THE PULSATION OF DELTA SCUTI STARS

HELEN MILLIGAN

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The Pulsation of Delta Scuti Stars

by

Helen Milligan

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A thesis submitted to the University of St. Andrews in
application for the degree of Doctor of Philosophy.

St. Andrews

1988

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ABSTRACT

A theoretical study of the pulsation of Delta Scuti stars is presented. A modified Henyey code is used to produce a grid of evolutionary models of population I composition, spanning the area of the Instability Strip on and near the Main Sequence (mass range 1.4 to 3.0 $M_{\odot}$; $\log T > 3.8$). Linear pulsation analysis of these models is carried out, using both adiabatic (Sturm Sequence) and non-adiabatic techniques. The results are found to agree closely, and to be consistent with the observed properties of real stars.

A number of the models are subjected to non-linear analysis, using a modified Christy hydrodynamic code and the Carson opacities. A series of models is given an initial velocity kick of surface amplitude 10 km/s and profile varying as the fifth power of the fractional radius. The light curves of these models are found to be dominated by the fundamental mode and first harmonic. The first five eigenfunctions produced by the Sturm Sequence method are used as initial velocity profiles for another series of models, again with an initial kick of 10 km/s. The light curves of these models are found to be dominated by the frequency corresponding to the initially imposed mode of pulsation. For both sets of models, the light amplitudes are found to be larger than those of real stars. However, the indication of a number of criteria is that the models have not settled down to a final (smaller) amplitude.

The first three Sturm Sequence eigenfunctions are used as initial velocity profiles for a smaller set of models with an initial kick of surface amplitude 3 km/s. Again, a final, settled amplitude is not achieved, but amplitudes are closer to those of real stars. The shapes of the light and velocity curves correspond closely to those of the same models with the larger kick. It is suggested that the form of the light curve may depend on the mode of pulsation. An attempt to use the shape of the light curves of real stars as a diagnostic for the mode of pulsation is unsuccessful, due to lack of data in the literature.

A number of models depleted of helium are studied. One set has $(X,Y,Z) = (0.88,0.10,0.02)$. The opacity table for this composition is obtained by interpolating between the Carson tables C312 and C402. The second set has $(X,Y,Z) = (0.98,0.00,0.02)$, corresponding to table C402. Both sets of models continue to pulsate, and it is concluded that the region of hydrogen ionisation is an important source of driving in these stars.

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INTRODUCTION

Delta Scuti stars are a species of pulsating variable star, located on and near the Main Sequence within the Instability Strip on the Hertzsprung-Russell Diagram. The object of this work was to model them using non-linear pulsation analysis, in the hope of obtaining light and velocity curves of similar amplitude and appearance to those of real stars. The Carson opacities, which have been used to model successfully a range of types of Cepheid variables, were also used here.

In Chapter One, a review of the observational properties of Delta Scuti stars and some of the other stars that share their region of the Hertzsprung-Russell Diagram is presented. Chapter Two discusses some of the results of linear analysis of models of these stars by various authors. The non-linear pulsation code is described in Chapter Three, with some comments on the equation of state. A brief review of other non-linear pulsation studies of stars within the Instability Strip is also given, to illustrate what may be achieved by this method.

In Chapter Four, an outline is given of the evolution code used to produce a grid of static models, spanning the area of interest in the Hertzsprung-Russell Diagram. Some comments are made on the models. Chapter Five presents linear analysis, both adiabatic and non-adiabatic, of the grid of models. The implementation of the adiabatic, Sturm Sequence method is described in some detail. The results are compared to each other and to real stars.

Chapter Six contains all the non-linear models, with comments on the sources of damping and driving within the models, and a comparison of the shapes of the light and velocity curves for different initial kicks. Some helium-depleted models are constructed, and compared to those of normal abundance. Finally, the results are compared to real Delta Scuti stars.

CHAPTER 1

OBSERVATIONS OF DELTA SCUTI STARS

1.1 HISTORY AND NAMING

In 1900, at the Lick Observatory, Wright discovered the variability of the star δ Scuti (Campbell & Wright 1900). This variability was confirmed in 1935, when Colacevich and Fath made simultaneous observations of radial velocity and brightness on two nights (Colacevich 1935, Fath 1935). They found a period of 0.1938 days, a light curve of variable amplitude (between $0^m.23$ and $0^m.15$) and a velocity curve also of variable amplitude (between 7 and 12 km/s). Because the period was similar to that of known β Cep type variables, Colacevich placed δ Scuti in that class. However, subsequent observations (Paddock & Struve 1954) suggested it belonged rather amongst the Cepheids. The discovery of the variability of the stars DQ Cep, CC And (Walker 1953, Lindblad & Eggen 1953), ρ Pup, and δ Del (Eggen 1956) prompted Eggen to suggest the existence of a new group of variable stars. In an unpublished dissertation (widely cited nonetheless), Smith (1955) named this group dwarf Cepheids.

With the 1960s came improved photoelectric techniques and systematic searches for more stars of this type (e.g. Breger 1966, Danziger & Dickens 1967, Dickens 1968, Chevalier et al. 1968, Bessell 1969, Breger 1969, Leung 1970, Jorgensen et al. 1971, Breger 1972). Naming was (and is) inconsistent - those stars with amplitudes greater than $0^m.3$ were called dwarf Cepheids, AI Vel stars or RRs stars, while most of those with amplitudes less than $0^m.3$ were called Delta Scuti stars. This division has been shown to have no astrophysical meaning, and it has been suggested (Breger 1979) that all these stars (with the possible exception of some with unusually low metal abundances) be called Delta Scuti stars.

There are now well over 200 of these stars known, although few have been observed for more than the minimum necessary to obtain confirmation of variability and an approximate period. A catalogue (Seeds & Yamchak 1972) lists references to many of them. Some most recent work includes Fourier analysis of multiply-periodic variables, for which long data strings are required.

1.2 DEFINITION

1.2.1 Spectral Type And Space Distribution

Delta Scuti stars are pulsating variable stars, with spectral type A or F, class V, IV or III. On the Hertzsprung-Russell Diagram, they are found within the Cepheid Instability Strip (see Figure 1.1), where it crosses the Main Sequence; the range is from slightly below the

population I zero-age Main Sequence to some 2.5 magnitudes above it. This corresponds to an absolute magnitude range of +3 to 0.

As most Delta Scuti stars are discovered using photometric rather than spectroscopic techniques, precise spectral classification of them is often unavailable. Pena et al. (1981) found substantial differences between their classification of a number of Delta Scuti stars and the classifications quoted in, for example, Breger (1979). However, they believed that the limits of the observed Instability Strip (A0 to F2 on the zero age Main Sequence) would not be much changed.

Figure 1.1. Positions of variable stars on the Hertzsprung-Russell Diagram (Cox 1980).

They are typically population I stars, but a few (notably SX Phe) have metal content and space velocity more appropriate to population II. They have been found in open clusters, including the Hyades (Horan 1979), and moving groups of a wide range of ages. Frolov & Irkaev (1982) reported a correlation between average period and cluster age for Delta Scuti stars in clusters, but this was shown by Breger (1983) to be an unjustified conclusion. The correlation was due rather to a combination of Main Sequence evolution with the known period-luminosity-colour relation for these stars.

1.2.2 Period And Amplitude Of Variation

The pulsation periods range from 20 minutes to a few hours. The longer periods correspond to the more luminous stars, the giants and subgiants; the typical period of a Main Sequence Delta Scuti star is about an hour. In general, the periods quoted in the literature are obtained simply by averaging the few cycles observed in the data. These can only be close to the real period if the star is monoperiodic. Using these cycle-count periods for a large number of stars, Breger (1979) has obtained an empirical correlation between period and luminosity:

$$M_v = -3.052 \log P + 8.456 (b-y)_0 - 3.121 \pm 0^m.31 \quad (1.1)$$

where M_v is absolute visual magnitude, P is period in days, and $(b-y)_0$ is a colour index.

This is illustrated in Figure 1.2.

Figure 1.2. A comparison of (mainly cycle-count) periods with absolute magnitude, corrected for colour, showing a loose correlation (Breger 1979).

Halprin & Moon (1983) found a smaller coefficient of $(b-y)_0$, 5.285, and Frolov & Irkaev (1984) gave the equation

$$M_V = -2.02 \log P + 5.39 (b-y)_0 - 1.33 \pm 0^m.33 \quad (1.2)$$

where coefficients for $\log P$ and $(b-y)_0$ are smaller than Breger's. This may indicate a smaller slope for the P-L-C relation for Main Sequence pulsators alone.

The light curve of a typical Delta Scuti star has an amplitude in the range a few thousandths to 0.3 magnitudes. The inverse radial velocity curve lags the light curve very slightly; maximum light occurs between 0.12 and 0.06 periods before minimum radial velocity (Breger et al. 1976). The radial velocity amplitude lies somewhere between 2 and 30 km/s. Many stars have irregular, inconstant, or multiple periods, with more than one radial mode excited and possibly non-radial modes as well. Beat phenomena are observed, and frequency modulation. Specific types of Delta Scuti star have been identified by Auvergne et al. (1980):

(a) Main Sequence (dwarf) Delta Scuti stars, which usually have small amplitudes, unstable periods, and non-repetitive light curves.

(b) Post Main Sequence Delta Scuti stars, with stable period(s) and repetitive light curves of more complex form (featuring bumps or beat phenomena).

The range of amplitudes is illustrated in Figure 1.3, which shows light curves for BN Cnc and AD CMi.

Other variations reported include colour changes (Badalia & Gurm 1983) and line profile variations (Auvergne et al. 1979, Yang & Walker 1983). Chromospheric activity, caused by shock waves at particular stages of the pulsation cycle, has also been observed (Dravins et al. 1977) for ρ Pup.

1.2.3 Multiperiod Analysis

Figure 1.3. Light curves for the Delta Scuti stars BN Cnc and AD CMi. A continuous range of amplitudes between these extremes is observed (Breger 1979).

With long data strings and careful analysis, it is sometimes possible to identify apparently unstable periods as superpositions of two or more radial harmonics. However, some stars, for example 1 Mon (Shobbrook & Stobie 1974, Balona & Stobie 1980), may need non-radial modes to account for their light curves. The vast majority remain unanalysed. This is unfortunate for theoreticians, because such analysis produces pure numbers (period ratios) which can be very useful for checking models and comparing them with real stars. The few double mode Delta Scuti stars that have been analysed show period ratios P_1/P_0 of about 0.77 (Andreassen 1983).

Dziembowski (1982) put forward a theory of resonance between modes which may be of particular relevance to Delta Scuti stars. This process excites low order modes which (according to a linear analysis) ought to be stable, and thus limits pulsational amplitudes. A possible observational example of this is HR 4684 (Antonello et al. 1985). Antonello et al. found that the light curve of this star could be fitted by a combination of three sinusoidal elements of similar amplitudes. The resultant light curve is shown in Figure 1.4, superimposed upon the observations.

Figure 1.4. The light curve of HR 4684, fitted (not convincingly) by a synthesised curve with periods 10.25, 15.02 and 25.23 cycles per day (Antonello et al. 1985).

1.2.4 Properties Derived From Observations

Values of surface gravity, effective temperature, absolute magnitude, mass and radius derived from observations of these stars suggest normal evolution (as it is currently understood).

Surface gravity g lies in the range $3.3 < \log g < 4.2$ (Breger 1979) for high amplitude Delta Scuti stars. Andreasen (1983) pointed out that individual measurements could be overestimated by as much as 0.1 (in $\log g$), because the calibration of g is based on standard stars of average rotational velocity, whereas the high amplitude Delta Scuti stars are slow rotators.

Effective temperature is in the range $3.84 < \log T_e < 3.93$, according to Andreasen et al (1983). This corresponds to (in Kelvin) $6900 < T_e < 8500$.

A mass range of $1.5 < M/M_\odot < 2.5$ has been derived both from colour indices (Petersen & Jorgensen 1972) and period ratios (Cox et al. 1979a). This is consistent with normal population I.

The mean radius of a pulsating star can, in principle, be obtained by the Baade-Wesselink method. Balona & Martin (1978) obtained a radius of $2.9 \pm 0.1 R_\odot$ for RS Gru. Meylan & Burki (1986) applied this method to a number of RR Lyrae stars and one Delta Scuti star (BS Aqr); the results suggested that the $\log R - \log P$ relation for population II stars also satisfied the short period pulsators. This is illustrated in Figure 1.5. The diagram itself does not seem

conclusive; the Delta Scuti stars might equally well fit on an extension of the classical Cepheid relation.

Figure 1.5. Mean Baade-Wesselink radii plotted against period for stars from a range of types of intrinsic variable (Meylan & Burki 1986).

1.3 A BROADER VIEW

1.3.1 Cohabitation

The Delta Scuti stars share their small portion of the Hertzsprung-Russell Diagram with (on the Main Sequence) non-pulsating stars of normal abundances, classical and mild Am stars, Ap stars, and (evolved from the Main Sequence) the Delta Delphini stars, which have abnormal abundances and may or may not pulsate. It is possible that some low mass, population II stars, of which SX Phe may be an example, are also present. One of the questions theoreticians have to answer is why there should be such diversity, particularly so near the Main Sequence.

Classical Am stars are stars of spectral type A, class V, which show a particular spectroscopic anomaly, first defined by Roman et al. (1948) and clarified by Conti

(1970), who states:

"The Am phenomenon is present in stars that have an apparent surface underabundance of calcium (and/or scandium) and/or an apparent overabundance of the iron group and heavier elements."

(Note that this definition implies the existence of three subgroups of Am stars). The majority of Am stars can be identified by means of a photometric test; they have $\delta(v-b) < 0$, where:

$$\delta(v-b) = (v-b)_{\text{standard}} - (v-b)_{\text{observed}} \quad (1.3)$$

Eggen (1976) applied this test to a large number of stars classified as Am and found only a few which did not have this property.

Mild Am stars, also referred to as "Am:" stars, exhibit the Am phenomenon, but to a lesser extent than do classical Am stars.

Delta Delphini stars are subgiants (class IV or III), found on the redder side of the Instability Strip. Unlike Am stars, they (mostly) have no Ca or Sc deficiency, but they do show enhancement of Fe (with respect to H) and of rare earths (with respect to Fe) (Kurtz 1978). However, some stars of class III, for example 22 Boo or α Cnc, have the same abundance anomalies as classical Am stars, so the definition of the class is rather uncertain (Valtier et al. 1979).

Ap stars, also known as spectrum variables, show anomalous intensities and variations in the strengths of certain spectral lines over a period of days, together with variations in luminosity of amplitude $0^m.1$. Strong lines of extremely rare elements, such as osmium, platinum and uranium, have been observed (Brandt & Jaschek 1970).

1.3.2 The Pulsation - Metallicity Exclusion

Along the Main Sequence, 30 percent of stars are Am, some 10 percent are Ap, and the majority of the rest are Delta Scuti stars. If these stars are of the same mass and evolutionary status, there must be some other reason for their differences. Not only do they feature widely differing spectra, but it also seems that only certain types pulsate. While no Main Sequence (Delta Scuti) pulsator is observed to have anomalous (Am) abundances, no classical Am star is observed to pulsate.

This division peters out towards the subgiant region, where both Delta Delphini and Delta Scuti pulsators are found. Indeed, it has been suggested (Saez et al. 1981) that all Delta Delphini stars pulsate, but that the majority (70 percent) have amplitudes too small to be detected by modern equipment.

It is possible that the situation may not be so clear-cut on the Main Sequence either; some mild Am stars are observed to pulsate with small amplitudes (Kurtz 1978).

1.3.3 Magnetic Fields

Ap stars are known to have large magnetic fields (of a few thousand gauss); this probably accounts for their peculiarities. Am stars have little (few hundred gauss) or no magnetic field. Very little work has been done on the magnetic fields of Delta Scuti stars; one paper (Romanov et al. 1984) reports a variable magnetic field of amplitude several hundred gauss for the Delta Scuti star 1 Mon. It was noted that not only the field strength, but also the field sign changed with pulsation phase.

The strong magnetic fields of the Ap stars and the lack of evidence of strong magnetic fields in Am, mild Am, Delta Scuti or Delta Delphini stars would seem to rule out Ap stars as close relatives of the Delta Scuti stars. This leaves the problem that Am and Delta Scuti stars occupy the same area of the Hertzsprung-Russell Diagram, and yet look very different.

1.3.4 Rotational Velocity

The only other observed physical difference between these two types is the rotational velocity. Am stars have lower average rotational velocities than do Delta Scuti stars. If all those "normal" Main Sequence A stars for which pulsation has not (yet) been detected are included as Delta Scuti stars, and if mild Am stars are excluded, it is found that (Abt and Moyd 1973):

$\langle v_{\text{ini}} \rangle$ for Am is 33 km/s

$\langle v_{\text{ini}} \rangle$ for Delta Scuti stars is 141 km/s

This result is illustrated in Figure 1.6. It should be noted that there do exist Am stars with high $\langle v_{\text{ini}} \rangle$; for example, α Cnc, 60km/s, and IW Per, 100km/s (Valtier et al. 1979). However, it does not appear that the degree of metallicity of an Am star depends on its rotational velocity (Burkhart 1979).

Figure 1.6. Frequency distributions of the rotational velocities of Am stars (dots) and normal A stars (circles). (Abt & Moyd 1973).

In the spectral classes IV and III, to the right of the Main Sequence, there appears to be a temperature cut-off. On the blue side are found non-pulsating Am stars; on the red side are found the Delta Delphini stars. Garrido et al. (1980) estimate the cut-off at $\log T_e = 3.89$ at $M_V = 1$.

Subgiant Delta Scuti stars with normal abundances are found throughout the region. Here, rotation has a different effect; amongst known pulsating stars, slow rotators have large amplitudes, and fast rotators have small amplitudes.

Another curious observation is that although the fraction of long-period binaries (greater than 1000 days) among Am and normal A stars is similar, there are very few normal A stars (i.e., Delta Scuti stars) in binary systems with periods between 2.5 and 1000 days; almost all such systems contain Am stars (Abt & Levy 1985). Exceptions include β Ari ($P = 106.9973$ days) and η Vir ($P = 309.673$ days), neither of which have Am spectra. Figure 1.7 illustrates the anomaly: the upper diagram shows the distribution of periods for Am binaries compared with equivalent data for F3 - G2 dwarfs (the histogram). The lower diagram combines data for Am and normal A4 - F1 stars in the ratio 33:67. The conclusion drawn is that there is the normal proportion of binary systems in this period range for A stars, but that for some reason stars in these systems are Am.

Current understanding is that slow rotational velocity is necessary for the Am phenomenon to develop. For systems with periods less than 10 days, it is possible that slowing of rotational velocity by tidal interaction has resulted in the stars becoming Am. Systems with periods in the range $10 < P(\text{days}) < 100$ would require pre-Main Sequence tidal braking for the same conclusion to be drawn (Abt & Levy 1985).

Figure 1.7. Upper panel: the distribution of periods of Am binaries (circles and error bars) compared with that for F3-G2 dwarfs (histogram).

Lower panel: data for Am stars and normal A4-F1 stars are combined (circles and error bars) in the ratio 33:67, and compared to F3-G2 dwarfs (histogram). (Abt & Levy 1985).

Figure 1.8 illustrates Abt & Levy's summary of the situation. Note that all Am stars are thought to be in binary systems; the area corresponding to single A stars of $\langle v \sin i \rangle$ less than 100 km/s is empty.

Figure 1.8. Illustrates the methods of reduction of the equatorial rotational velocity (vertically) for binaries (left) and single stars (right). Blank areas are those observed to be unpopulated (Abt & Levy 1985).

CHAPTER 2

MODELS OF DELTA SCUTI STARS

2.1 LINEAR MODELS

2.1.1 Early Models

The first published model of a Delta Scuti star (Fitch 1970) extended Schwarzschild's (1941) standard model computations to the first six radial modes of a set of seven polytropes with indices in the range $n = 2.25$ to 4.00 . The linear, adiabatic approximation to radial pulsation was used; the coefficients for the wave equation were calculated from the values obtained for the Lane-Emden variable y and its first derivative. The wave equation was then integrated from the centre out and surface in, using single-precision Runge-Kutta integration with a fixed step length in the polytropic radius variable x . The eigenvalues obtained were expressed as pulsation constants Q_i (of the i th radial mode), where Q_i is given by:

$$Q_i = P_i \sqrt{\frac{\langle \rho \rangle}{\langle \rho_0 \rangle}} \quad (2.1)$$

and a table was given of Q_i and P_{i+1}/P_i for a range of n

and i (where P_i is the period associated with the i th mode, and $\langle \rho \rangle$ is the mean density of the polytrope). Fitch went on to derive a theoretical dependence of Q_0 on P_1 and P_0 , and applied this to (amongst other double mode variables) the Delta Scuti star δ CVn, to obtain Q_0 (0.0329 days) and mean density $\langle \rho \rangle / \langle \rho_0 \rangle$ of 3.95×10^{-2} . He believed that the pulsational instability arose from the same causes as for RR Lyrae stars, but attempted to explain the complex light curves of Delta Scuti stars as due to one or more of the following:

- (1). Tidal modulation by a companion.
- (2). Magnetic coupling with a companion.
- (3). Non-radial mode excitation.
- (4). Simultaneous excitation of two radial modes.
- (5). Composite variation due to binary nature.

Chevalier (1971) treated radial adiabatic oscillations of models taken from two evolutionary sequences (masses 1.8 and $2.0 M_{\odot}$) up to the hydrogen-burning-shell phase, choosing an initial composition of $(X, Y, Z) = (0.70, 0.27, 0.03)$, typical of the young disk population. She used an envelope consisting of an atmosphere and a sub-photospheric region, interpolating opacities non-linearly from the Cox tables and calculating the temperature gradient in the presence of convection using the mixing-length theory, with mixing length l equal to local pressure scale height H_V . Only the fundamental mode and first harmonic were considered; she found that the period

ratio P_1/P_0 was approximately constant, lying in the range 0.756 to 0.762.

Figure 2.1. An early comparison of calculated and observed periods of Delta Scuti stars. The dotted lines represent constant periods for the models. The symbols representing real stars have the following meanings:

∇	$P < 1.5$ hours
$\circ$	$1.5 < P < 2.0$
$\square$	$2.0 < P < 2.5$
$\times$	$2.5 < P$

(Chevalier 1971).

When Chevalier plotted her evolutionary tracks on a Hertzsprung-Russell Diagram (see Figure 2.1), she found that the positions of observed Delta Scuti stars were consistent with the idea that these stars had masses in the range 1.6 to 2.2 $M_{\odot}$, some in a core-hydrogen-burning stage (on or near the Main Sequence) and others, slightly more evolved, in the beginning of a hydrogen-burning-shell phase. Comparing her predicted periods with those observed, she came to the conclusion that the less evolved stars pulsated in the fundamental mode, while the more luminous, evolved

stars could be pulsating in a mixture of modes (supported by beat phenomena, variations in light curve amplitude, etc.).

In order to discover the excitation mechanism, Chevalier integrated (numerically) the linear, non-adiabatic pulsation equations, disregarding convection. She found her $1.8 M_{\odot}$ model unstable to both the fundamental mode and the first overtone, due to an opacity mechanism working in the second helium ionisation zone. Driving contributions from the hydrogen ionisation zone were negligible. The rising time of the pulsation for the fundamental mode was about 2×10^4 years, longer than for Cepheids but still less than the evolutionary time scale.

2.1.2 Dwarf Cepheids

Up until quite recently, it was thought that there were two different types of pulsating star in this same region of the Hertzsprung-Russell Diagram: the Delta Scuti stars, and AI Vel or dwarf Cepheid stars. The latter were thought to be low mass, population II stars in a very advanced stage of evolution. However, it has since been recognised (Breger 1979) that most of the dwarf Cepheids are Delta Scuti stars; a few odd stars, including SX Phe and AI Vel itself, are still thought perhaps to be of low mass (about $0.5 M_{\odot}$). Nonetheless, over the years several models of low mass dwarf Cepheids have been put forward and contrasted with Delta Scuti stars. One of the first was described in the paper by Petersen & Jorgensen (1972). They constructed models in the mass range 0.5 to $1.6 M_{\odot}$, with a range of chemical compositions from $Z = 0.001$ to 0.03 . Again, Main Sequence

and immediate post-Main Sequence evolutionary models were used for Delta Scuti stars; for dwarf Cepheids a hydrogen-burning phase following the mixing of the star during a helium flash was considered appropriate. Models covered only the outermost few percent of radius. Figure 2.2 illustrates the range of models produced. Note that the lines limiting the Instability Strip cross the Main Sequence at spectral types A5 and F0.

Model numbers	Z	X	Interval in mass (unit: $M_{\odot}$)	Symbol
100—110	0.001	0.90	0.7	□
200—219	0.005	0.60	0.5—0.7	+
300—307	0.005	0.90	0.7	▣
400—409	0.02	0.70	0.5—0.7	◇
500—509	0.03	0.60	0.7	*
600—631	0.03	0.70	0.7—1.6	x

Figure 2.2. Models from Petersen & Jorgensen (1972), with a population I ZAMS. The parallel lines indicate the limits of the Instability Strip. Model parameters are identified by the symbols explained in the table to the left. (Petersen & Jorgensen 1972).

The pulsation periods of the fundamental mode and first three overtones were derived, using the linear adiabatic approximation to radial pulsation. It was found that the pulsation constant Q was not strictly constant but, for the fundamental mode, lay between 0.0311 and 0.0347 days.

Convection was treated by the mixing-length theory; a small change in period ratios was noted for change in the mixing length for models on the low temperature side of the instability Strip. For $\lambda = 2H_V$, period ratios were slightly (0.05 percent) lower than for $\lambda = H_V$ or $\lambda = 1.5H_V$.

Petersen & Jorgensen concluded that population I variable stars on or near the Main Sequence should have P_1/P_0 in the range 0.750 to 0.763; for population II composition in the same area, P_1/P_0 lay in the range 0.768 to 0.778. This supported the idea that dwarf Cepheids were of population II composition, but observational data were too sparse for definite conclusions to be reached.

In a later paper, Jorgensen & Petersen (1974) responded to a suggestion by Iben (at the IAU Symposium 59, on Stellar Evolution and Instability) that the Q values of Delta Scuti stars might be significantly affected by the structure of the inner regions. They studied models of mass $1.8 M_\odot$, assuming: (i) a rigid core, with zero pulsational amplitude; (ii) a linearly pulsating core, with amplitude growing proportionally with distance from the centre.

The equations describing the envelope were integrated down to a core mass (relative to the total mass) $q_c = 0.1$ or 0.4 ; radial pulsation periods were found by solving the linear, adiabatic pulsation equation. In most cases, a normal population I composition was used, but $(X, Y, Z) = (0.900, 0.095, 0.005)$ and $(X, Y, Z) = (0.600, 0.395, 0.005)$ were also tested.

They found that internal structure was important; Q values were slightly different with the different methods, and period ratios were affected. They concluded that for the period interval 0.046 to 0.08 days, it was necessary to use full stellar models to derive accurate period ratios, because of the effects of the internal structure on P_1 and P_0 did not cancel out in the period ratio.

Dziembowski & Kozlowski (1974) attempted to explain dwarf Cepheids as low mass stars, with most of the mass in a degenerate helium core. On top was a very thin hydrogen-burning shell, and an extended envelope. (They did not dismiss the possibility that dwarf Cepheids and Delta Scuti stars might form a homogeneous group, on and near the Main Sequence). As a star in such an evolutionary phase might be expected to have suffered extensive mass loss, they modelled extreme hydrogen-depleted envelopes ($X = 0.30$) as well as normal population II compositions; little difference in the results was found. They used the linear, non-adiabatic approximation as outlined by Castor (1971); opacities were interpolated from the Cox tables. Their models had pulsation growth rates of the same order of magnitude as those of RR Lyrae stars; η about 1×10^{-2} for the fundamental mode. This was not in agreement with values available for Delta Scuti stars; η about 1×10^{-6} . P_1/P_0 was found to be 0.75, conflicting with the observations available, which suggested $P_1/P_0 > 0.77$ for stars classified as dwarf Cepheids.

Percy (1975) also undertook a linear, non-adiabatic pulsation analysis of static models of dwarf Cepheids (mass $0.5 M_{\odot}$). His static models included treatment of convection and radiation pressure, and he used a surface boundary condition determined from model atmospheres. Convection was neglected in the pulsation analysis (again a version of Castor's code).

He found that, for a series of models with similar gravity and varying effective temperature (T_e), the hottest models were stable ($\log T_e = 3.92$), the cooler models unstable to the first overtone, and the coolest ($\log T_e = 3.82$) unstable to both the fundamental and the first overtone. A few of the very coolest were stable; this was probably a result of the neglect of convection, which could be more important in the cooler models. The main stabilising (damping) region was the fully ionised region below the second helium ionisation zone.

As expected, the position of the blue edge depended strongly on the hydrogen abundance. Figure 2.3 illustrates the effect of a reduction of X from 0.70 to 0.60. The uncertainty in the values of $\log T_e$ for the stars plotted in Figure 2.3 prevented Percy from drawing any definite conclusions about their hydrogen abundances, although he suggested that X was probably between 0.70 and 0.75.

Unhappy with the unsatisfactory agreement between models' period ratios and observed period ratios, Simon (1979) proposed a different explanation for the presence of multiple periods: that the observed complexities were the

Figure 2.3. Movement of the blue edges of the fundamental mode and first harmonic, subject to a reduction in hydrogen content from $X = 0.7$ to 0.6 (Percy 1975).

result of a resonance between modes. If his theory were correct, it would be impossible for dwarf Cepheids to have a normal population I composition and mass. In particular, SX Phe would require a mass less than $0.2 M_{\odot}$, and extremely low values of both Y and Z . However, Cox et al. (1979a), with deep-envelope models, new opacities, and linear, non-adiabatic analysis (again a modified version of Castor's (1971) code), concluded that dwarf Cepheids could be normal population I stars with masses between 1.4 and $2.2 M_{\odot}$. Even SX Phe, still considered as something of an exception, could be successfully modelled at a mass of $1.1 M_{\odot}$ with a population II composition, $Z=0.001$. This was one of the first papers confirming that dwarf Cepheids were only high-amplitude Delta Scuti stars; observers (Breger 1979)

had pointed out that the cut-off at an amplitude of $0^m.3$ between Delta Scuti stars and dwarf Cepheids was arbitrary, and that in most cases (except SX Phe) there was no other observed difference between the two types.

Cox et al. stressed the need to handle the central regions of such stars very carefully, pointing out that there was a significant contribution to the eigenfunctions at depths interior to a few thousand Kelvin. In their models, cores contained 10 percent or less of the total mass. Allowance was made for energy generation in the envelope, and in some cases (when modelling the more evolved stars) where all the luminosity was generated in the envelope, a layer with $Y = 0.98$, $Z = 0.02$ was inserted at the bottom of the envelope. Otherwise, a composition gradient in the energy generating region was not imposed; tests indicated that composition there did not significantly affect pulsation properties. Neither did convection (using mixing length theory) seem to affect period ratios.

2.1.3 Blue Limits

As well as attempting to reproduce observed period ratios and Q values, theoreticians have modelled the theoretical blue limit of the Instability Strip in the Delta Scuti region. The red limit is ill-defined observationally, and is thought to be due to the increasing importance of convection in the surface layers of F stars; the blue limit is sharper and perhaps more amenable to modelling.

Pamjatnykh (1974) conducted a linear, non-adiabatic pulsation analysis of evolutionary models of mass 1.5 and 2.25 $M_{\odot}$, taken from the sequences of Iben (1967). The models were of homogeneous composition, containing 70 percent of the stellar mass.

Figure 2.4. Comparing theoretical blue limits for the fundamental mode and third harmonic, for two different compositions, with the positions of Delta Scuti stars of known period and estimated pulsation mode (Pamjatnykh 1974).

He found the blue edge for higher modes to be hotter than that for the fundamental, and that decreasing the hydrogen abundance (from 0.7 to 0.6) further increased the temperatures of the blue edges. Figure 2.4 shows the evolution tracks, and a number of Delta Scuti stars, which

have been assigned modes of pulsation by comparing their observed periods with periods that would be expected for their positions on the Hertzsprung-Russell Diagram.

Percy (1977) examined a more extensive grid of models (including some representing Am stars, in an attempt to study the observed pulsation-metallicism exclusion). The chemical composition, masses, and luminosities of these models were consistent with population I. Opacity values were obtained from Cox & Stewart (1970); radiation pressure was included in the static model and in the pulsation analysis. The blue edges he obtained (illustrated in Figure 2.5) varied only slightly from those of Pamjatnykh.

Figure 2.5. Comparing theoretical blue limits for a population I composition with the positions of Delta Scuti stars, whose modes of pulsation have been estimated from their Q values (Percy 1977).

Stellingwerf (1979) extended the analysis to cover the first six radial modes of models of Delta Scuti stars in the mass range 0.2 to $2.0 M_{\odot}$, with composition $(X,Z) = (0.7,0.005)$, and, for mass $2.0 M_{\odot}$ only, composition $(X,Z) = (0.7,0.02)$. The idea of studying such a wide range was to determine the limits of a parameter space to be analysed using a new, non-linear relaxation technique (Stellingwerf 1974; the attempt to model Delta Scuti stars with this technique failed). Growth rates, periods, phase lags and period ratios were calculated. Blue edges for the fourth and fifth harmonics of the $(M = 2.0, Z = 0.02)$ star were found to curve right round and become red edges too (see Figure 2.6). The higher modes were also found to have larger growth rates than the lower ones, leading to the conclusion that Delta Scuti stars should start pulsating in higher modes. Interference between modes would produce the irregular-looking light curves observed.

Figure 2.6. Blue edges for a model of mass $2.0 M_{\odot}$ and Z of 0.02. Note that the blue edges for the fourth and fifth harmonics curve right round to become red edges (Stellingwerf 1979).

Stellingwerf found an additional driving zone at $T = 1.5 \times 10^5$ K, caused by a bump in the opacity (the "King" series Los Alamos tables).

Tsvetkov (1982) applied the linear, non-adiabatic approximation for radial pulsation to equilibrium models chosen with effective temperatures near that of the presumed blue limit. He used a mass of $1.8 M_{\odot}$, luminosity $M_{\text{bol}} = 1^{\text{m}}.50$ (i.e., $\log (L/L_{\odot}) = 1.296$). Opacity was calculated using cubic splines in the tables of Cox & Stewart (1970); convection was treated according to the mixing length theory ($l = H_V$) in the equilibrium models, and ignored in the pulsational analysis. Most models had a homogeneous chemical composition $(X, Y, Z) = (0.70, 0.28, 0.02)$; other compositions were modelled for comparison. Radiation pressure was included, but not turbulent pressure. The external boundary was set at $\tau = 0.1$; the internal boundary was where $M_r/M = 0.3 = q$. The envelope had about 250 layers.

Tsvetkov found that the position of the blue limit depended on helium abundance, choice of method of calculating opacity, and (to a much lesser extent) on the treatment of convection and the metal content Z .

For a fixed value of Z , the blue limit of the third harmonic shifted towards higher T_e with increasing He - not surprising, since the main driving zone is thought to be the second helium ionisation region, and increased helium in the envelope would thus increase the driving, such that the star would remain pulsationally unstable for higher T_e , where the

zone is nearer the surface.

2.1.4 Most Recent Models

Andreasen et al. (1983) conducted an extensive linear, adiabatic analysis of the first seven radial modes of population I, intermediate, and extreme population II Delta Scuti stars, in an attempt to provide calibration scales of mass and age for observed stars. Their computer program combined evolution with pulsation analysis, making it easy to construct evolution tracks in a number of useful diagrams, such as $\log P_0$ against $\log T_e$ and $\log A$ against $\log P_0$ (where A is age, in years). Examples of these for $Z = 0.02$ are shown in Figure 2.7. Dotted lines indicate periods spent within the Instability Strip; the numbers by the tracks indicate mass, in units of $\log M/M_\odot$. Similar diagrams can be produced for other harmonics, and for double period stars, plots of P_1/P_0 against P_0 were produced (Figure 2.8).

Andreasen (1983) went on to try to use the diagrams to determine mass, composition and age of observed stars; his main problem was the lack of precise period measurements available. Nevertheless, he was able to conclude that Delta Scuti stars were Main Sequence or immediate post-Main Sequence stars, evolving according to standard evolution theory.

2.2 THE PULSATION-METALLICISM EXCLUSION

Figure 2.7. Evolution tracks in the $(\log T, \log P_0)$ and $(\log P_0, \log A)$ planes, for a population I composition. Dotted lines indicate time spent within the Instability Strip. The tracks are labelled with $\log(\text{mass})$. (Andreasen et al. 1983).

Figure 2.8. Evolution tracks in the $(\log P_0, P_1/P_0)$ plane, illustrating the effect of changes in Z . The tracks are labelled with $\log(\text{mass})$; dotted lines indicate time spent within the Instability Strip. (Andreassen et al. 1983).

2.2.1 Diffusion

While working on the abundances of elements in Am stars, Praderie (1968) suggested that the observed differences between abundances in the Sun and Am stars might be the result of gravitational separation of the elements. Schatzman (1969), in response, allowed for gravitational separation in a stellar model comprising a surface convection zone and a lower radiative zone. For a non-rotating, $1.4 M_{\odot}$ star, he obtained the result that after half its life on the Main Sequence, the concentration of iron on its surface would have been reduced by 10^{+300} . Schatzman suggested that turbulence (induced by stellar rotation) would prevent such extreme separation; he thought that Am stars would have to have extensive mixing below the convective zone.

Watson (1971) was the first to point out that, in the study of particle diffusion, gravitation was not the only force acting on the elements. The force due to absorption of radiation was also important, particularly in the case of the heavier elements, where momentum is transferred from radiation to the particles by means of line absorption.

Watson estimated the total force on the elements in stellar atmospheres using a stellar model. He predicted that Zn and Ca would be over- and under-abundant respectively for a wide range of atmospheric conditions, whereas Sc was very sensitive to those conditions. In general, the lighter elements (with atomic number less than 20) diffused downwards because, being the most abundant,

there was not enough radiation in the relevant frequency intervals (corresponding to absorption lines) to force them upwards. Heavier elements diffused outwards.

Thus, from his model, an Am star would be expected to show under-abundances of C, O, Na, Mg, Ca, and Sc, and over-abundances of other metals. However, Na and Mg are normal (with respect to other metals; over-abundant with respect to hydrogen) in Am star spectra.

Watson did not attempt to calculate absolute values for element abundances; the effects of circulation, turbulence, possible mass loss or accretion were not known.

Vauclair et al. (1974) studied the diffusion of helium in the sub-photospheric part of the envelopes of population I Main Sequence A stars of masses $2 M_{\odot}$ and $1.5 M_{\odot}$, taking into account the reduction of helium in the envelope, rather than (as had been done before) assuming that diffusion did not affect the model. They assumed that the first and second convective zones (due to HI and HeI ionisation, and to HeII ionisation) were linked by convective overshooting, a phenomenon described by Bohm (1963a, 1963b). The diffusion of metals was ignored.

Their stellar envelope models included opacities interpolated linearly from the tables of Cox & Stewart (1970). Initial chemical composition was $(X, Y, Z) = (0.70, 0.28, 0.02)$; this was allowed to become non-uniform in the course of helium diffusion. The number of helium atoms going in and out of each shell below the combined convection zones was computed, and the dilution of helium in

the convective zones allowed for as a boundary condition. With the new helium abundance profile, the next time step was considered. Radiation pressure on helium was found to be negligible, and subsequently ignored.

They found that the second convective zone (due to HeII) disappeared over a timescale of the order of 1×10^6 years, and that subsequently the first convective zone (HI and HeI) also disappeared, after a further 1×10^3 years. This disappearance of helium eliminated the kappa-mechanism, thought to be mainly responsible for radial pulsation in Delta Scuti stars; helium-depleted stars would not pulsate.

2.2.2 Rotational Velocity Criterion

Vauclair et al. concluded that helium diffusion would take place if the envelope were stable enough, but that the meridional circulation due to rotation, proposed by Baglin (1972), would cause enough turbulence to prevent diffusion. Thus, slow rotators would suffer helium depletion (and show abundance anomalies) and would not pulsate; fast rotators would pulsate. Hence the Main Sequence pulsation-metallicism exclusion. With a crude model, Baglin had proposed a cut-off at $\langle v \sin i \rangle$ about 50 km/s.

Vauclair improved the analysis in a later paper (Vauclair 1976), obtaining a more detailed estimate of the cut-off rotational velocity (of 20 km/s) and proposing a novel interpretation of the distribution of Am and Delta Scuti stars on the Hertzsprung-Russell Diagram.

He pointed out that particle diffusion is a very slow process, involving particle velocities of about 1 cm/s at the surface of the star. Thus, it requires very stable conditions to operate: in particular, it requires the stellar outer layers to be rotating as a solid body. Otherwise, the shear flow instabilities induce turbulent velocities, with turbulent diffusion coefficients much higher than the particle diffusion coefficients. Thus, the slow rotation of Am stars explains (qualitatively) their abundance anomalies.

Vauclair's idea was that this solid-body rotation would, sooner or later, break down, due to increasing meridional circulation, and that the turbulence thus induced might allow an Am star to evolve into a Delta Scuti star. He pointed out that several fast-rotating Am stars had been discovered (e.g. HD 73619, $\langle v \sin i \rangle = 135$ km/s; HD 110500, $\langle v \sin i \rangle = 150$ km/s), although this did not convincingly disprove the theory that Am and Delta Scuti stars are distinguishable by their different mean rotational velocities. More interesting was his observation that Delta Scuti star rotational velocities might have been consistently over-estimated, due to pulsation effects (such as line-splitting) broadening the lines on long-exposure spectra. Mitton (1977) has also commented that the rotational broadening of the lines of fast-rotating stars would make it difficult to identify Am anomalies.

Using stellar models to test the time-scales of helium diffusion for various rotational velocities, Vauclair found that the time necessary for complete helium removal from the second convection zone was 1.5×10^6 years, for a rotational velocity of about 25 km/s. Meridional circulation would set in after about 8×10^5 years. Only for a rotational velocity below 20 km/s was there time for all the helium to diffuse before meridional circulation incited turbulence; such stars, Vauclair suggested, would be the classical Am stars, obeying the pulsation-metallicism exclusion, due to the absence of the main pulsational driving zone. At about 25 km/s, the second convection zone is present, and mild abundance anomalies are present, but there is not sufficient helium to allow the operation of the kappa-mechanism. These stars would appear as non-pulsating, mild Am stars.

After the stable phase, Vauclair found that turbulent motions developed on a timescale of 3×10^4 years, smoothing out abundance gradients. Thus, all stars would, sooner or later according to their rotational velocities, become Delta Scuti variables. How, then, could he explain the old (1×10^9 years) Am stars in the Hyades cluster? Only zero rotational velocity stars would preserve their abundance anomalies for such a length of time. Vauclair proposed instead that these stars had only recently become Am, having, for some unexplained reason, taken a long time to achieve a state of solid body rotation.

Thus Vauclair's theory makes the Am phenomenon dependent not only on rotational velocity, but also on time. He believed that stars might be alternately Am or Delta Scuti. Some quantitative analysis of the Am to Delta Scuti phase was presented, but none on the Delta Scuti to Am transition.

2.2.3 Models Of Am Stars

Percy (1977) investigated the pulsational stability of Delta Scuti stars and Am stars, assuming the latter to have undergone surface helium depletion. The linear, non-adiabatic approximation to radial pulsation that he used was described in an earlier paper (Percy 1975). His Am star models had the same mass, radius, luminosity, helium abundance profiles, etc. as those of Vauclair et al. (1974). For the fundamental mode, first, and second overtones, he found pulsational growth timescales of 1×10^2 years, but the models became stable after about 1×10^6 years, with the reduction of the main driving zone; agreeing with Vauclair (1976).

Vauclair et al. (1978) studied the competition between diffusion processes and turbulent motions in the atmospheres of cool Am stars. The observed over- and under-abundances of elements in such stars imposed strict restraints on their numerical model, but they were not able to decide conclusively for or against Vauclair's earlier (1976) model because values of radiation force on specific elements were not available.

Vauclair & Vauclair (1982) suggested that the diffusion process could also explain the observed abundance anomalies in Delta Delphini stars, if some helium were preserved in the outer layers (perhaps by a small amount of turbulence) in order to drive pulsations.

2.2.4 Delta Delphini Stars

Many Delta Delphini stars pulsate; they appear as Delta Scuti stars with mild abundance anomalies. If these anomalies are due to diffusion of the elements, then presumably helium has also diffused, downwards, leaving a reduced pulsation driving zone. Valtier et al. (1979) proposed that these stars were driven by the lesser driving zones, due to the hydrogen and first helium ionisation regions. A linear, non-adiabatic analysis of static envelope models (integrated in as far as $\log P = 15$, where P is pressure) allowed Valtier et al. to conclude that a vibrational instability due to the hydrogen ionisation zone was the reason for the existence of pulsating Delta Delphini stars, although they did admit that their models were rather crude.

Cox et al. (1979b) pointed out that the models of Valtier et al. did not include the strong pulsation damping in the deep interior; the hydrogen ionisation zone driving was shown not to be enough to cause pulsation. Therefore, pulsating Delta Delphini stars had to have helium in the second helium ionisation region. Cox et al. went on to consider just how much helium depletion was needed before pulsation ceased to be theoretically possible.

Their linear, non-adiabatic models covered all but the innermost few percent of the radius; the mass of the envelope was thus between 70 and 98 percent of the stellar mass. Energy generation in the deep layers of the envelope was allowed for by reducing the luminosity in deeper layers. When all the luminosity was produced in the envelope, an inner layer of $Y = 0.98$, $Z = 0.02$ was used.

Cox et al. found that their low-helium models agreed with those of Valtier et al. (1979), although Valtier et al. derived different growth rates, due to their neglect of deep damping. For a surface helium abundance Y of about 0.10, the second helium ionisation zone disappeared (the exact value of Y depended on the period of pulsation considered). They gave a table of blue edge limits (to the Instability Strip) for various values of Y , P_0 , and pulsation mode, in terms of temperature and luminosity.

They concluded that simultaneous pulsation and metallicism could exist on the red side of the Instability Strip, in a narrow temperature band. Such models had lost about two-thirds of the helium from their second convection zones. To obtain this result, Cox et al. assumed that there was no additional mixing process acting to reduce metallicism; the HeII convection zone was found to be so weakly convective that it caused no mixing.

Saez et al. (1981) attempted to explain why the envelopes of Delta Delphini stars were not completely devoid of helium. They used linear models with envelopes integrated as far in as $\log P = 18.8$, instead of 15.0 as in

their earlier paper, Valtier et al. (1979). Envelopes with no helium were found to be stable; the second helium ionisation zone was always the main source of driving. They interpreted the Delta Delphini phenomenon as due to partial mixing of an atmosphere already sorted by microscopic diffusion, i.e., with metal abundance anomalies and depleted helium. This mixing, they suggested, was caused by some process related to rotation; perhaps shear instabilities. In that case, mild Am and Delta Delphini stars would be expected to rotate faster than classical Am stars. In support of this theory they quoted mean rotational velocity values for Am, mild Am and Delta Delphini stars:

Am stars	$\langle v \sin i \rangle = 35 \pm 3 \text{ km/s}$
mild Am	$49 \pm 4 \text{ km/s}$
Delta Delphini	$50 \pm 6 \text{ km/s},$

although noting that these results were far from conclusive.

They were unable to give any quantitative analysis of how rotation might mix helium without destroying metal abundance anomalies at the surface.

CHAPTER 3

NON-LINEAR PULSATION

3.1 HISTORY OF PULSATION THEORY

The early attempts to explain the observed variability of certain stars included the ideas of radial pulsation, binary orbital motion, and (non-radial) oscillation between prolate and oblate form.

Ritter (1879), noting that some stars underwent periodic light variations, suggested that this arose from a combination of radial pulsation and surface temperature variation. Kelvin (1863), Emden (1907) and Moulton (1909) studied a model which retained constant volume while oscillating between the forms of prolate and oblate ellipsoids of revolution. That model had the disadvantage that radial velocity variations would be expected to have a period twice that of the brightness, whereas in real stars the two were observed to have the same period. The authors tried to get round the problem by invoking a complex system of light variation with shape.

Although the variations of some stars could be explained as the result of binary orbital motion, Shapley (1914) showed that radial pulsation was a more likely explanation for the Cepheid variables as, if they were binary systems, the dimensions of the orbits had to be less than the radii of the (giant) stars.

Eddington (1917) put the idea of radial pulsation onto a thermodynamic basis, and proceeded (1918a, 1918b, 1926) to study pulsation of stars in general and of Cepheids in particular. He recognised the need for a driving force to maintain the pulsation, and put forward two possibilities:

- (1). Variation of energy generation; i.e., increased generation of heat during compression.

- (2). Variation of the outward flow of energy; e.g. an increase of opacity during compression.

The latter seemed the more plausible, as the result of the former was to make any stellar model unstable to pulsation, whereas only a narrow range of models would be likely to be unstable in the latter instance. Indeed, later work (Epstein 1950, Cox 1955) showed that a variation in the nuclear energy generation rate in the stellar interior would produce an amplitude variation several orders of magnitude smaller than that at the surface.

Reviews of early pulsation theory can be found in Eddington (1926), Rosseland (1949), and Ledoux & Walraven (1958). More modern works include Cox (1980), which serves as a general reference for the subsequent sections of this chapter.

3.2 THE NON-LINEAR APPROACH: INTRODUCTION

The time-dependent equations of stellar structure form a system of non-linear, partial differential equations. In the linear theory of stellar pulsation, these equations are approximated by a set of linear, partial differential equations obtained by linearising the non-linear set about an equilibrium state, defined by the static model. Linear adiabatic theory further assumes purely adiabatic heat changes, and can predict pulsation periods satisfactorily. However, only by considering non-adiabatic heat changes can the stability or instability of the star against pulsation be determined. Linear, non-adiabatic methods can be used to find which parts of the star drive or damp the pulsations, and the growth rate of the pulsation.

Non-linear effects cause light and velocity curves to deviate from pure sine waves. Therefore, non-linear methods are necessary for modelling individual stars, and trying to reproduce observed light and velocity curves, which are rarely, if ever, sinusoidal. It is also necessary to consider non-linear effects when attempting to find the limiting amplitude of pulsation. Non-linear models can be followed for many periods on a fast computer to discover which modes grow or decay.

3.3 THE EQUATIONS OF STELLAR PULSATION

3.3.1 Basic Physical Assumptions

Three main assumptions are made when modelling pulsating stars. Firstly, that space is Euclidean, i.e.,

flat rather than curved. Secondly, that mass is conserved, i.e., that matter is always travelling at considerably less than the speed of light in vacuo. Thirdly, that gravitation is adequately expressed by Newtonian theory.

It is also assumed that the motion is purely radial. Magnetic fields and stellar rotation are ignored.

3.3.2 The Basic Equations

The equations of stellar pulsation are essentially those of static stellar structure, but with time dependence retained. Their derivation can be found in (for example) Cox & Giuli 1968; the equations are listed here for the sake of completeness. They are (in Lagrangian form):

Continuity of mass:

$$\frac{\partial r}{\partial M_r} = \frac{1}{4\pi r^2 \rho(r)} \quad (3.1)$$

Hydrodynamic Equilibrium:

$$\frac{\partial^2 r}{\partial t^2} = -\frac{GM_r}{r^2} - 4\pi r^2 \frac{\partial P}{\partial M_r} \quad (3.2)$$

Radiative Energy Transfer:

$$L_r = -(4\pi r^2)^2 \frac{4\sigma}{3\kappa(r)} \frac{\partial (T^4)}{\partial M_r} \quad (3.3)$$

Variables and constants are identified as follows:

r = space variable (radius)

t = time variable

M_r = mass contained within radius r

$\rho(r)$ = density of matter at radius r

L_r = luminosity (total heat flux) at radius r

P = total pressure at radius r

T = temperature at radius r

$\kappa(r)$ = opacity at radius r

σ = the Stefan-Boltzmann constant.

Note that Equation 3.3 ignores convection. The assumption of purely radiative energy transport is particularly unjustified near the red edge of the Instability Strip, where convection is thought to be responsible for the cessation of pulsation. However, for models near the Main Sequence within the Instability Strip, neglect of convective heat transport should not have much effect. Besides, inclusion of any of the various theories of convection would have greatly increased the computing time necessary.

The equation of energy conservation is:

$$\frac{d}{dt} \left[\frac{1}{2} \dot{r}^2 - \frac{GM_r}{r} + E \right] + \frac{d}{dM_r} [4\pi r^2 P + L_r] = 0 \quad (3.4)$$

This is obtained from equations 3.1 and 3.2, making use of the equation for heat flow:

$$\frac{\partial E}{\partial t} + P \frac{\partial V}{\partial t} + \frac{dL}{dM_r} = 0 \quad (3.5)$$

where E = internal energy per unit mass

P = pressure

V = specific volume.

Note that this equation omits energy generation; the nuclear energy generation rate ϵ on the right hand side of equation 3.5 has been set equal to zero. This assumes that only the stellar envelope, where there is no energy generation, takes part in the pulsation. This assumption is valid for many types of pulsating star. The core of the static model is not used in the pulsation analysis; instead, various inner boundary conditions are imposed on the envelope.

3.3.3 The Boundary Conditions

The inner boundary is chosen such that the radius of the innermost zone used in the pulsation calculations is fixed at its equilibrium value, which is usually about a tenth of the radius of the model. The core is assumed to be non-pulsating, and radiating a constant luminosity. Thus:

$$\left[\frac{dr}{dt} \right]_{R(\text{inner})} = 0 \quad (3.6)$$

and

$$L_{R(\text{inner})} = L_0 \quad (3.7)$$

At the outer boundary, it is assumed that the gas pressure, a component of the total pressure, is zero. This leaves:

$$P_{\text{tot}(\text{surface})} = P_{\text{rad}(\text{surface})} \quad (3.8)$$

where P_{rad} is the radiation pressure. The radiation flow at the surface should be described by the time-dependent radiative transport equation, but this is not possible to

implement, because it requires knowledge of the state of gridpoints several mean free path lengths away. Therefore, the radiative boundary condition is chosen to approximate the results of diffusion theory:

$$\left[\frac{d(T^4)}{d\tau} \right]_{\text{surface}} = \frac{3}{4} T_e^4 = \frac{3}{2} T_s^4 \quad (3.9)$$

Here, τ is the optical depth, T_e the effective temperature, and T_s the surface temperature. The Eddington approximation has been used.

3.3.4 The Method Of Solution

Equations 3.1, 3.2, 3.3 and 3.5 are treated as an initial value problem, to be integrated (in time) from some initial conditions, in the hope that the system will settle down to a periodic pulsation of finite amplitude. In order to integrate the equations, it is necessary to put them in difference form, in a stellar envelope of about fifty mass zones. This procedure is described in Christy (1967), and in considerable detail in Bridger (1983). Bridger wrote a computer program based on the outline given by Christy, and used it to study type II Cepheids. That program closely resembles the one written by Dr T.R. Carson, TRCVSP, which was used in this work. Some of the main features of the program are given here for the sake of completeness.

The first step is to read in a static model (typically of about 300 zones) and divide it up into fifty zones. The program allows two methods of division, producing models with zones of equal mass or of equal sound travel time. In this work, the latter method was used exclusively. In either case, the mass in each zone remains constant with time as the non-linear analysis proceeds.

The boundary between each zone is labelled i , where $i = 1$ is the innermost zone, and $i = N$ the outermost ($N = 50$). The centre of each zone is labelled $i-1/2$. Radius is taken at the boundary; R_i^n would be the radius of boundary i at time t^n , where n specifies the timestep. The mass of the zone, $\Delta M_{i-1/2}$, and the mass at the boundary, ΔM_i , are both useful quantities and are defined as follows:

$$\Delta M_{i-1/2} = M_i - M_{i-1} \quad (3.10)$$

$$\Delta M_i = \frac{1}{2} (\Delta M_{i-1/2} + \Delta M_{i+1/2}) \quad (3.11)$$

Specific volume V is taken within the zone:

$$V_{i-1/2}^n = \frac{4\pi}{3} \frac{(R_i^n)^3 - (R_{i-1}^n)^3}{\Delta M_{i-1/2}} \quad (3.12)$$

The re-zoned model extends inwards to about $0.1R_*$. At this point, the radius variations of the pulsation are vanishingly small, and (looking in the other direction) energy generation by nuclear processes has virtually ceased. This envelope now has to be relaxed onto the same difference equations as used in the pulsation analysis (but with time

derivatives set to zero), otherwise problems arise in the pulsation calculations. The relaxation is achieved by an iterative scheme, which calculates $P_{N-1/2}$ and R_{N-1} from the outer boundary conditions, then, for all $N-1$ zones, calculates $P_{i-1/2}$ from R_i and $P_{i+1/2}$, using the equation of hydrostatic equilibrium in its differenced form, solves for temperature and density, obtains a value for R_{i-1} , and compares that value with the stored one. If the corrections to the values of R are more than a set limit, the iteration is repeated. When the corrections are sufficiently small, the non-linear analysis can proceed.

Velocity u is taken at time $n+1/2$. Thus, the radius at the new time $n+1$ is given by:

$$R_i^{n+1} = R_i^n + \Delta t^{n+1/2} u_i^{n+1/2} \quad (3.13)$$

where:

$$\Delta t^{n+1/2} = t^{n+1} - t^n \quad (3.14)$$

Note that $\Delta t^n (= t^{n+1/2} - t^{n-1/2})$ is not necessarily the same as $\Delta t^{n+1/2}$, because the program changes the timestep as it goes, in order to facilitate convergence.

There are two restrictions on $\Delta t^{n+1/2}$ in TRCVSP:

(1). The Courant condition. This specifies that $\Delta t^{n+1/2}$ must be less than the expression $f \Delta R_i / c_s(i)$, where c_s is the speed of sound and f is less than one. ΔR_i is $R_i - R_{i-1}$. It requires that there be sufficient time for

sound to travel across each zone. The Courant condition is applied to all zones, and can be written in the more useful form:

$$\Delta t^{n+1/2} < f \frac{\Delta R_i}{\sqrt{P_{i-1/2} V_{i-1/2}}} \quad (3.15)$$

The term f is set equal to 0.5, to make sure ample time is allowed.

(2). A condition preventing any inner radius overtaking an outer one. This can be expressed as:

$$\Delta t^{n+1/2} < f \frac{\Delta R_i}{|u_i - u_{i-1}|} \quad (3.16)$$

for all zones.

It is necessary to introduce an artificial viscosity parameter, Q , to cope with the shock waves that develop in pulsating stars, causing rapid compression of some zones, particularly in the region of hydrogen ionisation. This is:

$$Q_{i-1/2}^{n-1/2} = C_Q P_{i-1/2}^{n-1/2} \left[\text{Min} \left[\frac{u_i^{n-1/2} - u_{i-1}^{n-1/2}}{P_{i-1/2}^{n-1/2} V_{i-1/2}^{n-1/2}} + \alpha_v, 0 \right] \right]^2 \quad (3.17)$$

where superscripts refer to time and subscripts to zone. C_Q , usually 1.0, is chosen to achieve as stable a model as possible, without sacrificing accuracy. The larger C_Q , the more zones the shock front is spread over. α_v is of the order of 0.1, in units of the speed of sound.

This is a modified version of the form used by Christy, which was found (Stellingwerf 1975a) to be producing significant amounts of damping in the inner regions of the envelope.

We now write the equations of pulsation in difference form, and consider the boundary conditions on them. The equation of hydrodynamic equilibrium, 3.2, can be written:

$$u_i^{n+1/2} = u_i^{n-1/2} - \Delta t^n \left[\frac{GM_i}{(R_i^n)^2} + \frac{4\pi(R_i^n)^2}{\Delta M_i} \left[P_{i+1/2}^n - P_{i-1/2}^n + Q_{i+1/2}^{n-1/2} - Q_{i-1/2}^{n-1/2} \right] \right] \quad (3.18)$$

The boundary values of this are:

$$\text{inner:} \quad u_1^{n+1/2} = 0 \quad (3.19)$$

outer:

$$u_N^{n+1/2} = u_N^{n-1/2} - \Delta t^n \left[\frac{GM_N}{(R_N^n)^2} - \frac{4\pi(R_N^n)^2}{\Delta M_N} \left[2(P_{N-1/2}^n - \frac{aW_{N-1/2}^n}{3}) + Q_{N-1/2}^{n-1/2} \right] \right] \quad (3.20)$$

where W is T^4 . The expression for the outer boundary condition is obtained by defining some fictitious pressure, $P_{N-1/2}^n$, beyond the outer surface of the star. For a surface gas pressure of zero, we have:

$$\frac{1}{2} (P_{N+1/2}^n + P_{N-1/2}^n) = P(\text{rad})_N^n \quad (3.21)$$

where $P(\text{rad})_N^n$ is the radiation pressure at the surface.

This can be approximated by:

$$P(\text{rad})_N^n = \frac{aW_{N-1/2}^n}{3} \quad (3.22)$$

This gives:

$$P_{N+1/2}^n - P_{N-1/2}^n = -2 \left[P_{N-1/2}^n - \frac{aW_{N-1/2}^n}{3} \right] \quad (3.23)$$

Note that an expression for ΔM_N is also required to evaluate 3.20. One way of approximating it is to assume that the mass ratio between successive (nearby) zones, α , will remain the same. Then:

$$\Delta M_N = \frac{1+\alpha}{2\alpha} \Delta M_{N-1/2} \quad (3.24)$$

The radiative energy transfer equation, 3.3, becomes:

$$L_1^n = \left[4\pi(R_i^n)^2 \right]^2 \left[W_{i-1/2}^n - W_{i+1/2}^n \right] 2F_i^n \quad (3.25)$$

where $2F_1^n$ is some differenced form of $4\sigma/3\kappa(r)\Delta M$. Bridger used the same expression for this as did Christy; with opacity in zone $i+1/2$ given by $K_{i+1/2}$, this is:

$$F_i^n = \frac{4\sigma}{3} \frac{\frac{W_{i+1/2}^n}{K_{i+1/2}^n} + \frac{W_{i-1/2}^n}{K_{i-1/2}^n}}{\left[\Delta M_{i+1/2}^n + \Delta M_{i-1/2}^n \right] \left[W_{i+1/2}^n + W_{i-1/2}^n \right]} \quad (3.26)$$

Expressions for the partial derivatives of F_1^n by $W_{i+1/2}^n$ and $W_{i-1/2}^n$ are also required; these are quoted in Bridger's thesis.

We can now write the heat flow equation, 3.5, in differenced form:

$$\left[E_{i+1/2}^{n+1} - E_{i+1/2}^n + \left[\frac{1}{2} [P_{i+1/2}^n + P_{i+1/2}^{n+1}] + Q_{i+1/2}^{n+1/2} \right] [V_{i+1/2}^{n+1} - V_{i+1/2}^n] \right] \Delta M_{i+1/2} \\ = - \frac{\Delta t^{n+1/2}}{2} [L_i^{n+1} + L_i^n - L_{i+1}^{n+1} - L_{i+1}^n] \quad (3.27)$$

This is solved (by iteration) at each time step, subject to inner and outer boundary conditions. The inner boundary condition is simply that the luminosity of the innermost zone is equal to the total luminosity of the model, since no energy generation takes place in the pulsating envelope. Thus, $W_{1/2}^{n+1}$ is determined from:

$$L_1^{n+1} = L_0 = \left[4\pi(R_1^{n+1})^2 \right]^2 [W_{1/2}^{n+1} - W_{3/2}^{n+1}] 2F_1^{n+1} \quad (3.28)$$

The outer boundary condition makes use of the Eddington approximation, $T_e^4 = 2T_s^4$, where T_s is the surface temperature. The surface luminosity is given by:

$$L_s = 4\pi(R_{\text{phot}})^2 2\sigma T_s^4 \quad (3.29)$$

It is necessary to find an estimate for R_{phot} , which is the radius at which the temperature is equal to the effective temperature, T_e . In TRCVSP, this is done by interpolation. Consider zone boundaries of radii R_{-1} , R_0 , and R_1 , enclosing zones of temperatures T_R and T_{R+1} . If $T = T_e$ is equal to $1/2 (T_R + T_{R+1})$, the program takes R_0 as R_{phot} . If $T_e = T_R$,

then R_{phot} is given by $1/2 (R_{-1} + R_0)$. A weighted average of the radii is taken when T_e tends towards one side or other of the zone.

We can now write down the outer boundary condition:

$$\begin{aligned} & \left[E_{N-1/2}^{n+1} - E_{N-1/2}^n + \left[\frac{1}{2} (P_{N-1/2}^{n+1} + P_{N-1/2}^n) + Q_{N-1/2}^{n+1/2} \right] \left[V_{N-1/2}^{n+1} - V_{N-1/2}^n \right] \right] \Delta M_{N-1/2} \\ &= \frac{\Delta t^{n+1/2}}{2} \left[L_{N-1}^{n+1} + L_{N-1}^n - 2\sigma 4\pi \left[(R_{\text{phot}}^n)^2 W_{N-1/2}^n + (R_{\text{phot}}^{n+1})^2 W_{N-1/2}^{n+1} \right] \right] \end{aligned} \quad (3.30)$$

Equation 3.27 can now be solved for new temperatures $W_{i-1/2}^{n+1}$ using a Newton-Raphson iteration procedure (in fact, we still need some method of calculating or otherwise obtaining such things as opacity, entropy, and pressure, given temperature, density and composition, but that will be dealt with later). Let a superscript prefix denote the number of the iteration, and let $\Delta W_{i+1/2}^{n+1}$ be the correction to the temperature at each iteration i:

$${}^i \Delta W_{i+1/2}^{n+1} = {}^{i+1} W_{i+1/2}^{n+1} - {}^i W_{i+1/2}^{n+1} \quad (3.31)$$

then we have:

$${}^{i+1} P_{i+1/2}^{n+1} = {}^i P_{i+1/2}^{n+1} + \frac{{}^i \partial P^{n+1}}{\partial W_{i+1/2}} {}^i \Delta W_{i+1/2}^{n+1} \quad (3.32)$$

$${}^{i+1} E_{i+1/2}^{n+1} = {}^i E_{i+1/2}^{n+1} + \frac{{}^i \partial E^{n+1}}{\partial W_{i+1/2}} {}^i \Delta W_{i+1/2}^{n+1} \quad (3.33)$$

$${}^{i+1}F_{i+1/2}^{n+1} = {}^iF_{i+1/2}^{n+1} + \frac{{}^i\partial F_{i+1/2}^{n+1}}{\partial W_{i+1/2}} {}^i\Delta W_{i+1/2}^{n+1} + \frac{{}^i\partial F_{i-1/2}^{n+1}}{\partial W_{i-1/2}} {}^i\Delta W_{i-1/2}^{n+1} \quad (3.34)$$

$${}^{i+1}L_i^{n+1} = \left[4\pi(R_i^{n+1})^2 \right]^2 \left[W_{i-1/2}^{n+1} - W_{i+1/2}^{n+1} \right] 2F_i^{n+1} \quad (3.35)$$

When 3.31-3.35 are substituted into the differenced form of the energy equation, 3.27, a three-term recurrence relation is obtained:

$$-\alpha_{i+1/2} {}^i\Delta W_{i+3/2}^{n+1} + \beta_{i+1/2} {}^i\Delta W_{i+1/2}^{n+1} - \gamma_{i+1/2} {}^i\Delta W_{i-1/2}^{n+1} = \delta_{i+1/2} \quad (3.36)$$

where:

$$\alpha_{i+1/2} = \Delta t^{n+1/2} \left[A_{i+1}^{n+1} \right]^2 \left[{}^iF_{i+1}^{n+1} - \frac{{}^i\partial F_{i+1}^{n+1}}{\partial W_{i+3/2}} \left[{}^iW_{i+1/2}^{n+1} - {}^iW_{i+3/2}^{n+1} \right] \right] \quad (3.37)$$

$$\beta_{i+1/2} = \frac{{}^i\partial E_{i+1/2}^{n+1}}{\partial W_{i+1/2}} + \frac{1}{2} \frac{{}^i\partial P_{i+1/2}^{n+1}}{\partial W_{i+1/2}} \left[V_{i+1/2}^{n+1} - V_{i+1/2}^n \right] \Delta M_{i+1/2} + \alpha_{i-1/2} + \gamma_{i+3/2} \quad (3.38)$$

$$\gamma_{i+1/2} = \Delta t^{n+1/2} \left[A_i^{n+1} \right]^2 \left[{}^iF_i^{n+1} + \frac{{}^i\partial F_i^{n+1}}{\partial W_{i-1/2}} \left[{}^iW_{i-1/2}^{n+1} - {}^iW_{i+1/2}^{n+1} \right] \right] \quad (3.39)$$

$$\delta_{i+1/2} = \frac{\Delta t^{n+1/2}}{2} \left[{}^iL_i^{n+1} + L_i^n - {}^iL_{i+1}^{n+1} - L_{i+1}^n \right] - \Delta M_{i+1/2} \left[{}^iE_{i+1/2}^{n+1} - E_{i+1/2}^n + \left[\frac{1}{2} \left[P_{i+1/2}^n + {}^iP_{i+1/2}^{n+1} \right] + Q_{i+1/2}^{n+1/2} \right] \left[V_{i+1/2}^{n+1} - V_{i+1/2}^n \right] \right] \quad (3.40)$$

and

$$A_1^{n+1} = 4\pi (R_1^{n+1})^2 \quad (3.41)$$

Equation 3.36 is a matrix equation of the form $\underline{\underline{M}}\underline{\underline{X}} = \underline{\underline{D}}$, where $\underline{\underline{M}}$ is a tridiagonal matrix, with elements α , β and γ , $\underline{\underline{X}}$ is the solution matrix containing new values of $^{i+1}\Delta W_{i+1/2}^{n+1}$, and $\underline{\underline{D}}$ is a column matrix with elements δ . The equation (subject to its boundary conditions) can be solved using a standard technique.

Equations 3.37-3.40 provide most of the elements of matrix $\underline{\underline{M}}$. The boundary conditions give the following values for the remaining elements:

$$\alpha_{1/2} = 2 (A_1^{n+1})^2 \left[i F_1^{n+1} - \frac{i \partial F_1^{n+1}}{\partial W_{3/2}} \left[i W_{1/2}^{n+1} - i W_{3/2}^{n+1} \right] \right] \quad (3.42)$$

$$\beta_{1/2} = 2 (A_1^{n+1})^2 \left[i F_1^{n+1} + \frac{i \partial F_1^{n+1}}{\partial W_{1/2}} \left[i W_{1/2}^{n+1} - i W_{3/2}^{n+1} \right] \right] \quad (3.43)$$

$$\gamma_{1/2} = 0 \quad (3.44)$$

$$\delta_{1/2} = -2 (A_1^{n+1})^2 \left[i F_1^{n+1} \left[i W_{1/2}^{n+1} - i W_{3/2}^{n+1} \right] \right] + L_0 \quad (3.45)$$

$$\alpha_{N-1/2} = 0 \quad (3.46)$$

$$\beta_{N-1/2} = \Delta t^{n+1/2} \left[(A_{N-1}^{n+1})^2 \left[i F_{N-1}^{n+1} - \frac{i \partial F_{N-1}^{n+1}}{\partial W_{N-1/2}} \left(i W_{N-3/2}^{n+1} - i W_{N-1/2}^{n+1} \right) \right] \right] + \sigma A_{\text{phot}}^{n+1} \quad (3.47)$$

$$\gamma_{N-1/2} = \Delta t^{n+1/2} \left[(A_{N-1}^{n+1})^2 \left[i F_{N-1}^{n+1} + \frac{i \partial F_{N-1}^{n+1}}{\partial W_{N-3/2}} (i W_{N-3/2}^{n+1} - i W_{N-1/2}^{n+1}) \right] \right] \quad (3.48)$$

$$\begin{aligned} \delta_{N-1/2} = & \frac{\Delta t^{n+1/2}}{2} \left[L_{N-1}^n + i L_{N-1}^{n+1} - 2\sigma (A_{\text{phot}}^n W_{N-1/2}^n + i A_{\text{phot}}^{n+1} W_{N-1/2}^{n+1}) \right] \\ & - \Delta M_{N-1/2} \left[i E_{N-1/2}^{n+1} - E_{N-1/2}^n + \left[\frac{1}{2} (P_{N-1/2}^n + i P_{N-1/2}^{n+1}) \right] + Q_{N-1/2}^{n+1/2} \right] \left[V_{N-1/2}^{n+1} - V_{N-1/2}^n \right] \end{aligned} \quad (3.49)$$

To solve the matrix, we let:

$$\Delta W_{i+1/2}^{n+1} = X_{i+1/2} \Delta W_{i+3/2}^{n+1} + Y_{i+1/2} \quad (3.50)$$

Dropping time superscripts, we get:

$$-\alpha_{i+1/2} i \Delta W_{i+3/2} = \left[\gamma_{i+1/2} X_{i-1/2} - \beta_{i+1/2} \right] i \Delta W_{i+1/2} + \delta_{i+1/2} + \gamma_{i+1/2} Y_{i-1/2} \quad (3.51)$$

So we obtain:

$$X_{i+1/2} = \frac{\alpha_{i+1/2}}{\beta_{i+1/2} - \gamma_{i+1/2} X_{i-1/2}} \quad (3.52)$$

and

$$Y_{i+1/2} = \frac{\delta_{i+1/2} + \gamma_{i+1/2} Y_{i-1/2}}{\beta_{i+1/2} - \gamma_{i+1/2} X_{i-1/2}} \quad (3.53)$$

Figure 3.1. Flow chart for a non-linear pulsation code (Bridger 1983).

Thus, starting at $i-1/2 = 1/2$, all of $X_{i+1/2}$ and $Y_{i+1/2}$ can be calculated, since $X_{1/2}$ is $\alpha_{1/2}/\beta_{1/2}$ and $Y_{1/2}$ is $\delta_{1/2}/\beta_{1/2}$. At the surface, $X_{N-1/2}$ is zero, so all the ${}^i\Delta w_{i+1/2}$ can also be found. Iteration proceeds until the temperature correction terms are as small as is desired; in this work, the criterion was 10^{-6} .

The problems of programming, including the size and number of mass zones, the choice and adjustment of the time step, the scaling of variables, and the identification of a period, are discussed in both Christy and Bridger. Bridger's flow chart, showing schematically the method of integration of the pulsation, is reproduced in Figure 3.1.

3.3.5 The Equation Of State

In order to solve the stellar structure equations it is necessary to have available values of pressure, P , entropy per unit mass, S , and internal energy, E , for the stellar material, as functions of density, temperature and chemical composition. A subroutine to calculate P , S and E , similar to that described by Bridger (1983) in his chapter 5, was available. However, repeated calls to this subroutine used up substantial amounts of computing time, and it was decided instead to use the subroutine to generate extensive tables of P and E , which could then be interpolated by the main program. The stellar envelope was assumed to be homogeneous.

The equation of state subroutine used here, STATES, is rather more sophisticated than Bridger's, which allowed for a mixture of radiation and a non-degenerate gas consisting of H, He, and two metals (N, representing the set C, N, O, etc.; and Mg, representing elements with low first ionisation energies). STATES allows for ten elements, namely H, He, C, N, O, Ne, Mg, Si, S and Fe. Negative ions, where appropriate, are dealt with, and up to eight stages of ionisation allowed for. It is assumed that the material is an ideal gas in local thermodynamic equilibrium, thus allowing the use of Boltzmann statistics throughout.

Let A_i be the atomic mass of element i , β_i be the abundance by mass of element i , α_i be the abundance by number of element i , and μ be the mean molecular weight. Then:

$$\alpha_i = \frac{\beta_i/A_i}{\sum_{i=1}^{\text{all}} \beta_i/A_i} \quad (3.54)$$

$$\mu = \sum_{i=1}^{\text{all}} \alpha_i A_i \quad (3.55)$$

Since the stars being modelled are fairly hot, with surface temperatures greater than $\log T_e = 3.8$, formation of molecules can be disregarded. Thus, the number density of free ions of element i is just n_i , the number density of element i . This is:

$$n_i = \frac{N_0 \rho \alpha_i}{\mu} \quad (3.56)$$

where N_0 is Avogadro's number.

Let n_{ij} be the number density of element i in ionisation stage j , and n_e be the electron density. Then:

$$n_e = \sum_{i=1}^{\text{all}} \sum_{j=1}^{\text{all}} j n_{ij}(n_e, T) \quad (3.57)$$

This expression is non-linear for most cases, and hence must be solved by iteration. STATES uses the geometric mean of converging extremes. The system of equations is as follows:

$$n_e(\text{min}) = \max \left[n_e(\text{min}), \min ({}^n n_e, {}^{n+1} n_e) \right] \quad (3.58)$$

$$n_e(\text{max}) = \min \left[n_e(\text{max}), \max ({}^n n_e, {}^{n+1} n_e) \right] \quad (3.59)$$

$$n_e = \sqrt{n_e(\text{min}) n_e(\text{max})} \quad (3.60)$$

The iterations are repeated until n_e converges to six figures.

The state variables we require are given by the following expressions:

$$P = {}^{\text{ion}}P + n_e kT + \frac{1}{3} aT^4 \quad (3.61)$$

$$E = {}^{\text{ion}}E + \frac{3}{2} n_e kT + \frac{aT^4}{P} \quad (3.62)$$

$$TS = P + U + \eta kT \quad (3.63)$$

where the prefix "ion" denotes the contribution from free ions, which remains to be calculated. Saha's equation is used to find the number density n_{ij} of ion i in state j :

$$\frac{n_{ij}}{n_i} = \frac{\prod_{j=\omega_i}^{J_i-1} n_e \Phi_{ij}(T)}{\sum_{i=1}^{\text{all}} \left[\prod_{j=\omega_i}^{J_i-1} n_e \Phi_{ij}(T) \right]} \quad (3.64)$$

in which we have:

$$\Phi_{ij}(T) = \frac{n_{ij}}{n_{i,j+1} n_e} = \frac{1}{2} \frac{B_{ij}}{B_{i,j+1}} \left[\frac{h^2}{2\pi m_e k T} \right]^{3/2} \rho^{\chi_{ij}/kT} \quad (3.65)$$

$$B_{ij} = \frac{g_i!}{(g_i - p_i)! p_i!} \quad (3.66)$$

In the above, J_i is the highest ionisation stage (of those considered) of ion i ; ω_i is zero unless the negative ion exists, in which case it is -1 ; B_{ij} is the statistical weight of the outer shell configuration of ionisation stage j of ion i ; χ_{ij} is the transition energy of ion i between states j and $j+1$; g_i is the statistical weight of the outer shell of ion i , and p_i is the number of electrons left in the outermost shell of ion i .

The total number density of free ions is given by:

$$n^{\text{ion}} = \sum_{i=1}^{\text{all}} \sum_j^{J_i-1} n_{ij} \quad (3.67)$$

The partial pressure due to free ions is thus:

$$P^{\text{ion}} = n^{\text{ion}} kT \quad (3.68)$$

the partial entropy per unit mass is :

$$S^{\text{ion}} = \frac{E^{\text{ion}}}{T} + \frac{1}{\rho T} \left[P^{\text{ion}} - \sum_{i=1}^{\text{all}} n_i^{\text{ion}} \eta_i^{\text{ion}} kT \right] \quad (3.69)$$

and the partial internal energy per unit mass is:

$${}^{\text{ion}}E = \frac{1}{\rho} \left[\sum_{i=1}^{\text{all}} \sum_{j=\alpha_i}^{J_i-1} n_{ij} \chi_{ij} + \frac{3}{2} {}^{\text{ion}}n k T \right] \quad (3.70)$$

in which the degeneracy factor ${}^{\text{ion}}\eta_i$ is defined by:

$$e {}^{\text{ion}}\eta_i = \frac{N}{\left[2\pi A_i k T / h^2 \right]^{3/2}} \quad (3.71)$$

3.3.6 Opacity

The rate (and method) of energy transport in stars is determined largely by the opacity, κ , of the stellar material. κ is the harmonic mean of the radiative and conductive opacities, while the radiative opacity is itself a harmonic mean of the monochromatic mass absorption coefficient, $\kappa(\nu)$, where ν is the frequency of the radiation. The value of $\kappa(\nu)$ depends upon the composition and the degree of ionisation of the stellar material.

The opacity tables used in this work were C312 and HM88. C312 is one of a series of tables produced by Dr Carson; the name refers to the composition of the material, which is $(X,Y,Z) = (0.73,0.27,0.02)$. The table HM88 has composition $(X,Y,Z) = (0.88,0.10,0.02)$ and was obtained by interpolating between C312 and C402, which has composition $(X,Z) = (0.98,0.02)$. The interpolation was carried out using a program provided by Dr Simon Jeffery.

C312 was used where a typical population I composition was being considered; HM88 and C402 were used when studying helium depletion in the envelope. The three tables can be compared in Figures 3.2, 3.3 and 3.4, which are 3-D plots of

Figure 3.2. The opacity table HM88, produced by interpolating between the Carson tables C402 and C312. The chemical composition is $(X, Y, Z) = (0.88, 0.10, 0.02)$.

Figure 3.3. The Carson opacity table C312. The chemical composition is $(X,Y,Z) = (0.73,0.27,0.02)$.

Figure 3.4. The Carson opacity table C402. The chemical composition is $(X,Y,Z) = (0.98,0.00,0.02)$.

TABLE 3.1

c312	0.73000	0.25000	0.02000								
3.3	-12.0	-2.16990	-2.27260	-2.56610	-2.98060	-3.33540	-2.97750	-2.04510	-1.05970	-0.06110	1.40550
3.4	-12.0	-2.13260	-2.13040	-2.13440	-2.17120	-2.30000	-2.29480	-1.64500	-0.65730	0.41480	1.98690
3.5	-12.0	-2.10540	-2.10130	-2.08700	-2.03700	-1.89300	-1.61780	-1.16440	-0.33520	0.61460	1.89190
3.6	-11.0	-2.06640	-2.04810	-1.95180	-1.65970	-1.11890	-0.42490	0.24890	0.99370	1.88490	4.85830
3.7	-11.0	-1.98130	-1.96920	-1.87230	-1.55210	-0.87690	-0.07800	0.70610	1.56520	2.15560	3.22710
3.8	-11.0	-1.39540	-1.44910	-1.23820	-0.90870	-0.26920	0.33740	1.03070	1.96570	2.31800	3.55320
3.9	-11.0	-0.23220	-0.13200	0.02170	0.26680	0.60110	1.00440	1.50240	2.08560	2.79860	3.84870
4.0	-11.0	-0.10270	0.52730	1.12240	1.41740	1.67430	1.94300	2.39930	3.04270	3.42670	4.42790
4.1	-10.0	0.21630	1.10650	1.93790	2.47680	2.77770	3.09070	3.48040	3.94080	4.44270	7.12430
4.2	-10.0	-0.05970	0.68100	1.65550	2.63170	3.30020	3.72010	4.03150	4.45530	4.92730	6.69170
4.3	-10.0	-0.23940	0.33310	1.26590	2.31740	3.30590	4.12310	4.52960	4.90010	5.06650	6.36760
4.4	-10.0	-0.27990	0.12470	0.98720	2.04090	3.14710	4.25520	4.93950	5.31850	5.21020	6.12260
4.5	-10.0	-0.19700	0.09000	0.83760	1.87350	2.99320	4.19270	4.98840	5.53900	5.32430	5.82840
4.6	-9.0	0.22550	0.99610	1.78940	2.84350	3.98600	4.86250	5.51580	5.48050	5.66240	7.40930
4.7	-9.0	-0.10860	0.84100	1.76480	2.69400	3.68480	4.52980	5.25900	5.42100	5.48870	6.81730
4.8	-9.0	-0.20030	0.66020	1.25530	2.36500	3.31100	4.17450	4.91170	5.22280	5.29400	6.33580
4.9	-9.0	-0.16430	0.26300	0.83730	1.97060	2.88650	3.86550	4.62020	4.97690	5.07930	5.91510
5.0	-9.0	-0.29570	0.04460	0.66720	1.58970	2.53030	3.55850	4.37480	4.72960	4.84940	5.53440
5.2	-8.0	-0.29740	0.20150	0.98510	1.79520	2.70150	3.57960	4.01980	4.32300	4.81910	5.44940
5.4	-8.0	-0.06750	0.09180	0.25290	0.94800	1.84370	2.43360	3.10700	3.54410	4.12560	4.68070
5.6	-7.0	0.66060	0.66170	0.53200	1.04400	1.65570	2.34080	2.89080	3.28200	3.65380	4.71080
5.8	-7.0	0.84430	0.50100	0.45890	0.66490	1.06340	1.74470	2.28930	2.80600	3.40680	3.87620
6.0	-7.0	0.35590	0.29310	0.31310	0.37650	0.44230	1.15630	1.95520	2.34410	2.78870	3.23690
6.2	-6.0	-0.03400	-0.01500	-0.06430	0.41720	0.50660	1.40390	1.91710	2.23810	2.70040	3.34600
6.4	-6.0	-0.34530	-0.31840	-0.25820	0.01160	0.03560	0.77650	1.35650	1.71660	2.15070	2.56700
6.6	-5.0	-0.43730	-0.42890	-0.41280	-0.29260	0.07350	0.69560	1.13560	1.56300	2.02500	0.55020
6.8	-5.0	-0.46280	-0.45630	-0.45480	-0.43870	-0.31520	0.09340	0.50090	0.96160	1.49710	1.67150
7.0	-5.0	-0.46680	-0.46510	-0.46440	-0.46600	-0.44660	-0.29960	0.01450	0.42720	0.96750	1.33770
7.2	-4.0	-0.47330	-0.47290	-0.47190	-0.46480	-0.42050	-0.25230	0.01600	0.45570	0.85220	0.09830
7.4	-4.0	-0.48000	-0.48000	-0.47990	-0.47930	-0.47110	-0.40500	-0.25530	0.02460	0.38090	-0.28130
7.6	-3.0	-0.48960	-0.48960	-0.48970	-0.48930	-0.47780	-0.42060	-0.27130	-0.00160	0.28910	0.48390
7.8	-3.0	-0.50380	-0.50380	-0.50390	-0.50380	-0.50280	-0.48970	-0.43250	-0.28190	-0.14430	-0.22810
8.0	-3.0	-0.52470	-0.52470	-0.52470	-0.52470	-0.52480	-0.52310	-0.50810	-0.44710	-0.38120	-0.56320
8.2	-2.0	-0.55440	-0.55440	-0.55440	-0.55450	-0.55460	-0.55160	-0.53610	-0.51300	-0.55250	-0.01090
9.4	-2.0	-0.59480	-0.59480	-0.59480	-0.59490	-0.59500	-0.59390	-0.59320	-0.59290	-0.64260	-0.50350
8.6	-1.0	-0.64670	-0.64670	-0.64680	-0.64680	-0.64710	-0.64760	-0.65170	-0.68610	-0.71750	0.26690
8.8	-1.0	-0.71010	-0.71010	-0.71010	-0.71020	-0.71030	-0.71060	-0.71380	-0.73440	-0.79610	-0.42810
9.0	-1.0	-0.78420	-0.78420	-0.78420	-0.78420	-0.78420	-0.78400	-0.78590	-0.79520	-0.82690	-0.75280

Table 3.1. The Carson opacity table C312. The first column gives the value of $\log T$ for each row; the second gives the initial value of $\log \rho$. The third column is the opacity for that temperature and density; the density is incremented in steps of 1.0 to give the opacities listed in subsequent columns.

TABLE 3.2

HM88	0.88000	0.10000	0.02000								
3.3	-12.0	-2.09710	-2.21270	-1.91040	-2.94350	-3.32320	-3.06030	-2.16350	-1.19360	-0.16000	1.51390
3.4	-12.0	-2.05660	-2.05450	-2.06010	-2.10560	-2.25340	-2.29440	-1.67900	-0.69830	0.37890	2.06360
3.5	-12.0	-2.02950	-2.02520	-2.01170	-1.96170	-1.82120	-1.56450	-1.15690	-0.35540	0.59890	1.93320
3.6	-11.0	-1.99090	-1.97290	-1.87720	-1.58650	-1.05300	-0.36650	0.28620	1.00770	1.91950	3.91860
3.7	-11.0	-1.90670	-1.89090	-1.79200	-1.47480	-0.80620	-0.01050	0.76650	1.56960	2.20700	3.33890
3.8	-11.0	-1.33290	-1.36710	-1.14260	-0.70660	-0.16650	0.43420	1.11710	1.83210	2.38930	3.53060
3.9	-11.0	-0.14300	-0.04820	0.10790	0.35990	0.69270	1.09280	1.58790	2.18660	2.91410	3.92620
4.0	-11.0	0.01300	0.65580	1.22790	1.50830	1.75580	2.01570	2.44690	2.96170	3.54480	4.49620
4.1	-10.0	0.34700	1.24890	2.06730	2.57080	2.85590	3.14850	3.56530	4.05900	4.45480	3.05060
4.2	-10.0	0.02480	0.79990	1.79110	2.75200	3.38860	3.78450	4.10660	4.57310	4.86710	3.29460
4.3	-10.0	-0.16180	0.43110	1.38390	2.45550	3.44210	4.25120	4.62000	4.80580	5.13600	3.64850
4.4	-10.0	-0.21310	0.21180	1.09590	2.16220	3.27670	4.40590	5.01800	5.23800	5.38650	3.98100
4.5	-10.0	-0.17420	0.14320	0.93330	1.98100	3.11900	4.32720	5.08510	5.48960	5.43750	4.34410
4.6	-9.0	0.25390	0.90500	1.84020	2.92440	4.02240	4.86290	5.43080	5.51300	4.90990	2.13950
4.7	-9.0	-0.09290	0.82370	1.72890	2.65720	3.60490	4.42200	5.10170	4.94790	4.95050	2.35000
4.8	-9.0	-0.21220	0.68830	1.34440	2.29070	3.15760	3.95620	4.66920	4.70760	4.89090	2.60240
4.9	-9.0	-0.15900	0.30400	0.80810	1.92170	2.74060	3.55110	4.26710	4.63250	4.82980	2.91810
5.0	-9.0	-0.26170	0.06580	0.51390	1.54460	2.38850	3.21710	3.91750	4.35980	4.63520	3.26210
5.2	-8.0	-0.28100	0.19680	0.89310	1.66730	2.47550	3.19070	3.65070	3.83440	4.22110	1.50780
5.4	-8.0	-0.06490	0.01400	0.21890	0.92020	1.73490	2.30920	2.90510	3.34770	3.98300	2.08910
5.6	-7.0	0.66340	0.71790	0.54460	0.94990	1.61650	2.26370	2.76040	3.31170	2.80120	0.43640
5.8	-7.0	0.84170	0.50020	0.46410	0.63800	1.02840	1.70230	2.25370	2.74800	3.19160	0.89660
6.0	-7.0	0.36660	0.28520	0.32610	0.37830	0.39300	1.15590	1.84810	2.31820	2.75450	1.55820
6.2	-6.0	-0.01570	0.00180	-0.04570	0.23170	0.52520	1.41970	1.91980	2.23890	2.35770	-0.24230
6.4	-6.0	-0.31550	-0.28860	-0.23280	-0.08960	0.05340	0.79250	1.36350	1.69820	2.08600	0.33190
6.6	-5.0	-0.40410	-0.39560	-0.37900	-0.25600	0.12950	0.71180	1.12770	1.56670	1.15390	-1.32840
6.8	-5.0	-0.42920	-0.42170	-0.42050	-0.40420	-0.29140	0.10970	0.52850	0.97500	1.35020	-0.87990
7.0	-5.0	-0.43210	-0.43050	-0.42990	-0.43110	-0.41010	-0.26780	0.03750	0.44430	0.95460	-0.21470
7.2	-4.0	-0.43940	-0.43800	-0.43710	-0.42990	-0.38730	-0.22580	0.04000	0.47340	0.53950	-2.02720
7.4	-4.0	-0.44510	-0.44510	-0.44500	-0.44420	-0.43640	-0.37140	-0.22740	0.04750	0.31430	-1.46430
7.6	-3.0	-0.45470	-0.45470	-0.45480	-0.45350	-0.44330	-0.38930	-0.24220	-0.00090	-0.56470	-3.08560
7.8	-3.0	-0.46890	-0.46890	-0.46900	-0.46890	-0.46790	-0.45540	-0.40060	-0.27520	-0.35040	-2.65320
8.0	-3.0	-0.48980	-0.48980	-0.48980	-0.48980	-0.48990	-0.48800	-0.47470	-0.42390	-0.43740	-1.96080
8.2	-2.0	-0.51950	-0.51950	-0.51950	-0.51960	-0.51970	-0.51770	-0.50670	-0.51820	-1.06700	-3.79200
8.4	-2.0	-0.55990	-0.55990	-0.55990	-0.56000	-0.56010	-0.55990	-0.56100	-0.60340	-0.93870	-3.17810
8.6	-1.0	-0.61180	-0.61180	-0.61190	-0.61190	-0.61230	-0.61410	-0.63790	-0.84560	-2.19220	-4.83110
8.8	-1.0	-0.67520	-0.67520	-0.67520	-0.67530	-0.67540	-0.67630	-0.68960	-0.95570	-1.69370	-4.40080
9.0	-1.0	-0.74930	-0.74930	-0.74930	-0.74930	-0.74940	-0.74980	-0.75660	-0.87640	-1.50200	-3.65720

Table 3.2. The opacity table HM88. The layout is the same as that for C312.

TABLE 3.3

	402	0.98000	0.00000	0.02000							
3.3	-12.0	-2.04860	-2.17270	-1.47320	-2.91880	-3.31510	-3.11550	-2.24250	-1.28280	-0.22600	1.58620
3.4	-12.0	-2.00600	-2.00390	-2.01060	-2.06180	-2.22240	-2.28120	-1.70160	-0.72570	0.35500	2.11480
3.5	-2.0	-1.97890	-1.97450	-1.96150	-1.91150	-1.77330	-1.52890	-1.15190	-0.36890	0.58850	1.96070
3.6	-1.0	-1.94060	-1.92270	-1.82740	-1.53770	-1.00900	-0.32760	0.31100	1.01700	1.94260	3.29210
3.7	-1.0	-1.85700	-1.83930	-1.73840	-1.42320	-0.75900	0.03450	0.80680	1.57250	2.24120	3.41350
3.8	-1.0	-1.29100	-1.31240	-1.07890	-0.63850	-0.09760	0.49890	1.17470	1.74310	2.43680	3.51560
3.9	-1.0	-0.08350	0.00760	0.16530	0.42200	0.75380	1.15170	1.64490	2.25390	2.99130	3.97780
4.0	-1.0	0.09010	0.74140	1.29820	1.56890	1.81020	2.06420	2.47870	2.90770	3.62350	4.55300
4.1	-0.0	0.43420	1.34380	2.15360	2.63350	2.90810	3.18700	3.62200	4.13780	4.46280	3.02380
4.2	-0.0	0.08440	0.87920	1.88150	2.83220	3.44760	3.82740	4.15690	4.65160	4.82700	3.23640
4.3	-0.0	-0.11010	0.49640	1.46250	2.54750	3.53290	4.33660	4.68080	5.08620	5.18230	3.59970
4.4	-0.0	-0.16860	0.26990	1.16840	2.24310	3.36310	4.50640	5.07110	5.27050	5.50400	3.93730
4.5	-0.0	-0.15900	0.19090	0.99580	2.05260	3.20290	4.41690	5.15030	5.52090	5.51290	4.26650
4.6	9.0	0.28290	0.91100	1.87480	2.97840	4.04660	5.41100	5.53470	4.84670	2.11050	
4.7	9.0	-0.08230	0.81220	1.70500	2.63260	3.55140	4.35030	5.01060	5.30080	4.91710	2.32190
4.8	9.0	-0.22020	0.70710	1.40380	2.24110	3.05530	3.81070	4.51190	4.63170	4.84000	2.55230
4.9	9.0	-0.15540	0.33130	0.78860	1.89910	2.64340	3.34150	4.03330	4.49680	4.66340	2.87160
5.0	9.0	-0.24570	0.07990	0.41170	1.51460	2.29390	2.98950	3.61340	4.14800	4.49240	3.20910
5.2	8.0	-0.27680	0.19360	0.83180	1.58200	2.32480	2.93150	3.40790	3.71680	4.21940	1.47810
5.4	8.0	-0.06310	0.02220	0.19620	0.90160	1.66240	2.22620	2.77070	3.23060	3.98140	2.04610
5.6	7.0	0.66530	0.76210	0.55300	0.88720	1.59040	2.21230	2.68160	3.34270	2.74320	0.40750
5.8	7.0	0.83990	0.49960	0.46750	0.62000	1.00510	1.67410	2.23010	2.72570	3.18490	0.85310
6.0	7.0	0.37370	0.28660	0.33480	0.37950	0.36020	1.15570	1.84350	2.30360	2.75680	1.51280
6.2	6.0	-0.00350	0.01300	-0.03300	0.10800	0.53760	1.42690	1.92220	2.24460	2.35230	-0.27300
6.4	6.0	-0.29560	-0.26980	-0.21580	-0.15710	0.06520	0.90310	1.36820	1.69040	2.09000	0.28750
6.6	5.0	-0.38190	-0.37340	-0.35650	-0.23160	0.16690	0.72260	1.12310	1.57700	1.07820	-1.35500
6.8	5.0	-0.40510	-0.39870	-0.39760	-0.38120	-0.27560	0.12730	0.54700	0.98900	1.35060	-0.92410
7.0	5.0	-0.40900	-0.40740	-0.40690	-0.40790	-0.38590	-0.24660	0.05290	0.45660	0.96370	-0.25890
7.2	4.0	-0.41520	-0.41480	-0.41390	-0.40670	-0.36510	-0.20780	0.05620	0.48840	0.53250	-2.05460
7.4	4.0	-0.42190	-0.42180	-0.42180	-0.42080	-0.41320	-0.34900	-0.20880	0.06520	0.31640	-1.50500
7.6	3.0	-0.43150	-0.43140	-0.43150	-0.43030	-0.42030	-0.36840	-0.22200	0.01010	-0.64550	-3.11030
7.8	3.0	-0.44570	-0.44570	-0.44570	-0.44560	-0.44460	-0.43250	-0.37900	-0.26010	-0.35760	-2.69640
8.0	3.0	-0.46660	-0.46660	-0.46660	-0.46660	-0.46660	-0.46460	-0.45230	-0.40460	-0.42920	-2.00340
8.2	2.0	-0.49620	-0.49630	-0.49630	-0.49630	-0.49650	-0.49500	-0.48520	-0.50370	-1.07720	-3.81620
8.4	2.0	-0.53660	-0.53670	-0.53670	-0.53670	-0.53690	-0.53710	-0.53860	-0.58710	-0.93500	-3.22160
8.6	1.0	-0.58860	-0.58860	-0.58860	-0.58870	-0.58910	-0.59120	-0.61790	-0.83870	-2.28420	-4.85220
8.8	1.0	-0.65200	-0.65200	-0.65200	-0.65200	-0.65220	-0.65310	-0.66760	-0.95680	-1.70020	-4.44420
9.0	1.0	-0.72600	-0.72600	-0.72600	-0.72600	-0.72610	-0.72680	-0.73390	-0.86460	-1.50790	-3.70260

Table 3.3. The opacity table C402. The layout is the same as that for C312.

$\log T$, $\log \rho$ and K (out of the paper). Note that the ridge corresponding to the second ionisation stage of helium is slightly reduced in HM88, and absent in C402. Tables 3.1, 3.2 and 3.3 give the actual grid points used in the non-linear analysis; the program used linear interpolation to obtain values of K for known temperature and density.

3.3.7 Output

The program TRCVSP defines the end of a period as the point where a chosen radius (deep in the star) crosses its equilibrium value while decreasing. For stars with multiple or non-repetitive periods, this definition of a period is not particularly useful, and in this work Fourier analyses were carried out on segments of light curve to find the periods actually present (see Chapter 6).

Each time the program finds what it believes to be the end of a period, it writes out various useful parameters, including the maximum inward and outward velocities of a chosen zone (usually that for which the optical depth, τ , is nearest to 0.2), the maximum and minimum radii of that zone, the full amplitude in radius, the maximum and minimum luminosities, and the full amplitude in magnitudes. These values are a good guide to what is going on, and to whether or not the model is going haywire, but for Delta Scuti stars, Fourier analyses of the light, radius and velocity variations are more informative.

The program also calculates the peak kinetic energy for each period, using the formula:

$$KE = \frac{1}{2} \sum_{i=1}^N \Delta M_{i-1/2} (u_{i-1/2})^2 \quad (3.72)$$

where

$$u_{i-1/2} = \frac{1}{2} [u_i + u_{i-1}] \quad (3.73)$$

This is a useful quantity, because it indicates whether the pulsation is growing (kinetic energy increases with time) or decaying (kinetic energy decreases with time). For a star pulsating at its final amplitude, the maximum kinetic energy per period should be constant. Here, the long growth-times of the Delta Scuti stars are a problem; it is difficult to tell whether the kinetic energy is increasing or decreasing over a small number of periods.

The work function for each period is also stored. The work done per zone is given by:

$$\Delta M_{i-1/2} \oint dW \quad (3.74)$$

where $dW = (P+Q)dV$. Q must be included, despite being artificial; it is a pressure, and therefore contributes to the work done. Note that the loop in the P - V plane may not close exactly, particularly when the point that the program insists is a period does not correspond well to the actual cycle. Hence, some estimate of the work required to close the loop is added to the sum of work done per zone. Plots

of the work function against zone number can be used to find which parts of the star are driving and which are damping the pulsation.

The program does not write out the entire luminosity, velocity and radius history of each zone over each period; this would consume excessive amounts of space. However, it can provide such histories. Plots of (for example) velocity for all zones over one period can be used to locate shock fronts or reflections which may be causes of particular features in the observed velocity curves of real stars.

3.4 SOME PREVIOUS ATTEMPTS

3.4.1 Introduction

It has been suggested that non-linear modelling of real stars is more of an art form than a science. In particular, the initial velocity profile chosen can have a very influential effect on the shape of light curve produced. Also of importance are the choices of opacity tables, boundary conditions, and artificial viscosity parameter, but with these there is less room to manoeuvre.

It should be possible to start a model without a "kick", and allow computer noise to initiate pulsation. However, the growth time of Delta Scuti star pulsations, at around 10^6 years, precludes this. Attempts to produce models this way resulted in light curves with many high harmonics present.

A brief review of published non-linear models is presented here, to illustrate the various methods researchers have used to achieve good copies of real light curves.

3.4.2 Review

One of the first applications of non-linear pulsation theory to real stars was a study of RR Lyrae variables by Christy (1966), using a method and program described in Christy 1964 and 1967. Integration for 20 to 30 periods was required to eliminate unwanted harmonics and allow development of the non-linear features found in real light curves. A model was considered to have settled down when its peak kinetic energy remained constant from period to period. In some cases, this final large-amplitude motion was found to be a mixture of fundamental and first harmonic, giving rise to a modulated (and not strictly periodic) light curve.

The behaviour of models towards the red edge of the Instability Strip was unsatisfactory; lack of convection in the models was put forward as the explanation.

Stobie (1969a,b,c) used a slightly modified form of Christy's method to study classical Cepheids. He paid particular attention to the initiation of motion in the non-linear model, using the following equations to represent the velocity distribution u as a function of the relative radius, ξ , where ξ is r/R :

$$\text{fundamental: } u = -a\xi^l \quad (3.75)$$

$$\text{first harmonic: } u = -a\xi^l + b\xi^m \quad (3.76)$$

$$\text{second harmonic: } u = -a\xi^l + b\xi^m - c\xi^n \quad (3.77)$$

where a , b , c , l , m , and n are constants which satisfy the relations $a > b > c > 0$ and $l > m > n > 1$, and which have to be different for each model, because the velocity distribution depends on the density distribution, which varies markedly over the length of the Instability Strip. For a mass of $2.5M_{\odot}$ and luminosity $2500L_{\odot}$, Stobie suggested $l = 4.6$ for the fundamental mode. In general, the values were chosen to give a node at temperature $T = 1.0 \times 10^5 \text{K}$.

To save computing time, an extra kick was given to models at the phase of maximum velocity of expansion if otherwise they would take more than 40 periods to reach a maximum amplitude. Again, the maximum kinetic energy per period was found to be a useful diagnostic for determining whether the pulsation was growing or decaying.

Stellingwerf (1974) proposed a method for speeding up the non-linear code, using a mathematical technique. He applied his program to RR Lyrae variables with success (Stellingwerf 1975a), improving on Christy's (1966) models. However, his attempt to resolve the discrepancy between evolution masses and pulsation masses for double-mode classical Cepheids failed (Stellingwerf 1975b), producing satisfactory light curves only for models with half the mass expected from evolutionary studies. The method itself proved unsuitable for Delta Scuti stars (Stellingwerf 1976),

giving velocity and luminosity amplitudes well above those observed. For example, a model intended to represent the star δ Scuti had mass $2.0 M_{\odot}$, $M_{\text{bol}} = 0.881$, $T_e = 7000\text{K}$, $Z = 0.02$, $P_0 = 0^{\text{d}}.187$ (cf. δ Scuti $0^{\text{d}}.194$), $P_2 = 0^{\text{d}}.116$ (cf. δ Scuti $0^{\text{d}}.117$). This model gave amplitudes of 162 km/s and $\Delta m = 2^{\text{m}}.7$ before calculations diverged due to ejection of the outer layers. Stellingwerf appealed to rotation and helium diffusion to resolve this problem.

Vemury & Stothers (1978) based their program on Christy's (1967) code. Carson, Stothers & Vemury (1981) used the code and the Carson opacities to study type II Cepheids in globular clusters, and achieved a successful model of the light curve of BL Her (Figure 3.5).

Figure 3.5. Comparison of a non-linear model with the type II Cepheid BL Herculis. (Carson et al. 1981).

The greater diversity of forms of light curves of type II Cepheid field variables (the BL Herculis stars; periods 1 to 3 days) was successfully treated by the same method by Carson & Stothers (1982), and they went on (Carson & Stothers 1984a,b) to cover classical Cepheids of periods greater than 13 days, and to reconcile the theory of classical bump Cepheids with observations (Carson & Stothers 1988).

Bridger's (1983) code has already been mentioned; he used it to study the W Virginis variables, type II Cepheids of periods 10 to 20 days. Since the e-folding times for these stars are very short, artificial amplification was not used - in most cases, maximum amplitude was reached after 10 periods. Two initial velocity profiles were used; one due to Christy (1967):

$$u = -13\xi^{10} - 7\xi^5 \quad (3.78)$$

and one due to formulae given in Stobie (1969a):

$$u = -18\xi^5 \quad (3.79)$$

The short e-folding time also allowed rapid elimination of unwanted harmonics.

CHAPTER 4

THE STATIC MODEL

4.1 INTRODUCTION

The non-linear pulsation code was designed to analyse a static model envelope. The code contains a subroutine which will generate an envelope, given the mass, luminosity and effective temperature of the star desired; it does this by integrating inward (using a numerical quadrature method) from the surface to a point approximately one tenth the radius of the star. However, this approach is rather crude, as it allows the analysis of combinations of M , L and T_e which could not be expected to occur in real stars. Therefore, it was decided to prepare a grid of static models taken from evolutionary tracks spanning the area of interest in the Hertzsprung-Russell Diagram. Each evolved model produced has about 300 zones; this is reduced to 50 in the non-linear code, and the static model is then relaxed to satisfy the dynamical difference equations with time derivatives set to zero, to avoid the introduction of any unwanted transients.

The use of complete stellar models from evolutionary tracks has some advantages over the envelope-only method, particularly for Delta Scuti stars, which are conveniently situated on or near the Main Sequence, where stellar evolution theory is not in dispute. The models can be examined, perhaps with reference to linear pulsation studies, to see whether pulsation amplitude becomes vanishingly small before regions of energy generation are encountered, and hence whether neglect of energy generation in the non-linear study is justified. Areas of convective energy transport can be identified, although as yet nothing can be done to allow for these in the non-linear analysis.

Perhaps most importantly, if "state-of-the-art" evolved models were used, the knowledge that the models formed a grid of points of known mass and age might be used to give a calibration scheme for real stars of known T_e , amplitude(s) and period(s). It must be remembered that the models in this study incorporate a number of unrealistic assumptions, such as no rotation, no magnetic field, and no element diffusion, which would prejudice their use in such a project.

4.2 THE METHOD

The program used to produce static evolutionary models, KWHMP, is an implementation (by Dr Carson) of the program outline given in Kippenhahn, Weigert & Hofmeister (1967). Kippenhahn et al. described the method in great detail, giving the equations, the techniques for solving them, and numerous intricacies of programming. Their flow chart of

Figure 4.1. Flow chart for an evolution code (Kippenhahn et al. 1967).

the program is reproduced in Figure 4.1.

The equations of stellar evolution can be found in any astrophysics text (e.g., Cox & Giuli 1968). With mass as the independent variable and identifications as in Chapter 3, we have:

Continuity of Mass:

$$\frac{\partial r}{\partial M_r} = \frac{1}{4\pi r^2 \rho(r)} \quad (4.1)$$

Hydrostatic Equilibrium:

$$\frac{\partial P}{\partial M_r} = \frac{GM_r}{4\pi r^2} \quad (4.2)$$

Energy Balance:

$$\frac{\partial L_r}{\partial M_r} = \epsilon - T \frac{\partial S}{\partial t} \quad (4.3)$$

Transport Equation:

$$\frac{\partial T}{\partial M_r} = - \frac{GM_r T}{4\pi r^4 P} \nabla \quad (4.4)$$

where ϵ is the rate of energy generation. The mechanism of radiation transport determines ∇ ; for radiation and conduction we have:

$$\nabla_R = \frac{3}{16\pi acG} \frac{\kappa_L P}{M_r T^4} \quad (4.5)$$

while for convective transport, ∇_{conv} must be determined

from a theory of convection.

Boundary conditions at the centre are obtained by consideration of solutions of these equations for M , r and L equal to zero. The program KWHMP allowed for three different surface boundary conditions. The one used in this work can be expressed as:

$$P_{\text{phot}} = \left[\frac{g}{\kappa} \tau \right]_{\tau=2/3} \quad \text{when } T = T_c \quad (4.6)$$

Time differentials in the equations are replaced by differences and, from some initial conditions, a model can be integrated in time, thus giving an evolutionary sequence of models. New chemical compositions in each layer are evaluated from energy generation rates for the p-p chain, the CNO cycle, etc. In convective regions, elements are mixed up to give a homogeneous composition.

To economise on computing time, the subroutines for the constitutive equations used in KWHMP were kept simple. Opacity is calculated using approximate formulae given in Christy (1967). Energy generation rates are from Fowler et al. (1967). The equation of state is simple but rapid; an approximate mean ionisation potential, V_{mean} , is calculated from:

$$V_{\text{mean}} = \mu \left[\frac{X}{1.0} V_x + \frac{Y}{4.0} V_y + \frac{Z}{16.0} V_z \right] \quad (4.7)$$

where μ is the mean molecular weight, V_x is the ionisation energy of hydrogen, V_y is the ionisation energy of helium,

and V_Z is the mean ionisation energy of the metals. V_{mean} is used to estimate the electron number density, N_e . The pressure is then calculated from the expression:

$$P = kT \left[N_a + H N_e \right] + \frac{aT^4}{3} \quad (4.8)$$

where

$$H = \frac{2}{3} \frac{F_{3/2}}{F_{1/2}} \quad (4.9)$$

H is related to the degeneracy parameter, η , which requires the estimate of N_e in its calculation. The F 's are Fermi-Dirac integrals. The subroutine also returns values for the internal energy, U , and the entropy, S .

The program KWHMP requires an approximate stellar model to begin with. For each shell of this initial model, mass, radius, luminosity, temperature, pressure and density are required. Values of energy generation rate and opacity are only required during the subsequent static relaxation to a new mass grid, while internal energy and entropy are needed for evolutionary models.

In this project, the initial model used was a $1.0 M_{\odot}$ (solar) model of homogeneous population I composition, obtained by solving the Lane-Emden equation (see, for example, Chandrasekhar 1939) numerically for a polytrope of index 3. The solution provides mass, radius, temperature, pressure and density, but not luminosity. Approximations to nuclear energy generation rates were obtained from Schwarzschild (1958) and used to calculate the energy

generated per unit mass, which was then multiplied by the shell mass to give the luminosity of each shell. The luminosities of each shell were summed to give the total luminosity as a function of the Lane-Emden variable x , and hence as a function of radius.

The resulting model was relaxed in KWHMP to give a homogeneous, population I, zero-age Main Sequence solar model. The mass was then incremented in steps of $0.2 M_{\odot}$ and the models re-relaxed to give zero-age Main Sequence models up to mass $3.0 M_{\odot}$.

4.3 THE STATIC GRID

4.3.1 Evolutionary Tracks

Each zero-age Main Sequence model was allowed to evolve for some 2 to 3 hours of cpu time, until an effective temperature of $\log T_e \sim 3.8$ (beyond the red edge) was reached. (Some unexplained bug in the program prevented the $2.2 M_{\odot}$ model from running properly; it did not proceed beyond $\log T_e \sim 3.95$.) The evolutionary tracks produced are shown in Figure 4.2, with real Delta Scuti stars indicated by asterisks (data from Appendix E). The tracks have a certain similarity in form: from a to b in the diagram corresponds to the period of core hydrogen burning. Between b and c, gravitational contraction takes place, after the near exhaustion of hydrogen at the core (and the cessation of hydrogen burning). At c, the burning of hydrogen in a shell around the core commences.

Initial composition was (by mass) $X = 0.73$, $Y = 0.25$,

KWHMP Evolution Tracks

Figure 4.2. The ZAMS and evolution tracks produced using KWHMP. The tracks are labelled with mass, in solar units. Some real Delta Scuti stars (data from appendix E) are plotted as asterisks.

$Z = 0.02$. At each timestep (initially five million years, but modified by the program), a complete stellar model suitable for input to the non-linear code was produced. The ones selected for linear pulsation analysis are indicated by asterisks in Figure 4.3. These comprise every tenth model early in evolution, but nearly every model over the Hertzsprung Gap, where evolution is rapid. Models with $\log T_e$ above that expected for the blue edge were retained in order to locate (using linear analysis) blue edges of harmonic modes. Appendix B lists details of each static model.

Figure 4.3. The ZAMS and evolution tracks produced using KWHMP, with the selected models indicated by asterisks.

4.3.2 Comparison

A recent paper by Vandenberg (1985) presented grids of stellar models of masses 0.7 to $3.0 M_{\odot}$ and various metal contents. Colour indices were predicted for the models, and the isochrones were transposed from the theoretical Hertzsprung-Russell Diagram to the more practical $(M_V, B-V)$ plane. Colour-magnitude diagrams were constructed and compared with those of open clusters, with satisfactory results.

VandenBerg's evolution code uses Los Alamos opacities (Huebner et al. 1977) and a mixing-length theory of convection (with mixing length $l = 1.6H_v$). The method used by Kippenhahn et al. (1967) to find the pressure at the outer boundary (the outer boundary condition) was replaced simply by interpolation in sophisticated model atmospheres, from (amongst others) Kurucz (1979). This removed the need for an assumed relationship between T and τ .

Models from VandenBerg's evolutionary sequences are indicated by asterisks in Figure 4.4, and are joined by lines for clarity. The KWHMP tracks are also plotted, for comparison.

The differences between the tracks are most probably due to the more sophisticated treatment of the equation of state and surface boundary condition in VandenBerg's models.

4.4 SOME ASPECTS OF THE MODELS

The non-linear code, TRCVSP, rezones the static model and discards the innermost tenth of the radius. It is then assumed that no energy generation takes place in the remaining envelope. To check that this is justified, plots were made of the energy generation term, ϵ , against scaled radius, for each of the static models in the evolution sequences. These are shown in Figure 4.5 and 4.6. Each line in these figures corresponds to one of the selected static models indicated in Figure 4.3. It can readily be seen that the energy generation is either zero or very small for all the models for $r > 0.1 R_*$, and hence that energy generation can be safely ignored in the non-linear analysis,

Figure 4.4. KWHMP ZAMS and tracks compared to VandenBerg's models (VandenBerg 1985), which are indicated by asterisks. The lines joining models of the same mass are merely for clarity; they do not represent evolution tracks.

so long as the pulsational amplitude is also negligible.

The non-linear code also makes the assumption that the models are of homogeneous composition. The static models are homogeneous initially, and plots of the abundance profiles of the last model in each sequence (Figures 4.7 and 4.8) show that, beyond the red edge of the Instability Strip, most of the star remains homogeneous. Again, the changes are within $0.1 R_*$, and can safely be ignored. It should be noted, however, that these models make no allowance for element diffusion, which may be an important aspect of the observed pulsation-metallicity exclusion.

Figure 4.5. Illustrating that energy generation has fallen almost to zero before a radius of $0.1 R_*$ is reached, for masses 1.4, 1.6, 1.8 and $2.0 M_{\odot}$. Each line corresponds to one model in the evolution sequence.

Figure 4.6. Illustrating that energy generation has fallen almost to zero before a radius of $0.1 R_*$ is reached, for masses 2.4, 2.6, 2.8 and $3.0 M_{\odot}$. Each line corresponds to one model in the evolution sequence; only models spanning the Instability Strip are shown.

Figure 4.7. Illustrating that, well beyond the red edge of the Instability Strip, the chemical composition of the models is homogeneous between $0.1 R_*$ and the surface for models of masses 1.4, 1.6, 1.8 and $2.0 M_{\odot}$.

Figure 4.8. Illustrating that, well beyond the red edge of the Instability Strip, the chemical composition of the models is homogeneous between $0.1 R_*$ and the surface for models of masses 2.4, 2.6, 2.8 and $3.0 M_{\odot}$.

CHAPTER 5

LINEAR MODELS

5.1 INTRODUCTION

In the linear theory of stellar pulsation, the non-linear, time-dependent equations of stellar structure are approximated by a set of linear, partial differential equations. The adiabatic theory assumes purely adiabatic heat changes, and is useful for obtaining pulsation periods alone; the non-adiabatic theory allows determination of whether or not a model is stable against pulsation. Both theories can be used to estimate the relative size of radius variations throughout the star for the various modes of pulsation, by consideration of the eigenfunctions, which give variation in radius, $\delta R/R$, against total radius R .

Two programs were used in this study. NEWCONV is a radial, linear, non-adiabatic code, written by Dr J.K. Worrell (Worrell 1985). STURM is a radial, linear, adiabatic code, which relies on the Sturm Sequence method of finding eigenvalue solutions to the matrix problem (see, for example, Castor 1971).

The main idea behind running a series of linear models was to use the eigenfunctions as initial velocity profiles (scaled by an appropriate velocity kick), so as to start the non-linear models off in a state presumably similar to their final (observed) state. This aim was achieved, and the results are discussed in Chapter 6. The linear models provided other useful results as well. The non-adiabatic code NEWCONV gave values for the growth-rates of modes, allowing identification of "blue edges" in the Hertzsprung-Russell Diagram. It also gave periods of the lower modes, which might have been used for mode identification in the non-linear models, but here the adiabatic models (obtained by the Sturm Sequence method) proved more useful, having been produced directly from the zoned initial static model for the non-linear code, rather than from the envelope generation subroutine that was an integral part of Dr Worrell's non-adiabatic code.

5.2 ADIABATIC THEORY: THE STURM SEQUENCE METHOD

5.2.1 The Equations

The difference equations of linear, non-adiabatic theory, as used by Castor (1971) and Worrell (1985), can be written as follows:

$$\frac{d^2 r_i}{dt^2} = -\frac{GM_i}{r_i^2} - 4\pi r_i^2 \frac{P_i - P_{i-1}}{DM2_i} \quad (5.1)$$

$$T_i \frac{dS_i}{dt} = \frac{L_i - L_{i+1}}{DM1_i} \quad (5.2)$$

$$V_i = \frac{4\pi}{3} \frac{r_{i+1}^3 - r_i^3}{DM1_i} \quad (5.3)$$

$$L_i = 2 \frac{4\sigma}{3} (4\pi r_i^2)^2 \frac{T_{i+1}^4 - T_i^4}{(\kappa_i DM1_i + \kappa_{i-1} DM1_{i-1})} \quad (5.4)$$

where the model is divided into N concentric spherical shells, of fixed mass. Zones are numbered 1 to N (where N is the surface), and interfaces from 1 to $N+1$. Mass differences are defined as follows:

$$\text{mass of shell } i: \quad DM1_i = M_{i+1} - M_i \quad (5.5)$$

$$\text{average mass of shell at interface } i: \quad DM2_i = \frac{1}{2} (M_{i+1} - M_{i-1}) \quad (5.6)$$

Radius and luminosity are defined on interfaces; temperature, pressure, density and other state variables are defined within shells.

The difference equations are perturbed, and all terms except those linear in the perturbation are discarded. The time dependence for perturbed quantities is assumed to be $e^{i\omega t}$, where w is complex. Thus we have:

$$-\omega^2 \delta R_i = 4 \frac{GM_i}{R_i^3} \delta R_i - \frac{4\pi R_i^2}{DM2_i} (\delta \rho_i - \delta \rho_{i-1}) \quad (5.7)$$

$$i\omega T_i \delta S_i = \frac{\delta L_i - \delta L_{i-1}}{DM1_{i-1}} \quad (5.8)$$

$$\frac{\delta \rho_i}{\rho_i} = -4\pi \frac{R_{i+1}^2 \delta R_{i+1} - R_i^2 \delta R_i}{DM1_{i-1}} \quad (5.9)$$

$$\begin{aligned} \frac{\delta L_i}{L_i} = & 4 \frac{\delta R_i}{R_i} + \left[4 \frac{T_{i-1}^4}{T_{i-1}^4 - T_i^4} - \frac{\kappa_{i-1} DM1_{i-1} \kappa_{T(i-1)}}{\kappa_{i-1} DM1_{i-1} + \kappa_i DM1_i} \right] \frac{\delta T_{i-1}}{T_{i-1}} \\ & + \left[4 \frac{T_i^4}{T_i^4 - T_{i-1}^4} - \frac{\kappa_i DM1_i \kappa_{T(i)}}{\kappa_{i-1} DM1_{i-1} + \kappa_i DM1_i} \right] \frac{\delta T_i}{T_i} \\ & - \frac{\kappa_{i-1} DM1_{i-1} \kappa_{\rho(i-1)}}{\kappa_{i-1} DM1_{i-1} + \kappa_i DM1_i} \frac{\delta \rho_{i-1}}{\rho_{i-1}} - \frac{\kappa_i DM1_i \kappa_{\rho(i)}}{\kappa_{i-1} DM1_{i-1} + \kappa_i DM1_i} \frac{\delta \rho_i}{\rho_i} \end{aligned} \quad (5.10)$$

The equation of state is given by:

$$\frac{\delta Q_i}{Q_i} = \left[\left[\frac{\partial \log Q}{\partial \log T} \right]_{\rho} \right]_i \frac{\delta S_i}{c_{v(i)}} + \left[\left[\frac{\partial \log Q}{\partial \log \rho} \right]_s \right]_i \frac{\delta \rho_i}{\rho_i} \quad (5.11)$$

where Q can be either P or T.

Equations 5.9, 5.10 and 5.11 are used to eliminate $(\delta P/P)_i$, $(\delta T/T)_i$, $(\delta \rho/\rho)_i$ and $(\delta L/L)_i$ from 5.7 and 5.8. New variables X_i and Y_i are defined by:

$$X_i = \sqrt{DM2_i} \delta R_i \quad (5.12)$$

$$Y_i = T_i \delta S_i \quad (5.13)$$

and, after some algebra and some tidying up, we obtain:

$$\omega^2 X_i = G11_i X_{i-1} + G12_i X_i + G13_i X_{i+1} + G21_i Y_{i-1} + G22_i Y_i \quad (5.14)$$

$$i\omega Y_i = K11_i X_{i-1} + K12_i X_i + K13_i X_{i+1} + K14_i X_{i+2} + K21_i Y_{i-1} + K22_i Y_i + K23_i Y_{i+1} \quad (5.15)$$

where $G11, G12, \dots, K22, K23$ are elements of matrices $\underline{G1}, \underline{G2}, \underline{K1}$ and $\underline{K2}$. In matrix form, these equations become:

$$\omega^2 X = \underline{G1} X + \underline{G2} Y \quad (5.16)$$

$$i\omega Y = \underline{K1} X + \underline{K2} Y \quad (5.17)$$

If Y is set equal to zero, then the adiabatic pulsation equation is obtained:

$$\underline{G1} X = \omega^2 X \quad (5.18)$$

where $\underline{G1}$ is a real, symmetric, tridiagonal matrix, and X is the eigenfunction corresponding to the eigenfrequency ω^2 . This can clearly be treated as a matrix eigenvalue problem for a zoned stellar envelope. Matrix $\underline{G1}$ is of the form:

$$\begin{bmatrix} G1_{1,2} & G1_{1,3} & 0 & \dots & 0 \\ G1_{2,1} & G1_{2,2} & G1_{2,3} & 0 & \dots & : \\ 0 & G1_{3,1} & G1_{3,2} & G1_{3,3} & 0 & \dots & : \\ : & .. & .. & .. & .. & .. & : \\ : & & .. & .. & .. & .. & : \\ : & & & .. & .. & .. & 0 \\ : & & & & G1_{1,1} & G1_{1,2} & G1_{1,3} \\ 0 & \dots & \dots & 0 & G1_{1+1,1} & G1_{1+1,2} \end{bmatrix} \quad (5.19)$$

We need expressions for $G1_{1,1}, G1_{1,2}$ and $G1_{1,3}$, so we now go

back through the algebra of the problem rather more carefully. The linear, adiabatic wave equation (LAW) is just 5.1, which becomes, in finite difference form,

$$-\omega^2 \delta R_i = \frac{2GM_i}{R_i^3} \delta R_i - \frac{4\pi}{DM2_i} \left[2R_i \delta R_i (P_i - P_{i-1}) + R_i^2 (\delta P_i - \delta P_{i-1}) \right] \quad (5.20)$$

We also have the expression for specific volume, V_i :

$$V_i = \frac{1}{\rho_i} = \frac{4\pi}{3} \frac{R_{i+1}^3 - R_i^3}{DM1_i} \quad (5.21)$$

which becomes, in finite difference form,

$$\frac{\delta \rho_i}{\rho_i} = \frac{4\pi \rho_i}{DM1_i} \left[R_i^2 \delta R_i - R_{i+1}^2 \delta R_{i+1} \right] \quad (5.22)$$

Now substitute for $X_i = (DM2_i)^{1/2} \delta r_i$:

$$\frac{\delta \rho_i}{\rho_i} = \frac{\rho_i}{DM1_i} \left[\frac{4\pi R_i^2 X_i}{\sqrt{DM2_i}} - \frac{4\pi R_{i+1}^2 X_{i+1}}{\sqrt{DM2_{i+1}}} \right] \quad (5.23)$$

$$= DR_{i,1} X_i + DR_{i,2} X_{i+1} \quad (5.24)$$

where we have defined:

$$DR_{i,1} = \frac{4\pi R_i^2 \rho_i}{DM1_i \sqrt{DM2_i}} \quad (5.25)$$

and

$$DR_{i,2} = \frac{-4\pi R_{i+1}^2 \rho_i}{DM1_i \sqrt{DM2_{i+1}}} \quad (5.26)$$

The values of δP_i are given by:

$$\delta P_i = \left[\Gamma_1 P \frac{\delta \rho}{\rho} \right]_i \quad (5.27)$$

where Γ_1 is an adiabatic exponent.

Thus we have:

$$\delta P_i = (\Gamma_1 P)_i (DR_{i,1} X_i + DR_{i,2} X_{i+1}) \quad (5.28)$$

$$\delta P_{i-1} = (\Gamma_1 P)_{i-1} (DR_{i-1,1} X_{i-1} + DR_{i-1,2} X_i) \quad (5.29)$$

and part of the finite difference form of the LAWE can be rewritten as:

$$\begin{aligned} & - \frac{4\pi R_i^2}{DM_{2,i}} (\delta P_i - \delta P_{i-1}) \\ &= - \frac{4\pi R_i^2}{DM_{2,i}} \left[-X_{i-1} ((\Gamma_1 P)_{i-1} DR_{i-1,1}) + X_i ((\Gamma_1 P)_i DR_{i,1} \right. \\ & \quad \left. - (\Gamma_1 P)_{i-1} DR_{i-1,2}) + X_{i+1} ((\Gamma_1 P)_i DR_{i,2}) \right] \end{aligned} \quad (5.30)$$

We still require an alternative expression for $(P_i - P_{i-1})$.
If we consider the LAWE to zero order, when $d^2 r / dt^2 = 0$,
then:

$$\frac{GM_i}{R_i^2} = -4\pi R_i^2 \frac{P_i - P_{i-1}}{DM_{2,i}} \quad (5.31)$$

$$\Rightarrow \frac{-4\pi}{DM2_i} 2R_i (P_i - P_{i-1}) \delta R_i = \frac{2GM_i}{R_i^3} \delta R_i \quad (5.32)$$

5.2.2 Elements Of The Matrix

In equation (5.20), the finite difference form of the LAWE, we substitute for X_i , and replace $(P_i - P_{i-1})$ and the δP_i with expressions (5.32), (5.28) and (5.29), to give:

$$\begin{aligned} \omega^2 X_i = & - \frac{4GM_i}{R_i^3} X_i \\ & + \frac{4\pi R_i^2}{\sqrt{DM2_i}} \left[-(\Gamma_1 P)_{i-1} DR_{i-1,1} X_{i-1} + ((\Gamma_1 P)_i DR_{i,1} \right. \\ & \left. - (\Gamma_1 P)_{i-1} DR_{i-1,2}) X_i + (\Gamma_1 P)_i DR_{i,2} X_{i+1} \right] \end{aligned} \quad (5.33)$$

Consider the matrix equation (5.18). This can be written:

$$\omega^2 X_i = G1_{i,1} X_{i-1} + G1_{i,2} X_i + G1_{i,3} X_{i+1} \quad (5.34)$$

and by comparison with (5.33), we have:

$$G1_{i,1} = \frac{-4\pi R_i^2}{\sqrt{DM2_i}} (\Gamma_1 P)_{i-1} DR_{i-1,1} \quad (5.35)$$

$$G1_{i,2} = -\frac{4GM_i}{R_i^3} + \frac{4\pi R_i^2}{\sqrt{DM2_i}} \left[(\Gamma_1 P)_i DR_{i,1} - (\Gamma_1 P)_{i-1} DR_{i-1,2} \right] \quad (5.36)$$

$$G1_{i,3} = \frac{4\pi R_i^2}{\sqrt{DM2_i}} (\Gamma_1 P)_i DR_{i,2} \quad (5.37)$$

For a symmetric matrix, we require:

$$G1_{i+1,1} = G1_{i,3} \quad (5.38)$$

It can easily be shown that this is the case, by expanding, and using the definitions of $DR_{1,1}$ and $DR_{1,2}$. We have:

$$\begin{aligned} G1_{i+1,1} &= -\frac{4\pi R_{i+1}^2}{\sqrt{DM2_{i+2}}} (\Gamma_1 P)_i DR_{i,1} \\ &= \frac{-4^2 \pi^2 R_i^2 R_{i+1}^2 \rho_i (\Gamma_1 P)_i}{DM1_i \sqrt{DM2_i} \sqrt{DM2_{i+1}}} \end{aligned}$$

and

$$\begin{aligned} G1_{i,3} &= \frac{4\pi R_i^2}{\sqrt{DM2_i}} (\Gamma_1 P)_i DR_{i,2} \\ &= \frac{-4^2 \pi^2 R_i^2 R_{i+1}^2 \rho_i (\Gamma_1 P)_i}{DM1_i \sqrt{DM2_i} \sqrt{DM2_{i+1}}} \\ &= G1_{i+1,1} \end{aligned}$$

as required.

5.2.3 Application Of Boundary Conditions

It is now possible to calculate most of the elements of the matrix G1. Exceptions are the values at the top and bottom of the diagonal, which require consideration of the boundary conditions. For clarity, we rewrite the matrix as follows:

$$\underline{\underline{G1}} = \begin{bmatrix} b_1 & -c_1 & 0 & \dots & 0 & \dots & 0 \\ -a_2 & b_2 & -c_2 & 0 & \dots & \dots & \vdots \\ 0 & -a_3 & b_3 & -c_3 & 0 & \dots & \vdots \\ \vdots & \dots & \dots & \dots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & 0 & \dots & \dots & 0 \\ \vdots & \dots & \dots & 0 & -a_n & b_n & -c_n \\ 0 & \dots & \dots & 0 & -a_{n+1} & b_{n+1} & \vdots \end{bmatrix}$$

$X_1 = 0$ is assumed to be a solution of this problem, so that we discard the first row and column of the matrix. Note that in any case the values of a_1 and b_1 are undefined, because $DM2_1$ is unknown.

c_n and a_{n+1} (which are equal) are calculated by approximating $DM2_{n+1}$:

$$DM2_{n+1} = \frac{1}{2}(M_{n+1} - M_n) \quad (5.40)$$

c_{n+1} is zero, subject to the boundary condition $P_{n+1} = 0$.

5.2.4 Procedure For Solution

We have n equations, $i = 2$ to $n+1$, for the n unknowns X_2 to X_{n+1} . These equations are homogeneous; all the right-hand-sides are equal to zero. We solve for the roots of the characteristic equation,

$$|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = 0 \quad (5.41)$$

where $\underline{\underline{I}}$ is the identity matrix and λ is ω^2 , the eigenvalue. The Sturm Sequence method (Castor 1971) allows us to obtain as many eigenvalues as desired (bearing in mind that the finite zoning introduces error into the eigenvalues of higher modes), in sequence (i.e., first the eigenvalue corresponding to the fundamental mode, then the first

harmonic, etc.).

The eigenvector $\underline{X}$ corresponding to each eigenvalue is obtained by setting $X_{n+1} = 1.0$, ignoring the equation for $i = n+1$, and putting $c_n X_{n+1} (= c_n)$ on the right-hand-side of the system:

$$\begin{bmatrix} b_2 - \lambda & -c_2 & 0 & \dots & 0 \\ -a_3 & b_3 - \lambda & -c_3 & 0 & \dots \\ 0 & -a_4 & b_4 - \lambda & -c_4 & 0 \\ \vdots & 0 & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \dots & \vdots \\ \vdots & \dots & 0 & -a_{n-1} & b_{n-1} - \lambda & -c_{n-1} \\ 0 & \dots & 0 & -a_n & b_n - \lambda \end{bmatrix} \begin{bmatrix} X_2 \\ X_3 \\ X_4 \\ \vdots \\ \vdots \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ c_n \end{bmatrix} \quad (5.42)$$

The $n-1$ equations are now inhomogeneous; the system can be solved for $\underline{X}$ by a standard tridiagonal method.

5.2.5 Method

A sequence of polynomials, $\{f_i(\lambda), i = 0, n\}$, is defined by the set of equations:

$$\left. \begin{aligned} f_0(\lambda) &= 1 \\ f_1(\lambda) &= b_2 - \lambda \\ f_{i+1}(\lambda) &= (b_{i+2} - \lambda)f_i(\lambda) - a_{i+2}c_{i+1}f_{i-1}(\lambda) \end{aligned} \right\} \quad (5.43)$$

Note that the subscripts of the a_i , b_i and c_i appear to be out of step with the f_i ; this is because the first row and

column of the matrix were not used, so the first entries correspond to $i = 2$. $f_1(\lambda)$ is the principal minor of order 1 of the matrix $(G1 - \lambda I)$, so $f_n(\lambda)$ is the characteristic polynomial of $G1$. It can be shown that the sequence $f_1(\lambda)$ is a Sturm Sequence, and therefore has the property that the number of sign changes in the sequence of values of the function is equal to the number of eigenvalues of the matrix that are less than λ .

A practical point to note here is that the elements of the matrix $G1$ should be multiplied by a scale factor such that all elements are of order one, otherwise the values of $f_1(\lambda)$ may well exceed the range of floating point exponents. It was also found that the program took longer to run, or failed to converge, if the scaling factor was not correct. A typical value (for cgs units) was 10^5 .

We assume that if λ is zero, the number of sign changes is zero; this is the case if the stellar model is dynamically stable, as indeed it should be. Then, having calculated the (scaled) elements of matrix $G1$, we hunt for an interval (from 0 to x) containing the number of eigenvalues required, using a fairly large step length in x (typically 0.01). Once a suitable value of x has been obtained, the step length is drastically reduced (divided by ten times the number of eigenvalues), and the program goes back to look for the eigenvalues one by one, in sequence, obtaining accurate values by means of a Newton-Raphson procedure. The derivatives of $f_1(\lambda)$ are given by:

$$\left. \begin{aligned} g_0(\lambda) &= 0 \\ g_1(\lambda) &= -1 \\ g_{i+1}(\lambda) &= -f_i(\lambda) + (b_{i+2} - \lambda) g_i(\lambda) - a_{i+2} c_{i+1} g_{i-1}(\lambda) \end{aligned} \right\} (5.44)$$

For each eigenvalue, the eigenvector is found by solving the tridiagonal matrix system, 5.42. This can be written as:

$$\underline{\underline{G'}} \underline{\underline{X}} = \underline{\underline{D}} \quad (5.45)$$

where $\underline{\underline{X}}$ is the eigenvector and $\underline{\underline{D}}$ has only one non-zero entry; the $(n-1)$ st, which is c_n .

The standard method of solving such matrices is to write the tridiagonal matrix as the product of two bidiagonal matrices:

$$\underline{\underline{G'}} = \underline{\underline{V}} \underline{\underline{U}} \quad (5.46)$$

where:

$$\underline{\underline{V}} = \begin{bmatrix} p(2) & 0 & \dots & \dots & \dots & 0 \\ q(3) & p(3) & 0 & \dots & \dots & \vdots \\ 0 & q(4) & p(4) & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & \dots & 0 & q(n-1) & p(n-1) & 0 \\ 0 & \dots & \dots & \dots & 0 & q(n) & p(n) \end{bmatrix}$$

and (5.47)

$$\underline{\underline{U}} = \begin{bmatrix} 1 & r(2) & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & r(3) & 0 & \dots & \dots & \vdots \\ \vdots & & 1 & r(4) & 0 & \dots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & \dots & \dots & 0 & 1 & r(n-1) \end{bmatrix} \quad (5.48)$$

Comparing $\underline{V} \underline{U}$ to $\underline{G}'$, we see that (letting $bb(i)$ be $b_i - \lambda$):

$$bb(2) = p(2) \quad (5.49)$$

$$-a_i = q(i) \quad \text{for } i=3 \text{ to } n \quad (5.50)$$

$$bb(i) = q(i)r(i-1) + p(i) \quad \text{for } i=3 \text{ to } n \quad (5.51)$$

$$-c_i = p(i)r(i) \quad \text{for } i=2 \text{ to } n-1 \quad (5.52)$$

These equations have to be inverted to obtain expressions for the elements of $\underline{V}$ and $\underline{U}$:

$$p(2) = bb(2) \quad (5.53)$$

$$q(i) = -a_i \quad \text{for } i=3 \text{ to } n \quad (5.54)$$

and for $i = 2$ to $n-1$;

$$r(i) = -\frac{c_i}{p(i)} \quad (5.55)$$

$$p(i+1) = bb(i+1) - q(i+1)r(i) \quad (5.56)$$

Now, to solve $\underline{G}'\underline{X} = \underline{D}$, we solve:

$$\underline{V}\underline{Z} = \underline{D} \quad (5.57)$$

then

$$\underline{\underline{U}} \underline{X} = \underline{Z} \quad (5.58)$$

Because $\underline{D}$ is almost all zero, $\underline{Z}$ is also all zero but for $z(n)$, which is given by:

$$z(n) = \frac{c_n}{p(n)} \quad (5.59)$$

Thus we have:

$$\begin{bmatrix} 1 & r(2) & 0 & \dots & 0 \\ 0 & 1 & r(3) & 0 & \dots \\ : & & \ddots & \ddots & \\ : & & & \ddots & \\ 0 & \dots & \dots & 1 & r(n-1) \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ : \\ : \\ : \\ : \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ : \\ : \\ : \\ : \\ : \\ z(n) \end{bmatrix} \quad (5.60)$$

and $\underline{X}$ is given by:

$$\left. \begin{aligned} X_n &= z(n) \\ X_i &= -r(i) X_{i+1} \quad \text{for } i = n-1 \text{ to } 2 \end{aligned} \right\} \quad (5.61)$$

Recall the definition of X_1 in Equation 5.12. Clearly, to obtain the eigenfunction in the desired form, $\delta R/R$, it is necessary to divide X_1 by $\sqrt{DM_2} \delta R_1$. Then the eigenfunction can be scaled such that X_{n+1} is 1.0 and X_1 is 0.0.

The period of pulsation is found from the eigenvalue, λ . We have:

$$\omega = 2\pi\nu \quad (5.62)$$

and

$$\lambda = \omega^2 \quad (5.63)$$

but we must remember to take into account the scale factor; this gives:

$$\lambda = \frac{\omega^2}{\text{scale factor}} \quad (5.64)$$

so period Π is given by:

$$\Pi = \frac{2\pi\sqrt{\text{scale factor}}}{\sqrt{\lambda}} \text{ seconds} \quad (5.65)$$

For the Delta Scuti stars, there should be little difference between adiabatic and non-adiabatic periods; this was borne out using Dr Worrell's code. Hence, the Sturm Sequence results should be very useful for identifying modes of pulsation in the non-linear models.

5.2.6 Results: Periods

The program STURM was designed either to read a model re-zoned and relaxed by TRCVSP, and provide a complete analysis, or to be called as a subroutine (STURMSUB) from TRCVSP once re-zoning was complete, in order to provide a

velocity profile (a scaled eigenfunction) for a selected mode of pulsation. In each case, the static model analysed was of about 50 zones, of equal sound travel time. Non-linear models produced using STURMSUB are described in Chapter 6; here, we consider only the linear pulsation analyses.

For each static evolutionary model, periods and eigenfunctions for the first five modes of pulsation were obtained. Tables of periods, period ratios, and Q values for each static model can be found in Appendix C. These quantities are useful for comparison with real stars: periods and period ratios can be directly observed, and Q values are obtained from colour index calibrations.

The periods range in size from nearly 10 hours for the fundamental modes of the more evolved, more massive stars, to about 30 minutes for the fourth harmonic of the lightest models. The period ratios lie within the following limits:

$$0.756 < P_1/P_0 < 0.767$$

$$0.605 < P_2/P_0 < 0.619$$

$$0.499 < P_3/P_0 < 0.516$$

$$0.425 < P_4/P_0 < 0.443$$

$$0.797 < P_2/P_1 < 0.808$$

$$0.658 < P_3/P_1 < 0.677$$

$$0.556 < P_4/P_1 < 0.582$$

$$0.825 < P_3/P_2 < 0.840$$

$$0.698 < P_4/P_2 < 0.722$$

$$0.846 < P_4/P_3 < 0.862$$

The Q values lie within the following limits, where values are given in days:

$$0.0324 < Q_0 < 0.0349$$

$$0.0245 < Q_1 < 0.0265$$

$$0.0197 < Q_2 < 0.0211$$

$$0.0165 < Q_3 < 0.0174$$

$$0.0142 < Q_4 < 0.0148$$

Figure 5.1 illustrates the slight variation in Q_0 with period P_0 for a range of masses within the Instability Strip.

Figure 5.1. Illustrating the slight variation in Q for masses $1.4M_{\odot}$ (triangles), $2.0M_{\odot}$ (squares) and $3.0M_{\odot}$ (stars).

Figure 5.2. Eigenfunctions of the first five modes of pulsation for four different models, produced by STURM.

5.2.7 Results: Eigenfunctions

In Figure 5.2, the first five eigenfunctions for four different static models within the Instability Strip are shown. Variation in radius, $\delta R/R$, is plotted against zone number; the zones are those produced by the non-linear code when re-zoning the static model. Note the similarity of the sets of eigenfunctions; all fall to zero within zone 10, and all feature a bump (marked by an arrow in Figure 5.2a) which is a result of the zoning.

5.3 THE WORRELL CODE, NEWCONV

5.3.1 Introduction

Dr John Worrell kindly made available his linear, non-adiabatic pulsation code, based on that outlined by Castor (1971) and described in great detail in his thesis (Worrell 1985). The program, NEWCONV, was written specifically for Worrell's work on giant stars, and as such was not entirely suitable for the study of the more compact Delta Scuti stars. However, some useful results were obtained from it, of which the most important was the identification of blue edges of the lower pulsation modes in the Hertzsprung-Russell Diagram. Blue edges cannot be obtained from a purely adiabatic analysis, so it was impossible to find them using STURM.

NEWCONV produces its own static equilibrium model, given mass, luminosity, effective temperature, chemical composition and a table of opacities. It would have been preferable to feed in the same static model as used by TRCVSP and STURM, but the procedure for creating a static model was such an integral part of the coding that this was impossible. Therefore, some comments on Worrell's static model are made in the next section.

5.3.2 The Equilibrium Model

The program NEWCONV creates a static model by integrating inwards until $T_i \geq 2 \times 10^6 \text{ K}$ and one of the following also applies:

$$M_i/M_* \leq 0.9$$

or

$$R_i/R_* \leq 0.02 \text{ and } M_i/M_* \leq 0.95$$

or

$$P_i \geq 10^{14}.$$

The eigenfunction can be listed against a pressure variable or a mass variable, but not against radius or density (at least, not without changing the independent variable within NEWCONV itself, which would be a major task). In principle, the eigenfunction could be fitted by a polynomial in a pressure variable, then the polynomial could be used within TRCVSP to provide a scaled initial velocity profile. It was hoped, however, that STURM would prove adequate to provide the velocity profile, since, using STURM, it would only be necessary to return the

eigenfunction to the main program TRCVSP, rather than use a polynomial fit.

5.3.3 Results: Periods

Although NEWCONV uses a Sturm Sequence routine to find approximate starting points for the eigenvalue iteration, it does not necessarily find the modes of pulsation of the star in a sequential order. Therefore, it was necessary to examine the eigenfunctions of each period found, in order to identify the mode. In most cases, the fundamental and first four harmonics were indeed found, and quite often in sequence. Figure 5.3 shows eigenfunctions for the model m20e05. Note the bump near the surface, probably caused by the second ionisation of helium.

Output files (eigenfunctions, work functions, etc.) were re-labelled with the correct mode. In some cases, particularly for the lowest masses in the survey, periods were found to correspond to very high harmonics; an example is m14e13, illustrated in Figure 5.4. These were discarded; hence the occasional gaps in the tables of periods, which can be found in Appendix D.

Despite the differences in the static model and the non-adiabatic rather than adiabatic analysis, the periods found by NEWCONV differ little from the STURM results. This is illustrated by Figure 5.5, which plots fundamental mode periods for models of the same M , L and T_e against each other. Figure 5.6 shows the same result for higher harmonics.

Figure 5.3. Eigenfunctions of the first five modes of pulsation for m20e05, produced by NEWCONV.

Figure 5.4. Eigenfunctions of the first five modes produced by NEWCONV for m14e13. Note that the program has converged to unwanted high harmonics.

Figure 5.5. A comparison of the periods of the fundamental modes of models of masses $1.4M_{\odot}$, $2.0M_{\odot}$ and $3.0M_{\odot}$. The straight line corresponds to equal periods.

5.3.4 Results: Phase Lag

It is possible to calculate the phase lag between maximum luminosity and velocity at the stellar surface using purely linear pulsation analysis (see Cox 1980, Chapter 11). The main cause of the phase lag is the movement of the narrow hydrogen ionisation zone through the star as it pulsates. Perhaps coincidentally, stars within the Instability Strip have exactly the internal structure required to cause a 90° phase lag: hotter stars have little or no phase lag, due to the proximity of the hydrogen ionisation zone to the surface, and cooler stars have a

Figure 5.6. A comparison of the periods of the higher modes of models of masses $1.4M_{\odot}$, $2.0M_{\odot}$ and $3.0M_{\odot}$. The straight line corresponds to equal periods.

phase lag of the order of 180° (i.e., maximum luminosity occurring at maximum radius), since the hydrogen ionisation zone is so deep within the envelope that the change in the mass distribution around it as it moves is negligible. The second helium ionisation zone, which drives the pulsation, introduces only a very slight phase lag.

In Appendix D, the values of the phase lags for each mode of each model are listed. Note that from these values, it is possible to determine approximately which models lie within the Instability Strip. Those with phase lags in excess of about 130° lie beyond the red edge, and those with phase lags less than about 50° lie to the hotter side of the blue edge.

5.3.5 Results: Blue Edges

In non-adiabatic theory, the stability coefficient is defined as the logarithm of the factor by which pulsation energy increases in a period. Thus:

$$\eta = -2 \operatorname{Im}(\omega) \Pi \quad (5.66)$$

since the pulsation energy increase in one period is given by:

$$|e^{i\omega\Pi}|^2 = |e^{-\operatorname{Im}(\omega)\Pi}|^2 \quad (5.67)$$

It follows that:

$$\eta = -4\pi \frac{\text{Im}(\omega)}{\text{Re}(\omega)} \quad (5.68)$$

which is the form used to calculate η in NEWCONV. This gives a total stability coefficient associated with a particular mode of pulsation. Negative values of η indicate pulsational stability; positive values indicate instability. The "blue edge" of the Instability Strip is where the value of η is zero. Values of η for the fundamental and first four harmonics can be found in Appendix D.

There is an option in NEWCONV that allows the identification of the effective temperature associated with $\eta = 0$ for the first eigenvalue encountered (normally the fundamental) for a sequence of models of constant mass and luminosity. For higher modes, it was necessary to interpolate in luminosity and effective temperature for each evolution sequence. The results, with real stars (data from Appendix E) plotted for comparison, are shown in Figure 5.7. The evolution tracks of the KWHMP models are also shown. Note that a large number of the real stars are situated to the bluer side of the fundamental blue edge, and would therefore be expected to be pulsating in a higher harmonic.

5.3.6 Results: Work Functions

It is also possible to calculate values of the stability coefficient per zone, averaged over one period. This allows the identification of regions within the envelope that are driving or damping the pulsations. The

Figure 5.7. Blue edges of the fundamental mode (0) and first four harmonics, showing a number of real Delta Scuti stars (asterisks) and the evolution tracks of the static models (with mass indicated in solar units).

work functions for a sequence of models are shown in Figure 5.8, plotted against $\log P$. Notice that the work does not fall to zero on the right hand side of the diagram (the stellar surface) for the higher harmonics. In fact it should do, and this feature makes the results somewhat suspect. The driving peaks correspond to the (HI, HeI) and HeII ionisation regions; clearly, driving by the former is important in these particular models.

5.4 COMPARISONS

Figure 5.8. Work functions for several models in the $2.0M_{\odot}$ sequence. Note the driving due to both (HI, HeI) and HeII regions, particularly for the higher modes. In each case the fourth harmonic is at the top at the right hand edge, and the fundamental at the bottom.

5.4.1 Comparison With Real Stars

Before proceeding with a non-linear analysis relying on the eigenfunctions produced here, it is necessary to pause and consider how well these linear models correspond to real Delta Scuti stars. If the linear models bore no resemblance to the real stars, there would be little point in assuming that an initial velocity profile based on these eigenfunctions would produce anything similar to the real stars.

We have chosen models with temperatures and luminosities appropriate to the observed positions of Delta Scuti stars on the Hertzsprung-Russell Diagram. We have assumed an early stage of evolution, and hence a mass range of 1.4 to 3.0 $M_{\odot}$. If the real stars have reached this position on the Hertzsprung-Russell Diagram by a longer route, the assumption of such a mass range might be unjustified. In that case, we would find that periods and Q values for the models would disagree with those observed.

The reported periods range from 20 minutes to several hours. This range is encompassed by the linear results. It is also observed that longer periods correspond to more luminous (and, under our assumptions, more evolved) stars. Glancing at the lists of periods in appendices C and D, we see that, indeed, the more evolved models have the longer periods.

Most reported period ratios fall within the ranges obtained for the linear models, and others may correspond to higher harmonics. Given that Delta Scuti stars are unstable to many small oscillations, it would not seem unreasonable to presume that non-radial modes may also be present, as claimed in the literature.

For the most part, the Q values derived from observations also fall within the limits defined by the linear models. Exceptions are those for which Q is unusually large; some of these misfits can be found amongst the data listed in Appendix E.

It is not easy to compare the phase lags produced by NEWCONV with those of real stars, for several reasons. Firstly, nonlinear effects cause the actual phase lag to be fairly independent of the position of a star on the Instability Strip. Secondly, both radial velocity and light curves are required for determining phase lag, and in general only the light curve is available. Thirdly, the non-repetitive nature of many Delta Scuti stars prevents accurate determination of the phase lag. Where values are available, phase lags are small. Breger et al. (1976) gave a range 0.06 to 0.12 periods, which is approximately 20° to 40° . This is not inconsistent with the results from NEWCONV.

In Figure 5.7, the blue edges produced by NEWCONV were plotted on a Hertzsprung-Russell Diagram, showing also the evolution sequences from which the models were taken, and some real stars. A substantial number of the stars lie

bluewards of the fundamental blue edge; models on the left of that edge have negative growth rates, η_0 , for the fundamental mode, and would thus be unlikely to pulsate in it. Hence, it is probable that these stars are pulsating in higher harmonics.

Only a very few of the stars listed in Appendix E lie beyond the blue edges of the third and fourth harmonics. This may be due to an absence of stars pulsating in higher harmonics, or to the pulsational amplitude of such stars being below the current observational limit. Whatever the reason for the dearth, it seems good grounds for restricting the non-linear analysis to the first five modes of pulsation.

Figure 5.9. Evolution tracks in a period-temperature diagram, labelled with mass. Asterisks with 10 points correspond to models; those with 5 points to real stars.

Figure 5.10. Evolution tracks in a period-temperature diagram, for the first and second harmonics. Asterisks with 10 points correspond to models; those with 5 points to real stars.

Period-Surface Gravity Relations

Figure 5.11. Period-gravity relations for a selection of models. Triangles correspond to the fundamental mode, squares to the second harmonic, and pentagons to the fourth.

Having established that the models and the real stars have the same range of periods, we can make some further attempts to estimate the masses of the real stars. In Figure 5.9, stars from Appendix E are plotted on a ($\log T_e, \log P_0$) plane, along with the tracks produced by the evolution sequences of constant mass. The limits we have chosen, 1.4 to 3.0 $M_\odot$, do seem to include most of the real stars. When first and second harmonics are considered, as in Figure 5.10, nearly all the real stars can be accommodated.

In Figure 5.11, the first five modes of a selection of models (specifically, those which were analysed using STURMSUB) are plotted on a ($\log g, \log P$) plane. In Figure 5.12, the modes are split up, and real stars thought to be pulsating in the appropriate mode (data from Appendix E) are

Figure 5.12. Period-gravity relations for a selection of models, compared to real stars (asterisks) thought to be pulsating in that particular mode.

also plotted. It can be seen that, for the most part, the real stars follow the log P-log g relation of the models. This suggests that the stars are pulsating in these particular modes.

In Chapter 1 (Section 1.2.4), the comment was made that the log R-log P relation for Delta Scuti stars was the same as that for population II (Meylan & Burki 1986). Burki (1986) gave the following linear regressions (for period P in days and radius in solar radii):

for classical Cepheids:

$$P < 10 \text{ days: } \log R = 1.16 + 0.72 \log P \quad (5.69)$$

$$P > 10 \text{ days: } \log R = 1.03 + 0.77 \log P \quad (5.70)$$

for population II Cepheids:

$$\log R = 0.87 + 0.54 \log P \quad (5.71)$$

Figure 5.13 shows log R-log P relations for the same selection of models as Figure 5.16. Linear regression gave the following relationships (same units as before):

$$F0: \log R = (1.193 \pm 0.008) + (0.777 \pm 0.010) \log P_0 \quad (5.72)$$

$$H1: \log R = (1.286 \pm 0.009) + (0.781 \pm 0.009) \log P_1 \quad (5.73)$$

$$H2: \log R = (1.355 \pm 0.010) + (0.779 \pm 0.009) \log P_2 \quad (5.74)$$

$$H3: \log R = (1.423 \pm 0.010) + (0.785 \pm 0.009) \log P_3 \quad (5.75)$$

$$H4: \log R = (1.465 \pm 0.016) + (0.778 \pm 0.012) \log P_4 \quad (5.76)$$

Figure 5.13. Period-radius relations for a selection of models. Triangles correspond to the fundamental mode, squares to the second harmonic, and pentagons to the fourth.

From these results, it would appear that the models follow the classical Cepheid relation rather than the population II star relation - as indeed might be expected, since the models are of population I composition.

One last parameter that might be considered is age. In Figure 5.14, a tentative age calibration scheme is presented; tracks of constant mass are plotted on a ($\log A, \log P$) plane. Bearing in mind the shortcuts taken by KWHMP, it is clear that this can only give a very rough approximation. However, it does illustrate that, with accurate observations and detailed stellar models, age

Figure 5.14. Evolution tracks, labelled with mass (in solar units), in an age-period diagram. The level parts correspond to the (brief) time spent in the Instability Strip.

Figure 5.15. A comparison of the blue edges produced by NEWCONV (labelled 0,1,2,3,4) with those produced by Stellingwerf (1979) for the same five modes. Stellingwerf's fourth harmonic blue edge curves right round to become a red edge also.

determination may not be an unattainable aspiration.

5.4.2 Comparison With Previous Models

Insofar as published linear studies all claim agreement with observed periods, period ratios, and Q values, as does this work, it must be concluded that the results from this work agree with the published surveys. The periods encompass the same range, and increase for more evolved models, as noted by other authors. Growth rates are of the same order of magnitude, and are larger for the higher harmonics, as has been observed elsewhere (Cox 1980). However, it does not follow from this that the real stars should pulsate in the higher harmonics; non-linear effects are important here.

No attempt is made here to compare a specific model directly with one of the same parameters (mass, luminosity, effective temperature) produced and analysed by another author. Such a study would illustrate only the differences between the equilibrium models, and none are claimed to be definitive.

In Figure 5.15, the blue edges given by NEWCONV are compared to those produced by Stellingwerf (1979). Here we do find some differences, due no doubt in part at least to differences in the grids of static models. Notice in particular that the curious feature found by Stellingwerf, the curving of the blue edges of the higher harmonics right round to become red edges, was not reproduced by NEWCONV. All five edges given by Stellingwerf lie within the second

harmonic edge given by NEWCONV. However, since all published blue edges differ to some extent, this result was not seen as proof of any great difference between the models.

5.4.3 Conclusions

We conclude that the models produced by STURM and NEWCONV agree with each other and with earlier models by other authors, and that they represent the observations quite well. We can, therefore, conclude that the grid of static models is of the appropriate evolutionary status, composition, and position in the Hertzsprung-Russell Diagram to represent Delta Scuti stars.

Given that STURM actually uses the same full stellar model as TRCVSP, while NEWCONV just integrates an envelope, and given that the results from the two programs are very similar, it seemed logical to use eigenfunctions produced by STURM to scale the initial velocity kick imposed upon the non-linear models. For the non-linear analysis, static models beyond the blue edge defined by the fourth harmonic and beyond the red edge defined by observation ($\log T_e$ about 3.8) were neglected; only models within the Instability Strip were analysed.

CHAPTER 6

NON-LINEAR MODELS

6.1 INTRODUCTION

In the previous five chapters, we have sought to establish the nature of the Delta Scuti stars, to become familiar with their properties, and to provide a grid of static models of known intrinsic radial pulsation periods, suitable for non-linear analysis by the method outlined in Chapter 3. We have seen that the observed periods are matched by those of the linear models, and that the adiabatic and non-adiabatic methods give very similar periods. Therefore, the linear, adiabatic routine STURM was modified such that it could be called directly by the non-linear code, returning the eigenfunction for the desired mode of pulsation, which was then used to scale the initial velocity profile. The results from this approach are presented in section 6.4.

Earlier work was carried out using a profile varying as some simple power of the radius, both for comparison with the STURMSUB results and as a direct extension of the work by previous authors on cepheids of various types. Results from this approach are presented in section 6.3.

In section 6.5, depletion of helium in the envelope is considered, with reference to a small selection of models analysed using STURMSUB. Some conclusions on the source of the driving of the pulsations are drawn in section 6.6. In section 6.7, some comments are made on how the models compare to real stars in general and to some stars in particular.

For all the models, it was necessary to find some method of analysing the light and velocity curves to determine periods and amplitudes, in much the same way (although with greater certainty) as for real stars. The classical Fourier Transform method, as implemented by Dr Jeffery within the local version of DIPSO, proved satisfactory for finding periods but, due to the multiperiodic nature of the light curves, did not give useful amplitudes. The method is described in the next section.

6.2 FOURIER ANALYSIS

The non-linear pulsation code generates light and velocity curves in the form of discrete samples at nearly equal intervals in time. The sampling frequency is much higher than an observer would be able to obtain for a real star, and is not interrupted by periods of daylight, the setting of the star, cloud, etc. It was, therefore, hoped that Fourier analysis of the light (or velocity) curve would yield periods and amplitudes with some accuracy.

The method used, within the local version of DIPSO, was to calculate the classical periodogram of the data, using a form of the discrete Fourier Transform appropriate for unequally spaced data (described in Deeming (1975) and Kurtz (1985)). The program is referred to as FTRAN in subsequent sections. For a monoperiodic, sinusoidal oscillation, the peak produced in the classical periodogram has its x-coordinate equal to the frequency of the oscillation, and y-coordinate, the power P, equal to:

$$P = \frac{A^2}{4} \quad (6.1)$$

where A is the semi-amplitude of the oscillation. Hence, the period and amplitude of the oscillation can be obtained from the periodogram.

It is always important, when using the classical periodogram, to examine the window function of the data. Figure 6.1 shows the type of window function associated with all of the periodograms illustrated in subsequent sections. Because the sampling frequency is so high, the side-lobes associated with the main peaks in the periodogram are very small and close together, so that the problem of distinguishing true peaks from side-lobes does not arise. Furthermore, since the "observation" of the models is not limited, spurious periodicities (such as one day) associated with real observing runs do not complicate the periodograms.

Figure 6.1. The form of window function typical of the classical Fourier transforms carried out in this chapter.

Therefore, periodicities present in the light curve can be read from the periodogram. Unfortunately, multiples (in frequency space) and differences (beats) of these periodicities also appear, and although they can usually be identified and discounted, their presence makes it impossible to determine amplitudes from the periodogram. The less sinusoidal the wave form in the light curve, the more power is distributed among beats and overtones of the frequencies that are really present. Equation 6.1 becomes less and less accurate.

Given the complicated nature of the light curves produced for the Delta Scuti models, it proved necessary to estimate amplitudes from the light and velocity curves themselves; these amplitudes were as much as a factor of ten larger than the amplitude suggested by the main peak in the periodogram.

6.3 SIMPLE VELOCITY PROFILES

In previous non-linear studies, authors have imposed velocity profiles corresponding to a simple equation in scaled radius. For the fundamental mode, Stobie (1969) suggested a profile of the form:

$$u \propto -\xi^1 \quad (6.2)$$

where u is velocity and ξ is the fractional radius. In his work on W Virginis stars, Bridger (1983) used the profile:

$$u \propto -\xi^5 \quad (6.3)$$

and a kick of 18 km/s. The bulk of the work in this section was done using this profile and a kick of surface amplitude 10 km/s. Some experiments were carried out with other profiles, and for one model, no initial velocity was imposed.

Allowing the pulsation to build up from noise alone produced a light curve with many high harmonics present (Figure 6.2); given that the growth times of the high harmonics produced by NEWCONV were much shorter than those

of the lowest modes, this result is not surprising. Each segment of light curve in Figure 6.2 corresponds to about one hour of CPU time. The amplitude can be estimated from the light curve as about 0.001 magnitudes, less than that of the lowest amplitude stars. In Figure 6.3, the periodograms for each of the four segments are shown. These feature many peaks, with very small amplitudes. Many hours of CPU time would be required before the high harmonics died away and the lowest modes (e.g., the fundamental, at 0.342 cycles per hour) replaced them. Since errors in the calculations would become important over very long runs, it is clear that this method is neither desirable nor practical.

Also shown in Figure 6.3 is a plot of the maximum kinetic energy per period (ergs divided by 10^{30}) against period number. There is no obvious overall increase or decrease. The change in maximum kinetic energy per period is usually taken to be an indication as to whether the pulsation is growing or decaying; a level graph would indicate that the pulsation had reached its limiting amplitude. Here, it probably suggests that the model is very close to stability.

In Figure 6.4, plots of scaled radius to the fifth and eighth powers are plotted against zone for four different models. The bump in the profile at approximately zone 30 is similar to that observed in the STURM eigenfunctions; clearly, this is a product of the zoning,

Models treated using the scaled radius to the fifth power are henceforth labelled "k10r5", where "k10" indicates the 10 km/s kick, and "r5" the profile. Similarly, "r8" indicates scaled radius to the eighth power, and different values after the "k" indicate initial kick, in km/s.

Table 6.1 lists all the models that were analysed, with the periods and amplitudes of the main features in the periodogram of the final run of each model. Note that, as with all the models, the amplitudes taken from the periodograms are far smaller than the amplitude an observer would interpret from the light curve. Estimates of the light and velocity amplitudes from the curves are also given in Table 6.1. Most models were run for eight hours, since there was no lack of available CPU time on the St. Andrews Observatory MicroVAX. The length of each run was in effect determined by the FTRAN routine, which can accept a limited number of data points. Each segment was analysed separately, and the sequences of periodograms compared, to find out whether the amplitudes of particular frequencies were growing or shrinking.

Typical curves of light, velocity and radius variations for a range of models in the "k10r5" series, spanning the Instability Strip, are illustrated in Figures 6.5 to 6.12. In each case, the curves are taken from near the end of long runs.

The light curves show evidence of high harmonics not present in the velocity or radius curves; these are probably not real. All three curves are basically sinusoidal, and

modulated. The same form is repeated for every model: two large peaks, followed by a smaller one. Estimates of amplitudes from the light curves suggest that the light amplitude depends on mass, since the larger masses have larger amplitudes. These range from about $0^m.25$ for m16e03 to about $0^m.5$ for m30e13. The velocity amplitude does not show the same progression; for each model it varies between 15 and 30 km/s.

Of course, the critical question is whether or not these curves represent the final, settled state of the model, and hence of the star. If we examine the light curves for the whole eight hour run, we see (Figures 6.13 to 6.16) that the pulsation is multiply periodic and, to some extent, decaying. The maximum kinetic energy per period (where the period is that interpreted by TRCVSP) decreases slowly but continuously over the runs, for every model, and this should also be an indicator that the pulsation is decaying. In Figure 6.17, maximum kinetic energy is plotted against period number for the same series of models. The decrease is evident, and amounts to some 25 percent over the eight hour run. Some of the plots (particularly the last two) are clearly curved, and perhaps would level off eventually.

In Figures 6.18 to 6.25, the periodograms for the sequences of runs for the same set of models are shown. The periodograms of earlier segments show many subsidiary peaks, some of which correspond to high harmonics, and some to aliases of the main peak, which can invariably be identified as the fundamental mode of radial pulsation. The first

harmonic is also important, with amplitude approximately half that of the main peak. As the run proceeds, the high frequency peaks fade away, and in every case the fundamental and first harmonic remain as the most important peaks, with a lesser peak to the left corresponding to the beat between these two, and lesser peaks to the right, mostly corresponding to multiples of the fundamental frequency.

If the amplitude of the the main peak given by the periodogram is plotted against period, the relation shown in the first part of Figure 6.26 is produced. The amplitudes are of the same order as those of the real stars, but the relation does not resemble the period-amplitude relation for real stars, shown in the lower half of Figure 6.26. Besides, as we have seen, the amplitudes estimated from the light curves are much larger than those produced by the Fourier analysis, due to the leakage of power into other modes.

In order to try and identify the sources of driving in the models, work functions for sequences of periods towards the end of each run were examined. Those for the same set of models as above are plotted in Figures 6.27 and 6.28. Another indication that the pulsation has not settled down is the variation of the work functions from period to period. Sometimes, the work done by a zone is negative, indicating damping; in subsequent periods it may be positive, indicating driving. However, some major features can be identified. Between zones 25 and 30 there is a large amount of damping, which cuts off sharply at its outer limit. Towards the surface of the star, between zones 42

and 45, there is a source of driving. If the work functions are plotted against temperature instead of zone, as in Figure 6.29, this can be identified as due to the second helium ionisation region, and the (HeI,HI) ionisation region.

In Figures 6.30 and 6.31, the pressure-volume variation over two or three periods is shown for the outermost zones of two of the models. Zones 39, 40 and 41 have clockwise loops, indicating damping; the rest (42-46) have anticlockwise loops, indicating driving. The loops are very irregular; this is another indication that the pulsation has not settled down.

The loops for zones 25-30 are not shown; they are similar in appearance to those for zones 39 and 40.

Figures 6.32 and 6.33 show the ionisation profiles for the same two models. Helium is equally split between its singly and doubly ionised forms at zone 42 or 43; hydrogen ionisation takes place at zone 45. Carbon, nitrogen and oxygen share a similar feature between zones 15 and 30. For m18e08, the damping feature cut off sharply at zone 29, which corresponds (in Figure 6.32) to CV beginning to ionise to CVI, and to NVI beginning to ionise to NVII. For m24e16, the feature ends sharply at zone 30, which (in Figure 6.33) again corresponds to the beginning of the transition between CV and CVI. No doubt the coincidence of ionisation edges of the more abundant metals C, N and O in the same region is responsible for the damping feature in the work function.

Yet another indication that the pulsation has not settled down is that the velocity profiles from period to period (taken when the model is contracting through its equilibrium radius) are not the same. Figures 6.34 and 6.35 show sequences of velocity profiles for the set of models. From the centre to about zone 25 the profiles are constant, but beyond that they vary. This is illustrated for two of the models in Figure 6.36, which shows complete velocity histories for (approximately) three periods. The velocity variations for each zone are displaced vertically, so as not to overlap. In the interior, the variations are constant, but towards the surface, the variation differs from period to period.

6.4 STURM PROFILES

Adapting the linear, adiabatic, Sturm Sequence program STURM into a subroutine to provide an initial velocity profile was straightforward. Once the non-linear code, TRCVSP, has read in the equilibrium model, re-zoned it, and relaxed the re-zoned model back onto the difference equations, it generates an initial velocity profile from parameters which are set in DATA statements at the beginning of the program. These dictate the magnitude of the kick (normally 10 km/s) and the power to which the scaled radius is raised. In fact, provision is made for the initial velocity profile to be a polynomial of several terms in scaled radius, but that option was not used in this work.

Instead of this, the modified TRCVSP calls the subroutine STURMSUB, transferring the quantities required to calculate the tridiagonal matrix. STURMSUB proceeds to calculate eigenfunctions until it has reached the mode of pulsation required, then returns that eigenfunction to TRCVSP. Since the eigenfunction is listed for the same gridpoints throughout the star as all the other variables, it is then simply a matter of multiplying the value of the eigenfunction at each point by the velocity kick to obtain the initial velocity profile. The eigenfunction and period are written out for examination, so that models not previously analysed using STURM can be compared to those that have been.

Henceforth, the following labels are used. "Stm00" denotes a model for which the the initial velocity kick was in the fundamental mode, "stm01" the first harmonic, "stm02" the second, "stm03" the third, and "stm04" the fourth harmonic. Examples of the eigenfunctions can be found in Chapter 5; see Figure 5.2.

Since each different static model required five different runs, one per mode, it was no longer practical for each run to be of eight hours CPU time. Moreover, it was hoped that these models would prove more settled, and not require long runs to determine trends in periods and amplitudes. To check this assumption, one model was run for eight hours, for the fundamental mode and first harmonic. The full light curves are shown in Figure 6.37. Apart from an initial decrease, these light curves do appear to be of constant amplitude, and they certainly show less decrease

than the "k10r5" series (Figures 6.13 to 6.16), and fewer modulations and harmonics. The periodograms (Figures 6.38 and 6.39), excepting perhaps the first in each sequence, are also quite consistent. Again, however, the maximum kinetic energy per period shows a small but relentless decrease (Figure 6.40).

If the maximum kinetic energy per period for m20e12_stm00 is compared to that for m20e14_k10r5 (see Figure 6.17), we see that the starting point for the former is about half that for the latter, and that the decrease of about 20 percent over eight hours for the former is somewhat less than that of the latter (25 percent). The starting point for m20e12_stm01 is a factor of ten smaller than that for m20e12_stm00, and the decrease over the eight hour run is nearly 50 percent. Neither shows any sign of levelling off.

It is, therefore, not possible to claim that these models are more settled than the "k10r5" series. Nevertheless, a series of models was produced for each of the five different initial velocity profiles. Some of these are presented in Figures 6.41 to 6.88. Each figure comprises segments of the light, velocity and radius variations, with the periodogram of the light curve, and a plot of the decrease of maximum kinetic energy per period. Table 6.2 lists the period of the dominant peak in the periodogram for the full set of models, together with the light amplitude given by the periodogram, and estimates of the amplitudes of the light and velocity curves. As before, the amplitude estimated from the light curve is much larger

than that obtained by Fourier analysis.

The higher the mode of pulsation of the initial velocity profile, the more sinusoidal is the appearance of the variation in radius, velocity and (to a lesser extent) light. In each case, the dominant peak in the periodogram corresponds to the mode of pulsation of the kick, but lesser peaks are also present, and the oscillations are clearly modulated.

There seems to be little or no dependence of the amplitude of pulsation on the mass or evolutionary status of the models, but some dependence on the mode of pulsation. The higher the mode, the lower the amplitude. The form of the velocity curve also depends to some extent on the mode of pulsation. The fundamental mode has a distinctive pattern of two small oscillations followed by one or two large oscillations. The first harmonic is usually characterised by three small oscillations followed by one larger one, or sometimes by an alternation of two small and two large. If the (probably spurious) high harmonics were smoothed out of the light curve, its form would be similar to that of the velocity curve.

In all cases, the maximum kinetic energy per period decreases with time. The average value is lowest for the highest modes. In very few cases is there any sign that it is levelling off. It must, therefore, be assumed that the pulsations have not settled down to their final state.

As for the "k10r5" models, the work functions for a sequence of periods vary, although those for the higher modes are more repetitive. Figure 6.89 shows sequences of work functions for the fundamental mode of four of the models. Those for the higher modes are illustrated in Figures 6.90 to 6.93.

The work functions for the fundamental mode have the same general features as those for the "k10r5" models, with a damping region ending sharply near zone 30, and driving in the outermost zones, associated with the HeII and (HeI,HI) ionisation regions. This pattern is repeated for the first harmonic, but beyond that, the main source of damping shifts to between zones 35 and 41. This can be followed in the pressure-volume loops (Figures 6.94 to 6.98) for the m20e12 model. Zones 42 to 46 have anticlockwise, driving loops, whereas the loops for zones 39 to 41 are clockwise, damping loops. As the mode of pulsation increases, the loop for zone 41 becomes fatter, and that for zone 42 becomes narrower. It is also noticeable that the loops for zones 44 to 46 become more repetitive from period to period as the mode of pulsation increases.

The ionisation profiles of the most important elements for model m20e12 are plotted in Figure 6.99. Both N and O have several ionisation edges between zones 35 and 41, which may be responsible for the damping. The second ionisation of helium takes place between zones 42 and 43, and beyond that is the (HeI,HI) ionisation region, at zone 45. It must be remembered that, to soften shocks, the pulsation code introduces a fictitious pressure which has the effect of

smearing out the H and He ionisation regions, so it is likely that a contribution at least to the driving seen in zones 44 outwards is made by HeII.

Sequences of velocity profiles for the five modes of four different models are shown in Figures 6.100 to 6.104. They become slightly more regular for the higher modes. Until approximately zone 30, they follow the imposed velocity profile, then become more erratic. The oscillation is larger further into the star for the lowest modes, as can be seen in Figures 6.105 to 6.107, which show complete velocity histories for several periods for the five modes for model m20e12.

6.4.1 A Smaller Kick

Giving an initial kick of 10 km/s invariably resulted in models with decreasing maximum kinetic energy per period and with light and velocity amplitudes much larger than those observed for the majority of Delta Scuti stars (although, for light at least, similar to the high amplitude stars, such as δ Scuti itself). Therefore, some attempts were made using a kick of only 3 km/s. We have already seen that a model with no kick exhibits many different pulsation frequencies, and an amplitude much smaller than that of an average Delta Scuti star. It was hoped that a kick of 3 km/s would be just enough to avoid all the high frequencies, without being so much as to generate excess kinetic energy.

Figures 6.108 to 6.116 present the light, velocity and radius variations, periodogram, and maximum kinetic energy per period for the first three modes of pulsation of models m16e03, m18e09 and m20e12, chosen because they lie amidst the real Delta Scuti stars on the Hertzsprung-Russell Diagram. Table 6.3 lists the period and amplitude of the dominant peak in the periodogram of the light curve of each star, together with estimates of the amplitudes of the light and velocity curves.

The figures can be compared directly to Figures 6.41 to 6.43 (for m16e03), Figures 6.61 to 6.63 (for m18e09), and 6.71 (for m20e12_stm02). In every case, the maximum kinetic energy per period is much less for these models, and the gradient of the decrease less steep. For m18e09_stm01, the curve even levels off.

For m16e03 in particular, it is clear that the high frequency oscillations were not suppressed. However, for all the models, the velocity curves are very similar in form to those with the larger initial kick. The characteristics of the fundamental mode, first harmonic and second harmonic persist, even though the amplitudes are much smaller than before and, indeed, only slightly higher than those of the real stars.

Evidence of the unsettled nature of the pulsation can, however, still be found in the work functions (Figures 6.117 to 6.119), the pressure-volume loops (Figures 6.120 to 6.122), and the velocity profiles (Figures 6.123 to 6.125). The sequences of work functions for the fundamental mode in

particular show a wide variation in work done by each zone from period to period. The zones associated with the HeII and (HeI,HI) ionisation regions are consistently driving zones.

Complete velocity histories of the model m20e12, shown in Figures 6.126 and 6.127, again show that the variation is confined to layers further from the centre of the star for the higher modes; the velocity profile for the innermost zones is constant from period to period.

The discovery that the form of the velocity and (smoothed) light curves remains the same for a smaller kick is perhaps the most important of this section, since it means that the light and velocity curves of the more extensive series of models with the larger kick can be compared with real stars, neglecting the actual amplitude. Given enough data, this might allow identification of the mode of pulsation of a star by a method independent of derived quantities, such as Q values.

6.5 THE EFFECT OF HELIUM DEPLETION

Perhaps the best way to study models with envelopes depleted of helium would have been to run a whole repeat static grid incorporating some assumption of the rate of inward diffusion, then to use those models, with their internal abundance gradients, in TRCVSP. However, this would have required an extensive modification to TRCVSP, which assumes constant abundances throughout. It would also have required a wide selection of opacity tables, and

interpolation between them. Moreover, since TRCVSP uses tables for pressure and internal energy to speed the calculations, it would have been necessary either to revert to using the full equation of state subroutine for every value of those quantities, or to generate a range of pressure and internal energy tables and interpolate between them.

Instead, the original grid of static models was used, and constant abundance gradients were assumed. Two helium-depleted opacity tables were used, as described in Chapter 3; HM88 has $(X,Y,Z) = (0.88,0.10,0.02)$ and C402 has $(X,Y,Z) = (0.98,0.00,0.02)$. Although the static model remained the same, once rezoning was complete the relaxation was carried out using the new abundances. STURMSUB was used, and eigenfunctions of the modified models are illustrated in Figure 6.128, where the unbroken line is the eigenfunction for models of normal abundance, the broken line that for models using HM88, and the dotted line that for models using C402. Use of the HM88 table is indicated by the notation "hed"; "hdd" denotes use of C402. As before, the numbers 00, 01, and 02 indicate the form of the initial velocity profile.

Only two models, m16e03 and m18e03, were studied. Both lie near the Main Sequence, which is where the Am-pulsation exclusion is observed. Both models were followed for two hours of CPU time for each of the first three modes of pulsation, for each level of helium depletion. Plots of light, velocity and radius variations, the periodogram of the light curve, and the decrease of maximum kinetic energy

per period are shown in Figures 6.129 to 6.140. Table 6.4 gives the period of the dominant peak in the periodogram, its amplitude, and light and velocity amplitudes estimated from the curves. The periods are the same as those for the models of normal abundance.

All these models were given an initial kick of 3 km/s, and if we compare the helium depleted m16e03 models to those of normal abundance (Figures 6.108 to 6.110), we see that the light amplitude is slightly higher for the "hed" models, and slightly higher still for the "hdd" models. In all instances, the maximum kinetic energy per period is slightly lower in the "hed" models than the "stm" models, and less in the "hdd" models. In most cases, the rate of decrease is about the same for all three sets; two exceptions to this are m16e03_hed02 and m16e03_hdd02, both of which level off.

The characteristic patterns of the velocity curves of each mode are still present; two smaller peaks followed by one or two larger ones for the fundamental, and three smaller peaks followed by one larger one for the first harmonic. As before, high frequencies complicate the light curves, particularly for the less massive model.

If the second helium ionisation zone were the source of all the driving, we would expect its removal to result in smaller amplitudes and a faster decrease of maximum kinetic energy per period. Neither of these is observed; in fact, for the second harmonic, the maximum kinetic energy per period even seems to level off in two cases. To try and find out what was causing the driving, the work functions

were examined. These are shown in Figure 6.141 (for the fundamental mode), Figure 6.142 (first harmonic), and Figure 6.143 (second harmonic). The sequences of work functions for the fundamental mode are too variable to draw any conclusions. For the first and second harmonics, "hed" and "hdd" work functions are very similar. If they are compared to those of the "stm" models (Figures 6.118 and 6.119), some evidence can be seen of increased damping around zone 40. Where the damping regions have a toothed look for m16e03_stm, they merge together into one feature for m16e03_hed and m16e03_hdd. It is not clear whether the driving region, presumably due entirely to hydrogen ionisation in the "hdd" models, is any larger in the "hed" and "hdd" models than in the "stm" ones.

The pressure-volume loops for the outermost zones of the m16e03 models are illustrated in Figures 6.144 to 6.149. The loops are erratic, but it can be seen that the driving present in zone 44 of the "hed" models is much less marked in the "hdd" ones. As for previous models, the damping in zones 39 to 41 is not visible in the pressure-volume loops, which are very narrow, if loops at all.

Figure 6.150 shows the ionisation profiles of the more important elements for m16e03 (for normal abundance). The second helium ionisation region lies between zones 43 and 44, and the (HeI, HI) ionisation region between zones 45 and 46. Given that the pulsation code smears out shock fronts over more than one zone, it is clear that driving in the "hdd" models is provided by the ionisation of hydrogen.

From this work, it is not possible to conclude that helium depletion might explain the lack of pulsation of Am stars. The models show only slight differences from those of normal abundance and, given the persistent indications in both sets that the pulsations have not settled down, it is not possible to decide whether or not the pulsations would decay to nothing over long periods of time.

6.6 THE CAUSE OF THE DRIVING

Most pulsating stars are generally understood to be driven by an envelope ionisation mechanism (Cox 1980), with most driving due to the second helium ionisation zone, and a contribution from the hydrogen ionisation zone. The first helium ionisation zone is very near to the hydrogen zone, resulting in a combined (HI,HeI) zone. Since the contribution from the hydrogen in this zone is far higher than that from the helium, the region is usually referred to as the hydrogen ionisation zone. The cause of the driving in Cepheids is the coincidence of the "transition region" (between the quasi-adiabatic interior and the non-adiabatic exterior of a star) with the helium ionisation zone. The hydrogen ionisation zone contributes to the phase lag. In RR Lyrae stars (Christy 1966), hydrogen is more important as a source of driving, but still the major contribution is from the helium ionisation zone.

The temperature (and hence position) of the transition region depends, most critically, on the radius of the star. The smaller the star, the higher the temperature at which the transition region is located. The Instability Strip

turns out to be that region of the Hertzsprung-Russell Diagram where the temperature of the transition region is equal to that of the helium ionisation zone; some 4×10^4 K. Cox (1980) describes the behaviour of the luminosity variations, $\delta L_r/L_r$, at temperatures above and below that of the transition region. Above, the luminosity variation is similar to its quasi-adiabatic value. Below, $\delta L_r/L_r$ becomes decoupled from the temperature and pressure variations, and remains essentially constant. When the transition region coincides with a major ionisation region, pulsational driving occurs because energy is absorbed at the moment of maximum compression, and $\delta L_r/L_r$ decreases. Normally it would increase again beyond the ionisation region, but here the transition region "freezes in" the decrease before damping can take place. So, the flow of radiation to the surface is dammed up at maximum compression, and is freed when the star expands; this combination exacerbates any incipient instabilities, and the star pulsates.

The models presented in the previous sections are driven by both the second helium ionisation zone and the hydrogen ionisation zone. The low growth rates suggest that the driving is inefficient; the fact that models depleted of helium still pulsate indicates that driving by hydrogen is important.

It was noted in Chapter 5 that the results for linear adiabatic and non-adiabatic analyses were virtually identical. It may be the case that, in Delta Scuti stars, the transition region marks the border between the

quasi-adiabatic interior and a not quite so quasi-adiabatic exterior, rather than a non-adiabatic region. Then the "freezing in" process would be much less efficient, and there would be scope for damping in the outer part of the second helium ionisation zone. Likewise, there would be considerable driving caused by the hydrogen ionisation zone, since the flux variation would still be loosely coupled to the pressure and temperature variations.

Since the zoning of the models is so coarse, with only one zone covering the hydrogen ionisation region, it is not possible to obtain any information on the detailed structure of the luminosity variation throughout each model. The calculations would have to be repeated with much finer zoning to discover whether the variations were really following the outline given in the paragraph above.

6.7 COMPARISON WITH REAL STARS

In Chapter 5, it was concluded that the models used were of the appropriate mass and evolutionary status, and that their periods of pulsation were in the same range as those of real stars. Since the non-linear models have been found to pulsate with the same periods (although with added high frequency components of low amplitude), it remains in this chapter only to compare the amplitudes and shapes of the light and velocity curves with those of real stars.

There are two major problems to be faced in making this comparison. Firstly, in very few cases were the pulsations of the models found to settle down, when judged by the

criterion of maximum kinetic energy per period. Hence, the amplitudes of the models cannot be compared directly with those of the real stars. Secondly, the light curves for real stars are fragmentary, due to practical limits on observation runs, and nowhere could sequences of periods be found long enough to use the form of the light curve as a diagnostic for the mode of pulsation. Radial velocity curves are rare, since they are more difficult to obtain than light curves. Nevertheless, some comments can be made on the range of amplitudes, and on the appearance of high frequency features in the light curves.

There are generally recognised to be two types of Delta Scuti star, although there is no sharp divide between them. There are the high amplitude stars, like δ Scuti itself, whose amplitudes lie between 0.3 and 0.6 magnitudes, and which are mostly monoperiodic. The majority of Delta Scuti stars, however, have amplitudes measured in hundredths of magnitudes rather than tenths, and have multiperiodic (or irregular) light curves. In the lower part of Figure 6.26, the general increase in amplitude with period for the low amplitude stars was illustrated. Where enough data are available and the frequencies of oscillation can be obtained by Fourier analysis, the periods are usually found to correspond to the lowest modes of radial pulsation. Authors often also identify non-radial modes which, of course, would not occur in these models.

It is encouraging to note that the light amplitudes of the "k10r5" models and those of the "stm" series given a 10 km/s kick are of the same order as the high and medium amplitude stars, i.e., from a tenth to several tenths of a magnitude. The extremely high amplitudes reported by Stellingwerf (1976) for his technique were not obtained in this work. The amplitudes of the "stm" models given a 3 km/s kick, some 0.05 to 0.1 magnitudes, include a large number of the low-amplitude stars, although not the lowest. A lower bound on the amplitude can be set by the model given no kick, which exhibited oscillations of about 0.002 magnitudes, with no indication that these would grow larger. Radial velocity amplitudes lie in the range 2 to 30 km/s quoted as typical of Delta Scuti stars (Baglin et al 1973, Breger 1979).

Although the range is correct, there is little to indicate that the pulsation in the models would not eventually die away to a few thousandths of a magnitude, leaving them visible as non-pulsating stars within the Instability Strip (of which, indeed, there are a fair number). It seems likely that something important has been missed out of the models, something that interacts with the incipient pulsation and preserves it from decay. A tentative suggestion here is that the rotation of the stars may be relevant to more than just the pulsation-metallicism exclusion.

Figure 6.151 shows the light curves of a number of stable, monop periodic, high amplitude stars. Such repetition from cycle to cycle was not found in the models, all of which show some degree of modulation. A phased diagram for sequences of periods for any of the models would give a broad band of points, rather than a line.

There is one aspect of the light curves of low amplitude stars that is found in the models and does not require long data strings to be evident in real stars. In Figure 6.152, the light curves of several low amplitude Delta Scuti stars are reproduced (see also Figure 1.4). All have still smaller amplitude features superimposed on the main pulsation. This might be noise in the data, but given the occurrence of high frequency oscillations of similar appearance in the models, it is more probably real.

It is frustrating to find so few long sequences of periods pictured in the literature. Figure 6.153 shows some of the longest data strings available, for the low amplitude star BD +28°1494. This star was analysed by Broglia & Conconi (1985), who identified periods of (in days) 0.064558, 0.062790, 0.056722 and 0.055163. The ratios between these four periods mostly do not agree with those for low radial modes, but that between the first and fourth periods, 0.854, is close to the ratio P_4/P_3 for the linear models. The light curve could perhaps be described as sinusoidal, with evidence of modulation; as such we would expect, from the "stm" models, that the dominant mode of pulsation would be one of the higher harmonics (the second harmonic, or above). Broglia & Conconi give a temperature

for the star, 7760K, but no estimate of its absolute magnitude. If we identify the first period as that of the third harmonic, and the fourth as that of the fourth harmonic, we can place BD +28^o1494 neatly between models m20e11 and m20e12. M20e11 has temperature 7782K, $P_3 = 0.060$ days, $P_4 = 0.051$ days, and m20e12 has temperature 7411K, $P_3 = 0.068$ days, and $P_4 = 0.058$ days.

TABLE 6.1

"name" refers to the static model. P1 and P2 are the periods, in hours, of the dominant and secondary peaks in the periodogram. The amplitudes of these peaks are amp1 and amp2 (in magnitudes) respectively. Amplitudes estimated from the light curve (in magnitudes) and the velocity curve (in km/s) are also listed.

"k10r5" models:

name	P1	P2	amp1	amp2	light	velocity
m14e01	1.09	0.83	0.047	0.027		
m14e04	1.61	1.24	0.035	0.026		
m14e07	1.84	1.41	0.046	0.027		
m16e03	1.46	1.10	0.058	0.031	0.25	25
m16e05	1.75	1.34	0.054	0.028	0.3	25
m16e09	2.57	1.98	0.037	0.025		
m18e08	1.84	1.70	0.065	0.034	0.35	25
m18e10	2.72	2.09	0.060	0.034	0.35	25
m20e03	1.49	1.15	0.092	0.050		
m20e14	3.73	2.89	0.057	0.029	0.3	25
m20e16	4.44	3.42	0.039	0.018		
m24e16	5.92	4.50	0.060	0.028	0.3	25
m24e17	6.58	5.00	0.045	0.018		
m26e17	6.62	5.05	0.062	0.026	0.4	20
m26e19	7.41	5.68	0.059	0.023		
m26e20	8.00	6.13	0.046	0.018		
m26e22	9.17	6.99	0.037	0.015		
m28e18	7.14	5.41	0.072	0.030		
m28e20	8.20	6.25	0.070	0.028		
m28e22	9.62	7.35	0.060	0.023		
m30e13	5.88	4.44	0.082	0.041	0.5	25
m30e18	11.1	8.47	0.057	0.021		
m30e19	11.6	8.85	0.053	0.016		

"k5r5" models:

m16e00	1.06	0.81	0.028	0.019
m16e07	2.11	1.62	0.023	0.015

"k5r8" model:

m16e00: 4 dominant periods of 1.06, 0.81, 0.65 and 0.55 hrs.
Amplitudes from periodogram 0.011, 0.009, 0.011, 0.006 mag.

"k10r8" model:

m26e15: 1 dominant period of 5.21 hours. Amplitude from periodogram 0.045 mag.

TABLE 6.2

"name" refers to the static model. P is the period (in hours) of the main peak in the periodogram; amp is its amplitude (in magnitudes). Amplitudes estimated from the light curve (in magnitudes) and the velocity curve (in km/s) are also listed. All models received an initial kick of 10 km/s. For "stm00" models, the initial velocity profile was the fundamental eigenfunction. For "stm01" models, it was the first harmonic, for "stm02" the second, for "stm03" the third, and for "stm04" the fourth harmonic.

"stm00" models:

name	P	amp	light	velocity
m14e01	1.09	0.038		
m14e07	1.85	0.031		
m16e03	1.45	0.036	0.3	20
m16e05	1.75	0.037	0.3	20
m18e03	1.52	0.053	0.35	20
m18e05	1.85	0.046	0.3	20
m18e09	2.33	0.052	0.35	17
m20e05	1.96	0.064		
m20e10	2.63	0.052	0.35	20
m20e12	3.13	0.059		
m20e14	3.70	0.041	0.25	15
m24e12	3.45	0.069		
m24e13	4.00	0.061	0.35	17
m24e14	4.55	0.048		
m24e15	5.00	0.044	0.3	17
m24e16	5.88	0.041		
m26e14	4.55	0.072		
m26e15	5.26	0.066		
m26e17	6.67	0.043		
m28e16	5.56	0.065		
m28e17	6.25	0.063		
m28e18	7.14	0.053		
m30e14	6.67	0.058		
m30e15	7.70	0.065		
m30e16	9.09	0.052		

long run (8 hrs):

m20e12 3.13 0.056

"stm01" models

name	P	amp	light	velocity
m14e01	0.83	0.033		
m14e04	1.23	0.022		
m14e07	1.41	0.032		
m16e01	0.86	0.032		
m16e03	1.11	0.034	0.2	13
m16e05	1.35	0.033	0.2	15
m18e03	1.15	0.044	0.25	15
m18e05	1.43	0.039	0.25	15
m18e09	1.79	0.054	0.25	15
m20e05	1.52	0.066		
m20e10	2.04	0.048	0.3	15
m20e12	2.44	0.062		
m20e14	2.86	0.040	0.2	12
m24e12	2.63	0.072		

m24e13	3.03	0.064	0.3	12
m24e14	3.57	0.046		
m24e15	3.85	0.042	0.3	15
m24e16	4.55	0.041		
m26e14	3.57	0.070		
m26e15	4.00	0.066		
m26e17	5.00	0.039		
m28e16	4.35	0.069		
m28e17	4.76	0.064		
m28e18	5.56	0.049		
m30e14	5.00	0.056		
m30e15	5.88	0.059		
m30e16	6.67	0.041		

long run (8 hrs):

m20e12	2.44	0.055
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"stm02" models

name	P	amp	light	velocity
m14e01	0.68	0.036		
m14e04	1.00	0.026		
m14e07	1.14	0.034		
m16e01	0.70	0.029		
m16e03	0.89	0.036	0.2	12
m16e05	1.09	0.030	0.15	12
m18e03	0.93	0.039	0.15	14
m18e05	1.15	0.036	0.15	12
m18e09	1.45	0.051	0.2	12
m20e05	1.22	0.069		
m20e10	1.64	0.047	0.25	12
m20e12	1.96	0.065	0.25	12
m20e14	2.33	0.039	0.15	12
m24e12	2.13	0.069		
m24e13	2.50	0.060	0.3	12
m24e14	2.86	0.038		
m24e15	3.13	0.036	0.2	12
m24e16	3.70	0.040		
m26e14	2.86	0.068		
m26e15	3.23	0.060		
m26e17	4.17	0.031		
m28e16	3.57	0.062		
m28e17	3.85	0.058		
m28e18	4.35	0.043		
m30e14	4.00	0.054		
m30e15	4.76	0.054		
m30e16	5.56	0.035		

"stm03" models

name	P	amp	light	velocity
m14e01	0.57	0.037		
m14e04	0.84	0.028		
m14e07	0.95	0.034		
m16e01	0.59	0.027		
m16e03	0.75	0.040	0.2	10
m16e05	0.91	0.029	0.15	10
m18e03	0.79	0.034	0.15	10
m18e05	0.97	0.032	0.15	10
m18e09	1.22	0.048	0.2	10
m20e05	1.02	0.068		
m20e10	1.37	0.045	0.2	10
m20e12	1.64	0.066	0.25	12
m20e14	1.96	0.038	0.15	10
m24e12	1.79	0.063		

m24e13	2.08	0.051	0.25	10
m24e14	2.44	0.031		
m24e15	2.63	0.031	0.015	12
m24e16	3.03	0.040		
m26e14	2.44	0.060		
m26e15	2.70	0.052		
m26e17	3.45	0.027		
m28e16	2.94	0.052		
m28e17	3.23	0.050		
m28e18	3.70	0.037		
m30e14	3.33	0.042		
m30e15	3.85	0.042		
m30e16	4.55	0.026		

"stm04" models

name	P	amp	light	velocity
m14e01	0.49	0.042		
m14e04	0.72	0.031		
m14e07	0.81	0.033		
m16e01	0.51	0.025		
m16e03	0.65	0.039	0.2	10
m16e05	0.78	0.028	0.15	10
m18e03	0.68	0.029	0.15	10
m18e05	0.83	0.028	0.15	10
m18e09	1.04	0.042	0.15	10
m20e05	0.88	0.062		
m20e10	1.18	0.041	0.2	10
m20e12	1.41	0.061	0.25	10
m20e14	1.67	0.038	0.15	10
m24e12	1.52	0.052		
m24e13	1.79	0.042	0.2	10
m24e14	2.04	0.025		
m24e15	2.27	0.028	0.15	10
m24e16	2.63	0.038		
m26e14	2.08	0.048		
m26e15	2.33	0.042		
m26e17	2.94	0.032		

TABLE 6.3

"name" refers to the static model. P is the period (in hours) of the main peak in the periodogram; amp is its amplitude (in magnitudes). Amplitudes estimated from the light curve (in magnitudes) and the velocity curve (in km/s) are also listed. All models received an initial kick of 3 km/s. For "stm00" models, the initial velocity profile was the fundamental eigenfunction. For "stm01" models, it was the first harmonic, for "stm02" the second.

"stm00" models:

name	P	amp	light	velocity
m16e03	1.45	0.009	0.1	6
m18e09	2.33	0.014	0.1	5
m20e12	3.13	0.016	0.1	5

"stm01" models:

m16e03	1.11	0.007	0.06	4
m18e09	1.79	0.014	0.07	4
m20e12	2.44	0.017	0.08	4

"stm02" models:

m16e03	0.89	0.008	0.05	3
m18e09	1.45	0.013	0.06	3
m20e12	1.96	0.018	0.07	3

TABLE 6.4

"name" refers to the static model. P is the period (in hours) of the main peak in the periodogram; amp is its amplitude (in magnitudes). Amplitudes estimated from the light curve (in magnitudes) and the velocity curve (in km/s) are also listed. All models received an initial kick of 3 km/s. "Hed" indicates use of the opacity table HM88; "hdd" indicates use of table C402. As before, 00, 01 and 02 refer to the eigenfunction used as an initial velocity profile.

"00" models:

name	P	amp	light	velocity
m16e03_hed	1.45	0.012	0.12	6
m16e03_hdd	1.45	0.016	0.15	6
m18e03_hed	1.52	0.016	0.12	6
m18e03_hdd	1.52	0.018	0.15	6

"01" models:

m16e03_hed	1.11	0.012	0.08	4
m16e03_hdd	1.11	0.018	0.1	4
m18e03_hed	1.15	0.015	0.09	4
m18e03_hdd	1.15	0.018	0.1	4

"02" models:

m16e03_hed	0.89	0.013	0.06	4
m16e03_hdd	0.89	0.020	0.1	4
m18e03_hed	0.93	0.015	0.07	3
m18e03_hdd	0.93	0.019	0.09	4

Figure 6.2. The light curve of model m16e10 produced when no initial velocity kick was imposed.

Figure 6.3. The four periodograms correspond to the four segments of light curve of model `m16e10` in Figure 6.2. The lowest diagram illustrates the variation in maximum kinetic energy with period number over the whole run.

Figure 6.4. Initial velocity profiles for four models. In each diagram, the upper line corresponds to fractional radius to the fifth power, and the lower to the eighth power.

Figure 6.5. Light, velocity and radius variations over several periods for the model m16e03_k10r5.

Figure 6.6. Light, velocity and radius variations over several periods for the model m16e05_k10r5.

Figure 6.7. Light, velocity and radius variations over several periods for the model m18e08_k10r5.

Figure 6.8. Light, velocity and radius variations over several periods for the model m18e10_k10r5.

Figure 6.9. Light, velocity and radius variations over several periods for the model m20e14_k10r5.

Figure 6.10. Light, velocity and radius variations over several periods for the model m24e16_k10r5.

Figure 6.11. Light, velocity and radius variations over several periods for the model m26e17_k10r5.

Figure 6.12. Light, velocity and radius variations over several periods for the model m30e13_k10r5.

Figure 6.13. Light curves for the complete 8 hour runs of models m16e03_k10r5 and m16e05_k10r5.

Figure 6.14. Light curves for the complete 8 hour runs of models m18e08_k10r5 and m18e10_k10r5.

Figure 6.15. Light curves for the complete 8 hour runs of models m20e14_k10r5 and m24e16_k10r5.

Figure 6.16. Light curves for the complete 8 hour runs of models m26e17_k10r5 and m30e13_k10r5.

Figure 6.17. The decrease in maximum kinetic energy with period number for the entire 8 hour runs of the eight models featured in Figures 6.5 to 6.16.

Figure 6.18. The sequence of periodograms, one per 1 hour run, for the model m16e03_k10r5. Note the reduction in power in the higher modes as the sequence progresses.

Figure 6.19. The sequence of periodograms, one per 1 hour run, for the model m16e05_k10r5. Note the reduction in power in the higher modes as the sequence progresses.

Figure 6.20. The sequence of periodograms, one per 1 hour run, for the model m18e08_k10r5. Note the reduction in power in the higher modes as the sequence progresses.

Figure 6.21. The sequence of periodograms, one per 1 hour run, for the model m18e10_k10r5. Note the reduction in power in the higher modes as the sequence progresses.

Figure 6.22. The sequence of periodograms, one per 1 hour run, for the model m20e14_k10r5. Note the reduction in power in the higher modes as the sequence progresses.

Figure 6.23. The sequence of periodograms, one per 1 hour run, for the model `m24e16_k10r5`. Note the reduction in power in the higher modes as the sequence progresses.

Figure 6.24. The sequence of periodograms, one per 1 hour run, for the model m26e17_k10r5. Note the reduction in power in the higher modes as the sequence progresses.

Figure 6.25. The sequence of periodograms, one per 1 hour run, for the model m30e13_k10r5: Note the reduction in power in the higher modes as the sequence progresses.

Figure 6.26. Upper diagram: the period-amplitude relation for the k10r5 models, produced using the amplitudes given by the periodograms. As explained in the text, these are much smaller than the amplitudes obtained from an inspection of the light curves. Lower diagram: the period-amplitude relation for the stars listed in Appendix E. Triangles represent those thought to be pulsating in the fundamental mode, crosses the first harmonic, pentagons the second harmonic, and asterisks the third harmonic.

Figure 6.27. Work done per zone for a sequence of about 15 periods for models m16e03_k10r5, m16e05_k10r5, m18e08_k10r5 and m18e10_k10r5.

Figure 6.28. Work done per zone for a sequence of about 15 periods for models m20e14_k10r5, m24e16_k10r5, m26e17_k10r5 and m30e13_k10r5.

Figure 6.29. Work done per zone, plotted against zone temperature, for models m16e05_k10r5, m18e08_k10r5, m24e16_k10r5 and m30e13_k10r5.

Figure 6.30. The pressure-volume variation over several periods for certain zones for the model m18e08_k10r5. In zones 42 to 46 the loops are anticlockwise, indicating driving. The loops in zones 39 to 41 run clockwise, indicating damping.

Figure 6.31. The pressure-volume variation over several periods for certain zones for the model m24e16_k10r5. In zones 42 to 46 the loops are anticlockwise, indicating driving. The loops in zones 39 to 41 run clockwise, indicating damping.

Figure 6.32. Ionisation profiles for the model m18e08. Helium is equally split between its singly and doubly ionised forms between zones 42 and 43; the ionisation of hydrogen and the first ionisation of helium both take place near zone 45.

Figure 6.33. Ionisation profiles for the model m24e16. Helium is equally split between its singly and doubly ionised forms between zones 42 and 43; the ionisation of hydrogen and the first ionisation of helium both take place near zone 45.

Figure 6.34. The velocity profiles, taken when the model is contracting through its equilibrium radius, for sequences of periods of the models m16e03_k10r5, m16e05_k10r5, m18e08_k10r5, and m18e10_k10r5.

Figure 6.35. The velocity profiles, taken when the model is contracting through its equilibrium radius, for sequences of periods of the models m20e14_k10r5, m24e16_k10r5, m26e17_k10r5, and m30e13_k10r5.

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Figure 6.47. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m16e05_stm01, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

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Figure 6.53. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m18e03_stm02, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

Figure 6.54. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m18e03_stm03, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

Figure 6.55. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m18e03_stm04, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

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Figure 6.57. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model `m18e05_stm01`, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

Figure 6.58. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m18e05_stm02, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

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Figure 6.63. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m18e09_stm02, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

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Figure 6.65. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m18e09_stm04, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

Figure 6.66. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m20e10_stm00, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

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Figure 6.72. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m20e12_stm03, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

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Figure 6.74. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model `m20e14_stm00`, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

Figure 6.75. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m20e14_stm01, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

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Figure 6.77. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m20e14_stm03, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

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Figure 6.79. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m24e13_stm00, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

Figure 6.80. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m24e13_stm01, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

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Figure 6.83. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m24e13_stm04, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

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Figure 6.87. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m24e15_stm03, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

Figure 6.88. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m24e15_stm04, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

Figure 6.89. Work done per zone for a sequence of periods for the stm00 models m16e03, m18e03, m20e12 and m24e13.

Figure 6.90. Work done per zone for a sequence of periods for the stm01 models m16e03, m18e03, m20e12 and m24e13.

Figure 6.91. Work done per zone for a sequence of periods for the stm02 models m16e03, m18e03, m20e12 and m24e13.

Figure 6.92. Work done per zone for a sequence of periods for the stm03 models m16e03, m18e03, m20e12 and m24e13.

Figure 6.93. Work done per zone for a sequence of periods for the stm04 models m16e03, m18e03, m20e12 and m24e13.

Figure 6.94. The pressure-volume variation over several periods for certain zones of the model m20e12_stm00. In zones 42 to 46, the loops are anticlockwise, indicating driving. The loops in zones 39 to 41 run clockwise, indicating damping.

Figure 6.95. The pressure-volume variation over several periods for certain zones of the model m20e12_stm01. In zones 42 to 46, the loops are anticlockwise, indicating driving. The loops in zones 39 to 41 run clockwise, indicating damping.

Figure 6.96. The pressure-volume variation over several periods for certain zones of the model m20e12_stm02. In zones 42 to 46, the loops are anticlockwise, indicating driving. The loops in zones 39 to 41 run clockwise, indicating damping.

Figure 6.97. The pressure-volume variation over several periods for certain zones of the model m20e12_stm03. In zones 42 to 46, the loops are anticlockwise, indicating driving. The loops in zones 39 to 41 run clockwise, indicating damping.

Figure 6.98. The pressure-volume variation over several periods for certain zones of the model m20e12_stm04. In zones 42 to 46, the loops are anticlockwise, indicating driving. The loops in zones 39 to 41 run clockwise, indicating damping.

Figure 6.99. Ionisation profiles for the model m20e12. The second ionisation of helium takes place between zones 42 and 43. The first ionisation of helium and the ionisation of hydrogen take place near zone 45.

Figure 6.100. The velocity profiles, taken when the model is contracting through its equilibrium radius, for the stm00 models m16e03, m18e03, m20e12 and m24e13.

Figure 6.101. The velocity profiles, taken when the model is contracting through its equilibrium radius, for the stm01 models m16e03, m18e03, m20e12 and m24e13.

Figure 6.102. The velocity profiles, taken when the model is contracting through its equilibrium radius, for the stm02 models m16e03, m18e03, m20e12 and m24e13.

Figure 6.103. The velocity profiles, taken when the model is contracting through its equilibrium radius, for the stm03 models m16e03, m18e03, m20e12 and m24e13.

Figure 6.104. The velocity profiles, taken when the model is contracting through its equilibrium radius, for the stm04 models m16e03, m18e03, m20e12 and m24e13.

Figure 6.105. Complete velocity histories over several periods for the models m20e12_stm00 and m20e12_stm01. The velocity curves for each zone have been displaced vertically. The tick marks on the velocity axis are 5 km/s apart.

Figure 6.106. Complete velocity histories over several periods for the models m20e12_stm02 and m20e12_stm03. The velocity curves for each zone have been displaced vertically. The tick marks on the velocity axis are 5 km/s apart.

Figure 6.107. Complete velocity histories over several periods for the model m20e12_stm04. The velocity curves for each zone have been displaced vertically. The tick marks on the velocity axis are 5 km/s apart.

Figure 6.108. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m16e03_stm00 (initial kick 3 km/s), along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

Figure 6.109. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m16e03_stm01 (initial kick 3 km/s), along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

Figure 6.110. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m16e03_stm02 (initial kick 3 km/s), along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

Figure 6.111. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m18e09_stm00 (initial kick 3 km/s), along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

Figure 6.112. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m18e09_stm01 (initial kick 3 km/s), along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

Figure 6.113. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m18e09_stm02 (initial kick 3 km/s), along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

Figure 6.114. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m20e12_stm00 (initial kick 3 km/s), along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

Figure 6.115. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m20e12_stm01 (initial kick 3 km/s), along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

Figure 6.116. The second segments of the light, velocity and radius variation curves of a 2 hour run of the model m20e12_stm02 (initial kick 3 km/s), along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number over the whole run.

Figure 6.117. Work done per zone for a sequence of periods for the 3 km/s kick, stm00 models m16e03, m18e09 and m20e12.

Figure 6.118. Work done per zone for a sequence of periods for the 3 km/s kick, stm01 models m16e03, m18e09 and m20e12.

Figure 6.119. Work done per zone for a sequence of periods for the 3 km/s kick, stm02 models m16e03, m18e09 and m20e12.

Figure 6.120. The pressure-volume variation over several periods for the model m20e12_stm00 (3 km/s kick). The loops in zones 42 to 46 run anticlockwise (driving), and those in zones 39 to 41 run clockwise (damping).

Figure 6.121. The pressure-volume variation over several periods for the model m20e12_stm01 (3 km/s kick). The loops in zones 42 to 46 run anticlockwise (driving), and those in zones 39 to 41 run clockwise (damping).

Figure 6.122. The pressure-volume variation over several periods for the model m20e12_stm02 (3 km/s kick). The loops for zones 42 to 46 run anticlockwise (driving), and those in zones 39 to 41 run clockwise (damping).

Figure 6.123. The velocity profiles, taken when the model is contracting through its equilibrium radius, for the stm00 models (3 km/s kick) m16e03, m18e09 and m20e12.

Figure 6.124. The velocity profiles, taken when the model is contracting through its equilibrium radius, for the stm01 models (3 km/s kick) m16e03, m18e09 and m20e12.

Figure 6.125. The velocity profiles, taken when the model is contracting through its equilibrium radius, for the stm02 models (3 km/s kick) m16e03, m18e09 and m20e12.

Figure 6.126. Complete velocity histories over several periods for the models m20e12_stm00 and m20e12_stm01 (3 km/s kick). The velocity curves for each zone have been displaced vertically. The tick marks on the velocity axis are 2 km/s apart.

Figure 6.127. Complete velocity histories over several periods for the model m20e12_stm02 (3 km/s kick). The velocity curves for each zone have been displaced vertically. The tick marks on the velocity axis are 2 km/s apart.

Figure 6.128. Eigenfunctions of the models m16e03 and m18e03. The unbroken line corresponds to the eigenfunction for normal abundance (opacity table C312), the broken line to helium-depleted models (HM88) and the dotted line to extreme helium-depleted models (C402).

Figure 6.129. The second segments of the light, velocity and radius variation curves of a 2 hour run of the helium-depleted model m16e03_hed00, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number for the whole run.

Figure 6.130. The second segments of the light, velocity and radius variation curves of a 2 hour run of the helium-depleted model m16e03_hed01, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number for the whole run.

Figure 6.131. The second segments of the light, velocity and radius variation curves of a 2 hour run of the helium-depleted model m16e03_hed02, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number for the whole run.

Figure 6.132. The second segments of the light, velocity and radius variation curves of a 2 hour run of the helium-depleted model m18e03 hed00, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number for the whole run.

Figure 6.133. The second segments of the light, velocity and radius variation curves of a 2 hour run of the helium-depleted model m18e03_hed01, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number for the whole run.

Figure 6.134. The second segments of the light, velocity and radius variation curves of a 2 hour run of the helium-depleted model m18e03_hed02, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number for the whole run.

Figure 6.135. The second segments of the light, velocity and radius variation curves of a 2 hour run of the helium-depleted model m16e03 hdd00, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number for the whole run.

Figure 6.136. The second segments of the light, velocity and radius variation curves of a 2 hour run of the helium-depleted model m16e03_hdd01, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number for the whole run.

Figure 6.137. The second segments of the light, velocity and radius variation curves of a 2 hour run of the helium-depleted model m16e03 hdd02, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number for the whole run.

Figure 6.138. The second segments of the light, velocity and radius variation curves of a 2 hour run of the helium-depleted model m18e03_hdd00, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number for the whole run.

Figure 6.139. The second segments of the light, velocity and radius variation curves of a 2 hour run of the helium-depleted model m18e03_hdd01, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number for the whole run.

Figure 6.140. The second segments of the light, velocity and radius variation curves of a 2 hour run of the helium-depleted model m18e03_hdd02, along with the periodogram of the second segment of the light curve, and the decrease in maximum kinetic energy with period number for the whole run.

Figure 6.141. Work done per zone for a sequence of periods for the models m16e03_hed00, m16e03_hdd00, m18e03_hed00 and m18e03_hdd00.

Figure 6.142. Work done per zone for a sequence of periods for the models m16e03_hed01, m16e03_hdd01, m18e03_hed01 and m18e03_hdd01.

Figure 6.143. Work done per zone for a sequence of periods for the models m16e03_hed02, m16e03_hdd02, m18e03_hed02 and m18e03_hdd02.

Figure 6.144. The pressure-volume variation over several periods for certain zones of the model m16e03_hed00. In zones 42 to 46 the loops are anticlockwise, indicating driving. The loops in zones 39 to 41 run clockwise, indicating damping. Note that the loops in zones 42 and 43 are much narrower than for models of normal helium abundance.

Figure 6.145. The pressure-volume variation over several periods for certain zones of the model m16e03 hed01. In zones 42 to 46 the loops are anticlockwise, indicating driving. The loops in zones 39 to 41 run clockwise, indicating damping. Note that the loops in zones 42 and 43 are much narrower than for models of normal helium abundance.

Figure 6.146. The pressure-volume variation over several periods for certain zones of the model m16e03_hed02. In zones 42 to 46 the loops are anticlockwise, indicating driving. The loops in zones 39 to 41 run clockwise, indicating damping. Note that the loops in zones 42 and 43 are much narrower than for models of normal helium abundance.

Figure 6.147. The pressure-volume variation over several periods for certain zones of the model m16e03_hdd00. In zones 42 to 46 the loops are anticlockwise, indicating driving. The loops in zones 39 to 41 run clockwise, indicating damping. Note that the loops in zones 42 and 43 are much narrower than for models of normal helium abundance.

Figure 6.148. The pressure-volume variation over several periods for certain zones of the model m16e03_hdd01. In zones 42 to 46 the loops are anticlockwise, indicating driving. The loops in zones 39 to 41 run clockwise, indicating damping. Note that the loops in zones 42 and 43 are much narrower than for models of normal helium abundance.

Figure 6.149. The pressure-volume variation over several periods for certain zones of the model m16e03_hdd02. In zones 42 to 46 the loops are anticlockwise, indicating driving. The loops in zones 39 to 41 run clockwise, indicating damping. Note that the loops in zones 42 and 43 are much narrower than for models of normal helium abundance.

Figure 6.150. Ionisation profiles for the model m16e03. The second helium ionisation region lies between zones 43 and 44. Hydrogen ionisation and the first ionisation of helium coincide, between zones 45 and 46.

Figure 6.151. The light curves of a number of stable, monopерiodic, high amplitude Delta Scuti stars. (Antonello et al. 1986).

Figure 6.152. The light curves of some low amplitude Delta Scuti stars. (McNamara & Horan 1984, Costa et al. 1984, Lopez de Coca et al. 1984, van der Linden & Sterken 1986).

Figure 6.153. The light curve of BD +28°1494.
(Broglia & Conconi 1985).

CONCLUSION

The object of this study was to obtain realistic models of Delta Scuti stars using non-linear pulsation analysis. Towards that end, an evolutionary grid of static models was produced and analysed by linear methods, in order to confirm the suitability of the parameters chosen. The result of this was confirmation of earlier work by other authors, who have concluded that Delta Scuti stars lie in the mass range 1.4 to 3.0 $M_{\odot}$, are of typical population I composition, and are fairly unevolved objects, on or near the Main Sequence. The models had periods in the same range as the real stars, and their properties were consistent with those derived for real stars (e.g., Q values and surface gravity). Long growth times, of the order of 10^6 years, were found. The blue edges of the lower harmonics of the models matched the observed blue edge. In all, the linear study confirmed the results from previous work, which has been extensive in this area.

It was found that the periods obtained from purely adiabatic analysis were virtually identical to those obtained from non-adiabatic analysis. This discovery allowed the eigenfunctions of static models to be used directly as initial velocity profiles for subsequent non-linear analysis, since the linear adiabatic code could be incorporated as a subroutine within the non-linear code. The non-adiabatic code had several disadvantages in this respect: it could not use the zoned models from the static

grid, but generated its own, with different zoning. Hence, it would have been necessary to devise a fitting procedure, to match the eigenfunctions it produced to the models from the grid.

A wide range of non-linear models were investigated. None could really be said to have reached a limiting amplitude, since the maximum kinetic energy continued to decrease with period number. A settled model would have had constant maximum kinetic energy per period.

When the eigenfunctions (from a linear analysis) were used as initial velocity profiles, it was found that the dominant peak in the periodogram of the light curve invariably corresponded to the mode of pulsation initiated. Furthermore, whether the kick itself was large (10 km/s surface amplitude) or small (3 km/s), the form of the light and velocity curves remained the same, and was dependent on the mode of pulsation. Higher modes produced more sinusoidal variations.

The driving was found to be due mainly to the second helium ionisation zone, but with an important contribution from the hydrogen ionisation zone. The coarseness of the zoning precluded any detailed examination of the driving. The number of zones (about 50) was determined by the requirements of speed and accuracy; a great increase in the number of zones would result in an enormous increase in computing time. However, the analysis of even one model might yield important information on the driving, and would be a priority in any continuation of this study.

The amplitudes of pulsation were found to be larger than those observed for real stars. The decrease in maximum kinetic energy with period number suggests that the final amplitudes of these models would be smaller. Attempts to start models with a very small kick resulted in light curves with very high harmonics present, rather than in the more settled models hoped for. However, the results were encouraging, since the enormous discrepancy between non-linear results and real stars found by a previous author was not repeated. With a more detailed knowledge of exactly which modes of pulsation are present for a particular Delta Scuti star, it might be possible to produce a sum of eigenfunctions which, when used as an initial velocity profile, would produce light and velocity curves mimicking the real ones, in shape if not in amplitude.

The comparison of the shapes of the light and velocity curves with those of real stars was hampered by the lack of data in the literature. Of course, it would be optimistic to expect to find light and velocity curves of long sequences of periods for real Delta Scuti stars, since it would only be possible to observe a few periods in a night. Possible high-frequency oscillations superimposed on light curves were noted for several stars, and the similarity of these features to the models was commented on. However, it was not possible to attempt to identify the mode of pulsation of real stars. If longer data sets could be obtained, it would be interesting to compare the mode of pulsation predicted by the shape of the light curve with that from (for example) the Q value derived from

observations. It is hoped that the wide selection of model light and velocity curves pictured in Chapter 6 might prove useful for this purpose.

APPENDIX A

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APPENDIX B
STATIC MODELS

name	mass	R/R ₀	L/L ₀	Tcen	Pcen	Te	Pe	log g
m14e00	1.40	1.38	4.45	0.17798E+08	0.18410E+18	0.71743E+04	0.51209E+05	4.31
m14e01	1.40	1.39	4.52	0.17962E+08	0.18678E+18	0.71760E+04	0.48893E+05	4.30
m14e02	1.40	1.53	5.08	0.19420E+08	0.20447E+18	0.70337E+04	0.50999E+05	4.21
m14e03	1.40	1.65	5.27	0.20370E+08	0.21208E+18	0.68372E+04	0.59350E+05	4.15
m14e04	1.40	1.81	5.50	0.22574E+08	0.25402E+18	0.66078E+04	0.71866E+05	4.07
m14e05	1.40	1.83	6.57	0.21955E+08	0.56808E+18	0.68674E+04	0.49476E+05	4.06
m14e06	1.40	1.95	6.83	0.20539E+08	0.12903E+19	0.67052E+04	0.56191E+05	4.00
m14e07	1.40	1.97	6.83	0.20527E+08	0.14522E+19	0.66723E+04	0.58010E+05	3.99
m14e08	1.40	2.05	6.76	0.20929E+08	0.26365E+19	0.65237E+04	0.67759E+05	3.96
m14e09	1.40	2.20	6.20	0.30287E+08	0.23190E+20	0.61720E+04	0.10492E+06	3.90
m14e10	1.40	2.19	5.61	0.44311E+08	0.98181E+20	0.60348E+04	0.13172E+06	3.90
m16e00	1.60	1.43	7.65	0.18935E+08	0.17743E+18	0.80638E+04	0.19022E+05	4.33
m16e01	1.60	1.49	8.04	0.19440E+08	0.18084E+18	0.79928E+04	0.18706E+05	4.29
m16e02	1.60	1.64	8.68	0.20398E+08	0.18405E+18	0.77650E+04	0.20972E+05	4.21
m16e03	1.60	1.76	8.95	0.21047E+08	0.18545E+18	0.75585E+04	0.24136E+05	4.15
m16e04	1.60	1.92	9.14	0.22073E+08	0.19287E+18	0.72841E+04	0.29768E+05	4.08
m16e05	1.60	2.00	10.79	0.22761E+08	0.41958E+18	0.74317E+04	0.23375E+05	4.04
m16e06	1.60	2.20	11.94	0.22279E+08	0.18409E+19	0.72703E+04	0.24890E+05	3.96
m16e07	1.60	2.26	11.87	0.22757E+08	0.26682E+19	0.71627E+04	0.27440E+05	3.93
m16e08	1.60	2.42	11.39	0.26876E+08	0.85383E+19	0.68429E+04	0.37787E+05	3.87
m16e09	1.60	2.58	10.83	0.33830E+08	0.22645E+20	0.65520E+04	0.52069E+05	3.82
m16e10	1.60	2.77	10.05	0.43802E+08	0.59413E+20	0.62028E+04	0.79723E+05	3.76
m16e11	1.60	2.81	9.42	0.51713E+08	0.10786E+21	0.60564E+04	0.98699E+05	3.74
m18e00	1.80	1.49	12.16	0.19757E+08	0.16580E+18	0.88563E+04	0.87415E+04	4.34
m18e01	1.80	1.59	12.97	0.20319E+08	0.16599E+18	0.87253E+04	0.87245E+04	4.29
m18e02	1.80	1.72	13.70	0.20987E+08	0.16590E+18	0.85163E+04	0.95798E+04	4.22
m18e03	1.80	1.89	14.35	0.21730E+08	0.16463E+18	0.82198E+04	0.11332E+05	4.14
m18e04	1.80	2.01	14.63	0.22366E+08	0.16674E+18	0.80015E+04	0.13084E+05	4.09
m18e05	1.80	2.16	14.88	0.23474E+08	0.17759E+18	0.77506E+04	0.15621E+05	4.02
m18e06	1.80	2.15	17.34	0.24280E+08	0.35641E+18	0.80674E+04	0.11004E+05	4.03
m18e07	1.80	2.32	19.21	0.22922E+08	0.11926E+19	0.79744E+04	0.10936E+05	3.96
m18e08	1.80	2.43	19.20	0.23382E+08	0.20836E+19	0.77857E+04	0.12631E+05	3.92
m18e09	1.80	2.52	18.99	0.24370E+08	0.32629E+19	0.76319E+04	0.14375E+05	3.89
m18e10	1.80	2.79	17.84	0.32069E+08	0.13599E+20	0.71373E+04	0.22830E+05	3.80
m18e11	1.80	3.10	16.46	0.43325E+08	0.43249E+20	0.66375E+04	0.38705E+05	3.71
m18e12	1.80	3.40	15.03	0.54315E+08	0.97319E+20	0.61885E+04	0.66557E+05	3.63

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original thesis

m20e00	2.00	1.56	18.29	0.20417E+08	0.15345E+18	0.95856E+04	0.47085E+04	4.35
m20e01	2.00	1.66	19.30	0.20945E+08	0.15324E+18	0.94387E+04	0.46254E+04	4.30
m20e02	2.00	1.81	20.71	0.21559E+08	0.15004E+18	0.91990E+04	0.49999E+04	4.22
m20e03	2.00	1.93	21.45	0.22102E+08	0.14910E+18	0.89761E+04	0.55408E+04	4.17
m20e04	2.00	2.11	22.22	0.22878E+08	0.14967E+18	0.86713E+04	0.65411E+04	4.09
m20e05	2.00	2.33	22.94	0.24519E+08	0.16505E+18	0.83190E+04	0.80010E+04	4.01
m20e06	2.00	2.28	27.55	0.25303E+08	0.39581E+18	0.88038E+04	0.51060E+04	4.02
m20e07	2.00	2.46	29.35	0.23600E+08	0.16421E+19	0.86623E+04	0.55574E+04	3.96
m20e08	2.00	2.60	29.45	0.24070E+08	0.18832E+19	0.83771E+04	0.64507E+04	3.91
m20e09	2.00	2.74	29.05	0.25873E+08	0.35824E+19	0.81254E+04	0.77940E+04	3.86
m20e10	2.00	2.82	28.72	0.27433E+08	0.49890E+19	0.79924E+04	0.86763E+04	3.84
m20e11	2.00	2.94	28.12	0.30521E+08	0.82364E+19	0.77822E+04	0.10356E+05	3.80
m20e12	2.00	3.17	26.84	0.37291E+08	0.18726E+20	0.74110E+04	0.14513E+05	3.74
m20e13	2.00	3.38	25.60	0.43722E+08	0.34190E+20	0.70896E+04	0.19861E+05	3.68
m20e14	2.00	3.55	24.68	0.48411E+08	0.49701E+20	0.68620E+04	0.25179E+05	3.64
m20e15	2.00	3.80	23.31	0.55096E+08	0.79223E+20	0.65350E+04	0.36271E+05	3.58
m20e16	2.00	3.97	22.42	0.59340E+08	0.10331E+21	0.63336E+04	0.46170E+05	3.54
m20e17	2.00	4.06	21.93	0.61736E+08	0.11900E+21	0.62253E+04	0.52882E+05	3.52
m20e18	2.00	4.16	21.40	0.64382E+08	0.13824E+21	0.61113E+04	0.61336E+05	3.50
m20e19	2.00	4.32	20.21	0.70536E+08	0.19152E+21	0.59159E+04	0.80849E+05	3.47

m24e00	2.40	1.71	36.81	0.21470E+08	0.13157E+18	0.10919E+05	0.20959E+04	4.35
m24e01	2.40	1.77	37.99	0.21732E+08	0.13088E+18	0.10833E+05	0.19544E+04	4.32
m24e02	2.40	1.91	40.95	0.22259E+08	0.12712E+18	0.10612E+05	0.19306E+04	4.26
m24e03	2.40	2.13	44.11	0.23025E+08	0.12361E+18	0.10248E+05	0.20162E+04	4.16
m24e04	2.40	2.28	45.65	0.23605E+08	0.12289E+18	0.99812E+04	0.21620E+04	4.10
m24e05	2.40	2.49	47.11	0.24538E+08	0.12534E+18	0.96338E+04	0.24572E+04	4.03
m24e06	2.40	2.66	50.34	0.27448E+08	0.16321E+18	0.94612E+04	0.25469E+04	3.97
m24e07	2.40	2.58	58.58	0.26021E+08	0.39541E+18	0.99890E+04	0.17941E+04	4.00
m24e08	2.40	2.86	61.87	0.24770E+08	0.11232E+19	0.96175E+04	0.20195E+04	3.91
m24e09	2.40	3.09	61.66	0.26426E+08	0.25718E+19	0.92378E+04	0.24691E+04	3.84
m24e10	2.40	3.24	60.89	0.28685E+08	0.41866E+19	0.90038E+04	0.28551E+04	3.80
m24e11	2.40	3.36	60.05	0.31148E+08	0.61794E+19	0.88071E+04	0.32560E+04	3.77
m24e12	2.40	3.56	58.51	0.35611E+08	0.10828E+20	0.85029E+04	0.40658E+04	3.72
m24e13	2.40	3.94	55.29	0.44641E+08	0.25781E+20	0.79609E+04	0.62735E+04	3.63
m24e14	2.40	4.33	52.36	0.52787E+08	0.47360E+20	0.74927E+04	0.95176E+04	3.54
m24e15	2.40	4.61	50.25	0.58429E+08	0.68036E+20	0.71897E+04	0.12771E+05	3.49
m24e16	2.40	5.06	46.98	0.66235E+08	0.10562E+21	0.67459E+04	0.20351E+05	3.41
m24e17	2.40	5.41	44.50	0.71098E+08	0.13515E+21	0.64366E+04	0.29093E+05	3.35
m24e18	2.40	5.62	43.04	0.73757E+08	0.15359E+21	0.62654E+04	0.35949E+05	3.32
m24e19	2.40	5.86	41.40	0.76643E+08	0.17557E+21	0.60744E+04	0.46078E+05	3.28

m26e00	2.60	1.78	49.89	0.21914E+08	0.12233E+18	0.11537E+05	0.17803E+04	4.35
m26e01	2.60	1.86	52.01	0.22230E+08	0.12116E+18	0.11429E+05	0.16315E+04	4.32
m26e02	2.60	1.95	54.64	0.22563E+08	0.11881E+18	0.11287E+05	0.15754E+04	4.27
m26e03	2.60	2.06	57.40	0.22934E+08	0.11634E+18	0.11111E+05	0.15286E+04	4.22
m26e04	2.60	2.21	60.22	0.23445E+08	0.11424E+18	0.10859E+05	0.15058E+04	4.16
m26e05	2.60	2.42	63.26	0.24156E+08	0.11302E+18	0.10519E+05	0.15359E+04	4.09
m26e06	2.60	2.71	66.21	0.25598E+08	0.11862E+18	0.10046E+05	0.17033E+04	3.99
m26e07	2.60	2.80	67.49	0.26478E+08	0.12690E+18	0.99282E+04	0.17553E+04	3.96
m26e08	2.60	2.74	78.44	0.26456E+08	0.28685E+18	0.10431E+05	0.13467E+04	3.98
m26e09	2.60	2.92	85.18	0.25317E+08	0.70943E+18	0.10308E+05	0.13091E+04	3.92
m26e10	2.60	3.27	86.30	0.26596E+08	0.21251E+19	0.97748E+04	0.15583E+04	3.82

m26e11	2.60	3.48	85.08	0.29488E+08	0.40801E+19	0.94413E+04	0.18419E+04	3.77
m26e12	2.60	3.67	83.59	0.33067E+08	0.68947E+19	0.91546E+04	0.21851E+04	3.72
m26e13	2.60	4.05	80.01	0.41180E+08	0.16428E+20	0.86137E+04	0.31604E+04	3.64
m26e14	2.60	4.46	76.01	0.49136E+08	0.31244E+20	0.81021E+04	0.47254E+04	3.55
m26e15	2.60	4.80	73.09	0.54900E+08	0.46318E+20	0.77364E+04	0.64671E+04	3.49
m26e16	2.60	5.10	67.85	0.63268E+08	0.75816E+20	0.71506E+04	0.11204E+05	3.39
m26e17	2.60	5.60	66.17	0.65698E+08	0.86404E+20	0.69906E+04	0.13334E+05	3.36
m26e18	2.60	5.70	64.30	0.68204E+08	0.93794E+20	0.68335E+04	0.15647E+05	3.33
m26e19	2.60	6.03	62.80	0.71103E+08	0.11385E+21	0.66484E+04	0.19356E+05	3.29
m26e20	2.60	6.33	60.46	0.74222E+08	0.13220E+21	0.64250E+04	0.25147E+05	3.25
m26e21	2.60	6.70	57.88	0.77623E+08	0.15455E+21	0.61787E+04	0.34174E+05	3.20
m26e22	2.60	6.90	56.48	0.79421E+08	0.16743E+21	0.60510E+04	0.40398E+05	3.17
m26e23	2.60	7.10	55.00	0.81301E+08	0.18170E+21	0.59235E+04	0.48137E+05	3.15
m26e24	2.60	7.33	53.40	0.83270E+08	0.19758E+21	0.57882E+04	0.56778E+05	3.12
m26e25	2.60	7.44	51.68	0.85327E+08	0.21521E+21	0.57001E+04	0.64298E+05	3.11

m28e00	2.80	1.86	66.04	0.22322E+08	0.11421E+18	0.12131E+05	0.16688E+04	4.35
m28e01	2.80	1.95	69.62	0.22702E+08	0.11254E+18	0.11996E+05	0.15212E+04	4.30
m28e02	2.80	2.08	74.13	0.23122E+08	0.10960E+18	0.11805E+05	0.14278E+04	4.25
m28e03	2.80	2.24	78.97	0.23642E+08	0.10669E+18	0.11546E+05	0.13364E+04	4.18
m28e04	2.80	2.48	84.08	0.24464E+08	0.10476E+18	0.11142E+05	0.12555E+04	4.09
m28e05	2.80	2.69	87.18	0.25147E+08	0.10444E+18	0.10806E+05	0.12427E+04	4.03
m28e06	2.80	2.90	89.88	0.26290E+08	0.11009E+18	0.10483E+05	0.12757E+04	3.96
m28e07	2.80	2.92	99.06	0.28262E+08	0.17335E+18	0.10708E+05	0.11527E+04	3.95
m28e08	2.80	2.92	111.18	0.26382E+08	0.41282E+18	0.11008E+05	0.10514E+04	3.95
m28e09	2.80	3.09	114.96	0.25758E+08	0.69689E+18	0.10802E+05	0.10359E+04	3.91
m28e10	2.80	3.37	116.86	0.26468E+08	0.15643E+19	0.10389E+05	0.10832E+04	3.83
m28e11	2.80	3.53	116.33	0.27953E+08	0.25076E+19	0.10139E+05	0.11582E+04	3.79
m28e12	2.80	3.68	115.25	0.30236E+08	0.39006E+19	0.98964E+04	0.12660E+04	3.75
m28e13	2.80	3.99	112.43	0.35789E+08	0.82080E+19	0.94574E+04	0.15667E+04	3.68
m28e14	2.80	4.25	109.63	0.40951E+08	0.13894E+20	0.91057E+04	0.19369E+04	3.63
m28e15	2.80	4.75	103.94	0.50401E+08	0.29719E+20	0.84925E+04	0.29811E+04	3.53
m28e16	2.80	5.21	98.83	0.57853E+08	0.48369E+20	0.80055E+04	0.44214E+04	3.45
m28e17	2.80	5.55	95.49	0.62606E+08	0.63724E+20	0.76902E+04	0.58273E+04	3.40
m28e18	2.80	6.02	90.97	0.68392E+08	0.86647E+20	0.72974E+04	0.84586E+04	3.33
m28e19	2.80	6.27	88.73	0.71612E+08	0.10174E+21	0.71048E+04	0.10246E+05	3.29
m28e20	2.80	6.61	85.57	0.75204E+08	0.12063E+21	0.68595E+04	0.13272E+05	3.24
m28e21	2.80	7.05	81.70	0.79237E+08	0.14462E+21	0.65635E+04	0.18503E+05	3.19
m28e22	2.80	7.31	79.49	0.81410E+08	0.15884E+21	0.64016E+04	0.22438E+05	3.16
m28e23	2.80	7.62	76.98	0.83720E+08	0.17500E+21	0.62212E+04	0.28099E+05	3.12
m28e24	2.80	8.00	73.92	0.86193E+08	0.19350E+21	0.60121E+04	0.37056E+05	3.08
m28e25	2.80	8.20	72.20	0.87483E+08	0.20367E+21	0.59009E+04	0.43036E+05	3.06

m30e00	3.00	1.93	85.57	0.22700E+08	0.10695E+18	0.12700E+05	0.16539E+04	4.34
m30e01	3.00	2.05	91.29	0.23168E+08	0.10487E+18	0.12526E+05	0.14969E+04	4.29
m30e02	3.00	2.21	98.39	0.23619E+08	0.10106E+18	0.12295E+05	0.13727E+04	4.23
m30e03	3.00	2.41	105.70	0.24230E+08	0.97920E+17	0.11969E+05	0.12399E+04	4.15
m30e04	3.00	2.58	109.97	0.24749E+08	0.96722E+17	0.11700E+05	0.11635E+04	4.09
m30e05	3.00	2.79	114.34	0.25506E+08	0.96928E+17	0.11353E+05	0.10961E+04	4.02
m30e06	3.00	3.06	119.18	0.26980E+08	0.10394E+18	0.10947E+05	0.10518E+04	3.94
m30e07	3.00	3.01	137.59	0.27568E+08	0.23076E+18	0.11444E+05	0.98843E+03	3.96
m30e08	3.00	3.35	153.27	0.26269E+08	0.84152E+18	0.11151E+05	0.90212E+03	3.87
m30e09	3.00	3.41	154.03	0.26403E+08	0.10109E+19	0.11053E+05	0.89527E+03	3.85

m30e10	3.00	3.90	152.90	0.30697E+08	0.36221E+19	0.10327E+05	0.95505E+03	3.73
m30e11	3.00	4.36	147.93	0.38595E+08	0.97260E+19	0.96885E+04	0.12057E+04	3.64
m30e12	3.00	4.89	141.12	0.47784E+08	0.21553E+20	0.90342E+04	0.17583E+04	3.54
m30e13	3.00	5.42	134.38	0.55480E+08	0.36559E+20	0.84780E+04	0.26147E+04	3.45
m30e14	3.00	5.83	129.40	0.60606E+08	0.49682E+20	0.80958E+04	0.35556E+04	3.38
m30e16	3.00	7.13	116.06	0.75068E+08	0.10390E+21	0.71260E+04	0.87050E+04	3.21
m30e17	3.00	7.76	109.48	0.80069E+08	0.12932E+21	0.67309E+04	0.13297E+05	3.13
m30e18	3.00	8.19	105.22	0.82839E+08	0.14507E+21	0.64869E+04	0.17637E+05	3.09
m30e19	3.00	8.44	102.85	0.84288E+08	0.15382E+21	0.63551E+04	0.20702E+05	3.06
m30e20	3.00	8.73	100.09	0.85800E+08	0.16333E+21	0.62060E+04	0.24991E+05	3.03
m30e21	3.00	9.07	96.93	0.87380E+08	0.17367E+21	0.60393E+04	0.31133E+05	3.00
m30e22	3.00	9.47	93.35	0.89029E+08	0.18492E+21	0.58568E+04	0.39515E+05	2.96
m30e23	3.00	9.90	89.26	0.90751E+08	0.19717E+21	0.56629E+04	0.50711E+05	2.92

APPENDIX C

STURM SEQUENCE RESULTS

Sturm Sequence: Periods

"Name" refers to the static model.
Periods are given in hours.

name	P0	P1	P2	P3	P4
m14e00	1.06647	0.80863	0.65170	0.54763	0.46969
m14e01	1.08018	0.81842	0.65935	0.55315	0.47611
m14e02	1.25499	0.95503	0.76970	0.64632	0.55591
m14e03	1.40619	1.07081	0.86163	0.72221	0.61894
m14e04	1.60657	1.22739	0.98814	0.82685	0.70767
m14e05	1.63175	1.24570	1.00308	0.83869	0.71767
m14e06	1.80427	1.37730	1.10639	0.92264	0.78705
m14e07	1.83278	1.39946	1.12448	0.93768	0.79970
m14e08	1.95034	1.48996	1.19731	0.99830	0.85332
m14e09	2.17257	1.66439	1.34000	1.11399	0.95152
m16e00	1.05127	0.79480	0.63901	0.53593	0.45946
m16e01	1.12143	0.84910	0.68281	0.57254	0.49136
m16e02	1.29565	0.98264	0.78842	0.65977	0.56660
m16e03	1.44132	1.09621	0.88015	0.73728	0.63223
m16e04	1.63760	1.24850	1.00266	0.83955	0.72021
m16e05	1.74659	1.33262	1.06921	0.89531	0.76602
m16e06	2.01508	1.53944	1.23552	1.03400	0.88254
m16e07	2.10024	1.60490	1.28858	1.07829	0.92009
m16e08	2.33134	1.78213	1.43276	1.19470	1.01936
m16e09	2.56825	1.96400	1.57999	1.31510	1.12171
m16e10	2.88371	2.21455	1.78459	1.48682	1.26618

m16e11	2.95648	2.26838	1.83080	1.52273	1.29803
m18e00	1.06261	0.80659	0.65061	0.54629	0.46922
m18e01	1.16624	0.88493	0.71175	0.59582	0.51318
m18e02	1.30802	0.99443	0.79952	0.67010	0.57748
m18e03	1.50408	1.14395	0.91749	0.76769	0.65876
m18e04	1.65549	1.26155	1.01082	0.84596	0.72530
m18e05	1.84862	1.41121	1.13300	0.94869	0.81346
m18e06	1.83312	1.40016	1.12476	0.94155	0.80537
m18e07	2.05420	1.57146	1.26128	1.05708	0.90408
m18e08	2.20703	1.68654	1.35305	1.13296	0.96731
m18e09	2.32894	1.78004	1.42866	1.19588	1.02030
m18e10	2.72397	2.08800	1.68228	1.40517	1.19917
m18e11	3.20149	2.45666	1.98109	1.65286	1.41009
m18e12	3.71694	2.84746	2.29439	1.90952	1.62455
m20e00	1.08169	0.82073	0.66181	0.55425	0.47786
m20e01	1.17768	0.89386	0.71899	0.60117	0.51779
m20e02	1.34300	1.02243	0.82203	0.68780	0.59270
m20e03	1.48570	1.13275	0.91048	0.76239	0.65694
m20e04	1.68705	1.28478	1.02947	0.86191	0.73874
m20e05	1.96023	1.49671	1.20073	1.00459	0.86072
m20e06	1.89916	1.45236	1.16627	0.97814	0.83848
m20e07	2.13300	1.63194	1.31139	1.09924	0.93804
m20e09	2.51985	1.92800	1.54811	1.29727	1.10922
m20e10	2.62604	2.00866	1.61220	1.34935	1.15061
m20e11	2.79985	2.14166	1.72155	1.43923	1.22775
m20e12	3.14472	2.40925	1.93951	1.62408	1.38417
m20e13	3.47332	2.65864	2.14112	1.78794	1.52580
m20e14	3.73369	2.85943	2.30244	1.92382	1.64110
m20e15	4.16071	3.18927	2.57269	2.14590	1.82606
m20e16	4.43994	3.40114	2.74680	2.28423	1.94735
m20e17	4.61251	3.52421	2.84275	2.36276	2.01053
m20e18	4.80038	3.67300	2.96426	2.45909	2.09194
m20e19	5.10071	3.90822	3.15111	2.61339	2.22282

m24e00	1.12447	0.85226	0.68510	0.57209	0.49123
m24e01	1.18329	0.89996	0.72567	0.60656	0.52218
m24e02	1.32880	1.01116	0.81332	0.67985	0.58533
m24e03	1.56421	1.19441	0.95996	0.80366	0.69153
m24e04	1.73749	1.32772	1.06715	0.89352	0.76862
m24e05	1.98168	1.51778	1.21828	1.02060	0.87686
m24e06	2.19646	1.67835	1.34302	1.12544	0.96290
m24e07	2.08262	1.58822	1.26990	1.06217	0.90750
m24e08	2.44005	1.86334	1.48990	1.24937	1.06654
m24e09	2.75319	2.09992	1.68314	1.40939	1.20259
m24e10	2.95063	2.25460	1.81207	1.52097	1.29563
m24e11	3.12006	2.38219	1.91134	1.59864	1.36603
m24e12	3.41106	2.60412	2.09309	1.75206	1.49612
m24e13	3.99091	3.04515	2.45319	2.04715	1.74519
m24e14	4.62436	3.52992	2.84217	2.37102	2.02239
m24e15	5.08931	3.87894	3.12785	2.60888	2.22491
m24e16	5.88268	4.48703	3.62515	3.01718	2.56480

m24e17	6.52478	4.96557	4.00618	3.32573	2.83217
m24e18	6.93534	5.28122	4.26348	3.53271	3.00685
m24e19	7.42403	5.66746	4.57456	3.79415	3.22656

m26e00	1.15669	0.88260	0.71264	0.59558	0.51204
m26e01	1.22531	0.93260	0.75121	0.62721	0.53992
m26e02	1.32143	1.00860	0.81346	0.67984	0.58572
m26e03	1.43089	1.08818	0.87314	0.72851	0.62700
m26e04	1.59250	1.21563	0.97580	0.81568	0.70225
m26e05	1.82335	1.39433	1.11906	0.93657	0.80553
m26e06	2.17059	1.66351	1.33569	1.11861	0.96018
m26e07	2.28158	1.74975	1.40480	1.17785	1.00938
m26e08	2.19526	1.68069	1.34772	1.13014	0.96794
m26e10	2.87750	2.19457	1.75716	1.47216	1.25619
m26e11	3.16382	2.41242	1.93191	1.61741	1.38158
m26e12	3.43190	2.61322	2.09957	1.75660	1.50081
m26e13	4.00236	3.04738	2.45221	2.04987	1.74759
m26e14	4.64301	3.53224	2.84646	2.37682	2.02661
m26e15	5.19968	3.95949	3.19632	2.66228	2.26910
m26e16	6.22963	4.73703	3.82784	3.18251	2.71360
m26e17	6.59101	5.01385	4.04712	3.37003	2.86664
m26e19	7.38499	5.61231	4.52373	3.75463	3.19870
m26e20	7.98764	6.07458	4.90541	4.07463	3.46830
m26e21	8.74446	6.64641	5.34474	4.43363	3.76987
m26e22	9.17554	6.98108	5.63068	4.66701	3.96936
m26e23	9.62668	7.31214	5.85827	4.83771	4.10218
m26e24	10.12710	7.69100	6.18268	5.11765	4.34798
m26e25	10.34870	7.87927	6.32740	5.23167	4.44365

m28e00	1.18228	0.90106	0.72617	0.60594	0.52097
m28e01	1.27189	0.96939	0.78043	0.65028	0.56025
m28e02	1.39919	1.06873	0.86185	0.71944	0.62028
m28e03	1.56250	1.19074	0.95541	0.79705	0.68581
m28e04	1.82931	1.39914	1.12298	0.93924	0.80735
m28e05	2.06454	1.57984	1.26826	1.06283	0.91335
m28e06	2.31619	1.77625	1.42478	1.19385	1.02426
m28e07	2.33179	1.78127	1.42480	1.19319	1.02061
m28e08	2.34101	1.78871	1.43238	1.20145	1.02699
m28e09	2.54126	1.93671	1.54875	1.29883	1.10728
m28e10	2.89943	2.21063	1.76733	1.48184	1.26490
m28e11	3.10813	2.36711	1.89702	1.58876	1.35393
m28e12	3.32046	2.52604	2.02526	1.69517	1.44401
m28e13	3.74568	2.84562	2.28443	1.90752	1.62507
m28e14	4.14207	3.15142	2.53197	2.11803	1.81157
m28e15	4.92169	3.73937	3.01183	2.51409	2.14357
m28e16	5.68409	4.32005	3.48662	2.90298	2.47534
m28e17	6.27001	4.75724	3.83941	3.19934	2.72336
m28e18	7.09081	5.38917	4.35228	3.61223	3.07932
m28e19	7.58354	5.75224	4.64467	3.86497	3.28812
m28e20	8.20293	6.21928	5.00675	4.15633	3.54462
m28e21	9.06715	6.86618	5.51832	4.57674	3.89480
m28e22	9.59984	7.28412	5.86743	4.86259	4.14471
m28e23	10.28460	7.81179	6.28595	5.21276	4.43946

m28e24	11.11550	8.43597	6.77101	5.61598	4.77052
m28e25	11.58270	8.80007	7.06162	5.85152	4.97362
m30e00	1.21018	0.92352	0.74459	0.62105	0.53349
m30e01	1.32304	1.01005	0.81352	0.67746	0.58342
m30e02	1.48029	1.13174	0.91125	0.76022	0.65536
m30e03	1.68906	1.28861	1.03380	0.86241	0.74212
m30e04	1.88459	1.42519	1.14222	0.95358	0.81958
m30e05	2.10782	1.61496	1.29713	1.08507	0.93241
m30e06	2.43155	1.86551	1.45427	1.25214	1.07324
m30e07	2.36227	1.80245	1.44163	1.20864	1.03310
m30e08	2.77673	2.11344	1.68975	1.41937	1.20967
m30e09	2.86262	2.17773	1.74054	1.46303	1.24724
m30e10	3.50148	2.65950	2.12917	1.78459	1.52144
m30e11	4.14953	3.14728	2.52537	2.10990	1.79987
m30e12	4.97690	3.77565	3.03667	2.53646	2.16773
m30e13	5.82960	4.41512	3.55881	2.96480	2.52763
m30e14	6.53175	4.94941	3.99484	3.32518	2.83068
m30e16	8.91092	6.74472	5.42950	4.50572	3.83986
m30e17	10.17020	7.68978	6.18072	5.13110	4.37025
m30e18	11.07930	8.38691	6.73283	5.57700	4.74691
m30e19	11.61170	8.77107	7.02248	5.82075	4.94835
m30e20	12.26540	9.27280	7.41843	6.14294	5.21434
m30e21	13.05400	9.86375	7.88731	6.51223	5.52065
m30e22	13.99800	10.60990	8.48164	7.02529	5.95656
m30e23	15.08600	11.43570	9.11801	7.52271	6.36073

Sturm Sequence: Period Ratios

name	P1/P0	P2/P0	P3/P0	P4/P0	P2/P1	P3/P1	P4/P1	P3/P2	P4/P2	P4/P3
m14e00	0.75823	0.61109	0.51350	0.44041	0.80594	0.67723	0.58085	0.84030	0.72071	0.85767
m14e01	0.75767	0.61041	0.51209	0.44077	0.80563	0.67587	0.58174	0.83894	0.72208	0.86072
m14e02	0.76098	0.61331	0.51500	0.44296	0.80595	0.67675	0.58209	0.83970	0.72224	0.86012
m14e03	0.76150	0.61274	0.51360	0.44016	0.80465	0.67445	0.57801	0.83819	0.71834	0.85701
m14e04	0.76398	0.61507	0.51467	0.44048	0.80508	0.67367	0.57656	0.83677	0.71616	0.85586
m14e05	0.76341	0.61473	0.51398	0.43982	0.80523	0.67327	0.57612	0.83611	0.71547	0.85571
m14e06	0.76336	0.61321	0.51136	0.43622	0.80330	0.66989	0.57144	0.83392	0.71137	0.85304
m14e07	0.76357	0.61354	0.51162	0.43633	0.80351	0.67003	0.57144	0.83388	0.71117	0.85285
m14e08	0.76395	0.61390	0.51186	0.43752	0.80359	0.67002	0.57271	0.83379	0.71269	0.85476
m14e09	0.76609	0.61678	0.51275	0.43797	0.80510	0.66931	0.57169	0.83134	0.71009	0.85415
m16e00	0.75604	0.60785	0.50979	0.43705	0.80399	0.67429	0.57808	0.83868	0.71901	0.85731
m16e01	0.75715	0.60888	0.51054	0.43815	0.80416	0.67429	0.57868	0.83850	0.71961	0.85821
m16e02	0.75841	0.60852	0.50922	0.43731	0.80235	0.67143	0.57661	0.83682	0.71865	0.85879
m16e03	0.76056	0.61066	0.51153	0.43865	0.80290	0.67257	0.57674	0.83767	0.71832	0.85752
m16e04	0.76240	0.61227	0.51267	0.43980	0.80309	0.67245	0.57686	0.83733	0.71830	0.85785
m16e05	0.76298	0.61217	0.51261	0.43858	0.80234	0.67184	0.57483	0.83736	0.71644	0.85559

m16e06	0.76396	0.61314	0.51313	0.43797	0.80258	0.67167	0.57329	0.83689	0.71431	0.85352
m16e07	0.76415	0.61354	0.51341	0.43809	0.80290	0.67187	0.57330	0.83680	0.71403	0.85328
m16e08	0.76442	0.61457	0.51245	0.43724	0.80396	0.67038	0.57199	0.83385	0.71147	0.85324
m16e09	0.76472	0.61520	0.51206	0.43676	0.80448	0.66960	0.57114	0.83235	0.70995	0.85295
m16e10	0.76495	0.61885	0.51559	0.43908	0.80585	0.67139	0.57175	0.83314	0.70951	0.85160
m16e11	0.76726	0.61925	0.51505	0.43905	0.80710	0.67129	0.57223	0.83173	0.70900	0.85244

m18e00	0.75907	0.61228	0.51410	0.44157	0.80662	0.67728	0.58173	0.83966	0.72120	0.85892
m18e01	0.75879	0.61029	0.51089	0.44003	0.80430	0.67329	0.57990	0.83712	0.72101	0.86130
m18e02	0.76025	0.61125	0.51230	0.44149	0.80400	0.67385	0.58072	0.83813	0.72228	0.86178
m18e03	0.76056	0.61000	0.51041	0.43798	0.80204	0.67109	0.57587	0.83673	0.71800	0.85811
m18e04	0.76204	0.61059	0.51101	0.43812	0.80125	0.67058	0.57493	0.83691	0.71753	0.85736
m18e05	0.76339	0.61289	0.51319	0.44004	0.80286	0.67226	0.57643	0.83733	0.71797	0.85746
m18e06	0.76381	0.61358	0.51363	0.43934	0.80331	0.67246	0.57520	0.83711	0.71604	0.85537
m18e07	0.76500	0.61400	0.51459	0.44012	0.80262	0.67267	0.57532	0.83810	0.71680	0.85527
m18e08	0.76417	0.61306	0.51334	0.43828	0.80226	0.67177	0.57355	0.83734	0.71491	0.85379
m18e09	0.76431	0.61344	0.51349	0.43810	0.80260	0.67183	0.57319	0.83706	0.71417	0.85318
m18e10	0.76653	0.61758	0.51585	0.44023	0.80569	0.67297	0.57432	0.83528	0.71282	0.85340
m18e11	0.76735	0.61880	0.51628	0.44045	0.80642	0.67281	0.57399	0.83432	0.71177	0.85312
m18e12	0.76608	0.61728	0.51373	0.43707	0.80577	0.67060	0.57053	0.83226	0.70805	0.85076

m20e00	0.75875	0.61183	0.51239	0.44178	0.80637	0.67531	0.58224	0.83747	0.72205	0.86218
m20e01	0.75900	0.61051	0.51047	0.43967	0.80436	0.67256	0.57928	0.83614	0.72017	0.86131
m20e02	0.76130	0.61208	0.51214	0.44133	0.80399	0.67272	0.57970	0.83672	0.72102	0.86173
m20e03	0.76244	0.61283	0.51315	0.44218	0.80377	0.67304	0.57995	0.83735	0.72154	0.86169
m20e04	0.76155	0.61022	0.51090	0.43789	0.80128	0.67086	0.57499	0.83723	0.71759	0.85710
m20e05	0.76354	0.61255	0.51249	0.43909	0.80225	0.67120	0.57507	0.83665	0.71683	0.85679
m20e06	0.76474	0.61410	0.51504	0.44150	0.80302	0.67348	0.57732	0.83869	0.71894	0.85722
m20e07	0.76509	0.61481	0.51535	0.43978	0.80358	0.67358	0.57480	0.83823	0.71530	0.85335
m20e09	0.76512	0.61437	0.51482	0.44019	0.80296	0.67286	0.57532	0.83797	0.71650	0.85504
m20e10	0.76490	0.61393	0.51383	0.43815	0.80262	0.67177	0.57282	0.83696	0.71369	0.85271
m20e11	0.76492	0.61487	0.51404	0.43851	0.80384	0.67202	0.57327	0.83601	0.71317	0.85306
m20e12	0.76613	0.61675	0.51645	0.44016	0.80503	0.67410	0.57452	0.83737	0.71367	0.85228
m20e13	0.76545	0.61645	0.51476	0.43929	0.80534	0.67250	0.57390	0.83505	0.71262	0.85338
m20e14	0.76585	0.61667	0.51526	0.43954	0.80521	0.67280	0.57393	0.83556	0.71277	0.85304
m20e15	0.76652	0.61833	0.51575	0.43888	0.80667	0.67285	0.57256	0.83411	0.70979	0.85095
m20e16	0.76603	0.61866	0.51447	0.43860	0.80761	0.67161	0.57256	0.83160	0.70895	0.85252
m20e17	0.76405	0.61631	0.51225	0.43589	0.80663	0.67044	0.57049	0.83115	0.70725	0.85092
m20e18	0.76515	0.61751	0.51227	0.43579	0.80704	0.66950	0.56955	0.82958	0.70572	0.85070
m20e19	0.76621	0.61778	0.51236	0.43579	0.80628	0.66869	0.56876	0.82936	0.70541	0.85055

m24e00	0.75792	0.60926	0.50877	0.43685	0.80386	0.67127	0.57638	0.83506	0.71702	0.85865
m24e01	0.76056	0.61327	0.51260	0.44130	0.80634	0.67398	0.58022	0.83585	0.71958	0.86089
m24e02	0.76096	0.61207	0.51162	0.44049	0.80435	0.67234	0.57887	0.83589	0.71968	0.86097
m24e03	0.76359	0.61371	0.51378	0.44210	0.80371	0.67285	0.57897	0.83718	0.72037	0.86047
m24e04	0.76416	0.61419	0.51426	0.44238	0.80375	0.67297	0.57890	0.83729	0.72026	0.86022
m24e05	0.76591	0.61477	0.51502	0.44248	0.80267	0.67243	0.57773	0.83774	0.71975	0.85916
m24e06	0.76412	0.61145	0.51239	0.43839	0.80020	0.67056	0.57372	0.83799	0.71697	0.85558
m24e07	0.76261	0.60976	0.51002	0.43575	0.79957	0.66878	0.57139	0.83642	0.71462	0.85438
m24e08	0.76365	0.61060	0.51203	0.43710	0.79959	0.67050	0.57238	0.83856	0.71585	0.85366
m24e09	0.76272	0.61134	0.51191	0.43680	0.80153	0.67116	0.57268	0.83736	0.71449	0.85327
m24e10	0.76411	0.61413	0.51547	0.43910	0.80372	0.67461	0.57466	0.83935	0.71500	0.85184
m24e11	0.76351	0.61260	0.51237	0.43782	0.80235	0.67108	0.57343	0.83640	0.71470	0.85450

m24e12	0.76343	0.61362	0.51364	0.43861	0.80376	0.67280	0.57452	0.83707	0.711479	0.85392
m24e13	0.76302	0.61469	0.51295	0.43729	0.80561	0.67227	0.57310	0.83448	0.71140	0.85250
m24e14	0.76333	0.61461	0.51272	0.43733	0.80517	0.67169	0.57293	0.83423	0.71157	0.85296
m24e15	0.76217	0.61459	0.51262	0.43717	0.80637	0.67258	0.57359	0.83408	0.71132	0.85282
m24e16	0.76275	0.61624	0.51209	0.43599	0.80792	0.67242	0.57160	0.83229	0.70750	0.85007
m24e17	0.76103	0.61399	0.50971	0.43406	0.80679	0.66976	0.57036	0.83015	0.70695	0.85159
m24e18	0.76149	0.61475	0.50938	0.43355	0.80723	0.66952	0.56935	0.82868	0.70520	0.85115
m24e19	0.76339	0.61618	0.51106	0.43461	0.80716	0.66946	0.56931	0.82940	0.70533	0.85040

m26e00	0.76304	0.61610	0.51490	0.44268	0.80743	0.67480	0.58015	0.83574	0.71852	0.85974
m26e01	0.76112	0.61308	0.51188	0.44064	0.80550	0.67253	0.57894	0.83493	0.71873	0.86083
m26e02	0.76326	0.61559	0.51447	0.44325	0.80652	0.67404	0.58073	0.83574	0.72004	0.86156
m26e03	0.76049	0.61020	0.50913	0.43819	0.80238	0.66947	0.57619	0.83436	0.71810	0.86066
m26e04	0.76335	0.61275	0.51220	0.44097	0.80271	0.67099	0.57768	0.83591	0.71966	0.86009
m26e05	0.76471	0.61374	0.51365	0.44179	0.80258	0.67170	0.57772	0.83692	0.71983	0.86004
m26e06	0.76639	0.61536	0.51535	0.44236	0.80293	0.67244	0.57720	0.83748	0.71887	0.85837
m26e07	0.76690	0.61571	0.51624	0.44240	0.80286	0.67315	0.57687	0.83845	0.71852	0.85697
m26e08	0.76560	0.61392	0.51481	0.44092	0.80188	0.67243	0.57592	0.83856	0.71820	0.85648
m26e10	0.76267	0.61066	0.51161	0.43656	0.80069	0.67082	0.57241	0.83781	0.71490	0.85330
m26e11	0.76250	0.61063	0.51122	0.43668	0.80082	0.67045	0.57269	0.83721	0.71514	0.85419
m26e12	0.76145	0.61178	0.51184	0.43731	0.80344	0.67220	0.57431	0.83665	0.71482	0.85438
m26e13	0.76140	0.61269	0.51217	0.43664	0.80469	0.67267	0.57347	0.83593	0.71266	0.85254
m26e14	0.76077	0.61306	0.51191	0.43649	0.80585	0.67289	0.57375	0.83501	0.71198	0.85266
m26e15	0.76149	0.61471	0.51201	0.43639	0.80726	0.67238	0.57308	0.83292	0.70991	0.85231
m26e16	0.76040	0.61446	0.51087	0.43560	0.80807	0.67184	0.57285	0.83141	0.70891	0.85266
m26e17	0.76071	0.61404	0.51131	0.43493	0.80719	0.67214	0.57174	0.83270	0.70832	0.85063
m26e19	0.75996	0.61256	0.50841	0.43314	0.80604	0.66900	0.56994	0.82999	0.70709	0.85193
m26e20	0.76050	0.61413	0.51012	0.43421	0.80753	0.67077	0.57095	0.83064	0.70704	0.85119
m26e21	0.76007	0.61121	0.50702	0.43112	0.80415	0.66707	0.56720	0.82953	0.70534	0.85029
m26e22	0.76084	0.61366	0.50864	0.43260	0.80656	0.66852	0.56859	0.82885	0.70495	0.85051
m26e23	0.75957	0.60855	0.50253	0.42613	0.80117	0.66160	0.56101	0.82579	0.70024	0.84796
m26e24	0.75945	0.61051	0.50534	0.42934	0.80389	0.66541	0.56533	0.82774	0.70325	0.84960
m26e25	0.76138	0.61142	0.50554	0.42939	0.80304	0.66398	0.56397	0.82683	0.70229	0.84938

m28e00	0.76214	0.61421	0.51252	0.44065	0.80591	0.67247	0.57818	0.83443	0.71742	0.85978
m28e01	0.76217	0.61360	0.51127	0.44048	0.80507	0.67081	0.57793	0.83323	0.71787	0.86155
m28e02	0.76382	0.61597	0.51418	0.44331	0.80643	0.67317	0.58039	0.83475	0.71970	0.86217
m28e03	0.76207	0.61146	0.51011	0.43892	0.80237	0.66937	0.57595	0.83425	0.71781	0.86044
m28e04	0.76485	0.61388	0.51344	0.44134	0.80262	0.67129	0.57703	0.83638	0.71893	0.85958
m28e05	0.76523	0.61431	0.51480	0.44240	0.80278	0.67275	0.57813	0.83802	0.72016	0.85936
m28e06	0.76688	0.61514	0.51544	0.44222	0.80213	0.67212	0.57664	0.83792	0.71889	0.85795
m28e07	0.76391	0.61103	0.51171	0.43769	0.79988	0.66985	0.57297	0.83744	0.71632	0.85536
m28e08	0.76408	0.61186	0.51322	0.43870	0.80079	0.67169	0.57415	0.83878	0.71698	0.85479
m28e09	0.76211	0.60944	0.51110	0.43572	0.79968	0.67064	0.57173	0.83863	0.71495	0.85252
m28e10	0.76244	0.60954	0.51108	0.43626	0.79947	0.67032	0.57219	0.83846	0.71571	0.85360
m28e11	0.76159	0.61034	0.51116	0.43561	0.80141	0.67118	0.57198	0.83750	0.71371	0.85219
m28e12	0.76075	0.60993	0.51052	0.43488	0.80175	0.67108	0.57165	0.83701	0.71300	0.85184
m28e13	0.75971	0.60988	0.50926	0.43385	0.80279	0.67034	0.57108	0.83501	0.71137	0.85193
m28e14	0.76083	0.61128	0.51135	0.43736	0.80344	0.67209	0.57484	0.83651	0.71548	0.85531
m28e15	0.75977	0.61195	0.51082	0.43554	0.80544	0.67233	0.57324	0.83474	0.71172	0.85262
m28e16	0.76002	0.61340	0.51072	0.43549	0.80708	0.67198	0.57299	0.83261	0.70995	0.85269
m28e17	0.75873	0.61235	0.51026	0.43435	0.80707	0.67252	0.57247	0.83329	0.70932	0.85123
m28e18	0.76002	0.61379	0.50942	0.43427	0.80760	0.67028	0.57139	0.82996	0.70752	0.85247

m28e19	0.75852	0.61247	0.50965	0.43359	0.80745	0.67191	0.57162	0.83213	0.70793	0.85075
m28e20	0.75818	0.61036	0.50669	0.43212	0.80504	0.66830	0.56994	0.83015	0.70797	0.85282
m28e21	0.75726	0.60861	0.50476	0.42955	0.80370	0.66656	0.56724	0.82937	0.70579	0.85100
m28e22	0.75878	0.61120	0.50653	0.43175	0.80551	0.66756	0.56901	0.82874	0.70639	0.85237
m28e23	0.75850	0.61120	0.50685	0.43100	0.80401	0.66729	0.56830	0.82927	0.70625	0.85165
m28e24	0.75894	0.60915	0.50524	0.42918	0.80264	0.66572	0.56550	0.82942	0.70455	0.84945
m28e25	0.75970	0.60967	0.50510	0.42940	0.80245	0.66494	0.56518	0.82864	0.70432	0.84927
m30e00	0.76313	0.61527	0.51319	0.44083	0.80625	0.67248	0.57766	0.83408	0.71648	0.85901
m30e01	0.76343	0.61488	0.51205	0.44097	0.80542	0.67072	0.57761	0.83276	0.71716	0.86118
m30e02	0.76454	0.61559	0.51356	0.44272	0.80517	0.67173	0.57907	0.83427	0.71919	0.86206
m30e03	0.76292	0.61206	0.51058	0.43937	0.80226	0.66925	0.57591	0.83421	0.71785	0.86052
m30e04	0.76434	0.61259	0.51142	0.43955	0.80145	0.66909	0.57507	0.83485	0.71753	0.85948
m30e05	0.76618	0.61539	0.51478	0.44236	0.80320	0.67189	0.57736	0.83652	0.71882	0.85930
m30e06	0.76631	0.61454	0.51496	0.44138	0.80194	0.67200	0.57599	0.83796	0.71824	0.85712
m30e07	0.76302	0.61027	0.51164	0.43733	0.79982	0.67055	0.57316	0.83838	0.71662	0.85476
m30e08	0.76113	0.60854	0.51117	0.43565	0.79953	0.67159	0.57237	0.83999	0.71589	0.85226
m30e09	0.76075	0.60802	0.51108	0.43570	0.79925	0.67181	0.57272	0.84056	0.71658	0.85250
m30e10	0.75954	0.60808	0.50967	0.43451	0.80059	0.67102	0.57208	0.83816	0.71457	0.85254
m30e11	0.75847	0.60859	0.50847	0.43375	0.80240	0.67039	0.57188	0.83548	0.71272	0.85306
m30e12	0.75863	0.61015	0.50965	0.43556	0.80428	0.67179	0.57413	0.83528	0.71385	0.85463
m30e13	0.75736	0.61047	0.50858	0.43359	0.80605	0.67151	0.57249	0.83309	0.71025	0.85255
m30e14	0.75775	0.61160	0.50908	0.43337	0.80713	0.67183	0.57192	0.83237	0.70858	0.85129
m30e16	0.75691	0.60931	0.50564	0.43092	0.80500	0.66804	0.56931	0.82986	0.70722	0.85222
m30e17	0.75611	0.60773	0.50452	0.42971	0.80376	0.66726	0.56832	0.83018	0.70708	0.85172
m30e18	0.75699	0.60769	0.50337	0.42845	0.80278	0.66496	0.56599	0.82833	0.70504	0.85116
m30e19	0.75536	0.60478	0.50128	0.42615	0.80064	0.66363	0.56417	0.82887	0.70464	0.85012
m30e20	0.75601	0.60483	0.50083	0.42513	0.80002	0.66247	0.56233	0.82806	0.70289	0.84883
m30e21	0.75561	0.60421	0.49887	0.42291	0.79963	0.66022	0.55969	0.82566	0.69994	0.84774
m30e22	0.75796	0.60592	0.50188	0.42553	0.79941	0.66214	0.56142	0.82829	0.70229	0.84787
m30e23	0.75803	0.60440	0.49866	0.42163	0.79733	0.65783	0.55622	0.82504	0.69760	0.84554

Sturm Sequence: Q Values

Values are given in days.

name	Q0	Q1	Q2	Q3	Q4
m14e00	0.03252	0.02466	0.01987	0.01670	0.01432
m14e01	0.03256	0.02467	0.01987	0.01667	0.01435
m14e02	0.03266	0.02486	0.02003	0.01682	0.01447
m14e03	0.03266	0.02487	0.02001	0.01678	0.01438
m14e04	0.03265	0.02494	0.02008	0.01680	0.01438
m14e05	0.03258	0.02487	0.02003	0.01674	0.01433
m14e06	0.03258	0.02487	0.01998	0.01666	0.01421
m14e07	0.03260	0.02489	0.02000	0.01668	0.01423

m14e08	0.03266	0.02495	0.02005	0.01672	0.01429
m14e09	0.03287	0.02518	0.02027	0.01686	0.01440
m16e00	0.03240	0.02450	0.01970	0.01652	0.01416
m16e01	0.03244	0.02456	0.01975	0.01656	0.01421
m16e02	0.03244	0.02460	0.01974	0.01652	0.01419
m16e03	0.03253	0.02474	0.01986	0.01664	0.01427
m16e04	0.03256	0.02482	0.01994	0.01669	0.01432
m16e05	0.03256	0.02484	0.01993	0.01669	0.01428
m16e06	0.03260	0.02490	0.01999	0.01673	0.01428
m16e07	0.03264	0.02494	0.02003	0.01676	0.01430
m16e08	0.03259	0.02491	0.02003	0.01670	0.01425
m16e09	0.03273	0.02503	0.02013	0.01676	0.01429
m16e10	0.03298	0.02533	0.02041	0.01700	0.01448
m16e11	0.03304	0.02535	0.02046	0.01702	0.01451
m18e00	0.03251	0.02468	0.01991	0.01671	0.01436
m18e01	0.03252	0.02468	0.01985	0.01661	0.01431
m18e02	0.03254	0.02474	0.01989	0.01667	0.01437
m18e03	0.03249	0.02471	0.01982	0.01658	0.01423
m18e04	0.03252	0.02478	0.01986	0.01662	0.01425
m18e05	0.03258	0.02487	0.01997	0.01672	0.01434
m18e06	0.03249	0.02482	0.01994	0.01669	0.01427
m18e07	0.03257	0.02491	0.02000	0.01676	0.01433
m18e08	0.03257	0.02489	0.01997	0.01672	0.01428
m18e09	0.03264	0.02495	0.02002	0.01676	0.01430
m18e10	0.03272	0.02508	0.02021	0.01688	0.01440
m18e11	0.03286	0.02522	0.02033	0.01697	0.01447
m18e12	0.03310	0.02536	0.02043	0.01700	0.01447
m20e00	0.03256	0.02471	0.01992	0.01669	0.01439
m20e01	0.03252	0.02468	0.01985	0.01660	0.01430
m20e02	0.03256	0.02479	0.01993	0.01668	0.01437
m20e03	0.03259	0.02485	0.01997	0.01672	0.01441
m20e04	0.03250	0.02475	0.01983	0.01660	0.01423
m20e05	0.03255	0.02485	0.01994	0.01668	0.01429
m20e06	0.03258	0.02491	0.02001	0.01678	0.01438
m20e07	0.03255	0.02491	0.02001	0.01678	0.01432
m20e09	0.03266	0.02499	0.02007	0.01681	0.01438
m20e10	0.03268	0.02499	0.02006	0.01679	0.01432
m20e11	0.03268	0.02499	0.02009	0.01680	0.01433
m20e12	0.03282	0.02515	0.02024	0.01695	0.01445
m20e13	0.03288	0.02517	0.02027	0.01692	0.01444
m20e14	0.03294	0.02523	0.02032	0.01697	0.01448
m20e15	0.03310	0.02537	0.02047	0.01707	0.01453
m20e16	0.03310	0.02535	0.02048	0.01703	0.01452
m20e17	0.03320	0.02536	0.02046	0.01700	0.01447
m20e18	0.03329	0.02547	0.02056	0.01705	0.01451
m20e19	0.03349	0.02566	0.02069	0.01716	0.01460
m24e00	0.03244	0.02459	0.01976	0.01650	0.01417
m24e01	0.03256	0.02476	0.01997	0.01669	0.01437
m24e02	0.03249	0.02472	0.01989	0.01662	0.01431

m24e03	0.03258	0.02488	0.01999	0.01674	0.01440
m24e04	0.03258	0.02490	0.02001	0.01675	0.01441
m24e05	0.03263	0.02500	0.02006	0.01681	0.01444
m24e06	0.03260	0.02491	0.01995	0.01670	0.01429
m24e07	0.03247	0.02476	0.01980	0.01656	0.01415
m24e08	0.03259	0.02489	0.01990	0.01669	0.01425
m24e09	0.03267	0.02492	0.01997	0.01672	0.01427
m24e10	0.03273	0.02501	0.02010	0.01687	0.01437
m24e11	0.03273	0.02499	0.02005	0.01677	0.01433
m24e12	0.03283	0.02506	0.02014	0.01686	0.01440
m24e13	0.03289	0.02509	0.02022	0.01687	0.01438
m24e14	0.03310	0.02527	0.02034	0.01697	0.01448
m24e15	0.03319	0.02530	0.02040	0.01701	0.01451
m24e16	0.03333	0.02542	0.02054	0.01710	0.01453
m24e17	0.03345	0.02545	0.02054	0.01705	0.01452
m24e18	0.03362	0.02560	0.02067	0.01712	0.01458
m24e19	0.03377	0.02578	0.02081	0.01726	0.01468

m26e00	0.03262	0.02489	0.02010	0.01680	0.01444
m26e01	0.03256	0.02478	0.01996	0.01667	0.01435
m26e02	0.03259	0.02488	0.02006	0.01677	0.01445
m26e03	0.03244	0.02467	0.01980	0.01652	0.01422
m26e04	0.03252	0.02482	0.01992	0.01665	0.01434
m26e05	0.03262	0.02494	0.02002	0.01675	0.01441
m26e06	0.03268	0.02505	0.02011	0.01684	0.01446
m26e07	0.03269	0.02507	0.02013	0.01688	0.01446
m26e08	0.03259	0.02495	0.02001	0.01678	0.01437
m26e10	0.03272	0.02495	0.01998	0.01674	0.01428
m26e11	0.03277	0.02498	0.02001	0.01675	0.01431
m26e12	0.03283	0.02500	0.02009	0.01681	0.01436
m26e13	0.03296	0.02510	0.02019	0.01688	0.01439
m26e14	0.03307	0.02516	0.02027	0.01693	0.01443
m26e15	0.03320	0.02528	0.02041	0.01700	0.01449
m26e16	0.03334	0.02535	0.02049	0.01703	0.01452
m26e17	0.03346	0.02545	0.02054	0.01711	0.01455
m26e19	0.03353	0.02548	0.02054	0.01705	0.01453
m26e20	0.03368	0.02561	0.02068	0.01718	0.01462
m26e21	0.03389	0.02576	0.02071	0.01718	0.01461
m26e22	0.03401	0.02588	0.02087	0.01730	0.01471
m26e23	0.03415	0.02594	0.02078	0.01716	0.01455
m26e24	0.03427	0.02603	0.02092	0.01732	0.01471
m26e25	0.03428	0.02610	0.02096	0.01733	0.01472

m28e00	0.03259	0.02484	0.02002	0.01670	0.01436
m28e01	0.03258	0.02483	0.01999	0.01666	0.01435
m28e02	0.03260	0.02490	0.02008	0.01676	0.01445
m28e03	0.03248	0.02475	0.01986	0.01657	0.01426
m28e04	0.03260	0.02494	0.02002	0.01674	0.01439
m28e05	0.03266	0.02500	0.02007	0.01682	0.01445
m28e06	0.03270	0.02507	0.02011	0.01685	0.01446
m28e07	0.03261	0.02491	0.01993	0.01669	0.01428
m28e08	0.03263	0.02493	0.01996	0.01675	0.01431
m28e09	0.03264	0.02487	0.01989	0.01668	0.01422

m28e10	0.03273	0.02495	0.01995	0.01673	0.01428
m28e11	0.03271	0.02491	0.01997	0.01672	0.01425
m28e12	0.03273	0.02490	0.01996	0.01671	0.01423
m28e13	0.03263	0.02494	0.02002	0.01672	0.01424
m28e14	0.03302	0.02512	0.02018	0.01688	0.01444
m28e15	0.03313	0.02517	0.02027	0.01692	0.01443
m28e16	0.03328	0.02530	0.02042	0.01700	0.01449
m28e17	0.03340	0.02534	0.02045	0.01704	0.01451
m28e18	0.03346	0.02543	0.02054	0.01705	0.01453
m28e19	0.03366	0.02553	0.02061	0.01715	0.01459
m28e20	0.03366	0.02552	0.02055	0.01706	0.01455
m28e21	0.03375	0.02556	0.02054	0.01704	0.01450
m28e22	0.03384	0.02568	0.02068	0.01714	0.01461
m28e23	0.03409	0.02589	0.02083	0.01728	0.01471
m28e24	0.03428	0.02601	0.02088	0.01732	0.01471
m28e25	0.03437	0.02612	0.02096	0.01737	0.01476

m30e00	0.03262	0.02490	0.02007	0.01674	0.01438
m30e01	0.03260	0.02489	0.02005	0.01669	0.01438
m30e02	0.03261	0.02493	0.02007	0.01675	0.01444
m30e03	0.03253	0.02482	0.01991	0.01661	0.01429
m30e04	0.03256	0.02489	0.01995	0.01665	0.01431
m30e05	0.03266	0.02503	0.02010	0.01682	0.01445
m30e06	0.03275	0.02510	0.02013	0.01687	0.01446
m30e07	0.03263	0.02490	0.01991	0.01670	0.01427
m30e08	0.03273	0.02491	0.01992	0.01673	0.01426
m30e09	0.03274	0.02490	0.01990	0.01673	0.01426
m30e10	0.03284	0.02494	0.01997	0.01674	0.01427
m30e11	0.03294	0.02499	0.02005	0.01675	0.01429
m30e12	0.03319	0.02518	0.02025	0.01691	0.01446
m30e13	0.03333	0.02524	0.02035	0.01695	0.01445
m30e14	0.03345	0.02535	0.02046	0.01703	0.01450
m30e16	0.03377	0.02556	0.02057	0.01707	0.01455
m30e17	0.03393	0.02565	0.02062	0.01712	0.01458
m30e18	0.03409	0.02580	0.02072	0.01716	0.01461
m30e19	0.03417	0.02581	0.02066	0.01713	0.01456
m30e20	0.03431	0.02594	0.02075	0.01718	0.01458
m30e21	0.03447	0.02604	0.02083	0.01719	0.01458
m30e22	0.03467	0.02628	0.02101	0.01740	0.01475
m30e23	0.03493	0.02648	0.02111	0.01742	0.01473

APPENDIX D

NEWCONV RESULTS

NEWCONV: Periods

"Name" refers to the static model from which the parameters were taken to create the model envelope analysed by NEWCONV.

Periods are given in minutes.

name	P0	P1	P2	P3	P4
m14e00	59.342	43.690	34.474	28.368	24.048
m14e01	60.034	44.222	34.891	28.714	24.350
m14e02	67.205	49.104	38.534	31.608	26.726
m14e03	79.790	59.386	46.987	38.692	32.760
m14e04	92.966	69.509	54.979	45.058	37.886
m14e05	88.819	65.102	51.091	41.861	35.309
m14e06	104.77	78.739	62.496	51.552	43.632
m14e07	106.55	80.107	63.619	52.416	44.338
m14e08	114.38	86.098	68.242	55.958	47.059
m14e09	135.53	99.374	75.456	60.077	50.112
m14e10	140.60	99.029	73.469	58.450	48.672
m14e11	143.58	97.646	71.683	57.067	47.318
m14e12	140.933	91.411	66.787	52.863	43.790
m14e13	128.30		58.694		38.261
m16e00	58.075	42.595	33.538	27.605	23.400
m16e01	62.366	45.893	36.158	29.779	25.243
m16e02	73.037	54.072	42.710	35.194	29.866

m16e03	81.547	60.566	47.894	39.485	33.509
m16e04	93.41	69.696	55.195	45.533	38.650
m16e05	100.15	74.938	59.429	49.046	41.645
m16e06	116.57	87.653	69.667	57.542	48.888
m16e07	121.51	91.397	72.662	60.005	50.962
m16e08	136.07	102.61	81.691	67.464	57.269
m16e09	150.77	113.98	90.778	74.851	63.33
m16e10	175.09	131.72	102.73	82.771	69.034
m16e11	186.91	137.146	104.26	83.146	69.509

m18e00	58.464	42.826	33.682	27.720	23.486
m18e01	64.584	47.491	37.382	30.787	26.093
m18e02	73.037	53.942	42.523	35.035	29.707
m18e03	85.190	63.274	49.997	41.242	34.992
m18e04	94.320	70.301	55.613	45.893	38.952
m18e05	105.84	79.200	62.770	51.826	44.006
m18e06	105.221	78.667	62.323	51.451	43.690
m18e07	118.66	89.078	70.733	58.435	49.637
m18e08	127.87	96.120	76.392	63.101	53.611
m18e09	135.03	101.64	80.842	66.787	56.750
m18e10	158.98	120.08	95.760	79.128	67.262
m18e11	187.92	142.36	113.70	93.902	79.704
m18e12	224.21	169.78	134.24	109.14	91.224

m20e00	59.24	43.387	34.085	28.066	23.774
m20e01	65.002	47.750	37.541	30.917	26.194
m20e02	74.966	55.354	43.603	35.957	30.470
m20e03	83.506	61.877	48.802	40.262	34.128
m20e04	96.005	71.453	56.444	46.613	39.542
m20e05	112.31	83.938	66.470	54.878	46.584
m20e06	108.85	81.403	64.454	53.251	45.187
m20e07	123.08	92.261	73.181	60.437	51.322
m20e08	134.40	101.03	80.280	66.326	56.347
m20e09	146.304	110.15	87.638	72.403	61.531
m20e10	152.78	115.04	91.584	75.658	64.310
m20e11	163.44	123.32	98.309	81.230	69.062
m20e12	183.60	138.82	110.82	91.570	77.861
m20e13	203.62	154.08	123.21	101.82	86.587
m20e14	219.46	166.32	133.13	110.04	93.557
m20e15	245.09	185.90	148.90	122.96	104.36
m20e16	263.66	200.30	160.27	132.13	111.72
m20e17	275.47	209.23	167.04	137.19	115.42
m20e18	290.16	219.89	174.38	141.95	118.70
m20e19	320.83	237.60	183.17	146.88	122.93

m24e00	61.488	44.971	35.251	29.030	24.581
m24e01	64.915	47.592	37.339	30.758	26.050
m24e02	73.814	54.346	42.696	35.208	29.808
m24e03	87.797	65.002	51.163	42.250	35.784
m24e04	98.352	73.066	57.614	47.592	40.334
m24e05	112.95	82.240	66.542	54.994	46.642
m24e06	126.22	94.406	74.707	61.762	52.402
m24e07	120.07	89.755	70.992	58.709	49.810

m24e08	141.49	106.17	84.211	69.638	59.112
m24e09	160.27	120.46	95.746	79.142	67.22
m24e10	171.79	129.30	102.89	85.003	72.202
m24e11	182.30	137.33	109.40	90.461	76.353
m24e12	199.44	150.48	120.10	99.202	84.312
m24e13	234.43	177.12	141.71	117.01	99.504
m24e14	271.87	205.78	165.02	136.31	115.96
m24e15	299.81	227.23	182.30	150.77	128.28
m24e16	347.76	263.81	211.82	175.10	149.04
m24e17	387.07	293.90	236.02	195.12	165.74
m24e18	404.93	302.69	239.47	196.27	165.31
m24e19	444.38	337.68	270.00	221.90	186.77

m26e00	62.856	45.965	36.014	29.664	25.114
m26e01	67.234	49.291	38.664	31.867	26.971
m26e02	73.022	53.698	42.149	34.762	29.434
m26e03	80.006	59.011	46.354	38.275	32.414
m26e04	89.597	66.326	52.171	43.099	36.518
m26e05	103.18	76.628	60.394	49.925	42.307
m26e06	124.07	92.635	73.195	60.523	51.336
m26e07	130.81	97.790	77.328	63.965	54.259
m26e08	126.29	94.378	74.606	61.733	52.373
m26e09	140.16	104.99	83.146	68.803	58.406
m26e10	167.47	125.81	99.965	82.656	70.200
m26e11	184.75	138.99	110.65	91.454	77.688
m26e12	200.74	151.20	120.57	99.619	84.629
m26e13	234.58	176.98	141.41	116.73	99.173
m26e14	273.02	206.21	165.17	136.40	115.96
m26e15	306.14	231.70	185.90	153.50	130.61
m26e16	368.50	279.07	224.21	185.47	157.97
m26e17	389.52	295.20	237.17	196.13	167.04
m26e18	410.69	311.33	250.13	206.93	176.26
m26e19	438.3	332.50	267.12	221.04	188.21
m26e20	474.48	360.00	289.01	239.18	203.33
m26e21	514.08	384.62	303.98	249.26	210.10
m26e22	548.78	416.59	333.22	274.46	231.70
m26e23	580.32	439.78	350.06	286.42	240.48
m26e24	622.08	467.57	367.63	297.94	249.26
m26e25	649.30	481.10	373.54	301.25	252.29

m28e00					
m28e01	69.797	51.192	40.133	33.077	28.008
m28e02	77.386	56.923	44.654	36.850	31.205
m28e03	87.826	64.930	51.019	42.163	35.712
m28e04	103.45	76.766	60.422	49.968	42.350
m28e05	117.71	87.725	69.192	57.269	48.557
m28e06	132.71	99.101	78.278	64.771	54.950
m28e07	134.21	100.27	79.214	65.563	55.642
m28e08	135.06	100.96	79.790	66.082	56.642
m28e09	147.31	110.25	87.264	72.245	61.344
m28e10	168.77	126.66	100.53	83.189	70.661
m28e11	181.73	136.53	108.53	89.784	76.277
m28e12	194.40	146.16	116.29	96.120	81.648

m28e13	220.03	165.60	132.13	109.15	92.736
m28e14	242.78	182.88	146.16	120.67	102.50
m28e15	289.73	218.59	175.10	144.58	122.83
m28e16	335.09	253.01	203.04	167.62	142.57
m28e17	369.94	279.65	224.64	185.62	157.97
m28e18	419.90	317.66	255.17	210.96	179.71
m28e19	447.98	339.12	272.59	225.50	192.10
m28e20	486.58	368.50	296.06	245.09	208.94
m28e21	539.86	409.25	328.61	272.16	231.84
m28e22	572.40	433.87	348.05	288.29	245.23
m28e23	612.29	464.40	372.24	308.02	261.65

m30e00

m30e01					
m30e02	82.339	60.682	47.621	39.326	33.307
m30e03	95.083	70.330	55.253	45.691	38.722
m30e04	105.75	78.480	61.733	51.091	43.315
m30e05	120.31	89.626	70.632	58.507	49.622
m30e06	139.64	104.37	82.440	68.299	57.960
m30e07	136.27	101.78	80.366	66.600	56.549
m30e08	161.42	120.92	95.774	79.373	67.450
m30e09	188.50	141.41	112.32	92.938	78.970
m30e10	205.06	153.94	122.53	101.33	86.112
m30e11	244.08	183.60	146.59	121.09	102.86
m30e12	292.75	220.46	176.69	145.73	123.81
m30e13	343.73	259.06	207.94	171.50	145.87
m30e14	385.49	290.88	233.71	192.82	164.02
m30e15	450.58	340.27	273.46	225.94	192.67
m30e16	528.62	399.60	320.98	265.68	226.66
m30e17	604.80	457.49	367.06	304.27	259.49
m30e18	659.52	499.25	400.03	331.78	282.53
m30e19	692.06	524.02	419.76	348.05	296.06
m30e20	731.52	554.11	443.23	367.34	312.05
m30e21	757.15	558.29	433.58	352.51	293.18
m30e22	842.11	637.06	506.02	416.02	350.35

NEWCONV: Growth Rates.

name	eta0	eta1	eta2	eta3	eta4
m14e00	0.1953e-05	0.2780e-04	0.1300e-03	0.3203e-03	0.3430e-03
m14e01	0.2042e-05	0.2884e-04	0.1333e-03	0.3011e-03	0.3060e-03
m14e02	0.5069e-05	0.6461e-04	0.2773e-03	0.6221e-03	0.7815e-03
m14e03	0.2685e-05	0.3557e-04	0.1754e-03	0.4866e-03	0.8997e-03
m14e04	0.6302e-05	0.7123e-04	0.2774e-03	0.5691e-03	0.7817e-03
m14e05	0.1083e-04	0.1208e-03	0.4742e-03	0.9826e-03	0.1167e-02
m14e06	0.5284e-05	0.5995e-04	0.2673e-03	0.6706e-03	0.1133e-02
m14e07	0.5867e-05	0.6916e-04	0.3128e-03	0.7808e-03	0.1305e-02

m14e08	0.1071e-04	0.1070e-03	0.3862e-03	0.7431e-03	0.9789e-03
m14e09	0.3369e-05	0.1237e-04	0.2115e-04	0.3901e-04	0.9358e-04
m14e10	0.1111e-05	0.2981e-05	0.6411e-05	0.1968e-04	0.5408e-04
m14e11	0.5429e-06	0.1353e-05	0.4015e-05	0.1435e-04	0.4173e-04
m14e12	0.1887e-06	0.5097e-06	0.2154e-05	0.8147e-05	0.2603e-04
m14e13	0.8001e-07		0.1230e-05		0.1591e-04
m16e00	0.1028e-05	0.3218e-04	0.2409e-04	0.8582e-03	0.1797e-02
m16e01	0.1298e-05	0.3689e-04	0.2687e-03	0.9502e-03	0.2021e-02
m16e02	0.2809e-05	0.5438e-04	0.3415e-03	0.1099e-02	0.2142e-02
m16e03	0.3945e-05	0.6490e-04	0.3743e-03	0.1108e-02	0.1898e-02
m16e04	0.5242e-05	0.7434e-04	0.3749e-03	0.9144e-03	0.9906e-03
m16e05	0.6656e-05	0.9956e-04	0.5542e-03	0.1605e-02	0.2775e-02
m16e06	0.9850e-05	0.1309e-03	0.6661e-03	0.1734e-02	0.2495e-02
m16e07	0.1121e-04	0.1394e-03	0.6565e-03	0.1523e-02	0.1644e-02
m16e08	0.1324e-04	0.1368e-03	0.5404e-03	0.1064e-02	0.9805e-03
m16e09	0.9792e-05	0.9481e-04	0.3761e-03	0.8442e-03	0.1296e-02
m16e10	0.1261e-04	0.6619e-04	0.1353e-03	0.1861e-03	0.2555e-03
m16e11	0.5102e-05	0.1683e-04	0.2891e-04	0.5730e-04	0.1373e-03
m18e00	-0.3112e-05	-0.3146e-04	-0.8077e-04	0.7677e-04	0.8934e-03
m18e01	-0.4037e-05	-0.4141e-04	-0.1193e-03	0.1119e-04	0.8734e-03
m18e02	-0.4218e-05	-0.3424e-04	-0.4131e-04	0.3303e-03	0.1601e-02
m18e03	-0.2054e-05	0.8892e-05	0.2234e-03	0.1106e-02	0.2878e-02
m18e04	0.1934e-06	0.4504e-04	0.4175e-03	0.1613e-02	0.3676e-02
m18e05	0.3196e-05	0.8200e-04	0.5747e-03	0.1946e-02	0.4117e-02
m18e06	-0.2845e-05	0.2457e-04	0.3937e-03	0.1747e-02	0.4263e-02
m18e07	-0.2888e-05	0.4254e-04	0.5377e-03	0.2223e-02	0.5194e-02
m18e08	0.2878e-06	0.8955e-04	0.7731e-03	0.2827e-02	0.6255e-02
m18e09	0.5742e-05	0.1421e-03	0.9557e-03	0.3130e-02	0.6468e-02
m18e10	0.1686e-04	0.2089e-03	0.1030e-02	0.2664e-02	0.4006e-02
m18e11	0.2309e-04	0.2118e-03	0.7986e-03	0.1675e-02	0.2202e-02
m18e12	0.2728e-04	0.1692e-03	0.3998e-03	0.5579e-03	0.6370e-03
m20e00	-0.6697e-05	-0.1026e-03	-0.5834e-03	-0.1779e-02	-0.3427e-02
m20e01	-0.8179e-05	-0.1226e-03	-0.6890e-03	-0.2051e-02	-0.3860e-02
m20e02	-0.9219e-05	-0.1210e-03	-0.6032e-03	-0.1539e-02	-0.2330e-02
m20e03	-0.1118e-04	-0.1405e-03	-0.6777e-03	-0.1642e-02	-0.2299e-02
m20e04	-0.1009e-04	-0.9886e-04	-0.3335e-03	-0.3294e-03	0.7819e-03
m20e05	-0.1051e-04	-0.7521e-04	-0.9963e-04	0.5650e-03	0.2756e-02
m20e06	-0.2254e-05	-0.2515e-03	-0.1142e-02	-0.2540e-02	-0.3272e-02
m20e07	-0.2649e-04	-0.2591e-03	-0.1002e-02	-0.1685e-02	-0.8539e-03
m20e08	-0.2579e-04	-0.2191e-03	-0.6840e-03	-0.6047e-03	0.1404e-02
m20e09	-0.1808e-04	-0.9376e-04	0.4115e-04	0.1411e-02	0.4809e-02
m20e10	-0.1532e-04	-0.4558e-04	0.3288e-03	0.2249e-02	0.6338e-02
m20e11	-0.4010e-05	0.8801e-04	0.9249e-03	0.3554e-02	0.7933e-02
m20e12	0.1608e-04	0.2631e-03	0.1509e-02	0.4563e-02	0.8891e-02
m20e13	0.2631e-04	0.3141e-03	0.5140e-02	0.3959e-02	0.6235e-02
m20e14	0.3146e-04	0.3293e-03	0.1373e-02	0.2978e-02	0.3221e-02
m20e15	0.3799e-04	0.3254e-03	0.1153e-02	0.2340e-02	0.2977e-02
m20e16	0.2590e-04	0.2020e-03	0.6647e-03	0.1323e-02	0.1857e-02
m20e17	0.4080e-04	0.3148e-03	0.9510e-03	0.1659e-02	0.2092e-02
m20e18	0.4470e-04	0.2667e-03	0.6101e-03	0.8442e-03	0.9574e-03

m20e19	0.1937e-04	0.6423e-04	0.1098e-03	0.1848e-03	0.3628e-03
m24e00	-0.1243e-04	-0.2323e-03	-0.1687e-02	-0.6842e-02	-0.1798e-01
m24e01	-0.1342e-04	-0.2433e-03	-0.1723e-02	-0.6784e-02	-0.1738e-01
m24e02	-0.1697e-04	-0.2911e-03	-0.2006e-02	-0.7510e-02	-0.1873e-01
m24e03	-0.2261e-04	-0.3553e-03	-0.2319e-02	-0.8047e-02	-0.1878e-01
m24e04	-0.2712e-04	-0.4104e-03	-0.2635e-02	-0.8847e-02	-0.2014e-01
m24e05	-0.3380e-04	-0.4705e-03	-0.2823e-02	-0.8708e-02	-0.1832e-01
m24e06	-0.4213e-04	-0.5500e-03	-0.3142e-02	-0.9114e-02	-0.1806e-01
m24e07	-0.5117e-04	-0.7081e-03	-0.4384e-02	-0.1380e-01	0.3007e-01
m24e08	-0.6769e-04	-0.8509e-03	-0.4834e-02	-0.1383e-01	-0.2745e-01
m24e09	-0.7923e-04	-0.9112e-03	-0.4704e-02	-0.1227e-01	-0.2209e-01
m24e10	-0.8007e-04	-0.8574e-03	-0.4065e-02	-0.9719e-02	-0.1574e-01
m24e11	-0.8516e-04	-0.8626e-03	-0.3846e-02	-0.8626e-02	-0.1279e-01
m24e12	-0.7675e-04	-0.6715e-03	-0.2484e-02	-0.4317e-02	-0.3840e-02
m24e13	-0.5502e-04	-0.2948e-03	-0.2472e-03	0.2050e-02	0.7712e-02
m24e14	-0.9755e-05	0.2387e-03	0.2087e-02	0.7544e-02	0.1604e-01
m24e15	0.4505e-04	0.6184e-03	0.3056e-02	0.8633e-02	0.1533e-01
m24e16	0.7515e-04	0.7562e-03	0.2973e-02	0.6747e-02	0.8345e-02
m24e17	0.9602e-04	0.7703e-03	0.2425e-02	0.4451e-02	0.4262e-02
m24e18	0.1145e-03	0.8895e-03	0.2628e-02	0.4741e-02	0.4892e-02
m24e19	0.7851e-04	0.5372e-03	0.1427e-02	0.2426e-02	0.2965e-02
m26e00	-0.1425e-04	-0.2717e-03	-0.2046e-02	-0.8782e-02	-0.2432e-01
m26e01	-0.1598e-04	-0.2943e-03	-0.2619e-02	-0.9058e-02	-0.2455e-01
m26e02	-0.1883e-04	-0.3394e-03	-0.2489e-02	-0.1020e-01	-0.2732e-01
m26e03	-0.2229e-04	-0.3874e-03	-0.2797e-02	-0.1108e-01	-0.2908e-01
m26e04	-0.2679e-04	-0.4440e-03	-0.3134e-02	-0.1188e-01	-0.3025e-01
m26e05	-0.3426e-04	-0.5279e-03	-0.3552e-02	-0.1247e-01	-0.2986e-01
m26e06	-0.4607e-04	-0.6413e-03	-0.3985e-02	-0.1260e-01	-0.2753e-01
m26e07	-0.5293e-04	-0.7263e-03	-0.4482e-02	-0.1401e-01	-0.3041e-01
m26e08	-0.5921e-04	-0.8328e-03	-0.5378e-02	-0.1755e-01	-0.3982e-01
m26e09	-0.7976e-04	-0.1075e-02	-0.6757e-02	-0.2125e-01	-0.4690e-01
m26e10	-0.1036e-03	-0.1258e-02	-0.7073e-02	-0.1993e-01	-0.3961e-01
m26e11	-0.1193e-03	-0.1367e-02	-0.7175e-02	-0.1907e-01	-0.3576e-01
m26e12	-0.1286e-03	-0.1381e-02	-0.6700e-02	-0.1664e-01	-0.2902e-01
m26e13	-0.1381e-03	-0.1277e-02	-0.5098e-02	-0.1045e-01	-0.1416e-01
m26e14	-0.1096e-03	-0.7551e-03	-0.1904e-02	-0.1236e-02	0.3329e-02
m26e15	-0.8329e-04	-0.3446e-03	0.2101e-03	0.4326e-02	0.1250e-01
m26e16	0.6218e-04	0.8856e-03	0.4196e-02	0.1168e-01	0.1987e-01
m26e17	0.8153e-04	0.9829e-03	0.4337e-02	0.1159e-01	0.1859e-01
m26e18	0.9567e-04	0.1053e-02	0.4361e-02	0.1099e-01	0.1606e-01
m26e19	0.1167e-03	0.1129e-02	0.4137e-02	0.9065e-02	0.1040e-01
m26e20	0.1446e-03	0.1168e-02	0.3567e-02	0.6348e-02	0.5046e-02
m26e21	0.1751e-03	0.1263e-02	0.3503e-02	0.5989e-02	0.5544e-02
m26e22	0.9911e-04	0.6012e-03	0.1486e-02	0.2525e-02	0.2882e-02
m26e23	0.1486e-03	0.8410e-03	0.1802e-02	0.2618e-02	0.2865e-02
m26e24	0.1406e-03	0.5714e-03	0.9303e-03	0.1187e-02	0.1499e-02
m26e25	0.9406e-04	0.2987e-03	0.4508e-03	0.6649e-03	0.1137e-02
m28e00					
m28e01	-0.1866e-04	-0.3444e-03	-0.2596e-02	-0.1131e-01	-0.3192e-01
m28e02	-0.2298e-04	-0.4094e-03	-0.3055e-02	-0.1284e-01	-0.3551e-01

m28e03	-0.2773e-04	-0.4703e-03	-0.3443e-02	-0.1394e-01	-0.3759e-01
m28e04	-0.3756e-04	-0.5970e-03	-0.4241e-02	-0.1592e-01	-0.4075e-01
m28e05	-0.4639e-04	-0.7009e-03	-0.4841e-02	-0.1729e-01	-0.4251e-01
m28e06	-0.5870e-04	-0.8386e-03	-0.5527e-02	-0.1856e-01	-0.4256e-01
m28e07	-0.6788e-04	-0.9605e-03	-0.6355e-02	-0.2110e-01	-0.4880e-01
m28e08	-0.7791e-04	-0.1098e-02	-0.7398e-02	-0.2501e-01	-0.5916e-01
m28e09	-0.9263e-04	-0.1262e-02	-0.8263e-02	-0.2687e-01	-0.6143e-01
m28e10	-0.1203e-03	-0.1542e-02	-0.9434e-02	-0.2850e-01	-0.6089e-01
m28e11	-0.1298e-03	-0.1618e-02	-0.9593e-02	-0.2825e-01	-0.5900e-01
m28e12	-0.1439e-03	-0.1734e-02	-0.9756e-02	-0.2740e-01	-0.5445e-01
m28e13	-0.1678e-03	-0.1841e-02	-0.9198e-02	-0.2336e-01	-0.4199e-01
m28e14	-0.1847e-03	-0.1924e-02	-0.8872e-02	-0.2121e-01	-0.3563e-01
m28e15	-0.2037e-03	-0.1756e-02	-0.6346e-02	-0.1207e-01	-0.1515e-01
m28e16	-0.1651e-03	-0.1080e-02	-0.2481e-02	-0.1339e-02	0.4449e-02
m28e17	-0.1131e-03	-0.3957e-03	0.5783e-03	0.6008e-02	0.1486e-01
m28e18	0.1544e-04	0.8048e-03	0.4613e-02	0.1385e-01	0.2325e-01
m28e19	0.7188e-04	0.1179e-02	0.5545e-02	0.1551e-01	0.2539e-01
m28e20	0.1350e-03	0.1458e-02	0.5762e-02	0.1420e-01	0.1942e-01
m28e21	0.1815e-03	0.1660e-02	0.5643e-02	0.1166e-01	0.1168e-01
m28e22	0.2033e-03	0.1659e-02	0.5016e-02	0.9002e-02	0.6943e-02
m28e23	0.2311e-03	0.1635e-02	0.4494e-02	0.7866e-02	0.7316e-02

m30e00					
m30e01					
m30e02	-0.2646e-04	-0.4575e-03	-0.3398e-02	-0.1455e-01	-0.4087e-01
m30e03	-0.3534e-04	-0.5835e-03	-0.4303e-02	-0.1750e-01	-0.4793e-01
m30e04	-0.4144e-04	-0.6560e-03	-0.4749e-02	-0.1848e-01	-0.4899e-01
m30e05	-0.5266e-04	-0.7981e-03	-0.5661e-02	-0.2095e-01	-0.5349e-01
m30e06	-0.6956e-04	-0.9954e-03	-0.6796e-02	-0.2344e-01	-0.5648e-01
m30e07	-0.7788e-04	-0.1111e-02	-0.7752e-02	-0.2739e-01	-0.6776e-01
m30e08	-0.1197e-03	-0.1604e-02	-0.1066e-01	-0.3494e-01	-0.8119e-01
m30e09	-0.1546e-03	-0.1975e-02	-0.1226e-01	-0.3736e-01	-0.8083e-01
m30e10	-0.1723e-03	-0.2120e-02	-0.1250e-01	-0.3650e-01	-0.7592e-01
m30e11	-0.2206e-03	-0.2481e-02	-0.1275e-01	-0.3346e-01	-0.6275e-01
m30e12	-0.2717e-03	-0.2686e-02	-0.1145e-01	-0.2613e-01	-0.4245e-01
m30e13	-0.2751e-03	-0.2270e-02	-0.7581e-02	-0.1359e-01	-0.1596e-01
m30e14	-0.2602e-03	-0.1813e-02	-0.4676e-02	-0.5350e-02	-0.8870e-03
m30e15	-0.1638e-03	-0.4986e-03	0.1060e-02	0.8377e-02	0.1826e-01
m30e16	0.8353e-04	0.1559e-02	0.7110e-02	0.1908e-01	0.2768e-01
m30e17	0.2192e-03	0.2207e-02	0.8017e-02	0.1832e-01	0.2144e-01
m30e18	0.2746e-03	0.2391e-02	0.7627e-02	0.1475e-01	0.1274e-01
m30e19	0.3096e-03	0.2392e-02	0.6806e-02	0.1143e-01	0.7400e-02
m30e20	0.3344e-03	0.2327e-02	0.6119e-02	0.9811e-02	0.6976e-02
m30e21	0.4678e-03	0.3104e-02	0.7923e-02	0.1039e-01	0.5456e-02
m30e22	0.2042e-03	0.1116e-02	0.2477e-02	0.3869e-02	0.3926e-02

NEWCONV: Phase Lags

Values are given in degrees.

name	0	1	2	3	4
m14e00	175.3	176.4	177.7	178.6	179.3
m14e01	175.3	176.3	177.7	178.6	179.3
m14e02	175.1	176.6	177.6	178.6	179.6
m14e03	175.4	176.6	177.8	178.8	179.8
m14e04	175.8	177.1	178.3	179.4	-179.5
m14e05	175.4	176.7	177.8	178.9	179.8
m14e06	175.6	176.8	177.8	178.8	179.8
m14e07	175.7	176.9	178.0	179.0	179.9
m14e08	175.9	177.1	178.3	179.4	-179.6
m14e09	176.3	177.9	179.6	-178.7	-177.2
m14e10	176.2	178.3	-179.5	-177.4	-175.5
m14e11	176.0	178.6	-178.7	-176.4	-174.2
m14e12	176.0	179.4	-177.3	-174.4	-171.6
m14e13	176.2		-175.8		-169.0
m16e00	92.87	103.2	105.2	97.27	80.69
m16e01	97.64	108.2	110.9	104.7	90.38
m16e02	121.4	130.5	131.9	126.3	116.5
m16e03	144.7	150.9	151.1	147.4	142.8
m16e04	168.6	169.6	167.7	165.7	165.1
m16e05	150.6	156.4	157.2	154.9	151.8
m16e06	162.6	165.8	165.2	162.8	160.8
m16e07	170.6	171.4	169.6	167.8	167.1
m16e08	176.1	176.4	177.3	178.3	178.9
m16e09	175.8	176.8	177.8	178.7	179.6
m16e10	176.3	177.6	178.9	-179.9	-178.7
m16e11	176.3	177.8	179.5	-178.9	-177.5
m18e00	42.30	50.43	55.68	56.82	52.11
m18e01	42.26	51.06	57.02	59.00	55.43
m18e02	47.57	56.94	62.98	64.41	59.29
m18e03	62.80	72.53	77.58	76.05	64.51
m18e04	78.32	89.30	94.62	92.19	78.87
m18e05	100.6	112.4	117.6	115.6	105.5
m18e06	67.62	78.00	83.58	82.49	71.44
m18e07	71.34	82.09	87.91	87.01	76.07
m18e08	85.52	97.89	104.6	104.7	96.01
m18e09	103.3	115.3	121.3	120.9	113.5
m18e10	164.2	167.7	167.7	165.9	164.1
m18e11	175.3	176.4	177.3	178.0	178.7
m18e12	176.3	177.4	178.6	179.7	-179.3
m20e00	24.90	32.05	38.14	42.32	43.79
m20e01	24.65	31.91	38.26	42.70	44.34
m20e02	29.50	37.20	43.42	47.15	47.43
m20e03	30.54	38.86	45.71	49.95	50.62
m20e04	39.28	47.80	53.97	56.52	53.89
m20e05	48.35	58.08	64.65	66.69	62.14

m20e06	30.18	38.62	45.59	49.87	50.27
m20e07	35.24	44.32	51.42	55.22	54.28
m20e08	40.84	50.50	57.67	61.02	58.86
m20e09	52.95	62.80	69.24	70.67	64.50
m20e10	59.98	71.09	78.19	79.93	73.53
m20e11	77.19	88.96	95.68	96.17	87.31
m20e12	122.1	133.1	138.7	139.5	135.8
m20e13	160.4	165.2	166.2	165.0	163.1
m20e14	175.7	175.7	174.3	173.4	173.5
m20e15	175.4	176.4	177.3	178.0	178.6
m20e16	176.0	177.0	177.8	178.7	179.5
m20e17	176.2	177.2	178.1	179.0	179.9
m20e18	176.3	177.4	178.4	179.5	-179.5
m20e19	176.1	177.5	178.9	-179.6	-178.3

m24e00	5.092	7.636	10.21	12.50	14.05
m24e01	6.227	9.261	12.43	15.20	17.06
m24e02	6.932	10.31	13.86	16.94	18.97
m24e03	9.198	13.49	17.92	21.68	24.02
m24e04	9.223	13.72	18.48	22.64	25.39
m24e05	12.36	18.04	23.82	28.66	31.62
m24e06	14.12	20.35	26.63	31.78	34.70
m24e07	6.753	10.92	15.39	19.27	21.67
m24e08	10.39	15.85	21.51	26.30	28.99
m24e09	15.15	21.98	28.56	33.69	35.95
m24e10	20.27	27.97	34.97	39.95	41.32
m24e11	23.08	31.52	38.92	44.02	45.23
m24e12	31.89	40.91	47.91	51.67	50.30
m24e13	50.99	61.67	68.69	70.83	65.00
m24e14	88.30	102.2	110.7	113.9	110.3
m24e15	130.9	141.3	146.9	148.5	146.6
m24e16	174.1	175.5	174.8	173.6	172.8
m24e17	176.1	176.5	177.2	177.9	178.5
m24e18	176.0	176.9	177.7	178.5	179.2
m24e19	176.2	177.1	178.0	178.8	179.7

m26e00	0.957	1.266	1.519	1.388	0.521
m26e01	1.600	2.480	3.163	3.454	2.984
m26e02	0.538	1.326	2.131	2.539	2.142
m26e03	0.782	1.820	2.907	3.587	3.412
m26e04	1.670	3.156	4.773	6.010	6.378
m26e05	4.152	6.714	9.523	11.92	13.28
m26e06	7.646	11.88	16.33	20.06	22.19
m26e07	6.577	10.86	15.37	19.20	21.42
m26e08	2.875	5.317	7.979	10.15	11.07
m26e09	1.649	4.120	6.755	8.837	9.534
m26e10	6.970	11.56	16.22	19.97	21.50
m26e11	9.792	15.57	21.22	25.68	27.40
m26e12	13.98	20.86	27.26	32.01	33.42
m26e13	24.39	33.28	40.63	45.20	45.00
m26e14	40.55	50.62	57.56	60.11	55.51
m26e15	57.06	69.45	77.52	80.22	74.48
m26e16	126.4	137.4	143.4	145.5	143.9

m26e17	147.8	156.2	160.3	161.4	160.5
m26e18	162.9	168.2	170.0	169.7	168.6
m26e19	175.4	176.4	175.7	174.6	174.1
m26e20	177.5	176.9	177.2	177.9	178.5
m26e21	176.2	177.0	177.8	178.5	179.2
m26e22	176.2	177.1	177.9	178.6	179.4
m26e23	176.0	177.0	178.0	178.9	179.8
m26e24	176.0	177.2	178.3	179.4	-179.5
m26e25	176.0	177.3	178.6	179.9	-178.9

m28e00					
m28e01	-1.460	-2.052	-3.767	-5.492	-8.054
m28e02	-1.773	-2.678	-3.760	-5.319	-7.774
m28e03	-1.649	-2.271	-3.020	-4.235	-6.354
m28e04	-0.516	-0.440	-0.380	-0.760	-2.046
m28e05	-0.339	0.407	1.286	1.799	1.447
m28e06	1.632	3.562	5.746	7.544	8.296
m28e07	0.133	1.418	2.822	3.740	3.553
m28e08	-2.123	-2.126	-2.169	-2.735	-4.351
m28e09	-1.583	-1.153	-0.720	-0.818	-2.033
m28e10	-0.242	1.633	3.605	4.997	4.968
m28e11	1.083	3.470	6.026	8.032	8.482
m28e12	3.525	7.070	10.81	13.92	15.08
m28e13	9.187	14.99	20.66	25.15	26.59
m28e14	12.80	19.77	26.38	31.45	32.85
m28e15	25.08	34.23	41.45	45.51	44.10
m28e16	40.65	51.25	58.40	60.90	55.70
m28e17	57.63	69.52	76.66	77.86	69.14
m28e18	97.53	110.8	117.9	119.4	113.3
m28e19	124.5	136.4	143.0	145.6	144.7
m28e20	155.0	161.6	164.5	164.9	163.8
m28e21	176.2	176.9	176.0	175.0	174.5
m28e22	178.3	177.3	177.1	177.5	178.1
m28e23	175.8	176.7	177.4	178.0	178.5

m30e00					
m30e01					
m30e02	-2.651	-4.451	-6.611	-9.277	-12.89
m30e03	-3.436	-5.268	-7.426	-10.09	-13.75
m30e04	-2.594	-3.971	-5.604	-7.753	-10.90
m30e05	-3.375	-4.293	-5.639	-6.962	-9.563
m30e06	-2.713	-2.767	-2.873	-3.511	-5.208
m30e07	-4.069	-5.556	-7.292	-9.597	-12.96
m30e08	-5.196	-6.237	-7.524	-9.508	-12.70
m30e09	-3.331	-2.996	-2.713	-3.061	-4.693
m30e10	-1.354	0.008	1.407	2.194	1.489
m30e11	3.884	8.077	12.34	15.80	16.73
m30e12	12.58	19.83	26.41	31.16	31.63
m30e13	25.20	34.45	41.51	45.10	42.47
m30e14	35.08	45.45	52.57	55.15	50.10
m30e15	59.47	72.04	79.28	80.13	70.44
m30e16	113.3	125.7	132.2	133.9	130.5
m30e17	162.2	167.4	169.2	169.1	168.1

m30e18	176.8	177.2	176.3	175.4	175.1
m30e19	178.3	177.3	176.9	177.3	177.9
m30e20	176.2	176.7	177.4	178.0	178.5
m30e21	176.0	176.9	177.6	176.6	179.3
m30e22	175.9	176.9	177.7	178.6	179.4

APPENDIX E
DELTA SCUTI STARS

Data on various Delta Scuti stars, taken from Breger (1979) and Tsvetkov (1985). The period is in days, amplitude in magnitudes, and Q in days. G is surface gravity, in cgs units; Te is the effective temperature, in Kelvin. Ways of estimating the mode of pulsation are discussed by Tsvetkov.

Period	Ampl	log g	log Te	Mbol	Q	(b-y)o	mode	name
0.104	0.03	3.56	3.844	1.12	0.0198	0.224	2	HD 432
0.070	0.035	3.88	3.879	1.36	0.0220	0.167	2	2628
0.049	0.06	3.94	3.920	1.39	0.0184	0.098	3	3112
0.10	0.04	3.81	3.888	0.56	0.0248	0.149	2	4490
0.136	0.025	4.14	3.881	2.32	0.0724	0.172		4818
0.105	0.04	3.80	3.858	1.62	0.0305	0.203	0	4919
0.065	0.015	4.14	3.889	1.98	0.0327	0.157	0	6870
0.032	0.02	4.12	3.892	2.11	0.0163	0.151	3	8511
0.090	0.03	3.94	3.862	1.55	0.0305	0.199	0	9065
0.136	0.02	3.62	3.916	0.38	0.0276	0.093	1	9100
0.219	0.02	3.71	3.891	0.83	0.0519	0.140		10845
0.10	0.07	3.55	3.850	0.7	0.0174	0.213	3	15165
0.080	0.026	4.06	3.889	1.95	0.0364	0.155	0	15550
0.037	0.02	4.21	3.905	2.30	0.0226	0.131	2	17093
0.035	0.01	4.25	3.888	2.68	0.0234	0.162	1	20919
0.030	0.02	4.26	3.944	2.16	0.0203	0.062	2	21553
0.037	0.01	4.27	3.883	2.86	0.0261	0.171	1	H 501
0.021	0.01	4.41	3.924	2.47	0.0174	0.105	3	HD23156
0.032	0.015	4.37	3.897	2.40	0.0235	0.151	1	23567
0.047	0.01	4.44	3.927	2.45	0.0408	0.101	0	23607
0.031	0.03	4.19	3.938	2.11	0.0191	0.074	2	23643
0.097	0.08	3.91	3.873	1.53	0.0325	0.180	0	23728
0.076	0.03	3.93	3.873	1.73	0.0272	0.180	1	24550

0.055	0.01	4.24	3.905	2.38	0.0351	0.133	0	24809
0.156	0.05	3.82	3.877	1.29	0.0450	0.169	0	24832
0.145	0.09	3.67	3.844	1.43	0.0338	0.226	0	26322
0.081	0.03	3.82	3.860	1.60	0.0241	0.201	1	26574
0.054	0.03	4.17	3.881	2.38	0.0302	0.172	0	27397
0.036	0.01	4.15	3.907	2.03	0.0192	0.126	2	27459
0.08	0.02	3.86	3.919	0.17	0.0205	0.096	3	28319
0.042	0.012	4.07	3.908	1.87	0.0198	0.122	2	30780
0.088	0.08	3.78	3.902	0.93	0.0235	0.120	1	33959
0.054	0.01	3.80	3.893	1.15	0.0153	0.139	3	40372
0.136	0.2	3.62	3.865	0.94	0.0278	0.186	1	40535
0.124	0.04	3.57	3.837	0.4	0.0200	0.238	2	V589 Mon
0.12	0.03	3.52	3.838	1.14	0.0218	0.234	1	HD50018
0.17	0.01	3.40	3.876	-0.2	0.0236	(0.16)	1	50420
0.12	0.03	3.79	3.861	1.52	0.0341	0.196	0	52788
0.10	0.04	3.70	3.870	1.09	0.0236	0.182	1	55057
0.095	0.06	3.80	3.909	1.06	0.0274	0.111	1	62437
0.141	0.51	3.78	3.846	1.70	0.0395	(0.17)	0	67523
0.076	0.029	3.95	3.868	2.04	0.0297	0.190	0	69997
0.096	0.02	3.84	3.895	1.21	0.0292	0.137	1	71496
0.07	0.01	3.76	3.882	1.11	0.0183	0.159	3	71935
0.038	0.02	4.05	3.893	2.10	0.0179	0.147	2	73175
0.051	0.02	4.13	3.884	2.35	0.0273	0.166	1	73450
0.102	0.07	3.67	3.891	0.50	0.0213	0.139	2	73575
0.06	0.02	3.90	3.908	1.50	0.0213	0.115	2	73576
0.074	0.01	3.91	3.869	2.04	0.0276	0.186	0	73729
0.037	0.012	4.01	3.900	1.64	0.0151	0.133	3	73763
0.072	0.01	4.02	3.877	2.33	0.0331	0.175	0	73798
0.053	0.02	4.04	3.908	1.79	0.0238	0.121	1	74028
0.058	0.01	3.95	3.908	1.74	0.0292	0.117	0	74050
0.065	0.015	3.80	3.889	1.25	0.0187	0.146	3	77140
0.12	0.03	4.15	3.915	2.02	0.0651	(0.113)		79439
0.132	0.05	3.77	3.864	1.24	0.0344	0.192	0	84999
0.05	0.02	4.23	3.876	2.22	0.0285	0.182	1	100363
0.04	0.02	4.23	3.884	2.50	0.0248	0.169	1	104513
0.07	0.025	3.89	3.877	1.45	0.0226	0.171	2	106384
0.055	0.04	4.07	3.923	1.80	0.0263	0.097	1	107131
0.209	0.1	3.38	3.842	0.62	0.0286	0.224	0	107904
0.05	0.01	3.53	3.819	1.56	0.0096	0.271		108506
0.05	0.01	4.09	3.907	1.90	0.0243	0.124	1	110379
0.119	0.04	3.59	3.852	0.94	0.0230	0.209	1	115308
0.135	0.035	3.83	3.885	1.42	0.0412	0.154	0	115604
0.066	0.03	3.72	3.904	0.84	0.0163	(0.116)	3	124675
0.063	0.02	4.12	3.878	2.19	0.0317	0.176	0	143466
0.100	0.02	3.76	3.844	1.69	0.0275	0.228	1	152830
0.077	0.025	4.01	3.900	1.72	0.0321	0.134	0	155514
0.188	0.03	3.52	3.861	0.40	0.0301	0.192	0	156697
0.088	0.025	3.89	3.922	1.11	0.0294	0.092	0	170625
0.194	0.29	3.79	3.864	1.51	0.0552	0.192		172748
0.109	0.04	3.76	3.876	1.10	0.0281	0.171	1	177392
0.150	0.06	3.49	3.876	0.53	0.0246	(0.164)	1	181333
0.05	0.02	3.94	3.910	1.45	0.0186	0.114	3	181577
0.088	0.02	3.97	3.860	1.75	0.0324	0.205	0	186357

0.08	0.02	3.91	3.870	1.57	0.0269	0.184	1	186786
0.10	0.04	3.77	3.871	1.37	0.0273	(0.180)	1	187764
0.11	0.03	3.95	3.844	2.43	0.0445	0.231		195961
0.153	0.05	3.78	3.866	1.23	0.0405	0.183	0	197461
0.095	0.07	4.04	3.879	1.90	0.0406	0.172	0	199124
0.079	0.08	3.77	3.867	1.30	0.0211	0.187	2	199908
0.097	0.07	3.95	3.886	1.44	0.0343	0.156	0	201707
0.044	0.01	4.34	3.905	2.68	0.0342	0.136	0	204188
0.042	0.02	4.06	3.875	2.06	0.0188	0.179	2	211336
0.056	0.015	3.97	3.915	1.48	0.0219	0.107	2	213534
0.087	0.02	3.90	3.899	1.27	0.0295	0.131	1	215874
0.054	0.01	3.82	3.913	1.58	0.0180	(0.104)	2	220061
0.06	0.01	4.14	3.940	1.77	0.0322	0.068	0	223781

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