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Induced subgraphs of zero-divisor graphs

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ABSTRACT

The zero-divisor graph of a finite commutative ring with unity is the graph whose vertex set is the set of zero-divisors in the ring, with a and b adjacent if $ab = 0$. We show that the class of zero-divisor graphs is universal, in the sense that every finite graph is isomorphic to an induced subgraph of a zero-divisor graph. This remains true for various restricted classes of rings, including boolean rings, products of fields, and local rings. But in more restricted classes, the zero-divisor graphs do not form a universal family. For example, the zero-divisor graph of a local ring whose maximal ideal is principal is a threshold graph; and every threshold graph is embeddable in the zero-divisor graph of such a ring. More generally, we give necessary and sufficient conditions on a non-local ring for which its zero-divisor graph to be a threshold graph. In addition, we show that there is a countable local ring whose zero-divisor graph embeds the *Rado graph*, and hence every finite or countable graph, as induced subgraph. Finally, we consider embeddings in related graphs such as the 2-dimensional dot product graph.

1. Introduction

In this paper, "ring" means "finite commutative ring with unity", while "graph" means "finite simple undirected graph" (except in the penultimate section, where finiteness will be relaxed). The *zero-divisor graph* $\Gamma(R)$ of a ring R has vertices the zero-divisors in R (the non-zero elements a for which there exists $b \neq 0$ such that $ab = 0$), with an edge $\{a, b\}$ whenever $ab = 0$. The zero-divisor graphs have been extensively studied in the past [1,3–7,9,10,20,22,25].

In this paper, we are interested in universal graphs. Let G be a graph. A graph H is said to be G-free if no subgraph of H is isomorphic to G. A countable G-free graph H is weakly universal if every countable G-free graph is isomorphic to a subgraph of H, and strongly universal if every such graph is isomorphic to an induced subgraph of H. Similarly, a class of graphs $\mathcal{C}$ is said to be universal if every graph is an induced subgraph of a graph in the class $\mathcal{C}$. Universal graphs are well-studied and there is a vast literature spreading over the past few decades [24,18].

Universal graphs for collections of graphs satisfying certain forbidden conditions were also explored in the past [8,14,21]. In [15], the search for a graph G on n vertices with the minimum number of edges such that every tree on n vertices is isomorphic to a spanning tree of G is carried out.

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Here we address the question: Which finite graphs are induced subgraphs of the zero-divisor graph of some ring? The answer turns out to be "all of them", but there are interesting questions still open when we restrict the class of rings or the class of graphs.

2. Zero divisor graphs are universal

In this section, we show that the zero divisor graphs associated to various classes of rings are universal.

2.1. The zero divisor graphs of Boolean rings are universal

In this subsection we show that intersection graphs are universal. Indeed we prove a stronger result: namely, every graph G can be identified as an intersection graph for a natural choice of sets associated to G. We need this result in a couple of proofs in later sections.

Definition 1. Let X be a family of subsets of a set A. The *intersection graph* on X, denoted by $\mathcal{G}(X)$, has as vertices the sets in the family, two vertices joined if they have non-empty intersection.

The following proposition is well known.

Proposition 2.1. *Let G be a finite graph. Then G can be identified as an intersection graph for a natural choice of sets associated to G.*

Proof. Let G be a finite graph with edge set E. Without loss of generality we assume that G has no isolated vertices or isolated edges (see Remark 1). The construction is simple. We represent a vertex v of G by the subset A_v of E consisting of all edges of G containing v. Then

$$A_v \cap A_w = \begin{cases} \{e\} \text{ if } v \text{ is adjacent to } w \text{ by the edge } e, \\ \emptyset \text{ otherwise.} \end{cases}$$

Moreover, $A_v \neq A_w$ for all $v \neq w$, since (in the absence of isolated vertices and edges) two vertices cannot be adjacent with the same set of edges. Now, it is immediate that the graph G and the intersection graph $\mathcal{G}(X)$ are isomorphic where $X = \{A_v : v \in V(G)\}$. □

Remark 1. If there are isolated vertices in G, they can be represented by non-empty sets disjoint from the graph obtained from the non-isolated vertices. If there are isolated edges in G we represent the vertices of such an edge by two distinct but intersecting sets disjoint from the sets representing the rest of the graph.

Let X be a finite set. The *power set* $P(X)$ of X is the set of all subsets of X. The *Boolean ring* on X has as element set $P(X)$; addition is symmetric difference, and multiplication is intersection. So the empty set is the zero element and $ab = 0$ if and only if a and b are disjoint.

Lemma 2.2. *Let $\Gamma(R)$ be the zero divisor graph of a Boolean ring R. A graph G is an induced subgraph of $\Gamma(R)$ if, and only if, the complement graph of G is isomorphic to an intersection graph $\mathcal{G}(X)$ for some collection of sets X.*

Proof. Suppose G is an induced subgraph of the zero divisor graph of a Boolean ring R. Now R is a subring of $(P(X), \triangle, \cap)$ for some X. Therefore the vertices of G are elements of $P(X)$ and two of them are adjacent if their intersection is empty. In other words, two of them are adjacent in the complement graph of G if their intersection is non-empty. This shows that the complement graph of G is an intersection graph.

Conversely, consider an arbitrary intersection graph $G := \mathcal{G}(X)$ for some collection of subsets X of a set A. Then the vertices of G are subsets of A and two of them are adjacent if their intersection is non-empty. In other words, two of them are adjacent in the complement graph of G if their intersection is empty. This shows that the complement graph of G is an induced subgraph of the zero divisor graph of the Boolean ring $(P(A), \triangle, \cap)$. □

Theorem 2.3. *The zero divisor graphs of Boolean rings are universal. That is, every finite graph is an induced subgraph of the zero-divisor graph of a Boolean ring.*

Proof. By Lemma 2.2, it is enough to show that the complement graph of any finite graph is an intersection graph. But this claim follows from Proposition 2.1. □

Corollary 2.4. *For every finite graph G, there exists a positive integer k such that G is an induced subgraph of the zero-divisor graph of the ring $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ (k-times).*

Proof. Note that every finite Boolean ring is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ for some positive integer k, and hence the result follows from Theorem 2.3. $\square$

2.2. The zero divisor graphs of ring of integers modulo n and the reduced rings are universal

Lemma 2.5. *Let $X = \{1, 2, \ldots, m\}$, and let $P(X)$ denote its power set (the set of all its subsets). Then the zero divisor graph of the Boolean ring $(P(X), \triangle, \cap)$ is an induced subgraph of the zero divisor graph of the ring $\mathbb{Z}_n$, the integers modulo n, where $n = p_1 p_2 \cdots p_m$ is a square-free integer with m prime divisors.*

Proof. Let A and B be two proper subsets of X. Then the following statements are equivalent:

- A and B are adjacent in the zero divisor graph of the Boolean ring $(P(X), \triangle, \cap)$;
- A and B are disjoint;
- $A^c \cup B^c = X$;
- the product $n_{A^c} n_{B^c}$ is divisible by n, where $n_A = \prod_{i \in A} p_i$.

Therefore the map which sends A to n_{A^c} defines an injective graph homomorphism of the zero divisor graph of the Boolean ring $(P(X), \triangle, \cap)$ and the zero divisor graph of the ring $\mathbb{Z}_n$, the integers modulo n. $\square$

Theorem 2.6. *Every finite graph is an induced subgraph of the zero-divisor graph of a product of distinct finite fields.*

Proof. By the Chinese Remainder Theorem, if $n = p_1 p_2 \cdots p_m$ where $p_1, p_2, \ldots, p_m$ are distinct primes, the product of the fields $\mathbb{Z}_{p_i}$ is isomorphic to $\mathbb{Z}_n$, so the result follows from the preceding theorem. $\square$

Corollary 2.7. *The class of zero-divisor graphs of the rings of integers modulo n is universal. Further, we can even restrict the n to be squarefree.*

2.3. The zero divisor graphs of local rings are universal

Further discussion of ring-theoretic concepts such as local, Noetherian and Artinian rings can be found in most ring-theory books, for example [19,23].

A *local ring* is a ring R with a unique maximal ideal M. Any non-zero element of M is a zero-divisor, and every element of $R \setminus M$ is a unit.

Lemma 2.8. *Let R be a local ring with a unique maximal ideal M. Then there exists r such that $M^r = \{0\}$.*

Proof. Since R is a finite local ring, it is a local Artinian ring and hence M is nilpotent. $\square$

From the above lemma we observe that if $a \in M^s$ and $b \in M^t$ with $s + t \geq r$, then $ab = 0$, and hence a and b are adjacent in the zero-divisor graph. This observation might suggest that the zero-divisor graph is a threshold graph (these graphs are defined in the next section), but we show that this is not so. The problem is complicated by the fact that the converse of this observation is false in general.

In the following theorem, we show that in fact the zero-divisor graphs of local rings are universal.

Theorem 2.9. *For every finite graph G, there is a finite commutative local ring R with unity such that G is an induced subgraph of the zero-divisor graph of R.*

Proof. Let the vertex set of G be $\{v_1, \ldots, v_n\}$. Take R to be the quotient of $F[x_1, \ldots, x_n]$ by the ideal I generated by all homogeneous polynomials of degree 3 in $x_1, \ldots, x_n$ together with all products $x_i x_j$ for which $\{v_i, v_j\}$ is an edge of G, where F is a finite prime field of order p and $x_1, \ldots, x_n$ are indeterminates.

Let M be the ideal of R generated by the elements $x_1, \ldots, x_n$ (We abuse notation by identifying polynomials in $x_1, \ldots, x_n$ with their images in R.) Thus M is the set of elements represented by polynomials with zero constant term. So M is nilpotent and every element of $R \setminus M$ is a unit, so M is the unique maximal ideal, and R is a local ring.

Now the elements $x_1, \ldots, x_m$ of M satisfy $x_i x_j = 0$ if and only if $\{v_i, v_j\}$ is an edge of G. To see this, note that the set

$$\{1, x_1, \ldots, x_n\} \cup \{x_i x_j : \{v_i, v_j\} \notin E(G)\}$$

is an F-basis for R. This completes the proof. $\square$

3. When is the zero-divisor graph threshold?

In preparation for the next section, we need some background on threshold graphs. These were introduced by Chvátal and Hammer [16] in 1977.

Definition 2. A graph G is a threshold graph if there exists $t \in \mathbb{R}$ and for each vertex v a weight $w(v) \in \mathbb{R}$ such that uv is an edge in G if, and only if, $w(u) + w(v) > t$.

Definition 3. A *split graph* is a graph whose vertex set is the disjoint union of an independent set and a clique, with arbitrary edges between them. A *nested split graph* (or NSG for short) is a split graph in which we add cross edges in accordance with partitions of U (the independent set) and V (the clique) into h cells (namely, $U = U_1 \cup U_2 \cup \ldots \cup U_h$ and $V = V_1 \cup V_2 \cup \ldots \cup V_h$) in the following way: each vertex $u \in U_i$ is adjacent to all vertices $v \in V_1 \cup V_2 \cup \ldots \cup V_i$. The vertices $U_i \cup V_i$ form the i-th level of the NSG, and h is the number of levels. The NSG as described can be denoted by $\mathrm{NSG}(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$, where $m_i = |U_i|$ and $n_i = |V_i|$ $(i = 1, 2, \ldots, h)$.

The following theorem is well-known.

Theorem 3.1 *([16]). For a finite graph G, the following three properties are equivalent:*

(a) *G is a threshold graph;*
(b) *G has no four-vertex induced subgraph isomorphic to C_4 (the cycle), P_4 (the path), or $2K_2$ (the matching);*
(c) *G can be built from the empty set by repeatedly adding vertices joined either to nothing or to all other vertices;*
(d) *G is a nested split graph.*

The reverse of the building-up procedure of part (d) of the theorem (that is, removing the vertices one at a time so that each removed vertex is joined to all or none of the remaining vertices) will be called “dismantling” the graph.

We thought originally that the zero-divisor graph of a local ring might be a threshold graph. This is not true in general, as shown by part (b) of the above theorem together with Theorem 2.9; Example 1 also demonstrates this (it was the first example we discovered).

Example 1. Let $A = \mathbb{Z}_4[x, y, z]/M$, where M is the ideal generated by $\{x^2 - 2, y^2 - 2, z^2, 2x, 2y, 2z, xy, xz, yz - 2\}$. In the zero-divisor graph of A, the induced subgraph on the vertices $\{x, z, x + y, x + y + 2\}$ is $2K_2$ and the induced subgraph of the vertices $\{x, z + 2, x + z, x + y\}$ is P_4.

But there is one class of local rings whose zero-divisor graphs are threshold, those whose maximal ideal is principal. If p is a generator of M as an ideal, then every element of M has the form $p^s u$ where u is a unit and $s > 0$. If u and v are units, then $p^s u . p^t v = 0$ if and only if $s + t \geq r$, where r is the nilpotent index of the ideal M (the smallest positive integer k such that $M^k = \{0\}$).

Definition 4. A collection $\mathcal{C}$ of graphs is said to be *threshold-universal* (for short, t-universal) if every threshold graph is an induced subgraph of a graph from $\mathcal{C}$.

Theorem 3.2. *Let R be a local ring whose maximal ideal M is principal.*

(a) *The zero-divisor graph of R is a threshold graph.*
(b) *Any threshold graph is an induced subgraph of some local ring whose maximal ideal is principal. In other words the set of all zero divisor graphs of local rings with principal maximal ideal form a t-universal collection of graphs.*

Proof. (a) For the first part, let R be a local ring with maximal ideal M. Let r be the smallest integer such that $M^r = \{0\}$. Take threshold $t = r$, and set $w(a) = i$ if $a \in M^i \setminus M^{i+1}$. By the remarks above, $ab = 0$ if and only if $w(a) + w(b) \geq r$, so the zero-divisor graph is a threshold graph.

(b) Let G be an arbitrary threshold graph. Choose a prime which is sufficiently large (larger than the number of vertices of G is certainly enough). Let m be the number of stages required to dismantle G; embed the vertices of G in $R = \mathbb{Z}/(p^{2m+1})$ as follows: if a is removed as an isolated vertex in round i, map it to an element of $p^i R \setminus p^{i+1} R$; if it is removed as a vertex joined to all others in round i, map it to an element of $p^{2m-i+1} R \setminus p^{2m-i+2} R$. (Each of these differences contains at least $p - 1$ elements, enough to embed all required vertices.) $\square$

The next result is a necessary condition for the zero divisor graph of a ring to be threshold. Here $\mathrm{Ann}(x) = \{y \in R : xy = 0\}$ is the *annihilator* of x: it is a non-zero ideal if x is a zero-divisor.

Theorem 3.3. *If R is a ring whose zero divisor graph is threshold, then for any two distinct zero-divisors $x, y \in R$, the following statements hold:*

(a) $\mathrm{Ann}(x) \cap \mathrm{Ann}(y) \neq \{0\}$;
(b) *either* $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(y)$ *or* $\mathrm{Ann}(y) \subseteq \mathrm{Ann}(x)$;
(c) *if* $\mathrm{Ann}(x) \subsetneq \mathrm{Ann}(y)$ *and* $xy \neq 0$, *then* $\langle \mathrm{Ann}(x) \rangle$ *is a clique;*
(d) *if* $\mathrm{Ann}(x) \subsetneq \mathrm{Ann}(y)$ *and* $xy \neq 0$, *then for any* $a \in \mathrm{Ann}(x)$ *and for any* $b \in \mathrm{Ann}(y) \setminus \mathrm{Ann}(x)$, $ab = 0$.

Proof. (a) Suppose $\mathrm{Ann}(x) \cap \mathrm{Ann}(y) = \{0\}$, for some zero-divisors $x, y \in R$. Then there exist $0 \neq a \in \mathrm{Ann}(x)$ and $0 \neq b \in \mathrm{Ann}(y)$ such that $a \notin \mathrm{Ann}(y)$ and $b \notin \mathrm{Ann}(x)$. Hence, the subgraph induced by $\{x, a, y, b\}$ is isomorphic to $2K_2$, P_4 or C_4, a contradiction.

(b) Suppose that there exist non-zero elements $x, y \in R$ (necessarily zero-divisors) such that $\mathrm{Ann}(x) \nsubseteq \mathrm{Ann}(y)$ and $\mathrm{Ann}(y) \nsubseteq \mathrm{Ann}(x)$. Then again there exist $0 \neq a \in \mathrm{Ann}(x)$ and $0 \neq b \in \mathrm{Ann}(y)$ such that $a \notin \mathrm{Ann}(y)$ and $b \notin \mathrm{Ann}(x)$. Hence we get a similar contradiction in (a).

(c) If $\mathrm{Ann}(x) \subsetneq \mathrm{Ann}(y)$ and $\langle \mathrm{Ann}(x) \rangle$ is not a clique, then there exist $a, b \in \mathrm{Ann}(x)$ such that $ab \neq 0$ and hence the subgraph induced by $\{x, a, y, b\}$ is isomorphic to C_4, a contradiction.

(d) Similar to (c). □

The following theorem is a characterization of non-local rings R whose zero-divisor graph is threshold. We use the fact that any finite ring is a direct sum of local rings, see [19, p.430], [13, p.110].

Theorem 3.4. *Let R be a non-local ring. Then $\Gamma(R)$ is threshold if and only if $R \cong \mathbb{F}_2 \times \mathbb{F}_q$, where $q \geq 3$ and $\mathbb{F}_q$ is the field with q elements.*

Proof. If $R \cong \mathbb{F}_2 \times \mathbb{F}_q$, then $\Gamma(R) \cong K_{1,\, q-1}$ (a star graph) and hence it is threshold. Suppose $\Gamma(R)$ is threshold and $R = R_1 \times R_2 \times \ldots \times R_n$, where $n \geq 2$. If $n \geq 3$, then $\mathrm{Ann}((1,1,0,0,\ldots,0)) = \{0\} \times \{0\} \times R_3 \times \ldots \times R_n$ and $\mathrm{Ann}((0,0,1,0,\ldots,0)) = R_1 \times R_2 \times \{0\} \times R_4 \times \ldots \times R_n$. By Theorem 3.3(b), we have $\Gamma(R)$ is not threshold. So $n = 2$. If one of the R_i is not a field, say R_1, then there exist non-zero elements $x, y \in R_1$ such that $xy = 0$. Hence $\mathrm{Ann}((x,0)) = \mathrm{Ann}(x) \times R_2$ and $\mathrm{Ann}((0,1)) = R_1 \times \{0\}$ and thus we have $\Gamma(R)$ is not threshold by Theorem 3.3(b). Therefore, both R_1 and R_2 are fields. If $|R_i| > 2$, for $i = 1, 2$ then, by the same argument, $\mathrm{Ann}((1,0))$ is not a subset of $\mathrm{Ann}((0,1))$ and $\mathrm{Ann}((0,1))$ is not a subset of $\mathrm{Ann}((1,0))$. □

Next, we ask the following question about local rings.

Problem 1. For which local rings (R, M) is the zero-divisor graph threshold?

For the rest of this section, we consider a local ring (R, M), where M is the maximal ideal.
Since every finite commutative ring R with unity is Noetherian, every ideal of R is finitely generated. In particular, the maximal ideal M in a local ring (R, M) is finitely generated.

Proposition 3.5. *Let M be the maximal ideal of R, and let M be generated by $\{x_1, x_2, \ldots, x_k\}$, where $k \geq 2$. If $x_i^2 = 0$ for $1 \leq i \leq k$ and $x_i x_j = 0$ for $1 \leq i < j \leq k$, then the zero-divisor graph of R is threshold.*

Proof. Let $X = \sum_{i=1}^{k} a_i x_i \in V(\Gamma(R))$ and $Y = \sum_{i=1}^{k} b_i x_i \in V(\Gamma(R))$. Then $XY = 0$ and hence $\Gamma(R)$ is complete. Therefore it is threshold. □

Proposition 3.6. *Let M be the maximal ideal of R, and let M be generated by $\{x_1, x_2, \ldots, x_k\}$, where $k \geq 2$, such that $x_i x_j = 0$ for $1 \leq i < j \leq k$. If $x_1^{n-1} \neq 0$, $x_1^n = 0$ where $n \geq 3$ and $x_i^2 = 0$ for $2 \leq i \leq k$, then $\Gamma(R)$ is threshold.*

Proof. Using the definition of nested split graph, we prove the result. Set the level $h = \frac{n}{2}$ if n is even and $h = \frac{n-1}{2}$ otherwise.
First we define, for $1 \leq i \leq h-1$,

$$U_i = \left\{ \begin{array}{c} a_{i,1}x_1^i + a_{i,2}x_2 + \ldots + a_{i,n}x_n \mid a_{i,1} \text{ is unit element of } R, \text{ and} \\ a_{i,j} \text{ is either zero or a unit element of } R, \text{ for } 2 \leq j \leq n \end{array} \right\}$$

and if n is odd, define

$$U_h = \left\{ \begin{array}{c} a_{h,1}x_1^h + a_{h,2}x_2 + \ldots + a_{h,n}x_n \mid a_{h,1} \text{ is unit element of } R, \text{ and} \\ a_{h,j} \text{ is either zero or a unit element of } R, \text{ for } 2 \leq j \leq n \end{array} \right\}$$

and if n is even, define $U_h = \emptyset$.
Next we define,

$$V_1 = \left\{ \begin{array}{c} b_{1,1}x_1^{n-1} + b_{1,2}x_2 + \ldots + b_{1,n}x_n \mid b_{1,j} \text{ is either zero or unit} \\ \text{element of } R, \text{ for } 1 \le j \le n \end{array} \right\},$$

and for $2 \le i \le h$,

$$V_i = \left\{ \begin{array}{c} b_{i,1}x_1^{n-i} + b_{i,2}x_2 + \ldots + b_{i,n}x_n \mid b_{i,1} \text{ is unit element of } R, \text{ and} \\ b_{i,j} \text{ is either zero or unit element of } R, \text{ for } 2 \le j \le n \end{array} \right\}.$$

Then clearly, $\bigcup_{i=1}^{h} U_i$ is an independent set and $\bigcup_{i=1}^{h} V_i$ is a complete subgraph of $\Gamma(R)$. Also they satisfy the definition of nested split graph and hence the graph is threshold. □

Proposition 3.7. *If M is the maximal ideal generated by $\{x_1, x_2, \ldots, x_k\}$, where $k \ge 2$ such that $x_1x_2 = 0$, $x_1^2 \ne 0 \ne x_2^2$, then $\Gamma(R)$ is not threshold.*

Proof. Clearly, $x_1 \in \mathrm{Ann}(x_2) \setminus \mathrm{Ann}(x_1)$ and $x_2 \in \mathrm{Ann}(x_1) \setminus \mathrm{Ann}(x_2)$ and hence $\mathrm{Ann}(x_1)$ is not a subset of $\mathrm{Ann}(x_2)$ and $\mathrm{Ann}(x_2)$ is not a subset of $\mathrm{Ann}(x_1)$. Thus $\Gamma(R)$ is not threshold, by Theorem 3.3. □

4. The zero-divisor graphs of local rings with countable cardinality are universal

It is natural to wonder about embedding infinite graphs in zero-divisor graphs of infinite rings. Here we consider the countable case.

Our tool will be the *Rado graph*, or *countable random graph* $\mathcal{R}$: this was first explicitly constructed by Rado, but about the same time Erdős and Rényi proved that if a countable graph was selected by choosing edges independently with probability $\frac{1}{2}$ from the 2-subsets of a countably infinite set, the resulting graph is isomorphic to $\mathcal{R}$ with probability 1. Among many beautiful properties of this graph (for which we refer to [11]), we require the following:

- $\mathcal{R}$ is the unique countable graph having the property that, given any two finite disjoint sets U and V of vertices, there is a vertex z joined to every vertex in U and to none in V.
- Every finite or countable graph is embeddable in $\mathcal{R}$ as an induced subgraph.

We refer to [17] for terminology and results on Model Theory, in particular the Compactness and Löwenheim–Skolem theorems.

Theorem 4.1. *There is a countable local ring with unity having the property that every finite or countable graph is an induced subgraph of its zero-divisor graph.*

Proof. It suffices to show that the Rado graph $\mathcal{R}$ can be embedded in the zero-divisor graph of a countable ring, since every finite or countable graph is an induced subgraph of the Rado graph.

We give two proofs of this. The first is simple and direct, using the method we used in Theorem 2.9. The second is non-constructive but shows a technique which we hope will be of wider use.

First proof. Let F be a field (for simplicity the field with two elements). Let $R = F[X]/S$, where the set X of indeterminates is bijective with the vertex set of $\mathcal{R}$ (with x_i corresponding to vertex i), and S is the ideal generated by all homogeneous polynomials of degree 3 in these variables together with all products x_ix_j for which $\{i, j\}$ is an edge of $\mathcal{R}$. Just as in the proof of Theorem 2.9, the set $\{x_i + S : i \in V(\mathcal{R})\}$ induces a subgraph isomorphic to $\mathcal{R}$ of the zero-divisor graph of R.

The ring R has the properties that it is a local ring (though its maximal ideal is not finitely generated) and its automorphism group contains the automorphism group of $\mathcal{R}$.

Second proof. We use basic results from model theory. We take the first-order language of rings with unity together with an additional unary relation S. Now consider the following set Σ of first-order sentences:

(a) the axioms for a commutative ring with unity;
(b) the statement that every element of S is a (non-zero) zero-divisor;
(c) for each pair (m, n) of non-negative integers, the sentence stating that for any given $m + n$ elements $x_1, \ldots, x_m, y_1, \ldots, y_n$, all satisfying S, there exists an element z such that z satisfies S, that $x_iz = 0$ for $i = 1, \ldots, m$, and that $y_jz \ne 0$ for $j = 1, \ldots, n$.

We claim that any finite set of these sentences has a model. Any finite subset of the sentences in (c) are satisfied in some finite graph (taking “product zero” to mean “adjacent”), with S satisfied by the vertices used in the embedding: indeed, a sufficiently large finite random graph will have this property. By our earlier results, this finite graph is embeddable as an induced subgraph in the zero-divisor graph of some finite commutative ring with unity.

By the First-Order Compactness Theorem, the entire set Σ has a model R. This says that the set of ring elements satisfying S induces a subgraph isomorphic to Rado's graph $\mathcal{R}$ in the zero-divisor graph of R. (The sentences under (c) are first-order axioms for $\mathcal{R}$.) But $\mathcal{R}$ contains every finite or countable graph as an induced subgraph.

Now the downward Löwenheim–Skolem theorem guarantees that there is a countable ring whose zero-divisor graph contains $\mathcal{R}$, and hence all finite and countable graphs, as induced subgraphs. $\square$

Problem 2. Is there a local ring whose maximal ideal is finitely generated as an ideal and whose zero-divisor graph embeds the Rado graph as an induced subgraph?

Problem 3. Find the smallest number N (in terms of n and m) such that, if G is a graph with n vertices and m edges, then G is an induced subgraph of the zero divisor graph of a ring (commutative with unity) of order at most N.

Let $f(n,m)$ be this number. Our construction using Boolean rings shows that $N \le 2^p$, where p is the least number of points in a representation of G as an intersection graph. Moreover, we constructed an intersection representation with $p = m + a + b$, where a and b are the numbers of isolated vertices and edges in G. This gives an upper bound for $f(n,m)$.

Problem 4. Is this best possible?

We can ask similar questions for subclasses of rings (such as local rings), or for variants of the zero-divisor graph.

5. Other graphs from rings

There are several natural classes of graphs containing the threshold graphs. These include split graphs (defined earlier), chordal graphs (containing no induced cycle of length greater than 3), cographs (containing no induced path on four vertices) and perfect graphs (containing no induced odd cycle of length at least 5 or complement of one). For each of these classes $\mathcal{C}$, we can ask:

Problem 5. For which commutative rings with unity does the zero-divisor graph belong to $\mathcal{C}$?

Another very general problem is to examine the induced subgraphs of various generalizations of zero-divisor graphs, such as the extended zero-divisor graphs and the trace graphs [2,26–29].

As a contribution to this problem, here is an example of a kind of universality question that can be asked when we have graphs defined from algebraic structures where one is a subgraph of the other. The pattern for this theorem is [12, Theorem 5.9], the analogous result for the enhanced power graph and commuting graph of a group.

Let A be a commutative ring with unity. We define the zero-divisor graph of A to have as vertices all the non-zero elements of A, two vertices a and b joined if $ab = 0$. (This is not the usual definition since we don't restrict just to zero-divisors: vertices which are not zero-divisors are isolated. This doesn't affect our conclusion.) Given a positive integer m, we define the m-dimensional dot product graph to have vertices all the non-zero elements of A^m with two vertices $(a_1, a_2, \dots, a_m)$ and $(b_1, b_2, \dots, b_m)$ joined if $a.b = 0$, where

$$a \cdot b = a_1 b_1 + a_2 b_2 + \cdots + a_m b_m.$$

Note that A^m is a ring, with the product $*$ given by

$$a * b = (a_1 b_2, a_2 b_2, \dots, a_m b_m).$$

It is clear that $a * b = 0$ implies $a.b = 0$, so the zero-divisor graph of A^m is a spanning subgraph of the m-dimensional dot product graph of A.

We prove the following:

Theorem 5.1. *Take the complete graph on a finite set X, with the edges coloured red, green and blue in any manner whatever. Then there is a ring A and an embedding of X into A^2 such that*

- *the red edges are edges of the zero-divisor graph of A^2;*
- *the green edges are edges of the 2-dimensional dot product graph of A but not of the zero-divisor graph of A^2;*
- *the blue edges are not edges of the 2-dimensional dot product graph of A.*

Proof. First, by enlarging X by at most four points joined to all others with blue or green edges, we can assume that neither the blue nor the green subgraphs have isolated vertices or edges.

Let P be the set of blue or green edges. For each vertex $v \in X$, define a pair $(S(v), T(v))$ of subsets of P by the rule that $S(v)$ is the set of green edges containing v, while $T(v)$ is the set of blue or green edges containing v. The assumption in the previous paragraph shows that the map $\theta : v \mapsto (S(v), T(v))$ is one-to-one.

Now let A denote the Boolean ring on P. Then θ is an embedding of X into $A \times A \setminus \{0\}$. We claim that this has the required property.

- Suppose that $e = \{v, w\}$ is a red edge. Then $S(v) \cap S(w) = \emptyset$ and $T(v) \cap T(w) = \emptyset$; so $\theta(v) * \theta(w) = 0$, whence $\theta(v)$ and $\theta(w)$ are joined in the zero divisor graph of A^2.
- Suppose that $e = \{v, w\}$ is green. Then $S(v) \cap S(w) = \{e\}$ and $T(v) \cap T(w) = \{e\}$; so $\theta(v) * \theta(w) \neq 0$ but $\theta(v).\theta(w) = 0$. Thus $\theta(v)$ and $\theta(w)$ are joined in the dot product graph but not the zero divisor graph.
- Suppose that $e = \{v, w\}$ is blue. Then $S(v) \cap S(w) = \emptyset$ and $T(v) \cap T(w) = \{e\}$, so $\theta(v).\theta(w) = \{e\} \neq 0$. So $\theta(v)$ and $\theta(w)$ are not joined in the dot product graph.

The theorem is proved. □

Remark 2. This theorem has several consequences:

- By ignoring the distinction between green and blue, we have another construction showing the universality of the zero-divisor graphs of rings.
- By ignoring the distinction between red and green, we have shown the universality of the 2-dimensional dot product graphs of rings.
- By ignoring the distinction between red and blue, we have shown that the graphs obtained from the 2-dimensional dot product graph of A by deleting the edges of the zero-divisor graph of A^2 are universal.

The theorem suggests several questions:

Problem 6.

- Can we restrict the ring A to a special class such as local rings?
- Can we prove similar results for other pairs of graphs?
- Can we prove similar results for more than two graphs?

Declaration of competing interest

The authors declare no conflict of interest.

Data availability

No data was used for the research described in the article.

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