Journal of Algebra and Its Applications

Superpower Graphs of Finite Groups

AJAY KUMAR
Department of Mathematical Sciences, Indian Institute of Technology (BHU),
Varanasi, Uttar Pradesh 221005, India
ajaykr.rs.mat17@iitbhu.ac.in

LAVANYA SELVAGANESH
Department of Mathematical Sciences, Indian Institute of Technology (BHU),
Varanasi, Uttar Pradesh 221005, India
lavanyas.mat@iitbhu.ac.in

PETER J CAMERON
School of Mathematics and Statistics
University of St Andrews, North Haugh
St Andrews, Fife, KY16 9SS, U.K.
pjc20@st-andrews.ac.uk

T. TAMIZH CHELVAM
Department of Mathematics
Manonmaniam Sundaranar University
Tirunelveli, Tamil Nadu, India
tamche59@gmail.com & tamizhchelvam@msuniv.ac.in

Received (Day Month Year)
Revised (Day Month Year)
Accepted (Day Month Year)

Communicated by (xxxxxxxxx)

For a finite group G, the superpower graph $S(G)$ of G is an undirected simple graph with vertex set G and two vertices are adjacent in $S(G)$ if and only if the order of one divides the order of the other in G. The aim of this paper is to provide tight bounds for the vertex connectivity, discuss Hamiltonian-like properties of superpower graph of finite non-Abelian groups having an element of exponent order.

We also give some general results about superpower graphs and their relation to other graphs such as the Gruenberg–Kegel graph.

Keywords: Superpower graph; Power graph; Hamiltonian cycle; Simple group; Vertex connectivity.

05C25, 05C45, 05C75, 20E45

1. Introduction

Graphs associated with groups and other algebraic structures have been actively investigated, since they have valuable applications are related to automata theory which can be seen in [17],[18] and the books [15],[16]. The research of graphical representation of semi-group and group has emerged as a promising study area in recent decades, generating a number of interesting results and problems. Some of the most popular graphs in this area, Cayley graph [25], [8], [21], commuting graph [23] and power graph [1],[19] are publicly available.

The superpower graphs of groups are a fairly recent development in the realm of graphs from groups, and it was firstly introduced by Hamzeh and Ashrafi, which they call the *order supergraph* $S(G)$ of the power graph $P(G)$ of a finite group, in 2018 [13]. They asked, "What is the structure of G with $|\pi(G)| = \alpha(S(G))$"? The objective of the study was to explain the connection between $P(G)$ and $S(G)$ as well as various structural properties of superpower graph. After that, Ma and Su [22], investigated the independence number of $S(G)$, and provided an answer to this question. In [11], Hamzeh and Ashrafi calculated the superpower graph's automorphism groups and full automorphism groups. They also established that the automorphism group of this graph might be represented as a combination of wreath and Cartesian products of various symmetric groups. In [12], for specific finite groups, the same authors have calculated the characteristic polynomial of these graphs. As a result, it was possible to calculate the spectrum and Laplacian spectrum of the graphs formed from dihedral, semi-dihedral, cyclic, and dicyclic groups. In [2], Asboei and Salehi proved that $G = PSL(2, p)$ or $PGL(2, p)$ if and only if $S(G) = S(PSL(2, p))$ or$S(PGL(2, p))$, respectively. They also proved that if M is a sporadic simple group, then $G = M$ if only if $S(G) = S(M)$. In [14], the authors investigated this graph's Hamiltonianity, Eulerian and 2-connectedness.

With these motivations we explore the super power graphs for any finite groups. In one of our earlier works, we have discussed finite Abelian groups. Hence, in this paper to avoid repetition we will focus mainly on finite non-Abelian groups and in particular non-ableian groups having an element of exponent order.

The paper is organized as follows: In Section 2, we present required definitions and fix notations from both group theory and graph theory. Section 3 develops a technique for studying the superpower graph of G based on a smaller graph, the order graph of G. We use the technique to show, among other things, that superpower graphs are perfect; we give a more elementary proof of this in Section 4. In Section 5, we present the main results of this paper by dividing it into two subsections, one on Hamiltonian and Hamiltonian-like properties while the other subsection discusses the vertex connectivity related findings. We conclude the article with some open problems in Section 6.

2. Preliminaries

The aim of this section is to provide some definitions and results from group theory and graph theory to achieve the goal of this paper. We use standard definitions and results from [10] for group theory and [3] for graph theory which we restate here to establish notations. Throughout this paper, by a group G, we mean a finite group of order n with the identity e. The order of an element x of G is denoted by $o(x)$. The *exponent* $\exp(G)$ of G is the least common multiple of the orders of the elements of G, in other words, the smallest positive integer m such that $x^m = e$ for all $x \in G$. We say that G *has an element of exponent order* if there exists $x \in G$ with $o(x) = \exp(G)$. Let $\pi(G)$ be the set of orders of elements of G. Thus, G has an element of exponent order if and only if $\pi(G)$ is the set of all divisors of m, for some integer m.

Note that any finite Abelian group, or any group of prime power order, contains an element of exponent order, so this class of groups generalises both Abelian groups and p-groups. Moreover, for any finite group G, there is an integer m such that G^m contains an element of exponent order.

For each positive divisor d of n, define $w_d(G) = \{x \in G\colon o(x) = d\}$. For any subset X of G, we let $X^\sharp = X \setminus w_1(G)$, and $\overline{G} = G^\sharp \setminus w_{\exp(G)}(G)$. As usual, $\mathbb{N}$ denotes the set of all natural numbers. Let $Z_n = \{\overline{0}, \overline{1}. \cdots, \overline{n-1}\}$ denote the cyclic group of order n. *Euler's function* ϕ is defined by the rule that, for a positive integer n, the number of positive integers less than n that are relatively prime to n is $\phi(n)$. Whenever we consider the prime factorization of a positive integer $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, it is assumed that $m \geq 2$, that $p_1 < p_2 < \cdots < p_m$ are primes and that $\alpha_i \in \mathbb{N}$ for $1 \leq i \leq m$.

By a graph $\Gamma(V, E)$, we mean an undirected simple finite graph. A vertex of a graph Γ is called a *dominant vertex* if it is adjacent to every other vertex of Γ. For a graph Γ, let $\mathrm{dom}(\Gamma)$ denote the set of all dominant vertices in Γ. A *connected component* of Γ is a maximal connected subgraph of Γ. If a graph Γ has a path P (or a cycle C) which contains all the vertices of Γ, then P (or C) is called *Hamiltonian path (or Hamiltonian cycle)* of the graph Γ. A *Hamiltonian graph* Γ is a graph which contains a Hamiltonian cycle. A graph Γ is called 1-*Hamiltonian* if it is Hamiltonian and all of its 1-vertex-deleted subgraphs are Hamiltonian. A graph Γ is called *Hamiltonian connected* if any two vertices of Γ can be join by a Hamiltonian path. A graph Γ on n vertices is called *pancyclic* if, for every ℓ with $3 \leq \ell \leq n$, there exists a cycle of length ℓ in Γ. If a graph Γ have the property that for any vertices u and v of it, there exist a path of each possible length $\ell, d(u, v) \leq \ell \leq n$, then Γ is called *pan connected.* A set of vertices T of a graph Γ is said to be a *separating set,* if its removal increases the number of connected components of Γ. T is called a *minimal separating set* if none of its non-empty proper subset is a separating set. If T is of least cardinality, then it is known as a *minimum separating set* of Γ. The cardinality of a minimum separating set is the *vertex connectivity* of Γ and denoted by $\kappa(\Gamma)$.

The *power graph* of a finite group G has vertex set G, with an edge from x to y if one of x and y is a power of the other, that is, if $x^m = y$ or $y^m = x$ for some integer m. Our main object of interest is the (order) *superpower graph*, which has vertex set G and an edge from x to y if the order of one of x and y divides the order of the other. Clearly it contains $P(G)$ as a spanning subgraph.

Theorem 2.1 ([13], Theorem 2.3). *Let G be a finite group. Then $S(G)$ is complete if and only if G is a p-group.*

Lemma 2.1. *Let G be a finite group having an element x of order* $\exp(G)$. *Then* $o(g) \mid o(x)$*for all* $g \in G$.

In one of our earlier works, we have proved similar results for finite Abelian groups. Hence, in this paper, we will focus mainly on finite non-Abelian groups. That is, for the rest of our paper G denotes a non-Abelian group having an element of exponent order.

3. The order graph

Recall that $\pi(G)$ denotes the set of orders of elements of G. We construct a graph, which we call the *order graph* of G: the vertex set is $\pi(G)$, and there is an edge joining m and n if one of them divides the other. We denote this graph by $\mathrm{Ord}(G)$. The *weighted order graph* is obtained from $\mathrm{Ord}(G)$ by labelling each vertex m with the number of elements of order n in G.

Note that the graph $S(G)$ can be reconstructed uniquely from the weighted order graph of G; simply blow up each vertex of $\mathrm{Ord}(G)$ to a complete graph whose number of vertices is equal to the weight of that vertex, and lift each edge to all edges between the corresponding sets. Note also that $\mathrm{Ord}(G)$ is isomorphic to an induced subgraph of $S(G)$, obtained by choosing one element of G of each possible order.

Let Γ and Δ be graphs. We say that Γ is Δ-*free* if it has no induced subgraph isomorphic to Δ. Two vertices v, w in a graph Γ are *twins* if they have the same neighbours, possibly excepting each other: that is, either v and w are not joined and have equal neighbourhoods, or they are joined and $N(v) \setminus \{w\} = N(w) \setminus \{v\}$, where $N(v)$ is the neighbourhood of v.

Theorem 3.1. *Let Δ be a graph containing no pair of twins, and G a finite group, then $S(G)$ is Δ-free if and only if* $\mathrm{Ord}(G)$ *is Δ-free.*

Proof. In an embedding of Δ into $S(G)$ as induced subgraph, two vertices of Δ mapping to elements of G with the same order must be twins. So the mapping taking an element of Δ to the order of its image must be an injection, and is an embedding in $\mathrm{Ord}(G)$. The converse is clear. □

Theorem 3.2. *For any finite group G, the following hold.*

(a) $S(G)$ *is perfect.*
(b) If $\mathrm{Ord}(G)$ *is Hamiltonian, then* $S(G)$ *is Hamiltonian.*

Proof. (a) The graph $\mathrm{Ord}(G)$ is the comparability graph of the partial order on $\omega(G)$ defined by divisibility, so by Dilworth's theorem [9] it is perfect. Now by the Strong Perfect Graph Theorem [7], a graph is perfect if and only if it contains neither an odd cycle of length at least 5 nor the complement of one. The odd cycles and their complements have no pairs of twins, so it follows from Theorem 3.1 that $S(G)$ is also perfect.

(b) Take a Hamiltonian cycle in $\mathrm{Ord}(G)$ and lift it to a Hamiltonian cycle of G by choosing a Hamiltonian path within the set of vertices of each possible order and joining their ends according to the edges in the cycle in $\mathrm{Ord}(G)$. □

We exploit this to show that, if G contains an element of exponent order, then $S(G)$ is Hamiltonian. This is a foretaste of stronger results which will be proved in Section 5 below, including another proof of this fact.

By the above theorem, it suffices to show that, for any integer n, the comparability graph of the poset of divisors of n (ordered by divisibility) is Hamiltonian. Our proof of this is by induction on the number s of prime divisors of n. If $s = 1$, then the graph is complete, so the result is true. So suppose that it holds for numbers m with $s-1$ prime divisors, and let $n = p^a m$, where p is prime and m has $s-1$ prime divisors. We can identify the elements of this lattice with ordered pairs (c, d), where c divides p^a and d divides m; we have $(c, d) \leq (c', d')$ if $c < c'$ and $d < d'$. The set of such pairs for fixed d carries a complete graph, which is Hamiltonian connected, while the set of such pairs for fixed c carries a graph isomorphic to the comparability graph of the lattice of divisors of m, which (by induction) has a Hamiltonian cycle $((c, x_0), (c, x_1), \ldots, (c, x_{m-1}), (c, x_0))$. Let $C = \{c_0, \ldots, c_{a-1}\}$. Now we construct a Hamiltonian cycle as follows. Start with a Hamiltonian path from (c_0, x_0) to (c_1, x_0), the follow the edge to (c_1, x_1). Then follow a Hamiltonian path from there to (c_2, x_1), and an edge to (c_2, x_2). Continue until we reach a vertex (c_{m-1}, x_{m-1}) (we take the indices of the c vertices modulo m if necessary). If $c_{m-1} \neq c_0$, then take a Hamiltonian path from (c_{m-1}, x_{m-1}) to (c_0, x_{m-1}) and finally an edge back to (c_0, m_0). Otherwise, modify the path on vertices with second component x_{m-2} so that its end point is, say, (c', x_{m-2}) with $c' \neq c_{m-1}$, and finish as before.

This suggests a general question:

Problem 3.1. *Which graph-theoretic properties can be lifted from* $\mathrm{Ord}(G)$ *to* $S(G)$*?*

Another well-studied graph which has a role to play here is the *Gruenberg–Kegel graph* of G (sometimes called the *prime graph*). We refer to [5] for a summary of its properties. We call it for short the GK-graph and denote it by $\mathrm{GK}(G)$. Its vertices are the prime divisors of $|G|$, with an edge from p to q if and only if G contains an element of order pq. Detailed information about the groups whose GK graph is disconnected is available from a theorem proved by Gruenberg and Kegel in an unpublished manuscript and refined by later authors.

Recall that $G^\sharp$ denotes $G \setminus \{e\}$, while $\mathrm{Ord}(G^\sharp)$ and $S(G^\sharp)$ denote the graphs $\mathrm{Ord}(G)$ or $S(G)$ with the number 1 or the identity element respectively removed.

Theorem 3.3. *For a finite group G, there are bijections between the connected components of the graphs* $\mathrm{GK}(G)$, $\mathrm{Ord}(G^\sharp)$ *and* $S(G^\sharp)$.

Proof. The bijection between the last two is clear from our earlier discussion.

Take a set of primes forming a connected component of the Gruenberg–Kegel graph. The prime-power orders form complete subgraphs of $\mathrm{Ord}(G^\sharp)$, and the edges of the GK-graph connect them up. Any number which is an order of an element of G is joined in $\mathrm{Ord}^\sharp(G)$ to an element of prime power order.

For the converse, note that the set of prime divisors of any element order in G induce a complete subgraph in $\mathrm{GK}(G)$, so any path in $\mathrm{Ord}(G^\sharp)$ can be replaced by a path in $\mathrm{GK}(G)$. □

We briefly mention a class of groups at the opposite extreme to the groups with elements of exponent order which are our main subject. These are the *EPPO groups*, the groups in which every element has prime power order. After preliminary results by Higman (on the solvable EPPO groups) and Suzuki (on the simple ones), all EPPO groups where classified by Brandl [4] in 1981; the result can be found in [5]. For an EPPO group G, the GK graph has no edges, while $\mathrm{Ord}(G^\sharp)$ and $S(G^\sharp)$ are disjoint unions of complete graphs, one for each prime divisor of $|G|$.

4. Perfectness of $S(G)$

In this section we show that $S(G)$ is the comparability graph of a partial order, and hence is perfect, and that there is no further restriction on the induced subgraphs. Unlike the proof in the preceding section, this does not depend on the Strong Perfect Graph Theorem.

A *partial preorder* on a set X is a reflexive and transitive relation R on X. Its *comparability graph* is the graph on the vertex set X in which x and y are joined if $x \mathrel{\mathrm{R}} y$ or $y \mathrel{\mathrm{R}} x$. A partial preorder is a *partial order* if $x \mathrel{\mathrm{R}} y$ and $y \mathrel{\mathrm{R}} x$ imply $x = y$. A partial order is a *total order* if, for any $x, y \in X$, either $x \mathrel{\mathrm{R}} y$ or $y \mathrel{\mathrm{R}} x$.

Proposition 4.1. *The classes of comparability graphs of partial preorders and of partial orders are equal.*

Proof. Every partial order is a partial preorder. Conversely, let R be a partial preorder. Define a relation $\equiv$ by the rule that $x \equiv y$ if and only if $x \mathrel{\mathrm{R}} y$ and $y \mathrel{\mathrm{R}} x$. It is easily seen that $\equiv$ is an equivalence relation. Now enlarge the relation R to a new relation $\leq$ by imposing a total order on each equivalence class of $\equiv$. The result is a partial order with the same comparability graph as R. □

Proposition 4.2. *The superpower graph of a group G is the comparability graph of a partial order.*

Proof. Define a relation R on G by the rule that $x \mathrel{R} y$ if $o(x) \mid o(y)$. This relation is reflexive and transitive, thus is a partial preorder; and the superpower graph is its comparability graph. Now the result follows using the preceding Proposition. □

Proposition 4.3. *Let X be the comparability graph of a partial order. Then there is a group G such that X is an induced subgraph of the superpower graph of G.*

Proof. Let $\leq$ be a partial order on $V(X)$ whose comparability graph is X. Put $\downarrow(x) = \{y \in V(X) : y \leq x\}$. Then we have

- $\downarrow(x) = \downarrow(y)$ if and only if $x = y$;
- $\downarrow(x) \subseteq \downarrow(y)$ if and only if $x \leq y$.

To see this, observe that in each case the reverse implication is true, since $\leq$ is a transitive relation. In the other direction, suppose that $\downarrow(x) \subseteq \downarrow(y)$; then $x \in \downarrow(y)$, so $x \leq y$. If $\downarrow(x) = \downarrow(y)$, then $x \leq y$ and $y \leq x$, so $x = y$, since $\leq$ is a partial order.

Now choose distinct primes p_x for each $x \in V(X)$, and let

$$N = \prod_{x \in V(X)} p_x.$$

Let G be the cyclic group of order N, and for each $x \in V(X)$ let g_x be an element of G whose order is

$$o(g_x) = \prod_{y \in \downarrow(x)} p_y.$$

If $x \sim y$ then, without loss, $x \leq y$, so $\downarrow(x) \subseteq \downarrow(y)$, whence $o(g_x) \mid o(g_y)$. The argument reverses. So the map $x \mapsto g_x$ is an embedding of X as an induced subgraph of the superpower graph of G. □

5. Main results

It is a well known fact that dominant vertices play an important role in characterization of graphs. In fact, if a graph contains a dominant vertex, then it is connected and diameter is at most two. Note that for any group G, the identity element, e, of G is a dominant vertex in $S(G)$. So, we say a graph $S(G)$ is *dominatable*, if it contains dominant vertices other than identity, that is $|\operatorname{dom}(S(G))| > 1$. Thus, it is interesting to find out the set of all dominant vertices in $S(G)$. In the following theorem, we find the number of dominant vertices in $S(G)$ for any finite non p-group G having an element of exponent order.

Proposition 5.1. *Suppose that G is not of prime power order. The element x is a dominant vertex in $S(G)$ if and only if either $x = e$ or $o(x) = \exp(G)$.*

Proof. Clearly e is a dominant vertex, so suppose that x is a dominant vertex with $x \neq e$. If $o(x) \neq \exp(G)$, then there is an element y whose order does not divide $o(x)$; so $o(x)$ divides $o(y)$. Then there is a prime power p^m which divides $o(y)$ but not $o(x)$, and an element of order p^m in the subgroup generated by y; so x is not dominant. Conversely, if $o(x) = \exp(G)$, then x is dominant, by definition. □

Theorem 5.2. *Let G be a finite non p-group and* $\mathrm{dom}(S(G))$ *be the set of all dominant vertices in the superpower graph $S(G)$ of G. Then*

$$|\mathrm{dom}(S(G))| = \begin{cases} t\phi(\exp(G)) + 1, & \text{if } G \text{ has an element of exponent order;} \\ 1, & \text{otherwise;} \end{cases}$$

where t is the number of distinct cyclic subgroups of order $\exp(G)$ *in G.*

Proof. If there is no element of exponent order, then e is the only dominant vertex. Otherwise, let $m = \exp(G)$. If $o(x) = m$, then $\langle x \rangle$ contains $\phi(m)$ elements of order m, all of which are dominant and generate the same cyclic group. So each cyclic group of order m contains $\phi(m)$ elements of order m, and there is no overlap between these sets of size $\phi(m)$. So the theorem is proved. □

Remark 5.1. For any non-Abelian finite simple group G, $|\mathrm{dom}(S(G))| = 1$. The proof follows from Theorem 5.2 and the fact that G does not contain an element or order $\exp(G)$, [24].

As mentioned in the proof of Theorem 5.2, if G contains an element of order $\exp(G)$ then $S(G)$ always contains a dominant vertex other than identity and hence we have the following corollary.

Corollary 5.1. *For any finite group G having an element of order* $\exp(G)$*, the superpower graph $S(G)$ is dominatable.*

5.1. *Hamiltonian-Like Properties in $S(G)$*

In [6], it was proved that power graph $P(G)$ of any cyclic group of order at least three is Hamiltonian, [see Theorem 4.13]. In [13], it was proved that $S(G) = P(G)$ if and only if G is a finite cyclic group. Thus, $S(G)$ is Hamiltonian for any cyclic group of order at least three. Can we extend this result for finite groups? Unfortunately, the result may not hold in general. For instance, in [20], we have shown that the superpower graph $S(D_{2n})$ of the dihedral group D_{2n} is Hamiltonian if and only if n is an even integer whereas $S(T_{4n})$ of dicyclic group is Hamiltonian for any integer n (one can prove this in similar lines as we proved for $S(D_{2n})$). In the following theorem we prove that $S(G)$ is Hamiltonian under certain condition.

Theorem 5.3. *Let G be a finite group G of order $n \geq 3$, having an element of order* $\exp(G)$*. Then $S(G)$ is Hamiltonian.*

Proof. Let G be a finite group with $a_k = \exp(G)$ and $1 = a_1 < a_2 < a_3 < \cdots < a_{k-1} < a_k$ are all orders of elements in G. Clearly, $a_i | a_k \quad \forall \quad i, 1 \leq i \leq k-1$ which gives that every vertex of the set $w_{a_k}(G)$ is adjacent to all other vertices of the graph $S(G)$. Also, the identity element e is adjacent to all other vertices of $S(G)$. For each element of order a, induced subgraph say H_a on $w_a(G)$ forms a clique in $S(G)$. We get Hamiltonian cycle in $S(G)$ as follows: Start from the vertex $v_1 \in \text{dom}(S(G))$. From v_1, go to any vertex of the clique H_{a_2} and traverse all vertices of H_{a_2}. Now we have a Hamiltonian path containing all the vertices in $H_{a_2} \cup \{v_1\}$. Note that the terminal vertex of this Hamiltonian path is adjacent to a vertex $v_2 \in \text{dom}(S(G))$ and $v_2 \neq v_1$. From v_2, go to any vertex of the clique H_{a_3} and traverse all vertices of H_{a_3}. Now the terminating vertex of the resulting Hamiltonian path is adjacent to a vertex $v_3 \in w_{a_k(G)}$ and $v_3 \notin \{v_1, v_2\}$. Repeat this until all the cliques $H_{a_i}(G), 2 \leq i \leq k-1$ are covered. Since $k-2 < |\text{dom}(S(G))|$, there are sufficient number of vertices in $\text{dom}(S(G))$ to connect all disjoint cliques. Finally, complete the cycle by joining all the uncovered vertices of $\text{dom}(S(G))$ by path to v_1. The entire process of identifying a Hamiltonian cycle is given in Fig. 1. □

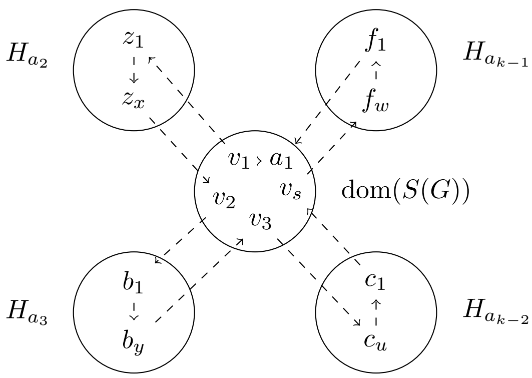

Fig. 1. Hamiltonian Cycle in $S(G)$

Corollary 5.2. *For any finite group G with $o(G) \geq 4$ having an element of order* $\exp(G)$*, $S(G)$ is 1-Hamiltonian.*

Proof. Let $a_k = \exp(G)$ be the largest order of an element in the group G and let $g \in G$. If $o(g) = a_i, 2 \leq i \leq k-1$, then g is a vertex in the clique induced by $w_{a_i}(G)$ for the divisor a_i of $o(G)$. Further $H_{a_i} \setminus \{g\}$ remains as a clique and so it has a spanning path whose initial and terminal vertices can be joined by two different vertices of $\text{dom}(S(G))$. Now, the proof can be completed as in the case of Theorem 5.3.

If $o(g) = a_k$, then $g \in \text{dom}(S(G))$. As seen in the proof of Theorem 5.3, there are sufficient number of vertices in $\text{dom}(S(G)) \setminus \{g\}$ to connect all the disjoint cliques corresponding to all proper divisors of $o(G)$. Hence the required Hamiltonian cycle can be obtained as in Theorem 5.3. Thus, $S(G) \setminus \{g\}$ contains a Hamiltonian cycle implying that $S(G)$ is 1-Hamiltonian. □

Corollary 5.3. *For any finite group G of order at least three having an element of order $\exp(G)$, $S(G)$ is pancyclic.*

Proof. Let $a_k = \exp(G)$. Clearly, $\mathrm{dom}(S(G))$ forms a clique in $S(G)$ and $|\mathrm{dom}(S(G))| = t\phi(a_k) + 1$, where t is the number of cyclic subgroups of order a_k in the group G. So we have cycles of length 3 to $t\phi(a_k) + 1$. Also, Theorem 5.3 implying that $S(G)$ contains a cycle of length n. For any $g_1 \in V(S(G))$, by Corollary 5.2, $S(G) \setminus \{g_1\}$ is Hamiltonian and thus $S(G)$ contains a cycle of length $n - 1$. Note that, in the proof of Corollary 5.2, we see that as long as we keep choosing a vertex $g \in w_{a_i} \subset V(G) \setminus \{\mathrm{dom}(S(G))\}$, obtaining a cycle containing remaining vertices is immediate. Choose $g_2 \in w_{a_i}(G)$ (if exists), otherwise choose $\{g_2\} \in w_{a_j}(G)$ for some $2 \leq i, j \leq \ell$ and we immediately get that $S(G) \setminus \{g_1, g_2\}$ is Hamiltonian. So $S(G)$ contains a cycle of length $n - 2$. Recursively deleting the vertices of w_{a_i} for each i, $2 \leq i \leq l$, we can get cycles of length $n - 2$ to $t\phi(a_k) + 2$. Thus $S(G)$ contains cycles of all length ℓ for $3 \leq \ell \leq n$ and hence $S(G)$ is pancyclic. □

It is not always true that there exists a Hamiltonian path between any pair of vertices in a graph even if it a Hamiltonian. However, this happens in the case of the superpower graph $S(G)$ of any finite group G which contains an element of $\exp(G)$ and hence we have the following result.

Corollary 5.4. *For any finite group G having an element of order $\exp(G)$, $S(G)$ is Hamiltonian connected.*

Proof. Let $u.v \in V(S(G))$ be two distinct vertices in $S(G)$. Without loss of generality, one can take $u = v_1 \in w_{a_2}(G)$ and $v = v_2 \in w_{a_3}(G)$ where a_2 and a_3 are two non-trivial distinct divisors of a_k. Start from the vertex v_1 and traverse along the spanning path in H_{a_2} and join it with a vertex v_3 of $\mathrm{dom}(S(G))$. From v_3 go to any vertex of H_{a_4} and repeat the process until all vertices of the cliques $H_{a_i}(G) \cup \mathrm{dom}(S(G)), 5 \leq i \leq k - 1$ belongs to the path such that $v_\ell \in \mathrm{dom}(S(G))$ is the last vertex of this path. Now, join v_{k-1} to a vertex $x \neq v_2$ of H_{a_3}. Upon completing the path from x to v_2 in H_{a_3}, we obtain the required Hamiltonian path between u and v in $S(G)$. □

Corollary 5.5. *For any finite group G having an element of order $\exp(G)$, $S(G)$ is pan connected.*

Proof. Let $u, v \in V(S(G))$ be two distinct non-adjacent vertices in $S(G)$. Then there always exists a path of every length from 2 to n. Now the required path can be obtained by inserting the vertices from $H_{a_i} \cup \mathrm{dom}(S(G)), 2 \leq i \leq k - 1$ in such a way that any two vertices of cliques H_{a_i}, H_{a_j} can be joined through a vertex of $\mathrm{dom}(S(G))$. $\square$

What will be the effect on Hamiltonianity of the graph $S(G)$, if we remove all dominant vertices from it? The following theorem answers this question. For a finite p-group G, $S(\overline{G})$ is a null graph, so in the following theorem, we consider class of finite non p-groups.

Theorem 5.4. *Let G be a finite non p-group having an element of order $\exp(G)$. Let $w_{\exp(G)}(G) = \{x \in G : o(x) = \exp(G)\}$. Then the induced subgraph $H_{\overline{G}}$ of the superpower graph $S(G)$ induced by $\overline{G} = G \setminus (\{e\} \cup w_{\exp(G)}(G))$ is Hamiltonian if and only if $\exp(G)$ is not a product of two distinct primes.*

Proof. Let $a_k = \exp(G)$. Assume that $H_{\overline{G}}$ is Hamiltonian. If $a_k = pq$ for two distinct primes p and q, then $H_{\overline{G}} = H_p \cup H_q$ is disconnected as there is no path connecting the vertices of H_p and H_q, a contradiction.

Conversely, assume that a_k is not a product of two distinct primes. i.e., $a_k = p_1^{\beta_1} p_2^{\beta_2} \cdots p_m^{\beta_m}$ and $\beta_i \in \mathbb{N}, m \geq 2$ and $p_1 < \cdots < p_m$ are distinct primes. Since G is not a p-group, $m \geq 2$. By the assumption on a_k, we have either $\beta_1 > 1$ or $\beta_2 > 1$ or $m \geq 3$. Since G is not a p-group and by the assumption on a_k, the largest order of an element in the set $\overline{G} = G \setminus (\{e\} \cup w_{a_k}(G))$ will be $\dfrac{a_k}{p_1}(= \overline{a_k}$, say).

Let $w_{\overline{a_k}}(\overline{G}) = \{b_1, \cdots, b_s\}$ be the set of all elements of order $\overline{a_k}$. Let $\{\overline{a_1}, \ldots, \overline{a_\ell}\}$ be the set of all non trivial divisors of $\overline{a_k}$ with $1 < \overline{a_1} < \overline{a_2} < \cdots < \overline{d_\ell} < \overline{a_k}$. Let $w_{\overline{a_i}}(\overline{G})$ be the set of all elements of order $\overline{a_i}$ in $\overline{G}$ and let $H_{\overline{a_i}}$ be the subgraph induced by $w_{\overline{a_i}}(\overline{G})$. For each $i, 1 \leq i \leq \ell$, let $P_{H_{\overline{a_i}}} = \langle v_i, u_i \cdots, x_i \rangle$ be the Hamiltonian path in $H_{\overline{a_i}}$. Then the induced subgraph H of $H_{\overline{G}}$ on the vertices of $\bigcup\limits_{1 \leq i \leq \ell} w_{\overline{a_i}} \cup w_{\overline{a_k}}$ is Hamiltonian, since $C = \langle b_1, P_{H_{\overline{a_1}}}, b_2, P_{H_{\overline{a_2}}}, b_3, \cdots, b_\ell, P_{H_{\overline{a_\ell}}}, b_{\ell+1}, \cdots, b_s \rangle$ is a Hamiltonian cycle in H.

It remains for us to include remaining vertices from $H_{\overline{G}} \setminus H$ into C appropriately to get Hamiltonian cycle in $H_{\overline{G}}$. Based on the condition on $\overline{a_k}$, we observe that the only possible subsets of different orders in $\overline{G} \setminus \{\bigcup\limits_{1 \leq i \leq \ell} w_{\overline{a_i}}(\overline{G}) \cup w_{\overline{a_k}}(\overline{G})\}$ are of the form $w_{{p_1}^{\beta_1}}(\overline{G})$ and $w_{p_1^{\beta_1} r}(\overline{G})$, where $r = \overline{a_i}$, for some $i, 1 < i \leq \ell$. If cliques $H_{p_1^{\beta_1}}$, $H_{{p_1}^{\beta_1} \overline{a_j}}$ exist in $H_{\overline{G}} \setminus H$, then the spanning paths $P(v_1', u_1')$ of $H_{p_1^{\alpha_1}}$ and $P(v_j', u_j')$ for $1 < j \leq \ell$ of $H_{{p_1}^{\alpha_1} a_j}$ are inserted into the spanning path of $H_{\overline{a_i}}$ and $H_{\overline{a_j}}$ respectively, as shown in Fig. 2. That is, the required Hamiltonian cycle $C_{H_{\overline{G}}}$ in $H_{\overline{G}}$ is given by $\langle b_1, P_1, b_2, P_2, \cdots, b_\ell, P_l, b_{\ell+1}, \cdots, b_s \rangle$, where $P_j = \langle v_j, v_j', P(v_j', u_j'), u_j, P(u_j, x_j) \rangle$ (if it exists). $\square$

Corollary 5.6. *Let G be a finite non p-group having an element of order $\exp(G)$, which is not a product of two primes. Then $H_{\overline{G}}$ is 2-connected.*

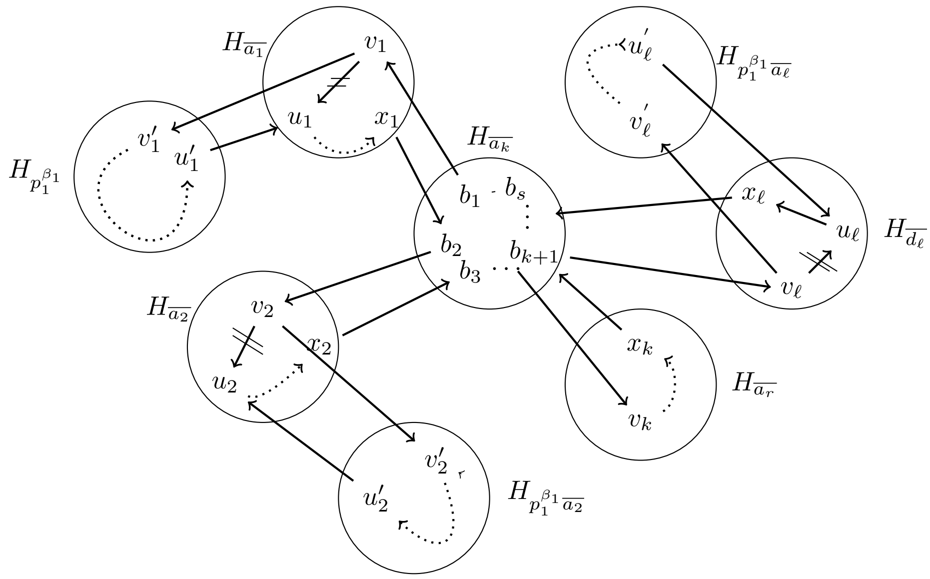

Fig. 2. Hamiltonian Cycle in $H_{\overline{G}}$

Corollary 5.7. *Let G be finite non p-group which is either Abelian or nilpotent. Then the induced subgraph $H_{\overline{G}}$ of the superpower graph $S(G)$ induced by $\overline{G} = G \setminus (\{e\} \cup w_{\exp(G)}(G))$ is Hamiltonian if and only if $\exp(G)$ is not a product of two distinct primes.*

Proof. Result follows from the fact that both groups mention above contains an element of order $\exp(G)$. □

So far, we have discussed the Hamiltonian properties of $S(G)$ when G has an element of order $\exp(G)$. What will happen if G does not have an element of order $\exp(G)$? The following examples shows that it may or may not be Hamiltonian.

Example 5.1. Let D_{2n} denotes the dihedral group D_{2n} of order $2n$. Then $S(D_{2n})$ does not have elements of order $\exp(G)$ if and only if n is odd. Also, $S(D_{2n})$ is not Hamiltonian if and only if n is odd, [20].

Example 5.2. Consider the dicyclic group T_{4n} of order $4n$. It can be seen that $S(T_{4n})$ does not have elements of order $\exp(G)$ if and only if n is odd. Also, when n is odd, $S(T_{4n})$ is Hamiltonian.

In [14], Hamzeh and Ashrafi proposed a conjecture which states that for any non-Abelian finite simple groups G, $S(G)$ is not Hamiltonian. Consider the Mathieu groups M_{11} and M_{22}. It can be seen that corresponding graphs $S(M_{11})$ and $S(M_{22})$ are not Hamiltonian. Since these graphs contain cliques $H_{11} \subset S(M_{11})$ and $H_7 \subset$

$S(M_{22})$ such that no vertex of these cliques are adjacent to any vertex in $S(M_{11}) \setminus \{H_{11}, e\}$ and $S(M_{22}) \setminus \{H_7, e\}$, respectively.

In the following theorem we use this idea and prove that $S(A_n)$ is not Hamiltonian for $n \geq 5$, where A_n denotes non-Abelian finite simple alternating group on n symbols.

Theorem 5.5. *For any alternating group $A_n, n \geq 5$ of permutation group S_n, $S(A_n)$ is not Hamiltonian.*

Proof. For given integer $n \geq 5$, there always exist a prime number $p \in (\lfloor \frac{n}{2} \rfloor, n)$ and cycle of order p in A_n. Let P_p be a spanning path in $S(A_n)$ corresponding to $w_p(A_n)$. Clearly, one of the last vertex of P_p will not adjacent to any other vertex implying that $S(A_n)$ is not Hamiltonian. □

Consider the following question posted by Hamzeh and Ashrafi in [14]:

Problem 5.1. *In a finite non-Abelian simple group G, can we always find an integer r, such that elements in $w_r(G)$ of order r (as we have done in case of A_n) are not adjacent to any vertex of $S(G) \setminus H_r$ other than $e \in G$, where H_r is the induced subgraph on $w_r(G)$?*

Note that, when such a subgraph H_r exists, then $S(G)$ is not Hamiltonian.

From the above remarks, can we generalize these observations to any finite non-Abelian simple groups. If so, then we can state an interesting property for any simple group as follows:

Problem 5.2. *If G is a non-Abelian, non-p finite simple group then $S(G)$ is non-Hamiltonian. In other words, if $S(G)$ is Hamiltonian, then G cannot be a non-Abelian simple group.*

With this observation, we next move on to study when the graph $S(G)$ becomes Eulerian. In [14], it was proved that $S(G)$ is Eulerian if and only if G is a group of odd order. Now what will be the effect on order of G if we removed all dominant vertices from $S(G)$. Following theorem gives answer of this question.

Theorem 5.6. *Let G be a finite non p-group. Then $H_{\overline{G}}$ is Eulerian if and only if $O(G)$ is even integer.*

Proof. Suppose $H_{\overline{G}}$ is Eulerian. Let $\pi(\overline{G}) = \{\overline{a_1}, \overline{a_2}, \cdots, a_{\overline{\ell}}\}$ be the set of all element orders in $\overline{G}$. Then for any $x \in \overline{G}$ with $o(x) = a_i$

$$deg_{H_{\overline{G}}}(x) = w_{a_i}(\overline{G}) + \sum_{a_i | a_j \text{ or } a_j | a_i \quad a_i \neq a_j} w_{a_j}(\overline{G}) - 1. \tag{5.1}$$

For each $i, 1 \leq i \leq \overline{\ell}$, number of elements of order $a_i = t_i \phi(a_i)$, where t_i is the number of cyclic subgroups of order a_i in $\overline{G}$. Also, it is well known fact that $\phi(k)$ is odd if and only if $k \in \{1, 2\}$. Clearly, $deg_{H_{\overline{G}}}(x)$ is even if and only if expression in

(5.1) is even which is possible if and only if G must have odd numbers of involutions elements. Thus, $o(G)$ is even. Conversely, If n is even then by Cauchy theorem, G and hence $\overline{G}$ must have an involutions and these are odd in numbers. By expression in 5.1, degree of any element in $H_{\overline{G}}$ is even. Thus, $H_{\overline{G}}$ is Eulerian. □

5.2. *Vertex connectivity of $S(G)$*

It is well known that any graph containing Hamiltonian cycle is 2-connected and hence $S(G)$ is 2-connected for any finite group G having an element of order $\exp(G)$. In the following theorem, we are giving the tight lower bound for the vertex connectivity of $S(G)$ for any finite group G having an element of order $\exp(G)$, which extend the results [14, Theorem 2.7] and [13, Theorems 2.11].

Theorem 5.7. *Let G be a finite group having an element of order $\exp(G)$. Then $\kappa(S(G)) \geq t\phi(\exp(G)) + 1$, where t is the number of distinct cyclic subgroups of order a_k in G. Further, $\kappa(S(G)) = t\phi(\exp(G)) + 1$ if and only if $\exp(G) = pq$, where p, q are different primes.*

Proof. Let G be a finite group having an element of order $a_k = \exp(G)$. Then to disconnect $S(G)$, we need to remove at least all vertices of $\text{dom}(S(G))$, since these vertices are adjacent to all other vertices of $S(G)$. This implies that $\kappa(S(G)) \geq t\phi(\exp(G)) + 1$.

Next we prove the second part of the statement. Let $\exp(G)$ is product of two distinct primes, that is $a_k = pq$ and $\overline{G} = G \setminus (\{e\} \cup w_{a_k}(G))$. Since there is no path between elements of order p and q, the induced subgraph $H_{\overline{G}}$ is disconnected. Thus $\kappa(S(G)) = t\phi(a_k) + 1$.

Conversely, assume that $\kappa(S(G)) = t\phi(a_k) + 1$. Suppose G has an element of order $\exp(G) = a_k$ which satisfies $a_k = p^{\alpha_1} q^{\beta_1}$ and α_1, β_1 are integers, $\alpha_1, \beta_1 \geq 1$ and either $\alpha_1 > 1$ or $\beta_1 > 1$. We will prove that $\kappa(S(G)) > t\phi(a_k) + 1$ by showing that $H_{\overline{G}}$ is connected. For $u, v \in V(H_{\overline{G}})$, take $w, x, z \in V(H_{\overline{G}})$ with $o(w)$ is the least prime divisor of $o(u)$ and $o(z)$ is the least prime divisor of $o(v)$ and $o(x)$ is the product of least prime divisors of $o(u)$ and $o(v)$. Then $P :=< u, w, x, z, v >$ is a path between u and v. Also, by the same way it can be shown that $H_{\overline{G}}$ is connected if $\exp(G)$ has at least three prime divisors. Otherwise, $H_{\overline{G}}$ is connected. Thus, $\exp(G)$ has at most two prime divisors with $a_k = pq$ in G. □

Let G be a finite group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}, m \geq 2$ having an element of order $a_k = \exp(G)$. Let the prime decomposition of $a_k = p_1^{\beta_1} p_2^{\beta_2} \cdots p_s^{\beta_s}, 1 \leq s \leq m, \beta_i \geq 0, 1 \leq i \leq s$. Let $a_0 = p_1^{\beta_1}, a_1 = \dfrac{a_k}{a_0}$ and $\pi(G) = \{a_0, a_1, a_2, \cdots, a_k\}$ be the set of all orders of elements in G.

In the following theorem, we find the tight upper bound for the vertex connectivity of $S(G)$ using the notations defined above.

Theorem 5.8. *Let G be a finite group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}, m \geq 2$. Then* *there exist a minimal separating set T of $S(G)$ with*

$$\kappa(S(G)) \leq |T| = \sum_{(a_i|a_1 \quad or \quad a_1|a_i, a_i \neq a_1)} t_i \phi(a_i), \tag{5.2}$$

where t_i is the number of distinct cyclic subgroups of order a_i in G. Also, this bound is tight.

Proof. Consider the graph $S(G)$ and define a set T in $S(G)$ as follows:

$$T = \{w_{a_i}(G)| \quad a_i|a_1 \quad or \quad a_1|a_i, \quad 2 \leq i \leq k\},$$

Clearly, T is a separating set of $S(G)$, since there is no path between any vertices of the cliques $w_{a_0}(G)$ and $w_{a_1}(G)$. Let A and B are two connected component of $S(G) \setminus T$ such that $w_{a_0}(G) \in A$ and $w_{a_1}(G) \in B$. Now, we prove that this set is a minimal separating set by showing that for any non empty subset $T^\dagger$ of T, there is a path, connecting $u \in w_{a_0}(G)$ and $v \in w_{a_1}(G)$ in $S(G) \setminus T^\dagger$. Without loss of generality assume that $T \setminus T^\dagger = \{x\}$ with $x \in w_{a_r}(G)$. Since either $a_r|a_1$ or $a_1|a_r$, there exist a path $P_1(u, x)$, connecting u and x in $A \cup w_r(G)$. Similarly, let $y \in B$ such that $o(y) = a_r p_s^{\beta_s}$, then there exist a path $P_2(y, v)$, connecting y to v in B. Now, consider the path $P =:< P_1(u, x), x, y, P_2(y, v) >$ which connect u to v in $S(G) \setminus T^\dagger$. Thus, T is a minimal separating set of $S(G)$. If t_i, number of cyclic subgroups of order a_i in G, $1 \leq i \leq r$. Then

$$|T| = \sum_{(a_i|a_1 \ or \ a_1|a_i), \ a_i \neq a_1} t_i \phi(a_i).$$

Thus, $\kappa(S(G)) \leq |T|$.

Now, we show that the obtained bound is in fact tight. That is, there exists a minimum separating set T, with $|T|$ given above serve the $\kappa(S(G))$ for the group $G \simeq D_{2n}$ when $n = 2^\alpha p^\beta$. Clearly, in line with the above proof, the separating set given by $T = \{w_{p^i}(G) : 0 \leq i \leq \beta - 1\} \cup \{w_{2^j p^\beta}(G) : 1 \leq j \leq \alpha\}$ with $|T| = p^{\beta-1} + (2^\alpha - 1)(p^\beta - p^{\beta-1})$ is minimal. Next we show that T is, in fact, minimum. Let $T^\dagger$ be any other minimal separating set of $S(D_{2n})$.

Claim $|T^\dagger| \geq |T|$: Without loss of generality assume that $S(G) \setminus T^\dagger$ has two components say A and B. Let $x \in A$ and $y \in B$ be such that there is no path joining x and y in $S(G) \setminus T^\dagger$. Clearly, both x and y are not of odd order, otherwise they will be adjacent. Also, both x are y are not of even order, otherwise they can be joined through an element of $w_2(G)$. Note that if $w_2(G) \subset T^\dagger$, then $|T^\dagger| \geq |T|$. Finally, let us assume that $o(x) = p^\ell$ and $o(y) = 2^s p^t$ for some $1 \leq s \leq \alpha, 1 \leq t < \ell \leq \beta$. Therefore, the minimal separating set $T^\dagger$, that separates x and y, is then given by

$$T^\dagger = \{w_{p^i}(G) : 0 \leq i < \ell\} \bigcup \{w_{2^j p^\ell}(G) : 1 \leq j \leq \alpha\}$$
$$\bigcup \{w_{2^j p^i(G)} : 1 \leq j \leq \alpha, \ell + 1 \leq i \leq \beta\}$$

and with

$$|T^\dagger| = \sum_{i=0}^{\ell-1}\phi(p^i) + \sum_{j=1}^{\alpha}\phi(2^j p^\ell) + \sum_{j=1}^{\alpha}\sum_{i=\ell+1}^{\beta}\phi(2^j p^i) = p^{\ell-1} + (2^\alpha - 1)(p^\beta - p^{\ell-1}).$$

Thus, we get that $|T^\dagger| - |T| = (2^\alpha - 2)(p^{\beta-1} - p^{\ell-1}) \geq 0$. Hence, T is a minimum separating set of $S(G)$. $\square$

Corollary 5.8. *Let G be an finite non p-group of order n having an element of order $= a_k = \exp(G)$. If a_k is not a product of two distinct prime, then the bounds of $\kappa(S(G))$ is given by*

$t\phi(\exp(G)) + 1 \leq \kappa(S(G)) \leq |T|$, *when* $\exp(G)$ *is a product of two primes and*

$t\phi(\exp(G)) + 3 \leq \kappa(S(G)) \leq |T|$, *otherwise*

where t denotes the number of cyclic subgroups of order a_k in G and T is the minimal separating set given in theorem 5.8 and $|T|$ is given by equation (5.2).

Proof. Note that the required lower bound follows from Corollary 5.6 and Theorem 5.7 while the upper bound follows from Theorem 5.8. $\square$

6. Conclusion

In this paper, we have mostly studied the finite non-Abelian groups having an element of exponent order and their superpower graphs. We have studied the structure of superpower graphs in terms of their separating sets and have given tight bounds for the vertex connectivity. Further, we have also discussed the Hamiltonian-like properties of these superpower graphs. Overall, these results extend the scope for the superpower graphs defined on finite Abelian groups to finite non-Abelian groups.

We conclude this article with an open problem:

Problem 6.1. *Characterize non-simple groups which do not have any element of exponent order such that their superpower graph is non-Hamiltonian.*

References

[1] Jemal Abawajy, Andrei Kelarev, and Morshed Chowdhury, Power graphs: A survey. *Electronic Journal of Graph Theory and Applications (EJGTA)*, 1(2):125–147, 2013.

[2] Alireza Khalili Asboei and Seyed Sadegh Salehi, Some results on the main supergraph of finite groups. *Algebra and Discrete Mathematics*, 30(2):172–178, 2020.

[3] J. A. Bondy and U. S. R Murty, *Graph Theory.* Springer Publishing Company, Incorporated, 1st edition, 2008.

[4] Rolf Brandl, Finite groups all of whose elements are of prime power order, *Bollettino UMI* (5) **18(A)** (1981), 491–493.

[5] Peter J. Cameron and Natalia Maslova, Criteria of unrecognizability of a finite group by its Gruenberg-Kegel graph, *J. Algebra* **607** (2022), 186–213.

[6] Ivy Chakrabarty, Shamik Ghosh, and M K Sen, Undirected power graphs of semigroups. In *Semigroup Forum*, volume 78, pages 410–426. Springer, 2009.

[7] Maria Chudnovsky, Neil Robertson, Paul Seymour and Robin Thomas, The strong perfect graph theorem, *Ann. Math.* 164 (2006), 51–229.
[8] Stephen J Curran and Joseph A Gallian, Hamiltonian cycles and paths in cayley graphs and digraphs—a survey. *Discrete Mathematics*, 156(1-3):1–18, 1996.
[9] Robert P. Dilworth, A decomposition theorem for partially ordered sets, *Ann. Math.* **51** (1950), 161–166.
[10] Joseph A. Gallian, *Contemporary abstract algebra.* Narosa publishing house, fourth edition, 1999.
[11] Asma Hamzeh and Ali-Reza Ashrafi, Automorphism groups of supergraphs of the power graph of a finite group. *Europ. J. Combinatorics*, 60:82–88, 2017.
[12] Asma Hamzeh and Ali Reza Ashrafi, Spectrum and $l-$ spectrum of the power graph and its main supergraph for certain finite groups. *Filomat*, 31(16):5323–5334, 2017.
[13] Asma Hamzeh and Ali Reza Ashrafi, The order supergraph of the power graph of a finite group. *Turk J Math*, 42(4):1978–1989, 2018.
[14] Asma Hamzeh and Ali Reza Ashrafi, Some remarks on the order supergraph of the power graph of a finite group. *International Electronic Journal of Algebra*, 26(26):1–12, 2019.
[15] Andrei Kelarev. *Graph algebras and automata*, CRC Press, 2003.
[16] Andrei V Kelarev, *Ring constructions and applications*, volume 9. World Scientific, 2002.
[17] Andrei Kelarev, Labelled cayley graphs and minimal automata. *Australasian J. Combinatorics*, 30:95–101, 2004.
[18] Andrei Kelarev, Joe Ryan, and John Yearwood, Cayley graphs as classifiers for data mining: the influence of asymmetries. *Discrete Mathematics*, 309(17):5360–5369, 2009.
[19] Ajay Kumar, Lavanya Selvaganesh, Peter J Cameron, and T Tamizh Chelvam, Recent developments on the power graph of finite groups–a survey. *AKCE International Journal of Graphs and Combinatorics*, 18(2):65–94, 2021.
[20] Ajay Kumar, Lavanya Selvaganesh, and T Tamizh Chelvam, Structural Properties of Super PowerGraph of Dihedral Group D_{2n}. *preprint: submitted for Publication.*
[21] Cai Heng Li, On isomorphisms of finite cayley graphs—a survey. *Discrete Mathematics*, 256(1-2):301–334, 2002.
[22] Xuanlong Ma and Huadong Su, On the order supergraph of the power graph of a finite group. *Ricerche di Matematica*, pages 1–10, 2020.
[23] Mahsa Mirzargar, A survey on the automorphism groups of the commuting graphs and power graphs. *Facta Universitatis, Series: Mathematics and Informatics*, pages 729–743, 2019.
[24] A. V. Vasiliev and Evgeny P. Vdovin, An adjacency criterion for the prime graph of a finite simple group. *Algebra and Logic*, 44(6):381–406, 2005.
[25] David Witte and Joseph A Gallian, A survey: Hamiltonian cycles in cayley graphs. *Discrete Mathematics*, 51(3):293–304, 1984.

Acknowledgements

The research work of

(a) Ajay Kumar is supported by CSIR-UGC JRF, New Delhi, India, through Ref No.: 19/06/2016(i) EU-V/Roll No. 417267.
(b) Lavanya Selvaganesh is partially supported by SERB, India, through Grant No. MTR/2018/000254 under the scheme MATRICS.

(c) T. Tamizh Chelvam is supported by CSIR Emeritus Scientist Scheme of Council of Scientific and Industrial Research (No.21(1123)/20/EMR-II), Government of India.