_p))\,\} \\ =\ &\{\, \{\sigma, \mathrm{rev}(\sigma)\} : \sigma \in \mathcal{L}_n \cap B_p \,\} \\ =\ &\{\, \{\sigma, \mathrm{rev}(\sigma)\} : \sigma \in \mathcal{L}(p)\,\}\,. \end{aligned} \tag{7.3.4}$$

Consider the case $l(abcd) \in \mathcal{F}'(p)$. Then we can exclude the elements of $\mathrm{rev}(B_p)$ which $l(abcd)$ does not intersect: we have

$$\mathrm{rev}(l(abcd)), y_1(\mathrm{rev}(l(abcd))), y_{n-1}(\mathrm{rev}(l(abcd))), \mathrm{rev}(p) \notin \mathcal{F}'(p).$$

Suppose, for a contradiction, that $\mathrm{rev}(l(badc)) \in \mathcal{F}'(p)$, then

$$l(badc), y_1(l(badc)), y_{n-1}(l(badc)), p \notin \mathcal{F}'(p).$$

There are now 10 elements of $B_p \cup B_{\mathrm{rev}(p)}$ we have not yet chosen or excluded; among them

$$l(bacd), \mathrm{rev}(l(bacd)), l(abdc), \mathrm{rev}(l(abdc)).$$

Since a linear order does not intersect its own reverse this implies $|\mathcal{F}'(p)| \leq 2+10-2 = 8 < 9 = |B_p|$. We conclude that $\mathrm{rev}(l(badc)) \notin \mathcal{F}'(p)$.

Since $\mathcal{F}'(p)$ contains a transversal of (7.3.4), we must therefore have $l(badc) \in \mathcal{F}'(p)$, which excludes $y_1(\mathrm{rev}(l(badc)))$ and $y_{n-1}(\mathrm{rev}(l(badc)))$ from $\mathcal{F}'(p)$. We have now narrowed $\mathcal{F}'(p)$ down to

$$\mathcal{F}'(p) \subseteq B_p \cup \{\mathrm{rev}(l(bacd)), \mathrm{rev}(l(abdc))\}.$$

Again since $\mathcal{F}'(p)$ contains a transversal of (7.3.4), $\mathcal{F}'(p)$ contains precisely one of $l(bacd), \mathrm{rev}(l(bacd))$ and $l(abdc), \mathrm{rev}(l(abdc))$ respectively. Including either of $\mathrm{rev}(l(bacd)), \mathrm{rev}(l(abdc))$ would exclude at least two elements of B_p and thus force $|\mathcal{F}'(p)| < |B_p|$; hence we conclude $\mathcal{F}'(p) = B_p$.

We have shown that $l(abcd) \in \mathcal{F}'(p)$ implies $\mathcal{F}'(p) = B_p$. It follows by symmetry that $\mathrm{rev}(l(abcd)) \in \mathcal{F}'(p)$ implies $\mathcal{F}'(p) = \mathrm{rev}(B_p)$. $\square$

Corollary 7.3.5. *If $\mathcal{F} \subseteq \Upsilon_n$ is intersecting then*

$$|\mathcal{F}| \leq \frac{|\Upsilon_n|}{2} = \frac{9n!}{8}.$$

Proof. Recall from the proof of Theorem 7.3.3 that $|B_p| = 9$ for all $p \in Z_n$. It thus follows from Theorem 7.3.3 that $|\mathcal{F}| \leq |\Upsilon_n|/2$. Since the B_p partition Υ_n by Lemma 7.3.2,

$$|\Upsilon_n| = \sum_{p \in Z_n} |B_p| = |Z_n| \cdot 9.$$

Each $p \in Z_n$ has four linear extensions, so

$$|\Upsilon_n| = \frac{9 \cdot |\mathcal{L}_n|}{4} = \frac{9 \cdot n!}{4}$$

and the result follows. □

It seems that Υ_n does indeed share various properties with the isomorphism classes $Y_{k,n}$ considered in Section 7.1: not only did we apply a very similar proof method to obtain the maximal size of an intersecting subset, but we will now show that, mirroring the situation in $Y_{k,n}$, saturation is optimal in Υ_n while fixing is not. As before, $v_n \in \mathcal{P}_n$ is the poset $v_n = \{ (i, n) : 1 \leq i \leq n-1 \}$ and

$$G(\Upsilon_n) = \{ p \in \Upsilon_n : |p \cap v_n| \geq n/2 \}$$

is intersecting by the pigeonhole principle.

Proposition 7.3.6. *$G(\Upsilon_n)$ is a maximum intersecting in Υ_n.*

Proof. It suffices to show that $G(\Upsilon_n)$ is the union of a transversal of

$$\left\{ \{B_p, B_{\mathrm{rev}(p)}\} : p \in Z_n \right\}$$

by Theorem 7.3.3. For $p \in Z_n$ we have $|p \cap v_n| \geq n/2$ if, and only if, at least $n/2$ points are less than n under p, i.e. fewer than $n/2$ points are larger than n under p which is equivalent to the statement that fewer than $n/2$ points are smaller than n under $\mathrm{rev}(p)$. We have shown that

$$|p \cap v_n| \geq n/2 \iff |\,\mathrm{rev}(p) \cap v_n| < n/2, \tag{7.3.7}$$

i.e. p is in $G(\Upsilon_n)$ if and only if $\mathrm{rev}(p)$ is not in $G(\Upsilon_n)$.

Since p is contained in every element q of B_p, $|p \cap v_n| \geq n/2$ implies $|q \cap v_n| \geq n/2$. Conversely, suppose for a contradiction that $p \notin G(\Upsilon_n)$ but some distinct $q \in B_p$ is in $G(\Upsilon_n)$. Then any linear extension $\sigma = x_1 \dots x_n$ of q is also in $G(\Upsilon_n)$, and σ is a linear extension of p by transitivity. In particular,

$$p = \sigma \setminus \{(x_1, x_2), (x_{n-1}, x_n)\}$$

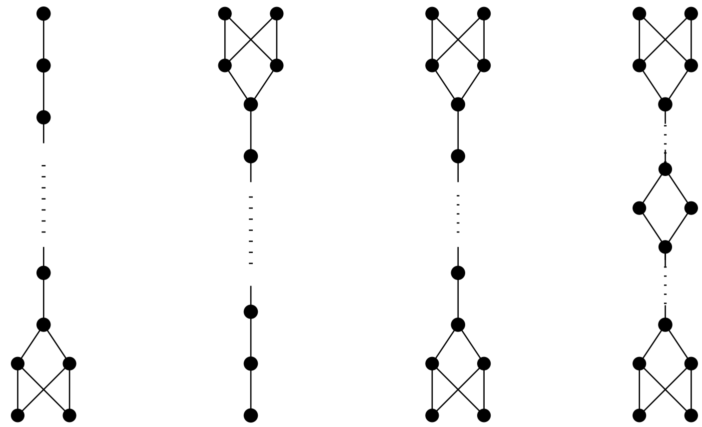


Figure 7.3.2: Let Ψ_n be the union of Υ_n with the isomorphism classes represented by the above Hasse diagrams. Mimicking the partition of Υ_n, blocks in Ψ_n would be indexed by labelled versions of the far right poset above, since these have smallest size among elements of Ψ_n.

and since σ shares $n/2$ points with v_n but p does not, this forces $n \in \{x_2, x_n\}$.

Now $x_n = n$ would imply

$$|p \cap v_n| = n - 2 \geq n/2$$

since $n \geq 5$, contradicting $p \notin G(\Upsilon_n)$. On the other hand, $x_2 = n$ would imply

$$|\sigma \cap v_n| = 1 < n/2$$

for $n \geq 5$, contradicting $\sigma \in G(\Upsilon_n)$. We conclude that each B_p is either contained in or disjoint from $G(\Upsilon_n)$. Combining this with (7.3.7) completes the proof. $\square$

Remark 7.3.8. Observe that for $a, b \in [n]$, the fix family

$$F_{a,b}(\Upsilon_n) = \{\, p \in \Upsilon_n : (a, b) \in p \,\}$$

does not have maximal size. To demonstrate this, it suffices to show that $F_{a,b}$ is not the union of a transversal of

$$\{\, \{B_p, B_{\mathrm{rev}(p)}\} : p \in Z_n \,\}$$

by Theorem 7.3.3. Clearly Z_n contains an element p with $a||_p b$. Now $a \not<_p b$ and $a \not<_{\mathrm{rev}(p)} b$, so $F_{a,b}$ does not entirely contain either B_p or $B_{\mathrm{rev}(p)}$, as required.

Considering Remarks 7.3.8 and 7.1.16, we do not expect fixing to be optimal in any classes where the maximum size of an intersecting family can be determined by a partition into blocks which are closed under the map N from Section 7.1. It seems likely that the arguments in this section would extend to classes larger than Υ_n, for instance, the union of Υ_n with the labelled versions of the posets in Figure 7.3.2.

However, we do not classify these 'reverse pairing' classes here. Instead, we now turn our attention to the opposite end of $\mathcal{P}_n$: in this chapter, we studied posets with lots of comparisons, i.e. posets which are almost linear. The next chapter is concerned with posets with very few comparisons instead, i.e. posets which are close to the antichain.

CHAPTER 8

POSETS CLOSE TO THE ANTICHAIN

Having investigated maximum intersecting families of posets which are almost linear in Chapter 7, we turn our attention to the other end of the spectrum now by studying posets which are close to the antichain. Recall from Chapter 6 that the height $h_p(x)$ of a point x under the poset p is one less than the greatest number of elements in a chain whose largest member is x.

Definition 8.0.1. The height of a poset $p \in \mathcal{P}_n$ is the maximum height of any of its points, that is,

$$h(p) = \max_{x \in [n]} h_p(x).$$

We say that a point $x \in [n]$ is *maximal* in p if $h_p(x) = h(p)$, and *minimal* if $h_p(x) = 0$.

8.1 Posets of Height 1

The antichain is the only poset on n points with height 0. In order for two posets to intersect, each of them has to have height at least 1, so we begin by investigating these. In Section 8.1.1 we fix a poset in which points of different heights are comparable and consider the class arising from all permutations of the labels. Then we consider the union of all such posets in Section 8.1.2. We will see that the classifications of maximum intersecting families in the height 1 classes we consider are consequences of known results. In Section 8.2 we investigate the corresponding posets of height 2.

8.1.1 Fixing a Complete Poset of Height 1

Definition 8.1.1. We define *level i* of p as

$$L_i(p) \quad = \quad \{ \, x \in [n] : h_p(x) = i \, \} \, .$$

If $p \in \mathcal{P}_n$ has height 1, we may refer to $L_0(p)$ and $L_1(p)$ as the *lower* and *upper* level of p, respectively. In a graph-theoretic analogy, we call a poset $p \in \mathcal{P}_n$ *complete* if for all $x, y \in [n]$, $h_p(x) < h_p(y)$ implies $x <_p y$. If X, Y are subsets of $[n]$, we use the notation $X <_p Y$ to indicate that for all $x \in X$, $y \in Y$, we have $x <_p y$.

For $1 \le k \le n-1$, let $\mathcal{C}_{k,n-k}$ be the set of all complete height 1 posets on $[n]$ with lower levels of size k:

$$\mathcal{C}_{k,n-k} = \{\, p \in \mathcal{P}_n \;\mid\; h(p) = 1,\ |L_0(p)| = k,\ L_0(p) <_p L_1(p) \}\,.$$

This subsection investigates maximum intersecting subsets of $\mathcal{C}_{k,n-k}$. Clearly, if $p \in \mathcal{P}_n$ has height 1 then $L_0(p) \dot\cup L_1(p) = [n]$, where the symbol $\dot\cup$ denotes the disjoint union of two sets. Moreover, for $p \in \mathcal{C}_{k,n-k}$, all points in $L_0(p)$ are less than all points in $L_1(p)$ under p, so any element of $\mathcal{C}_{k,n-k}$ is determined by either of its levels. This means that

$$|\mathcal{C}_{k,n-k}| = \binom{n}{k} = \binom{n}{n-k}.$$

Furthermore, the intersecting structure of $\mathcal{C}_{k,n-k}$ is determined by the levels of its elements, as described in the following lemma.

Lemma 8.1.2. *Two posets $p, q \in \mathcal{C}_{k,n-k}$ intersect if, and only if, $L_i(p)$ and $L_i(q)$ intersect, where $i = 0$ if $k \le n/2$ and $i = 1$ if $k > n/2$.*

Proof. Clearly two elements $p, q \in \mathcal{C}_{k,n-k}$ intersect if, and only if, their upper levels intersect and their lower levels intersect.

If $k < n/2$ then $L_1(p)$ and $L_1(q)$ both have size $n-k > n/2$, so they have non-empty intersection by the pigeonhole principle. This proves the lemma for $k < n/2$, and for $k > n/2$ by symmetry.

Now two $(n/2)$-subsets of $[n]$ intersect if, and only if, one is not the complement of the other. Thus if $L_0(p)$ and $L_0(q)$ intersect, then $L_0(p) \ne \overline{L_0(q)}$, implying

$$L_1(q) = \overline{L_0(q)} \ne L_0(p) = \overline{L_1(p)}.$$

In other words, if the lower levels of p and q intersect, then their upper levels must also intersect. $\square$

It is not difficult to see from Lemma 8.1.2 that for all but one value of k, the characterisation of intersecting subsets of $\mathcal{C}_{k,n-k}$ is simply a corollary to the Erdős-Ko-Rado Theorem. However, we will see in this chapter that in the context of posets close to the antichain, the Erdős-Ko-Rado Theorem is usually applied to deduce the optimality of a *saturation* family such as

$$\begin{aligned} G_0(\mathcal{C}_{k,n-k}) &= \{ p \in \mathcal{C}_{k,n-k} : 1 \in L_0(p) \} , \\ G_1(\mathcal{C}_{k,n-k}) &= \{ p \in \mathcal{C}_{k,n-k} : n \in L_1(p) \} , \end{aligned}$$

as opposed to a *fix-family* such as

$$F_{1,n}(\mathcal{C}_{k,n-k}) = \{ p \in \mathcal{C}_{k,n-k} : (1, n) \in p \} .$$

We refer to $G_1(\mathcal{C}_{k,n-k})$ as a saturation family because it is obtained by saturating over the poset $v_n = \{ (i, n) : 1 \leq i \leq n-1 \}$ from Chapter 7. Clearly $G_0(\mathcal{C}_{k,n-k})$ is based on the same idea as $G_1(\mathcal{C}_{k,n-k})$.

Proposition 8.1.3. *Let k and n be natural numbers with $1 \leq k \leq n-1$ and $k \neq n/2$. If $\mathcal{F}$ is an intersecting subset of $\mathcal{C}_{k,n-k}$ then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Moreover, equality holds if, and only if, there exists $a \in [n]$ such that

$$\mathcal{F} = \{ (a\ b)p : p \in G_i(\mathcal{C}_{k,n-k}) \}$$

where $(a\ \ b)p$ is obtained from p by swapping the labels a and b; $k < n/2$ implies $i = 0$, $b = 1$; and if $k > n/2$ then $i = 1$, $b = n$.

Proof. Suppose $k < n/2$, then $L_0(\mathcal{F}) = \{ L_0(p) : p \in \mathcal{F} \}$ is intersecting by Lemma 8.1.2. Moreover, elements of $L_0(\mathcal{F})$ are k-subsets of $[n]$. Therefore, by Theorem 1.2.1,

$$|L_0(\mathcal{F})| \leq \binom{n-1}{k-1}$$

and equality implies that there exists $a \in [n]$ such that for all $p \in \mathcal{F}$, we have $a \in L_0(p)$. Since elements of $\mathcal{C}_{k,n-k}$ are determined by their lower levels, the result follows. The case $k > n/2$ follows by symmetry. $\square$

To complete the classification of maximum intersecting subsets of $\mathcal{C}_{k,n-k}$ for all k, it remains to consider the case $k = n/2$.

Proposition 8.1.4. *Let k be a natural number. If $\mathcal{F}$ is an intersecting subset of $\mathcal{C}_{k,k}$ then*

$$|\mathcal{F}| \leq \frac{|\mathcal{C}_{k,k}|}{2} = \frac{1}{2}\binom{2k}{k}.$$

Moreover, equality holds if, and only if, $\mathcal{F}$ is a transversal of

$$\{ \{p, \mathrm{rev}(p)\} : p \in \mathcal{C}_{k,k} \} .$$

Proof. Let $p, q \in \mathcal{C}_{k,k}$ with $L_0(p) \cap L_0(q) = \emptyset$. Since upper and lower levels of elements of $\mathcal{C}_{k,k}$ all have size k, this means $L_0(q) = L_1(p)$ and $L_1(q) = L_0(p)$ which is equivalent to $q = \mathrm{rev}(p)$. Thus, by Lemma 8.1.2, we have

$$p \cap q = \emptyset \iff q = \mathrm{rev}(p)$$

for any $p, q \in \mathcal{C}_{k,k}$. The rev operator is clearly well-defined and bijective, so the result follows. □

8.1.2 The Union of all Complete Posets of Height 1

We now consider the union of the classes studied in the previous section: let

$$\mathcal{H}_n^{(1)} = \{\, p \in \mathcal{P}_n \;\mid\; h(p) = 1,\ L_0(p) <_p L_1(p) \} = \bigcup_{k=1}^{n-1} \mathcal{C}_{k,n-k}.$$

Any complete poset of height 1 is determined by either of its levels, and neither level may be empty, otherwise the poset would have height 0. Thus $\mathcal{H}_n^{(1)}$ has two elements less than the power set of $[n]$, i.e.

$$|\mathcal{H}_n^{(1)}| = 2^n - 2.$$

The bound on the size of intersecting subsets of this class arises from the following theorem.

Theorem 8.1.5. *(Marica, Schönheim [MS69]).*

If $\mathcal{A}$ is a finite collection of sets, then $|\{\, X \setminus Y : X, Y \in \mathcal{A} \,\}| \geq |\mathcal{A}|$.

In [DL76], Daykin and Lovász use Theorem 8.1.5 to establish a result on set families equivalent to the bound for intersecting subsets of $\mathcal{H}_n^{(1)}$. The following proposition and its proof are a replication of their argument in the language of height 1 posets.

Theorem 8.1.6. *(Daykin, Lovász [DL76]).*

If $\mathcal{F}$ is an intersecting subset of $\mathcal{H}_n^{(1)}$ then

$$|\mathcal{F}| \leq 2^{n-2}.$$

Proof. [DL76]

Let $L = \{\, L_0(p) : p \in \mathcal{F} \,\}$, then L is an intersecting family of subsets of $[n]$ of size $|L| = |\mathcal{F}|$. Setting

$$D = \{\, X \setminus Y : X, Y \in L \,\},$$

we have $|D| \geq |L|$ by Theorem 8.1.5, so

$$|D| \geq |\mathcal{F}|.$$

If D and L have non-empty intersection, then there exist $X_1, X_2, X_3 \in L$ with $X_1 = X_2 \setminus X_3$. But this implies $X_1 \cap X_3 = \emptyset$, contradicting the fact that L is intersecting. Thus $D \cap L = \emptyset$, giving

$$|D \cup L| = |D| + |L| \geq 2|\mathcal{F}|. \tag{8.1.7}$$

Let $L' = \{ L_1(p) : p \in \mathcal{F} \}$, then L' is intersecting. So let $p_1, p_2 \in \mathcal{F}$ and $x \in L_1(p_1) \cap L_1(p_2)$. Since both posets have height 1, we have $L_0(p) = [n] \setminus L_1(p)$ for $p = p_1, p_2$, and so $x \notin L_0(p_1) \cup L_0(p_2)$. We have shown that no two elements of L can have union $[n]$; therefore no elements of $D \cup L$ can have union $[n]$. In other words, no subset Y of $[n]$ and its complement $\overline{Y} = [n] \setminus Y$ are both in $D \cup L$, giving

$$|D \cup L| \leq 2^n/2 = 2^{n-1}.$$

Combining this with (8.1.7) yields the result. $\square$

The class $\mathcal{H}_n^{(1)}$ has the nice property that each of the traditional saturation families, including the fix-family, is maximum: for $0 \leq r \leq (n-2)/2$, define the poset $v(r)$ by

$$v(r) = \{ (i, n) : 1 \leq i \leq 2r + 1 \}$$

and set

$$G_r(\mathcal{H}_n^{(1)}) = \left\{ p \in \mathcal{H}_n^{(1)} : |p \cap v(r)| \geq r + 1 \right\}.$$

As usual, we abbreviate $G_r(\mathcal{H}_n^{(1)})$ by G_r in the context of $\mathcal{H}_n^{(1)}$. Note that G_r is intersecting by the pigeonhole principle. In particular, G_0 is the fix-family in $\mathcal{H}_n^{(1)}$:

$$G_0(\mathcal{H}_n^{(1)}) = \left\{ p \in \mathcal{H}_n^{(1)} : (1, n) \in p \right\} = F_{1,n}(\mathcal{H}_n^{(1)}).$$

Proposition 8.1.8. *Each of the $n/2$ intersecting families G_r is maximum in $\mathcal{H}_n^{(1)}$.*

Proof. We need to show that G_r attains the bound of Theorem 8.1.6. Since posets of height 1 are determined by either of their levels, we have

$$\begin{aligned} G_r &= \{p \in \mathcal{H}_n^{(1)} \ : \ n \in L_1(p),\ \text{at least } r+1 \text{ elements of } [2r+1] \text{ are in } L_0(p)\} \\ &= \{p \in \mathcal{H}_n^{(1)} \ : \ n \in L_1(p),\ \text{at most } r \text{ elements of } [2r+1] \text{ are in } L_1(p)\} \end{aligned}$$

and so

$$|G_r| = 2^{n-(2r+2)} \cdot \sum_{i=0}^{r} \binom{2r+1}{i}.$$

A basic property of binomial coefficients is $\binom{n}{k} = \binom{n}{n-k}$ and so

$$2 \cdot \sum_{i=0}^{r} \binom{2r+1}{i} = \sum_{i=0}^{2r+1} \binom{2r+1}{i} = 2^{2r+1},$$

since the middle sum is clearly the size of the power set of $[2r + 1]$. Together, the previous two equations give

$$|G_r| = 2^{n-(2r+2)} \cdot \frac{1}{2} \cdot 2^{2r+1} = 2^{n-2}$$

as required. □

A Classification due to Frankl

Let $\mathcal{F}$ be an intersecting family of subsets of $[n]$ such that the family of complements

$$\overline{\mathcal{F}} = \left\{ X \subseteq [n] : \overline{X} \in \mathcal{F} \right\}$$

is also intersecting. Clearly these set families $\mathcal{F}$ are equivalent to intersecting subsets of $\mathcal{H}_n^{(1)}$. They have been studied, for instance, in [DL76, Fra88a]. Indeed, Frankl claims in [Fra88a] that the largest such $\mathcal{F}$ can be classified as follows: let $\{A, B\}$ be a partition of $[n]$ into two parts, so

$$[n] = A \mathbin{\dot\cup} B.$$

Let $\mathcal{A}$ be an intersecting family of subsets of A, of maximal size $|\mathcal{A}| = 2^{|A|-1}$, and let $\mathcal{B}$ be a family of subsets of B, of size $|\mathcal{B}| = 2^{|B|-1}$, such that no two elements of $\mathcal{B}$ have union B. Frankl defines a family $\mathcal{F}'(\mathcal{A}, \mathcal{B})$ as follows:

$$\mathcal{F}'(\mathcal{A}, \mathcal{B}) = \{ X \cup Y : X \in \mathcal{A},\ Y \in \mathcal{B} \}.$$

The intersection property of $\mathcal{A}$ implies that $\mathcal{F}'(\mathcal{A}, \mathcal{B})$ is intersecting, and it follows from the special property of $\mathcal{B}$ that $\overline{\mathcal{F}'(\mathcal{A}, \mathcal{B})}$ is also intersecting. Finally,

$$|\mathcal{F}'(\mathcal{A}, \mathcal{B})| = 2^{|A|-1} \cdot 2^{|B|-1} = 2^{|A|+|B|-2} = 2^{n-2},$$

so $\mathcal{F}'(\mathcal{A}, \mathcal{B})$ is maximum by Theorem 8.1.6. Moreover, Frankl tells us:

> "One can show that all extremal families can be obtained in this way." [Fra88a]

For a specific family, it is usually clear how the corresponding 'Frankl classification sets' A, B, $\mathcal{A}$ and $\mathcal{B}$ are obtained, but it is not obvious how this generalises. To illustrate this in the language of posets, we will consider the familiar examples of the fix-family $F_{1,n}(\mathcal{H}_n^{(1)})$ and the saturation family $G_r(\mathcal{H}_n^{(1)})$.

Note that both the upper and lower levels of the fix-family in $\mathcal{H}_n^{(1)}$ are fix-families of sets:

$$\begin{aligned} F_{1,n} &= \left\{ p \in \mathcal{H}_n^{(1)} : (1, n) \in p \right\} \\ &= \left\{ p \in \mathcal{H}_n^{(1)} : 1 \in L_0(p),\ n \in L_1(p) \right\}. \end{aligned}$$

So setting $A_0 = [n-1]$, $B_0 = \{n\}$ and

$$\begin{aligned} \mathcal{A}_0 &= \{ X \subseteq A_0 : 1 \in X \}, \\ \mathcal{B}_0 &= \{\emptyset\}, \end{aligned}$$

it is easily confirmed that the set of lower levels $L_0(F_{1,n})$ of the poset family is equal to $\mathcal{F}'(\mathcal{A}_0, \mathcal{B}_0)$. Recalling that the lower level of an element of $\mathcal{H}_n^{(1)}$ determines the poset, we see that $L_0(F_{1,n})$ determines $F_{1,n}$. Moreover, the sets A_0, B_0, $\mathcal{A}_0$ and $\mathcal{B}_0$ satisfy the properties described in Frankl's classification: $[n]$ is the disjoint union of A_0 and B_0; $\mathcal{A}_0$ is an intersecting family of subsets of A_0 of size $2^{|A_0|-1}$, and $\mathcal{B}_0$ is a family of subsets of B, of size $|\mathcal{B}| = 1 = 2^{|B_0|-1}$, with

$$X \cup Y \neq B_0, \qquad \forall X, Y \in \mathcal{B}_0.$$

As a second example, let us consider

$$\begin{aligned} G_1(\mathcal{H}_6^{(1)}) &= \left\{ p \in \mathcal{H}_6^{(1)} : |p \cap \{(1,6),(2,6),(3,6)\}| \geq 2 \right\} \\ &= \left\{ p \in \mathcal{H}_6^{(1)} : |L_0(p) \cap \{1,2,3\}| \geq 2,\ 6 \in L_1(p) \right\}. \end{aligned}$$

Note that one level of $G_1(\mathcal{H}_6^{(1)})$ is obtained by fixing and the other by saturation, and using the same approaches in the associated set families will lead to the desired outcome: set $A_1 = [5]$, $B_1 = \{6\}$ and $\mathcal{B}_1 = \{\emptyset\}$ as before, with

$$\mathcal{A}_1 = \{ X \subseteq A_1 : |X \cap \{1,2,3\}| \geq 2 \}.$$

Then $L_0(G_1(\mathcal{H}_6^{(1)})) = \mathcal{F}'(\mathcal{A}_1, \mathcal{B}_1)$ and all sets involved satisfy Frankl's requirements from [Fra88a]. This example could easily be generalised to describe saturation families $G_r(\mathcal{H}_n^{(1)})$ in terms of Frankl's classification for arbitrary r and n, or more general saturation families where both the upper and lower levels are obtained by saturation: for instance, if

$$\mathcal{F} = \left\{ p \in \mathcal{H}_8^{(1)} : |L_0(p) \cap [5]| \geq 3,\ |L_1(p) \cap \{6,7,8\}| \geq 2 \right\},$$

we simply set $A_2 = [5]$, $B_2 = \{6,7,8\}$,

$$\begin{aligned} \mathcal{A}_2 &= \{ X \subseteq A_2 : |X \cap [5]| \geq 3 \}, \\ \mathcal{B}_2 &= \{ X \subseteq B_2 : |X \cap \{6,7,8\}| \geq 2 \} \end{aligned}$$

to obtain the desired classification. Indeed, given a maximum intersecting subset $\mathcal{F}$ of $\mathcal{H}_n^{(1)}$, it is clear that $L_0(\mathcal{F})$ is intersecting and, since $L_1(\mathcal{F})$ is also intersecting, we have

$$X \cup Y \neq [n], \qquad \forall X, Y \in \overline{L_1(\mathcal{F})},$$

suggesting that $\mathcal{A}$ and $\mathcal{B}$ should generally be obtained from $L_0(\mathcal{F})$ and $L_1(\mathcal{F})$ respectively. However, whilst this may seem intuitively plausible, it is not clear how the existence of a partition of $[n]$ into disjoint sets A and B could be guaranteed in general, in such a way that elements of $L_0(\mathcal{F})$ intersect not simply anywhere, but in elements of A, while elements of $L_1(\mathcal{F})$ intersect in elements of B.

Thus it seems that we have said as much as we can about the intersection structure of $\mathcal{H}_n^{(1)}$ and its subsets $\mathcal{C}_{k,n-k}$. There are two natural directions for further inquiry now: either we consider more general height 1 posets, i.e. posets which are not necessarily complete, or else we attempt to generalise our results in Section 8.1 to posets of height 2. We choose the latter avenue at this stage, as it seems slightly more accessible to begin with.

8.2 Posets of Height 2

Let us turn our attention to posets of height 2. Using the notation from the previous section, set

$$\mathcal{H}_n^{(2)} = \{\, p \in \mathcal{P}_n \;\mid\; h(p) = 2,\ L_0(p) <_p L_1(p) <_p L_2(p)\} \,.$$

Let a, b, c be positive natural numbers which sum to n. We define the set of all complete height 2 posets with lower, middle and upper levels having sizes a, b and c, respectively, as follows:

$$\mathcal{C}_{a,b,c} = \left\{ p \in \mathcal{H}_n^{(2)} \;\mid\; |L_0(p)| = a,\ |L_1(p)| = b,\ |L_2(p)| = c \right\} .$$

Similarly to our initial investigation of height 1 posets, we fix a poset of height 2 and consider maximum intersecting subsets of the class arising from permutations of the labels. Section 8.2.1 begins by specifying for which values of a, b, c the classification of maximum intersecting subsets of $\mathcal{C}_{a,b,c}$ is easily obtained. Towards the end of the section we will see that some of the remaining classes are quite complicated; indeed, we do not have results on intersecting subsets of $\mathcal{H}_n^{(2)}$ as a whole.

8.2.1 Fixing a Complete Poset of Height 2

Posets with Large Upper or Lower Levels

We begin by classifying maximum intersecting subsets of isomorphism classes of $\mathcal{H}_n^{(2)}$ with large upper or lower levels, because their intersection structure is fairly simple, as we will see in the next lemma. In order to generalise the reasoning from Section 8.1.1 to height 2 posets, we need the following definition.

Definition 8.2.1. Let a, b, c be natural numbers with $a + b + c = n$ and either $a \geq n/2$ or $c \geq n/2$. For a poset $p \in \mathcal{C}_{a,b,c}$, let $U(p)$ be the union of its two smaller levels:

$$U(p) = L_1(p) \cup L_j(p), \quad j = \begin{cases} 2 & \text{if } a \geq n/2 \\ 0 & \text{if } c \geq n/2 \end{cases}.$$

For a set $X \subseteq \mathcal{C}_{a,b,c}$, we set $U(X) = \{ U(p) : p \in X \}$.

Lemma 8.2.2. *Let a, b, c be natural numbers with $a + b + c = n$ and either $a \geq n/2$ or $c \geq n/2$. Then $p, q \in \mathcal{C}_{a,b,c}$ intersect if, and only if, $U(p)$ and $U(q)$ intersect.*

Proof. Suppose $a \geq n/2$. If $U(p) \cap U(q) = \emptyset$, then $U(p) \subseteq L_0(q)$ and so p and q cannot intersect. Conversely, it suffices to show that if $U(p)$ and $U(q)$ intersect, then $L_0(p)$ and $L_0(q)$ must also intersect. If $a = n/2$, this follows from the fact that two $(n/2)$-subsets of $[n]$ intersect if, and only if, they are not each other's complements; and $L_0(p)$ is the complement of $U(p)$. If $a > n/2$, then $L_0(p)$ intersects $L_0(q)$ for any $p, q \in \mathcal{C}_{a,b,c}$ by the pigeonhole principle.

The case $c \geq n/2$ follows by symmetry. □

We use the standard notation

$$\binom{[n]}{k} = \{ A \subseteq [n] : |A| = k \}.$$

The following proposition specifies how we can use known results about intersecting families of sets to deduce results about intersecting families in isomorphism classes of $\mathcal{H}_n^{(2)}$.

Proposition 8.2.3. *Let a, b, c, m, n be natural numbers with $n = a + b + c$ and $m = \max(a, c) \geq n/2$. If $\mathcal{Z}$ is a maximum intersecting family of $\binom{[n]}{n-m}$, then for an intersecting family $\mathcal{F} \subseteq \mathcal{C}_{a,b,c}$ we have*

$$|\mathcal{F}| \leq |\mathcal{Z}| \cdot \binom{n-m}{b},$$

and equality holds if, and only if,

$$\mathcal{F} = \{ p \in \mathcal{C}_{a,b,c} : U(p) \in \mathcal{Z}_0 \}$$

for some maximum intersecting family $\mathcal{Z}_0$ of $\binom{[n]}{n-m}$.

Proof. Suppose, without loss of generality, that $m = a$, so $U(p) = L_1(p) \cup L_2(p)$ for $p \in \mathcal{C}_{a,b,c}$. Clearly any intersecting family $\mathcal{F}$ satisfies

$$\mathcal{F} \subseteq \{ p \in \mathcal{C}_{a,b,c} : U(p) \in U(\mathcal{F}) \} \tag{8.2.4}$$

and any element of $\mathcal{H}_n^{(2)}$ is determined by any two of its levels. Thus given $L_1(p) \cup L_2(p)$, there are $\binom{b+c}{b}$ ways of choosing $L_1(p)$, and this determines p. Conversely, two distinct $(b+c)$-subsets of $[n]$ clearly cannot give rise to the same poset in this way, and so

$$|\mathcal{F}| \leq |\{\, p \in \mathcal{C}_{a,b,c} : U(p) \in U(\mathcal{F}) \,\}| = |U(\mathcal{F})| \cdot \binom{b+c}{b}.$$

By Lemma 8.2.2, $U(\mathcal{F})$ is an intersecting subset of $\binom{[n]}{b+c}$, so

$$|U(\mathcal{F})| \leq |\mathcal{Z}|, \tag{8.2.5}$$

giving

$$|\mathcal{F}| \leq |\mathcal{Z}| \cdot \binom{b+c}{b} = |\mathcal{Z}| \cdot \binom{n-m}{b}.$$

Equality in this bound requires equality in both (8.2.4) and (8.2.5), implying that $U(\mathcal{F})$ is a maximum intersecting subset of $\binom{[n]}{n-m}$, and the result follows. □

To interpret this proposition in the context of $\mathcal{C}_{a,b,c}$, we will apply the Erdős-Ko-Rado Theorem. Therefore the case $m < n/2$ differs from the case $m = n/2$, and we have two separate corollaries.

Corollary 8.2.6. *Let a, b, c, m, n be natural numbers with $n = a + b + c$ and $m = \max(a, c) > n/2$.*

If $\mathcal{F}$ is an intersecting subset of $\mathcal{C}_{a,b,c}$ then

$$|\mathcal{F}| \leq \binom{n-1}{n-m-1} \cdot \binom{n-m}{b}.$$

Moreover, equality holds if, and only if, there exists $x \in [n]$ such that

$$\mathcal{F} = \{\, p \in \mathcal{C}_{a,b,c} : x \notin L_i(p) \,\}$$

where $i = 0$ if $a > n/2$ and $i = 2$ if $c > n/2$.

Proof. Since $n - m < n/2$, this result follows from the Erdős-Ko-Rado Theorem 1.2.1 together with Proposition 8.2.3.

The bound is clear. In the case of equality, let $(i, j) = (0, 2)$ if $m = a$ and $(i, j) = (2, 0)$ if $m = c$, then

$$\mathcal{F} = \{\, p \in \mathcal{C}_{a,b,c} : L_1(p) \cup L_j(p) \in \mathcal{Z} \,\}$$

where

$$\mathcal{Z} = \left\{ A \in \binom{[n]}{n-m} : x \in A \right\}$$

for some fixed $x \in [n]$. This gives

$$\mathcal{F} = \{\, p \in \mathcal{C}_{a,b,c} : x \in L_1(p) \cup L_j(p) \,\}$$

and the result follows. □

Corollary 8.2.7. *Let a, b, c, k be natural numbers with $2k = a + b + c$ and $\max(a, c) = k$.*

If $\mathcal{F}$ is an intersecting subset of $\mathcal{C}_{a,b,c}$ then

$$|\mathcal{F}| \leq \frac{1}{2}\binom{2k}{k} \cdot \binom{k}{b},$$

and equality holds if, and only if, $\mathcal{F} = \{\, p \in \mathcal{C}_{a,b,c} : U(p) \in \mathcal{Z} \,\}$ where $\mathcal{Z}$ is some transversal of

$$\mathcal{A} = \left\{ \{A, \overline{A}\} : A \in \binom{[2k]}{k} \right\}.$$

Proof. As previously noted, two elements A, B of $\binom{[2k]}{k}$ intersect if, and only if, $B \neq \overline{A}$. Thus the maximum intersecting subsets of $\binom{[2k]}{k}$ are unions of transversals of $\mathcal{A}$, and clearly such a transversal has size $\frac{1}{2}\binom{2k}{k}$. Hence the result follows from Proposition 8.2.3. $\square$

Symmetrical Posets

There is one more set of isomorphism classes in $\mathcal{H}_n^{(2)}$ whose maximum intersecting subsets are easily described. These are the classes whose elements satisfy $p \cong \mathrm{rev}(p)$, i.e. the classes $\mathcal{C}_{a,b,a}$. Note that these do not overlap with the classes $\mathcal{C}_{a,b,c}$ considered in Corollaries 8.2.6 and 8.2.7: since $b \neq 0$, either of $a \geq n/2$ or $c \geq n/2$ implies $a \neq c$.

Proposition 8.2.8. *If $\mathcal{F}$ is an intersecting subset of $\mathcal{C}_{a,b,a}$ then*

$$|\mathcal{F}| \leq \frac{|\mathcal{C}_{a,b,a}|}{2} = \frac{(2a+b)!}{2 \cdot (a!)^2 \cdot b!}.$$

Moreover, equality holds if, and only if, $\mathcal{F}$ is a transversal of

$$\{\, \{p, \mathrm{rev}(p)\} : p \in \mathcal{C}_{a,b,a} \,\}.$$

Proof. It suffices to show that two posets $p, q \in \mathcal{C}_{a,b,a}$ intersect if, and only if, one is not the reverse of the other. It follows from Definition 6.2.1 that p and $\mathrm{rev}(p)$ do not intersect, so suppose $q \neq \mathrm{rev}(p)$. Clearly $L_0(p) = L_2(q)$ together with $L_2(p) = L_0(q)$ would imply $p = \mathrm{rev}(q)$, so we may assume that at least one of $L_0(p) \setminus L_2(q)$ and $L_2(p) \setminus L_0(q)$ is non-empty.

If there exists $x \in L_0(p) \setminus L_2(q)$, then $L_2(q)$ cannot be contained in $L_0(p)$ since $L_0(p)$ and $L_2(q)$ have equal size. So there exists $y \in L_2(q) \setminus L_0(p)$, and $y \in L_1(p) \cup L_2(p)$ since $y \notin L_0(p)$. Similarly, $x \in L_0(q) \cup L_1(q)$, and so the intersection of p and q contains (x, y).

The case $L_2(p) \setminus L_0(q) \neq \emptyset$ follows by symmetry. $\square$

Some Observations on the Remaining Cases

So far we have classified the maximum intersecting subsets of height 2 isomorphism classes with large upper or lower levels, or equal size upper and lower levels. In this subsection, we illustrate the fact that some of the remaining classes are quite complicated.

Consider a general isomorphism class of height 2 posets whose maximum intersecting subsets are not determined by results in the previous sections, noting that results on $\mathcal{C}_{c,b,a}$ will follow by symmetry from results concerning $\mathcal{C}_{a,b,c}$. That is, let $\mathcal{C}_{a,b,c} \subseteq \mathcal{H}_n^{(2)}$ with $a < c < n/2$.

We would like to find the maximum intersecting subsets of $\mathcal{C}_{a,b,c}$. One obvious candidate is the traditional fix-family: setting

$$F_{1,n}(\mathcal{C}_{a,b,c}) = \{\, p \in \mathcal{C}_{a,b,c} : (1,n) \in p \,\},$$

it is clear that $F_{1,n}(\mathcal{C}_{a,b,c})$ is intersecting.

However, the use of the Erdős-Ko-Rado Theorem in Corollary 8.2.6 suggests that there is an alternative concept of fixing for height 2 posets: in view of Corollary 8.2.6, we define a family of posets by taking all elements of $\mathcal{C}_{a,b,c}$ in which some fixed point has large enough height to guarantee the intersection of all posets in the family: let

$$G(\mathcal{C}_{a,b,c}) = \{\, p \in \mathcal{C}_{a,b,c} : n \in L_2(p) \,\}.$$

To check that n does indeed have sufficiently large height in $p, q \in G(\mathcal{C}_{a,b,c})$ to guarantee their intersection, note that $c < n/2$ implies $a + b > n/2$, so there exists

$$x \;\in\; (L_0(p) \cup L_1(p)) \;\cap\; (L_0(q) \cup L_1(q))$$

by the pigeonhole principle. This guarantees $(x, n) \in (p \cap q)$, so $G(\mathcal{C}_{a,b,c})$ is indeed intersecting. Note that $G(\mathcal{C}_{a,b,c})$ can be obtained by saturating over

$$v_n = \{\, (i,n) : 1 \leq i \leq n-1 \,\}$$

from Chapter 7, a poset which is equal to $v((n-2)/2)$ from Section 8.1.2. Thus $G(\mathcal{C}_{a,b,c})$ should be regarded as yet another instance of the saturation families which we have encountered throughout Part III.

We might expect that either the fix-family $F_{1,n}(\mathcal{C}_{a,b,c})$ or the saturation family $G(\mathcal{C}_{a,b,c})$ are maximum. However, computational investigations of the intersection structure of $\mathcal{C}_{a,b,c}$ soon point towards a different family, which in some sense generalises the idea behind $G(\mathcal{C}_{a,b,c})$: the family $R(\mathcal{C}_{a,b,c})$ contains posets in which 1 is minimal, posets in which i is minimal and no element of $[i-1]$

is maximal for $2 \leq i \leq b$, and finally the posets which have middle level $[b]$: using the convention $[0] = \emptyset$, set

$$R(\mathcal{C}_{a,b,c}) = \{ p \in \mathcal{C}_{a,b,c} : \exists i \in [b] \text{ such that } i \in L_0(p) \text{ and } [i-1] \subseteq L_0(p) \cup L_1(p) \} \cup \{ p \in \mathcal{C}_{a,b,c} : L_1(p) = [b] \}.$$

It is easy to verify computationally that $R(\mathcal{C}_{a,b,c})$ is intersecting for $5 \leq n \leq 7$, but we do not claim here that $R(\mathcal{C}_{a,b,c})$ is intersecting for any a, b, c with $a < c < n/2$. Our justification for drawing the reader's attention to the family $R(\mathcal{C}_{a,b,c})$ is Table 8.2.1, which demonstrates that for $5 \leq n \leq 7$, $R(\mathcal{C}_{a,b,c})$ is at least as large as the fix-family $F_{1,n}(\mathcal{C}_{a,b,c})$ and the saturation family $G(\mathcal{C}_{a,b,c})$ in all classes which are not covered by Corollaries 8.2.6 and 8.2.7 or Proposition 8.2.8, except $\mathcal{C}_{1,3,3}$.

The Class $\mathcal{D}_n$

As Table 8.2.1 suggests, we have not completed the classification of intersecting families in $\mathcal{C}_{a,b,c}$. The remainder of this chapter aims to illustrate the difficulties we encountered in trying to do so, by concentrating on one specific case.

Up to permutations of the labels, the families listed in Table 8.2.1 were the only large ones found in the classes listed there during various computational searches, so we hazard the guess that $R(\mathcal{C}_{a,b,c})$ is maximum in $\mathcal{C}_{1,n-3,2}$. (Though we remark that even if this is the case, $R(\mathcal{C}_{a,b,c})$ is not the only maximum family since when $n \in \{6, 7\}$, these classes contain another intersecting family of size $|R(\mathcal{C}_{1,n-3,2})|$ which is half way between F and R, in the sense that it has many elements in common with both of these families. We will not further specify these families here.)

For $n \geq 4$, set

$$\mathcal{D}_n = \mathcal{C}_{1,n-3,2}$$

Table 8.2.1: Sizes of some intersecting families in height 2 isomorphism classes not covered by previous results for $5 \leq n \leq 7$.

$\mathcal{C}_{a,b,c}$	$\mathcal{C}_{1,2,2}$	$\mathcal{C}_{1,3,2}$	$\mathcal{C}_{1,4,2}$	$\mathcal{C}_{1,3,3}$	$\mathcal{C}_{2,2,3}$
$\lvert\mathcal{C}_{a,b,c}\rvert$	30	60	105	140	210
$\lvert F_{1,n}(\mathcal{C}_{a,b,c})\rvert$	12	22	35	50	80 (max. to 90)[a]
$\lvert G(\mathcal{C}_{a,b,c})\rvert$	12	20	30	60	90
$\lvert R(\mathcal{C}_{a,b,c})\rvert$	12	22	37	38 (max. to 50)[b]	90

[a] $F_{1,n}(\mathcal{C}_{2,2,3})$ has size 80 but $F_{1,n}(\mathcal{C}_{2,2,3}) \cup \{ p \in \mathcal{C}_{2,2,3} : L_0(p) = \{1, 7\} \}$ is an intersecting subset of $\mathcal{C}_{2,2,3}$ of size 90.
[b] $|R(\mathcal{C}_{1,3,3})| = 38$ but there is an intersecting family of size 50 in $\mathcal{C}_{1,3,3}$ which contains $R(\mathcal{C}_{1,3,3})$.

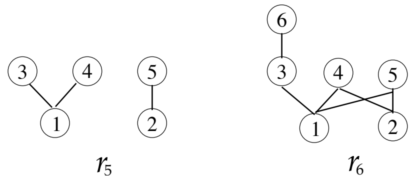


Figure 8.2.1: Hasse diagrams of saturation bases for $R(5)$ and $R(6)$.

and denote by $\Diamond_x^{y,z}$ the element of $\mathcal{D}_n$ which has $L_0(p) = \{x\}$, $L_2(p) = \{y, z\}$ and $L_1(p) = [n] \setminus \{x, y, z\}$. We introduce some further notation for the components of $R(\mathcal{C}_{1,n-3,2})$ and, since the remainder of this chapter is concerned with $\mathcal{D}_n$ only, we abbreviate $R(\mathcal{C}_{1,n-3,2})$ itself by $R(n)$: for $1 \leq i \leq n-3$, set

$$R_i(n) \;=\; \{\, p \in \mathcal{D}_n : L_0(p) = \{i\},\ [i-1] \subseteq L_1(p)\, \},$$
$$R_{n-2}(n) = \left\{\Diamond_{n-2}^{n-1,n}\right\}, \qquad R_{n-1}(n) = \left\{\Diamond_{n-1}^{n-2,n}\right\}, \qquad R_n(n) = \left\{\Diamond_{n}^{n-2,n-1}\right\}.$$

Then $R(n) = \bigcup_{i=1}^{n} R_i(n)$.

Lemma 8.2.9. *The family $R(n) \subseteq \mathcal{D}_n$ is intersecting for $n \geq 4$.*

Proof. If $p, q \in R_i(n)$ then $(i, j) \in p \cap q$ for all $j \neq i, j \in [n]$. It is also easy to see that for any two elements of

$$R_{n-2}(n) \cup R_{n-1}(n) \cup R_n(n) = \left\{\Diamond_{n-2}^{n-1,n},\ \Diamond_{n-1}^{n-2,n},\ \Diamond_{n}^{n-2,n-1}\right\},$$

there exists a point which is maximal in both posets, and the two middle levels have $n - 3 \geq 1$ points in common. So let $p \in R_i(n)$ and $q = \Diamond_j^{x,y} \in R_j(n)$ for some $i < j \leq n$ with $i < n-2$. Then $i \in L_0(p) \cap L_1(q)$, so $(i, x) \in p \cap q$. $\square$

It is easily shown that $R(n)$ is not a fix-family: since

$$R_1(n) = \{\, p \in \mathcal{D}_n : L_0(p) = \{1\}\, \} \subseteq R(n),$$

a fixed comparison contained in all elements of $R(n)$ would need to be of the form $(1, x)$ for some $x \in [n]$. However, for any such x there exists

$$p \in \left(\bigcup_{i=n-2}^{n} R_i(n)\right) \subseteq R(n)$$

with $(1, x) \notin p$. So is R a saturation family? It can be shown that

$$R(5) = \{\, p \in \mathcal{D}_5 : |p \cap r_5| \geq 2\, \}, \quad R(6) = \{\, p \in \mathcal{D}_6 : |p \cap r_6| \geq 4\, \},$$

where r_5 and r_6 are defined by their Hasse diagrams in Figure 8.2.1. However, it is not entirely clear how these saturation bases would generalise for $n > 6$. Note in particular that neither of the posets r_5 or r_6 are elements of $\mathcal{D}_n$, or even of $\mathcal{H}_n^{(2)}$ or $\mathcal{H}_n^{(1)}$, since they are not complete. The alternative saturation bases for $R(5)$ and $R(6)$ look very similar to r_5 and r_6. Thus $R(n)$ is not a fix-family, and does not arise naturally as a saturation family either.

Recall that we are trying to determine the intersecting structure of $\mathcal{D}_n$, which the following lemma considers in some more detail.

Lemma 8.2.10. *Let $p = \Diamond_b^{t_1,t_2} \in \mathcal{D}_n$. The elements of $\mathcal{D}_n$ which do not intersect p are given by*

$$N(p) = \left\{ \Diamond_{t_i}^{b,x} \in \mathcal{D}_n : x \in [n] \setminus \{b, t_1, t_2\},\ i = 1, 2 \right\}.$$

Proof. Let $q \in \mathcal{D}_n$ such that $p \cap q = \emptyset$. Then b must be maximal in q. Moreover, the only label which may be less than t_1 under q is t_2, and vice versa, so we have one of t_1, t_2 in $L_0(q)$ and the other in $L_1(q)$. $\square$

Our aim is to map an arbitrary intersecting subset of $\mathcal{D}_n$ injectively into $R(n)$, but investigating the situation for small values of n suggests that no such general injection exists. In view of Lemma 8.2.10, we therefore define a map Φ which assigns to each element p of $\mathcal{D}_n$ a set of posets in $R(n)$ as follows: for $p = \Diamond_b^{t_1,t_2} \in \mathcal{D}_n$ with $t_1 < t_2$, we set

$$\Phi(p) = \begin{cases} \{p\} & \text{if } p \in R(n) \\ \left\{ \Diamond_{t_1}^{b,x} \in \mathcal{D}_n : x \in [n] \setminus \{b, t_1, t_2\},\ x > t_1 \right\} & \text{if } p \notin R(n) \end{cases}$$

and for a set $X \subseteq \mathcal{D}_n$, set $\Phi(X) = \bigcup_{p \in X} \Phi(p)$. The following lemma proves our claim that Φ maps into $R(n)$.

Lemma 8.2.11. *For $X \subseteq \mathcal{D}_n$ we have $\Phi(X) \subseteq R(n)$.*

Proof. For $p = \Diamond_b^{t_1,t_2} \in X$ with $t_1 < t_2$, we need to show that $\Phi(p) \subseteq R(n)$. If $p \in R(n)$, this holds trivially. On the other hand, $p \notin R(n)$ implies that either $b \in [n-3]$ and

$$\{t_1, t_2\} \cap [b-1] \neq \emptyset, \tag{8.2.12}$$

or else $b \in \{n-2, n-1, n\}$ and

$$\{t_1, t_2\} \cap [n-3] \neq \emptyset. \tag{8.2.13}$$

Let $q = \Diamond_{t_1}^{b,x} \in \Phi(p)$. Using either (8.2.12) or (8.2.13) together with $t_1 < t_2$ implies $t_1 < b$. Moreover, by the definition of Φ, we have $t_1 < x$. Thus both b and x are elements of $\{t_1 + 1, \ldots, n\}$, giving $q \in R_{t_1}(n) \subseteq R(n)$. $\square$

To indicate why Φ might be a sensible way of mapping an arbitrary intersecting subset $\mathcal{F}$ of $\mathcal{D}_n$ into $R(n)$, note that Φ at least maps $\mathcal{F}$ closer to $R(n)$, rather than giving us elements of $R(n)$ which we already had in $\mathcal{F}$: for $p = \lozenge_b^{t_1,t_2} \in \mathcal{F} \setminus R(n)$, the set $\Phi(p)$ is a subset of $N(p)$ in Lemma 8.2.10, which implies

$$\Phi(\mathcal{F} \setminus R(n)) \cap \mathcal{F} = \emptyset, \tag{8.2.14}$$

since $\mathcal{F}$ is intersecting.

Moreover, investigating examples of various maximal intersecting families $\mathcal{F}$ of $\mathcal{D}_n$ for small n suggests that, denoting the element $\lozenge_n^{n-2,n-1}$ of $R_n(n)$ by p_n, the following facts hold:

- if $p_n \in \mathcal{F}$ then $|\Phi(\mathcal{F})| \geq |\mathcal{F}|$, and
- if $p_n \notin \mathcal{F}$ then $|\Phi(\mathcal{F})| \geq |\mathcal{F}| - 1$ and $p_n \notin \Phi(\mathcal{F})$.

In view of Lemma 8.2.11, it is clear that establishing the validity of these two claims in general would suffice to prove that $R(n)$ is maximum in $\mathcal{D}_n$, and a proof of these facts when $n = 5$ is included in the appendix. Unfortunately however, the proof of Proposition A.1.1 amounts to little more than a detailed case analysis, i.e. it gives hardly any further insight into the intersection structure of $\mathcal{D}_5$ and therefore does not inspire ideas for a proof for general n.

Conclusion

This section has discussed the intersection structure of some of the isomorphism classes $\mathcal{C}_{a,b,c}$ which make up $\mathcal{H}_n^{(2)}$, the set of complete posets of height 2. If the top and bottom levels of posets in the class either have the same size, or one of them contains at least half of the points, then the intersection structure of $\mathcal{C}_{a,b,c}$ is very straightforward: see Lemma 8.2.2 and Proposition 8.2.8. In these cases, the classifications of intersecting families in $\mathcal{C}_{a,b,c}$ follow from the corresponding classifications in Chapter 1, as we saw in Corollaries 8.2.6 and 8.2.7 as well as Proposition 8.2.8. However, some of the remaining cases are much more complicated, as the present subsection has demonstrated: considering the class $\mathcal{D}_n = \mathcal{C}_{1,n-3,2}$ for instance, the best analogue of Lemma 8.2.2 we can achieve is Lemma 8.2.10, revealing a much more complex intersection structure. Moreover, numerous computational investigations found no intersecting family which is maximum in all remaining classes simultaneously when $5 \leq n \leq 7$. We conclude that the intersection structure of poset classes of small height can be much more complicated than one might expect.

Part IV

Appendix

CONCLUSION

This thesis began with a brief account of what we call Erdős-Ko-Rado Theory in Part I.

Following this introduction, Part II classified the maximum 1-intersecting injection families: all optimal intersecting families of injections from $[k]$ to $[n]$ are equivalent to the fix-family. Concerning t-intersecting subsets of $\mathcal{I}_n^k$, we proved two limit results in Chapter 3: for fixed k and t, increasing n will ensure that fixing is eventually the unique optimal strategy. On the other hand, if we fix the differences $k-t$ and $n-k$ as we increase k, then one particular saturation family becomes the unique maximum t-intersecting subset of $\mathcal{I}_n^k$, so the fix-family is not maximum in this scenario. Finally we showed that, among so-called exemplary injection families with $k<n$, one of the saturation families $\mathcal{K}_r$ is always optimal. Whether there are any injection families which cannot be standardised in this way remains an open question. We have also made considerable progress towards a function determining which of the saturation families $\mathcal{K}_r$ is the optimal one.

One of the main objectives of this thesis was to obtain results on t-intersecting injection families, and many have been achieved. Note, however, that our results are not concerned with permutations: our classification of 1-intersecting injection families takes Cameron & Ku's corresponding result on permutations from [CK03] as given, and our bound on exemplary families requires $k<n$. Our limit result regarding the case where k is large in terms of its differences with t and n was a generalisation of Frankl & Deza's earlier result on permutations in [DF77], while the result which guarantees that fixing is eventually optimal requires n to be large in terms of k, and therefore does not apply to permutations. Indeed, the main open problem in this area, first conjectured in [DF77], is still open: does there exist a function $n_0(t)$ such that for $n>n_0(t)$, every maximum t-intersecting subset of $\mathcal{S}_n$ is equivalent to the fix-family? The answer to this question is widely believed to be yes, but noone has yet been able to demonstrate this.

The theme of fixing versus saturation was continued in Part III, where we introduced the concept of intersecting posets. However, our results on the intersection structure of partial orders are — well, partial. The lack of previous literature to put this work into context adds to the difficulty in giving a meaningful overview of what has been achieved. Given the current state of knowledge, it seems that a complete classification of maximum intersecting families of partial orders is out of the question at present. Even the study of subclasses of partial orders requires a variety of methods, as a comparison of Chapter 7 with Chapter 8 will illustrate. Therefore we present a factual summary of our results here, without endeavouring to assess their significance within Erdős-Ko-Rado Theory.

In Chapter 6 we started our investigation of the intersection structure of $\mathcal{P}_n$, the set of partial orders, by classifying the maximum intersecting families of linear orders. After defining poset intersection in a way that is compatible with the view of posets as sets with additional properties, our initial observations concluded that the fix-family has a good chance of being maximum in $\mathcal{P}_n$. Indeed, we showed that the fix-family is maximal in terms of set inclusion, and established sufficient conditions for the optimality of the fix-family in $\mathcal{P}_n$. We proved that any maximal intersecting family of partial orders contains a maximum family of linear orders, and if the latter is a fix-family, then the former must be a fix-family also. However, our investigation of the relationship between partial orders and their linear extensions did not yield the desired classification of maximum intersecting families, so we turned our attention to subclasses of $\mathcal{P}_n$ as a more feasible object of study in the subsequent chapters.

Chapter 7 classified the maximum intersecting subsets of two poset classes whose elements are almost linear, and obtained a bound for intersecting families in a third such class: elements of $Y_{k,n}$ are constructed from linear orders by removing the comparison between the k^{th} and $k+1^{st}$ smallest points. Obtained as the union of the classes $Y_{k,n}$, the class $\mathcal{M}_n$ consists of all orders on $[n]$ which are one comparison away from being linear. Finally, Υ_n contains linear orders as well as partial orders obtained from linear ones by replacing the two highest and/or the two lowest points in their Hasse diagrams by an antichain of length two. Our classification results on $Y_{k,n}$ and Υ_n rely on the approach of reverse pairings which we first used to classify intersecting families of linear orders at the beginning of Chapter 6. To obtain our bound on intersecting subsets of $\mathcal{M}_n$, we adapted the popular method of cyclic arrangements to the poset scenario.

Chapter 8 investigated so-called complete posets of small height; those are posets where points of different height are necessarily comparable in the poset. We classified the intersecting subsets of individual isomorphism classes of complete height 1 posets, as well as presenting a bound on the size of intersecting families in their union, i.e. the set of all complete height 1 posets. Concerning complete height 2 posets, we obtained classifications for some isomorphism classes, but illustrated that the problem is difficult in general by concentrating on one specific height 2 class towards the end of the chapter, as well as in the appendix.

The author of this thesis hopes that the initial insights of Part III will provide some first steps towards a unified Erdős-Ko-Rado Theory of combinatorial structures, which honours its origins in extremal set theory as well as being intuitively compatible with our perception of relational structures.

Appendix A

One Particular Class of Complete Height 2 Posets

A.1 Maximum Intersecting Subsets of $\mathcal{D}_5$

It was conjectured in Chapter 8 that the family $R(n)$ is maximum intersecting in the poset class $\mathcal{D}_n$ (see page 164 for definitions). The following proposition establishes this for the case $n = 5$.

Proposition A.1.1. *If $\mathcal{F} \subseteq \mathcal{D}_5$ is intersecting then $|\mathcal{F}| \leq |R(5)|$.*

Proof. Let $\mathcal{F}$ be a maximum intersecting subset of $\mathcal{D}_5$, then $|\mathcal{F}| \geq |R(5)| = 12$. Let 1 be the label occurring most frequently at the bottom of elements of $\mathcal{F}$ and set

$$\mathcal{F}_1 = \{\, p \in \mathcal{F} : L_0(p) = \{1\}\,\},$$

then $|\mathcal{F}_1| \geq 3$ by the pigeonhole principle. Let $\alpha = \Diamond_1^{a_1,a_2}$, $\beta = \Diamond_1^{b_1,b_2}$, $\gamma = \Diamond_1^{c_1,c_2}$ be distinct elements of $\mathcal{F}_1$, then

$$\{a_1, a_2, b_1, b_2, c_1, c_2\} \subseteq \{2,3,4,5\},\ a_1 \neq a_2,\ b_1 \neq b_2,\ c_1 \neq c_2. \tag{A.1.2}$$

To fit the six element set on the LHS into the four element set on the RHS, we must have at least two equalities among elements of the LHS.

If there exists a label occurring in the top levels of all three of α, β, γ, then the other three labels in the top levels must all be different to guarantee that the three posets are distinct. This gives

$$\alpha = \Diamond_1^{a_1,a_2}, \beta = \Diamond_1^{a_1,b_2}, \gamma = \Diamond_1^{a_1,c_2},\ \ |\{a_2, b_2, c_2\}| = 3.$$

Otherwise, we have $a_1 = b_1$ without loss of generality, and $a_1 = b_1$ cannot be equal to any of the other top level labels. Thus (A.1.2) reduces to

$$\{a_2, b_2, c_1, c_2\} \subseteq (\{2,3,4,5\} \setminus \{a_1\}),\ a_2 \neq b_2,\ c_1 \neq c_2.$$

Since nothing in our discussion so far distinguishes a_2 from b_2, or indeed c_1 from c_2, we may take $a_2 = c_1$ without loss of generality. This leaves us with

$$\alpha = \Diamond_1^{a_1,a_2}, \beta = \Diamond_1^{a_1,b}, \gamma = \Diamond_1^{a_2,c}. \tag{A.1.3}$$

Suppose firstly that $b = c$. Recall that in the case under consideration, there does not exist a label which occurs in the top levels of three elements of $\mathcal{F}_1$, so $\mathcal{F}_1$ cannot contain any other posets in which a_1, a_2 or $b = c$ are maximal in addition to the three listed in (A.1.3), since $n = 5$. This forces $\mathcal{F}_1 = \{\alpha, \beta, \gamma\}$.

Now consider the case $b \neq c$. Since again, no label occurs in the top levels of three elements of $\mathcal{F}_1$, we see from (A.1.3) that $\mathcal{F}_1$ cannot contain any posets in which a_1 or a_2 are maximal, other than those in (A.1.3). This gives

$$\mathcal{F}_1 \subseteq \left\{\Diamond_1^{a_1,a_2}, \Diamond_1^{a_1,b}, \Diamond_1^{a_2,c}, \Diamond_1^{b,c}\right\}.$$

Note we may use the labels $2, 3, 4, 5$ instead without loss of generality. Thus, in summary, we must have one of the following three cases:

$$\left\{\Diamond_1^{2,3}, \Diamond_1^{2,4}, \Diamond_1^{2,5}\right\} \subseteq \mathcal{F}_1 \tag{A.1.4}$$

$$\left\{\Diamond_1^{2,3}, \Diamond_1^{2,4}, \Diamond_1^{3,4}\right\} = \mathcal{F}_1 \tag{A.1.5}$$

$$\left\{\Diamond_1^{2,3}, \Diamond_1^{2,4}, \Diamond_1^{3,5}\right\} \subseteq \mathcal{F}_1 \subseteq \left\{\Diamond_1^{2,3}, \Diamond_1^{2,4}, \Diamond_1^{3,5}, \Diamond_1^{4,5}\right\}. \tag{A.1.6}$$

Mirroring the definition of $\mathcal{F}_1$, we set

$$\mathcal{F}_i = \{\, p \in \mathcal{F} : L_0(p) = \{i\}\,\}$$

for $2 \leq i \leq n$ also. Denoting $R(5)$ and $R_i(5)$ simply by R and R_i respectively, we have

$$\begin{aligned} R_1 &= \left\{\Diamond_1^{2,3},\ \Diamond_1^{2,4},\ \Diamond_1^{2,5},\ \Diamond_1^{3,4},\ \Diamond_1^{3,5},\ \Diamond_1^{4,5}\right\}, \\ R_2 &= \left\{\Diamond_2^{3,4},\ \Diamond_2^{3,5},\ \Diamond_2^{4,5}\right\}, \\ R_3 &= \left\{\Diamond_3^{4,5}\right\},\ R_4 = \left\{\Diamond_4^{3,5}\right\},\ R_5 = \left\{\Diamond_5^{3,4}\right\}. \end{aligned}$$

To prove the proposition, we need to show that $|R \setminus \mathcal{F}| \geq |\mathcal{F} \setminus R|$ or, equivalently,

$$\sum_{i=1}^{5} |R_i \setminus \mathcal{F}_i| \geq \sum_{i=1}^{5} |\mathcal{F}_i \setminus R_i|. \tag{A.1.7}$$

Since the subset R_1 of R consists of all posets in $\mathcal{D}_5$ with 1 at the bottom, we have $\mathcal{F}_1 \setminus R_1 = \emptyset$.

For $p \in \mathcal{F}_2 \setminus R_2$ we have $L_0(p) = \{2\}$ and $1 \in L_2(p)$ since $p \notin R_2$. Because $p = \Diamond_2^{1,x}$ intersects $\Diamond_1^{2,3} \in \mathcal{F}$, we must have $x = 3$. But then p does not intersect $\Diamond_1^{2,4} \in \mathcal{F}$ which contradicts the intersecting property of $\mathcal{F}$. Thus $\mathcal{F}_2 \setminus R_2 = \emptyset$, and we have reduced (A.1.7) to

$$\sum_{i=1}^{5} |R_i \setminus \mathcal{F}_i| \geq \sum_{i=3}^{5} |\mathcal{F}_i \setminus R_i|. \tag{A.1.8}$$

Since any element p of $\mathcal{F}_i \setminus R_i$ for $3 \leq i \leq 5$ is not in R, at least one of the labels 1, 2 must occur in $L_2(p)$. Moreover, p must intersect all elements of $\mathcal{F}_1$. Going through cases (A.1.4)-(A.1.6), it is therefore easily verified that

$$\begin{array}{rcl}
(A.1.4) \implies \mathcal{F}_i \setminus R_i & \subseteq & X_i^{(A.1.4)} := \left\{ \Diamond_i^{2,x} : x \in [5] \setminus \{2, i\} \right\},\ 3 \leq i \leq 5; \\
(j) \implies \mathcal{F}_3 \setminus R_3 & \subseteq & X_3^{(j)} := \left\{ \Diamond_3^{2,4}, \Diamond_3^{2,5} \right\},\ A.1.5 \leq j \leq A.1.6; \\
(A.1.5) \implies \mathcal{F}_4 \setminus R_4 & \subseteq & X_4^{(A.1.5)} := \left\{ \Diamond_4^{2,3}, \Diamond_4^{2,5} \right\}, \\
\mathcal{F}_5 \setminus R_5 & \subseteq & X_5^{(A.1.5)} := \{ \Diamond_5^{x,y} : \{x, y\} \neq \{4, 5\} \}; \\
(A.1.6) \implies \mathcal{F}_4 \setminus R_4 & \subseteq & X_4^{(A.1.6)} := \left\{ \Diamond_4^{2,1}, \Diamond_4^{2,3}, \Diamond_4^{2,5} \right\}, \\
\mathcal{F}_5 \setminus R_5 & \subseteq & X_5^{(A.1.6)} := \left\{ \Diamond_5^{1,3}, \Diamond_5^{2,3}, \Diamond_5^{2,4} \right\}.
\end{array}$$

The remainder of the proof will be split into four cases: since $|R_2| = 3$, we clearly have $|R_2 \cap \mathcal{F}| \in \{0, 1, 2, 3\}$. Note that all $X_5^{(j)}$ are contained in $X_5^{(A.1.5)}$. Similarly, for $i = 3, 4$, we have $X_i^{(j)} \subseteq X_i^{(A.1.4)}$. Thus, in the context of considering implications of $R_2 \cap \mathcal{F}$ on $\mathcal{F} \setminus R$ and hence (A.1.8), we are simply considering different sub-graphs of Figures A.1.1 and A.1.2, according to which of cases (A.1.4)-(A.1.6) we are in.

Case $|R_2 \cap \mathcal{F}| = 3$

Every element $p \in X_i^{(A.1.4)}$, $i = 3, 4, 5$, is incident to at least one red edge in Figures A.1.1 and A.1.2, meaning that there exists $q \in R_2$ which does not intersect p. Thus if $R_2 \subset \mathcal{F}$ in Case (A.1.4) then $\mathcal{F}_i \setminus R_i = \emptyset$ for $i = 3, 4, 5$, so $\sum_{i=3}^{5} |\mathcal{F}_i \setminus R_i| = 0$ and (A.1.8) holds.

In Cases (A.1.5) and (A.1.6) we have $|\mathcal{F}_1| \leq 4$, giving $\sum_{i=1}^{5} |R_i \setminus \mathcal{F}_i| \geq |R_1 \setminus \mathcal{F}_1| \geq 2$, so it suffices to show that $\sum_{i=3}^{5} |\mathcal{F}_i \setminus R_i| \leq 2$. This is indeed the case, since the only black posets in Figures A.1.1 and A.1.2 which are not incident to a red edge are $\Diamond_5^{1,3}$ and $\Diamond_5^{1,4}$.

Case $|R_2 \cap \mathcal{F}| = 2$

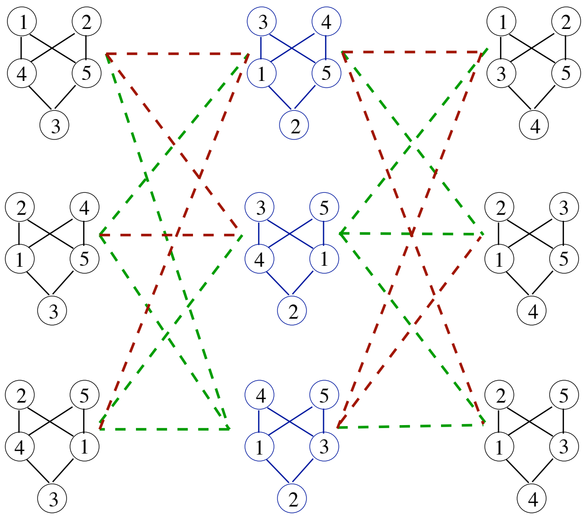


Figure A.1.1: Intersection graph of R_2 with $X_3^{(A.1.4)}$ and $X_4^{(A.1.4)}$.
From left to right, columns show elements of $X_3^{(A.1.4)}$, R_2 and $X_4^{(A.1.4)}$ respectively.
Posets are joined by a green edge if they intersect, and a red edge if they do not.

Then $|R_2 \setminus \mathcal{F}_2| = 1$, so (A.1.5) implies $\sum_{i=1}^{5} |R_i \setminus \mathcal{F}_i| \geq 4$ and (A.1.6) implies $\sum_{i=1}^{5} |R_i \setminus \mathcal{F}_i| \geq 3$. Examining Figures A.1.1 and A.1.2, it is easily checked that

$$(A.1.5) \implies \sum_{i=3}^{5} |\mathcal{F}_i \setminus R_i| \quad \leq \quad \begin{cases} 0+1+3 & \text{if } R_2 \cap \mathcal{F} = \left\{\Diamond_2^{3,4},\ \Diamond_2^{3,5}\right\} \\ 1+0+3 & \text{if } R_2 \cap \mathcal{F} = \left\{\Diamond_2^{3,4},\ \Diamond_2^{4,5}\right\} \\ 1+1+2 & \text{if } R_2 \cap \mathcal{F} = \left\{\Diamond_2^{3,5},\ \Diamond_2^{4,5}\right\} \end{cases},$$

$$(A.1.6) \implies \sum_{i=3}^{5} |\mathcal{F}_i \setminus R_i| \quad \leq \quad \begin{cases} 0+1+2 & \text{if } R_2 \cap \mathcal{F} = \left\{\Diamond_2^{3,4},\ \Diamond_2^{3,5}\right\} \\ 1+0+2 & \text{if } R_2 \cap \mathcal{F} = \left\{\Diamond_2^{3,4},\ \Diamond_2^{4,5}\right\} \\ 1+1+1 & \text{if } R_2 \cap \mathcal{F} = \left\{\Diamond_2^{3,5},\ \Diamond_2^{4,5}\right\} \end{cases},$$

so (A.1.8) holds.

Table A.1.1: Case (A.1.4) with $|R_2 \cap \mathcal{F}| = 2$.

i	1	2	s	t	u
$\lvert\mathcal{F}_i \setminus R_i\rvert$	0	0	0	≤ 1	≤ 1
$\lvert R_i \setminus \mathcal{F}_i\rvert$		1	≥ 1 if $\lvert\mathcal{F}_i \setminus R_i\rvert = 1$ for $i = t$ or $i = u$		

Now consider Case (A.1.4). For some assignment $\{s,t,u\} = \{3,4,5\}$, we then have $\Diamond_2^{s,t}, \Diamond_2^{s,u} \in \mathcal{F}$ and $\Diamond_2^{t,u} \notin \mathcal{F}$, giving $|R_2 \setminus \mathcal{F}_2| = 1$ in Table A.1.1. Note from Figures A.1.1 and A.1.2 that this implies $\mathcal{F}_s \setminus R_s = \emptyset$, and for both $z = t$ and $z = u$, we have $\mathcal{F}_z \setminus R_z \subseteq \{\Diamond_z^{2,s}\}$. This completes the $|\mathcal{F}_i \setminus R_i|$ line of Table A.1.1.

Moreover, if one of $\Diamond_t^{2,s}, \Diamond_u^{2,s}$ is actually in $\mathcal{F}$, then the element $\Diamond_s^{t,u}$ of R cannot be in $\mathcal{F}$, since it does not intersect either of them. Now we do not need to fill in the rest of Table A.1.1, since the existing entries clearly ensure that (A.1.8) holds.

Case $|R_2 \cap \mathcal{F}| = 1$

This case is very tedious. Although the conclusion is always the same, the minor details of subcases (A.1.4)-(A.1.6) differ sufficiently to force us to deal with them separately if we wish to avoid an explosion of notation. For this reason, however, this case does at least serve the purpose of

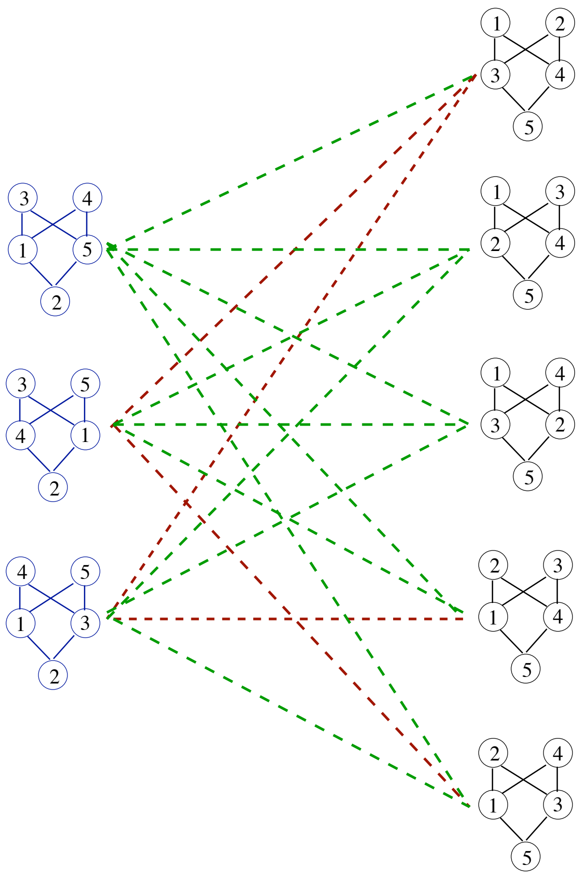


Figure A.1.2: Intersection graph of R_2 (blue) with $X_5^{(A.1.5)}$ (black).

demonstrating why the author believes that a proof of Proposition A.1.1 for general n would be difficult.

Case $|R_2 \cap \mathcal{F}| = 1$ with (A.1.4)

Here we have $\lozenge_2^{s,t} \in \mathcal{F}$ and $\lozenge_2^{t,u}, \lozenge_2^{s,u} \notin \mathcal{F}$ for some $\{s,t,u\} = \{3,4,5\}$, so $|R_2 \setminus \mathcal{F}_2| = 2$ in Table A.1.2. Figures A.1.1 and A.1.2 then imply that $\mathcal{F}_s \setminus R_s \subseteq \{\lozenge_s^{2,t}\}$ and $\mathcal{F}_t \setminus R_t \subseteq \left\{\lozenge_t^{2,s}\right\}$, whereas $\lozenge_2^{s,t}$ does not exclude any of $X_u^{(I)}$ from $\mathcal{F}$. This completes the $|\mathcal{F}_i \setminus R_i|$ line of Table A.1.2.

Table A.1.2: Case (A.1.4) with $|R_2 \cap \mathcal{F}| = 1$.

i	1	2	s	t	u
$\lvert\mathcal{F}_i \setminus R_i\rvert$	0	0	≤ 1	≤ 1	$\left\lvert X_u^{(I)} \cap \mathcal{F}\right\rvert$
$\lvert R_i \setminus \mathcal{F}_i\rvert$	≥ 2 if $\lozenge_u^{2,1} \in \mathcal{F}$	2	≥ 1 if $\lozenge_u^{2,s} \in \mathcal{F}$	≥ 1 if $\lozenge_u^{2,t} \in \mathcal{F}$	

Note that $|\mathcal{F}_s \setminus R_s| + |\mathcal{F}_t \setminus R_t|$ cancels with $|R_2 \setminus \mathcal{F}_2|$, so consider the elements of $X_u^{(I)} = \{\lozenge_u^{2,1}, \lozenge_u^{2,s}, \lozenge_u^{2,t}\}$. Each of them does not intersect at least one of

$$Y = \{\lozenge_s^{t,u}, \lozenge_t^{s,u}, \lozenge_1^{u,s}, \lozenge_1^{u,t}\} \subset (R_1 \cup R_s \cup R_t),$$

and each element of $X_u^{(I)}$ excludes a different element of Y from $\mathcal{F}$. Thus (A.1.8) holds.

Case $|R_2 \cap \mathcal{F}| = 1$ with (A.1.5)

If $R_2 \cap \mathcal{F} = \left\{\lozenge_2^{3,4}\right\}$, then it follows from Figure A.1.1 that $\mathcal{F}_3 \setminus R_3 \subseteq \left\{\lozenge_3^{2,4}\right\}$ and $\mathcal{F}_4 \setminus R_4 \subseteq \left\{\lozenge_4^{2,3}\right\}$. Now $\lozenge_3^{2,4}$ does not intersect $\lozenge_4^{3,5} \in R_4$ and, similarly, $\lozenge_4^{2,3} \in \mathcal{F}$ increases $|R_3 \setminus \mathcal{F}|$ by 1. This information is summarised in Table A.1.3, showing that A.1.8 holds.

Table A.1.3: Case (A.1.5) with $R_2 \cap \mathcal{F} = \left\{\lozenge_2^{3,4}\right\}$.

i	1	2	3	4	5
$\lvert\mathcal{F}_i \setminus R_i\rvert$	0	0	$\left\lvert\left\{\lozenge_3^{2,4}\right\} \cap \mathcal{F}\right\rvert$	$\left\lvert\left\{\lozenge_4^{2,3}\right\} \cap \mathcal{F}\right\rvert$	≤ 5
$\lvert R_i \setminus \mathcal{F}_i\rvert$	3	2	$\left\lvert\left\{\lozenge_4^{2,3}\right\} \cap \mathcal{F}\right\rvert$	$\left\lvert\left\{\lozenge_3^{2,4}\right\} \cap \mathcal{F}\right\rvert$	

If $R_2 \cap \mathcal{F} = \left\{\lozenge_2^{s,5}\right\}$ for $\{s,t\} = \{3,4\}$, then neither $\lozenge_5^{1,2}$ nor $\lozenge_5^{2,t}$ can be elements of $\mathcal{F}$, giving $|\mathcal{F}_5 \setminus R_5| \leq 3$. Also, $\lozenge_2^{s,5}$ does not intersect $\lozenge_s^{2,t} \in X_s$, giving $\mathcal{F}_s \setminus R_s = \{\lozenge_s^{2,5}\} \cap \mathcal{F}$. Since that poset does not intersect $\Diamond_5^{s,t} \in R_5$, this completes Table A.1.4.

Table A.1.4: Case (A.1.5) with $R_2 \cap \mathcal{F} = \left\{\Diamond_2^{s,5}\right\}$.

i	1	2	s	t	5
$\lvert\mathcal{F}_i \setminus R_i\rvert$	0	0	$\left\lvert\{\Diamond_s^{2,5}\} \cap \mathcal{F}\right\rvert$	≤ 2	≤ 3
$\lvert R_i \setminus \mathcal{F}_i\rvert$	3	2			$\left\lvert\{\Diamond_s^{2,5}\} \cap \mathcal{F}\right\rvert$

Case $|R_2 \cap \mathcal{F}| = 1$ with (A.1.6)

We have $|R_2 \setminus \mathcal{F}_2| = 2$, so

$$\begin{aligned}\sum_{i=1}^{5} |R_i \setminus \mathcal{F}_i| &= |R_1 \setminus \mathcal{F}_1| + 2 + \sum_{i=3}^{5} |R_i \setminus \mathcal{F}_i| \\ &= \begin{cases} 4 + \sum_{i=3}^{5} |R_i \setminus \mathcal{F}_i| & \text{if } \Diamond_1^{4,5} \in \mathcal{F} \\ 5 + \sum_{i=3}^{5} |R_i \setminus \mathcal{F}_i| & \text{if } \Diamond_1^{4,5} \notin \mathcal{F} \end{cases}.\end{aligned} \tag{A.1.9}$$

Now it is easily seen from Figures A.1.1 and A.1.2 that if $R_2 \cap \mathcal{F} = \left\{\Diamond_2^{3,4}\right\}$, then

$$\mathcal{F}_3 \setminus R_3 \subseteq \left\{\Diamond_3^{2,4}\right\}, \quad \mathcal{F}_4 \setminus R_4 \subseteq \left\{\Diamond_4^{2,3}\right\}, \quad \mathcal{F}_5 \setminus R_5 \subseteq \left\{\Diamond_5^{1,3}, \Diamond_5^{2,3}, \Diamond_5^{2,4}\right\},$$

and similarly, if $R_2 \cap \mathcal{F} = \left\{\Diamond_2^{4,5}\right\}$, then

$$\mathcal{F}_3 \setminus R_3 \subseteq \left\{\Diamond_3^{2,4}, \Diamond_3^{2,5}\right\}, \quad \mathcal{F}_4 \setminus R_4 \subseteq \left\{\Diamond_4^{2,5}\right\}, \quad \mathcal{F}_5 \setminus R_5 \subseteq \left\{\Diamond_5^{1,3}, \Diamond_5^{2,4}\right\}.$$

Note that in either case, $\Diamond_5^{1,3} \in \mathcal{F}_5 \setminus R_5$, and this poset does not intersect $\Diamond_1^{4,5}$. Thus

$$\sum_{i=3}^{5} |\mathcal{F}_i \setminus R_i| \leq \begin{cases} 4 & \text{if } \Diamond_1^{4,5} \in \mathcal{F} \\ 5 & \text{if } \Diamond_1^{4,5} \notin \mathcal{F} \end{cases}$$

and so (A.1.8) holds by (A.1.9).

We are therefore left with the case $R_2 \cap \mathcal{F} = \left\{\Diamond_2^{3,5}\right\}$ where

$$\mathcal{F}_3 \setminus R_3 \subseteq \left\{\Diamond_3^{2,5}\right\}, \quad \mathcal{F}_4 \setminus R_4 \subseteq \left\{\Diamond_4^{1,2}, \Diamond_4^{2,3}, \Diamond_4^{2,5}\right\}, \quad \mathcal{F}_5 \setminus R_5 \subseteq \left\{\Diamond_5^{1,3}, \Diamond_5^{2,3}\right\}. \tag{A.1.10}$$

Since neither $\Diamond_4^{1,2}$ nor $\Diamond_5^{1,3}$ intersect $\Diamond_1^{4,5}$, we have $\sum_{i=3}^{5} |\mathcal{F}_i \setminus R_i| \leq 4$ if $\Diamond_1^{4,5} \in \mathcal{F}$, so (A.1.8) holds again by (A.1.9).

If $\Diamond_1^{4,5} \notin \mathcal{F}$, then $\sum_{i=3}^{5} |\mathcal{F}_i \setminus R_i| \leq 6$. Clearly, if this bound of 6 is not attained then (A.1.8) holds by (A.1.9). If $\sum_{i=3}^{5} |\mathcal{F}_i \setminus R_i| = 6$ then $R \setminus \mathcal{F} \supset \left\{\Diamond_3^{4,5}, \Diamond_4^{3,5}, \Diamond_5^{3,4}\right\}$ since these three posets do not intersect the posets in (A.1.10) in general. Thus $\sum_{i=1}^{5} |R_i \setminus \mathcal{F}_i| = 5 + 3 = 8 > 6$, and (A.1.8) holds as usual.

Case $|R_2 \cap \mathcal{F}| = 0$

Recall that $\mathcal{F}_2 \setminus R_2 = \emptyset$, so we have $\mathcal{F}_2 = \emptyset$ in this case. To prove the proposition, we wish to show that $|\mathcal{F}| \leq 12$ so assume, for a contradiction, that $|\mathcal{F}| > 12$. Showing that (A.1.8) holds will suffice, since this implies $|\mathcal{F}| \leq |R| = 12$.

Recall that 1 is the label occurring most frequently at the bottom of elements of $\mathcal{F}$ and $\mathcal{F}_2 = \emptyset$, so we must have $|\mathcal{F}_1| \geq 4$ to guarantee $|\mathcal{F}| > 12$. Note that in case (A.1.4), nothing in our previous discussion distinguishes the labels 3, 4 and 5. Thus by (A.1.4)-(A.1.6), we must have one of the following:

$$\mathcal{F}_1 \supseteq \left\{\Diamond_1^{2,3}, \Diamond_1^{2,4}, \Diamond_1^{2,5}, \Diamond_1^{3,4}\right\} \tag{A.1.11}$$

$$\mathcal{F}_1 = \left\{\Diamond_1^{2,3}, \Diamond_1^{2,4}, \Diamond_1^{3,5}, \Diamond_1^{4,5}\right\}. \tag{A.1.12}$$

In case (A.1.12), we have $|R_1 \setminus \mathcal{F}_1| = 2$, and $|R_2 \setminus \mathcal{F}_2| = 3$ since $|R_2 \cap \mathcal{F}| = 0$; hence $\sum_{i=1}^{5} |R_i \setminus \mathcal{F}_i| \geq 5$. On the other hand, since we have (A.1.6) and $\Diamond_1^{4,5}$ does not intersect $\Diamond_4^{1,2}$ or $\Diamond_5^{1,3}$, the definitions of $X_i^{(A.1.6)}$ yield

$$\mathcal{F}_i \setminus R_i \subseteq \left\{\Diamond_i^{2,j}, \Diamond_i^{2,k}\right\}, \text{ for } \{i,j,k\} = \{3,4,5\}. \tag{A.1.13}$$

Therefore $\sum_{i=3}^{5} |\mathcal{F}_i \setminus R_i| \leq 6$, and if this bound is attained then $\Diamond_j^{i,k} \in R \setminus \mathcal{F}$ for $\{i,j,k\} = \{3,4,5\}$. This would imply $\sum_{i=1}^{5} |R_i \setminus \mathcal{F}_i| = 5 + 3 = 8$, so (A.1.8) holds.

In case (A.1.11), we use the fact that $\Diamond_1^{3,4} \in \mathcal{F}$ together with the definition of $X_i^{(A.1.4)}$ to deduce that $\mathcal{F}_i \setminus R_i$ for $i = 3, 4$ are as in (A.1.13), and

$$\mathcal{F}_5 \setminus R_5 \subseteq \left\{\Diamond_5^{2,1}, \Diamond_5^{2,3}, \Diamond_5^{2,4}\right\}.$$

Now $\Diamond_5^{2,1}$ does not intersect $\Diamond_1^{5,i}$ for $i = 3, 4$. Therefore $\Diamond_5^{2,1} \in \mathcal{F}$ would imply $|\mathcal{F}_1| = 4$, giving $\sum_{i=1}^{5} |R_i \setminus \mathcal{F}_i| \geq 5$ as above. This time $\sum_{i=3}^{5} |\mathcal{F}_i \setminus R_i| \leq 7$ but, again by the arguments in the previous paragraph, adding any poset in (A.1.13) to $\mathcal{F}$ increases $\sum_{i=1}^{5} |R_i \setminus \mathcal{F}_i|$ by 2, so (A.1.8) holds.

Finally, suppose that $\Diamond_5^{2,1} \notin \mathcal{F}$. Then we have (A.1.13), so once again considering the elements of R_i for $3 \leq i \leq 5$, we see that $\sum_{i=3}^{5} |\mathcal{F}_i \setminus R_i| \geq 2$ implies $\sum_{i=1}^{5} |R_i \setminus \mathcal{F}_i| \geq 6 \geq \sum_{i=3}^{5} |\mathcal{F}_i \setminus R_i|$. $\square$

BIBLIOGRAPHY

[AAK98] R. Ahlswede, H. Aydinian, and L. H. Khachatrian, *The intersection theorem for direct products*, European J. Combin. **19** (1998), no. 6, 649–661. Cited on page 12.

[Ahl01] Rudolf Ahlswede, *Advances on extremal problems in number theory and combinatorics*, European Congress of Mathematics, Vol. I (Barcelona, 2000), Progr. Math., vol. 201, Birkhäuser, Basel, 2001, pp. 147–175. Cited on pages 5 and 12.

[AK77] R. Ahlswede and G. O. H. Katona, *Contributions to the geometry of Hamming spaces*, Discrete Math. **17** (1977), no. 1, 1–22. Cited on page 18.

[AK96] Rudolf Ahlswede and Levon H. Khachatrian, *The complete nontrivial-intersection theorem for systems of finite sets*, J. Combin. Theory Ser. A **76** (1996), no. 1, 121–138. Cited on page 10.

[AK97] R. Ahlswede and L.H. Khachatrian, *The complete intersection theorem for systems of finite sets*, European Journal of Combinatorics **18** (1997), 125–136. Cited on pages 8 and 19.

[AK98] R. Ahlswede and L.H. Khachatrian, *The diametric theorem in hamming spaces — optimal anticodes*, Advances in Applied Mathematics **20** (1998), 429–449. Cited on pages 19, 60, 71, 79, 80, and 81.

[AK99] Rudolf Ahlswede and Levon H. Khachatrian, *A pushing-pulling method: new proofs of intersection theorems*, Combinatorica **19** (1999), no. 1, 1–15. Cited on page 7.

[AK05] R. Ahlswede and L. H. Khachatrian, *Katona's intersection theorem: four proofs*, Combinatorica **25** (2005), no. 1, 105–110. Cited on page 7.

[And87] Ian Anderson, *Combinatorics of finite sets*, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1987. Cited on page 12.

[BE00] Christian Bey and Konrad Engel, *Old and new results for the weighted t-intersection problem via AK-methods*, Numbers, information and complexity (Bielefeld, 1998), Kluwer Acad. Publ., Boston, MA, 2000, pp. 45–74. Cited on page 12.

[Ber74] Claude Berge, *Nombres de coloration de l'hypergraphe h-parti complet*, Hypergraph Seminar (Proc. First Working Sem., Ohio State Univ., Columbus, Ohio, 1972; dedicated to Arnold Ross), Springer, Berlin, 1974, pp. 13–20. Lecture Notes in Math., Vol. 411. Cited on page 14.

[BK08] G. Brockman and B. Kay, *Elementary Techniques for Erdos-Ko-Rado-like Theorems*, ArXiv e-prints (2008). Cited on pages 12, 14, 15, 34, and 133.

[BL97] B. Bollobás and I. Leader, *An Erdős-Ko-Rado theorem for signed sets*, Comput. Math. Appl. **34** (1997), no. 11, 9–13, Graph theory in computer science, chemistry, and other fields (Las Cruces, NM, 1991). Cited on pages 12 and 21.

[BM08] József Balogh and Dhruv Mubayi, *A new short proof of a theorem of Ahlswede and Khachatrian*, J. Combin. Theory Ser. A **115** (2008), no. 2, 326–330. Cited on page 10.

[Bor08] Peter Borg, *Intersecting and cross-intersecting families of labeled sets*, Electron. J. Combin. **15** (2008), no. 1, Note 9, 7. Cited on page 21.

[BSP60] R. C. Bose, S. S. Shrikhande, and E. T. Parker, *Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture*, Canad. J. Math. **12** (1960), 189–203. Cited on page 99.

[CF92] A. R. Calderbank and P. Frankl, *Improved upper bounds concerning the Erdős-Ko-Rado theorem*, Combin. Probab. Comput. **1** (1992), no. 2, 115–122. Cited on page 8.

[CGFS86] F. R. K. Chung, R. L. Graham, P. Frankl, and J. B. Shearer, *Some intersection theorems for ordered sets and graphs*, J. Combin. Theory Ser. A **43** (1986), no. 1, 23–37. Cited on page 11.

[CK03] P.J. Cameron and C.Y. Ku, *Intersecting families of permutations*, European Journal of Combinatorics **24** (2003), 881–890. Cited on pages 20, 31, 34, 78, 79, 80, 81, 85, 86, 87, 88, 89, 90, 94, 100, 101, and 169.

[Cza99] Éva Czabarka, *Intersecting chains in finite vector spaces*, Combin. Probab. Comput. **8** (1999), no. 6, 509–528. Cited on pages 24 and 107.

[DEF78] M. Deza, P. Erdős, and P. Frankl, *Intersection properties of systems of finite sets*, Proc. London Math. Soc. (3) **36** (1978), no. 2, 369–384. Cited on page 11.

[DER03] Richard A. Duke, Paul Erdős, and Vojtěch Rödl, *On large intersecting subfamilies of uniform setfamilies*, Random Structures Algorithms **23** (2003), no. 4, 351–356. Cited on page 11.

[DF77] M. Deza and P. Frankl, *On the maximum number of permutations with given maximal or minimal distance*, Journal of Combinatorial Theory (A) **22** (1977), 352–360. Cited on pages 14, 20, 31, 33, 42, 58, 78, 86, 101, and 169.

[DF83] M. Deza and P. Frankl, *Erdős-Ko-Rado theorem—22 years later*, SIAM J. Algebraic Discrete Methods **4** (1983), no. 4, 419–431. Cited on pages 5 and 12.

[DL76] D. E. Daykin and L. Lovász, *The number of values of a Boolean function*, J. London Math. Soc. (2) **12** (1975/76), no. 2, 225–230. Cited on pages 154 and 156.

[DR94] Richard Duke and Vojtěch Rödl, *The Erdős-Ko-Rado theorem for small families*, J. Combin. Theory Ser. A **65** (1994), no. 2, 246–251. Cited on page 11.

[DS05] Irit Dinur and Samuel Safra, *On the hardness of approximating minimum vertex cover*, Ann. of Math. (2) **162** (2005), no. 1, 439–485. Cited on page 8.

[EFK92] Péter L. Erdős, Ulrich Faigle, and Walter Kern, *A group-theoretic setting for some intersecting Sperner families*, Combin. Probab. Comput. **1** (1992), no. 4, 323–334. Cited on page 107.

[EKR61] P. Erdős, C. Ko, and R. Rado, *Intersection theorems for systems of finite sets*, Quart. J. Math. Oxford Ser. (2) **12** (1961), 313–320. Cited on pages 4, 5, 6, 7, 8, 9, 10, 11, and 84.

[Erd87] Paul Erdős, *My joint work with Richard Rado*, Surveys in combinatorics 1987 (New Cross, 1987), London Math. Soc. Lecture Note Ser., vol. 123, Cambridge Univ. Press, Cambridge, 1987, pp. 53–80. Cited on page 4.

[Erd90] Paul Erdős, *Some of my favourite unsolved problems*, A tribute to Paul Erdős, Cambridge Univ. Press, Cambridge, 1990, pp. 467–478. Cited on pages 5 and 8.

[ES00] Péter L. Erdős and László A. Székely, *Erdős-Ko-Rado theorems of higher order*, Numbers, information and complexity (Bielefeld, 1998), Kluwer Acad. Publ., Boston, MA, 2000, pp. 117–124. Cited on page 23.

[ESS94] Péter L. Erdős, Ákos Seress, and László A. Székely, *On intersecting chains in Boolean algebras*, Combin. Probab. Comput. **3** (1994), no. 1, 57–62. Cited on page 107.

[ESS00] Péter L. Erdős, Ákos Seress, and László A. Székely, *Erdős-Ko-Rado and Hilton-Milner type theorems for intersecting chains in posets*, Combinatorica **20** (2000), no. 1, 27–45. Cited on page 107.

[FF80] P. Frankl and Z. Füredi, *The Erdős-Ko-Rado theorem for integer sequences*, SIAM J. Algebraic Discrete Methods **1** (1980), no. 4, 376–381. Cited on pages 18, 60, 69, 79, and 81.

[FF86] P. Frankl and Z. Füredi, *Nontrivial intersecting families*, J. Combin. Theory Ser. A **41** (1986), no. 1, 150–153. Cited on pages 9 and 10.

[FF91] Peter Frankl and Zoltán Füredi, *Beyond the Erdős-Ko-Rado theorem*, J. Combin. Theory Ser. A **56** (1991), no. 2, 182–194. Cited on page 8.

[FG89] P. Frankl and R. L. Graham, *Old and new proofs of the Erdős-Ko-Rado theorem*, Sichuan Daxue Xuebao **26** (1989), no. Special Issue, 112–122. Cited on page 133.

[Fra78a] P. Frankl, *The Erdős-Ko-Rado theorem is true for $n = ckt$*, Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. I, Colloq. Math. Soc. János Bolyai, vol. 18, North-Holland, Amsterdam, 1978, pp. 365–375. Cited on pages 5 and 7.

[Fra78b] P. Frankl, *A Sperner-type theorem for families of finite sets*, Period. Math. Hungar. **9** (1978), no. 4, 257–267. Cited on page 11.

[Fra78c] Péter Frankl, *On intersecting families of finite sets*, J. Combinatorial Theory Ser. A **24** (1978), no. 2, 146–161. Cited on pages 9 and 10.

[Fra87] Peter Frankl, *The shifting technique in extremal set theory*, Surveys in combinatorics 1987 (New Cross, 1987), London Math. Soc. Lecture Note Ser., vol. 123, Cambridge Univ. Press, Cambridge, 1987, pp. 81–110. Cited on page 79.

[Fra88a] P. Frankl, *Intersection and containment problems without size restrictions*, Algebraic, extremal and metric combinatorics, 1986 (Montreal, PQ, 1986), London Math. Soc. Lecture Note Ser., vol. 131, Cambridge Univ. Press, Cambridge, 1988, pp. 62–111. Cited on pages 12, 156, and 157.

[Fra88b] P. Frankl, *Old and new problems on finite sets*, Congr. Numer. **67** (1988), 243–256, Nineteenth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Baton Rouge, LA, 1988). Cited on pages 5 and 12.

[Fra96] P. Frankl, *An Erdős-Ko-Rado theorem for direct products*, European J. Combin. **17** (1996), no. 8, 727–730. Cited on page 12.

[FT99] Peter Frankl and Norihide Tokushige, *The Erdős-Ko-Rado theorem for integer sequences*, Combinatorica **19** (1999), no. 1, 55–63. Cited on page 19.

[FT02] Peter Frankl and Norihide Tokushige, *Weighted 3-wise 2-intersecting families*, J. Combin. Theory Ser. A **100** (2002), no. 1, 94–115. Cited on page 11.

[FT03] P. Frankl and N. Tokushige, *Weighted multiply intersecting families*, Studia Sci. Math. Hungar. **40** (2003), no. 3, 287–291. Cited on page 11.

[FT05] Peter Frankl and Norihide Tokushige, *Random walks and multiply intersecting families*, J. Combin. Theory Ser. A **109** (2005), no. 1, 121–134. Cited on page 11.

[FT06] Peter Frankl and Norihide Tokushige, *Weighted non-trivial multiply intersecting families*, Combinatorica **26** (2006), no. 1, 37–46. Cited on page 11.

[FW86] P. Frankl and R. M. Wilson, *The Erdős-Ko-Rado theorem for vector spaces*, J. Combin. Theory Ser. A **43** (1986), no. 2, 228–236. Cited on page 27.

[GAP07] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.10*, 2007. Cited on pages 78, 91, 96, and 98.

[HKS01] Ralph Howard, Gyula Károlyi, and László Székely, *Towards a Katona type proof for the 2-intersecting Erdős-Ko-Rado theorem*, Electron. J. Combin. **8** (2001), no. 1, Research Paper 31, 8 pp. (electronic). Cited on page 133.

[HM67] A. J. W. Hilton and E. C. Milner, *Some intersection theorems for systems of finite sets*, Quart. J. Math. Oxford Ser. (2) **18** (1967), 369–384. Cited on pages 9 and 11.

[HR73] A. Hajnal and Bruce Rothschild, *A generalization of the Erdős-Ko-Rado theorem on finite set systems*, J. Combinatorial Theory Ser. A **15** (1973), 359–362. Cited on pages 11 and 37.

[Hsi75] W. N. Hsieh, *Intersection theorems for systems of finite vector spaces*, Discrete Math. **12** (1975), 1–16. Cited on page 27.

[HST05] Fred Holroyd, Claire Spencer, and John Talbot, *Compression and Erdős-Ko-Rado graphs*, Discrete Math. **293** (2005), no. 1-3, 155–164. Cited on page 12.

[Kat64] Gy. Katona, *Intersection theorems for systems of finite sets*, Acta Math. Acad. Sci. Hungar **15** (1964), 329–337. Cited on pages 6, 7, 20, and 78.

[Kat72] G. O. H. Katona, *A simple proof of the Erdős-Chao Ko-Rado theorem*, J. Combinatorial Theory Ser. B **13** (1972), 183–184. Cited on page 133.

[Kat00] Gyula O. H. Katona, *The cycle method and its limits*, Numbers, information and complexity (Bielefeld, 1998), Kluwer Acad. Publ., Boston, MA, 2000, pp. 129–141. Cited on page 133.

[KL06] C.Y. Ku and I. Leader, *An Erdős-Ko-Rado theorem for partial permutations*, Discrete Math. **306** (2006), no. 1, 74–86. Cited on pages 21 and 133.

[Kle66a] Daniel J. Kleitman, *Families of non-disjoint subsets*, J. Combinatorial Theory **1** (1966), 153–155. Cited on page 80.

[Kle66b] Daniel J. Kleitman, *On a combinatorial conjecture of Erdős*, J. Combinatorial Theory **1** (1966), 209–214. Cited on pages 14, 79, and 80.

[KR08] Cheng Yeaw Ku and David Renshaw, *Erdős-Ko-Rado theorems for permutations and set partitions*, J. Combin. Theory Ser. A **115** (2008), no. 6, 1008–1020. Cited on pages 21 and 24.

[KW07] Cheng Yeaw Ku and Tony W. H. Wong, *Intersecting families in the alternating group and direct product of symmetric groups*, Electron. J. Combin. **14** (2007), no. 1, Research Paper 25, 15 pp. (electronic). Cited on page 21.

[Liv79] M. L. Livingston, *An ordered version of the Erdős-Ko-Radó theorem*, J. Combin. Theory Ser. A **26** (1979), no. 2, 162–165. Cited on pages 14 and 18.

[LM04] Benoit Larose and Claudia Malvenuto, *Stable sets of maximal size in Kneser-type graphs*, European J. Combin. **25** (2004), no. 5, 657–673. Cited on page 20.

[Lub66] D. Lubell, *A short proof of Sperner's lemma*, J. Combinatorial Theory **1** (1966), 299. Cited on page 133.

[LW07] Yu-Shuang Li and Jun Wang, *Erdős-Ko-Rado-type theorems for colored sets*, Electron. J. Combin. **14** (2007), no. 1, Research Paper 1, 9 pp. (electronic). Cited on pages 21 and 133.

[MM05] Karen Meagher and Lucia Moura, *Erdős-Ko-Rado theorems for uniform set-partition systems*, Electron. J. Combin. **12** (2005), Research Paper 40, 12 pp. (electronic). Cited on pages 24, 25, 32, and 37.

[Moo82] Aeryung Moon, *An analogue of the Erdős-Ko-Rado theorem for the Hamming schemes* $H(n, q)$, J. Combin. Theory Ser. A **32** (1982), no. 3, 386–390. Cited on page 18.

[MS69] J. Marica and J. Schönheim, *Differences of sets and a problem of Graham*, Canad. Math. Bull. **12** (1969), 635–637. Cited on page 154.

[MT89] Makoto Matsumoto and Norihide Tokushige, *A generalization of the Katona theorem for cross t-intersecting families*, Graphs Combin. **5** (1989), no. 2, 159–171. Cited on page 11.

[Ran82] B. M. I. Rands, *An extension of the Erdős, Ko, Rado theorem to t-designs*, J. Combin. Theory Ser. A **32** (1982), no. 3, 391–395. Cited on page 27.

[Sta80] Dennis Stanton, *Some Erdős-Ko-Rado theorems for Chevalley groups*, SIAM J. Algebraic Discrete Methods **1** (1980), no. 2, 160–163. Cited on page 27.

[Szw03] J. L. Szwarcfiter, *A survey on clique graphs*, Recent advances in algorithms and combinatorics, CMS Books Math./Ouvrages Math. SMC, vol. 11, Springer, New York, 2003, pp. 109–136. Cited on page 22.

[Tal04] John Talbot, *The number of k-intersections of an intersecting family of r-sets*, J. Combin. Theory Ser. A **106** (2004), no. 2, 277–286. Cited on page 12.

[Tok05] Norihide Tokushige, *Intersecting families—uniform versus weighted*, Ryukyu Math. J. **18** (2005), 89–103. Cited on page 11.

[Tok06] Norihide Tokushige, *Extending the Erdős-Ko-Rado theorem*, J. Combin. Des. **14** (2006), no. 1, 52–55. Cited on page 11.

[Tok07a] Norihide Tokushige, *EKR type inequalities for 4-wise intersecting families*, J. Combin. Theory Ser. A **114** (2007), no. 4, 575–596. Cited on page 11.

[Tok07b] Norihide Tokushige, *The maximum size of 3-wise t-intersecting families*, European J. Combin. **28** (2007), no. 1, 152–166. Cited on page 11.

[Tok07c] Norihide Tokushige, *Multiply-intersecting families revisited*, J. Combin. Theory Ser. B **97** (2007), no. 6, 929–948. Cited on page 11.

[Wil84] Richard M. Wilson, *The exact bound in the Erdős-Ko-Rado theorem*, Combinatorica **4** (1984), no. 2-3, 247–257. Cited on pages 5 and 8.

[WZ08] Jun Wang and Sophia J. Zhang, *An Erdős-Ko-Rado-type theorem in Coxeter groups*, European J. Combin. **29** (2008), no. 5, 1112–1115. Cited on page 21.